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# Differential Galois Theory and Darboux transformations for Integrable Systems

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## Abstract

We apply the Differential Galois Theory of linear partial differential systems to the Bäcklund–Darboux transformations of the AKNS solitonic partial differential equations. We prove that the Galois group of the transformed system is isomorphic to a subgroup of the Galois group of the initial system. As an example, we study the integrability in closed form of the linear systems corresponding to the solitonic solutions of KdV equation.

*Keywords:* Differential Galois Theory, Picard-Vessiot Theory, Integrability, Darboux Transformations, Solitons

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## Introduction

In the last years the differential Galois theories, i.e., the Galois theories of differential equations, has undergone an important renaissance, partially due to their relevance in the applications to other areas, like integrability of dynamical systems, connections with asymptotic theory (Stokes multipliers), some special spectral problems and to the integrability–reducibility of the Painlevé transcendents.

Picard–Vessiot theory is the Galois theory for linear differential equations. This will be essentially the only differential Galois theory relevant in our paper. It was initiated by Picard and Vessiot at the end of the nineteen century for ordinary linear differential equations. Then it was formalized by Kolchin in the last century. In particular, Picard–Vessiot theory for fields with several derivations, necessary for systems of linear differential equations, was constructed by Kolchin in the fifties of the last century ([17]). Some years later, Kolchin himself introduced a more general differential Galois theory, given by the so-called strongly normal extensions, where this Picard-Vessiot theory is included in a natural way. This culminated in the publication in the seventies of the monograph [18]. In his constructions Kolchin uses the algebraic geometry of Weil, but in the first decade of the present century, Kovacic reformulates Kolchin’s strongly normal extensions in the language of differential schemes, instead of Weil’s geometry (see [19] and references therein). Three

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monographs on the Picard–Vessiot theory are [8], [9] and [23]. In particular, in one appendix in [23] are given some indications of how to apply in a direct way the Picard–Vessiot theory from ordinary linear systems to (compatible or flat) systems of linear partial differential equations. Along our paper we will assume some familiarity with this Picard–Vessiot theory of flat systems.

We remark that we consider the Picard–Vessiot theory as an integrability theory. For instance, the integrability by quadratures of a differential linear system is characterized by the solvability of the identity component,  $G^0$ , of the Galois group  $G$  of the system. We say then that the linear system is integrable. For a survey of results in several areas of the Mathematical Physics supporting this point of view, see [22].

Also at the end of the nineteenth century, Darboux discovered a transformation for second order ordinary linear differential equations with spectral parameter, using a particular solution of the associated Riccati equation for a concrete value of the spectral parameter. We call this the classical Darboux transformation [11, 12]. In 1955 Crum generalized the Darboux transformations considering the iteration of the classical Darboux transformation and he also obtained that the spectrum remains (essentially) invariant under these transformations. Two monographs on Darboux transformations are [20] and [15].

One of our main guiding ideas is to study the invariance of the Galois group of the linear partial differential systems, under relevant transformations of the associated solitonic partial differential equations, like the Darboux transformations and the flow. Moreover two additional motivations for our paper were the followings:

- In a work of one of the authors with J.M. Peris it is arrived to prove that for some special Schrödinger equations coming from the dynamics the fact that the potential is a reflection-less one is equivalent to the commutativity of the identity component of the Galois group ([21], see also [14])
- In the article [2] (see also [1]) was applied the Picard–Vessiot theory to the stationary one-dimensional Schrödinger equation and it was proved that *all the known cases of integrability in closed form of this equation* can be handled by Picard–Vessiot theory. Furthermore, an important role was played by the invariance of the Lie algebra of the Galois group of the Schrödinger equation under classical Darboux transformation.

This paper is devoted to study how the Galois groups of the linear systems of (Zhakharov–Shabat) AKNS type are transformed under Darboux transformations. After some definitions in section 2, our main result is that the Galois group of the transformed system is isomorphic to a subgroup of the Galois group of the initial system (sections 2 and 3). Thus, from our point of view that means that by Darboux transformations we obtain systems at least as integrable as the initial systems: recall that integrability is characterized by the solvability of the identity component of the Galois group. In particular, if the initial system is integrable, the transformed system is integrable too, in complete agreement with Darboux ideas (see [12], p. 210). As an illustration of our result we study the classical solitonic solutions of the KdV equation (section 4). In the Appendix it is given a self-contained proof of the equivalence between the Crum transformation and the higher order matrix Darboux transformations.

We remark that it seems clear to obtain similar results for other systems not of AKNS type. For instance, for KP equations or for Drinfeld–Sokolov systems. Our idea was only to illustrate the power of the Galoisian methods in Darboux transformations with an important family of solitonic equations, not to obtain the more general results.

There are some previous works with applications of the Picard–Vessiot theory to the solitonic equations but, as far as we know, all of them only consider algebro–geometric (or finite–gap) solutions ([3, 13, 5, 6, 7]).

## 1. Darboux-Crum Transformations for AKNS Systems

Let  $K$  be a  $\Delta$ -field with  $\Delta = \{\partial_x, \partial_t\}$  (i.e., a differential field with two compatible derivations) such that its field of constants is the field  $\mathbf{C}$  of complex numbers. In fact in the examples  $K$  will be always a field of (local) complex meromorphic functions of two variables  $(x, t)$ . Let  $u \in K$  be a fixed element of  $K$ .

Along the paper  $\lambda$  will denote a complex parameter.

We will have the AKNS system:

$$\begin{cases} \Phi_x = U\Phi = (\lambda J + P)\Phi, \\ \Phi_t = V\Phi = \sum_{j=0}^n V_j \lambda^j \Phi. \end{cases} \quad (\mathfrak{s})$$

where  $J = \text{diag}(c_1, \dots, c_m) \in \mathbb{M}_{m,m}(\mathbf{C})$ , such that  $c_i \neq c_k$  if  $i \neq k$ ,  $\text{tr}(J) = \sum_i c_i = 0$  and  $\det(J) \neq 0$ ,  $V_j(x, t) := V_j[u] \in \mathbb{M}_{m,m}(K)$  of trace zero,  $j = 0, \dots, n$ , and  $P(x, t) := P[u] \in \mathbb{M}_{m,m}(K)$  whose diagonal entries are all zero.

From now on,  $P$  will be a  $m \times m$  matrix with entries in the differential ring  $\mathbf{C}[u, u_x, u_t, \dots]$ . We will denote  $P = P[u]$  to express this fact.

If the system  $(\mathfrak{s})$  satisfies the *zero curvature condition*:

$$U_t - V_x + [U, V] = 0, \quad (\iota)$$

we say that it is a *flat system*. In this case, we say that  $(\mathfrak{s})$ , or  $(U, V)$ , is a *Lax pair* for the partial differential system equations  $(\iota)$ . We denote by  $(\mathfrak{s})_{(U,V)}$  the equation  $(\mathfrak{s})$ , if it were necessary to specify the Lax pair.

It is well-known that the entries of  $V_j$ ,  $0 \leq j \leq n$ , are differential polynomials in the entries of  $P$ , whose coefficients may depend on  $t$  (see for instance, [15]). We write  $V_j[P]$  for the  $V_j$  to specify the dependence on  $P$ .

From now on, we will assume that all the systems are flat systems.

**Definition 1.1.** Let  $\mathcal{K}$  be a differential extension of  $K$ . A *Darboux transformation* for the system  $(\mathfrak{s})_{(U,V)}$  is an invertible matrix  $D$  which is a rational function of  $\lambda$  with coefficients in  $\mathcal{K}$ , such that if  $\Phi$  is a solution of  $(\mathfrak{s})_{(U,V)}$ , then  $\tilde{\Phi} := D\Phi$  is a solution of  $(\mathfrak{s})_{(\tilde{U}, \tilde{V})}$ , where

$$\tilde{U} = DUD^{-1} + D_x D^{-1}, \quad \tilde{V} = DVD^{-1} + D_t D^{-1}. \quad (1.1)$$

The matrix  $D$  is called a *Darboux matrix*. When  $D$  is a polynomial in  $\lambda$ , its degree is of course the degree of that polynomial. The degree one Darboux transformations are also called elementary Darboux transformations.

We remark that a Darboux transformation is nothing but a special Gauge transformation and it will be covariant with respect to the AKNS structure. This means that the transformed system is also an AKNS system, with the same  $J$ ,  $P$  and  $V$  matrices, but depending on a new function  $\tilde{u} \in \mathcal{K}$ .

Applying the Liouville's theorem on Wronskians to  $\Phi$  and  $\tilde{\Phi} = D\Phi$ :

$$\begin{aligned} (\det \Phi)_x &= \operatorname{tr} U \det \Phi, & (\det D \det \Phi)_x &= \operatorname{tr} \tilde{U} \det D \det \Phi, \\ (\det \Phi)_t &= \operatorname{tr} V \det \Phi, & (\det D \det \Phi)_t &= \operatorname{tr} \tilde{V} \det D \det \Phi, \end{aligned}$$

we can compute:

$$\frac{(\det D)_x}{\det D} = \operatorname{tr} \tilde{U} - \operatorname{tr} U, \quad \frac{(\det D)_t}{\det D} = \operatorname{tr} \tilde{V} - \operatorname{tr} V,$$

and, since  $\operatorname{tr} U = \operatorname{tr} V = \operatorname{tr} \tilde{U} = \operatorname{tr} \tilde{V} = 0$ , we have that  $\det \Phi, \det D, \det \tilde{\Phi} \in \mathbf{C}$ .

From the above it is not difficult to prove that the set of Darboux transformations form a subgroup of the group of Gauge transformations.

## 2. Elementary Darboux transformations

First, we consider the Darboux matrix of degree one, i.e.,  $D$  is linear in  $\lambda$ . We will consider  $D = D_0 + \lambda D_1$ , where  $D_1 = \operatorname{diag}(d_1, \dots, d_m) \in GL_m(\mathbf{C})$ . Then we have (see for example [15], Theorem 1.8, where the hypothesis of  $(\mathfrak{s})$  to be unreduced is irrelevant in the proof):

**Proposition 2.1.** *The matrix  $D$  is a Darboux matrix for the AKNS system  $(\mathfrak{s})$  if and only if  $\Sigma = -D_1^{-1}D_0$  satisfies*

$$\Sigma_x + [\Sigma, P + J\Sigma] = 0, \quad \Sigma_t + \sum_{p=0}^n [\Sigma, V_p] \Sigma^p = 0. \quad (2.1)$$

Moreover,  $\tilde{P} = D_1(P + [J, \Sigma])D_1^{-1}$ .

To solve the system (2.1) we proceed as follows:

We choose  $\lambda_1, \dots, \lambda_m$  in  $\mathbf{C} \setminus \{0\}$  with  $\lambda_i \neq \lambda_j$  if  $i \neq j$ . Take  $\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$ . For each  $\lambda_i$  let us take  $\vec{\psi}_i$  a solution of the system  $(\mathfrak{s})$  with coefficients in some differential field extension  $\mathcal{K}$  of  $K$ . Assume that  $\vec{\psi}_1, \dots, \vec{\psi}_m$  are linearly independent over  $\mathbf{C}$ . Let us put  $\Psi = (\vec{\psi}_1 \ \dots \ \vec{\psi}_m)$ . Then the matrix  $\Sigma = \Psi \Lambda \Psi^{-1} = (\sigma_{kl})$  satisfies the system (2.1) and  $\lambda I - \Sigma$  is an elementary Darboux transformation (see Theorem 1.3 in [24], and also [4]).

The family given by  $\lambda \rightarrow D = D(\lambda) = \lambda I - \Sigma$ , for  $\Sigma$  constructed as before, has the evolution given by (2.1), since  $D_x = -\Sigma_x$  and  $D_t = -\Sigma_t$ .

Moreover, when  $\Lambda$  is of trace zero the characteristic polynomial of  $\Lambda$ , say  $p_\Lambda(X) = \det(XI - \Lambda) = a_0 + a_1X + \dots + a_{m-2}X^{m-2} + X^m \in \mathbf{C}[X]$ , has the property  $p_\Lambda(\lambda) = \det(D)$ . Hence the characteristic polynomial of  $D$  is

$$p_D(X) = (-1)^m p_\Lambda(\lambda - X). \quad (2.2)$$

So,  $\det D|_{\lambda=\lambda_i} = 0$ , but  $p_D(\lambda_i) \neq 0$ . Furthermore, we have:

$$p_D(X) = \lambda^m X^m - m\lambda^{m-1} X^{m-1} + \sum_{p=0}^{m-2} c_p(\lambda) X^p, \quad \deg_\lambda c_p(\lambda) \leq m - p - 2. \quad (2.3)$$

Since  $p_D(D) = 0$ , we get the formula for the inverse of  $D$ :

$$0 = \lambda^m D^{m-1} - m\lambda^{m-1} D^{m-2} + \sum_{p=1}^{m-2} c_p(\lambda) D^{p-1} + c_0(\lambda) D^{-1}. \quad (2.4)$$

Let  $D$  be a Darboux transformation of degree one for the AKNS system  $(\mathfrak{s})$ . Let  $\Sigma$  be a solution of (2.1) constructed for some diagonal constant matrix  $\Lambda$  as before. Hence we get the new system:

$$\begin{cases} \tilde{\Phi}_x = \tilde{U}\tilde{\Phi} = (\lambda J + \tilde{P})\tilde{\Phi}, \\ \tilde{\Phi}_t = \tilde{V}\tilde{\Phi} = \sum_{j=0}^n \tilde{V}_j \lambda^j \tilde{\Phi}. \end{cases} \quad (\tilde{\mathfrak{s}})$$

Now we choose  $\lambda \in \mathbf{C}$  such that  $p_\Lambda(\lambda) \neq 0$ . Let us take  $\Sigma = (\sigma_{kl})$  a solution of (2.1). Then the field  $\tilde{K} = K(\sigma_{kl}) = K(\Sigma)$  is the coefficients field of the system  $(\tilde{\mathfrak{s}})$ . Let  $\Phi = (\phi_{kl})$  be a fundamental solution of  $(\mathfrak{s})$  and let  $L = K(\phi_{kl}) = K(\Phi)$  be a Picard-Vessiot field for  $(\mathfrak{s})$ . Let  $\tilde{\Phi} = (\tilde{\phi}_{kl})$  be a fundamental solution of  $(\tilde{\mathfrak{s}})$  and let  $\tilde{L} = K(\tilde{\phi}_{kl}) = K(\tilde{\Phi})$  be a Picard-Vessiot field for  $(\tilde{\mathfrak{s}})$ . Then we have the field commutative diagram:

$$\begin{array}{ccccc} & & \tilde{L} & & \\ & \nearrow & & \nwarrow & \\ L & & & & \tilde{K} \\ & \nwarrow & & \nearrow & \\ & & L \cap \tilde{K} & & \\ & \nearrow & \uparrow & \nwarrow & \\ & & K & & \end{array} \quad \begin{matrix} \\ \\ H \\ G \\ \end{matrix} \quad (2.5)$$

Let  $G$  be the differential Galois group of the system  $(\mathfrak{s})$ , and consider the differential Galois group of the differential field extension  $L \cap \tilde{K} \subset L$ , say  $H$ . Let be  $\tilde{G}$  the Galois group of  $(\tilde{\mathfrak{s}})$ . Then we have the following statement:

**Theorem 2.2.** *The differential Galois group  $\tilde{G}$  is isomorphic to the subgroup  $H = \text{Gal}(L/L \cap \tilde{K})$  of the differential Galois group  $G$ .*

In order to prove this statement, we need the following Lemma. For the convenience of the reader, we reproduce it.

**Lemma 2.3** (Lemma 5.10 in [16]). *Let  $L$  be a Picard-Vessiot extension of  $K$  (characteristic 0 and algebraically closed constant field). Let  $L_1 = L \langle z \rangle$  be an extension of  $L$  with no new constants. Write  $K_1 = K \langle z \rangle$ . Then  $L_1$  is a Picard-Vessiot extension of  $K_1$ , and its differential Galois group,  $G_1$ , is isomorphic to an algebraic subgroup of the differential Galois group of  $L$  over  $K$ ,  $G$ , namely the subgroup leaving  $L \cap K_1$  fixed.*

*Proof.* First, we note that  $\tilde{K}(D) = \tilde{K}(\Sigma) = \tilde{K}$ . Hence, we have

$$\tilde{L} = \tilde{K}(\tilde{\Phi}) = \tilde{K}(D\Phi) = \tilde{K}(\Phi) = K(\Sigma, \Phi) = K(\Phi)(\Sigma) = L(\Sigma) = L(\sigma_{kl}).$$

As the derivatives of the entries of matrix  $\Sigma$  are defined rationally by means of the equation (2.1),  $\partial_x \sigma_{kl} \in L(\sigma_{kl})$ ,  $\partial_t \sigma_{kl} \in L(\sigma_{kl})$ , thus  $L(\sigma_{kl}) = L\langle \sigma_{kl} \rangle$ . Then, we apply Lemma 2.3 and using induction over  $\sigma_{kl}$  we obtain that (see diagram (2.5))

$$\tilde{G} \simeq H \simeq \text{Gal}(L/L \cap \tilde{K}) \subset G$$

and this concludes the proof.  $\square$

**Corollary 2.4.** *We have the equivalence:  $G \simeq \tilde{G}$  if and only if  $K = L \cap \tilde{K}$ .*

If  $G$  is an algebraic group, as usual,  $G^0$  denotes the identity group of  $G$ .

**Corollary 2.5.** *If  $K \subset \tilde{K}$  is an algebraic extension, then  $G^0 \simeq \tilde{G}^0$ .*

*Proof.* If  $K \subset \tilde{K}$  is an algebraic extension,  $K \subset L \cap \tilde{K}$  is also algebraic. Then,  $G_0 \subseteq H$  and since,  $H \simeq \tilde{G}$ , we conclude that  $G^0 \simeq \tilde{G}^0$  (see again diagram 2.5).  $\square$

*Remark 2.6.* (This remark is mainly addressed to people interested in Algebraic Geometry) A geometric interpretation of the Theorem 2.2 can be done by means of torsors (see page 430 in Appendix D in [23]). Let  $R$  and  $\tilde{R}$  be Picard-Vessiot rings of  $(\mathfrak{s})$  and  $(\tilde{\mathfrak{s}})$  respectively. Consider  $S$  the multiplicative closed system of  $R[z]$  given by the powers of  $p_\Lambda(z) = \det(zI - \lambda)$ , and also  $S_\lambda = \{p_\Lambda(\lambda)^n : n \in \mathbb{N}\}$ . Then

$$(S^{-1}R[z])/S^{-1}(z - \lambda) \simeq S^{-1}(R[z]/(z - \lambda)) \simeq S_\lambda^{-1}R, \quad (2.6)$$

since  $(z - \lambda) \cap S = \emptyset$  because  $p_\Lambda(\lambda) \neq 0$ . Now we have:

$$S_\lambda^{-1}R \otimes_K \tilde{K} \simeq S_\lambda^{-1}\tilde{R},$$

because  $S_\lambda^{-1}\tilde{K} = \tilde{K}$ , since it is a field. But,  $\tilde{R}$  is a domain, then  $S_\lambda^{-1}\tilde{R} \simeq \tilde{R}$ , because they are simple rings. So, at the end we get the following formula:

$$S_\lambda^{-1}R \otimes_K \tilde{K} \simeq \tilde{R}. \quad (2.7)$$

Hence  $Z_{\tilde{L}} \simeq Z_L \otimes_K \tilde{K}$ , where  $Z$  and  $\tilde{Z}$  are the maximal spectra of  $R$  and  $\tilde{R}$  respectively.

### 3. Darboux matrices of higher degree

For the construction of the higher degree Darboux matrices we follow [15]. Then we will obtain some Galoisian consequences of the constructions.

The composition of  $d$  Darboux transformations of degree one is a Darboux transformation of degree  $d$ . Although, we can construct Darboux transformations of degree  $d$  directly. To do this, let us consider a  $m \times m$  matrix of the form:

$$D(x, t, \lambda) = \sum_{j=0}^d \tilde{D}_j(x, t) \lambda^j = \tilde{D}_d \left( \sum_{j=0}^{d-1} D_j(x, t) \lambda^j + \lambda^d I \right), \quad (3.1)$$

where  $\tilde{D}_d = \text{diag}(d_1, \dots, d_m) \in GL_m(\mathbf{C})$ . As seen before, a Darboux matrix of degree one of the form  $D(x, t, \lambda) = D_1(\lambda I - \Sigma)$  can be constructed. Suppose  $\Sigma = \Psi\Lambda\Psi^{-1}$ , then  $\Sigma\Psi = \Psi\Lambda$  is equivalent to  $D(x, t, \lambda_i)\psi_i = 0$ . This can be generalized to Darboux matrices of degree  $d$  as follows.

For  $i = 1, \dots, md$ , let us take a solution  $\vec{\psi}_i$  of the system  $(\mathfrak{s})$  for  $\lambda = \lambda_i \in \mathbf{C} \setminus \{0\}$ . As before, assume that  $\vec{\psi}_1, \dots, \vec{\psi}_{md}$  are linearly independent over  $\mathbf{C}$ . Let us put  $\Psi = (\vec{\psi}_1 \ \dots \ \vec{\psi}_{md})$ .

In order to simplify the notation, hereafter we write  $\Psi = (\psi_1 \ \dots \ \psi_{md})$ . Consider the  $md \times md$  matrix:

$$\Gamma_d = \begin{pmatrix} \psi_1 & \psi_2 & \dots & \psi_{md} \\ \lambda_1\psi_1 & \lambda_2\psi_2 & \dots & \lambda_{md}\psi_{md} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{d-1}\psi_1 & \lambda_2^{d-1}\psi_2 & \dots & \lambda_{md}^{d-1}\psi_{md} \end{pmatrix}. \quad (3.2)$$

The system  $D(x, t, \lambda_i)\psi_i = 0$ ,  $i = 1, \dots, md$ , is equivalent to:

$$\sum_{j=0}^{d-1} D_j(x, t) \lambda_i^j \psi_i = -\lambda_i^d \psi_i, \quad (3.3)$$

for  $i = 1, \dots, md$ , which can be written as:

$$(D_0, D_1, \dots, D_{d-1})\Gamma_d = -(\lambda_1^d\psi_1, \lambda_2^d\psi_2, \dots, \lambda_{md}^d\psi_{md}). \quad (3.4)$$

When  $\det \Gamma_d \neq 0$ , this system has a unique solution  $(D_0, D_1, \dots, D_{d-1})$ , i.e., a unique  $m \times m$  matrix  $D(x, t, \lambda)$  in the form (3.1) such that  $D(x, t, \lambda_i)\psi_i = 0$ ,  $i = 1, \dots, md$ .

We denote this matrix by  $D(\psi_1, \dots, \psi_{md}, \lambda)$  to indicate that it is constructed from  $\psi_1, \dots, \psi_{md}$ .

The following result states that this matrix is a Darboux matrix and that it can be decomposed, under some assumptions, as a product of two Darboux matrices of lower degree (see [15], page 26). Let us denote:

$$\begin{cases} \tilde{\Phi}_x &= \tilde{U}\tilde{\Phi} = (\lambda J + \tilde{P})\tilde{\Phi}, \\ \tilde{\Phi}_t &= \tilde{V}\tilde{\Phi} = \sum_{j=0}^n \tilde{V}_j \lambda^j \tilde{\Phi}. \end{cases} \quad (\tilde{5})$$

for the transformed system.

**Proposition 3.1.** *Given  $\lambda_1, \dots, \lambda_{md} \in \mathbf{C}$ . Let  $\psi_i$  be a column solution of  $(\mathfrak{s})$ ,  $1 \leq i \leq md$  and  $\Gamma_d$  be defined by (3.2). Suppose that  $\det \Gamma_d \neq 0$ , then the following statements hold:*

1. *There exist a unique matrix  $D(\psi_1, \dots, \psi_{md}, \lambda)$  in the form (3.1) such that*

$$D(\psi_1, \dots, \psi_{md}, \lambda_i)\psi_i = 0, \quad 1 \leq i \leq md. \quad (3.5)$$

*In this case, we say that  $D(\psi_1, \dots, \psi_{md}, \lambda)$  is a Darboux matrix of degree  $d$  for  $(\mathfrak{s})$ .*



2. If  $\det \Gamma_{d-1} \neq 0$ , then this Darboux matrix of degree  $d$  can be decomposed as:

$$D(\psi_1, \dots, \psi_{md}, \lambda) = \quad (3.6)$$

$$D(D(\psi_1, \dots, \psi_{m(d-1)}, \lambda_{m(d-1)+1})\psi_{m(d-1)+1}, \dots, D(\psi_1, \dots, \psi_{m(d-1)}, \lambda_{md})\psi_{md}, \lambda) \cdot D(\psi_1, \dots, \psi_{m(d-1)}, \lambda).$$

Where the first matrix on the right hand side of this equality is a Darboux matrix of degree one and the second one is a Darboux matrix of degree  $d - 1$ .

3. If  $\det \Gamma_i \neq 0$  for  $i = 1, \dots, d - 1$ , the Darboux matrix  $D(\psi_1, \dots, \psi_{md}, \lambda)$  of degree  $d$  can be decomposed as a product of  $d$  Darboux matrices of degree one.
4. Let us take  $D$  a decomposable Darboux transformation, that is  $D = \tilde{D}_d(\lambda I - \Sigma_d) \cdots (\lambda I - \Sigma_1)$ . Then the matrix  $\tilde{P} = \tilde{D}_d(P - [J, D_{d-1}])\tilde{D}_d^{-1}$  satisfies  $\tilde{U} = \lambda J + \tilde{P}$  in  $(\tilde{\mathfrak{s}})$ .

The proof of this proposition is an easy extension of the argument in [15].

From the above proposition and theorem 2.2, we arrive to the following

**Theorem 3.2.** Let be  $G$  the differential Galois group of the system  $(\mathfrak{s})$ . Let be  $D = D_d \cdots D_1$  a decomposable Darboux transformation of degree  $d$ . If  $G_i$  is the differential Galois group of the system  $(\mathfrak{s})$  transformed by  $\tilde{D}_i = D_i \cdots D_1$ , then we have the chain of groups:

$$G_d \subset G_{d-1} \subset \cdots \subset G_1 \subset G.$$

*Remark 3.3.* Let us observe that  $G_d$  will be the Galois group of the transformed system under  $D$  for  $\lambda \in \mathbf{C} \setminus \{\det(D(\lambda)) = 0\}$ .

Now we will consider the case  $d = 2$ .

Suppose  $D(\lambda) = D(\psi_1, \dots, \psi_{2m}, \lambda)$  is a Darboux matrix of degree two with  $\tilde{D}_2 = I$  and decomposable as a product of two Darboux matrices of degree one of the form (3.6):

$$D(\lambda) = D(\tilde{\psi}_{m+1}, \dots, \tilde{\psi}_{2m}, \lambda) \cdot D(\psi_1, \dots, \psi_m, \lambda),$$

where  $\tilde{\psi}_i = D(\psi_1, \dots, \psi_m, \lambda_i)\psi_i$ , for  $i = m+1, \dots, 2m$ . The matrix  $D^{(1)} = D(\psi_1, \dots, \psi_m, \lambda)$  has the form:

$$D^{(1)} = \lambda I - \Sigma_1 = \lambda I - \Psi^{(1)}\Lambda_1(\Psi^{(1)})^{-1},$$

where  $\Psi^{(1)} = (\psi_1, \dots, \psi_m)$  for some  $\Lambda_1 = \text{diag}(\lambda_1, \dots, \lambda_m)$  (see (2.1) and the construction of a Darboux transformation for  $d = 1$ ).

For  $i = m + 1, \dots, 2m$ , we denote  $\psi_i$  the transformed of  $\psi_i$ , that is:

$$\tilde{\psi}_i = D(\psi_1, \dots, \psi_m, \lambda_i)\psi_i = (\lambda_i I - \Sigma_1)\psi_i.$$

So,

$$\tilde{\Psi}^{(2)} = (\tilde{\psi}_{m+1}, \dots, \tilde{\psi}_{2m}) = \Psi^{(2)}\Lambda_2 - \Sigma_1\Psi^{(2)} = \Sigma_2\Psi^{(2)} - \Sigma_1\Psi^{(2)} = (\Sigma_2 - \Sigma_1)\Psi^{(2)},$$

where  $\Psi^{(2)} = (\psi_{m+1}, \dots, \psi_{2m})$ , and  $\Lambda_2 = \text{diag}(\lambda_{m+1}, \dots, \lambda_{2m})$  such that  $p_{\Lambda_1}(\lambda_i) \neq 0$ , for the new considered  $\lambda_i$ . Hence:

$$\tilde{D}^{(2)} = D(\tilde{\psi}_{m+1}, \dots, \tilde{\psi}_{2m}, \lambda) = \lambda I - \tilde{\Sigma}_2 = \lambda I - \tilde{\Psi}^{(2)} \Lambda_2 (\tilde{\Psi}^{(2)})^{-1}.$$

Therefore

$$D(\lambda) = \tilde{D}^{(2)} D^{(1)}.$$

The previous procedure can be done exchanging the matrices  $\Lambda_1$  and  $\Lambda_2$ , whenever their characteristic polynomials are coprime. More precisely, if we consider the Darboux matrix  $D'(\lambda) = D(\psi_{m+1}, \dots, \psi_{2m}, \psi_1, \dots, \psi_m, \lambda)$  of degree two with  $\tilde{D}'_2 = I$ . This matrix is also decomposable as a product of two Darboux matrices of degree one:

$$D'(\lambda) = D(\tilde{\psi}_1, \dots, \tilde{\psi}_m, \lambda) \cdot D(\psi_{m+1}, \dots, \psi_{2m}, \lambda),$$

where  $\tilde{\psi}_i = D(\psi_{m+1}, \dots, \psi_{2m}, \lambda_i) \psi_i$  for  $i = 1, \dots, m$ . In this case, we have that:

$$\begin{aligned} D^{(2)} &= D(\psi_{m+1}, \dots, \psi_{2m}, \lambda) = \lambda I - \Sigma_2 = \lambda I - \Psi^{(2)} \Lambda_2 (\Psi^{(2)})^{-1}, \\ \tilde{D}^{(1)} &= D(\tilde{\psi}_1, \dots, \tilde{\psi}_m, \lambda) = \lambda I - \tilde{\Sigma}_1 = \lambda I - \tilde{\Psi}^{(1)} \Lambda_2 (\tilde{\Psi}^{(1)})^{-1}, \end{aligned}$$

where  $\tilde{\Psi}^{(1)} = (\tilde{\psi}_1, \dots, \tilde{\psi}_m) = \Psi^{(1)} \Lambda_1 - \Sigma_2 \Psi^{(1)} = \Sigma_1 \Psi^{(1)} - \Sigma_2 \Psi^{(1)} = (\Sigma_1 - \Sigma_2) \Psi^{(1)}$ . Therefore

$$D'(\lambda) = \tilde{D}^{(1)} D^{(2)}.$$

Both matrices  $D(\lambda)$  and  $D'(\lambda)$  satisfy:

$$D(\lambda_i) \psi_i = 0, \quad D'(\lambda_i) \psi_i = 0,$$

for  $i = 1, \dots, 2m$ . So, by part (1) of Proposition 3.1, we have that  $D(\lambda) = D'(\lambda)$  and

$$\begin{aligned} \tilde{D}^{(2)} D^{(1)} &= \tilde{D}^{(1)} D^{(2)}, \\ (\lambda I - \tilde{\Sigma}_2)(\lambda I - \Sigma_1) &= (\lambda I - \tilde{\Sigma}_1)(\lambda I - \Sigma_2), \\ \lambda^2 I - \lambda(\Sigma_1 + \tilde{\Sigma}_2) + \tilde{\Sigma}_2 \Sigma_1 &= \lambda^2 I - \lambda(\Sigma_2 + \tilde{\Sigma}_1) + \tilde{\Sigma}_1 \Sigma_2. \end{aligned}$$

Since  $\lambda = 0 \in \mathbf{C} \setminus \{\det(D(\lambda)) = 0\}$ , we have  $\tilde{\Sigma}_2 \Sigma_1 = \tilde{\Sigma}_1 \Sigma_2$ , and then  $\Sigma_1 + \tilde{\Sigma}_2 = \tilde{\Sigma}_1 + \Sigma_2$ . Then we have the exchange formula:

$$\tilde{\Sigma}_2 \Sigma_1 = (\Sigma_1 - \Sigma_2) \Sigma_2 (\Sigma_1 - \Sigma_2)^{-1} \Sigma_1 = (\Sigma_1 - \Sigma_2) \Sigma_1 (\Sigma_1 - \Sigma_2)^{-1} \Sigma_2 = \tilde{\Sigma}_1 \Sigma_2. \quad (3.7)$$

That gives us the following classical result (see, for example, [15]):

**Corollary 3.4** (Theorem of permutability). *Let  $\Lambda_1 = \text{diag}(\lambda_1, \dots, \lambda_m)$  and  $\Lambda_2 = \text{diag}(\lambda_{m+1}, \dots, \lambda_{2m})$  be two invertible matrix, with  $\lambda_i \neq \lambda_j$  if  $i \neq j$ . Let be  $\Phi[\lambda_i] = (\psi_{i1} \dots \psi_{im})$  a solution of  $(\mathfrak{s})$  for  $\lambda = \lambda_i$ ,  $1 \leq i \leq 2m$ . For  $i_0 \in \{1, \dots, m\}$  consider  $\Psi_{i_0}^{(1)} = (\psi_{1i_0} \dots \psi_{mi_0})$  and  $\Psi_{i_0}^{(2)} = (\psi_{m+1,i_0} \dots \psi_{2m,i_0})$ . Let us put  $D^{(j)} = \lambda I - \Psi_{i_0}^{(j)} \Lambda_j (\Psi_{i_0}^{(j)})^{-1} = \lambda I - \Sigma_j$  and the corresponding transformed system  $(\tilde{\mathfrak{s}})^{(j)}$ ,  $j = 1, 2$ .*

Let be  $\tilde{\Phi}^{(j)}[\lambda_i] = (\tilde{\psi}_{i1} \ \dots \ \tilde{\psi}_{im})$  a solution of  $(\tilde{\mathfrak{s}})^{(j)}$  for  $\lambda = \lambda_i$ ,  $1 \leq i \leq 2m$ . For  $i_0 \in \{1, \dots, m\}$  consider  $\tilde{\Psi}_{i_0}^{(2)} = (\tilde{\psi}_{1i_0} \ \dots \ \tilde{\psi}_{mi_0})$  and  $\tilde{\Psi}_{i_0}^{(1)} = (\tilde{\psi}_{m+1,i_0} \ \dots \ \tilde{\psi}_{2m,i_0})$ . Let us put  $\tilde{D}^{(j)} = \lambda I - \tilde{\Psi}_{i_0}^{(j)} \Lambda_j (\Psi_{i_0}^{(j)})^{-1} = \lambda I - \tilde{\Sigma}_j$ ,  $j = 1, 2$ . Suppose that:

$$\det \begin{pmatrix} \Psi_{i_0}^{(1)} & \Psi_{i_0}^{(2)} \\ \Psi_{i_0}^{(1)} \Lambda_1 & \Psi_{i_0}^{(2)} \Lambda_2 \end{pmatrix} \neq 0.$$

Then, for every fundamental solution  $\Phi$  of  $(\mathfrak{s})$  we have:

$$\tilde{D}^{(2)} D^{(1)} \Phi = \tilde{D}^{(1)} D^{(2)} \Phi, \quad P^{(1,2)} = P^{(2,1)},$$

where  $P^{(1,2)} = P - [J, -\Sigma_1 - \tilde{\Sigma}_2]$  and  $P^{(2,1)} = P - [J, -\Sigma_2 - \tilde{\Sigma}_1]$ .

The theorem of permutability can be expressed by the following diagram:

$$\begin{array}{ccc} & (P^{(1)}, \Phi^{(1)}) & \\ \nearrow^{D^{(1)} \atop \Lambda_1} & & \searrow^{\tilde{D}^{(2)} \atop \Lambda_2} \\ (P, \Phi) & & (P^{(1,2)}, \Phi^{(1,2)}) = (P^{(2,1)}, \Phi^{(2,1)}) \\ \searrow_{D^{(2)} \atop \Lambda_2} & & \nearrow_{\Lambda_1 \atop \tilde{D}^{(1)}} \\ & (P^{(2)}, \Phi^{(2)}) & \end{array} \quad (3.8)$$

Now, if  $G$  is the differential Galois group of  $(\mathfrak{s})$ , and  $\tilde{G}_1$  (resp.  $\tilde{G}_2$ ) is the differential Galois group of  $(\tilde{\mathfrak{s}})$  for  $D = \tilde{D}^{(1)}$  (resp.  $D = \tilde{D}^{(2)}$ ), we can compare the exchanging effect of the diagram (3.8) on Galois groups. In fact we have,

**Corollary 3.5.** *The differential Galois group,  $G$ , of the system  $(\mathfrak{s})$  after the Darboux transform  $\tilde{D}^{(2)} D^{(1)}$ , say  $\tilde{G}_{12}$ , is isomorphic to differential Galois group of the system  $(\mathfrak{s})$  after the Darboux transform  $\tilde{D}^{(1)} D^{(2)}$ , say  $\tilde{G}_{21}$ , both of them as subgroups of the differential Galois group  $G$  of  $(\mathfrak{s})$ .*

Moreover we have the following commutative diagram:

$$\begin{array}{ccc} & \tilde{G}_2 & \\ \swarrow_{D^{(1)}} & & \searrow^{\tilde{D}^{(2)}} \\ G & & \tilde{G}_{12} = \tilde{G}_{21} \\ \swarrow_{D^{(2)}} & & \searrow^{\tilde{D}^{(1)}} \\ & \tilde{G}_1 & \end{array} \quad (3.9)$$

#### 4. The KdV equation

We apply the above general results to the classical solitonic solutions of the Korteweg de Vries equation (KdV).

Let  $K$  be a  $\Delta$ -field, with  $\Delta = \{\partial_x, \partial_t\}$  and  $E$  a complex parameter.

Consider the system:

$$\Phi_x^0 = U^0 \Phi^0 = \begin{pmatrix} 0 & 1 \\ u - E & 0 \end{pmatrix} \Phi^0, \quad \Phi_t^0 = V^0 \Phi^0 = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} \Phi^0, \quad (4.1)$$

where  $A = A[u], B = B[u], C = C[u]$  are differential polynomials of  $u$  and the parameter  $E$ .

The zero curvature condition of the system is

$$U_t^0 - V_x^0 + [U^0, V^0] = 0,$$

hence

$$A = -\frac{1}{2}B_x, \quad C = (u - E)B - \frac{1}{2}B_{xx}, \quad (4.2)$$

and so we have:

$$u_t = 2(u - E)B_x + u_x B - \frac{1}{2}B_{xxx}. \quad (\iota)$$

In particular, if

$$A = -u_x, \quad B = 2u + 4E, \quad C = 2u^2 + 2uE - 4E^2 - u_{xx}, \quad (4.3)$$

the zero curvature condition  $(\iota)$  is the KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0. \quad (4.4)$$

Thus, the linear system associated to the KdV equation is:

$$\begin{cases} \Phi_x^0 = U^0 \Phi^0 = \begin{pmatrix} 0 & 1 \\ u - E & 0 \end{pmatrix} \Phi^0, \\ \Phi_t^0 = V^0 \Phi^0 = \begin{pmatrix} -u_x & 2u + 4E \\ 2u^2 + 2uE - 4E^2 - u_{xx} & u_x \end{pmatrix} \Phi^0. \end{cases} \quad (4.5)$$

Now take  $\Phi^0 = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$  a solution of (4.1), then it satisfies the Schrödinger equation with spectral parameter  $E$ :

$$(-\partial_x^2 + u)\phi_1 = E\phi_1. \quad (4.6)$$

Now, if  $\lambda^2 + E = 0$ , then the system (4.1) is transformed into

$$\begin{cases} \Phi_x = U\Phi, \\ \Phi_t = V\Phi, \end{cases} \quad (4.7)$$

where

$$\begin{aligned} R &= \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix}, \\ U &= RU^0R^{-1} = \begin{pmatrix} \lambda & -u \\ -1 & -\lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -u \\ -1 & 0 \end{pmatrix}, \\ V &= RV^0R^{-1} = \begin{pmatrix} \lambda B - A & \lambda^2 B - 2\lambda A - C \\ -B & -\lambda B + A \end{pmatrix}, \end{aligned}$$

and  $\Phi = R\Phi^0$ . It is an AKNS system. Moreover its zero curvature condition are equations (4.2) and (4.3) with  $E = -\lambda^2$ .

Then, for

$$A = -u_x, \quad B = 2u - 4\lambda^2, \quad C = 2u^2 - 2u\lambda^2 - 4\lambda^4 - u_{xx},$$

we have the system:

$$\begin{cases} \Phi_x = U\Phi = \begin{pmatrix} \lambda & -u \\ -1 & -\lambda \end{pmatrix} \Phi, \\ \Phi_t = V\Phi = \begin{pmatrix} 2\lambda u - 4\lambda^3 + u_x & 4\lambda^2 u + u_{xx} - 2u^2 + 2\lambda u_x \\ 4\lambda^2 - 2u & -2\lambda u + 4\lambda^3 - u_x \end{pmatrix} \Phi, \end{cases} \quad (4.8)$$

where (4.3) is the KdV equation.

Let  $\phi_0$  be a solution of (4.6) for  $E = E_0$ , then, for  $E_0 + \lambda_0^2 = 0$  we have:

$$\Phi_0 = R_0\Phi_0^0 = \begin{pmatrix} \lambda_0\phi_0 + \phi_{0,x} \\ -\phi_0 \end{pmatrix}, \quad (4.9)$$

with  $\Phi_0^0 = \begin{pmatrix} \phi_0 \\ \phi_{0,x} \end{pmatrix}$  a solution of (4.5) and  $R_0 = \begin{pmatrix} \lambda_0 & 1 \\ -1 & 0 \end{pmatrix}$ . Hence, from a solution of (4.6) we can obtain a solution of (4.7) whenever  $E_0 + \lambda_0^2 = 0$ . And it is easy to check the following remark:

*Remark 4.1.* Let  $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$  be a solution of (4.7) for  $\lambda = \lambda_0$  and  $\beta \neq 0$ . Then  $\begin{pmatrix} \alpha + 2\lambda_0\beta \\ \beta \end{pmatrix}$  is a solution of (4.7) for  $\lambda = -\lambda_0$ .

Let put for  $\lambda_0 \neq 0$  :

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \lambda_0\phi_0 + \phi_{0,x} & \phi_{0,x} - \lambda_0\phi_0 \\ -\phi_0 & -\phi_0 \end{pmatrix}$$

for  $\phi_0$  a non trivial solution of the Schrödinger equation (4.6) for  $E = E_0 = -\lambda_0^2 \neq 0$ . A Darboux matrix  $D = J(\lambda I - \Sigma)$ , with  $\Sigma = \Psi\Lambda\Psi^{-1}$ , for the system (4.7) is:

$$D = J(\lambda I - \Sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \lambda - \sigma_0 & -\sigma_0^2 + \lambda_0^2 \\ 1 & \lambda + \sigma_0 \end{pmatrix} = \begin{pmatrix} \lambda - \sigma_0 & -\sigma_0^2 + \lambda_0^2 \\ -1 & -\lambda - \sigma_0 \end{pmatrix}, \quad (4.10)$$

where  $\sigma_0 = \frac{\phi_{0,x}}{\phi_0}$  (see [15] page 38). We notice that  $\sigma_0$  is a solution of the Riccati equations for  $\lambda = \lambda_0$ :

$$\sigma_x = (u - E) - \sigma^2 = (u + \lambda^2) - \sigma^2, \quad (4.11)$$

$$\begin{aligned} \sigma_t &= C - 2A\sigma - B\sigma^2 = (u + \lambda^2 - \sigma^2)B - \frac{1}{2}B_{xx} + \sigma B_x = \sigma_x B + \sigma B_x - \frac{1}{2}B_{xx} = \\ &= 2u\sigma_x - 4\lambda^2\sigma_x + 2\sigma u_x - u_{xx}. \end{aligned} \quad (4.12)$$

Then for  $\tilde{\Phi} = D\Phi$ :

$$\tilde{\Phi} = D\Phi = \begin{pmatrix} (\lambda^2 - \lambda\sigma_0 + \sigma_0^2 - \lambda_0^2)\phi + (\lambda - \sigma_0)\phi_x \\ \sigma_0\phi - \phi_x \end{pmatrix}, \quad (4.13)$$

we have the following AKNS system:

$$\begin{cases} \tilde{\Phi}_x = \tilde{U}\tilde{\Phi} &= (DUD^{-1} + D_x D^{-1})\tilde{\Phi} \\ &= \begin{pmatrix} \lambda & u - 2(\sigma_0^2 - \lambda_0^2) \\ -1 & -\lambda \end{pmatrix} \tilde{\Phi} = \begin{pmatrix} \lambda & -\tilde{u} \\ -1 & -\lambda \end{pmatrix} \tilde{\Phi}, \\ \tilde{\Phi}_t = \tilde{V}\tilde{\Phi} &= (DV D^{-1} + D_t D^{-1})\tilde{\Phi} \\ &= \begin{pmatrix} \lambda B[\tilde{u}] - A[\tilde{u}] & \lambda^2 B[\tilde{u}] - 2\lambda A[\tilde{u}] - C[\tilde{u}] \\ -B[\tilde{u}] & -\lambda B[\tilde{u}] + A[\tilde{u}] \end{pmatrix} \tilde{\Phi}, \end{cases} \quad (4.14)$$

where  $\tilde{u} = -u + 2(\sigma_0^2 - \lambda_0^2) = -u + 2(u - \sigma_{0,x}) = u - 2\sigma_{0,x}$  (i.e., the classical Darboux transformation) and  $A[\tilde{u}], B[\tilde{u}], C[\tilde{u}]$  satisfy the relations (4.2) and ( $\iota$ ).

Now take the trivial potential for (4.6),  $u = 0$ . Then, for  $E_0 = -\lambda_0^2 \neq 0$ , the solution of (4.6):

$$\phi_0 = \cosh(\lambda_0 x - 4\lambda_0^3 t) \quad (4.15)$$

gives the solutions of (4.8):

$$\psi_{0,1} = \begin{pmatrix} \lambda_0 \phi_0 + \phi_{0,x} \\ -\phi_0 \end{pmatrix} \quad \text{and} \quad \psi_{0,2} = \begin{pmatrix} -\lambda_0 \phi_0 + \phi_{0,x} \\ -\phi_0 \end{pmatrix}, \quad (4.16)$$

for  $\lambda = \lambda_0$  and  $\lambda = -\lambda_0$  respectively. So, applying the Darboux transform (4.10) with

$$\Lambda_0 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix} \quad \text{and} \quad \Psi_0 = (\psi_{0,1} \quad \psi_{0,2})$$

the system (4.8) becomes:

$$\begin{cases} \tilde{\Phi}_x = \tilde{U}\tilde{\Phi} &= \begin{pmatrix} \lambda & -2(\sigma_0^2 - \lambda_0^2) \\ -1 & -\lambda \end{pmatrix} \tilde{\Phi} = \begin{pmatrix} \lambda & -\tilde{u} \\ -1 & -\lambda \end{pmatrix} \tilde{\Phi}, \\ \tilde{\Phi}_t = \tilde{V}\tilde{\Phi} &= \begin{pmatrix} 2\lambda\tilde{u} - 4\lambda^3 + \tilde{u}_x & 4\lambda^2\tilde{u} + \tilde{u}_{xx} - 2\tilde{u}^2 + 2\lambda\tilde{u}_x \\ 4\lambda^2 - 2\tilde{u} & -2\lambda\tilde{u} + 4\lambda^3 - \tilde{u}_x \end{pmatrix} \tilde{\Phi}, \end{cases} \quad (4.17)$$

where  $\tilde{u} = 2(\sigma_0^2 - \lambda_0^2) = -2\lambda_0^2 \operatorname{sech}^2(\lambda_0 x - 4\lambda_0^3 t)$  is the one soliton solution of the KdV equation.

Now, we consider the field of coefficients  $K = \mathbf{C}(x, t)$  and the solution (4.15) of (4.6), then

$$\sigma_0 = \frac{\phi_{0,x}}{\phi_0} = \lambda_0 \tanh(\lambda_0 x - 4\lambda_0^3 t)$$

and also  $\tilde{K} = K(\Sigma) = K(\sigma_0) = K(e^{\lambda_0 x - 4\lambda_0^3 t})$ . The solutions of (4.6) for  $u = 0$  are  $\phi_1 = \cosh(\lambda x - 4\lambda^3 t)$  and  $\phi_2 = \sinh(\lambda x - 4\lambda^3 t)$  for  $\lambda \neq \pm\lambda_0$ . So, for  $\lambda \neq \pm\lambda_0$ , we have

$$L = K(e^{\lambda x - 4\lambda^3 t}).$$

Thus,  $L \cap \tilde{K} = K$ , since  $e^{\lambda_0 x - 4\lambda_0^3 t}$  and  $e^{\lambda x - 4\lambda^3 t}$  are rationally independent when  $\lambda_0 \neq \pm\lambda$ .

Let  $G$  be the Galois group of system (4.8) and  $\tilde{G}$  the Galois group of the Darboux transformed system (4.17) for  $\lambda \neq \pm\lambda_0$ . Therefore, by corollary 2.4, we have that

$$G \simeq \tilde{G}, \quad (4.18)$$

and both Galois groups are isomorphic to the multiplicative group  $G_m$ , i.e.,

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbf{C}^* \right\}.$$

Now, take  $\lambda_1 \neq \pm\lambda_0$ ,  $\lambda_1 \neq 0$  in (4.6) for  $u = 0$ . Since  $\phi_1 = \sinh(\lambda_1 x - 4\lambda_1^3 t)$  is a particular solution of such equation, then

$$\psi_{1,1} = \begin{pmatrix} \lambda_1 \phi_1 + \phi_{1,x} \\ -\phi_1 \end{pmatrix} \quad \text{and} \quad \psi_{1,2} = \begin{pmatrix} -\lambda_1 \phi_1 + \phi_{1,x} \\ -\phi_1 \end{pmatrix}, \quad (4.19)$$

are solutions of (4.8) for  $\lambda = \lambda_1$  and  $\lambda = -\lambda_1$  respectively. Thus,

$$\tilde{\psi}_{1,1} = D|_{\lambda=\lambda_1} \psi_{1,1} = \begin{pmatrix} (\lambda_1^2 - \lambda_1 \sigma_0 + \sigma_0^2 - \lambda_0^2) \phi_1 + (\lambda_1 - \sigma_0) \phi_{1,x} \\ \sigma_0 \phi_1 - \phi_{1,x} \end{pmatrix}, \quad (4.20)$$

$$\tilde{\psi}_{1,2} = D|_{\lambda=-\lambda_1} \psi_{1,2} = \begin{pmatrix} (\lambda_1^2 + \lambda_1 \sigma_0 + \sigma_0^2 - \lambda_0^2) \phi_1 + (-\lambda_1 - \sigma_0) \phi_{1,x} \\ \sigma_0 \phi_1 - \phi_{1,x} \end{pmatrix}, \quad (4.21)$$

are particular solutions of (4.17) for  $\lambda = \lambda_1$  and  $\lambda = -\lambda_1$  respectively. Now, we can construct the Darboux transformation  $\tilde{D} = J(\lambda I - \tilde{\Sigma})$ , where  $\tilde{\Sigma} = \Psi_1 \Lambda_1 \Psi_1^{-1}$  for

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix} \quad \text{and} \quad \Psi_1 = \begin{pmatrix} \tilde{\psi}_{1,1} & \tilde{\psi}_{1,2} \end{pmatrix}.$$

Since

$$\begin{aligned} \tilde{\tilde{P}} &= J(\tilde{P} + [J, \tilde{\Sigma}])J^{-1} = P + [J, \Sigma - \tilde{\Sigma}] \\ &= \begin{pmatrix} 0 & 2(\lambda_1^2 - \lambda_0^2) \frac{(\lambda_1^2 - \lambda_0^2 + \sigma_0^2) \phi_1^2 - \phi_{1,x}^2}{-(\phi_{1,x} - \sigma_0 \phi_1)^2} \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\tilde{u} \\ -1 & 0 \end{pmatrix} \end{aligned}$$

by proposition 3.1 part (4), then:

$$\begin{aligned}\tilde{u} &= 2(\lambda_1^2 - \lambda_0^2) \frac{(\lambda_1^2 - \lambda_0^2 + \sigma_0^2)\phi_1^2 - \phi_{1,x}^2}{(\phi_{1,x} - \sigma_0\phi_1)^2} \\ &= \frac{-2(\lambda_1^2 - \lambda_0^2)[\lambda_1^2 \cosh^2(z_0) + \lambda_0^2 \sinh^2(z_1)]}{[\lambda_1 \cosh(z_0) \cosh(z_1) - \lambda_0 \sinh(z_0) \sinh(z_1)]^2},\end{aligned}\tag{4.22}$$

where  $z_i = \lambda_i x - 4\lambda_i^3 t$ ,  $i = 0, 1$ , is the two soliton solution of the KdV equation.

We point out that we can rewrite the expression (4.22) as:

$$\tilde{u} = u[2] = u - 2\partial_x^2 \ln W(\phi_0, \phi_1) = -2\partial_x^2 \ln W(\phi_0, \phi_1),\tag{4.23}$$

where  $W(\phi_0, \phi_1)$  is the Wronskian, hence we get the Crum transformation, as it is stated in [10] (see also [20]).

By the Appendix, we can iterate the above procedure and obtain the  $d$ -soliton. In each iteration the Galois groups remain invariant, i.e.,  $G_i \simeq G_{i+1}$ ,  $i = 1, \dots, d-1$ , because of theorem 3.2, and isomorphism (4.18).

*Remark 4.2.* For  $t = 0$  and  $\lambda_i = (i+1)\lambda_0$ ,  $i = 1, \dots, d-1$ , then  $u[d]$  becomes:

$$u[d] = \frac{-d(d+1)\lambda_0^2}{\cosh^2(\lambda_0 x)}.$$

Rosen and Morse found that the Schrödinger equation with the above potential is solvable in closed form (in fact they studied a more general potential). A Galoisian interpretation of this fact is given in [22].

## Appendix A. Equivalence between Crum transformations and decomposable Darboux transformations

Let  $\phi_i \neq 0$  be a solution of (4.6) for  $E_i = -\lambda_i^2$ ,  $E_i \neq E_j$  if  $i \neq j$ , for  $1 \leq i \leq d$ . Let  $\phi$  be a general solution of (4.6). Now, we consider the potential

$$u[d] = u - 2\partial_x^2 \ln W(\phi_1, \dots, \phi_d)\tag{A.1}$$

and the equation

$$-\phi_{xx} + u[d]\phi = E\phi.\tag{A.2}$$

Let be the linear differential operator:

$$\mathcal{D}[d] = \partial_x^{(d)} + s_1 \partial_x^{(d-1)} + \dots + s_{d-1} \partial_x + s_d,\tag{A.3}$$

with  $s_i = -W_i/W(\phi_1, \dots, \phi_d)$ , for  $i = 1, \dots, d$ , where  $W_j$  is obtained by replacing the  $(d-i)$ -th column of  $W(\phi_1, \dots, \phi_d)$  by  $(\partial_x^{(d)} \phi_1, \dots, \partial_x^{(d)} \phi_d)^t$ . We observe that  $\mathcal{D}[d]\phi_i = 0$  for  $i = 1, \dots, d$ .

Then Crum theorem ([10]) states that the function

$$\phi[d] = \mathcal{D}[d]\phi\tag{A.4}$$



is a solution of (A.2). Moreover:

$$s_1 = -\partial_x \ln W(\phi_1, \dots, \phi_d),$$

and then  $u[d] = u + 2s_{1,x} = u - 2\partial_x^2 \ln W(\phi_1, \dots, \phi_d)$ .

We notice that, since  $\phi$  is a solution of (4.6), expression (A.4) can be rewritten as

$$\phi[d] = Q_1\phi + Q_2\phi_x, \quad (\text{A.5})$$

where  $Q_1, Q_2$  are differential polynomials of  $u, (u + \lambda^2), s_1, \dots, s_d$ .

Let us consider the  $2d \times 2d$  matrix

$$\Gamma_d = \begin{pmatrix} \vec{\psi}_1 & \vec{\psi}_1^* & \cdots & \vec{\psi}_d & \vec{\psi}_d^* \\ \lambda_1 \vec{\psi}_1 & (-\lambda_1) \vec{\psi}_1^* & \cdots & (\lambda_d) \vec{\psi}_d & (-\lambda_d) \vec{\psi}_d^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1^{d-1} \vec{\psi}_1 & (-\lambda_1)^{d-1} \vec{\psi}_1^* & \cdots & (\lambda_d)^{d-1} \vec{\psi}_d & (-\lambda_d)^{d-1} \vec{\psi}_d^* \end{pmatrix}$$

where

$$\vec{\psi}_i = \begin{pmatrix} \lambda_i \phi_i + \phi_{i,x} \\ -\phi_i \end{pmatrix} \quad \text{and} \quad \vec{\psi}_i^* = \begin{pmatrix} -\lambda_i \phi_i + \phi_{i,x} \\ -\phi_i \end{pmatrix} \quad \text{for } i = 1, \dots, d.$$

We observe that  $\det \Gamma_d \neq 0$  since, by remark 4.1,  $\vec{\psi}_i$  is a fundamental solution of (4.7) for  $\lambda = \lambda_i$  and  $\vec{\psi}_i^*$  is a fundamental solution of (4.7) for  $\lambda = -\lambda_i$ , so setting  $\Psi_i = (\vec{\psi}_i \quad \vec{\psi}_i^*)$  and  $\Lambda = \text{diag}(\lambda_i, -\lambda_i)$  we can rewrite  $\Gamma_d$  as:

$$\tilde{\Gamma}_d = \begin{pmatrix} \Psi_1 & \cdots & \Psi_d \\ \Psi_1 \Lambda_1 & \cdots & \Psi_d \Lambda_d \\ \vdots & \ddots & \vdots \\ \Psi_1 \Lambda_1^{d-1} & \cdots & \Psi_d \Lambda_d^{d-1} \end{pmatrix},$$

which is an invertible matrix.

And, by proposition 3.1 part (1), if

$$D_{\lambda_i}[d] \cdot \vec{\psi}_i = 0 \quad \text{and} \quad D_{-\lambda_i}[d] \cdot \vec{\psi}_i^* = 0 \quad \text{for } i = 1, \dots, d,$$

then

$$D_\lambda[d] := D(\vec{\psi}_1, \vec{\psi}_1^*, \dots, \vec{\psi}_d, \vec{\psi}_d^*, \lambda) = \widetilde{D}_d \left( \sum_{j=0}^{d-1} D_j(x, t) \lambda^j + \lambda^d I \right)$$

is a Darboux matrix of degree  $d$  for the system (4.7). Moreover, since  $\det \Gamma_{d-1} \neq 0$ , we have

$$D_\lambda[d] = D_1 \cdot D_\lambda[d-1]$$

with  $D_\lambda[d-1] = D(\vec{\psi}_1, \vec{\psi}_1^*, \dots, \vec{\psi}_{d-1}, \vec{\psi}_{d-1}^*, \lambda)$  a Darboux matrix of degree  $d-1$  and  $D_1 = D_\lambda[1] = D(\vec{f}_1, \vec{f}_2, \lambda)$  a Darboux matrix of degree one where

$$\vec{f}_1 = D_{\lambda_d}[d-1] \cdot \vec{\psi}_d \quad \text{and} \quad \vec{f}_2 = D_{-\lambda_d}[d-1] \cdot \vec{\psi}_d^*.$$

Let  $(\mathfrak{s}_d)$  be the corresponding system to the Darboux transform  $D_\lambda[d]$ . Because of proposition 3.1 we have that the systems (4.7) and  $(\mathfrak{s}_d)$  satisfy

$$\widetilde{P} = \widetilde{D}_d(P - [J, D_{d-1}])\widetilde{D}_d^{-1}. \quad (\text{A.6})$$

With the same notations as above, we have the following results:

**Proposition AppendixA.1.** *Let us take  $\lambda_i \in \mathbf{C}^*$ ,  $\lambda_i \neq \lambda_j$  if  $i \neq j$ , and  $\phi_i \neq 0$  solutions of (4.6) for  $E_i = -\lambda_i^2$ , for  $i = 1, \dots, d$ . The Darboux matrix of degree  $d$ , say  $D_\lambda[d]$ , satisfies:*

$$D_\lambda[d] \cdot \Phi = \Phi[d] = \begin{pmatrix} \lambda\phi[d] + \phi[d]_x \\ -\phi[d] \end{pmatrix},$$

where  $\Phi = \begin{pmatrix} \lambda\phi + \phi_x \\ -\phi \end{pmatrix}$  is a general solution of (4.7) and  $\phi$  is the general solution of (4.6).

Before starting the proof we make the following remark.

*Remark AppendixA.2.* If we apply a Crum transformation of order one to  $\phi[d]$  we must obtain  $\phi[d+1] = \mathcal{D}[d+1]\phi$ , so:

$$\phi[d+1] = \phi[d]_x - \frac{\phi[d]_{d+1,x}}{\phi[d]_{d+1}}\phi[d] = \phi[d]_x - \sigma_{d+1}\phi[d], \quad (\text{A.7})$$

where  $\phi[d]_{d+1}$  is a solution of (A.2) for  $E_{d+1}$ .

*Proof.* We prove it by induction on the degree  $d$ . If  $d = 1$ ,  $D_\lambda[1]\Phi = \Phi[1]$  has the form:

$$\Phi[1] = \begin{pmatrix} (\lambda^2 - \lambda\sigma_0 + \sigma_0^2 - \lambda_0^2)\phi + (\lambda - \sigma_0)\phi_x \\ \sigma_0\phi - \phi_x \end{pmatrix},$$

by (4.13). Since  $\phi[1] = \phi_x - \sigma_0\phi$  it follows that

$$\Psi[1] = \begin{pmatrix} \lambda\phi[1] + \phi[1]_x \\ -\phi[1] \end{pmatrix}.$$

Assume it holds for  $d$ :

Let

$$D_\lambda[d] = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \quad (\text{A.8})$$

be the Darboux matrix of degree  $d$ , then we have that:

$$D_\lambda[d]\Phi = \Phi[d] = \begin{pmatrix} (\lambda d_{11} - d_{12})\phi + \phi_x d_{11} \\ (\lambda d_{21} - d_{22})\phi + \phi_x d_{21} \end{pmatrix} = \begin{pmatrix} \lambda\phi[d] + \phi[d]_x \\ -\phi[d] \end{pmatrix}. \quad (\text{A.9})$$

Let us see it holds for  $d+1$ . Let  $D_\lambda[d+1]$  be the Darboux matrix of degree  $d+1$ . As we have seen we can decompose it as a product of a Darboux matrix  $D_\lambda[d]$  of degree  $d$  of the form (A.8) and a Darboux matrix of degree one of the form:

$$D_\lambda[1] = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} - \begin{pmatrix} \sigma_{d+1} & \sigma_{d+1}^2 - \lambda_{d+1}^2 \\ 1 & \sigma_{d+1} \end{pmatrix},$$

with  $\sigma_{d+1} = \frac{\phi[d]_{d+1,x}}{\phi[d]_{d+1}}$ , where  $\phi[d]_{d+1} \neq 0$  is a solution for  $E_{d+1} = -\lambda_{d+1}^2$ ,  $E_{d+1} \neq E_i$ ,  $1 \leq i \leq d$ , of the equation (A.2). Thus, we have:

$$D[d+1]_\lambda = D_\lambda[1]D_\lambda[d] = \lambda \begin{pmatrix} d_{11} & d_{12} \\ -d_{21} & -d_{22} \end{pmatrix} - \begin{pmatrix} \sigma_{d+1}d_{11} + d_{21}(\sigma_{d+1}^2 - \lambda_{d+1}^2) & \sigma_{d+1}d_{12} + d_{22}(\sigma_{d+1}^2 - \lambda_{d+1}^2) \\ d_{11} + \sigma_{d+1}d_{21} & d_{12} + \sigma_{d+1}d_{22} \end{pmatrix}.$$

So,

$$\begin{aligned} D[d+1]_\lambda \Phi &= \Phi[d+1] = \lambda \begin{pmatrix} (\lambda d_{11} - d_{12})\phi + \phi_x d_{11} \\ (-\lambda d_{21} + d_{22})\phi - \phi_x d_{21} \end{pmatrix} + \\ &+ \begin{pmatrix} -\sigma_{d+1}[(\lambda d_{11} - d_{12})\phi + \phi_x d_{11}] - (\sigma_{d+1}^2 - \lambda_{d+1}^2)[(\lambda d_{21} - d_{22})\phi + \phi_x d_{21}] \\ -\sigma_{d+1}[(\lambda d_{21} - d_{22})\phi + \phi_x d_{21}] - (\lambda d_{11} - d_{12})\phi - \phi_x d_{11} \end{pmatrix} \\ &= \begin{pmatrix} \lambda(\lambda\phi[d] + \phi[d]_x) - \sigma_{d+1}(\lambda\phi[d] + \phi[d]_x) + (\sigma_{d+1}^2 - \lambda_{d+1}^2)\phi[d] \\ \lambda\phi[d] + \sigma_{d+1}\phi[d] - \lambda\phi[d] - \phi[d]_x \end{pmatrix} \\ &= \begin{pmatrix} \lambda\phi[d+1] + \phi[d+1]_x \\ -\phi[d+1] \end{pmatrix} \end{aligned}$$

by (A.7) and (A.9). □

**Proposition AppendixA.3.** *The function  $\Phi[d] = \begin{pmatrix} \lambda\phi[d] + \phi[d]_x \\ -\phi[d] \end{pmatrix}$  is a solution of the AKNS system*

$$\begin{cases} \tilde{\Phi}_x = \tilde{U}\tilde{\Phi} = \begin{pmatrix} \lambda & -\tilde{u} \\ -1 & -\lambda \end{pmatrix} \tilde{\Phi}, \\ \tilde{\Phi}_t = \tilde{V}\tilde{\Phi} = \begin{pmatrix} \lambda B[\tilde{u}] - A[\tilde{u}] & \lambda^2 B[\tilde{u}] - 2\lambda A[\tilde{u}] - C[\tilde{u}] \\ -B[\tilde{u}] & -\lambda B[\tilde{u}] + A[\tilde{u}] \end{pmatrix} \tilde{\Phi}, \end{cases} \quad (\text{A.10})$$

where  $E + \lambda^2 = 0$ , if and only if  $\tilde{u} = u[d]$ .

*Proof.* Since  $\phi[d]$  satisfies equation (A.2), we have that  $\Phi[d] = \begin{pmatrix} \phi[d] \\ \phi[d]_x \end{pmatrix}$  is a solution of the following flat system:

$$\begin{cases} \Phi_x = U_d \Phi = \begin{pmatrix} 0 & 1 \\ u[d] + \lambda^2 & 0 \end{pmatrix} \Phi, \\ \Phi_t = V_d \Phi = \begin{pmatrix} A[u[d]] & B[u[d]] \\ C[u[d]] & -A[u[d]] \end{pmatrix} \Phi. \end{cases} \quad (\text{A.11})$$

Now, we apply the gauge transformation  $R = \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix}$  to system (A.11) and we obtain that

$$\tilde{\Phi}[d] = R \cdot \Phi[d] = \begin{pmatrix} \lambda\phi[d] + \phi[d]_x \\ -\phi[d] \end{pmatrix}$$

is a solution of the system:

$$\begin{cases} \tilde{\Phi}_x = \tilde{U}_d \tilde{\Phi} = R U_d R^{-1} \tilde{\Phi} = \begin{pmatrix} \lambda & -u[d] \\ -1 & -\lambda \end{pmatrix} \tilde{\Phi}, \\ \tilde{\Phi}_t = \tilde{V}_d \tilde{\Phi} = R V_d R^{-1} \tilde{\Phi} \\ = \begin{pmatrix} \lambda B[u[d]] - A[u[d]] & \lambda^2 B[u[d]] - 2\lambda A[u[d]] - C[u[d]] \\ -B[u[d]] & -\lambda B[u[d]] + A[u[d]] \end{pmatrix} \tilde{\Phi}. \end{cases} \quad (\text{A.12})$$

So, if  $\tilde{u} = u[d]$ , systems (A.10) and (A.12) are the same, thus  $\Psi[d]$  is a solution of (A.10).

Conversely, if  $\Psi[d]$  is a solution of (A.10), from the first equation we have that

$$\phi[d]_{xx} = (\tilde{u} + \lambda^2)\phi[d],$$

hence  $\tilde{u} = u[d]$ . □

Once we have the equivalence between the classical Crum transformation and the matrix Darboux transformation, then using the construction found for example in [20], we can recover the  $d$ -soliton by means of decomposable Darboux matrices.

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- [1] P. B. Acosta-Humánez, *Galoisian Approach to Supersymmetric Quantum Mechanics*, Phd Thesis, Technical Univ. of Catalonia, 2009 (<http://arxiv.org/pdf/1008.3445.pdf>)
- [2] P. B. Acosta-Humánez, J. J. Morales-Ruiz, J. A. Weil, Galoisian Approach to Integrability of Schrödinger Equation, *Reports on Mathematical Physics* **67**, (2011) 305-374.
- [3] A. Braverman, P. Etingoff, D. Gaitsgory, Quantum Integrable Systems and Differential Galois Theory, *Transformation Groups* **2**, No. 1, (1997) 31–56.
- [4] R. Beals, R. R. Coifman, Scattering and Inverse Scattering for First Order Systems: II, *Inverse Problems* **3**, (1987) 577–593.
- [5] Yu. V. Brezhnev, What does integrability of finite-gap or soliton potentials mean?, *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **366**, no. 1867 (2008) 923–945.
- [6] Yu. V. Brezhnev, Spectral/quadrature duality: Picard-Vessiot theory and finite-gap potentials. Algebraic aspects of Darboux transformations, quantum integrable systems and supersymmetric quantum mechanics, *Contemp. Math.* **563**, Amer. Math. Soc., Providence, RI, (2012) 1–31.
- [7] Yu. V. Brezhnev, Elliptic solitons, Fuchsian equations, and algorithms, *St. Petersburg Math. J.* **24**, (2013) 555–574.

- [8] J. Cano, J.-P. Ramis , Théorie de Galois différentielle, multisommabilité et phénomènes de Stokes, in preparation.
- [9] T. Crespo, H.Hajto, *Differential Galois Theory and Algebraic Groups*, American Mathematical Society, Providence, Rhode Island 2011.
- [10] M.M. Crum, Associated Sturm-Liouville Systems, *Quart. J. Math. Oxford* (2), **6**, (1955), 121–127.
- [11] G. Darboux, Sur une proposition relative aux équations linéaires, *Comptes Rendus Acad. Sci.* **94**, (1882) 1456–1459.
- [12] G. Darboux, *Théorie des Surfaces, II*, Gauthier- Villars, Paris 1889.
- [13] N. V. Grigorenko, Algebraic–Geometric Operators and Galois Differential Theory, *Ukrainian Mathematical Journal* **61**, (2009) 14–29.
- [14] C.Grotta–Ragazzo, Nonintegrability of some Hamiltonian Systems, Scattering and Analytic Continuation, *Commun. Math. Phys.* **166**, (1994) 255–277.
- [15] C. Gu, H. Hu, Z. Zhou, *Darboux Transformations in Integrable Systems. Theory and their Applications to Geometry, Mathematical Physics Studies*, Vol. 26, Springer, (2005)
- [16] I. Kaplansky, An introduction to differential algebra, Hermann (Actualites scientifiques et industrielles); Enlarged 2nd edition (1976).
- [17] E.R. Kolchin, Picard-Vessiot Theory of Partial Differential Fields, Proceedings of the A.M.S. **3**, (1952) 596–603.
- [18] E. R. Kolchin, *Differential Algebra and Algebraic Groups*, Pure and Applied Mathematics, Vol. 54, Academic Press, (1973).
- [19] J.J. Kovacic, Geometric Characterization of Strongly Normal Extensions, *Transactions of A.M.S.* **358** , (2006) 4135–4157.
- [20] V. B. Matveev, M. A. Salle, *Darboux Transformations and Solitons*, Springer Series in Nonlinear Dynamics. Springer-Verlag, Berlin, (1991).
- [21] J.J. Morales-Ruiz, J.M. Peris, On a Galosian Approach to the Splitting of Separatrices, *Annales de la Faculté des Sciences de Toulouse* **VIII**, (1999) 125–141.
- [22] J. J. Morales-Ruiz, Differential Galois Theory and Integrability, *Journal of Geometry and Physics* **87**, (2015) 314–343.
- [23] M. van der Put, M. F. Singer, *Galois Theory of Linear Differential Equations*, Grundlehren der mathematischen Wissenschaften, Volume 328, Springer Verlag, (2003).
- [24] D.H. Sattinger, V.D. Zurkowski, Gauge Theory of Bäcklund Transformations II, *Physica* **26D**, (1987) 225–250.