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A note on generalized laplacians and minimal surfaces

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Abstract

In these notes we give an interdisciplinary result which links the geometric concept of minimal surfaces with generalized harmonic functions.

Introduction

Let f be a locally integrable function defined on an open domain of the euclidean space \mathbb{R}^n , its generalized laplacian $\widehat{\Delta}f$ is given by the following limit (provided it exists):

$$\widehat{\Delta}f(x) := \lim_{r \rightarrow 0^+} \frac{2(n+2)}{r^2} \int_{B_r^{(n)}(x)} (f(y) - f(x)) dy.$$

Here the symbol $\int_{B_r^{(n)}(x)}$ denotes the average on the ball $B_r^{(n)}(x) \subset \mathbb{R}^n$ centered at x and with radius r ; $dy = d\mu_n(y)$ is the Lebegue measure in \mathbb{R}^n and $\mu_n(A) = |A| = \text{vol}(A)$ is the measure of the measurable set A .

It is well known ([1], [2]) that a continuous generalized harmonic function (i.e. $\widehat{\Delta}f(x) = 0$ for every x) must be smooth and, therefore, harmonic in the ordinary sense: $\Delta f = 0$.

There are, however, discontinuous functions which are harmonic in the generalized sense. An example is given by:

$$f(x', x_n) = \begin{cases} \alpha_+ & \text{if } x_n > 0 \\ \frac{1}{2}(\alpha_+ + \alpha_-), & \text{if } x_n = 0 \\ \alpha_- & \text{if } x_n < 0 \end{cases}$$

which is clearly discontinuous when $\alpha_+ \neq \alpha_-$ and, nevertheless, satisfies $\widehat{\Delta}f(x) = 0$ everywhere.

Suppose that S is a C^1 -hypersurface separating the domain Ω in two non empty components: $\Omega = \Omega^+ \cup \Omega^- \cup S$,

$$S = \partial\Omega^+ \cap \Omega = \partial\Omega^- \cap \Omega.$$

Let the function f_S be equal to α_+ inside Ω^+ , α_- inside Ω^- and $\frac{1}{2}(\alpha_+ + \alpha_-)$ in S .

The main purpose of this note is to give a proof of the following:

Theorem 1. *The function f_S (with $\alpha_+ \neq \alpha_-$) is generalized harmonic if and only if S is minimal.*

In the proof we will make use of the modern notion of viscosity solution of uniformly elliptic equations. Namely, we will show that if $\widehat{\Delta}f_S = 0$, then, locally, S will be given as the graph of a viscosity solution of the minimal surface equation and, therefore, it has to be smooth. We shall use also several well-known properties of minimal surfaces and elliptic equations for which [3], [4] and [5] are appropriate references.

A basic calculus lemma

Let S be a smooth (C^4) hypersurface in \mathbb{R}^n with unit normal vector field $\nu(x)$. Given $x \in S$ and $r > 0$, small enough, the ball $B_r^{(n)}(x)$ is separated by S in two connected components, $S_r^+(x)$, $S_r^-(x)$, where $S_r^+(x)$ (respectively $S_r^-(x)$) consists of the points inside $B_r^{(n)}(x)$ which are placed above (respect. below) S in the given normal direction.

Then we have:

Lemma 2. *As $r \rightarrow 0^+$,*

$$\text{vol}(S_r^+(x)) - \text{vol}(S_r^-(x)) = -c_n H_S(x) \cdot r^{n+1} + O(r^{n+3}).$$

Here $H_S(x)$ denotes the mean curvature of S at the point x and $c_n > 0$ is a universal constant that only depends on the dimension n .

Proof. Without loss of generality we can assume that $x = 0$ is the origin of a coordinate system such that the tangent space of S at x is horizontal, i.e., the normal vector is $\nu(0) = (0, \dots, 0, 1)$. Hence, near $x = 0$, S is the graph of a smooth (C^4) function $x_n = \varphi(x_1, \dots, x_{n-1})$ satisfying:

1. $\varphi(0) = 0$.
2. $\nabla \varphi(0) = 0$.

Then, inside the cylinder $B_r^{(n-1)}(0) \times \mathbb{R}$, r small enough, we have the inclusion

$$S \cap \left(B_r^{(n-1)}(0) \times \mathbb{R} \right) \subset \{x \in \mathbb{R}^n : |x_n| \leq c_1 r^2\}$$

for a positive constant c_1 depending upon the size of the second derivatives of φ .

An elementary computation shows that the vertical projection of $S \cap B_r^{(n)}(0)$ onto $B_r^{(n-1)}(0)$ must contain the ball

$$B_{r-c_2 r^3}^{(n-1)}(0)$$

for a fixed constant c_2 .

Let

$$\begin{aligned} D_r^+ &= B_r^{(n)}(0) \cap \Omega^+ \cap \{|x_n| \leq c_1 r^2\}, \\ D_r^- &= B_r^{(n)}(0) \cap \Omega^- \cap \{|x_n| \leq c_1 r^2\}. \end{aligned}$$

Then, since S is contained in the strip, by symmetry we get the first equality

$$\begin{aligned} \text{vol}(S_r^+(0)) - \text{vol}(S_r^-(0)) &= \text{vol}(D_r^+) - \text{vol}(D_r^-) \\ &= \int_{B_{r-c_2 r^3}^{(n-1)}(0)} (c_1 r^2 - \varphi(x)) \, dx - \int_{B_{r-c_2 r^3}^{(n-1)}(0)} (c_1 r^2 + \varphi(x)) \, dx + I \end{aligned}$$

where I is the correction of restricting the domain of integration.

A direct computation of the term I shows that

$$|I| \lesssim r^2 \cdot (r^{n-1} - (r - c_2 r^3)^{n-1}) \lesssim r^{n+3} = O(r^{n+3}).$$

Hence

$$\text{vol}(S_r^+(0)) - \text{vol}(S_r^-(0)) = -2 \int_{B_{r-c_2 r^3}^{(n-1)}(0)} \varphi(x) \, dx + O(r^{n+3}).$$

With another computation with respect to the volume of the cylinders, we obtain that

$$\left| \int_{B_r^{(n-1)}(0)} \varphi(x) dx - \int_{B_{r-c_2r^3}^{(n-1)}(0)} \varphi(x) dx \right| \lesssim r^{n+3} = O(r^{n+3}).$$

Then Taylor's expansion yields

$$\int_{B_r^{(n-1)}(0)} \varphi(y) dy = \frac{1}{2(n+1)} \Delta \varphi(0) \cdot r^2 + O(r^4).$$

This allows us to finish the proof of the lemma, because we know that

$$H_S(x', \varphi(x')) = \frac{1}{n-1} \operatorname{div} \left(\frac{\nabla \varphi}{\sqrt{1 + |\nabla \varphi|^2}} \right) (x')$$

And since $\nabla \varphi(0) = 0$, we have

$$H_S(0) = \frac{1}{n-1} \Delta \varphi(0).$$

□

Proof of Theorem 1

First, without loss of generality, one can assume that $\alpha_+ = +1$ and $\alpha_- = -1$. Next, let us observe that one of the two implications of the theorem follows immediately: namely if S is minimal and C^1 then, by the classical theorem of de Giorgi-Nash, S has to be smooth and we can apply Lemma 2 to observe that at any point $x \in S$ we have:

$$\begin{aligned} \widehat{\Delta} f(x) &= \lim_{r \rightarrow 0^+} \frac{2(n+2)}{r^2} \int_{B_r^{(n)}(x)} (f(y) - 0) dy \\ &= \lim_{r \rightarrow 0^+} \frac{2(n+2)}{|B_r^{(n)}(x)| r^2} [\operatorname{vol}(S_r^+(x)) - \operatorname{vol}(S_r^-(x))] \\ &= 0, \end{aligned}$$

because of the minimality condition $H_S(x) = 0$.

Note, that a similar argument with Lemma 2 also works to prove that f_S being generalized harmonic implies that S is minimal. Therefore, to finish the proof we just need to prove the regularity (C^4) of S .

To continue the proof let us recall now that a real continuous function F defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n \times M^n$, where M^n denotes the vector space of $n \times n$ symmetric matrices, yields an elliptic equation $F(x, u, Du, D^2u) = 0$ if

$$F(x, u, \eta, \delta) \leq F(x, u, \eta, \delta + \sigma)$$

for all $(x, u, \eta, \delta) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times M^n$ and $\sigma \in M^n$ non-negative.

The elliptic equation is called uniformly elliptic if there exist positive constant λ, Λ satisfying the estimate:

$$0 < \lambda \|\sigma\| < F(x, u, \eta, \delta + \sigma) - F(x, u, \eta, \delta) \leq \Lambda \|\sigma\|$$

where $\|\sigma\|$ denotes the (L^2, L^2) -norm (i.e. $\|\sigma\| = \sup_{\|x\|=1} \|\sigma x\| = \text{maximum of the eigenvalues of } \sigma$).

Definition: A continuous function u is called a viscosity subsolution (respectively supersolution) of $F(x, u, Du, D^2u) = 0$ if, for any quadratic polynomial $\Psi \in C^2(\Omega)$ and local maximum (respectively local minimum) x_0 of $u - \Psi$ we have

$$F(x_0, u(x_0), D\Psi(x_0), D^2\Psi(x_0)) \geq 0 \quad (\text{respect. } \leq 0).$$

Finally u is a viscosity solution if it is both a viscosity subsolution and supersolution.

Reference [5] contains the result about regularity (Corollary 5.7) and uniqueness (Corollary 5.4) of viscosity solutions of uniformly elliptic equations, which we shall invoke to conclude the proof.

More precisely, under the hypothesis that $\widehat{\Delta}f \equiv 0$, let P be a paraboloid tangent from below to our hypersurface $S = \{x_n = \varphi(x_1, \dots, x_{n-1})\}$ at a point $x = (x_1, \dots, x_{n-1}, x_n) \in S \cap P$.

Let f_P be its corresponding function defined in the introduction, we have the inequality

$$f_P \geq f_S \quad \text{on } B_r^{(n)}(x)$$

On the other hand, lemma 2 applied to the hypersurface P yields

$$f_P = -c_n H_P(x) \cdot r^{n+1} + O(r^{n+2}) \quad \text{on } B_r^{(n)}(x)$$

which together with the hypothesis

$$\lim_{r \rightarrow 0^+} \frac{1}{r^2} f_S = 0 \quad \text{on } B_r^{(n)}(x)$$

implies that $H_P(x) \leq 0$.

Similarly if P is now a paraboloid tangent to S from above at the point x , then we must have $H_P(x) \geq 0$. Therefore φ is a viscosity solution of the equation

$$\operatorname{div} \left(\frac{\nabla \varphi}{\sqrt{1 + |\nabla \varphi|^2}} \right) = 0$$

whose uniform ellipticity is ensured by the hypothesis that S is of class C^1 .

The regularity theory of such solutions ([5]) allows us to conclude the smoothness of φ (and S) and, therefore, its minimality.

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