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Departamento de Matemáticas,
Universidad Autónoma de Madrid and
Instituto de Ciencias Matemáticas,
CSIC-UAM-UC3M-UCM, Madrid, Spain

Correspondence

Andrei Jaikin-Zapirain, Departamento de
Matemáticas, Universidad Autónoma de
Madrid and Instituto de Ciencias
Matemáticas, CSIC-UAM-UC3M-UCM,
28049 Madrid, Spain.
Email: andrei.jaikin@uam.es

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Abstract

For a given group G , Wolfgang Lück asked whether twisting a chain complex of finitely generated free $\mathbb{C}[G]$ -modules with a finite-dimensional complex representation V of G before passing to the L^2 -completion has no other effect on L^2 -Betti numbers than a scaling by the factor $\dim_{\mathbb{C}} V$. The purpose of the article is to answer this question affirmatively provided G is sofic.

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1 | INTRODUCTION

Given a group G we denote by rk_G the von Neumann Sylvester matrix rank function of the group algebra $\mathbb{C}[G]$. Although rk_G is defined in an analytic way (we recall its definition in Subsection 2.2), it may be characterized algebraically for some groups (see, for example, the case of locally indicable groups [5, Corollary 6.2]). Therefore, we think that it is plausible that in general there is also a pure algebraic way to define rk_G . If this is true, then rk_G should be “rigid” under natural algebraic manipulations. For example, in [3], the following conjecture was proposed.

Conjecture 1 (The independence conjecture). *Let G be a group. Let K be a field and let $\phi_1, \phi_2 : K \rightarrow \mathbb{C}$ be two embeddings of K into \mathbb{C} . Then for every matrix $A \in \text{Mat}_{n \times m}(K[G])$*

$$\text{rk}_G(\phi_1(A)) = \text{rk}_G(\phi_2(A)).$$

This conjecture was proved for sofic groups in [3] and for locally indicable groups in [5].

In this paper, we consider a similar problem. Given a representation $\sigma : G \rightarrow \mathrm{GL}_k(\mathbb{C})$, we define $\tilde{\sigma} : \mathbb{C}[G] \rightarrow \mathrm{Mat}_k(\mathbb{C}[G])$ by sending $g \in G$ to $\sigma(g)g$ and then extending by linearity. The following conjecture is a rephrasing of a question raised by Lück in [8, Question 0.1].

Conjecture 2 (The Lück twisted conjecture). *Let G be a group and $\sigma : G \rightarrow \mathrm{GL}_k(\mathbb{C})$ a homomorphism. Then for every matrix $A \in \mathrm{Mat}_{n \times m}(\mathbb{C}[G])$*

$$\mathrm{rk}_G(\tilde{\sigma}(A)) = k \cdot \mathrm{rk}_G(A).$$

Lück noticed that a positive solution of this conjecture for a group G would allow to calculate L^2 -Betti numbers of some fibrations of connected finite CW -complexes $F \rightarrow E \rightarrow B$ where $\pi_1(B) \cong G$ and the map $\pi_1(E) \rightarrow \pi_1(B)$ induced by the fibration is an isomorphism. More details on this application can be found in [8, subsection 5.2] and [6, section 4]. Conjecture 2 was proved for torsion-free elementary amenable groups in [8] and for locally indicable groups in [6]. In our main result, we prove the conjecture for sofic groups.

Theorem 1.1. *The Lück twisted conjecture holds for sofic groups.*

We introduce the notion of sofic group in Subsection 2.3. In particular, amenable and residually finite groups are sofic.

Notice that the sofic case of the independence conjecture follows immediately from the sofic Lück approximation proved in [3] (see Theorem 2.1). However, in the case of the Lück twisted conjecture, this implication is not so direct.

Let us outline the proof of Theorem 1.1. The equality is shown by approximation techniques in two different ways: the multiplication operator is approximated using the “almost” G -sets of a sofic approximation of G and by approximating the complex matrix coefficients by algebraic numbers.

In more detail, we assume first that

$$\sigma : G \rightarrow \mathrm{GL}_k(\overline{\mathbb{Q}}) \text{ and } A \in \mathrm{Mat}_{n \times m}(\overline{\mathbb{Q}}[G]).$$

We may moreover assume that G is finitely generated so that the image of σ is determined by finitely many matrices which together with A have all coefficients in a ring of S -integers $\mathcal{O}_{K,S}$ in some number field K . This ring can be approximated by finite fields $\mathbb{F}_i = \mathcal{O}_{K,S}/P_i$ where P_i runs through different maximal ideals (so that necessarily $|\mathbb{F}_i| \rightarrow \infty$). An arbitrary sofic approximation $\{X_i\}$ of $G = F/N$ is now replaced by a sofic approximation $\{Y_i\}$ such that any $f \in F$ fixing a point of Y_i also lies in the kernel of the composition

$$F \rightarrow G \rightarrow \mathrm{GL}_k(\mathcal{O}_{K,S}) \rightarrow \mathrm{GL}_k(\mathbb{F}_i).$$

Consider a matrix B over $\mathbb{Q}[F]$ that maps to A under the canonical homomorphism $F \rightarrow G$. By abuse of notation we denote by σ also the composition $F \rightarrow G \rightarrow \mathrm{GL}_k(\mathcal{O}_{K,S})$ and by $\tilde{\sigma}$ the map $\mathbb{C}[F] \rightarrow \mathrm{Mat}_k(\mathbb{C}[F])$ defined as before for $\mathbb{C}[G]$. It is an elementary consideration that

$$\mathrm{rk}_{Y_i, P_i}(\tilde{\sigma}(B)) = k \cdot \mathrm{rk}_{Y_i, P_i}(B)$$

holds for the associated finite analogues rk_{Y_i, P_i} . Using previously developed methods from [3] and the sofic Lück approximation over $\overline{\mathbb{Q}}$, we show that the equality passes to rk_G instead of rk_{Y_i, P_i} .

For general complex coefficients, we specialize any possibly occurring transcendental elements to nearby algebraic numbers with the same algebraic dependencies and show that the rk_G is “continuous” with respect to this operation. To do so, we use that $\overline{\mathbb{Q}}$ -points lie dense (in the Euclidean topology) in complex algebraic varieties defined over $\overline{\mathbb{Q}}$, and deduce an approximation statement for rk_G from weak convergence of spectral measures, similarly as one does to obtain Kazhdan’s inequality. The final argument involves the sofic Lück approximation over \mathbb{C} .

2 | PRELIMINARIES

2.1 | General notation

The linear operators on vector spaces will act on the right. If \mathcal{H} is a Hilbert space, $U(\mathcal{H})$ denotes the group of unitary operators on \mathcal{H} and $\mathcal{B}(\mathcal{H})$ is the algebra of bounded operators.

In this paper, the action of a group on a set is on the right side. If F acts on X and $w \in F$, then $\text{Fix}_X(w)$ denotes the set of fixed points of w in X .

We will use the language of Sylvester matrix rank functions. They are maps from the set of matrices over a ring to the set of nonnegative real numbers. For a precise definition, see [4, section 5].

2.2 | Von Neumann Sylvester matrix rank function

In this subsection, we recall the definition of the von Neumann Sylvester matrix rank function rk_G on $\mathbb{C}[G]$.

First, consider G to be countable. Let $\rho_G : G \rightarrow U(l^2(G))$ and $\lambda_G : G \rightarrow U(l^2(G))$ be the right and left regular representation of G , respectively:

$$\left(\sum_{h \in G} a_h h \right) \rho_G(g) = \sum_{h \in G} a_h h g, \quad \left(\sum_{h \in G} a_h h \right) \lambda_G(g) = \sum_{h \in G} a_h g^{-1} h \quad (a_h \in \mathbb{C}, g \in G).$$

By linearity, we can extend ρ_G and λ_G to the homomorphisms

$$\rho_G, \lambda_G : \mathbb{C}[G] \rightarrow \mathcal{B}(l^2(G)).$$

A finitely generated Hilbert G -module is a closed subspace $V \leq l^2(G)^n$, which is invariant under the actions of elements of $\lambda_G(G)$. We denote by $\text{proj}_V : l^2(G)^n \rightarrow l^2(G)^n$ the orthogonal projection onto V and we define

$$\dim_G V := \text{Tr}_G(\text{proj}_V) := \sum_{i=1}^n \langle (\mathbf{1}_i) \text{proj}_V, \mathbf{1}_i \rangle_{(l^2(G))^n},$$

where $\mathbf{1}_i$ is the element of $l^2(G)^n$ having 1 in the i th entry and 0 in the rest of the entries. The number $\dim_G V$ is the *von Neumann dimension* of V .

Let $A \in \text{Mat}_{n \times m}(\mathbb{C}[G])$ be a matrix over $\mathbb{C}[G]$. The action of A by right multiplication on $l^2(G)^n$ induces a bounded linear operator $\rho_G(A) : l^2(G)^n \rightarrow l^2(G)^m$. Now we can define the *von Neumann Sylvester matrix rank function* rk_G :

$$\text{rk}_G(A) := \dim_G \overline{\text{Im } \rho_G(A)} = n - \dim_G \ker \rho_G(A).$$

If $G = F/N$ is a quotient of a group F and $A \in \text{Mat}_{n \times m}(\mathbb{C}[F])$ is a matrix over $\mathbb{C}[F]$, by abuse of notation, we will also write $\text{rk}_G(A)$ instead of $\text{rk}_G(\bar{A})$, where \bar{A} is the image of A in $\text{Mat}_{n \times m}(\mathbb{C}[F/N])$.

If G is not a countable group then rk_G is defined as follows. Take a matrix A over $\mathbb{C}[G]$. Then the group elements that appear in A are contained in a finitely generated group H . We will put $\text{rk}_G(A) = \text{rk}_H(A)$. One easily checks that the value $\text{rk}_H(A)$ does not depend on the subgroup H .

2.3 | Sofic groups and the sofic Lück approximation

Let F be a free finitely generated group and assume that it is freely generated by a set S . An element w of F has *length* n if w can be expressed as a product of n elements from $S \cup S^{-1}$ and n is the smallest number with this property. We denote by $B_n(1)$ the set of elements of length at most n .

Let N be a normal subgroup of F . We put $G = F/N$. We say that G is *sofic* if there is a family $\{X_k : k \in \mathbb{N}\}$ of finite F -sets such that for any $w \in F$,

$$\lim_{k \rightarrow \infty} \frac{|\text{Fix}_{X_k}(w)|}{|X_k|} = 1 \text{ if } w \in N \text{ and } \lim_{k \rightarrow \infty} \frac{|\text{Fix}_{X_k}(w)|}{|X_k|} = 0 \text{ if } w \notin N.$$

The family of F -sets $\{X_k\}$ is called a *sofic approximation* of F/N . For an arbitrary group G we say that G is *sofic* if every finitely generated subgroup of G is sofic. Amenable groups and residually finite groups are sofic. At this moment, no nonsofic group is known.

Let $B \in \text{Mat}_{n \times m}(\mathbb{C}[F])$ be a matrix over $\mathbb{C}[F]$. By multiplication on the right side, B induces a linear operator $\rho_{X_k}(B) : l^2(X_k)^n \rightarrow l^2(X_k)^m$. We put

$$\text{rk}_{X_k}(B) := \frac{\dim_{\mathbb{C}} \text{Im } \rho_{X_k}(B)}{|X_k|}.$$

The sofic Lück approximation is the following result.

Theorem 2.1 [3, Theorem 1.3]. *Let $\{X_k\}$ be a sofic approximation of $G = F/N$. Then for every $B \in \text{Mat}_{n \times m}(\mathbb{C}[F])$,*

$$\lim_{k \rightarrow \infty} \text{rk}_{X_k}(B) = \text{rk}_G(B).$$

This result has its origin in the paper of Lück [7] (see [4] for more details).

3 | THE PROOF OF THE MAIN RESULT

In this section, we prove Theorem 1.1. As we have explained in the introduction, we consider first the case where the matrix A and the homomorphism σ are realized over algebraic numbers. This will be done in the first subsection. In the second subsection, we prove the general case.

3.1 | The algebraic case

In this subsection, we prove the following result.

Theorem 3.1. *Let G be a sofic group and $\sigma : G \rightarrow \mathrm{GL}_k(\overline{\mathbb{Q}})$ a homomorphism. Then for every matrix $A \in \mathrm{Mat}_{n \times m}(\overline{\mathbb{Q}}[G])$*

$$\mathrm{rk}_G(\tilde{\sigma}(A)) = k \cdot \mathrm{rk}_G(A).$$

First we need to prove an auxiliary result. Let Y be a finite right F -set, \mathbb{F} a field and V a right $\mathbb{F}[F]$ -module of dimension k over \mathbb{F} . Denote two structures of $\mathbb{F}[F]$ -module on the F -space $V \otimes_{\mathbb{F}} \mathbb{F}[Y]$:

$$(v \otimes y) \cdot_1 f = v \otimes (y \cdot f) \text{ and } (v \otimes y) \cdot_2 f = (v \cdot f) \otimes (y \cdot f), \quad (v \in V, y \in Y, f \in F).$$

Denote these two modules by $(V \otimes_{\mathbb{F}} \mathbb{F}[Y])_1$ and $(V \otimes_{\mathbb{F}} \mathbb{F}[Y])_2$, respectively. The following lemma resembles [8, Lemma 1.1].

Lemma 3.2. *Assume that for any $y \in Y$, $v \in V$ and $f \in \mathrm{Stab}_F(y)$, $vf = v$. Then*

$$(V \otimes_{\mathbb{F}} \mathbb{F}[Y])_2 \cong (V \otimes_{\mathbb{F}} \mathbb{F}[Y])_1 \cong \mathbb{F}[Y]^k \text{ as } \mathbb{F}[F] \text{ -- modules.}$$

Proof. It is clear that $(V \otimes_{\mathbb{F}} \mathbb{F}[Y])_1 \cong \mathbb{F}[Y]^k$. Without loss of generality, we can assume that F acts transitively on Y . Hence $Y = x \cdot F$ for some $x \in Y$. Define the following \mathbb{F} -map

$$\alpha : (V \otimes_{\mathbb{F}} \mathbb{F}[Y])_1 \rightarrow (V \otimes_{\mathbb{F}} \mathbb{F}[Y])_2, \quad \alpha(v \otimes x \cdot f) = v \cdot f \otimes x \cdot f \quad (v \in V, f \in F).$$

Observe that, as $\mathrm{Stab}_F(x)$ acts trivially on V , α is well-defined. On the other hand, if $g \in F$ we obtain

$$\alpha((v \otimes x \cdot f) \cdot_1 g) = \alpha(v \otimes x \cdot fg) = v \cdot fg \otimes x \cdot fg = (v \cdot f \otimes x \cdot f) \cdot_2 g = \alpha(v \otimes x \cdot f) \cdot_2 g.$$

Thus, α is a $\mathbb{F}[F]$ -homomorphism. As α is clearly bijective, we are done. \square

Proof of Theorem 3.1. As the matrix A involves only a finite number of elements of G , we can assume that G is finitely generated. Therefore, there are a finite extension K of \mathbb{Q} and a finite collection S of valuations of \mathcal{O}_K such that $\sigma(G) \leq \mathrm{GL}_k(\mathcal{O}_{K,S})$ and $A \in \mathrm{Mat}_{n \times m}(\mathcal{O}_{K,S}[G])$.

As G is a finitely generated sofic group, there exist a finitely generated free group F , a normal subgroup N of F and a family of F -sets $\{X_i : i \in \mathbb{N}\}$ such that $G \cong F/N$ and $\{X_i : i \in \mathbb{N}\}$ is a sofic approximation of F/N .

We fix an infinite collection $\{P_i : i \in \mathbb{N}\}$ of maximal ideals of $\mathcal{O}_{K,S}$ and for each $i \in \mathbb{N}$ we put $\mathbb{F}_i = \mathcal{O}_{K,S}/P_i$. Then clearly we have that

$$\lim_{i \rightarrow \infty} |\mathbb{F}_i| \rightarrow \infty. \quad (1)$$

Denote by $\sigma_i : F \rightarrow \text{GL}_k(\mathbb{F}_i)$ the composition of the natural map $F \rightarrow G$, the map $\sigma : G \rightarrow \text{GL}_k(\mathcal{O}_{K,S})$ and the canonical map $\text{GL}_k(\mathcal{O}_{K,S}) \rightarrow \text{GL}_k(\mathbb{F}_i)$. Put $N_i = \ker \sigma_i$. We consider $Y_i = X_i \times F/N_i$ with diagonal action of F .

Claim 3.3. The collection $\{Y_i : i \in \mathbb{N}\}$ is a sofic approximation of F/N .

Proof. Let $w \in F$. Observe that if $w \in N$, then $\text{Fix}_{X_k}(w) \times F/N_i \subseteq \text{Fix}_{Y_k}(w)$ and if $w \notin N$, then $\text{Fix}_{Y_k}(w) \subseteq \text{Fix}_{X_k}(w) \times F/N_i$. Thus, as $\{X_i\}$ is a sofic approximation of F/N , $\{Y_i\}$ is also a sofic approximation of F/N . \square

Let $B \in \text{Mat}_{n \times m}(\mathcal{O}_{K,S}[F])$. For each $i \in \mathbb{N}$, let $\rho_{Y_i, P_i}(B) : \mathbb{F}_i[Y_i]^n \rightarrow \mathbb{F}_i[Y_i]^m$ be the map induced by multiplication by B on the right side:

$$(v_1, \dots, v_n) \rho_{Y_i, P_i}(B) = (v_1, \dots, v_n) B.$$

We define the Sylvester matrix rank function rk_{Y_i, P_i} over $\mathcal{O}_{K,S}[F]$ by means of

$$\text{rk}_{Y_i, P_i}(B) := \frac{\text{rk}_{\mathbb{F}_i}(\rho_{Y_i, P_i}(B))}{|Y_i|}.$$

The following result explains why it is more convenient to work with the approximation $\{Y_i\}$ than $\{X_i\}$.

Claim 3.4. For each $i \in \mathbb{N}$ and any matrix B over $\mathcal{O}_{K,S}[F]$, we have that

$$\text{rk}_{Y_i, P_i}(\tilde{\sigma}(B)) = k \cdot \text{rk}_{Y_i, P_i}(B).$$

Proof. Let $V = \mathbb{F}_i^k$. It becomes an $\mathbb{F}_i[F]$ -module if we define $v \cdot f = v \sigma_i(f)$. Then we obtain that F acts on $(V \otimes_{\mathbb{F}_i} \mathbb{F}_i[Y_i])_1$ and $(V \otimes_{\mathbb{F}_i} \mathbb{F}_i[Y_i])_2$ as follows.

$$(v \otimes y) \cdot_1 f = v \otimes (y \cdot f), \quad (v \otimes y) \cdot_2 f = (v) \sigma_i(f) \otimes (y \cdot f).$$

Therefore, if we identify $\mathbb{F}_i^k \otimes_{\mathbb{F}_i} \mathbb{F}_i[Y_i]$ with $\mathbb{F}_i[Y_i]^k$ we obtain that

$$m \cdot_1 b = (m) \rho_{Y_i, P_i}(b I_k) \text{ and } m \cdot_2 b = (m) \rho_{Y_i, P_i}(\tilde{\sigma}(b)) \quad (m \in \mathbb{F}_i[Y_i]^k, b \in \mathbb{F}_i[F]).$$

Here I_k denotes the identity k by k matrix. Thus, the claim follows from Lemma 3.2. \square

Now we compare $\text{rk}_{Y_i}(B)$ and $\text{rk}_{Y_i, P_i}(B)$.

Claim 3.5. Let $B \in \text{Mat}_{n \times m}(\mathcal{O}_{K,S}[F])$. Then there exists a constant C depending only on B such that

$$|\text{rk}_{Y_i}(B) - \text{rk}_{Y_i, P_i}(B)| \leq \frac{C}{\log_2 |\mathbb{F}_i|}.$$

Proof. We prove the claim using results from [3, section 8].

Let $\alpha \in K$ and let $\alpha_1, \dots, \alpha_s$ be the roots of the minimal polynomial of α over \mathbb{Q} . We put

$$[\alpha] = \max_i |\alpha_i|.$$

For any element $b = \sum_{h \in F} a_h h$ ($a_h \in K$) of the group algebra $K[F]$, we put

$$[b] = \sum_{h \in F} [a_h].$$

We also define

$$[B] = \max_j \sum_i [b_{ij}].$$

For each $i \in \mathbb{N}$, let us define

$$U_i = (\mathcal{O}_K[Y_i]^m + \mathcal{O}_K[Y_i]^n \rho_{Y_i}(B)) / \mathcal{O}_K[Y_i]^n \rho_{Y_i}(B).$$

Observe that

$$U_i \cong (U_i / U_i^{\text{tor}}) \oplus U_i^{\text{tor}},$$

where U_i^{tor} is the torsion part of the \mathcal{O}_K -module U_i . Therefore,

$$\begin{aligned} |\text{rk}_{Y_i}(B) - \text{rk}_{Y_i, P_i}(B)| &= \text{rk}_{Y_i}(B) - \text{rk}_{Y_i, P_i}(B) \\ &= \left(m - \frac{\dim_K(K \otimes_{\mathcal{O}_K} U_i)}{|Y_i|} \right) - \left(m - \frac{\dim_{\mathbb{F}_i}(\mathbb{F}_i \otimes_{\mathcal{O}_K} U_i)}{|Y_i|} \right) \\ &= \frac{\dim_{\mathbb{F}_i}(\mathbb{F}_i \otimes_{\mathcal{O}_K} U_i^{\text{tor}})}{|Y_i|} \leq \frac{\log_{|\mathbb{F}_i|} |U_i^{\text{tor}}|}{|Y_i|} = \frac{\log_2 |U_i^{\text{tor}}|}{|Y_i| \log_2 |\mathbb{F}_i|}. \end{aligned}$$

By [3, Lemmas 8.6 and 8.7],

$$\log_2 |U_i^{\text{tor}}| \leq m |Y_i| |K : \mathbb{Q}| \log_2 [B].$$

Therefore, we can take $C = m |K : \mathbb{Q}| \log_2 [B]$. □

We are ready to finish the proof of the theorem. Let $\epsilon > 0$. Let B be a matrix over $\mathcal{O}_{K,S}[F]$ that maps to A . By (1) and Claim 3.5, there exists $j_1 \in \mathbb{N}$ such that for every $i \geq j_1$,

$$|\mathrm{rk}_{Y_i}(B) - \mathrm{rk}_{Y_i, P_i}(B)| \leq \frac{\epsilon}{4k} \text{ and } |\mathrm{rk}_{Y_i}(\tilde{\sigma}(B)) - \mathrm{rk}_{Y_i, P_i}(\tilde{\sigma}(B))| \leq \frac{\epsilon}{4}. \quad (2)$$

By Claim 3.3 and Theorem 2.1, there exists $j_2 \in \mathbb{N}$ such that for every $i \geq j_2$,

$$|\mathrm{rk}_G(B) - \mathrm{rk}_{Y_i}(B)| \leq \frac{\epsilon}{4k} \text{ and } |\mathrm{rk}_G(\tilde{\sigma}(B)) - \mathrm{rk}_{Y_i}(\tilde{\sigma}(B))| \leq \frac{\epsilon}{4}. \quad (3)$$

Therefore, putting together Claim 3.4, (2) and (3), we obtain that for every $i \geq j_1, j_2$,

$$\begin{aligned} |\mathrm{rk}_G(\tilde{\sigma}(B)) - k \cdot \mathrm{rk}_G(B)| &\leq |\mathrm{rk}_G(\tilde{\sigma}(B)) - \mathrm{rk}_{Y_i}(\tilde{\sigma}(B))| + \\ &|\mathrm{rk}_{Y_i}(\tilde{\sigma}(B)) - \mathrm{rk}_{Y_i, P_i}(\tilde{\sigma}(B))| + |\mathrm{rk}_{Y_i, P_i}(\tilde{\sigma}(B)) - k \cdot \mathrm{rk}_{Y_i, P_i}(B)| + \\ &k \cdot |\mathrm{rk}_{Y_i, P_i}(B) - \mathrm{rk}_{Y_i}(B)| + k \cdot |\mathrm{rk}_{Y_i}(B) - \mathrm{rk}_G(B)| \leq \epsilon. \end{aligned}$$

This finishes the proof. \square

3.2 | Proof of Theorem 1.1

As the matrix A involves only a finite number of elements of G , we can assume that G is finitely generated. Therefore, there are $t_1, \dots, t_l \in \mathbb{C}$ such that if we put $R = \overline{\mathbb{Q}}[t_1, \dots, t_l]$, then $\sigma(G) \leq \mathrm{GL}_k(R)$ and A is a matrix over $R[G]$.

Let I be the kernel of the map $\overline{\mathbb{Q}}[x_1, \dots, x_l] \rightarrow R$ that sends x_i to t_i ($1 \leq i \leq l$). Then $R \cong \overline{\mathbb{Q}}[x_1, \dots, x_l]/I$. If $p = (p_1, \dots, p_l) \in V(I)$ and C is a matrix over $R[G]$, then we denote by $C(p)$ the image of C after sending t_i to p_i ($1 \leq i \leq l$). We put $t = (t_1, \dots, t_l)$. Thus, $A = A(t)$ and $\tilde{\sigma}(A) = \tilde{\sigma}(A)(t)$.

Claim 3.6. Let $s_i \in V(I)$ and assume that $\lim_{i \rightarrow \infty} s_i = t$. Let $C \in \mathrm{Mat}_{n \times m}(R[G])$ be a matrix over $R[G]$. Then

$$\liminf_{i \rightarrow \infty} \mathrm{rk}_G(C(s_i)) \geq \mathrm{rk}_G(C).$$

Proof. The proof of this claim is analogous to the proof of Kazhdan's inequality (see [4, Proposition 10.7] for details).

First we construct a spectral measure μ associated with C . Let $T = \rho(C)\rho(C)^* \in \mathrm{Mat}_n(\mathcal{B}(l^2(G)))$. Then μ is defined as the measure with support in $[0, \|T\|]$, whose moments are calculated as

$$\int x^l d\mu = \mathrm{Tr}_G(T^l) = \sum_{i=1}^n \langle (\mathbf{1}_i)T^l, \mathbf{1}_i \rangle_{(l^2(G))^n}.$$

Similarly, we define a spectral measure μ_i associated with $C(s_i)$.

The condition $\lim_{i \rightarrow \infty} s_i = t$ implies that the measures μ_i converge weakly to μ . Therefore, by the Portmanteau theorem (see, e.g., [2, Theorem 11.1.1]),

$$\limsup_{i \rightarrow \infty} \mu_i(\{0\}) \leq \mu(\{0\}).$$

However, we have that

$$\mu(\{0\}) = \dim_G(\ker(\rho(C)\rho(C)^*)) = \dim_G(\ker \rho(C)) \text{ and}$$

$$\mu_i(\{0\}) = \dim_G(\ker(\rho(C(s_i))\rho(C(s_i))^*)) = \dim_G(\ker \rho(C(s_i))).$$

Therefore,

$$\begin{aligned} \liminf_{i \rightarrow \infty} \text{rk}_G(C(s_i)) &= \liminf_{i \rightarrow \infty} (n - \dim_G(\ker \rho(C(s_i)))) \\ &= \liminf_{i \rightarrow \infty} (n - \mu_i(\{0\})) = n - \limsup_{i \rightarrow \infty} \mu_i(\{0\}) \geq n - \mu(\{0\}) = \text{rk}_G(C). \end{aligned}$$

□

Claim 3.7. Let $s \in V(I)$ and C a matrix over $R[G]$. Then $\text{rk}_G(C(s)) \leq \text{rk}_G(C)$.

Proof. As G is a finitely generated sofic group, there exist a finitely generated free group F , a normal subgroup N of F and a family of F -sets $\{X_i : i \in \mathbb{N}\}$ such that $G \cong F/N$ and $\{X_i : i \in \mathbb{N}\}$ is a sofic approximation of F/N . Let B be a matrix over $R[F]$ that maps to C . Then $B(s)$ maps to $C(s)$.

As for each $i \in \mathbb{N}$, the matrix associated to $\rho_{X_i}(B(s))$ (with respect to some bases) is an image of the matrix associated to $\rho_{X_i}(B)$, we obtain that $\text{rk}_{X_i}(B(s)) \leq \text{rk}_{X_i}(B)$. Now applying Theorem 2.1, we conclude that

$$\text{rk}_G(C(s)) = \text{rk}_G(B(s)) = \lim_{i \rightarrow \infty} \text{rk}_{X_i}(B(s)) \leq \lim_{i \rightarrow \infty} \text{rk}_{X_i}(B) = \text{rk}_G(B) = \text{rk}_G(C).$$

□

Now we are ready to finish the proof of the theorem. By [1, Lemma 3.2], there are $\{s_i \in V(I) \cap \overline{\mathbb{Q}}\}$ such that $\lim_{i \rightarrow \infty} s_i = t$. By Claims 3.6 and 3.7,

$$\lim_{i \rightarrow \infty} \text{rk}_G(A(s_i)) = \text{rk}_G(A) \text{ and } \lim_{i \rightarrow \infty} \text{rk}_G(\tilde{\sigma}(A)(s_i)) = \text{rk}_G(\tilde{\sigma}(A)).$$

Denote by $\sigma_i : G \rightarrow \text{GL}_k(\overline{\mathbb{Q}})$ the composition of $\sigma : G \rightarrow \text{GL}_k(R)$ and the map $R \rightarrow \overline{\mathbb{Q}}$ sending the l -tuple t to the l -tuple s_i . Then $\tilde{\sigma}(A)(s_i) = \tilde{\sigma}_i(A(s_i))$. Applying Theorem 3.1, we obtain that

$$\text{rk}_G(\tilde{\sigma}(A)(s_i)) = \text{rk}_G(\tilde{\sigma}_i(A(s_i))) = k \cdot \text{rk}_G(A(s_i)).$$

Thus, $\text{rk}_G(\tilde{\sigma}(A)) = k \cdot \text{rk}_G(A)$.

Remark 3.8. In [3, Theorem 1.1], it was proved that for a sofic group G , the strong Atiyah conjecture over \mathbb{Q} implies the strong Atiyah conjecture over \mathbb{C} . We would like to notice that the previous argument provides an alternative way to obtain this result from the sofic Lück approximation.

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ORCID

Andrei Jaikin-Zapirain  <https://orcid.org/0000-0003-0958-2695>

REFERENCES

1. E. Breuillard and T. Gelander, *Uniform independence in linear groups*, *Invent. Math.* **173** (2008), 225–263.
2. R. M. Dudley, *Real analysis and probability*, Revised reprint of the 1989 original. Cambridge Studies in Advanced Mathematics, vol. 74, Cambridge University Press, Cambridge, 2002.
3. A. Jaikin-Zapirain, *The base change in the Atiyah and the Lück approximation conjectures*, *Geom. Funct. Anal.* **29** (2019), 464–538.
4. A. Jaikin-Zapirain, *L^2 -Betti numbers and their analogues in positive characteristic*, Groups St Andrews 2017 in Birmingham, London Math. Soc. Lecture Note Ser., vol. 455, Cambridge University Press, Cambridge, 2019, pp. 346–405.
5. A. Jaikin-Zapirain and D. López-Álvarez, *The strong Atiyah and Lück approximation conjectures for one-relator groups*, *Math. Ann.* **376** (2020), 1741–1793.
6. D. Kielak and B. Sun, *Agrarian and L^2 -Betti numbers of locally indicable groups, with a twist*, *Math. Ann.* (2024). <https://doi.org/10.1007/s00208-024-02835-7>
7. W. Lück, *Approximating L^2 -invariants by their finite-dimensional analogues*, *Geom. Funct. Anal.* **4** (1994), 455–481.
8. W. Lück, *Twisting L^2 -invariants with finite-dimensional representations*, *J. Topol. Anal.* **10** (2018), 723–816.