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This is an **author produced version** of a paper published in:

**Tunisian Journal of Mathematics 5.3 (2023): 405-456**

**DOI:** <https://doi.org/10.2140/tunis.2023.5.405>

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# Cartier transform and prismatic crystals

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Version of October 27, 2023

## Abstract

We show that the abstract equivalence of categories, called Cartier transform, between crystals on the  $q$ -crystalline and prismatic sites can be locally identified with the explicit local  $q$ -twisted Simpson correspondence. This establishes four equivalences that are all compatible with the relevant cohomology theories. We restrict ourselves for simplicity to the one-dimensional case.

## Contents

<b>Introduction</b>	<b>2</b>
<b>1 Prismatic and <math>q</math>-crystalline sites</b>	<b>3</b>
<b>2 Derived and completed envelopes</b>	<b>10</b>
<b>3 Prismatic stratifications</b>	<b>14</b>
<b>4 Prismatic differential operators</b>	<b>18</b>
<b>5 Prismatic crystals and twisted calculus</b>	<b>24</b>
<b>6 Cohomology</b>	<b>27</b>
<b>A Appendix</b>	<b>33</b>
A.1 Complete flatness . . . . .	33
A.2 Derived completeness . . . . .	35
A.3 Example: the monogenic case . . . . .	38
A.4 Example: the “bigenic” case . . . . .	42
<b>References</b>	<b>45</b>

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\*Supported by grant PGC2018-095392-B-I00 (MCI/AEI/FEDER, UE).

# Introduction

This article is the continuation of [GLQ23] and [GLQ22b]. It is devoted to giving the final arguments realizing our project, first outlined in [Gro20], of putting the local  $q$ -twisted Simpson correspondence constructed in [GLQ22a] into the perspective of the  $q$ -crystalline and prismatic sites theories. These two sites, introduced by Bhargav Bhatt and Peter Scholze in [BS22], and their beautiful properties are the framework allowing the whole picture to take form. Indeed, we interpret our correspondence, very local in nature, as the explicit description of something much deeper, global and free from arbitrary choices: the Cartier transform, a canonical map between the  $q$ -crystalline and prismatic topoi ([GLQ22b], Definition 6.8). This transform had appeared already in the first lines of the proof of Theorem 16.18 of [BS22] that was, in some sense, considering the case of the trivial crystals, and we started to look at it systematically in [GLQ22b]. Our main result takes the form of the commutative diagram of Theorem 5.5, that establishes four compatible equivalences between categories of crystals on both the  $q$ -crystalline and prismatic sites and more down-to-earth categories of modules with a certain type of “twisted” connections. Under two of these equivalences, the local  $q$ -twisted Simpson correspondence corresponds to the functor induced by the general Cartier transform. Moreover, we show that these four equivalences are all compatible with cohomology. Along the way, we explain in particular what kind of twisted de Rham complex computes locally the cohomology of crystals on these two sites.

Before describing the contents of this work, let us warn the reader that, starting from Section 2, most of the results are written only in the one-dimensional case. We have done this for clarity but, although the higher dimensional case will admittedly require a few more constructions to be written in order to develop the  $q$ -calculus, no conceptual obstacle prevents the extension in a natural way of the present results.

Let us come now to the general organization of the article, referring the reader to the main body of the text for the precise setting and statements. In Section 1, we prove that the Cartier transform, a map from the  $q$ -crystalline topos of a smooth formal scheme  $\mathcal{X}$  to the prismatic topos of its base change  $\mathcal{X}'$  by the Frobenius of the base is an equivalence of categories (Theorem 1.5). This result, and the strategy to prove it using ideas of Hidetoshi Oyama ([Oya17]) and Daxin Xu ([Xu19]), was foreseen since the writing of [GLQ23]. Soon after, a complete proof following this strategy was written by Kimihiko Li ([Li22]), but using an additional smart factorization, more precisely via higher level prismatic topoi, of some arrow in our construction we had not noticed. We include here a proof, using the perspective of Frobenius descent for the prismatic topoi, but focusing only on the aspects we need for our own goal (i.e. we only introduce the level 1 prismatic topos) and improving on [Li22] by showing also a cohomological compatibility. In Section 2, we prove (Theorem 2.2) a slight enhancement of a result to Bhatt and Scholze about the complete *faithful* flatness of some  $\delta$ -envelope. In revisiting their argument in our setting, we have tried to avoid, as far as possible, the heavy machinery they used to develop their general theory. In Section 3, still in a local situation, we show (Proposition 3.7) how to cover the final object of the prismatic topos and deduce (Proposition 3.9) from this a description of locally free prismatic crystals as finite projective modules endowed with a *prismatic stratification* (Definition 3.8), a notion that uses crucially the existence of the  $\delta$ -envelope discussed in the previous section. The reader familiar with classical crystalline cohomology will notice that this mimics exactly the description of crystals as modules endowed with a stratification. Pursuing the analogies with the classical theory, we introduce in Section

4, the *linearization* of modules (Definition 4.2) and of *prismatic differential operators* (Definition 4.9) and prove (Proposition 4.12) that linearization provides a functor from the category of finite projective modules and prismatic differential operators to the category of prismatic sheaves. In Section 5, we come to less formal and much more explicit results that allow us to describe (locally finite free) prismatic crystals as (finite projective) modules endowed with a topologically quasi-nilpotent *twisted connection of level -1*. This result (Theorem 5.1) includes the description of the prismatic envelope of the diagonal as a *twisted divided polynomial algebra of level -1*. The same kind of description works also for the  $q$ -crystalline site and makes striking the analogy with the “classical” theory of [Shi15]. It is then easy to prove (Theorem 5.5) the four equivalences of categories we mentioned before. In Section 6, we prove the natural compatibilities of our equivalences with the computation of the respective cohomologies, showing in particular that the cohomology of a prismatic crystal is computed by a *twisted de Rham complex of level -1*. An analogous result for the  $q$ -crystalline cohomology is shown to be true using the same technique, and recovers a result of Bhatt and Scholze that they proved in a different way. Finally, an appendix collects some results about complete flatness (Section A.1) and derived completeness (Section A.2) that are not due to us. They are part of the general folklore and can certainly be found here and there but, since we need them at several places, we have thought useful to gather them for the comfort of the reader and have garnished them with some useful examples (Sections A.3, A.4) to keep in mind.

We are very thankful to the referee for her/his meticulous reading, allowing us to improve on the first version of the article.

Throughout the article, we fix a prime  $p$ .

**Caveat:** In the core of the paper, we will denote by

$$\widehat{M}_{\{ \} } = \varprojlim_{n \in \mathbb{N}} M/I^n$$

(or  $M^\wedge$ ) the *classical* completion of a module  $M$  for an  $I$ -adic topology that should be understood from the context. In contrast, in the appendix,  $\widehat{M}$  will denote the *derived completion*. These choices are made to lighten the notation according to the context.

## 1 Prismatic and $q$ -crystalline sites

We briefly recall here the formalism of prismatic and  $q$ -crystalline sites introduced by Bhatt and Scholze in [BS22] and enhance some comparison theorems of Li from [Li22] with a discussion of cohomology.

### Prismatic crystals

We refer the reader to [Joy85] or to Section 1 of [GLQ22b] for the basics on  $\delta$ -rings, and in particular the fact that they carry a Frobenius  $\phi$ . If  $B$  is a  $\delta$ -ring and  $I \subset B$  is an ideal, then  $(B, I)$  is called a  $\delta$ -pair. The  $\delta$ -pairs form a category in an obvious way. A  $\delta$ -pair  $(B, I)$  is called a *prism* if  $I$  is an invertible ideal,  $B$  is derived complete (see Definition A.6 in the appendix) for the  $(p, I)$ -adic topology and  $p \in I + \phi(I)B$ . Unless otherwise specified, we will always use the  $(p, I)$ -adic topology. We will also consider below the corresponding *Nygaard ideal*  $\mathfrak{N}_B := \phi^{-1}(I) \subset B$ . A remarkable feature of prisms is that,

if  $(B, I) \rightarrow (B', I')$  is a morphism, then necessarily  $I' = IB'$ . The prism is said to be *bounded* if  $B/I$  has bounded  $p^\infty$ -torsion (see Definition A.12 in the appendix). A prism  $(B, I)$  is said to be *orientable* if  $I$  is principal. The choice of a generator  $d$  is called an *orientation* and we may then simply write  $(B, d)$  instead of  $(B, (d))$ . In this situation, the conditions on  $I$  are equivalent to  $B$  being  $d$ -torsion free and  $d$  being *distinguished* (i.e.  $\delta(d) \in B^\times$ ). We will call a  $B$ -module  $M$  *bounded* (with respect to  $p$  modulo  $d$ ) when it is  $d$ -torsion free and  $M/dM$  has bounded  $p^\infty$ -torsion. Its derived completion is then automatically classically complete (see Proposition A.22 in the appendix).

The underlying category of the *absolute prismatic site*  $\mathbb{A}$  is the category opposite to the category of bounded prisms (but one still writes morphisms in the natural direction). The category  $\mathbb{A}$  does not have a final object or fibered products in general.

Recall (see Definition A.5 in the appendix) that, when  $R$  is an adic ring, an  $R$ -module is said to be *formally (faithfully) flat* if it becomes (faithfully) flat modulo any ideal of definition of  $R$ . A morphism of rings  $R \rightarrow S$  is said to be *formally (faithfully) flat* if it turns  $S$  into a formally (faithfully) flat  $R$ -module. A morphism of bounded prisms is formally (faithfully) flat if and only if it is completely (faithfully) flat (see Proposition A.23 in the appendix for the orientable case, which is the one we will use, or Proposition 1.4 in [Tia21] in a more general situation). Pulling back in  $\mathbb{A}$  along a formally flat morphism is always possible (see the proof of corollary 3.12 in [BS22] or lemma 3.8 of [MT20] for example). It follows that formally faithfully flat morphisms define a pretopology that will actually spread to all the categories that we consider below. This is formalized as follows: if we call a fibered category  $T$  over  $\mathbb{A}$  a *prismatic site*, then  $T$  inherits a pretopology from  $\mathbb{A}$  (the coarsest topology making the fibration cocontinuous). Moreover, any morphism of prismatic sites (i.e. of fibered categories over  $\mathbb{A}$ )  $u : T' \rightarrow T$  is automatically continuous and cocontinuous, and therefore induces a series of adjoint functors  $u_!, u^{-1}, u_*$ , the last two of them defining a morphism of topoi.

One may always consider an object of a prismatic site  $T$  as a morphism  $(B, I) \rightarrow T$  if we identify the bounded prism  $(B, I)$  with the fibered category that it represents. A presheaf  $E$  on  $T$  (which we will call a *prismatic presheaf*) is then a family of its *realizations*  $E_B$  for all bounded prisms  $(B, I)$  over  $T$ , together with a compatible family of transition maps  $E_B \rightarrow E_{B'}$  when  $(B, I) \rightarrow (B', I')$  is a morphism of bounded prisms over  $T$ . By definition, it is a sheaf if

$$E_B \simeq \ker \left( E_{B'} \rightrightarrows E_{B' \widehat{\otimes}_B B'} \right) \quad (1)$$

whenever the morphism  $(B, I) \rightarrow (B', I')$  is cartesian over  $T$  and the map  $B \rightarrow B'$  is formally faithfully flat. Note that  $E$  is then also a sheaf for the Zariski topology (see remark 2.16 of [BS22]). As an example, we can consider the sheaf of rings  $\mathcal{O}_T$  whose realization on  $(B, I)$  is the ring  $B$ . A morphism of prismatic sites  $u : T' \rightarrow T$  is automatically a flat morphism of ringed sites: even better, we have  $u^{-1}\mathcal{O}_T = \mathcal{O}_{T'}$ . If we are given a presheaf of  $\mathcal{O}_T$ -modules  $E$ , then we can linearize the transition maps as  $B' \otimes_B E_B \rightarrow E_{B'}$ . The presheaf is called a *crystal* if all linearized transition maps are bijective (be careful that this notion depends on the site and *not* only on the topos). Note that we do not require that a crystal is a sheaf *a priori*, even if this will be the case in practice. The inverse image of a crystal by a morphism of prismatic sites is automatically a crystal.

Before going any further, let us recall that formal faithfully flat descent holds for classically complete modules:

**Lemma 1.1.** *Let  $R$  be an adic ring with respect to a finitely generated ideal  $I$ . Let  $B \rightarrow C$  be a formally faithfully flat morphism of classically complete  $R$ -algebras. Then the*

classical complete pullback functor  $M \mapsto M' := C \hat{\otimes}_B M$  induces an equivalence between classically complete  $B$ -modules  $M$  and classically complete  $C$ -modules  $M'$  endowed with an isomorphism  $C \hat{\otimes}_B M' \simeq M' \hat{\otimes}_B C$  satisfying the usual cocycle condition. Moreover,  $M$  is finite projective if and only if  $M'$  is.

*Proof.* It follows from [Sta19, Tag 09B8] that the category of classically complete  $R$ -modules is equivalent to the category of inverse systems  $\{M_n\}_{n \in \mathbb{N}}$  of  $R/I^{n+1}$ -modules endowed with a compatible family of isomorphisms  $M_{n+1}/I^{n+1}M_{n+1} \simeq M_n$ . The first assertion therefore follows from classical faithfully flat descent ([Sta19, Tag 023N]) modulo  $I^{n+1}$ . The second one follows from classical faithfully flat descent for projective modules ([Sta19, Tag 058S]) and the fact that an  $A$ -module is finite projective if and only if it is so modulo  $I^{n+1}$  for all  $n$  ([Sta19, Tag 0D4B]).  $\square$

**Remark.** Yichao Tian gives a derived version of Lemma 1.1 in Proposition 1.8 of [Tia21].

In the situation of Lemma 1.1, there exists an exact sequence

$$0 \rightarrow M \rightarrow C \hat{\otimes}_B M \rightrightarrows C \hat{\otimes}_B C \hat{\otimes}_B M.$$

When  $M$  is finite projective<sup>1</sup>, we actually have  $M' = C \otimes_B M$  and the sequence reads

$$0 \rightarrow M \rightarrow C \otimes_B M \rightrightarrows (C \hat{\otimes}_B C) \otimes_B M. \quad (2)$$

**Proposition 1.2.** *For a presheaf  $E$  on a prismatic site  $T$ , the following conditions are equivalent:*

1.  *$E$  is a locally finite free sheaf of  $\mathcal{O}_T$ -modules,*
2.  *$E$  is a crystal with finite projective realizations.*

We will then call  $E$  a *locally finite free prismatic crystal*.

*Proof.* Let  $(B, I)$  be a bounded prism over  $T$ . Assume that  $E_B$  is finite projective. It follows from the remark after Lemma 1.1 that, if  $C$  is a complete bounded  $\delta$ - $R$ -algebra and  $B \rightarrow C$  is a formally faithfully flat morphism, then the sequence

$$0 \rightarrow E_B \rightarrow C \otimes_B E_B \rightrightarrows (C \hat{\otimes}_B C) \otimes_B E_B$$

is left exact. Moreover, we can find a complete bounded  $\delta$ - $R$ -algebra  $C$  and a formally faithfully flat morphism  $B \rightarrow C$  such that  $C \otimes_B E_B$  is finite free (use a Zariski covering). This shows that the second condition implies the first. Assume conversely that  $E$  is a locally finite free prismatic sheaf of  $\mathcal{O}_T$ -modules. By definition (of locally finite freeness), there exists a formally faithfully flat morphism of bounded prisms  $B \rightarrow C$  such that  $E_C \simeq C^r$  as *sheaves over  $C$* : this implies that any transition map  $C_1 \otimes_C E_C \rightarrow E_{C_1}$  for a given  $C \rightarrow C_1$  is bijective. In particular,  $E_C$  comes with a descent datum

$$(C \hat{\otimes}_B C) \otimes_C E_C \simeq E_{C \hat{\otimes}_B C} \simeq E_C \otimes_C (C \hat{\otimes}_B C)$$

as in Lemma 1.1 and therefore defines a finite projective  $B$ -module  $M$ . Since  $E$  is a sheaf, we must have  $E_B = M$  (compare the short exact sequences (1) and (2)). It remains to check that the transition maps  $B_1 \otimes_B E_B \rightarrow E_{B_1}$  are bijective for all maps  $B \rightarrow B_1$ , but this follows by formal faithfully flat descent from the analogous assertion on  $C$ .  $\square$

<sup>1</sup>This is equivalent to finitely presented flat.

A prismatic presheaf  $E$  will be called *complete* (resp. *formally (faithfully) flat*) if  $E_B$  is classically complete (resp. formally (faithfully) flat) for all bounded prisms  $(B, I)$ . Note that both conditions together are equivalent to  $E_B$  being derived complete and completely (faithfully) flat thanks to Theorem A.25 in the appendix. One may also consider the alternative notion of a *complete prismatic crystal* with the requirement  $B' \hat{\otimes}_B E_B \simeq E_{B'}$  as in [Li22] or [Tia21] for example. It then follows from Lemma 1.1 that a complete prismatic crystal is automatically a prismatic sheaf.

## Level changing

We let  $R$  be a complete  $\delta$ -ring and  $d \in R$  a distinguished element such that  $R$  is bounded (with respect to  $d$  modulo  $p$ ). Actually, it is not strictly necessary to assume that  $R$  is complete since this will be our base ring and that any morphism with value in a complete ring will extend uniquely to the completion  $\hat{R}$  of  $R$  (compatibly with the extra structures) and  $\hat{R}$  is bounded by proposition A.22. We may then consider the oriented<sup>2</sup> bounded prism  $(R, d)$ , the relative prismatic site  $\Delta(/R)$  that it represents and denote by  $/R_\Delta$  the corresponding topos. The site  $\Delta(/R)$  is (anti) equivalent to the category of complete bounded  $\delta$ - $R$ -algebras  $B$  and we do not need to mention the ideal (which is automatically equal to  $dB$ ) anymore. There exist obvious functors

$$\begin{aligned} \Delta(/R) &\rightarrow \mathbf{FS}_{/\mathrm{Spf}(R/d)}, & B &\mapsto \mathrm{Spf}(B/dB), \\ \Delta(/R) &\rightarrow \mathbf{FS}_{/\mathrm{Spf}(R/\mathfrak{N}_R)}, & B &\mapsto \mathrm{Spf}(B/\mathfrak{N}_B), \end{aligned}$$

where  $\mathfrak{N}_B = \phi^{-1}(d)B$  denotes the Nygaard ideal introduced above. If  $\mathcal{X}$  is a formal scheme over  $R/d$  (resp.  $R/\mathfrak{N}_R$ ), then we can pull back the fibered category that it represents and we obtain a prismatic site  $\Delta(\mathcal{X}/R)$  (resp.  $\Delta^{(1)}(\mathcal{X}/R)$ ). An object is a complete bounded  $\delta$ - $R$ -algebra  $B$  endowed with a structural map  $\mathrm{Spf}(B/dB) \rightarrow \mathcal{X}$  (resp.  $\mathrm{Spf}(B/\mathfrak{N}_B) \rightarrow \mathcal{X}$ ). A morphism in this site is a morphism of  $\delta$ - $R$ -algebras  $B \rightarrow C$  such that the induced map

$$\mathrm{Spf}(C/dC) \rightarrow \mathrm{Spf}(B/dB) \quad (\text{resp. } \mathrm{Spf}(C/\mathfrak{N}_C) \rightarrow \mathrm{Spf}(B/\mathfrak{N}_B))$$

is compatible with the structural maps. We will denote by  $(\mathcal{X}/R)_\Delta$  (resp.  $(\mathcal{X}/R)_{\Delta^{(1)}}$ ) the corresponding topos<sup>3</sup>.

The Frobenius of  $R$  induces a morphism  $\bar{\phi} : R/\mathfrak{N}_R \hookrightarrow R/d$ . If  $\mathcal{X}$  is a formal scheme over  $R/\mathfrak{N}_R$  and we set  $\mathcal{X}' := \mathcal{X} \hat{\otimes}_{\{R/\mathfrak{N}_R \xrightarrow{\bar{\phi}} R/d\}} R/d$ , then there exists a *level changing* functor

$$\rho : \Delta^{(1)}(\mathcal{X}/R) \rightarrow \Delta(\mathcal{X}'/R).$$

This functor does nothing at the prism level and sends the structural map  $\mathrm{Spf}(B/\mathfrak{N}_B) \rightarrow \mathcal{X}$  to

$$\mathrm{Spf}(B/dB) \rightarrow \mathrm{Spf}(B/\mathfrak{N}_B) \hat{\otimes}_{\{R/\mathfrak{N}_R \xrightarrow{\bar{\phi}} R/d\}} R/d \rightarrow \mathcal{X} \hat{\otimes}_{\{R/\mathfrak{N}_R \xrightarrow{\bar{\phi}} R/d\}} R/d = \mathcal{X}',$$

where the first map is induced by  $\bar{\phi} : B/\mathfrak{N}_B \hookrightarrow B/dB$ . This change of level is a morphism of prismatic sites which is functorial in  $\mathcal{X}/R$  and fully faithful if  $\mathcal{X}$  is affine (see Proposition 1.6 in [Li22]). But there is more (the theorem is actually valid if we only assume  $\mathcal{X}$  separated but it is then necessary to modify a bit the site that gives rise to the topos):

<sup>2</sup>Everything below generalizes to nonoriented bounded prisms, but we are only interested in this case.

<sup>3</sup>When  $\mathcal{X}$  is not affine, our definition of the sites are slightly different from those in [Li22] but the topos are always identical.

**Theorem 1.3** (Li). *If  $(R, d)$  is an oriented bounded prism,  $\mathcal{X}$  is a smooth<sup>4</sup> affine formal scheme over  $R/\mathfrak{N}_R$  and  $\mathcal{X}' := \mathcal{X} \widehat{\otimes}_{\{R/\mathfrak{N}_R \nearrow \overline{\phi}\}} R/d$ , then there exists an equivalence of topoi*

$$\rho : (\mathcal{X}/R)_{\Delta^{(1)}} \simeq (\mathcal{X}'/R)_{\Delta}.$$

*Proof.* This is essentially Theorem 1.15 of [Li22] and we will only sketch it. According to Proposition 4.2.1 of [Oya17], it is sufficient to show that the functor is locally surjective at the site level. It means that any object in  $\Delta(\mathcal{X}'/R)$  can be covered by an object in  $\Delta^{(1)}(\mathcal{X}/R)$ . In other words, given a  $\delta$ - $R$ -algebra  $B'$  and a morphism  $\mathrm{Spf}(B'/dB') \rightarrow \mathcal{X}'$ , we have to show that there exists a formally faithfully flat morphism of  $\delta$ - $R$ -algebras  $B' \rightarrow B$  and a morphism  $\mathrm{Spf}(B/\mathfrak{N}_B) \rightarrow \mathcal{X}$  such that the diagram

$$\begin{array}{ccc} \mathrm{Spf}(B/dB) & \longrightarrow & \mathrm{Spf}(B/\mathfrak{N}_B) \widehat{\otimes}_{\{R/\mathfrak{N}_R \nearrow \overline{\phi}\}} R/d \\ \downarrow & & \downarrow \\ \mathrm{Spf}(B'/dB') & \longrightarrow & \mathcal{X}' = \mathcal{X} \widehat{\otimes}_{\{R/\mathfrak{N}_R \nearrow \overline{\phi}\}} R/d \end{array}$$

is commutative. This is a local question and we may therefore assume that there exists a completely regular sequence  $\underline{f}$  in  $R[\underline{x}]$  such that  $\mathcal{X} = \mathrm{Spf}(A)$  with  $A = (R/\mathfrak{N}_R)[\underline{x}]/(\underline{f})$ . We may then consider, as in [BS22], Proposition 3.13, the *prismatic envelope*  $S$  (resp.  $S'$ ) of

$$1 \otimes \underline{f} \in R_{\phi} \widehat{\otimes}_R R[\underline{x}]^{\delta} \quad \left( \text{resp. } \phi(\underline{f}) \in R[\underline{x}]^{\delta} \right) \{$$

and set  $B := S \widehat{\otimes}_{S'} B'$ . □

As an immediate consequence of Theorem 1.3, we obtain from Proposition 1.2 an equivalence between the category of locally finite free 1-prismatic crystals  $E$  on  $\mathcal{X}/R$  (i.e. crystals on  $\Delta^{(1)}(\mathcal{X}/R)$ ) and the category of locally finite free prismatic crystals  $E'$  on  $\mathcal{X}'/R$  (i.e. crystals on  $\Delta(\mathcal{X}'/R)$ ). There also exists an isomorphism on cohomology

$$\mathrm{R}\Gamma((\mathcal{X}/R)_{\Delta^{(1)}}, E) \simeq \mathrm{R}\Gamma((\mathcal{X}'/R)_{\Delta}, E')$$

and therefore

$$\forall k \in \mathbb{Z}, \quad \mathrm{H}^k((\mathcal{X}/R)_{\Delta^{(1)}}, E) \simeq \mathrm{H}^k((\mathcal{X}'/R)_{\Delta}, E').$$

## $q$ -crystals

We fix now an indeterminate  $q$  and consider the local ring  $\mathbb{Z}[q]_{(p, q-1)}$  endowed with the unique  $\delta$ -structure such that  $\delta(q) = 0$ . We will denote by  $(n)_q$  the  $q$ -analog of an integer  $n \in \mathbb{Z}$ . We define a  $q$ -PD-pair as a  $\delta$ -pair  $(B, J)$ , where  $B$  is a  $(p)_q$ -torsion free  $\delta$ - $\mathbb{Z}[q]_{(p, q-1)}$ -algebra and  $J$  an ideal of  $B$  such that

$$\forall f \in J, \quad \phi(f) - (p)_q \delta(f) \in (p)_q J.$$

In order to comply with the existing literature, we will also assume that  $q - 1 \in J$  and that  $B/(q - 1)$  is  $p$ -torsion free with finite complete Tor amplitude. The  $q$ -PD-pair is said to be *complete* if  $B$  is derived complete (automatically classically complete when  $B$  is bounded) and, moreover,  $J$  is closed in  $B$ . We may then consider the category  $q$ -CRIS

<sup>4</sup>We mean formally smooth and locally finitely presented (or, equivalently here, completely smooth).



whose objects are complete bounded  $q$ -PD-pairs  $(B, J)$  and morphisms again go in the other direction, even if we still write them in the usual way. There exists a final object  $(\mathbb{Z}_p[[q-1]], q-1)$  in this category.

The forgetful functor

$$q\text{-CRIS} \rightarrow \Delta, \quad (B, J) \mapsto (B, (p)_q)$$

is a fibration (in ordered sets) and  $q\text{-CRIS}$  is therefore a prismatic site called the *absolute  $q$ -crystalline site*. A morphism  $(B, J) \rightarrow (B', J')$  of complete bounded  $q$ -PD-pairs is cartesian when  $\overline{JB'}^{\text{cl}} = J'$  (we use the bar here to denote the closure). In particular, this is a covering when the map  $B \rightarrow B'$  is formally faithfully flat and  $\overline{JB'}^{\text{cl}} = J'$ . One defines more generally a  *$q$ -crystalline site* as a fibered category over  $q\text{-CRIS}$  (this is then automatically a prismatic site) and then talk about  $q$ -presheaves and so on.

We fix a (complete) bounded  $q$ -PD-pair  $(R, \mathfrak{r})$  and we consider the  $q$ -crystalline site  $q\text{-CRIS}(R)$  which is represented by  $(R, \mathfrak{r})$ . Then, the above forgetful functor (the fibration) induces a morphism of prismatic sites  $q\text{-CRIS}(R) \rightarrow \Delta(R)$  (where by  $\Delta(R)$  we mean the prismatic site over  $(R, (p)_q)$ ), but the situation is even nicer now. This forgetful functor has an adjoint  $B \mapsto (B, \overline{\mathfrak{r}B}^{\text{cl}})$  which is a fully faithful morphism of prismatic sites, and also a coadjoint  $B \mapsto (B, \mathfrak{N}_B)$  which is fully faithful and cocontinuous (but is not a morphism of prismatic sites because  $\overline{\mathfrak{N}_B B'}^{\text{cl}} \subsetneq \mathfrak{N}_{B'}$  in general). Anyway, we obtain a sequence of adjoint morphisms of topoi

$$R_\Delta \hookrightarrow R_{q\text{-CRIS}}, \quad R_{q\text{-CRIS}} \rightarrow R_\Delta, \quad R_\Delta \hookrightarrow R_{q\text{-CRIS}}.$$

As above, there also exists a functor

$$q\text{-CRIS}(R) \rightarrow \mathbf{FS}_{/\text{Spf}(R/\mathfrak{r})}, \quad (B, J) \mapsto \text{Spf}(B/J).$$

If  $\mathcal{X}$  is a formal scheme over  $R/\mathfrak{r}$ , we can then consider its fiber  $q\text{-CRIS}(\mathcal{X}/R)$  as we did above and we will denote by  $(\mathcal{X}/R)_{q\text{-CRIS}}$  the corresponding topos. An object of  $q\text{-CRIS}(\mathcal{X}/R)$  is a complete bounded  $q$ -PD-pair  $(B, J)$  over  $(R, \mathfrak{r})$  endowed with a structural map  $\text{Spf}(B/J) \rightarrow \mathcal{X}$  over  $R/\mathfrak{r}$  and morphisms are defined as usual. If we write  $\mathcal{X}_1 := \mathcal{X} \hat{\otimes}_{R/\mathfrak{r}} R/\mathfrak{N}_R$ , then there exists a couple of adjoint functors

$$\Delta^{(1)}(\mathcal{X}_1/R) \rightarrow q\text{-CRIS}(\mathcal{X}/R) \quad \text{and} \quad q\text{-CRIS}(\mathcal{X}/R) \rightarrow \Delta^{(1)}(\mathcal{X}_1/R)$$

such that  $\circ = \text{Id}$  (and this is natural in  $\mathcal{X}/R$ ). The functor  $\rightarrow$  sends the  $\delta$ - $R$ -algebra  $B'$  to the  $q$ -PD-pair  $(B', \mathfrak{N}_{B'})$  and the structural map  $\text{Spf}(B'/\mathfrak{N}_{B'}) \rightarrow \mathcal{X}_1$  to the composite map  $\text{Spf}(B'/\mathfrak{N}_{B'}) \rightarrow \mathcal{X}_1 \hookrightarrow \mathcal{X}$ . The functor  $\leftarrow$  sends the  $q$ -PD-pair  $(B, J)$  to the  $\delta$ - $R$ -algebra  $B$  and the structural map  $\text{Spf}(B/J) \rightarrow \mathcal{X}$  to the composite

$$\text{Spf}(B/\mathfrak{N}_B) \rightarrow \text{Spf}(B/J) \hat{\otimes}_{R/\mathfrak{r}} R/\mathfrak{N}_R \rightarrow \mathcal{X} \hat{\otimes}_{R/\mathfrak{r}} R/\mathfrak{N}_R = \mathcal{X}_1.$$

Both functors are morphisms of prismatic sites (and the first one is fully faithful). Thus, there exists a pair of adjoint morphisms of topoi

$$(\mathcal{X}_1/R)_{\Delta^{(1)}} \rightarrow (\mathcal{X}/R)_{q\text{-CRIS}} \quad \text{and} \quad (\mathcal{X}/R)_{q\text{-CRIS}} \rightarrow (\mathcal{X}_1/R)_{\Delta^{(1)}}$$

such that  $\circ = \text{Id}$  giving rise to a sequence of adjoint functors

$$!, \quad {}^{-1} = !, \quad * = {}^{-1}, \quad *.$$

The last part of the next lemma is also due to Li ([Li22], Theorem 3.2):

**Lemma 1.4.** *If  $\mathcal{X}$  is a formal scheme over  $R/\mathfrak{r}$  and  $\mathcal{X}_1 := \mathcal{X} \hat{\otimes}_{R/\mathfrak{r}} R/\mathfrak{N}_R$ , then the morphism of topoi*

$$(\cdot)_* : (\mathcal{X}_1/R)_{\Delta^{(1)}} \rightarrow (\mathcal{X}/R)_{q\text{-CRIS}}$$

*is an exact embedding ( $(\cdot)_*$  is exact and fully faithful). Moreover, it induces an equivalence between locally finite free crystals on both sides.*

*Proof.* After the previous discussion, only the last assertion needs a proof. If  $B'$  is the underlying  $\delta$ - $R$ -algebra of an object of  $\Delta^{(1)}(\mathcal{X}_1/R)$  and  $E$  is a sheaf on  $q\text{-CRIS}(\mathcal{X}/R)$ , then we have

$$(\cdot)^{-1}E)_{B'} = E_{(B', \mathfrak{N}_{B'})}.$$

On the other hand, if  $(B, J)$  is the underlying  $q$ -PD-pair of an object of  $q\text{-CRIS}(\mathcal{X}/R)$  and  $E'$  is a sheaf on  $\Delta^{(1)}(\mathcal{X}_1/R)$ , then we have

$$(\cdot)_*E')_{(B, J)} = (\cdot)^{-1}E')_{(B, J)} = E'_B.$$

The assertion relative to locally finite free crystals is therefore clear.  $\square$

If  $E'$  corresponds to  $E$  in Lemma 1.4, then  $R_*E' = {}_R^*E' = E$  and therefore also

$$R_*E = R_*R_*E' = R(\circ)_*E' = E'.$$

In particular, we have

$$R\Gamma((\mathcal{X}/R)_{q\text{-CRIS}}, E) \simeq R\Gamma((\mathcal{X}_1/R)_{\Delta^{(1)}}, E')$$

and

$$\forall k \in \mathbb{Z}, \quad H^k((\mathcal{X}/R)_{q\text{-CRIS}}, E) \simeq H^k((\mathcal{X}_1/R)_{\Delta^{(1)}}, E').$$

## Cartier transform

We now come to the Cartier transform that we introduced in Definition 6.8 of [GLQ22b]. We fix a bounded  $q$ -PD-pair  $(R, \mathfrak{r})$ , we let  $\mathcal{X}$  be a formal scheme over  $R/\mathfrak{r}$  and we set

$$\mathcal{X}' := \mathcal{X} \hat{\otimes}_{\mathfrak{f}_{R/\mathfrak{r}} \nearrow \overline{\phi}} R/(p)_q.$$

We consider the functor

$$C : q\text{-CRIS}(\mathcal{X}/R) \rightarrow \Delta(\mathcal{X}'/R)$$

that sends the  $q$ -PD-pair  $(B, J)$  to the  $\delta$ - $R$  algebra  $B$  and the structural map  $\mathrm{Spf}(B/J) \rightarrow \mathcal{X}$  to

$$\mathrm{Spf}(B/(p)_q) \rightarrow \mathrm{Spf}(B/J) \hat{\otimes}_{\mathfrak{f}_{R/\mathfrak{r}} \nearrow \overline{\phi}} R/(p)_q \rightarrow \mathcal{X} \hat{\otimes}_{\mathfrak{f}_{R/\mathfrak{r}} \nearrow \overline{\phi}} R/(p)_q,$$

where the first map is induced by  $\overline{\phi} : B/J \rightarrow B/(p)_q B$ . This is a morphism of prismatic sites that provides us with a morphism of topoi

$$C : (\mathcal{X}/R)_{q\text{-CRIS}} \rightarrow (\mathcal{X}'/R)_{\Delta}.$$

The next theorem (which is actually valid for  $\mathcal{X}$  separated) is essentially due to Li ([Li22]).

**Theorem 1.5.** *Let  $(R, \mathfrak{r})$  be a bounded  $q$ -PD pair,  $\mathcal{X}$  a smooth affine formal scheme over  $R/\mathfrak{r}$  and  $\mathcal{X}' := \mathcal{X} \hat{\otimes}_{R/\mathfrak{r}} R/(p)_q$ . Then  $\mathrm{RC}_*$  and  $C^{-1}$  induce an equivalence between the category of locally finite free  $q$ -crystals  $E$  on  $\mathcal{X}$  over  $R$  and the category of locally finite free prismatic crystals  $E'$  on  $\mathcal{X}'$  over  $R$ .*

*Proof.* We have the following decomposition

$$C : q\text{-CRIS}(\mathcal{X}/R) \rightarrow \Delta^{(1)}(\mathcal{X}_1/R) \xrightarrow{\rho} \Delta(\mathcal{X}'/R),$$

where  $\mathcal{X}_1 := \mathcal{X} \hat{\otimes}_{R/\mathfrak{r}} R/\mathfrak{N}_R$ , is as in Lemma 1.4 and  $\rho$  is as in Theorem 1.3 (but with  $\mathcal{X}_1$  instead of  $\mathcal{X}$ ). We know from Proposition 1.2 that a finite locally free crystal is the same thing as a finite locally free sheaf and therefore a topos-theoretic notion. Thanks to Theorem 1.3, we are therefore reduced to proving our assertion with  $C$  replaced by  $\rho$  and this then follows from Lemma 1.4.  $\square$

As a consequence, we will have in the setting of Theorem 1.5,

$$\mathrm{R}\Gamma((\mathcal{X}/R)_{q\text{-CRIS}}, E) \simeq \mathrm{R}\Gamma((\mathcal{X}'/R)_{\Delta}, E')$$

(for the trivial crystal, this is due to Bhatt and Scholze ([BS22], Theorem 16.18)) and therefore also

$$\forall k \in \mathbb{Z}, \quad H^k((\mathcal{X}/R)_{q\text{-CRIS}}, E) \simeq H^k((\mathcal{X}'/R)_{\Delta}, E').$$

## 2 Derived and completed envelopes

We review some material from [BS22] that will be needed afterwards. We shall try to stay in the realm of “classical” constructions and avoid arguments from nonabelian derived algebra (see however the thesis [Mao21] of Zouhang Mao for a detailed approach using animated rings).

We let  $R$  be a commutative ring and fix some  $d \in R$ . An  $R$ -module  $M$  will always be endowed with its  $(p, d)$ -adic topology and we will write  $\overline{M} = M/dM$ .

At some point, we will assume that  $R$  is a  $\delta\text{-}\mathbb{Z}_{(p)}$ -algebra and that  $d$  is distinguished in  $R$ .

We refer the reader to the appendix for basic results concerning complete flatness and derived completions. Let us just recall from Definition A.3 in the appendix that a complex  $M^\bullet$  of  $R$ -modules is said to be *completely (faithfully) flat* if the complex  $\overline{R}/p\overline{R} \otimes_R^L M^\bullet$  of  $\overline{R}/p\overline{R}$ -modules is discrete (faithfully) flat.

**Definition 2.1.** An element  $g$  in an  $R$ -algebra  $B$  is *completely (faithfully) regular over  $R$*  if the complex  $[B \xrightarrow{g} B]$  is completely (faithfully) flat over  $R$ .

**Remarks.** 1. The complex  $[B \xrightarrow{g} B]$  is the Koszul complex  $\mathrm{Kos}(B, g)$  (see [Sta19, Tag 0623]). We have

$$[B \xrightarrow{g} B] \simeq \mathbb{Z} \otimes_{\mathbb{Z}[x]}^L B,$$

with  $x \mapsto 0$  on the left and  $x \mapsto g$  on the right. In particular, Definition 2.1 generalizes in a straightforward way to the case of a complex of  $R$ -modules  $M^\bullet$  endowed with a sequence of commuting endomorphisms  $g_1, \dots, g_d$  (which may then be seen as a complex of  $\mathbb{Z}[x_1, \dots, x_d]$ -modules).

2. It follows from the last remarks after Definition A.4 in the appendix that the property of being completely (faithfully) regular is stable under derived pullback (and complete regularity is detected by derived pullback under a completely faithfully flat morphism). It also follows from Proposition A.10 in the appendix that it is invariant under derived completion.
3. Assume  $B$  is completely flat over  $R$ . Then,  $g$  is completely (faithfully) regular over  $R$  if and only if  $g$  becomes a nonzero divisor in  $\overline{B}/p\overline{B}$  and  $(\overline{B}/g\overline{B})/p(\overline{B}/g\overline{B})$  is (faithfully) flat over  $\overline{R}/p\overline{R}$ . In particular, the property then only depends on the class of  $g$  modulo  $(p, d)$ .
4. If  $B$  is completely flat over  $R$  and  $g \in B$  is completely (faithfully) regular over  $R$ , then any power  $g^k$  of  $g$  is also completely (faithfully) regular over  $R$  (use the previous remark and induction). As a consequence (and using the previous remark again),  $\phi(g)$  will also be completely (faithfully) regular over  $R$  when  $R$  is a  $\delta$ -ring.
5. We will need this notion of complete regularity only in the following very simple case: if  $B$  is a  $R$ -algebra, then the variable  $\xi$  is completely faithfully regular over  $B$  in the polynomial ring  $B[\xi]$ .

So far, we have not used the fact that  $R$  is a  $\delta$ -ring (nor that  $d$  is distinguished). We will keep our notation from [GLQ23] and denote<sup>5</sup> by  $R[x]^\delta$  the  $\delta$ -envelope of  $R[x]$ , that is, the polynomial ring

$$R[\{x_k\}_{k \in \mathbb{N}}]$$

endowed with the unique  $\delta$ -structure such that  $\delta(x_k) = x_{k+1}$  for  $k \in \mathbb{N}$ . We will systematically use the fact ([BS22], Lemma 2.11) that the Frobenius  $\phi : R[x]^\delta \rightarrow R[x]^\delta$  is faithfully flat.

If  $B$  is a  $\delta$ - $R$ -algebra and  $g \in B$ , we will write

$$B[g/d]^\delta := R[w]^\delta \otimes_{R[x]^\delta}^L B$$

with  $x \mapsto dw$  on the left and  $x \mapsto g$  on the right (note that this is the derived version). Observe also that, if  $B \rightarrow C$  is a morphism of  $\delta$ - $R$ -algebras, then

$$C \otimes_B^L B[g/d]^\delta \simeq C[g/d]^\delta$$

if we still denote by  $g$  the image of  $g$  in  $C$ .

We will need the following particular case of Proposition 3.13 of [BS22] (actually we need the faithful version which is not stated in loc. cit.):

**Theorem 2.2** (Bhatt-Scholze). *Let  $R$  be a  $\delta$ - $\mathbb{Z}_{(p)}$ -algebra and  $d \in R$  distinguished. If  $B$  is a  $\delta$ - $R$ -algebra and  $g \in B$  is completely (faithfully) regular over  $R$ , then the complex  $B[g/d]^\delta$  is completely (faithfully) flat over  $R$ .*

*Proof.* Since the Frobenius  $\phi : \mathbb{Z}_{(p)}[x]^\delta \rightarrow \mathbb{Z}_{(p)}[x]^\delta$  is faithfully flat, we can first replace  $R$  with

$$\mathbb{Z}_{(p)}[x]^\delta \frown_{\phi} \otimes_{\mathbb{Z}_{(p)}[x]^\delta} R,$$

---

<sup>5</sup>This is the same thing as  $R\{y\}$  in [BS22], but we'd rather keep curly brackets for convergent series.

where, on the right,  $x \mapsto d$ , and assume that there exists  $e \in R$  such that  $d = \phi(e)$ . For the same reason, we can replace  $B$  with

$$R[x]^\delta_{\phi} \otimes_{R[x]^\delta} B,$$

where, on the right,  $x \mapsto g$  and assume that there exists  $h \in B$  such that  $g = \phi(h)$ .

If we denote by  $\mathbb{Z}_{(p)}[x]^{\text{PD}}$  the divided power envelope of  $(x)$  in  $\mathbb{Z}_{(p)}[x]$ , we can then consider the obvious factorization of the projection:

$$\mathbb{Z}_{(p)}[x] \rightarrow \mathbb{Z}_{(p)}[x]^{\text{PD}} \rightarrow \mathbb{F}_p[x]^{\text{PD}} \rightarrow \mathbb{F}_p[x]/(x^p).$$

We can pull it back (classically) to  $R$  (and to  $\bar{R}/p\bar{R}$  on the right) along the map  $x \mapsto e$  and, writing

$$S := R \otimes_{\mathbb{Z}_{(p)}[x]} \mathbb{Z}_{(p)}[x]^{\text{PD}},$$

obtain a factorisation of the projection:

$$R \rightarrow S \rightarrow \bar{S}/p\bar{S} \rightarrow \bar{R}/p\bar{R}.$$

It is then sufficient to prove that the complex  $S \otimes_R^{\mathbb{L}} B[g/d]^\delta$  is completely (faithfully) flat over  $S$ . We now denote by  $\mathbb{Z}_{(p)}[x]^{\delta, \text{PD}}$  the divided power envelope of  $(x)$  in  $\mathbb{Z}_{(p)}[x]^\delta$ . Since the formation of divided power envelope commutes with flat base extension, we have

$$\mathbb{Z}_{(p)}[x]^{\delta, \text{PD}} \simeq \mathbb{Z}_{(p)}[x]^\delta \otimes_{\mathbb{Z}_{(p)}[x]} \mathbb{Z}_{(p)}[x]^{\text{PD}}.$$

On the other hand, we know from Lemma 2.36 and of [BS22] that

$$\mathbb{Z}_{(p)}[x]^{\delta, \text{PD}} \simeq \mathbb{Z}_{(p)}[x, \phi(x)/p]^\delta.$$

It follows that

$$S \simeq H^0 \left( \mathbb{R}[d/p]^\delta \right) \{$$

If we set  $u := d/p \in S$ , then  $\delta(d) = \delta(pu) = \delta(p)u^p + p\delta(u)$ . Since  $\delta(d), \delta(p) \in R^\times$ , we can write  $u^p = v(1 - f)$  with  $v \in R^\times$  and  $f \equiv 0 \pmod{p}$ . It follows that  $u$  acts invertibly modulo  $p$  and therefore

$$\bar{S}/p\bar{S} \otimes_R^{\mathbb{L}} B[g/d]^\delta \simeq \bar{S}/p\bar{S} \otimes_R^{\mathbb{L}} B[g/p]^\delta.$$

It is thus sufficient to show that  $S \otimes_R^{\mathbb{L}} B[g/p]^\delta$  is completely (faithfully) flat over  $S$  or even that  $B[g/p]^\delta$  is completely (faithfully) flat over  $R$ . Since we were careful enough to make sure that we can write  $g = \phi(h)$ , this follows from Lemma 2.3 below (with  $h$  in the place of  $g$ ).  $\square$

**Lemma 2.3.** *If  $B$  is a  $\delta$ - $R$ -algebra and  $g \in B$  is such that  $\phi(g)$  is completely (faithfully) regular over  $R$ , then the complex  $B[\phi(g)/p]^\delta$  is completely (faithfully) flat over  $R$ .*

*Proof.* Let us choose flat simplicial resolutions  $R_\bullet$  over  $\mathbb{Z}_{(p)}$  and  $B_\bullet$  over  $R_\bullet[x]^\delta$  of  $R$  and  $B$  respectively in the category of  $\delta$ -rings. Fix some  $i \in \mathbb{N}$ . The image  $g_i$  of  $x$  in  $B_i$  is a nonzero divisor modulo  $p$  and  $B_i$  is  $p$ -torsion free. Therefore, it follows from Corollary 2.39 of [BS22] that there exists an isomorphism with the divided power envelope of  $(g_i)$  in  $B_i$ :

$$B_i[\phi(g_i)/p]^\delta \simeq B_i^{\text{PD}}.$$

Now, we consider the free  $R$ -module  $F := \bigoplus_{n \in \mathbb{N}} Rv_n$  and send  $v_n \in F$  to  $g_i^{[np]} \in B_i^{\text{PD}}$ . Thanks to Lemma 2.4 below, this induces an isomorphism:

$$B_i/\phi(g_i)B_i \otimes_R F/pF \simeq B_i^{\text{PD}}/pB_i^{\text{PD}}.$$

If we identify a simplicial complex with the corresponding chain complex via the Dold-Kan correspondence, then we have

$$B_\bullet/\phi(g_\bullet)B_\bullet \simeq [B \xrightarrow{\phi(g)} B] \quad \text{and} \quad B_\bullet[\phi(g_\bullet)/p]^\delta \simeq B[\phi(g)/p]^\delta,$$

and therefore

$$\overline{R}/p\overline{R} \otimes_R^L [B \xrightarrow{\phi(g)} B] \otimes_R F \simeq \overline{R}/p\overline{R} \otimes_R^L B[\phi(g)/p]^\delta.$$

Our hypothesis implies that the left hand side is (faithfully) flat over  $\overline{R}/p\overline{R}$  and the same therefore also holds for the right hand side.  $\square$

We used above the following standard result:

**Lemma 2.4.** *Assume  $B$  is a  $p$ -torsion free  $\mathbb{Z}_{(p)}$ -algebra and  $g \in B$  is a nonzero divisor modulo  $p$ . If we denote by  $B^{\text{PD}}$  the PD-envelope of  $(g)$  in  $B$ , then there exists an isomorphism*

$$B/g^p B \otimes \bigoplus_{n \in \mathbb{N}} \mathbb{F}_p v_n \simeq B^{\text{PD}}/pB^{\text{PD}}, \quad v_n \rightarrow g^{[np]}.$$

*Proof.* Thanks to Lemma 2.38 of [BS22] for example, we may assume that  $B = \mathbb{Z}_{(p)}[x]$  and  $g = x$ . It is then sufficient to notice that the map

$$\bigoplus_{n \in \mathbb{N}} (\mathbb{F}_p[x]/x^p) v_n \simeq \bigoplus_{m \in \mathbb{N}} \mathbb{F}_p x^{[m]}, \quad v_n \rightarrow x^{[np]}$$

is a bijection with inverse sending  $x^{[m]}$  to  $r! \binom{m}{r} \mathbb{F}_p v_n$  if  $m = np + r$  with  $0 \leq r < p$ .  $\square$

**Remarks.** 1. The proof of Theorem 2.2 (and Lemma 2.3) follows exactly the same pattern as in [BS22] (Lemma 2.43, Corollary 2.44 and Proposition 3.13). It is however self-contained and only requires the interpretation of classical divided powers in term of  $\delta$ -structures.

2. Lemma 2.3 is still valid if we replace the condition that  $\phi(g)$  is completely (faithfully) regular by the same condition on  $g$  as long as  $B$  is completely flat, because the condition is then stronger.
3. Of course, Theorem 2.2, as well as the lemma, hold more generally for a completely (faithfully) regular sequence, but we shall only need this simple case. They also hold in the nonoriented situation: we can replace  $(d)$  with an invertible ideal.
4. For a more general statement than Theorem 2.2, the reader may refer to Proposition 5.49 of [Mao21].

If  $B$  is a  $\delta$ - $R$ -algebra and  $g \in B$ , we will set  $B[g/d]_0 = R[w] \otimes_{R[x]} B$ , with  $x \mapsto dw$  on the left and  $x \mapsto g$  on the right (we use the subindex 0 to make clear that we do not mean a

derived version). Then, if  $B[g/d]_0^\delta$  denotes the  $\delta$ -envelope of  $B[g/d]_0$  (in contrast with the derived version), we have

$$B[g/d]_0^\delta \simeq H^0 \left( B[g/d]^\delta \right) \lrcorner R[w]^\delta \otimes_{R[x]^\delta} B.$$

We will denote by  $B[g/d]_0^{\delta, \wedge}$  its *classical* completion. If we denote derived completion by  $L^\wedge$  (in contrast with classical completion  $\wedge$ ), then we have:

**Corollary 2.5.** *In the situation of Theorem 2.2, if  $R$  is bounded (with respect to  $p$  modulo  $d$ : see Definition A.19 in the appendix), then*

$$B[g/d]_0^{\delta, \wedge} \simeq B[g/d]^{\delta, L^\wedge}$$

*is a formally (faithfully) flat  $\delta$ - $R$ -algebra. In particular, it is bounded.*

*Proof.* It follows from Theorem 2.2 that  $B[g/d]^{\delta, L^\wedge}$  is completely (faithfully) flat and derived complete. Since  $R$  is bounded, Theorem A.25 in the appendix tells us that it is discrete formally (faithfully) flat and classically complete. Lemma A.11 in the appendix applied to  $A = R[x]^\delta$ ,  $M = R[w]^\delta$  and  $N = B$  then provides an isomorphism

$$B[g/d]^{\delta, L^\wedge} \simeq H^0 \left( B[g/d]_0^{\delta, L^\wedge} \right) \{$$

Since the left hand side is classically complete, so is the right hand side, and it therefore follows from remark (5) after Proposition A.9 in the appendix that we also have

$$B[g/d]_0^{\delta, \wedge} \simeq H^0 \left( B[g/d]_0^{\delta, L^\wedge} \right) \{$$

□

### 3 Prismatic stratifications

We will introduce in the end the notion of a prismatic stratification on a module and show that, in a local situation, it corresponds to that of a prismatic crystal (see also Section 3.1 of [MT20]).

We keep the assumptions and notation from the previous section. More precisely, we let  $R$  be a (complete)  $\delta$ - $\mathbb{Z}_{(p)}$ -algebra with a distinguished element  $d$  so that  $(R, d)$  will become our base prism. An  $R$ -module  $M$  will always be endowed with its  $(p, d)$ -adic topology and we will write  $\overline{M} = M/dM$ .

At some point, we will assume that  $R$  is bounded, we will consider a complete  $R$ -algebra  $A$  with a topologically étale<sup>6</sup> coordinate  $x$  (we mean that there exists a topologically finitely presented formally étale morphism  $R[x] \rightarrow A$ ) and we will let  $\mathcal{X} := \mathrm{Spf}(\overline{A})$ .

**Definition 3.1.** The (*bounded*) *prismatic envelope* of an ideal  $J$  in a  $\delta$ - $R$ -algebra  $B$  is a complete bounded  $\delta$ - $R$ -algebra  $C$  endowed with a morphism  $B \rightarrow C$  which is universal for the complete bounded  $\delta$ - $R$ -algebra  $JC \subset dC$ .

We may also say that the prism  $(C, dC)$  is the prismatic envelope of  $(B, J)$ . It is unclear for the authors under which generality such an envelope exists, but we will show its existence and explicit description in our case of interest (see Proposition 3.5 below).

<sup>6</sup>Or equivalently completely étale, since  $R$  is bounded.

**Example.** If  $R$  is bounded in Theorem 2.2, then  $B[g/d]^{\delta, \wedge}$  is the prismatic envelope of  $(g)$  in  $B$ . More precisely, this is a complete bounded  $\delta$ - $R$ -algebra which is isomorphic to  $B[g/d]_0^{\delta, \wedge}$  thanks to Corollary 2.5. The assertion therefore follows from the universal property of  $B[g/d]_0^{\delta, \wedge}$ .

Let us recall the following notation from Section 1 of [GLQ23]: if  $A$  is a  $\delta$ -ring and  $I \subset A$  any ideal, then we denote by  $I_\delta$  the  $\delta$ -ideal generated by  $I$ .

**Definition 3.2.** If  $B$  is a  $\delta$ - $R$ -algebra and  $x \in B$ , then the *ring of prismatic polynomials* on  $B$  (with respect to  $x$ ) is

$$B[\omega]_d^\delta := B[\omega]^\delta / (\delta(x + d\omega))_\delta$$

(where  $\omega$  is an indeterminate).

**Remark.** The ring of prismatic polynomials is universal among  $\delta$ - $B$ -algebras  $C$  endowed with some  $h \in C$  such that  $\delta(x + dh) = 0$  (if we still denote by  $x$  the image of  $x$  in  $C$ ).

**Proposition 3.3.** *If  $B$  is a bounded  $\delta$ - $R$ -algebra and  $x \in B$ , then  $B[\omega]_d^{\delta, \wedge}$  is a complete formally faithfully flat  $\delta$ - $B$ -algebra. Moreover, if we endow the polynomial ring  $B[\xi]$  with the unique structure<sup>7</sup> of  $\delta$ - $B$ -algebra such that  $\delta(x + \xi) = 0$ , then  $B[\omega]_d^{\delta, \wedge}$  is the prismatic envelope of  $\xi$  in  $B[\xi]$ .*

*Proof.* Recall that we have defined for  $\xi \in B[\xi]$  the rings  $B[\xi][\xi/d]_0$  and  $B[\xi][\xi/d]_0^\delta$ , where the subindex 0 means that we do not consider the derived version. Since  $B$  is  $d$ -torsion free, there exists a distinguished element  $\xi/d \in B[\xi][\xi/d]_0$  and  $(B[\xi][\xi/d]_0^\delta, \xi/d)$  shares the same universal property with  $(B[\omega]_d^\delta, \omega)$ . In other words, there exists a canonical isomorphism

$$B[\omega]_d^\delta \simeq B[\xi][\xi/d]_0^\delta, \quad \omega \leftrightarrow \xi/d.$$

It then follows from Corollary 2.5 that  $B[\omega]_d^{\delta, \wedge}$  is completely faithfully flat over  $B$  (and in particular bounded). The last assertion follows from the universal property of  $B[\xi][\xi/d]_0$ .  $\square$

We will need the following elementary result (recall that we implicitly use the  $(p, d)$ -adic topology):

**Lemma 3.4.** *Let  $A' \rightarrow A$  be a formally étale morphism of  $\delta$ - $R$ -algebras and  $B$  a classically complete  $\delta$ - $R$ -algebra. For a morphism  $A \rightarrow B$  of  $R$ -algebras to be a morphism of  $\delta$ -rings, it is sufficient that the composite map  $A' \rightarrow B$  is a morphism of  $\delta$ -rings.*

*Proof.* Recall that if we denote by  $W_1(S)$  the ring of Witt vectors of length 2 on a ring  $S$ , then it is equivalent to give a  $\delta$ -structure on  $S$  or a section of the projection  $W_1(S) \rightarrow S$ . In our situation, we have

$$\begin{array}{ccccc} W_1(A') & \longrightarrow & W_1(A) & \longrightarrow & W_1(B) \\ \uparrow \downarrow & & \uparrow \downarrow & & \uparrow \downarrow \\ A' & \longrightarrow & A & \longrightarrow & B. \end{array}$$

<sup>7</sup>Note that  $x + \xi$  is an alternative indeterminate for  $B[\xi]$ .



We want to show that the right hand side square involving the sections is commutative and we know that both other squares are. Since  $A' \rightarrow A$  is formally étale and both maps  $A \rightarrow W_1(B)$  coincide, not only on  $A'$ , but also modulo  $p$ , they must be equal on  $A$ .  $\square$

We assume now that  $R$  is bounded and consider a complete  $R$ -algebra  $A$  with a topologically étale coordinate  $x$ . We let  $\mathcal{X} := \mathrm{Spf}(\overline{A})$ . We endow  $A$  with the unique structure of  $\delta$ - $R$ -algebra such that  $\delta(x) = 0$  (use Lemma 2.18 of [BS22] for example). This is a bounded prism on  $\mathcal{X}$  over  $R$ .

**Proposition 3.5.** *If  $B$  is a complete bounded  $\delta$ - $R$ -algebra and  $A \rightarrow B$  is a morphism of  $R$ -algebras, then  $B[\omega]_d^{\delta, \wedge}$  is the prismatic envelope of the kernel  $J$  of multiplication  $B \otimes_R A \rightarrow B$ .*

It is *not* assumed here that  $A \rightarrow B$  is a morphism of  $\delta$ -rings. Also the “ $x$ ” hidden in the definition of the ring of prismatic polynomials is the image of  $x$  in  $B$ .

*Proof.* Since  $x$  is a topologically étale coordinate, the morphism  $R[x] \rightarrow A$  is formally étale and the morphism

$$R[x] \rightarrow B[\omega]_d^{\delta}, \quad x \mapsto x + d\omega$$

therefore extends uniquely to a morphism  $\theta : A \rightarrow B[\omega]_d^{\delta, \wedge}$ . It follows from Lemma 3.4 that this is a morphism of  $\delta$ - $R$ -algebras. This provides a  $\delta$ -morphism  $B \otimes_R A \rightarrow B[\omega]_d^{\delta, \wedge}$ . Since  $\overline{\theta}$  factors through  $\overline{B}$ , the induced map

$$\overline{B} \otimes_{\overline{R}} \overline{A} \rightarrow \overline{B[\omega]_d^{\delta, \wedge}}$$

also factors through  $\overline{B}$ . It means that the image of  $J$  under the morphism  $B \otimes_R A \rightarrow B[\omega]_d^{\delta, \wedge}$  is contained in  $dB[\omega]_d^{\delta, \wedge}$ . Assume now that we are given a complete bounded  $\delta$ - $R$ -algebra  $C$  and a morphism  $B \otimes_R A \rightarrow C$  of  $\delta$ - $R$ -algebras such that  $JC \subset dC$ . We endow  $B[\xi]$  with the unique  $\delta$ -structure such that  $\delta(x + \xi) = 0$  and consider the  $\delta$ -morphism

$$B[\xi] \rightarrow B \otimes_R A, \quad \xi \mapsto 1 \otimes x - x \otimes 1.$$

Proposition 3.3 implies that the composite map  $B[\xi] \rightarrow B \otimes_R A \rightarrow C$  extends uniquely to a  $\delta$ -morphism  $B[\omega]_d^{\delta, \wedge} \rightarrow C$  (with  $\xi = d\omega$ ). Since the image  $1 \otimes x - x \otimes 1$  of  $\xi$  in  $B \otimes_R A$  is a topologically étale coordinate for  $B \otimes_R A$  over  $B$ , the usual argument shows that this morphism does extend the original one.  $\square$

**Remarks.** 1. It is important to notice that the  $\delta$ -morphism  $\theta : A \rightarrow B[\omega]_d^{\delta, \wedge}$  that we consider here is the *Taylor map*  $x \mapsto x + d\omega$ . We will systematically write  $B[\omega]_d^{\delta, \wedge} \otimes'_A -$  when we want to insist on the fact that we use this Taylor map as structural map.

2. In the particular case  $B = A$ , we see that  $A[\omega]_d^{\delta, \wedge}$  is the prismatic envelope of the diagonal in the product  $P := A \otimes_R A$ . There exists a (left) structure  $A \rightarrow A[\omega]_d^{\delta, \wedge}$  which is the naive map  $x \mapsto x$  and a right structure  $\theta : A \rightarrow A[\omega]_d^{\delta, \wedge}$  which is the *Taylor map*  $x \mapsto x + d\omega$ .

3. If, besides the map  $A \rightarrow B$ , we have any morphism  $B \rightarrow C$  of complete bounded  $\delta$ - $R$ -algebras, then

$$C[\omega]_d^{\delta} \simeq C \otimes_B B[\omega]_d^{\delta}.$$

It follows that

$$B[\omega]_d^{\delta, \wedge} [\omega']_d^{\delta, \wedge} \simeq B[\omega]_d^{\delta, \wedge} \widehat{\otimes}_B B[\omega]_d^{\delta, \wedge} \simeq B[\omega]_d^{\delta, \wedge} \widehat{\otimes}'_A A[\omega]_d^{\delta, \wedge}$$

(using both structural maps).

**Corollary 3.6.** *If  $B$  is a bounded prism on  $\mathcal{X}$  over  $R$ , then  $B[\omega]_d^{\delta, \wedge}$  is the product of  $B$  by  $A$  in the prismatic site of  $\mathcal{X}$  over  $R$ .*

*Proof.* We choose a lifting in  $B$  of the image in  $\overline{B}$  of the class  $\overline{x}$  of  $x$  in  $\overline{A}$  and we apply Proposition 3.5.  $\square$

**Proposition 3.7.** *The prism  $A$  is a covering of (the final object of the topos associated to)  $\Delta(\mathcal{X}/R)$  for the flat topology.*

*Proof.* It means that, in the prismatic site of  $\mathcal{X}$  over  $R$ , if we are given any  $B$ , then there exists a formally faithfully flat morphism of bounded prisms  $B \rightarrow B'$  and a morphism  $A \rightarrow B'$ . Since we know from Corollary 2.5 that the product  $B[\omega]_d^{\delta, \wedge}$  of  $B$  by  $A$  is formally faithfully flat over  $B$ , we can simply choose  $B' := B[\omega]_d^{\delta, \wedge}$ .  $\square$

**Remarks.** 1. The result will hold more generally with more coordinates and gives a general way to describe locally the prismatic site of any smooth formal scheme  $\mathcal{X}$ .

2. The result also holds when  $(R, I)$  is not orientable: one may always find an orientation after a formally faithfully flat extension of the base.

3. As we shall see in a forthcoming article, the same proof with  $d$  replaced by  $p$  shows that  $(\mathbb{Z}_p[[x]], x - p)$ , endowed with the unique  $\delta$ -structure such that  $\delta(x) = 0$ , covers the absolute prismatic site.

4. There exist more sophisticated versions of Proposition 3.7, such as Proposition 5.56 of [Mao21] or Lemma 3.2 of [Tia21], for example. The strategy is always the same.

In the next definition, we should perhaps say prismatic hyper-stratification, but we drop the prefix because there is nothing like a non-hyper-stratification in this theory.

**Definition 3.8.** A *prismatic stratification* on a finite projective  $A$ -module  $M$  is an  $A[\omega]_d^{\delta, \wedge}$ -linear isomorphism

$$\epsilon_M : A[\omega]_d^{\delta, \wedge} \otimes'_A M \simeq M \otimes_A A[\omega]_d^{\delta, \wedge}$$

(in which  $\otimes'$  indicates again that we use  $\theta$  as structural map) satisfying the cocycle condition

$$(\epsilon_M \widehat{\otimes} \text{Id}_{A[\omega]_d^{\delta, \wedge}}) \circ (\text{Id}_{A[\omega]_d^{\delta, \wedge}} \widehat{\otimes}' \epsilon_M) \circ (\Delta \otimes' \text{Id}_M) = (\text{Id}_M \otimes \Delta) \circ \epsilon_M$$

where

$$\Delta : A[\omega]_d^{\delta, \wedge} \rightarrow A[\omega]_d^{\delta, \wedge} \widehat{\otimes}'_A A[\omega]_d^{\delta, \wedge}, \quad \omega \mapsto \omega \otimes' 1 + 1 \otimes' \omega$$

is the comultiplication map (see Definition 4.8 below).

**Proposition 3.9.** *The category of locally finite free prismatic crystals on  $\mathcal{X}$  is equivalent to the category of finite projective  $A$ -modules endowed with a prismatic stratification.*

*Proof.* Follows from Proposition 3.7. □

**Remarks.** 1. As we shall see in a forthcoming article, one can show in the same way that the category of *absolute* prismatic crystals is equivalent to the category of  $\mathbb{Z}_p[[x]]$ -modules endowed with a “prismatic stratification”, using this time the ring

$$\mathbb{Z}[x, \omega]_{x-p}^\delta := \mathbb{Z}[x, \omega]^\delta / (\delta(x + (x - p)\omega))_\delta.$$

2. Proposition 3.7 in [Tia21] provides a generalization of Proposition 3.9.

3. All these results are rather theoretical: it is in general very hard to do any computation with the description we have given of a prismatic envelope. We will see how to remedy this in a more specific situation in Section 5.

## 4 Prismatic differential operators

We will explain a standard technique (see for example construction 6.9 in [BO78]) that can be used in order to compute the cohomology of a prismatic crystal. For this purpose, we need to introduce the notion of a prismatic differential operator. We keep the notation as in the previous sections. Thus, we assume that  $R$  is a (complete) bounded  $\delta$ - $\mathbb{Z}_{(p)}$ -algebra and that  $d \in R$  is distinguished. Any  $R$ -module  $M$  is endowed with its  $(p, d)$ -adic topology and we write  $\overline{M}$  for  $M/dM$ .

At some point, we will consider a complete bounded  $\delta$ - $R$ -algebra  $A$  and write  $\mathcal{X} := \mathrm{Spf}(\overline{A})$ . At some point, we will also assume that  $A$  has a topologically étale coordinate  $x$ .

We begin with some very general considerations. Let  $T$  be any site (for example a prismatic site) and  $\tilde{T}$  denote the corresponding topos. If we denote by  $\mathbb{1} := \{0\}$  the final category, then there exists (by definition) a unique functor  $e_T : T \rightarrow \mathbb{1}$ . It is obviously cocontinuous and, if we identify the category of (pre-) sheaves on  $\mathbb{1}$  with the category  $\mathbf{Sets}$  of sets, then we obtain the (final) morphism of topoi

$$e_T : \tilde{T} \rightarrow \mathbf{Sets}.$$

We have  $e_{T*}(E) = \Gamma(T, E)$  and what we want to compute is  $H^k(T, E) = R^k e_{T*}(E)$ . One may also notice that  $e_T^{-1}(S) = \underline{S}$  is the sheaf associated to the set  $S$ .

We consider now a complete bounded  $\delta$ - $R$ -algebra  $A$  and the final morphism of topoi

$$e_A : A_\Delta \rightarrow \mathbf{Sets}.$$

The category  $\Delta/A$  has the final object  $A$  so that now  $e_{A*}(E) = E_A$  is simply the realization of the sheaf  $E$  on  $A$ . The site  $\Delta/A$  comes with its structural ring  $\mathcal{O}_{\Delta/A}$  given by  $\mathcal{O}_{\Delta/A}(B) = B$ . In particular, we have  $\mathcal{O}_{\Delta/A}(A) = A$  and we can promote our final morphism of topoi to a morphism of ringed topoi

$$e_A : (A_\Delta, \mathcal{O}_{\Delta/A}) \rightarrow (\mathbf{Sets}, A).$$

We still have  $e_{A*}(E) = E_A$ , but now we also have  $e_A^*(M) = \mathcal{O}_{\Delta/A} \otimes_A \underline{M}$  and we will simply write  $\mathcal{O}_{\Delta/A} \otimes_A M$ : this is the sheaf associated to the presheaf  $B \mapsto B \otimes_A M$ . It follows from Lemma 1.1 that, if  $M$  is finite projective, then this presheaf is already a sheaf. More precisely, we have the fundamental result:

**Proposition 4.1.** *The functors  $\mathrm{Re}_{A*}$  and  $\mathrm{Le}_A^*$  induce an equivalence between locally finite free prismatic crystals on  $\Delta(\!/A)$  and finite projective  $A$ -modules.*

*Proof.* One shows as in Proposition 1.2 (but it is actually easier here) that  $e_{A*}$  and  $e_A^*$  induce an equivalence between locally finite free prismatic crystals on  $A$  and finite projective  $A$ -modules. Moreover, if  $M$  is finite projective, then it is flat and therefore  $\mathrm{Le}_A^* M = e_A^* M$ . It only remains to show that  $\mathrm{R}^i e_{A*} E = 0$  for  $i > 0$  when  $E$  is locally finite free. The question is local and reduces to proving that  $\mathrm{H}^i(A_\Delta, \mathcal{O}_{\Delta(\!/A)}) = 0$  for *any* complete bounded  $\delta$ -algebra  $A$ , but this is done in the proof of Corollary 3.12 in [BS22].  $\square$

**Remark.** 1. As a consequence, we get Theorem A ( $e_A^* e_{A*} E = E$ ) and Theorem B ( $\mathrm{R}^i e_{A*} E = 0$  for  $i > 0$ ) for locally finite free prismatic crystals on  $A$ .

2. In the proof of the proposition, it is actually sufficient to show that Čech cohomology  $\check{\mathrm{H}}^i(A_\Delta, \mathcal{O}_{\Delta(\!/A)})$  vanishes for any complete bounded  $\delta$ -algebra  $A$  (see [Sta19, Tag 03F9]), which is in fact what Bhatt and Scholze do.

We consider now the formal scheme  $\mathcal{X} := \mathrm{Spf}(\overline{A})$  and the localization functor

$$j_A : \Delta(\!/A) \rightarrow \Delta(\mathcal{X}/R)$$

sending a complete bounded  $\delta$ - $A$ -algebra  $B$  to the complete bounded  $\delta$ - $R$ -algebra  $B$  endowed with the morphism  $\mathrm{Spf}(\overline{B}) \rightarrow \mathrm{Spf}(\overline{A}) = \mathcal{X}$ . This is a morphism of prismatic sites and we will still denote by

$$j_A : A_\Delta \rightarrow (\mathcal{X}/R)_\Delta$$

the corresponding morphism of topoi.

**Definition 4.2.** If  $M$  is an  $A$ -module, then the *linearization* of  $M$  is the prismatic sheaf  $L(M) := j_{A*} e_A^*(M)$ .

**Remark.** If  $M$  is a finite projective  $A$ -module, and we denote by  $e_{\mathcal{X}/R}$  the final morphism on  $\Delta(\mathcal{X}/R)$ , then we have

$$\Gamma((\mathcal{X}/R)_\Delta, L(M)) = e_{\mathcal{X}/R*} L(M) = e_{\mathcal{X}/R*} j_{A*} e_A^* M = e_{A*} e_A^* M = M.$$

It will be essential to prove later in Corollary 4.5 a derived version of this statement.

We assume from now on that  $A$  is endowed with a topologically étale coordinate  $x$  and we recall that we introduced in Definition 3.2 the notion of a ring of prismatic polynomials. Then, we have:

**Proposition 4.3.** *If  $B$  is a bounded prism on  $\mathcal{X}$  over  $R$ , then  $j_A^{-1}(B)$  is representable and we have*

$$j_A^{-1}(B) = B[\omega]_d^{\delta, \wedge}.$$

*Proof.* Formally follows from the description of the product in Corollary 3.6.  $\square$

**Corollary 4.4.** *We have  $\mathrm{R}^i j_{A*} E = 0$  for  $i > 0$  when  $E$  is locally finite free.*

*Proof.* The sheaf  $R^i j_{A*} E$  is associated to the presheaf  $B \mapsto H^i(B'_\Delta, E|_{B'})$  with  $B' = B[\omega]_d^{\delta, \wedge}$ . But it follows from Proposition 4.1 applied to the case  $A = B'$  that  $H^i(B'_\Delta, E|_{B'}) = R^i e_{B'*} E|_{B'} = 0$ .  $\square$

**Corollary 4.5.** *If  $M$  is a finite projective  $A$ -module, then*

$$R\Gamma((\mathcal{X}/R)_\Delta, L(M)) = M.$$

*Proof.* It follows from Corollary 4.4 and Proposition 4.1 that

$$\begin{aligned} R\Gamma((\mathcal{X}/R)_\Delta, L(M)) &= Re_{\mathcal{X}/R*} L(M) \\ &= Re_{\mathcal{X}/R*} j_{A*} e_A^* M \\ &= Re_{\mathcal{X}/R} Rj_{A*} e_A^* M \\ &= R(e_{\mathcal{X}/R*} j_{A*}) e_A^* M \\ &= Re_{A*} e_A^* M \\ &= M. \end{aligned} \quad \square$$

In the previous corollaries, we only used the representability property, but we now turn to an explicit description of  $L(M)$ :

**Lemma 4.6.** *Let  $M$  be a finite projective  $A$ -module.*

1. *If  $B$  is a bounded prism on  $\mathcal{X}$  over  $R$ , then*

$$L(M)_B = B[\omega]_d^{\delta, \wedge} \otimes'_A M.$$

2. *If  $B \rightarrow C$  is a morphism of bounded prisms on  $\mathcal{X}$  over  $R$ , then*

$$L(M)_C = C \hat{\otimes}_B L(M)_B = C \hat{\otimes}_B^L L(M)_B.$$

*Proof.* It follows from Lemma 1.1 that if  $B$  is a bounded prism, then  $(e_A^* M)_B = B \otimes_A M$ . Using Proposition 4.3, we deduce that  $L(M) = B[\omega]_d^{\delta, \wedge} \otimes'_A M$ . The second assertion then follows from Proposition 3.3.  $\square$

**Remark.** The sheaf  $L(M)$  is complete and formally faithfully flat in the sense that  $L(M)_B$  is always complete and formally faithfully flat. This is *not* a crystal *stricto sensu*, but can be called a *complete crystal* taking into account the second statement in Lemma 4.6.

The following is a standard intermediate result in a linearization process:

**Lemma 4.7.** *If  $E$  is a locally finite free prismatic crystal on  $\mathcal{X}$  over  $R$  and  $M$  is a finite projective  $A$ -module, then*

$$E \otimes_{\mathcal{O}_{\mathcal{X}/R}} L(M) \simeq L(E_A \otimes_A M).$$

*Proof.* If  $B$  is a bounded prism over  $A$ , then

$$(j_A^{-1} E)_B = B \otimes_A (j_A^{-1} E)_A = B \otimes_A E_A = (e_A^* E_A)_B$$

because  $j_A^{-1}E$  is a locally finite free prismatic crystal on  $A$ . It follows that  $j_A^{-1}E = e_A^*E_A$ . Now, we extend the adjunction map  $j_A^{-1}j_{A*}e_A^*M \rightarrow e_A^*M$  in order to get

$$\begin{aligned} j_A^{-1}(E \otimes_{\mathcal{O}_{\mathcal{X}/R}} j_{A*}e_A^*M) &= j_A^{-1}E \otimes_{\mathcal{O}_{\Delta(\mathcal{X}/A)}} j_A^{-1}j_{A*}e_A^*M \\ &= e_A^*E_A \otimes_{\mathcal{O}_{\Delta(\mathcal{X}/A)}} j_A^{-1}j_{A*}e_A^*M \\ &\rightarrow e_A^*E_A \otimes_{\mathcal{O}_{\Delta(\mathcal{X}/A)}} e_A^*M \\ &= e_A^*(E_A \otimes_A M). \end{aligned}$$

By adjunction, we obtain a natural map

$$E \otimes_{\mathcal{O}_{\mathcal{X}/R}} L(M) = E \otimes_{\mathcal{O}_{\mathcal{X}/R}} j_{A*}e_A^*M \rightarrow j_{A*}(e_A^*(E_A \otimes_A M)) = L(E_A \otimes_A M).$$

We want to show that this is an isomorphism. By additivity, since  $M$  is finite projective, we may assume that  $M = A$ . It is then sufficient to prove that for any bounded prism  $B$  over  $\mathcal{X}/R$ , we have

$$E_B \otimes_B L(A)_B \simeq L(E_A)_B.$$

But, since  $E$  is a crystal, if we write  $B' := B[\omega]_d^{\delta, \wedge}$ , we have by Lemma 4.6

$$E_B \otimes_B L(A)_B \simeq E_B \otimes_B B' \simeq E_{B'} \simeq B' \otimes'_A E_A \simeq L(E_A)_B. \quad \square$$

**Definition 4.8.** If  $B$  is a bounded prism on  $\mathcal{X}$  over  $R$ , then *comultiplication* is the unique  $\delta$ - $B$ -morphism

$$\Delta_B : B[\omega]_d^{\delta, \wedge} \rightarrow B[\omega]_d^{\delta, \wedge} \hat{\otimes}'_A A[\omega]_d^{\delta, \wedge}, \quad \omega \mapsto 1 \otimes' \omega + \omega \otimes' 1.$$

Dually (in the category of bounded prisms with morphisms going the right way),  $\Delta_B$  corresponds to the projection  $p_{13} : B \times A \times A \rightarrow B \times A$  that forgets the middle term. In other words,  $\Delta_B$  is the canonical morphism induced by the maps

$$B \rightarrow B[\omega]_d^{\delta, \wedge} \hat{\otimes}'_A A[\omega]_d^{\delta, \wedge}, \quad b \mapsto b \otimes' 1$$

and

$$A \rightarrow B[\omega]_d^{\delta, \wedge} \hat{\otimes}'_A A[\omega]_d^{\delta, \wedge}, \quad x \mapsto 1 \otimes' x + d \otimes' \omega.$$

In the case  $B = A$ , we shall simply write  $\Delta$  as in Definition 3.8.

In the next definition, we should perhaps say hyper-differential operator, but we will again drop the prefix because there is nothing like a non-hyper-differential operator in this theory.

**Definition 4.9.** If  $M$  and  $N$  are two finite projective  $A$ -modules, then a *prismatic differential operator*  $D : M \rightarrow N$  is an  $A$ -linear map

$$\tilde{D}_{\zeta} : A[\omega]_d^{\delta, \wedge} \otimes'_A M \rightarrow N.$$

We will write  $D(s) := \tilde{D}(1 \otimes' s)$  if  $s \in M$ . Note however that  $\tilde{D}_{\zeta}$  is not uniquely determined by the map  $D$ .

Composition of prismatic differential operators  $D : M \rightarrow N$  and  $E : N \rightarrow P$  is obtained as follows

$$\begin{array}{ccc} A[\omega]_d^{\delta, \wedge} \otimes'_A M & \xrightarrow{\widetilde{E \circ D}} & P \\ \downarrow \Delta \otimes' \text{Id}_M & & \uparrow \tilde{E} \\ A[\omega]_d^{\delta, \wedge} \hat{\otimes}'_A A[\omega]_d^{\delta, \wedge} \otimes'_A M & \xrightarrow{\text{Id}_{A[\omega]_d^{\delta, \wedge}} \hat{\otimes}' \tilde{D}_{\zeta}} & A[\omega]_d^{\delta, \wedge} \otimes'_A N. \end{array}$$

We will denote by

$$\Delta\text{-Diff}(M, N) := \text{Hom}_A(A[\omega]_d^{\delta, \wedge} \otimes'_A M, N)$$

the  $A$ -module of prismatic differential operators from  $M$  to  $N$ , and simply write  $\Delta\text{-Diff}(M)$  when  $M = N$ . Multiplication turns  $\Delta\text{-Diff}(M)$  into an  $R$ -algebra. If  $M$  is a finite projective  $A$ -module, we may then consider the adjunction map (see the proof of Proposition 6.16 in [GLQ23])

$$M \otimes_A A[\omega]_d^{\delta, \wedge} \rightarrow \text{Hom}_A(\text{Hom}_A(A[\omega]_d^{\delta, \wedge}, A), M) = \text{Hom}_A(\Delta\text{-Diff}(A), M).$$

If we are given a prismatic stratification  $\epsilon$  on  $M$ , then we can consider the composite map

$$A[\omega]_d^{\delta, \wedge} \otimes'_A M \xrightarrow{\epsilon} M \otimes_A A[\omega]_d^{\delta, \wedge} \rightarrow \text{Hom}_A(\Delta\text{-Diff}(A), M).$$

By adjunction again, it provides an  $A$ -linear map

$$u : \Delta\text{-Diff}(A) \rightarrow \text{Hom}_A(A[\omega]_d^{\delta, \wedge} \otimes'_A M, M) = \Delta\text{-Diff}(M)$$

so that, if  $D$  is a prismatic differential operator on  $A$ , then

$$\widetilde{u(D)} = (\text{Id}_M \otimes \tilde{D}) \circ \epsilon.$$

**Lemma 4.10.** *If a finite projective  $A$ -module  $M$  is endowed with a prismatic stratification  $\epsilon$ , then the corresponding morphism  $u : \Delta\text{-Diff}(A) \rightarrow \Delta\text{-Diff}(M)$  is a morphism of rings.*

*Proof.* This is completely standard and we will simply write  $B := A[\omega]_d^{\delta, \wedge}$  in order to make notation lighter. We will need the cocycle formula from Definition 3.8 that we recall here:

$$(\epsilon \hat{\otimes}' \text{Id}_B) \circ (\text{Id}_B \hat{\otimes}' \epsilon) \circ (\Delta \otimes' \text{Id}_M) = (\text{Id}_M \otimes \Delta) \circ \epsilon.$$

Then, if  $E$  is another prismatic differential operator on  $A$ , we have

$$\begin{aligned} u(\widetilde{E \circ D}) &= (\text{Id}_M \otimes (\widetilde{E \circ D})) \circ \epsilon \\ &= (\text{Id}_M \otimes (\tilde{E} \circ (\text{Id}_B \hat{\otimes}' \tilde{D}))) \circ \epsilon \\ &= (\text{Id}_M \otimes \tilde{E}) \circ (\text{Id}_M \otimes \text{Id}_B \hat{\otimes}' \tilde{D}) \circ (\text{Id}_M \otimes \Delta) \circ \epsilon \\ &= (\text{Id}_M \otimes \tilde{E}) \circ (\text{Id}_M \otimes \text{Id}_B \hat{\otimes}' \tilde{D}) \circ (\epsilon \hat{\otimes}' \text{Id}_B) \circ (\text{Id}_B \hat{\otimes}' \epsilon) \circ (\Delta \otimes' \text{Id}_M) \\ &= (\text{Id}_M \otimes \tilde{E}) \circ ((\text{Id}_M \otimes \text{Id}_B) \circ \epsilon) \hat{\otimes}' \tilde{D} \circ (\text{Id}_B \hat{\otimes}' \epsilon) \circ (\Delta \otimes' \text{Id}_M) \\ &= (\text{Id}_M \otimes \tilde{E}) \circ ((\epsilon \circ (\text{Id}_B \otimes' \text{Id}_M)) \hat{\otimes}' \tilde{D}) \circ (\text{Id}_B \hat{\otimes}' \epsilon) \circ (\Delta \otimes' \text{Id}_M) \\ &= (\text{Id}_M \otimes \tilde{E}) \circ \epsilon \circ (\text{Id}_B \otimes' \text{Id}_M \otimes \tilde{D}) \circ (\text{Id}_B \hat{\otimes}' \epsilon) \circ (\Delta \otimes' \text{Id}_M) \\ &= (\text{Id}_M \otimes \tilde{E}) \circ \epsilon \circ (\text{Id}_B \hat{\otimes}' ((\text{Id}_M \otimes \tilde{D}) \circ \epsilon)) \circ (\Delta \otimes' \text{Id}_M) \\ &= \widetilde{u(E)} \circ (\text{Id}_B \hat{\otimes}' \widetilde{u(D)}) \circ (\Delta \otimes' \text{Id}_M) \\ &= u(E) \circ u(D). \end{aligned}$$

□

In other words, a prismatic stratification provides an action on  $M$  by prismatic differential operators of the ring of prismatic differential operators on  $A$ . This turns  $M$  into a  $\Delta\text{-Diff}(A)$ -module via

$$\forall D \in \Delta\text{-Diff}(A), \forall s \in M, \quad Ds = (\text{Id}_M \otimes \tilde{D})(\epsilon(1 \otimes s)).$$

Recall that an  $A$ -module is called *torsionless* if it injects into a product of copies of  $A$  (this is equivalent to being *semi-reflexive*, which means that it injects canonically into its double dual). We do not know if  $A[\omega]_d^{\delta, \wedge}$  is always torsionless, but this is the case, for example, in the situation of Section 5.

**Proposition 4.11.** *Assume  $A[\omega]_d^{\delta, \wedge}$  is torsionless. Then, the functor that associates a  $\Delta\text{-Diff}(A)$ -module to a finite projective  $A$ -module  $M$  endowed with a stratification  $\epsilon$  is fully faithful.*

*Proof.* This is again standard and we shall write as above  $B := A[\omega]_d^{\delta, \wedge}$ . If  $M_1$  and  $M_2$  are two finite projective  $A$ -modules endowed with a prismatic stratification, then  $M := \text{Hom}_A(M_1, M_2)$  inherits a prismatic stratification  $\epsilon$ . A morphism  $s : M_1 \rightarrow M_2$  is compatible with the stratifications if and only if  $\epsilon(1 \otimes s) = s \otimes 1$ . On the other hand, the morphism  $s$  is compatible with the action of  $\Delta\text{-Diff}(A)$  if and only if

$$\forall D \in \Delta\text{-Diff}(A), \quad Ds = D(1)s.$$

We are therefore reduced to showing that, if  $M$  is a finite projective  $A$ -module endowed with a prismatic stratification and  $s \in M$ , then

$$\forall D \in \Delta\text{-Diff}(A), Ds = D(1)s \quad \Leftrightarrow \quad \epsilon(1 \otimes s) = s \otimes 1.$$

We may assume that  $M$  is free with basis  $\{s_i\}_{i=1}^r$ . We can then uniquely write  $s = \sum_{i=1}^r a_i s_i$  with  $a_i \in A$  and  $\epsilon(1 \otimes s) = \sum_{i=1}^r s_i \otimes f_i$  with  $f_i \in B$ . We have

$$Ds = \sum_{i=1}^r \{\tilde{D}(f_i)s_i\}$$

and we can therefore rewrite our expected equivalence:

$$\forall D \in \Delta\text{-Diff}(A), \sum_{i=1}^r \tilde{D}(f_i)s_i = \sum_{i=1}^r \{\tilde{D}(a_i)s_i\} \quad \Leftrightarrow \quad \sum_{i=1}^r s_i \otimes f_i = \sum_{i=1}^r a_i s_i \otimes 1.$$

We see that  $M$  gets out of the picture and we are reduced to showing that, given  $f \in B$ , we always have

$$\forall D \in \Delta\text{-Diff}(A), \tilde{D}(f) = 0 \quad \Leftrightarrow \quad f = 0.$$

This exactly means that  $B$  is semi-reflexive or, equivalently, torsionless.  $\square$

Now, we define the linearization  $L(D)_B$  of a prismatic differential operator  $D : M \rightarrow N$  as follows

$$\begin{array}{ccc} L(M)_B & \xrightarrow{L(D)_B} & L(N)_B \\ \parallel & & \parallel \\ B[\omega]_d^{\delta, \wedge} \otimes'_A M & \xrightarrow{\Delta_B \otimes' \text{Id}_M} & B[\omega]_d^{\delta, \wedge} \otimes'_A N \\ & \searrow \text{Id}_{B[\omega]_d^{\delta, \wedge}} \otimes \tilde{D} \nearrow & \\ & B[\omega]_d^{\delta, \wedge} \hat{\otimes}'_A A[\omega]_d^{\delta, \wedge} \otimes'_A M & \end{array}$$

Note that we recover composition of differential operators as  $\widetilde{E \circ D} = \tilde{E} \circ L(D)_A$ .



**Proposition 4.12.** *Linearization provides a functor from the category of finite projective  $A$ -modules and prismatic differential operators to the category of prismatic sheaves on  $\mathcal{X}$  over  $R$ .*

*Proof.* The map  $L(D)_B$  is clearly  $B$ -linear and one easily checks that the construction is compatible with completed base extension and composition. Thanks to Lemma 4.6, we are done.  $\square$

**Remarks.** 1. Note that if  $D$  is a prismatic differential operator, then

$$\mathrm{R}\Gamma((\mathcal{X}/R)_\Delta, L(D)) = D.$$

2. Again, the results of this section are difficult to use in practice unless we are in a specific situation such as in Section 6.

## 5 Prismatic crystals and twisted calculus

We recall some constructions from our previous articles [GLQ23] and [GLQ22b], and give a local proof that Cartier transform defines an equivalence between prismatic crystals and  $q$ -crystals.

We endow the local ring  $\mathbb{Z}[q]_{(p,q-1)}$  with the unique  $\delta$ -structure such that  $\delta(q) = 0$  and we denote by  $(n)_q$  the  $q$ -analog of an integer  $n$ .

We let  $R$  be a (complete) bounded  $\delta$ - $\mathbb{Z}[q]_{(p,q-1)}$ -algebra (with respect to  $p$  modulo  $(p)_q$ ). Then (the image of)  $(p)_q$  is a distinguished element in  $R$  and  $(R, (p)_q)$  will be our base prism (so that  $d = (p)_q$  now). An  $R$ -module  $M$  is systematically endowed with the  $(p, q-1)$ -adic (or equivalently  $(p, (p)_q)$ -adic) topology.

We let  $A$  be a complete  $R$ -algebra with a topologically étale coordinate  $x$  and we endow it with the unique structure of a  $\delta$ - $R$ -algebra on  $A$  such that  $\delta(x) = 0$ .

### Prismatic case

Let us denote by  $\sigma$  the unique endomorphism of the  $R$ -algebra  $A$  such that  $\sigma(x) = qx$  and  $\sigma \equiv \mathrm{Id}_A \pmod{q-1}$ . Recall from Proposition 2.4 of [LQ18] (applied to  $\sigma^p$ ) that there exists a  $q^p$ -analog  $d_{q^p} : A \rightarrow {}_{A/R, q^p}$  of the universal differential map. We may then recall from Definition 3.1 of [GLQ22b] that a *twisted connection of level  $-1$*  on an  $A$ -module  $M$  is an  $R$ -linear map  $\nabla : M \rightarrow M \otimes {}_{A/R, q^p}$  such that

$$\forall f \in A, \forall s \in M, \quad \nabla(fs) = (p)_q s \otimes d_{q^p} f + \sigma^p(f) \nabla(s).$$

Actually,  ${}_{A/R, q^p}$  is free on one generator  $d_{q^p} x$  and we may always write<sup>8</sup>  $\nabla(s) =: \partial_{q(-1)}(s) d_{q^p} x$ . The connection is said to be *topologically quasi-nilpotent* if

$$\forall s \in M, \quad \partial_{q(-1)}^k(s) \rightarrow 0.$$

**Theorem 5.1.** *The category of locally finite free prismatic crystals  $E$  on  $\mathcal{X} := \mathrm{Spf}(A/(p)_q)$  over  $R$  is equivalent to the category of finite projective  $A$ -modules  $M$  endowed with a topologically quasi-nilpotent twisted connection of level  $-1$ .*

<sup>8</sup>We wrote  $\partial_q^{\langle 1 \rangle}$  instead of  $\partial_{q(-1)}$  in [GLQ22b], but we'd rather specify the level here.

*Proof.* We introduced in [GLQ22b], Definition 1.6, the ring  $A\langle\omega\rangle_{q(-1)}$  of twisted divided polynomials of level  $-1$  and showed in Proposition 5.7 of loc. cit. that its completion is the prismatic envelope of the diagonal. In other words, we have  $A[\omega]_d^{\delta,\wedge} = \widehat{A\langle\omega\rangle_{q(-1)}}$ , meaning that both  $\omega$ 's correspond to each other, with the notation of the previous sections. In this situation, a prismatic stratification is what we called a *twisted hyper-stratification of level  $-1$* . Proposition 3.9 then tells us that the category of locally finite free prismatic crystals on  $\mathcal{X}$  is equivalent to the category of projective  $A$ -modules  $M$  endowed with a twisted hyper-stratification of level  $-1$ . The conclusion then follows from Proposition 3.10 of loc. cit., which lets us interpret a twisted hyper-stratification of level  $-1$  as a twisted connection of level  $-1$  when the  $A$ -module is flat and finitely presented (i.e. finite projective).  $\square$

**Remarks.** 1. As a consequence of this theorem (which also holds in higher dimension) we get a local interpretation of prismatic crystals with respect to  $(p)_q$  on any smooth formal scheme.

2. Conversely, the theorem also provides a method to glue  $q$ -difference equations of negative level, which is a real challenge due to the non-commutative nature of  $q$ -geometry (see also [Cha20], Corollary 2.3.3 for a  $q$ -crystalline version).

### $q$ -crystalline case

We now want to investigate the  $q$ -crystalline side. Thus, we assume from now on that  $(R, \mathfrak{r})$  is a bounded  $q$ -PD-pair.

There exists a  $q$ -crystalline variant of Theorem 2.2 which is also due to Bhatt and Scholze, for which we give a short alternative proof:

**Proposition 5.2** (Bhatt-Scholze). *If  $B$  is a completely flat  $\delta$ - $R$ -algebra and  $g \in B$  is completely (faithfully) regular over  $R$ , then the complex  $B[\phi(g)/(p)_q]^\delta$  is completely (faithfully) flat over  $R$ .*

*Proof.* Note first that  $\phi(g)$  is automatically completely (faithfully) regular because  $g$  is completely (faithfully) regular and  $B$  is completely flat. Also, it follows from Proposition A.18 in the appendix that  $R/(q-1) \hat{\otimes}_R^L B$  (where the notation follows the one in the appendix) is discrete because  $B$  is completely flat over  $R$  (and  $R$  has bounded  $p^\infty$ -torsion). After complete derived pullback along  $R \rightarrow R/(q-1)$ , we fall back onto Lemma 2.3.  $\square$

**Remark.** This proposition is essentially a particular case of the first part of Lemma 16.10 in [BS22]. In the second part, they also show that  $B[\phi(g)/(p)_q]^{\delta,\wedge}$  is the complete  $q$ -PD-envelope of  $(\mathfrak{r}B, g)$  in  $B$ . We shall come back to this matter in Proposition 6.5.

We assume from now on that  $A$  is endowed with a closed  $q$ -PD-ideal  $\mathfrak{a}$  such that  $\mathfrak{r}A \subset \mathfrak{a}$ .

**Proposition 5.3.** *If  $\mathcal{X} := \mathrm{Spf}(A/\mathfrak{a})$ , then  $A$  is a covering of (the final object of the topos associated to)  $q\text{-CRIS}(\mathcal{X}/R)$ .*

*Proof.* This is analogous to Proposition 3.7 and the proof follows the same arguments as in Proposition 3.5. We give ourselves a complete bounded  $q$ -PD-pair  $(B, J)$  over  $\mathcal{X}/R$  and we lift the image of  $x$  under  $A/\mathfrak{a} \rightarrow B/J$  to some (still written)  $x \in B$ . We endow

the polynomial ring  $B[\xi]$  with the unique  $\delta$ -structure such that  $\delta(x + \xi) = 0$ . We let  $B' := B[\xi][\phi(\xi)/(p)_q]^{\delta, \wedge}$ . Since  $x$  is a topologically étale coordinate and  $B'$  is classically complete, the morphism of  $\delta$ -rings  $x \mapsto x + \xi$  from  $R[x]$  to  $B'$  extends uniquely to a morphism of  $\delta$ -rings  $\theta : A \rightarrow B'$ . If we endow  $B'$  with the closure  $J'$  of  $\theta(\mathfrak{a})B' + JB'$  in  $B'$ , then  $\theta$  is a morphism of  $q$ -PD-pairs (use Lemma 3.2 of [GLQ23] for example). The point now is that the map  $B \rightarrow B'$  is formally faithfully flat thanks to Proposition 5.2.  $\square$

Note that we don't need to know in this proof that  $(B', J')$  is indeed the product of  $(B, J)$  by  $(A, \mathfrak{a})$  in the  $q$ -crystalline site (a fact that will follow from Proposition 6.5 below).

We may now consider, as we did above in the case of level  $-1$ , the  $q$ -analog  $d_q : A \rightarrow {}_{A/R, q}$  of the universal differential map and recall from Definition 2.8 of [LQ18] that a *twisted connection (of level 0)* on an  $A$ -module  $M$  is an  $R$ -linear map  $\nabla : M \rightarrow M \otimes {}_{A/R, q}$  such that

$$\forall f \in A, \forall s \in M, \quad \nabla(fs) = s \otimes d_q f + \sigma(f) \nabla(s).$$

Again,  ${}_{A/R, q}$  is free on  $d_q x$ . We can write  $\nabla(s) =: \partial_q(s) d_q x$  and call the connection *topologically quasi-nilpotent* when

$$\forall s \in M, \quad \partial_q^k(s) \rightarrow 0.$$

**Corollary 5.4.** *The category of locally finite free  $q$ -crystals  $E$  on  $\mathcal{X} := \mathrm{Spf}(A/\mathfrak{a})$  over  $R$  is equivalent to the category of finite projective  $A$ -modules  $M$  endowed with a topologically quasi-nilpotent twisted connection.*

*Proof.* This is similar to the proof of Proposition 5.1 using the ring  $A\langle \xi \rangle_q$  of twisted divided polynomials of level 0 and Theorem 7.3 of [GLQ23] instead of Proposition 5.7 of [GLQ22b].  $\square$

**Remark.** Using the same techniques, Andre Chatzistamatiou shows in [Cha20] (Theorem 1.3.3 and Proposition 2.1.4) that Corollary 5.4 holds in a more general setting.

## Comparison

Recall now from Definition 4.3 of [GLQ22b] that, if we denote by  $\mathrm{MIC}_q$  (resp.  $\mathrm{MIC}_q^{(-1)}$ ) the category of modules endowed with a twisted connection (resp. a twisted connection of level  $-1$ ), then there exists a *level raising functor*

$$F^* : \mathrm{MIC}_q^{(-1)}(A'/R) \rightarrow \mathrm{MIC}_q(A/R)$$

where  $A' := R_{\phi} \widehat{\otimes}_R A$ .

**Theorem 5.5.** *Let  $(R, \mathfrak{r}) \rightarrow (A, \mathfrak{a})$  be a morphism of bounded  $q$ -PD-pairs with  $A$  complete. Assume that there exists a topologically étale coordinate  $x$  on  $A/R$  with  $\delta(x) = 0$ . If  $\mathcal{X} := \mathrm{Spf}(A/\mathfrak{a})$  and  $\mathcal{X}' := \mathrm{Spf}(A'/(p)_q)$  with  $A' := R_{\phi} \widehat{\otimes}_R A$ , then there exists a commutative diagram of equivalences*

$$\begin{array}{ccc} \{\text{prismatic crystals on } \mathcal{X}'/R\} & \xrightarrow[\simeq]{C_{\mathcal{X}'/R}^{-1}} & \{q\text{-crystals on } \mathcal{X}/R\} \\ \simeq \downarrow & & \simeq \downarrow \\ \mathrm{MIC}_q^{(-1)}(A'/R) & \xrightarrow[\simeq]{F^*} & \mathrm{MIC}_q^{(0)}(A/R) \end{array}$$

if we stick to locally finite free crystals, finite projective modules and topologically quasi-nilpotent connections.

*Proof.* We showed in Proposition 6.9 of [GLQ22b] that there exists such a commutative diagram. We proved in Theorem 4.8 of loc. cit. that the bottom map is an equivalence and we have proved in Proposition 5.1 and Corollary 5.4 that both vertical maps are equivalences.  $\square$

**Remarks.** 1. Our proof relies on Frobenius descent for twisted modules endowed with a connection and is independent of the results in [Li22] or in [Cha20]. In particular, we recover Theorem 1.5 in the particular setting of this section.

2. Theorem 5.5 can be seen as a  $q$ -deformation of Proposition 9.17 in [Xu19].

## 6 Cohomology

We will show that prismatic cohomology (resp.  $q$ -crystalline cohomology) agrees with twisted de Rham cohomology of level  $-1$  (resp. of level  $0$ ) and prove a comparison theorem for the de Rham cohomology.

We keep the assumptions and notation of the previous section. More precisely, we consider a base prism of the form  $(R, (p)_q)$  where  $R$  is a (complete) bounded  $\delta\text{-}\mathbb{Z}[q]_{(p,q-1)}$ -algebra and  $\delta(q) = 0$ . We let  $A$  be a complete  $R$ -algebra with a topologically étale coordinate  $x$  (and we set  $\delta(x) = 0$ ).

### Prismatic case

If  $M$  is an  $A$ -module endowed with a twisted connection  $\nabla : M \rightarrow M \otimes_{A/R, q^p}$  of level  $-1$ , then its *de Rham cohomology*  $H_{\mathrm{dR}, q(-1)}^i(M)$  is simply the cohomology of the de Rham complex

$$\left[ M \xrightarrow{\nabla} M \otimes_{A/R, q^p} \right] \{ \quad \quad \quad (3)$$

The twisted connection  $\nabla$  provides a *twisted hyper-differential operator of level  $-1$  and order 1* in the sense that  $\nabla$  extends canonically to

$$\begin{array}{ccc} \widehat{A\langle \omega \rangle}_{\mathfrak{q}(-1)} \otimes'_A M & \xrightarrow{\tilde{\nabla}} & M \otimes_{A/R, q^p} \\ \omega \otimes s \mapsto & \longrightarrow & s \otimes d_{q^p} x + (q^p - 1)x \nabla(s) \end{array} \quad (4)$$

(and  $\omega^{\{k\}} \otimes s \mapsto 0$  when  $k > 1$ .) We know from Proposition 5.7 in [GLQ22b] that  $A[\omega]_d^{\delta, \wedge} = \widehat{A\langle \omega \rangle}_{\mathfrak{q}(-1)}$  is the completed ring of twisted divided polynomials of level  $-1$ . It follows that a prismatic differential operator as in Definition 4.9 is then the same thing as a *twisted hyper-differential operator of level  $-1$* .

The de Rham complex (3) is endowed with a semilinear Frobenius  $\phi$  induced by

$$\phi : A/R, q^p \rightarrow A/R, q^p, \quad d_{p^q} x \mapsto (p)_q x^{p-1} d_{p^q} x.$$

This formula comes from the explicit description

$$\phi(\omega) = \sum_{k=1}^p (k-1)_{q^p}! \binom{p-1}{k-1}_{q^p} (p)_q^k x^{p-k} \omega^{\{k\}}$$

of the absolute Frobenius on  $A\langle\omega\rangle_{q(-1)}$  that can be deduced from the computations in the proof of Proposition 1.12 of [GLQ22b].

**Proposition 6.1** (Prismatic Poincaré Lemma). *If  $E$  is a locally finite free prismatic crystal on  $\mathcal{X} := \mathrm{Spf}(A/(p)_q)$  over  $R$  and  $\nabla : E_A \rightarrow E_A \otimes_{A/R, q^p} A/R, q^p$  denotes the corresponding twisted connection of level  $-1$ , then the sequence*

$$0 \rightarrow E \rightarrow L(E_A) \xrightarrow{L(\nabla)} L(E_A \otimes_{A/R, q^p} A/R, q^p) \rightarrow 0$$

is exact.

*Proof.* Thanks to Lemma 4.7 applied to both  $M = A$  and  $M = A/R, q^p$ , we may assume that  $E = \mathcal{O}_{\mathcal{X}/R}$  is the structural sheaf and  $\nabla = (p)_q d_{q^p}$ . We have to show that, for any bounded prism  $B$  on  $\mathcal{X}/R$ , then up to a formally faithfully flat map, the sequence

$$0 \rightarrow B \rightarrow L(A)_B \xrightarrow{L(\nabla)_B} L(A/R, q^p)_B \rightarrow 0$$

is exact. Actually, since  $A$  is a covering of the site, we may assume that there exists a  $\delta$ -morphism  $A \rightarrow B$ . Now, thanks to assertion 2) of Lemma 4.6, it is sufficient to show that augmented de Rham complex

$$0 \rightarrow A \rightarrow \widehat{A\langle\omega\rangle_{q(-1)}} \xrightarrow{L(\nabla)^A} \widehat{A\langle\omega\rangle_{q(-1)}} \otimes_{A/R, q^p} A/R, q^p \rightarrow 0 \quad (5)$$

is split exact. Let us recall the standard formulas in twisted calculus:

$$\Delta(\omega^{\{k\}}) = \sum_{i+j=k} \omega^{\{i\}} \otimes \omega^{\{j\}}, \quad \tilde{\nabla}(\omega) = d_{q^p} x \quad \text{and} \quad \tilde{\nabla}(\omega^{\{k\}}) = 0 \text{ for } k > 1.$$

The first one may be derived from Theorem 3.5 in [LQ18] and the other ones come from (4). By definition of linearization, we see that

$$\forall k \in \mathbb{N}, \quad L(\nabla)_A(\omega^{\{k+1\}}) = \omega^{\{k\}} \otimes d_{q^p} x.$$

This shows that the sequence (5) is split exact.  $\square$

The Poincaré lemma provides an isomorphism on cohomology:

**Theorem 6.2.** *If  $E$  is a locally finite free prismatic crystal on  $\mathcal{X} := \mathrm{Spf}(A/(p)_q)$  over  $R$ , then*

$$\mathrm{R}\Gamma((\mathcal{X}/R)_{\Delta}, E) \simeq \left[ E_A \xrightarrow{\nabla} E_A \otimes_{A/R, q^p} A/R, q^p \right] \{$$

*Proof.* With the notation of Section 4, we have

$$\begin{aligned} \mathrm{R}\Gamma((\mathcal{X}/R)_{\Delta}, E) &\simeq \mathrm{Re}_{\mathcal{X}/R*} E \\ &\simeq \mathrm{Re}_{\mathcal{X}/R*} \left[ L(E_A) \xrightarrow{L(\nabla)} L(E_A \otimes_{A/R, q^p} A/R, q^p) \right] \\ &\simeq \left[ E_A \xrightarrow{\nabla} E_A \otimes_{A/R, q^p} A/R, q^p \right] \{ \end{aligned} \quad \square$$

As a consequence of this theorem, we see that

$$\forall i \in \mathbb{N}, \quad H^i((\mathcal{X}/R)_\Delta, E) \simeq H_{\mathrm{dR}, q(-1)}^i(E_A).$$

We also obtain a canonical *Hodge decomposition*: if as usual we denote reduction modulo  $(p)_q$  with a bar and let  $\bar{A}/\bar{R}$  be the usual module of differentials, then:

**Corollary 6.3.** *There exists a canonical isomorphism*

$$\mathrm{R}\Gamma((\mathcal{X}/R)_\Delta, \bar{\mathcal{O}}) \simeq \bar{A} \oplus \bar{A}/\bar{R}[-1]. \quad (6)$$

*Proof.* Since a bounded prism over  $R$  is  $(p)_q$ -torsion free, we have  $[\mathcal{O} \xrightarrow{(p)_q} \mathcal{O}] \simeq \bar{\mathcal{O}}$  and it therefore follows from Theorem 6.2 that

$$\mathrm{R}\Gamma((\mathcal{X}/R)_\Delta, \bar{\mathcal{O}}) \simeq \left[ \bar{A} \xrightarrow{0} \bar{A}/\bar{R} \right]. \quad \square$$

**Remarks.** 1. Let us briefly review construction 4.9 of [BS22] in our setting. The short exact sequence

$$0 \longrightarrow \bar{R} \xrightarrow{(p)_q} R/(p)_q^2 \longrightarrow \bar{R} \longrightarrow 0$$

provides a Bockstein boundary homomorphism

$$: \bar{A} \simeq H^0((\mathcal{X}/R)_\Delta, \bar{\mathcal{O}}) \rightarrow \bar{A}/\bar{R} \simeq H^1((\mathcal{X}/R)_\Delta, \bar{\mathcal{O}}).$$

If we denote by  $d : \bar{A} \rightarrow \bar{A}/\bar{R}$  the usual differential, then  $\eta$  extends to a morphism

$$\eta : \bar{A}/\bar{R} \rightarrow \bar{A}/\bar{R}, \quad f dg \mapsto f \cdot g.$$

It is shown in proposition 4.11 of [BS22] that  $\eta$  is an isomorphism and we may therefore identify  $\eta$  with  $d$ . It follows that our Hodge decomposition theorem is the incarnation in the derived category of Theorem 6.3 of [BS22].

2. This Hodge decomposition is also obtained by Tian (remark 4.16 in [Tia21]).
3. The isomorphism of Theorem 6.2 is functorial: it sends the semilinear Frobenius endomorphism  $\phi$  of  $\mathrm{R}\Gamma((\mathcal{X}/R)_\Delta, E)$  to the semilinear endomorphism  $\phi$  of  $[E_A \xrightarrow{\nabla} E_A \otimes_A A/R, q^p]$ .

### $q$ -crystalline case

We will now consider the  $q$ -crystalline case and assume that  $(R, \mathfrak{r})$  is a bounded  $q$ -PD-pair. We first recall the following:

**Definition 6.4.** The *(complete bounded)  $q$ -PD-envelope* of an ideal  $J$  in a  $\delta$ - $R$ -algebra  $B$  is a complete bounded  $q$ -PD-pair  $(C, K)$  over  $R$  which is universal for morphisms  $B \rightarrow C$  sending  $J$  into  $K$ .

We may also say that  $C$  is the  $q$ -PD-envelope, meaning that there exists such a  $K$ , which is then unique.

**Proposition 6.5** (Bhatt-Scholze). *Assume that  $B$  is a formally flat bounded  $\delta$ - $R$ -algebra. If  $g \in B$  is completely regular over  $R$ , then  $B' := B[\phi(g)/(p)_q]^{\delta, \wedge}$  is the complete  $q$ -PD-envelope of  $J := (\tau B, g)$ . Moreover, if  $J'$  denotes the  $q$ -PD-ideal of  $B'$ , then  $B/J \simeq B'/J'$ .*

*Proof.* This is shown in Lemma 16.10 of [BS22].  $\square$

Let us assume now that  $A$  is endowed with a closed  $q$ -PD-ideal  $\mathfrak{a}$  such that  $\tau A \subset \mathfrak{a}$ . If  $M$  is an  $A$ -module endowed with a twisted connection  $\nabla : M \rightarrow M \otimes_A {}_A/R, q$ , then its de Rham cohomology  $H_{\mathrm{dR}, q}^i(M)$  is the cohomology of the de Rham complex

$$\left[ M \xrightarrow{\nabla} M \otimes_A {}_A/R, q \right] \{$$

Based on Proposition 6.5, we can then apply the prismatic method to the  $q$ -crystalline situation and show that:

**Theorem 6.6.** *If  $E$  is a locally finite free  $q$ -crystal on  $\mathcal{X} := \mathrm{Spf}(A/\mathfrak{a})$  over  $R$ , then*

$$\mathrm{R}\Gamma((\mathcal{X}/R)_{q\text{-CRIS}}, E) \simeq \left[ E_A \xrightarrow{\nabla} E_A \otimes_A {}_A/R, q \right] \{ \quad \square$$

As a consequence, we will have

$$\forall i \in \mathbb{N}, \quad H^i((\mathcal{X}/R)_{q\text{-CRIS}}, E) \simeq H_{\mathrm{dR}, q}^i(E_A).$$

## Comparison

We finish by proving that raising level provides an isomorphism on de Rham complexes. We write  $A' := R_{\phi} \widehat{\otimes}_R A$  and we consider  $A$  as an  $A'$ -algebra via the relative Frobenius  $F : A' \rightarrow A$ .

**Theorem 6.7.** *If  $M'$  is a complete formally flat  $A'$ -module endowed with a topologically quasi-nilpotent twisted connection of level  $-1$  and  $M := A \otimes_{A'} M'$  is endowed by level rising with its topologically quasi-nilpotent twisted connection of level  $0$ , then Frobenius induces a quasi-isomorphism*

$$[M' \rightarrow M' \otimes_{{}_A/R, q^p}] \simeq [M \rightarrow M \otimes_{{}_A/R, q}].$$

*Proof.* Our map is induced by Frobenius

$$F : M' \rightarrow M := A \otimes_{A'} M', \quad s \mapsto 1 \otimes s,$$

but we have to be careful because the Frobenius on differentials is induced by the divided Frobenius

$$[F] : {}_A/R, q^p \rightarrow {}_A/R, q, \quad d_{q^p} x' \mapsto x^{p-1} d_q x,$$

where we write  $x' := 1_{\phi} \widehat{\otimes} x \in A'$ . Let us denote respectively by  $s \mapsto \theta(s) \otimes d_q x$  and  $s' \mapsto \theta'(s) \otimes d_{q^p} x'$  the twisted connections (of level  $0$ ) of  $M$  and (of level  $-1$ ) of  $M'$ . We also set

$$[F] : M' \rightarrow M, \quad s \mapsto x^{p-1} F(s).$$

Let us also recall from Proposition 4.2 in [GLQ22a] that

$$\forall s \in M', \quad \theta(1 \otimes s) = x^{p-1} \otimes \theta'(s).$$

We have to prove that the vertical map of complexes

$$\begin{array}{ccc} M' & \xrightarrow{\theta'} & M' \\ \downarrow F & & \downarrow [F] \\ M & \xrightarrow{\theta} & M \end{array}$$

is a quasi-isomorphism. Since  $A$  is free on  $A'$  with generators  $1, x, \dots, x^{p-1}$ , we have

$$M := A \otimes_{A'} M' \simeq \bigoplus_{k=0}^{p-1} A' x^k \otimes_{A'} M' = \bigoplus_{k=0}^{p-1} \{x^k M'\}$$

as an  $A'$ -module. On the one hand,

$$\theta(s) = \theta(1 \otimes s) = x^{p-1} \otimes \theta'(s) = x^{p-1} \theta'(s),$$

and there exists therefore a vertical isomorphism of complexes

$$\begin{array}{ccc} M' & \xrightarrow{\theta'} & M' \\ \downarrow F & & \downarrow [F] \\ M' & \xrightarrow{\theta} & x^{p-1} M'. \end{array}$$

On the other hand, we have for all  $1 \leq k < p$ ,

$$\begin{aligned} \theta(x^k s) &= \theta(x^k \otimes s) \\ &= x^k \theta(1 \otimes s) + (k)_q x^{k-1} \otimes s \\ &= x^k x^{p-1} \otimes \theta'(s) + x^{k-1} \otimes (k)_q s \\ &= x^{k-1} \otimes x' \theta'(s) + x^{k-1} \otimes (k)_q s \\ &= x^{k-1} (x' \theta'(s) + (k)_q s). \end{aligned}$$

Our assertion therefore reduces to showing that, for  $1 \leq k < p$ , the map

$$M' \rightarrow M', \quad s \mapsto x' \theta'(s) + (k)_q s$$

is bijective. We may then invoke Derived Nakayama Lemma (see Proposition A.8) and assume that  $q = 1$  and  $p = 0$ , in which case we fall back on classical Cartier isomorphism (Corollary 2.28 of [OV07]). We could also only assume that  $(p)_q = 0$  and rely on Corollary 8.10 of [GLQ22a], which is more in the spirit of this article.  $\square$

**Remarks.** 1. As a consequence of this theorem, there exist isomorphisms

$$\forall k \in \mathbb{N}, \quad H_{\mathrm{dR}, q(-1)}^k(M') \simeq H_{\mathrm{dR}, q}^k(M).$$

2. In the case  $q = 1$ , Theorem 6.7 is not new. In this situation, it is in fact just an improvement, already noticed by Arthur Ogus, of a result of Atsushi Shiho ([Shi15], Theorem 4.4).
3. When  $M'$  is finitely presented and the connection is topologically quasi-nilpotent, one can recover this result from Theorem 1.5 thanks to Theorem 6.2 and Theorem 6.6.



4. Conversely, Theorem 6.7 provides us with a new proof of theorem 1.5 in our particular setting.

**Corollary 6.8.** *If  $E'$  is a locally finite free prismatic crystal on  $\mathcal{X}' := \mathrm{Spf}(A'/(p)_q)$  over  $R$ , then Frobenius induces an isomorphism*

$$\mathrm{R}\Gamma((\mathcal{X}'/R)_\Delta, E') \left[ \frac{1}{(p)_q} \right] \left\{ \simeq \mathrm{R}\Gamma((\mathcal{X}/R)_\Delta, F^*E') \left[ \frac{1}{(p)_q} \right] \right\}.$$

*Proof.* If we let  $M' := E'_{A'}$  and  $M := A \otimes_{A'} M'$ , then we already know that

$$\begin{aligned} \mathrm{R}\Gamma((\mathcal{X}'/R)_\Delta, E') &\simeq [\mathcal{M}' \rightarrow M' \otimes_{A'} A/R, q^p] \\ &\simeq [M \rightarrow M \otimes_{A/R, q}] \end{aligned}$$

(with a twisted connection of level 0). But we also have  $(F^*E')_A = A \otimes_{A'} M' = M$  and therefore

$$\mathrm{R}\Gamma((\mathcal{X}/R)_\Delta, F^*E') = [\mathcal{M} \rightarrow M \otimes_A A/R, q^p]$$

(with a twisted connection of level  $-1$ ). Now, the canonical map

$$[M \rightarrow M \otimes_{A/R, q}] \rightarrow [M \rightarrow M \otimes_{A/R, q^p}]$$

is the identity on  $M$  and the map induced on differentials is the “Verschiebung”

$$A/R, q \longrightarrow A/R, q^p, \quad d_q x \mapsto (p)_q d_{q^p} x$$

(see Section 1 of [GLQ22b]). □

**Remark.** In the case of the trivial crystal, Corollary 6.8 is due to Bhatt and Scholze (see Theorem 1.8 (6) in the introduction of [BS22]).

There exists a variant to this last corollary:

**Corollary 6.9.** *If  $E$  is a locally finite free prismatic crystal on  $\mathcal{X} := \mathrm{Spf}(A/(p)_q)$  over  $R$ , then Frobenius induces an isomorphism*

$$R_{\phi^*} \widehat{\otimes}_R^L \mathrm{R}\Gamma((\mathcal{X}/R)_\Delta, E) \left[ \frac{1}{(p)_{q^p}} \right] \left\{ \simeq \mathrm{R}\Gamma((\mathcal{X}/R)_\Delta, \phi^*E) \left[ \frac{1}{(p)_{q^p}} \right] \right\}.$$

*Proof.* If we pull back  $E$  along the Frobenius morphism  $\phi : (R, (p)_q) \rightarrow (R, (p)_{q^p})$ , we get some  $E'$  and we have

$$\begin{aligned} R_{\phi^*} \widehat{\otimes}_R^L \mathrm{R}\Gamma((\mathcal{X}/R)_\Delta, E) &\simeq R_{\phi^*} \widehat{\otimes}_R^L \left[ \mathcal{E}_A \xrightarrow{\nabla} E_A \otimes_A A/R, q^p \right] \{ \\ &\simeq \left[ \mathcal{E}'_{A'} \xrightarrow{\nabla} E'_{A'} \otimes_{A'} A'/R', q^{p^2} \right] \{ \\ &\simeq \mathrm{R}\Gamma((\mathcal{X}'/R)_\Delta, E') \{ \end{aligned}$$

and

$$\mathrm{R}\Gamma((\mathcal{X}/R)_\Delta, F^*E') \simeq \mathrm{R}\Gamma((\mathcal{X}/R)_\Delta, \phi^*E).$$

We may then apply Corollary 6.8. □

## A Appendix

We gather some results found here and there about complete flatness and derived completion that we use in the core of the article. We however do not discuss the simplicial approach. Our aim is mostly to explain how complete flatness and derived completion boil down in practice to the discrete notions of classical completion and formal flatness.

**Caveat:** In contrast with the rest of the article, we use in this appendix the *hat* notation to denote derived completion (and not classical completion).

We let  $A$  be a (not necessarily complete) adic ring. All  $A$ -modules are endowed with their adic topology.

At some point, we will assume that  $A$  has a finitely generated ideal of definition.

### A.1 Complete flatness

When we say that  $M^\bullet$  is a *complex of  $A$ -modules*, we mean that it is an object of the derived category  $D(A)$  of the category of  $A$ -modules. We consider the latter as a full subcategory of  $D(A)$ .

**Definition A.1.** 1. A complex of  $A$ -modules  $M^\bullet$  is *discrete* if  $H^k(M^\bullet) = 0$  whenever  $k \neq 0$ .

2. A complex of  $A$ -modules  $M^\bullet$  is *discrete (faithfully) flat* (over  $A$ ) if it is *discrete* and  $H^0(M^\bullet)$  is (faithfully) flat over  $A$ .

**Remarks.** 1. The inclusion of the category of  $A$ -modules into the category of complexes of  $A$ -modules induces an equivalence between the category of  $A$ -modules (resp. (faithfully) flat  $A$ -modules) and the category of discrete (resp. discrete (faithfully) flat) complexes of  $A$ -modules. An inverse is given by the functor  $M^\bullet \mapsto H^0(M^\bullet)$ .

2. Since we will have to move back and forth between  $A$ -modules and complexes of  $A$ -modules, we also recall the following:

(a) If  $L^\bullet \rightarrow M^\bullet \rightarrow N^\bullet \rightarrow$  is a distinguished triangle and two of the complexes are discrete, then so is the third.

(b) A sequence  $0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$  of  $A$ -modules is exact if and only if the triangle  $L \rightarrow M \rightarrow N \rightarrow$  is distinguished.

3. A complex of  $A$ -modules  $M^\bullet$  is discrete (resp. discrete (faithfully) flat) if and only if  $M^\bullet \simeq H^0(M^\bullet)$  (resp.  $M^\bullet \simeq H^0(M^\bullet)$  and  $H^0(M^\bullet)$  is (faithfully) flat).

4. A complex of  $A$ -modules  $M^\bullet$  is discrete flat (resp. discrete faithfully flat) if and only if for any  $A$ -module  $N$ , the complex  $M^\bullet \otimes_A^L N$  is discrete (resp. the complex  $M^\bullet \otimes_A^L N$  is discrete and  $\neq 0$  unless  $N = 0$ ). When this is the case, we have

$$M^\bullet \otimes_A^L N \simeq H^0(M^\bullet \otimes_A^L N) \simeq H^0(M^\bullet) \otimes_A N.$$

**Lemma A.2.** Let  $M^\bullet$  be a complex of  $A$ -modules and  $I, J \subset A$  two ideals. If  $M^\bullet \otimes_A^L A/I$  is discrete (faithfully) flat over  $A/I$  and  $I^{n+1} \subset J$ , then  $M^\bullet \otimes_A^L A/J$  is discrete (faithfully) flat over  $A/J$ .

*Proof.* It is sufficient to do the cases  $I = 0$  and  $J = I^2 = 0$ .

1. Assume  $I = 0$ . Then  $M^\bullet$  is discrete (faithfully) flat over  $A$  so that

$$M^\bullet \otimes_A^L A/J \simeq H^0(M^\bullet \otimes_A^L A/J) \simeq H^0(M^\bullet) \otimes_A A/J$$

and we know that scalar extension preserves (faithful) flatness.

2. Assume  $J = I^2 = 0$ . If  $N$  is any  $A$ -module, then the short exact sequence  $0 \hookrightarrow IN \rightarrow N \rightarrow N/IN \rightarrow 0$  provides us with a distinguished triangle

$$M^\bullet \otimes_A^L IN \rightarrow M^\bullet \otimes_A^L N \rightarrow M^\bullet \otimes_A^L N/IN \rightarrow .$$

Of course,  $N/IN$  is an  $A/I$ -module, but since  $I^2 = 0$ ,  $IN$  is also an  $A/I$ -module and we may rewrite our triangle:

$$(M^\bullet \otimes_A^L A/I) \otimes_{A/I}^L IN \rightarrow M^\bullet \otimes_A^L N \rightarrow (M^\bullet \otimes_A^L A/I) \otimes_{A/I}^L N/IN \rightarrow .$$

Since  $M^\bullet \otimes_A^L A/I$  is discrete flat over  $A/I$ , we see that both sides are discrete and  $M^\bullet \otimes_A^L N$  must also be discrete. We also see that, when  $M^\bullet \otimes_A^L N = 0$ , we have

$$(M^\bullet \otimes_A^L A/I) \otimes_{A/I}^L IN = (M^\bullet \otimes_A^L A/I) \otimes_{A/I}^L N/IN = 0.$$

Therefore, if we assume that  $M^\bullet \otimes_A^L A/I$  is actually faithfully flat, we will have  $IN = N/IN = 0$  and finally  $N = 0$ .  $\square$

**Definition A.3.** A complex of  $A$ -modules  $M^\bullet$  is *completely (faithfully) flat* if  $M^\bullet \otimes_A^L A/I$  is discrete (faithfully) flat over  $A/I$  for *some* ideal of definition  $I$ .

**Warning :** In this definition, the complex  $M^\bullet$  itself need not be discrete.

**Remarks.** 1. By Lemma A.2, if a complex of  $A$ -modules  $M^\bullet$  is completely (faithfully) flat, then  $M^\bullet \otimes_A^L A/I$  is discrete (faithfully) flat over  $A/I$  for *any* ideal of definition  $I$ .

2. A complex of  $A$ -modules  $M^\bullet$  is discrete (faithfully) flat if and only if it is completely (faithfully) flat for the discrete topology.
3. Complete (faithful) flatness is inherited by any coarser adic topology (bigger ideal of definition).
4. If a complex of  $A$ -modules  $M^\bullet$  is discrete (faithfully) flat, then it is completely (faithfully) flat.

We now let  $I$  be an ideal of definition for  $A$ .

5. If  $J \subset I$ , then  $M^\bullet$  is completely (faithfully) flat if and only if  $M^\bullet \otimes_A^L A/J$  is completely (faithfully) flat for the  $I/J$ -adic topology.
6. (a) A complex of  $A$ -modules  $M^\bullet$  is completely (faithfully) flat if and only if  $H^0(M^\bullet \otimes_A^L A/I)$  is (faithfully) flat over  $A/I$  and  $H^k(M^\bullet \otimes_A^L A/I) = 0$  for  $k \neq 0$ .  
 (b) An  $A$ -module  $M$  is completely flat if and only if  $M/IM$  is flat over  $A/I$  and  $M \otimes_A^L A/I = M \otimes_A A/I$  (or equivalently  $\text{Tor}_k^A(M, A/I) = 0$  for  $k \neq 0$ ).

7. (a) If a complex of  $A$ -modules  $M^\bullet$  is completely flat, then

$$M^\bullet \otimes_A^L A/I \simeq H^0(M^\bullet \otimes_A^L A/I).$$

- (b) If an  $A$ -module  $M$  is completely flat, then

$$M \otimes_A^L A/I \simeq M/IM.$$

8. (a) A complex of  $A$ -modules  $M^\bullet$  is completely flat if and only if for all  $A/I$ -module  $N$ , the complex  $M^\bullet \otimes_A^L N$  is discrete.  
(b) An  $A$ -module  $M$  is completely flat if and only if for all  $A/I$ -module  $N$ ,  $M \otimes_A^L N = M \otimes_A N$  (i.e.  $\text{Tor}_k^A(M, N) = 0$  for  $k \neq 0$ ).

**Definition A.4.** An adic morphism of adic rings  $u : A \rightarrow B$  is said to be *completely (faithfully) flat* if  $B$  is completely (faithfully) flat as an  $A$ -module.

**Remarks.** Let  $u : A \rightarrow B$  is an adic morphism of adic rings.

1. If  $M^\bullet$  is a completely (faithfully) flat complex of  $A$ -modules, then  $B \otimes_A^L M^\bullet$  is a completely (faithfully) flat complex of  $B$ -modules.
2. Conversely, if  $u$  is completely faithfully flat and  $B \otimes_A^L M^\bullet$  is completely flat, then  $M^\bullet$  was already completely flat.

**Definition A.5.** 1. An  $A$ -module  $M$  is *formally (faithfully) flat*<sup>9</sup> if  $M/IM$  is (faithfully) flat over  $A/I$  for *any* ideal of definition  $I$ .  
2. A complex of  $A$ -modules  $M^\bullet$  is *formally (faithfully) flat* if it is *discrete* and  $H^0(M^\bullet)$  is formally (faithfully) flat.

**Remarks.** 1. If  $I$  is an ideal of definition of  $A$ , then an  $A$ -module  $M$  is formally (faithfully) flat if and only if  $M/I^{n+1}M$  is (faithfully) flat over  $A/I^{n+1}$  for all  $n \in \mathbb{N}$ .  
2. It is clearly not sufficient that  $M/IM$  is (faithfully) flat over  $A/I$  for *some* ideal of definition  $I$ .  
3. A (faithfully) flat  $A$ -module  $M$  is automatically completely (faithfully) flat and a completely (faithfully) flat  $A$ -module  $M$  is automatically formally (faithfully) flat.

## A.2 Derived completeness

In order to go further, we need to recall some results about derived completeness (we refer the reader to [Sta19, Tag 091N] for the details).

**Definition A.6.** 1. An  $A$ -module  $M$  is said to be *classically complete*<sup>10</sup> if

$$M \simeq \varprojlim_{n \in \mathbb{N}} M/I^n$$

for some (or any) ideal of definition.

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<sup>9</sup>Also called *adically flat*.

<sup>10</sup>Also called *adically complete*.

2. A complex of  $A$ -modules  $M^\bullet$  is said to be *classically complete* if it is *discrete* and  $H^0(M^\bullet)$  is classically complete.
3. A complex of  $A$ -modules  $M^\bullet$  is said to be *derived complete* if

$$\forall f \in I, \quad \mathbf{R}\varprojlim_f M^\bullet = 0 \tag{7}$$

for *some* ideal of definition  $I$  of  $A$ .

- Remarks.**
1. If a complex of  $A$ -modules  $M^\bullet$  is derived complete, then (7) holds for any ideal of definition  $I$  of  $A$  (and even for  $\sqrt{I}$ ).
  2. Derived completeness may be checked on generators of  $I$ .
  3. Derived completeness is inherited by any finer adic topology (smaller ideal of definition).
  4. If a complex of  $A$ -modules  $M^\bullet$  is classically complete, then  $M^\bullet$  is automatically derived complete.
  5. If  $M^\bullet$  is a complex of  $A$ -modules, we shall denote by

$$M^{\bullet, \text{sep}} := H^0(M^\bullet) / \bigcap_{n \geq 1} \mathfrak{f}^n H^0(M^\bullet)$$

the *separated quotient* of  $H^0(M^\bullet)$ . Assume  $I$  is finitely generated. If a complex  $M^\bullet$  is derived complete, it then follows from assertion 2) of Proposition A.7 below and assertion 2) of [Sta19, Tag 091R] that  $M^{\bullet, \text{sep}}$  is classically complete.

Derived completeness is a remarkably stable notion:

- Proposition A.7.**
1. *Derived complete complexes of  $A$ -modules form a saturated<sup>11</sup> full triangulated subcategory of the category of all complexes of  $A$ -modules*
  2. *A complex of  $A$ -modules is derived complete if and only if it has derived complete cohomology.*
  3. *Derived complete  $A$ -modules form a full abelian subcategory of the category of all  $A$ -modules which is stable under kernels, cokernels and extensions (weak Serre subcategory).*

*Proof.* [Sta19, Tag 091U]. □

We assume from now on that  $A$  has a finitely generated ideal of definition.

Let us first mention the important *Derived Nakayama Lemma*:

**Proposition A.8.** *If  $M^\bullet$  is derived complete, then*

$$M^\bullet = 0 \Leftrightarrow M^\bullet \otimes_A^L A/I = 0.$$

*Proof.* [Sta19, Tag 0G1U]. □

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<sup>11</sup>That is, stable under direct factor.

**Proposition A.9.** *The inclusion of the category of derived complete complexes of  $A$ -modules into the category of all complexes of  $A$ -modules has a left adjoint  $M^\bullet \mapsto \widehat{M}^\bullet$ .*

*Proof.* [Sta19, Tag 0920]. □

The complex  $\widehat{M}^\bullet$  is then called the *derived completion* of  $M^\bullet$ .

**Warning:** If  $M$  is an  $A$ -module, then its derived completion may not be discrete, and even when it is discrete, it may differ from (classical) adic completion.

**Remarks.** 1. Assume  $(f_1, \dots, f_r)$  is an ideal of definition for  $A$  and denote as above by  $\text{Kos}$  the corresponding Koszul complex (see [Sta19, Tag 0623]). Then, if  $M^\bullet$  is a complex of  $A$ -modules, we have

$$\widehat{M}^\bullet = \mathbf{R} \varprojlim (M^\bullet \otimes_A^{\mathbf{L}} \text{Kos}(A, f_1^n, \dots, f_r^n)).$$

Moreover,  $M^\bullet$  is derived complete if and only if  $M^\bullet = \widehat{M}^\bullet$ .

2. As a particular case, the *derived completion* of an  $A$ -module  $M$  will be the complex

$$\widehat{M} = \mathbf{R} \varprojlim \text{Kos}(M, f_1^n, \dots, f_r^n).$$

Notice that  $\widehat{M} \neq \varprojlim M/(f_1^n, \dots, f_r^n)$  in general.

3. The inclusion of the category of classically complete  $A$ -modules into the category of all  $A$ -modules also has a left adjoint given by

$$M \mapsto \varprojlim_{n \in \mathbb{N}} M/I^n$$

and the latter is called the *classical completion* of  $M$ .

4. If  $M$  is an  $A$ -module, then the *classical completion* of  $M$  is  $\widehat{M}^{\text{sep}}$ . This is shown as follows. Assume that we are given an  $A$ -linear map  $M \rightarrow N$  with  $N$  classically complete. It extends uniquely to  $\widehat{M} \rightarrow N$  and since, by [Sta19, Tag 0AAJ], we have  $H^i(\widehat{M}) = 0$  for  $i > 0$ , it factors through  $H^0(\widehat{M})$ , and then through  $\widehat{M}^{\text{sep}}$ .

5. As a consequence, if  $M$  is an  $A$ -module, then  $\widehat{M}$  is classically complete if and only if  $\widehat{M}$  is isomorphic to the classical completion  $\widehat{M}^{\text{sep}}$  of  $M$ .

**Proposition A.10.** 1. *If  $M^\bullet$  is any complex of  $A$ -modules and  $I$  denotes an ideal of definition for  $A$ , then*

$$M^\bullet \otimes_A^{\mathbf{L}} A/I \simeq \widehat{M}^\bullet \otimes_A^{\mathbf{L}} A/I.$$

2. *A complex of  $A$ -modules  $M^\bullet$  is completely flat if and only if  $\widehat{M}^\bullet$  is completely flat.*

*Proof.* The second assertion is a consequence of the first one. Note now that any complex of  $A$ -modules quasi-isomorphic to a complex of  $A/I$ -modules is automatically derived complete. Assume that we are given a morphism  $M^\bullet \otimes_A^{\mathbf{L}} A/I \rightarrow N^\bullet$  with  $N^\bullet$  derived complete. Then, the composite map  $M^\bullet \rightarrow M^\bullet \otimes_A^{\mathbf{L}} A/I \rightarrow N^\bullet$  extends uniquely to a morphism  $\widehat{M}^\bullet \rightarrow N^\bullet$  which factors uniquely through  $\widehat{M}^\bullet \otimes_A^{\mathbf{L}} A/I$ . This shows that  $\widehat{M}^\bullet \otimes_A^{\mathbf{L}} A/I$  is the derived completion of  $M^\bullet \otimes_A^{\mathbf{L}} A/I$  and they must therefore be the same. □

We will also write

$$M^{\bullet, \wedge} := \widehat{M^{\bullet}} \quad \text{and} \quad M^{\bullet} \widehat{\otimes}_A^L N^{\bullet} := (M^{\bullet} \otimes_A^L N^{\bullet})^{\wedge},$$

as well as, if  $M$  and  $N$  are two  $A$ -modules,

$$M \widehat{\otimes}_A N := H^0(M \widehat{\otimes}_A^L N).$$

Note that  $M \widehat{\otimes}_A N$  is *not* the derived completion (which may not be discrete) of  $M \otimes_A N$  *nor* its classical completion (because it may not be separated).

**Lemma A.11.** *If  $M$  and  $N$  are two  $A$ -modules, then*

$$M \widehat{\otimes}_A N \simeq H^0((M \otimes_A N)^{\wedge}).$$

*Proof.* Let  $M^{\bullet}$  be a complex of  $A$ -modules. Recall from [Sta19, Tag 0AAJ] that if we assume that  $H^j(M^{\bullet}) = 0$  for  $j > 0$ , then the same holds for  $H^j(M^{\bullet, \wedge})$ . Now, there exists a bounded (in the sense of spectral sequences) spectral sequence ([Sta19, Tag 0BKE])

$$E_2^{i,j} = H^i(H^j(M^{\bullet})^{\wedge}) \Rightarrow H^{i+j}(M^{\bullet, \wedge}).$$

It follows that we will always have  $E_2^{i,j} = 0$  for  $i > 0$  or  $j > 0$ . As a consequence,

$$H^0(H^0(M^{\bullet})^{\wedge}) = H^0(M^{\bullet, \wedge})$$

and we can apply this to  $M \otimes_A^L N$ . □

**Remark.** Set

$$M^{\bullet} \widehat{\otimes}_{\mathfrak{A}}^{\text{sep}} N^{\bullet} := (M^{\bullet} \widehat{\otimes}_A^L N^{\bullet})^{\text{sep}}.$$

Then, if  $M$  and  $N$  are two  $A$ -modules,  $M \widehat{\otimes}_{\mathfrak{A}}^{\text{sep}} N$  is the classical completion of  $M \otimes_A N$ . Moreover,  $M \widehat{\otimes}_A^L N$  is discrete and classically complete if and only if

$$M \widehat{\otimes}_A^L N \simeq M \widehat{\otimes}_{\mathfrak{A}}^{\text{sep}} N,$$

in which case also  $M \widehat{\otimes}_A^L N \simeq M \widehat{\otimes}_A N$ .

### A.3 Example: the monogenic case

We assume here that  $A$  is endowed with the  $f$ -adic topology for some  $f \in A$ .

**Remark.** If  $A$  is  $f$ -torsion free and  $M$  is an  $A$ -module, then

1.  $M \otimes_A^L A/fA \simeq [M \xrightarrow{f} M]$ ,
2.  $M \otimes_A^L A/fA$  is discrete if and only if  $M$  is  $f$ -torsion free,
3.  $M$  is completely flat if and only if  $M$  is  $f$ -torsion free and  $M/fM$  is flat over  $A/fA$ .
4. If  $M$  is formally flat and separated, then  $M$  is  $f$ -torsion free.

Also, if  $M$  is an  $f$ -torsion free  $A$ -module, then  $\widehat{M}$  is classically complete and  $f$ -torsion free.

It is however too much to require torsion freeness and we shall introduce a weaker condition. We will denote the  $f$ -torsion of an  $A$ -module  $M$  by<sup>12</sup>

$${}_fM = \{s \in M, fs = 0\}.$$

**Definition A.12.** 1. The  $f^\infty$ -torsion of an  $A$ -module  $M$  is

$$f^\infty M := \bigcup_{n \in \mathbb{N}} \{f^n M\}.$$

2. An  $A$ -module  $M$  has  $f^\infty$ -torsion *bounded* by  $N \in \mathbb{N}$  if  $f^\infty M = f^N M$ . It has *bounded  $f^\infty$ -torsion* if it has  $f^\infty$ -torsion bounded by  $N$  for some  $N \in \mathbb{N}$ .
3. A complex of  $A$ -modules  $M^\bullet$  has *bounded  $f^\infty$ -torsion* if it is discrete and  $H^0(M^\bullet)$  has bounded  $f^\infty$ -torsion.

For example, an  $A$ -module  $M$  has bounded  $f^\infty$ -torsion when  $M$  is noetherian, when  $M$  is  $f$ -torsion free or when  $M$  is  $f^N$ -torsion for some  $N$ .

Recall that an  $A$ -module  $M$  is said to be completely flat if the corresponding complex of  $A$ -modules placed in degree zero satisfies the condition of Definition A.3 with  $I = (f)$ .

**Lemma A.13.** *If  $M$  is an  $A$ -module, then the following conditions are equivalent:*

1.  $M$  is a completely flat  $A$ -module,
2.  $M/fM$  is flat over  $A/fA$  and  $M \otimes_A^L A/fA \simeq {}_fM$ .

*When this is the case, if  $A$  has  $f^\infty$ -torsion bounded by  $N$ , then so does  $M$ .*

*Proof.* We know from remark (6b) after Definition A.3 that  $M$  is completely flat if and only if  $M/fM$  is flat over  $A/fA$  and  $M \otimes_A^L A/fA \simeq M \otimes_A A/fA$ . Now, by definition, there always exists a distinguished triangle

$${}_fM[1] \rightarrow [M \xrightarrow{f} M] \rightarrow M/fM \rightarrow .$$

From the case  $M = A$ , we deduce another distinguished triangle

$$(M \otimes_A^L A/fA)[1] \rightarrow [M \xrightarrow{f} M] \rightarrow M \otimes_A^L A/fA \rightarrow .$$

It follows that

$$M \otimes_A^L A/fA \simeq {}_fM \quad \Leftrightarrow \quad M \otimes_A^L A/fA \simeq M/fM$$

and the first assertion is proved.

Assume now that  $M$  is completely flat and  $A$  has  $f^\infty$ -torsion bounded by  $N$ . Then, for  $n \geq N$ , we can apply the first assertion to both  $f^n$  and  $f^N$  and obtain

$$f^n M \simeq M \otimes_A^L f^n A \simeq M \otimes_A^L f^N A \simeq f^N M. \quad \square$$

**Lemma A.14.** 1. *If  $M$  is an  $A$ -module with bounded  $f^\infty$ -torsion, then there exists an isomorphism of pro-complexes*

$$\left\{ [M \xrightarrow{f^n} M] \right\}_{n \in \mathbb{N}} \simeq \{M/f^n M\}_{n \in \mathbb{N}}.$$

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<sup>12</sup>We want to avoid the notation  $M[f]$  which may be confused with a twist or a shift.



2. If  $A$  has bounded  $f^\infty$ -torsion and  $M^\bullet$  is any complex of  $A$ -modules, then there exists an isomorphism of pro-complexes

$$\left\{ [M^\bullet \xrightarrow{f^n} M^\bullet] \right\}_{n \in \mathbb{N}} \simeq \left\{ M^\bullet \otimes_A^L A/f^n A \right\}_{n \in \mathbb{N}}.$$

*Proof.* Recall that there exists a distinguished triangle

$$f^n M[1] \rightarrow [M \xrightarrow{f^n} M] \rightarrow M/f^n M \rightarrow .$$

If the  $f^\infty$ -torsion of  $M$  is bounded by  $N$ , we will have

$$\{f^n M\}_{n \in \mathbb{N}} \simeq \{f^N M\}_{n \in \mathbb{N}} \simeq 0$$

because the transition map is multiplication by  $f$  (and therefore eventually 0 on  $f^N M$ ). We obtain the first isomorphism. If we assume now that  $A$  has bounded  $f^\infty$ -torsion and apply the first result to  $A$ , we obtain an isomorphism

$$\left\{ [M^\bullet \xrightarrow{f^n} M^\bullet] \right\}_{n \in \mathbb{N}} = \left\{ M^\bullet \otimes_A^L [A \xrightarrow{f^n} A] \right\}_{n \in \mathbb{N}} \simeq \{M^\bullet \otimes_A^L A/f^n A\}_{n \in \mathbb{N}}. \quad \square$$

**Remark.** If  $M^\bullet$  is a complex of  $A$ -modules, then its derived completion is

$$\widehat{M^\bullet} = \varprojlim_n [M^\bullet \xrightarrow{f^n} M^\bullet].$$

**Proposition A.15.** *If  $M$  has  $f^\infty$ -torsion bounded by  $N$ , then its derived completion  $\widehat{M}$  is classically complete with  $f^\infty$ -torsion bounded by  $N$ .*

*Proof.* It follows from the first assertion of Lemma A.14 and the remark after it (as well as a Mittag-Leffler argument) that derived and usual completions coincide. Assume now that  $f^m s_n \rightarrow 0$  when  $n \rightarrow \infty$  for some  $m \geq N$  and that the  $f^\infty$ -torsion of  $M$  is bounded by  $N$ . Then, given  $k \geq N$ , if we write  $k' = k + m - N$ , there exists  $n_0 \in \mathbb{N}$  such that for all  $n \geq n_0$ , we can write  $f^m s_n = f^{k'} t_n$  with  $t_n \in M$ . But then  $f^m(s_n - f^{k'-m} t_n) = 0$  and therefore we already have  $f^N(s_n - f^{k'-m} t_n) = 0$  so that  $f^N s_n = f^k t_n$ . This shows that  $f^N s_n \rightarrow 0$ . Thus we see that  $\widehat{M}$  also has  $f^\infty$ -torsion bounded by  $N$ .  $\square$

**Proposition A.16.** *If  $A$  has bounded  $f^\infty$ -torsion and  $M$  is an  $A$ -module, then the following are equivalent*

1.  $M$  is completely flat,
2.  $M$  is formally flat with bounded  $f^\infty$ -torsion.

*Proof.* Since complete flatness implies formal flatness, we already know from Lemma A.13 that condition 1 implies condition 2. Conversely, using both assertions in Lemma A.14, we will have an isomorphism of pro-complexes

$$\{(M \otimes_A^L A/f^n A) \otimes_{A/f^n A}^L A/f A\}_{n \in \mathbb{N}} \simeq \{M/f^n M \otimes_{A/f^n A}^L A/f A\}_{n \in \mathbb{N}}.$$

But

$$(M \otimes_A^L A/f^n A) \otimes_{A/f^n A}^L A/f A \simeq M \otimes_A^L A/f A$$

and, since  $M/f^n M$  is flat over  $A/f^n A$ ,

$$M/f^n M \otimes_{A/f^n A}^L A/f A \simeq M/f^n M \otimes_{A/f^n A} A/f A = M/f M.$$

It follows that  $M \otimes_A^L A/f A \simeq M/f M$  is discrete flat over  $A/f A$ .  $\square$

**Proposition A.17.** *If  $A$  has  $f^\infty$ -torsion bounded by  $N$  and  $M$  is a formally flat  $A$ -module which is separated, then  $M$  also has  $f^\infty$ -torsion bounded by  $N$ .*

*Proof.* Since  $M$  is separated, it is contained in its classical completion. Since formal flatness of  $M$  only depend on its classical completion, we may assume from the beginning that  $M$  is actually classically complete. There exists, for  $k, n \geq N$ , a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
& & & 0 & & 0 & \\
& & & \downarrow & & \downarrow & \\
& & & f^N A & \xlongequal{\quad} & f^N A & \\
& & & \downarrow & & \downarrow & \\
0 \longrightarrow & f^N A & \longrightarrow & A & \xrightarrow{f^n} & A & \longrightarrow A/f^n \longrightarrow 0 \\
& \parallel & & \parallel & & \downarrow f^k & \downarrow f^k \\
0 \longrightarrow & f^N A & \longrightarrow & A & \xrightarrow{f^{n+k}} & A & \longrightarrow A/f^{n+k} \longrightarrow 0 \\
& & & \downarrow & & \downarrow & \\
& & & A/f^k & \xlongequal{\quad} & A/f^k & \\
& & & \downarrow & & \downarrow & \\
& & & 0 & & 0 & 
\end{array}$$

In particular, there exists an exact sequence of  $A/f^{n+k}$ -modules

$$0 \rightarrow f^N A \rightarrow A/f^n \xrightarrow{f^k} A/f^{n+k} \rightarrow A/f^k \rightarrow 0.$$

Since  $M$  is formally flat,  $M/f^{n+k}M$  is flat over  $A/f^{n+k}$ , and therefore, the sequence

$$0 \rightarrow M \otimes_A f^N A \rightarrow M/f^n M \xrightarrow{f^k} M/f^{n+k} M \rightarrow M/f^k M \rightarrow 0$$

is also exact. Taking inverse limit provides us with a left exact sequence

$$0 \rightarrow M \otimes_A f^N A \rightarrow M \xrightarrow{f^k} M,$$

showing that  $f^k M = M \otimes_A f^N A$  does not depend on  $k$ . □

**Proposition A.18.** *Assume  $A$  has bounded  $f^\infty$ -torsion. If  $M^\bullet$  is a complex of  $A$ -modules, then, the following conditions are equivalent*

1.  $M^\bullet$  is derived complete and completely flat,
2.  $M^\bullet$  is classically complete and formally flat.

*Proof.* Assume that  $M^\bullet$  is derived complete and completely flat. Since  $A$  has bounded  $f^\infty$ -torsion, it follows from Lemma A.14 and the fact that  $M^\bullet$  is derived complete, that there exists an isomorphism

$$M^\bullet = \mathbf{R}\varprojlim_n [M^\bullet \xrightarrow{f^n} M^\bullet] \simeq \mathbf{R}\varprojlim_n (M^\bullet \otimes_A^L A/f^n A) \{$$

Since  $M^\bullet$  is completely flat and  $f^n A/f^{n+1} A$  is an  $A/fA$ -module, the complex

$$M^\bullet \otimes_A^L (f^n A/f^{n+1} A)$$

is discrete, and this implies the surjectivity of the map

$$H^0(M^\bullet \otimes_A^L A/f^{n+1} A) \twoheadrightarrow H^0(M^\bullet \otimes_A^L A/f^n A).$$

In particular, this is a Mittag-Leffler system. Using again the fact that  $M^\bullet$  is completely flat, we also know that

$$M^\bullet \otimes_A^L A/f^n A \simeq H^0(M^\bullet \otimes_A^L A/f^n A).$$

It follows that

$$M^\bullet \simeq \varprojlim H^0(M^\bullet \otimes_A^L A/f^n A)$$

is discrete. The complex  $M^\bullet$  being discrete and completely flat, we have

$$H^0(M^\bullet \otimes_A^L A/f^n A) = H^0(M^\bullet)/f^n H^0(M^\bullet).$$

It follows that

$$H^0(M^\bullet) \simeq M^\bullet \simeq \varprojlim H^0(M^\bullet)/f^n H^0(M^\bullet)$$

is classically complete.

Everything else follows from the previous results. More precisely, we mentioned in the last remark after Definition A.5 that a completely flat morphism is automatically formally flat and this finishes the proof of the direct implication. Assume conversely that  $M^\bullet$  is classically complete and formally flat. Then we know from the remark 4 after Definition A.6 that  $M^\bullet$  is derived complete. We also know from Proposition A.17 that  $M^\bullet$  as bounded  $f^\infty$ -torsion. It then follows from Proposition A.16 that  $M^\bullet$  is completely flat.  $\square$

#### A.4 Example: the “bigenic” case

We assume here that  $A$  is endowed with the  $(f, g)$ -adic topology for some fixed  $f, g \in A$ .

**Definition A.19.** 1. An  $A$ -module  $M$  is said to be *bounded* (with respect to  $f$  modulo  $g$ ) if  $M$  is  $g$ -torsion free and  $M/gM$  has bounded  $f^\infty$ -torsion.

2. A complex of  $A$ -modules  $M^\bullet$  is said to be *bounded* if it is discrete and  $H^0(M^\bullet)$  is bounded.

We will also say that  $A$  is *bounded* when it is bounded as an  $A$ -module.

**Remarks.** 1. This notion obviously depends on the ordered pair  $(f, g)$ , but the properties will also hold if we replace any of them by some power.

2. If  $M$  is a  $g$ -torsion free  $A$ -module, then we have

$$\text{Kos}(M, f, g) := \left[ \begin{array}{ccc} \{M & \xrightarrow{g} & M\} \\ \{f \uparrow & & \uparrow f\} \\ \{M & \xrightarrow{g} & M\} \end{array} \right] \simeq [M/gM \xrightarrow{f} M/gM].$$

**Lemma A.20.** 1. If  $M$  is a bounded  $A$ -module, then there exists an isomorphism of pro-complexes

$$\{\mathrm{Kos}(M, f^n, g^m)\}_{n,m \in \mathbb{N}} \simeq \{M/(f^n, g^m)\}_{n,m \in \mathbb{N}}.$$

2. If  $A$  is bounded and  $M$  is any  $A$ -module, then there exists an isomorphism of pro-complexes

$$\{\mathrm{Kos}(M, f^n, g^m)\}_{n,m \in \mathbb{N}} \simeq \{M \otimes_A^L A/(f^n, g^m)\}_{n,m \in \mathbb{N}}.$$

*Proof.* Using the last remark, this follows from Lemma A.14.  $\square$

**Lemma A.21.** If  $M$  is an  $A$ -module such that  $M/gM$  has bounded  $f^\infty$ -torsion, then the  $(f, g)$ -adic topology on  $gM$  is identical to the topology induced by the  $(f, g)$ -adic topology of  $M$ .

*Proof.* It is actually sufficient to show that the  $f$ -adic topology on  $gM$  is identical to the topology induced by the  $f$ -adic topology of  $M$ . In other words, we have to show that, given  $n \in \mathbb{N}$ , we can find  $m \in \mathbb{N}$  such that

$$gM \cap f^m M \subset f^n gM.$$

If  $M/gM$  has  $f^\infty$ -torsion bounded by  $N \in \mathbb{N}$ , then we can choose  $m := n + N$ : if  $s \in M$  satisfies  $f^m s = gt$  for some  $t \in M$ , then there also exists  $r \in M$  such that  $f^N s = gr$ , and therefore  $f^m s = f^n gr$ .  $\square$

**Proposition A.22.** If an  $A$ -module  $M$  is bounded, then its derived completion  $\widehat{M}$  is classically complete and bounded.

*Proof.* It follows from the first assertion of Lemma A.20 that both completions coincide and it remains to show that  $\widehat{M}$  is bounded. It follows from Lemma A.21 that the sequence

$$0 \longrightarrow M \xrightarrow{g} M \longrightarrow M/gM \longrightarrow 0$$

is strict exact, and then the sequence

$$0 \longrightarrow \widehat{M} \xrightarrow{g} \widehat{M} \longrightarrow \widehat{M/gM} \longrightarrow 0$$

is also strict exact. As a consequence, we see that  $\widehat{M}$  is  $g$ -torsion free and  $\widehat{M}/g\widehat{M} = \widehat{M/gM}$  has bounded  $f^\infty$ -torsion thanks to Proposition A.15!  $\square$

**Proposition A.23.** If  $A$  is bounded and  $M$  is an  $A$ -module, then the following are equivalent

1.  $M$  is  $g$ -torsion free and completely flat,
2.  $M$  is bounded and formally flat.

*Proof.* We already know that a completely flat module is always formally flat. Also, by definition, a bounded module is  $g$ -torsion free. Our assertion is therefore obtained by applying Proposition A.16 to  $M/gM = M \otimes_A^L A/g$ .  $\square$

**Proposition A.24.** If  $A$  is bounded and  $M$  is a classically complete formally flat  $A$ -module, then  $M$  also is bounded.

*Proof.* If  $A/g$  has  $f^\infty$ -torsion bounded by  $N$ , then it follows from the snake-lemma applied to the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A/g^n & \xrightarrow{g} & A/g^{n+1} & \longrightarrow & A/g \longrightarrow 0 \\ & & \downarrow f^k & & \downarrow f^k & & \downarrow f^k \\ 0 & \longrightarrow & A/g^n & \xrightarrow{g} & A/g^{n+1} & \longrightarrow & A/g \longrightarrow 0 \end{array}$$

that there exists for  $k \geq N$  and any  $n$ , a long exact sequence of  $A/(f^k, g^{n+1})$ -modules

$$\cdots \rightarrow {}_{f^N}(A/g) \xrightarrow{k} A/(f^k, g^n) \xrightarrow{g} A/(f^k, g^{n+1}) \rightarrow \cdots$$

and a quick look shows that  $k = f^{k-N} \cdot N$ . Since  $M$  is formally flat,  $M/(f^k, g^{n+1})M$  is flat over  $A/(f^k, g^{n+1})$ , and therefore, the sequence

$$\cdots \rightarrow M \otimes_A {}_{f^N}(A/g) \rightarrow M/(f^k, g^n)M \xrightarrow{g} M/(f^k, g^{n+1})M \rightarrow \cdots$$

is also exact. Moreover, the image of the first map is contained in  $f^{k-N}M/(f^k, g^n)M$ . It follows that, whenever  $s \in M$  satisfies  $gs \in (f^k, g^{n+1})M$ , we have  $s \in (f^{k-N}, g^n)M$ . In particular, if  $gs = 0$ , then, necessarily,  $s = 0$  because  $M$  is separated. This shows that  $M$  is  $g$ -torsion free. Moreover, we have

$$gM \cap (f^k, g^{n+1})M \subset (f^{k-N}, g^n)gM.$$

This implies that the topology induced on  $gM$  is identical to the image topology and, since  $M$  is classically complete,  $gM$  is necessarily closed in  $M$ , or equivalently  $M/gM$  is separated. It then follows from Proposition A.17 that  $M$  is bounded.  $\square$

**Theorem A.25.** *Assume  $A$  is bounded and let  $M^\bullet$  be a complex of  $A$ -modules. Then, the following conditions are equivalent*

1.  $M^\bullet$  is derived complete and completely flat,
2.  $M^\bullet$  is classically complete and formally flat.

*Proof.* We already know that the second condition implies the first. Assume conversely that  $M^\bullet$  is derived complete and completely flat. Then,  $M^\bullet$  is automatically formally flat. Moreover, the fact that  $M^\bullet$  is completely flat implies that  $M^\bullet \otimes_A^L A/g^n A$  is completely flat (for the  $f$ -adic topology). On the other hand, since  $A$  is  $g$ -torsion free, we have

$$M^\bullet \otimes_A^L A/g^n A \simeq [M^\bullet \xrightarrow{g^n} M^\bullet].$$

Since  $M^\bullet$  is derived complete, Proposition A.7 implies that  $M^\bullet \otimes_A^L A/g^n A$  also is derived complete (for the  $f$ -adic topology). Therefore, it follows from Proposition A.18 and Proposition A.16 that

$$M^\bullet \otimes_A^L A/g^n A \simeq H^0(M^\bullet \otimes_A^L A/g^n A)$$

is (discrete) classically complete and formally flat with bounded  $f^\infty$ -torsion. We have the following commutative diagram (in which we simply write  $A/g^n$  and  $A/g^{n+1}$  in order to

lighten the notation)

$$\begin{array}{ccc}
H^0(M^\bullet \otimes_A^L A/g^{n+1}) \otimes_{A/g^{n+1}}^L A/g^n & \longrightarrow & H^0(M^\bullet \otimes_A^L A/g^{n+1}) \otimes_{A/g^{n+1}} A/g^n \\
\downarrow \simeq & & \downarrow \simeq \\
(M^\bullet \otimes_A^L A/g^{n+1}) \otimes_{A/g^{n+1}}^L A/g^n & & H^0(M^\bullet \otimes_A^L A/g^{n+1})/g^n H^0(M^\bullet \otimes_A^L A/g^{n+1}) \\
\downarrow \simeq & & \downarrow \\
M^\bullet \otimes_A^L A/g^n & \xrightarrow{\simeq} & H^0(M^\bullet \otimes_A^L A/g^n).
\end{array}$$

In particular, the top left derived tensor product is discrete, which implies that the top morphism also is an isomorphism and it follows that the remaining map is an isomorphism too. In other words, we obtain the surjectivity of the map

$$H^0(M^\bullet \otimes_A^L A/g^{n+1} A) \twoheadrightarrow H^0(M^\bullet \otimes_A^L A/g^n A).$$

In particular, this is a Mittag-Leffler system. Since  $A$  is  $g$ -torsion free, we have

$$M^\bullet \otimes_A^L [A \xrightarrow{g^n} A] \simeq M^\bullet \otimes_A^L A/g^n A \simeq H^0(M^\bullet \otimes_A^L A/g^n A),$$

and since  $M^\bullet$  is derived complete (for the  $g$ -adic topology), we obtain that

$$M^\bullet = R\varprojlim M^\bullet \otimes_A^L [A \xrightarrow{g^n} A] \simeq \varprojlim H^0(M^\bullet \otimes_A^L A/g^n A)$$

is discrete. It follows that

$$[H^0(M^\bullet) \xrightarrow{g} H^0(M^\bullet)] \simeq M^\bullet \otimes_A^L A/gA$$

is discrete. Thus,  $H^0(M^\bullet)$  is  $g$ -torsion free and

$$H^0(M^\bullet)/gH^0(M^\bullet) \simeq M^\bullet \otimes_A^L A/gA$$

has bounded  $f^\infty$ -torsion. Since  $M^\bullet$  is discrete, bounded and derived complete, it is classically complete by Proposition A.22.  $\square$

**Remarks.** 1. Theorem A.25 is the same as Tian's Proposition 1.4 in [Tia21]. We are very thankful to him for clarifying some points in his proof allowing us to improve on our original statement.

2. As a consequence of Theorem A.25, we see that if  $B$  is a bounded  $A$ -algebra and  $M$  is a completely flat  $A$ -module, then the derived completed tensor product  $B\hat{\otimes}_A^L M$  is identical to the classical completed tensor product  $B\hat{\otimes}_A M$ .

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