Universidad Autónoma de Madrid

Facultad de Ciencias

Departamento de Física Teórica

Ultraviolet Divergences in Higher-Dimensional Field Theories

Antón F. Faedo,

Madrid, Mayo 2009.

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Memoria de Tesis Doctoral realizada por
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presentada ante el Departamento de Física Teórica
de la Universidad Autónoma de Madrid
para la obtención del Título de Doctor en Ciencias.

Tesis Doctoral dirigida por
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Madrid, Mayo 2009.

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Introduction

The origin of four-dimensional gauge symmetries is one of the deepest mysteries of particle physics. The idea of Theodor Kaluza, improved by Oskar Klein in the twenties (cf. for example [1] and references therein) that higher dimensional spacetime symmetries imply low energy gauge symmetries in four dimensions provided the extra dimensions are curled up in an appropriate way has proved quite fruitful and worth pursuing.

In the simplest setting, the Einstein–Hilbert gravitational action defined in a five-dimensional manifold which is a product of four-dimensional Minkowski spacetime with a circle of radius R, looks at energies $E \ll M \equiv \frac{1}{R}$ like a four-dimensional Einstein–Hilbert theory coupled to an Abelian Maxwell field.

With the realization in the early eighties that a consistent string theory will necessarily include extra dimensions the idea of a spacetime with more than four dimensions received a new impulse. In parallel to developments in the fundamental theory, studies along more phenomenological lines have recently lead to new insights on whether and how extra dimensions manifest themselves, and whether and how they may help to solve long standing problems of particle physics, such as the Hierarchy and the Cosmological Constant problems. These phenomenological studies are commonly based on field-theoretical models and treated with the help of the notion of effective field theory.

An important issue in higher dimensional models is the mechanism by means of which extra dimensions are hidden, in the sense that the spacetime we experiment is effectively four-dimensional. Traditionally, extra dimensions are supposed to be compact and with a characteristic size extremely small so that we would need energies unattainable in present colliders in order to directly detect them. Compactness of the extra dimensions allows us to expand fields propagating in the whole spacetime in harmonics and perform integrals over the extra coordinates. In that way we find a four-dimensional theory, but with an infinite number of fields corresponding to modes of the expansion: the so-called Kaluza–Klein modes.

There are then two complementary viewpoints, the higher dimensional one, and the four-dimensional with the Kaluza–Klein tower; and if we want to make explicit statements on exactly when the tower begins to be relevant in an effective treatment, we have to relate not only the classical parts, but also the quantum contributions on both sides. Unfortunately, as it is well known, quantum corrections often lead to divergent values for quantities of the theory that in principle are measurable, making necessary the process of renormalization.

In order to be more precise, if we believe that extra dimensions are *real*, we got to renormalize the theory. Even if we do not embed the extra-dimensional theory in some supposedly consistent framework, such as superstrings, (which would provide a cutoff of sorts), at one loop order, the fact that the higher dimensional (sometimes called the mother) theory is not renormalizable is not directly relevant, in the sense that we still can study and classify all divergences. For example, the six-dimensional electric charge is dimensionful with negative mass dimension, which allows for an unbounded number of candidate counterterms.

However, to any given order in perturbation theory this number is finite, and the theory can in principle be renormalized, although it is still true that always appear new operators in the counterterms which were not present in the original Lagrangian. This is then essentially a low energy approximation, because we can only expect it to be good (in the example of QED_6, in which we are going to concentrate upon) when it is verified that the dimensionless quantity $\alpha_{d=6}E^2 \ll 1$, where $\alpha_{d=6}$ is the six-dimensional fine structure constant. Given the fact that the six and four-dimensional coupling constants are related by $\alpha_{d=6}M^2 \equiv \alpha_{d=4} \equiv \frac{1}{137}$, in terms of the usual four-dimensional fine-structure constant, this means $E \ll \frac{M}{\sqrt{\alpha}} \sim 10M$. It follows that one can compute reliably for energies $E \sim M$, but not much bigger. Our viewpoint will thus be that the theory is *defined* in higher dimensions by means of the necessary counterterms, in a sense that we shall try to make more precise in what follows.

At any rate, and in order to dissipate any doubts, we shall repeat in due course the same analysis on QED_4 on a two-torus. In this case the extra dimensional theory is well defined (forgetting the Landau pole), and our results are essentially the same.

Besides the six-dimensional viewpoint we are going to favor, there is always the possibility of computing directly in four dimensions once the expansion and the integrals over the compact space have been done. It seems quite intuitive that provided we keep track of the infinite set of modes, this four dimensional theory should be equivalent to the full extra-dimensional one; their respective divergences, in particular, should match. The main purpose of this Thesis is to check this intuition with some explicit computations. Although it is not the goal of this work, it should be possible to express our results in the language of effective low energy field theories.

Some steps in this direction have been already given in [2, 3].

Curiously enough, in the case where the fields only interact through the universal coupling to an external gravitational field, the two viewpoints, with some qualifications, are *exactly* equivalent. This was proved by Duff and Toms [4], and provided a strong motivation for our research.

The organization of this Thesis is as follows:

Chapter 1 is a brief history of the extra-dimensional hypothesis in high energy physics, starting in 1914 and culminating at the end of the century when it became an extremely popular paradigm. The organization of the Chapter is based on the logic usually followed by the reviews on the subject, in particular [1, 5, 6]. More phenomenological points of view can be found in [7–13].

Chapter 2 includes some basic notions and formulas of the heat kernel. They will be extensively used in the rest of the Thesis. We also check the validity of our formulas with the well known example of four-dimensional Quantum Electrodynamics.

Chapter 3 reviews an essential result due to Duff and Toms [4] stating that in the case of a free field in curved spacetime, the divergences calculated in the entire manifold and in the dimensionally-reduced with the whole tower of modes can be made to coincide if one is careful enough. We have tried to emphasize the main points in the argumentation in order to compare with more complicated situations.

Chapter 4 contains the main computations of the Thesis. In order to exemplify the case of an interacting theory, we considered six-dimensional Quantum Electrodynamics compactified on a two-torus. We calculated the one-loop counterterms corresponding to both manners of defining the theory and we pointed out the origin of the discrepancies. This Chapter is based on [14].

Chapter 5 explores some possible consequences of the preceding results, in particular the implications for the masses of the Kaluza–Klein modes. We illustrate and interpret several ways of performing the computations from both the higher and the four-dimensional points of view. This Chapter is based on [15].

Chapter 6 finally sets the conclusions and comments on some possible directions to continue.

Introducción

El origen de la simetrías gauge es uno de los mayores misterios de la física de partículas. La idea de Theodor Kaluza, desarrollada por Oskar Klein en los años veinte (véase por ejemplo [1] y las referencias en el mismo) de que las simetrías espaciotemporales en dimensiones superiores implican simetrías gauge cuadridimensionales a bajas energías, siempre que las dimensiones extra estén compactificadas apropiadamente, se ha revelado como muy interesante y fructífera.

En su versión más sencilla, la acción gravitatoria de Einstein–Hilbert definida en una variedad de cinco dimensiones producto de Minkowski cuadridimensional con un círculo de radio R se ve a energías $E \ll M \equiv \frac{1}{R}$ como una teoría de Einstein–Hilbert cuadridimensional acoplada a un campo de Maxwell.

Con el descubrimiento a principios de los ochenta de que una teoría de cuerdas consistente incluye necesariamente dimensiones extra la idea de un espaciotiempo con más de cuatro dimensiones recibió un nuevo impulso. Paralelamente a desarrollos en teorías fundamentales, estudios en una línea más fenomenológica han llevado recientemente a un nuevo entendimiento acerca de cómo las dimensiones extra se manifiestan y de qué manera pueden ayudar a solucionar problemas de la física de partículas, como el problema de las jerarquías o el de la constante cosmológica. Estos estudios fenomenológicos se basan usualmente en modelos considerados como teorías efectivas de campos a bajas energías.

Una cuestión importante en modelos en dimensiones superiores es el mecanismo por el cual las dimensiones extra están escondidas, en el sentido de que el espaciotiempo que experimentamos es cuadridimensional. Tradicionalmente, se supone que estas dimensiones son compactas con un tamaño característico extremadamente pequeño, de manera que se necesitarían energías inalcanzables en los aceleradores actuales para detectarlas directamente. El hecho de que las dimensiones extra sean compactas permite expandir los campos que se propagan en el espaciotiempo completo en armónicos e integrar las coordenadas extra. De ese modo, se encuentra una teoría en cuatro dimensiones pero con un número infinito de campos que corresponden a los modos de la expansión: son los llamados modos de Kaluza–Klein.

Existen entonces dos puntos de vista complementarios, el de la teoría en dimensiones superiores y el cuadridimensional con la torre de Kaluza–Klein; y si se quieren hacer afirmaciones explícitas sobre cuando la torre empieza a ser relevante en un tratamiento efectivo, se tienen que relacionar no solo las partes clásicas sino también las contribuciones cuánticas en ambos lados. Desafortunadamente, como es bien sabido las correcciones cuánticas con frecuencia producen valores divergentes para cantidades que en principio deberían poder medirse, haciendo necesario el proceso de renormalización.

Para ser más precisos, si consideramos que la dimensiones extra son *reales*, es necesario renormalizar la teoría. Incluso si no se embebe el modelo extra dimensional en un marco consistente como supercuerdas (que de todas maneras proporcionaría un cutoff) a orden un loop este hecho no es directamente relevante en el sentido de que aún se pueden estudiar y clasificar todas las divergencias. Por ejemplo, la carga eléctrica en seis dimensiones tiene dimensión de masa negativa, lo que permite un número no acotado de contratérminos.

En cualquier caso, a un orden dado en teoría de perturbaciones este número es finito y la teoría puede ser renormalizada, aunque es cierto que siempre aparecerán nuevos operadores en los contratérminos que no estaban presentes en el Lagrangiano original. Lo que se tiene entonces es esencialmente una aproximación de bajas energías, que solo se espera que sea correcta (en el ejemplo de la Electrodinámica Cuántica en seis dimensiones en el que nos vamos a fijar) cuando la constante de estructura fina verifica $\alpha_{d=6}E^2 \ll 1$. Dado que las constantes de acoplo en cuatro y seis dimensiones están relacionadas por $\alpha_{d=6}M^2 \equiv \alpha_{d=4} \equiv \frac{1}{137}$, en términos de la constante de estructura fina usual esto se traduce en $E \ll \frac{M}{\sqrt{\alpha}} \sim 10M$. Se obtiene entonces que los cálculos son fiables para energías $E \sim M$, pero no mucho mayores. Nuestro punto de vista será entonces que la teoría se define en dimensiones superiores mediante los contratérminos necesarios en un sentido que trataremos de precisar en lo que sigue.

De todas maneras y con el fin de disipar cualquier duda, repetiremos el mismo análisis para la Electrodinámica en cuatro dimensiones compactificada en un toro bidimensional. En este caso la teoría en dimensiones extra está perfectamente definida (obviando el polo de Landau) y nuestros resultados son esencialmente los mismos.

Aparte del punto de vista en seis dimensiones que vamos a favorecer, siempre existe la posibilidad de calcular directamente en cuatro una vez que se ha hecho la expansión en modos y las integrales sobre las dimensiones compactas. Parece natural pensar que siempre que se tengan en cuenta los infinitos modos, esta teoría cuadridimensional debería ser perfectamente equivalente a la teoría en el espaciotiempo completo. En particular sus divergencias deberían coincidir. El objetivo principal de esta Tesis será comprobar esta intuición con algunos cálculos concretos. Aunque no es la meta de este trabajo, nuestros resultados deberían poder expresarse en el lenguaje de teorías efectivas de bajas energías. Algunos pasos en esta dirección han sido dados en [2, 3].

Curiosamente, en el caso de campos libres interaccionando a través del acoplo universal a un campo gravitatorio externo, los dos puntos de vista, bajo algunos supuestos, son *exactamente* equivalentes. Esto fue demostrado por Duff y Toms [4] y motivó en gran medida nuestra investigación.

La Tesis está organizada como sigue:

El **Capítulo 1** es una breve historia de las teorías en dimensiones superiores a cuatro en física de altas energías, comenzando en 1914 y culminando a finales de siglo cuando se convirtieron en un paradigma extremadamente popular. La organización del Capítulo se basa en la que siguen habitualmente los estudios al respecto, particularmente [1, 5, 6]. El aspecto fenomenológico se enfatiza en mayor medida en [7–13].

El **Capítulo 2** incluye algunas nociones básicas así como ecuaciones del heat kernel. Serán utilizadas frecuentemente a lo largo de la Tesis. Adicionalmente se comprueba la validez de las expresiones con el conocido ejemplo de la Electrodinámica Cuántica en cuatro dimensiones.

El **Capítulo 3** es un repaso al resultado, debido a Duff y Toms [4], de que en el caso de un escalar libre en espaciotiempo curvo si uno es suficientemente cuidadoso puede hacer que las divergencias calculadas en la variedad completa y en la reducida con la torre infinita coincidan. Se han tratado de enfatizar los puntos principales de la argumentación para comparar con situaciones más complicadas.

El **Capítulo 4** contiene los principales cálculos de la Tesis. Con el fin de ejemplificar el caso de una teoría con interacciones, se toma la Electrodinámica Cuántica compactificada en un toro bidimensional. Se calculan las divergencias a un loop para ambas formas de definir la teoría y se mencionan las causas de la discrepancia. Este Capítulo se basa en [14].

El **Capítulo 5** explora posibles consecuencias de los resultados anteriores, en particular las implicaciones para las masas de los modos de Kaluza–Klein. Se ilustran e interpretan varias maneras de hacer los cálculos desde ambos puntos de vista. Este Capítulo está basado en [15].

El **Capítulo 6** finalmente establece las conclusiones y apunta posibles direcciones en las que continuar.

1 Historical notes

1.1 The origin of extra dimensions: the Kaluza–Klein idea

The fascinating possibility of increasing the dimensionality of space(time) in order to unify in the same theory different physical phenomena has almost a century. Even before the birth of what we modernly call Kaluza–Klein theories, a major unification had been achieved by means of considering a new dimension. Until the work of Minkowski, it seemed natural to think of time as an external entity not related essentially to the three dimensions of space. In 1909, he showed that there is a natural geometrical interpretation of Maxwell's unified electromagnetic theory along with Special Relativity if space and time were treated on equal footing as part of a four-dimensional manifold. On this manifold, it is straightforward to describe jointly electric and magnetic interactions if both three-dimensional fields are packed together into a single four-dimensional antisymmetric tensor $F_{\mu\nu}$, and the corresponding scalar and vector potentials become the components of a four-vector.

The next step was even more radical in the sense that it was taken to unify two interactions that were not at all related before. Additionally, it postulated an extra dimension besides the usual four of the (recently appeared) spacetime. It was 1914, and Nordström had written a theory in which gravity was explained with a scalar field coupled to the trace of the energy momentum tensor of matter. In a tremendous exercise of imagination, he then wrote a Maxwell like theory in five dimensions, introducing an abelian five-vector field obeying the same equations as the usual electromagnetic field, including a conserved five-current. The extra component corresponding to the fifth dimension was identified with the scalar responsible for gravity while the remaining four components formed the usual electromagnetic potentials. In the particular case in which the fields are independent of the fifth coordinate, five-dimensional Maxwell's equations boil down to the coupled electromagnetic-scalar gravity system in four dimensions. This was the beginning of the extremely interesting higher-dimensional unification idea, in the sense that gravity emerges as the residue of an abelian gauge theory in five dimensions. Curiously, incorporating consistently gravity into the scheme of the rest of the gauge interactions is still nowadays the main motivation for considering higher-dimensional theories in the form of superstrings.

That was of course before Einstein formulated his masterpiece, General Relativity, after which the dynamical nature of spacetime was very well established as described by a metric in a Riemannian manifold. Here is where Kaluza appeared, apparently not aware of Nordström's work. Contrary to him, his goal was to explain four-dimensional electromagnetism as the remnant of gravity in higher dimensions. He demonstrated that General Relativity, when taken in five dimensions and, very importantly, in vacuum, contained the usual four-dimensional gravity coupled to an electromagnetic field that verifies Maxwell's equations. An additional scalar is also generated, but in the original work of Kaluza it was suppressed. Let us show in more detail the mechanism involved. Consider the usual Einstein–Hilbert action but defined in five dimensions

$$S = -\frac{1}{2\kappa_5^2} \int d^5x \sqrt{g}\, R \tag{1.1}$$

where κ_5 is related to a five-dimensional gravitational constant. Notice the essential point that there is no matter action, our physic is described only by pure geometry. Also it is a minimal extension of General Relativity in the sense that we have just written 5 instead of 4 in the volume element. The field equation following from this action is the vanishing of the Einstein tensor[1]

$$G_{MN} = 0 \tag{1.2}$$

Now, we are free to parametrize the 15 components of metric in the following convenient way

$$g_{MN} = \begin{pmatrix} g_{\mu\nu} + \kappa^2\phi^2 A_\mu A_\nu & \kappa\phi^2 A_\mu \\ \kappa\phi^2 A_\nu & \phi^2 \end{pmatrix} \tag{1.3}$$

that is, as 10 components of a four-dimensional metric, 4 components of a four-dimensional vector and a scalar. Moreover, under the assumption that the metric is independent of the coordinate in the extra dimension (the so called cylinder condition) substitution of this ansatz in (1.2) gives the equations

$$\begin{aligned} G_{\mu\nu} &= \frac{\kappa^2\phi^2}{2}\left[g_{\mu\nu}\frac{F^{\rho\sigma}F_{\rho\sigma}}{4} - F_\mu{}^\rho F_{\rho\nu}\right] - \frac{1}{\phi}\left[\nabla_\mu\nabla_\nu\phi - g_{\mu\nu}\nabla^2\phi\right] \\ \nabla_\mu F^{\mu\nu} &= -3\frac{\partial_\mu\phi}{\phi}F^{\mu\nu} \\ \nabla^2\phi &= \frac{\kappa^2\phi^3}{4}F^{\mu\nu}F_{\mu\nu} \end{aligned} \tag{1.4}$$

[1] From now on, capital letters from the middle of the alphabet $M, N, \ldots$ label the entire manifold while greek ones $\mu, \nu, \ldots$ refer to the usual four-dimensional spacetime. For example, in the case at hand $M, N = 0, \ldots, 4$ and $\mu, \nu = 0, \ldots, 3$. For later use let us just mention that small letters from the beginning of the alphabet $a, b, \ldots$ will label the extra dimensions.

where we have defined $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The crucial observation already made by Kaluza is that if one further imposes the field ϕ to be a constant, the last equations turn up to be Einstein's equations in four dimensions coupled to an electromagnetic field that verifies Maxwell's equations. That $g_{\mu\nu}$ is a metric in four dimensions and that A_μ is an abelian gauge field can be seen from the invariance of the five-dimensional Einstein-Hilbert action under linearized diffeomorphisms

$$x^M \to x^M + \xi^M(x^N) \quad \Longrightarrow \quad \delta g_{MN} = \partial_M \xi^R g_{RN} + \partial_N \xi^R g_{MR} + \xi^R \partial_R g_{MN} \tag{1.5}$$

If we take the infinitesimal parameter $\xi^M(x^\nu)$ not to depend on the fifth coordinate, from its components along the usual four dimensions we deduce that the different fields are a metric, a vector and a scalar, that is

$$\begin{aligned} \delta g_{\mu\nu} &= \partial_\mu \xi^\rho g_{\rho\nu} + \partial_\nu \xi^\rho g_{\mu\rho} + \xi^\rho \partial_\rho g_{\mu\nu} \\ \delta A_\mu &= \partial_\mu \xi^\rho A_\rho + \xi^\rho \partial_\rho A_\mu \\ \delta \phi &= \xi^\rho \partial_\rho \phi \end{aligned} \tag{1.6}$$

while, amazingly, the fifth component of the infinitesimal parameter corresponds to an abelian symmetry for the vector

$$\delta A_\mu = \partial_\mu \xi^4 \equiv \partial_\mu \alpha \tag{1.7}$$

This is all the point and the beauty of Kaluza–Klein theory: it is not only that the electromagnetic field appeared from a purely geometrical action involving only a metric, but also that its abelian gauge symmetry is a consequence of invariance under higher-dimensional diffeomorphisms. In this particular sense we say that gravity and gauge theories are unified. The theory admits as ground state solution four-dimensional Minkowski times a circle $M_4 \times S^1$, that is

$$\begin{aligned} < g_{\mu\nu} > &= \eta_{\mu\nu} \\ < A_\mu > &= 0 \\ < \phi > &= \phi_0 \end{aligned} \tag{1.8}$$

so that the symmetry of the vacuum is $U(1)$ times four-dimensional Poincaré[2].

Obviously, the symmetry of the original action is greater than that of four-dimensional gravity plus electromagnetism, since we have imposed all the fields as

[2]The action has an additional global scale invariance broken by the vacuum, so that the scalar degree of freedom is the Goldstone boson of this breaking and thus is massless.

well as the infinitesimal parameter to be independent of the coordinate in the fifth dimension. Additionally, there is an unwanted constraint once we impose the scalar to be a constant: according to the last equation in (1.4) the square of the abelian field strength is forced to vanish. But the most important question should be where is the extra dimension and why we do not experience it neither in everyday life nor in laboratories. Kaluza did not consider these observations a drawback. According to him, the fifth dimension was some sort of mathematical trick or theoretical construct with no physical entity.

The resolution to the problems was given by Klein in 1926. In his revisiting of the question of the extra dimension, he took it as a real part of spacetime but proposed it to be circular, that is, periodic

$$0 \leq y < 2\pi R \tag{1.9}$$

where R is the radius of the circle and y its coordinate. The topology of the entire spacetime was consequently $\mathbb{R}^4 \times S^1$, i.e., there is a circle attached to every point of the usual four dimensions. Then, the cylinder condition was dropped and due to the periodicity of the manifold every field can be expanded in Fourier modes

$$\begin{aligned} g_{\mu\nu}(x^\rho, y) &= \sum_{n=-\infty}^{\infty} g^n_{\mu\nu}(x^\rho)\ e^{in\frac{y}{R}} \\ A_\mu(x^\rho, y) &= \sum_{n=-\infty}^{\infty} A^n_\mu(x^\rho)\ e^{in\frac{y}{R}} \\ \phi(x^\rho, y) &= \sum_{n=-\infty}^{\infty} \phi^n(x^\rho)\ e^{in\frac{y}{R}} \end{aligned} \tag{1.10}$$

with $\phi^{*n}(x^\rho) = \phi^{-n}(x^\rho)$ for real fields and the $n = 0$ mode corresponds to the one verifying cylindricity. Substituting the expansion into the action one is able to perform the integral over the compact dimension, giving rise to a purely four-dimensional theory with an infinite number of fields, the so called Kaluza–Klein towers. Generically, the interactions between the modes are very involved, as we will see in the body of this thesis.

Another important features are the following. Notice that actions in the full five-dimensional manifold do contain derivatives with respect to the fifth coordinate. When expanding the fields as in (1.10), this derivatives yield powers of $\frac{n}{R}$ that after integration are seen in four dimensions as mass terms for the Kaluza–Klein modes

$$m_n^2 = \frac{n^2}{R^2} \tag{1.11}$$

so harmonics other than $n = 0$ are massive, the spectrum being equally spaced. Moreover, reading from the action the interaction of the modes with the four-dimensional photon A^0_μ we can obtain the value of the corresponding charges

$$e_n = n\,\frac{4\sqrt{\pi G_4}}{R} \tag{1.12}$$

where we have defined the usual four-dimensional gravitational coupling as $\kappa_5^2 = 2\pi R\kappa_4^2$ and $\kappa_4^2 = 8\pi G_4$. The remarkable fact is that, like the masses, as a result of the compactness of the fifth dimension, the charges of the Kaluza–Klein modes are quantized in units of the inverse radius. In fact, the fine structure constant can be calculated from the unit of charge

$$\alpha = \frac{e_1^2}{4\pi} = \frac{4G_4}{R^2} \tag{1.13}$$

Were this fine structure constant the one we measure $\alpha \sim \frac{1}{100}$, the corresponding radius would be approximately

$$R = \sqrt{\frac{4G_4}{\alpha}} \sim 20\sqrt{G_4} \sim 20 \times 10^{-19} GeV^{-1} \tag{1.14}$$

that is, around twenty times the Planck length. Incidentally, that would explain why the extra dimension is unobserved: it is far too small to be seen, not only in the experiments of Klein's time but also nowadays and even in the near future. The masses of the Kaluza–Klein modes would also be given by multiples of the Planck mass, so there is no chance to observe the tower beyond the zero mode.

1.2 Non-Abelian generalization

The history of higher-dimensional unification continued after the advent of Yang–Mills theories in 1954 and its subsequent blossoming in the 70's, when it became clear that the structure of the Standard Model is that of a $SU(3) \times SU(2) \times U(1)$ non-Abelian gauge theory. Then, the search for a unified description of all the interactions in terms of merely a gravitational theory had to go beyond five dimensions. It is direct to see that $4+n$-dimensional Einstein's equations in the absence of matter admit as a ground state $M_4 \times T^n$. However, the resulting gauge group is certainly $U(1)^n$, so to have any chance of reproducing the gauge group of the Standard Model additional structure has to be imposed on the extra dimensions. This problem[3] has been studied by many authors, notably [17–21].

[3]The term "problem" is literal, since this natural question seems to have appeared for the first time as an exercise in de Witt's book [16].

Suppose that we have an extra-dimensional space with a metric $\gamma_{ab}(y)$, where $y \equiv y^a$ are the coordinates in the extra space. Moreover, assume that this metric admits a set of N linearly independent Killing vectors, $k_i(y) \equiv k_i^a(y)\partial_a$, so by definition they verify

$$\partial_a k_i^c \gamma_{cb} + \partial_b k_i^c \gamma_{ac} + k_i^c \partial_c \gamma_{ab} = 0 \qquad ; \qquad i = 1, \ldots, N \tag{1.15}$$

and they form the isometry group G of the extra-dimensional manifold

$$[k_i, k_j] = f_{ij}{}^l k_l \tag{1.16}$$

with $f_{ij}{}^l$ the structure constants of G. Recall that for T^n the metric would be $\gamma_{ab} = \delta_{ab}$ and the corresponding isometry group $U(1)^n$. Therefore there is a chance that the isometry group G could be somehow related to the gauge symmetry of the reduced theory. Actually that is the case as deduced generalizing (1.3) to the following reduction ansatz

$$g_{MN} = \begin{pmatrix} g_{\mu\nu}(x) + \gamma_{ab}(y)k_i^a(y)k_j^b(y)A_\mu^i(x)A_\nu^j(x) & \gamma_{ab}(y)k_i^a(y)A_\mu^i(x) \\ \gamma_{ab}(y)k_i^b(y)A_\nu^i(x) & \gamma_{ab}(y) \end{pmatrix} \tag{1.17}$$

Plugging this metric into the $4+n$-dimensional Einstein–Hilbert action and defining suitably both four-dimensional Newton's constant

$$\frac{1}{16\pi G_4} = \frac{1}{16\pi G_{4+n}} \int d^n y \sqrt{\gamma} \equiv \frac{\Omega_n}{16\pi G_{4+n}} \tag{1.18}$$

as well as the Yang–Mills coupling

$$\int d^n y \sqrt{\gamma}\, \gamma_{ab}\, k_i^a\, k_j^b \equiv \frac{16\pi G_4\, \Omega_n}{g^2}\, \delta_{ij} \tag{1.19}$$

one finds the usual four dimensional gravitational theory coupled to pure Yang–Mills with precisely G as gauge group. Once again the non-Abelian gauge symmetry emerges as a consequence of higher dimensional diffeomorphism invariance, as it is easy to see from (1.5) considering as infinitesimal parameter $\xi^R = (\xi^\rho\,,\, \xi^a) = (0\,,\, \epsilon^i(x)k_i^a(y))$. Owing to Killingness of k_i the transformation for the four-dimensional vector results

$$\delta A_\mu^i(x) = \partial_\mu \epsilon^i(x) - f_{jk}{}^i\, A_\mu^j(x)\, \epsilon^k(x) \tag{1.20}$$

as it should for a non-Abelian gauge field. At this point, we have succeeded in obtaining a non-Abelian gauge theory in four dimensions starting from pure gravity in a curved internal manifold and with the guiding principle that the gauge group is identified with the isometry group of the compact space.

There is nevertheless an important drawback in this construction. As we have mentioned in the case of five dimensions, it was trivial to find a ground state solution for Einstein's equation in the form of $M_4 \times S^1$. Unfortunately, in the non-Abelian generalization that is no longer the case: it is not possible to find a solution to the $4+n$-dimensional gravitational equations in vacuum consisting in the product of M_4 and an internal space with Euclidean signature. The argument is very simple and goes as follows. Einstein's equations in the complete manifold, with the addition of a possible cosmological constant, read

$$R_{MN} - \frac{1}{2}\, g_{MN}\ (R + \Lambda) = 0 \tag{1.21}$$

Were the usual four dimensions flat $R_{\mu\nu} = 0$, that would imply also $R + \Lambda = 0$ and therefore the extra-dimensional equations would force

$$R_{ab} = 0 \tag{1.22}$$

But as we have emphasized one needs compact curved extra dimensions with non-trivial isometry group in order to obtain non-Abelian field theories in four dimensions. There are of course ways to circumvent this problem, essentially by altering the equations of motion. One could for example introduce torsion in the problem [22–25] or even adding higher-derivative invariants of the curvature [26]. An important phenomenon uncovered by Cremmer and Scherk [27, 28] is that of "spontaneous compactification" of the extra dimensions, achieved by considering a suitable chosen energy-momentum tensor for matter already in the complete manifold. Thus, scalar and Yang–Mills fields in the higher-dimensional theory support classical solutions in the factorized form M_4 times a compact internal space of constant curvature.

1.3 Kaluza–Klein supergravity

Introducing matter in the complete spacetime was however a step backwards in the Kaluza–Klein program. Nevertheless, it soon became clear that there are ways in which this violation of the "unification philosophy" is minimal, in the sense that a symmetry can be imposed so that the matter and gravity content of the theory are unique. The symmetry we are talking about is of course local supersymmetry, which gives rise to supergravity. Supersymmetry is a global spacetime symmetry that relates in a unique way bosonic and fermionic degrees of freedom. This relation between bosons and fermions fits very well with the kind of ideas we are pursuing since, as far as we now, the quanta interchanged in the interactions are bosonic while usual matter is Fermi like[4]. Having a symmetry relating them is exactly the kind of thing that goes with the unification program.

If one wants to include in the scheme gravity, then it is necessary to make supersymmetry local. The first supergravities had nothing to do with Kaluza–Klein models and started as purely four-dimensional theories [29, 30] in 1976 but quickly jumped to arbitrary higher dimensions in a discipline called "Kaluza–Klein supergravity". In this context, eleven stands out as candidate for the dimensionality of spacetime for several reasons.

First, Nahm showed that eleven was the maximum number of dimensions for a consistent supergravity with only one graviton [31]. The reason behind is that in greater dimensions the fermionic companion of the graviton, called gravitino (a Rarita–Schwinger field of spin $\frac{3}{2}$) has more than 128 degrees of freedom. After reducing to four dimensions compactifying on a torus one would get a supergravity containing fields with spin greater than two that generically are believed to lead to inconsistencies.

On the other hand, Witten provided a phenomenological argument pointing to eleven also as the minimal number of dimensions [32]. He showed that if the gauge group of the Standard Model originates from the isometries of a compact manifold as explained above, then the internal space must be at least seven-dimensional, leading to a complete manifold with eleven or more dimensions. The crucial observation is that there are no manifolds of dimension smaller than four with more than six isometries. Since $SU(3)$ has dimension eight, the minimum space one needs to accommodate it is four-dimensional, like the complex projective space $\mathbb{CP}^2$. The rest of the gauge group can be easily obtained from the isometries of $S^2 \times S^1$, giving a total of seven dimensions. Combining both arguments one is able to single out eleven as the dimensionality of spacetime.

Amazingly, while in lower dimensions there are several possible supergravities, the one constructed by Cremmer, Julia and Scherk [33] in eleven dimensions seems to be unique. In fact, supergravities in lower dimensions can be constructed from eleven-dimensional supergravity by dimensional reduction. The theory contains besides the graviton and the gravitino a three-index gauge field A_{MNR}, that matches the remaining degrees of freedom (128 fermionic of the gravitino correspond to 44 of the metric and 84 of the gauge field). It appears in the action in terms of the field

[4] Of course that is not the case for the hypothetical Higgs boson.

strength

$$F_{MNRS} = \partial_M A_{NRS} + \partial_S A_{MNR} + \partial_R A_{SMN} + \partial_N A_{RSM} \tag{1.23}$$

which has the gauge invariance

$$\delta A_{MNR} = \partial_M \alpha_{NR} + \partial_N \alpha_{RM} + \partial_R \alpha_{MN} \tag{1.24}$$

with α_{MN} antisymmetric.

Finally, Freund and Rubin [34] found in 1980 a mechanism by which four is (almost) selected as the number of non-compact dimensions in eleven-dimensional supergravity. Suppose that we seek a solution to the classical equations of motion in the factorized form $M_{11} = M_d \times M_{11-d}$ with M_d the usual spacetime while M_{11-d} is the compact piece. Suppose moreover that we want M_d to be maximally symmetric. If the field strength F or its dual is to have a non-vanishing vacuum expectation value on M_d without destroying maximality of the symmetry, then the number of indices of F must equal the dimensionality of one of the products, that is, either $d = 4$ or $d = 11 - 4 = 7$. Thus, one can obtain four (or seven) non-compact dimensions using symmetry arguments. It is nevertheless true that there is a degeneracy with the $d = 7$ possibility and therefore some ambiguity in the process.

For all those reasons that we have exposed, that is, unicity of the theory as well as the dimensionality of the spacetime where it lives and the existence of a dynamical mechanism to explain why the observable world is four-dimensional, eleven-dimensional supergravity was in the early 80's a prominent candidate for a "Theory of Everything". Unfortunately it has several blemishes, as was clear already at that time.

Some of the drawbacks were phenomenological, most notably that the original compact manifolds considered by Witten containing the Standard Model in their isometry group are incompatible with supersymmetry and therefore are not relevant for supergravity. Moreover, Witten himself showed that the spectrum of fermions cannot be chiral on any eleven-dimensional manifold that is a product of the usual four-dimensional and a compact seven-dimensional admitting the Standard Model gauge group [32]. On the other hand, the fermionic content of the Standard Model is famous to be chiral. Once again, introducing additional structures can alleviate the tension. One may for example introduce extra gauge fields in the higher dimensional theory in such a way that if the compactification involves topologically non-trivial configurations of the gauge fields then the fermionic spectrum can be chiral [35, 36]. Relaxing the hypothesis of compactness of the extra dimensions can also help [37, 38]. Additionally, the four-dimensional anti-de Sitter space resulting from the compactification mechanism has unrealistically large cosmological constant that must cancel with other sources of vacuum energy, leading to a problem of severe fine-tuning.

The theory also has another problem more related to consistency at the quantum level. It is the presence of ultraviolet divergences that render the theory non-renormalizable. This is usually considered an indication that in any case eleven-dimensional supergravity cannot be the ultimate fundamental description of gravity.

1.4 Superstrings

The majority of these problems are known to be avoided by moving to a theory involving superstrings in ten dimensions. Superstrings are still nowadays the main motivation for going beyond four dimensions. They gained the favor of a considerable part of the theoretical community worried about the "Theory of Everything" after what is called the "first superstring revolution" in the mid 80's. First, Green and Schwarz [39] uncovered that the gravitational and gauge anomalies that plagued ten-dimensional superstring theories all cancel provided the gauge group is either $SO(32)$ or $E_8 \times E_8$. Soon later, Gross, Harvey, Martinec and Rohm [40] constructed the heterotic string with precisely those gauge groups. Finally, Candelas, Horowitz, Strominger and Witten [41] discovered Calabi–Yau compactifications, in particular that the $E_8 \times E_8$ heterotic string admits "spontaneous compactification" to four dimensions on a six-dimensional Calabi–Yau. Interestingly, the gauge theory obtained is based on E_6 and has chiral families of fermions.

It is however fair to admit that a great part of the Kaluza–Klein picture is lost in these theories, since even if extra spacetime dimensions are needed, in fact in a very precise number, the way of getting the gauge group is not from diffeomorphism invariance on the entire manifold. Indeed, the low energy limit of superstrings are particular versions of ten-dimensional supergravities and thus Yang-Mills fields are present already in the ten-dimensional formulation. Constructing compactifications for the superstrings and understanding dualities occupied many of the efforts during the years between revolutions.

Initial excitement about ten-dimensional theories, and the superstrings that provide their ultraviolet completion, ended by the beginning of the 90's. It became clear that there were too many Calabi–Yaus to compactify on, and although none quite gave the Standard Model as four-dimensional gauge theory, it seemed that one could get close with enough effort in many distinct manners. This led nevertheless to a sort of vacuum degeneracy problem.

On the theoretical side, there was a lack of understanding of the theory beyond perturbation theory. In the first years of the 90's several important tools were developed. For example, it became apparent that the various superstring theories were not completely independent but related by "string dualities", some of which relate perturbative physics (that is, weak string coupling) to strongly coupled physics.

The major breakthrough that led to the "second superstring revolution" was Polchinski's realization that obscure string theory objects, called D-branes, are string-theoretical versions of the p-branes that were known in supergravity [42]. The important point was that the knowledge of these p-branes was not restricted to perturbation theory. In fact, due to supersymmetry, p-branes in supergravity were understood well beyond the limits in which string theory was understood. Using this new nonperturbative tool, string theorists were able to show that all of the perturbative string theories were connected by dualities. Moreover, apparently they are different descriptions of a single theory which Witten named M-theory. He also argued that in some particular limit M-theory should be described by the eleven-dimensional supergravity that we have been discussing along with its corresponding 2- and 5-branes.

The D-branes of string theory are extremely important objects. On the formal side, their analysis played a prominent role in the discovery of the AdS/CFT correspondence and the microscopic understanding of the thermodynamic properties of black holes. Furthermore, their name comes from Dirichlet-branes, and that is because open strings, describing for example gauge fields, may have their ends attached to them and verifying Dirichlet boundary conditions. At the end, this means that one can easily confine its gauge fields to live exclusively on such a brane without the need to propagate in the whole higher-dimensional spacetime. In particular, the four-dimensional world we experience may be just a 3-brane embedded in a ten-dimensional spacetime. The applications of this possibility to model-building turned out to be enormous.

1.5 Phenomenology in higher dimensions

In the last years of the 90's and the first of the new century, and leaving apart string theory, the dominant motivation for physicists to study theories in higher dimensions turned to be phenomenological. Traditionally, the philosophy directing the investigation in extra dimensions was that of the old Kaluza-Klein program, that is, unification of all the interactions in a single theory. Then, higher-dimensional theories become more a general arena in which several mechanisms and scenarios were considered to cure some of the problems encountered in particle physics and cosmology.

The kind of ideas discussed in the literature were often influenced by the advances in string theory. We have mentioned the great importance of branes in string theory and beyond (although there were earlier attempts to confine matter on subspaces using much less sophisticated domain-walls [43]). Another example is compactification on orbifolds [44], some sort of manifolds with fixed points. However, the kind of models and phenomena invoked are often considered low energy effective theories rather than definitive, fully consistent theories. That is mainly because it is extremely difficult to write down a renormalizable model in higher dimensions, and ultraviolet divergencies would spoil the consistency. Thus, it is supposed that an ultraviolet completion of the theory exists (essentially string theory) that renders the model coherent.

We will comment in a moment the most popular (indeed we could say "famous") extra-dimensional scenarios. But first, let us review some of the most noticeable problems particle physicists have to face and that the richness of an extra-dimensional point of view may help to alleviate.

Probably the most pressing misunderstood question of the Standard Model is the so called Hierarchy Problem, that is, the extreme weakness of the gravitational interaction with respect to the gauge ones. In other words, we do not comprehend why there is such a large gap in energies between the scale at which gravity is important (for particle physics of course) marked by the Planck scale $M_P = \sqrt{\frac{\hbar c}{G}} \sim 10^{19}\ GeV$ and the scale at which the Electroweak symmetry is broken, around the vacuum expectation value of the Higgs boson $v \sim 246\ GeV$. This leads to the problem of naturalness and fine-tuning of the Higgs mass, i.e., one would expect large quantum corrections to the mass of a scalar such as the Higgs (since it is not protected by any symmetry) that has to be fine-tuned in order to concord with measurements. This problem has directed much of the higher-dimensional model-building effort. There is an associated Little Hierarchy Problem, that is, the discrepancy between the mass of the Higgs and the scale at which new physic is supposed to stabilize it.

As well, there is the hope of understanding the structure behind the parameters of the Standard Model, in the sense that we do not have a clear reason to explain neither the observed hierarchy of masses nor the mixing angles. An additional extremely puzzling discrepancy is the Cosmological Constant Problem, which is also some sort of hierarchy problem between the tiny curvature of spacetime we measure and the large one we would expect from the dynamics of particle physics. In particular, the vacuum energy estimated from zero-point energies of the fields describing the particles and the different symmetry breakings of the Standard Model would produce a Cosmological Constant several tens of orders of magnitude bigger than the one we seem to observe.

Considering higher dimensions offers also new ways out for the problems of models designed to solve the mentioned misunderstandings. For example, supersymmetry in the form of the MSSM is an outstanding candidate to solve the Hierarchy Problem, but there is the obvious drawback that supersymmetry is not an exact symmetry of Nature at the scales we have explored and thus we must break it. We know several extra-dimensional mechanisms that can provide such breaking, notably compactification on orbifolds, the Scherk–Schwarz mechanism [45] and the Hosotani mechanism [46]. To explain in some detail this different ways of symmetry breaking, let us recall how does compactification work in the context of field theories.

Consider a generic $4+n$-dimensional flat space theory

$$S = \int d^{4+n}x \; \mathcal{L}[\phi_i(x)] \tag{1.25}$$

where L is a functional of the several matter fields $\phi_i(x)$ and it must be a scalar of any symmetry in the model. It is said that the theory is compactified on $M_4 \times C$ with C a compact manifold if the coordinates of the entire spacetime can be split as $x^M = (x^\mu, y^a)$ with x^μ describing four-dimensional Minkowski and y^a, $a = 1, \ldots, n$ corresponding to C. After integrating the compact coordinates one ends up with a four-dimensional Lagrangian that describes complicate dynamics of towers of massless and massive fields.

On the other hand, a compact space can be thought as a coset $C = M/G$ where M is a non-compact manifold and G is a discrete group acting freely on M (that is, only the identity in G has fixed points) by operators $\tau_g : M \to M$ for $g \in G$. It is said that M is the covering space of C, and the compact space is constructed by identifying points belonging to the same orbit

$$y \equiv \tau_g(y) \tag{1.26}$$

For example, take $M = \mathbb{R}$ the real numbers and $G = \mathbb{Z}$ the integers. An element of the group is represented by

$$\tau_n(y) = y + 2\pi n R \tag{1.27}$$

Identification of y and $\tau_n(y)$ leads to the circle S^1 with length $2\pi R$ and fundamental domain $[y, y + 2\pi R)$. Now, after the identification, physics should not depend on points in the entire covering space M but only on orbits, i.e., points in C and thus

$$\mathcal{L}[\phi_i(x, y)] = \mathcal{L}[\phi_i(x, \tau_g(y))] \tag{1.28}$$

It is clear that a sufficient condition to verify this last equation is

$$\phi(x, \tau_g(y)) = \phi(x, y) \tag{1.29}$$

Nonetheless, it is not a necessary one. Suppose that the theory has a global or local symmetry and let T_g be an element of the symmetry group. The necessary condition to fulfill (1.28) is in this case

$$\phi(x, \tau_g(y)) = T_g \phi(x, y) \tag{1.30}$$

It can be seen that T_g, often called twists, are also a representation of the group G acting on field space. When all the twists are trivial we have the ordinary compactification, while $T_g \neq \mathbb{I}$ for some g corresponds to Scherk–Schwarz compactification, that can be used for symmetry breaking and mass generation. If the symmetry is a local one the Scherk–Schwarz breaking is equivalent to a Hosotani breaking, where the extra dimensional components of gauge fields acquire a constant background or vacuum expectation value and the symmetry is then broken by a Wilson line.

Compactification on orbifolds can be defined similarly. Let C be a compact manifold and H a discrete group represented by operators $\chi_h : C \to C$ for $h \in H$ but now acting non-freely on C. The resulting space $O = C/H$ that one gets by identifying points which differ by the action of χ_h is not a smooth manifold since it has singularities at the fixed points. Compactification is achieved by requiring that fields evaluated at the points y and $\chi_h(y)$ differ by some transformation Z_h that again must be a symmetry of the theory

$$\phi(x, \chi_h(y)) = Z_h \phi(x, y) \tag{1.31}$$

In the example of the circle, we can consider $H = \mathbb{Z}_2$ and the action of the only non-trivial element (the inversion) is represented by

$$\chi(y) = -y \tag{1.32}$$

which obviously squares to unity $\chi^2 = 1$. This means that for the field space representation

$$\phi(x, \chi(y)) = \phi(x, -y) = Z\phi(x, y) \tag{1.33}$$

it is also satisfied $Z^2 = 1$. Thus, Z is a matrix that can be diagonalized with eigenvalues ± 1 and the resulting orbifold $S^1/\mathbb{Z}_2$ has fixed points that act as co-dimension one boundaries. In this type of compactification, with appropriate boundary conditions, left and right components of a spinor can be chosen not to behave equal at the singular points, therefore projecting out different modes and obtaining chirality. The very same mechanism may be used to induce gauge symmetry breaking choosing boundary conditions acting differently on distinct components of the extra dimensional gauge bosons.

Another very interesting proposals like the idea of the Standard Model group being a subgroup of a larger one such as $SU(5)$ or $SO(10)$, called Grand Unification, also suffer from severe problems (doublet-triplet splitting, conflict with the experimental bounds on proton decay rates...) that can be treated in a higher-dimensional context.

What we are trying to emphasize is that the conceptual and mathematical tools offered by just extending the number of space dimensions makes it possible to rethink several specific problems that up to date have not received a satisfactory answer when considered in a four-dimensional spacetime. Next, we will comment on the most famous scenarios people have studied. The volume of bibliography devoted tu study these models is enormous.

A geometrical reformulation of the hierarchy problem can be given in the context of "large extra dimensions" [47]. In this particular scenario there is only one fundamental energy scale postulated for particle interactions, including gravity, and it is in the range of the TeV. Gravity describes the dynamics of a $D = 4 + n$ dimensional spacetime with n compactified dimensions and the other four describing a 3-brane. By hypothesis the D-dimensional Planck mass is of the order $M_D \sim 1\ TeV$ so that there is no energy gap with respect to the Electroweak scale. All the degrees of freedom of the Standard Model are assumed to be confined to the brane, that is, they do not propagate in the extra dimensions (nevertheless they could feel the compact space trough interaction with gravitons). Because gravity feels the complete D-dimensional manifold, the gravitational potential $V(r)$ between two massive particles at a distance r has two regimes. Let R denote the typical size of the extra dimensions (for example the radius of a torus). Then, for $r << R$, the extra dimensions enter in the computation on the same footing as the non-compact ones and the lines of the gravitational field extend isotropically in all directions, giving

$$V(r) \sim \frac{1}{M_D^{2+n}} \frac{1}{r^{1+n}} \tag{1.34}$$

which is essentially Gauss theorem in D dimensions. Instead, when $r >> R$, the lines are squeezed along the usual four dimensions and we get the usual fall off

$$V(r) \sim \frac{1}{M_D^{2+n} V_n} \frac{1}{r} \tag{1.35}$$

from which one can immediately identify the four-dimensional Planck mass

$$M_P^2 = M_D^{2+n} V_n \tag{1.36}$$

with V_n the volume of the extra dimensions. A huge hierarchy between M_P and $M_D \sim 1\ TeV$ is explained if the volume of the compact space is much larger than naively expected, and thus the name of the scenario. Physically, one may understand the result as the dilution of the graviton wave function into the extra-dimensions. The problem of explaining the hierarchy in energies is traduced into the problem of explaining why the extra dimensions are so large, or in other words why $V_n >> M_D^{-n}$. Due to gravity, this is however a dynamical question since the volume under study is the vacuum expectation value of a field. For instance in a five-dimensional theory, the compactification radius is determined by the dynamics of the radion field $\phi(x)$, which is the purely extra-dimensional component of the metric. This is a general feature of extra-dimensional theories where the size and shape of the compact dimensions are determined by the dynamics of gravity. In string theory it is even a formidable task since the number of such fields, called moduli, is huge, and fixing them to their vacuum value is not a easy problem. It is nevertheless necessary and a delicate question in order to give them an elevated mass such that they could not have been yet observed in experiments.

The hypothesis of large extra dimensions has experimental consequences. If the hierarchy is to be explained by only one extra dimension it would have to measure 10^8 Km, which is of course ruled out. In the case of a two-dimensional space the gap is explained by a radius $R \sim 0.1\ mm$. This size is already accessible to gravitational experiments in the form of deviations from Newton's law due to a Yukawa additional term predicted by large extra dimensions. For instance, in the case of a two torus a bound in the size of the common radius can be given [48]: $2\pi R < 150\ \mu m$ at the 95% confidence level.

There are other kinds of signals that could be explored in colliders [49]. There is the possibility of direct production of Kaluza–Klien gravitons in association with a photon or a jet, giving rise to a signal characterized by missing energy plus a single photon or a jet. A second type of effect is that induced by virtual Kaluza–Klein graviton exchange. Since the amplitudes are ultraviolet divergent, the best that can be done is to parametrize these effects in terms of effective higher-dimensional operators that contribute to processes at LEP and Tevatron. The most restrictive bounds come from astrophysics, though, in particular from processes that can influence supernova formation and the evolution of the daughter neutron star [50]. If we insist in maintaining $M_D \sim 1\ TeV$, these data seem to push the number of extra dimensions to $n \geq 4$ in the particular version of the scenario discussed here.

The other largely studied phenomenological scenario is that of Randall-Sundrum or warped compactification [51]. In this setup, the fifth dimension is an orbifold with branes sitting in both fixed points, say at $y = 0$ and $y = \pi R$, the Standard Model living in this las one. The branes are supposed to carry some tension, i.e., some energy density. The metric solution to the gravitatory equations does no longer factorize, but takes a warped form that is a slice of anti-de Sitter (AdS) in five dimensions

$$ds^2 = e^{-2ky}\, \eta_{\mu\nu} dx^\mu dx^\nu + dy^2 \tag{1.37}$$

where k^{-1} is the AdS radius and $\eta_{\mu\nu}$ is four-dimensional Minkowski, being the metric in every section of constant y, in particular on the branes. The tension on the branes is balanced by a bulk cosmological constant so that a flat solution is supported on them, but at the cost of fine tuning. Once again, it is an easy exercise to deduce the effective four-dimensional Newton's constant which reads

$$M_P^2 = M_D^3\, \frac{1 - e^{-2k\pi R}}{k} \tag{1.38}$$

Notice that the warp factor has taken the place of the volume in large extra dimensions, measured in units used by the observer at the Standard Model brane. Now, since the dependence on the R is exponential, we do not need a huge radius to achieve a large hierarchy between M_P and $M_D \sim 1\ TeV$. If k and M_D are comparable, then it is enough a radius $R \sim 10k^{-1}$. Another important remark is that the limit $R \to \infty$ in (1.38) is smooth, that is, one can remove the Standard Model brane. This means essentially that gravity is localized near the $y = 0$ brane so that, due to the warp factor, the coupling between gravitons and the Standard Model is weak, explaining the apparent hierarchy.

Concerning observational signatures, the astrophysical and cosmological processes that gave restrictive bounds on large extra dimensions are not a problem for warped compactification. The typical explosion of a supernova has characteristic temperature of approximately 50 MeV. The first graviton Kaluza–Klein levels are then kinematically accessible for the compactification scales in large extra dimensions. In the case of Randall-Sundrum, the Kaluza–Klein graviton levels start naturally at the TeV scale, so that bounds coming from supernovae are not relevant. On the other hand, signals at colliders are quite different from those already discussed because the Kaluza–Klein gravitons have couplings suppressed by the TeV scale instead of M_P. Their levels are not uniformly spaced and they are expected to produce resonance enhancements in Drell-Yan processes [52]. The parameter space is explored by searching for heavy graviton decays into dilepton and dijet final states.

The phenomenology of this two particular scenarios is very rich and has been meticulously studied in the literature. We do not intend to enter into more detail nor give the precise bounds encountered. Let us nevertheless mention that there are other possibilities for theories in extra dimensions, the seemingly most simple one is to let fields other than the graviton propagate into the compact manifold. This kind of model is usually called "Universal Extra Dimensions", and permits mechanisms such as gauge-Higgs unification [53, 54] which curiously comes closest to the original idea of Kaluza and Klein. The working hypothesis is that the Higgs field, which is a four dimensional scalar, is formed with the extra-dimensional components of a higher-dimensional gauge field, much like the scalar graviton in the Nordström model. The masses of scalar fields are very sensitive to radiative corrections and as we have mentioned this fact is behind the hierarchy problem. The interesting thing is that gauge particles are famous for being massless fields because of gauge invariance. Thus, being part of a higher-dimensional gauge field protects the mass of the Higgs from dangerous radiative corrections. We mention the case of universal extra dimensions because all the computations presented in this thesis were performed under this particular assumption.

Another characteristic feature of models in which fields propagate in the bulk is that of power-law running of the couplings. It seems an unavoidable consequence of the presence of n extra dimensions that the gauge couplings run with the renormalization scale as μ^2, cf. [55–57]. A simple dimensional analysis argument can be given. Consider for example Quantum Electrodynamics, then in $\overline{MS}$ regularization schemes the β-function is proportional to the number of fields lighter than μ so that just by counting the modes below the scale one is lead to the above result $\beta \sim \mu^n$. When combined with the idea of Grand Unification this opens the exciting possibility of the unification of gauge couplings at accessible energies [58]. Nevertheless the interpretation of this results in a non-renormalizable theory like the one at hand has raised some controversy [2, 59].

The lesson to learn from this section is that, nowadays, the extra-dimensional hypothesis is much more than a requirement for superstring consistency. Considering field theories in higher-dimensions opens a vast amount of model-building possibilities. They are normally considered effective theories with the need of an ultraviolet completion and as so are not intended to be a definitive proposal for a theory of everything but just provide explanations for many open questions.

2 The heat kernel and ζ-functions

Along this thesis, we shall work to one-loop order only since this will be enough to illustrate all the main points in our argumentation. As it is well known, to this order the effective action, after expanding around on-shell background fields, is given in terms of a functional determinant. At one loop the effective action is already divergent, so we shall regularize it through the heat kernel approach, which is very convenient because it respects all gauge invariances, including the geometrical ones. Also the computations are straightforward though frequently lengthy. Let us quickly review our notation and remark on some potential ambiguities. For a classical review on heat kernel techniques (under the name Schwinger–de Witt techniques) see [60]. A more modern view with updated references is [61].

2.1 The heat kernel and effective actions

The geometric setting is given by a Riemannian n-dimensional manifold, with a metric g_{MN}. This manifold will usually be of a factorized form $\mathbb{R}^4 \times K$ where K is a compact $(n-4)$-dimensional manifold , and $\mathbb{R}^4$ represents the Euclidean version of Minkowski space. More generally, like in the models popularized by Randall and Sundrum [51], this structure is present only locally, i.e., we have a fiber bundle (warped space) based on Minkowski.

As we have mentioned, we are exclusively interested in the set of one-loop diagrams. A practical way to implement computations is to shift the fields into classical and quantum parts, where classical fields are forced to satisfy the corresponding equations of motion and the quantum fields are integrated over in the path integral. The action is then expanded around the background of the classical field. Since by assumption the background fields are on-shell, the linear term in quantum fields vanish. The quadratic term gives the propagators for the quantum fields as well as vertices with two quantum fields and other background fields. Higher order terms can be safely neglected because they generate vertices with three or more quantum fields, and so cannot contribute to one-loop graphs. The Gaussian path integral in the generating functional can be performed, giving a functional determinant which defines the effective action

$$W = \frac{1}{2} \log \det \Delta \tag{2.1}$$

where Δ is the operator representing the quadratic part of the action. As we shall see, this functional can be related to the heat kernel. On the other hand, all our operators will enjoy the form

$$\Delta \equiv -D_M D^M + Y \tag{2.2}$$

with the covariant derivative

$$D_M \equiv \partial_M + X_M \tag{2.3}$$

containing both Riemann and gauge (bundle) parts. This form turns out to be extremely general and it is often reachable, possibly after fixing symmetries, for the systems of interest (gauge theories, gravitational theories with diffeomorphism invariance, bosonic strings...). For simplicity we are omitting indices in the internal field space. The quantities Y and X_M generically depend on the background fields. The operator defining the heat kernel satisfies the heat equation

$$\Delta K(\tau) = -\frac{\partial}{\partial \tau} K(\tau) \tag{2.4}$$

and is formally given by

$$K(\tau) \equiv e^{-\tau\Delta} \tag{2.5}$$

acting on a convenient functional space in such a way that

$$(Kf)(x) \equiv \int d^n y \sqrt{|g|}\; K(x, y; \tau)\; f(y) \tag{2.6}$$

The heat kernel function above is similarly a solution to the problem

$$\Delta_x K(x, y; \tau) = -\frac{\partial}{\partial \tau} K(x, y; \tau) \tag{2.7}$$

with the initial condition $K(x, y; \tau = 0) = \delta(x - y)$. Resolving this equation for a generic operator is by no means an easy task. However, for many interesting purposes like computing anomalies and ultraviolet divergences, the exact result is not necessary. The important point is that this kernel function admits a short time off-diagonal [16] expansion defined (for manifolds without boundary) by

$$K(x, y; \tau) = K_0(x, y; \tau) \sum_{p=0}^{\infty} b_{2p}(x, y)\; \tau^p \tag{2.8}$$

where the function $K_0(x, y; \tau)$ is a sort of "reference" heat kernel and the coefficients in the expansion can be viewed as "perturbations" to it. It will turn out that the diagonal part of these coefficients, to be defined below, it is given by a combination of local operators of the appropriate dimension, constructed with Y, X_M and their derivatives in a covariant way. The reference kernel is given in flat space by

$$K_0(x, y; \tau) = \frac{1}{(4\pi\tau)^{\frac{n}{2}}}\; e^{-\frac{\sigma^2}{4\tau}} \tag{2.9}$$

with $\sigma = (x-y)^2$ the distance between the two points and for consistency

$$b_0(x,x) = 1. \tag{2.10}$$

This kernel is a solution to (2.7) when the operator is simply a flat Laplacian. It can be easily generalized to a curved situation by replacing the flat distance defined above with the geodesic distance. The coefficients thus parametrize deviations from this simple Laplacian form. When boundaries are present, odd powers of $\tau^{\frac{1}{2}}$ do appear, which can formally be incorporated in the former expansion by allowing non vanishing odd coefficients, $b_{2p+1}\tau^{p+\frac{1}{2}} \neq 0$.

It is sometimes useful to consider the integrated quantity:

$$Y(\tau, f) \equiv \mathrm{tr}\,(f e^{-\tau\Delta}) = \sum_{k=0}^{\infty} \tau^{\frac{k-n}{2}} a_k(f) \tag{2.11}$$

where the trace involves whatever finite rank indices the operator might posses, and

$$a_k(f) = \frac{1}{(4\pi)^{\frac{n}{2}}} \int d^n x \sqrt{|g|}\ \mathrm{tr}\, b_k(x,x)\, f(x) \tag{2.12}$$

Once again, in the absence of a boundary odd coefficients a_{2p+1} vanish. The mass dimension of a_k is $k-n$, whereas the one of b_k is simply k. It follows that

$$a_0 = \frac{\mathrm{tr}\ \mathbb{I}}{(4\pi)^{\frac{n}{2}}}\ V \equiv \frac{1}{(4\pi)^{\frac{n}{2}}} \int d^n x \sqrt{|g|}\ \mathrm{tr}\, \mathbb{I} \tag{2.13}$$

As usual, we shall denote

$$a_k \equiv a_k(f=1) \tag{2.14}$$

Note in particular that

$$Y(\tau) \equiv Y(\tau, f=1) = \mathrm{tr}\ e^{-\tau\Delta} = \sum_{k=0}^{\infty} \tau^{\frac{k-n}{2}} a_k \tag{2.15}$$

After all these prolegomena, the determinant is defined as:

$$\log\ \det\ \Delta = -\int_0^{\infty} \frac{d\tau}{\tau^{1+n/2}} \sum_{p=0}^{\infty} a_p\, \tau^{p/2} \tag{2.16}$$

This definition may seem a bit arbitrary. It is nevertheless motivated by the following consideration. For each positive eigenvalue λ_n of the operator, up to an infinite constant not dependent of λ_n and thus irrelevant we have

$$\log \lambda_n = -\int_0^{\infty} \frac{d\tau}{\tau}\ e^{-\tau\lambda_n} \tag{2.17}$$

If we use the famous identity $\log\det\Delta = \mathrm{Tr}\log\Delta$ and extend the integral above to the whole operator, substituting the expansion then we get naturally (2.16). The effective action defined in this way is potentially divergent. There are several possible viewpoints on this question. One of them is to analytically continue on the dimension n. The integral over the proper time τ, cut off in the infrared by $\tau_{max} = \mu^{-2}$ produces poles in the complex variable n, given by:

$$\log\,\det\,\Delta = -\sum_{p=0}^{\infty} a_p\,\frac{2\mu^{n-p}}{p-n} + \text{finite part.} \tag{2.18}$$

which when n approaches the physical dimension, say d,

$$n = d + \epsilon \tag{2.19}$$

yields the divergent piece of the determinant (a dimensionless quantity):

$$\log\,\det\,\Delta|_{div} = \frac{2\mu^{\epsilon}}{\epsilon}\,a_d(\Delta). \tag{2.20}$$

Recall that in the absence of a boundary $a_{2p+1} = 0$, and this particular prescription yields a finite answer for odd dimensions. It is completely equivalent to perform the diagrammatical computation in the dimensional regularization scheme, for example using 't Hooft's algorithm [62].

A different, and in some sense more physical possibility is to introduce a cutoff in the lower end of the proper time integral, $\Lambda/\mu \to \infty$. In that way we get, for example in six dimensions[5]

$$\log\,\det\,\Delta|_{div} = \frac{1}{3}\,a_0\,\Lambda^6_{(d=6)} + \frac{1}{2}\,a_2\,\Lambda^4_{(d=6)} + a_4\,\Lambda^2_{(d=6)} + a_6\,\log\frac{\Lambda^2_{(d=6)}}{\mu^2_{(d=6)}} \tag{2.21}$$

where the heat kernel coefficients are obviously in six dimensions. In four dimensions instead

$$\log\,\det\,\Delta|_{div} = \frac{1}{2}\,a_0\,\Lambda^4_{(d=4)} + a_2\,\Lambda^2_{(d=4)} + a_4\,\log\frac{\Lambda^2_{(d=4)}}{\mu^2_{(d=4)}} \tag{2.22}$$

where now the coefficients are the corresponding ones in four dimensions. The dominant divergence (sixth power and fourth power of the cutoff) is vacuum energy and therefore universal and independent of the particular operator under consideration (up to numerical factors). We shall not study it further here.

[5] Although we shall try our best to avoid cluttering the notation unnecessarily, we are forced to distinguish between quantities bearing identical names, but coming from different dimensions.

In spite of the fact that it is often pointed out that there is no way of imposing a cutoff in a gauge invariant way, we would like to stress that, at least to the one loop order, this procedure respects all gauge invariances, abelian and non abelian, as well as general covariance in its case. This is obvious, because we are *not* cutting off the momentum, but rather the *proper time*, a covariant as well as gauge invariant concept. If we remember that the proper time in the sense we are employing it, has mass dimension $[\tau] = -2$, we are neglecting in the evaluation of the one loop determinants proper times smaller than Λ^{-1}. This fact, which was probably first pointed out by Schwinger [63] in 1951, has been exploited by Bryce de Witt [16] to get covariant expansions in quantum gravity and also by Fujikawa [64] to get the covariant anomaly.

We shall denote these two procedures *dimensional regularization* and *cutoff*, respectively. Both respect all gauge invariances of the theory but only the cutoff theory yields information on the divergences in the odd dimensional case.

2.2 The heat kernel as a superdeterminant

In order to get acquainted with the heat kernel techniques and to "calibrate" the formulas we are going to use, let us repeat a well-known computation, namely, the divergent part of the effective action of Quantum Electrodynamics (QED) in $d = 4$ dimensions. In doing so, we shall employ a technique first introduced by I. Jack and H. Osborn [65] (cf. also [66]), which is exceedingly convenient in case there are non-vanishing fermionic background fields and which will be extensively used in the following. The main idea will be to represent the combination of fermionic and bosonic determinants as a single superdeterminant, or Berezinian, as is sometimes referred to.

We shall compute in the dimensional regularization scheme. The Euclidean version of the QED action reads

$$\mathcal{L} = \bar{\chi}\gamma^M \partial_M \chi - e\bar{\chi}\gamma^M A_M \chi + m\bar{\chi}\chi + \frac{1}{4} F_{MN} F^{MN} \tag{2.23}$$

As has been explained, we now split the fields in a classical and quantum parts

$$\begin{aligned} A_M &= \bar{A}_M + \phi_M \\ \chi &= \eta + \psi \end{aligned} \tag{2.24}$$

where the backgrounds do obey the classical equations of motion, i.e.,

$$\begin{aligned}\left(\gamma^M(\partial_M - e\bar{A}_M) + m\right)\eta &= 0 \\ \partial_M \bar{F}^{MN} + e\bar{\eta}\gamma^N\eta &= 0\end{aligned} \tag{2.25}$$

Keeping only the terms quadratic in the quantum fields and sticking to Feynman's gauge leads to:

$$\mathcal{L} = \bar{\psi}\gamma^M\partial_M\psi - e\bar{\psi}\gamma^M\bar{A}_M\psi + m\bar{\psi}\psi - e\bar{\eta}\gamma^M\phi_M\psi - e\bar{\psi}\gamma^M\phi_M\eta - \frac{1}{2}\phi_M\partial_N\partial^N\phi^M \tag{2.26}$$

which can be written as:

$$\mathcal{L} = \frac{1}{2}\phi_M A_{MN}\phi_N + \bar{\psi}B\psi + \phi_N\bar{\Gamma}_N\psi + \bar{\psi}\Gamma_M\phi_M \tag{2.27}$$

with

$$\begin{aligned}A_{MN} &= -\partial_R\partial^R\delta_{MN} \\ B &= \gamma^M\partial_M - e\gamma^M\bar{A}_M + m \\ \Gamma_N &= -e\gamma_N\eta \\ \bar{\Gamma}_M &= -e\bar{\eta}\gamma_M\end{aligned} \tag{2.28}$$

This can equally well be expressed (cf. [66]) in terms of the supermatrix

$$\Delta = \begin{pmatrix} A_{MN} & \sqrt{\frac{2}{\mu}}\bar{\Gamma}_M\gamma_5 B\gamma_5 \\ \sqrt{2\mu}\Gamma_N & B\gamma_5 B\gamma_5\end{pmatrix} \tag{2.29}$$

as

$$S = \int d^4x\, \bar{\xi}\Delta\xi \tag{2.30}$$

with the newly defined field combining bosonic and fermionic degrees of freedom $\xi = (\phi_M, \psi)$. We have introduced an arbitrary mass scale μ for dimensional reasons. Our main interest is the computation of the one loop effective action

$$Z \equiv e^{-W} \sim \int \mathcal{D}\xi\; e^{-S[\bar{\xi}+\xi]} \tag{2.31}$$

which after performing the functional Gaussian integral is given by

$$W = \frac{1}{2}\log \operatorname{sdet}\Delta \tag{2.32}$$

The superdeterminant, or Berezinian of a supermatrix M involving bosonic (+) and fermionic (−) entries

$$M = \begin{pmatrix} M_{++} & M_{+-} \\ M_{-+} & M_{--}\end{pmatrix} \tag{2.33}$$

is defined by

$$\text{ber}\, M \equiv \text{sdet}\, M \equiv \det\, M_{++} \det{}^{-1} \left(M_{--} - M_{-+} M_{++}^{-1} M_{+-}\right) \tag{2.34}$$

In a similar way the supertrace is given by

$$\text{str}\, M = \text{tr}\, M_{++} - \text{tr}\, M_{--} \tag{2.35}$$

In the present situation, the operator reads

$$\Delta = \begin{pmatrix} -\partial_R \partial^R \delta_{MN} & \sqrt{\frac{2}{\mu}} e \left(\bar{\eta}\gamma^M \gamma^R \bar{D}_R - m\bar{\eta}\gamma^M\right) \\ -\sqrt{2\mu} e \gamma_N \eta & -\bar{D}_M \bar{D}^M + \frac{e}{2}\gamma^M \gamma^N \bar{F}_{MN} + m^2 \end{pmatrix} \tag{2.36}$$

This supermatrix operator enjoys the form we mentioned (2.2) with the supermatrices

$$X_M = \begin{pmatrix} 0 & \frac{-e}{\sqrt{2\mu}} \bar{\eta}\gamma^R \gamma_M \\ 0 & -e\bar{A}_M \end{pmatrix} \tag{2.37}$$

and

$$Y = \begin{pmatrix} 0 & \frac{-e}{\sqrt{2\mu}} \left(2m\bar{\eta}\gamma^M + \bar{D}_R \bar{\eta}\gamma^M \gamma^R\right) \\ -\sqrt{2\mu} e \gamma_N \eta & \frac{e}{2}\gamma^M \gamma^N \bar{F}_{MN} + m^2 \end{pmatrix} \tag{2.38}$$

Once we have reduced our problem to the computation of the determinant of a supermatrix the divergent part of the effective action is given by the $a_4(\Delta)$ coefficient in the heat kernel expansion, which is well known and can be read in the literature. For a flat manifold without boundaries it is given by

$$a_4(\Delta) = \int \frac{d^d x}{(4\pi)^{d/2}}\, \text{str} \left(\frac{1}{2} Y^2 + \frac{1}{12} X_{MN}^2\right) \tag{2.39}$$

where as usual X_{MN} is the field strength associated with X_M. In our case the field strength supermatrix is

$$X_{MN} = \begin{pmatrix} 0 & \frac{-e}{\sqrt{2\mu}} \left(\bar{D}_M \bar{\eta}\gamma^R \gamma_N - \bar{D}_N \bar{\eta}\gamma^R \gamma_M\right) \\ 0 & -e\bar{F}_{MN} \end{pmatrix} \tag{2.40}$$

which after squaring and tracing gives a contribution

$$\frac{1}{12}\, \text{str}\; X_{MN}^2 = -\frac{2^{[d/2]}}{12} e^2 \bar{F}_{MN}^2 \tag{2.41}$$

While the contribution from Y^2 is

$$\frac{1}{2}\, \text{str}\; Y^2 = e^2(d-2)\bar{\eta}\gamma^M \partial_M \eta - e^3(d-2)\bar{\eta}\gamma^M \bar{A}_M \eta + 2de^2 m\bar{\eta}\eta + \frac{2^{[d/2]}}{4} e^2 \bar{F}_{MN}^2 \tag{2.42}$$

Finally we can write the coefficient in four dimensions

$$a_4(\Delta) = \int \frac{d^4x}{(4\pi)^2}\left(\frac{2}{3}e^2\bar{F}^2_{MN} + 2e^2\bar{\eta}\gamma^M\bar{D}_M\eta + 8e^2m\bar{\eta}\eta\right) \tag{2.43}$$

which coincides with the result obtained through the application of the classical 't Hooft algorithm [62, 67]. We have then a non-trivial test that the supermatrices Y and X_M needed for a QED computation (in any dimensionality) are the ones we have found.

2.3 Generalized ζ-functions

Let us include here, for completeness, some well-known facts on ζ-functions. Using ζ-functions is yet another way of regularizing the effective action at one loop [68,69]. Some of the formulas will be used when discussing the case of a free scalar in higher dimensional curved space. The ζ-function associated to a particular operator $\mathcal{O}$ is defined in terms of its eigenvalues (assumed discrete and strictly positive, though it can be generalized), λ_n by means of

$$\zeta(\mathcal{O}, s) \equiv \sum_n \lambda_n^{-s} \tag{2.44}$$

This is the Mellin transform of the integrated heat kernel

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty d\tau\, \tau^{s-1}\, Y(\tau) \tag{2.45}$$

and trivially can be split into

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^1 d\tau\, \tau^{s-1}\, Y(\tau) + \frac{1}{\Gamma(s)}\int_1^\infty d\tau\, \tau^{s-1}\, Y(\tau) \tag{2.46}$$

Now the second integral is finite, provided $Y \sim e^{-\lambda_{min}\tau}$ for large τ. On the other hand, the first integral converges as long as

$$\text{Re } s > d/2 \tag{2.47}$$

in such a way that the potentially divergent part is given by:

$$\zeta(s) = \frac{1}{\Gamma(s)}\sum_{k=0}^\infty \frac{a_k}{s - \frac{d-k}{2}} \tag{2.48}$$

that is, there is an infinite set of poles located at

$$s = \frac{d-k}{2} \tag{2.49}$$

for $k \in 2\mathbb{N}$ (remember that in the absence of a boundary there are only even k coefficients). The structure of the poles depends on the dimensionality of the manifold. To be specific, for even dimension, poles appear at a finite number of positive integers, i.e. when

$$0 \leq s = \bar{n} < d/2 \in \mathbb{N} \tag{2.50}$$

The structure of the zeta function is in this case

$$\zeta(s) = \frac{1}{\Gamma(\bar{n})} \frac{a_{d-2\bar{n}}}{s - \bar{n}} \tag{2.51}$$

There are, besides that, apparent poles for all negative integers that are cancelled by similar poles of the Gamma function; when $s \in -\mathbb{N}$, say $s = -n_0$, then

$$\Gamma(s) \sim \frac{(-1)^{n_0}}{n_0!} \frac{1}{s + n_0} \tag{2.52}$$

so that

$$\zeta(s = -n_0) = (-1)^{n_0}\ n_0!\ a_{d+2n_0} \tag{2.53}$$

which vanishes for odd dimension in the absence of boundary. This cancellation does not work for odd dimension, where there are an infinite number of poles, corresponding to

$$s = \frac{d - k}{2} \tag{2.54}$$

with $k \in 2\mathbb{N}$. Otherwise

$$\zeta(s = 0) = a_d \tag{2.55}$$

yields a regularized count of the number of eigenvalues of the operator. When there are N_0 zero modes, those are not counted by the ζ-function, and they have to be added by hand; that is

$$a_d = N_0 + \zeta(0) \ . \tag{2.56}$$

We will comment on an extremely interesting application of all this technology to the case of a free scalar in curved space in the next section. There, we will see that in this particular case it is possible to recover the correct counterterms from the point of view of the four-dimensional reduced theory if the divergent sums which arise by adding up the counterterms for each of the infinite Kaluza–Klein modes are regularized precisely using generalized ζ-functions. It is important to mention that even in this case unconventional counterterms from the standpoint of the four-dimensional theory are necessary.

3 The tower of divergences for free fields

The simplest situation that can be contemplated is as usual the one of free scalar fields in curved space. This is still non trivial, owing to the universal gravitational coupling, and counterterms related to the spacetime curvature will appear. The problem has been solved by Duff and Toms [4] a long time ago using dimensional regularization, and we present their results here as an introduction to more complicated issues.

Their main motivation was to study the renormalizability properties of higher dimensional field theories. In particular, they noticed that even in the simplest case of a scalar living in $M_4 \times S^1$ some puzzling observations can be made. As we have tried to emphasize in the previous section, dimensional regularization at one loop gives a finite answer in odd dimensions. Nevertheless, once the theory has been dimensionally reduced, it consists on an infinite tower of scalar fields on a curved four-dimensional background. A non-vanishing counterterm can then be found for each Kaluza–Klein mode. The question is how do we recover the one-loop finite answer since any single mode produces its own divergencies. The situation in even dimensions is not better. For example, a scalar in curved six-dimensional space would produce divergencies and the counterterms involve curvature invariants of dimension six. But dimension six invariants cannot appear in four-dimensional counterterms.

In the following, we will review their answer to this intriguing remark. Let us start with the setting. Consider a physical (Riemannian) spacetime represented as a product manifold

$$M_d = V_4 \times C_{d-4} \tag{3.1}$$

with coordinates[6] $x^M \equiv (x^\mu, y^a)$. The metric is moreover assumed to be of the direct product form

$$ds^2 = G_{MN}(x^M)dx^M dx^N = g_{\mu\nu}(x)dx^\mu dx^\nu + \gamma_{ab}(y)dy^a dy^b \tag{3.2}$$

[6] The full spacetime is coordinatized by $0 \leq M \leq d-1$, whereas the four dimensional spacetime is represented by $0 \leq \mu \leq 3$ and the manifold of extra dimensions as $4 \leq a \leq d-1$.

A free scalar field propagating in this manifold enjoys an action[7]

$$S = \frac{1}{2}\int \sqrt{|G|}\; d^d x\; \Phi \Box \Phi \tag{3.3}$$

where

$$\Box \equiv -G^{AB}\nabla_A \nabla_B \tag{3.4}$$

Because of the assumed form of the metric this operator splits into the direct sum of the corresponding operators on the manifolds V and C as

$$\Box = \Box_x + \Box_y \tag{3.5}$$

And $\Box_x$ ($\Box_y$) acts trivially on C (V). This will turn out to be a crucial property. An arbitrary field configuration can now be formally expanded in harmonics of C

$$\Phi(x,y) \equiv \sum_n \phi_n(x)\, H_n(y) \tag{3.6}$$

where the eigenfunctions

$$\Box_y H_n(y) = \lambda_n\, M^2\, H_n(y) \tag{3.7}$$

are orthonormalized

$$\int d^{d-4}y\, \sqrt{|\gamma|}\, H_n(y)\, H_m(y) = \delta_{nm} \tag{3.8}$$

We have introduced a mass dimension 1 parameter (related to the compactification scale $M \sim \frac{1}{R}$) in order to make λ_n dimensionless. The action after dimensional reduction then reads

$$S = \sum_n S_n = \frac{1}{2}\sum_n \int d^4x \sqrt{|g|}\left(\phi_n \Box_x \phi_n + \lambda_n M^2 \phi_n \phi_n\right) \tag{3.9}$$

That is, the quadratic operator relevant for each piece is

$$\Box_n \equiv \Box_x + M^2 \lambda_n \tag{3.10}$$

It follows that the traced heat kernel takes the form

$$Y(\Box_n, \tau) = e^{-M^2 \lambda_n \tau}\, Y(\Box_x, \tau) \tag{3.11}$$

[7]This action can be easily generalized to allow for a non-minimal coupling to the curvature scalar, but the conclusions would not change. Since the scalar is a free one it is not needed to expand around a background.

which could be thought to imply, by expanding the exponential and following the reasonings of the previous chapter, that in four dimensions the relevant heat kernel coefficient is not a_4 anymore, but

$$\sum_{k=0}^{4} a_k(\Box_x)\frac{(-M^2\lambda_n)^{(4-k)/2}}{(\frac{4-k}{2})!} = a_4 - M^2\lambda_n a_2 + \frac{1}{2}M^4\lambda_n^2 a_0 \tag{3.12}$$

On the other hand due to the property (3.5) the integrated heat kernel factorizes:

$$Y(\Box,\tau) = Y(\Box_x,\tau)\,Y(\Box_y,\tau) \tag{3.13}$$

which implies that

$$a_k(\Box) = \sum_{l=0}^{k} a_l(\Box_x)\,a_{k-l}(\Box_y) \tag{3.14}$$

This expression makes sense because the dimension of the left hand side is $k-6$ and the dimensions of the terms in the right hand side are $l-4$ and $(k-l)-2$. This means that the one-loop divergent part of the effective action in d-dimensions is given by:

$$\Gamma_{div} = \frac{1}{\epsilon}\;a_d(\Box) = \frac{1}{\epsilon}\sum_{l=0}^{d} a_l(\Box_x)\,a_{d-l}(\Box_y) \tag{3.15}$$

and the question is, in what sense do we recover this result by adding the infinite tower of four-dimensional divergent parts. It is very easy to convince oneself that with the prescription implicit in the equation (3.12), this is simply not true, *even in the free case we are considering*

$$\Gamma^{(div)} \neq \sum_n \Gamma_n^{(div)} \tag{3.16}$$

This fact has been realized in the past (cf. for example [70], where the difference between both expressions was called "dimensional reduction anomaly"). The important point from our present perspective is whether there is a four-dimensional renormalization prescription that ensures the recover of the six-dimensional counterterm. Otherwise, there would be an essential ambiguity in the four-dimensional analysis.

As noticed in [4], the problem lies in the sum over the exponential

$$\sum_n e^{-M^2\lambda_n\tau} \tag{3.17}$$

In order to get the correct result through the infinite Kaluza–Klein tower, we need to make sure that

$$\sum_n e^{-M^2\lambda_n\tau} = Y(\Box_y,\tau) \sim \sum_{k=0}^{\infty} \tau^{\frac{k-(d-4)}{2}} a_k(\Box_y) \tag{3.18}$$

and therefore (3.13) is recovered. Please observe that there are negative powers of the proper time in the expansion. To be specific, this means that were we to perform the naive expansion (3.12)

$$\sum_n e^{-M^2\lambda_n\tau} = \sum_n \sum_k (-M^2\lambda_n)^k \frac{\tau^k}{k!} \tag{3.19}$$

and use the zeta-function result

$$\sum_n (M^2\lambda_n)^k = \zeta(-k,\Box_y) \tag{3.20}$$

as well as

$$\zeta(-k) = (-1)^k\, k!\; a_{d-4+2k} \tag{3.21}$$

we would get instead

$$\sum_n e^{-M^2\lambda_n\tau} = \sum_{k=0}^{\infty} a_{d-4+2k}\,\tau^k \tag{3.22}$$

i.e., we would have missed the negative powers of the proper time. If we reinstate those negative powers, we are led in $4+\epsilon$ dimensions to the dimensionless expression

$$\begin{aligned}\Gamma_n^{div} &= \frac{1}{\epsilon}\sum_{k=0}^{d} a_k(\Box_x)\frac{(-M^2\lambda_n)^{\frac{4+\epsilon-k}{2}}}{\Gamma(\frac{4+\epsilon-k}{2}+1)} = \\ &= \frac{1}{\epsilon}\left(\frac{1}{2}M^4\lambda_n^2 a_0 - M^2\lambda_n a_2 + a_4 - \frac{1}{M^2\lambda_n\Gamma(\frac{\epsilon}{2})}a_6 + \ldots\right)\end{aligned} \tag{3.23}$$

in such a way that we now have

$$\sum_n \Gamma_n^{div} = \frac{1}{\epsilon}\sum_{k=0}^{d} a_k(\Box_x)\;\zeta\left(\frac{k-4-\epsilon}{2}\right)\frac{1}{\Gamma(\frac{4-\epsilon-k}{2}+1)}(-1)^{\frac{4-k}{2}} \tag{3.24}$$

Let us first examine the terms corresponding to $0 \le k \le 4$. Denoting $-n_0 \equiv \frac{k-4}{2}$, the structure is

$$\frac{\zeta\left(-n_0-\frac{\epsilon}{2}\right)}{\Gamma\left(1+n_0-\frac{\epsilon}{2}\right)} = (-1)^{n_0}\,\Gamma(1+n_0)\,a_{d-4+2n_0}\,\frac{1}{\Gamma(1+n_0)} = (-1)^{\frac{4-k}{2}}\,a_{d-k}(\Box_y) \tag{3.25}$$

On the other hand, for $4 \leq k \leq d$ we can denote $\bar{n} \equiv \frac{k-4}{2}$ and the structure is

$$\zeta\left(\bar{n}-\frac{\epsilon}{2}\right)\frac{1}{\Gamma\left(1-\bar{n}-\frac{\epsilon}{2}\right)} = \frac{a_{d-4-2\bar{n}}}{\Gamma\left(\bar{n}-\frac{\epsilon}{2}\right)\left(-\frac{\epsilon}{2}\right)}\;\frac{1}{\Gamma\left(1-\bar{n}-\frac{\epsilon}{2}\right)} = a_{d-k}(\Box_y)\,(-1)^{\frac{k-4}{2}} \tag{3.26}$$

where we have used

$$\Gamma\left(\bar{n}-\frac{\epsilon}{2}\right)\Gamma\left(1-\bar{n}+\frac{\epsilon}{2}\right) = \frac{\pi}{\sin\,\pi\left(\bar{n}-\frac{\epsilon}{2}\right)} = \frac{2}{\epsilon}\,(-1)^{1+\bar{n}} \tag{3.27}$$

Substitution of this formulas into (3.24) allows us to recover (3.15). This shows that the four dimensional reduction can be renormalized in such a way that the sum of the divergences is the divergence of the sum, at least in the free case. A glance at (3.23) shows that we have to add finite counterterms. For example, when the mother theory lives in six dimensions, the only new one is

$$\Delta\Gamma_n\Big|_{fin} = -\frac{1}{2M^2\lambda_n}\,a_6(\Box_x) \tag{3.28}$$

The whole infinite series of finite renormalizations has to be resummed using the appropriate zeta function:

$$\sum_n \Delta\Gamma_n\Big|_{fin} = -\frac{1}{2}\zeta_{\Box_y}(s=1)\,a_6(\Box_x) \tag{3.29}$$

and the pole in the zeta function can be resolved by working in $d_2 = 2+\epsilon$ dimensions, yielding the mass dimension -2 result

$$\zeta_{\Box_y}(s=1) = -2\frac{1}{\epsilon}\,a_0(\Box_y) \tag{3.30}$$

Altogether, the whole sum of finite four-dimensional counterterms reproduces the higher dimensional divergence (of vanishing mass dimension: $2-2=0$)

$$\frac{1}{\epsilon}a_6(\Box_x)\,a_0(\Box_y) \tag{3.31}$$

This counterterm is certainly not expected from a four-dimensional viewpoint. The inverse problem is also solved, i.e., we recover the four-dimensional counterterms from the six dimensional one, through the factorization formula, as well as the interpretation

$$a_{d+2n_0} = \frac{(-1)^{n_0}}{n_0!}\zeta(-n_0) \equiv \frac{(-1)^{n_0}}{n_0!}\sum \lambda_n^{2n_0} \tag{3.32}$$

and the aforementioned resolution of the pole in $\zeta(1)$ in $2+\epsilon$ dimensions in terms of a set of higher dimension operators with finite coefficients.

This procedure is quite consistent. For example, if the mother theory lives in odd dimension (for example, five) the renormalization procedure we are advocating does not yield any counterterm. This result can be recovered from the four-dimensional sum just by realizing that the adequate ζ-function in (3.20) is given in terms of Riemann's zeta function through

$$\zeta_{S^1}(\Box_y) \equiv \sum_n (n^2)^{-s} = \zeta_R(2s) \tag{3.33}$$

$$0 = \zeta_R(-2m) \tag{3.34}$$

for $m \in \mathbb{Z}^+$. By means of the regularization, the sum of the tower of divergences gives a vanishing result.

To conclude, in this case there is a complete consistency of the high and low dimensional divergences, even in the strictest dimensional regularization. It is worth emphasizing that if one takes the point of view of the reduced theory, the divergent sums arising by adding up the counterterms for each mode *must* be regularized with the appropriate ζ-function and with this prescription unconventional counterterms are generated. Exactly equivalent statements can be made concerning the axial anomaly [4]. Notice that the splitting of the operator into purely four-dimensional and extra-dimensional pieces is crucial in the argumentation. When it is not the case, there is no natural candidate for the regularization of the divergent sums and thus one cannot relate it with a heat kernel coefficient.

4 Renormalization in higher-dimensional gauge theories

As we have seen, when the theory considered is that of a free field in a curved factorizable background, or more generally when it is verified (3.5), there is a way to relate the counterterms both from the higher dimensional and the dimensionally-reduced points of view. In this section, we will study a somewhat more complicated case, that of a interacting theory. Since we will be also interested in exploring possible phenomenological consequences of our findings, we will take a gauge theory as an example. As we will see, the lack of operator splitting makes the computations and the reasonings much more obscure.

4.1 Six-dimensional quantum electrodynamics compactified on a torus

Let us now consider an example not altogether trivial, namely quantum electrodynamics (QED) on a six-dimensional manifold which is topologically four-dimensional Minkowski space times a two-torus, that is, $\mathbb{R}^4 \times S^1 \times S^1$. This example avoids the complications of interacting gravitational sectors, but in some sense is not representative of the whole Kaluza–Klein philosophy, because we are introducing gauge fields already in the extra dimensions. We are using it as a toy model and to study some of the features of the universal extra dimensional scenario.

We will first establish our setup. The metric for the time being is assumed to be

$$ds^2 = \delta_{\mu\nu} dx^\mu dx^\nu + R_5^2 d\theta_5^2 + R_6^2 d\theta_6^2 \tag{4.1}$$

that is, $y_5 = R_5\theta_5$ and $y_6 = R_6\theta_6$. We shall follow consistently the above mentioned convention that capital indices, like $M, N, \ldots$ run over the full dimensions, in our case from 1 to 6; greek indices, $\mu, \nu, \ldots$ run over the ordinary Minkowski coordinates, from 1 to 4; and small roman letters, $a, b, \ldots$, over the extra dimensions, that is, from 5 to 6.

The (euclidean version of the) action for a single Dirac fermion coupled to an Abelian gauge field then reads

$$S = \int d^6x \left[\frac{1}{4} F_{MN}^2 + \bar{\psi}(\not{D} + m)\psi\right] \tag{4.2}$$

which will be our six-dimensional toy version of QED. The covariant derivative is simply:

$$D_M\psi \equiv (\partial_M - eA_M)\,\psi \tag{4.3}$$

Let us recall here for later use that, for vanishing curvature, the general formulas [71] for the first few heat kernel coefficients of an operator of the form (2.2) are:

$$a_2 = -\int \frac{d^n x}{(4\pi)^{\frac{n}{2}}}\ \mathrm{tr}\, Y \tag{4.4}$$

$$a_4 = \int \frac{d^n x}{(4\pi)^{\frac{n}{2}}}\ \mathrm{tr}\left[\frac{1}{12}X_{MN}^2 + \frac{1}{2}Y^2 - \frac{1}{6}Y_{;MM}\right] \tag{4.5}$$

$$\begin{aligned} a_6 = \frac{1}{360}\int \frac{d^n x}{(4\pi)^{\frac{n}{2}}}\ \mathrm{tr}\Big[&8X_{MN;R}^2 + 2X_{MN;N}^2 + 12X_{MN;RR}X^{MN} \\ &-12X_{MN}X^{NR}X_R{}^M - 6Y_{;MMNN} \\ &+60YY_{;MM} + 30Y_{;M}^2 - 60Y^3 - 30YX_{MN}^2\Big] \end{aligned} \tag{4.6}$$

where ; denotes covariant derivative, and

$$X_{MN} = \partial_M X_N - \partial_N X_M + [X_M, X_N] \tag{4.7}$$

is the field strength associated to the gauge connection. Since we have fermions interacting with the gauge field, in order to perform the explicit computation it is exceedingly useful to combine the fermionic and bosonic sectors in a full supermatrix. The precise formulas needed were reviewed in the previous sections, where we found also the particular form of the relevant supermatrices for a QED like theory (in any dimension).

Computing the coefficients is then straightforward albeit somewhat laborious. In terms of the background fields $\bar{A}_M, \eta, \bar{\eta}$

$$a_2 = \int \frac{d^6 x}{(4\pi)^3}\ 8m^2 \tag{4.8}$$

as well as

$$a_4 = \int \frac{d^6 x}{(4\pi)^3}\ \left[\frac{4}{3}e^2\bar{F}_{MN}^2 + 4e^2\bar{\eta}\bar{\not{D}}\eta + 12me^2\bar{\eta}\eta\right] \tag{4.9}$$

Finally we get, using the background equations of motion

$$\begin{aligned} a_6 = \int \frac{d^6 x}{(4\pi)^3}\ &\Big[-\frac{1}{12}e^4\bar{\eta}\Sigma_{MNL}\eta\bar{\eta}\Sigma^{MNL}\eta + \frac{19}{15}e^2m\bar{\eta}\bar{D}_M\bar{D}^M\eta \\ &+\frac{2}{15}e^3\bar{\eta}\gamma_N\bar{D}_M\eta\bar{F}^{MN} - e^3m\bar{\eta}\gamma^M\gamma^N\eta\bar{F}_{MN} - 2e^2m^2\bar{\eta}\gamma^M\bar{D}_M\eta - 6e^2m^3\bar{\eta}\eta \\ &-\frac{11}{45}e^2\bar{D}_R\bar{F}_{MN}\bar{D}^R\bar{F}^{MN} + \frac{23}{9}e^2\bar{D}_M\bar{F}^{MN}\bar{D}^R\bar{F}_{RN} - \frac{4}{3}e^2m^2\bar{F}_{MN}\bar{F}^{MN}\Big] \end{aligned} \tag{4.10}$$

where Σ_{MNL} is the totally antisymmetric product of three gammas. Remember that in dimensional regularization the counterterm is given by

$$\Delta S = \frac{1}{\epsilon}\, a_6 \tag{4.11}$$

plus a possible finite part. With a cutoff, these are the logarithmic divergences, and we have in addition both quadratic and quartic divergences, on which more to follow.

The first conclusion we can draw from this analysis is that quantum effects, besides renormalizing the six-dimensional couplings, induce a set of non-minimal interactions which are generated with arbitrary coefficients. Actually, due to the fact that the mass dimension of the coupling constant is $[e] = -1$, there is no finite closed set of operators in the counterterms. Let us be more specific concerning this point.

First of all, there is a dimension five operator, which becomes a potential counterterm in the massive case:

$$\mathcal{O}_{(5)} = \left(\bar{\psi}\psi\right) \tag{4.12}$$

The set of gauge-invariant dimension six operators is given by:

$$\mathcal{O}^i_{(6)} = \left(\bar{\psi}\not{D}\psi,\, F^2_{MN}\right) \tag{4.13}$$

To the next order, that is, dimension seven, the list reads:

$$\mathcal{O}^i_{(7)} = \left(\bar{\psi}\not{D}\not{D}\psi\right) \tag{4.14}$$

The dimension eight operators are:

$$\mathcal{O}^i_{(8)} = \left(\bar{\psi}\not{D}\not{D}\not{D}\psi,\, \bar{\psi}\sigma_{MN}\psi F^{MN},\, D^M F_{MN} D_R F^{RN},\, F_{NL} D^2 F^{NL}\right) \tag{4.15}$$

And finally, to dimension nine we have to consider:

$$\mathcal{O}^i_{(9)} = \left(\bar{\psi}\gamma_M D_N \psi F^{MN},\, \bar{\psi} D_A D_B D_C D_D \psi t^{ABCD}\right) \tag{4.16}$$

In the massive case the dimension of this operators can be increased by introducing powers of m. Amongst the operators that actually appear as counterterms only the $\mathcal{O}^2_{(8)}$ is absent. At any rate it should be plain that we can claim results only to first nontrivial order in the six-dimensional fine structure constant, and that we have really no right to keep the e^3 and e^4 terms in the counterterm.

The non renormalizability of the theory manifests itself in the fact that if we were to include all those dimension seven and dimension eight operators in the bare Lagrangian, they would generate more and more higher dimension operators as counterterms since their bare coupling would also have negative mass dimension. There is no closed set, unless we assume, as is natural to the order we are working, that the effect of all those couplings is of higher order in the six-dimensional fine structure constant.

In any case, keeping in mind that we are not performing a fully consistent computation, if we define the renormalization constants as is usually done

$$\begin{aligned} A_0 &= Z_3^{1/2} A \\ \psi_0 &= Z_2^{1/2} \psi \\ e_0 &= Z_1 Z_2^{-1} Z_3^{-1/2} e \\ m_0 &= Z_m m \end{aligned} \tag{4.17}$$

we easily get $Z_1 = Z_2$ which conveys the fact that the theory is gauge invariant, and

$$\begin{aligned} Z_2 &= 1 - \frac{e^2 m^2}{32\pi^3 \epsilon} \\ Z_3 &= 1 - \frac{e^2 m^2}{12\pi^3 \epsilon} \\ Z_m &= 1 - \frac{e^2 m^2}{16\pi^3 \epsilon} \end{aligned} \tag{4.18}$$

A simple calculation then leads to the renormalization group functions:

$$\begin{aligned} \beta_e &\equiv \frac{\partial e}{\partial \log \mu} = -\frac{1}{24\pi^3} e^3 m^2 \\ \beta_m &\equiv \frac{\partial m}{\partial \log \mu} = \frac{1}{16\pi^3} e^2 m^3 \end{aligned} \tag{4.19}$$

The renormalization of the fermion mass is entangled with the charge renormalization. The behavior of the coupling constants reads

$$\begin{aligned} e &= e_0 - \frac{1}{24\pi^3}\, m_0^2\, e_0^3\, \log \frac{\mu}{\mu_0} \\ m &= m_0 \left(1 - \frac{1}{24\pi^3}\, m_0^2\, e_0^2\, \log \frac{\mu}{\mu_0} \right)^{-3/2} \end{aligned} \tag{4.20}$$

The dimensionful charge vanishes at the scale

$$\mu = \mu_0 e^{\frac{24\pi^3}{m_0^2 e_0^2}} \tag{4.21}$$

If we define the dimensionless couplings

$$\begin{aligned}\hat{e} &\equiv e\mu \\ \hat{m} &\equiv \frac{m}{\mu}\end{aligned} \tag{4.22}$$

then the renormalization group equations read

$$\begin{aligned}\beta_{\hat{e}} &= \hat{e} - \frac{1}{24\pi^3}\hat{m}^2\hat{e}^3 \\ \beta_{\hat{m}} &= -\hat{m} + \frac{1}{16\pi^3}\hat{e}^2\hat{m}^3\end{aligned} \tag{4.23}$$

Notice however that this is not the expected power-law running mentioned earlier. This is because the power-law behaviour comes from the higher order terms in the Vacuum Polarization Function that we have not taken into account in the charge renormalization. For a critical study of this question from the point of view of effective field theory see [2, 59]. Our results do support their conclusions but the calculation presented here does not rely on summing over Kaluza–Klein modes.

4.2 The four-dimensional point of view

The other viewpoint one can have on the model is that of the dimensionally reduced theory. The extra dimensions describe a torus, so the fields must be periodic with respect to the corresponding coordinates and we expand in Fourier series

$$\phi(x,y) = \frac{1}{2\pi\sqrt{R_5R_6}}\sum_n \phi_n(x)e^{i\frac{n}{R}\cdot y} \tag{4.24}$$

where $n \equiv (n_5, n_6)$, and we have included a convenient factor in front to take care of the difference of canonical dimensions of the fields in six and four dimensions. Real fields (such as the photon) obey

$$\phi_n^*(x) = \phi_{-n}(x) \tag{4.25}$$

In order to dimensionally reduce fermions, one must choose a representation of the six-dimensional Clifford algebra

$$\{\gamma^M, \gamma^N\} = \delta^{MN} \tag{4.26}$$

A possible way to construct such a representation is using a four-dimensional one. The six-dimensional gamma matrices can be chosen as

$$\begin{aligned}\gamma_\mu^{(6)} &= \sigma_3 \otimes \gamma_\mu^{(4)} \\ \gamma_5^{(6)} &= \sigma_1 \otimes 1 \\ \gamma_6^{(6)} &= \sigma_2 \otimes 1\end{aligned} \tag{4.27}$$

where σ_i are the usual Pauli matrices and $\gamma_\mu^{(4)}$ form a representation of the four-dimensional algebra $\{\gamma^\mu, \gamma^\nu\} = \delta^{\mu\nu}$. In that way, six-dimensional spinors split in two four-dimensional ones

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{4.28}$$

It is a simple matter to perform the integrals over the angular variables and obtain the gauge fixed action (still exact) in the four dimensional form

$$\begin{aligned} S \;=\; \int d^4x \sum_{n_5,n_6} \Big[& \bar\psi^1_n \partial\!\!\!/ \psi^1_n + \bar\psi^2_n \partial\!\!\!/ \psi^2_n + N \bar\psi^1_n \psi^2_n + N^* \bar\psi^2_n \psi^1_n \\ & + m\left(\bar\psi^1_n \psi^1_n - \bar\psi^2_n \psi^2_n\right) - \frac{1}{2}(A^n_\mu)^* \left(\Box - |N|^2\right) A^\mu_n \\ & - \frac{1}{2}(A^n_5)^* \left(\Box - |N|^2\right) A^n_5 - \frac{1}{2}(A^n_6)^* \left(\Box - |N|^2\right) A^n_6 \\ & - e \sum_m \left(\bar\psi^1_m A\!\!\!/_{m-n} \psi^1_n + \bar\psi^2_m A\!\!\!/_{m-n} \psi^2_n + \bar\psi^1_m A^{m-n}_5 \psi^2_n \right. \\ & \left. - \bar\psi^2_m A^{m-n}_5 \psi^1_n - i \bar\psi^1_m A^{m-n}_6 \psi^2_n - i \bar\psi^2_m A^{m-n}_6 \psi^1_n \right) \Big] \end{aligned} \tag{4.29}$$

where we have defined the complex mass number $N = \frac{n_6}{R_6} + i\frac{n_5}{R_5}$ and the dimensionless four-dimensional coupling constant is

$$e \equiv \frac{e^{(6)}}{2\pi\sqrt{R_5 R_6}} \equiv e^{(6)} M \tag{4.30}$$

Here we see clearly a generic feature of interacting theories, namely that there is no consistent truncation in the sense that all massive fields interact among themselves and with the massless fields.

It is important to study the symmetries of the system since they normally dictate the form of the counterterms. Six-dimensional QED has of course a $U(1)$ gauge symmetry. It is interesting to see how this invariance is traduced in the lower dimensional theory. Before gauge fixing, the four-dimensional action enjoys the infinite set of symmetries [72]

$$\begin{aligned} \delta A^n_\mu &= i\partial_\mu \Lambda_n \\ \delta A^n_5 &= -\frac{n_5}{R_5}\Lambda_n \\ \delta A^n_6 &= -\frac{n_6}{R_6}\Lambda_n \end{aligned} \tag{4.31}$$

where Λ_n are the modes of the expansion of the abelian transformation parameter. All those gauge symmetries Λ_{n_5,n_6} are spontaneously broken, except for the zero mode, corresponding to $\Lambda_{0,0}$. The A^n_μ are the massive vector bosons, and the A^n_5 and A^n_6 the scalar Higgses.

There is a curious fact, however, and this is the appearance of two singlets in four dimensions, namely A^0_5 and A^0_6. Those singlets are massless at tree level, but no symmetry protects them from getting massive through quantum corrections. This is similar to the observation made in [72] with respect to the scalar (the purely extra-dimensional component) contained in the metric of five-dimensional Kaluza–Klein gravity, which is a pseudo-Goldstone boson of a global scale transformation.

The same fields are protected from getting masses in six dimensions, through gauge invariance and six dimensional Lorentz covariance. The point is that the breaking

$$O(1,5) \to O(1,3) \times O(2) \times O(2) \tag{4.32}$$

of the symmetry group of the vacuum is an instance of spontaneous compactification; i.e., the equations of motion enjoy the full $O(1,5)$ symmetry, and only the solution breaks it. Given that it is supposed to be the visible one, a very important part of the theory is the massless sector. The zero mode of the above action is

$$\begin{aligned} S_{zm} &= \int d^4x \Big[\bar\psi^1 \partial\!\!\!/ \psi^1 + \bar\psi^2 \partial\!\!\!/ \psi^2 + m\left(\bar\psi^1\psi^1 - \bar\psi^2\psi^2\right) - \frac{1}{2} A_\mu \Box A^\mu \\ &\quad - \frac{1}{2}\phi^* \Box \phi - e\left(\bar\psi^1 A\!\!\!/ \psi^1 + \bar\psi^2 A\!\!\!/ \psi^2 + \bar\psi^1 \phi \psi^2 - \bar\psi^2 \phi^* \psi^1\right)\Big] \end{aligned} \tag{4.33}$$

where we have represented the zero modes of all fields by the same letter without any subindex

$$A^0_5 - iA^0_6 \equiv \phi^0 \equiv \phi \tag{4.34}$$

It must be stressed that this is *not* a consistent truncation, (in the sense of the word usually employed in supergravity and superstrings) owing to the fact that both A^0_μ and ϕ couple diagonally to the whole fermionic tower; it is expected, however, to be a physically sensible one at energies $E \ll M$.

Let us perform the corresponding computations with the usual algorithm. Denoting $\bar\phi$ the background for ϕ the quadratic part of the action is

$$\begin{aligned} S_{zm} &= \int d^4x \Big[\bar\psi^1 \partial\!\!\!/ \psi^1 + \bar\psi^2 \partial\!\!\!/ \psi^2 + m\left(\bar\psi^1\psi^1 - \bar\psi^2\psi^2\right) - \frac{1}{2} \phi_\mu \Box \phi^\mu \\ &\quad - \frac{1}{2}\phi^* \Box \phi - e\left(\bar\psi^1 \bar{A\!\!\!/} \psi^1 + \bar\psi^2 \bar{A\!\!\!/} \psi^2 + \bar\eta^1 \gamma^\mu \phi_\mu \psi^1 + \bar\eta^2 \gamma^\mu \phi_\mu \psi^2 \right. \\ &\quad + \bar\psi^1 \gamma^\mu \phi_\mu \eta^1 + \bar\psi^2 \gamma^\mu \phi_\mu \eta^2 + \bar\psi^1 \bar\phi \psi^2 - \bar\psi^2 \bar\phi^* \psi^1 + \bar\eta^1 \phi \psi^2 \\ &\quad \left. - \bar\eta^2 \phi^* \psi^1 + \bar\psi^1 \phi \eta^2 - \bar\psi^2 \phi^* \eta^1 \right)\Big] \end{aligned} \tag{4.35}$$

where e is now the four dimensional coupling. The first coefficients in the heat kernel expansion are

$$a_2^{(zm)} = \int \frac{d^4x}{(4\pi)^2}\ 8\ \left(m^2 - e^2|\bar{\phi}|^2\right) \tag{4.36}$$

and

$$\begin{aligned} a_4^{(zm)} \ = \ \int \frac{d^4x}{(4\pi)^2} & \left[\frac{4}{3}e^2\bar{F}_{\mu\nu}^2 - 4e^2\bar{\phi}^*\Box\bar{\phi} + 8e^2m^2|\bar{\phi}|^2 - 4e^4|\bar{\phi}|^4 \right. \\ & + 4e^2\left(\bar{\eta}^1\bar{\not{D}}\eta^1 + \bar{\eta}^2\bar{\not{D}}\eta^2\right) + 12me^2\left(\bar{\eta}^1\eta^1 - \bar{\eta}^2\eta^2\right) \\ & \left. + 8e^3\bar{\eta}^2\bar{\phi}^*\eta^1 - 8e^3\bar{\eta}^1\bar{\phi}\eta^2\right] \end{aligned} \tag{4.37}$$

This is the logarithmically divergent counterterm that arises when renormalizing the zero mode of the four dimensional action.

It should be remarked that the resulting four dimensional model is superficially very similar to the Coleman-Weinberg setup, in which radiative spontaneous symmetry breaking was first discovered. There is a crucial difference though, and this is that the scalar field is not charged, in spite of being complex. The reason is that it remembers its gauge origin, and six-dimensional gauge invariance manifests here as a Kac-Moody transformation acting on the full tower of massive states. In addition to that, the quartic coupling is here a quantum effect, because it was not present in the bare four-dimensional lagrangian. Also the scalar field gets massive, with a mass proportional to the fermion mass (times the four- dimensional fine structure constant)[8].

4.3 Comparison of the divergences in the massless sector

After all this work, we are finally in a position to study our main concern, namely, how the divergent part of the six-dimensional effective action is related to the cor-

[8]At any rate, this yields (twice) the usual beta function for the four dimensional fine structure constant

$$\beta_e = \frac{1}{6\pi^2}e^3 \tag{4.38}$$

The behavior of the charge is

$$e^2 = \frac{e_0^2}{1 - \frac{e_0^2}{3\pi^2}\log\mu/\mu_0} \tag{4.39}$$

which blows up at a Landau pole located at

$$\Lambda \equiv \mu_0 e^{3\pi^2/e_0^2} \tag{4.40}$$

responding four-dimensional quantity. Let us first analyze the problem from the viewpoint of the cutoff theory. As we have seen previously, in six dimensions the divergent part of the effective action is given through the equation

$$\log\,\det\,\Delta|_{div} = \frac{1}{3}\,a_0\,\Lambda^6_{(d=6)} + \frac{1}{2}\,a_2\,\Lambda^4_{(d=6)} + a_4\,\Lambda^2_{(d=6)} + a_6\,\log\frac{\Lambda^2_{(d=6)}}{\mu^2_{(d=6)}} \tag{4.41}$$

while from the four-dimensional viewpoint the corresponding formula stems from equation

$$\log\,\det\,\Delta|_{div} = \frac{1}{2}\,a_0\,\Lambda^4_{(d=4)} + a_2\,\Lambda^2_{(d=4)} + a_4\,\log\frac{\Lambda^2_{(d=4)}}{\mu^2_{(d=4)}} \tag{4.42}$$

When we are interested in the zero mode, i.e., the piece in six dimensions where all fields are independent of the extra dimensions, the measure clearly factorizes

$$d^6x \to \frac{1}{M^2}d^4x \tag{4.43}$$

It is however plain that the divergences cannot coincide exactly. The *only* way to make the divergences related to the fourth heat kernel coefficient similar in six and in four dimensions is to choose different proper time cutoffs in both dimensions in such a way that

$$\frac{\Lambda^2_{(d=6)}}{M^2} \equiv \log\frac{\Lambda^2_{(d=4)}}{\mu^2_{(d=4)}} \tag{4.44}$$

This choice is justified by the fact that those coefficients are *almost* identical, so that the logarithmic divergences are as similar as possible. This identification leads to the reinterpretation of the six-dimensional quartic divergences as $\log^2$:

$$\Lambda^4_{(d=6)} \to M^4\left(\log\frac{\Lambda^2_{(d=4)}}{\mu^2_{(d=4)}}\right)^2 \tag{4.45}$$

and finally, the six-dimensional logarithmic divergences appear in the guise of $\log\log$.

$$\log\frac{\Lambda^2_{(d=6)}}{\mu^2_{(d=6)}} \to \log\left(\frac{M^2}{\mu^2_{(d=6)}}\,\log\frac{\Lambda^2_{(d=4)}}{\mu^2_{(d=4)}}\right) \tag{4.46}$$

This reinterpretation gives rise to a few more four-dimensional nonstandard counterterms, which we will comment upon in a moment. Let us stress, for the time being, that the logarithmic divergence, when renormalizing (correctly) from six dimensions is not identical to the one (4.37). It comes from the restriction of the six-dimensional a_4 to the zero mode approximation and reads

$$\begin{aligned}\Delta S_{\log} &= \int \frac{d^4x}{(4\pi)^3}\, e^2 \left[\frac{4}{3}\left(\bar{F}^{\mu\nu}\bar{F}_{\mu\nu} - 2\bar{\phi}^*\Box\bar{\phi}\right)\right. \\ &\quad +4\left(\bar{\eta}^1 \not{\partial}\eta^1 + \bar{\eta}^2\not{\partial}\eta^2\right) + 12m\left(\bar{\eta}^1\eta^1 - \bar{\eta}^2\eta^2\right) \\ &\quad \left. -4e\left(\bar{\eta}^1\bar{\not{A}}\eta^1 + \bar{\eta}^2\bar{\not{A}}\eta^2 + \bar{\eta}^1\bar{\phi}\eta^2 - \bar{\eta}^2\bar{\phi}^*\eta^1\right)\right] \log\frac{\Lambda^2_{d=4}}{\mu^2_{d=4}} \end{aligned} \tag{4.47}$$

The scalars A_5 and A_6 are now protected by the six dimensional symmetries, as they should be. In particular, we do not obtain neither a mass term nor a quartic interaction.

Were we to stick to dimensional regularization, we would have to compare the four dimensional counterterm with the massless sector of the six-dimensional one, which was previously determined in equation (4.10). There are then two types of terms.

First of all, those terms which have negative dimension constants in front, which are precisely the ones not present in the original six-dimensional Lagrangian, yield in four dimensions counterterms with dimension six operators, suppressed by two powers of the Kaluza–Klein scale

$$\begin{aligned}\Delta S_{(1)} &= \frac{e^2}{64\pi^3 M^2\epsilon}\int d^4x \left[-\frac{1}{12}e^2\left(\bar{\eta}\Sigma_{\mu\nu\rho}\eta\right)^2 + \frac{19}{15}m\bar{\eta}\bar{D}_\mu\bar{D}^\mu\eta\right. \\ &\quad +\frac{2}{15}e\bar{\eta}\gamma_\nu\bar{D}_\mu\eta\bar{F}^{\mu\nu} - em\bar{\eta}\gamma^\mu\gamma^\nu\eta\bar{F}_{\mu\nu} - \frac{11}{45}\left(\bar{D}_\lambda\bar{F}_{\mu\nu}\right)^2 \\ &\quad \left. +\frac{23}{9}(\bar{D}_\mu\bar{F}_{\mu\nu})^2 + \ldots\right] \end{aligned} \tag{4.48}$$

where the dots stand for terms with contractions of index in the extra dimensions and e is the four-dimensional coupling. Then, there are the usual four-dimensional counterterms in the guise

$$\Delta S_{(2)} = -\frac{2e^2m^2}{64\pi^3M^2\epsilon}\int d^4x\left(\bar{\eta}\bar{\not{D}}\eta + 3m\bar{\eta}\eta + \frac{2}{3}\bar{F}^2_{\mu\nu}\right) \tag{4.49}$$

The six-dimensional mass m^2 can clearly be tuned so as to survive in the limit in which the Kaluza–Klein scale is pushed to infinity. We simply have to tune the dimensionless quantity

$$\frac{e^2m^2}{64\pi^3M^2\epsilon} \tag{4.50}$$

towards the true four-dimensional $\frac{e^2}{16\pi^2\epsilon}$, while keeping the six-dimensional mass m in its four-dimensional value. In such a way we recover *almost* all four dimensional counterterms, albeit with a different sign, which could be accounted for by changing the direction of the analytical continuation: $\epsilon_{d=6} = -\epsilon_{d=4}$.

We say almost, because it can easily be seen from these results that there is no room for the $|\phi|^2$ and $|\phi|^4$ counterterms, which appear when working upwards from four-dimensions, but do not appear in the zero mode of the six-dimensional counterterm.

The only (dim) hope is that these four-dimensional counterterms are actually cancelled when the full tower of Kaluza–Klein states is considered. The next subsection is devoted to see if this is possible in a natural way as was in the case of a curved spacetime scalar.

It seems indeed strange that no quartic interaction is generated when coming from six dimensions. No definite conclusions can be draw, however, because those effects are of order $O(\lambda^2)$, where λ is que quartic coupling constant, which means order $O(e^8)$ in our case. We have no right to keep those terms.

There is a very simple mapping from six-dimensional operators to four-dimensional ones, namely

$$\mathcal{O}_{(n)} \to \mathcal{O}_{(n-N)} \tag{4.51}$$

where N is the number of fields involved in the operator.

In that way it is seen that the reduction works at follows:

$$\begin{aligned} \mathcal{O}_{(5)} &\to \mathcal{O}_{(3)} \\ \mathcal{O}_{(6)} &\to \mathcal{O}_{(4)} \\ \mathcal{O}_{(7)} &\to \mathcal{O}_{(5)} \\ \mathcal{O}_{(8)} &\to \mathcal{O}_{(6)} \end{aligned} \tag{4.52}$$

except for

$$\mathcal{O}^2_{(8)} \to \mathcal{O}^2_{(5)} \tag{4.53}$$

In four dimensions, all operators with dimension higher than four appear necessarily with coefficients which get inverse powers of the compactification scale, M. We should be then pretty confident of all results gotten in the limit in which this scale goes to infinity.

Another question is what happens in the chiral limit. If the mass of the fermion vanishes, then the six-dimensional counterterms do *not* include the four-dimensional ones. If we think about it, the conclusion is almost unavoidable, because there is no parameter in the lagrangian with the dimension of mass. The inverse coupling constant does not qualify for this, because it is never going to appear in a perturbative computation.

4.4 The full tower of four-dimensional divergences

Let us consider now the problem of the divergences of the four-dimensional theory with the whole Kaluza–Klein tower. We intend to compute the counterterm associated with the full four-dimensional Lagrangian (4.29). We let the index $n = (n_5, n_6)$ run over the whole tower of each field. Notice that the bosonic fields are now complex (except the one corresponding to $n = 0$). N is the complex mass number $N = \frac{n_6}{R_6} + i\frac{n_5}{R_5}$. We have also defined $\bar{\phi}_n \equiv \bar{A}_5^n - i\bar{A}_6^n$ and $\bar{\phi}_n^* \equiv \bar{A}_5^n + i\bar{A}_6^n \neq (\bar{\phi}_n)^* = \bar{A}_5^{-n} + i\bar{A}_6^{-n}$. Each element of this matrices has to be understood as a $m \times n$ matrix, m and n running over the tower (from $-\infty$ to $+\infty$).

As we have said the massive ($n \neq 0$) modes are complex. In order to use the algorithm explained in the appendix we have to double this modes into real and imaginary parts. However it is also possible to do the calculations with the complex fields and introduce at the end some extra factors in the adequate terms. After squaring the matrices and performing the supertraces we get with some labor the following counterterms in four dimensions

$$a_2 = \int \frac{d^4x}{(4\pi)^2} \sum_l \left[8m^2 - 4|L|^2 - 8e^2 \sum_n \bar{\phi}_n^* \bar{\phi}_{-n} + 8e \left(L^* \bar{\phi}_0 - L \bar{\phi}_0^* \right) \right] \tag{4.54}$$

The mode sum can be regularized and performed with the help of a zeta function. We shall do it in the next section, when working out the reduction of four-dimensional QED on a two-torus. It will turn out that there is no clear way of discriminating between the different possible regularizations. The fourth heat kernel coefficient is quite messy indeed

$$\begin{aligned}
a_4 \;=\; & \int \frac{d^4x}{(4\pi)^2} \sum_l \Bigg[\frac{4}{3}e^2 \sum_n \bar{F}^n_{\mu\nu}\bar{F}^{\mu\nu}_{-n} + 4e^2 \sum_n |N|^2 \bar{A}^n_\mu \bar{A}^\mu_{-n} \\
& -4e^2 \sum_n N^* \partial_\mu \bar{\phi}_n \bar{A}^\mu_{-n} + 4e^2 \sum_n N \partial_\mu \bar{\phi}^*_n \bar{A}^\mu_{-n} - 4e^2 \sum_n \bar{\phi}^*_n \Box \bar{\phi}_{-n} \\
& -8e \left(m^2 + |L|^2\right)\left(L\bar{\phi}^*_0 - L^* \bar{\phi}_0\right) - 4e^2 \sum_n (N+L)\, L \bar{\phi}^*_n \bar{\phi}^*_{-n} \\
& -4e^2 \sum_n (N^* + L^*)\, L^* \bar{\phi}_n \bar{\phi}_{-n} + 8e^2 \sum_n \left(m^2 + |L+N|^2 + |N|^2\right) \bar{\phi}^*_n \bar{\phi}_{-n} \\
& +8e^3 \sum_{m,n} \bar{\phi}^*_{m-l} \bar{\phi}_{l-n} \left(M \bar{\phi}^*_{n-m} - N^* \bar{\phi}_{n-m}\right) + 4e^2 \sum_{m,n,s} \bar{\phi}^*_{m-l} \bar{\phi}_{l-s} \bar{\phi}^*_{s-n} \bar{\phi}_{n-m} \\
& +8e^2 \sum_{n\neq 0} \left(\bar{\eta}^1_{l-n} \partial\!\!\!/ \eta^1_{l-n} + \bar{\eta}^2_{l-n} \partial\!\!\!/ \eta^2_{l-n}\right) + 24me^2 \sum_{n\neq 0} \left(\bar{\eta}^1_{l-n}\eta^1_{l-n} - \bar{\eta}^2_{l-n}\eta^2_{l-n}\right) \\
& -8e^3 \sum_{m\neq 0,n} \left(\bar{\eta}^1_{l-m} \bar{A\!\!\!/}_{l-n} \eta^1_{n-m} + \bar{\eta}^2_{l-m} \bar{A\!\!\!/}_{l-n} \eta^2_{n-m}\right) + 16e^3 \sum_{m\neq 0,n} \bar{\eta}^2_{l-m} \bar{\phi}^*_{l-n} \eta^1_{n-m} \\
& -16e^3 \sum_{m\neq 0,n} \bar{\eta}^1_{l-m} \bar{\phi}_{l-n} \eta^2_{n-m} + 16e^2 \sum_{n\neq 0} L^* \bar{\eta}^2_{l-n} \eta^1_{l-n} + 16e^2 \sum_{n\neq 0} L \bar{\eta}^1_{l-n} \eta^2_{l-n} \\
& +4e^2 \left(\bar{\eta}^1_l \partial\!\!\!/ \eta^1_l + \bar{\eta}^2_l \partial\!\!\!/ \eta^2_l\right) - 4e^3 \sum_n \left(\bar{\eta}^1_n \bar{A\!\!\!/}_{n-l} \eta^1_l + \bar{\eta}^2_n \bar{A\!\!\!/}_{n-l} \eta^2_l\right) + 12me^2 \left(\bar{\eta}^1_l \eta^1_l \right. \\
& \left. -\bar{\eta}^2_l \eta^2_l\right) + 8e^3 \sum_n \bar{\eta}^2_n \bar{\phi}^*_{n-l} \eta^1_l - 8e^3 \sum_n \bar{\eta}^1_n \bar{\phi}_{n-l} \eta^2_l + 8e^2 L^* \bar{\eta}^2_l \eta^1_l + 8e^2 L \bar{\eta}^1_l \eta^2_l \\
& + \left(2|L|^4 - 4m^4 - 8m^2|L|^2\right) \Bigg]
\end{aligned} \tag{4.55}$$

At least, one thing is clear: there is no natural way to perform a clever resummation (like the one Duff and Toms did in the free case) in order to cancel the four dimensional counterterms for both $|\phi|^2$ and $|\phi|^4$, for the simple reason that there is no generalized ζ-function associated with a purely extra-dimensional operator. We will comment on other possible regularizations for the divergent sums in the following sections. This fact was not obvious *a priori* and the doubt about it was the main reason why this computation was performed.

4.5 The true four-dimensional renormalization

From our point of view, in which the full theory is defined in six dimensions, the true renormalization is the one that is obtained via an harmonic expansion of the six-dimensional counterterm(s). With the interpretation of the six-dimensional cutoff we have advocated, the four-dimensional logarithmic divergences read

$$
\begin{aligned}
\Delta S_{\log} \equiv & \int \frac{d^4x}{(4\pi)^3} e^2 \sum_n \Big[4\left(\bar{\eta}^1_n \partial\!\!\!/ \eta^1_n + \bar{\eta}^2_n \partial\!\!\!/ \eta^2_n + N \bar{\eta}^1_n \eta^2_n + N^* \bar{\eta}^2_n \eta^1_n\right) \\
& + 12m\left(\bar{\eta}^1_n \eta^1_n - \bar{\eta}^2_n \eta^2_n\right) + \frac{4}{3}\left(\bar{F}^{\mu\nu}_{-n} \bar{F}^n_{\mu\nu} + 2|N|^2 \bar{A}^{\mu}_{-n} \bar{A}^n_{\mu}\right) \\
& - 4i\partial_\mu \bar{A}^\mu_{-n}\left(\frac{n_5}{R_5}\bar{A}^n_5 + \frac{n_6}{R_6}\bar{A}^n_6\right) + 2\bar{A}^{-n}_5\left(-\Box + \frac{n_6^2}{R_6^2}\right)\bar{A}^n_5 \\
& + 2\bar{A}^{-n}_6\left(-\Box + \frac{n_5^2}{R_5^2}\right)\bar{A}^n_6 - 4\frac{n_5 n_6}{R_5 R_6}\bar{A}^{-n}_5 \bar{A}^n_6 - 4e\sum_m \left(\bar{\eta}^1_m \bar{A\!\!\!/}_{m-n} \eta^1_n\right. \\
& \left. + \bar{\eta}^2_m \bar{A\!\!\!/}_{m-n} \eta^2_n + \bar{\eta}^1_m \bar{\phi}_{m-n} \eta^2_n - \bar{\eta}^2_m \bar{\phi}^*_{m-n} \eta^1_n\right) \Big] \log \frac{\Lambda^2_{d=4}}{\mu^2_{d=4}}
\end{aligned} \tag{4.56}
$$

In addition to that, there are the $\log^2$ divergences, coming from the quartic divergences in six dimensions. Those are trivial in our case, because they do not depend on the background fields.

Finally, there are the log log divergences, stemming from the logarithmic divergence in six dimensions. This divergence is suppressed by the scale of compactification. The result of a somewhat heavy computation, keeping terms up to cubic order in the four-dimensional electric charge, is

$$\begin{aligned}\Delta S_{\log\log} \equiv & \int \frac{d^4x}{(4\pi)^3}\,\frac{e^2}{M^2}\,\sum_n \Big[-\bar{F}^{-n}_{\mu\nu}\left(\frac{11}{45}\left(-\Box+|N|^2\right)+\frac{4}{3}m^2\right)\bar{F}^{\mu\nu}_n \\ & -m\bar{\eta}^1_n\left(\frac{19}{15}\left(-\Box+|N|^2\right)+2m\partial\!\!\!/+6m^2\right)\eta^1_n \\ & +m\bar{\eta}^2_n\left(\frac{19}{15}\left(-\Box+|N|^2\right)-2m\partial\!\!\!/+6m^2\right)\eta^2_n \\ & -2m^2\left(N\bar{\eta}^1_n\eta^2_n+N^*\bar{\eta}^2_n\eta^1_n\right)+\frac{23}{9}\partial_\mu\bar{F}^{\mu\nu}_{-n}\left(\partial^\rho\bar{F}^n_{\rho\nu}-2|N|^2\bar{A}^n_\nu\right) \\ & +\bar{A}^\mu_{-n}\left(\frac{31}{15}\left(-|N|^2\Box+|N|^4\right)-\frac{8}{3}m^2|N|^2\right)\bar{A}^n_\mu \\ & -i\partial_\mu\bar{A}^\mu_{-n}\left(\frac{62}{15}\left(-\Box+|N|^2\right)-\frac{16}{3}m^2\right)\left(\frac{n_5}{R_5}\bar{A}^n_5+\frac{n_6}{R_6}\bar{A}^n_6\right) \\ & +\bar{A}^{-n}_5\left(\frac{31}{15}\left(-\Box+|N|^2\right)-\frac{8}{3}m^2\right)\left(-\Box+\frac{n_6^2}{R_6^2}\right)\bar{A}^n_5 \\ & +\bar{A}^{-n}_6\left(\frac{31}{15}\left(-\Box+|N|^2\right)-\frac{8}{3}m^2\right)\left(-\Box+\frac{n_5^2}{R_5^2}\right)\bar{A}^n_6 \\ & -\frac{n_5n_6}{R_5R_6}\bar{A}^{-n}_5\left(\frac{62}{15}\left(-\Box+|N|^2\right)-\frac{16}{3}m^2\right)\bar{A}^n_6\Big] \\ & \times\log\left(\frac{M^2}{\mu^2_{(d=6)}}\log\frac{\Lambda^2_{(d=4)}}{\mu^2_{(d=4)}}\right)+O(e^3)\end{aligned} \tag{4.57}$$

In the case of dimensional regularization, the true divergences only come from the sixth coefficient, which yields the log log divergences we just wrote down. This means that in addition to the already mentioned counterterms to the zero modes there are a full tower of counterterms involving six-dimensional operators.

It is of interest to specialize to the massless case ($m=0$), in which, as we have already noticed, no ordinary dimension four operator is recovered:

$$\begin{aligned}a_6 = & \int \frac{d^4x}{(4\pi)^3}\frac{e^2}{M^2}\Big[\frac{23}{9}\partial_\mu\bar{F}^{\mu\nu}_{-n}\left(\partial^\rho\bar{F}^n_{\rho\nu}-2|N|^2\bar{A}^n_\nu\right)-\frac{11}{45}\bar{F}^{-n}_{\mu\nu}\left(-\Box+|N|^2\right)\bar{F}^{\mu\nu}_n \\ & +\frac{31}{15}|N|^2\bar{A}^\mu_{-n}\left(-\Box+|N|^2\right)\bar{A}^n_\mu-i\frac{62}{15}\partial_\mu\bar{A}^\mu_{-n}\left(-\Box+|N|^2\right)\left(\frac{n_5}{R_5}\bar{A}^n_5+\frac{n_6}{R_6}\bar{A}^n_6\right) \\ & +\frac{31}{15}\bar{A}^{-n}_5\left(-\Box+|N|^2\right)\left(-\Box+\frac{n_6^2}{R_6^2}\right)\bar{A}^n_5+\frac{31}{15}\bar{A}^{-n}_6\left(-\Box+|N|^2\right)\left(-\Box+\frac{n_5^2}{R_5^2}\right)\bar{A}^n_6 \\ & -\frac{62}{15}\frac{n_5n_6}{R_5R_6}\bar{A}^{-n}_5\left(-\Box+|N|^2\right)\bar{A}^n_6\Big]+O(e^3)\end{aligned} \tag{4.58}$$

That is, in the chiral case there is no renormalization of the fermionic tower (at this order) whatsoever, which is *not* what happens from the four-dimensional point of view of the previous paragraph.

4.6 A renormalizable example

Let us now repeat this exercise in a situation that, although probably much less interesting from the physical point of view, is much better defined as a quantum theory, namely four-dimensional quantum electrodynamics (QED_4) on a two-torus. In this example everything in the mother theory is well defined (as long as we keep shy of the Landau pole). There is no ambiguity associated to how we define the cutoff scale in the extra-dimensional theory, or about the energy scale above which perturbation theory is not expected to be valid anymore. The reduced theory is a two-dimensional one, where all divergences are more or less trivial (essentially normal ordering). It is nevertheless possible to analyze it with the very same general techniques. We perform this computation in order to get a template to compare with our previous six-dimensional example and to be sure that non-renormalizability is not playing the crucial role in our results.

Let us then consider QED_4 on a manifold $R^2 \times S^1 \times S^1$. The action is now

$$S = \int d^4x \left[\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(\not{D} + m)\psi \right] \tag{4.59}$$

where the abelian covariant derivative is simply:

$$D_\mu \psi \equiv (\partial_\mu - eA_\mu)\,\psi \tag{4.60}$$

The gauge coupling is dimensionless and the theory is famous to be renormalizable. In dimensional regularization the counterterm is the well known fourth coefficient in the small-time heat kernel expansion that we have already computed

$$a_4 \;=\; \int \frac{d^4x}{(4\pi)^2} \left[\frac{2}{3} e^2 \bar{F}_{\mu\nu}^2 + 2e^2 \bar{\eta}\gamma^\mu \bar{D}_\mu \eta + 8e^2 m \bar{\eta}\eta \right] \tag{4.61}$$

In the cutoff theory, this is precisely the coefficient of the logarithmic divergence, but there is a quadratic divergence as well

$$\Delta S = \frac{1}{\epsilon} \left(a_2 \Lambda_{d=4}^2 + a_4 \log \frac{\Lambda_{d=4}^2}{\mu_{d=4}^2} \right) \tag{4.62}$$

where

$$a_2 = \int \frac{d^4x}{(4\pi)^2} 4m^2 \tag{4.63}$$

In order to dimensionally reduce the theory we consider the following representation of the Clifford algebra ($a = 1, 2$)

$$\gamma_a^{(4)} = \sigma_3 \otimes \sigma_a$$
$$\gamma_3^{(4)} = \sigma_1 \otimes 1$$
$$\gamma_4^{(4)} = \sigma_2 \otimes 1 \tag{4.64}$$

In that way, four-dimensional spinors split in two two-dimensional ones

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \tag{4.65}$$

It is direct to perform the integrals over the angular variables and obtain the gauge fixed action in two dimensions

$$\begin{aligned} S &= \int d^2x \sum_{n_3,n_4} \Big[\bar{\psi}_n^1 \partial\!\!\!/ \psi_n^1 + \bar{\psi}_n^2 \partial\!\!\!/ \psi_n^2 + N \bar{\psi}_n^1 \psi_n^2 + N^* \bar{\psi}_n^2 \psi_n^1 \\ &\quad + m\left(\bar{\psi}_n^1 \psi_n^1 - \bar{\psi}_n^2 \psi_n^2\right) + \frac{1}{2}(A_a^n)^* \left(-\Box + |N|^2\right) A_n^a \\ &\quad + \frac{1}{2}(A_3^n)^* \left(-\Box + |N|^2\right) A_3^n + \frac{1}{2}(A_4^n)^* \left(-\Box + |N|^2\right) A_4^n \\ &\quad - e \sum_m \left(\bar{\psi}_m^1 A\!\!\!/_{m-n} \psi_n^1 + \bar{\psi}_m^2 A\!\!\!/_{m-n} \psi_n^2 + \bar{\psi}_m^1 A_3^{m-n} \psi_n^2 \right. \\ &\quad \left. - \bar{\psi}_m^2 A_3^{m-n} \psi_n^1 - i \bar{\psi}_m^1 A_4^{m-n} \psi_n^2 - i \bar{\psi}_m^2 A_4^{m-n} \psi_n^1\right) \Big] \end{aligned} \tag{4.66}$$

Where we have defined again a complex mass number $N = \frac{n_4}{R_4} + i\frac{n_3}{R_3}$. The two-dimensional coupling constant has positive mass dimension

$$e \equiv \frac{e^{(4)}}{2\pi\sqrt{R_3 R_4}} \equiv e^{(4)} M \tag{4.67}$$

In two dimensions, gauge fields are dimensionless and so are scalar fields. Fermionic fields enjoy mass dimension $[\psi] = 1/2$. We hope that there would arise no confusion for the use of the same symbol e for both coupling constants. The zero mode of this action is

$$\begin{aligned} S = \int d^2x \Big[&\bar{\psi}^1 \partial\!\!\!/ \psi^1 + \bar{\psi}^2 \partial\!\!\!/ \psi^2 + m\left(\bar{\psi}^1 \psi^1 - \bar{\psi}^2 \psi^2\right) - \frac{1}{2} A_a \Box A^a - \\ &- \frac{1}{2}\phi^* \Box \phi - e\left(\bar{\psi}^1 A\!\!\!/ \psi^1 + \bar{\psi}^2 A\!\!\!/ \psi^2 + \bar{\psi}^1 \phi \psi^2 - \bar{\psi}^2 \phi^* \psi^1\right)\Big] \end{aligned} \tag{4.68}$$

where we have represented as in the previous paragraph the zero modes of all fields by the same letter without any subindex

$$A_3^0 - i A_4^0 \equiv \phi^0 \equiv \phi \tag{4.69}$$

If we define the theory by dimensional renormalization, the counterterm associated to the above action is

$$\Delta S_{zm} = \frac{1}{\epsilon} a_2^{(0)} = \frac{1}{\epsilon} \int \frac{d^2x}{4\pi} 4 \left[m^2 - e^2 |\phi|^2 \right] \tag{4.70}$$

If instead we consider the whole tower the corresponding counterterm is given in terms of the complex mass parameter

$$L \equiv \frac{l_4}{R_4} + i \frac{l_3}{R_3} \tag{4.71}$$

$$\Delta S_{tower} = \frac{1}{\epsilon} a_2 = \frac{1}{\epsilon} \int \frac{d^2x}{4\pi} \sum_l 4 \left[m^2 - |L|^2 - e^2 \sum_n \bar{\phi}_n^* \bar{\phi}_{-n} + e \left(L^* \bar{\phi}_0 - L \bar{\phi}_0^* \right) \right] \tag{4.72}$$

Once again radiative corrections generate a mass term for the unprotected scalars and also a tadpole. We have a sum of contributions from all higher modes. This is a divergent sum which needs regularization. In the expression for the tadpole, for example, we are forced to compute the sum

$$T(R) \equiv \sum_{n \in \mathbb{Z}} \frac{n}{R} \equiv \frac{1}{R} \sum_{n \in \mathbb{Z}} n \tag{4.73}$$

This can be regularized, for example, by imposing a cutoff [73]

$$\begin{aligned} \sum_{n=1} n &\equiv \lim_{\epsilon \to 0} \sum_{n=1} n e^{-\epsilon n} = \lim_{\epsilon \to 0} \sum_{n=1} - \frac{\partial}{\partial \epsilon} e^{-\epsilon n} = - \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \sum_{n=1} e^{-\epsilon n} = \\ &= - \lim_{\epsilon \to 0} \frac{\partial}{\partial \epsilon} \frac{1}{e^\epsilon - 1} = \lim_{\epsilon \to 0} \frac{e^\epsilon}{(e^\epsilon - 1)^2} = \lim_{\epsilon \to 0} \left(\frac{1}{\epsilon^2} - \frac{1}{12} \right) \end{aligned} \tag{4.74}$$

This clearly shows the divergence of the sum. When adopting a finite prescription, it is important to keep this in mind. One such finite prescription, quite natural, stems from a consideration of the Laplacian operator on the extra dimensions, Δ_y, whose eigenvalues are precisely

$$\lambda_l \equiv |L|^2 \tag{4.75}$$

The corresponding generalized ζ function is

$$\zeta(s) \equiv \sum_{l \neq 0} \left(|L|^2 \right)^{-s} \tag{4.76}$$

which happens to be a particular instance of Epstein's zeta function. This would lead to definite values for

$$\sum_l 1 \equiv \zeta(s = 0) + 1 = 0 \tag{4.77}$$

and

$$\sum_l |L|^2 \equiv \zeta(-1) = 0 \tag{4.78}$$

In order to evaluate the coefficient of the tadpole, it is not possible to use this same ζ function. One possibility is to use Riemann's ζ function

$$\zeta_R(s) \equiv \sum_{n=1} n^{-s} \tag{4.79}$$

so that, for example,

$$T(R) = \frac{1}{R}(\zeta_R(-1) - \zeta_R(-1)) = 0 \tag{4.80}$$

Actually this is a unavoidable consequence of any definition in which the first of Hardy's properties of the sum of a divergent series is satisfied, namely, if $\sum a_n = S$ then $\sum \lambda a_n = \lambda S$ (cf. [74], and the discussion in [75]).

It has to be acknowledged that the need to use two different zeta functions greatly diminishes the attractiveness of this whole procedure of resummation.

Ay any rate, in order to eliminate the tadpole, one would have in its case to shift the field:

$$\bar{\phi}_0 \to \bar{\phi}_0 - \frac{T}{m^2} \tag{4.81}$$

This shift would in turn affect the fermionic masses through the Yukawa couplings and convey another contribution to the fermion mass renormalization.

To compare the counterterms needed in both viewpoints, let us first concentrate upon dimensional renormalization. The mode expansion of the four-dimensional counterterm (4.61) is

$$\begin{aligned} a_4 \;=\; & \int \frac{d^2x}{(4\pi)^2}\frac{e^2}{M^2}\sum_n \Big[2\left(\bar{\eta}^1_n \partial\!\!\!/ \eta^1_n + \bar{\eta}^2_n \partial\!\!\!/ \eta^2_n + N\bar{\eta}^1_n\eta^2_n + N^*\bar{\eta}^2_n\eta^1_n\right) + 8m\left(\bar{\eta}^1_n\eta^1_n\right. \\ & \left. -\bar{\eta}^2_n\eta^2_n\right) + \frac{2}{3}\left(\bar{F}^{ab}_{-n}\bar{F}^n_{ab} + 2|N|^2\bar{A}^a_{-n}\bar{A}^n_a - 4i\partial_a\bar{A}^a_{-n}\left(\frac{n_3}{R_3}\bar{A}^n_3 + \frac{n_4}{R_4}\bar{A}^n_4\right) + \right. \\ & \left. +2\bar{A}^{-n}_3\left(-\Box + \frac{n_4^2}{R_4^2}\right)\bar{A}^n_3 + 2\bar{A}^{-n}_4\left(-\Box + \frac{n_3^2}{R_3^2}\right)\bar{A}^n_4 - 4\frac{n_3 n_4}{R_3R_4}\bar{A}^{-n}_3\bar{A}^n_4\right) \\ & -2e\sum_m\left(\bar{\eta}^1_m \bar{A}\!\!\!/_{m-n}\eta^1_n + \bar{\eta}^2_m \bar{A}\!\!\!/_{m-n}\eta^2_n + \bar{\eta}^1_m\bar{\phi}_{m-n}\eta^2_n - \bar{\eta}^2_m\bar{\phi}^*_{m-n}\eta^1_n\right)\Big] \end{aligned} \tag{4.82}$$

which has a zero mode

$$\begin{aligned} a_4^{(0)} \;=\; & \int \frac{d^2x}{(4\pi)^2}\frac{e^2}{M^2}\Big[2\left(\bar{\eta}^1\partial\!\!\!/\eta^1 + \bar{\eta}^2\partial\!\!\!/\eta^2\right) + 8m\left(\bar{\eta}^1\eta^1 - \bar{\eta}^2\eta^2\right) + \frac{2}{3}\bar{F}^{ab}\bar{F}_{ab} \\ & -\frac{4}{3}\bar{\phi}^*\Box\bar{\phi} - 2e\left(\bar{\eta}^1\bar{A}\!\!\!/\eta^1 + \bar{\eta}^2\bar{A}\!\!\!/\eta^2 + \bar{\eta}^1\bar{\phi}\eta^2 - \bar{\eta}^2\bar{\phi}^*\eta^1\right)\Big] \end{aligned} \tag{4.83}$$

In this case, it is plain that there are many differences between the detailed forms of the mode expansion of the renormalized four dimensional theory (4.83) and the renormalization of the two-dimensional mode expansion of the bare four-dimensional theory (4.72). In particular, the tadpole and the mass term for the scalars are absent, but we have shown that there is a way of regularizing the sums so that the contribution of the whole tower cancels out. Nevertheless, here we have a bunch of higher-dimensional operators that are needed to renormalize QED_4 and that there is no way to obtain from a purely two-dimensional computation.

Concerning the cutoff theory, the detailed analysis points to the same conclusions. When the dimensionally reduced theory is defined through a proper time cutoff, the counterterm is given precisely by

$$\Delta S = \frac{1}{\epsilon} a_2 \log \frac{\Lambda^2_{d=2}}{\mu^2_{d=2}} \tag{4.84}$$

To make the counterterms as similar as possible we could be tempted to identify

$$\frac{\Lambda^2_{d=4}}{M^2} \equiv \log \frac{\Lambda^2_{d=2}}{\mu^2_{d=2}} \tag{4.85}$$

If one is willing to do this, there are two things that happen. First of all, one never recovers the two dimensional correction to the mass of the scalar field,

$$e^2 |\phi|^2 \tag{4.86}$$

The reason is exactly the same as it was when reducing from six to four dimensions in the previous subsection, namely, the spontaneously nature of the breaking of Lorentz symmetry of the mother theory

$$O(1,3) \rightarrow O(1,1) \times O(2) \times O(2) \tag{4.87}$$

It is true that this correction vanishes when one considers the full tower and one is willing to regularize the sum using the zeta function approach. As we have pointed out, there is an implicit renormalization of the scalar mass involved in this regularization. It is nevertheless true that one can regularize the sum in such a way as to get essentially the same result for the dominant (logarithmic) divergence in both the mother and the daughter theories.

The second thing that happens, and this seems unavoidable, is that there are $\log \log \Lambda^2$ divergences coming from the a_4 four-dimensional counterterm (4.83), suppressed by appropriate powers of the Kaluza-Klein scale.

To conclude, even in this example, the two-dimensional theory never forgets its mother. We have tried to avoid the complications that arise when dealing with a non-renormalizable theory by moving to four-dimensional QED. This exercise fully supports the general conclusions of the previous subsections.

4.7 Concluding remarks

Two radically different ways to define QED_6 at a one-loop level have been explored. The lessons of this exercise seem to be as follows.

When the fundamental theory is defined in dimensions higher than four using dimensional regularization, the divergences calculated from the whole tower of four dimensional fields do not match the ones of the extra-dimensional (mother) one. This is true even in the zero volume limit, when the volume of the extra dimensions is shrunk to zero, and the Kaluza–Klein scale correspondingly goes to infinity, as we have shown in detail in an explicit six-dimensional example. This had been already observed in [4].

In other words, the theory never forgets its higher dimensional origin. This is most clearly seen in the chiral limit, but appears also in the massive case, with the need of taking into account counterterms involving higher dimensional operators, whose coefficients can be computed in an unambiguous and straightforward way. We understand that a need for those counterterms has been hinted at in [2] and [3].

The full set of four-dimensional counterterms can be easily recovered from the six-dimensional one by performing an harmonic expansion. This yields what is, in our opinion, *the* correct way of renormalizing Kaluza–Klein theories.

In the chiral case (as well as when coming from an odd number of spacetime dimensions) the four-dimensional counterterms are simply not contained in the higher dimensional ones. The appropriate procedure in those cases would be, from our point of view, to compute in the mother theory (in which finite results are obtained through the use of dimensional regularization), and then perform the mode expansion.

Alternatively, when the quantum theory is defined through a proper time cutoff, we recover the four dimensional logarithmic divergences via a tuning of the six-dimensional cutoff. There are then calculable $\log\,\log\,\Lambda^2$ divergences coming from the six-dimensional logarithmic divergences as a reminder of the sicknesses of the mother theory. Those divergences are, however, suppressed by appropriate powers of the compactification scale, which means that they are multiplied by a small coefficient at energies at which six-dimensional perturbation theory is reliable (essentially $E/M \ll \alpha_{d=4}^{-1}$).

In neither case do we find from six dimensions corrections to the potential energy of the four-dimensional singlet scalars associated to the zero modes of the extra-dimensional legs of the gauge field. This being true for the zero mode, is clearly a low energy effect, well within the range of validity of the one-loop six-dimensional calculation. Those corrections are found in four dimensions because there is no gauge symmetry to prevent that to happen.

We have repeated the analysis for QED_4 on a two-torus, getting similar results. This is very important, because there is now no ambiguity as to how to define the extra-dimensional theory. This shows that our main results do not stem from the ambiguities inherent in any practical approach to a non-renormalizable theory.

There are no special difficulties with either odd-dimensional spaces (cf. for example [76]) or massless fermions from the viewpoint of the cutoff theory. Let us finally stress that the strictest equivalence can be achieved for free theories coupled to the gravitational field in dimensional regularization. In that case the operator describing the action at one loop does split into a four-dimensional and an extra-dimensional piece. This factorization makes the reasonings much more transparent and the origin of the mismatch clearer. According to the results of [4], it may happen that we are not taking into account finite contributions from each of the modes that, when adding up the infinite tower, can give a divergent contribution.

Our results have obvious applications to the study of the range of validity of the low energy effective four-dimensional models when studying Kaluza–Klein theories (cf. for example [77]) because our framework is consistent by construction (that is, to the extent that the six-dimensional model is consistent). Although a very simple abelian model has been studied as an example, there is no reason to expect our main results to change in more complicated non-abelian situations.

5 Renormalized masses

In this section we will explore some possible physical consequences of the inequivalence between renormalizing the theory directly in higher dimensions and naively doing it in four dimensions with the tower.

In order to do that we will study the vacuum polarization of an Abelian gauge theory, which is a valuable source of information. It conveys information on the running of the corresponding gauge coupling and in principle it can be used to compute would-be radiative corrections to the mass of the gauge bosons. These corrections vanish in the usual four dimensional theory owing to gauge invariance; that is, the Ward–Takahashi identity. However, this last statement cannot be directly applied to a theory defined in dimensions greater than four because of its peculiarities, in particular, the presence of an infinite tower of Kaluza–Klein states from the four dimensional point of view.

This has motivated a vast number of studies on these issues, including the possibility of a power law behaviour of the couplings [2, 58] and finiteness of the radiative Higgs mass in gauge-Higgs unification models [78, 79]. As we have mentioned, in this models the Higgs is identified with the extra components of a gauge field in higher dimensions. We want to focus our attention on the calculation of the radiative mass of the extra dimensional gauge boson with trivial holonomy, i.e., we will not consider noncontractible Wilson loops. This sector is believed to give operators that are non-local in the complete spacetime and therefore cannot contribute to the divergent piece.

The physical intuition behind these ideas is that higher dimensional gauge invariance somewhat protects the Higgs from getting radiative contributions to its mass. And for this to be true, it is plain that at very short distances, physics must be really higher- dimensional.

There are essentially three different ways to compute these corrections. We shall comment on them in turn, and argue eventually that if we want to formally implement the aforementioned ideas, a full higher dimensional computation is mandatory.

The correction has been often computed diagrammatically once the mode expansion and the integral over the compact manifold had been performed, which means that in some sense this computation is purely four-dimensional because the Feynman rules applied correspond to a theory with an infinite number of KK modes and their corresponding interactions (see for example [80, 81]). The result of this kind of calculations is a one loop finite mass for the Higgs field proportional to the compactification scale.

5.1 Four-dimensional vacuum polarization.

We will start by reviewing the four-dimensional calculation in order to illustrate its inherent difficulties. We will follow closely the computation done in [80] but performing the sum over the extra dimensional momentum at the end. Consider the vacuum polarization function of quantum electrodynamics in five dimensions (QED_5). If one of the dimensions corresponds to a circle S^1 with radius R then the momentum in that dimensions is quantized in units of $R^{-1} \equiv M$ and the integral has to be replaced by a sum. Taking into account the Feynman rules the vacuum polarization has the form ($p^2 = p_\mu p^\mu$)

$$\begin{aligned} i\Pi_{\mu\nu}(p^2, p_5^2) &= -e^2 \sum_{k_5} \int \frac{d^4k}{(2\pi)^4} \operatorname{tr}\left(\gamma_\mu \frac{1}{\not{k} + i\gamma^5 k_5} \gamma_\nu \frac{1}{(\not{k} - \not{p}) + i\gamma^5 (k_5 - p_5)}\right) \\ &= -4e^2 \sum_{k_5} \int \frac{d^4k}{(2\pi)^4} \frac{k_\mu (k_\nu - p_\nu) + k_\nu (k_\mu - p_\mu) - g_{\mu\nu} k (k-p) + g_{\mu\nu} k_5 (k_5 - p_5)}{(k^2 - k_5^2)\left((k-p)^2 - (k_5 - p_5)^2\right)} \end{aligned} \tag{5.1}$$

Introducing a Feynman parameter and doing the usual shift in the four-momentum $k'_\mu = k_\mu - \alpha p_\mu$ as well as a shift in the compact dimension $k'_5 = k_5 - \alpha p_5$ we get

$$i\Pi_{\mu\nu} = -4e^2 \sum_{k_5} \int_0^1 d\alpha \int \frac{d^4k}{(2\pi)^4} \frac{N_{\mu\nu}}{\left(k^2 - k_5'^2 + \alpha(1-\alpha)(p^2 - p_5^2)\right)^2} \tag{5.2}$$

Where the numerator is

$$N_{\mu\nu} = 2k_\mu k_\nu + g_{\mu\nu}\left(-k^2 + \alpha(1-\alpha)\left(p^2 - p_5^2\right) + (2\alpha - 1) p_5 k'_5 + k_5'^2\right) - 2\alpha(1-\alpha) p_\mu p_\nu \tag{5.3}$$

And we have neglected terms linear in k_μ which vanish because of the angular integral. Let us then split the vacuum polarization into two pieces.

$$\Pi_{\mu\nu} \equiv g_{\mu\nu}\Pi_1 - p_\mu p_\nu \Pi_2 \tag{5.4}$$

After Wick rotation to Euclidean space

$$\begin{aligned} \Pi_1 &= -4e^2 \sum_{k_5} \int_0^1 d\alpha \int \frac{d^4k}{(2\pi)^4} \frac{\frac{k^2}{2} + \alpha(\alpha - 1)(p^2 + p_5^2) + (2\alpha - 1) p_5 k'_5 + k_5'^2}{\left(k^2 + k_5'^2 + \alpha(1-\alpha)(p^2 + p_5^2)\right)^2} \\ \Pi_2 &= 8e^2 \sum_{k_5} \int_0^1 d\alpha (1-\alpha)\alpha \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left(k^2 + k_5'^2 + \alpha(1-\alpha)(p^2 + p_5^2)\right)^2} \end{aligned} \tag{5.5}$$

Using a proper time parametrization the first piece can be put into the form

$$\Pi_1 = -4e^2 \sum_{k_5} \int_0^1 d\alpha \int_0^\infty d\tau\tau \int \frac{d^4k}{(2\pi)^4} \left(\frac{k^2}{2} + \alpha(\alpha-1)\left(p^2+p_5^2\right) + (2\alpha-1)\, p_5 k_5' + k_5'^2 \right) e^{-\tau\left(k^2+k_5'^2+\alpha(\alpha-1)\left(p^2+p_5^2\right)\right)} \tag{5.6}$$

The integral in momentum space is obviously quadratically divergent, but it can be computed in dimensional regularization:

$$\Pi_1 = -\frac{4e^2\pi^{n/2}}{(2\pi)^n} \sum_{k_5} \int_0^1 d\alpha \int_0^\infty d\tau\tau \left(\frac{\alpha(\alpha-1)(p^2+p_5^2) + (2\alpha-1)\, p_5 k_5' + k_5'^2}{\tau^{\frac{n}{2}}} + \frac{n}{4\tau^{\frac{n}{2}+1}} \right) e^{-\tau\left(k_5'^2+\alpha(\alpha-1)\left(p^2+p_5^2\right)\right)} \tag{5.7}$$

It is now easy to perform the integral in proper time and particularize to $n = 4+\epsilon$ dimensions to get

$$\begin{aligned} \Pi_1 &= -\frac{4e^2\pi^{\frac{n}{2}}}{(2\pi)^n}\Gamma\left(2-\frac{n}{2}\right) \sum_{k_5} \int_0^1 d\alpha \left(\frac{n}{4\left(1-\frac{n}{2}\right)} \left(k_5'^2 + \alpha(\alpha-1)\left(p^2+p_5^2\right)\right)^{\frac{n}{2}-1} \right. \\ &\quad \left. + \left(\alpha(\alpha-1)\left(p^2+p_5^2\right) + (2\alpha-1)\, p_5 k_5' + k_5'^2\right)\left(k_5'^2 + \alpha(1-\alpha)\left(p^2+p_5^2\right)\right)^{\frac{n}{2}-2} \right) \\ &= \frac{e^2}{12\pi^2}\Gamma\left(-\frac{\epsilon}{2}\right)\left(p^2+p_5^2+\frac{1}{2}p_5^2\right)\sum_{k_5} 1 \end{aligned} \tag{5.8}$$

Analogous manipulations with Π_2 yield

$$\Pi_{\mu\nu}(p^2, p_5^2) = \frac{e^2}{12\pi^2}\Gamma\left(-\frac{\epsilon}{2}\right)\left(\left(p^2+p_5^2+\frac{1}{2}p_5^2\right) g_{\mu\nu} - p_\mu p_\nu\right)\sum_{k_5} 1 \tag{5.9}$$

Note that the vacuum polarization of the four-dimensional photon $A_\mu^{(0)}$ (which means $p_5 = 0$) verifies the Ward–Takahashi identity (see the Appendix)

$$p^\mu \Pi_{\mu\nu}(p^2, p_5 = 0) = 0 \tag{5.10}$$

From a four dimensional point of view this result is not surprising at all. For fixed k_5 it corresponds to the contribution of a single fermionic loop. If we now consider an infinite number of fermions coupled with the same strength to the gauge bosons we have an additional divergence coming from the sum over the whole tower[9]. The

[9]It is nevertheless true that this divergent sum can be regularized in such a way that it gives a vanishing contribution, for example

$$\sum_{k_5=-\infty}^{\infty} 1 = 1 + 2\zeta_R(0) = 0 \tag{5.11}$$

computation done in the Appendix with the tower of modes in four dimensions seems to support this conclusion. For this model, if we define the renormalized field and mass ($m_n^2 = \frac{n^2}{R^2}$)

$$A^n_{\mu(0)} = Z_3^{1/2} A^n_\mu \tag{5.14}$$
$$m^2_{n(0)} = Z_m m_n^2 \tag{5.15}$$

Then we get

$$Z_3 = 1 + \frac{e^2}{3\pi^2\epsilon}\sum_l 1 \tag{5.16}$$
$$Z_m = 1 + \frac{e^2}{6\pi^2\epsilon}\sum_l 1 \tag{5.17}$$

With the renormalization group functions

$$\beta_e \equiv \frac{\partial e}{\partial \log \mu} = \frac{e^3}{12\pi^2}\sum_l 1 \tag{5.18}$$
$$\beta_{m_n} \equiv \frac{\partial m_n}{\partial \log \mu} = -\frac{e^2 m_n}{12\pi^2}\sum_l 1 \tag{5.19}$$

Notice that the beta function of the fine structure constant embodies an infinite number of identical fermion contributions. The behavior of the couplings is

$$\begin{gathered} e^2 = \frac{e_0^2}{1-\frac{e_0^2}{6\pi^2}\sum_l 1 \log\frac{\mu}{\mu_0}} \\ m_n = m_n^0\left(1-\frac{e_0^2}{6\pi^2}\sum_l 1\log\frac{\mu}{\mu_0}\right)^{\frac{1}{2}} \end{gathered} \tag{5.20}$$

The case of the scalar A^n_5, whose zero mode would play the role of the Higgs, is much more complicated. For technical aspects we refer to the Appendix. In any case one

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Unfortunately, we do not have an interpretation of this ζ as the one associated with the eigenvalues of a relevant operator. Consider the Laplacian on the circle, being the eigenvalues $\lambda_n = \frac{n^2}{R^2}$ and the corresponding ζ-function

$$\zeta(\Delta, s) = \sum_{n=-\infty}^{\infty}\left(\frac{n}{R}\right)^{-2s} = 2R^{2s}\zeta_R(2s) \tag{5.12}$$

Taking this "more natural" function would give the regularized value

$$\sum_{k_5=-\infty}^{\infty} 1 = \zeta(\Delta, 0) = -1 \tag{5.13}$$

thing seems clear: up to the subtleties in the regularization of the divergent sum, the correction is in principle not finite even for the zero mode in the chiral theory $m_f = 0$. In fact, since A_5^0 is massless at tree level we cannot absorb the divergence at one loop. For consistency of the theory one should include a mass term in the original Lagrangian

$$\mathcal{L}_m = \frac{1}{2} m_B^2 A_M A^M \supset \frac{1}{2} m_B^2 A_5^0 A_5^0 \tag{5.21}$$

But this is clearly non gauge-invariant (except precisely for the zero mode). Another possibility is to include by hand a mass term only for the zero mode in the compactified Lagrangian but it would make the theory lose all the advantages of gauge-Higgs unification coming from extra-dimensional gauge invariance and the usual problems associated with the mass of a scalar would reappear.

This interpretation is in contrast with the (also four-dimensional) one in [80] where a totally finite result was obtained[10]. In particular the correction to the mass of the Kaluza–Klein modes is universal and given by

$$\delta m^2 = -\frac{e^2 \zeta(3)}{4\pi^4} M^2 \tag{5.22}$$

In the approximation $p^2 = p_5^2$. The reason of the difference is of course the point where the sum over the extra-dimensional momentum is performed[11].

Suppose we are trying to do a purely five-dimensional calculation of the diagram. Before the compactification of the theory, let us say to $\mathbb{R}^4 \times S^1$, we have a full $O(1,4)$ invariance. In that case the momentum integral has trivially the property

$$\int \frac{d^5k}{(2\pi)^5} f(k^2) = \int \frac{d^4k}{(2\pi)^4} \int \frac{dk_5}{2\pi} f(k^2) = \int \frac{dk_5}{2\pi} \int \frac{d^4k}{(2\pi)^4} f(k^2) \tag{5.23}$$

Which means that it is strictly equivalent to perform the integral first over the extra dimension and then the four dimensional one or vice versa. If we now compactify the theory the full five-dimensional Lorentz invariance is spontaneously broken to $O(1,3) \times O(2)$. An essential ambiguity [12] appears then if we insist in interpreting

[10] Some authors [82,83] have found quadratically divergent corrections with similar calculations, which suggests that this kind of computation may be not very well established.

[11] In [80] a Poisson resummation is done before the proper time integral. Also the divergent piece coming from loops that do not wrap completely the compact dimension was subtracted. On the other hand, this divergence should vanish in dimensional regularization since we are in an odd dimensional space

[12] The ambiguity is related to considering k_5 as a component of the five-momentum, but usually

the diagrams as five-dimensional because clearly

$$\sum_{k_5} \int d^4k f(k_\mu, k_5) \neq \int d^4k \sum_{k_5} f(k_\mu, k_5) \tag{5.24}$$

When the integral (or the sum) is divergent. Those two alternatives are then the two different four-dimensional calculations we were referring to above.

This observation is not new and a lot of effort has been put into studying its possible consequences, also when the expressions are not formally divergent. In [77] a brane Gaussian distribution along the extra dimension was used to regularize the theory while Kaluza–Klein modes were not truncated. The integral can be performed and after the infinite sum the result is claimed to be finite. Similar conclusions were reached in [84] using Pauli–Villars and an adapted version of dimensional regularization. Both regulators are supposed to preserve the symmetries. The most explicit study of the validity of (5.24) is that of [85] were a method to dimensionally regularize KK sums using Mellin transform and analytic extension of special functions is proposed. With this procedure it is believed that the ambiguity is resolved. Works with a similar philosophy can be found in [86] where the tower is summed using a pole function and in [87] were the sum is regularized using a ζ-function.

In any case, we believe that none of these works is fully satisfactory and the controversy is not solved. Indeed, this mismatch in the results may be intimately related with the other questions raised in this Thesis, since clearly the computation we have done corresponds to the prescription

$$\sum_n \Gamma_n \tag{5.25}$$

On the other hand, the second possibility might be associated with a higher dimensional computation, since the divergent piece respects Lorentz invariance. The results in [2] corroborate this interpretation.

5.2 Six-dimensional vacuum polarization.

Now that we know the potential ambiguities of a four dimensional calculation we will try to avoid them by computing directly in the higher dimensional space. This

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it is treated as a mass term for the higher Kaluza–Klein modes. Then, it is natural to do the summation after the evaluation of a single diagram because in that case k_5 simply labels fermions with different masses.

alternative point of view is not only interesting but necessary because as we have already advertised its conclusions will turn out to be quite different from the ones drawn from a four-dimensional computation.

Suppose we have quantum electrodynamics on a six-dimensional manifold, as in the previous section. The theory is of course non-renormalizable because the coupling constant has mass dimension $[e] = -1$. Nevertheless it is possible to identify and study all divergences appearing at one loop order (or $O(e^2)$) as we have seen.

In dimensional regularization the one loop divergences are given by (see (4.10))

$$\begin{aligned} a_6 \;=\; \int \frac{d^6x}{(4\pi)^3} \Big[& -\frac{4}{3}e^2 m^2 \bar{F}_{MN}\bar{F}^{MN} - 2e^2m^2\bar{\eta}\gamma^M \bar{D}_M \eta - 6e^2m^3\bar{\eta}\eta \\ & - \frac{11}{45}e^2 \bar{D}_R \bar{F}_{MN} \bar{D}^R \bar{F}^{MN} + \frac{23}{9} e^2 \bar{D}_M \bar{F}^{MN} \bar{D}^R \bar{F}_{RN} \\ & + \frac{19}{15} e^2 m \bar{\eta} \bar{D}_M \bar{D}^M \eta \Big] + O(e^3) \end{aligned} \tag{5.26}$$

In addition to the counterterms corresponding to operators that were already present in the original Lagrangian higher order operators have been generated radiatively. The appearance or this terms was discussed in [2, 3]. If we want to absorb their divergences we must include them in the bare Lagrangian

$$\mathcal{L} = \mathcal{L}_0 + \mu D_M F^{MN} D^R F_{RN} + \lambda D_R F_{MN} D^R F^{MN} + \dots \tag{5.27}$$

Where $[\mu] = [\lambda] = -2$. We have written explicitly only the extra terms that are quadratic in the gauge field and therefore the ones that modify the extra-dimensional vacuum polarization. Once we perform the mode expansion the same operators will yield the mass of the tower coming from the gauge field. If we define

$$A^0_M = Z_3^{1/2} A_M \tag{5.28}$$

We get $Z_3 = 1 - \frac{e^2m^2}{12\pi^3\epsilon}$. It is easy to see then that the pole in F^2_{MN} is absorbed in the wave function renormalization of the gauge field so from an extra dimensional point of view there is no renormalization of the mass of the gauge boson. This is expected in some sense due to gauge invariance. In four dimensions it is well known that even if we include a mass term for the photon in the bare Lagrangian its mass does not renormalize. Nevertheless, gauge invariance is not enough to ensure a massless photon as we know from the Schwinger model in two dimensions. The lesson to learn from this is that the number of dimensions is crucial. Since in gauge-Higgs unification the Higgs boson is identified with the extra-dimensional components of the gauge field once the mode expansion has been performed its mass does not renormalize either.

Concerning higher order terms, its divergences can be absorbed in arbitrary dimensionful couplings like μ and λ in (5.27) if we define

$$\begin{aligned} \mu_0 &= Z_\mu \mu \\ \lambda_0 &= Z_\lambda \lambda \end{aligned} \tag{5.29}$$

The conclusion is the very same as for F_{MN}^2: once we have renormalized the theory in six dimensions the mass coming from the mode expansions does not renormalize because the divergences are absorbed in Z_3 and Z_μ,Z_λ. Of course, to all orders of perturbation theory we would need an infinite number of arbitrary couplings to fit with experiments and this is precisely the benchmark for a non renormalizable field theory.

Similar conclusions are obtained with a proper time cutoff. In that case the extra dimensional counterterm comes from a_4 and a_6

$$a_4 \Lambda_{(d=6)}^2 + a_6 \log \frac{\Lambda_{(d=6)}^2}{\mu_{(d=6)}^2} \tag{5.30}$$

where the fourth coefficient is

$$a_4 = \int \frac{d^6x}{(4\pi)^3} \left[\frac{4}{3} e^2 \bar{F}_{MN}^2 + 4e^2 \bar{\eta} \bar{\not{D}} \eta + 12 m e^2 \bar{\eta} \eta \right] \tag{5.31}$$

Again the possible pole in F_{MN}^2 is absorbed defining $A_M^0 = Z_3^{1/2} A_M$ with

$$Z_3 = 1 + \frac{e^2}{12\pi^3} \Lambda_{(d=6)}^2 - \frac{e^2 m^2}{12\pi^3} \log \frac{\Lambda_{(d=6)}^2}{\mu_{(d=6)}^2} \tag{5.32}$$

And divergences coming from higher order terms present in a_6 are absorbed in the same way.

It is interesting to study the effects of this extra operators at tree level. First of all they induce corrections to the mass of the gauge bosons once the compactification has been performed. For example in six dimensions compactification of (5.27) yields terms like

$$(\mu + 2\lambda) |N|^4 A_\mu^{-n} A_n^\mu \tag{5.33}$$

And similar ones (i.e. of order $(2\lambda + \mu) M^4$) for the scalar field. This last statement is not true in the five dimensional case. Observe that at one loop we find a renormalization of the dimensionful couplings μ and λ that induces a running for the masses through (5.33) which is suppressed by M^{-2} (with respect to the usual mass).

Concerning the propagator suppose now that we include higher order terms in the form

$$F_{MN}^2 + \frac{c_1}{\Lambda^2} F_{MN} \partial^2 F^{MN} + \frac{c_2}{\Lambda^2} F_{MN} \partial^M \partial_R F^{RN} \tag{5.34}$$

Where Λ is a parameter (naturally of the order of the compactification mass) in order to make c_1 and c_2 dimensionless. Then the propagator of the gauge field is

$$A_M D_{MN}^{-1} A_N = A_M \left(1 - \frac{2c_1 + c_2}{\Lambda^2} p^2\right) \left(p^2 \delta_{MN} - p_M p_N\right) A_N \tag{5.35}$$

It has the usual pole in $p = 0$, but also depending on the sign of the couplings c_1 and c_2 it can have another one

$$p^2 \sim \frac{\Lambda^2}{2c_1 + c_2} \tag{5.36}$$

It may be possible to use arguments [88] concerning superluminal fluctuations around non-trivial backgrounds to fix the sign of the couplings and avoid this second pole. In any case possible poles coming from this higher order terms can be absorbed in dimensionful coupling constants introduced in the bare Lagrangian in the form (5.27). Therefore, in some sense, the mass of the gauge field is protected from renormalization.

5.3 Concluding remarks.

In this section, we have tried to argue that a four dimensional calculation is at least ambiguous when one considers the theory at one loop. There are two different ways of computing diagrams according to the place where the sum is performed. When the sum is performed after the momentum integral (which seems to correspond to the calculation of the Appendix with the whole tower) usual four-dimensional divergences are found along with extra divergences coming from infinite sums that should be regularized. Also we find many problems with the divergence of the mass of the zero mode scalar because it is massless at tree level.

If we adopt the higher dimensional point of view with the purpose of renormalizing the theory then the possible counterterms are dictated by gauge and Lorentz invariance in the extra-dimensional manifold. This fixes the form of the possible mass terms for the four-dimensional gauge boson as well as the Higgs in gauge-Higgs unification. Therefore, it is easy to convince oneself that every divergence may be absorbed in the wave function renormalization of A_M and the renormalization of the couplings of higher dimension operators such as μ and λ in (5.27).

This approach is embodies in a straightforward manner the physical intuition, which we believe correct, that at very short distances all dimensions should appear at the same foot, and physics should be higher dimensional.

Conclusions

In the past few years there has been a strong interest in field theories defined in spacetimes of dimension greater than four. Such models, seen as low energy effective theories of a more fundamental consistent theory like superstrings, provide a new variety of very interesting mechanisms in order to solve long standing problems of the Standard Model.

Interesting possibilities are the idea of the Higgs particle originated from extra-dimensional components of gauge fields [53,54], often called gauge-Higgs unification, and alternative mechanisms for symmetry breaking [45,46]. The best known of this kind of proposals are probably Large Extra Dimensions [47] and warped scenarios [51]. These are only the original references, although the literature on the matter is incredibly large.

A common problem in higher dimensional models is the necessity to explain why extra dimensions are hidden, in the sense that the spacetime we experiment is effectively four-dimensional. Traditionally, extra dimensions are supposed to be compact and with a characteristic size extremely small so that we would need energies unattainable in present colliders in order to directly detect them. Compactness of the extra dimensions allows to expand fields propagating in the whole spacetime in harmonics and perform integrals over the extra coordinates. In that way we find a four-dimensional theory, but with an infinite number of fields corresponding to modes of the expansion: the so-called Kaluza–Klein modes.

We can then distinguish two viewpoints, the higher-dimensional and the four-dimensional with the tower. They are of course completely equivalent at the classical level. The question we have tried to answer in this Thesis was if this last statement remains true, and if so under what conditions, when one consider quantum corrections on both points of view.

All the effects we are interested in will show up already at one loop order. To this order, the effective action is given in terms of a functional determinant for the operator representing the quadratic part of the action. In many interesting theories, for instance the Standard Model, this last quantity is divergent. Extraction of the divergent part in a consistent way is the process of renormalization, in this case to one loop. There are several ways of identifying the divergences, for example diagrammatically in the sense of 't Hooft's algorithm [62] generalized to the appropriate dimension[13]. A more effective approach, specially on curved backgrounds, is the heat kernel [16, 60, 61].

Concerning higher-dimensional theories, it is then obvious that quantum equivalence requires the matching of the divergences on both points of view. The aim of this work was to explore whether this matching is possible or not.

It is important to say that in the particular case of a scalar interacting only through the universal coupling to an external gravitational field, after solving some subtleties, it is possible to perform a clever resummation of the modes in a way that divergences do coincide, although it is true that one finds counterterms that we should not expect in a purely four-dimensional computation, as shown by Duff and Toms in [4]. A crucial point in the argument is that the operators considered can be split into the form

$$\Delta = \Delta_1 + \Delta_2 \tag{5.37}$$

where Δ_1 acts trivially on the extra dimensional coordinates and Δ_2 acts trivially on the usual four-dimensional ones. In fact, their results can be *automatically* applied to any theory whose corresponding operator splits in such a way. Nevertheless, the result is not valid when the splitting does not take place, as happens on a warped background as well as for a general interacting theory. The fact that the divergences do not match if the geometry has warpings was called "dimensional reduction anomaly" in [70]. Later, it was noticed in [89] that the anomaly was due to a naive interpretation of the divergences so that using an approach similar to the one for the factorizable geometry the divergences do match[14]. It seems that none of the authors was aware of the Duff–Toms result.

Given the results presented in this work, a relevant question would be what are the necessary and sufficient conditions the operator must verify in order to have coincidence of the radiative corrections. Also, here we have only dealt with the ultraviolet divergent part of the action. There must be also a finite contribution coming from the non-contractible loops wrapping the extra dimensions that we are not seeing with the actual computation. In principle, this finite contribution could be different in both schemes. However, an explicit computation with the particular

[13]This algorithm is designed to give the poles in dimensional regularization in four dimensions, but it can be easily generalized to an arbitrary *even* dimension. If one uses a proper time cutoff instead of dimensional regularization it can also be applied to odd dimensions.

[14]The proof is perturbative in the heat kernel coefficients. They obtain coincidence in the first two and conjecture all orders matching.

extra-dimensional symmetry breaking model of [90] shows that, at least in this case, the finite parts do coincide [91], though we must point out that in this case the operator enjoys the above mentioned splitting. Calculating finite pieces for arbitrary operators with the heat kernel is not an easy task. Moreover, as far as we now, additional tools should be developed to take into account the periodicity of the extra-dimensions and the relevant non-contractible loops. The general case seems to be hardly tractable.

On the other hand, our main intention was to extend the analysis to an interacting theory, although without the complications of a curved background. We have focused our attention on a simple gauge theory, in particular Quantum Electrodynamics defined on a six-dimensional manifold $\mathbb{R}^4 \times S^1 \times S^1$. This theory is interesting because it is representative of the Universal Extra Dimensions paradigm. It also permits the study, from an alternative perspective, of a couple of questions relevant for the phenomenology of extra dimensions: the protection of the Higgs mass due to a gauge symmetry and the possibility of a power-law running for the gauge coupling.

An informed reader may notice that the corresponding action is non-renormalizable, since the gauge coupling has mass dimension $[e_6] = -1$. However, up to one loop this fact is not crucial in the sense that we can still identify and study all the divergences.

Let us think a moment what should we expect to find when one considers the one loop correction. As it is well known, the counterterms of the theory will be the most general six-dimensional operators compatible with the symmetries of the system, in this case a $U(1)$ gauge symmetry and Lorentz invariance. Thus the dimensionality of the coupling allows us to write terms like

$$e^2 D_M F^{MN} D^R F_{RN} \qquad ; \qquad e^2 D_R F_{MN} D^R F^{MN} \tag{5.38}$$

Despite they were not present in the original Lagrangian, the radiative generation of these operators is unavoidable: the power of the coupling shows that is a one loop effect, they are of the right dimension and have the correct invariance. The appearance or this terms was discussed in [2,3]. Another important point is that the symmetry forbids a mass term for the gauge field, so the bosonic zero modes remain massless at the quantum level. This is the implementation of the idea of gauge-Higgs unification. The explicit six-dimensional computation performed in [14, 15] agrees with these intuitions.

In order to perform a four-dimensional computation with the whole Kaluza–Klein tower one has to expand the fields in modes. Compactification of six-dimensional

QED on a two-torus gives the (gauge-fixed) action (4.29). One has to double the number of fermions because in d dimensions they have $2^{[d/2]}$ components (eight in six dimensions, four in four dimensions). Also the extra components of the gauge field A_5^n and A_6^n appear as four-dimensional scalars[15]. It is important to note that the spacetime symmetry is spontaneously broken to

$$O(6) \longrightarrow O(4) \times O(2) \times O(2) \tag{5.39}$$

While the extra-dimensional gauge symmetry traduces into the infinite set of four-dimensional symmetries

$$\begin{aligned} \delta A_\mu^n &= i\partial_\mu \Lambda_n \\ \delta A_5^n &= -\frac{n_5}{R_5}\Lambda_n \\ \delta A_6^n &= -\frac{n_6}{R_6}\Lambda_n \end{aligned} \tag{5.40}$$

Please note that the scalar zero modes are singlets under a gauge transformation. Finally the coupling is now dimensionless, as it is defined by

$$e \equiv \frac{e_6}{\sqrt{R_5 R_6}} \equiv e_6 M \tag{5.41}$$

Let us repeat the exercise done with the previous action and ask ourselves what kind of corrections one would expect. First of all, the coupling is dimensionless so we cannot use it to reduce the dimension of higher order operators. Therefore, terms like the ones in (5.38) are in principle forbidden, at least in perturbation theory. Nevertheless, there are pieces in the four-dimensional computation coming from higher order coefficients like a_6. For a single mode they are certainly finite, but when we sum over the infinite tower divergences can appear, and therefore generate higher order terms in the counterterms that in principle we do not expect by power-counting. A very specific resolution of the pole with generalized *zeta*-functions, valid when the operator enjoys (5.37), ensures that indeed this is the case. It may be worth exploring this possibility in detail for the interacting case at hands.

Next, since the scalar zero mode is singlet there are no symmetries to protect its mass against radiative corrections, as it happens with the Standard Model Higgs. Then we expect a mass term for it (in fact there is no reason not to expect operators of higher power, i.e. cubic or quartic interactions).

Another important point is that the gauge zero mode A_μ^0, which plays the role of the usual photon, couples diagonally to an infinite tower of fermions, with the

[15]They are identified with the Higgs in gauge-Higgs unification.

same strength as in four-dimensional QED and to every fermion. The only difference between the fermions of the tower is their masses, which are labelled by a pair of integers. Now, the pole in the Vacuum Polarization Function does not depend on the mass of the fermion running in the loop. We should have then the same contribution to the β-function as in QED for every fermion. Since the number of fermions is infinite, one has to sum the same quantity an infinite number of times. This gives rise to an additional divergence coming from the sum. One can think that this is the expected effect of an infinite number of fields interacting all to each other. When the operator splits like in the free case, there is again a natural regularization of the divergent sums in terms of the generalized ζ-function associated to the operator acting on the extra dimensions. This ζ-function may be related to the heat kernel coefficients of the operator and that is the reason behind the precise matching of the divergences. When the divergent sums are not directly related to the eigenvalues of the operator, an ambiguity in the definition appears and the counterterms of the different viewpoints do not necessarily match.

Again all these expectations are confirmed with standard computations and the explicit result can be found in [14, 15]. Naively, it seems impossible to reconcile both points of view. A natural question is to what extent this is the consequence of the non renormalizability of the model. Unfortunately studies along these lines but with a renormalizable theory (in particular four dimensional QED) show that the inequivalence has nothing to do with renormalizability.

In fact, the case of QED_4 on the four-dimensional manifold $\mathbb{R}^2 \times S^1 \times S^1$ provides a very transparent example of this kind of effects, so it is worth to study it. The counterterm calculated in the whole spacetime is the usual one of QED, which yields the well known β-function

$$\beta = \frac{e^3}{12\pi^2} \tag{5.42}$$

Or in terms of the two-dimensional coupling $\bar{e} = eM$

$$\bar{e}^2 = \frac{\bar{e}_0^2}{1 - \frac{\bar{e}_0^2}{6\pi^2 M^2} \log \frac{\mu}{\mu_0}} \tag{5.43}$$

On the other hand, symmetry forbids again a mass term form the gauge boson. Moreover, in four dimensions even if we include explicitly a mass term for the gauge boson in the bare Lagrangian its mass does not receive radiative corrections and remains unrenormalized [92]. From a two-dimensional perspective the situation is radically different. The superficial degree of divergence of a diagram is now

$$D = 2 - \frac{1}{2} E_f - V \tag{5.44}$$

where V is the number of vertices and E_f is the number of fermionic external lines. This means that diagrams with fermions in external lines cannot be primitively divergent. Thus, there are no counterterms for the fermionic sector, a fact that is impossible to justify thinking in four dimensions. The primitively divergent diagrams involve only bosons as external states.

Moreover, the Vacuum Polarization Function is known to be finite in two dimensions (remember the Schwinger model). This means that the only divergent correction, apart from a tadpole, is the two point function of the two-dimensional scalars A_3^n and A_4^n. The zero mode was massless at tree level, but now since it is a gauge singlet it gets mass through radiative corrections. This was impossible in four dimensions as we have said. Also there is no running of the coupling at all, in clear contradiction with (5.43), although possible deviations from $e^2 \approx e_0^2$ can be seen only in energies exponential in the compactification mass

$$\frac{\mu}{\mu_0} \gg e^{\frac{6\pi^2 M^2}{e_0^2}} \tag{5.45}$$

The explicit counterterm is given in [14] but its properties are basically the ones explained here.

The conclusion is that there may be some sort of quantum inequivalence between Kaluza–Klein models when one considers loop corrections in the whole spacetime or in the dimensionally reduced theory. Therefore one has to take care when computing in extra-dimensional field theories, at least when dealing with radiative corrections. In that case, since one considers effects at energies much higher than the compactification scale, the compact dimensions should be seen in the same way as the usual four and the spacetime must be higher-dimensional. The natural way of computing is then in the whole manifold performing next the mode expansion to get four-dimensional quantities. In some known particular cases, namely when the operator splits as in (5.37) and probably also for a free scalar propagating in a warped geometry, one can force the reduced theory to give the correct answer, but certainly including some peculiar facts with respect to the four-dimensional viewpoint we are used to. One has to be prudent when computing with an infinite number of fields.

Conclusiones

En los últimos años ha habido un creciente interés por las teorías de campos definidas en espaciotiempos con dimensión mayor que cuatro. Tales modelos, considerados teorías efectivas a bajas energías de una teoría fundamental consistente como supercuerdas, proporcionan una gran variedad de mecanismos nuevos cuya finalidad es resolver algunos de los problemas del Modelo Estándar.

Posibilidades interesantes son la idea del Higgs originado a partir de las componentes extra de campos gauge en dimensiones superiores, llamada unificación gauge-Higgs [53, 54], o mecanismos alternativos de ruptura de simetrías [45, 46]. Los más conocidos de entre los modelos en dimensiones superiores son sin duda las "Large Extra Dimensions" [47] y los escenarios "warped" [51]. Estas son solo las referencias originales, aunque la literatura al respecto es muy extensa.

Un problema común en los modelos en dimensiones superiores es la necesidad de explicar por qué las dimensiones extra están ocultas, en el sentido de que tan solo experimentamos las cuatro usuales. Tradicionalmente, las dimensiones extra se consideran compactas y de un tamaño tan pequeño que la energía necesaria para explorarlas aún no se ha alcanzado. Dado que las dimensiones adicionales son compactas, es posible expandir los campos en armónicos e integrar sobre las coordenadas extra, de manera que se obtiene una teoría en cuatro dimensiones pero con infinitos campos que corresponden a lo modos de la expansión. Los campos de esta torre reciben el nombre de modos Kaluza–Klein.

Se pueden distinguir entonces dos puntos de vista, el del espaciotiempo completo y el de cuatro dimensiones con la torre infinita de campos. A nivel clásico son claramente equivalentes. La pregunta que hemos tratado de responder a lo largo de esta Tesis es si la equivalencia se mantiene, y bajo qué condiciones, cuando se consideran correcciones cuánticas desde ambos puntos de vista.

Todos los efectos que nos interesan aparecen ya a orden un loop. A este orden, la acción efectiva viene dada por un determinante funcional asociado al operador que representa la parte cuadrática de la acción. En muchas teorías de interés, empezando por el Modelo Estándar, esta cantidad es divergente. La substracción de la parte divergente de manera consistente es el proceso de renormalización, en este caso a un loop. Existen varias maneras de identificar las divergencias, por ejemplo diagramáticamente con el algoritmo de 't Hooft [62] generalizado a las dimensiones apropiadas. Una aproximación al problema más efectiva, especialmente en espacios curvos, la proporciona el "heat kernel" [16, 60, 61].

Con respecto a las teorías en dimensiones superiores, es entonces obvio que la equivalencia a nivel cuántico requiere que las divergencias, calculadas desde los dos puntos de vista mencionados, coincidan. El propósito de este trabajo ha sido explorar si la coincidencia es posible.

Es importante mencionar que en el caso particular de un escalar interaccionando exclusivamente a través del acoplo universal a la gravitación, tras solventar algunas sutilezas, es posible hacer una suma de los modos de manera que las divergencias coinciden. Sin embargo se encuentran contratérminos que no se esperan en una teoría en cuatro dimensiones, como demostraron Duff y Toms [4]. Un punto esencial en la argumentación es que sus operadores verificaban

$$\Delta = \Delta_1 + \Delta_2 \tag{5.46}$$

donde Δ_1 actúa de manera trivial sobre las coordenadas extra y Δ_2 lo hace sobre las coordenadas ordinarias. De hecho, sus resultados pueden aplicarse *automáticamente* a cualquier teoría cuyo operador factoriza de esta manera. Lamentablemente cuando no se verifica lo anterior, por ejemplo en geometrías con "warping" o cuando existe interacción, la coincidencia no está asegurada. La no coincidencia de las divergencias en geometrías warped se denominó "anomalía de reducción dimensional" en [70]. Más tarde, se descubrió en [89] que la anomalía se debía a una interpretación poco satisfactoria de la divergencias, de manera que utilizando una aproximación similar a la de Duff y Toms para la geometría factorizable las divergencia se podían hacer coincidir[16]. Parece que ninguno de los autores conocía el teorema de [4].

Dados los resultados presentes en esta Tesis, una pregunta relevante es cuales son las condiciones necesarias y suficientes que debe cumplir el operador para que las correcciones radiativas coincidan. Además, aquí hemos tratado tan solo la parte divergente de la acción efectiva. Tiene que haber también contribuciones finitas que vienen de los loops no contractibles alrededor de las dimensiones extra que no estamos viendo en nuestro cálculo. En principio esta contribución finita podría ser diferente en ambos esquemas. Sin embargo, un cálculo explícito en el modelo de ruptura de simetrías extra dimensional de [90] muestra que, al menos en este caso particular, las partes finitas coinciden [91], aunque debemos mencionar que el

[16] La prueba es perturbativa en los coeficientes del heat kernel. Obtienen coincidencia de los dos primeros coeficientes y conjeturan que se cumple a todo orden.

operador verifica la factorizción que mencionamos. Calcular la parte finita de un operador arbitrario utilizando el heat kernel no es una tarea fácil. Más aún, hasta donde sabemos, se necesitaría desarrollar herramientas adicionales para tener en cuenta la periodicidad de la dimensión extra y los loops no contractibles relevantes. El caso general parece difícilmente tratable.

Por otra parte, nuestro objetivo principal es extender el análisis a una teoría con interacción, aunque sin las complicaciones de un espaciotiempo curvo. Nos hemos concentrado en una teoría gauge, en particular Electrodinámica Cuántica definida en una variedad seis dimensional $\mathbb{R}^4 \times S^1 \times S^1$. Este modelo es interesante porque es representativo del paradigma de las "Universal Extra Dimensions". Adicionalmente permite el estudio, desde una perspectiva alternativa, de un par de cuestiones relevantes para la fenomenología de dimensiones extra: la protección de la masa del Higgs mediante una simetría gauge y la posibilidad de que las constantes de acoplo corran con la energía cumpliendo una ley de potencias.

Un lector atento habrá notado que la acción correspondiente no es renormalizable, ya que la constante de acoplo gauge tiene dimensión de masa negativa $[e_6] = -1$. Sin embargo, este hecho no es crucial a primer orden en el sentido de que aún es posible identificar y estudiar todas las divergencias.

Reflexionemos un momento que es lo que se espera cuando se considera la corrección a un loop. Según es fama, los contratérminos serán los operadores seis dimensionales más generales compatibles con las simetrías del sistema, en este caso una simetría interna $U(1)$ y la invariancia Lorentz. De ese modo, la dimensión negativa de la constante de acoplo nos permite escribir términos como

$$e^2 D_M F^{MN} D^R F_{RN} \qquad ; \qquad e^2 D_R F_{MN} D^R F^{MN} \tag{5.47}$$

A pesar de que no estaban presentes en el Lagrangiano original, es inevitable que se generen radiativamente: la potencia de la constante de acoplo muestra que es un efecto a un loop, tienen la dimensión correcta y las simetrías adecuadas. La aparición de estos términos se discute en [2, 3]. Otro punto importante es que la simetría prohibe un término de masa para el campo gauge, de manera que el modo cero escalar permanece sin masa a nivel cuántico. Esta es la implementación de la idea de unificación gauge-Higgs. El cálculo explícito llevado a cabo en [14, 15] confirma estas intuiciones.

Para llevar a cabo un cálculo cuadridimensional con la torre de Kaluza–Klein es necesario expandir los campos en modos. Compactificación en el toro resulta en la acción (4.29). Se tienen que duplicar el número de fermiones ya que en d dimensiones tienen $2^{[d/2]}$ componentes (ocho en seis dimensiones y cuatro en cuatro dimensiones). Además las componentes extra del campo gauge A_5^n y A_6^n se manifiestan en cuatro dimensiones como escalares, que se identifican con el Higgs en unificación gauge-Higgs. Debemos mencionar que la simetría se rompe espontáneamente a

$$O(6) \longrightarrow O(4) \times O(2) \times O(2) \tag{5.48}$$

mientras que la simetría gauge extra dimensional se traduce en un conjunto infinito de simetrías cuadridimensionales

$$\begin{aligned} \delta A_\mu^n &= i\partial_\mu \Lambda_n \\ \delta A_5^n &= -\frac{n_5}{R_5}\Lambda_n \\ \delta A_6^n &= -\frac{n_6}{R_6}\Lambda_n \end{aligned} \tag{5.49}$$

Nótese que los modos cero escalares son singletes bajo la transformación gauge. Finalmente la constante de acoplo no tiene dimensiones de masa y viene dada por

$$e \equiv \frac{e_6}{\sqrt{R_5 R_6}} \equiv e_6 M \tag{5.50}$$

Repitámos el análisis realizado en el caso anterior y preguntémonos que tipo de correcciones se esperan. En primer lugar, la constante de acoplo es adimensional y por lo tanto no se puede utilizar para reducir la dimensión de los operadores de orden superior. Así, términos como los mencionados en (5.47) en principio no deberían aparecer, al menos en teoría de perturbaciones. Sin embargo, en el cálculo cuadridimensional se tienen coeficientes de orden superior en la expansión del heat kernel que contienen operadores de orden superior. Si bien para un único modo estas contribuciones son finitas, cuando se suma la torre infinita de modos pueden aparecer divergencias que generen términos de orden superior que no se esperan por conteo de potencias. Si el polo generado por la suma divergente se resuelve de una manera muy precisa, utilizando funciones ζ generalizadas, válidas cuando el operador verifica (5.46), entonces los operadores de orden superior adecuados efectivamente aparecen. Podría ser interesante explorar esta posibilidad para una teoría interactuante como la considerada aquí.

En segundo lugar, dado que el modo cero escalar es un singlete, su masa no está protegida frente a correcciones radiativas, al igual que sucede con el Higgs del Modelo Estándar. De este modo se esperan términos de masa para el singlete

(de hecho no hay razón para no esperar operadores con una potencia superior, i.e. cúbicos, cuárticos...)

Otra cuestión importante es que el modo cero del campo gauge A^0_μ, que toma el papel del fotón usual, se acopla diagonalmente a una torre infinita de fermiones, con la misma intensidad que en la Electrodinámica Cuántica. La única diferencia entre fermiones es su masa, etiquetada por un par de enteros. Ahora bien, el polo en la función de polarización del vacío no depende de la masa del fermión presente en el loop. Se debería entonces encontrar la misma contribución a la función β por cada uno de los fermiones, y dado que son infinitos, se tendría que encontrar la misma cantidad sumada infinitas veces. Esto produce una divergencia adicional, que se podría considerar el efecto de calcular con infinitos campos interaccionando entre ellos. Cuando el operador factoriza como en el caso libre, existe de nuevo una regularización natural de la suma divergente en términos de la función ζ generalizada asociada al operador que actúa sobre las coordenadas extra. Esta función ζ se puede relacionar con los coeficientes del heat kernel del operador y esa es la razón detrás de la coincidencia en las divergencias. Cuando la suma divergente no se puede relacionar directamente con los autovalores del operador, aparece una ambigüedad en su definición y los contratérminos de ambos puntos de vista no necesariamente coinciden.

De nuevo estas intuiciones se confirman con los cálculos adecuados y los resultados explícitos se pueden encontrar en [14, 15]. Superficialmente parece que los dos puntos de vista son irreconciliables. Una pregunta natural es hasta que punto esto es consecuencia de la no renormalizabilidad de la teoría original. Desafortunadamente estudios al respecto con un modelo renormalizable (en particular, Electrodinámica Cuántica en cuatro dimensiones) muestra que la desigualdad no tiene nada que ver con la renormalizabilidad.

De hecho, el caso de la Electrodinámica cuadridimensional compactificada en $\mathbb{R}^2 \times S^1 \times S^1$ proporciona un ejemplo transparente de esta clase de efectos, de manera que merece la pena detenerse a examinarlo con un poco de detalle. El contratérmino calculado en el espaciotiempo completo es el habitual de QED, que nos da una función β

$$\beta = \frac{e^3}{12\pi^2} \tag{5.51}$$

o en términos del acoplo bidimensional $\bar{e} = eM$

$$\bar{e}^2 = \frac{\bar{e}_0^2}{1 - \frac{\bar{e}_0^2}{6\pi^2 M^2} \log \frac{\mu}{\mu_0}} \tag{5.52}$$

Por otra parte, las simetrías prohiben de nuevo un término de masa para el bosón gauge. Es más, en cuatro dimensiones si se incluye explícitamente tal término de masa en el Lagrangiano desnudo, ésta no recibe correcciones radiativas y permanece sin renormalizar [92]. Desde un punto de vista bidimensional la situación es radicalmente distinta. El grado superficial de divergencia de un diagrama es ahora

$$D = 2 - \frac{1}{2}E_f - V \tag{5.53}$$

donde V es el número de vértices y E_f es el número de patas fermiónicas externas. Esto significa que cualquier diagrama con fermiones en las patas externas no puede ser primitivamente divergente, de manera que no hay contratérminos para el sector fermiónico, lo cual es difícilmente justificable si pensamos en cuatro dimensiones. Los diagramas primitivamente divergentes involucran tan solo bosones como estados externos.

Es más, es conocido que la función de polarización del vacío es finita en dos dimensiones (recuérdese el modelo de Schwinger). Esto quiere decir que la única contribución divergente, dejando aparte el "tadpole", es la función a dos puntos de los escalares bidimensionales A_3^n y A_4^n. El modo cero no tenía masa a orden árbol, pero dado que es un singlete gauge la adquiere a través de correcciones cuánticas. Esto mismo es imposible en cuatro dimensiones como hemos mencionado. Además las constantes de acoplo no corren, contradiciendo claramente (5.52), aunque desviaciones de $e^2 \approx e_0^2$ serían visibles tan solo a energías exponenciales en la masa de compactificación

$$\frac{\mu}{\mu_0} \gg e^{\frac{6\pi^2 M^2}{e_0^2}} \tag{5.54}$$

El contratérmino se da explícitamente en [14], pero sus propiedades son básicamente las mencionadas aquí.

La conclusión es que la equivalencia de ambos puntos de vista, el de la teoría definida en el espaciotiempo completo y la reducida dimensionalmente, parece romperse cuando se consideran correcciones cuánticas. Así, uno debe ser cuidadoso cuando hace cálculos en teorías de campos con dimensiones extra, al menos cuando involucran correcciones radiativas. En ese caso, dado que se consideran efectos a energías mayores que la escala de compactificación, el espacio compacto debería verse del mismo modo que las dimensiones usuales y el espaciotiempo debería ser de dimensión superior. El modo natural de calcular sería entonces en el espacio completo, seguido del desarrollo en modos con el fin de obtener cantidades cuadridimensionales. En algunos casos particulares, esencialmente cuando el operador factoriza según (5.46) y también probablemente para un escalar propagándose en una geometría warped, se puede forzar a la teoría en cuatro dimensiones a dar el resultado correcto, si bien es necesario incluir algunos elementos inusuales con respecto a la intuición cuadridimensional a la que estamos habituados. La prudencia es aconsejable al calcular con un número infinito de campos.

A Appendix

A.1 Ward–Takahashi identities

In this Appendix we will derive the Ward–Takahashi identities deduced from the Abelian gauge symmetry in the complete manifold as well as the ones obtained from the infinite Kac–Moody symmetries of the reduced theory. Consider QED defined on a manifold of arbitrary dimension n. The following Lagrangian

$$\mathcal{L} = \frac{1}{4}F_{MN}^2 + \bar{\psi}\not{D}\psi + m\bar{\psi}\psi \tag{A.1}$$

where $D_M \equiv \partial_M - eA_M$ is invariant under the infinitesimal $U(1)$ symmetry

$$\begin{aligned} \delta A_M &= i\partial_M \Lambda \\ \delta\psi &= ie\Lambda\psi \\ \delta\bar{\psi} &= -ie\Lambda\bar{\psi} \end{aligned} \tag{A.2}$$

Adding a gauge-fixing term and source terms for all the fields

$$\mathcal{L}_{gf} + \mathcal{L}_{sources} = \frac{1}{2\alpha}\left(\partial_M A^M\right)^2 + J_M A^M + \bar{\eta}\psi + \bar{\psi}\eta \tag{A.3}$$

Then the complete Lagrangian is no longer invariant. Moreover, performing a transformation (A.2) it changes

$$\delta\mathcal{L} \equiv \delta\mathcal{L}_{gf} + \delta\mathcal{L}_{sources} = \frac{i}{\alpha}\partial_M A^M \Box\Lambda + iJ^M\partial_M\Lambda + ie\Lambda\bar{\eta}\psi - ie\Lambda\bar{\psi}\eta \tag{A.4}$$

If we demand the generating functional

$$Z \equiv N\int \mathcal{D}A_M \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left(-\int d^n x \left(\mathcal{L} + \mathcal{L}_{gf} + \mathcal{L}_{sources}\right)\right) \tag{A.5}$$

to be gauge-invariant then it has to verify

$$\left(\frac{1}{\alpha}\Box\partial_M A^M - \partial_M J^M + e\bar{\eta}\psi - e\bar{\psi}\eta\right) Z = 0 \tag{A.6}$$

In terms of the effective action

$$\Gamma[A_M, \psi, \bar{\psi}] \equiv W[J_M, \eta, \bar{\eta}] - \int d^n x \left(J_M A^M + \bar{\eta}\psi + \bar{\psi}\eta\right) \tag{A.7}$$

Where $Z = e^{iW}$ the equation has the form

$$\frac{1}{\alpha}\Box\partial_M A^M + \partial_M \frac{\delta\Gamma}{\delta A_M} + e\psi\frac{\delta\Gamma}{\delta\psi} - e\bar{\psi}\frac{\delta\Gamma}{\delta\bar{\psi}} = 0 \tag{A.8}$$

And we have done the substitutions

$$\begin{aligned}
\frac{\delta\Gamma}{\delta A_M} &= -J^M \\
\frac{\delta\Gamma}{\delta\psi} &= -\bar{\eta} \\
\frac{\delta\Gamma}{\delta\bar{\psi}} &= -\eta
\end{aligned} \tag{A.9}$$

Suppose now that the space-time is the manifold $\mathbb{R}^4 \times S^1 \times S^1$. We can perform the integrals corresponding to the torus to get a compactified action ($N = \frac{n_6}{R_6} + i\frac{n_5}{R_5}$)

$$\begin{aligned}
S &= \int d^4x \sum_{n_5,n_6} \Big[\bar{\psi}^1_n \partial\!\!\!/ \psi^1_n + \bar{\psi}^2_n \partial\!\!\!/ \psi^2_n + N\bar{\psi}^1_n\psi^2_n + N^*\bar{\psi}^2_n\psi^1_n + m\left(\bar{\psi}^1_n\psi^1_n - \bar{\psi}^2_n\psi^2_n\right) \\
&\quad +\frac{1}{4}\Bigg(F^{-n}_{\mu\nu}F^{\mu\nu}_n + 2|N|^2 A^{-n}_\mu A^\mu_n - 4i\partial_\mu A^\mu_{-n}\left(\frac{n_5}{R_5}A^n_5 + \frac{n_6}{R_6}A^n_6\right) \\
&\quad +2A^{-n}_5\left(-\Box + \frac{n_6^2}{R_6^2}\right)A^n_5 + 2A^{-n}_6\left(-\Box + \frac{n_5^2}{R_5^2}\right)A^n_6 - 4\frac{n_5 n_6}{R_5 R_6}A^{-n}_5 A^n_6\Bigg) \\
&\quad -e\sum_m \big(\bar{\psi}^1_m A\!\!\!/_{m-n}\psi^1_n + \bar{\psi}^2_m A\!\!\!/_{m-n}\psi^2_n + \bar{\psi}^1_m A^{m-n}_5\psi^2_n - \bar{\psi}^2_m A^{m-n}_5\psi^1_n \\
&\quad -i\bar{\psi}^1_m A^{m-n}_6\psi^2_n - i\bar{\psi}^2_m A^{m-n}_6\psi^1_n\big)\Big]
\end{aligned} \tag{A.10}$$

This action is now invariant under ($a = 1, 2$)

$$\begin{aligned}
\delta A^n_\mu &= i\partial_\mu\Lambda_n \\
\delta A^n_5 &= -\frac{n_5}{R_5}\Lambda_n \\
\delta A^n_6 &= -\frac{n_6}{R_6}\Lambda_n \\
\delta\psi^a_n &= ie\sum_m \Lambda_m\psi^a_{n-m} \\
\delta\bar{\psi}^a_n &= -ie\sum_m \Lambda_m\bar{\psi}^a_{m-n}
\end{aligned} \tag{A.11}$$

Compactification of (A.3) gives

$$\begin{aligned}
S_{gf} + S_{sources} &= \int d^4x \Bigg(\frac{1}{2\alpha}\Bigg(\partial_\mu A^\mu_{-n}\partial_\nu A^\nu_n + 2i\partial_\mu A^\mu_{-n}\left(\frac{n_5}{R_5}A^n_5 + \frac{n_6}{R_6}A^n_6\right) + \\
&\quad +\frac{n_5^2}{R_5^2}A^{-n}_5 A^n_5 + \frac{n_6^2}{R_6^2}A^{-n}_6 A^n_6 + 2\frac{n_5 n_6}{R_5 R_6}A^{-n}_5 A^n_6\Bigg) + J^\mu_{-n}A^n_\mu + \\
&\quad +J^5_{-n}A^n_5 + J^6_{-n}A^n_6 + \bar{\eta}^1_n\psi^1_n - \bar{\eta}^2_n\psi^2_n + \bar{\psi}^1_n\eta^1_n - \bar{\psi}^2_n\eta^2_n\Bigg)
\end{aligned} \tag{A.12}$$

A gauge transformation (A.11) changes the complete Lagrangian by

$$\begin{aligned}\delta\mathcal{L} &= \sum_n \Bigg(\frac{1}{\alpha}\Bigg(\left(\Box - |N|^2\right)\left(i\partial_\mu A^\mu_{-n} + \frac{n_5}{R_5}A_5^{-n} + \frac{n_6}{R_6}A_6^{-n}\right) - \\ &\quad -\frac{n_5 n_6}{R_5 R_6}\left(\frac{n_5}{R_5}A_6^{-n} + \frac{n_6}{R_6}A_5^{-n}\right)\Bigg) - i\partial_\mu J^\mu_{-n} - \frac{n_5}{R_5}J^5_{-n} - \frac{n_6}{R_6}J^6_{-n} + \\ &\quad + ie\sum_m \left(\bar{\eta}^1_m \psi^1_{m-n} - \bar{\eta}^2_m \psi^2_{m-n} - \bar{\psi}^1_{n-m}\eta^1_m + \bar{\psi}^2_{n-m}\eta^2_m\right)\Bigg)\Lambda_n \qquad \text{(A.13)}\end{aligned}$$

Again if we demand the generating functional to be invariant then it has to verify

$$\begin{aligned}&\Bigg(\frac{1}{\alpha}\Bigg(\left(\Box - |N|^2\right)\left(i\partial_\mu A^\mu_{-n} + \frac{n_5}{R_5}A_5^{-n} + \frac{n_6}{R_6}A_6^{-n}\right) - \\ &-\frac{n_5 n_6}{R_5 R_6}\left(\frac{n_5}{R_5}A_6^{-n} + \frac{n_6}{R_6}A_5^{-n}\right)\Bigg) - i\partial_\mu J^\mu_{-n} - \frac{n_5}{R_5}J^5_{-n} - \frac{n_6}{R_6}J^6_{-n} + \\ &+ ie\sum_m \left(\bar{\eta}^1_m \psi^1_{m-n} - \bar{\eta}^2_m \psi^2_{m-n} - \bar{\psi}^1_{n-m}\eta^1_m + \bar{\psi}^2_{n-m}\eta^2_m\right)\Bigg) Z = 0 \qquad \text{(A.14)}\end{aligned}$$

Or in terms of the (compactification of the) effective action (A.7)

$$\begin{aligned}&\Gamma[A^n_\mu, A^n_5, A^n_6, \psi^1_n, \psi^2_n, \bar{\psi}^1_n, \bar{\psi}^2_n] \equiv W[J^n_\mu, J^5_n, J^6_n, \eta^1_n, \eta^2_n, \bar{\eta}^1_n, \bar{\eta}^2_n] \qquad \text{(A.15)} \\ &-\int d^4x \sum_n \left(J^{-n}_\mu A^\mu_n + J^5_{-n}A^n_5 + J^6_{-n}A^n_6 + \bar{\eta}^1_n\psi^1_n - \bar{\eta}^2_n\psi^2_n + \bar{\psi}^1_n\eta^1_n - \bar{\psi}^2_n\eta^2_n\right)\end{aligned}$$

one can obtain

$$\begin{aligned}&\frac{1}{\alpha}\Bigg[\left(\Box - |N|^2\right)\left(i\partial_\mu A^\mu_{-n} + \frac{n_5}{R_5}A_5^{-n} + \frac{n_6}{R_6}A_6^{-n}\right) \qquad \text{(A.16)} \\ &-\frac{n_5 n_6}{R_5 R_6}\left(\frac{n_5}{R_5}A_6^{-n} + \frac{n_6}{R_6}A_5^{-n}\right)\Bigg] + i\partial_\mu \frac{\delta\Gamma}{\delta A^n_\mu} + \frac{n_5}{R_5}\frac{\delta\Gamma}{\delta A^n_5} + \frac{n_6}{R_6}\frac{\delta\Gamma}{\delta A^n_6} \\ &+ ie\sum_m \left(\psi^1_{m-n}\frac{\delta\Gamma}{\delta\psi^1_m} - \psi^2_{m-n}\frac{\delta\Gamma}{\delta\psi^2_m} - \bar{\psi}^1_{n-m}\frac{\delta\Gamma}{\delta\bar{\psi}^1_m} + \bar{\psi}^2_{n-m}\frac{\delta\Gamma}{\delta\bar{\psi}^2_m}\right) = 0\end{aligned}$$

This equation expanded in powers of the fields yields the corresponding Ward–Takahashi identities for proper vertices.

A.2 Five-dimensional QED

As it is well known it is not possible to define a matrix that anticonmutes with all Dirac matrices in a space-time of odd dimensions. But the algorithm used to calculate the heat-kernel coefficients [65, 66] makes use of such object. In order to avoid this problem we have to double the fermion content of the theory and define new matrices $(i, j = 1, 2)$

$$\gamma^M_{ij} \equiv \gamma^M \otimes \sigma^2_{ij} \tag{A.17}$$

That satisfy a modified Clifford algebra

$$\{\gamma^M, \gamma^N\}_{ij} = 2\delta^{MN} \otimes \delta_{ij} \tag{A.18}$$

We are now able to construct a matrix that verifies the desired property, in particular

$$\begin{aligned} \bar{\gamma}_{ij} &\equiv \mathbb{I} \otimes \sigma^3_{ij} \\ \{\bar{\gamma}, \gamma^M\} &= 0 \end{aligned} \tag{A.19}$$

Consider now a QED type action on a five-dimensional manifold $\mathbb{R}^4 \times S^1$

$$S = \int d^5x \left[\frac{1}{4}F^2_{MN} + \bar{\psi}^i \left(\not{D}_{ij} + m_f \delta_{ij}\right) \psi^j\right] \tag{A.20}$$

where $M, N = 1, \ldots, 5$ are five-dimensional indices. If we now split the fields in classical and quantum parts and organize them on a column

$$\xi = \begin{pmatrix} \phi_N \\ \psi^j \end{pmatrix} \tag{A.21}$$

Then we have to deal with a second order operator

$$\Delta_{ij} = \begin{pmatrix} -\Box\delta_{MN} & \sqrt{\frac{2}{\mu}}\left(\bar{\eta}^k \gamma^M_{kh}\gamma^R_{hj}\bar{D}_R - m_f \bar{\eta}^k \gamma^M_{kj}\right) \\ -\sqrt{2\mu} e \gamma^N_{ik}\eta^k & \left(-\bar{D}_R \bar{D}^R + m_f^2\right)\delta_{ij} + \frac{e}{2}\gamma^R_{ik}\gamma^S_{kj}\bar{F}_{RS} \end{pmatrix} \tag{A.22}$$

That gives the following supermatrices

$$X^{ij}_{MN} = \begin{pmatrix} 0 & * \\ o & -e\bar{F}_{MN}\delta_{ij} \end{pmatrix} \tag{A.23}$$

$$Y_{ij} = \begin{pmatrix} 0 & \frac{-e}{\sqrt{2\mu}}\left(\bar{D}_R \bar{\eta}^k \gamma^M_{kh}\gamma^R_{hj} + 2m_f \bar{\eta}^k \gamma^M_{kj}\right) \\ -\sqrt{2\mu} e \gamma^N_{ik}\eta^k & \frac{e}{2}\gamma^R_{ik}\gamma^S_{kj}\bar{F}_{RS} + m_f^2 \delta_{ij} \end{pmatrix} \tag{A.24}$$

So the fourth heat-kernel coefficient in five dimensions is

$$a_4 = \int \frac{d^5x}{(4\pi)^{\frac{5}{2}}}\left[\frac{4}{3}e^2 \bar{F}^2_{MN} + 3e^2 \bar{\eta}^i \bar{\not{D}}_{ij}\eta^j + 10e^2 m_f \bar{\eta}^i \eta^i\right] \tag{A.25}$$

The Clifford algebra (A.18) and the anticommutativity condition (A.19) are satisfied by

$$\gamma^M = \left(\gamma^\mu, \gamma^5\right) \tag{A.26}$$

Fourier expanding the fields

$$\phi(x^\mu, y) = \frac{1}{\sqrt{2\pi R}} \sum_n \phi_n(x^\mu) e^{i\frac{n}{R}y} \tag{A.27}$$

Where R is the radius of the circle it is easy to perform the integral over the compact dimension and get in the Feynman gauge

$$\begin{aligned} S \;\; = \;\; \int d^4x \sum_n \Bigg[& \frac{1}{2} \left(A_\mu^n\right)^* \left(-\Box + \frac{n^2}{R^2}\right) A_n^\mu + \frac{1}{2} \left(A_5^n\right)^* \left(-\Box + \frac{n^2}{R^2}\right) A_5^n \\ & + \bar{\psi}_n^i \partial\!\!\!/_{ij} \psi_n^j + i\frac{n}{R} \bar{\psi}_n^i \gamma_{ij}^5 \psi_n^j + m_f \bar{\psi}_n^i \psi_n^i \\ & - e \sum_m \left(\bar{\psi}_m^i A\!\!\!/_{ij}^{m-n} \psi_n^j + \bar{\psi}_m^i \gamma_{ij}^5 A_5^{m-n} \psi_n^j \right) \Bigg] \end{aligned} \tag{A.28}$$

The four-dimensional coupling is related to the five-dimensional one by

$$e \equiv \frac{e^{(5)}}{\sqrt{2\pi R}} \tag{A.29}$$

After the split of the fields in classical and quantum parts and organizing them on a column

$$\xi = \begin{pmatrix} \phi_\mu^n \\ A_5^n \\ \psi_n^j \end{pmatrix} \tag{A.30}$$

we are lead to the operators

$$\begin{aligned} A_{mn} &= \begin{pmatrix} \left(-\Box + \frac{n^2}{R^2}\right) \delta_{\mu\nu} \delta_{mn} & 0 \\ 0 & \left(-\Box + \frac{n^2}{R^2}\right) \delta_{mn} \end{pmatrix} \\ B_{mn}^{ij} &= \left(\partial\!\!\!/_{ij} + i\frac{n}{R} \gamma_{ij}^5 + m_f \delta_{ij} \right) \delta_{mn} - e\bar{A\!\!\!/}_{ij}^{m-n} - e\gamma_{ij}^5 \bar{A}_5^{m-n} \\ \Gamma_{mn} &= \begin{pmatrix} -e\bar{\eta}_{n-m}^k \gamma_{kj}^\mu \\ -e\bar{\eta}_{n-m}^k \gamma_{kj}^5 \end{pmatrix} \\ \Gamma_{mn} &= \begin{pmatrix} -e\gamma_{ik}^\mu \eta_{m-n}^k & -e\gamma_{ik}^5 \eta_{m-n}^k \end{pmatrix} \end{aligned} \tag{A.31}$$

Following the same steps as in the precedent sections we get the relevant supermatrices

$$\begin{aligned}
(X_{\mu\nu}^{--})_{mn}^{ij} &= -e\bar{F}_{\mu\nu}^{m-n}\delta_{ij} \\
(Y_{--})_{mn}^{ij} &= \frac{e}{2}\gamma_{ik}^{\mu}\gamma_{kj}^{\nu}\bar{F}_{\mu\nu}^{m-n} + ie\frac{m-n}{R}\gamma_{ik}^{5}\gamma_{kj}^{\mu}\bar{A}_{\mu}^{m-n} - e\gamma_{ik}^{5}\gamma_{kj}^{\mu}\partial_{\mu}\bar{A}_{5}^{m-n} \\
&\quad +ie\frac{m+n}{R}\bar{A}_{5}^{m-n}\delta_{ij} - e^{2}\bar{A}_{5}^{m-l}\bar{A}_{5}^{l-n}\delta_{ij} + \left(m_f^2 + \frac{n^2}{R^2}\right)\delta_{mn}\delta_{ij} \\
(Y_{+-}^{11})_{mn}^{j} &= \frac{-e}{\sqrt{2\mu}}\left(\partial_{\nu}\bar{\eta}_{n-m}^{k}\gamma_{kh}^{\mu}\gamma_{hj}^{\nu} + e\bar{\eta}_{l-m}^{k}\gamma_{kh}^{\mu}\gamma_{hj}^{\nu}\bar{A}_{\nu}^{l-n} + 2m_f\bar{\eta}_{n-m}^{k}\gamma_{kj}^{\mu}\right. \\
&\quad \left. +2e\bar{\eta}_{l-m}^{k}\gamma_{kh}^{\mu}\gamma_{hj}^{5}\bar{A}_{5}^{l-n} - 2i\frac{n}{R}\bar{\eta}_{n-m}^{k}\gamma_{kh}^{\mu}\gamma_{hj}^{5}\right) \\
(Y_{+-}^{21})_{mn}^{j} &= \frac{-e}{\sqrt{2\mu}}\left(\partial_{\nu}\bar{\eta}_{n-m}^{k}\gamma_{kh}^{5}\gamma_{hj}^{\nu} + e\bar{\eta}_{l-m}^{k}\gamma_{kh}^{5}\gamma_{hj}^{\nu}\bar{A}_{\nu}^{l-n} + 2m_f\bar{\eta}_{n-m}^{k}\gamma_{kj}^{5} +\right. \\
&\quad \left. +2e\bar{\eta}_{l-m}^{j}\bar{A}_{5}^{l-n} - 2i\frac{n}{R}\bar{\eta}_{n-m}^{j}\right) \\
(Y_{-+})_{mn}^{i} &= \begin{pmatrix} -\sqrt{2\mu}e\gamma_{ik}^{\nu}\eta_{m-n}^{k} & -\sqrt{2\mu}e\gamma_{ik}^{5}\eta_{m-n}^{k} \end{pmatrix}
\end{aligned} \tag{A.32}$$

The fourth heat kernel coefficient associated with this operator is

$$\begin{aligned}
a_4 &= \int\frac{d^4x}{(4\pi)^2}\sum_{l}\left[\frac{4}{3}e^2\sum_{n}\bar{F}_{\mu\nu}^{n}\bar{F}_{-n}^{\mu\nu} - 4e^2\sum_{n}\bar{A}_{5}^{-n}\Box\bar{A}_{5}^{n} - 16ie\frac{l}{R}\left(\frac{l^2}{R^2}+m_f^2\right)\bar{A}_{5}^{0}\right. \\
&\quad +4e^2\sum_{n}\left(2m_f^2 + \frac{2n^2+(n+l)^2}{R^2}\right)\bar{A}_{5}^{n-l}\bar{A}_{5}^{l-n} + 4ie^3\sum_{n,m}\frac{2m+l+n}{R}\bar{A}_{5}^{m-l}\bar{A}_{5}^{l-n}\bar{A}_{5}^{n-m} \\
&\quad -4e^4\sum_{n,m,s}\bar{A}_{5}^{m-l}\bar{A}_{5}^{l-s}\bar{A}_{5}^{s-n}\bar{A}_{5}^{n-m} + 8ie^2\sum_{n}\frac{n}{R}\partial_{\mu}\bar{A}_{5}^{n}\bar{A}_{-n}^{\mu} + 4e^2\sum_{n}\frac{n^2}{R^2}\bar{A}_{n}^{\mu}\bar{A}_{\mu}^{-n} \\
&\quad +6e^2\sum_{n\neq 0}\bar{\eta}_{l-n}^{i}\partial\!\!\!/_{ij}\eta_{l-n}^{j} - 6e^3\sum_{n,m\neq 0}\bar{\eta}_{l-m}^{i}A\!\!\!/_{ij}^{l-n}\eta_{n-m}^{j} - 12e^3\sum_{n,m\neq 0}\bar{\eta}_{l-m}^{i}\gamma_{ij}^{5}\bar{A}_{5}^{l-n}\eta_{n-m}^{j} \\
&\quad +12i\sum_{n\neq 0}\frac{l}{R}\bar{\eta}_{l-n}^{i}\gamma_{ij}^{5}\eta_{l-n}^{j} + 20m_f\sum_{n\neq 0}\bar{\eta}_{l-n}^{i}\eta_{l-n}^{i} + 3e^2\bar{\eta}_{l}^{i}\partial\!\!\!/_{ij}\eta_{l}^{j} - 3e^3\sum_{n}\bar{\eta}_{n}^{i}A\!\!\!/_{ij}^{n-l}\eta_{l}^{j} \\
&\quad \left.-6e^3\sum_{n}\bar{\eta}_{n}^{i}\gamma_{ij}^{5}\bar{A}_{5}^{n-l}\eta_{l}^{j} + 6i\frac{l}{R}\bar{\eta}_{l}^{i}\gamma_{ij}^{5}\eta_{l}^{j} + 10m_f\bar{\eta}_{l}^{i}\eta_{l}^{i} + \left(6\frac{l^4}{R^4} - 8m_f^2\frac{l^2}{R^2} - 4m_f^4\right)\right]
\end{aligned} \tag{A.33}$$

A.3 The scalar masses

Here we treat in more detail the renormalization of the scalar masses coming from the four-dimensional computation with the whole tower. Defining $A^n_{5(0)} = Z_5^{1/2} A^n_5$ the wave function renormalization is

$$Z_5 = 1 + \frac{e^2}{2\pi^2\epsilon}\sum_l 1 \tag{A.34}$$

Is easy to see that the renormalization of the mass is in this case additive

$$Z_5 m_0^2 = m_n^2 + \delta m_n^2 \tag{A.35}$$

Where δm_n^2 is a complicated function of the form

$$m_0^2 = m_n^2 + \frac{e^2}{\pi^2\epsilon}\left(S_0 m_f^2 + S_0 m_n^2 + \frac{4}{R}S_1 m_n + \frac{3}{R^2}S_2\right) \tag{A.36}$$

And we have defined the sums

$$S_0 = \sum_l 1 \tag{A.37}$$

$$S_1 = \sum_l l \tag{A.38}$$

$$S_2 = \sum_l l^2 \tag{A.39}$$

In order to get the renormalization group functions

$$\begin{aligned} 0 \;=\; & 2m_0\frac{\partial m_0}{\partial\log\mu} = 2m_n\frac{\partial m_n}{\partial\log\mu} + \frac{2e}{\pi^2\epsilon}\frac{\partial e}{\partial\log\mu}\left(S_0 m_f^2 + S_0 m_n^2 + \frac{4}{R}S_1 m_n + \frac{3}{R^2}S_2\right) + \\ & + \frac{2e^2}{\pi^2\epsilon}\left(S_0 m_f\frac{\partial m_f}{\partial\log\mu} + S_0 m_n\frac{\partial m_n}{\partial\log\mu} + \frac{2}{R}S_1\frac{\partial m_n}{\partial\log\mu}\right) \end{aligned} \tag{A.40}$$

Defining

$$\frac{\partial m_n}{\partial\log\mu} = \alpha_0 + \alpha_1\epsilon \tag{A.41}$$

$$\frac{\partial e}{\partial\log\mu} = \beta_0 + \beta_1\epsilon \tag{A.42}$$

$$\frac{\partial m_f}{\partial\log\mu} = \gamma_0 + \gamma_1\epsilon \tag{A.43}$$

This yields to order $O(\epsilon)$ the condition

$$m_n\alpha_1 = 0 \tag{A.44}$$

So supposing $m_n \neq 0$ we get $\alpha_1 = 0$ and to order $O(1)$ (we can see also that $\gamma_1 = 0$)

$$m_n \alpha_0 = -\frac{e^2}{2\pi^2}\left(S_0 m_f^2 + S_0 m_n^2 + \frac{4}{R}S_1 m_n + \frac{3}{R^2}S_2\right) \tag{A.45}$$

Therefore we must solve the equation

$$m_n \frac{\partial m_n}{\partial \log \mu} = -\frac{e^2}{2\pi^2}\sum_l \left(m_f^2 + m_n^2 + 4m_n\frac{l}{R} + 3\frac{l^2}{R^2}\right) \tag{A.46}$$

where

$$\begin{aligned} \frac{\partial e}{\partial \log \mu} &= \frac{e^3}{12\pi^2}S_0 \\ \frac{\partial m_f}{\partial \log \mu} &= -\frac{7e^2 m_f}{8\pi^2}\left(S_0 - \frac{1}{2}\right) \end{aligned} \tag{A.47}$$

The renormalization of the scalar masses is entangled with the running of the coupling and the fermion mass. After solving (A.47) we get

$$\begin{aligned} e^2 &= \frac{e_0^2}{1 - \frac{e_0^2}{6\pi^2}S_0 \log\frac{\mu}{\mu_0}} \\ m_f &= m_{f(0)}\left(1 - \frac{e_0^2}{6\pi^2}S_0 \log\frac{\mu}{\mu_0}\right)^{\frac{21}{4}\frac{2S_0-1}{S_0}} \end{aligned} \tag{A.48}$$

And the running of the scalar masses is governed by

$$\frac{dm_n^2}{du} = -6e_0^2\left(S_0 m_n^2 + S_0 m_{f(0)}^2\left(\frac{S_0 e_0^2}{6\pi^2}e^{-uS_0e_0^2}\right)^{\frac{21}{4}\frac{2S_0-1}{S_0}} + \frac{4}{R}S_1 m_n + \frac{3}{R^2}S_2\right) \tag{A.49}$$

where we have performed the change of variables

$$\frac{dt}{6\pi^2 - S_0 e_0^2 (t - t_0)} = du \tag{A.50}$$

With $t \equiv \log \mu$. This differential equation is hard to solve. Nevertheless for the chiral case $m_f = 0$ a solution may be given in terms of a trascendent equation

$$\begin{aligned} &\frac{-1}{12e_0^2 S_0}\log\frac{|S_0 m_n^2 + \frac{4}{R}S_1 m_n + \frac{3}{R^2}S_2|}{|S_0 m_{n(0)}^2 + \frac{4}{R}S_1 m_{n(0)} + \frac{3}{R^2}S_2|} + \frac{2S_1}{RS_0}\big(F(e_0^2, m_n, S_i, R) - \\ &- F(e_0^2, m_{n(0)}, S_i, R)\big) = u - u_0 \end{aligned} \tag{A.51}$$

Where $F(e_0^2, m_n, S_i, R)$ is a function depending on the sign of $4S_1 - 3S_0S_2$. In particular for $4S_1 - 3S_0S_2 < 0$ it reads

$$F(e_0^2, m_n, S_i, R) = \frac{R}{6e_0^2(3S_0S_2 - 4S_1)^{\frac{1}{2}}}\arctan\frac{RS_0 m_n + 2S_1}{(3S_0S_2 - 4S_1)^{\frac{1}{2}}} \tag{A.52}$$

While for $4S_1 - 3S_0S_2 > 0$ the function is

$$F(e_0^2, m_n, S_i, R) = \frac{R}{12e_0^2\,(4S_1 - 3S_0S_2)^{\frac{1}{2}}} \log\left|\frac{RS_0m_n + 2S_1 - (4S_1 - 3S_0S_2)^{\frac{1}{2}}}{RS_0m_n + 2S_1 + (4S_1 - 3S_0S_2)^{\frac{1}{2}}}\right| \tag{A.53}$$

Finally when $4S_1 - 3S_0S_2 = 0$

$$F(e_0^2, m_n, S_i, R) = -\frac{1}{6e_0^2S_0m_n + \frac{12}{R}e_0^2S_1} \tag{A.54}$$

Agradecimientos

Me gustaría agradecer en primer lugar a Enrique las inestimables, en valor y número, conversaciones que hemos compartido a lo largo de estos años, ya tantos. No me refiero únicamente a la física que he aprendido. La vida del aspirante a investigador es cuanto menos inestable. Es importante que alguien te haga notar que las cosas pueden ir mejor, te muestre lo que es accesorio y te cuente que, a veces, las revistas aceptan los artículos a la primera.

Quiero acordarme también de mi familia, siempre tan pendiente de que no me falte nada. Cuando esta ciudad se vuelve demasiado amenazadora, sois mi refugio, especialmente mi madre, a la que sin duda nunca dejaré de necesitar.

No me veo capaz de mencionar, individualmente, al resto de la gente que merece estar aquí. Es mi esperanza que cada una de esas personas entienda a quien me refiero, he tratado de hacérselo saber, seguiré intentándolo. Gracias sinceras a los que habéis compartido mis rincones todo este tiempo, los personales y los profesionales, los públicos y los privados, los exteriores y, esencialmente, los interiores.

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