

## Research Article

José Carlos Bellido, Javier Cueto, and Carlos Mora-Corral\*

# Non-local gradients in bounded domains motivated by continuum mechanics: Fundamental theorem of calculus and embeddings

<https://doi.org/10.1515/anona-2022-0316>

received April 12, 2022; accepted April 12, 2023

**Abstract:** In this article, we develop a new set of results based on a non-local gradient jointly inspired by the Riesz  $s$ -fractional gradient and peridynamics, in the sense that its integration domain depends on a ball of radius  $\delta > 0$  (horizon of interaction among particles, in the terminology of peridynamics), while keeping at the same time the singularity of the Riesz potential in its integration kernel. Accordingly, we define a functional space suitable for non-local models in calculus of variations and partial differential equations. Our motivation is to develop the proper functional analysis framework to tackle non-local models in *continuum mechanics*, which requires working with bounded domains, while retaining the good mathematical properties of Riesz  $s$ -fractional gradients. This functional space is defined consistently with Sobolev and Bessel fractional ones: we consider the closure of smooth functions under the natural norm obtained as the sum of the  $L^p$  norms of the function and its non-local gradient. Among the results showed in this investigation, we highlight a non-local version of the fundamental theorem of calculus (namely, a representation formula where a function can be recovered from its non-local gradient), which allows us to prove inequalities in the spirit of Poincaré, Morrey, Trudinger, and Hardy as well as the corresponding compact embeddings. These results are enough to show the existence of minimizers of general energy functionals under the assumption of convexity. Equilibrium conditions in this non-local situation are also established, and those can be viewed as a new class of non-local partial differential equations in bounded domains.

**Keywords:** Riesz fractional gradient, non-local gradient, non-local fundamental theorem of calculus, non-local Poincaré inequality, non-local embeddings, non-local calculus of variations, peridynamics

**MSC 2020:** Primary: 26A33, 35R11, 46E35, 49J45, 74A70, Secondary: 35Q74, 42B20, 49K21, 74B20, 74G65

## 1 Introduction

In the last decades, models based on differential equations are increasingly sharing their prominence with those based on integral or integro-differential equations. This is due to the fact that they can catch some information that the local ones cannot, such as long-range interactions or multi-scale behaviour; they usually require less regularity of the functions, allowing for more general admissible solutions. In fact, they can

---

\* **Corresponding author: Carlos Mora-Corral**, Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain; Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM, 28049 Madrid, Spain, e-mail: Carlos.Mora@uam.es

**José Carlos Bellido:** Department of Mathematics, E.T.S.I. Industriales, Universidad de Castilla-La Mancha, 13071-Ciudad Real, Spain, e-mail: JoseCarlos.Bellido@uclm.es

**Javier Cueto:** Department of Mathematics, University of Nebraska-Lincoln, Lincoln NE 68588-0130, US, e-mail: jcuetogarcia2@unl.edu

overcome some drawbacks of the local models since a typical feature is to be able to provide an effective modelling for discontinuities or singularities.

One such example is peridynamics, a non-local alternative model in solid mechanics proposed by Silling [68]; see also [47,69,70]. One of its goals was to unify elastic and singularity phenomena, such as fracture or cavitation. The development of this theory in the last years has been impressive. As general expositions, we can mention the review paper [43], the two books [36,48], and the collaborative handbooks [12,76]. Several aspects of these models have been studied such as localization [34,52,54], existence and regularity [35], computational issues [18,19], function spaces involved [50,53], or linear theories [23,27,29,51,65,75,79].

Accordingly, non-local models are gaining attention in the modelling of various phenomena in physics, biology, geometry, and more. As a consequence, it is required a more thorough mathematical analysis of the new objects and operators involved. Some of those objects are of diffusion type, where the fractional Laplacian stands out: this is an operator that generalizes the standard Laplacian to a degree of differentiability beyond derivatives of integer order (see, e.g. [1,14,15,22,56,60,62,63] among hundreds of possible references). Others, on the other hand, are of gradient type, which may provide a better scope for non-local vector calculus [7,16,20,21,24,52,54,66,67]. One of the features of both kinds of operators is that they require less regularity of the functions than their classical counterparts. In this article, we focus on the latter type, in particular, those called one-point gradients, or weighted non-local gradients in the terminology of [21]. These non-local gradients are usually written in terms of a kernel  $\rho$ , typically with a singularity at the origin. In general, for a function  $u : \Omega \rightarrow \mathbb{R}$ , they are defined as follows:

$$\mathcal{G}_\rho u(x) = \int_{\Omega} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \rho(x - y) dy. \quad (1)$$

A particular case of non-local gradient where this analysis has experienced a great interest since the works of Shieh and Spector [66,67] is Riesz'  $s$ -fractional gradient. For  $0 < s < 1$  and  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  a smooth enough function, its  $s$ -fractional gradient is defined as follows:

$$D^s u(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} dy, \quad (2)$$

where  $c_{n,s}$  is a suitable normalizing constant. It follows the same formula as in (1), where the integration domain is considered to be  $\mathbb{R}^n$  and

$$\rho(x) = c_{n,s} \frac{1}{|x|^{n-1+s}}. \quad (3)$$

This object has attracted a great interest in the last years, as will be reported below, mainly due to its better properties from a functional analysis perspective. However, the fact of being defined over the whole space is a remarkable drawback for applications in realistic physical models; furthermore, this feature obviously leads to some extra difficulties from a computational or numerical point of view.

Thus, given the interest of working with bounded domains, and inspired by (1) and (2), we propose the operator

$$D_\delta^s u(x) = c_{n,s} \int_{B(x,\delta)} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \frac{w_\delta(x - y)}{|x - y|^{n-1+s}} dy, \quad (4)$$

as a particular case of (1), with

$$\rho(x) = c_{n,s} \frac{1}{|x|^{n-1+s}} w_\delta(x) \quad (5)$$

and  $w_\delta \in C_c^\infty(B(0, \delta))$  a cut-off function, except for the fact that in this case,  $u$  is assumed to be defined, at least, in  $\Omega_\delta = \Omega + B(0, \delta)$  so that the integral in (4) is completely defined in the whole ball. This entails dealing with the “collar”  $\Omega_\delta \setminus \Omega$  as a non-local boundary. The non-local gradient (4) is defined so that it keeps the fractional index  $s$  of differentiability, while at the same time the interaction of particles is restricted to a distance smaller than  $\delta$  (the *horizon* parameter in peridynamics).

At this point, we ought to mention that referring  $\rho$  as the kernel is not a universal convention. Indeed, some authors would state that the kernel in (1) is

$$\tilde{\rho}(x) = -\frac{1}{|x|} \frac{x}{|x|} \rho(x).$$

For example,

$$\tilde{\rho}(x) = -c_{n,s} \frac{x}{|x|} \frac{1}{|x|^{n+s}}$$

in (2), and

$$\tilde{\rho}(x) = -c_{n,s} \frac{x}{|x|} \frac{1}{|x|^{n+s}} w_\delta(x) \quad (6)$$

in (4). With this terminology, a natural classification of kernels arises, namely, those which are integrable (or locally integrable) and those which are not. Sometimes they are called kernels with a strong singularity or with a weak singularity (or no singularity at all), respectively. In our case, the operator  $D_\delta^s$  of (4) with kernel  $\tilde{\rho}$  in (6) fits into the structure described in several works addressing non-local gradients [21,28,51,52,54]. For numerical simulations, strongly singular kernels may be seen as a drawback, but some previous studies have addressed the necessity of gradients with strongly singular kernels; for example, in [30] to avoid some instabilities, or in [27] to achieve coercivity of the Dirichlet energy. This, together with the fact that the fractional singularity provides this operator with a derivative-like structure (with consequences including a Poincaré inequality) has motivated us to pursue this direction.

Another way of avoiding the instabilities while not necessarily considering singular kernels is provided by [46] through the analysis of non-radially symmetric kernels (in that case, defined over half-balls). This approach has been recently continued in [40] studying a Poincaré inequality, and in [25,42] for the traffic flow ahead of a vehicle. In [33], Poincaré inequalities are also studied under some assumptions on non-homogeneous kernels. Some other references for general heterogeneous kernels are [51,78], and in particular those with a variable horizon [26,71,74].

To put it into context, the fractional gradient (2), as recently addressed by several authors [7,8,16,44,61,64], seems to be the suitable notion, from a merely mathematical perspective, for such a differential object. In particular, it has been proved in [77] that formula (2) determines up to a multiplicative constant the unique object fulfilling some minimal consistency requirements from the physical and mathematical point of view, such as invariance under rotations and translations,  $s$ -homogeneity under dilations, and some weak continuity properties. Moreover, the classical gradient can be recovered when  $s$  goes to 1 in (2), [8]. This operator is closely related to the Riesz potential,  $I_{1-s}(x) = \frac{c_{n,s}}{n-1+s} |x|^{-(n-1+s)}$ , and particularly in the case of smooth functions  $u \in C_c^\infty(\mathbb{R}^n)$ , it can be written as a convolution of this kernel with the classical gradient:  $D^s u = I_{1-s} * \nabla u$ . This implies that its Fourier transform can be computed as  $\widehat{D^s u}(\xi) = \frac{2\pi i \xi}{|2\pi \xi|^s} |2\pi \xi|^s \hat{u}(\xi)$ , which gives another insight to the unfamiliar reader.

In the case of the non-local gradient  $D_\delta^s u$ , among all the properties mentioned earlier and systematized in [77] characterising the fractional gradient (2), the one that is not fulfilled by (4) is the  $s$ -homogeneity under dilations, in favour of considering bounded domains (equivalently, a compactly supported kernel). Similar operators have been studied in works like [52,54], where (4) could fit after normalizing its kernel. In particular, it was shown in [54, Th. 1.1] that these operators converge to the classical gradient when the non-locality  $\delta$  goes to zero, and this localization result applies to (4). Furthermore, we will see that a function  $Q_\delta^s$  plays the role of the Riesz potential in this framework, sharing the same singularity at zero than the Riesz potential, but with compact support, making  $Q_\delta^s$  integrable. Thus, the non-local gradient can be written as a convolution with the classical one:  $D_\delta^s u = Q_\delta^s * \nabla u$  for sufficiently smooth functions. As it will be seen later, this shows that the non-local gradient (defined or extended to every point in  $\mathbb{R}^n$  if necessary) of test functions such as  $C_c^\infty(\mathbb{R}^n)$  or  $\mathcal{S}$  (the Schwartz space) remains in such spaces, as opposed to the fractional gradient, making it more suitable for defining the non-local gradient of a distribution.

Regarding functional spaces, essential for calculus of variations, the one associated with Riesz fractional gradients is the Bessel space  $H^{s,p}(\mathbb{R}^n)$ . Among the several equivalent definitions, the most intuitive in this context is that based on the completion of  $C_c^\infty(\mathbb{R}^n)$  functions under the norm

$$\|u\|_{H^{s,p}(\mathbb{R}^n)} = (\|u\|_{L^p(\mathbb{R}^n)}^p + \|D^s u\|_{L^p(\mathbb{R}^n)}^p)^{\frac{1}{p}}.$$

There are, of course, many spaces between  $L^p(\mathbb{R}^n)$  and the Sobolev space  $W^{1,p}(\mathbb{R}^n)$  that possess a degree of differentiability of order  $s$ . The most familiar one is possibly the Gagliardo space  $W^{s,p}(\mathbb{R}^n)$ , which is equipped with the seminorm

$$[u]_{W^{s,p}(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy \right)^{\frac{1}{p}}$$

and the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^n)} = (\|u\|_{L^p(\mathbb{R}^n)}^p + [u]_{W^{s,p}(\mathbb{R}^n)}^p)^{\frac{1}{p}}.$$

A great difference between  $H^{s,p}(\mathbb{R}^n)$  and  $W^{s,p}(\mathbb{R}^n)$  is that in the latter there is no suitable concept of fractional gradient, even though it possibly is the natural space to define the fractional Laplacian. Moreover, despite the analogy of the seminorms in those spaces,

$$\|D^s u\|_{L^p(\mathbb{R}^n)} = c_{n,s} \left( \int_{\mathbb{R}^n} \left| \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}} \frac{x - y}{|x - y|} dy \right|^p dx \right)^{\frac{1}{p}} \quad \text{and} \quad [u]_{W^{s,p}(\mathbb{R}^n)},$$

the fact that in  $\|D^s u\|_{L^p(\mathbb{R}^n)}$  the absolute value affects the inner integral, while in  $[u]_{W^{s,p}(\mathbb{R}^n)}$  the absolute value affects the integrand, reveals that the inclusions between these spaces are not obvious. We mention, in passing, that in [2,66], it is shown the embeddings  $H^{s_2,p}(\mathbb{R}^n) \subset W^{s,p}(\mathbb{R}^n) \subset H^{s_1,p}(\mathbb{R}^n)$  for  $0 < s_1 < s < s_2 < 1$ , as well as the equality  $H^{s,2}(\mathbb{R}^n) = W^{s,2}(\mathbb{R}^n)$ . This feature of the absolute value affecting the inner integral in  $\|D^s u\|_{L^p(\mathbb{R}^n)}$  has several consequences in the proofs of properties of  $H^{s,p}(\mathbb{R}^n)$ , since, in general, it hampers a direct application of the elementary inequality that the absolute value of the integral is less than the integral of the absolute value, since that inequality cannot be reversed. For example, one cannot apply directly the techniques of [57,58], which are suitable for seminorms in the style of  $W^{s,p}$ , but with general kernels. Although the definitions of the Riesz gradient and the Bessel spaces are rather old, it was the study [66] that initiated the attention in the community of non-local problems in partial differential equations and calculus of variations. In fact, in [66,67], it was shown the relationship between Riesz gradients and Bessel spaces, as well as a series of inequalities and embeddings mimicking those of Sobolev spaces, which constitute the basis for an analysis of the equations and minimization problems naturally related to the fractional gradient.

While [67] treated the existence of minimizers for convex scalar problems using the direct method of the calculus of variations, the following two papers make extensions and applications for vectorial problems: in [7], we showed that the concept of polyconvexity is also suitable in these problems, while in [44], it was shown the analogue for the concept of quasiconvexity. These notions, polyconvexity and quasiconvexity, are classical in the calculus of variations (see, e.g. [17]). In these three works, the functional to minimize is of the form

$$\int_{\mathbb{R}^n} W(x, u(x), D^s u(x)) dx, \quad (7)$$

with the integrand  $W$  satisfying similar assumptions as in local problems.

This article can be seen as a follow-up to other works in the search of a non-local model suitable for hyperelasticity but also introducing a theory applicable to other phenomena. Following previous works [9–11] by some of the authors of this article, we showed in [6] that (bond-based) peridynamics models based on energy functionals of the form

$$\int \int_{\Omega \cap B(x,\delta)} w(x - y, u(x) - u(y)) dy dx,$$

although defined for bounded domains, do not fit in non-linear solid mechanics, since very few local non-linear models are limit of non-local ones when  $\delta \rightarrow 0$ . As mentioned earlier,  $\delta$  is the *horizon*: the interaction distance between the particles. On the other hand, going back to the analysis in  $H^{s,p}(\mathbb{R}^n)$ , we showed in [8] that the limit when  $s \rightarrow 1$  of integral (7) based on the Riesz gradient is the local model

$$\int_{\mathbb{R}^n} W(x, u(x), \nabla u(x)) dx.$$

In this article, we propose a model that combines the good properties of the Riesz gradient and the space  $H^{s,p}$  with the requirement that the energy is defined in a bounded domain of  $\mathbb{R}^n$ , since it is in this case where its interpretation of an elastic energy is physically meaningful, and so we think it could fit in state-based peridynamics.

To reproduce similar existence results for energy functionals like

$$\int_{\Omega} W(x, u(x), D_{\delta}^s u(x)) dx,$$

it is necessary to study the functional space associated with the non-local gradient (4). To be precise, we define the space  $H^{s,p,\delta}(\Omega)$  in concordance with Bessel and Sobolev spaces, as the completion of  $C_c^{\infty}(\mathbb{R}^n)$  under the norm

$$\|u\|_{H^{s,p,\delta}(\Omega)} = (\|u\|_{L^p(\Omega_{\delta})}^p + \|D_{\delta}^s u\|_{L^p(\Omega)}^p)^{\frac{1}{p}},$$

where  $\Omega_{\delta}$  is the union of  $\Omega$  with a tubular neighbourhood of the boundary of radius  $\delta$ . A subspace  $H_0^{s,p,\delta}(\Omega)$  representing roughly  $H^{s,p,\delta}(\Omega)$  functions with zero “boundary” conditions (in truth, with zero values in another tubular neighbourhood of the boundary) is also studied.

We highlight as one of the main contributions of this work a non-local version of the fundamental theorem of calculus, which is obtained despite the lack of homogeneity and semigroup properties of the kernels involved. As in other frameworks, given that it allows us to recover a function from its non-local gradient, we see it as a versatile tool that may prove itself useful in other situations. In particular, it helps in overcoming the aforementioned problem of an absolute value affecting the inner integral in the seminorms we are considering, and therefore, it is a key ingredient in the process of obtaining several inequalities and embeddings.

Thus, this article can be regarded as a first step to explore properties in  $H^{s,p,\delta}$  that are known in  $W^{1,p}$  and  $H^{s,p}$ , such as integration by parts, fundamental theorem of calculus, Poincaré inequalities, and compact embeddings. In fact, it is illustrative to compare those definitions and properties in the three contexts: classical, fractional, and non-local. In what follows, *classical* will typically refer to properties for Sobolev  $W^{1,p}$  or even smooth functions involving the (classical or distributional) gradient  $\nabla$ , *fractional* to properties in  $H^{s,p}$  involving Riesz’  $s$ -fractional gradient  $D^s$ , and *non-local* to properties in  $H^{s,p,\delta}$  involving the non-local gradient  $D_{\delta}^s$ . A partial list of this comparison is as follows:

- Gradient: The classical gradient is just the pointwise or distributional gradient. The fractional gradient is (2), while the non-local gradient is (4).
- Divergence: The classical divergence is the pointwise divergence. The fractional divergence is

$$\operatorname{div}^s \phi(x) = c_{n,s} \int_{\mathbb{R}^n} \frac{\phi(x) - \phi(y)}{|x - y|^{n+s}} \cdot \frac{x - y}{|x - y|} dy,$$

while the non-local divergence is

$$\operatorname{div}_{\delta}^s \phi(x) = c_{n,s} \int_{B(x,\delta)} \frac{\phi(x) - \phi(y)}{|x - y|} \cdot \frac{x - y}{|x - y|} \frac{w_{\delta}(x - y)}{|x - y|^{n-1+s}} dy.$$

- Integration by parts: For  $u \in C_c^1(\Omega)$  and  $\phi \in C_c^1(\Omega, \mathbb{R}^n)$ ,

$$\begin{aligned} \text{Classical:} \quad & \int_{\Omega} \nabla u \cdot \phi = - \int_{\Omega} u \operatorname{div} \phi. \\ \text{Fractional:} \quad & \int_{\mathbb{R}^n} D^s u \cdot \phi = - \int_{\mathbb{R}^n} u \operatorname{div}^s \phi. \\ \text{Non-local:} \quad & \int_{\Omega} D_{\delta}^s u \cdot \phi = - \int_{\Omega} u \operatorname{div}_{\delta}^s \phi. \end{aligned}$$

- Fundamental theorem of calculus:

$$\text{Classical: } u(x) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \nabla u(y) \cdot \frac{x-y}{|x-y|^n} dy.$$

$$\text{Fractional: } u(x) = c_{n,s} \int_{\mathbb{R}^n} D^s u(y) \cdot \frac{x-y}{|x-y|^{n-s+1}} dy.$$

$$\text{Non-local: } u(x) = \int_{\mathbb{R}^n} D_\delta^s u(y) \cdot V_\delta^s(x-y) dy.$$

At this point, we mention the attempt to unify fractional and non-local theories recently explored in [21], in the context of a general vector calculus (following the earlier works [24,39]) and in [20], which focuses on non-local gradients. We also point out the work [31], where a different approach to the fractional fundamental theorem of calculus in dimension one is addressed, as well as a study of the function spaces involved.

The role of the Fourier transform in this analysis deserves a special mention. It was pointed out in [66] that the fractional gradient behaves nicely under Fourier transform:  $\widehat{D^s u}(\xi) = 2\pi i \xi |\xi|^{s-1} \hat{u}(\xi)$ . This fact was used in [8] to obtain some properties that would otherwise require a much longer argument. In this article, we also use Fourier transform, which is of no surprise having in mind the aforementioned fundamental theorems of calculus expressing  $u$  as a convolution, and, in fact, the constant presence of convolutions in this work. Again in [66], the properties of the Riesz potential and its Fourier transform were used in connection with the Riesz gradient. In this article, we also use a potential playing the role of Riesz'. In our case, this potential will no longer have the semigroup property, but yet we will succeed in capturing its main features to prove the non-local fundamental theorem of calculus.

The similarities with the fractional case show that a parallel theory could potentially be developed. Actually, it turns out that when restricted to  $\Omega_\delta$ , functions from Bessel spaces belong to  $H^{s,p,\delta}(\Omega)$  (Proposition 3.5). This leads to wonder if functions in Bessel spaces  $H^{s,p}(\mathbb{R}^n)$  are always an extension to  $\mathbb{R}^n$  of functions in  $H^{s,p,\delta}(\Omega)$ . This inclusion also implies that functions exhibiting fracture or cavitation phenomena are admitted in  $H^{s,p,\delta}(\Omega)$  [7, Sect. 2.1].

The outline of this article is the following. Section 2 fixes some notation used throughout the article. In Section 3, the new versions of non-local gradient, divergence, and integration by parts are established. We also define the associated function space  $H^{s,p,\delta}(\Omega)$  and state its basic properties. Section 4 proves the non-local version of the fundamental theorem of calculus. Its proof, nevertheless, depends on the existence of the kernel  $V_\delta^s$ , which is addressed in Section 5. Then, in Section 6, we first define the space  $H_0^{s,p,\delta}(\Omega)$  and then use the non-local fundamental theorem of calculus to prove versions in this context of the inequalities by Poincaré, Morrey, Trudinger, and Hardy. In Section 7, we establish the compact embeddings from  $H_0^{s,p,\delta}(\Omega)$  to  $L^q(\Omega)$ . Section 8 shows the existence of minimizers of scalar convex variational problems involving  $D_\delta^s$ , as well as the corresponding Euler-Lagrange equation. The article finishes with two appendices: in Appendix A, we point out the necessary changes needed in Section 5 for the case  $n = 1$ , while in Appendix B, we state some Fourier analysis facts used throughout the article for which we have not found a reference.

## 2 Notation

### 2.1 General notation

In all this work, we fix the dimension  $n \in \mathbb{N}$  of the space ( $n \geq 1$ ), an open bounded set  $\Omega$  of  $\mathbb{R}^n$  representing the body, a number  $0 < s < 1$  quantifying the degree of differentiability, a  $\delta > 0$  indicating the horizon (the interaction distance between the particles of the body), and an exponent  $1 \leq p < \infty$  of integrability. Sometimes we will additionally require  $p > 1$ . The Hölder conjugate exponent of  $p$  is  $p' = \frac{p}{p-1}$ .



The notation for Sobolev  $W^{1,p}$  and Lebesgue  $L^p$  spaces is standard. So is the notation for functions that are continuous  $C$ , and of class  $C^k$  for  $k$  an integer or infinity. Their version of compact support are  $C_c^k$ . The set of continuous functions vanishing at infinity is  $C_0$ . We will indicate the domain of the functions, as in  $C^1(\Omega)$ ; the target is indicated only if it is not  $\mathbb{R}$ . When using the norm in those spaces, the target is omitted, as in  $\|\cdot\|_{L^p(\Omega)}$ .

We write  $B(x, r)$  for the open ball centred at  $x \in \mathbb{R}^n$  of radius  $r > 0$ . The complement of a subset  $A \subset \mathbb{R}^n$  is denoted by  $A^c$ , its closure by  $\bar{A}$ , and its boundary by  $\partial A$ .

We denote by  $\sigma_{n-1}$  the area of the unit sphere, while the surface area in integrals is indicated by  $\mathcal{H}^{n-1}$ .

We will use the multi-index notation: for  $\alpha \in \mathbb{N}^n$ , we give the standard meaning to the partial derivative  $\partial^\alpha$ , the size  $|\alpha|$ , the monomial  $x^\alpha$  for  $x \in \mathbb{R}^n$ , the ordering  $\beta \leq \alpha$ , and the combinatorial number  $\binom{\alpha}{\beta}$ ; see, e.g. [38, Sect. 2.2].

The vectors of the canonical basis of  $\mathbb{R}^n$  are  $e_j$ ,  $j = 1, \dots, n$ .

The operation of convolution is denoted by  $*$ . We indicate the duality product between tempered distributions and Schwartz functions as  $\langle \cdot, \cdot \rangle$ .

## 2.2 Fourier transform

The convention for the Fourier transform of a function  $f$  is

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx$$

for  $f \in L^1(\mathbb{R}^n)$ . This definition is extended by continuity and duality to other function and distribution spaces, notably, as isomorphisms in the Schwartz space  $\mathcal{S}$  and in the space of tempered distributions  $\mathcal{S}'$ . Sometimes we will also use the alternative notation  $\mathcal{F}(f)$  for  $\hat{f}$ . The variable in the Fourier space is generically designed by  $\xi$ . The reflection of the function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is  $\tilde{f}(x) = f(-x)$ , and one has  $\tilde{\tilde{f}} = f$ , in principle, for functions  $f \in L^1(\mathbb{R}^n)$  for which  $\hat{f} \in L^1(\mathbb{R}^n)$ , but then by continuity and duality this property is extended to a larger class of functions and distributions. Classical texts in Fourier analysis are presented in [28,38].

## 2.3 Radial functions

We recall the following definitions regarding radial functions.

**Definition 2.1.** We will say that

- (a) a function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is *radial* if there exists  $\bar{f}: [0, \infty) \rightarrow \mathbb{R}$  such that  $f(x) = \bar{f}(|x|)$  for every  $x \in \mathbb{R}^n$ . In such a case,  $\bar{f}$  is the radial representation of  $f$ .
- (b) a radial function  $f: \mathbb{R}^n \rightarrow \mathbb{R}$  is *radially decreasing* if its radial representation  $\bar{f}: [0, \infty) \rightarrow \mathbb{R}$  is a decreasing function.
- (c) a function  $\phi: \mathbb{R}^n \rightarrow \mathbb{R}^n$  is *vector radial* if there exists a radial function  $\bar{\phi}: [0, \infty) \rightarrow \mathbb{R}$  such that  $\phi(x) = \bar{\phi}(|x|)x$  for every  $x \in \mathbb{R}^n$ .

It is known (see, e.g. [38, App. B.5]) that the Fourier transform of a radial (respectively, vector radial) function is radial (respectively, vector radial).

## 3 Function space: non-local gradient, divergence, and integration by parts

In this section, we define the non-local gradient and divergence, and state their basic properties, notably, the integration by parts. We also set the natural functional space associated with the non-local gradient. The

framework is the following. As typical in non-local models [3–5,24,39,41], “boundary” conditions are usually of volumetric type. In our case, we fix a distance  $\delta > 0$  and consider a bounded, open domain  $\Omega \subset \mathbb{R}^n$ . The set  $\Omega$  itself is regarded as a non-local interior domain, while  $\Omega_\delta := \Omega + B(0, \delta)$  is considered as its non-local closure. Accordingly, the set  $\Omega_{B,\delta} := \Omega_\delta \setminus \Omega$  plays the role of non-local boundary; see Figure 1. The set  $\Omega_{-\delta} = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$  will also be relevant along this work. Thus, we consider  $\delta$  small enough so that  $\Omega_{-\delta}$  is not empty.

Let  $w_\delta : \mathbb{R}^n \rightarrow [0, +\infty)$  be a cut-off function, and  $\rho_\delta : \mathbb{R}^n \rightarrow [0, +\infty)$  is defined as follows:

$$\rho_\delta(x) = \frac{1}{\gamma(1-s)|x|^{n-1+s}} w_\delta(x),$$

where the constant  $\gamma(s)$  is given by

$$\gamma(s) = \frac{\pi^{\frac{n}{2}} 2^s \Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{n-s}{2}\right)} \quad (8)$$

and  $\Gamma$  is Euler’s gamma function. We assume the following conditions over  $w_\delta$ :

- (a)  $w_\delta$  is radial and nonnegative;  $\bar{w}_\delta$  is its radial representation.
- (b)  $w_\delta \in C_c^\infty(B(0, \delta))$ .
- (c) There are constants  $a_0 > 0$  and  $0 < b_0 < 1$  such that  $0 \leq w_\delta \leq a_0$ , with  $w_\delta = a_0$  in  $B(0, b_0\delta)$ .
- (d)  $\bar{w}_\delta$  is decreasing.

In fact, it will be apparent in the proof of Lemma 5.3 that condition (d) can be considerably weakened. Note, in addition, that  $\rho_\delta \in L^1(\mathbb{R}^n)$ .

Given a function  $f : \Omega \rightarrow \mathbb{R}$  and  $x \in \Omega$  such that  $f \in L^1(\Omega \setminus B(x, r))$  for every  $r > 0$ , the principal value of  $\int_\Omega f$ , denoted by

$$\text{pv} \int_\Omega f$$

is defined as follows:

$$\lim_{r \rightarrow 0} \int_{\Omega \setminus B(x, r)} f,$$

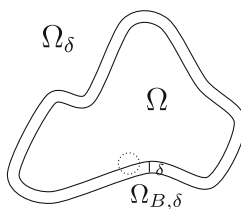
whenever this limit exists.

The definitions of the non-local gradient and divergence for smooth functions are the following.

**Definition 3.1.** Set

$$c_{n,s} := \frac{n-1+s}{\gamma(1-s)}.$$

- (a) Let  $u \in C_c^\infty(\mathbb{R}^n)$ . The non-local gradient  $D_\delta^s u$  is defined as follows:



**Figure 1:** The sets  $\Omega$ ,  $\Omega_\delta$ , and  $\Omega_{B,\delta}$ , together with the distance  $\delta$ .



$$D_\delta^s u(x) = c_{n,s} \int_{B(x,\delta)} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \frac{w_\delta(x - y)}{|x - y|^{n-1+s}} dy, \quad x \in \mathbb{R}^n. \quad (9)$$

(b) Let  $\phi \in C_c^1(\mathbb{R}^n, \mathbb{R}^n)$ . The non-local divergence is defined as follows:

$$\operatorname{div}_\delta^s \phi(x) = -\operatorname{pv} c_{n,s} \int_{B(x,\delta)} \frac{\phi(x) + \phi(y)}{|x - y|} \cdot \frac{x - y}{|x - y|} \frac{w_\delta(x - y)}{|x - y|^{n-1+s}} dy, \quad x \in \mathbb{R}^n.$$

Notice that the integral in (9) is absolutely convergent because  $u$  is Lipschitz and  $\rho_\delta \in L^1(\mathbb{R}^n)$ . It is also immediate from the definition that  $\operatorname{supp} D_\delta^s u \subset \operatorname{supp} u + B(0, \delta)$ . On the other hand, by odd symmetry,

$$-\operatorname{pv} \int_{B(x,\delta)} \frac{\phi(x) + \phi(y)}{|x - y|} \cdot \frac{x - y}{|x - y|} \frac{w_\delta(x - y)}{|x - y|^{n-1+s}} dy = \int_{B(x,\delta)} \frac{\phi(x) - \phi(y)}{|x - y|} \cdot \frac{x - y}{|x - y|} \frac{w_\delta(x - y)}{|x - y|^{n-1+s}} dy, \quad (10)$$

and this last integral is absolutely convergent. Similarly,  $D_\delta^s u$  can be written as follows:

$$\begin{aligned} D_\delta^s u(x) &= -\operatorname{pv} c_{n,s} \int_{B(x,\delta)} \frac{u(y)}{|x - y|} \frac{x - y}{|x - y|} \frac{w_\delta(x - y)}{|x - y|^{n-1+s}} dy = \operatorname{pv} c_{n,s} \int_{\mathbb{R}^n} \frac{u(x + h)h}{|h|^{n+1+s}} w_\delta(h) dh \\ &= c_{n,s} \int_{\mathbb{R}^n} \frac{[u(x + h) - u(x)]h}{|h|^{n+1+s}} w_\delta(h) dh. \end{aligned} \quad (11)$$

Variants of these expressions have been used to study properties of non-local operators of this style [16,21,66,77]; notably, the Riesz fractional gradient  $D^s u$ , which corresponds to the choice  $w_\delta = 1$ . Comparing (2) with (9) and (11), we can see that the operator  $D_\delta^s$  can be regarded as a truncated version near zero of the fractional operator  $D^s$ .

Note also that, for each  $x \in \Omega$ ,

$$\int_{B(x,\delta)} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \frac{w_\delta(x - y)}{|x - y|^{n-1+s}} dy = \int_{\Omega_\delta} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \frac{w_\delta(x - y)}{|x - y|^{n-1+s}} dy,$$

and similarly for the integral in (10), since  $B(x, \delta) \subset \Omega_\delta$  and  $\operatorname{supp} w_\delta \subset B(0, \delta)$ .

The operators of Definition 3.1 are dual operators in the sense of integration by parts. Actually, several versions of integration by parts formulas for related fractional or non-local operators have already appeared in the literature [16,24,54,77]. For the purposes of this work, we will use a particular case of [54, Th. 1.4], which, for convenience, we restate here in our context.

**Proposition 3.1.** Assume  $u \in C_c^\infty(\mathbb{R}^n)$  and  $\phi \in C_c^1(\Omega, \mathbb{R}^n)$ . Then

$$\iint_{\Omega \times \Omega} \frac{u(x) - u(y)}{|x - y|} \frac{x - y}{|x - y|} \cdot \phi(x) \rho_\delta(x - y) dy dx = \int_{\Omega} u(x) \operatorname{pv} \int_{\Omega} \frac{\phi(x) + \phi(y)}{|x - y|} \cdot \frac{x - y}{|x - y|} \rho_\delta(x - y) dy dx.$$

The integration by parts formula of interest in this investigation is the following. Notice the presence of a boundary term, which is due to the fact that  $u$  is not assumed to have compact support in  $\Omega$ . Note that the minus sign in the boundary term makes sense since the vector  $x - y$  points inwards.

**Theorem 3.2.** Suppose that  $u \in C_c^\infty(\mathbb{R}^n)$  and  $\phi \in C_c^1(\Omega, \mathbb{R}^n)$ . Then  $D_\delta^s u \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$  and  $\operatorname{div}_\delta^s \phi \in L^\infty(\mathbb{R}^n)$ . Moreover,

$$\int_{\Omega} D_\delta^s u(x) \cdot \phi(x) dx = - \int_{\Omega} u(x) \operatorname{div}_\delta^s \phi(x) dx - (n - 1 + s) \iint_{\Omega \times \Omega_{B,\delta}} \frac{u(y) \phi(x)}{|x - y|} \cdot \frac{x - y}{|x - y|} \rho_\delta(x - y) dy dx,$$

and these three integrals are absolutely convergent.

**Proof.** Denoting by  $L > 0$  the Lipschitz constant of  $u$ , we have, for each  $x \in \mathbb{R}^n$ ,

$$|D_\delta^s u(x)| \leq c_{n,s} L \int_{B(x,\delta)} \frac{w_\delta(x-y)}{|x-y|^{n-1+s}} dy = (n-1+s)L \|\rho_\delta\|_{L^1(\mathbb{R}^n)},$$

so  $D_\delta^s u \in L^\infty(\mathbb{R}^n, \mathbb{R}^n)$ . Analogously, the integral of the right-hand side of (10) is absolutely convergent and  $\operatorname{div}_\delta^s \phi \in L^\infty(\mathbb{R}^n)$ .

We have

$$\int_{\Omega} D_\delta^s u(x) \cdot \phi(x) dx = (n-1+s) \iint_{\Omega \Omega_\delta} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \cdot \phi(x) \rho_\delta(x-y) dy dx$$

with

$$\begin{aligned} & \iint_{\Omega \Omega_\delta} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho_\delta(x-y) \cdot \phi(x) dy dx \\ &= \iint_{\Omega \Omega} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \cdot \phi(x) \rho_\delta(x-y) dy dx + \iint_{\Omega \Omega_{B,\delta}} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \cdot \phi(x) \rho_\delta(x-y) dy dx. \end{aligned}$$

By Proposition 3.1,

$$\iint_{\Omega \Omega} \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \cdot \phi(x) \rho_\delta(x-y) dy dx = \int_{\Omega} u(x) \operatorname{pv} \int_{\Omega} \frac{\phi(x)+\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy dx.$$

On the other hand,

$$-\int_{\Omega} u(x) \operatorname{div}_\delta^s \phi(x) dx = (n-1+s) \int_{\Omega} u(x) \operatorname{pv} \int_{B(x,\delta)} \frac{\phi(x)+\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy dx.$$

Now, for each  $x \in \Omega$ ,

$$\begin{aligned} & \operatorname{pv} \int_{B(x,\delta)} \frac{\phi(x)+\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy \\ &= \operatorname{pv} \int_{\Omega} \frac{\phi(x)+\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy + \operatorname{pv} \int_{\Omega_{B,\delta}} \frac{\phi(x)+\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy \end{aligned}$$

and, since  $\phi$  vanishes in  $\Omega_{B,\delta}$ ,

$$\operatorname{pv} \int_{\Omega_{B,\delta}} \frac{\phi(x)+\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy = \int_{\Omega_{B,\delta}} \frac{\phi(x)-\phi(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy,$$

and this last integral is absolutely convergent, as explained in (10).

Putting together the aforementioned formulas, we have obtained that

$$\begin{aligned} & \int_{\Omega} D_\delta^s u(x) \cdot \phi(x) dx + \int_{\Omega} u(x) \operatorname{div}_\delta^s \phi(x) dx \\ &= (n-1+s) \iint_{\Omega \Omega_{B,\delta}} \left[ \frac{u(x)-u(y)}{|x-y|} \phi(x) - u(x) \frac{\phi(x)-\phi(y)}{|x-y|} \right] \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy dx, \end{aligned}$$

being the three integrals absolutely convergent. Finally, for each  $x \in \Omega$  and  $y \in \Omega_{B,\delta}$ ,

$$\frac{u(x)-u(y)}{|x-y|} \phi(x) - u(x) \frac{\phi(x)-\phi(y)}{|x-y|} = \frac{-u(y)}{|x-y|} \phi(x) + u(x) \frac{\phi(y)}{|x-y|} = \frac{-u(y)}{|x-y|} \phi(x) \quad (12)$$

since  $\phi \in C_c^1(\Omega, \mathbb{R}^n)$ . This concludes the proof.  $\square$

Although we have proved  $L^\infty$  regularity for  $D_\delta^s u$ , much more is true, since Proposition 4.3 will show that  $D_\delta^s u \in C_c^\infty(\mathbb{R}^n)$ .

We now extend Definition 3.1(a) to a broader class of functions, namely,  $L_{loc}^1(\Omega_\delta)$  functions admitting  $L_{loc}^1(\Omega, \mathbb{R}^n)$  gradients. As usual, when  $U \subset \mathbb{R}^n$  is open,  $L_{loc}^1(U)$  stands for the space of functions defined in  $U \subset \mathbb{R}^n$  that are in  $L^1(K)$  for each compact  $K \subset U$ . Convergence in  $L_{loc}^1(U)$  means convergence in  $L^1(K)$  for each compact  $K \subset U$ . The fact that the function is defined in  $\Omega_\delta$ , while the gradient in  $\Omega$  can be visualised in Figure 1.

**Definition 3.2.**

- (a) Let  $u \in L_{loc}^1(\Omega_\delta)$  be such that there exists a sequence  $\{u_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n)$  converging to  $u$  in  $L_{loc}^1(\Omega_\delta)$  and for which  $\{D_\delta^s u_j\}_{j \in \mathbb{N}}$  converges to some  $U$  in  $L_{loc}^1(\Omega, \mathbb{R}^n)$ . We define  $D_\delta^s u$  as  $U$ .
- (b) Let  $\phi \in L_{loc}^1(\Omega_\delta, \mathbb{R}^n)$  be such that there exists a sequence  $\{\phi_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$  converging to  $\phi$  in  $L_{loc}^1(\Omega_\delta, \mathbb{R}^n)$  and for which  $\{\text{div}_\delta^s \phi_j\}_{j \in \mathbb{N}}$  converges to some  $\Phi$  in  $L_{loc}^1(\Omega)$ . We define  $\text{div}_\delta^s \phi$  as  $\Phi$ .

The following result shows that the aforementioned definitions are independent of the sequence chosen.

**Lemma 3.3.**

- (a) Let  $u \in L_{loc}^1(\Omega_\delta)$  be such that there exist sequences  $\{u_j\}_{j \in \mathbb{N}}$  and  $\{v_j\}_{j \in \mathbb{N}}$  in  $C_c^\infty(\mathbb{R}^n)$  such that  $u_j \rightarrow u$  and  $v_j \rightarrow u$  in  $L_{loc}^1(\Omega_\delta)$ , and for which  $D_\delta^s u_j \rightarrow U$  and  $D_\delta^s v_j \rightarrow V$  in  $L_{loc}^1(\Omega, \mathbb{R}^n)$  as  $j \rightarrow \infty$ . Then  $U = V$  a.e. in  $\Omega$ .
- (b) Let  $\phi \in L_{loc}^1(\Omega_\delta, \mathbb{R}^n)$  be such that there exist sequences  $\{\phi_j\}_{j \in \mathbb{N}}$  and  $\{\theta_j\}_{j \in \mathbb{N}}$  in  $C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$  such that  $\phi_j \rightarrow \phi$  and  $\theta_j \rightarrow \phi$  in  $L_{loc}^1(\Omega_\delta, \mathbb{R}^n)$ , and for which  $\text{div}_\delta^s \phi_j \rightarrow \Phi$  and  $\text{div}_\delta^s \theta_j \rightarrow \Theta$  in  $L_{loc}^1(\Omega)$  as  $j \rightarrow \infty$ . Then  $\Phi = \Theta$  a.e. in  $\Omega$ .

**Proof.** We prove (a), the proof of (b) being analogous. Let  $\phi \in C_c^1(\Omega, \mathbb{R}^n)$ . As  $D_\delta^s u_j \rightarrow U$  in  $L_{loc}^1(\Omega, \mathbb{R}^n)$  and  $\text{supp } \phi \subset \Omega$ , we have

$$\int_{\Omega} U \cdot \phi = \lim_{j \rightarrow \infty} \int_{\Omega} D_\delta^s u_j \cdot \phi.$$

By Theorem 3.2, for each  $j \in \mathbb{N}$ ,

$$\int_{\Omega} D_\delta^s u_j \cdot \phi = - \int_{\Omega} u_j \text{div}_\delta^s \phi - (n-1+s) \int_{\Omega} \int_{\Omega_{B,\delta}} \frac{u_j(y)\phi(x)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy dx.$$

Now,  $\text{supp div}_\delta^s \phi \subset \Omega_\delta$ , so, thanks to the convergence  $u_j \rightarrow u$  in  $L_{loc}^1(\Omega_\delta)$ , we have

$$\lim_{j \rightarrow \infty} \int_{\Omega} u_j \text{div}_\delta^s \phi = \int_{\Omega} u \text{div}_\delta^s \phi.$$

Assume that the limit

$$\lim_{j \rightarrow \infty} \int_{\Omega} \int_{\Omega_{B,\delta}} \frac{u_j(y)\phi(x)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy dx = \int_{\Omega} \int_{\Omega_{B,\delta}} \frac{u(y)\phi(x)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy dx \quad (13)$$

holds. Then, we would have proved that

$$\int_{\Omega} U \cdot \phi = - \int_{\Omega} u \text{div}_\delta^s \phi - (n-1+s) \int_{\Omega} \int_{\Omega_{B,\delta}} \frac{u(y)\phi(x)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy dx.$$

Since the same reasoning can be done for  $V$ , we would conclude that

$$\int_{\Omega} U \cdot \phi = \int_{\Omega} V \cdot \phi$$

for all  $\phi \in C_c^1(\Omega, \mathbb{R}^n)$ , whence  $U = V$  a.e. in  $\Omega$ .

It remains to justify limit (13). Due to the convergence  $u_j \rightarrow u$  in  $L^1_{\text{loc}}(\Omega_\delta)$ , it is enough to show that the function

$$F(y) := \int_{\Omega} \frac{\phi(x)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dx, \quad y \in \Omega_{B,\delta}$$

is in  $L^\infty(\Omega_{B,\delta})$  and has support in  $\Omega_\delta$ . It is in  $L^\infty(\Omega_{B,\delta})$  since, denoting by  $L$  the Lipschitz constant of  $\phi$  and using that  $\phi$  vanishes in  $\Omega_{B,\delta}$ , we have

$$|F(y)| \leq \int_{\Omega} \frac{|\phi(x) - \phi(y)|}{|x-y|} \rho_\delta(x-y) dx \leq L.$$

Finally,  $\text{supp} F \subset \text{supp} \phi + B(0, \delta) \subset \Omega_\delta$ . □

It is quite natural to consider the space of  $L^p$  functions whose non-local gradient is also an  $L^p$  function. Taking into account the previous definitions, it is also natural to define such a space as the closure of smooth, compactly supported functions. Notice that this is analogous to the definition of the Bessel space  $H^{s,p}(\mathbb{R}^n)$  [8,16].

**Definition 3.3.** We define the space  $H^{s,p,\delta}(\Omega)$  as follows:

$$H^{s,p,\delta}(\Omega) := \overline{C_c^\infty(\mathbb{R}^n)}^{\|\cdot\|_{H^{s,p,\delta}(\Omega)}}$$

equipped with the norm

$$\|u\|_{H^{s,p,\delta}(\Omega)} = (\|u\|_{L^p(\Omega_\delta)}^p + \|D_\delta^s u\|_{L^p(\Omega)}^p)^{\frac{1}{p}}.$$

Notice that functions in  $H^{s,p,\delta}(\Omega)$  are defined a.e. in  $\Omega_\delta$ , while their gradient in  $\Omega$ . As can be seen in Figure 1,  $\Omega_\delta$  is needed for  $D_\delta^s u$  to be defined in  $\Omega$ .

Similarly to Sobolev spaces, this space satisfies reflexivity and separability properties.

**Proposition 3.4.** *Let  $1 \leq p < \infty$ . Then the space  $H^{s,p,\delta}(\Omega)$  is a separable Banach space. Moreover, when  $p > 1$ , it is reflexive.*

**Proof.** That  $H^{s,p,\delta}(\Omega)$  is a Banach space is immediate since it has been defined as a closure.

For the rest of the proof, we apply a standard argument; see, e.g. [52, Th. 2.1] for the non-local case and [13, Prop. 8.1] for the local case.

We have that the space  $F_p = L^p(\Omega_\delta) \times L^p(\Omega, \mathbb{R}^n)$  is separable and, if  $p > 1$ , it is reflexive. Now we define the map  $T : H^{s,p,\delta}(\Omega) \rightarrow F_p$  by  $T(u) = (u, D_\delta^s u)$ . Then  $T$  is an isometry since

$$\|T(u)\|_{F_p}^p = \|u\|_{L^p(\Omega_\delta)}^p + \|D_\delta^s u\|_{L^p(\Omega)}^p = \|u\|_{H^{s,p,\delta}(\Omega)}^p.$$

By Definitions 3.2 and 3.3, it is clear that  $T(H^{s,p,\delta}(\Omega))$  is a closed subspace of  $F_p$ . Since every closed subspace of a reflexive space is reflexive (see, e.g. [13, Prop. 3.20]) and every subset of a separable space is separable (e.g. [13, Prop. 3.25]), it follows that  $T(H^{s,p,\delta}(\Omega))$  is separable and, if  $p > 1$ , it is reflexive. The conclusion follows since  $T$  is an isometry. □

In the next result, we compare the spaces  $H^{s,p,\delta}(\Omega)$  for different exponents  $p$ , as well as with the better-known Bessel space  $H^{s,p}(\mathbb{R}^n)$ .

**Proposition 3.5.** *Let  $1 \leq q \leq p < \infty$ . Then:*

- (a)  $H^{s,p,\delta}(\Omega) \subset H^{s,q,\delta}(\Omega)$ .
- (b)  $H^{s,p}(\mathbb{R}^n) \subset H^{s,p,\delta}(\Omega)$ , with continuous embedding.

**Proof.** The proof of (a) is obtained in a straightforward manner applying the known inclusions  $L^p(\Omega_\delta) \subset L^q(\Omega_\delta)$  and  $L^p(\Omega) \subset L^q(\Omega)$  to the norms of  $u$  and  $D_\delta^s u$ .

Regarding (b), we first prove the corresponding inequality for smooth functions. Thus, let  $u \in C_c^\infty(\mathbb{R}^n)$ . We have that, for  $x \in \Omega$ ,

$$\begin{aligned} D_\delta^s u(x) &= c_{n,s} \int_{B(x,\delta)} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{w_\delta(x-y)}{|x-y|^{n-1+s}} dy \\ &= c_{n,s} a_0 \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{1}{|x-y|^{n-1+s}} dy - c_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{a_0 - w_\delta(x-y)}{|x-y|^{n-1+s}} dy \\ &= a_0 D^s u(x) - c_{n,s} \int_{B(x,b_0\delta)^c} \frac{u(x) - u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{a_0 - w_\delta(x-y)}{|x-y|^{n-1+s}} dy \\ &= a_0 D^s u(x) + c_{n,s} \int_{B(x,b_0\delta)^c} \frac{u(y)}{|x-y|} \frac{x-y}{|x-y|} \frac{a_0 - w_\delta(x-y)}{|x-y|^{n-1+s}} dy, \end{aligned}$$

where the last equality is obtained thanks to the odd symmetry of the integrand.

We recall that  $a_0$  and  $b_0$  are the constants from the definition of  $w_\delta$  and that  $w_\delta = a_0$  in  $B(0, b_0\delta)$ . Therefore, applying Hölder inequality, we have that

$$|D_\delta^s u(x)| \leq a_0 |D^s u(x)| + a_0 c_{n,s} \int_{B(x,b_0\delta)^c} \frac{|u(y)|}{|x-y|^{n+s}} dy \leq a_0 |D^s u(x)| + c_1 \|u\|_{L^p(B(x,b_0\delta)^c)}$$

for some constant  $c_1 > 0$ . Consequently,

$$\|D_\delta^s u\|_{L^p(\Omega)} \leq a_0 \|D^s u\|_{L^p(\Omega)} + c_1 |\Omega|^{\frac{1}{p}} \|u\|_{L^p(B(x,b_0\delta)^c)} \leq c_2 \|u\|_{H^{s,p}(\mathbb{R}^n)}$$

for some constant  $c_2 > 0$ . Since we also have that  $\|u\|_{L^p(\Omega_\delta)} \leq \|u\|_{L^p(\mathbb{R}^n)}$ , we obtain that there exists  $C > 0$  such that for every  $u \in C_c^\infty(\mathbb{R}^n)$ ,

$$\|u\|_{H^{s,p,\delta}(\Omega)} \leq C \|u\|_{H^{s,p}(\mathbb{R}^n)}.$$

Being the spaces  $H^{s,p,\delta}(\Omega)$  and  $H^{s,p}(\mathbb{R}^n)$  defined as the closure of  $C_c^\infty(\mathbb{R}^n)$  with their respective norms, the result follows.  $\square$

The inclusion of Proposition 3.5(b) is one of the main motivations for the definition of our space, since it roughly suggests that  $H^{s,p,\delta}(\Omega)$  consists of Bessel  $H^{s,p}$  functions defined only on  $\Omega$  and without any integrability requirement at infinity. As a matter of fact, the examples of [7, Sect. 2.1] in the context of solid mechanics, together with the inclusion  $H^{s,p}(\mathbb{R}^n) \subset H^{s,p,\delta}(\Omega)$  show that  $H^{s,p,\delta}(\Omega, \mathbb{R}^n)$  contains deformations exhibiting fracture or cavitation, for some range of  $s$  and  $p$ . One of the advantages of the space  $H^{s,p,\delta}(\Omega)$  is that, contrary to  $H^{s,p}(\mathbb{R}^n)$ , it contains linear functions, such as the identity, which would be relevant in a future linearization process.

## 4 Non-local fundamental theorem of calculus

We start by recalling the following classical representation theorem, which can be seen in [37, Lemma 7.14] or [59, Prop. 4.14].

**Proposition 4.1.** *For every  $\varphi \in C_c^\infty(\mathbb{R}^n)$  and every  $x \in \mathbb{R}^n$ , we have*

$$\varphi(x) = \frac{1}{\sigma_{n-1}} \int_{\mathbb{R}^n} \nabla \varphi(y) \cdot \frac{x-y}{|x-y|^n} dy.$$

This result may be understood as a fundamental theorem of calculus, in the sense that we recover a function from its gradient by integration; more precisely, by convolution. A fractional version of it, involving

the Riesz fractional gradient, is also known [59,66]. This section is devoted to a novel non-local version of Proposition 4.1, where a function can be recovered from its non-local gradient  $D_\delta^s$  through a convolution with a suitable kernel  $V_\delta^s$ .

Our approach is inspired by the proofs of the fractional fundamental theorem of calculus previously referred in [59,66]. However, those partly rely on the semigroup properties of Riesz potentials, which our kernels do not enjoy. Therefore, our procedure is much more involved.

To begin with, we show that the kernel in the definition of  $D_\delta^s u$  (9) can be seen, in a certain sense, as the gradient of a certain function. To do so, we introduce the following kernels.

**Definition 4.1.** Define

$$\bar{q}_\delta : [0, \infty) \rightarrow \mathbb{R}, \quad q_\delta : \mathbb{R}^n \rightarrow \mathbb{R} \quad \text{and} \quad Q_\delta^s : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$$

as

$$\bar{q}_\delta(t) = (n-1+s)t^{n-1+s} \int_t^\delta \frac{\bar{w}_\delta(r)}{r^{n+s}} dr, \quad q_\delta(x) = \bar{q}_\delta(|x|) \quad \text{and} \quad Q_\delta^s(x) = \frac{1}{\gamma(1-s)|x|^{n-1+s}} q_\delta(x).$$

In the following result,  $a_0$  and  $b_0$  are the constants from the definition of  $w_\delta$ .

**Lemma 4.2.**

(a)  $\bar{q}_\delta$  is  $C^\infty((0, \infty))$ ,  $C^{n-1}([0, \infty))$ ,  $\text{supp} \bar{q}_\delta \subset [0, \delta)$ , and there exists a constant  $z_0 \in \mathbb{R}$  such that for every  $t \in [0, b_0\delta]$ ,

$$\bar{q}_\delta(t) = a_0 + z_0 t^{n-1+s}.$$

(b)  $Q_\delta^s$  is radially decreasing,  $Q_\delta^s \in L^1(\mathbb{R}^n)$ ,  $\text{supp} Q_\delta^s \subset B(0, \delta)$  and

$$\frac{-1}{n-1+s} \nabla Q_\delta^s(x) = \frac{\rho_\delta(x)}{|x|} \frac{x}{|x|}. \quad (14)$$

**Proof.** We start with (a). The function  $\bar{q}_\delta$  is clearly  $C^\infty$  in  $(0, \infty)$  as a product of  $C^\infty$  functions in  $(0, \infty)$ . We have that

$$\left( \frac{\bar{q}_\delta(t)}{t^{n-1+s}} \right)' = -(n-1+s) \frac{\bar{w}_\delta(t)}{t^{n+s}}, \quad t > 0. \quad (15)$$

Since  $\bar{q}_\delta(\delta) = 0$  and  $\text{supp} \bar{w}_\delta \subset [0, \delta)$ , we obtain that  $\text{supp} \bar{q}_\delta \subset [0, \delta)$ . Now, for  $0 < t < \delta b_0$ , we have that

$$\left( \frac{\bar{q}_\delta(t)}{t^{n-1+s}} \right)' = -(n-1+s) \frac{a_0}{t^{n+s}} = \left( \frac{a_0}{t^{n-1+s}} \right)',$$

so the existence of  $z_0$  in the statement follows. In particular,  $\bar{q}_\delta$  is  $C^{n-1}([0, \infty))$ .

We now show (b). We get immediately from (15) that  $Q_\delta^s$  is radially decreasing and

$$\nabla Q_\delta^s(x) = -\frac{n-1+s}{\gamma(1-s)} \frac{\bar{w}_\delta(|x|)}{|x|^{n+s}} \frac{x}{|x|}, \quad x \in \mathbb{R}^n \setminus \{0\},$$

so (14) holds. As  $\text{supp} \bar{q}_\delta \subset [0, \delta)$ , we obtain that  $\text{supp} Q_\delta^s \subset B(0, \delta)$ . Consequently,  $Q_\delta^s \in L^1(\mathbb{R}^n)$  because of the boundedness of  $\bar{q}_\delta$ .  $\square$

In the following proposition, we write the non-local gradient as a convolution of the classical one with the kernel  $Q_\delta^s$ . Its fractional version can be found in [59, Lemma 15.9] and [66, Th. 1.2]. In fact, our proof is inspired by that of [59]: its idea is based on an integration by parts starting with Definition 3.1 and (14).

**Proposition 4.3.** For every  $u \in C_c^\infty(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ , we have

$$D_\delta^s u(x) = \int_{\mathbb{R}^n} \nabla u(y) Q_\delta^s(x-y) dy \quad (16)$$

and  $D_\delta^s u \in C_c^\infty(\mathbb{R}^n)$ .

**Proof.** Let  $K$  be a ball containing  $\text{supp} u$  and let  $K_\delta = K + B(0, \delta)$ . If  $x \in K_\delta^c$  then both terms of (16) are zero since  $\text{supp} D_\delta^s u \subset \text{supp} u + B(0, \delta) \subset K_\delta$  and  $\text{supp} Q_\delta^s \subset B(0, \delta)$ . Thus, we consider  $x \in K_\delta$ ,  $e \in \mathbb{R}^n$  with  $|e| = 1$  and the vector field

$$\beta : K_\delta \setminus \{x\} \rightarrow \mathbb{R}^n$$

defined by

$$\beta(y) = (u(x) - u(y)) Q_\delta^s(x-y) e.$$

Let  $\varepsilon > 0$  be such that  $\bar{B}(x, \varepsilon) \subset K_\delta$ . From Lemma 4.2 we have that

$$\text{div} \beta(y) = (n-1+s) \frac{u(x) - u(y)}{|x-y|} \rho_\delta(x-y) \frac{x-y}{|x-y|} \cdot e - Q_\delta^s(x-y) \nabla u(y) \cdot e, \quad y \in K_\delta \setminus B(x, \varepsilon) \quad (17)$$

and notice that  $\text{div} \beta$  is integrable in  $K_\delta \setminus B(x, \varepsilon)$ . By applying the divergence theorem, we obtain

$$\int_{K_\delta \setminus B(x, \varepsilon)} \text{div} \beta(y) dy = \int_{\partial K_\delta} \beta(y) \cdot \nu_y d\mathcal{H}^{n-1}(y) + \int_{\partial B(x, \varepsilon)} \beta(y) \cdot \frac{x-y}{|x-y|} d\mathcal{H}^{n-1}(y),$$

where  $\nu_y$  is the outer normal vector to  $K_\delta$ . Now we show that  $\beta(y) = 0$  for all  $y \in \partial K_\delta$ . Indeed, if  $x \in K_\delta \setminus K$ , then  $u(x) = u(y) = 0$  for all  $y \in \partial K_\delta$ , whereas if  $x \in K$ , then  $Q_\delta^s(x-y) = 0$  for every  $y \in \partial K_\delta$ . Thus,

$$\int_{K_\delta \setminus B(x, \varepsilon)} \text{div} \beta(y) dy = \int_{\partial B(x, \varepsilon)} \beta(y) \cdot \frac{x-y}{|x-y|} d\mathcal{H}^{n-1}(y).$$

We estimate the integrand on the right-hand side. As  $u$  is Lipschitz, using the mean value theorem and the definition of  $Q_\delta^s$  (Definition 4.1 and Lemma 4.2), we find that, for all  $y \in \partial B(x, \varepsilon)$ ,

$$\left| \beta(y) \cdot \frac{x-y}{|x-y|} \right| \leq |\beta(y)| \leq \|\nabla u\|_\infty |x-y| |Q_\delta^s(x-y)| \leq \|\nabla u\|_\infty \frac{c}{|x-y|^{n+s-2}} = \|\nabla u\|_\infty \frac{c}{\varepsilon^{n+s-2}}$$

for some constant  $c > 0$ , so

$$\left| \int_{\partial B(x, \varepsilon)} \beta(y) \cdot \frac{x-y}{|x-y|} d\mathcal{H}^{n-1}(y) \right| \leq \|\nabla u\|_\infty c \sigma_{n-1} \varepsilon^{1-s},$$

which goes to 0 when  $\varepsilon$  goes to 0. Therefore,

$$\lim_{\varepsilon \rightarrow 0} \int_{K_\delta \setminus B(x, \varepsilon)} \text{div} \beta(y) dy = 0.$$

As a result, by using (17), we obtain that

$$\lim_{\varepsilon \rightarrow 0} \int_{K_\delta \setminus B(x, \varepsilon)} (n-1+s) \frac{u(x) - u(y)}{|x-y|} \rho_\delta(x-y) \frac{x-y}{|x-y|} \cdot e dy = \lim_{\varepsilon \rightarrow 0} \int_{K_\delta \setminus B(x, \varepsilon)} Q_\delta^s(x-y) \nabla u(y) \cdot e dy,$$

provided that both limits exists, which is actually true as both integrals are absolutely convergent in  $K_\delta$ ; see the comment after Definition 3.1 for the left integral and notice that  $Q_\delta^s \in L^1(\mathbb{R}^n)$  (see Lemma 4.2) for the right integral. Thus,



$$\int_{K_\delta} (n-1+s) \frac{u(x)-u(y)}{|x-y|} \rho_\delta(x-y) \frac{x-y}{|x-y|} \cdot e dy = \int_{K_\delta} Q_\delta^s(x-y) \nabla u(y) \cdot e dy.$$

As this is true for every  $e \in \mathbb{R}^n$  with  $|e| = 1$ , we conclude that

$$\int_{K_\delta} (n-1+s) \frac{u(x)-u(y)}{|x-y|} \frac{x-y}{|x-y|} \rho_\delta(x-y) dy = \int_{K_\delta} \nabla u(y) Q_\delta^s(x-y) dy,$$

and formula (16) is proved.

We have thus shown that  $D_\delta^s u = \nabla u * Q_\delta^s$ . As  $Q_\delta^s \in L^1(\mathbb{R}^n)$ ,  $\nabla u \in C^\infty(\mathbb{R}^n)$  and both have compact support, we conclude that  $D_\delta^s u \in C_c^\infty(\mathbb{R}^n)$ .  $\square$

Proposition 4.3 shows that for  $u \in C_c^\infty(\mathbb{R}^n)$ , its non-local gradient  $D_\delta^s u$  is also in  $C_c^\infty(\mathbb{R}^n)$ , unlike its fractional gradient  $D^s u$ , which is  $C^\infty$  (see [77, Prop. 5.2] or [49]) but not of compact support (an easy counterexample can be found in [44, Sect. 2.2]). The reason for this difference is that, while Proposition 4.3 shows that  $D_\delta^s u = \nabla u * Q_\delta^s$  with  $Q_\delta^s$  of compact support, the fractional analogue states that  $D^s u = \nabla u * I_{1-s}$ , with  $I_{1-s}$  the Riesz potential (see (21) for the definition), which is not of compact support.

We continue by restating the main result of Section 5 (Theorem 5.9).

**Proposition 4.4.** *There exists a function  $V_\delta^s \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$  such that*

$$\int_{\mathbb{R}^n} V_\delta^s(z) Q_\delta^s(y-z) dz = \frac{1}{\sigma_{n-1}} \frac{y}{|y|^n}, \quad y \in \mathbb{R}^n \setminus \{0\}. \quad (18)$$

Moreover,  $V_\delta^s \in L_{\text{loc}}^1(\mathbb{R}^n, \mathbb{R}^n)$ , and for every  $R > 0$ , there exists  $M > 0$  such that

$$|V_\delta^s(x)| \leq \frac{M}{|x|^{n-s}}, \quad x \in B(0, R) \setminus \{0\}.$$

We will study further properties of  $V_\delta^s$  in Theorem 5.9. The proof of Proposition 4.4 is long and comprises the whole of Section 5. With this, the main result of this section (a non-local version of the fundamental theorem of calculus) reads as follows. Its proof follows the lines from [59, Prop. 15.8], whereas the main differences are gathered in Proposition 4.4.

**Theorem 4.5.** *Let  $V_\delta^s$  be the function of Proposition 4.4. Then, for every  $u \in C_c^\infty(\mathbb{R}^n)$  and  $x \in \mathbb{R}^n$ ,*

$$u(x) = \int_{\mathbb{R}^n} D_\delta^s u(y) \cdot V_\delta^s(x-y) dy. \quad (19)$$

**Proof.** Let  $F(x)$  denote the right-hand side of (19). This integral is absolutely convergent since  $V_\delta^s \in L_{\text{loc}}^1(\mathbb{R}^n, \mathbb{R}^n)$  (Proposition 4.4) and  $D_\delta^s u$  is bounded with compact support (Proposition 4.3). In fact, Proposition 4.3 allows us to write the equality

$$F(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \nabla u(z) Q_\delta^s(y-z) \cdot V_\delta^s(x-y) dz dy.$$

Next we make the changes of variables  $\eta = x - y$  and  $\xi = x - z$  to obtain

$$F(x) = \int_{\mathbb{R}^n} \nabla u(x-\xi) \cdot \int_{\mathbb{R}^n} V_\delta^s(\eta) Q_\delta^s(\xi-\eta) d\eta d\xi.$$

By Proposition 4.4,

$$\int_{\mathbb{R}^n} V_\delta^s(\eta) Q_\delta^s(\xi-\eta) d\eta = \frac{1}{\sigma_{n-1}} \frac{\xi}{|\xi|^n}.$$

Thus, thanks to Proposition 4.1,

$$F(x) = \int_{\mathbb{R}^n} \nabla u(x - \xi) \cdot \frac{\xi}{\sigma_{n-1} |\xi|^n} d\xi = u(x)$$

and the proof is complete.  $\square$

## 5 Existence of $V_\delta^s$

This section is devoted to the proof of Proposition 4.4, as well as to the derivation of further properties of  $V_\delta^s$ . Readers more interested in applications may omit this section, since only the statement of Theorem 5.9 will be used in the following sections.

The idea of the proof is to convert equation (18) into

$$\hat{V}_\delta^s(\xi) \hat{Q}_\delta^s(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi\xi|}$$

through Fourier transform (see Lemma B.1(c) for the Fourier transform of the right-hand side of (18)). Thus, the candidate for  $V_\delta^s$  should satisfy

$$\hat{V}_\delta^s(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi\xi|} \frac{1}{\hat{Q}_\delta^s(\xi)}.$$

In the first half of the section, we show that  $\hat{Q}_\delta^s$  is positive and, consequently, the aforementioned formula is well defined. Taking inverse Fourier transform, we then conclude that  $V_\delta^s$  is at least a tempered distribution. In the second half, we see that  $V_\delta^s$  is actually a function.

As seen Section 1, it is illustrative to compare formula (19) with the fractional fundamental theorem of calculus in  $H^{s,p}$  (see [16, Th. 3.11], [59, Prop. 15.8] or [66, Th. 1.12]):

$$u(x) = c_{n,s} \int_{\mathbb{R}^n} D^s u(y) \cdot \frac{x - y}{|x - y|^{n-s+1}} dy. \quad (20)$$

Here,  $D^s u$  is Riesz'  $s$ -fractional gradient. Thus,  $D_\delta^s u$  and  $V_\delta^s$  in our context play the role of  $D^s u$  and  $\frac{x}{|x|^{n-s+1}}$  in  $H^{s,p}$ , respectively. In fact, in the analysis in  $H^{s,p}$ , an essential part is performed by the *Riesz potential*. We recall (see [66,72]) that given  $0 < s < n$ , the Riesz potential  $I_s : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$  and its Fourier transform are

$$I_s(x) = \frac{1}{\gamma(s)} \frac{1}{|x|^{n-s}} \quad \text{and} \quad \hat{I}_s(\xi) = |2\pi\xi|^{-s}, \quad (21)$$

where  $\gamma(s)$  is defined in (8). A key study in a great part of this and the following section will be the comparison, first, of  $\hat{I}_s$  with  $\hat{V}_\delta^s$ , and, then, of  $I_s$  with  $V_\delta^s$ . In fact, in the comment after Proposition 4.3, it was hinted that here  $Q_\delta^s$  plays the role of  $I_{1-s}$  in the fractional case, so it becomes natural that in this section we also compare  $\hat{Q}_\delta^s$  with  $\hat{I}_{1-s}$ .

We present another parallelism between  $D^s$  and  $D_\delta^s$  in terms of Calderón-Zygmund operators. Having in mind the expression (11) for  $D_\delta^s u$ , as well as its analogue for  $D^s u$ ,

$$D^s u(x) = c_{n,s} \text{pv} \int_{\mathbb{R}^n} \frac{u(x+h)h}{|h|^{n+s+1}} dy,$$

(recall (2)), we can write

$$D_\delta^s u(x) = c_{n,s} \text{pv} \int_{\mathbb{R}^n} \frac{u(x+h)h}{|h|^{n+1+s}} (a_0 - (a_0 - w_\delta(h))) dh = a_0 D^s u(x) - T_\delta^s u(x),$$

where

$$T_\delta^s u(x) = c_{n,s} \text{pv} \int_{\mathbb{R}^n} \frac{u(x+h)h}{|h|^{n+1+s}} (a_0 - w_\delta(h)) dh = c_{n,s} \int_{B(0, \beta_0 \delta)^c} \frac{u(x+h)h}{|h|^{n+1+s}} (a_0 - w_\delta(h)) dh$$

is a (vectorial) Calderón-Zygmund operator of order zero. Thus, we have the decomposition

$$\frac{1}{a_0} D^s u = D_\delta^s u + T_\delta^s u,$$

which splits  $D^s$  into a part near zero ( $D_\delta^s$ ) and a part near infinity ( $T_\delta^s$ ). Therefore, it is natural that most properties of  $D^s$  are transferred to  $D_\delta^s$ , since they differ by a zeroth-order operator supported in the complement of a ball at the origin. In this sense, a comparison between the fractional and non-local fundamental theorem of calculus (see Section 1, and also (19) and (20)) suggests that this property is encapsulated in the local part of  $D^s$  only; in other words, only the behaviour at the origin of the kernel  $\rho$  (see (3) and (5)) seems to be relevant.

## 5.1 Positivity of $\hat{Q}_\delta^s$ and existence of $V_\delta^s$ as a distribution

We start with an analysis of the Fourier transform of the function  $q_\delta$  of Definition 4.1.

**Lemma 5.1.** *The function  $\hat{q}_\delta$  is analytic,  $C_0(\mathbb{R}^n)$  and  $L^1(\mathbb{R}^n)$ .*

**Proof.** Given that  $\bar{q}_\delta \in C_c^{n-1}([0, \infty))$  (Lemma 4.2), we have that  $q_\delta \in L^1(\mathbb{R}^n)$  and has compact support. Therefore, by known facts in Fourier analysis,  $\hat{q}_\delta$  belongs to  $C_0(\mathbb{R}^n)$  and is analytic.

It remains to show that  $\hat{q}_\delta \in L^1(\mathbb{R}^n)$ , and for this, we will previously check that  $q_\delta \in W^{2n-1,1}(\mathbb{R}^n)$ . Indeed, as a consequence of Lemma 4.2, for  $1 \leq j \leq 2n-1$ , there exists  $z_j \in \mathbb{R}$  such that

$$\bar{q}_\delta^{(j)}(t) = z_j t^{n-1+s-j}, \quad t \in (0, b_0 \delta),$$

where the superindex  $j$  indicates the  $j$ th derivative. On the other hand,  $\bar{q}_\delta^{(j)}$  is bounded in  $[b_0 \delta, \delta]$  and vanishes in  $[\delta, \infty)$ . This implies that the a.e. and weak derivative of order  $j$  of  $q_\delta$  coincide and they satisfy, for some constant  $C_j > 0$ ,

$$|D^{(j)} q_\delta(x)| \leq C_j |x|^{n-1+s-j}, \quad x \in B(0, b_0 \delta) \setminus \{0\},$$

while  $|D^{(j)} q_\delta|$  is bounded in  $B(0, \delta) \setminus B(0, b_0 \delta)$  and vanishes in  $B(0, \delta)^c$ . This implies that  $q_\delta \in W^{2n-1,1}(\mathbb{R}^n)$ . In particular, the Fourier transform of any partial derivative of order  $2n-1$  of  $q_\delta$  is bounded, so there exists  $C > 0$  such that for any multi-index  $\alpha$  of order  $2n-1$ , we have

$$|(2\pi i \xi)^\alpha \hat{q}_\delta(\xi)| = |\widehat{\partial^\alpha q_\delta}(\xi)| \leq C,$$

and, hence,

$$|\hat{q}_\delta(\xi)| \leq \frac{C}{|2\pi \xi|^{2n-1}}.$$

This decay at infinity of  $\hat{q}_\delta$ , together with the fact that  $\hat{q}_\delta$  is continuous, implies that  $\hat{q}_\delta \in L^1(\mathbb{R}^n)$  for  $n \geq 2$ .

In the rest of the proof, we assume that  $n = 1$ . In this case,  $q_\delta$  is the even extension of  $\bar{q}_\delta$ . As shown earlier, there exists  $z_1 \in \mathbb{R}$  such that  $q_\delta'(x) = \frac{z_1}{|x|^{1-s}}$  for  $x \in B(0, b_0 \delta)$ . If  $z_1 = 0$ , then  $q_\delta$  is  $C_c^\infty(\mathbb{R})$ , so  $\hat{q}_\delta$  is in  $\mathcal{S}$  and, in particular, in  $L^1(\mathbb{R})$ . We assume from now on that  $z_1 \neq 0$ .

Consider a  $\varphi \in C_c^\infty(\mathbb{R})$  with  $\varphi|_{B(0, \frac{1}{4})} = 1$  and  $\varphi|_{B(0, \frac{1}{2})^c} = 0$ . Then,

$$|2\pi \xi|^{-s} - \frac{1}{z_1 \gamma(s)} \hat{q}_\delta'(\xi) = \mathcal{F} \left[ \frac{1}{\gamma(s)|x|^{1-s}} - \frac{1}{z_1 \gamma(s)} q_\delta'(x) \right] = \mathcal{F} \left[ \frac{\varphi}{\gamma(s)|x|^{1-s}} - \frac{1}{z_1 \gamma(s)} q_\delta'(x) \right] + \mathcal{F} \left[ \frac{1-\varphi}{\gamma(s)|x|^{1-s}} \right].$$

Looking at the expression of  $q'_\delta$ , we notice that the functions  $\frac{\varphi}{\gamma(s)|x|^{1-s}}$  and  $\frac{1}{z_1\gamma(s)}q'_\delta(x)$  coincide in  $B(0, \min\{b_0\delta, \frac{1}{4}\})$ , and both have compact support. Therefore, its difference is a smooth function of compact support. In particular, it is in the Schwartz space, as well as its Fourier transform:

$$\mathcal{F}\left(\frac{\varphi}{\gamma(s)|x|^{1-s}} - \frac{1}{z_1\gamma(s)}q'_\delta(x)\right) \in \mathcal{S}.$$

On the other hand, the function  $\mathcal{F}\left(\frac{1-\varphi}{\gamma(s)|x|^{1-s}}\right)$  is treated in [38, Ex. 2.4.9], and it is concluded that its decay at infinity is faster than any negative power of  $|\xi|$ . Consequently, the decay at infinity of

$$|2\pi\xi|^{-s} - \frac{1}{z_1\gamma(s)}\hat{q}'_\delta(\xi)$$

is also faster than any negative power of  $|\xi|$ . In particular, there exists  $C'_1 > 0$  such that

$$\left|\frac{2\pi\xi}{z_1\gamma(s)}\hat{q}_\delta(\xi)\right| = \left|\frac{1}{z_1\gamma(s)}\hat{q}'_\delta(\xi)\right| \leq \frac{C'_1}{|2\pi\xi|^s},$$

which allows us to conclude that  $\hat{q}_\delta \in L^1(\mathbb{R})$ .  $\square$

In the following result, we obtain relevant properties about  $\hat{Q}_\delta^s$ . Recall that  $a_0$  is the constant from the definition of  $w_\delta$ .

**Proposition 5.2.**

(a)  $\hat{Q}_\delta^s$  is analytic, bounded, radial, and  $\hat{Q}_\delta^s(0) = \|Q_\delta^s\|_{L^1(\mathbb{R}^n)}$ .

(b)  $\partial^\alpha \hat{Q}_\delta^s$  is bounded for every multi-index  $\alpha$ .

(c)  $\lim_{|\xi| \rightarrow \infty} \frac{\hat{Q}_\delta^s(\xi)}{|2\pi\xi|^{-(1-s)}} = a_0$ .

**Proof.** The proof of part (a) comes directly from known facts in Fourier analysis. Indeed, as  $Q_\delta^s \in L^1(\mathbb{R}^n)$ , we have  $\hat{Q}_\delta^s \in L^\infty(\mathbb{R}^n)$ . As  $Q_\delta^s$  has compact support,  $\hat{Q}_\delta^s$  is analytic. Since  $Q_\delta^s$  is radial, so is  $\hat{Q}_\delta^s$ . Finally, the equality  $\hat{Q}_\delta^s(0) = \|Q_\delta^s\|_{L^1(\mathbb{R}^n)}$  is a straightforward consequence of the formula of the Fourier transform.

Regarding (b), we have that  $\partial^\alpha \hat{Q}_\delta^s = \mathcal{F}((-2\pi i\xi)^\alpha Q_\delta^s)$ . Thus,  $\partial^\alpha \hat{Q}_\delta^s$  is the Fourier transform of an  $L^1(\mathbb{R}^n)$  function (since  $Q_\delta^s \in L^1(\mathbb{R}^n)$  has compact support). Therefore,  $\partial^\alpha \hat{Q}_\delta^s$  is bounded.

To show (c), we apply the Fourier transform to the expression  $Q_\delta^s = I_{1-s}q_\delta$  (see Definition 4.1). Since the Riesz potential  $I_{1-s}$  is not an  $L^1(\mathbb{R}^n)$  function and  $q_\delta$  is not Schwartz, the Fourier transform is, in principle, in the sense of tempered distributions. To wit, as  $I_{1-s} \in L^1(B(0, 1)) + L^\infty(B(0, 1)^c)$ , both factors  $I_{1-s}$  and  $q_\delta$  can be seen as distributions; in addition,  $q_\delta$  has compact support, so we can use Lemma B.2 and obtain that

$$\hat{Q}_\delta^s(\xi) = |2\pi\xi|^{-(1-s)} * \hat{q}_\delta(\xi) \quad (22)$$

in the sense of distributions. Actually, by Young's inequality for the convolution, we have that

$$\|\hat{I}_{1-s} * \hat{q}_\delta\|_{L^\infty(\mathbb{R}^n)} \leq \|\hat{I}_{1-s}\|_{L^1(B(0,1))} \|\hat{q}_\delta\|_{L^\infty(\mathbb{R}^n)} + \|\hat{I}_{1-s}\|_{L^\infty(B(0,1)^c)} \|\hat{q}_\delta\|_{L^1(\mathbb{R}^n)}.$$

Therefore, the integral defining  $(\hat{I}_{1-s} * \hat{q}_\delta)(\xi)$  is absolutely convergent for a.e.  $\xi \in \mathbb{R}^n$ . Consequently, equality (22) holds a.e.

Then, we consider  $\xi = \lambda\xi_0$  with  $\xi_0 \in B(0, 1)^c$  fixed and  $\lambda > 0$ . By using the change of variables  $x = \lambda x'$ , we have

$$\begin{aligned} \hat{Q}_\delta^s(\lambda\xi_0) &= \int_{\mathbb{R}^n} |2\pi(x - \lambda\xi_0)|^{-(1-s)} \hat{q}_\delta(x) dx = \int_{\mathbb{R}^n} |2\pi(\lambda x - \lambda\xi_0)|^{-(1-s)} \hat{q}_\delta(\lambda x) \lambda^n dx \\ &= \lambda^{-(1-s)} \int_{\mathbb{R}^n} |2\pi(\xi_0 - x)|^{-(1-s)} \hat{q}_\delta(\lambda x) \lambda^n dx. \end{aligned}$$

As the function  $\xi \mapsto \frac{\hat{Q}_\delta^s(\xi)}{|2\pi\xi|^{-(1-s)}}$  is radial, in order for (c) to hold, it is enough that

$$\lim_{\lambda \rightarrow \infty} \frac{\hat{Q}_\delta^s(\lambda\xi_0)}{|2\pi\lambda\xi_0|^{-(1-s)}} = a_0,$$

equivalently,

$$\lim_{\lambda \rightarrow \infty} \frac{\int_{\mathbb{R}^n} |2\pi(\xi_0 - x)|^{-(1-s)} \hat{q}_\delta(\lambda x) \lambda^n dx}{|2\pi\xi_0|^{-(1-s)}} = a_0.$$

Define now  $g_\lambda(x) = \frac{1}{a_0} \hat{q}_\delta(\lambda x) \lambda^n$  and  $f(\xi) = |2\pi\xi|^{-(1-s)}$ . The aforementioned limit is equivalent to

$$\lim_{\lambda \rightarrow \infty} \frac{\int_{\mathbb{R}^n} f(\xi_0 - x) g_\lambda(x) dx}{f(\xi_0)} = 1,$$

and, in turn, equivalent to

$$\lim_{\lambda \rightarrow \infty} \int_{\mathbb{R}^n} f(\xi_0 - x) g_\lambda(x) dx = f(\xi_0),$$

in other words,

$$\lim_{\lambda \rightarrow \infty} f * g_\lambda(\xi_0) = f(\xi_0). \quad (23)$$

We recall from Lemma 4.2 that  $a_0 = \bar{q}_\delta(0) = q_\delta(0) = \int_{\mathbb{R}^n} \hat{q}_\delta$ ; note that  $\hat{q}_\delta \in L^1(\mathbb{R}^n)$  thanks to Lemma 5.1. Thus,  $\int_{\mathbb{R}^n} g_\lambda = 1$  for each  $\lambda > 0$ . Then, by construction,  $g_\lambda$  is a mollifier family tending to the Dirac delta at 0, when  $\lambda \rightarrow \infty$  in the sense of distributions. Thus,

$$|f * g_\lambda(\xi_0) - f(\xi_0)| = \left| \int_{\mathbb{R}^n} [f(\xi_0 - x) - f(\xi_0)] g_\lambda(x) dx \right| \leq \int_{\mathbb{R}^n} |f(\xi_0 - x) - f(\xi_0)| g_\lambda(x) dx.$$

Let  $\varepsilon > 0$ . Since  $f$  is uniformly continuous in  $B(0, 1/2)^c$ , there exists  $0 < r < \frac{1}{2}$  such that

$$|f(\xi_0 - x) - f(\xi_0)| < \varepsilon, \quad \text{for all } \xi_0 \in B(0, 1)^c \quad \text{and} \quad x \in B(0, r).$$

Therefore, as  $f \in L^\infty(B(0, 1/2)^c)$ ,

$$\begin{aligned} |f * g_\lambda(\xi_0) - f(\xi_0)| &\leq \int_{B(0, r)} |f(\xi_0 - x) - f(\xi_0)| g_\lambda(x) dx + \int_{B(0, r)^c} |f(\xi_0 - x) - f(\xi_0)| g_\lambda(x) dx \\ &\leq \varepsilon \int_{B(0, r)} g_\lambda(x) dx + 2\|f\|_{L^\infty(B(0, 1/2)^c)} \int_{B(0, r)^c} g_\lambda(x) dx. \end{aligned}$$

Finally, we use that  $\lim_{\lambda \rightarrow \infty} \int_{B(0, r)^c} g_\lambda(x) dx = 0$ . As a result, there exists  $\lambda_0 > 0$  such that for every  $\lambda > \lambda_0$ , the inequality  $\int_{B(0, r)^c} g_\lambda(x) dx < \varepsilon$  holds. Consequently,

$$|f * g_\lambda(\xi_0) - f(\xi_0)| \leq (\|g_\lambda\|_{L^1(\mathbb{R}^n)} + 2\|f\|_{L^\infty(B(0, 1/2)^c)})\varepsilon.$$

As  $\|g_\lambda\|_{L^1(\mathbb{R}^n)} = \|g_1\|_{L^1(\mathbb{R}^n)}$ , this proves convergence (23), and, hence, statement (c).  $\square$

The following calculation is useful for showing the positivity of  $\hat{Q}_\delta^s$ .

**Lemma 5.3.** For all  $j \in \{1, \dots, n\}$  and  $r > 0$ ,

$$\int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi r x_j) dx > 0.$$

**Proof.** The integral is absolutely convergent since

$$\left| \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi r x_j) \right| \leq \frac{1}{|x|^{n-1+s}} w_\delta(x) 2\pi r \quad (24)$$

and  $w_\delta$  as compact support. By a change of variables, we have

$$\int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi r x_j) dx = r^s \int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta\left(\frac{x}{r}\right) \sin(2\pi x_j) dx.$$

Recall that  $\bar{w}_\delta$  is the radial representation of  $w_\delta$ . By symmetry, the co-area formula, and Fubini's theorem we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta\left(\frac{x}{r}\right) \sin(2\pi x_j) dx &= 2 \int_{\{x_j > 0\}} \frac{x_j}{|x|^{n+s+1}} w_\delta\left(\frac{x}{r}\right) \sin(2\pi x_j) dx \\ &= 2 \int_0^\infty \frac{\bar{w}_\delta\left(\frac{t}{r}\right)}{t^{n+s+1}} t^{n-1} \int_{\mathbb{S}_j^+} t z_j \sin(2\pi t z_j) d\mathcal{H}^{n-1}(z) dt \\ &= 2 \int_{\mathbb{S}_j^+} z_j \int_0^\infty \frac{\bar{w}_\delta\left(\frac{t}{r}\right)}{t^{s+1}} \sin(2\pi t z_j) dt d\mathcal{H}^{n-1}(z), \end{aligned}$$

where  $\mathbb{S}_j^+ = \{z \in \mathbb{R}^n : |z| = 1, z_j > 0\}$ . Finally, let us show that

$$\int_0^\infty \frac{\bar{w}_\delta\left(\frac{t}{r}\right)}{t^{s+1}} \sin(2\pi t z_j) dt > 0 \quad (25)$$

for each  $z \in \mathbb{S}_j^+$ . For this, consider the function  $f(t) = \frac{\bar{w}_\delta\left(\frac{t}{r}\right)}{t^{s+1}}$  and express

$$\int_0^\infty \frac{\bar{w}_\delta\left(\frac{t}{r}\right)}{t^{s+1}} \sin(2\pi t z_j) dt = \sum_{k=0}^\infty \int_{\frac{k}{z_j}}^{\frac{k+1}{z_j}} f(t) \sin(2\pi t z_j) dt.$$

We have that each term in the sum is positive; indeed, by splitting the integral in two through point  $\frac{k+\frac{1}{2}}{z_j}$  and making the change of variables  $t = t' + \frac{1}{2z_j}$  in one of them, it is easy to obtain

$$\int_{\frac{k}{z_j}}^{\frac{k+1}{z_j}} f(t) \sin(2\pi t z_j) dt = \int_{\frac{k}{z_j}}^{\frac{k+\frac{1}{2}}{z_j}} \left[ f(t) - f\left(t + \frac{1}{2z_j}\right) \right] \sin(2\pi t z_j) dt \geq 0,$$

since  $\sin(2\pi t z_j) > 0$  and  $f$  is decreasing (as so is  $\bar{w}_\delta$ ). In fact,

$$\int_0^{\frac{1}{2z_j}} \left[ f(t) - f\left(t + \frac{1}{2z_j}\right) \right] \sin(2\pi t z_j) dt > 0,$$

as  $f$  is strictly decreasing in  $[0, rb_0\delta]$ , so (25) holds, which concludes the proof.  $\square$

As can be seen from the aforementioned proof, the assumption that  $\bar{w}_\delta$  is decreasing can be weakened to the following: the function  $f$  of the proof is decreasing in  $t$  for all  $r > 0$ . This is equivalent to  $f'(t) \leq 0$  for all  $t > 0$  and  $r > 0$ , which in turn, is equivalent to the differential inequality

$$\bar{w}_\delta'(t) \leq (s+1) \frac{\bar{w}_\delta(t)}{t}, \quad t \geq b_0\delta.$$

Therefore, assumption (d) on  $\bar{w}_\delta$  (Section 3) could have been replaced with the aforementioned inequality, but, for ease of reading, we preferred to state that  $\bar{w}_\delta$  is decreasing.

The following result shows the convergence of the truncations of  $\nabla Q_\delta^s$ .

**Lemma 5.4.** The function  $\nabla Q_\delta^s$  can be identified with the tempered distribution defined componentwise as follows:

$$\langle \partial_j Q_\delta^s, \varphi \rangle = -c_{n,s} \int_{\{x_j > 0\}} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) (\varphi(x) - \varphi(-x)) dx, \quad j \in \{1, \dots, n\}, \quad (26)$$

and we have the convergence

$$\nabla Q_\delta^s \chi_{B(0,\varepsilon)^c} \rightarrow \nabla Q_\delta^s \quad \text{in } \mathcal{S}' \quad \text{as } \varepsilon \rightarrow 0.$$

**Proof.** We recall from Lemma 4.2 that

$$\nabla Q_\delta^s(x) = -(n-1+s) \frac{\rho_\delta(x)}{|x|} \frac{x}{|x|} = -c_{n,s} \frac{x}{|x|^{n+s}} w_\delta(x).$$

Thus,  $\nabla Q_\delta^s \chi_{B(0,\varepsilon)^c}$  is in  $L^1(\mathbb{R}^n)$  for each  $\varepsilon > 0$ , so that it can be identified with a tempered distribution. Let  $j \in \{1, \dots, n\}$ ; we shall prove the desired convergence for the  $j$ th component of  $\nabla Q_\delta^s$ .

By using the notation  $B_j^\pm(0, \varepsilon)^c = \{x \in B(0, \varepsilon)^c : \pm x_j > 0\}$ , we have

$$\begin{aligned} \int_{B(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \varphi(x) dx &= \int_{B_j^-(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \varphi(x) dx + \int_{B_j^+(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \varphi(x) dx \\ &= \int_{B_j^+(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) (\varphi(x) - \varphi(-x)) dx. \end{aligned}$$

By the mean value theorem,

$$\left| \frac{x_j}{|x|^{n+s+1}} w_\delta(x) (\varphi(x) - \varphi(-x)) \chi_{B_j^+(0,\varepsilon)^c}(x) \right| \leq \frac{2 \|\nabla \varphi\|_\infty \|w_\delta\|_\infty}{|x|^{n-1+s}} \chi_{B(0,\delta)}(x).$$

This shows that formula (26) defines a tempered distribution; moreover, by dominated convergence, we obtain that

$$\int_{B_j^+(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) (\varphi(x) - \varphi(-x)) dx \rightarrow \int_{\{x_j > 0\}} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) (\varphi(x) - \varphi(-x)) dx$$

as  $\varepsilon \rightarrow 0$ . This proves the desired convergence.  $\square$

We now show the positivity of  $\hat{Q}_\delta^s$ .

**Proposition 5.5.**  $\hat{Q}_\delta^s(\xi) > 0$  for all  $\xi \in \mathbb{R}^n$ .

**Proof.** Since  $\hat{Q}_\delta^s(0) = \|Q_\delta^s\|_{L^1(\mathbb{R}^n)} > 0$ , we have to show that  $\hat{Q}_\delta^s(\xi) > 0$  for every  $\xi \in \mathbb{R}^n \setminus \{0\}$ . For this, we fix any  $j \in \{1, \dots, n\}$  and claim that, despite  $\frac{\partial Q_\delta^s}{\partial x_j} \notin L^1(\mathbb{R}^n)$ ,

$$\widehat{\frac{\partial Q_\delta^s}{\partial x_j}}(\xi) = \frac{(n-1+s)}{\gamma(1-s)} i \int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi \xi \cdot x) dx. \quad (27)$$

This is shown at the end of the proof. Assuming the validity of (27), by Lemma 5.3, we obtain

$$\frac{1}{i} \frac{\partial \widehat{Q}_\delta^s}{\partial x_j}(\xi_j e_j) > 0, \quad \xi_j > 0.$$

Now, the formula

$$2\pi i \xi_j \hat{Q}_\delta^s(\xi_j e_j) = \frac{\partial \widehat{Q}_\delta^s}{\partial x_j}(\xi_j e_j)$$



holds in the sense of tempered distributions. Since both terms are actually functions, the equality holds as functions for almost every point. Moreover, since both functions are continuous, the equality holds everywhere. We then conclude that

$$\xi_j \hat{Q}_\delta^s(\xi_j e_j) > 0, \quad \xi_j > 0.$$

Consequently, since  $\hat{Q}_\delta^s$  is radial,  $\hat{Q}_\delta^s(\xi) > 0$  for all  $\xi \in \mathbb{R}^n$ .

It remains to prove (27). By Lemma 4.2,

$$\frac{\partial Q_\delta^s}{\partial x_j}(x) = -(n-1+s) \frac{\rho_\delta(x)}{|x|} \frac{x_j}{|x|}.$$

We have  $\frac{\partial Q_\delta^s}{\partial x_j} \chi_{B(0,\varepsilon)^c} \in L^1(\mathbb{R}^n)$  for all  $\varepsilon > 0$ , and by Lemma 5.4,

$$\frac{\partial Q_\delta^s}{\partial x_j} \chi_{B(0,\varepsilon)^c} \rightarrow \frac{\partial Q_\delta^s}{\partial x_j} \quad \text{in } S' \quad \text{as } \varepsilon \rightarrow 0,$$

so

$$\mathcal{F}\left(\frac{\partial Q_\delta^s}{\partial x_j} \chi_{B(0,\varepsilon)^c}\right) \rightarrow \mathcal{F}\left(\frac{\partial Q_\delta^s}{\partial x_j}\right) \quad \text{in } S' \quad \text{as } \varepsilon \rightarrow 0.$$

We now compute

$$\begin{aligned} \mathcal{F}\left(\frac{\partial Q_\delta^s}{\partial x_j} \chi_{B(0,\varepsilon)^c}\right)(\xi) &= -\frac{(n-1+s)}{i(1-s)} \int_{B(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) e^{-2\pi i \xi \cdot x} dx \\ &= \frac{(n-1+s)}{i(1-s)} i \int_{B(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi \xi \cdot x) dx, \end{aligned}$$

where we have used the odd symmetry. Now, by dominated convergence,

$$\int_{B(0,\varepsilon)^c} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi \xi \cdot x) dx \rightarrow \int_{\mathbb{R}^n} \frac{x_j}{|x|^{n+s+1}} w_\delta(x) \sin(2\pi \xi \cdot x) dx$$

because of the same argument as in (24). This proves (27).  $\square$

With this, we can conclude the existence of  $V_\delta^s$  as a distribution.

**Proposition 5.6.** *There exists a tempered distribution  $V_\delta^s$  whose Fourier transform is given by*

$$\hat{V}_\delta^s(\xi) = -\frac{i\xi}{2\pi |\xi|^2} \frac{1}{\hat{Q}_\delta^s(\xi)}. \quad (28)$$

**Proof.** Denote by  $W_\delta^s$  the right-hand side of (28), which is well defined for  $\xi \in \mathbb{R}^n \setminus \{0\}$  since  $\hat{Q}_\delta^s$  is positive (Proposition 5.5). Next, we subtract from  $W_\delta^s$  a multiple of the function (and distribution)  $\frac{-i\xi}{|\xi|} \frac{1}{2\pi\xi}$  of Lemma B.1(c):

$$\overline{W}_\delta^s(\xi) - \frac{-i\xi}{2\pi |\xi|^2} \frac{1}{\hat{Q}_\delta^s(0)} = -\frac{i\xi}{2\pi |\xi|^2} \left( \frac{1}{\hat{Q}_\delta^s(\xi)} - \frac{1}{\hat{Q}_\delta^s(0)} \right). \quad (29)$$

This function is in  $L^\infty(B(0,1)^c)$ , as a difference of functions in  $L^\infty(B(0,1)^c)$  (Proposition 5.2). Let us see that it is also in  $L^\infty(B(0,1))$ . By the mean value theorem, there exists  $c > 0$  such that for all  $\xi \in B(0,1)$ ,

$$\left| \frac{1}{\hat{Q}_\delta^s(\xi)} - \frac{1}{\hat{Q}_\delta^s(0)} \right| \leq c|\xi|.$$

As a result,

$$\left| W_{\delta}^s(\xi) - \frac{-i\xi}{2\pi |\xi|^2} \frac{1}{\hat{Q}_{\delta}^s(0)} \right| \leq \frac{c}{2\pi}, \quad (30)$$

so the function in (29) is in  $L^{\infty}(B(0, 1))$ , and, hence, in  $L^{\infty}(\mathbb{R}^n)$ . In particular, this function is a tempered distribution, and, by Lemma B.1(c), so is  $W_{\delta}^s$ . As the Fourier transform is an isomorphism from  $\mathcal{S}'$  into itself, there exists  $V_{\delta}^s \in \mathcal{S}'$  such that (28) holds.  $\square$

## 5.2 Existence of $V_{\delta}^s$ as a function

In this subsection, we prove that the distribution  $V_{\delta}^s$  of Proposition 5.6 is actually a function. First, we notice that  $\hat{V}_{\delta}^s$  does not belong to any space where we can conclude directly that its Fourier transform is a function. The main drawback comes from the fact that the tail of  $\hat{V}_{\delta}^s$  is not integrable enough. So as to tackle this, we exploit the fact that, at infinity,  $\hat{V}_{\delta}^s$  behaves like a homogeneous function with a known Fourier transform (namely, like  $\hat{I}_s$ ). Thus, we adapt the proof of [38, Prop. 2.4.8] (homogeneous function) to the non-homogeneous function  $\hat{V}_{\delta}^s$ .

We first need the following decay estimate for the derivatives of  $\hat{V}_{\delta}^s$ .

**Lemma 5.7.** *For every  $\alpha \in \mathbb{N}^n$ , there exists  $C_{\alpha} > 0$  such that for any  $|\xi| \geq 1$ ,*

$$|\partial^{\alpha} \hat{V}_{\delta}^s(\xi)| \leq \frac{C_{\alpha}}{|\xi|^{s(|\alpha|+1)}}.$$

**Proof.** In this proof, we use the letter  $C$  with some subindices to denote a generic positive constant independent of  $\xi$ ; the relevant dependence is included in the subindices. The value of the constant may vary from line to line.

Express  $\hat{V}_{\delta}^s = \frac{-i}{2\pi} g f$  with

$$g(\xi) = \frac{\xi}{|\xi|}, \quad f = f_1 \circ g_1, \quad f_1(t) = t^{-1}, \quad g_1(\xi) = |\xi| \hat{Q}_{\delta}^s(\xi).$$

By Leibniz' formula,

$$\partial^{\alpha}(gf) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\beta} g \partial^{\alpha-\beta} f.$$

Let  $\beta \in \mathbb{N}^n$ . By induction, it is easy to see that  $\partial^{\beta} g(\xi)$  can be expressed as follows:

$$\frac{P(\xi)}{|\xi|^{2|\beta|+1}}$$

for some  $\mathbb{R}^n$ -valued polynomial  $P$ , all of which components are of degree  $|\beta| + 1$ . Therefore,

$$|\partial^{\beta} g(\xi)| \leq \frac{C_{\beta}}{|\xi|^{|\beta|}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \quad (31)$$

We apply Faà di Bruno's formula for the higher-order derivatives of a composition, and obtain that

$$\partial^{\gamma} f = \sum_{k=1}^{|\gamma|} f_1^{(k)} \circ g_1 G_k,$$

where  $G_k$  is a linear combination of products of  $k$  partial derivatives of  $g_1$ , the order of which adds up  $|\gamma|$ .

We estimate the partial derivatives of  $g_1$ . We express  $g_1 = h\hat{Q}_\delta^s$  with  $h(\xi) = |\xi|$ . Since  $\nabla h = g$ , we have, by (31), that

$$|\partial^\beta h(\xi)| \leq \frac{C_\beta}{|\xi|^{\|\beta\|-1}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \quad (32)$$

Now we show that

$$|\partial^\beta \hat{Q}_\delta^s(\xi)| \leq \frac{C_\beta}{|\xi|^{\|\beta\|}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}. \quad (33)$$

From Definition 4.1 and Lemma 4.2, we have that  $Q_\delta^s$  is an  $L^1$  function of compact support, smooth outside the origin, and that in a ball  $B$  centred at the origin, one has

$$Q_\delta^s(x) = \lambda_0 + \frac{\lambda_1}{|x|^{n-1+s}}, \quad x \in B \setminus \{0\}$$

for some  $\lambda_0, \lambda_1 \in \mathbb{R}$ . With this expression, it is easy to see that

$$|\partial^\beta(x^\beta Q_\delta^s(x))| = \left| \sum_{\gamma \leq \beta} \binom{\beta}{\gamma} \partial^\gamma(x^\beta) \partial^{\beta-\gamma} Q_\delta^s(x) \right| \leq \frac{C_\beta}{|x|^{n-1+s}}, \quad x \in B \setminus \{0\}$$

for some  $C_\beta > 0$ . Moreover, since  $\partial^\beta(x^\beta Q_\delta^s)$  is smooth outside the origin and has compact support, we conclude that it is in  $L^1(\mathbb{R}^n)$ . Consequently,  $\mathcal{F}(\partial^\beta(x^\beta Q_\delta^s))$  is bounded. But

$$\mathcal{F}(\partial^\beta((-2\pi i x)^\beta Q_\delta^s)) = (2\pi i \xi)^\beta \mathcal{F}((-2\pi i x)^\beta Q_\delta^s) = (2\pi i \xi)^\beta \partial^\beta \hat{Q}_\delta^s(\xi),$$

which shows (33).

Now, by Leibniz' formula, (32), and (33),

$$|\partial^\alpha g_1(\xi)| \leq C_\alpha \sum_{\beta \leq \alpha} |\partial^\beta h(\xi)| |\partial^{\alpha-\beta} \hat{Q}_\delta^s(\xi)| \leq \frac{C_\alpha}{|\xi|^{\|\alpha\|-1}}, \quad \xi \in \mathbb{R}^n \setminus \{0\},$$

for some constant  $C_\alpha > 0$ . Hence, if we multiply  $k$  partial derivatives of  $g_1$ , the order of which adds up  $|\gamma|$ , we obtain that

$$|G_k(\xi)| \leq \frac{C_{\gamma,k}}{|\xi|^{|\gamma|-k}}, \quad \xi \in \mathbb{R}^n \setminus \{0\},$$

for some constants  $C_{\gamma,k} > 0$ . On the other hand, by induction,

$$|f_1^{(k)}(t)| = \frac{C_k}{t^{k+1}}, \quad k \in \mathbb{N}, \quad t > 0,$$

for some constants  $C_k > 0$ , and, hence,

$$|f_1^{(k)} \circ g_1(\xi)| \leq \frac{C_k}{(|\xi| \hat{Q}_\delta^s(\xi))^{k+1}}, \quad k \in \mathbb{N}, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$

From Proposition 5.2, we know that, for  $|\xi| \geq 1$ ,

$$\left| \frac{1}{\hat{Q}_\delta^s(\xi)} \right| \leq C |\xi|^{1-s},$$

so

$$\frac{1}{(|\xi| \hat{Q}_\delta^s(\xi))^{k+1}} \leq \frac{C}{|\xi|^{s(k+1)}}.$$

Thus,

$$|\partial^y f(\xi)| \leq \sum_{k=1}^{|y|} |f_1^{(k)} \circ g_1(\xi)| |G_k(\xi)| \leq C_y \sum_{k=1}^{|y|} \frac{1}{|\xi|^{s(k+1)+|y|-k}} \leq \frac{C_y}{|\xi|^{s(|y|+1)}}.$$

We conclude that, for  $|\xi| \geq 1$ ,

$$|\partial^a \hat{V}_\delta^s(\xi)| \leq C_\alpha \sum_{\beta \leq a} |\partial^\beta g(\xi)| |\partial^{a-\beta} f(\xi)| \leq C_\alpha \sum_{\beta \leq a} \frac{1}{|\xi|^{s(|a|+1)+|\beta|(1-s)}} \leq \frac{C_\alpha}{|\xi|^{s(|a|+1)}},$$

as desired.  $\square$

The decay estimate of Lemma 5.7 is not optimal. In fact, a more refined argument can possibly improve decay (33) and show that the bound

$$|\partial^\beta \hat{Q}_\delta^s(\xi)| \leq \frac{C_\alpha}{|\xi|^{|\beta|+1-s}}, \quad \xi \in \mathbb{R}^n \setminus \{0\}$$

holds. With that estimate, an adaptation of the proof of Lemma 5.7 would yield

$$|\partial^a \hat{V}_\delta^s(\xi)| \leq \frac{C_\alpha}{|\xi|^{a+1-s}}, \quad |\xi| \geq 1.$$

Nevertheless, the bound of Lemma 5.7 is enough for our purposes in Theorem 5.9. Before that, we need the following inverse Lipschitz estimate of the function  $\frac{x}{|x|^{n+1-s}}$ .

**Lemma 5.8.** *For every  $R_1, R_2 > 0$ , there exists  $m > 0$  such that for all  $x \in B(0, R_1) \setminus \{0\}$  and  $h \in B(0, R_2) \setminus \{x\}$ ,*

$$m|h| \leq \left| \frac{x}{|x|^{n+1-s}} - \frac{x-h}{|x-h|^{n+1-s}} \right|. \quad (34)$$

**Proof.** We divide the proof into four cases, according to the position of the points  $x$  and  $h$ . Let us define  $G(x) = \frac{x}{|x|^{n+1-s}}$ .

*Case 1:*  $2|x| \leq |x-h|$ . Taking

$$m \leq \frac{1 - \frac{1}{2^{n-s}}}{R_1^{n-s} R_2},$$

we have

$$|G(x) - G(x-h)| \geq \frac{1}{|x|^{n-s}} - \frac{1}{|x-h|^{n-s}} \geq \left(1 - \frac{1}{2^{n-s}}\right) \frac{1}{|x|^{n-s}} \geq \frac{1 - \frac{1}{2^{n-s}}}{R_1^{n-s}} \geq m R_2 \geq m|h|.$$

*Case 2:*  $G(x) \cdot G(x-h) \leq 0$ . Taking

$$m \leq \frac{1}{R_1^{n-s} R_2},$$

we have

$$|G(x) - G(x-h)| = (|G(x)|^2 + |G(x-h)|^2 - 2G(x) \cdot G(x-h))^{\frac{1}{2}} \geq |G(x)| = \frac{1}{|x|^{n-s}} \geq \frac{1}{R_1^{n-s}} \geq m R_2 \geq m|h|.$$

*Case 3:*  $|x-h| \leq 2|x|$  and

$$\min\{|G(x)|^2, |G(x-h)|^2\} \leq G(x) \cdot G(x-h). \quad (35)$$

We observe that the inverse of  $G$  is  $G^{-1}(y) = \frac{y}{|y|^{\frac{n+1-s}{n-s}}}$ , with derivative

$$DG^{-1}(y) = |y|^{-\frac{n-s+1}{n-s}} I - \frac{n+1-s}{n-s} |y|^{-\frac{n-s+1}{n-s}-2} y \otimes y,$$

where  $\otimes$  denotes the tensor product, so

$$|DG^{-1}(y)| \leq d_{n,s} |y|^{-\frac{n-s+1}{n-s}} \quad (36)$$

for some constant  $d_{n,s} > 0$ . By the mean value theorem,

$$|h| = |G^{-1}(G(x)) - G^{-1}(G(x-h))| \leq \|DG^{-1}\|_{L^\infty([G(x), G(x-h)])} |G(x) - G(x-h)|. \quad (37)$$

Now, by using (36),

$$\|DG^{-1}\|_{L^\infty([G(x), G(x-h)])} \leq d_{n,s} \max_{y \in [G(x), G(x-h)]} |y|^{-\frac{n-s+1}{n-s}} = d_{n,s} \left( \min_{y \in [G(x), G(x-h)]} |y| \right)^{-\frac{n-s+1}{n-s}}.$$

Elementary geometry (see Figure 2) shows that

$$\min_{y \in [G(x), G(x-h)]} |y| = \begin{cases} |G(x-h)| & \text{if } (G(x-h) - G(x)) \cdot G(x-h) \leq 0, \\ |G(x)| & \text{if } (G(x-h) - G(x)) \cdot G(x) \geq 0. \end{cases} \quad (38)$$

Assumption (35) asserts that one of the two options of (38) occurs, so

$$\min_{y \in [G(x), G(x-h)]} |y| \geq \min\{|G(x-h)|, |G(x)|\}$$

and, hence,

$$\left( \min_{y \in [G(x), G(x-h)]} |y| \right)^{-\frac{n-s+1}{n-s}} \leq (\min\{|G(x-h)|, |G(x)|\})^{-\frac{n-s+1}{n-s}} = \max\{|x|^{n-s+1}, |x-h|^{n-s+1}\}.$$

Finally, since  $|x-h| \leq 2|x|$ ,

$$\max\{|x|^{n-s+1}, |x-h|^{n-s+1}\} \leq 2^{n-s+1} |x|^{n-s+1} \leq 2^{n-s+1} R_1^{n-s+1}. \quad (39)$$

Going back to (37), we find that  $|h| \leq 2^{n-s+1} d_{n,s} R_1^{n-s+1} |G(x) - G(x-h)|$ , so inequality (34) holds for

$$m \leq \frac{1}{2^{n-s+1} d_{n,s} R_1^{n-s+1}}.$$

Case 4:  $|x-h| \leq 2|x|$  and

$$0 < G(x) \cdot G(x-h) < \min\{|G(x)|^2, |G(x-h)|^2\}. \quad (40)$$

Note first that inequality (40) cannot occur in dimension  $n = 1$ .

Let  $\gamma : [0, 1] \rightarrow \mathbb{R}^n$  be any piecewise  $C^1$  curve such that  $\gamma(0) = G(x)$  and  $\gamma(1) = G(x-h)$ . By the fundamental theorem of calculus,

$$|h| = |G^{-1}(\gamma(0)) - G^{-1}(\gamma(1))| = \left| \int_0^1 (G^{-1} \circ \gamma)'(t) dt \right| \leq \max_{\gamma([0,1])} |DG^{-1}| \ell(\gamma), \quad (41)$$

where  $\ell$  denotes the length of the curve.

Assumption (40) implies that none of the cases of (38) occurs (hence none of the situations depicted in Figure 2), but the distance from the origin to the segment  $[G(x), G(x-h)]$  is attained at a point  $P$  in the interior of the segment. Assume that  $|G(x) - P| \leq |G(x-h) - P|$ , although the construction is totally analogous in the symmetric case  $|G(x) - P| \geq |G(x-h) - P|$ . Let  $Q$  be the point in the segment  $[G(x), G(x-h)]$  such that  $P$  is the



**Figure 2:** Position of the points  $G(x)$ ,  $G(x-h)$  and origin  $O$  when  $(G(x-h) - G(x)) \cdot G(x-h) \leq 0$  (left) and when  $(G(x-h) - G(x)) \cdot G(x) \geq 0$  (right).

middle point between  $G(x)$  and  $Q$ . We define the curve  $\gamma$  as follows. The curve  $\gamma$  starts at  $G(x)$  and describes the arc of circumference of centre the origin  $O$  and radius  $|G(x)|$  joining  $G(x)$  with  $Q$ ; among the two possible arcs, we choose that which subtends an angle of less than  $\pi$  radians. Then,  $\gamma$  continues joining  $Q$  and  $G(x-h)$  with a straight line. See Figure 3.

For this particular  $\gamma$ , we estimate the right hand-side of (41). First, by using (36),

$$\max_{\gamma([0,1])} |DG^{-1}| \leq d_{n,s} \max_{y \in \gamma([0,1])} |y|^{-\frac{n-s+1}{n-s}} = d_{n,s} |G(x)|^{-\frac{n-s+1}{n-s}}, \quad (42)$$

since, by construction of  $\gamma$ , the shortest distance of  $\gamma([0,1])$  to the origin is  $|G(x)|$ . To estimate  $\ell(\gamma)$ , let  $\theta$  be the angle  $\widehat{G(x)OP}$  if it is positive, or else the opposite angle  $\widehat{POG(x)}$ , so that

$$\sin \theta = \frac{\ell([G(x), P])}{|G(x)|}$$

and  $\theta \in \left[0, \frac{\pi}{2}\right]$  because  $0 \leq G(x) \cdot G(x-h)$ . Then

$$\ell(\gamma) = 2\theta|G(x)| + \ell([Q, G(x-h)]).$$

Now we use the elementary inequality

$$t \leq \frac{\pi}{2} \sin t, \quad t \in \left[0, \frac{\pi}{2}\right]$$

to obtain that

$$2\theta|G(x)| \leq \pi \sin \theta |G(x)| = \pi \ell([G(x), P]) = \frac{\pi}{2} \ell([G(x), Q]),$$

so

$$\ell(\gamma) \leq \frac{\pi}{2} \ell([G(x), Q]) + \ell([Q, G(x-h)]) \leq \frac{\pi}{2} \ell([G(x), G(x-h)]). \quad (43)$$

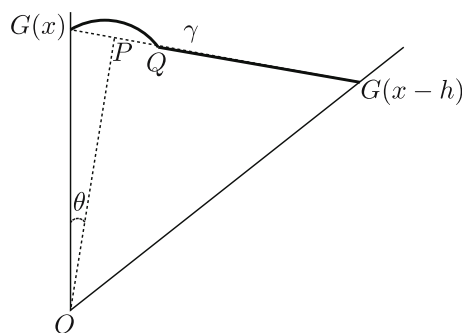
By using (42) and (43), inequality (41) becomes

$$|h| \leq \frac{\pi}{2} d_{n,s} |G(x)|^{-\frac{n-s+1}{n-s}} \ell([G(x), G(x-h)]).$$

If we had assumed  $|G(x) - P| \geq |G(x-h) - P|$  instead of  $|G(x) - P| \leq |G(x-h) - P|$ , we would have obtained

$$|h| \leq \frac{\pi}{2} d_{n,s} |G(x-h)|^{-\frac{n-s+1}{n-s}} \ell([G(x), G(x-h)]),$$

so, in either case,



**Figure 3:** The curve  $\gamma$  (in thick line), the points  $G(x)$ ,  $P$ ,  $Q$ ,  $G(x-h)$  (aligned, in dotted line), the origin  $O$ , and the angle  $\theta$ .

$$\begin{aligned}
|h| &\leq \frac{\pi}{2} d_{n,s} \max \left\{ |G(x)|^{-\frac{n-s+1}{n-s}}, |G(x-h)|^{-\frac{n-s+1}{n-s}} \right\} \ell([G(x), G(x-h)]) \\
&= \frac{\pi}{2} d_{n,s} \max \{|x|^{n-s+1}, |x-h|^{n-s+1}\} |G(x) - G(x-h)|.
\end{aligned}$$

Now we use (39) and find that inequality (34) holds for

$$m \leq \frac{1}{2^{n-s} \pi d_{n,s} R_1^{n-s+1}}. \square$$

Finally, we present the main result of this section: its statement includes that of Proposition 4.4, shows that  $V_\delta^s$  is actually a function, and exhibits its main properties.

**Theorem 5.9.** *There exists a vector radial function  $V_\delta^s \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$  such that*

$$\hat{V}_\delta^s(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi\xi| Q_\delta^s(\xi)}. \quad (44)$$

Furthermore, the following properties hold:

(a)  $V_\delta^s$  is the only  $L_{\text{loc}}^1$  function that satisfies

$$\int_{\mathbb{R}^n} V_\delta^s(z) Q_\delta^s(y-z) dz = \frac{1}{\sigma_{n-1}} \frac{y}{|y|^n}, \quad y \in \mathbb{R}^n \setminus \{0\}. \quad (45)$$

(b) There exists a Lipschitz bounded  $W : \mathbb{R}^n \rightarrow \mathbb{R}^n$  (actually,  $W \in C_0(\mathbb{R}^n, \mathbb{R}^n)$  when  $n \geq 2$ ) such that

$$V_\delta^s(x) = W(x) + \frac{c_{n,s}}{a_0} \frac{x}{|x|^{n+1-s}}. \quad (46)$$

(c) For any  $R > 0$  there exists  $M > 0$  such that for all  $x \in B(0, R) \setminus \{0\}$ ,

$$|V_\delta^s(x)| \leq \frac{M}{|x|^{n-s}}.$$

(d) For any  $R_1, R_2 > 0$  there exists  $M > 0$  such that for all  $x \in B(0, R_1) \setminus \{0\}$  and  $h \in B(0, R_2) \setminus \{x\}$ ,

$$|V_\delta^s(x) - V_\delta^s(x-h)| \leq M \left| \frac{x}{|x|^{n+1-s}} - \frac{x-h}{|x-h|^{n+1-s}} \right|.$$

**Proof.** We first prove that there exists  $V_\delta^s \in C^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^n)$  such that (44) holds.

We start as in the proof of [38, Prop. 2.4.8]. To see that  $V_\delta^s$  is  $C^\infty$  away from the origin we note that  $\mathcal{F}(\hat{V}_\delta^s) = \tilde{V}_\delta^s$  and shall see that  $\mathcal{F}(\hat{V}_\delta^s)$  is  $C^M$  in  $\mathbb{R}^n \setminus \{0\}$  for all  $M$ . Thus, fix  $M \in \mathbb{N}$  and let  $\alpha \in \mathbb{N}^n$  be any multi-index such that

$$s(|\alpha| + 1) - n \geq M. \quad (47)$$

We take  $\varphi \in C^\infty(\mathbb{R}^n)$  such that  $\varphi = 1$  in  $B(0, 2)^c$  and  $\varphi = 0$  in  $B(0, 1)$ . Write

$$u = \hat{V}_\delta^s, \quad u_0 = (1 - \varphi)u \quad \text{and} \quad u_\infty = \varphi u.$$

On the one hand,  $\partial^\alpha u = \partial^\alpha u_0 + \partial^\alpha u_\infty$  in the sense of distributions and also in  $\mathbb{R}^n \setminus \{0\}$ . On the other hand, as  $u$  is smooth outside the origin, we have that  $\partial^\alpha u_\infty$  is smooth and can calculate

$$\partial^\alpha u_\infty = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta} \varphi \partial^\beta u.$$

Write



$$v = \partial^a u_0 + \sum_{\substack{\beta \leq a \\ \beta \neq a}} \binom{a}{\beta} \partial^{a-\beta} \varphi \partial^\beta u.$$

Then  $v$  is a distribution with support in  $B(0, 2)$ , so  $\hat{v}$  is  $C^\infty$ . Moreover,  $\partial^a u = v + \varphi \partial^a u$ . Thus, to see that  $\widehat{\partial^a u}$  is  $C^M$ , it remains to show that  $\widehat{\varphi \partial^a u}$  is  $C^M$ . The function  $\varphi \partial^a u$  is  $C^\infty$  and, by Lemma 5.7,

$$|\varphi(\xi) \partial^a u(\xi)| \leq \frac{C_a}{1 + |\xi|^{s(|a|+1)}}, \quad \xi \in \mathbb{R}^n,$$

Having in mind (47), a classical result relating the decay of a function at infinity with the regularity of its Fourier transform (see, e.g. [38, Exercise 2.4.1]) shows that  $\widehat{\varphi \partial^a u}$  is  $C^M$ .

Once we have shown that  $\widehat{\partial^a u}$  is  $C^M$ , we note that  $\widehat{\partial^a u}(\xi) = (2\pi i \xi)^a \hat{u}(\xi)$ . Let  $\xi \in \mathbb{R}^n \setminus \{0\}$ ; then  $\xi_j \neq 0$  for some  $j \in \{1, \dots, M\}$ . Let  $V$  be a neighbourhood of  $\xi$  such that every  $\eta \in V$  satisfies  $\eta_j \neq 0$ . Let  $m \in \mathbb{N}$  be such that  $s(m+1) - n \geq M$  and let  $\alpha$  be the multi-index  $(0, \dots, 0, m, 0, \dots, 0)$ , with the component  $m$  in position  $j$ . Then  $\alpha$  satisfies (47). Moreover, for any  $\eta \in V$ ,

$$\hat{u}(\eta) = \frac{\widehat{\partial^\alpha u}(\eta)}{(2\pi i \eta_j)^m},$$

so  $\hat{u}$  is of class  $C^M$  in  $\mathbb{R}^n \setminus \{0\}$  for every  $M \in \mathbb{N}$ , and therefore, so is  $V_\delta^s$ .

Once we have that  $V_\delta^s$  is a function, since  $\hat{V}_\delta^s$  is radial and imaginary valued, standard properties of the Fourier transform show that  $V_\delta^s$  must be radial and real valued.

Next, we show that the function

$$Z(\xi) = \hat{V}_\delta^s(\xi) - \frac{-i\xi}{a_0|\xi|} \frac{1}{|2\pi\xi|^s}$$

decays to zero at infinity faster than any negative power of  $|\xi|$ . For this, we observe that

$$Z(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi\xi| \hat{Q}_\delta^s(\xi)} - \frac{-i\xi}{a_0|\xi|} \frac{1}{|2\pi\xi|^s} = -i \frac{\xi}{|\xi|} \frac{a_0 |2\pi\xi|^{-1+s} - \hat{Q}_\delta^s(\xi)}{a_0 |2\pi\xi|^s \hat{Q}_\delta^s(\xi)}. \quad (48)$$

The terms  $|2\pi\xi|^s$  and  $\hat{Q}_\delta^s(\xi)$  in the aforementioned denominator only contribute as a power of  $|\xi|$  in the growth at infinity (Proposition 5.2). Therefore, it remains to show that the aforementioned numerator  $a_0 |2\pi\xi|^{-1+s} - \hat{Q}_\delta^s(\xi)$  decays faster at infinity than any negative power of  $|\xi|$ . Consider a  $\varphi \in C_c^\infty(\mathbb{R}^n)$  with  $\varphi_{B(0, \frac{1}{4})} = 1$  and  $\varphi_{B(0, \frac{1}{2})^c} = 0$ . Then, recalling (21),

$$a_0 |2\pi\xi|^{-1+s} - \hat{Q}_\delta^s(\xi) = \mathcal{F} \left( \frac{a_0}{\gamma(1-s)|x|^{n-1+s}} - Q_\delta^s(x) \right) = \mathcal{F} \left( \frac{a_0 \varphi}{\gamma(1-s)|x|^{n-1+s}} - Q_\delta^s(x) \right) + \mathcal{F} \left( \frac{a_0(1-\varphi)}{\gamma(1-s)|x|^{n-1+s}} \right).$$

Looking at the expression of  $Q_\delta^s$  (Definition 4.1 and Lemma 4.2), we notice that the difference between  $\frac{a_0 \varphi}{\gamma(1-s)|x|^{n-1+s}}$  and  $Q_\delta^s(x)$  coincides with the constant  $\frac{-z_0}{\gamma(1-s)}$  in  $B(0, \min\{b_0\delta, \frac{1}{4}\})$ , and both have a compact support. Therefore, its difference is a smooth function of the compact support. In particular, it is in the Schwartz space, as well as its Fourier transform:

$$\mathcal{F} \left( \frac{a_0 \varphi}{\gamma(1-s)|x|^{n-1+s}} - Q_\delta^s(x) \right) \in \mathcal{S}.$$

On the other hand, the function  $\mathcal{F} \left( \frac{1-\varphi}{\gamma(1-s)|x|^{n-1+s}} \right)$  is treated in [38, Example 2.4.9], and it is shown that its decay at infinity is faster than any negative power of  $|\xi|$ .

From expression (48), we can see that  $Z$  is in  $L_{loc}^1$  when  $n \geq 2$ . Because of its decay at infinity,  $Z$  is in  $L^1$  when  $n \geq 2$ , so it has a Fourier transform  $\hat{Z}$ , which is  $C_0$ . When  $n = 1$ , in Lemma A.1, it will be shown that  $Z$  is a tempered distribution, so it has a Fourier transform  $\hat{Z}$ , which, in principle, is a tempered distribution. But

since both  $\hat{V}_\delta^s(\xi)$  and  $\frac{i\xi}{|\xi|} \frac{1}{|2\pi\xi|^s}$  are Fourier transforms of functions (see Lemma B.1(b) for the latter), we conclude that  $\hat{Z}$  is a function.

We continue by proving that  $\hat{Z}$  is Lipschitz. We have  $-\nabla \hat{Z} = \mathcal{F}(2\pi i \xi Z(\xi))$ . The function  $2\pi i \xi Z(\xi)$  is in  $L^1_{\text{loc}}$ , as can be seen from expression (48). Due to the decay of  $Z$  at infinity,  $2\pi i \xi Z(\xi)$  is in  $L^1(\mathbb{R}^n)$ , so  $\mathcal{F}(2\pi i \xi Z(\xi))$  is bounded, and, hence,  $\hat{Z}$  is Lipschitz.

We define  $W$  as  $W(x) = \hat{Z}(-x)$ . Taking inverse Fourier transforms to the expression

$$\hat{V}_\delta^s(\xi) = Z(\xi) + \frac{-i\xi}{a_0|\xi|} \frac{1}{|2\pi\xi|^s},$$

we obtain equality (46) (see Lemma B.1 for the inverse Fourier transform of the last term). That expression, together with the fact that  $W$  is continuous, shows that  $\hat{V}_\delta^s$  is in  $L^1_{\text{loc}}$ .

To show (a), we note that equality (45) is equivalent to the equality of its Fourier transforms. More precisely, the functions  $V_\delta^s$  and  $Q_\delta^s$  can also be seen as tempered distributions, and, in particular,  $Q_\delta^s$  with compact support. Hence, by Lemmas B.2 and B.1(c), we have that equality (45) is equivalent to

$$\hat{V}_\delta^s(\xi) \hat{Q}_\delta^s(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi\xi|^s}, \quad (49)$$

which holds due to (44). The uniqueness of  $\hat{V}_\delta^s$  also follows from this argument, since  $\hat{V}_\delta^s$  is uniquely determined by equality (49). Thus, (a) is proved.

Fact (b) has been proved when  $n \geq 2$ , while for the case  $n = 1$ , it only remains to show that  $W$  is bounded: this is tackled in Appendix A. With this, we have that given  $R > 0$ , for all  $x \in B(0, R) \setminus \{0\}$ ,

$$|V_\delta^s(x)| \leq \|W\|_{L^\infty(B(0,R))} + \frac{|c_{n,-s}|}{a_0} \frac{1}{|x|^{n-s}} \leq \left( \|W\|_{L^\infty(\mathbb{R}^n)} R^{n-s} + \frac{|c_{n,-s}|}{a_0} \right) \frac{1}{|x|^{n-s}},$$

which shows (c).

As for inequality (d), since  $W$  is Lipschitz, we estimate

$$\begin{aligned} |V_\delta^s(x) - V_\delta^s(x-h)| &\leq \|DW\|_{L^\infty(\mathbb{R}^n)} |h| + \frac{|c_{n,-s}|}{a_0} \left| \frac{x}{|x|^{n+1-s}} - \frac{x-h}{|x-h|^{n+1-s}} \right| \\ &\leq M \left| \frac{x}{|x|^{n+1-s}} - \frac{x-h}{|x-h|^{n+1-s}} \right|, \end{aligned}$$

for a suitable constant  $M > 0$  coming from Lemma 5.8. The proof is complete.  $\square$

Part (b) of Theorem 5.9 shows that  $V_\delta^s$  behaves like  $\frac{x}{|x|^{n+1-s}}$  around 0. It can also be seen that it behaves like  $\frac{x}{|x|^n}$  at infinity. Comparing these facts with the classical and fractional fundamental theorem of calculus (Section 1), we have the following picture of  $V_\delta^s$ : at 0, it behaves like the kernel of the fractional fundamental theorem of calculus, while, at infinity, like that of the classical fundamental theorem of calculus.

## 6 Poincaré, Morrey, Trudinger, and Hardy inequalities

In this section, we will use the non-local fundamental theorem of calculus (Theorem 4.5) to prove inequalities in the spirit of Poincaré-Sobolev, Morrey, Trudinger, and Hardy. In those inequalities, we will need a boundary condition implying that the function vanishes in a tubular neighbourhood of  $\partial\Omega$ .

To describe more precisely that boundary condition, we recall the set  $\Omega_{-\delta} = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}$  and define the subspace  $H_0^{s,p,\delta}(\Omega_{-\delta})$  as the closure of  $C_c^\infty(\Omega_{-\delta})$  in  $H^{s,p,\delta}(\Omega)$ :

$$H_0^{s,p,\delta}(\Omega_{-\delta}) = \overline{C_c^\infty(\Omega_{-\delta})}^{H^{s,p,\delta}(\Omega)}.$$

It is immediate to check that any  $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$  satisfies  $u = 0$  a.e. in  $\Omega_\delta \setminus \Omega_{-\delta}$  and  $D_\delta^s u = 0$  a.e. in  $\Omega_{B,\delta}$ . We leave for a future work the issue of whether  $H_0^{s,p,\delta}(\Omega_{-\delta})$  actually coincides with the set of  $u \in H^{s,p,\delta}(\Omega)$  such that  $u = 0$  a.e. in  $\Omega_\delta \setminus \Omega_{-\delta}$ , so that  $H_0^{s,p,\delta}(\Omega_{-\delta})$  can be regarded as a volumetric-type condition. Finally, given  $g \in H^{s,p,\delta}(\Omega)$ , we define the affine subspace  $H_g^{s,p,\delta}(\Omega_{-\delta})$  as  $g + H_0^{s,p,\delta}(\Omega_{-\delta})$ .

As in the space  $H^{s,p}(\mathbb{R}^n)$  (and  $W^{s,p}$ , too), the Sobolev conjugate exponent of a  $p \in [1, \frac{n}{s})$  is

$$p_s^* = \frac{np}{n - sp}. \quad (50)$$

The Poincaré-Sobolev inequality in  $H_0^{s,p,\delta}(\Omega_{-\delta})$  is as follows. Its analogue in the fractional case can be found in [66, Th. 1.8]. As theirs, our proof is based on (our version of) the non-local fundamental theorem of calculus and the Hardy-Littlewood-Sobolev inequality, but we also take advantage of the comparison between the kernel  $V_\delta^s$  and the Riesz potential given by Theorem 5.9(c).

**Theorem 6.1.** *Let  $1 < p < \infty$  be with  $sp < n$ . Then, there exists  $C > 0$  such that for all  $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$ ,*

$$\|u\|_{L^q(\Omega)} \leq C \|D_\delta^s u\|_{L^p(\Omega)}$$

for every  $q \in [1, p_s^*]$ .

**Proof.** By density, it is enough to prove the inequality for  $u \in C_c^\infty(\Omega_{-\delta})$ .

Fix  $x \in \Omega$  and let  $C > 0$  denote a constant whose value may vary through this process. Notice that  $\text{supp } D_\delta^s u \subset \Omega$  and, by Proposition 4.3,  $D_\delta^s u \in C^\infty(\mathbb{R}^n)$ . By Theorem 4.5 and Proposition 4.4,

$$|u(x)| \leq \int_\Omega |D_\delta^s u(y)| |V_\delta^s(x - y)| dy \leq C \int_\Omega \frac{|D_\delta^s u(y)|}{|x - y|^{n-s}} dy = C(I_s * |D_\delta^s u|)(x), \quad (51)$$

where  $I_s$  is the Riesz potential (21). On the other hand, by the Hardy-Littlewood-Sobolev inequality (e.g. [55, Ch. 4, Th. 2.1]), we have that

$$\|I_s * |D_\delta^s u|\|_{L^{p_s^*}(\mathbb{R}^n)} \leq C \|D_\delta^s u\|_{L^p(\mathbb{R}^n)}.$$

Therefore, for every  $q \in [1, p_s^*]$ ,

$$\|u\|_{L^q(\Omega)} \leq C \|u\|_{L^{p_s^*}(\Omega)} \leq C \|I_s * |D_\delta^s u|\|_{L^{p_s^*}(\mathbb{R}^n)} \leq C \|D_\delta^s u\|_{L^p(\mathbb{R}^n)} = C \|D_\delta^s u\|_{L^p(\Omega)}. \quad \square$$

A non-local Poincaré inequality is obtained as a corollary.

**Theorem 6.2.** *Let  $1 < p < \infty$ . Then there exists  $C > 0$  such that for all  $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$ ,*

$$\|u\|_{L^p(\Omega)} \leq C \|D_\delta^s u\|_{L^p(\Omega)}.$$

**Proof.** If  $sp < n$ , the result is a particular case of Theorem 6.1. If  $sp \geq n$ , we take any  $q$  satisfying

$$1 < q \leq p, \quad sq < n \quad \text{and} \quad p \leq q_s^*, \quad (52)$$

which is easily seen to exist. Indeed, if  $n \geq 2$ , we can take  $q = \frac{np}{n + sp}$ , while if  $n = 1$ , we choose any  $q$  such that

$$1 < q < \frac{1}{s}, \quad \frac{p}{1 + sp} \leq q,$$

which exists because  $1 < \frac{1}{s}$  and  $\frac{p}{1 + sp} < \frac{1}{s}$ .

Once  $q$  is chosen, by Theorem 6.1 and (52), we have for some  $c_1, c_2, c_3 > 0$ ,

$$\|u\|_{L^p(\Omega)} \leq c_1 \|u\|_{L^{q_s^*}(\Omega)} \leq c_2 \|D_\delta^s u\|_{L^q(\Omega)} \leq c_3 \|D_\delta^s u\|_{L^p(\Omega)}. \quad \square$$

Next we introduce a non-local analogue of Morrey's inequality, whose fractional version was shown in [66, Th. 1.11]. Unlike their proof, which uses Morrey-type estimates of the Riesz transform, ours is based on the non-local fundamental theorem of calculus in this context (Theorem 4.5) together with the estimates of the kernel  $V_\delta^s$ .

**Theorem 6.3.** *Let  $1 < p < \infty$  be such that  $sp > n$ . Then there exists  $C > 0$  such that for all  $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$ ,*

$$|u(x) - u(y)| \leq C |x - y|^{s-\frac{n}{p}} \|D_\delta^s u\|_{L^p(\Omega)}, \quad \text{a.e. } x, y \in \Omega \quad (53)$$

and

$$\|u\|_{L^\infty(\Omega)} \leq C \|D_\delta^s u\|_{L^p(\Omega)}. \quad (54)$$

In addition, any  $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$  has a representative that is Hölder continuous of exponent  $s - \frac{n}{p}$ , and the continuous inclusion  $H_0^{s,p,\delta}(\Omega_{-\delta}) \subset C^{0,s-\frac{n}{p}}(\overline{\Omega})$  holds.

**Proof.** The core of the proof consists in showing that

$$|u(x) - u(y)| \leq C |x - y|^{s-\frac{n}{p}} \|D_\delta^s u\|_{L^p(\Omega)}, \quad x, y \in \Omega \quad (55)$$

for all  $u \in C_c^\infty(\Omega_{-\delta})$ . Fix  $x, y \in \Omega$  and  $u \in C_c^\infty(\Omega_{-\delta})$ . By Theorems 4.5 and 5.9 (d), there exists  $C > 0$  such that

$$\begin{aligned} |u(x) - u(y)| &= \left| \int_{\mathbb{R}^n} D_\delta^s u(z) \cdot [V_\delta^s(x - z) - V_\delta^s(y - z)] dz \right| \\ &\leq \int_{\Omega} |V_\delta^s(x - z) - V_\delta^s(y - z)| |D_\delta^s u(z)| dz \\ &\leq C \int_{\Omega} \left| \frac{x - z}{|x - z|^{n+1-s}} - \frac{y - z}{|y - z|^{n+1-s}} \right| |D_\delta^s u(z)| dz. \end{aligned} \quad (56)$$

Now define  $r = |x - y|$ . Continuing with (56), we have

$$\begin{aligned} |u(x) - u(y)| &\leq C \int_{B(x,2r)} |x - z|^{s-n} |D_\delta^s u(z)| dz + C \int_{B(y,2r)} |y - z|^{s-n} |D_\delta^s u(z)| dz \\ &\quad + C \int_{B(x,2r)^c} \left| \frac{x - z}{|x - z|^{n+1-s}} - \frac{y - z}{|y - z|^{n+1-s}} \right| |D_\delta^s u(z)| dz. \end{aligned} \quad (57)$$

For the first term, we have that by Hölder's inequality,

$$\begin{aligned} \int_{B(x,2r)} |x - z|^{s-n} |D_\delta^s u(z)| dz &\leq \left( \int_{B(x,2r)} |x - z|^{(s-n)p'} dz \right)^{\frac{1}{p'}} \left( \int_{B(x,2r)} |D_\delta^s u(z)|^p dz \right)^{\frac{1}{p}} \\ &\leq (2r)^{s-\frac{n}{p}} \left( \frac{\sigma_{n-1}(p-1)}{sp-n} \right)^{\frac{1}{p'}} \|D_\delta^s u\|_{L^p(\mathbb{R}^n)}, \end{aligned} \quad (58)$$

since  $n + (s - n)p' = \frac{sp-n}{p-1} > 0$ . With respect to the second term, we use the inclusion  $B(x, 2r) \subset B(y, 3r)$  and an analogous calculation as in (58) allows us to obtain

$$\int_{B(y,3r)} |y - z|^{s-n} |D_\delta^s u(z)| dz \leq \int_{B(y,3r)} |y - z|^{s-n} |D_\delta^s u(z)| dz \leq (3r)^{s-\frac{n}{p}} \left( \frac{\sigma_{n-1}(p-1)}{sp-n} \right)^{\frac{1}{p'}} \|D_\delta^s u\|_{L^p(\mathbb{R}^n)}. \quad (59)$$

Finally, so as to tackle the last term, by the fundamental theorem of calculus,

$$\begin{aligned}
\left| \frac{x-z}{|x-z|^{n+1-s}} - \frac{y-z}{|y-z|^{n+1-s}} \right| &= \left| \int_0^1 \frac{d}{dt} \left[ \frac{tx + (1-t)y - z}{|tx + (1-t)y - z|^{n+1-s}} \right] dt \right| \\
&= \left| \int_0^1 \frac{x-y}{|tx + (1-t)y - z|^{n+1-s}} \right. \\
&\quad \left. - (n+1-s) \frac{[tx + (1-t)y - z][(tx + (1-t)y - z) \cdot (x-y)]}{|tx + (1-t)y - z|^{n+3-s}} dt \right| \\
&\leq \int_0^1 \left[ \frac{r}{|tx + (1-t)y - z|^{n+1-s}} + (n+1-s) \frac{r}{|tx + (1-t)y - z|^{n+1-s}} \right] dt \\
&= (n+2-s)r \int_0^1 \frac{1}{|tx + (1-t)y - z|^{n+1-s}} dt,
\end{aligned}$$

so

$$\int_{B(x, 2r)^c} \left| \frac{x-z}{|x-z|^{n+1-s}} - \frac{y-z}{|y-z|^{n+1-s}} \right| |D_\delta^s u(z)| dz \leq (n+2-s)r \int_0^1 \int_{B(x, 2r)^c} |tx + (1-t)y - z|^{s-n-1} |D_\delta^s u(z)| dz dt.$$

By Hölder's inequality,

$$\int_{B(x, 2r)^c} |tx + (1-t)y - z|^{s-n-1} |D_\delta^s u(z)| dz \leq \left( \int_{B(x, 2r)^c} |tx + (1-t)y - z|^{(s-n-1)p'} dz \right)^{\frac{1}{p'}} \|D_\delta^s u\|_{L^p(\mathbb{R}^n)}.$$

Since  $B(tx + (1-t)y, r) \subset B(x, 2r)$  for all  $t \in [0, 1]$ , we have

$$\int_{B(x, 2r)^c} |tx + (1-t)y - z|^{(s-n-1)p'} dz \leq \int_{B(tx + (1-t)y, r)^c} |tx + (1-t)y - z|^{(s-n-1)p'} dz = \frac{\sigma_{n-1}}{(n+1-s)p' - n} r^{n+(s-n-1)p'},$$

since  $n + (s-n-1)p' = -\frac{(1-s)p+n}{p-1} < 0$ . Putting together the last three inequalities, we can see that there exists  $\tilde{C} = \tilde{C}(s, n, p) > 0$  such that

$$\int_{B(x, 2r)^c} \left| \frac{x-z}{|x-z|^{n+1-s}} - \frac{y-z}{|y-z|^{n+1-s}} \right| |D_\delta^s u(z)| dz \leq \tilde{C} r^{n+(s-n-1)p' \frac{1}{p} + 1} \|D_\delta^s u\|_{L^p(\mathbb{R}^n)} = \tilde{C} r^{s-\frac{n}{p}} \|D_\delta^s u\|_{L^p(\mathbb{R}^n)}. \quad (60)$$

Then, inequality (55) follows combining (57), (58), (59), and (60), as well as the inclusion  $\text{supp } D_\delta^s u \subset \Omega$ , which implies  $\|D_\delta^s u\|_{L^p(\mathbb{R}^n)} = \|D_\delta^s u\|_{L^p(\Omega)}$ . Once (55) is established, inequality (53) follows from a standard density argument.

To show inequality (54) and the continuous inclusion  $H_0^{s,p,\delta}(\Omega_{-\delta}) \subset C^{0,s-\frac{n}{p}}(\overline{\Omega})$ , by a density argument, it is enough to prove (54) for  $u \in C_c^\infty(\Omega_{-\delta})$ . Let  $x \in \Omega$  and  $x_0 \in \Omega \setminus \Omega_{-\delta}$ . By (55),

$$|u(x)| = |u(x) - u(x_0)| \leq C|x - x_0|^{s-\frac{n}{p}} \|D_\delta^s u\|_{L^p(\Omega)} \leq C(\text{diam } \Omega)^{s-\frac{n}{p}} \|D_\delta^s u\|_{L^p(\Omega)},$$

where  $\text{diam}$  stands for the diameter of a set. The proof is concluded.  $\square$

The limiting case  $sp = n$  is covered by the following version of Trudinger's inequality. Its proof is a straightforward adaptation of the classical one (see, e.g. [37, Th. 7.15]) but using inequality (51). Its fractional version can be found in [66, Th. 1.10]. We denote by  $|\Omega|$  the measure of  $\Omega$ .

**Theorem 6.4.** *Let  $1 < p < \infty$  be such that  $sp = n$ . Then there exist  $c_1, c_2 > 0$  such that for all  $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$ ,*

$$\int_{\Omega} \exp\left(\frac{|u(x)|}{c_1 \|D_{\delta}^s u\|_{L^p(\Omega)}}\right)^{p'} dx \leq c_2 |\Omega|.$$

**Proof.** By a standard density argument, it is enough to prove the inequality for  $C_c^{\infty}(\Omega_{-\delta})$  functions, so let  $u \in C_c^{\infty}(\Omega_{-\delta})$  and set

$$g(x) = \int_{\Omega} \frac{|D_{\delta}^s u(y)|}{|x - y|^{n-s}} dy.$$

By (51),

$$|u| \leq Cg.$$

while by [37, Lemma 7.13], for some constants  $c'_1, c_2 > 0$ ,

$$\int_{\Omega} \exp\left(\frac{g(x)}{c'_1 \|D_{\delta}^s u\|_{L^p(\Omega)}}\right)^{p'} dx \leq c_2 |\Omega|. \quad (61)$$

Putting together these two inequalities, we obtain the conclusion.  $\square$

We mention that the constants  $c'_1, c_2$  of (61) do not depend on  $\Omega$ , but the constant  $C$  of (51) does. That is why the constant  $c_1$  of Theorem 6.4 depends on  $\Omega$ , but not the constant  $c_2$ .

We end this section with the analogue of Hardy's inequality. Its fractional version can be found in [66, Th. 1.9], whose proof (as well as the classical one [73]) is easily adapted to our context.

**Theorem 6.5.** *Let  $1 < p < \infty$  be with  $sp < n$ . Then, there exists  $C > 0$  such that for all  $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$ ,*

$$\left( \int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} dx \right)^{\frac{1}{p}} \leq C \|D_{\delta}^s u\|_{L^p(\Omega)}.$$

**Proof.** As before, it is enough to establish the inequality for  $u \in C_c^{\infty}(\Omega_{-\delta})$ . The proof is just a combination of inequality (51) together with the classical Hardy inequality for Riesz potentials due to Stein and Weiss [73] (see also [66, Lemma 2.8]):

$$\int_{\Omega} \frac{|u(x)|^p}{|x|^{sp}} dx \leq c_1 \int_{\Omega} \frac{(I_s * |D_{\delta}^s u|)(x)^p}{|x|^{sp}} dx \leq c_1 \int_{\mathbb{R}^n} \frac{(I_s * |D_{\delta}^s u|)(x)^p}{|x|^{sp}} dx \leq c_2 \|D_{\delta}^s u\|_{L^p(\mathbb{R}^n)}^p = c_2 \|D_{\delta}^s u\|_{L^p(\Omega)}^p,$$

for some constants  $c_1, c_2 > 0$ .  $\square$

## 7 Compact embeddings

In this section, we will use the non-local fundamental theorem of calculus (Theorem 4.5) to prove compact embeddings of the spaces  $H_g^{s,p,\delta}(\Omega_{-\delta})$  into  $L^q(\Omega)$  spaces.

We start with the following Hölder estimate of the function  $\frac{x}{|x|^{n+1-s}}$ . In fact, this result is included in the proof of [16, Prop. 3.14], but we provide a full proof for the comfort of the reader.

**Lemma 7.1.** *There exists a constant  $C > 0$ , such that for every  $s \in (0, 1)$  and  $h \in \mathbb{R}^n$ ,*

$$\int_{\mathbb{R}^n} \left| \frac{z}{|z|^{n+1-s}} - \frac{z-h}{|z-h|^{n+1-s}} \right| dz \leq \frac{C|h|^s}{s(1-s)}.$$

**Proof.** We first show that there exists a constant  $C > 0$ , such that for every  $s \in (0, 1)$ , we have

$$\int_{\mathbb{R}^n} \left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| dw \leq \frac{C}{s(1-s)}. \quad (62)$$

On the one hand, we have that

$$\int_{B(0,2)} \left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| dw \leq C \int_{B(0,2)} \frac{1}{|w|^{n-s}} dw \leq C \frac{2^s}{s} \leq \frac{C}{s}.$$

On the other hand, for a fixed  $w \in B(0, 2)^c$ ,

$$\begin{aligned} \left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| &= \left| \int_0^1 \frac{d}{dt} \frac{w - te_1}{|w - te_1|^{n+1-s}} dt \right| = \left| \int_0^1 (n+1-s) \frac{[(w - te_1) \cdot e_1](w - te_1)}{|w - te_1|^{n+3-s}} - \frac{e_1}{|w - te_1|^{n+1-s}} dt \right| \\ &\leq C \int_0^1 \frac{1}{|w - te_1|^{n+1-s}} dt. \end{aligned}$$

Now, for  $w \in B(0, 2)^c$  and  $t \in [0, 1]$ , we have

$$|w - te_1| \geq |w| - t \geq |w| - 1 \geq \frac{1}{2}|w|,$$

so

$$\int_0^1 \frac{1}{|w - te_1|^{n+1-s}} dt \leq 2^{n+1-s} \frac{1}{|w|^{n+1-s}} \leq 2^{n+1} \frac{1}{|w|^{n+1-s}}.$$

By integration, we obtain that

$$\int_{B(0,2)^c} \left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| dw \leq C \int_{B(0,2)^c} \frac{1}{|w|^{n+1-s}} dw \leq C \frac{2^{-1+s}}{1-s} \leq \frac{C}{1-s}.$$

This yields (62).

To complete the proof, we take a rotation  $R$  such that  $R^T h = |h|e_1$ . Then, making the change of variables  $z = |h|Rw$  and using (62), we arrive at

$$\int_{\mathbb{R}^n} \left| \frac{z}{|z|^{n+1-s}} - \frac{z - h}{|z - h|^{n+1-s}} \right| dz = |h|^s \int_{\mathbb{R}^n} \left| \frac{w}{|w|^{n+1-s}} - \frac{w - e_1}{|w - e_1|^{n+1-s}} \right| dw \leq |h|^s \frac{C}{s(1-s)}. \quad \square$$

A key ingredient of the desired compactness result is the application of the Fréchet-Kolmogorov criterion, for which the following estimate on the translations is crucial. The analogous result in the Sobolev case is classical [13, Prop. 9.3]. The next result is inspired by [16, Prop. 3.14], where they proved a fractional version when  $p = 1$ .

**Proposition 7.2.**

(a) Let  $1 < p < \infty$ . Then there exists  $C > 0$  such that for all  $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$  and  $h \in \mathbb{R}^n$ ,

$$\left( \int_{\Omega} |u(x+h) - u(x)|^p dx \right)^{\frac{1}{p}} \leq C |h|^s \|D_{\delta}^s u\|_{L^p(\Omega)}. \quad (63)$$

(b) Let  $p = 1$ . Then for all  $M > 0$  there exists  $C > 0$  such that for all  $u \in H_0^{s,p,\delta}(\Omega_{-\delta})$  and  $h \in B(0, M)$ , inequality (63) holds.



**Proof.** By a standard density argument, it is enough to prove the result for  $u \in C_c^\infty(\Omega_{-\delta})$ .

We start with case (a). Let us fix  $M > 0$  such that  $x + h \notin \Omega_{-\delta}$  for all  $x \in \Omega$  and  $h \in B(0, M)^c$ . Then, by Theorem 6.2,

$$\left( \int_{\Omega} |u(x+h) - u(x)|^p dx \right)^{\frac{1}{p}} = \|u\|_{L^p(\Omega)} \leq C \|D_{\delta}^s u\|_{L^p(\Omega)} \leq \frac{C}{M^s} |h|^s \|D_{\delta}^s u\|_{L^p(\Omega)},$$

and the proof is concluded in this case.

In the rest of the proof, we consider  $h \in B(0, M)$ . As  $\text{supp} D_{\delta}^s u \subset \Omega$ , there exists  $R > 0$  such that  $D_{\delta}^s u(x-z) = 0$  for all  $x \in \Omega$  and  $z \in B(0, R)^c$ . Let  $x \in \Omega$ . By Theorem 4.5,

$$\begin{aligned} |u(x+h) - u(x)| &= \left| \int_{\mathbb{R}^n} (V_{\delta}^s(z) - V_{\delta}^s(z+h)) \cdot D_{\delta}^s u(x-z) dz \right| \\ &\leq \int_{B(0,R)} |V_{\delta}^s(z) - V_{\delta}^s(z+h)| |D_{\delta}^s u(x-z)| dz. \end{aligned} \quad (64)$$

By Theorem 5.9 (d), there exists  $C > 0$  such that

$$|V_{\delta}^s(z) - V_{\delta}^s(z+h)| \leq C \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right|, \quad (65)$$

for all  $z \in B(0, R)$ . Thus, applying Hölder's inequality to the right-hand side of (64),

$$\begin{aligned} |u(x+h) - u(x)| &\leq C \left( \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right| |D_{\delta}^s u(x-z)|^p dz \right)^{\frac{1}{p}} \left( \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right| dz \right)^{\frac{1}{p'}} \\ &\leq \left( \frac{C |h|^s}{s(1-s)} \right)^{\frac{1}{p'}} \left( \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right| |D_{\delta}^s u(x-z)|^p dz \right)^{\frac{1}{p}}, \end{aligned}$$

where we have used Lemma 7.1. Next, we integrate and apply Fubini's theorem to obtain

$$\begin{aligned} \int_{\Omega} |u(x+h) - u(x)|^p dx &\leq \left( \frac{C |h|^s}{s(1-s)} \right)^{p/p'} \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right| \int_{\Omega} |D_{\delta}^s u(x-z)|^p dx dz \\ &\leq \left( \frac{C |h|^s}{s(1-s)} \right)^{p/p'+1} \|D_{\delta}^s u\|_{L^p(\mathbb{R}^n)}^p = \left( \frac{C |h|^s}{s(1-s)} \right)^p \|D_{\delta}^s u\|_{L^p(\Omega)}^p, \end{aligned}$$

where we have applied Lemma 7.1 again. This completes the proof of (a).

Now we prove (b) by following the same lines as in case (a). As  $\text{supp} D_{\delta}^s u \subset \Omega$ , there exists  $R > 0$  such that  $D_{\delta}^s u(x-z) = 0$  for all  $x \in \Omega$  and  $z \in B(0, R)^c$ . Fix  $M > 0$  and consider  $h \in B(0, M)$ . Let  $x \in \Omega$ . As in (64)–(65),

$$|u(x+h) - u(x)| \leq C \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right| |D_{\delta}^s u(x-z)| dz.$$

By using Lemma 7.1 we find that, for some  $C_1 > 0$ ,

$$\int_{\Omega} |u(x+h) - u(x)| dx \leq C \int_{B(0,R)} \left| \frac{z}{|z|^{n+1-s}} - \frac{z+h}{|z+h|^{n+1-s}} \right| \int_{\Omega} |D_{\delta}^s u(x-z)| dx dz \leq \frac{C_1 |h|^s}{s(1-s)} \|D_{\delta}^s u\|_{L^1(\Omega)},$$

which concludes the proof.  $\square$

The main result of this section is the following compact embedding, which is an analogue of the Rellich-Kondrachov theorem. See [67, Th. 2.2] for the fractional case; their proof uses Ascoli-Arzelà's theorem to

mollifiers of the sequence, while we prefer to invoke directly the Fréchet-Kolmogorov criterion. Recall the notation  $p_s^*$  from (50).

**Theorem 7.3.** Let  $g \in H^{s,p,\delta}(\Omega)$ . Then, for any sequence  $\{u_j\}_{j \in \mathbb{N}} \subset H^{s,p,\delta}_g(\Omega_{-\delta})$ , such that

$$u_j \rightharpoonup u \quad \text{in } H^{s,p,\delta}(\Omega),$$

for some  $u \in H^{s,p,\delta}(\Omega)$ , one has  $u \in H^{s,p,\delta}_g(\Omega_{-\delta})$  and:

(a) if  $p > 1$ ,

$$u_j \rightarrow u \quad \text{in } L^q(\Omega),$$

for every  $q$  satisfying

$$\begin{cases} q \in [1, p_s^*) & \text{if } sp < n, \\ q \in [1, \infty) & \text{if } sp = n, \\ q \in [1, \infty] & \text{if } sp > n. \end{cases}$$

(b) if  $p = 1$ ,

$$u_j \rightarrow u \quad \text{in } L^1(\Omega).$$

**Proof.** Clearly,  $u \in H^{s,p,\delta}_g(\Omega_{-\delta})$ , since  $H^{s,p,\delta}_g(\Omega_{-\delta})$  is a closed affine subspace of  $H^{s,p,\delta}(\Omega_{-\delta})$ . By linearity, we can assume  $g = 0$ .

The case  $sp > n$  implies  $p > 1$  and follows from Theorem 6.3 and the Ascoli-Arzelà theorem. The case  $sp = n$  reduces to the case  $sp < n$  thanks to Proposition 3.5(a). Thus, we focus on the case  $sp < n$ . Moreover, the case  $q < p$  reduces to the case  $q \geq p$  thanks to the inclusions  $L^{p_2}(\Omega) \subset L^{p_1}(\Omega)$  for all  $1 \leq p_1 \leq p_2$ , so we can assume that  $q \in [p, p_s^*)$ .

Let  $M > 0$  be such that  $\|u_j\|_{H^{s,p,\delta}(\Omega)} \leq M$  for each  $j \in \mathbb{N}$ .

We start with (a), so we assume  $p > 1$ . By Proposition 7.2, we have that for  $j \in \mathbb{N}$  and  $h \in \mathbb{R}^n$ ,

$$\|\tau_h u_j - u_j\|_{L^p(\Omega)} \leq C |h|^s \|D_\delta^s u_j\|_{L^p(\Omega)}, \quad (66)$$

with  $\tau_h u_j = u_j(\cdot - h)$  and some  $C > 0$ . Next, as  $p \leq q < p_s^*$ , we can write

$$\frac{1}{q} = \frac{\alpha}{p} + \frac{1-\alpha}{p_s^*} \quad \text{for some } \alpha \in (0, 1].$$

By using the interpolation inequality, (66), the triangular inequality and Theorem 6.1,

$$\begin{aligned} \|\tau_h u_j - u_j\|_{L^q(\Omega)} &\leq \|\tau_h u_j - u_j\|_{L^p(\Omega)}^\alpha \|\tau_h u_j - u_j\|_{L^{p_s^*}(\Omega)}^{1-\alpha} \\ &\leq (C |h|^s)^\alpha \|D_\delta^s u_j\|_{L^p(\Omega)}^\alpha (2 \|u_j\|_{L^{p_s^*}(\Omega)})^{1-\alpha} \\ &\leq (2C_1)^{1-\alpha} (C |h|^s)^\alpha \|D_\delta^s u_j\|_{L^p(\Omega)} \leq (2C_1)^{1-\alpha} M (C |h|^s)^\alpha, \end{aligned}$$

for some  $C_1 > 0$ . Thus,

$$\limsup_{h \rightarrow 0} \sup_{j \in \mathbb{N}} \|\tau_h u_j - u_j\|_{L^q(\Omega)} = 0.$$

As a result, the Fréchet-Kolmogorov criterion leads to the compactness of  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^q(\Omega)$ .

Now we show (b), so we assume  $p = 1$ . Fix  $M_1 > 0$ . By Proposition 7.2, we have that for  $j \in \mathbb{N}$  and  $h \in B(0, M_1)$

$$\|\tau_h u_j - u_j\|_{L^1(\Omega)} \leq C |h|^s \|D_\delta^s u_j\|_{L^1(\Omega)} \leq CM |h|^s,$$

for some  $C > 0$ . Again the Fréchet-Kolmogorov criterion concludes the compactness of  $\{u_j\}_{j \in \mathbb{N}}$  in  $L^1(\Omega)$ .  $\square$

Of course, Theorem 6.3 yields additionally the compact inclusion of  $H_0^{s,p,\delta}(\Omega_{-\delta})$  into  $C^{0,\beta}(\overline{\Omega})$  for any  $0 < \beta < s - \frac{n}{p}$  under the range  $sp > n$ .

## 8 Existence of minimizers and the Euler-Lagrange equation

In this final section, we prove the existence of minimizers of functionals of the form

$$I(u) = \int_{\Omega} W(x, u(x), D_{\delta}^s u(x)) dx \quad (67)$$

under coercivity and convexity conditions. We also show the corresponding (non-local) Euler-Lagrange equations satisfied by the minimizers.

From now on,  $\mathcal{L}^n$  denotes the Lebesgue sigma-algebra in  $\mathbb{R}^n$ , whereas  $\mathcal{B}$  and  $\mathcal{B}^n$  denote the Borel sigma-algebras in  $\mathbb{R}$  and  $\mathbb{R}^n$ , respectively. The result on the existence of minimizers, which is a standard application of the direct method of the calculus of variations, is as follows.

**Theorem 8.1.** *Let  $1 < p < \infty$ . Let  $u_0 \in H^{s,p,\delta}(\Omega)$ . Let  $W : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R} \cup \{\infty\}$  satisfy the following conditions:*

- (a)  *$W$  is  $\mathcal{L}^n \times \mathcal{B} \times \mathcal{B}^n$ -measurable.*
- (b)  *$W(x, \cdot, \cdot)$  is lower semicontinuous for a.e.  $x \in \Omega$ .*
- (c) *For a.e.  $x \in \Omega$  and every  $y \in \mathbb{R}$ , the function  $W(x, y, \cdot)$  is convex.*
- (d) *There exist  $c > 0$  and  $a \in L^1(\Omega)$  such that*

$$W(x, y, z) \geq a(x) + c|z|^p$$

*for a.e.  $x \in \Omega$ , all  $y \in \mathbb{R}$  and all  $z \in \mathbb{R}^n$ .*

*Define  $I$  as in (67), and assume that  $I$  is not identically infinity in  $H_{u_0}^{s,p,\delta}(\Omega_{-\delta})$ . Then there exists a minimizer of  $I$  in  $H_{u_0}^{s,p,\delta}(\Omega_{-\delta})$ .*

**Proof.** Assumption (d) shows that the functional  $I$  is bounded below by  $\int_{\Omega} a$ . As  $I$  is not identically infinity in  $H_{u_0}^{s,p,\delta}(\Omega_{-\delta})$ , there exists a minimizing sequence  $\{u_j\}_{j \in \mathbb{N}}$  of  $I$  in  $H_{u_0}^{s,p,\delta}(\Omega_{-\delta})$ . Then, assumption (d) implies that  $\{D_{\delta}^s u_j\}_{j \in \mathbb{N}}$  is bounded in  $L^p(\Omega, \mathbb{R}^n)$ . By Theorem 6.2 applied to  $u_j - u_0$ , we obtain that  $\{u_j - u_0\}_{j \in \mathbb{N}}$  and, hence,  $\{u_j\}_{j \in \mathbb{N}}$  are bounded in  $L^p(\Omega)$ . Therefore,  $\{u_j\}_{j \in \mathbb{N}}$  is bounded in  $H^{s,p,\delta}(\Omega)$ . As  $H^{s,p,\delta}(\Omega)$  is reflexive (Proposition 3.4), we can extract a weakly convergent subsequence. By using Theorem 7.3, we obtain that there exists  $u \in H^{s,p,\delta}(\Omega)$  such that for a subsequence (not relabelled),

$$u_j \rightharpoonup u \text{ in } H^{s,p,\delta}(\Omega) \quad \text{and} \quad u_j \rightarrow u \text{ in } L^p(\Omega).$$

Moreover,  $u \in H_{u_0}^{s,p,\delta}(\Omega_{-\delta})$ .

A standard lower semicontinuity result for convex functionals (see, e.g. [32, Th. 7.5]) shows that

$$I(u) \leq \liminf_{j \rightarrow \infty} I(u_j).$$

Therefore,  $u$  is a minimizer of  $I$  in  $H_{u_0}^{s,p,\delta}(\Omega_{-\delta})$  and the proof is concluded.  $\square$

We finally show the Euler-Lagrange equation satisfied by any minimizer. The notation for partial derivatives is as follows:  $D_y W(x, \cdot, z)$  is the derivative of  $W(x, \cdot, z)$ , and  $D_z W(x, y, \cdot)$  is the derivative of  $W(x, y, \cdot)$ .

**Theorem 8.2.** *Let  $1 < p < \infty$ . Let  $u_0 \in H^{s,p,\delta}(\Omega)$ . Let  $W : \Omega \times \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  satisfy the following conditions:*

- (a)  *$W(\cdot, y, z)$  is  $\mathcal{L}^n$ -measurable for each  $y \in \mathbb{R}$  and  $z \in \mathbb{R}^n$ .*
- (b)  *$W(x, \cdot, \cdot)$  is of class  $C^1$  for a.e.  $x \in \Omega$ .*
- (c) *There exist  $c > 0$  and  $a \in L^1(\Omega)$  such that*

$$|W(x, y, z)| + |D_y W(x, y, z)| + |D_z W(x, y, z)| \leq a(x) + c(|y|^p + |z|^p),$$

for a.e.  $x \in \Omega$ , all  $y \in \mathbb{R}$  and all  $z \in \mathbb{R}^n$ .

Define  $I$  as in (67). Let  $u$  be a minimizer of  $I$  in  $H_{u_0}^{s,p,\delta}(\Omega_{-\delta})$ . Then, for every  $\varphi \in C_c^\infty(\Omega_{-\delta})$ ,

$$\int_{\Omega} [D_y W(x, u(x), D_\delta^s u(x)) \varphi(x) + D_z W(x, u(x), D_\delta^s u(x)) \cdot D_\delta^s \varphi(x)] dx = 0. \quad (68)$$

If, in addition,  $D_z W(\cdot, u(\cdot), D_\delta^s u(\cdot)) \in C^1(\overline{\Omega_{-\delta}}, \mathbb{R}^n)$ , then

$$D_y W(x, u(x), D_\delta^s u(x)) = \operatorname{div}_\delta^s D_z W(x, u(x), D_\delta^s u(x)) \quad (69)$$

for a.e.  $x \in \Omega_{-\delta}$ .

**Proof.** By using a standard argument, to show (68), it is enough to check that one can differentiate under the integral sign in the function  $t \mapsto I(u + t\varphi)$ , since  $u + t\varphi \in H_{u_0}^{s,p,\delta}(\Omega_{-\delta})$ . Assumption (c) shows that this is the case (see, e.g. [45, Ch. 13, §2, Lemma 2.2]), so (68) is proved.

Now we make the assumption  $D_z W(\cdot, u(\cdot), D_\delta^s u(\cdot)) \in C^1(\overline{\Omega_{-\delta}}, \mathbb{R}^n)$ . Then there exists a  $C_c^1(\Omega, \mathbb{R}^n)$  extension of this function; we denote by  $W_z$  any such extension. To derive (69) from (68), we use Theorem 3.2 to obtain

$$\int_{\Omega} W_z(x) \cdot D_\delta^s \varphi(x) dx = - \int_{\Omega} \varphi(x) \operatorname{div}_\delta^s W_z(x) dx - (n-1+s) \int_{\Omega} \int_{\Omega_{B,\delta}} \frac{\varphi(x) W_z(y)}{|x-y|} \cdot \frac{x-y}{|x-y|} \rho_\delta(x-y) dy dx.$$

This last integral is zero; indeed,  $\varphi(x) = 0$  for  $x \in \Omega \setminus \Omega_{-\delta}$ , while for  $x \in \Omega_{-\delta}$  and  $y \in \Omega_{B,\delta}$ , we have  $\rho_\delta(x-y) = 0$ . Therefore,

$$\int_{\Omega} W_z(x) \cdot D_\delta^s \varphi(x) dx = - \int_{\Omega} \varphi(x) \operatorname{div}_\delta^s W_z(x) dx.$$

We combine this equality with (68) and apply the fundamental lemma of the calculus of variations to obtain that equality (69) holds for a.e.  $x \in \Omega_{-\delta}$ .  $\square$

Note that (69) imposes an a.e. equality in  $\Omega_{-\delta}$  and not in  $\Omega$ , which is natural since in  $\Omega_\delta \setminus \Omega_{-\delta}$  we already have the condition  $u = u_0$ . Even though (69) prescribes a pointwise condition, it is non-local because of the presence of  $D_\delta^s u$ .

**Acknowledgements:** We thank Davide Barbieri for useful discussions concerning the Fourier transform, and Hidde Schönberger for some comments on a previous version of the manuscript. We also thanks the referees for their comments.

**Funding information:** This work has been supported by the Agencia Estatal de Investigación of the Spanish Ministry of Research and Innovation, through projects PID2020-116207GB-I00 (J.C.B. and J.C.), PID2021-124195NB-C32 and the Severo Ochoa Programme for Centres of Excellence in R&D CEX2019-000904-S (C.M.-C.), by Junta de Comunidades de Castilla-La Mancha through project SBPLY/19/180501/000110 and European Regional Development Fund 2018/11744 (J.C.B. and J.C.), by the Madrid Government (Comunidad de Madrid, Spain) under the multi-annual Agreement with UAM in the line for the Excellence of the University Research Staff in the context of the V PRICIT (Regional Programme of Research and Technological Innovation) (C.M.-C.), by the ERC Advanced Grant 834728 (C.M.-C.), and by Fundación Ramón Areces (J.C.).

**Conflict of interest:** The authors declare no conflict of interest.

## References

- [1] N. Abatangelo and E. Valdinoci, *Getting acquainted with the fractional Laplacian*, in Contemporary research in elliptic PDEs and related topics, vol. 33 Springer INdAM Series, Springer, Cham, 2019, pp. 1–105.
- [2] R. A. Adams, *Sobolev spaces*, Pure and Applied Mathematics, vol. 65, Academic Press, New York-London, 1975.
- [3] B. Aksoylu and T. Mengesha, *Results on non-local boundary value problems*, Numer. Funct. Anal. Optim. **31** (2010), 1301–1317.
- [4] B. Aksoylu and M. L. Parks, *Variational theory and domain decomposition for non-local problems*, Appl. Math. Comput. **217** (2011), 6498–6515.
- [5] F. Andreu, J. M. Mazón, J. D. Rossi, and J. Toledo, *A non-local p-Laplacian evolution equation with non-homogeneous Dirichlet boundary conditions*, SIAM J. Math. Anal. **40** (2009), 1815–1851.
- [6] J. C. Bellido, J. Cueto, and C. Mora-Corral, *Bond-based peridynamics does not converge to hyperelasticity as the horizon goes to zero*, J. Elasticity **141** (2020), 273–289.
- [7] J. C. Bellido, J. Cueto, and C. Mora-Corral, *Fractional Piola identity and polyconvexity in fractional spaces*, Ann. I. H. Poincaré - AN **37** (2020), 955–981.
- [8] J. C. Bellido, J. Cueto, and C. Mora-Corral,  *$\Gamma$ -convergence of polyconvex functionals involving s-fractional gradients to their local counterparts*, Calc. Var. Partial Differential Equations **60** (2021), Paper no. 7, 29.
- [9] J. C. Bellido and C. Mora-Corral, *Existence for non-local variational problems in peridynamics*, SIAM J. Math. Anal. **46** (2014), 890–916.
- [10] J. C. Bellido and C. Mora-Corral, *Lower semicontinuity and relaxation via Young measures for non-local variational problems and applications to peridynamics*, SIAM J. Math. Anal. **50** (2018), 779–809.
- [11] J. C. Bellido, C. Mora-Corral, and P. Pedregal, *Hyperelasticity as a  $\Gamma$ -limit of peridynamics when the horizon goes to zero*, Calc. Var. Partial Differential Equations **54** (2015), 1643–1670.
- [12] F. Bobaru, J. T. Foster, P. H. Geubelle, and S. A. Silling, eds., *Handbook of peridynamic modeling*, Advances in Applied Mathematics, CRC Press, Boca Raton, FL, 2017.
- [13] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.
- [14] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), 1245–1260.
- [15] W. Chen, Y. Li, and P. Ma, *The fractional Laplacian*, World Scientific Publishing Co. Pte. Ltd. Hackensack, NJ, 2020.
- [16] G. E. Comi and G. Stefani, *A distributional approach to fractional Sobolev spaces and fractional variation: existence of blow-up*, J. Funct. Anal. **277** (2019), 3373–3435.
- [17] B. Dacorogna, *Direct methods in the calculus of variations*, Applied Mathematical Sciences, 2nd ed., vol. 78, Springer, New York, 2008.
- [18] M. D’Elia, Q. Du, C. Glusa, M. Gunzburger, X. Tian and Z. Zhou, *Numerical methods for non-local and fractional models*, Acta Numer. **29** (2020), 1–124.
- [19] M. D’Elia, Q. Du and M. D. Gunzburger, *Recent progress in mathematical and computational aspects of peridynamics*, in: Handbook of Nonlocal Continuum Mechanics for Materials and Structures, G. Z. Voyiadjis, ed., Springer, 2019.
- [20] M. D’Elia, M. Gulian, T. Mengesha, and J. M. Scott, *Connections between non-local operators: From vector calculus identities to a fractional Helmholtz decomposition*, Fract. Calc. Appl. Anal. **25** (2022), 2488–2531.
- [21] M. D’Elia, M. Gulian, H. Olson, and G. E. Karniadakis, *Towards a unified theory of fractional and non-local vector calculus*, Fract. Calc. Appl. Anal. **24** (2021), 1301–1355.
- [22] M. D’Elia and M. Gunzburger, *The fractional Laplacian operator on bounded domains as a special case of the non-local diffusion operator*, Comput. Math. Appl. **66** (2013), 1245–1260.
- [23] Q. Du, M. Gunzburger R. B. Lehoucq, and K. Zhou, *Analysis of the volume-constrained peridynamic Navier equation of linear elasticity*, J. Elasticity **113** (2013), 193–217.
- [24] Q. Du, M. Gunzburger, R. B. Lehoucq, and K. Zhou, *A non-local vector calculus, non-local volume-constrained problems, and non-local balance laws*, Math. Models Methods Appl. Sci. **23** (2013), 493–540.
- [25] Q. Du, K. Huang J. Scott, and W. Shen, *A Space-time Nonlocal Traffic Flow Model: Relaxation Representation and Local Limit*, 2022. ArXiv: <http://arXiv.org/abs/2211.00796>.
- [26] Q. Du, T. Mengesha, and X. Tian, *Fractional Hardy-type and trace theorems for non-local function spaces with heterogeneous localization*, Anal. Appl. (Singap.) **20** (2022), 579–614.
- [27] Q. Du and X. Tian, *Stability of non-local Dirichlet integrals and implications for peridynamic correspondence material modeling*, SIAM J. Appl. Math. **78** (2018), 1536–1552.
- [28] J. Duoandikoetxea, *Fourier analysis*, vol. 29 of Graduate Studies in Mathematics, American Mathematical Society, Providence, RI, 2001.
- [29] E. Emmrich and O. Weckner, *On the well-posedness of the linear peridynamic model and its convergence towards the Navier equation of linear elasticity*, Commun. Math. Sci. **5** (2007), 851–864.
- [30] A. Evgrafov and J. C. Bellido, *From non-local Eringen’s model to fractional elasticity*, Math. Mech. Solids, **24** (2019), 1935–1953.
- [31] X. Feng and M. Sutton, *A new theory of fractional differential calculus*, Analysis and Applications **19** (2021), 715–750.
- [32] I. Fonseca and G. Leoni, *Modern Methods in the Calculus of Variations:  $L^p$  Spaces*, Springer, New York, 2007.
- [33] M. Foss, *Nonlocal Poincaré inequalities for integral operators with integrable non-homogeneous kernels*, 2019, ArXiv preprint 1911.10292.

- [34] M. Foss, P. Radu, and Y. Yu, *Convergence analysis and numerical studies for linearly elastic peridynamics with Dirichlet-type boundary conditions*, J. Peridyn. Nonlocal Model. **5** (2023) 275–310.
- [35] M. D. Foss, P. Radu, and C. Wright, *Existence and regularity of minimizers for non-local energy functionals*, Differential Integral Equations **31** (2018), 807–832.
- [36] W. H. Gerstle, *Introduction to practical peridynamics*, Frontier Research in Computation and Mechanics of Materials and Biology, vol. 1, World Scientific Publishing Co Pte. Ltd. Hackensack, NJ, 2016.
- [37] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
- [38] L. Grafakos, *Classical Fourier analysis*, Graduate Texts in Mathematics, 2nd edn., vol. 249, Springer, New York, 2008.
- [39] M. Gunzburger and R. B. Lehoucq, *A non-local vector calculus with application to non-local boundary value problems*, Multiscale Model. Simul. **8** (2010), 1581–1598.
- [40] Z. Han and X. Tian, *Nonlocal Half-ball Vector Operators on Bounded Domains: Poincaré Inequality and Its Applications*, 2022, ArXiv:<http://arXiv.org/abs/2212.13720>.
- [41] B. Hinds and P. Radu, *Dirichlet's principle and wellposedness of solutions for a non-local  $p$ -Laplacian system*, Appl. Math. Comput. **219** (2012), 1411–1419.
- [42] K. Huang and Q. Du, *Asymptotically Compatibility of a Class of Numerical Schemes for a Nonlocal Traffic Flow Model*, 2023, ArXiv:<http://arXiv.org/abs/2301.00803>.
- [43] A. Javili, R. Morasata, E. Oterkus, and S. Oterkus, *Peridynamics Review*, Math. Mech. Solids **24** (2019), 3714–3739.
- [44] C. Kreisbeck and H. Schönberger, *Quasiconvexity in the fractional calculus of variations: characterization of lower semicontinuity and relaxation*, Nonlinear Anal. **215** (2022), 112625, 26.
- [45] S. Lang, *Real Analysis*, 2nd edn., Addison-Wesley, Reading MA, 1983.
- [46] H. Lee and Q. Du, *Nonlocal gradient operators with a nonspherical interaction neighborhood and their applications*, ESAIM Math. Model. Numer. Anal. **54** (2020), 105–128.
- [47] R. B. Lehoucq and S. A. Silling, *Force flux and the peridynamic stress tensor*, J. Mech. Phys. Solids **56** (2008), 1566–1577.
- [48] E. Madenci and E. Oterkus, *Peridynamic Theory and Its Applications*, Springer, New York, NY, 2014.
- [49] C. Martínez, M. Sanz, and F. Periago, *Distributional fractional powers of the Laplacean. Riesz potentials*, Studia Math. **135** (1999), 253–271.
- [50] T. Mengesha, *Nonlocal Korn-type characterization of Sobolev vector fields*, Commun. Contemp. Math. **14** (2012), 1250028, 28.
- [51] T. Mengesha and Q. Du, *Nonlocal constrained value problems for a linear peridynamic Navier equation*, J. Elasticity **116** (2014), 27–51.
- [52] T. Mengesha and Q. Du, *On the variational limit of a class of non-local functionals related to peridynamics*, Nonlinearity **28** (2015), 3999–4035.
- [53] T. Mengesha and Q. Du, *Characterization of function spaces of vector fields and an application in non-linear peridynamics*, Nonlinear Anal. **140** (2016), 82–111.
- [54] T. Mengesha and D. Spector, *Localization of non-local gradients in various topologies*, Calc. Var. Partial Differential Equations **52** (2015), 253–279.
- [55] Y. Mizuta, *Potential Theory in Euclidean Spaces*, GAKUTO International Series, vol. 6, Mathematical Sciences and Applications, Gakkōtoshō Co., Ltd, Tokyo, 1996.
- [56] G. Molica Bisci, V. D. Radulescu and R. Servadei, *Variational methods for non-local fractional problems*, vol. 162 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2016.
- [57] A. C. Ponce, *An estimate in the spirit of Poincaré's inequality*, J. Eur. Math. Soc. **6** (2004), 1–15.
- [58] A. C. Ponce, *A new approach to Sobolev spaces and connections to  $\Gamma$ -convergence*, Calc. Var. Partial Differential Equations **19** (2004), 229–255.
- [59] A. C. Ponce, *Elliptic PDEs, measures and capacities. From the Poisson equations to non-linear Thomas-Fermi problems*, vol. 23 of EMS Tracts in Mathematics European Mathematical Society (EMS), Zürich, 2016.
- [60] C. Pozrikidis, *The Fractional Laplacian*, CRC Press, Boca Raton, FL, 2016.
- [61] J. F. Rodrigues and L. Santos, *On non-local variational and quasi-variational inequalities with fractional gradient*, Appl. Math. Optim. **80** (2019), 835–852.
- [62] X. Ros-Oton, *Nonlocal elliptic equations in bounded domains: A survey*, Publ. Mat. **60** (2016), 3–26.
- [63] X. Ros-Oton and J. Serra, *The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary*, J. Math. Pures Appl. **101** (2014), 275–302.
- [64] A. Schikorra, D. Spector, and J. Van Schaftingen, *An  $L^1$ -type estimate for Riesz potentials*, Rev. Mat. Iberoam. **33** (2017), 291–303.
- [65] J. Scott and T. Mengesha, *A fractional Korn-type inequality*, Discrete Contin. Dyn. Syst. **39** (2019), 3315–3343.
- [66] T.-T. Shieh and D. E. Spector, *On a new class of fractional partial differential equations*, Adv. Calc. Var. **8** (2015), 321–336.
- [67] T.-T. Shieh and D. E. Spector, *On a new class of fractional partial differential equations II*, Adv. Calc. Var. **11** (2018), 289–307.
- [68] S. A. Silling, *Reformulation of elasticity theory for discontinuities and long-range forces*, J. Mech. Phys. Solids **48** (2000), 175–209.
- [69] S. A. Silling, M. Epton, O. Weckner, J. Xu, and E. Askari, *Peridynamic states and constitutive modeling*, J. Elasticity **88** (2007), 151–184.
- [70] S. A. Silling and R. B. Lehoucq, *Peridynamic theory of solid mechanics*, in: Advances in Applied Mechanics, H. Aref and E. van der Giessen, Eds., vol. 44 Elsevier, Amsterdam, 2010, pp. 73–168.
- [71] S. A. Silling, D. J. Littlewood, and P. Seleson, *Variable horizon in a peridynamic medium*, J. Mech. Mater. Struct. **10** (2015), 591–612.

- [72] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970.
- [73] E. M. Stein and G. Weiss, *Fractional integrals on  $n$ -dimensional Euclidean space*, J. Math. Mech. **7** (1958), 503–514.
- [74] Y. Tao, X. Tian, and Q. Du, *Nonlocal models with heterogeneous localization and their application to seamless local-non-local coupling*, Multiscale Model. Simul. **17** (2019), 1052–1075.
- [75] X. Tian and Q. Du, *Analysis and comparison of different approximations to non-local diffusion and linear peridynamic equations*, SIAM J. Numer. Anal. **51** (2013), 3458–3482.
- [76] G. Z. Voyiadjis, ed., *Handbook of Nonlocal Continuum Mechanics for Materials and Structures*, Springer, Cham, 2019.
- [77] M. Šilhavý, *Fractional vector analysis based on invariance requirements (critique of coordinate approaches)*, Contin. Mech. Thermodyn. **32** (2020), 207–228.
- [78] X. Yu, Y. Xu, and Q. Du, *Numerical simulation of singularity propagation modeled by linear convection equations with spatially heterogeneous non-local interactions*, J. Sci. Comput. **92** (2022), Paper No. 59, 24.
- [79] K. Zhou and Q. Du, *Mathematical and numerical analysis of linear peridynamic models with non-local boundary conditions*, SIAM J. Numer. Anal. **48** (2010), 1759–1780.

## Appendix

### $A V_{\delta}^s$ when $n = 1$

In this appendix, we explain the necessary changes in the proof of Theorem 5.9(b) when  $n = 1$ . We first need a result regarding the function  $Z$  appearing therein.

**Lemma A.1.** *Let  $n = 1$ . Then:*

(a) *The function  $Z$  of (48) can be identified with the tempered distribution*

$$\langle Z, \varphi \rangle = \int_0^{\infty} Z(\xi)(\varphi(\xi) - \varphi(-\xi))d\xi, \quad \varphi \in \mathcal{S}, \quad (\text{A1})$$

*and we have the convergence*

$$Z\chi_{B(0,\varepsilon)^c} \rightarrow Z \quad \text{in } \mathcal{S}' \quad \text{as } \varepsilon \rightarrow 0. \quad (\text{A2})$$

*The function*

$$Y(\xi) = -\frac{i\xi}{2\pi|\xi|^2} \frac{1}{\hat{Q}_{\delta}^s(0)} \chi_{B(0,1)}(\xi) \quad (\text{A3})$$

*can be identified with the tempered distribution*

$$\langle Y, \varphi \rangle = \int_0^{\infty} Y(\xi)(\varphi(\xi) - \varphi(-\xi))d\xi, \quad \varphi \in \mathcal{S},$$

*we have the convergence*

$$Y\chi_{B(0,\varepsilon)^c} \rightarrow Y \quad \text{in } \mathcal{S}' \quad \text{as } \varepsilon \rightarrow 0 \quad (\text{A4})$$

*and*

$$\hat{Y}(x) = \frac{-1}{\pi \hat{Q}_{\delta}^s(0)} \int_0^1 \frac{1}{\xi} \sin(2\pi\xi x) d\xi. \quad (\text{A5})$$

**Proof.** We start with (a). Let us see that formula (A1) defines a tempered distribution. By Propositions 5.2 and 5.5, there exists  $C > 0$  such that

$$|Z(\xi)| \leq \frac{1}{2\pi|\xi|} \frac{1}{\hat{Q}_{\delta}^s(\xi)} + \frac{1}{a_0} \frac{1}{|2\pi\xi|^s} \leq \frac{C}{|\xi|}, \quad |\xi| \leq 1.$$

Thus, by the mean value theorem,

$$\left| \int_0^1 Z(\xi)(\varphi(\xi) - \varphi(-\xi))d\xi \right| \leq 2C\|\varphi'\|_{\infty}. \quad (\text{A6})$$

On the other hand, in the proof of Theorem 5.9(b), we saw that  $Z$  decays to 0 at infinity faster than any negative power of  $|\xi|$ . In particular, there exists  $C > 0$  such that

$$|Z(\xi)| \leq \frac{C}{\xi^2}, \quad |\xi| \geq 1.$$

Consequently,



$$\left| \int_1^\infty Z(\xi)(\varphi(\xi) - \varphi(-\xi))d\xi \right| \leq 2C \int_1^\infty \frac{1}{\xi^2} d\xi \|\varphi\|_\infty. \quad (\text{A7})$$

Estimates (A6) and (A7) show that  $Z$  defined by (A1) is in  $S'$ .

As before, the fact that  $Z$  decays to 0 at infinity faster than any negative power of  $|\xi|$  implies that  $Z\chi_{B(0,\varepsilon)^c} \in L^1(\mathbb{R})$  for all  $\varepsilon > 0$ . In particular,  $Z\chi_{B(0,\varepsilon)^c}$  considered as a distribution acts as follows: for each  $\varphi \in S$ ,

$$\langle Z\chi_{B(0,\varepsilon)^c}, \varphi \rangle = \int_{B(0,\varepsilon)^c} Z(\xi) \varphi(\xi) d\xi = \int_\varepsilon^\infty Z(\xi)(\varphi(\xi) - \varphi(-\xi))d\xi.$$

As we saw in (A6)–(A7), the function  $Z(\xi)(\varphi(\xi) - \varphi(-\xi))$  is in  $L^1((0, \infty))$ , so, by dominated convergence, we have that

$$\int_\varepsilon^\infty Z(\xi)(\varphi(\xi) - \varphi(-\xi))d\xi \rightarrow \int_0^\infty Z(\xi)(\varphi(\xi) - \varphi(-\xi))d\xi \quad \text{as } \varepsilon \rightarrow 0,$$

which justifies the identification of the function  $Z$  with the distribution (A1) and shows the convergence (A2).

Thus, (a) is proved. The proof of (b) is analogous, and we only write the details for the expression of  $\hat{Y}$ . The function  $Y\chi_{B(0,\varepsilon)^c}$  is in  $L^1(\mathbb{R}^n)$  for  $0 < \varepsilon < 1$  and

$$\mathcal{F}(Y\chi_{B(0,\varepsilon)^c}) = \int_{B(0,\varepsilon)^c} Y(\xi) e^{-2\pi i x \xi} d\xi = \frac{-1}{\pi \hat{Q}_\delta^s(0)} \int_\varepsilon^1 \frac{1}{\xi} \sin(2\pi \xi x) d\xi,$$

because of odd symmetry. As the function  $\xi \mapsto \frac{1}{\xi} \sin(2\pi \xi x)$  is in  $L^1(0, 1)$ , and convergence (A4) holds, we obtain expression (A5).  $\square$

We are in a position to prove Theorem 5.9(b).

**Proof of Theorem 5.9(b) when  $n = 1$ .** It remains to show that  $W$ , or, equivalently,  $\hat{Z}$  is bounded. To that end, it is useful to introduce the function  $Y$  of (A3) and express

$$\hat{Z} = \mathcal{F}(Z\chi_{B(0,1)} - Y) + \hat{Y} + \mathcal{F}(Z\chi_{B(0,1)^c}).$$

Since  $Z\chi_{B(0,1)^c} \in L^1(\mathbb{R})$ , by the Riemann-Lebesgue Lemma  $\mathcal{F}(Z\chi_{B(0,1)^c}) \in C_0(\mathbb{R})$ . Now we study the function

$$Z\chi_{B(0,1)}(\xi) - Y(\xi) = \left[ -\frac{i\xi}{2\pi |\xi|^2} \left( \frac{1}{\hat{Q}_\delta^s(\xi)} - \frac{1}{\hat{Q}_\delta^s(0)} \right) - \frac{-i\xi}{a_0 |\xi|} \frac{1}{|2\pi \xi|^s} \right] \chi_{B(0,1)}(\xi).$$

Now, as we saw in (29)–(30),

$$\sup_{\xi \in B(0,1)} \left| -\frac{i\xi}{2\pi |\xi|^2} \left( \frac{1}{\hat{Q}_\delta^s(\xi)} - \frac{1}{\hat{Q}_\delta^s(0)} \right) \right| < \infty,$$

and, on the other hand,

$$\left| \frac{-i\xi}{a_0 |\xi|} \frac{1}{|2\pi \xi|^s} \right| \leq \frac{1}{a_0} \frac{1}{|2\pi \xi|^s},$$

which is integrable in  $B(0, 1)$ . Therefore,  $Z\chi_{B(0,1)} - Y \in L^1(\mathbb{R})$ , so  $\mathcal{F}(Z\chi_{B(0,1)} - Y) \in C_0(\mathbb{R})$ .

It remains to show that  $\hat{Y}$  is bounded. By Lemma A.1(b),

$$\hat{Y}(x) = \frac{-1}{\pi \hat{Q}_\delta^s(0)} \int_0^1 \frac{1}{\xi} \sin(2\pi \xi x) d\xi = \frac{-1}{\pi \hat{Q}_\delta^s(0)} \int_0^x \frac{1}{\xi} \sin(2\pi \xi) d\xi.$$

This latter function is known to be bounded. Therefore,  $\hat{Z}$  is bounded, and the proof is concluded.  $\square$

## B Fourier analysis results

In this appendix, we collect several Fourier analysis results needed throughout the article. They are possibly known to experts, but we have not found a precise reference.

First, we compute the Fourier transform of the vectorial version of the Riesz potential.

**Lemma B.1.**

(a) If  $n \geq 2$  and  $0 < n - 1$ , then

$$\mathcal{F}\left(\frac{n - \alpha - 1}{\gamma(1 + \alpha)} \frac{x}{|x|^{n-\alpha+1}}\right)(\xi) = -i \frac{\xi}{|\xi|} |2\pi\xi|^{-\alpha} = -i \frac{\xi}{|\xi|} \hat{I}_\alpha.$$

(b) If  $n = 1$  and  $0 < s < 1$ , then

$$\mathcal{F}\left(c_{1,-s} \frac{x}{|x|^{2-s}}\right) = -\frac{i\xi}{|\xi|} \frac{1}{|2\pi\xi|^s}.$$

(c)  $\mathcal{F}\left(\frac{1}{\sigma_{n-1}} \frac{x}{|x|^n}\right)(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi\xi|^1}.$

**Proof.** Fix  $j \in \{1, \dots, n\}$ . On the one hand, we have that

$$\frac{1}{\gamma(1 + \alpha)} \frac{\partial}{\partial x_j} \frac{1}{|x|^{n-(\alpha+1)}} = -\frac{n - \alpha - 1}{\gamma(1 + \alpha)} \frac{x_j}{|x|^{n-\alpha+1}}.$$

Thus,

$$\frac{1}{\gamma(1 + \alpha)} \mathcal{F}\left(\frac{\partial}{\partial x_j} \frac{1}{|x|^{n-(\alpha+1)}}\right)(\xi) = -\frac{n - \alpha - 1}{\gamma(1 + \alpha)} \mathcal{F}\left(\frac{x_j}{|x|^{n-\alpha+1}}\right)(\xi), \quad (\text{A8})$$

On the other hand, by standard properties of the Fourier transform, and, in particular, by (21),

$$\frac{1}{\gamma(1 + \alpha)} \mathcal{F}\left(\frac{\partial}{\partial x_j} \frac{1}{|x|^{n-(\alpha+1)}}\right)(\xi) = 2\pi i \xi_j \hat{I}_{1+\alpha} = 2\pi i \xi_j |2\pi\xi|^{-(1+\alpha)} = i \frac{\xi_j}{|\xi|} |2\pi\xi|^{-\alpha}. \quad (\text{A9})$$

Putting together (A8) and (A9) we obtain the conclusion of (a).

Now we present the proof of (b). We recall formula (20) of the fractional version of the fundamental theorem of calculus: for every  $u \in C_c^\infty(\mathbb{R})$ ,

$$u(x) = c_{1,-s} \int_{\mathbb{R}} D^s u(y) \frac{x - y}{|x - y|^{2-s}} dy.$$

Next, we take Fourier transform and use the formula for the convolution of a distribution with a Schwartz function:

$$\hat{u}(\xi) = \widehat{D^s u}(\xi) \mathcal{F}\left(c_{1,-s} \frac{x}{|x|^{2-s}}\right) = \frac{2\pi i \xi}{|2\pi\xi|^s} \hat{u}(\xi) \mathcal{F}\left(c_{1,-s} \frac{x}{|x|^{2-s}}\right),$$

where we have used the explicit expression for Fourier transform of  $D^s u$  (see [67, Th. 1.4] or [8, Lemma 3.1]). Now, we multiply both terms by  $-i2\pi\xi$  and obtain that

$$-i2\pi\xi \hat{u}(\xi) = |2\pi\xi|^{1+s} \hat{u}(\xi) \mathcal{F}\left(c_{1,-s} \frac{x}{|x|^{2-s}}\right).$$

Since that equality holds for every  $u \in C_c^\infty(\mathbb{R})$ , statement (b) follows.

Finally, we prove (c). Since  $\frac{x}{|x|^n} \in L^1(B(0, 1)) + L^\infty(B(0, 1)^c)$ , we have that  $\frac{x}{|x|^n}$  belongs to  $\mathcal{S}'$ , and so does its Fourier transform. Let  $\varphi \in C_c^\infty(\mathbb{R}^n)$ . We apply the Fourier transform to the representation formula of Proposition 4.1, obtaining that

$$\hat{\varphi}(\xi) = \widehat{\nabla \varphi}(\xi) \cdot \mathcal{F}\left(\frac{x}{\sigma_{n-1} |x|^n}\right)(\xi) = 2\pi i \xi \hat{\varphi}(\xi) \cdot \mathcal{F}\left(\frac{x}{\sigma_{n-1} |x|^n}\right)(\xi).$$

Since this is true for every  $\varphi \in C_c^\infty(\mathbb{R}^n)$ , we infer that

$$1 = 2\pi i \xi \cdot \mathcal{F}\left(\frac{x}{\sigma_{n-1} |x|^n}\right)(\xi).$$

Therefore, there exists a function  $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that  $\xi \cdot g(\xi) = 0$  and

$$\mathcal{F}\left(\frac{x}{\sigma_{n-1} |x|^n}\right)(\xi) = -i \frac{\xi}{|\xi|} \frac{1}{|2\pi \xi|} + g(\xi).$$

On the other hand,  $\mathcal{F}\left(\frac{x}{\sigma_{n-1} |x|^n}\right)$  must be a vector radial function, as the Fourier transform of a vector radial function. Consequently (recall Definition 2.1), there exists  $\bar{g} : [0, \infty) \rightarrow \mathbb{R}$  such that  $g(\xi) = \xi \bar{g}(|\xi|)$ . Thus,  $|\xi|^2 \bar{g}(|\xi|) = 0$ , so  $\bar{g} = 0$  and, hence,  $g = 0$  a.e. The proof is concluded.  $\square$

Now, we recall the following definitions and properties about the convolution and Fourier transform of tempered distributions. Recall from Section 2.2 the notation  $\tilde{f}$  for the reflection of  $f$ .

**Remark B.1.** Let  $u, v \in \mathcal{S}'$ .

(a)  $\tilde{v} \in \mathcal{S}'$  is defined as follows:

$$\langle \tilde{v}, \varphi \rangle = \langle v, \tilde{\varphi} \rangle \quad \forall \varphi \in \mathcal{S}.$$

(b) Assume that  $\tilde{v} * \varphi \in \mathcal{S}$  for every  $\varphi \in \mathcal{S}$ . Then the tempered distribution  $v * u$  is defined as follows:

$$\langle v * u, \varphi \rangle = \langle u, \tilde{v} * \varphi \rangle \quad \forall \varphi \in \mathcal{S}.$$

(c) We have that  $\mathcal{F}(\tilde{v}) = \tilde{v}$ .

Finally, we show that the product property of the Fourier transform of a convolution also holds for tempered distributions.

**Lemma B.2.** Let  $V, Q \in \mathcal{S}'$  be such that  $Q$  is a distribution with compact support. Then,  $\hat{V}\hat{Q}$  is well defined as a tempered distribution and

$$\widehat{V * Q} = \hat{V}\hat{Q}.$$

**Proof.** Firstly, we recall that the convolution  $V * Q$  is well defined since  $\tilde{Q} * \varphi \in \mathcal{S}$  for every  $\varphi \in \mathcal{S}$  (see [39, Th. 2.3.20]), and its action is defined as in Remark B.1(b):

$$\langle V * Q, \varphi \rangle = \langle V, \varphi * \tilde{Q} \rangle \quad \text{for every } \varphi \in \mathcal{S}.$$

Now, by definition of the Fourier transform in the sense of distributions, for every  $\varphi \in \mathcal{S}$ ,

$$\langle \widehat{V * Q}, \varphi \rangle = \langle V * Q, \hat{\varphi} \rangle = \langle V, \hat{\varphi} * \tilde{Q} \rangle. \quad (\text{A10})$$

Next, by the Fourier transform of a convolution (of a distribution with a Schwartz function),

$$\hat{\varphi} * \tilde{Q} = \mathcal{F}(\varphi \mathcal{F}^{-1}(\tilde{Q})) = \mathcal{F}(\varphi \hat{Q}),$$

since  $\mathcal{F}^{-1}(\tilde{Q}) = \hat{Q}$  (Remark B.1(c)). This also tells us that  $\varphi\hat{Q}$  belongs to  $\mathcal{S}$ , because so does  $\hat{\varphi} * \tilde{Q}$  (by the bijection of the Fourier transform in  $\mathcal{S}$ ). Actually, it is known that  $\hat{Q}$  is a smooth function [39, Th. 2.3.21]. Therefore, continuing with (A10) and using again the duality of the Fourier transform,

$$\langle \widehat{V * Q}, \varphi \rangle = \langle V, \hat{\varphi} * \tilde{Q} \rangle = \langle V, \mathcal{F}(\varphi\hat{Q}) \rangle = \langle \hat{V}, \varphi\hat{Q} \rangle.$$

As  $\varphi\hat{Q} \in \mathcal{S}$ , the product  $\hat{V}\hat{Q}$  is well defined in a distributional sense and

$$\langle \hat{V}\hat{Q}, \varphi \rangle = \langle \hat{V}, \varphi\hat{Q} \rangle = \langle \widehat{V * Q}, \varphi \rangle.$$

Consequently, the desired formula holds. □