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A GENERALIZATION OF LIVINGSTON'S COEFFICIENT INEQUALITIES FOR FUNCTIONS WITH POSITIVE REAL PART

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ABSTRACT. For functions $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ holomorphic in the unit disk, satisfying $\operatorname{Re} p(z) > 0$, we generalize two inequalities proved by Livingston [10, 11] and simplify their proofs. One of our results states that $|p_n - wp_k p_{n-k}| \leq 2 \max\{1, |1 - 2w|\}$, $w \in \mathbb{C}$. Another result involves certain determinants whose entries are the coefficients p_n . Both results are sharp. As applications we provide a simple proof of a theorem of Brown [2] and various inequalities for the coefficients of holomorphic self-maps of the unit disk.

1. INTRODUCTION

Let \mathcal{P} denote the class of functions of the form $p(z) = 1 + p_1 z + p_2 z^2 + \dots$ which are analytic in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy $\operatorname{Re} p(z) > 0$ for all $z \in \mathbb{D}$. As early as 1911 Carathéodory proved that coefficients of functions in \mathcal{P} satisfy $|p_n| \leq 2$ (Theorem A below). Livingston [10] proved that $|p_n - p_k p_{n-k}| \leq 2$, for all $0 \leq k \leq n$. He used this inequality in his study of multivalent close-to-convex functions, while more applications were later found in [3], [9] and [12]. In this note we generalize this inequality by finding the sharp bound of $|p_n - wp_k p_{n-k}|$, $w \in \mathbb{C}$, in Theorem 1. In particular, the bound 2 is still valid whenever w lies in the disk $\{w : |1 - 2w| \leq 1\}$.

For $w \in \mathbb{C}$ and $p \in \mathcal{P}$ we define the $(k+1) \times (k+1)$ determinant

$$A_{k,n}(w) = \begin{vmatrix} p_{n+k} & p_{n+k-1} & p_{n+k-2} & \cdots & p_{n+1} & p_n \\ wp_1 & 1 & 0 & \cdots & 0 & 0 \\ wp_2 & wp_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ wp_{k-1} & wp_{k-2} & wp_{k-3} & \cdots & 1 & 0 \\ wp_k & wp_{k-1} & wp_{k-2} & \cdots & wp_1 & 1 \end{vmatrix}.$$

Livingston [11] defined this for $w = 1$ and proved that $|A_{k,n}(1)| \leq 2$. In Theorem 2 we find the sharp bound of $|A_{k,n}(w)|$ for all $w \in \mathbb{C}$. When no confusion arises we will suppress w and write $A_{k,n}$ for $A_{k,n}(w)$. Here are some examples of initial

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$A_{k,n}$'s:

$$\begin{aligned} A_{0,n} &= p_n, & A_{1,n} &= p_{n+1} - wp_1 p_n, \\ A_{2,n} &= p_{n+2} - wp_1 p_{n+1} - wp_2 p_n + w^2 p_1^2 p_n. \end{aligned}$$

In order to fix the notation let $n \in \mathbb{N}$ and denote by $U_n = \{e^{2k\pi i/n} : k = 1, 2, \dots, n\}$ the set of n -th roots of unity. For $n = 0$ we understand U_0 as $\mathbb{T} = \partial\mathbb{D}$. Also, for a set $E \subset \mathbb{C}$ and a number $a \in \mathbb{C}$ we write $aE = \{az : z \in E\}$.

The Herglotz representation [5, p.22] asserts that for every $p \in \mathcal{P}$ there is a unique probability measure μ supported on \mathbb{T} , such that

$$p(z) = \int_{\mathbb{T}} \frac{1 + \lambda z}{1 - \lambda z} d\mu(\lambda), \quad z \in \mathbb{D}.$$

We call μ the *Herglotz measure* of p and write $\text{supp}(\mu)$ for its support. One can readily see that the coefficients satisfy $p_n = 2 \int_{\mathbb{T}} \lambda^n d\mu(\lambda)$.

We now state Carathéodory's Theorem [14, p.41] and our two main theorems 1 and 2.

Theorem A. *If $p \in \mathcal{P}$ then $|p_n| \leq 2$ for all $n \geq 1$. For a fixed n , equality holds if and only if $\text{supp}(\mu) \subseteq e^{i\varphi}U_n$ for some $\varphi \in [0, 2\pi)$.*

Theorem 1. *If $p \in \mathcal{P}$ and $w \in \mathbb{C}$ then*

$$|p_n - wp_k p_{n-k}| \leq 2 \max\{1, |1 - 2w|\}$$

for all $1 \leq k \leq n - 1$.

Let μ be the Herglotz measure of p . In the case $|1 - 2w| < 1$, equality holds if and only if $p_k = 0$ and $\text{supp}(\mu) \subseteq e^{i\varphi}U_n$, for some $\varphi \in [0, 2\pi)$. In the case $|1 - 2w| > 1$, equality holds if and only if $\text{supp}(\mu) \subseteq e^{i\vartheta}U_k \cap e^{i\varphi}U_n$, for some $\vartheta, \varphi \in [0, 2\pi)$. In the case $|1 - 2w| = 1$, if $\text{supp}(\mu)$ consists of one point then equality holds.

Theorem 2. *If $p \in \mathcal{P}$ and $w \in \mathbb{C}$ then*

$$|A_{k,n}(w)| \leq 2 \max\{1, |1 - 2w|^k\}$$

for all $k \geq 0$ and $n \geq 1$.

Let μ be the Herglotz measure of p . In the case $|1 - 2w| < 1$, equality holds if and only if $\text{supp}(\mu) \subseteq e^{i\varphi}U_{n+k}$, for some $\varphi \in [0, 2\pi)$ and $p_1 = p_2 = \dots = p_k = 0$. In the case $|1 - 2w| \geq 1$, if $\text{supp}(\mu)$ consists of one point then equality holds.

The condition for equality in Theorem 1, in the case $|1 - 2w| = 1$, is far from being necessary. To illustrate this consider $w = 1$, $n = 2k$ and a Herglotz measure supported on two arbitrary points λ_1, λ_2 on \mathbb{T} having equal point masses, $1/2$ each. Then the coefficients of the corresponding function in \mathcal{P} are $p_j = \lambda_1^j + \lambda_2^j$ and one easily computes

$$|p_{2k} - p_k^2| = |\lambda_1^{2k} + \lambda_2^{2k} - (\lambda_1^k + \lambda_2^k)^2| = 2.$$

The complete characterization of equality when $|1 - 2w| = 1$ is given in Theorem 3. Note that in the special case where $w = 1$, the form of the extremal functions was not explicitly stated in [10].

Since the set $e^{i\vartheta}U_k \cap e^{i\varphi}U_n$ in Theorem 1 cannot be empty, as $\text{supp}(\mu) \neq \emptyset$, the number of points it contains is equal to the greatest common divisor of k and n .

Both Theorems 1 and 2 have a version for non-normalized functions $p(z) = \sum_{n=0}^{\infty} p_n z^n$ with positive real part. For such a function p , let $p_0 = x + iy$, ($x > 0$) and $q(z) = (p(z) - iy)/x$, which is obviously a function in \mathcal{P} . To this q , having coefficients $q_n = p_n/x$, we can apply Theorems 1 and 2. Then multiply both inequalities by $x/|p_0|$ and set $w x/p_0$ in place of w . What results is

$$\frac{p_n}{p_0} - w \frac{p_k p_{n-k}}{p_0^2} \leq 2 \frac{\text{Re } p_0}{|p_0|} \max \left\{ 1, 1 - \frac{2w \text{Re } p_0}{p_0} \right\}$$

and

$$|A_{k,n}| \leq 2 \frac{\text{Re } p_0}{|p_0|} \max \left\{ 1, 1 - \frac{2w \text{Re } p_0}{p_0} \right\}^k$$

for the modified $A_{k,n}$, having p_j/p_0 in place of p_j (for all j). Note that for $w = 1$ the two entries in the maximum are equal and what one gets is Livingston's original results.

An alternative proof for the inequality in Theorem 2 under the additional condition $n \geq k + 1$ can be given via the method of Delsarte and Genin [4]. Their approach relies on the observation that $A_{k,n}(1)$ is related to a truncation of the reciprocal of a function in \mathcal{P} . With the aid of Herglotz' formula they get a substantially simpler proof of Livingston's result. The proof, which will be presented in section 2, is an adaptation of their arguments to our case of $A_{k,n}(w)$, for any $w \in \mathbb{C}$.

Finally, we turn to a question raised by Goodman ([6, p.104]) about the sharp bound of $|p_{n+1} - p_n|$ for functions in \mathcal{P} with prescribed p_1 . Using extreme point theory, Brown [2] proves the following theorem.

Theorem B. *Let $p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k$ be in \mathcal{P} , $m, n \in \mathbb{N}$ and $\nu \in \mathbb{R}$. Then*

$$|e^{i\nu} p_{n+m} - p_n| \leq 2 \sqrt{2 - \text{Re}(e^{i\nu} p_m)}.$$

The result is sharp.

In this note we provide a simpler proof.

We proceed with section 2 where the proofs of Theorems 1, 2 and B are presented. In section 3 we carry out a detailed study of the equality case in a special case of Theorem 1, namely the case $|1 - 2w| = 1$. In section 4 we deduce a simple corollary of Theorems 1 and 2 for initial coefficients of self-maps of \mathbb{D} that fix the origin.

2. PROOFS OF THEOREMS 1, 2 AND B

Proof of Theorem 1. First we note that $|1 - 2w| \leq 1$ if and only if $|w|^2 \leq \operatorname{Re} w$. We compute

$$\begin{aligned}
|p_n - wp_k p_{n-k}| &= 2 \int_{\mathbb{T}} \lambda^n d\mu(\lambda) - 2wp_k \int_{\mathbb{T}} \lambda^{n-k} d\mu(\lambda) \\
&\leq 2 \int_{\mathbb{T}} |\lambda^n - wp_k \lambda^{n-k}| d\mu(\lambda) \\
&\leq 2 \left(\int_{\mathbb{T}} |\lambda^k - wp_k|^2 d\mu(\lambda) \right)^{1/2} \\
&= 2 \left(\int_{\mathbb{T}} 1 - 2\operatorname{Re}(wp_k \lambda^{-k}) + |wp_k|^2 d\mu(\lambda) \right)^{1/2} \\
&= 2 \left(1 - 2\operatorname{Re}(wp_k \overline{p_k}/2) + |wp_k|^2 \right)^{1/2} \\
&= 2 \left(1 + (|w|^2 - \operatorname{Re} w) |p_k|^2 \right)^{1/2} \\
&\leq 2 \max\{1, |1 - 2w|\}.
\end{aligned}$$

Here we used the triangle and Cauchy-Schwarz inequalities. At the last step, in the case $|1 - 2w| > 1$, we made use of Theorem A.

Now suppose that equality holds. If $|1 - 2w| < 1$ then equality in the last of the above inequalities yields $p_k = 0$. Hence the second term in $p_n - wp_k p_{n-k}$ vanishes and we have $|p_n| = 2$. By Theorem A, $\operatorname{supp}(\mu) \subseteq e^{i\varphi} U_n$ for some $\varphi \in [0, 2\pi)$.

In the case $|1 - 2w| > 1$, the last inequality yields $|p_k| = 2$. Hence $\operatorname{supp}(\mu) \subseteq e^{i\vartheta} U_k$ for some $\vartheta \in [0, 2\pi)$. Now $p_k = 2e^{ik\vartheta}$ and

$$p_{n-k} = 2 \int_{\mathbb{T}} \lambda^{n-k} d\mu(\lambda) = 2e^{-ik\vartheta} \int_{\mathbb{T}} \lambda^n d\mu(\lambda) = e^{-ik\vartheta} p_n.$$

Hence $2|1 - 2w| = |p_n - 2wp_n|$, which implies that $|p_n| = 2$. Again by Theorem A we have $\operatorname{supp}(\mu) \subseteq e^{i\varphi} U_n$ for some $\varphi \in [0, 2\pi)$ and thus $\operatorname{supp}(\mu)$ must form a subset of the intersection $e^{i\vartheta} U_k \cap e^{i\varphi} U_n$.

It is elementary to check that in all three cases the conditions are sufficient for equality. \square

Proof of Theorem 2. Let $n \geq 1$ and $w \in \mathbb{C}$ be fixed. The case $k = 0$ follows from Theorem A. For $k \geq 1$ we define

$$Q_{k,n}(\lambda) = \begin{vmatrix} \lambda^{n+k-1} & p_{n+k-1} & p_{n+k-2} & \cdots & p_n \\ w & 1 & 0 & \cdots & 0 \\ w\lambda & wp_1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w\lambda^{k-1} & wp_{k-1} & wp_{k-2} & \cdots & 1 \end{vmatrix}.$$

Expanding $A_{k,n}$ along the first column, using the Herglotz formula and the linearity of the integral, and finally putting the determinant back together, we get $A_{k,n} = 2 \int_{\mathbb{T}} \lambda Q_{k,n}(\lambda) d\mu(\lambda)$.

We will now show by induction that

$$\int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda) \leq \max\{1, |1 - 2w|^{2k}\} \quad (1)$$

for all $k \geq 1$. Then the desired inequality will follow since

$$\begin{aligned} |A_{k,n}| &\leq 2 \int_{\mathbb{T}} |Q_{k,n}(\lambda)| d\mu(\lambda) \\ &\leq 2 \left(\int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda) \right)^{1/2} \\ &\leq 2 \max\{1, |1 - 2w|^k\}, \end{aligned} \quad (2)$$

by the triangle and Cauchy-Schwarz inequalities.

We first prove (1) for $k = 1$. (Recall that $|1 - 2w| < 1$ iff $|w|^2 < \operatorname{Re} w$.)

$$\begin{aligned} \int_{\mathbb{T}} |Q_{1,n}(\lambda)|^2 d\mu(\lambda) &= \int_{\mathbb{T}} 1 + |wp_n|^2 - 2\operatorname{Re}(wp_n\lambda^{-n}) d\mu(\lambda) \\ &= 1 + (|w|^2 - \operatorname{Re} w)|p_n|^2 \\ &\leq \max\{1, |1 - 2w|^2\}. \end{aligned} \quad (3)$$

Next, let us assume that (1) holds for k and let us prove it for $k + 1$ instead of k . Expanding $Q_{j,n}(\lambda)$ along the second row it is not difficult to see that

$$\begin{aligned} Q_{j,n}(\lambda) &= \begin{array}{cccc|cccc} \lambda^{n+j-1} & p_{n+j-2} & \cdots & p_n & & p_{n+j-1} & p_{n+j-2} & \cdots & p_n \\ w\lambda & 1 & \cdots & 0 & & wp_1 & 1 & \cdots & 0 \\ w\lambda^2 & wp_1 & \cdots & 0 & -w & wp_2 & wp_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\ w\lambda^{j-1} & wp_{j-2} & \cdots & 1 & & wp_{j-1} & wp_{j-2} & \cdots & 1 \end{array} \\ &= \lambda Q_{j-1,n}(\lambda) - w A_{j-1,n}. \end{aligned}$$

For $j = k + 1$ we have $Q_{k+1,n}(\lambda) = \lambda Q_{k,n}(\lambda) - w A_{k,n}$. Hence

$$\begin{aligned} \int_{\mathbb{T}} |Q_{k+1,n}(\lambda)|^2 d\mu(\lambda) &= \\ &= \int_{\mathbb{T}} \left[|Q_{k,n}(\lambda)|^2 - 2\operatorname{Re} \left(w \overline{\lambda Q_{k,n}(\lambda)} A_{k,n} \right) \right] d\mu(\lambda) + |w|^2 |A_{k,n}|^2 \\ &= \int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda) + (|w|^2 - \operatorname{Re} w) |A_{k,n}|^2. \end{aligned} \quad (4)$$

We distinguish two cases. If $|1 - 2w| < 1$ then (4) and (1) show that

$$\int_{\mathbb{T}} |Q_{k+1,n}(\lambda)|^2 d\mu(\lambda) \leq \int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda) \leq 1. \quad (5)$$

For the case $|1-2w| \geq 1$, we make a further use of the Cauchy-Schwarz inequality to obtain $|A_{k,n}|^2 \leq 4 \int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda)$. Now by (4) and (1) we get that

$$\begin{aligned} \int_{\mathbb{T}} |Q_{k+1,n}(\lambda)|^2 d\mu(\lambda) &\leq (1 + 4|w|^2 - 4\operatorname{Re} w) \int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda) \\ &\leq |1-2w|^2 |1-2w|^{2k} \\ &= |1-2w|^{2k+2}. \end{aligned}$$

Hence (1) has been proved for all $k \geq 1$.

We now turn to the case of equality. Suppose that $|1-2w| < 1$ and $|A_{k,n}| = 2$. Then inequalities (2) become equalities and in particular $\int_{\mathbb{T}} |Q_{k,n}(\lambda)|^2 d\mu(\lambda) = 1$. The inductive step (5) shows that $\int_{\mathbb{T}} |Q_{j,n}(\lambda)|^2 d\mu(\lambda) = 1$ for all $j = 1, 2, \dots, k$. This is true in particular for $j = 1$, which by (3) implies that $p_n = 0$. This in turn is easily seen to imply that $A_{k,n} = A_{k-1,n+1}$. Hence we may repeat the above argument to get that $\int_{\mathbb{T}} |Q_{j,n+1}(\lambda)|^2 d\mu(\lambda) = 1$ for all $j = 1, 2, \dots, k-1$. Again from $j = 1$ we get by (3) that $p_{n+1} = 0$. We repeat this argument until we get $p_n = p_{n+1} = \dots = p_{n+k-1} = 0$. Now $A_{k,n} = A_{0,n+k} = p_{n+k}$ is a number of modulus 2 and therefore Theorem A yields $\operatorname{supp}(\mu) \subseteq e^{i\varphi} U_{n+k}$, for some $\varphi \in [0, 2\pi)$. Finally, for all $j = 1, 2, \dots, k$ we have

$$p_j = 2 \int_{\mathbb{T}} \lambda^j d\mu(\lambda) = 2e^{i(n+k)\varphi} \int_{\mathbb{T}} \lambda^{j-n-k} d\mu(\lambda) = e^{i(n+k)\varphi} \overline{p_{n+k-j}} = 0.$$

In both cases the sufficiency for equality is easy to verify. \square

Alternative proof of Theorem 2 (case $n \geq k+1$). Let $w \in \mathbb{C}$ be fixed. The case $k = 0$ follows from Theorem A. Let $k \geq 1$ and consider the perturbation

$$p^*(z) = 1 + w(p_1 z + \dots + p_k z^k) + p_{k+1} z^{k+1} + \dots$$

Let $Q_k(z) = 1 + q_1 z + \dots + q_k z^k$ be the k^{th} partial sum of $(p^*)^{-1}$, the reciprocal of p^* . We define $v_k(z) = \sum_{m=0}^{\infty} v_{k,m} z^m$, analytic at the origin, via the identity

$$Q_k(z)p^*(z) = 1 + 2z^{k+1}v_k(z).$$

Computing the coefficient of z^{k+m+1} , for $m \geq k$, we get that

$$2v_{k,m} = \sum_{j=0}^k q_j p_{k+m+1-j}. \quad (6)$$

Note that for $k_1 \neq k_2$ the coefficients q_j coincide for $1 \leq j \leq \min\{k_1, k_2\}$, hence formula (6) readily implies that

$$2v_{k,m} = q_k p_{m+1} + 2v_{k-1,m+1}. \quad (7)$$

We now proceed with induction on $k \geq 1$ to prove that

$$2v_{k,m} = A_{k,m+1}(w), \quad \text{for all } m \geq k. \quad (8)$$

For $k = 1$ it is easy to verify that $2v_{1,m} = p_{m+2} - wp_1 p_{m+1} = A_{1,m+1}$ for all $m \geq 1$.

Next we suppose that (8) holds for some k . We shall prove it for $k+1$ instead of k . Expanding with respect to the last column we see that

$$\begin{aligned}
 A_{k+1,m+1} &= A_{k,m+2} + p_{m+1}(-1)^{k+1} \begin{vmatrix} wp_1 & 1 & \dots & 0 \\ wp_2 & wp_1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ wp_k & wp_{k-1} & \dots & 1 \\ wp_{k+1} & wp_k & \dots & wp_1 \end{vmatrix} \\
 &= A_{k,m+2} + p_{m+1}q_{k+1},
 \end{aligned}$$

where we made use of Wronski's formula [7, p.17] for the coefficients of the reciprocal of a power series. Therefore by (7) we get that

$$A_{k+1,m+1} = 2v_{k,m+1} + p_{m+1}q_{k+1} = 2v_{k+1,m}$$

for $m \geq k+1$. Thus (8) has been proved. We set $m = n-1$ and write $A_{k,n}(w) = 2v_{k,n-1}$, for $n \geq k+1$.

We proceed as in [4] using the Herglotz formula in (6) and the Cauchy-Schwarz inequality to get

$$|v_{k,n-1}|^2 = \left| \int_{\mathbb{T}} \lambda^{k+n} Q_k(\bar{\lambda}) d\mu(\lambda) \right|^2 \leq \int_{\mathbb{T}} |Q_k(\bar{\lambda})|^2 d\mu(\lambda).$$

Now, we show that

$$\int_{\mathbb{T}} |Q_k(\bar{\lambda})|^2 d\mu(\lambda) \leq \max\{1, |1-2w|^{2k}\} \quad (9)$$

by induction on $k \geq 1$.

For $k=1$ we compute

$$\int_{\mathbb{T}} |Q_1(\bar{\lambda})|^2 d\mu(\lambda) = 1 + |p_1|^2(|w|^2 - \operatorname{Re} w) \leq \max\{1, |1-2w|^2\}.$$

Now we suppose that (9) is true for k . We shall prove it for $k+1$ instead of k . We compute

$$\begin{aligned}
 \int_{\mathbb{T}} |Q_{k+1}(\bar{\lambda})|^2 d\mu(\lambda) &= \int_{\mathbb{T}} \sum_{j,m=0}^{k+1} q_j \bar{q}_m \lambda^{m-j} d\mu(\lambda) \\
 &= \sum_{j=0}^{k+1} |q_j|^2 + \operatorname{Re} \left(\sum_{j < m} q_j \bar{q}_m p_{m-j} \right),
 \end{aligned}$$

where $j = 0, 1, \dots, k$ and $m = 1, 2, \dots, k+1$ at the last summation. Therefore

$$\begin{aligned}
 \int_{\mathbb{T}} |Q_{k+1}(\bar{\lambda})|^2 d\mu(\lambda) &= \int_{\mathbb{T}} |Q_k(\bar{\lambda})|^2 d\mu(\lambda) + |q_{k+1}|^2 + \operatorname{Re} \left(\bar{q}_{k+1} \sum_{j=0}^k q_j p_{k+1-j} \right) \\
 &= \int_{\mathbb{T}} |Q_k(\bar{\lambda})|^2 d\mu(\lambda) + (|w|^2 - \operatorname{Re} w) \sum_{j=0}^k q_j p_{k+1-j}^2,
 \end{aligned}$$

since $q_{k+1} + w \sum_{j=0}^k q_j p_{k+1-j} = 0$ by the definition of Q_{k+1} . If $|1 - 2w| < 1$ then

$$\int_{\mathbb{T}} |Q_{k+1}(\bar{\lambda})|^2 d\mu(\lambda) \leq \int_{\mathbb{T}} |Q_k(\bar{\lambda})|^2 d\mu(\lambda) \leq 1$$

and we are done. If $|1 - 2w| \geq 1$ then we make a further use of the Herglotz formula to get

$$\sum_{j=0}^k q_j p_{k+1-j} = 2 \int_{\mathbb{T}} \lambda^{k+1} Q_k(\bar{\lambda}) d\mu(\lambda) \leq 4 \int_{\mathbb{T}} |Q_k(\bar{\lambda})|^2 d\mu(\lambda).$$

Hence

$$\int_{\mathbb{T}} |Q_{k+1}(\bar{\lambda})|^2 d\mu(\lambda) \leq (1 + 4|w|^2 - 4\operatorname{Re} w) \int_{\mathbb{T}} |Q_k(\bar{\lambda})|^2 d\mu(\lambda) \leq |1 - 2w|^{2k+2}$$

and (9) has been established.

It is not clear how one can make the above argument work when $n \leq k$. \square

Proof of Theorem B. The proof relies on a further generalization of Theorem 1. Let $w \in \mathbb{C}$ and compute

$$\begin{aligned} |p_{n+m} - wp_n| &\leq 2 \int_{\mathbb{T}} |\lambda^m - w| d\mu(\lambda) \\ &\leq 2 \left(\int_{\mathbb{T}} |\lambda^m - w|^2 d\mu(\lambda) \right)^{1/2} \\ &= 2 \left(1 + |w|^2 - \operatorname{Re}(\bar{w} p_m) \right)^{1/2}. \end{aligned}$$

Choosing $w = e^{-i\nu}$ we obtain the desired inequality. Equality evidently holds for the half-plane function $\frac{1+z}{1-z}$. \square

3. CASE OF EQUALITY FOR THEOREM 1

We now consider the case of equality for Theorem 1 when $|1 - 2w| = 1$. Since our result is more general than Livingston's, it is not surprising that the conditions for equality and their proofs are lengthy.

Theorem 3. *Let $p \in \mathcal{P}$, μ be its representing Herglotz measure, $1 \leq k \leq n - 1$ and $w = (1 + e^{i\vartheta})/2$ with $|\vartheta| < \pi$. Then $p_n - wp_k p_{n-k} = 2e^{ic}$ for some c in $[0, 2\pi)$ if and only if either*

- (i) $p_k = 0$ and $\operatorname{supp}(\mu) \subseteq e^{ic/n} U_n$; or
- (ii) $p_k \neq 0$ and

$$\operatorname{supp}(\mu) \subseteq (e^{i\frac{\psi}{n-2k}} U_{n-2k} \cap e^{i(\frac{\varphi}{k} + \frac{c-\psi}{2k})} U_k) \cup (e^{i\frac{\psi}{n-2k}} U_{n-2k} \cap e^{i(\frac{\pi-\varphi}{k} + \frac{c-\psi}{2k})} U_k) \quad (10)$$

for some ψ in $[0, 2\pi)$ and $|\varphi| \leq \pi/2$. Except for the degenerate case where the support of μ consists of only one point, the total mass of the measure in each of the two sets of the union is (respectively) equal to

$$\frac{1}{2} \left(1 + \frac{\sin \vartheta}{1 + \cos \vartheta} \tan \varphi \right) \quad \text{and} \quad \frac{1}{2} \left(1 - \frac{\sin \vartheta}{1 + \cos \vartheta} \tan \varphi \right).$$

Proof. We observe that without loss of generality we may assume that $2k \leq n$, since otherwise, we may set $m = n - k$ and see that the functional $p_n - wp_k p_{n-k}$ remains unchanged while the new pair of integers (m, n) satisfies $2m < n$. Therefore the second condition makes sense.

We will prove the necessity of the two conditions, since the sufficiency is elementary, although laborious in the case (ii).

We assume that $c = 0$. Having proved the assertion in this case we apply it to the rotated function $p(e^{-ic/n}z)$ in order to obtain the general result.

Retracing the equalities in the proof of Theorem 1 we see that

$$\lambda^n - wp_k \lambda^{n-k} = 1, \quad \lambda \in \text{supp}(\mu), \quad (11)$$

since equality in the triangle inequality yields constant argument and equality in the Cauchy-Schwarz inequality yields constant modulus. Formula (11) is equivalent to $\lambda^k - wp_k = \lambda^{k-n}$, which we integrate with respect to μ in order to get

$$p_{n-k} = (1 - 2\bar{w})\overline{p_k} = -e^{-i\vartheta}\overline{p_k}. \quad (12)$$

It is now evident that if one of the coefficients p_k, p_{n-k} is zero, then both of them are zero. If $p_k = 0$, case (i) clearly follows from Theorem A, but it can also be seen from (11) which becomes $\lambda^n = 1$.

Suppose that $p_k \neq 0$. In order to prove condition (ii) we begin with the additional assumption that $n = 2k$. Equation (11) is then equivalent to $\lambda^k - \lambda^{-k} = wp_k$. From this we deduce that $\text{Im } \lambda^k$ is constant on $\text{supp}(\mu)$ and that $\text{Re}(wp_k) = 0$. The former implies that for some $\zeta = e^{i\varphi}$ (we may assume that $|\varphi| \leq \pi/2$), the support of μ consists of the k -th roots of ζ and $-\bar{\zeta}$, having point masses, say, m_j and m_j^* , respectively, $1 \leq j \leq k$. In other words

$$\text{supp}(\mu) \subseteq e^{i\frac{\varphi}{k}}U_k \cup e^{i\frac{\pi-\varphi}{k}}U_k, \quad (13)$$

with total mass in each of the two sets of the union $M = \sum_{j=1}^k m_j$ and $M^* = \sum_{j=1}^k m_j^*$, respectively. The fact that μ is a probability measure means that $M + M^* = 1$. Next, we easily see that $p_k = 2 \int_{\mathbb{T}} \lambda^k d\mu(\lambda) = 2(\zeta M - \bar{\zeta} M^*) = 2((\zeta + \bar{\zeta})M - \bar{\zeta})$. Hence

$$\begin{aligned} 0 &= \text{Re}(wp_k) = \text{Re} \left[(1 + e^{i\vartheta}) ((\zeta + \bar{\zeta})M - \bar{\zeta}) \right] \\ &= (1 + \cos \vartheta) \cos \varphi (2M - 1) - \sin \varphi \sin \vartheta. \end{aligned} \quad (14)$$

If $|\varphi| = \pi/2$, i.e. if ζ is either i or $-i$, then ζ and $-\bar{\zeta}$ coincide and therefore we may choose to divide the total mass of μ into two parts in any possible way, and in particular as asserted in (ii). Otherwise, if $|\varphi| < \pi/2$, equation (14) implies

$$M = \frac{1}{2} \left(1 + \frac{\sin \vartheta}{1 + \cos \vartheta} \tan \varphi \right).$$

Hence, to see that (10) has been proved, recall that we regard U_0 as \mathbb{T} and therefore, since $n = 2k$, we may choose ψ freely. The choice $\psi = 0$ completes the proof of (10) in case $n = 2k$.

For the remaining case $n > 2k$ in the case (ii), we repeat the arguments used to prove (11) to get

$$\lambda^n - wp_{n-k}\lambda^k = 1, \quad \lambda \in \text{supp}(\mu). \quad (15)$$

A combination of (11) and (15) shows that $p_k\lambda^{n-k} = p_{n-k}\lambda^k$. Hence, by (12),

$$\lambda^{n-2k} = -e^{-i\vartheta} \overline{p_k}/p_k, \quad \lambda \in \text{supp}(\mu).$$

This yields

$$\text{supp}(\mu) \subseteq e^{i\frac{t}{n-2k}} U_{n-2k}, \quad (16)$$

for some $t \in [0, 2\pi)$. Hence $p_n = e^{it}p_{2k}$, $p_{n-k} = e^{it}p_k$ and $2 = p_n - wp_k p_{n-k} = e^{it}(p_{2k} - wp_k^2)$. It follows that the function $p(e^{it/2k}z)$ must satisfy condition (13) and, therefore, $p(z)$ satisfies the corresponding rotation of (13). Together with (16) this is

$$\text{supp}(\mu) \subseteq (e^{i\frac{t}{n-2k}} U_{n-2k} \cap e^{i(\frac{\varphi}{k} - \frac{t}{2k})} U_k) \cup (e^{i\frac{t}{n-2k}} U_{n-2k} \cap e^{i(\frac{\pi-\varphi}{k} - \frac{t}{2k})} U_k),$$

which is (10) in case $c = 0$. If $c \neq 0$ then a further rotation by $e^{ic/n}$ and the substitution $\psi = t + c(1 - 2k/n)$ yield (10). \square

We wish to remark that in the special case where $w = 1$, Theorem 3 has the following simpler form (note that this was not explicitly stated in [10]): It holds that $|p_n - p_k p_{n-k}| = 2$ if and only if either

- (i) $p_k = 0$ and $\text{supp}(\mu) \subseteq e^{i\varphi} U_n$ for some φ in $[0, 2\pi)$; or
- (ii) $p_k \neq 0$ and

$$\text{supp}(\mu) \subseteq (e^{i\varphi} U_{n-2k} \cap e^{i\vartheta_1} U_k) \cup (e^{i\varphi} U_{n-2k} \cap e^{i\vartheta_2} U_k)$$

for some φ, ϑ_1 and ϑ_2 in $[0, 2\pi)$. Except for the degenerate case where the support of μ consists of only one point, the total mass of the measure in each of the two sets of the union is equal to $1/2$.

4. APPLICATION TO THE SELF-MAPS OF \mathbb{D}

There is a close connection between the class \mathcal{P} and self-maps of \mathbb{D} via conformal maps of \mathbb{D} to the right half plane, namely, $p = \frac{1+\varphi}{1-\varphi}$ is in \mathcal{P} for a function φ analytic in \mathbb{D} if and only if $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ and $\varphi(0) = 0$. Writing $\varphi(z) = \sum_{n=1}^{\infty} a_n z^n$ we may relate the first few coefficients of the two functions by

$$\begin{aligned} p_1 &= 2a_1, & p_2 &= 2(a_2 + a_1^2), & p_3 &= 2(a_3 + 2a_1a_2 + a_1^3), \\ p_4 &= 2(a_4 + 2a_1a_3 + a_2^2 + 3a_1^2a_2 + a_1^4). \end{aligned}$$

For functions φ of this form, the Schwarz lemma states that $|a_1| \leq 1$ while the Schwarz-Pick lemma says that $|a_2| \leq 1 - |a_1|^2$. One then easily computes

$$|a_2 + \lambda a_1^2| \leq |a_2| + |\lambda||a_1|^2 \leq 1 + (|\lambda| - 1)|a_1|^2 \leq \max\{1, |\lambda|\}.$$

(See [8] for this calculation and an application of it.) The same inequality can be obtained from our Theorem 1 with $\lambda = 1 - 2w$ and $n = k + 1 = 2$.

For higher order coefficients one has F.W. Wiener's generalization of the Schwarz-Pick lemma $|a_n| \leq 1 - |a_1|^2$ (see [1] or problem 9 in p.172 of [13]). However, even

if we use this inequality, it does not seem easy to get the following corollary in a different way, without applying our Theorems 1 and 2.

Corollary. *If $\varphi : \mathbb{D} \rightarrow \mathbb{D}$ is holomorphic, $\varphi(0) = 0$ and $\lambda \in \mathbb{C}$ then*

$$|a_3 + (1 + \lambda)a_1a_2 + \lambda a_1^3| \leq \max\{1, |\lambda|\} \quad (17)$$

$$|a_3 + 2\lambda a_1a_2 + \lambda^2 a_1^3| \leq \max\{1, |\lambda|^2\} \quad (18)$$

and

$$|a_4 + (1 + \lambda)a_1a_3 + a_2^2 + (1 + 2\lambda)a_1^2a_2 + \lambda a_1^4| \leq \max\{1, |\lambda|\} \quad (19)$$

$$|a_4 + 2a_1a_3 + \lambda a_2^2 + (1 + 2\lambda)a_1^2a_2 + \lambda a_1^4| \leq \max\{1, |\lambda|\} \quad (20)$$

$$|a_4 + (1 + \lambda)a_1a_3 + \lambda a_2^2 + \lambda(2 + \lambda)a_1^2a_2 + \lambda^2 a_1^4| \leq \max\{1, |\lambda|^2\} \quad (21)$$

$$|a_4 + 2\lambda a_1a_3 + \lambda a_2^2 + 3\lambda^2 a_1^2a_2 + \lambda^3 a_1^4| \leq \max\{1, |\lambda|^3\}. \quad (22)$$

Proof. Set $\lambda = 1 - 2w$ and apply Theorem 1 with $n = k + 2 = 3$ to get (17), with $n = k + 3 = 4$ to get (19) and with $n = k + 2 = 4$ to get (20). Apply Theorem 2 with $k = n + 1 = 2$ to get (18), with $k = n = 2$ to get (21) and with $k = n + 2 = 3$ to get (22). \square

To the best of our knowledge, these inequalities appear to be new.

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