The Nature of Dark Energy: Theory and Observations

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Though my soul may set in darkness, it will rise in perfect light;
I have loved the stars too fondly to be fearful of the night.

– The Old Astronomer, by Sarah Williams
A mi familia, por su apoyo y cariño incondicional
a lo largo de los años.

A Manuel Fontán, porque, aunque no lo sepa,
sus palabras me hicieron recorrer este camino.
Declaration of Authorship

The results presented in this dissertation (Chapters 1-4) are based on original work done in collaboration with other researchers during the course of my PhD, from February 2009 to June 2013. The content of these Chapters is based on the released publications [1, 2], as well as another ongoing project which is at a mature stage [3]. These references are listed below for convenience. In particular, Chapters 1, 2 and 4 are based on Reference [1] and Chapter 3 on References [2, 3]. The introductory Chapters 1 and 2 are mainly a review. The exposition there has been very influenced by References [4, 5], as well as the works cited in the text. Although they represent my own vision about some of the topics in which the results are framed, no claim of originality is made about the introductory Chapters. Appendices A and C describe some of the tools necessary for the analysis. Appendix B contains some extra material needed for the analysis performed in [2].

Alicia Bueno Belloso


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Resumen y Conclusiones

La cosmología es el estudio del universo y de sus componentes, de cómo se formó, cómo ha evolucionado y cuál es su futuro. Lo que entendemos hoy en día como cosmología ha surgido de ideas que estaban ya presentes en la historia antigua cuando la humanidad intentaba responder a preguntas como “¿qué es lo que nos rodea?”, que se desarrollaron en preguntas más amplias como “¿cómo funciona el universo?”. Ésta es la pregunta fundamental que la cosmología se pregunta y trata de contestar.

Muchas de las primeras observaciones científicas de las que tenemos constancia son sobre cosmología y la búsqueda de conocimiento en este campo ha continuado durante más de 5000 años. La cosmología ha efectuado un salto monumental en los últimos 20 años a través de nueva información de la estructura, la evolución y el origen del universo, información obtenida a través de recientes avances tecnológicos en telescopios y misiones espaciales. Se ha convertido en una ciencia en búsqueda de conocimiento no sólo sobre los constituyentes del universo, sino sobre su arquitectura global. La cosmología moderna se encuentra entre la ciencia y la filosofía, cercana a la filosofía porque busca respuestas a las preguntas fundamentales sobre el universo y cercana a la ciencia en su búsqueda de respuestas a través de conocimiento empírico de observaciones y explicaciones racionales. De este modo, las teorías cosmológicas operan con una tensión entre la necesidad filosófica de la simplicidad y el deseo de incluir todas las complejas características del universo.

Sin embargo, cuando uno habla de cosmología moderna, se está refiriendo a la ciencia que nació a inicios de los años 20 para testar la teoría de la relatividad general de Einstein a gran escala. Cuando Einstein publicó sus famosas ecuaciones sobre relatividad general en 1926, creía que el universo era un ente estático, pero descubrió que la formulación original de su teoría no permitía un universo tal. Para superar este “problema”, descubrió que podía añadir una constante a sus ecuaciones que contrarrestaba la acción atractiva de la gravedad en escalas cósmicas, produciendo un universo estático. Su primer artículo sobre
La cosmología en 1917 fue el primero en incluir la que posteriormente se conocería como constante cosmológica en su modelo: un universo estático en el cual el espacio no tiene fronteras pero sigue siendo finito. No sería hasta 1929, cuando Edwin Hubble descubrió que el universo se estaba expandiendo, cuando el modelo de Einstein de un universo estático fue abandonado y su constante cosmológica apodada como “el mayor error de su vida”.

La cosmología observacional dio otro gran salto con el descubrimiento de la radiación cósmica de microondas en el año 1964 por Arno Penzias y Robert Wilson, radiación cuya existencia ya había sido predicha en los años 40. Este hallazgo permitió abrir una ventana al universo primitivo, y continuó haciéndolo a través de diversos experimentos. Dichos experimentos fueron los responsables de increíbles avances observacionales en cosmología y la catapultaron a ser una ciencia de precisión.

Con el descubrimiento de la aceleración cósmica a finales de los años 90 a través de observaciones de supernovas lejanas, la cosmología ha cerrado el círculo, retomando de nuevo la constante cosmológica propuesta por Einstein tantos años atrás. Dicha aceleración aparece como una fuerza contraria a la gravedad que, por su carácter atractivo, tiende a frenar paulatinamente la separación entre galaxias. La observación de esta aceleración ha sido, desde entonces, corroborada por numerosos experimentos cosmológicos pero, a día de hoy, seguimos sin tener una teoría satisfactoria que explique el origen y la naturaleza de dicha aceleración.

A pesar de que las últimas observaciones cosmológicas parecen favorecer lo que se conoce como el “modelo estándar de la cosmología”: un universo con materia oscura fría y una constante cosmológica, muchas nuevas teorías han surgido para explicar la aceleración del universo observada. Aparte del esfuerzo teórico, las observaciones cosmológicas han avanzado y ahora contamos con experimentos presentes y futuros que no sólo se basan en observaciones de supernovas, sino que usan datos de formación de estructura o del fondo de radiación cósmico de microondas. Sin embargo, de los muchos modelos, pocos han podido ser excluidos definitivamente y sólo se han precisado los diversos parámetros de los que dependen.

El objetivo de esta tesis consiste en el estudio de diversos modelos cosmológicos teóricos que explican dicha aceleración (comúnmente conocida como energía oscura) y su comparación con observaciones. Para ello, se deben clasificar los distintos modelos a través de observables medibles por los experimentos. De esta manera, podemos descartar o confirmar modelos o clases de modelos con observaciones presentes y futuras. También es importante investigar nuevos observables que nos puedan ayudar en el futuro a distin-
guir entre modelos y a caracterizar el origen y la naturaleza de esta elusiva componente de nuestro universo.

Esta tesis ha querido afrontar el problema de la energía oscura uniendo teoría y observaciones. Para ello, se han estudiado diversos modelos cosmológicos y sus parametrizaciones y cómo observaciones futuras podrán distinguir entre ellos. También se ha estudiado un nuevo método usando pares de galaxias aislados que puede ser utilizado para precisar modelos cosmológicos. Por último, se ha investigado la escala de homogeneidad del universo con vistas a medirla con experimentos con gran precisión angular pero poca resolución radial. Un resumen de estos proyectos se encuentra a continuación.

**Pronosticando el crecimiento de estructuras con Euclid**

Euclid es una misión espacial de la ESA de clase M que despegará aproximadamente el 2020. Es un experimento espacial de amplio rango cuyos objetivos principales son la investigación de la energía y la materia oscura a través del estudio de galaxias hasta un redshift del orden \( z \sim 2 \). Entre otros experimentos, incluirá una sonda para medir cúmulos de galaxias y otra para medir la desviación gravitacional de fotones al atravesar grandes acumulaciones de materia oscura. Para poder aprovechar al máximo este nuevo experimento de alta precisión, es necesario saber cómo de bien podrá constreñir parámetros cosmológicos para poder distinguir diferentes modelos teóricos. Sabemos que a nivel de background, diversos modelos pueden reproducir los datos experimentales de forma similar, así que buscamos un observable claramente diferente para cada modelo. Para ello, estudiamos la adecuación de Euclid a la hora de precisar una cantidad conocida como growth index.

Calculamos este growth index \( (\gamma) \) para varios modelos cosmológicos y encontramos una muy buena aproximación para \( \gamma \) con una ecuación de estado de la energía oscura variable \( (w = w_0 + w_a(1 - a)) \) para un amplio rango de valores de \( w_0 \) y \( w_a \). Posteriormente hicimos un análisis estadístico usando las matrices de Fisher para pronosticar cómo de bien podrá distinguir Euclid entre los diferentes modelos considerados usando el growth index, parametrizado como \( \gamma = \gamma_0 + \gamma_a(1 - a) \), usando tanto el espectro de potencias de la materia como aquel de la desviación de fotones gravitacional. Concluimos que un experimento como Euclid podrá en efecto distinguir entre diferentes modelos teóricos de energía oscura, en particular entre los modelos DGP, \( f(R) \) and ΛCDM, mientras que será más difícil distinguir este último con uno con ecuación de estado variable o un modelo inhomogéneo del estilo de LTB.
PARES AISLADOS DE GALAXIAS COMO TESTIGOS COSMOLÓGICOS

El efecto de Alcock Paczynski (AP) aprovecha el hecho de que, cuando se analiza con la geometría correcta, la estructura del universo debería observarse isotrópicamente distribuida. Para la materia que se está expandiendo con el background, este efecto construye el producto del parámetro de Hubble $H(z)$ y la distancia diámetro angular $d_A(z)$. El efecto se suele tener en cuenta cuando se analiza estructura a grandes escalas, sobre todo para corregir el espectro de potencias de la materia observado. Recientemente, ha habido una propuesta de usar pares de galaxias aislados como testigos del efecto AP a pequeñas escalas. Sin embargo, la expansión del universo es inhomogénea a dichas escalas y la curvatura espacial depende de la densidad del medio. Arguimos que este hecho distorsiona el efecto AP a pequeñas escalas.

Después de analizar la dinámica de pares de galaxias en la simulación Millennium, encontramos una relación entre velocidades peculiares, propiedades galácticas y densidad local que afecta a como dichos pares siguen la expansión cosmológica. Encontramos que sólo pares de baja masa y aislados pueden seguir esta expansión con una corrección mínima de sus velocidades peculiares. Otros pares requieren correcciones mucho mayores que dependen de la cosmología y de la distancia a la cual se encuentran. Además, en el límite de muy pequeñas escalas en las cuales las galaxias se encuentran ligadas gravitacionalmente, estos pares no contienen ninguna información cosmológica. Si fuésemos capaces de modelar dichas dependencias, podríamos usar estos pares de galaxias aislados para constreñir parámetros cosmológicos.

LA ESCALA DE HOMOGENEIDAD DEL UNIVERSO

El universo que observamos a nuestro alrededor es claramente inhomogéneo. A pequeñas escalas distinguimos galaxias y cúmulos agrupados en estructuras con grandes zonas de espacio vacío alrededor. Sin embargo, el principio cosmológico y la gran isotropía del fondo de radiación de microondas nos dicen que, a grandes escalas, el universo debería de ser homogéneo. Para estudiar esta transición hacia la homogeneidad y averiguar a qué escala el universo se puede considerar homogéneo, hemos desarrollado un método basado en lo que se conoce como la “dimensión fractal”.

El estudio de dicha dimensión fractal ha sido realizado en el pasado para experimentos que miden la distribución de galaxias con redshifts espectroscópicos, encontrando una escala de homogeneidad de $R_H \sim 100$ Mpc. Sin embargo, este análisis no se puede re-
alizar para experimentos fotométricos en los cuales la incertidumbre en la distancia radial de las galaxias es grande. Por ello, hemos diseñado un método para obtener la escala de homogeneidad para dichos experimentos usando la dimensión fractal angular. De este modo, podemos observar la transición hacia la homogeneidad sobre la esfera. Este método tiene como ventaja que, al contrario del caso en tres dimensiones, podemos usar los datos (la posición angular de las galaxias) directamente sin necesidad de convertir los redshifts medidos a distancias físicas usando un modelo. Consecuentemente, el análisis de homogeneidad sobre la esfera está libre de la dependencia con el modelo.

Encontramos que la transición a homogeneidad teórica en el caso angular es similar a la obtenida por trabajos precedentes pero que depende de efectos de proyección, la incertidumbre en la medida del redshift fotométrico y en el parámetro de bias galáctico. Aplicando nuestro análisis a futuras observaciones, seremos capaces de distinguir si el universo a gran escala tiende a la homogeneidad o sigue un modelo fractal.

Conclusiones

Con la llegada del siglo XXI, hemos entrado en una nueva era en la cual las observaciones cosmológicas han mejorado hasta un nivel en el que podemos empezar a definir un modelo estándar de la cosmología basado en materia oscura fría más una energía de vacío responsable de la reciente aceleración del universo. La naturaleza tanto de la materia oscura como de la energía oscura sigue siendo un misterio, a pesar de la precisa determinación de su contribución a la densidad total del universo.

Mientras la naturaleza de la materia oscura parece menos desconocida (la mayoría de los cosmólogos están a favor de un origen basado en la física de partículas), la naturaleza de la energía oscura sigue siendo un territorio desconocido. En la última década ha surgido una pléyada de teorías para explicar la observada aceleración del universo. Todas estas propuestas pueden clasificarse dentro de cuatro grandes categorías: i) la inclusión de un campo externo al modelo estándar de física de partículas, ii) la extensión de la relatividad general de Einstein a través de la suma de términos a la acción de Einstein-Hilbert, iii) la modificación de la gravedad a gran escala mediante el uso de dimensiones extras y iv) la reinterpretación de la aceleración en términos de una geometría espacial no trivial.

Todas las teorías constan de predicciones muy específicas para la evolución de cantidades cosmológicas a nivel de background y muchas de ellas se ajustan a las observaciones con unos pocos parámetros fenomenológicos: la ecuación de estado, la velocidad del sonido, el acoplo entre la materia oscura y la energía oscura, la viscosidad, etc. Sin em-
bargo, si queremos distinguir entre las diferentes alternativas, es necesario ir más allá de la evolución a nivel de background y considerar la evolución de parámetros a primer orden en teoría de perturbaciones.

Actualmente, las observaciones usadas para constreñir la materia y la energía oscura son las magnitudes de cefeidas y supernovas para la determinación de la expansión como función del redshift, el espectro de potencias de materia y las oscilaciones acústicas de bariones para la determinación del contenido de materia como función del redshift, junto con el fondo de radiación de microondas para la determinación de la curvatura espacial global y valores asintóticos de los parámetros cosmológicos. Muchas de estas observaciones muestran una detección de la energía oscura al nivel de 2 o 3 sigmas. Sin embargo, con los experimentos que se han planificado para el futuro, se espera obtener una detección mucho más rotunda.

En esta tesis, hemos estudiado la posibilidad de que un experimento futuro como Euclid pueda distinguir entre las 4 clases de teorías de energía oscura. En el proceso, hemos encontrado soluciones exactas y aproximadas para el “growth index” en términos de funciones simples para modelos como ΛCDM con una ecuación de estado tanto constante como variable. Hemos propuesto una parametrización simple para el growth index que se ajusta bien a la mayoría de modelos cosmológicos excepto casos extremos como $f(R)$. También encontramos que, aunque las cantidades a nivel de background son similares para los diferentes modelos estudiados, el growth index puede ser muy diferente para la mayoría de las distintas clases de modelos.

Para ver si un experimento como la futura misión espacial Euclid sería capaz de distinguir entre modelos cosmológicos usando el growth index, calculamos dicha cantidad para cuatro modelos diferentes: wCDM, DGP, $f(R)$ y LTB, cuyo comportamiento es muy diferente para $z \sim 1$. Además, calculamos los errores esperados para un experimento como Euclid, y obtenemos que seremos capaces de distinguir entre los modelos DGP, $f(R)$ y ΛCDM, mientras será más difícil distinguir este último de un modelo con ecuación de estado variable o un modelo LTB usando sólo el growth index como parámetro.

A lo largo de esta tesis, también hemos estudiado la posibilidad de usar pares aislados de galaxias para precisar parámetros cosmológicos. Hemos encontrado que dichos pares presentan una corrección mínima para separaciones a partir de $1\ Mpc$. También hemos investigado la evolución de dichos pares con el redshift, obteniendo que la región de interés cosmológico no parece depender de la historia de la expansión. Por ello, argüimos que estos pares probablemente puedan ser utilizados como testigos cosmológicos con una corrección de “redshift space distortions” mínima.
Dado que los experimentos de galaxias están limitados por la luminosidad de las mismas, hemos estudiado cómo depende la velocidad de pares aislados de galaxias con propiedades galácticas como luminosidad o masa estelar. Encontramos que los únicos pares que tienen alguna esperanza de poder ser usados para observaciones cosmológicas en este contexto son aquellos menos pesados y menos luminosos, haciendo difícil su detección y su uso para futuros experimentos. A pesar de que estos pares parecen prometedores por la pequeña corrección a su velocidad necesaria, vemos que sus velocidades dependen de la masa de las galaxias y de la luminosidad, propiedades que necesitan ser modeladas para poder usar este análisis en el futuro.

También hemos explorado la transición a la homogeneidad en una distribución de galaxias medida a través de redshifts fotométricos. Hemos descubierto que dicha transición se observa a una escala de aproximadamente 100 Mpc, pero que depende de factores como efectos de proyección, el parámetro de bias o la incertidumbre en la determinación del redshift fotométrico. Vemos que, a mayor bias, la transición a la homogeneidad sucede a mayores escalas. Esto se debe a que galaxias con un bias mayor se encuentran más atrai das gravitacionalmente al potencial de materia oscura subyacente y, por tanto, su distribución es menos homogénea. En el caso de la incertidumbre en el redshift, a mayor incertidumbre, antes se alcanza la escala de homogeneidad, ya que más galaxias cercanas acababan siendo catalogadas dentro del mismo análisis, homogeneizando la distribución.
Cosmology is the study of the Universe and its components, how it formed, how it has evolved and what is its future. What we understand today as cosmology has grown from ideas already present in ancient history when humankind tried to answer questions such as “what is it that surrounds me?”, which developed into “how does the Universe work?”, the key question that cosmology asks, and tries to answer.

Many of the earliest recorded scientific observations were about cosmology, and the pursuit of understanding has continued for over 5000 years. Cosmology has leaped in the last 20 years with radically new information about the structure, origin and evolution of the Universe obtained through recent technological advances in telescopes and space surveys, and basically has become a search for the understanding of not only what makes up the Universe, but also its overall architecture. Modern cosmology is between science and philosophy, close to philosophy because it asks fundamental questions about the Universe, close to science since it looks for answers in the form of empirical understanding by observation and rational explanation. Thus, theories about cosmology operate with a tension between a philosophical urge for simplicity and a wish to include all the Universe’s features versus the total complexity of it all.

However, when one speaks of modern cosmology, one is usually referring to the science that was reborn in the early 1920s to test Einstein’s theory of General Relativity (GR) on large scales. When he finally published the field equations of General Relativity in 1916
Einstein, together with the rest of mankind, believed that the Universe was static, but discovered that the original formulation of his theory did not permit it. In order to overcome this “problem”, he found that he could add a constant to his equations that would counteract the attractive force of gravity on cosmic scales, thus producing a static universe. His first paper on cosmology in 1917 [7] was the first to include the cosmological constant into the so-called Einstein model: a static universe in which space is unbounded but finite. It would not be until 1929, when Edwin Hubble discovered that the Universe was actually expanding [8], that Einstein’s static model of the Universe was abandoned and his cosmological constant dubbed “his biggest blunder” by its creator, although this may be apocryphal, since it was only recalled by G. Gamow and only after Einstein’s death.

Observational cosmology had another major leap with the discovery of cosmic background radiation (CMB) in the microwave spectrum discovered in 1964 by Arno Penzias and Robert Wilson [9], which was already predicted in the 1940s. This finding shed light on the early universe, and has continued to do so through the many surveys that have been conducted and are still being conducted (see for example: [10–12]). Such experiments made cosmology advance observationally and thus it begun its path as a precise science.

With the discovery of cosmic acceleration in the late 1990s [13, 14], cosmology has done a full circle back to the cosmological constant $\Lambda$. With the cold dark matter plus cosmological constant model (hereafter $\Lambda$CDM) in close agreement with current observations, we could say that we might have found a standard model of cosmology, analogous to the standard model of particle physics. However, there are several reasons for which we might not think that $\Lambda$ is the end of the story. Let us first look at some equations of the background evolution with a cosmological constant.

### 1.1 Einstein’s equations in an accelerated universe

Since Einstein’s equations satisfy energy-momentum conservation, $\nabla_\nu T^{\mu\nu} = 0$ and $\nabla_\alpha g_{\mu\nu} = 0$, $\forall\alpha$, it is possible to add the term $\Lambda g_{\mu\nu}$ to the equations:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu}. \quad (1.1)$$

If we consider a Friedmann-Lemaître-Robertson-Walker (FLRW) metric $ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right)$, where $a(t)$ is the scale factor and $k$ characterises
the spatial curvature, Einstein’s equations read:

\[ H^2 = \frac{8\pi G}{3} \rho - \frac{k}{a^2} + \frac{\Lambda}{3} \]  
(1.2)

\[ \frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3}, \]  
(1.3)

where \( H = \dot{a}/a \) is known as the Hubble parameter and the covariant conservation equation \( T_{\mu;\nu} = 0: \)

\[ \dot{\rho} + 3H(\rho + p) = 0, \]  
(1.4)

where a dot denotes a derivative with respect to cosmic time \( t \). It is clear from Eq. (1.3) that the cosmological constant term acts as a repulsive force against gravity at the background level. If we consider a pressureless, matter dominated, static universe \((p = 0, \dot{a} = \ddot{a} = 0)\), Eq. (1.2) and Eq. (1.3) become

\[ \rho = \frac{\Lambda}{4\pi G}, \quad \Lambda = \frac{k}{a^2}. \]  
(1.5)

Einstein believed that through this model you could link mass \((\rho)\) to inertia via the spacetime metric \( g_{\mu\nu} \). However, this solution is unstable under perturbations in the density \( \rho \). In fact, if \( \Lambda/3 > (4\pi G\rho)/3 \), Eq. (1.3) shows a departure from the static point with an increase in \( a \). If, on the other hand, \( \Lambda/3 < (4\pi G\rho)/3 \), Eq. (1.3) also departs from the static solution with decreasing \( a \). Even though the cosmological constant is indeed the “simplest” way to get an accelerated universe, it still has 2 main problems which are described below.

### 1.2 The fine tuning problem

Going back to Eq. (1.2), we notice that, if \( \Lambda \) is the cause of cosmic acceleration today, it must be of the order of the Hubble parameter today squared \( H_0^2 \), i.e.:

\[ \Lambda \sim H_0^2 = (2.13h \times 10^{-42} \text{ GeV})^2. \]

We can also interpret this as an energy density as:

\[ \rho_\Lambda \sim \frac{\Lambda m_{\text{pl}}^2}{8\pi} \sim 10^{-47} \text{ GeV}^4 \sim (10^{-3} \text{ eV})^4, \]  
(1.6)
where we have set \( h = 0.7 \) and \( m_{\text{pl}} = 10^{19} \text{ GeV} \). We could assume that this corresponds to vacuum energy of some field with mass \( m \), momentum \( k \) (not to be mistaken with spatial curvature from previous sections) and frequency \( \omega \), whose zero-point energy is given by \( E_0 = \omega/2 = \sqrt{k^2 + m^2}/2 \). The vacuum energy of this field would then correspond to the sum of all the zero-point energies up to a cut-off scale \( k_{\text{max}} \):

\[
\rho_{\text{vac}} = \int_0^{k_{\text{max}}} \frac{d^3k}{2(2\pi)^3} \sqrt{k^2 + m^2}.
\]

(1.7)

For cut-off scales much larger than the field mass, \( k \gg m \), Eq. (1.7) is approximately \( k_{\text{max}}^4/16\pi^2 \). Since gravity is supposed to be valid up to Planck scales \((m_{\text{pl}})\), this yields a vacuum energy of \( \rho_{\text{vac}} \sim 10^{74} \text{ GeV}^4 \). This is 121 orders of magnitude larger than the observed energy density of the cosmological constant. If we assume that gravity is only valid up to a smaller cut-off scale, for example, that of QCD, we obtain a vacuum energy value of \( \rho_{\text{vac}} \sim 10^{-3} \text{ GeV}^4 \), still much larger than the observed value of \( \rho_\Lambda \).

As the vacuum energy from QFT arises from an ambiguity in the ordering of fields, one could think of a specific order which could give us the desired \( \rho_\Lambda \). However, whether or not the zero point energy in field theory is realistic is still a debatable question.

Supersymmetry (SUSY) is a solution that has appeared to solve this problem in a more elegant way than introducing a specific ordering of the fields. SUSY introduces a bosonic partner for every fermion and vice versa. This means that the vacuum energy contribution of normal standard model particles is cancelled by the contribution from their supersymmetric partners. This can be seen in the following expression for the vacuum energy of a field with spin \( j > 0 \),

\[
\rho_{\text{vac}} = \frac{1}{2} (-1)^{2j}(2j + 1) \int_0^\infty \frac{d^3k}{(2\pi)^3} \sqrt{k^2 + m^2}
\]

\[
= \frac{(-1)^{2j}(2j + 1)}{4\pi^2} \int_0^\infty dk \frac{k^2}{\sqrt{k^2 + m^2}}.
\]

(1.8)

Exact supersymmetry implies an equal number of fermionic and bosonic degrees of freedom for a given value of the mass \( m \) such that the net contribution to the vacuum energy vanishes, which means that, if supersymmetry is not broken at some scale, the cosmological constant it predicts is zero.

However, we know that we do not live in a supersymmetric vacuum state and hence SUSY should be broken today. For a viable supersymmetric scenario, the supersymmetry
breaking scale should be around $M_{\text{SUSY}} \sim 10^3 \text{ GeV}$.

With a broken SUSY, one of course wants to ensure that no new scales are introduced between the electroweak scale of about 246 GeV and the Planck scale. The superpartners of the Standard Model particles are thus expected to have masses of the order of TeV. Masses much lower than this are ruled out from null experimental results in present day accelerators and specific bounds for the masses for the various superpartners of standard model particles are available from analysis of experimental data.

However, even with this new cut-off introduced by SUSY, the value of $\Lambda$ still differs in many orders of magnitude. At present we do not know how the Planck or the SUSY breaking scales are really related to the observed vacuum scale.

1.3 The coincidence problem

The second problem the cosmological constant suffers is that its value is, in theory, almost identical to a totally unrelated quantity: the present matter energy density $\rho_m^0$. This means that, not only is the current value of $\rho_m^0$ too small to be understood as a vacuum energy, but it also has the same order of magnitude of $\rho_m^0$, for no apparent reason. If the matter density evolves as $\rho_m = \rho_m^0 (1 + z)^3$, we can find the value of the coincidence redshift by equating this to the current cosmological constant density $\rho_\Lambda^0$ in a flat universe to obtain:

$$z_c = \left( \frac{\Omega_\Lambda^0}{1 - \Omega_\Lambda^0} \right)^{1/3} - 1.$$

Substituting $\Omega_\Lambda^0 = 0.7$, we get $z_c \sim 0.3$. This is what is known as the coincidence problem. Unlike the fine tuning problem, most of the theoretical models of dark energy that will be described in Chapter 2 suffer from this coincidence problem yielding a $z_c$ close to zero.

There are several solutions to this problem:

- Models in which $\rho_{\text{DE}}$ responds to the behaviour of $\rho_m$ and tends to it, no matter what the initial conditions for $\rho_{\text{DE}}$ are. In this case, $\rho_{\text{DE}}$ can be non-zero before catching up with $\rho_m$, which alleviates the coincidence problem. However, in such models, acceleration starts only very recently and a coincidence arises again. This behaviour is exhibited by “tracker” models such as quintessence, which will be described in Chapter 2.

- Models in which there is no coincidence. In this case, the two fluids (dark energy and dark matter) have always had similar energy densities, however, one of them
clusters while the other does not because of a large sound speed. This is achieved by regulating their equations of state so that $\rho_m$ and $\rho_{DE}$ are always similar. These models exhibit two problems: (i) the common equation of state always produces acceleration, making it hard to satisfy observational constraints and (ii) the equation of state changes to an accelerated phase when it is needed to fit observations and another coincidence arises. Along these lines, it is possible to have a model with several eras of acceleration by which it is just chance that we are experiencing one now. The problem then is to have sufficient structure formation.

- Backreaction. The coincidence between $\rho_m$ and $\rho_{DE}$ arises due to a coincidence between acceleration and structure formation. This can be the case if the clustering of matter causes acceleration through non-linear effects. In this case, there is no dark energy or acceleration, it arises from the fact that we are interpreting observations with the wrong background metric (i.e., FLRW). This is also the case for big void models such as the Lemaître-Tolman-Bondi case, in which distant objects appear to be receding from us in an accelerated manner due to the fact that we are inside an underdense region which broadens the light geodesics, making distant objects appear dimmer.

A quick summary of this section is that the coincidence problem is far from solved. A convincing explanation for dark energy must thus provide a solution to both the fine tuning and the coincidence problem.

As we have seen, even though the addition of a cosmological constant to Einstein’s field equations may be the simplest way to obtain acceleration, it has its caveats. For this reason, theorists have explored other possibilities ranging from modifications to Einstein’s theory of gravity to inhomogeneous universe models. The success of General Relativity on solar system scales is impressive, for example, the strong equivalence principle has been tested in one part in $10^{13}$ [15]. However, on larger, cosmological, scales, this success is less clear. On such large scales, General Relativity becomes harder to test due to the presence of components like dark matter or the possibility of clustering dark energy, which could be difficult to disentangle from a modification to GR. For these reasons and our general ignorance on the properties and nature of dark energy, many different theoretical models have arisen and their number is ever growing. It is therefore important to be able to constrain the parameter space of the different theories and propose observables that make them distinguishable with current and future observations. In order to do this, let us begin by reviewing the background and perturbation equations in a general setting, thus introducing parameters common to most models that will be studied in Chapter 2.
1.4 Background evolution

Let us consider a universe dominated by a single component with an equation of state defined as \( w \equiv p/\rho \). If \( w \) is constant, then one can solve Eq. (1.2) and Eq. (1.4) to find the evolution of \( \rho \) and \( a \) analytically (for a flat universe):

\[
\rho \propto a^{-3(1+w)}, \quad \rho \propto (t - t_i)^{2/3(1+w)}; \quad (1.9)
\]

where \( t_i \) is a constant. For relativistic species \( w = 1/3 \), leading to \( \rho \propto a^{-4} \) and \( a \propto (t - t_i)^{1/2} \) during the radiation dominated era. In the case of non-relativistic species, their pressure is negligible compared to their energy density, i.e. \( w \approx 0 \) and hence \( \rho \propto a^{-3} \) and \( a \propto (t - t_i)^{2/3} \) for the matter dominated phase. It is worth noting that, in order to get acceleration, \( \ddot{a} > 0 \) is needed; imposing this on Eq. (1.3), one obtains:

\[
p < -\frac{\rho}{3}, \quad \Rightarrow \quad w < -\frac{1}{3}. \quad (1.10)
\]

In the case that \( w = -1 \), i.e. \( p = -\rho \), we get from Eq. (1.4) that \( \rho \) is constant. This is the cosmological constant case. In the flat universe scenario with a constant energy density, Eq. (1.2) shows that the Hubble parameter \( H \) is constant, which leads to an exponential growth of the scale factor \( (a \propto e^H) \), producing cosmic acceleration.

Let us now consider a more general case in which the equation of state of dark energy is not constant but given by \( \rho_{DE} = p_{DE}/\rho_{DE} \) satisfying the continuity equation Eq. (1.4). Integrating this equation and using the relationship \( dt = -dz/[H(1 + z)] \), we obtain

\[
\rho_{DE} = \rho_{DE}^0 \exp \left( \int_0^z \frac{3(1 + \omega_{DE})}{1 + z'} dz' \right); \quad (1.11)
\]

which can be expressed in a similar form to Eq. (1.9) by defining \( \omega_{DE} \):

\[
\rho_{DE} \propto a^{-3(1+\omega_{DE})}, \quad \omega_{DE} = \frac{1}{\ln(1 + z)} \int_0^z \frac{\omega_{DE}}{1 + z'} dz'. \quad (1.12)
\]

With this in mind, Eq. (1.2) can be written in the form:

\[
H^2(z) = H_0^2 \left[ \Omega_r^0 (1 + z)^4 + \Omega_m^0 (1 + z)^3 \right. \\
\left. + \Omega_k^0 (1 + z)^2 + \Omega_{DE}^0 \exp \left\{ \int_0^z \frac{3(1 + \omega_{DE})}{1 + z'} d' \right\} \right]. \quad (1.13)
\]
Solving for \( w_{\text{DE}} \), we get

\[
w_{\text{DE}} = \frac{2(1 + z)E(z)E'(z) - 3E^2(z) - \Omega^0_0(1 + z)^4 + \Omega^0_k(1 + z)^2}{3 \left[ E^2(z) - \Omega^0_r(1 + z)^4 - \Omega^0_m(1 + z)^3 - \Omega^0_k(1 + z)^2 \right]}, \tag{1.14}
\]

where \( E(z) \equiv H(z)/H_0 \) and a prime denotes a derivative with respect to \( z \). Now, the relationship of \( E(z) \) to observations is done via the luminosity distance \( d_L(z) \) (for an FLRW metric):

\[
E^2(z) = \frac{(1 + z)^2 \left[ c^2(1 + z)^2 + \Omega^0_k H_0^2 d_L^2(z) \right]}{[(1 + z)H_0 d'_L(z) - H_0 d_L(z)]^2}.
\]

The expansion history is usually reconstructed with supernovae observations at \( z \lesssim 1 \), in which radiation can be neglected. We can also safely assume that the Universe is spatially flat, as can be seen from the latest constraints in which \( 100\Omega^0_k = -0.1^{0.062}_{0.05} \) \cite{16}. Taking this into account, the dark energy equation of state as a function of \( E(z) \) becomes:

\[
w_{\text{DE}} = \frac{2(1 + z)E(z)E'(z) - 3E^2(z)}{3 \left[ E^2(z) - \Omega^0_m(1 + z)^3 \right]}.
\tag{1.15}

This relationship is frequently used to constrain observationally \( w_{\text{DE}} \). However, it is important to note that the truly observable quantity is \( E(z) \) (or the so-called deceleration parameter: \( q(z) = -1 + d \ln H(z)/d \ln (1 + z) \)) and not \( w_{\text{DE}} \). In fact, one cannot determine \( w_{\text{DE}} \) fully from \( E(z) \) since it also depends on \( \Omega^0_m \), which can only be obtained if one assumes a parameterisation of \( w_{\text{DE}} \) or a particular known evolution for \( \rho_{\text{DE}} \) \cite{17}. Using gravitational lensing, one can estimate the values of \( \Omega^0_m \) using virialised objects that have already decoupled from the expansion.

### 1.4.1 Parameterisation of the Background Evolution

In order to study the nature of dark energy, it is convenient to parameterise its equation of state to be able to produce forecasts and to constrain theoretical models. In order to do this, there are three main approaches. The first one is to take a set of theories and find parameters to describe \( w_{\text{DE}} \) as accurately as possible. After having done this, one can try to include as many theoretical models as possible within each parameterisation. This has been done extensively for scalar-field models of dark energy, which will be explained in more detail in Chapter 2 (for several of these parameterisations, see \cite{18–26}).

The second approach is to start from a simple parameterisation of \( w_{\text{DE}} \) without assuming a particular theoretical model that can however fit the predictions. The most widely
used parameterisation of this kind was introduced in \([27, 28]\) and is linear in the scale factor \(a\):

\[
wx(a) = w_0 + wa(1 - a),
\]

where the subscript \(X\) denotes a generic form of dark energy. While this parameterisation is useful (due to its simplicity) to produce forecasts for different dark energy surveys, as we did in \([1]\), it contains no physical information since it is not based on any physical theory. Another downside to this class of parameterisations is that one could oversimplify the evolution of \(w_{DE}\) by assuming a particular form. Other similar parameterisations found in the literature are a linear and logarithmic parameterisation in \(z\) \([29–31]\), a Taylor expansion in \(a\) \([27, 28]\), a rapidly varying \(w_{DE}\) \([32]\) or a polynomial parameterisation \([33]\).

The third possibility is what is known as the principal component approach (PCA) \([34]\) which avoids assuming a form of \(w_{DE}\) by taking it to be constant or linear in each redshift bin and then derives which set of parameters is best constrained by each survey.

As we have seen in Sec. 1.4, Eq. (1.11) gives us a relationship between \(\rho_{DE}\) and \(w_{DE}\), so one could consider measuring \(\rho_{DE}\) instead of the dark energy equation of state since it is more directly related to observables and can be better constrained for equal number of redshift bins \([35]\). An interesting aspect of Eq. (1.11) is that, by parameterising dark energy with \(w_{DE}\), we are implicitly assuming that \(\rho_{DE}\) does not change sign. This can clash with some dark energy models in which \(\rho_{DE}\) becomes negative in the future (for example, “doomsday models” \([36]\)).

1.5 Cosmological perturbation theory

Now that we have seen how to parameterise dark energy on the background level, let us look into the linear perturbation equations in a dark energy dominated universe for a general fluid focusing on scalar perturbations. Since dark energy dominates at late times, let us, for simplicity, consider a spatially flat universe with only dark matter and dark energy components. In this case, Eq. (1.13) reduces to

\[
H^2(z) = H_0^2 \left[ \Omega_m^0 (1 + z)^3 + (1 - \Omega_m^0) \exp \left\{ \int_0^z \frac{3(1 + w_{DE})}{1 + z'} dz' \right\} \right].
\]

(1.17)
The most general linearly perturbed metric one can construct is

\[
    ds^2 = a^2(\eta) \left\{ -(1 + 2\psi)d\eta^2 + 2(B_{ij} - S_i) d\eta dx^i \\
    + \left[ (1 + 2\varphi)\gamma_{ij} + 2E_{(ij)} + 2F_{(ij)} + h_{ij} \right] dx^i dx^j \right\},
\]

(1.18)

where \(a(\eta)\) is the scale factor as a function of conformal time \((d\eta = dt/a)\), \(\varphi, \psi, E\) and \(B\) are all scalar perturbations, \(S\) and \(F\) are transverse vector perturbations and \(h_{ij}\) is a symmetric transverse traceless tensor perturbation. \(\gamma_{ij}\) corresponds to the spatial metric and \(\nabla\) is a covariant derivative with respect to that metric, whereas parenthesis indicate symmetrisation. As regards perturbations in the matter sector, we shall consider perfect fluids with anisotropic stresses such that their energy momentum tensor is given by

\[
    T^{\mu\nu} = (\rho + p) u^\mu u^\nu g^{\mu\nu} + \Pi^{\mu\nu},
\]

(1.19)

where \(\rho\) and \(p\) are the energy density and the pressure of the fluid respectively, \(u^\mu\) is the fluid’s four-velocity and \(\Pi^{\mu\nu}\) its anisotropic stress tensor which corresponds to the traceless component of \(T^i_j\). The perturbed fluid’s four-velocity can be written in terms of the metric variables as

\[
    u^\mu = \frac{1}{a} \left[ (1 + \varphi), v^i + \nu^i \right] \\
    u_\mu = a \left[ -(1 + \varphi), (\nu + B)_i + \nu_i - S_i \right],
\]

which satisfies \(u^\mu u_\mu = -1\). Here we have introduced two new perturbative quantities: a scalar irrotational velocity potential \(\nu\) and a transverse vector field \(v^i_i = 0\). Component-wise, the perturbed energy momentum tensor can be broken up into:

\[
    T_0^0 = -(\rho_0 + \delta\rho) \\
    T_i^0 = (\rho_0 + p_0)(B^i_i + v^i_i + \nu_i - S_i) \\
    T_0^j = -(\rho_0 + p_0)(v^j_i + \nu^j) \\
    T_j^i = (p_0 + \delta p)\delta^i_j + p_0 \Pi^i_j,
\]

where \(\rho_0\) and \(p_0\) are the energy density and pressure of the homogeneous and isotropic background universe respectively, \(\delta\rho\) and \(\delta p\) are the density and pressure perturbations. Coordinate transformations affect the splitting between spatial and timelike components of matter fields and, thus, quantities like density or pressure are gauge dependent. For this
reason, we must choose a gauge in which to pose our perturbed Einstein equations. Our
gauge of choice is the conformal Newtonian gauge (or longitudinal or zero-shear gauge),
characterised by zero off-diagonal metric terms, i.e. $S_i = B_{ij} = E_{ij} = F_{ij} = h_{ij} = 0$ in
Eq. (1.18). The metric described in Eq. (1.18) then reduces to
\[
\begin{align*}
    ds^2 &= a^2(\eta) \left[ -(1 + 2\Psi) d\eta^2 + (1 - 2\Phi) dx_i dx^i \right]. 
\end{align*}
\]
The energy-momentum components in this gauge are, in Fourier space:
\[
\begin{align*}
    T_0^0 &= -\left( \rho_0 + \delta \rho \right) \\
    ik_i T_0^i &= -ik_i T_0^0 = (\rho_0 + p_0) \theta \\
    T_i^i &= (p_0 + \delta p) \delta_i^i + p_0 \Pi_i^i,
\end{align*}
\]
where $\theta = ik_i \psi_i$ represents the divergence of the velocity field. In this gauge, the potential
$\Psi$ plays the part of the gravitational potential $\psi$ in the Newtonian limit and thus has a simple physical interpretation, while $\Phi$ is the Newtonian curvature perturbation equivalent. One of the advantages of working in the Newtonian gauge is that the metric $g_{\mu\nu}$ is diagonal, which greatly simplifies calculations. Moreover, the choice of working in the Newtonian gauge influences the evolution of perturbations on large scales (especially those larger than the Hubble horizon) while, for smaller scales, the gauge choice is less important and perturbations are expected to evolve independently of such a choice.

### 1.5.1 Einstein equations in the Newtonian gauge

In the Newtonian gauge and in Fourier space, the Einstein equations for the first-order perturbations give [37]:
\[
\begin{align*}
    k^2 \Phi + 3\frac{\dot{a}}{a} \left( \Phi + \frac{\dot{\Psi}}{a} \right) &= -4\pi G a^2 \sum_a \rho_{a0} \delta_a, \\
    k^2 \left( \dot{\Phi} + \frac{\dot{a}}{a} \Phi \right) &= 4\pi G a^2 \sum_a (\rho_{a0} + p_{a0}) \theta_a, \\
    \ddot{\Phi} + \frac{\dot{a}}{a} (\dot{\Psi} + 2\dot{\phi}) + \left( 2\frac{\dot{a}}{a} - \frac{\dot{a}^2}{a^2} \right) \Psi + \frac{k^2}{3} (\Phi - \Psi) &= 4\pi G a^2 \sum_a \delta p_a, \\
    k^2 (\Phi - \Psi) &= 12\pi G a^2 \sum_a (\rho_{a0} + p_{a0}) \pi_a, 
\end{align*}
\]
where a dot denotes a derivative with respect to conformal time, $a$ is an index that runs
over all species, $\delta = \delta \rho / \rho_0$ and $\pi$ is related to the fluid’s anisotropic stress by

$$\left(\rho_0 + \rho_0'\right) \pi = -\left(\hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij}\right) \Pi_{ij}.$$  

The energy momentum conservation equation $T_{\mu;\nu}^\mu = 0$ yields

$$\dot{\delta} = -(1 + w)(\theta - 3 \Phi) - \frac{3}{a} \left(\frac{\delta \rho}{\rho_0} - w \delta\right) \quad \text{for} \quad v = 0 \quad (1.22)$$

$$\dot{\theta} = -\frac{\dot{a}}{a} (1 - 3w) \theta - \frac{w}{1 + w} \theta + k^2 \frac{\delta \rho / \rho_0}{1 + w} + k^2 \Psi - k^2 \pi \quad \text{for} \quad v = i. \quad (1.23)$$

So far, the perturbation equations presented here are valid for any type of fluid. In order
to apply them to a particular species, one must specify the equation of state parameter $w$,
the pressure perturbation $\delta \rho$ and the anisotropic stress $\pi$. Let us look in more detail at
matter perturbations, since they are especially important to define the growth of structure.

1.5.2 The growth of structure

As will be seen later, it is not enough to measure the background quantities such as the ex-
pansion rate to be able to distinguish between different dark energy models. The growth
of structure is thus particularly important for this purpose. Due to the many dark energy
models that currently exist, it is useful to parameterise the possible departures from a par-
ticular fiducial model. For this purpose, two different approaches exist:

- **Model parameters.** These contain the degrees of freedom of the dark energy or mod-
ified gravity theory and modify the evolution equations of $T^{\mu\nu}$ of the fiducial model.
They can be associated with physical quantities and have generally different behaviour for the various theoretical models.

- **Trigger relations.** These are quantities or expressions derived from observations and
only hold in the fiducial model used. They, thus, break down if the fiducial model
is not the correct description of the Universe.

Since the current observational constraints favour the $\Lambda$CDM model, this is usually taken
to be the fiducial model. For large scale structure (LSS) surveys, the important quantities
to study are the matter perturbations. Let us now look into the equations that describe
them. In this case, \( \delta p = w = \pi = 0 \), and equations (1.22) and (1.23) become

\[
\dot{\delta} = -\theta + 3\dot{\Phi} \tag{1.24}
\]
\[
\dot{\theta} = -\frac{\dot{a}}{a}\theta + k^2\Psi. \tag{1.25}
\]

However, it is not enough to specify \( \delta p \), \( w \) and \( \pi \), since, as can be seen from the equations above Eq. (1.24), Eq. (1.25), these depend on the evolution of the metric potentials \( \Phi \) and \( \Psi \), which in turn depend on the evolution of the other species, see Eq. (1.21). Hence, if one wants to study the evolution of perturbations in a universe filled with dark matter and dark energy, one must also specify the equation of state of the dark energy, the pressure perturbation and the anisotropic stress for the dark energy perturbations. One can choose to ignore dark energy perturbations and just include acceleration at the background level. This assumption is usually violated, since dark energy perturbations affect the evolution of matter ones, especially if the sound speed of the dark energy perturbations is small \([38–40]\), since then dark energy is able to cluster on small scales. However, under the assumption of a high value of the dark energy sound speed (as it is the case for the quintessence model, where \( c_s^2 = 1 \), see Chapter 2) we expect this assumption still to hold because dark energy is able to cluster only on very large scales.

Eq. (1.24) and Eq. (1.25) are linked together via the Poisson equation:

\[
k^2\Phi = -4\pi G a^2 \rho_m \left( \delta_m + \frac{3aH}{k^2} \theta_m \right), \quad \text{and} \quad \Phi = \Psi. \tag{1.26}
\]

Joining Eq. (1.24) and Eq. (1.25) for sub-Hubble scales \( k \gg aH \), the master equation for the matter density contrast becomes:

\[
a^2 \delta_m''(a) + (3 - \varepsilon(a))a \delta_m'(a) - \frac{3}{2} \Omega_m(a)\delta_m(a) = 0, \tag{1.27}
\]

where \( \varepsilon(a) = -d\ln H(a)/d\ln a = 1 + q(a) \) and a prime denotes a derivative with respect to the scale factor \( a \). The exact growing mode solution of the above differential equation, for a constant dark energy equation of state parameter, \( w \), is \([1, 41]\)

\[
\delta_m(a) = a \cdot _2F_1 \left[ \frac{w - 1}{2w}, \frac{-1}{3w}, 1 - \frac{5}{6w}; 1 - \Omega_m^{-1}(a) \right] \tag{1.28}
\]
The full solution is reported in the Appendix A. Eq. (1.28) can be further simplified using transformation formulae for the Hypergeometric functions into

\[
\delta_m(a) = a \Omega_m(a)^{-\frac{1}{3w}} F_1\left[ -\frac{1}{3w}, \frac{1}{2}, -\frac{1}{3w}, 1 - \frac{5}{6w}; 1 - \Omega_m(a) \right].
\] (1.29)

It is worth having a further look at the structure of the solution found. Eq. (1.29) is composed of two terms: the first one is the usual scale factor \(a\), which is the solution we expect if dark energy had been neglected also at background level; furthermore, Eq. (1.29) reduces to the classical solution in the matter domination era; this can be clearly seen if we set \(\Omega_m(a) \rightarrow 1\), then the Hypergeometric function is 1. The second term, instead, contains all the information about the dark energy fluid: via a direct dependence on \(\Omega_m(a)\) and the Hypergeometric function, whose dependence is not straightforward. It is worth noticing that the slowed contribution to the dark matter comes from the term \(\Omega_m(a)^{-\frac{1}{3w}}\) but this is too big and it suppresses too much the growth of matter perturbations. The Hypergeometric function in Eq. (1.29), instead, is a function that slowly increases with the scale factor, from 1 for \(a \ll 1\) to just 1.16 for \(a \sim 1\). This compensates for the over suppression of the \(\Omega_m(a)^{-\frac{1}{3}}\) term.

Other useful quantities derived from the matter density contrast are the growth rate

\[
f(z) \equiv \frac{d \log \delta_m}{d \log a},
\] (1.30)

the growth factor

\[
G(z) = \Delta / \Delta_0,
\] (1.31)

where \(\Delta\) is the gauge-invariant comoving density contrast

\[
\Delta_m \equiv \delta_m + 3aH\theta_m/k^2,
\] (1.32)

and the growth index

\[
\gamma = \frac{\log f(z)}{\log \Omega(z)}.
\] (1.33)

For a constant equation of state, the growth rate \(f\) is related to \(\Omega_m\) and \(w\) exactly as follows [1]

\[
f(a) = \frac{d \log \delta_m}{d \log a} = \Omega_m^{1/2}(a) \frac{P_{5/6w}^{5/6w} \left[ \Omega_m^{-1/2}(a) \right]}{P_{-1/6w}^{5/6w} \left[ \Omega_m^{-1/2}(a) \right]}.\] (1.34)
It is also possible to find a similar relationship with a varying equation of state parameterised linearly in $a$ as in Eq. (1.16). In this case there are no exact analytical solutions for the matter density contrast as for the case of a constant $w$; the main problem being that there is no direct transformation between the scale factor $a$ and the new variable $u$, see Appendix A. For $w$ given by Eq. (1.16), the matter density can still be integrated to give

$$\Omega_m(a) = \left(1 + \frac{\Omega_{DE}}{\Omega_m} a^{-3(w_0+w_a)} e^{3w_a(a-1)}\right)^{-1}. \tag{1.35}$$

We can also integrate the master equation (1.27) in this case, and, although an analytical solution does not exist, we found an approximate one which is within 0.1% of the numerical solution for the whole range of $(w_0, w_a)$ values [1]. In this case, the growth rate reads

$$f(a) = \Omega_m^{1/2}(a) \left(\frac{P_{1/6w(a)}^{5/6w(a)+w_a/6w^2(a)}}{P_{-1/6w(a)}^{5/6w(a)+w_a/6w^2(a)}} \right) \left[\frac{\Omega_m^{-1/2}(a)}{\Omega_m^{-1/2}(a)}\right]. \tag{1.36}$$

### 1.5.2.1 The Gamma Parameterisation

One can conduct a similar analysis for the growth index $\gamma$. The original parameterisation $f(a) = \Omega_m(a)^\gamma$ [42] assumed $\gamma$ to be constant and, moreover, with a value that was very approximately $\gamma \simeq 0.6$ for general relativity plus a cosmological constant $\Lambda$, i.e. for a constant equation of state parameter $w = p/\rho = -1$. Furthermore, detailed studies went further and computed the $\gamma$ parameter for a constant but arbitrary $w$, [43]:

$$\gamma = \frac{3(w - 1)}{(6w - 5)} \tag{1.37}$$

which reduces to $\gamma = 6/11 \simeq 0.55$ for $w = -1$. Eq. (1.37) does not take into account the dependence of $\gamma$ on the matter density parameter. There is an expression for the growth index which does depend on $\Omega_m(a) < 1$, to first order, see [44],

$$\gamma = \frac{3(w - 1)}{(6w - 5)} + \frac{3}{2} \frac{(1 - w)(2 - 3w)}{(5 - 6w)^3}(1 - \Omega_m(a)) + \mathcal{O}(1 - \Omega_m)^2, \tag{1.38}$$

which reduces to the well known result Eq. (1.37) in the limit $a \to 0$. However, we have found in Eq. (1.34) an exact solution for all $a$. Substituting this result into the definition
of $\gamma$, we obtain

\[
\gamma(a) = \frac{1}{2} + \frac{1}{\ln \Omega_m(a)} \ln \left[ \frac{P_{1/6w}^{5/6w} \left( \Omega_m^{-1/2}(a) \right)}{P_{-1/6w}^{5/6w} \left( \Omega_m^{-1/2}(a) \right)} \right].
\]

(1.39)

In particular, Eq. (1.39) depends not only on $w$, but also on $\Omega_m$. For the fiducial values $w = -1$ and $\Omega_m = 0.25$, one finds $\gamma(a = 1) = 0.556$ instead of $\gamma(a \to 0) = 0.545$. For present day purposes, with galaxy surveys providing at most a few percent accuracy on the growth parameter, this difference — of order 3% — may seem academic. However, for future surveys like PAU [45], LSST [46] or Euclid [47], where we will have tomographic reconstruction of the past history in both the matter distribution and the expansion rate, up to redshift $z \simeq 2$, these differences may begin to play an important role as a discriminator between standard GR with a cosmological constant and, for example, modified gravity theories like $f(R)$, or quintessence models, which will be described in Chapter 2.

The same arguments apply when we want to evaluate the growth index in the case of the equation of state parameter is a linear function of the scale factor,

\[
\gamma(a) = \frac{1}{2} + \frac{1}{\ln \Omega_m(a)} \ln \left[ \frac{P_{1/6w(a)}^{5/6w(a) + w_a/6w^2(a)} \left( \Omega_m^{-1/2}(a) \right)}{P_{-1/6w(a)}^{5/6w(a) + w_a/6w^2(a)} \left( \Omega_m^{-1/2}(a) \right)} \right].
\]

(1.40)

As we shall see later on, the value of $\gamma$ can be very different for various dark energy and modified gravity theories, which provides us with a powerful tool to study departures from GR or clustering of dark energy. Additionally, using the growth index to characterise the linear growth of structure is a simple way to develop forecasts for future surveys and to compare theory with observations. However, there are a few shortcomings of this method:

- Since only one parameter is introduced (we have so far assumed no anisotropic stresses, in which case $\Phi = \Psi$), an extra parameter is needed to close the system and be able to characterise all possible modifications.
- The solution we have presented here is only valid on sub-Hubble scales and hence is not general enough to analyse the growth of structure on all scales.
- We have only considered matter perturbations, leaving all other species at the back-
ground level. This is fine as long as the onset of dark energy happens at late times when radiation is not important and if dark energy does not cluster, in which case one would also have to include dark energy perturbations when calculating the growth rate of matter.

In order to characterise how different models modify the perturbation equations, let us now look at the introduction of two new parameters.

1.5.3 Modifying the perturbation equations

As we have seen before, it is not enough to describe the growth of structure with only one parameter. We also need to include deviations from an isotropic fluid. In order to characterise deviations in the scalar perturbations from the simple scenario of a smooth dark energy component, two degrees of freedom are introduced in the perturbative constraint equations. This is done by modifying Eq. 1.26:

\[ k^2 \Phi = -4\pi G Q(a, k) a^2 \rho_m \Delta_m; \quad \Phi = \eta(a, k) \Psi. \]  

Modifications to the standard ΛCDM model are thus codified into \( Q \) and \( \eta \) (note that for ΛCDM or non clustering dark energy in an Einsteinian background \( Q = \eta = 1 \)). Function \( Q \) characterises modifications to gravity, like in modified gravity theories, in which this function can be understood as a modification to Newton’s constant, or modifications to the way matter clusters, be it because of a clustering dark energy component, or because of some different clustering, like in coupled dark energy models. On the other hand, the function \( \eta \) captures modifications to the anisotropic stress. Given a model, one can extract these two functions and thus compare the different theories.

The modified evolution equation for \( \Delta_m \) in the sub-Hubble limit reads [48]:

\[ \Delta_m'' + [2 + (\ln H)'] \Delta_m' = \frac{3Q}{2\eta} \Omega_m(a) \Delta_m; \quad \text{and} \quad \theta_m = -aH\Delta_m'. \]

As we can see in Eq. (1.42), the modifications to a standard ΛCDM model go into the \( Q/\eta \) factor in the right hand side.

It is worth noting that these two functions have been given different names and used in different combinations. For example, if one wants to study how modifications affect the
Newtonian potential $\Psi$, the usual parameter combination used is

$$\mu(a, k) \equiv \frac{Q(a, k)}{\eta(a, k)}, \quad -k^2 \Psi = 4\pi G a^2 \mu(a, k) \rho_m \Delta_m. \quad (1.43)$$

On the other hand, if one is interested in comparing weak lensing (WL) observations with theory, the quantity of interest is $(\Phi + \Psi)/2$ and the combination of parameters usually employed to relate this to the density field are

$$\Sigma(a, k) \equiv \frac{1}{2} Q(a, k) \left( 1 + \frac{1}{\eta(a, k)} \right) \quad -k^2 (\Phi + \Psi) = 8\pi G a^2 \Sigma(a, k) \rho_m \Delta_m. \quad (1.44)$$

To close the system, any two parameters out of $\{Q, \eta, \mu, \Sigma, \ldots\}$ can be chosen. The combination $(\mu, \Sigma)$ is widely used when one is interested in combining observations from WL and LSS. Of course, the aim is to measure the departure from $\Lambda$CDM codified in these two new functions. However, $Q$ and $\eta$ are not directly measurable and have to be inferred via their modifications on observable quantities. In order to do this, what is usually done is to introduce a parameterisation of $Q$ and $\eta$ which hopes to capture all the relevant physical behaviour using the least number of parameters. Useful parameterisations are motivated by specific predictions from different dark energy or modified gravity models $[38]$. Another alternative is to characterise these modification parameters as a function of redshift $z$ and scale $k$. To do this, one can take them constant in each bin or conduct a PCA of the binned modified growth functions.

Let us now look at the predictions from different theoretical dark energy and modified gravity models. This thesis is structured as follows: in Chapter 2 we will explore how different dark energy models solve the problem of late time cosmic acceleration and what their different predictions are, concerning observable quantities. In Chapter 3 we will discuss what the actual data measure and how to link these observables to theoretically predicted quantities in order to constrain model parameters. We will also see what the current data constraints are and what are the allowed values of the parameters of some dark energy theories. In Chapter 4 we will see what future surveys are available and how they will narrow down the error bars of current observations looking at different forecasts. Finally, Chapter 5 will be devoted to the conclusions.
“Curiosity demands that we ask questions, that we try to put things together and try to understand this multitude of aspects as perhaps resulting from the action of a relatively small number of elemental things and forces acting in an infinite variety of combinations.”

– Richard P. Feynman

2

Modelling acceleration

If we want to achieve a late time acceleration to match observations without using the cosmological constant that was discussed in Chapter 1, we have to consider three possible alternatives. The first two approaches consist of modifying Einstein’s equations Eq. (1.1), the right hand side or the left hand side respectively. The modifications of the right hand side, i.e. the energy-momentum tensor $T^{\mu\nu}$, involve achieving a negative pressure without a cosmological constant. The most representative models that fall within this category are quintessence [49], k-essence [50] and perfect fluid models [51]. In quintessence models, the agent responsible for late time acceleration is a scalar field slowly rolling down its near-flat potential whereas in k-essence it is the field’s kinetic term that causes the acceleration. In the case of perfect fluid models, it is a fluid with a specific equation of state, such as the Chaplygin gas model [51]. Constructing a viable scalar field theory of dark energy is not an easy task. In the context of inflation, where the energy scales are high, it is natural to propose a scalar field as the agent responsible for the quasi-exponential acceleration at early times. For dark energy, however, the energy scale is much lower and it would require a scalar field with mass $m_\phi \approx 10^{-33}$ eV rolling down a near-flat potential (or massless with an exponential potential) to accomplish the present cosmic acceleration. This extremely light field would mediate a “fifth force” with ordinary matter (if it couples non-gravitationally to matter); something that is not observed in local tests of gravity and needs to somehow be screened. In spite of the above-mentioned caveats for scalar field
theories of dark energy, it is still possible to construct viable models, as we will see further in Sec. 2.1.

The second approach is to modify the left hand side of Einstein’s equations. This implies changing the theory of gravity, especially on large scales. As mentioned in Chapter 1, Einstein’s GR has been tested to high accuracy on solar system scales. However, one could think that GR is not the ultimate theory of gravity and a modification to it on large scales could be responsible for dark energy. The main models that fall in the category of modified gravity theories are $f(R)$ gravity \cite{52}, scalar-tensor theories \cite{53} and braneworld scenarios \cite{54}. $f(R)$ theories change the Lagrangian density (which for a $\Lambda$CDM model is $f(R) = R - 2\Lambda$) with a non-linear function of the Ricci scalar $R$. In the case of scalar-tensor theories, acceleration is produced by the coupling of a scalar field $\phi$ to the Ricci scalar, in the form of $F(\phi)R$. Finally, in braneworld scenarios, gravity leaks from a 4 dimensional brane (where all the other matter fields “live”) into a 5th dimension, known as the bulk, on Hubble distances. We shall explore these type of models further in Sec. 2.2.

It is worth noting that these first two approaches are not fundamentally different, since a modification of matter can be rephrased as a gravity modification by finding an energy-momentum tensor equivalent to the Einstein tensor. However, this rephrasing is only possible at the classical level, since no viable theory of quantum gravity has been found yet.

The third approach consists on considering an inhomogeneous universe. The first evidence of late-time acceleration was the measurement of the dimming of light coming from distant supernovae. Through these observations it was inferred that distant objects recede from us slower than expected in an Einstein de Sitter spacetime, calibrated with nearby objects. This difference in expansion rates can be explained by an acceleration with a late onset in time or it could be that we lived in a inhomogeneous universe such that the light from distant sources would appear to be dimmed due to its inhomogeneous path across a large underdense region towards us through local open universe congruent geodesics. Since we only observe events that happen in our light cone where $dt^2 = 0$, time and distance are intrinsically related. Hence, it is always possible to interpret a homogeneous expansion rate $H(z)$ as an inhomogeneous one $H(r)$. It is also possible to arrange matter in such a fashion that in a local region of the Universe matter seems to be accelerating away from each other although, on large enough scales, the expansion is decelerated. The obvious caveat of such models is obtaining a large enough inhomogeneity preserving a high degree of isotropy of the CMB. The main representative models that fall within this category are large void models, such as the Lemaître-Tolman-Bondi model \cite{55–57} and
backreaction models \[58\]. We will see their details in Sec. 2.3. Let us now take a closer look at each class of models.

2.1 Dark Energy models

2.1.1 Quintessence

In Ref. [49], quintessence was born as a canonical scalar field \(\phi\) rolling down a potential \(V(\phi)\) responsible for the cosmic acceleration at late times. Scalar fields are often used in cosmology for many reasons, such as: their lack of internal degrees of freedom, they do not have a preferred direction and are usually weakly clustered. The main difference with the cosmological constant is that the equation of state of quintessence is dynamical, i.e. varies with time. As we shall see, in the case of quintessence, it is not necessary to have a large energy density of the field with respect to radiation or matter and in some cases, the field exhibits a “tracker” behaviour which means that it corresponds to an attractor solution in which the field energy density tracks the background fluid density regardless of the initial conditions.

If we consider a field with kinetic energy in the canonical form, the only degrees of freedom left are the potential and the initial conditions. The quintessence models is therefore described by the action and Lagrangian:

\[
S = \int d^4x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + \mathcal{L}_\phi \right] + S_M, \quad \mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi),
\]

where \(\kappa^2 = 8\pi G\), \(R\) is the Ricci scalar and \(S_M\) is the action of matter fields. We assume that it is a perfect fluid which satisfies the energy conservation Eq. (1.4). Using the action in Eq. (2.1), one can derive the energy momentum tensor for quintessence as:

\[
T^{(\phi)}_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L}_\phi)}{\delta g^{\mu\nu}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left[ \frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi + V(\phi) \right].
\]

From this energy momentum tensor and assuming a FLRW background, we can extract
the energy density and pressure corresponding to a homogeneous quintessence field:

$$\rho_\varphi = - T_0^{(\varphi)} = \frac{1}{2} \dot{\varphi}^2 + V(\varphi), \quad p_\varphi = \frac{1}{3} T_i^{(\varphi)} = \frac{1}{2} \dot{\varphi}^2 - V(\varphi),$$  \hspace{1cm} (2.3)

which gives the equation of state:

$$w_\varphi = \frac{p_\varphi}{\rho_\varphi} = \frac{\dot{\varphi}^2 - 2V(\varphi)}{\dot{\varphi}^2 + 2V(\varphi)}. \hspace{1cm} (2.4)$$

Substituting Eq. (2.2) into Einstein’s equation for a flat universe, one gets the following equations of motion:

$$H^2 = \frac{\kappa^2}{3} \left[ \frac{1}{2} \dot{\varphi}^2 + V(\varphi) + \rho_m \right],$$

$$\dot{H} = - \frac{\kappa^2}{2} \left[ \dot{\varphi}^2 + \rho_M + p_M \right], \hspace{1cm} (2.5)$$

while the evolution equation for the field itself can be obtained from varying the action in Eq. (2.1) with respect to the field itself, $\varphi$:

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{dV}{d\varphi} = 0. \hspace{1cm} (2.6)$$

The physics is straightforward: the field $\varphi$ will seek to roll towards the minimum of its potential $V$, according to Eq. (2.6). The rate of evolution is driven by the slope of the potential and damped by the cosmic expansion through the Hubble parameter $H$.

Since we do not measure an energy density corresponding to the field during matter and radiation domination, $\rho_M \gg \rho_\varphi$ during these epochs. As we have stated before, we would like $\rho_\varphi$ to track $\rho_M$ so that acceleration arises at late times and we can solve the coincidence problem. Whether we have such a tracking behaviour depends on the shape of the potential. If the potential is steep, i.e. $\dot{\varphi}^2 \gg V(\varphi)$, at all times, then the equation of state is $w_\varphi \approx 1$ and the energy density of the field evolves as $\rho_\varphi \propto a^{-6}$. This means that the energy density of quintessence decreases much faster than any other background fluid. Since we require $w_\varphi < -1/3$ to realise acceleration at late times, a steep potential is not what we are interested in.

Imposing $w_\varphi < -1/3$, one obtains from Eq. (2.4) the condition $\dot{\varphi}^2 < V(\varphi)$. This implies that the field is slow-rolling down a shallow potential. This is very similar to the
requirements to achieve inflation in early times and the equation of state can be expressed as a function of the slow-roll parameters defined in [59]:

\[ \varepsilon \equiv \frac{1}{2\kappa^2} \left( \frac{V_{,\phi}}{V} \right)^2, \quad \eta \equiv \frac{V_{,\phi\phi}}{\kappa^2 V}. \] (2.7)

The equation of state in terms of the field evolution is given by

\[ 1 + w_{,\phi} = \frac{V_{,\phi}^2}{9H^2(\xi + 1)^2\rho_{\phi}}, \] (2.8)

where \( \xi = \dot{\phi}/(3H\dot{\phi}) \). This means that \( w_{,\phi} \) is always larger than \(-1\) as long as the potential is positive. In the slow-roll limit, and assuming \( \rho_{\phi} \gg \rho_M, 1 + w_{,\phi} \approx 2\varepsilon / 3 \), which means that the deviation of \( w_{,\phi} \) from \(-1\) is determined by the slow-roll parameter \( \varepsilon \).

The evolution of perturbations does not require to know the full evolution of the quintessence fields explicitly since one can rewrite the perturbation equations of the field in terms of the perturbations of the density contrast \( \delta_{\phi} \) and velocity \( \theta_{\phi} \) to find that they correspond to those of a fluid with \( \pi = 0 \) and

\[ \delta p = c_s^2 \delta \rho + 3aH(c_s^2 - c_a^2)(1 + w)\rho \theta / k^2, \] (2.9)

where \( c_a^2 \) is the adiabatic sound speed [60]. In [60], they prove that this models have \( c_s^2 = 1 \). This large value of the sound speed means that quintessence models do not cluster and hence it is not necessary to consider their perturbations inside the horizon.

As regards the potential, there are many proposed forms in the literature. These can be classified mainly into two groups: the “freezing” models and the “thawing” models.

- **Freezing models:** In these models, the field which was already rolling towards its potential minimum, prior to the onset of acceleration, but which slows down and creeps to a halt as it comes to dominate the Universe. For these “freezing” models, initially \( w > -1 \) and \( w' < 0 \). These are essentially tracking models, but may be described more generally as vacuumless fields (in the sense that the minimum is attained as \( \phi \to \infty \)) or runaway potentials characterised by a potential with curvature that slows the field evolution as it rolls down towards the minimum. It follows that there is some value of the field beyond which the evolution is critically damped by the cosmic expansion, whence the field is frozen and \( w \to -1, \ w' \to 0 \). The potentials that belong to this class are:
- $V(\phi) = M^{4+n} \phi^{-n} \quad n > 0,$
- $V(\phi) = M^{4+n} \phi^{-n} \exp(a \phi^2 / m_{pl}^2).$

The first of these potentials does not have a minimum and the field simply rolls down eternally. The second has a minimum which corresponds to $w_\phi = -1$ at which the potential becomes eventually trapped.

- **Thawing models:** In this class of models the field has a mass $m_\phi$ lower than $H$ and has been frozen by Hubble damping $3H\dot{\phi}$ in Eq. (2.6) at a value displaced from its minimum until recently, when $H$ drops below $m_\phi$ and the field starts to roll down to the minimum. The equation of state evolves from $w_\phi \simeq -1$ at early times and then starts growing. The potentials that belong to this class are:

  - $V(\phi) = V_0 + M^{4-n} \phi^n \quad n > 0,$
  - $V(\phi) = M^4 \cos^2(\phi/f), \text{ where } f \text{ is a symmetry restoration energy scale.}$

The first of the potentials is similar to the one in chaotic inflation with $n = 2, 4,$ however the mass $M$ is different. The second potential is based on the potential of the Pseudo-Nambu-Goldstone Boson (PNGB) and was introduced in [61] as an alternative to the cosmological constant. In this case, the field is frozen at the potential maximum when $m_\phi < H$ and begins to roll down around the present, when $m_\phi \simeq H_0.$ We can see some of the models and their bounds in Fig. 2.1.1.

### 2.1.1.1 A word on early dark energy

One of the main differences between $\Lambda$CDM and quintessence models is that the energy density of the quintessence field $\rho_\phi$ is significant also during radiation and matter domination. It can be shown that for exponential potentials, there exists a scaling solution for $\Omega_\phi$ during radiation and matter domination. As one might imagine, there are observational constraints on such early dark energy models coming from early universe measurements. The ones that impose the tighter constraints are those from Big Bang Nucleosynthesis (BBN), which provide an upper bound to $\Omega_\phi$ from observations at a temperature $T \simeq 1$ MeV. The early presence of the quintessence field leads to an extra expansion of the Universe. This, on the other hand, leads to a change in the ratio of neutrons to protons at
freeze-out, and hence in the abundance of light elements. Bearing this in mind, the constraint on $\Omega^{\text{BBN}}_{\phi}$ (the energy density parameter of quintessence at the BBN time) is \cite{63}:

$$\Omega^{\text{BBN}}_{\phi} < 0.045. \quad (2.10)$$

The presence of this scaling solution also modifies the Cosmic Microwave Background power spectrum, which further constrains the energy density parameter of the quintessence field at BBN epoch \cite{63}: $\Omega^{\text{BBN}}_{\phi} < 0.39$, weaker than the constraint in Eq. (2.10). With these bounds, the quintessence potential can be constrained. If we naively assume an exponential potential of the form $V(\phi) = A e^{-\lambda \phi}$, we find $\lambda > 9.4$, assuming the bound from Eq. (2.10).
2.1.2 K-ESSENCE

So far, we have looked at quintessence, a scalar field model of dark energy that achieves late-time cosmic acceleration through its potential. Let us now explore the possibility of realising such an acceleration through a non-canonical kinetic term. The action of a field with a non-canonical kinetic term is usually given by

\[ S = \int d^4 x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + P(\varphi, X) \right] + S_M, \]  

(2.11)

where \( P(\varphi, X) \) is a function in terms of the scalar field \( \varphi \) and its canonical kinetic energy: \( X = -(1/2)g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \). The application of such an action to explain dark energy was first introduced in \([50]\). This was later extended and dubbed “k-essence” in \([64]\). Such models can have not only dynamical equations of state, but also clustering properties significantly different from quintessence. Clustering dark energy would contribute to density perturbation growth on scales larger than its sound horizon, leading to observable effects in large scale CMB and its correlation with large scale structure and weak lensing surveys.

There are four main models that fall under the category of k-essence:

- **Low-energy effective string theory.**
  In string theory, when one considers the low-energy effective theory, higher order derivative terms coming from the non-perturbative corrections, encoded in \( \alpha \) and in the coupling functions \( B_i(\varphi) \), and loop corrections to the tree-level action appear \([65, 66]\). The action in this case is

\[ S = \frac{1}{2\kappa^2} \int d^4 x \sqrt{-\tilde{g}} \left[ B_\varphi(\varphi) \tilde{R} + B_\varphi^0(\varphi)(\tilde{\nabla} \varphi)^2 + ac_1 B_\varphi^{(1)}(\varphi)(\tilde{\nabla} \varphi)^4 + O(a^2) \right], \]

(2.12)

where \( V(\varphi) = S_M = 0 \). Performing a conformal transformation of the metric, we can express the action in Eq. (2.12) in the Einstein frame \( g_{\mu\nu} = B_\varphi \tilde{g}_{\mu\nu} \):

\[ S_E = \int d^4 x \sqrt{-g} \left[ \frac{1}{2\kappa^2} R + K(\varphi)X + L(\varphi)X^2 + \ldots \right], \]

(2.13)

where

\[ K(\varphi) = 3 \left( \frac{1}{B_\varphi} \frac{dB_\varphi}{d\varphi} \right)^2 - 2 \frac{B_\varphi^0}{B_\varphi}, \quad L(\varphi) = 2c_1 \frac{\alpha}{\kappa^2} B_\varphi^{(1)}(\varphi). \]
It is easy to see that the action Eq. (2.13) belongs to the k-essence models with function $P$ given by

$$P = K(\phi)X + L(\phi)X^2.$$  \hfill (2.14)

**Ghost condensate model.**

As we shall see later in Chapter 3, current observations allow for an equation of state smaller than $-1$. This kind of equation of state can be explained by a scalar field $\phi$ with a negative kinetic energy term $-X$ and a potential $V(\phi)$, or by two scalar fields [67]. A field with this property is normally called a “ghost” or “phantom”. Such a scalar field model could in theory generate acceleration by the field evolving up the potential toward the maximum. Phantom fields are plagued by catastrophic UV instabilities, as particle excitations have a negative mass and their energy density is not bounded from below. This can be fixed by adding an extra $X^2$ term which stabilises the vacuum against the production of such ghosts. This is the idea behind the ghost condensate model [68] with Lagrangian density:

$$P = -X + X^2/M^4,$$  \hfill (2.15)

where $M$ is a constant with mass dimensions. This can be seen as a particular case of Eq. (2.14) with $K = -1$ and $L = 1/M^4$. There is another version of ghost condensates, dubbed the dilatonic ghost condensate model [69], with

$$P = -X + e^{\lambda\phi}X^2/M^4.$$  \hfill (2.16)

Such an exponential correction can appear in dilatonic theories with higher-order corrections.

**Tachyon field.**

There are some string theories in which the existence of unstable branes produce a single tachyonic mode with negative mass. The effective 4-dimensional Lagrangian density of this tachyon is given by [70]

$$P = -V(\phi) \sqrt{-\det(g_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi)}.$$  \hfill (2.17)

In open string theory, the field rolls down the potential starting at $\phi = 0$ and evolves towards its ground state at $|\phi| \rightarrow \infty$. The typical potential of such theories is $V(\phi) = V_0/\cosh(\beta\phi/2)$. However, this potential is too steep to power
late-time acceleration. Only tachyon potentials shallower than $\varphi^{-2}$ are flat enough to account for dark energy. Another possibility is based on the potential of a massive scalar field with mass $m$ on the anti D3-brane: $V(\varphi) = V_0 e^{\frac{1}{2} m^2 \varphi^2}$ [71]. In this case, when $\varphi = 0$, $V(\varphi) = V_0$, and late-time acceleration is possible.

- **Dirac-Born-Infeld (DBI) theories.**
  This model is based on the Anti de Sitter/Conformal Field Theory (AdS-CFT) correspondence whereby the field parametrises a direction on a Coulomb branch of the system in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [72]. The speed of the field is restricted by the causality in the gravity side of the AdS-CFT correspondence. The Lagrangian density of this theory is given by [72]

$$P = -f(\varphi)^{-1} \sqrt{1 - 2f(\varphi)X + f(\varphi)^{-1} - V(\varphi)}, \quad (2.18)$$

where $F(\varphi)$ is a warp factor. To realise cosmic acceleration, one needs $f(\varphi) \varphi^2 \simeq 1$. The application of these theories in the framework of dark energy has been studied in [73].

### 2.1.2.1 The equation of state for k-essence

The general energy-momentum tensor for a k-essence type field with action Eq. (2.11) is given by

$$T_{\mu\nu}^{(\varphi)} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}P)}{\delta g^{\mu\nu}} = P_{,\chi} \partial_{\mu} \varphi \partial_{\nu} \varphi + g_{\mu\nu}P. \quad (2.19)$$

The energy momentum tensor of k-essence as a perfect fluid, then we can relate the energy density and pressure of the field to that of the fluid by:

$$p_\varphi = P, \quad \rho_\varphi = 2XP_{,\chi} - P. \quad (2.20)$$

Then, the equation of state becomes:

$$w_\varphi = \frac{P}{2XP_{,\chi} - P}. \quad (2.21)$$
From Eq. (2.21) we see that if \(|2XP, X| \ll |P|\), then \(w_\phi \simeq -1\). The Friedmann and continuity equations in a flat FLRW background including a matter fluid are

\[
\begin{align*}
3H^2 &= \kappa^2 (\rho_\phi + \rho_m) \\
2\dot{H} &= -\kappa^2 (2XP, X + \rho_m + p_m) \\
\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) &= 0.
\end{align*}
\] (2.22)

Let us now look at what the different models of k-essence predict as an equation of state. In the case of the ghost condensate model, we have

\[ w_\phi = \frac{1 - X/M^4}{1 - 3X/M^4}; \] (2.23)

which gives values between \((-1, -1/3)\) for \(1/2 < X/M^4 < 2/3\). The solution for \(w_\phi = -1\) is \(X/M^4 = 1/2\). The field energy density for this value of the equation of state is \(\rho_\phi = M^4\), so, to achieve present time acceleration, we would need \(M \sim 10^{-3} \text{ eV}\) (see Eq. (1.6)).

In the case of tachyonic k-essence, the equation of state is given by

\[ w_\phi = -1 + \dot{\phi}^2, \] (2.24)

so we see that to achieve \(w_\phi \simeq -1\) we need \(\dot{\phi}^2 \ll 1\). This means that the kinetic energy is suppressed and is not the cause for the late time acceleration. Even though the action of the tachyon field belongs in theory to the general k-essence actions, its kinetic term is not responsible for acceleration.

Finally, for DBI models, whose Lagrangian is given by Eq. (2.17), the equation of state is given by

\[ w_\phi = \frac{1 - \gamma^{-1} - f(\phi)V(\phi)}{\gamma - 1 + f(\phi)V(\phi)}, \] (2.25)

where \(\gamma = 1/\sqrt{1 - f(\phi)\dot{\phi}^2}\). In the slow roll limit, \(f(\phi)\dot{\phi}^2 \ll 1\), \(w_\phi \rightarrow -1\). It can also be shown that in the ultra-relativistic region where \(\gamma \rightarrow \infty\), there is a fixed point which can cause acceleration \([74]\), even if the field is not in the slow roll regime.
Since the measured energy density for dark energy is similar to that of matter, it is reasonable to suppose that they are somehow related. If they were indeed coupled, that would help solve the coincidence problem and explain why dark energy started to dominate at late times. The way to achieve this coupling comes naturally when considering a scalar field as the source for acceleration. The presence of the scalar field alters the gravitational attraction of matter, giving rise to a fifth force. This makes this type of models distinguishable from others in which this extra force is not present. In order to evade solar system tests in which this fifth force has not been detected, these couplings have to be suppressed by a screening mechanism. We shall discuss some of these in more detail in Sec. 2.2. Since these models introduce this extra force, they can be also classified as modifications of Einstein gravity. However, the difference is that here their anisotropic stress is still zero in the Einstein frame while, in the case of modified gravity theories, this is generally not the case. In the case of scalar-tensor theories, they can be viewed as a modification of gravity in the Jordan frame while a conformal transformation makes them become a coupled dark energy model in the Einstein frame. In such models, if the field couples to all forms of matter, the interaction can be recast as a non-minimal coupling to gravity by a redefinition of the metric and matter fields. Another interesting motivation for such interacting models is the possibility that the non-linear evolution of coupled dark energy could help solve problems such as the excess in the number of observed massive clusters by introducing new features [75, 76].

Let us now look at the mathematical framework of such interacting models. At the level of the Lagrangian, this coupling is usually achieved by letting the mass of the matter fields $m$ depend on the field $i.e.$ $m(\phi)$ whose choice specifies the interaction. The action is then [77]

$$S = \int d^4x \sqrt{-g} \left[ -\frac{1}{2} \partial^\mu \phi \partial_\mu \phi - V(\phi) - m(\phi) \bar{\psi} \psi + \mathcal{L}_{\text{kin}}[\psi] \right], \quad (2.26)$$

where $\psi$ represents the matter fields. Here, the energy-momentum for each species is not conserved, although the total one still is. This means that

$$T^\mu_{(a)\nu;\mu} = Q_{(a)\nu}, \quad (2.27)$$

where the index $a$ runs through all of the matter species and $Q_{(a)\nu}$ encodes the coupling of each species to the scalar field $\phi$. The conservation of the total energy-momentum tensor
implies that
\[ \sum_a Q_{(a)_\nu} = 0. \tag{2.28} \]

If we set \( \nu = 0 \) in Eq. (2.27), we get the background conservation equations [78, 79]:
\begin{align*}
\frac{d\rho_\phi}{d\eta} &= -3\mathcal{H}(1 + w_\phi)\rho_\phi + \beta(\phi)\frac{d\phi}{d\eta}(1 - 3w_a)\rho_a, \\
\frac{d\rho_a}{d\eta} &= -3\mathcal{H}(1 + w_a)\rho_a - \beta(\phi)\frac{d\phi}{d\eta}(1 - 3w_a)\rho_a, \tag{2.29}
\end{align*}

where \( \mathcal{H} \) is the Hubble parameter in conformal time i.e. \( \mathcal{H} = aH \). Specifying the dependence \( m(\phi) \) is equivalent to a choice of \( \beta(\phi) \) or the source term \( Q_{(a)_\nu} \) and they are related by [65]
\[ Q_{(\varphi)_\nu} = \frac{\partial \ln m(\varphi)}{\partial \varphi} T_a \partial_\nu \varphi, \quad m_a = \bar{m}_a e^{-\beta(\varphi)\varphi}. \tag{2.31} \]

As regards perturbations, the modified Euler equation and the acceleration equation of particles at position \( \mathbf{r} \) read
\begin{align*}
\frac{d\mathbf{v}_a}{d\eta} + \left( \mathcal{H} - \beta(\varphi)\frac{d\phi}{d\eta} \right) \mathbf{v}_a - \nabla [\Phi_a + \beta(\varphi)\varphi] &= 0, \\
\ddot{\mathbf{v}}_a &= -\ddot{H}\mathbf{v}_a - \nabla \tilde{G}_a m_a, \tag{2.32}
\end{align*}

where \( \tilde{G}_a = G_N[1 + 2\beta^2(\varphi)] \) and \( \ddot{H} = H(1 - \beta(\varphi)\dot{\varphi}/H) \). In Eq. (2.33) we can see interesting characteristics of interacting dark energy models. Firstly, we see the term that causes the fifth force \( \nabla [\Phi_a + \beta(\varphi)] \). This fifth force has an effective gravitational Newton’s constant which depends on the coupling: \( \tilde{G}_a \). The acceleration equation Eq. (2.33) also has a distinct velocity dependent term \( \ddot{H}\mathbf{v}_a \). They also exhibit a time dependent mass that evolves according to Eq. (2.31). These features may have an impact on observations such as structure formation that can make them distinguishable from other dark energy models. Let us now look at the different species the scalar field \( \varphi \) can couple to.

### 2.1.3.1 Dark Energy Coupled to Baryons

In this case, we consider the scalar field \( \varphi \) only coupled to baryons, i.e. \( a = b \). This coupling is constrained by the equivalence principle and solar system tests [80] and by the variation of fundamental constants over cosmological scales [81]. According to GR, pho-
tons are deflected and delayed by mass curving spacetime by a factor $\propto 1 + \gamma$, where $\gamma$ here is the PPN parameter, not to be mistaken with the growth index introduced in previous chapters. In Newtonian gravity, $\gamma = 0$, while in GR $\gamma = 1$. If $\gamma$ deviates from 1, such effects could be due to gravity not being a purely geometric effect but being affected by the coupling to other fields. In [80], measurements of the frequency shift of radio photons to and from the Cassini spacecraft as they pass near the Sun, constrain $\gamma$ to be $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$. It is presumably very difficult to have significant cosmological effects due to a coupling of the field to baryons only.

2.1.3.2 Dark energy coupled to dark matter

In this case the field $\varphi$ couples only to dark matter: $a = c$. This model is usually known as “coupled quintessence”. Since we cannot observe dark matter, the coupling is not affected by equivalence principle or solar system tests, hence the constraints are weaker than in the case of coupling to baryons. Allowed values of $\beta$ by observations are $0 < \beta < 0.06$ at two sigma confidence level for a constant coupling and an exponential potential [82]. Various choices of the coupling $\beta(\varphi)$ both constant and time dependent have been studied extensively in the literature (see [79, 83] amongst others). As we have seen in Sec. 2.1.3, there are 3 main effects that a coupling induces in the perturbation equations. The fifth force and the coupling dependent velocity term have an effect on structure formation. Depending on the form of the coupling $\beta(\varphi)$, these two effects can partially balance: while the fifth force increases the gravitational attraction, the velocity term tries to dilute the concentration of virialised dark matter halos. In [76], the interplay of these effects was studied for a constant and time-dependent couplings leading to the result that, in the case of constant couplings, the dilution of virial halos is only caused by the velocity term. However, for varying couplings, the time evolution of the gravitational constant that mediates the fifth force can also affect the virialisation of halos by increasing (for increasing couplings) or decreasing (for decreasing couplings) their concentration.

The main distinguishable features that these models predict are

- enhanced integrated Sachs-Wolfe (ISW) effect [79],
- increase in massive clusters at high redshift [84],
- scale-dependent bias arising from the coupling only affecting dark matter and not baryons [75],
- shallower inner core halo profiles [75],

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• lower concentration of halos [76],

• emptier voids [85].

2.1.3.3 Dark energy coupled to neutrinos

Since the energy density of neutrinos is so small, a coupling between dark energy and neutrinos can be even stronger than the one with dark matter. The coupling will play a role once the neutrinos have decoupled and become non-relativistic. Typical values of the coupling $\beta$ are of the order 50-100 or even higher [86]. This means that even if neutrinos account for a small fraction of the total energy density, they have a substantial influence on the evolution of the scalar field $\phi$. When the coupling is in play, the fifth force can be up to $10^2 - 10^4$ orders stronger than gravity. In growing neutrino cosmologies, the function $m_\nu(\phi)$ is such that the neutrino mass grows with time from low, nearly massless values (when neutrinos are non-relativistic) up to present masses in a range in agreement with current observations. The key feature of growing neutrino models is that the amount of dark energy today is triggered by a cosmological event, corresponding to the transition from relativistic to non-relativistic neutrinos at redshift $z_{\text{NR}} \sim 5 - 10$ (for neutrino masses of the order $M_\nu \sim 10^{-3}$ eV). As long as neutrinos are relativistic, the coupling plays no role on the dynamics of the scalar field. From there on, the evolution of dark energy resembles that of a cosmological constant, plus small oscillations of the coupled dark energy-neutrino fluid. As a consequence, when a coupling between dark energy and neutrinos is active, the amount of dark energy and its equation of state today are strictly connected to the present value of the neutrino mass. The neutrino contribution to the potential affects the growth of dark matter and the CMB (via the ISW and the non-linear Rees-Sciama effect). Backreaction, as we will see later in Sec. 2.3.7 can modify the growth of neutrino lumps much more than in the dark matter case.

Some of the observable features that these couplings predict are listed below:

• existence of very large structures, order $10 - 500$ Mpc [86, 87],

• enhanced ISW effect, drastically reduced when taking into account non-linearities: information on the gravitational potential is a good mean to constrain the range of allowed values for the coupling $\beta$ [87],

• large-scale anisotropies and enhanced peculiar velocities [88],
the influence on the gravitational potential induced by the neutrino inhomogeneities can affect the baryon acoustic oscillation (BAO) scale in the dark-matter correlation function and power spectrum [89].

2.1.3.4 Scalar-tensor theories

Finally, there exists the possibility that dark energy couples to all matter species. This models are known as “scalar-tensor theories” or “extended quintessence”. They are based on the theory proposed by Brans and Dicke in [90], where the scalar field $\phi$ exhibits a non-minimal coupling to gravity. The Lagrangian for their model in the Jordan frame is

$$L_{BD} = \sqrt{-g} \left[ \frac{1}{\phi} R - \omega g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi \right] + L_{\text{matter}}(\psi), \quad (2.34)$$

where $\omega$ is a dimensionless constant. We notice that the first term of the right hand side of Eq. (2.34) is the non-minimal coupling to gravity. The effective gravitational constant in this theory is related to the scalar field by $1/(16\pi G_{\text{eff}}) = \phi$, as long as $\phi$ varies sufficiently slowly. The second term on the right hand side of Eq. (2.34) represents the field’s kinetic term, which, under redefinition of variables, can be brought back into its canonical form. Since in this case $\phi$ couples to all species, the allowed strength of the interaction is very weak [91]. It is possible to tune the coupling to the required small values by choosing a sufficiently flat potential, however, this leads back to fine tuning and naturalness problems. Another possibility is to screen the coupling so that they agree with observations, thus evading fine tuning. This is done via screening mechanisms which we shall discuss further in Sec. 2.2. The main observable effects of scalar-tensor theories are:

- enhanced ISW effect [91],

- violation of the strong equivalence principle: objects like galaxies or dark matter halos fall at different rates [92].

2.2 Modified gravity models

As we have seen in previous sections, a scalar field can be coupled to gravity in the Jordan frame while it can be coupled to matter in the Einstein frame. So, in a sense, dark energy models can be viewed as modifications of gravity too. Currently, there’s no consensus
on where to draw the line between the two classes. Let us name modified gravity models those that from the start change Einstein’s equations. We have already seen some of them in previous sections, like scalar-tensor theories, in which the Einstein-Hilbert action is explicitly modified by a non-minimal coupling to gravity (the $\varphi R$ term in Eq. (2.34)). Other models that fall under this category are $f(R)$ type theories, braneworld cosmologies and massive gravity among others. General properties of this class of models is that they usually change the clustering of structure and induce a non-zero anisotropic stress. This means that if we look back at Eq. (1.41), these models in general will have functions $Q$ and $\eta$ different from 1.

2.2.1 $f(R)$ Gravity

The simplest way to modify general relativity is by substituting the Ricci scalar $R$ with a function $f(R)$, in the Einstein-Hilbert action, [93]:

$$S = \frac{1}{2\kappa^2} \int d^4x \sqrt{-g} f(R) + S_m(g_{\mu\nu}, \Psi_m), \quad (2.35)$$

where $S_m$ is the matter action with matter fields $\Psi_m$. There are 2 approaches to calculate the field equations from the action in Eq. (2.35):

- **The metric formalism**

  In this formalism, the field equations are obtained by varying the action Eq. (2.35) with respect to the metric $g_{\mu\nu}$ and assuming that the metric tensor and the Chirstoffel symbols are not independent. This yields

  $$F(R)R_{\mu\nu}(g) - \frac{1}{2} f(R) g_{\mu\nu} - \nabla_\mu \nabla_\nu f(R) + g_{\mu\nu} \Box F(R) = \kappa^2 T_{\mu\nu}, \quad (2.36)$$

  where $F(R) \equiv \partial f/\partial R$. The trace equation reads

  $$3 \Box F(R) + F(R) R - 2f(R) = \kappa^2 (-\rho + 3p). \quad (2.37)$$

- **The Palatini formalism**

  In this approach, the metric tensor $g_{\mu\nu}$ is treated independently of the Christoffel symbols $\Gamma^a_{\mu\nu}$. In this case, varying action Eq. (2.35) with respect to the metric gives

  $$F(R)R_{\mu\nu}(\Gamma) - \frac{1}{2} f(R) g_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad (2.38)$$
where $R_{\mu\nu}(\Gamma)$ is the Ricci tensor corresponding to the affine connections $\Gamma^a_{\mu\nu}$. In general, this is different to $R_{\mu\nu}(g)$. The trace equation is

$$F(R)R - 2f(R) = \kappa^2 (-\rho + 3p). \quad (2.39)$$

Varying the action Eq. (2.35) with respect to the affine connection and using Eq. (2.38) we obtain

$$R_{\mu\nu}(g) - \frac{1}{2} g_{\mu\nu} R(g) = \frac{\kappa^2 T_{\mu\nu}}{F} - \frac{FR(T) - f}{2F} g_{\mu\nu} + \frac{1}{F} (\nabla_\mu \nabla_\nu F - g_{\mu\nu} \Box F)$$

$$- \frac{3}{2F^2} \left[ \partial_\mu F \partial_\nu F - \frac{1}{2} g_{\mu\nu} (\nabla F)^2 \right]. \quad (2.40)$$

In GR with a cosmological constant we have $f(R) = R - 2\Lambda$, which means that $F(R) = 1$ and $\Box F(R) = 0$. In this case, both the metric and Palatini formalisms give $R + 2\Lambda = \kappa^2 (\rho - 3p)$ which means that $R$ is fully determined by the matter content of the Universe. In modified gravity theories, however, $F(R)$ will be a function of $R$ and $\Box F(R)$ will, in general, not vanish. We can see this $F(R)$ as a scalar degree of freedom commonly dubbed "scalaron" [93]. In the Palatini formalism, the kinetic term $\Box F(R)$ is not present, meaning that the scalaron does not propagate freely. As a consequence, the dynamics of $f(R)$ is different depending on the formalism one chooses.

Let us consider the dynamics of $f(R)$ gravity in the metric formalism. In standard flat FLRW spacetime, the Ricci scalar is given by

$$R = 6(2H^2 + \dot{H}). \quad (2.41)$$

The corresponding Friedmann equations for a perfect non-relativistic matter fluid for this action are given by

$$3FH^2 = \rho_m + \frac{FR - f(R)}{2} - 3H\ddot{F}, \quad (2.42)$$

$$-2F\dot{H} = \rho_m + \ddot{F} - H\dot{F}.$$ 

It is convenient to introduce a set of dimensionless variables to study the dynamics of $f(R)$ gravity [94]:

$$x_1 = -\frac{\dot{F}}{HF}, \quad x_2 = -\frac{f(R)}{6FH^2}, \quad x_3 = \frac{R}{6H^2}, \quad x_4 = aH. \quad (2.43)$$
The matter density parameter is then given by

$$\tilde{\Omega}_m \equiv \frac{\rho_m}{3FH^2} = 1 - x_1 - x_2 - x_3$$  \hspace{1cm} (2.44)

With these variables we obtain the following closed system of differential equations

$$x_1' = -1 - x_3 - 3x_2 + x_1^2 - x_1x_3 ,$$
$$x_2' = \frac{x_1x_3}{m(r)} - x_2(2x_3 - 4 - x_1) ,$$
$$x_3' = -\frac{x_1x_3}{m(r)} - 2x_3(x_3 - 2) ,$$
$$x_4' = (x_3 - 1)x_4 .$$  \hspace{1cm} (2.45)

where a prime represents a derivative with respect to $N = \ln a$ and

$$m(r) \equiv \frac{RF_{,RR}}{f_{,R}} , \quad r \equiv -\frac{RF_{,R}}{f} = \frac{x_3}{x_2} .$$  \hspace{1cm} (2.46)

The linear perturbation equation for the matter density contrast in the comoving gauge (where the matter velocity vanishes), in terms of the dimensionless variables, $x_1$, $x_2$ and $x_3$, is [95]

$$\delta_m'' + (x_1 + x_3)\delta_m' - 3(1 - x_1 - x_2 - x_3)\delta_m = \left[ 3 \left( x_1 + x_3 - \frac{x_3}{m} - 1 \right) - \frac{k^2}{x_4^2} \right] \tilde{\delta}F - 3\tilde{\delta}F' ,$$  \hspace{1cm} (2.47)

where the differential equation for $\tilde{\delta}F = \delta F/F$ is

$$\tilde{\delta}F'' + (2x_1 - x_3 - 1)\tilde{\delta}F' + \left[ \frac{k^2}{x_4^2} - x_3 + \frac{2x_3}{m} + 3x_2 - x_1 + 1 \right] \tilde{\delta}F = 0 .$$  \hspace{1cm} (2.48)

Solving the coupled differential equations (2.45), we can obtain the background functions

$$H(a) = \frac{x_4(a)}{a} ,$$
$$\Omega_m(a) = F(a) \left( 1 - x_1(a) - x_2(a) - x_3(a) \right) ,$$
$$w(a) = \frac{1 - 2x_3(a)}{3(1 - \Omega_m(a))} .$$  \hspace{1cm} (2.49)
Then, solving equations (2.47) and (2.48) numerically for a given scale $k$, we can find a solution for $\delta_m$, and compute from it the growth index $\gamma$, which we studied in [1]. It is also possible to find simplified analytical expressions for the modified growth parameters mentioned in Sec. 1.5.2 on sub-Hubble scales

$$Q = 1 - \frac{k^2}{3(a^2M^2 + k^2)},$$

$$\eta = 1 - \frac{2k^2}{3a^2M^2 + 4k^2}, \quad (2.50)$$

where

$$M^2 = \frac{1}{3f_{RR}}. \quad (2.51)$$

In [1] we studied the Starobinsky model [96] as an example of an $f(R)$ model in which

$$f(R) = R + \lambda R_0 \left[ \left( 1 + \frac{R^2}{R_0^2} \right)^{-n} - 1 \right] \quad (2.52)$$

where $\lambda$ and $n$ are two positive constants and $R_0$ corresponds to the present value of the Ricci scalar. To be in agreement with observations, we take $n = 2$ and $\lambda = 2$ [96]. We also take $k = 0.16 \, h/\text{Mpc}$ in Eq. (2.47) to calculate the density contrast and the growth index. The results can be seen in Sec. 2.4.

2.2.2 The Dvali-Gabadadze-Porrati model

The brane-world scenario offers an alternative to explain the current acceleration of the Universe without the need of invoking a new scalar degree of freedom. The new degrees of freedom in this kind of scenarios belong to the gravitational sector instead. As an example of a higher-dimensional theory, let us look more closely at the Dvali-Gabadadze-Porrati (DGP) model [54]. In the DGP model, gravity lives in a $3 + 1$ dimensional brane with the rest of matter fields for scales $\lambda < r_c$ while it seeps into a fifth dimensional bulk for $\lambda > r_c$, effectively weakening its strength in our brane. This cross-over distance is given by the ratio of the Planck masses in 4 and 5 dimensions $r_c = M_{\text{pl}4}/M_{\text{pl}5}$. For such models, the Hubble parameter is given by [97]

$$H(a) = H_0 \left[ \sqrt{\Omega_{r_c}} + \sqrt{\Omega_{r_c} + \Omega_m a^{-3}} \right] \quad (2.53)$$
where $\Omega_{rc} = 1/(4r_c^2 H_0^2) = (1 - \Omega_m)^2/4$. The modified Friedmann equation in a homogeneous and isotropic background reads [98]

$$H^2 \pm mH = \frac{k^2}{3}\rho, \quad (2.54)$$

where $m$ is the effective soft mass of the graviton, related to the cross-over scale by $m = r_c^{-1}$. This means that at higher energies we recover the usual 4 dimensional limit in which $H^2 \propto \rho$, while at late times, the correction coming from the extra dimension kicks in. As we can see in Eq. (2.54), there are two branches of solutions, one with the minus sign, the self-accelerated one in which acceleration is achieved without the need of a cosmological constant (but showing ghost instabilities), and the normal branch with the plus sign in which expansion is slowed down and which needs the presence of $\Lambda$ to achieve late time acceleration. The Poisson equation is modified in these models as follows:

$$k^2 \Phi = -\frac{k^2}{2} \left(1 - \frac{1}{3\beta}\right) \rho_m \delta_m \quad (2.55)$$

where $\beta = 1 - \frac{2(Hr_c)^2}{2Hr_c - 1}$. For the growth index studied in [1], we have the approximation found by [99, 100]:

$$\gamma(a) = \frac{7 + 5\Omega_M(a) + 7\Omega_M^2(a) + 3\Omega_M^3(a)}{[1 + \Omega_M^2(a)][11 + 5\Omega_M(a)]}, \quad (2.56)$$

where $\Omega_M$ is given by

$$\Omega_m(a) = 1 - \frac{1}{Hr_c} = \frac{1 + \frac{4\Omega_m}{a^3(1-\Omega_m)^2} \left[1 + \frac{4\Omega_m}{a^3(1-\Omega_m)^2}\right]^{1/2} - 1}{\left[1 + \frac{4\Omega_m}{a^3(1-\Omega_m)^2}\right]^{1/2} + 1}, \quad (2.57)$$

while the effective equation of state reads

$$w(a) = \frac{Hr_c}{1 - 2Hr_c} = \frac{-1}{1 + \Omega_m(a)}. \quad (2.58)$$
2.2.3 Screening mechanisms

As we have seen in previous sections, both dark energy models and modified gravity theories introduce a fifth force from extra degrees of freedom. This fifth force is tightly constrained by solar system and weak equivalence principle tests. In order to pass these tests, the coupling is constrained to be much weaker than gravity, leading to naturalness problems. In this section we discuss how these new scalar degrees of freedom can naturally couple to standard model fields while still being in agreement with observations. This is done by making some property of the field dependent on the background environment. These “screening” mechanisms, as they are commonly known, fall into two main classes: either the field alters its mass and becomes massive in a dense environment, so that the fifth force is suppressed because the Compton wavelength of the interaction is small, or the coupling to matter becomes weaker in dense environments to ensure that the dynamical effect of the scalar field is suppressed.

- Density dependent masses: The chameleon mechanism
  
  This mechanism is based on a coupled quintessence field whose mass can be different depending on its environment \[ [101] \]. If the matter density of the surroundings is high, the field acquires a heavy mass, making it sit near the minimum of its potential and, hence, not being able to propagate freely. On the other hand, the field can propagate on lighter environments, where its mass is also lighter and is not constrained to sit at the minimum of the potential. Thus, the effective potential for such a mechanism is given in general by

  \[
  \nu_{\text{eff}}(\phi) = V(\phi) + \rho A(\phi),
  \]

  where \( V(\phi) \) is the bare potential, \( \rho \) the energy density of the environment and \( A(\phi) \) the conformal coupling, which usually takes the form \( e^{Q\phi} \). Such an effective potential is illustrated in Fig. 2.2.1. The environmental dependence of the mass of the field allows the chameleon to avoid fifth force and equivalence principle constraints through what is known as the “thin-shell” effect. This means that the scalar field \( \phi \) is approximately constant everywhere inside the body except for a small region near the surface, where large (\( \mathcal{O}(1) \)) changes in the value of \( \phi \) occur. Hence, inside a body with such a mechanism, \( \nabla \phi \) vanishes everywhere except in the thin-shell. Since the force exerted by such a field is proportional to \( \nabla \phi \), it is only the surface layer that feels and contributes to the fifth force mediated by \( \phi \). We can
see the effective potential inside and outside a spherically symmetric body in Fig. 2.2.2. This screening mechanism, even though it ensures that theories like coupled quintessence pass solar system and equivalence principle tests, also has problems. Recently in [102], it was discovered that this coupling could excite highly energetic quantum fluctuations in the early universe, questioning the validity of the effective theory itself.

**Density dependent couplings: The Vainshtein and symmetron mechanisms**

In higher-dimension models like DGP, the effects of the scalar field are screened by the Vainshtein mechanism [103]. This occurs when higher derivative operators are present in the Lagrangian for the scalar field, arranged in such a way that the equations of motion are still second order. In the presence of a massive source, the non-linear terms force the suppression of the scalar field in the vicinity of a massive object. The radius within which the fifth force is suppressed is known as the Vainshtein radius. For example, in the DGP model, this radius is given by

$$r_* \propto \left( \frac{M}{4\pi m_{\text{pl}}} \right)^{1/3} \frac{1}{\Lambda},$$

(2.60)
where $\Lambda$ is the strong coupling scale: $\Lambda = (m_{4\text{pl}} m^2)^{1/3}$. For the Sun, if $m \sim 10^{-33}$ eV, then the Vainshtein radius is $r_x \sim 10^2$ pc. Inside this radius, the high derivative terms become important and cause the kinetic terms for scalar fluctuations to be large. This can be interpreted as a weakening of the coupling between the scalar field and matter. Thus, the interaction is suppressed in the vicinity of massive objects.

The symmetron model introduced in [104] is similar to the chameleon mechanism described earlier. It requires a conformal coupling between the scalar field and matter with a certain potential. This leads to an effective potential for the scalar field

$$V_{\text{eff}} = \frac{1}{2} \left( \frac{\rho}{M^2} - \mu^2 \right) + \frac{1}{4} \lambda \phi^4, \quad (2.61)$$

where $M$, $\mu$ and $\lambda$ are parameters of the model. This is a Higgs type potential with two non-zero vacua. In highly dense environments where $\rho > \mu^2 M^2$, the field sits in a minimum of the potential at the origin. As we go down in energy density, the symmetry of the field breaks spontaneously and the field can fall into one of the non-zero vacua. In the high density regime, the symmetry is restored and the scalar
field vacuum expectation value should be near zero in such a way that fluctuations of the field do not couple to matter. The fifth force is thus suppressed in the exterior of a massive object since the field does not couple to it.

- **The Olive-Pospelov model**
  Finally, the Olive-Pospelov model \cite{105} also uses a scalar field coupled to matter. In this model, both the coupling and the field potential exhibit quadratic minima. If the field takes the minimum value of the coupling, then it effectively decouples from matter. In high density environments, the field sits close to the minimum of the coupling, while in low density ones the field relaxes to the minimum of the bare potential. In this way, this model suppresses interactions of the scalar field in dense environments.

### 2.3 Inhomogeneous universe models

Since Hubble discovered in 1929 that the Universe was undergoing an expansion and galaxies were in fact receding with a velocity proportional to their distance, the models that have been most favoured by both theory and observation are those that follow the Copernican principle. This basically means that there are no special observers in the Universe, *i.e.* we do not observe a different configuration of matter, expansion, etc. as compared to any other observer in any other position. Mathematically this translates into rotational and translational invariance: isotropy and homogeneity.

However, how safe is it to stick to the Copernican principle and assume homogeneity? These fundamental symmetries are supposed to hold on large scales but on small scales our Universe is clearly not homogeneous. Because general relativity is a non-linear theory, even relatively small local inhomogeneities with a sufficiently large density contrast could, in principle, give rise to cosmological evolution that is not accessed by the usual cosmological perturbation theory in an FLRW background. In fact, the potentially interesting consequences of the inhomogeneities were already recognised at the time when the homogeneous and isotropic models of the Universe were first studied, but their impact on the global dynamics of the Universe is still largely unknown, as we shall see in Sec. 2.3.7. At present, they are averaged out and everything seems to follow homogeneity.

In the case of large void models, the acceleration that we observe would be just a trick that light plays on us as it travels through an inhomogeneous Universe. As photons travel through the Universe towards us, they experiment the variations of the Hubble expansion rate rather than the average of the rate itself. If we were to live in a big enough underdense
region of the Universe, these corrections to the redshift of distant photons could amount to the apparent acceleration we observe today.

2.3.1 Synchronous comoving coordinates

Normal spacetime coordinates are not the best choice to describe an inhomogeneous spherical Universe. In order to simplify the equations and use the symmetries, the best coordinates to use are the synchronous comoving coordinates. Comoving coordinates can be introduced whenever there is a timelike vector field $u^\mu$, such as the matter flow. Furthermore, the contravariant version of this vector field has only a time component, i.e.: $u^\mu \propto \delta^\mu_0$. If this vector field is also rotation free, the comoving coordinates can be chosen to be as well synchronous. This means that, in these coordinates, the metric tensor has no time-space components.

2.3.1.1 Spherically symmetric inhomogeneous metrics and Einstein equations

The most general spherically symmetric metric is given by

$$ds^2 = a(r, t)dt^2 + 2\beta(r, t)dt dr + \gamma(r, t)dr^2 + \delta(r, t)d\Omega^2. \quad (2.62)$$

If the source in Einstein’s equation is a perfect fluid in a spherically symmetric spacetime, then its rotation must be necessarily zero. Hence, we can use the synchronous comoving coordinates that were previously introduced and rewrite the metric in Eq. (2.62) as

$$ds^2 = e^{C(r, t)}dt^2 - e^{B(r, t)}dr^2 - R^2(r, t)d\Omega^2. \quad (2.63)$$
Plugging Eq. (2.63) into the Einstein equations and assuming that we have a perfect fluid with a cosmological constant \( \Lambda \), the four equations we get are

\[
\begin{align*}
0-0 \text{ component} & \quad e^{-C} \left( \frac{\dot{R}^2}{R^2} + \frac{\dot{B}R}{R} \right) - e^{-B} \left( \frac{2R''}{R} + \frac{R'^2}{R^2} - \frac{B'R'}{R} \right) + \frac{1}{R^2} = 8\pi G \rho - \Lambda, \\
0-r \text{ component} & \quad e^{-B} \left( 2 \frac{\dot{R}'}{R} - \frac{\dot{B}R'}{R} - \frac{\dot{C}R'}{R} \right) = 0, \\
r-r \text{ component} & \quad e^{-C} \left( 2 \frac{\dot{R}}{R} + \frac{\dot{R}^2}{R^2} - \frac{\dot{C}R}{R} \right) - e^{-B} \left( \frac{R'^2}{R^2} + \frac{C'R'}{R} \right) + \frac{1}{R^2} = -8\pi G p - \Lambda, \\
\theta-\theta \text{ component} & \quad \frac{1}{4} \left( R + e^{-C} \dot{R}^2 - e^{-B} R R'^2 + \frac{1}{3} \Lambda R^3 \right)' = 8\pi G p R^2 R'.
\end{align*}
\]

Here, a dot means a partial derivative with respect to time and a prime is a partial derivative with respect to the radial coordinate \( r \). The \( \phi-\phi \) component is the same as the \( \theta-\theta \) one.

Multiplying the 0-0 component by \( R^2 \dot{R}' \) and using the 0-r component we have

\[
\left( R + e^{-C} \dot{R}^2 - e^{-B} R R'^2 + \frac{1}{3} \Lambda R^3 \right)' = 8\pi G p R^2 R'.
\]

From this equation we can recognise the quantity we usually associate with mass

\[
m \equiv \frac{c^2}{2G} \left( R + e^{-C} \dot{R}^2 - e^{-B} R R'^2 + \frac{1}{3} \Lambda R^3 \right),
\]

where the \( c^2/2G \) factor has been added to have the right units. This definition of mass was first introduced by Lemaître but is usually credited to Misner and Sharp [106].

Using the same method (this time multiplying by \( R^2 \dot{R} \)), we can rewrite the \( r-r \) component as

\[
\left( R + e^{-C} \dot{R}^2 - e^{-B} R R'^2 + \frac{1}{3} \Lambda R^3 \right)' = -8\pi G p R^2 \dot{R}
\]

which represents the energy-conservation equation for spherically inhomogeneous space-
To tackle the task of studying an inhomogeneous yet isotropic Universe, we will concentrate on the easiest toy model. To construct a metric for such model, let us think that we are, for the time being, in the centre of the void and consider a spherically symmetric dust Universe with radial inhomogeneities as seen from our location. Choosing comoving spatial coordinates and the time coordinate as the proper time of the fluid, the metric then takes the general form

\[ ds^2 = -dt^2 + X^2(r, t)dr^2 + A^2(r, t)d\Omega^2, \tag{2.66} \]

Note that the metric signature here is \((-+++\)) and the functions \(A(r, t)\) and \(X(r, t)\) have both temporal and spatial (radial) dependence and are, in principle, arbitrary. These functions correspond to the ones introduced previously in Eq. \((55-57)\) taking

\[ C(r, t) = 0, \quad \epsilon^{B(r,t)} = X(r, t), \quad R(r, t) = A(r, t). \]

Notice that this metric contains the FLRW one in the limit

\[ X(r, t) \rightarrow \frac{a(t)}{\sqrt{1 - kr^2}}, \quad A(r, t) \rightarrow a(t)r. \]

Let us now see what the Einstein equations tell us about the functions \(A(r, t)\) and \(X(r, t)\). We define our energy momentum tensor as

\[ T^\mu_v = -\rho_M(r, t)\delta_0^\mu\delta_0^v - \rho_\Lambda\delta^\mu_v, \tag{2.67} \]

where \(\rho_M(r, t)\) is the matter density, \(u^\mu = \delta_0^\mu\) represent the components of the 4-velocity-field of the fluid, and we have kept the vacuum energy \(\rho_\Lambda\) for generality. Note that, although the fluid is staying at fixed spatial coordinates \((u^\mu u_\mu = 1)\), it can physically move in the radial direction. Now introducing Eq. \((2.70)\) into the Einstein equation, one finds
the set of equations for the different components

0-0 component
\[
-2 \frac{A''}{AX^2} + 2 \frac{A'X'}{AX^3} + 2 \frac{\dot{X}\dot{A}}{AX} + \frac{1}{A^2} + \left( \frac{\dot{A}}{A} \right)^2 - \left( \frac{A'}{AX} \right)^2 = 8\pi G (\rho_M + \rho_{DE}),
\]

0-r component
\[
\dot{A}' = A' \frac{\dot{X}}{X},
\]

r-r component
\[
2 \frac{\ddot{A}}{A} + \frac{1}{A^2} + \left( \frac{\dot{A}}{A} \right)^2 - \left( \frac{A'}{AX} \right)^2 \rho_{DE} = 8\pi G \rho_{DE},
\]

θ-θ component
\[
- \frac{A''}{AX^2} + \frac{\dot{A}}{A} + \frac{\dot{X}}{AX} + \frac{A'X'}{AX^3} + \frac{\ddot{X}}{X} = 8\pi G \rho_{DE}.
\]

Again the φ-φ component is the same as the θ-θ one. Solving the second of these equations (0-r component), we find a relationship between \(X(r, t)\) and \(A(r, t)\):
\[
X(r, t) = C(r)A' (r, t),
\]

where now the function \(C(r)\) depends only on the coordinate \(r\). We can choose this function to be \(C(r) \equiv 1/\sqrt{1 - k(r)}\), where \(k(r) < 1\). This is convenient when we want to establish the FLRW limit, as \(k(r)\) will become the spatial curvature. Replacing this into Eq. (2.66), we finally express the LTB metric in its usual form
\[
ds^2 = -dt^2 + \frac{A'^2(r, t)}{1 - k(r)} dr^2 + A^2(r, t) d\Omega^2.
\]

Now the FLRW limit is obvious: \(A(r, t) \to a(t) r\) and \(k(r) \to kr^2\).

Out of the three remaining components of the Einstein equations, only 2 are independent. Expressing these in terms of the functions appearing in Eq. (2.70), they become
\[
\frac{\dot{A}^2 + k(r)}{A^2} + \frac{2\dot{A}\ddot{A} + k'(r)}{AA'} = 8\pi G (\rho_M + \rho_{DE}),
\]

\[
\dot{A}^2 + 2\ddot{A} + k(r) = 8\pi G \rho_{DE} A^2.
\]

Integrating Eq. (2.72) once yields
\[
\frac{\dot{A}^2}{A^2} = \frac{F(r)}{A^3} + \frac{8\pi G}{3} \rho_{DE} - \frac{k(r)}{A^2},
\]
where $F(r)$ is another arbitrary function playing the role of effective matter content which, when substituted back into Eq. (2.71), gives

$$\frac{F'}{A'A^2} = 8\pi G \rho_M'. \tag{2.74}$$

Combining these last two equations we can obtain an acceleration equation for LTB models (equivalent to the second Friedmann equation we are more familiar with).

$$\frac{2 \ddot{A}}{3A} + \frac{1}{A} = -\frac{4\pi G}{3} (\rho_M - 2\rho_{DE}). \tag{2.75}$$

Let us stop and look more closely at this modified acceleration equation. We can see that the total acceleration (given in this case by the left hand side of Eq. (2.75)) is always negative unless the contribution of $\rho_{DE}$ is large enough, in fact $\rho_{DE} > \rho_M/2$. However, if we look at radial acceleration $\ddot{A}(r, t)$, this can be positive if the function $A(r, t)$ is decreasing fast enough.

Let us now go back and see the actual meaning of the arbitrary functions we have introduced: $k(r)$ and $F(r)$. To do this, we must define similar parameters to the FLRW model. If we take the Einstein equation for the homogeneous case expressed in terms of relative densities

$$H^2(t) = H_0^2 \left[ \Omega_M \left(\frac{a_0}{a}\right)^3 + \Omega_{DE} + \Omega_k \left(\frac{a_0}{a}\right)^2 \right],$$

comparing this equation with Eq. (2.73), it is straightforward to establish analogies between quantities. This motivates us to define the local Hubble rate and the effective matter density as

$$H(r, t) \equiv \frac{\dot{A}(r, t)}{A(r, t)}, \tag{2.76}$$

$$F(r) \equiv H_0^2(r) \Omega_M(r) A_0^3(r). \tag{2.77}$$

Naturally, the new curvature parameter may be defined as

$$k(r) \equiv H_0^2(r) (\Omega_M(r) + \Omega_{DE}(r) - 1) A_0^3(r), \tag{2.78}$$

where now the boundary values are a function of the coordinate $r$: $A_0(r) \equiv A(r, t_0)$, $H_0(r) \equiv H(r, t_0)$, and $\Omega_{DE}(r) \equiv 8\pi G \rho_{DE}/3H_0^2(r)$. With these definitions, equation Eq.
acquires the more physical form

\[ H^2(r, t) = H_0^2(r) \left[ \Omega_M(r) \left( \frac{A_0}{A} \right)^3 + \Omega_{DE}(r) + \Omega_k(r) \left( \frac{A_0}{A} \right)^2 \right], \quad (2.79) \]

where \( \Omega_k(r) \equiv 1 - \Omega_{DE}(r) - \Omega_M(r) \).

At first glance, we observe that the difference between FLRW models and LTB ones is that, in the latter case, the quantities depend on the radial coordinate as well as on time. This new coordinate dependence means that inhomogeneities come in two different independent classes: in the Hubble rate and in the matter distribution. Although they are linked via the Einstein equations, their boundary conditions are completely independent. This makes it possible to have a homogeneous present \( \Omega_M \) while living in an inhomogeneous Universe. This does not mean, however, that the real matter distribution given by \( \rho_M \) itself has no spatial dependence, provided that \( H_0(r) \neq 0 \).

This spatial dependence also holds when considering the gauge freedom that also exists here for the present value of the transverse scale factor. In the FLRW model we recall that we have the freedom to assign the value of \( a_0 \) to that of any positive number. This translates to the LTB scenario the possibility to choose any smooth invertible positive function for \( A(r, t_0) \). It is usually chosen for convenience to be

\[ A(r, t_0) = r. \]

This is done to make the LTB model resemble an FLRW Universe locally and fit the galaxy surveys.

Now, we have all the ingredients to solve the equations and determine analytically \( A(r, t) \) and all its derivatives. To do this, we first start with integrating equation Eq. (2.79) to get a relationship between \( A(r, t) \) and the coordinates \( r \) and \( t \).

By doing this, we get

\[ t_0 - t = \frac{1}{H_0(r)} \int_{\frac{A(r,t)}{A_0(r)}}^1 \frac{dx}{\sqrt{\Omega_M(r)x^{-1} + \Omega_{DE}(r)x^2 + \Omega_k(r)}}. \quad (2.80) \]

This equation specifies the function \( A(r, t) \) and all of its derivatives. It can be analytically
solved for the cases $\Omega_{\text{DE}} = 0$ or $\Omega_k = 0$. For the first case we obtain [107]

$$H_0(r) t(r) = \frac{A(r, t)}{A_0(r) \sqrt{\Omega_k(r)}} \sqrt{1 + \frac{\Omega_M(r) A_0(r)}{\Omega_k(r) A(r, t)}} - \frac{\Omega_M(r)}{\sqrt{\Omega_k^3(r)}} \sinh^{-1} \frac{\Omega_k(r) A(r, t)}{\Omega_M(r) A_0(r)},$$

(2.81)

which can be solved parametrically to give

$$A(r, t) = \frac{\Omega_M(r)}{2\Omega_k(r)} [\cosh(\eta) - 1] A_0(r),$$

$$H_0(r) t = \frac{\Omega_M(r)}{2\Omega_k(r)^{3/2}} [\sinh(\eta) - \eta].$$

Here $\eta$ can be interpreted as the conformal time and is defined as $\sqrt{-k(r)} dt = A(r, t) d\eta$ and can be found given $r$ and $t$.

In the case of $\Omega_k = 0$ the solution is [108]

$$(t - t_0) H_0(r) = \frac{2}{3 \sqrt{\Omega_{\text{DE}}(r)}} \left[ \sinh^{-1} \omega(r) \left( \frac{A(r, t)}{A_0(r)} \right)^3 - \sinh^{-1} \omega(r) \right],$$

(2.83)

where

$$\omega(r) = \frac{\Omega_{\text{DE}}(r)}{\Omega_M(r)}.$$ 

(2.84)

In this particular case, $A(r, t)$ can be found explicitly as

$$A(r, t) = A_0(r) \left[ \cosh(\tau) + \sqrt{\frac{3}{8\pi G \rho_{\text{DE}}} H_0(r) \sinh(\tau)} \right],$$

(2.85)

where $\tau = \sqrt{6\pi G \rho_{\text{DE}}} (t - t_0)$.

### 2.3.3 Formation of structure in LTB Universes

Lemaître was the first to notice that an initial mass distribution may exist in such a way that a region of size $r = r_0$ will recollapse, while a region with $r > r_0$ will expand forever. In this case, the curvature of space is positive everywhere and the expansion of the outer region is caused by the cosmological constant.

Bonnor proposed 1956 a model to see if galaxies could form in an LTB scenario [109].
He considered a Friedmann dust tube around the centre of symmetry, surrounded by an LTB transition zone, and that in turn surrounded by another Friedmann dust region so that, at any time \( t = t_0 = \text{constant} \), the density in the outer region is different from that in the inner region. The boundaries of the Friedmann regions were assumed to be comoving. If both Friedmann regions have positive spatial curvature and the density in the inner region is higher than that in the outer region, then the inner region will start to recollapse earlier than the background and will form a condensation. Bonnor assumed that the condensation has the mass of a typical galaxy, that is, it contains \( N \approx 3 \times 10^{67} \) nucleons and it was formed roughly 1000 years after the Big Bang.

However, there was a problem with his proposal; namely, if we assume that such a condensation is a statistical fluctuation in a homogeneous background, then the initial density contrast is

\[
\frac{\delta \rho}{\rho} = \left| \frac{\rho_c - \rho_b}{\rho_b} \right| \sim N^{-1/2} \approx 10^{-34},
\]

where \( \rho_c \) is the density of the condensation and \( \rho_b \) the density of the background. However, for a perturbation to develop into a galaxy, the initial density contrast has to be of order \( 10^{-5} \). On the other hand, if such a perturbation were to arise as a statistical fluctuation, it would only give condensates of \( 10^{10} \) particles.

One can also play around with the spatial curvature of the different regions and ask if galaxies would form. If the outer region has negative curvature while the inner region has positive curvature, the initial perturbation has to be about 10 times larger than in the case shown above. If both Friedmann regions have negative curvature, no galaxies can form at all. Adding a cosmological constant does not help either in a model that starts with the Big Bang. Only two possibilities remain: either \( \Lambda \neq 0 \) and the Universe begins as an instability in the Einstein Universe in the asymptotic past, or there exists a mechanism for producing large perturbations. Considering inflation, the density fluctuations coming from quantum fluctuations of the inflaton are of the order \( 10^{-5} \).

2.3.4 Luminosity distance and redshift

Let us now deduce the equations for some of the actual observational quantities so that we can compare the model with observations. The first and easiest to compute is the luminosity distance. This quantity relates the redshift and the energy flux of photons travelling towards us. We will first deduce the equations for an observer in the centre of the void and then we will see how this changes if we are a distance \( d \) away from this centre.

We can intuitively say, considering the symmetries of the model, that there will exist
geodesics in the radial direction, since isotropy is conserved. This means that, along these geodesics, \( d\theta = d\phi = 0 \). Now, as we are interested in the path followed by photons, i.e.: null geodesics with \( ds^2 = 0 \), let us take these conditions and substitute them in Eq. (2.70), obtaining the geodesic equation for light rays

\[
\frac{dt}{dr} = -X(r, t),
\]

(2.86)

where there is a minus sign because we are considering incoming light rays, i.e.: decreasing radial coordinate with increasing time. We now want to obtain an expression that relates \( t \) and \( r \) with redshift. For this purpose, let us consider 2 light rays which follow geodesics. Let their path be defined by \( t_1 = t(\lambda) \) and \( t_2 = t(\lambda) + \tau(\lambda) \) with \( \lambda \) being the affine parameter. The equation of their paths will be given by Eq. (2.86)

\[
\begin{align*}
\frac{dt_1}{d\lambda} &= \frac{dt(\lambda)}{d\lambda} = -\frac{dr}{d\lambda} X(r, t), \\
\frac{dt_2}{du} &= \frac{dt(\lambda)}{d\lambda} + \frac{d\tau(\lambda)}{d\lambda} = -\frac{dr}{d\lambda} X(r, t) + \frac{d\tau(\lambda)}{d\lambda} = -\frac{dr}{d\lambda} X(r, t(\lambda) + \tau(\lambda)).
\end{align*}
\]

(2.87)

Performing a Taylor expansion up to first order in \( \tau \):

\[
\frac{dt_2}{d\lambda} = -\frac{dr}{d\lambda} \frac{A'(r, t) + \dot{A}'(r, t)\tau(\lambda)}{\sqrt{1 - k(r)}}.
\]

(2.88)

Combining these equations we get a differential equation for \( \tau \)

\[
\frac{d\tau(\lambda)}{d\lambda} = -\frac{dr}{d\lambda} \dot{X}(r, t) \tau(\lambda).
\]

(2.89)

Now, redshift is defined as \[110\]

\[
z \equiv \frac{[\tau(0) - \tau(\lambda)]}{\tau(\lambda)}.
\]

If we differentiate this with respect to \( u \) and use Eq. (2.89) we get an expression that relates \( z \) with \( \lambda \)

\[
\frac{dz}{d\lambda} = -\frac{d\tau(\lambda)}{d\lambda} \frac{\tau(0)}{\tau^2(\lambda)} = \frac{dr}{d\lambda} (1 + z) \dot{X}(r, t).
\]

(2.90)

Finally, combining Eq. (2.86), Eq. (2.90) and using Eq. (2.79) we find the equations we
were after
\[
\frac{dt}{dz} = \frac{-X(r, t)}{(1 + z)\dot{X}(r, t)}, \quad (2.91)
\]
\[
\frac{dr}{dz} = \frac{\sqrt{1 + H_0^2(r)(1 - \Omega_M(r) - \Omega_{DE}(r))A_0^2(r)}}{(1 + z)A'(r, t)}, \quad (2.92)
\]
which determine the relations between the coordinates and the observable redshift, i.e.: \(t(z)\) and \(r(z)\).

However, the quantities we measure in cosmology are the luminosity and angular distance as a function of redshift. The luminosity distance depends on the energy flux \(F\) and the total power radiated by the source \(L\) as \(d_L \equiv \sqrt{L/4\pi F}\). This relation is given by [111]

\[
d_L(z) = (1 + z)^2A(r(z), t(z)). \quad (2.93)
\]
Likewise, the angular distance diameter is given by

\[
d_A(z) = A(r(z), t(z)). \quad (2.94)
\]

Now we have a complete set of equations to determine all the parameters: \(r(z)\) and \(t(z)\) are given by Eq. (2.92) and Eq. (2.91) and the scale factor \(A(r, t)\) by Eq. (2.80). Taken all this into account, we can now calculate \(d_L\) and \(d_A\) for a given redshift.

### 2.3.5 Inhomogeneities in Matter Distribution and the Cosmic Microwave Background

The effect of inhomogeneities on the path of photons coming from the cosmic microwave background (CMB) was first studied for LTB Universes in 1993 by Arnau and Saez [112, 113]. To approach this subject, let us first start describing the way we observe temperature anisotropies in the CMB. We know that CMB radiation has a black-body spectrum and, in the case of LTB dust dominated models (those that are applicable after the photons have decoupled from matter), the photon number is conserved. Then, the temperature of the photons from the time of emission \(t_e\), to the time of observation \(t_o\), will be given by the relation:

\[
T(t_o) = \frac{T(t_e)}{1 + z}. \quad (2.95)
\]
To generalise it to the LTB case, we use the fact that LTB models behave like FLRW Universes in small neighbourhoods. This means that we can apply Eq. (2.95) to neighbouring
points on the same null geodesic (we are considering photons) at positions \( r \) and \( r + dr \). The difference in temperature of these two photons will be given by

\[
\frac{d \ln T}{dr} = -\frac{d[\ln(1 + z)]}{dr}.
\]

A more useful quantity is the temperature contrast, defined as \( \Delta T / T \), where \( T \) here is the temperature along a geodesic completely through a Friedmann region and \( \Delta T \) is the difference between \( T \) and the measured temperature in a given direction.

What Arnau and Saez did in [112, 113], was to compute numerically the dependence of the temperature contrast on the direction of observation for a model consisting of an arbitrary relative density on a FLRW background with a localised LTB perturbation superimposed on it. Their model consisted on three parameters (density profile of the condensation, background density and Hubble rate) which they then fitted to the observations coming from the Great Attractor and the Virgo cluster. With these best fit parameters, they calculated the temperature anisotropy caused by these condensations. They calculated the effects of the following:

- Varying velocity profiles with a fixed background density parameter and distance from observer.
- Varying background density parameters with a fixed velocity profile and the distance adjusted to produce the largest effect.
- Varying distance to the condensation with fixed velocity profile and background density parameter.

They concluded that the maximum anisotropy to be expected was \( 3 \times 10^{-5} \) with \( \Omega = 0.15 \) at an angular scale of 10°. This is in agreement with current measurements of temperature anisotropies from WMAP7 data in which the measured anisotropies are of the order \( 2 \times 10^{-5} \) [114].

2.3.6 Light geodesics and the CMB for off-centre observers

We will now study how light geodesics and temperature anisotropies from the CMB change if the observer is not positioned at the centre of the void. If we are not at the centre, the matter distribution we observe will be anisotropic. This will intuitively affect the way we observe CMB anisotropies and thus constrain the location of the observer when comparing with observations.
In this section we will first calculate, as we did in section Sec. 2.3.4, the path followed by photons following [115].

2.3.6.1 Photon path and redshift

We start with the geodesic equation. Even though we have lost a degree of freedom by abandoning homogeneity, photons are not constricted to travel in the radial direction. However, since the void is spherical, axial symmetry is not broken, meaning that photon paths must be independent of the coordinate $\phi$. The geodesic equation is thus transformed into

$$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{av} \frac{dx^a}{d\lambda} \frac{dx^v}{d\lambda} = 0. \quad (2.96)$$

Breaking this down into components:

$$\mu = t : \quad \frac{d^2 t}{d\lambda^2} + \frac{A' \dot{A}'}{1 - k(r)} \left( \frac{dr}{d\lambda} \right)^2 + A \dot{A} \left( \frac{d\theta}{d\lambda} \right)^2 = 0,$$

$$\mu = r : \quad \frac{d^2 r}{d\lambda^2} + \left( \frac{A''}{A'} + \frac{k(r)'}{2(1 - k(r))} \right) \left( \frac{dr}{d\lambda} \right)^2 + \frac{2 \dot{A}' \frac{dr}{d\lambda} \frac{dt}{d\lambda}}{A' \frac{d\lambda}{d\lambda}} - \frac{A(1 - k(r))}{A'} \left( \frac{d\theta}{d\lambda} \right)^2 = 0,$$

$$\mu = \theta : \quad \frac{d^2 \theta}{d\lambda^2} + \frac{A'}{A} \frac{d\theta}{d\lambda} \frac{dr}{d\lambda} + \frac{\dot{A} \frac{d\theta}{d\lambda} \frac{dt}{d\lambda}}{A d\lambda} = 0. \quad (2.97)$$

The $\mu = \theta$ component can be written as the conservation of angular momentum $J$, as in the FLRW case

$$\frac{d}{d\lambda} \left( A^2 \frac{d\theta}{d\lambda} \right) \equiv \frac{d}{d\lambda} J = 0.$$

To these equations we must add the 4-velocity identity for photons, $u^\mu u_\mu = 0$:

$$- \left( \frac{dt}{d\lambda} \right)^2 + X^2 \left( \frac{dr}{d\lambda} \right)^2 + \frac{f^2}{A^2} = 0. \quad (2.98)$$

Now, let us define some angles and vectors that indicate our position as seen in Fig. 2.3.1. For simplicity, we specify the initial conditions at the time $t_0$ when the photon arrives at the observer’s position, which is given by $r = r_0$ and $\theta = 0$. The path of the photon is shown in Fig. 2.3.1. It hits the observer at an angle $\xi$ relative to the $z$-axis. The spatial
components of the unit vector along this axis are
\[ u^i = \frac{1}{X}(1, 0, 0), \]
where the three components are in the \( r, \theta \) and \( \varphi \) direction, respectively.

The direction of \( u^1 \) is tangent to the photon path when it arrives at \((t_0, r_0, \varphi)\)
\[
u' = \frac{d\lambda}{dt} \left( \frac{dr}{d\lambda}, \frac{d\theta}{d\lambda}, \frac{d\varphi}{d\lambda} \right) = -\frac{1}{u} (p, J/A^2, 0),
\]
where the first factor ensures the normalisation, \( g_{ij} u^i u^j = 1 \), and we have defined \( u \equiv dt/d\lambda \) and \( p \equiv dr/d\lambda \). The reason for the minus sign, as in Sec. 2.3.4, is, once again, that we are considering the path of the photon backwards in time.

The angle \( \xi \) between the photon path and the \( z \)-axis is given by the inner product of \( u^j \) and \( u^i \)
\[
\cos \xi = g_{ij} u^i u^j = -\frac{p}{u}.
\]  \hspace{1cm} (2.99)

Since the parameterisation of the path in terms of the parameter \( \lambda \) is arbitrary, we can choose this such that \( \lambda = 0 \) when \( t = t_0 \) and \( u_0 = u(\lambda = 0) = -1 \). Using Eqs. (2.98) and (2.99), these translate into the following initial conditions
\[
p_0 = \frac{1}{X} \cos \xi, \hspace{1cm} (2.100)
\]
\[
J_0 = J = A \sin \xi. \hspace{1cm} (2.101)
\]
As done before, we now need to know how the coordinates depend on the affine parameter $\lambda$ and relate this to the redshift $z$. We also consider 2 photons separated by a time interval $\tau(\lambda)$. The path of the first photon will be thus given by $t_1(\lambda) = t(\lambda)$. For the second photon we will have $t_2(\lambda) = t(\lambda) + \tau(\lambda)$. Both photons must satisfy the 4-velocity identity (2.98). For the first photon this reads

$$\left(\frac{dt}{d\lambda}\right)^2 = \frac{X}{2}(r,t) \left(\frac{dr}{d\lambda}\right)^2 + A(r,t) \left(\frac{d\theta}{d\lambda}\right)^2,$$

(2.102)

while for the second photon, this becomes

$$\left(\frac{d(t + \tau)}{d\lambda}\right)^2 = \frac{X}{2}(r,t + \tau) \left(\frac{dr}{d\lambda}\right)^2 + A(r,t + \tau) \left(\frac{d\theta}{d\lambda}\right)^2.$$

(2.103)

Expanding Eq. (2.103) to first order in $\tau$ and using Eq. (2.102), we arrive at a similar expression as the one for centred observers

$$\frac{dt}{d\lambda} \frac{d\tau}{d\lambda} = \tau(\lambda) \left[ \frac{A^2}{1 - k(r)} \left(\frac{dr}{d\lambda}\right)^2 + A^2 \left(\frac{d\theta}{d\lambda}\right)^2 \right].$$

(2.104)

Now we can take Eq. (2.90) and substitute our expression Eq. (2.104) to get

$$\frac{dz}{d\lambda} = -(1 + z) \frac{d\lambda}{dt} \left[ \frac{A^2}{1 - k(r)} \left(\frac{dr}{d\lambda}\right)^2 + A^2 \left(\frac{d\theta}{d\lambda}\right)^2 \right].$$

(2.105)

This equation determines the change in redshift measured by the observer along an infinitesimal distance $d\lambda$. To find the redshift as a function of $\lambda$ for a photon hitting the observer today, we can sum up the infinitesimal contributions along the past light cone,

$$\frac{d\ln(1 + z)}{d\lambda} = -u^{-1} \left[ \frac{A^2}{1 - k(r)} p^2 + \frac{\dot{A}}{A^3} \dot{p}^2 \right],$$

(2.106)

with the initial condition $z(\lambda = 0) = z_0 = 0$.

Consequently, to determine the redshift we must now solve five equations for $(t, r, \theta, p$ and $z)$, with the corresponding initial conditions $(t_0, r_0, \theta_0, p_0$ and $z_0)$, under the constraints (2.98) and (2.101).
Now let us look at how being displaced from the centre of the void affects the temperature anisotropies we measure from the CMB. To study how this displacement affects CMB observations, we will disregard, for simplicity, any other intrinsic anisotropies apart from those caused by the location of the observer. Thus, we assume the temperature of the last-scattering surface to be isotropic and account for any anisotropy measured by observers due to their displacement.

The temperature of the background radiation in a given direction is determined by measuring the intensity of incident photons from this direction. Assuming the radiation to be black-body radiation, the intensity will be given by a Planck spectrum with a corresponding characteristic temperature. As it propagates through spacetime, it will retain a black-body nature and will exhibit a Planck spectrum with a different temperature at each instant.

In our specific case, the CMB temperature seen today by an observer as that specified by Fig. 2.3.1 will not be given by Eq. (2.95) as in the FLRW case, but by

\[
T(\xi) = \frac{T_*}{1 + z(\xi)},
\]

where \(T_*\) is the temperature at the last-scattering surface. The average temperature \(\hat{T}\) measured by the observer is then

\[
\hat{T} \equiv \frac{1}{4\pi} \int d\Omega T(\xi) = \frac{T_*}{2} \int_0^\pi d\xi \frac{\sin \xi}{1 + z(\xi)}. \tag{2.108}
\]

According to measurements made by the COBE satellite, this temperature is \(\hat{T} = 2.725\). We can now use Eq. (2.107) and Eq. (2.108) to define an average redshift to the last-scattering surface:

\[
1 + z_* \equiv \frac{T_*}{\hat{T}} = 2 \left[ \int_0^\pi d\xi \frac{\sin \xi}{1 + z(\xi)} \right]^{-1}. \tag{2.109}
\]

The relative temperature variation measured by the observer today will then be

\[
\Theta(\xi) \equiv \frac{\Delta T}{\hat{T}} = \frac{T(\xi) - \hat{T}}{\hat{T}} = \frac{z_* - z(\xi)}{1 + z(\xi)}. \tag{2.110}
\]
What is usually interesting is to have the angular dependence of this temperature variation, we Fourier expand Eq. (2.110) in spherical harmonics $Y_{lm}$:

$$
\Theta(\xi) = \sum_{l,m} a_{lm} Y_{lm},
$$

(2.111)

where the coefficients $a_{lm}$ are given by

$$
a_{lm} = \int_0^{2\pi} \int_0^\pi \Theta Y_{lm}^* \sin \xi \, d\xi \, d\phi.
$$

(2.112)

These amplitudes measure the level of anisotropy at different angular scales, with larger $l$ values corresponding to smaller scales. Since the relative temperature field does not depend on the axial angle $\phi$, all the $a_{lm}$ will vanish, except those with $m = 0$.

The observed dipole in the CMB is of the order $|a_{10}| \sim 10^{-3}$. This will put a natural constraint on how far away from the origin the observer can be located, since a farther off-centre position usually means a larger dipole.

What was found in [115] is that, for a void of size 1500 Mpc to still be in agreement with the dipole measurements from COBE, the observer must be displaced by a distance $d \lesssim 15 \text{ Mpc}$.

This was done via a numerical computation to fit both supernovae and CMB data for a best-fit on both the size of the void and the distance from its centre. This means that, for the void size mentioned above, the probability that we are placed within this permitted distance is $10^{-6}$. Although this can sound as a very small probability, it is still much higher than the highly improbable case of being placed exactly at the centre of the underdensity. Another option to account for a too large induced dipole, would be to consider that the observer that is displaced has a nonzero peculiar velocity towards the centre of the void. However, such a coincidence would be very difficult to justify as much as the fact that we are moving in the opposite direction to the local group.

### 2.3.6.3 Perturbation theory in LTB cosmologies

Perturbation theory in LTB models is not as simple as it is in other alternative theories of dark energy. Due to the loss of a degree of symmetry, the decomposition theorem does no longer hold. This means that, in general, our perturbations will no longer decouple
into scalar, vector and tensor modes. A study of the perturbation equations in this scenario using a 1+1+2 decomposition of spacetime can be found in \[ \ldots \]. However, if the normalized shear \( \varepsilon = (H_T - H_L)/(2H_T + H_L) \) is small, as observations seem to confirm \[ \ldots \], we can use the ADM formalism and express our perturbed LTB metric as

\[
ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Psi)\gamma_{ij}dx^i dx^j
\] (2.113)

where \( \gamma_{ij} = \text{diag}\{X^2(r, t), A^2(r, t), A^2(r, t) \sin^2 \theta\} \). Within this formalism, the evolution equation for a pressure-less fluid in the conformal Newtonian gauge (where the absence of anisotropic stresses gives \( \Phi = \Psi \)) is given by

\[
\ddot{\Phi} + 4H_T\dot{\Phi} + (4H_T + 6H_T^2)\Phi = 0
\] (2.114)

where we now have two different expansion rates \( H_T(r, t) = \dot{A}/A \) and \( H_L(r, t) = \dot{A}'/A' \) which correspond respectively to the transverse and longitudinal expansion rates. The growing mode solution of equation Eq. (2.114) is given by

\[
\Phi(r, t) = \Phi_0(r) F_1\left[ 1, 2, \frac{7}{2}; u \right],
\] (2.115)

where \( u = k(r)A(r, t)/F(r) \) and \( F(r) = H_0^2(r)\Omega_M(r)A^3(r, t_0) \) specifies the local matter density today. With this solution we find the density contrast,

\[
\delta(r, t) = \frac{A(r, t)}{r} \Phi(r, t)
\] (2.116)

We can calculate the growth index as before, noting that now the matter density parameter is a function of redshift via both time \( t \) and the radial coordinate \( r \). In LTB models, this is in principle an arbitrary function which must be fixed in each case. In the case of the constrained GBH model \[ \ldots \] the parameters are given by

\[
\Omega_M(r) = 1 + (\Omega_M^{(0)} - 1) \left( \frac{1 - \tanh[(r - r_0)/2\Delta r]}{1 + \tanh[r_0/\Delta r]} \right)
\] (2.117)

\[
H_0(r) = H_0 \left[ \frac{1}{1 - \Omega_M(r)} - \frac{\Omega_M(r)}{(1 - \Omega_M(r))^{3/2}} \arcsinh \left( \frac{1 - \Omega_M(r)}{\Omega_M(r)} \right) \right]
\] (2.118)
with

\[ r_0 = 3.0 \text{ Gpc} \,, \quad \Delta r = 1.5 \, r_0 \,, \quad \Omega_M^{(0)} = 0.15 \,, \quad (2.119) \]

where these values have been chosen to best fit the supernovae data \([117, 119]\). Within this model, the growth rate, i.e. the logarithmic derivative of the density contrast, is given by

\[ f(z) = 1 + \frac{4}{7} \left( 1 - \Omega_m^{-1}(z) \right) \frac{\gamma_1 \left[ 2, 3, \frac{9}{2}; 1 - \Omega_m^{-1}(z) \right]}{\gamma_1 \left[ 1, 2, \frac{7}{2}; 1 - \Omega_m^{-1}(z) \right]} \,, \quad (2.120) \]

where \( \Omega_m(z) \) is the fraction of matter density to critical density, as a function of redshift. The matter density in LTB model is given by \( \rho(r, t) = F'(r)/A'(r, t)A^2(r, t) \). Note that this is different from \( \Omega_M(r) = F(r)/A^3(r, t_0)H_0^2(r) \), which gives the mass radial function today, see \([107]\). This function Eq. (2.120) is identical to the instantaneous growth function of matter density in an open universe, where the local matter density \( \Omega_M \) is given by \( \Omega_m(z) \) at that redshift. This is a good approximation only in LTB models with small cosmic shear, see \([120]\).

Alternatively, we can write the growth function in terms of Legendre polynomials,

\[ f(z) = \Omega_m^{1/2}(z) \frac{P_{-5/2}^{-5/2} \left[ \Omega_m^{-1/2}(z) \right]}{P_{1/2}^{-5/2} \left[ \Omega_m^{-1/2}(z) \right]} \,, \quad (2.121) \]

which can also be written in terms of ordinary functions,

\[ f(z) = \frac{9u(1 - u^2) + 6(1 + 2u^2)\sqrt{u^2 - 1} \text{arcsinh} \sqrt{(u - 1)/2}}{2u(u^4 + u^2 - 1) - 12u^2 \sqrt{u^2 - 1} \text{arcsinh} \sqrt{(u - 1)/2}} \,, \quad (2.122) \]

where \( u \equiv \Omega_m^{-1/2}(z) > 1 \).

The equation of state \( w(z) \) has been obtained for this model from the expression

\[ w(a) = \frac{d \log(\Omega_m^{-1}(a) - 1)^{-1}}{d \log a^3} = \frac{a \Omega_m'(a)/\Omega_m(a)}{3(1 - \Omega_m(a))} \,, \quad (2.123) \]

which was used in Fig. 2.4.1. Note that the rate of expansion \( H(z) \) for this model is similar to that of \( \Lambda \text{CDM} \), which explains why it fits the SNIa data \([110, 121]\).
2.3.7 Backreaction

Void models like LTB try to explain observations with an apparent acceleration caused by a large inhomogeneity. It is also possible to generate a real acceleration rearranging small scale inhomogeneities. The motivation for such model is twofold: Einstein’s equations of GR are highly non-linear and the Universe around us is highly inhomogeneous, at least on small-scales and maybe on super-horizon scales. Because of these two facts, averaging over the inhomogeneities and then solving the GR equations (as is usually done) may not be the same as first solving the fully inhomogeneous GR equations and then averaging. In other words, the expected value of a non-linear function of a variable is not the same as the non-linear function of the expected value of that variable. This effect is commonly known as “backreaction” because it studies the effect of inhomogeneities on the background expansion [122, 123]. Let us take a closer look at how this non-commutability of averaging over non-linear functions can produce acceleration.

If we start with the usual Einstein equations:

\[ G_{\mu\nu} = \kappa^2 T_{\mu\nu}, \quad (2.124) \]

we can expand both sides up to first order (note that this splitting is not gauge independent):

\[ G_{\mu\nu} = G^{(0)}_{\mu\nu} + G^{(1)}_{\mu\nu}, \quad T_{\mu\nu} = T^{(0)}_{\mu\nu} + T^{(1)}_{\mu\nu}. \quad (2.125) \]

For pressureless matter, the \(t-t\) component of Einstein’s equations can be written as

\[ G^{(0)}_{00} = \kappa^2 (T^{(0)}_{00} + T^{(1)}_{00}) - G^{(1)}_{00} \quad (2.126) \]

The average energy density for pressureless matter can then be expressed as a function of the energy-momentum tensor as

\[ \langle \rho \rangle = T^{(0)}_{00} + T^{(1)}_{00}, \quad (2.127) \]

and by averaging Eq. (2.126) we get

\[ \langle G^{(0)}_{00} \rangle = \kappa^2 \langle \rho \rangle - \langle G^{(1)}_{00} \rangle. \quad (2.128) \]

If we take as our background metric a flat FLRW, then \( G^{(0)}_{00} = 3H^2 \) and we notice that in Eq. (2.128) \( 3H^2 \neq \kappa^2 \langle \rho \rangle \), as the normal approach gives. This is due to the fact that we normally first average the metric, obtaining \( \langle g^{(1)}_{\mu\nu} \rangle = 0 \) and then calculating \( G_{\mu\nu} \). This
standard procedure gives $G_{00}^{(1)} = G_{00}^{(1)}(\langle g_{00}^{(1)} \rangle) = 0$. We need not stop at first order, but can continue the expansion to arbitrary order. At second order, for example, Eq. 2.128 would be

$$\langle G_{00}^{(0)} \rangle = \kappa^2 \langle \rho \rangle - \langle G_{00}^{(1)} + G_{00}^{(2)} \rangle .$$

(2.129)

In this case, acceleration is caused by the extra terms not present in the usual average first, compute equations after approach, i.e. terms like $\langle G_{00}^{(1)} \rangle$ and $\langle G_{00}^{(1)} + G_{00}^{(2)} \rangle$ in Eq. (2.128) and Eq. (2.129). This would link the perturbative formation of structure with acceleration without the need of dark energy, however, it is not an easy task to accomplish. There are three main questions to be answered with this approach:

- how do we take the average?
- Up to which perturbative order do we expand our equations?
- If each perturbative order can induce a big correction to the background expansion, shouldn’t such large inhomogeneities be seen by observations?

There have been several attempts in the literature to introduce a physically reasonable way of averaging quantities. In [124], the average of a function $f(t, x_i)$ on constant time-like hypersurfaces using a $3+1$ splitting was defined as

$$\langle f \rangle (t) = \frac{\int d^3 x \sqrt{\gamma(t, x_i)} f(t, x_i)}{\int d^3 x \sqrt{\gamma(t, x_i)}},$$

(2.130)

where $\gamma$ is the determinant of the spatial part of the perturbed metric on constant time hypersurfaces. By using this averaging technique, one obtains second-order terms that could contribute to the expansion rate with an order of magnitude of $10^{-5}$, assuming standard power spectra. Even though this contribution seems small, it is much larger than what one naively gets with $\delta^2 \sim 10^{-10}$ and it could even be larger if higher-order terms were to be included. However, it must be noted that this average is not performed on the lightcone which is where observations are made and that we are assuming that the shift function $N_i = 0$. Changing the averaging algorithm can change the amplitude of the contribution, as was noted in [125]. If second-order contributions are important, one should study the successive perturbative orders. In [126], it is shown that, when higher-order corrections are considered, effects may cancel each other and it is not trivial where one should cut the expansion. Regarding the third problem, since observations do not reflect a high degree of inhomogeneity, any model which predicts a large degree of inhomogeneities must
have some mechanism to hide them. This can be done assuming strong peculiar velocities instead of strong density fluctuations. However, there are strong constraints on peculiar velocities coming from the kinematic Sunyaev-Zel'dovich effect \[127\]. Depending on the inhomogeneous distribution, one could also end up with an anisotropy which is not detected in current data.

### 2.4 The Growth Index for Different Dark Energy Models

In \[1\] we explored how the growth index varied for different dark energy models: DGP, \(f(R)\), LTB, \(\Lambda\)CDM and a varying equation of state (wCDM) that can be representative of a quintessence model. In Fig. 2.4.1 we can see the Hubble rate, the matter density parameter, the equation of state and the growth index for all the models studied. In the case of wCDM we have taken \(w_0 = -0.9\) and \(w_a = 0.2\). As we can see, even though the expansion history, matter density parameter and equation of state are quite similar for the different models considered, most of them have a very different growth index, being the biggest difference between DGP, \(f(R)\) and the rest.

This difference in growth indices comes from how linear perturbations in the matter density are described in each of the models. In the case of \(f(R)\), if we neglect the oscillation mode of \(\delta F\) relative to the mode induced by matter perturbations \(\delta_m\), we can obtain the following approximate equation for matter perturbations:

\[
\ddot{\delta}_m + 2H\dot{\delta}_m - 4\pi G_{\text{eff}}\rho_m\delta_m \simeq 0
\]

where

\[
G_{\text{eff}} \equiv \frac{G}{F} \left( \frac{1 + 4\frac{k^2F'}{a^2F}}{1 + 3\frac{k^2F'}{a^2F}} \right)
\]

where a prime here means a derivative with respect to \(R\), and \(G\) is Newton’s gravitational constant. The quantity \(G_{\text{eff}}\) encodes the modification of gravity due to the presence of the scalaron field. Thus, the Poisson equation in Fourier space is transformed by replacing Newton’s gravitational by the effective one in Eq. (2.132). The transition from the GR regime to the scalar-tensor one, occurs when \(m \sim (aH/k)^2\). For the wave numbers relevant to the observable linear region of the matter power spectrum, we require \(m(z = 0) \gtrsim 3 \times 10^{-6}\) for the transition to have occurred by today. The Starobinsky model in particular, allows for a rapid growth of \(m\) from \(R \gg H_0^2\) \((m \lesssim 10^{-15})\) to
Figure 2.4.1: Hubble rate, matter density parameter, equation of state and growth index as a function of redshift for different DE models. The solid lines represent the $\Lambda$CDM model, the short dashed lines the $w$CDM one, the dotted lines the DGP model, the dotted dashed lines the $f(R)$ theory and the long dashed lines, the LTB model.

\[ R \simeq H_0^2 (m = \mathcal{O}(0.1)) \]

Another way to understand the evolution of $\gamma$ in the Starobinsky model is analysing the evolution equation [128]. At high redshifts, we can approximate this evolution equation to [129]

\[
(1 + z)(1 - \Omega_m) \frac{d\gamma}{dz} = \frac{3}{2} \left( \frac{G_{\text{eff}}}{G} - 1 \right) + (1 - \Omega_m) \left[ \frac{11}{2} \left( \gamma - \frac{6}{11} \right) - \frac{3}{2} (1 - \gamma) \left( \frac{G_{\text{eff}}}{G} - 1 \right) - \frac{3}{2} (2\gamma - 1) (w_{\text{DE}} + 1) \right]
\]

(2.133)

At early stages, the first term of the right hand side of Eq. (2.133) dominates. This is why $\gamma$ decreases as $G_{\text{eff}}/G$ increases, even becoming negative. As we approach the present era, the second term starts to dominate making $\gamma$ increase again.
In the case of the DGP model, the effect of the extra dimension affects both the friction term in the evolution equation for $\delta_m$ and the source term as, similarly to $f(R)$ gravity, we can define an effective gravitational constant as

$$G_{\text{eff}} = G \left( 1 - \frac{1}{3\beta} \right)$$

where here $\beta$ is the same as the parameter described in Eq. (2.55). As we can see in Fig. 2.4.1b, the parameter $\Omega_M$ is smaller for the DGP model than for the standard $\Lambda$CDM one. To compensate for this lack of matter density, the growth index is higher than for the rest of the models, as it is clear from Fig. 2.4.1d.

In the case of the LTB model, we see in Fig. 2.4.1c that the equation of state parameter $w$ differs significantly from all the other models. The reason for this can be seen by looking at the formula used to calculate $w(a)$, Eq. (2.123). The expansion rate is similar to the $\Lambda$CDM case since the parameters selected for the void model considered have been chosen to fit the supernovae data. As we can see, the growth index for LTB is not significantly different from the flat universe scenario and also tends to an asymptotic value given by $\gamma = 0.573$, very close to the predicted one in the $\Lambda$CDM case.

As regards the $w$CDM model studied, we have used the values of $w_0 = -0.9$ and $w_a = 0.2$ to calculate the growth index using Eq. (1.40). We have chosen these values to see how a considerably different model from the standard $\Lambda$CDM one can have a growth index that is barely distinguishable from it, as it can be seen in Fig. 2.4.1d.

### 2.4.0.1 Ansatz for the Growth Index

In this subsection we compare the redshift evolution of the growth index for the different models studied with the Ansatz [1, 130, 131]

$$\gamma(a) = \gamma_0 + \gamma_a(1 - a) .$$  \hspace{1cm} (2.135)

We have plotted in Fig. 2.4.2 the known redshift dependence of the growth index for the different models together with the corresponding values derived with the above Ansatz. As can be seen very clearly, for most models this is an excellent approximation. However, for $f(R)$ models it fails to work at relatively nearby redshifts.
Figure 2.4.2: The redshift evolution of the growth index compared to the linear approximation in Eq. \( (2.135) \) (continuous lines), for all four classes of models.
The existence of dark energy is supported by a number of observations, namely: the age of the Universe compared to the oldest stars, supernovae observations, the CMB, baryon acoustic oscillations (BAO) and large-scale structure (LSS).

Even before the discovery of the acceleration in 1998 through supernovae observations, it was known that in a CDM universe the cosmic age can be smaller than the age of the oldest stars. Dark energy can alleviate this discrepancy because its presence can make the cosmic age longer. It is also important to note that, when inflation predicted a flat universe $k \sim 0$ and $\Omega_m < 1$, many cosmologists had a dark energy component in mind \[132\]. Light from distant type Ia supernovae was the definite proof of this acceleration. CMB observations are also consistent with the presence of dark energy, although the constraint coming from the CMB alone is not so strong. The measurements of the BAO have provided another independent test for the existence of dark energy, as does the matter power spectrum.

In the desire to acquire more knowledge on the origin and nature of dark energy, new probes are being sought for. One of such probes makes use of small scale isolated galaxy pairs by modelling their pairwise velocity. It is also interesting to develop consistency checks that might allow us in the future to test fundamental principles such as the Copernican one. This is the idea behind finding the homogeneity scale of our observable universe. Such endeavours will be further discussed in subsequent sections of this chapter.
In this chapter we will look in detail at the different ways to constrain dark energy observationally and what is the state of the current observations. We will also look at using isolated galaxy pairs as a test for the Alcock Paczynski effect on small scales. Finally, we will look into how to find the homogeneity scale of large-scale structure.

### 3.1 The age of the Universe

The inverse of the Hubble constant, $H_0$, is a rough estimate of the age of the Universe $t_0$. Let us calculate this more exactly to be able to compare it to the age of the oldest objects we know of, such as stars in globular clusters [133]. Consider a universe with radiation, matter, dark energy and curvature. The Hubble parameter $H(z)$ normalised by $H_0$ in this case is given by Eq. (1.13). Using the relationship $dt = -dz/[(1 + z)H(z)]$, the age of the Universe is expressed as

$$t_0 = H_0^{-1} \int_0^\infty \frac{dz}{E(z)(1 + z)}.$$  \hspace{1cm} (3.1)

The integral in Eq. (3.1) is dominated by the terms at low redshifts. Since $\Omega^r_0$ is of the order of $10^{-5} - 10^{-4}$, radiation becomes only important at high redshifts ($z > 1000$). Hence, to evaluate Eq. (3.1), it is safe to neglect the contribution coming from radiation. Let us consider the case of the cosmological constant. In this case, Eq. (3.1) becomes

$$t_0 = H_0^{-1} \int_1^\infty \frac{dx}{x [\Omega^0_m x^3 + \Omega^0_{DE} + \Omega^0_k x^2]^{1/2}},$$  \hspace{1cm} (3.2)

where $x \equiv 1 + z$, and $\Omega^0_m + \Omega^0_{DE} + \Omega^0_k = 1$. In the case of a flat universe, Eq. (3.2) can be integrated analytically to give

$$t_0 = \frac{H_0^{-1}}{3 \sqrt{1 - \Omega^0_m}} \ln \left( \frac{1 + \sqrt{1 - \Omega^0_m}}{1 - \sqrt{1 - \Omega^0_m}} \right),$$  \hspace{1cm} (3.3)

where we have used the fact that, in this case, $\Omega^0_m + \Omega^0_{DE} = 1$. In the limit of no dark energy ($\Omega^0_{DE} \to 0$) we have

$$t_0 = \frac{2}{3} H_0^{-1}.$$  \hspace{1cm} (3.4)

Using the value for $h = 0.743 \pm 0.021$ [134], we find that the age of the Universe is constrained to be between $8.5 < t_0 < 9.0$ Gyr. The age of the globular clusters in the
Milky Way is estimated to be \( 12.9 \pm 2.9 \) Gyr \cite{133}, whereas in \cite{135} they find an upper bound of \( 13.5 \pm 2 \) Gyr. The age of the globular cluster M4 has been constrained to be \( 12.7 \pm 0.7 \) Gyr, \cite{136}. In most cases, the ages of globular clusters are larger than 11 Gyr. This means that the cosmic age estimated using Eq. (3.4) is inconsistent with the ages of the oldest objects we know.

This problem can be solved by taking into account dark energy. In fact, Eq. (3.2) shows that \( t_0 \) becomes larger for decreasing \( \Omega_m^0 \). Taking \( \Omega_m^0 = 0.315 \pm 0.017 \) \cite{16}, we get an age of the Universe between \( 13.4 < t_0 < 14.3 \) Gyr, consistent with the globular cluster ages.

It is also interesting to note that one can also make the cosmic age of the Universe longer than \( t_0 = (2/3)H_0^{-1} \), even in the absence of dark energy. Setting \( \Omega_{DE}^0 = 0 \) in Eq. (3.2) we obtain

\[
  t_0 = \frac{H_0^{-1}}{1 - \Omega_m^0} \left[ 1 + \frac{\Omega_m^0}{2\sqrt{1 - \Omega_m^0}} \ln \left( \frac{1 + \sqrt{1 - \Omega_m^0}}{1 - \sqrt{1 - \Omega_m^0}} \right) \right], \tag{3.5}
\]

where now \( \Omega_m^0 + \Omega_k^0 = 1 \). In the limit \( \Omega_m^0 \to 0 \), we have that \( t_0 \to H_0^{-1} \), which means that the cosmic age in an open universe does not become so large compared to the case of the flat universe with the cosmological constant, but is still larger than \( t_0 = (2/3)H_0^{-1} \). Since observations constrain \( |\Omega_k^0| \) to be much smaller than unity \cite{16}, it is not possible to attain \( t_0 > 11 \) Gyr in an open universe without dark energy. All these arguments show that the presence of dark energy is crucial to solve the cosmic age problem.

### 3.2 Supernovae observations

When a supernova explodes, it is extremely luminous and gives off a burst of radiation. Supernovae can be classified according to the absorption lines of chemical elements. If the spectrum of a supernova includes the spectral line of hydrogen, it is classified as Type II. Otherwise, it is known as Type I. If a supernova contains an absorption line of singly ionised silicon, it is further classified as Type Ia (SN Ia). The explosion of Type Ia supernovae occurs when the mass of a white dwarf in a binary system exceeds the Chandrasekhar limit by absorbing gas for another nearby star \cite{137}. Since the absolute luminosity of Type Ia supernovae is almost constant at the peak of brightness (and believed to be independent of redshift), the distance to a SN Ia can be determined by measuring its observed (apparent) luminosity. Thus, the SN Ia are said to be “standard candles”, by
which luminosity distance can be measured observationally.

In reality, as is always the case with data, it is not as simple as this. The intrinsic spread in absolute magnitudes is actually too large to produce any stringent cosmological constraints. However, at the end of the 1990s, a high-quality sample of local \((z \ll 1)\) supernovae allowed the absolute magnitude to be correlated with the width of the light curve \([138]\): brighter supernovae have a broader light curve. Therefore, it is possible to extract the absolute magnitude of the supernovae by measuring the apparent magnitude and the light curve.

The apparent magnitude of a luminous object is used as a measure of brightness of stars observed on Earth. If we think of two objects with apparent fluxes given by \(F_1\) and \(F_2\), their apparent magnitudes \(m_1\) and \(m_2\) are related to their fluxes by

\[
m_1 - m_2 = \frac{5}{2} \log_{10} \left( \frac{F_1}{F_2} \right) .
\]

This implies that a star with \(m_1 = 1\) is about 100 times brighter than one with \(m_2 = 6\). We define the absolute magnitude of an object \(M\) in terms of its apparent magnitude \(m\) and luminosity distance \(d_L\) as

\[
m - M = 5 \log_{10} \left( \frac{d_L}{10 \text{pc}} \right) .
\]

If the distance is expressed in Megaparsecs, then the relation can be written as

\[
\mu = m - M = 5 \log_{10} d_L + 25 ,
\]

where \(\mu\) is often referred to as the "distance modulus". Eq. \((3.8)\) tells us that the absolute magnitude corresponds to the apparent magnitude an object would have if it were located at the luminosity distance \(d_L = 10\) pc from the observer. There is also an additional correction, known as \(K\)-correction, due to the fact that as the redshift increases, we observe different parts of the source spectrum. We assume this correction has been already taken into account on the estimation of \(m\).

The absolute magnitude of Type Ia supernovae is known to be around \(M = -19\) at the peak of brightness. If we consider two SN Ia with apparent magnitudes \(m_1\) and \(m_2\) and luminosity distances \(d_{L_1}\) and \(d_{L_2}\) respectively, then, using Eq. \((3.7)\), we obtain the following
relationship:

\[ m_1 - m_2 = 5 \log_{10} \left( \frac{d_{L_1}}{d_{L_2}} \right). \]  

(3.9)

Under the assumption that SN Ia are indeed standard candles, the corrected peak absolute magnitude \( M \) is the same for any SN Ia. Then, the luminosity distance \( d_L(z) \) can be obtained by observing the apparent magnitude \( m \) via Eq. (3.7) or Eq. (3.8). The theoretical luminosity distance for an FLRW metric is given by

\[ d_L(z) = \frac{c(1 + z)}{H_0 \sqrt{\Omega_k}} \sinh \left( \sqrt{\Omega_k} \int_0^z \frac{dz'}{E(z')} \right). \]  

(3.10)

Comparing the observational distance with the theoretical one, it is possible to know the expansion history of the Universe for redshifts \( z \ll \mathcal{O}(1) \). If we consider a Universe dominated by non-relativistic matter and dark energy with an equation of state \( w_{DE} \) and allowing for curvature, then the luminosity distance in the region \( z \ll 1 \) is given by

\[ d_L(z) = \frac{c}{H_0} \left[ z + \frac{1}{4} \left( 1 - w_{DE} \Omega_{DE}^0 + \Omega_k^0 \right) z^2 + \mathcal{O}(z^3) \right]. \]  

(3.11)

In Fig. 3.2.1, we plot this luminosity distance for several cases. We notice that in the presence of dark energy, the luminosity distance gets larger, especially for smaller (negative) \( w_{DE} \) and larger \( \Omega_{DE}^0 \). In an open universe \((k < 0)\) the effect of the cosmic curvature also leads to a larger luminosity distance. However, since observations limit the curvature to be \( \Omega_k^0 \ll 1 \), it is difficult to give rise to a significant difference relative to a flat universe without dark energy.

To conclude that the Universe is accelerating, we need a large dataset, since observations are prone to statistical and systematic errors. In Fig. 3.2.2 we can see the distance modulus as a function of redshift for a collection of supernovae compilations known as Union 2.1 [139]. It consists of 833 SN Ia coming from 19 different datasets. We see that the best fit line is one with a flat ΛCDM cosmology. In figures 3.2.3a, 3.2.3b, 3.2.3c and 3.2.3d, we see how dark energy is confirmed by supernovae and an equation of state \( w_{DE} = -1 \) is favoured. Finally, in Fig. 3.2.4, we see a combined constraint on the equation of state linear parameterisation described in Eq. (1.16). Supernovae alone cannot offer stringent constraints on the redshift evolution of \( w_{DE} \) due to their still large error.
bars and the fact that $d_L(z)$ involves an integral of $w_{DE}(z)$ and not the function itself.

3.3 The Cosmic Microwave Background

The observations of temperature anisotropies in the CMB provide another independent test for the existence of dark energy. The oldest sky we can see is the “last scattering” surface, at which electrons are trapped by hydrogen to form atoms. The photons are tightly coupled to baryons and electrons before the decoupling epoch at $z \approx 1100$, but they can after move freely to us in what is known as “free streaming”. In 1964, Penzias and Wilson [9] first detected the CMB photons thermalised to an almost uniform temperature in the sky. The CMB temperature anisotropies were first measured at large angular separations by the COBE satellite in 1992 [142]. Precision measurements of the temperature anisotropies by experiments like WMAP [11] opened a new opportunity to determine cosmological parameters to high precision.

The presence of dark energy affects the CMB anisotropies in two main aspects. The first is to change the position of the acoustic peaks coming from the modification of the angular diameter distance. The second effect is the ISW caused by the time variation of the gravitational potential due to the presence of dark energy. The latter effect is limited
to very large scales and the first is typically more relevant.

It is known that the comoving wavelength corresponding to acoustic peaks in the CMB can be approximately estimated as \( \lambda_c = 2\pi/k = (2/n)r_s \), where \( n \) is an integer and \( r_s \) is the sound horizon defined as

\[
    r_s(\eta) \equiv \int_0^\eta d\eta' c_s(\eta'),
\]

and here the sound speed squared is given by \( c_s^2 = 1/3(1 + 3\rho_b/4\rho_\gamma) \). The characteristic angle for the location of the peaks can be then defined as

\[
    \theta_A \equiv \frac{r_s(z_{\text{dec}})}{d_A^*(z_{\text{dec}})},
\]

where \( z_{\text{dec}} \) is the redshift at the decoupling epoch and \( d_A^* \) is the comoving angular diameter distance defined as

\[
    d_A^*(z) \equiv \frac{d_A(z)}{a} = (1 + z)d_A(z).
\]
Then, the multipole $\ell$ corresponding to the angle defined in Eq. (3.13) is

$$
\ell_A = \frac{\pi}{\theta_A} = \frac{d_A(z_{\text{dec}})}{r_s(z_{\text{dec}})}, \quad (3.15)
$$
Figure 3.2.4: 68.3%, 95.4%, and 99.7% confidence regions of the \((w_0, w_a)\) plane from SNe combined with the constraints from BAO [140], CMB [114], and \(H_0\) [141], both with (solid contours) and without (shaded contours) systematic errors. Zero curvature has been assumed. Points above the dotted line \((w_0 + w_a > 0)\) violate early matter domination and are disfavoured by the data. Figure taken from [139].

So far, we have not seen any dependence with cosmology on the position of the peaks. This is encoded in the comoving angular diameter distance and the sound horizon as follows:

\[
d_A^c(z_{\text{dec}}) = \frac{c}{H_0} \frac{1}{\sqrt{\Omega_m^0}} \mathcal{R},
\]

where \(\mathcal{R}\) is the so-called CMB shift parameter, defined by

\[
\mathcal{R} = \sqrt{\frac{\Omega_m^0}{\Omega_k^0}} \sinh \left( \sqrt{\Omega_k^0} \int_0^{z_{\text{dec}}} \frac{dz}{E(z)} \right).
\]

The sound horizon at decoupling is given by

\[
r_s(z_{\text{dec}}) = \frac{c}{\sqrt{3a_0 H_0}} \int_{z_{\text{dec}}}^{\infty} \frac{dz}{\sqrt{1 + 3 \rho_b/4 \rho_\gamma E(z)}}.
\]
To calculate the decoupling redshift $z_{\text{dec}}$, there exists a fitting formula \[143\]:

$$z_{\text{dec}} = 1048 \left( 1 + 0.00124\omega_b^{-0.738} \right) \left( 1 + g_1 \omega_m^{0.238} \right),$$

where $\omega_m \equiv \Omega_m^0 h^2$, $\omega_b \equiv \Omega_b^0 h^2$ and

$$g_1 = 0.0783\omega_b^{-0.238} / (1 + 39.5\omega_b^{0.763}), \quad g_2 = 0.560 / (1 + 21.1\omega_b^{1.81}).$$

The latest bounds from Planck give a decoupling redshift of $z_{\text{dec}} = 1090.43 \pm 0.54$ \[16\].

Since the contribution of dark energy to $E(z)$ in Eq. (3.18) is negligible for $z > z_{\text{dec}}$, one can estimate this quantity as $E = \sqrt{a + a_{\text{eq}}/a^2} \sqrt{\Omega_m^0}$, where $a_{\text{eq}}$ is the scale factor at radiation-matter equality. Putting this all together, the multipole $\ell_A$ is given by

$$\ell_A = \frac{3\pi}{4} \sqrt{\frac{\tilde{\omega}_b}{\tilde{\omega}_\gamma}} \mathcal{R} \left[ \ln \left( \frac{\sqrt{R_s^{(\text{dec})} + R_s^{(\text{eq})}} + \sqrt{1 + R_s^{(\text{dec})}}}{1 + \sqrt{R_s^{(\text{eq})}}} \right) \right]^{-1},$$

where $R_s^{(\text{dec})} = R_s(a_{\text{dec}}) = (3\omega_b/4\omega_\gamma)a_{\text{dec}}$ and $R_s^{(\text{eq})} = R_s(a_{\text{eq}}) = (3\omega_b/4\omega_\gamma)a_{\text{eq}}$. The CMB shift parameter is affected by the cosmic expansion history from the time of decoupling to the present. The presence of dark energy thus leads to a shift in $\mathcal{R}$ compared to a CDM model, thereby changing the value of $\ell_A$. Hence, $\mathcal{R}$ can be used to place constraints on dark energy. The bound on $\theta_A$ from the Planck experiment is $(1.04148 \pm 0.00066) \times 10^{-2}$ at the $1\sigma$ level. This leads to a value of the multipole $\ell_A \simeq 301$. This value is different from the location of the first acoustic peak from the data: $\ell_1 \sim 220$ (see Fig. 3.3.1). This difference in the position of the first peak is due to several effects such as free streaming of photons after decoupling and the dipole contribution. In order to take into account these effects, the general relation for all the observed peaks in the CMB anisotropies is given by

$$\ell_m = \ell_A(m - \varphi_m),$$

where $m$ represents the peak numbers and $\varphi_m$ is the shift of the multipoles. According to the fits in \[145\], the shift of the first peak is about $\varphi_1 = 0.265$, which gives $\ell_1 = 221$ for $\ell_A = 301$. For larger values of $\Omega_{\text{DE}}^0$, the CMB shift parameter gets smaller. Taking $w_{\text{DE}} = -1$, the dark energy density parameter is constrained to be $0.686 \pm 0.020$ from the bound in $\theta_A$ cited above. Since the CMB shift parameter depends only weakly on $w_{\text{DE}}$, the equation of state of dark energy is not strongly constrained by CMB data alone. How-
Figure 3.3.1: The CMB power spectrum $D_\ell = \ell(\ell + 1)C_\ell/2\pi$ versus the multipole moment $\ell$ and the angular size $\theta$. The curve represents the best fit model and the green shaded areas are the errors from cosmic variance. The dots correspond to the data points from the Planck experiment. Figure taken from [144].

ever, combining with different observations can help constrain $w_{DE}$ (see Fig. 3.3.2) and we see that evidence for accelerated expansion ($w_{DE} < -1/3$) is present, even considering just the CMB data.

3.4 BARYON ACOUSTIC OSCILLATIONS

Baryon Acoustic Oscillations (BAO) originate before decoupling when the Universe was not transparent. At this epoch, baryons and dark matter interacted gravitationally but this gravitational pull was in competition with the pressure from the photons trapped inside. The counteracting forces of gravity and pressure give rise to acoustic oscillations, leaving an imprint when the photons finally decouple at $z \sim 1100$ in the sound horizon at recombination: $L_S \sim 100\text{Mpc}/h$.

The relevant BAO quantity to test against observations is the observed peak compared to the size of the sound horizon:

$$\theta \equiv \frac{L_S}{D_V(z)}.$$  \hfill (3.23)
Here $L_S$ is the *comoving* sound horizon scale at recombination and $D_V$ is a combination of angular and “radial distance”, defined as follows:

$$D_V(z) \equiv [(1 + z)^2 d_A^2 D_z]^{1/3}.$$  \tag{3.24}

In the FLRW case the radial distance is simply given by

$$D_z \equiv \frac{z}{H(z)},$$  \tag{3.25}

and the angular diameter distance is (see Eq. (3.16) and Eq. (3.17))

$$d_A = \frac{1}{1 + z} \frac{c}{H_0} \frac{1}{\sqrt{\Omega_0^k}} \sinh \left( \sqrt{\frac{\Omega_0^k}{E(z)}} \int_0^z \frac{dz'}{E(z')} \right).$$  \tag{3.26}

Substituting these quantities into (3.24) one finds

$$D_V = \frac{1}{H_0} \left[ \frac{z \sinh^2 \left( \sqrt{\frac{\Omega_0^k}{E(z)}} \int \frac{dz}{E(z)} \right)}{\Omega_0^k E(z)} \right]^{1/3},$$  \tag{3.27}
which, for a flat Universe, $\Omega_0 = 0$, is

$$D_{V,\text{flat}} = \frac{1}{H_0} \left[ \frac{z \left( \int \frac{dz}{E(z)} \right)^2}{E(z)} \right]^{1/3}. \quad (3.28)$$

Observationally, it is more useful to consider the ratio of this horizon scale at two different redshifts:

$$D \equiv \frac{D_V(z_1)}{D_V(z_2)}. \quad (3.29)$$

The measured value for this ratio is $1.812 \pm 0.060$ [146] for $z_1 = 0.2$ and $z_2 = 0.35$ and can only be fitted if dark energy is present. In a $\Lambda$CDM scenario, this value is 1.67 (with $\Omega_{DE} = 0.75$). An open empty Universe gives about 1.5. In Fig. 3.4.1 we can see the BAO distance-redshift relationship divided by the best fit flat $\Lambda$CDM prediction from WMAP for three different cosmological models and some of the latest BAO measurements [147]. We see that the BAO data agrees with $\Lambda$CDM.

![Figure 3.4.1: The BAO distance-redshift relation divided by the best-fit flat, $\Lambda$CDM prediction from WMAP. The grey band indicates the 1σ prediction range from WMAP [114]. The BAO results agree with the best-fit WMAP model at the few percent level. If $\Omega_m h^2$ were 1σ higher than the best-fit WMAP value, then the prediction would be at the upper edge of the grey region, which matches the BAO data very closely. For example, the dashed line is the best-fit CMB+LRG+CMASS flat $\Lambda$CDM model, which clearly is a good fit to all data sets. Also shown are the predicted regions from varying the spatial curvature to $\Omega_k = 0.01$ (blue band) or varying the equation of state to $w_{DE} = 0.7$ (red band). Figure taken from [147].](image-url)
3.5 Large scale structure

The observations of large-scale structure such as the galaxy clustering properties, provide another test for the existence of dark energy. The wavenumber at the peak position of the matter power spectrum corresponds to \( k_{\text{eq}} \), given by

\[
k_{\text{eq}} = H_0 \sqrt{\frac{2\Omega_m^0}{a_{\text{eq}}}}.
\]

(3.30)

This shows that \( k_{\text{eq}} \) decreases for decreasing values of \( \Omega_m^0 \). In the presence of dark energy, the peak position shifts towards larger scales (i.e., smaller wavenumber \( k \)). Hence, the scale of the peak position can be used as a probe of dark energy.

The matter power spectrum is related to the observed galaxy power spectrum via de galaxy bias \( b \):

\[
P_{\text{gal}}(k) = b^2 P_m(k).
\]

(3.31)

In Fig. 3.5.1, the galaxy power spectra of luminous red galaxies (LRGs) and main galaxy samples of the SDSS survey are plotted, showing how the power spectra for different galaxy populations is affected by bias. The position of the peak, around the scale of \( 0.01 < k < 0.02 h \text{Mpc}^{-1} \), shows that the \( \Lambda \text{CDM} \) model is favoured over the CDM model (which predicts \( k_{\text{eq}} = 0.051 h \text{Mpc}^{-1} \)). Although the galaxy power spectra alone do not provide tight bounds on the density parameter \( \Omega_{\text{DE}}^0 \) as well as \( w_{\text{DE}} \), the important point is that the observations of large scale structure are consistent with the existence of dark energy.

3.6 The small scale Alcock Paczynski test

Alcock and Paczynski [150] proposed a cosmological test (hereafter denoted AP) based on the assumption of statistical isotropy around any comoving location. For regions of space-time that expand with the background, observed angles and redshifts can be translated into proper distances using the angular diameter distance \( d_A(z) \) and the reciprocal of the Hubble parameter \( H(z) \). Requiring isotropy in proper distance, after translating from angle and redshift measurements, leads to measurements of the product \( d_A(z)H(z) \).

Because radial information comes from redshifts, AP measurements are traditionally limited by peculiar velocities, also known as comoving velocities [151, 152]. These add to expansion-driven redshifts, leading to apparent anisotropic clustering if redshifts are
Figure 3.5.1: Measured power spectra with error bars for the full LRG and main galaxy samples of the SDSS survey. The solid curves correspond to the linear theory $\Lambda$CDM normalised to galaxy bias $b = 1.9$ (top) and $b = 1.1$ (bottom) relative to the $z = 0$ matter power spectrum. The dashed curves include the nonlinear correction of [148]. The onset of nonlinear corrections is clearly visible for $k > 0.09h\text{Mpc}^{-1}$ (vertical line). Figure taken from [149].

assumed to be completely cosmological in origin, even if the correct $d_A(z)$ and $H(z)$ are used to analyse redshifts. These redshift-space distortions (hereafter RSD) are degenerate with the AP effect, removing signal [151, 152], unless assumptions are made such as the Universe following a FLRW metric [153]. In fact, it is simply standard convention that makes us split redshift into cosmological and peculiar velocity components: considering that pairs of galaxies move due to local space-time curvature shows that the expansion rate and the RSD component can be strongly correlated. In the extreme case of bound systems, for example, the combined pairwise velocity is not dependent on background evolution, i.e. the expansion-driven redshift difference across a pair is exactly canceled by the RSD signal (see Appendix B).

Marinoni and Buzzi [154] recently proposed a method to derive cosmological constraints from pairs of galaxies for which peculiar velocities can be modelled. They provided a fitting formula for the observed distribution of velocities, which can then be used
to help break the AP-RSD degeneracy. See Fig. 3.6.1 for their constraints on cosmological parameters using this technique. They assume the normalisation of the galaxy velocity distribution to be redshift and cosmology independent, whereas a more recent work by [155] questioned this statement using N-body simulations. In [2] we investigate this further, considering how well pairs of galaxies, selected using different properties, trace the cosmological expansion.

![Figure 3.6.1](image-url) Cosmological constraints on the abundance of dark energy $\Omega_X$ and on its equation of state $w_X$. The solid blue contours represent the 1σ, 2σ and 3σ confidence regions using the method described in [154]. The black solid line represents a flat geometry, while the dotted lines are constraints coming from BAO observations. The coloured contours correspond to the constraints resulting from the combination of both data sets. Figure taken from [154].

We use the Millennium simulation [156] to test how the pairwise velocity of galaxy pairs may contain information about the background expansion of the Universe. We argue that the local density in which the pairs are found may affect the amount of information these pairs carry on cosmology, because each patch of the Universe expands in a way that depends on the local density. Our analysis suggests that selecting isolated pairs, as considered by Marinoni and Buzzi, can result in average pairwise velocities more in line with the Hubble expansion, i.e. they need smaller, less cosmology dependent peculiar velocity corrections. We also find a better match if low-mass tracers are used.
3.6.1 The Alcock Paczynski effect

Consider a distribution of particles expanding with the Hubble flow, in the redshift interval \((z - \Delta z/2, z + \Delta z/2)\) and subtended by an angle \(\Delta \theta\). Assuming a FLRW cosmology, the proper size of the object perpendicular to our line of sight is given by

\[
d_1 = d_A(z) \Delta \theta, \tag{3.32}
\]

where \(d_A(z)\) is the angular diameter distance to the object. The size of the object parallel to the line of sight is given by

\[
d_2 = \frac{\Delta z}{(1 + z)H(z)}, \tag{3.33}
\]

where \(\Delta z\) is the difference in the redshift of objects closest and furthest away from the observer and \(H(z)\) is the Hubble parameter at the central redshift of the distribution.

Assuming that the collection of particles statistically does not have a preferred direction with respect to one line of sight, then \(\langle d_1 \rangle = \langle d_2 \rangle\), allowing a statistical cosmological measurement \([150]\), from a sufficient number of pairs, of

\[
H(z)d_A(z) = \frac{\Delta z}{(1 + z)\Delta \theta}. \tag{3.34}
\]

Note that \(\Delta z\), \(z\) and \(\Delta \theta\) are all directly observable quantities. The AP effect, as described above, assumes that \(\Delta z\) as measured only depends on the cosmological expansion. In fact, the relative velocity of pairs of particles depends on the local curvature of space, so this is not necessarily a good approximation.

3.6.2 The Millennium simulation and semi-analytic galaxy models

In order to quantify how the dynamics of galaxy pairs may be affected by factors like redshift, isolation radius, mass of the halo etc., we have considered a population of galaxies from the Millennium simulation \([156]\). This traces the evolution of \(2160^3\) dark matter particles of mass \(1.18 \times 10^9 \, M_\odot\) from redshift 127 to the present day inside a periodic box of side \(500 \, h^{-1} \, \text{Mpc}\). The simulation assumes a \(\Lambda\)CDM cosmology with parameters based on a combined analysis of the 2dFGRS \([157]\) and the first-year WMAP data \([158]\). The parameters are \(\Omega_m = 0.25\), \(\Omega_b = 0.045\), \(\Omega_\Lambda = 0.75\), \(n = 1\), \(\sigma_8 = 0.9\) and \(H_0 = 73 \, \text{km s}^{-1} \, \text{Mpc}^{-1}\).
Data on dark matter particles were stored at 64 different times. At each output time, the post-processing pipeline produced a friends-of-friends (FOF) catalog by linking particles with a separation less than 0.2 of the mean value. The SUBFIND algorithm [159] was then applied to each FOF group to identify all the substructures. The merger trees, vital for galaxy formation modelling, were then constructed by linking each subhalo found in a given “snapshot” to one and only one descendant in the subsequent output time-slice.

We used the data from two semi-analytic models of galaxy formation based on the Millennium simulation: [160] and [161]. Most of our analysis uses the semi-analytic model developed by [160] which is based on the growth and merging of the population of subhaloes. Within this catalog, each FOF group hosts a central galaxy which sits in the minimum of the potential of the main subhalo. Other galaxies associated to the same FOF group may sit at the potential minima of smaller subhaloes or may no longer correspond to a resolved dark matter substructure, the latter being known as ‘orphans’. The last two collectives of galaxies are referred to as satellites, although in [160], the physical processes affecting satellite galaxies only begin to differ from those affecting central galaxies when a satellite first enters within the virial radius of the larger system. This is the radius of the largest sphere with its centre at the centre of the FOF group and a mean overdensity exceeding 200 times the critical value.

For our analysis, we have varied several parameters from the galaxy catalogs, namely redshift, mass of the subhalo that hosts the galaxy, stellar mass and $r$-band rest-frame magnitude. Unless differently stated, the redshift shown in our plots corresponds to $z = 0.989$.

### 3.6.3 All galaxy pairs

In this section we study the average pairwise velocity of galaxies regardless of local density and galaxy properties. We shall compare our findings with predictions from linear theory to examine general trends, and test the possibility of using randomly selected galaxy pairs to trace cosmological expansion.

#### 3.6.3.1 Method

For each galaxy pair, we compute the comoving separation $d$, the pairwise velocity $v_{12}$ and its square $v_{12}^2$. We define the pairwise velocity as:

$$v_{12} = \frac{d}{dt} = \frac{(v_1 - v_2) \cdot (x_1 - x_2)}{d},$$

(3.35)
where \( t \) is cosmic time, and \( x \) and \( v \) are galaxy positions and velocities. Note that, following the Millennium simulation, we work in coordinates that are comoving with the Hubble flow, hence, \( v_{12} \) represents the peculiar, non-Hubble component of the pairwise velocity. In the plots that follow, we shall always show the \( -H(z) d \) curve and denote it as “static solution”. We shall also highlight the zero line, which in these plots represents the Hubble flow, and denote it as “comoving solution”. Any data point above the comoving solution represents pairs where the two galaxies are receding from each other faster than the Hubble expansion, while below they are moving towards each other in comoving coordinates.

We group pairs in bins according to their separation \( d \) and, for each bin, compute the average \( \langle v_{12} \rangle \) and the variance \( \text{var}(v_{12}) = \langle v_{12}^2 \rangle - \langle v_{12} \rangle^2 \) of the pairwise velocity. Our definition of expectation value is:

\[
\langle v^n \rangle = \frac{1}{N_{\text{pairs}}} \sum_{i=1}^{N_{\text{pairs}}} (v_i)^n ,
\]

where \( N_{\text{pairs}} \) is the number of pairs in the separation bin. In all the plots in this paper, we shall always show \( \langle v_{12} \rangle \) as a function of pair separation, with the error bars for each bin taken as \( \sqrt{\text{var}(v_{12})/N_{\text{pairs}}} \).

By default, we employ a logarithmic binning in galaxy separation. In order to better visualise the data, however, in some plots we combine underpopulated bins together so that each bin represents at least a minimum number of galaxy pairs, \( N_{\text{min}} \). In this section we use \( N_{\text{min}} \geq 1000 \) while, due to poor statistics, we shall employ \( N_{\text{min}} \geq 2 \) for some of the “isolated” plots in Sections 3.6.4 and 3.6.5.

3.6.3.2 Results

In Fig. 3.6.2 we show the average pairwise velocity \( v_{12} \) of all the galaxies within the [160] semi-analytic model at redshift \( z = 0 \), as a function of separation \( d \). The velocity curve is represented by the red dots with one-sigma error bars, while the blue and black lines are, respectively, the static and comoving solutions.

We also plot the prediction of linear perturbation theory for \( v_{12} \) as the green dashed line, obtained using the prescription from [162, 163]:

\[
v_{12}(d) = -\frac{fb}{\pi^2} \int dk k P_m(k) j_1(kd),
\]

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where we have chosen a bias of $b = 1$, which is in agreement with that measured from clustering within the galaxy catalogue.

For separations larger than $d > 10 \, h^{-1}\text{Mpc}$, we can see that the average peculiar velocity is correctly predicted by linear theory. As we follow the velocity curve into the nonlinear regime, galaxy pairs approach the static line until they cross it at $d \sim 3 \, h^{-1}\text{Mpc}$. This crossing marks the beginning of the infall regime, where the galaxies in the pairs get closer to each other, but with smaller velocities as their separation decreases. On the smallest scales, the pairs asymptote to the static solution.

To use galaxy pairs as tracers of the cosmological expansion, we need their peculiar velocity to be small with respect to the Hubble flow or modellable. A smaller correction is required if $v_{12}$ is closer to zero comoving velocity than to the static solution. As we noted above, only for $d > 10 \, h^{-1}\text{Mpc}$ are the velocities closer to the comoving solution than the static solution, which is the regime of linear perturbation theory. On scales $d \lesssim 3 \, h^{-1}\text{Mpc}$, galaxy pairs follow closely the static solution on average. Their peculiar velocity component is equal and opposite to the Hubble flow, therefore they do not carry any cosmological information (refer to Appendix B).
3.6.4 Isolated pairs

In this section we investigate how the dynamics of galaxy pairs changes when an isolation criterion is imposed and how this depends on the isolation radius, the allowed number of galaxies within this radius, and redshift. We use the same galaxy sample as in Sec. 3.6.3.

3.6.4.1 Method

We initially define a galaxy pair to be isolated within a radius $r_{\text{iso}}$ if each galaxy in the pair has exactly one neighbour within $r_{\text{iso}}$, and that neighbour is the other galaxy in the pair. This is equivalent to drawing two spheres of radius $r_{\text{iso}}$ centred on the galaxies, and imposing the absence of galaxies extraneous to the pair in each of the spheres. We shall weaken this requirement, allowing for the maximum number of neighbours $N_{\text{neigh}}$ in each sphere to be larger than 1. Thus we can interpolate between the dynamics of galaxy pairs in the fully isolated case ($N_{\text{neigh}} = 1$) and in the unconstrained case ($N_{\text{neigh}} \rightarrow \infty$). We implement such isolation criterion with arbitrary number of neighbours in a two-step process. We first determine the number of neighbours for each galaxy in the simulation, and later use this information to select only those pairs where each galaxy has less than $N_{\text{neigh}}$ neighbours.

We fix the isolation radius to $r_{\text{iso}} = 4 \, \text{h}^{-1}\text{Mpc}$, which matches the definition of isolation adopted by [154]. Note that in the plots that follow, we only look at separations less than the isolation radius to ensure that the pair is truly isolated.

3.6.4.2 Results

In the upper panel of Fig. 3.6.3 we present the relative motion of galaxy pairs as a function of their separation for galaxies that are isolated within a $4 \, \text{h}^{-1}\text{Mpc}$ radius. The galaxies are taken from the [160] semi-analytic model applied to the Millennium simulation at redshift $z = 0$. The data points are plotted in red with one-sigma error bars. The blue line is the static solution and the black line the comoving solution – see Sec. 3.6.3.1 for their definition.

Our first comment on Fig. 3.6.3 regards the error bars, which are much larger than those in the non-isolated case of Fig. 3.6.2. The reason is that the imposition of an isolation criterion results in a drastic reduction of the galaxy pairs found, which in the case of Fig. 3.6.3 are only 694, equivalent to one isolated pair every $10^6$ other pairs for $d = 1 \, \text{h}^{-1}\text{Mpc}$. This number is in line with [154], who find 721 pairs for their low-redshift SDSS sample,
Figure 3.6.3: Upper panel: average pairwise velocity for isolated galaxy pairs at redshift \( z = 0 \). The isolation radius is taken to be \( 4 \, h^{-1} \text{Mpc} \) for each member of the pair. The blue solid line represents the static solution, showing the virialisation of pairs. The three shaded regions represent three different regimes. The left blue area represents the virialisation regime: galaxies within these separations have virialised and do not experience the background expansion. The middle red area shows the infall regime, where galaxies start to collapse to form bound systems. The right green region shows what we denote as the ‘void effect’: on average, the isolated pair feels a stronger gravitational pull separating the pair rather than making it closer. The error bars shown are the one sigma confidence limit, assuming Poisson statistics. Lower panel: ratio of average pairwise velocities to the static solution \( H_d \). Note that the y axis of this panel is in logarithmic scale. The error bars shown are the propagated one-sigma errors from the upper panel.

and S09 for their DEEP2 sample.

The most striking feature about the dynamics of isolated pairs is the roughly logarithmic growth of the peculiar velocity \( v_{12} \) for scales larger than \( \sim 0.2 \, h^{-1} \text{Mpc} \). The behaviour of \( v_{12} \) can be explained when we recognise that the dynamics is determined by the combined effect of two competing forces: the mutual attraction between the two galaxies, dominant for small separations, and the disrupting outflow from the void – the void effect, dominant for separations close to the isolation radius. For small separations, the mutual attraction of the members of the pair overcomes the void effect and we see an infall regime. As we study objects with larger separations, the void effect becomes dominant and we see a logarithmically growing pairwise velocity \( v_{12} \).

In the lower panel of Fig. 3.6.3 we plot \( v_{12}/H_d \), that is the ratio of peculiar velocity to Hubble flow. For separations \( 0.4 < d < 4 \, h^{-1} \text{Mpc} \) we find an almost comoving regime where the peculiar velocities are less than 20% of the Hubble flow. In such a regime, the RSD corrections are small, and one could hope that they are easier to model so that \( v_{12} \)
becomes a proxy of the expansion rate. In the following subsections we shall investigate how the comoving regime depends on the isolation radius, the local density and redshift.

### 3.6.4.3 Varying the isolation radius

It is interesting to investigate whether the void effect, giving the approximately logarithmic growth of \( v_{12}(d) \) for isolated pairs, is still present when we allow the size of the isolation radius to vary. We demonstrate this in Fig. 3.6.4, where we show the peculiar velocities of galaxy pairs with \( r_{\text{iso}} \) varying from 0.1 h\(^{-1}\)Mpc to 4 h\(^{-1}\)Mpc, at redshift \( z = 0.989 \).

![Figure 3.6.4: Average pairwise velocity for isolated galaxy pairs at redshift \( z = 0.989 \) for different isolation radii ranging from 0.1 h\(^{-1}\)Mpc to 4 h\(^{-1}\)Mpc.](image)

For \( r_{\text{iso}} \geq 0.6 \) h\(^{-1}\)Mpc the presence of the void severely affects the dynamics of galaxy pairs. The logarithmic growth of the peculiar velocity is visible, even though it is just a hint for the \( r_{\text{iso}} = 0.6 \) h\(^{-1}\)Mpc data points. The cosmological scale \( d_0 \), defined as the separation where the peculiar velocity \( v_{12} \) vanishes, occurs at \( d_0 \approx 0.8 \) h\(^{-1}\)Mpc and appears to be independent of the isolation radius.

At small separations, we cannot see any noticeable differences between the various \( r_{\text{iso}} \) datasets. They all follow the static solution line for \( d < 0.05 \) h\(^{-1}\)Mpc, suggesting that isolated galaxy pairs tend to virialise on the smallest scales just as non-isolated ones do.

The number of isolated galaxy pairs at \( z = 0.989 \) increases from 435 to 71, 201 when reducing the isolation radius from 4 to 2 h\(^{-1}\)Mpc, a factor of roughly 160. As we have noted above, the \( r_{\text{iso}} = 2 \) h\(^{-1}\)Mpc pairs still experience a regime where peculiar velocities are negligible with respect to the Hubble flow. Hence, their velocity difference still traces
the cosmological expansion, although the maximum separation for which this is the case is halved with respect to the $r_{\text{iso}} = 4\ h^{-1}\text{Mpc}$ case. We conclude that using pairs isolated within a $2\ h^{-1}\text{Mpc}$ radius as cosmological tracers would drastically reduce the statistical error with respect to the $4\ h^{-1}\text{Mpc}$ case, while still needing minimal corrections for RSD, provided that the cosmological dependence of the RSD could be modelled. We shall discuss this in more detail later.

3.6.4.4 VARYING THE ISOLATION DENSITY CRITERION

Here, we investigate how the dynamics of galaxy pairs changes if we relax the isolation criterion by increasing the allowed number of galaxies $N_{\text{neigh}}$ within the isolation sphere of $4\ h^{-1}\text{Mpc}$.

![Figure 3.6.5: Average pairwise velocity for isolated galaxy pairs at redshift $z = 0.989$ for different number of neighbours within the isolation sphere of $4\ h^{-1}\text{Mpc}$. The blue solid line represents the static solution, showing the virialisation of pairs. The green dashed line represents the average pairwise velocity for non-isolated pairs. The error bars shown are the one sigma confidence limit, assuming Poisson statistics.](image)

In the linear plot of Fig. 3.6.5, we present the average peculiar velocity $v_{12}$ at $z = 0.989$ for 7 values of $N_{\text{neigh}}$ ranging from $N_{\text{neigh}} = 1$ (equivalent to the pure isolated case of Fig. 3.6.3) to $N_{\text{neigh}} = 5000$. We also plot $v_{12}$ for the non-isolated galaxy pairs, as already shown in Fig. 3.6.2, as a dashed green curve. As $N_{\text{neigh}}$ increases, the different $v_{12}$ curves monotonically fill the gap between the pure isolated case and the non-isolated case. A good agreement between the dynamics of pairs with and without isolation criterion is reached once we allow each galaxy in the pair to have 5000 other neighbours.
In the \( N_{\text{neigh}} = 10 \) case, we found 250,670 pairs, roughly a factor 600 more pairs than in the fully isolated case of \( N_{\text{neigh}} = 1 \). Nonetheless, the \( v_{12} \) curve for \( N_{\text{neigh}} = 10 \) is strikingly similar to the \( N_{\text{neigh}} = 1 \) one. In particular, the void effect still seems to trigger the logarithmic growth of the peculiar velocity, with \( v_{12} \) crossing the zero line at \( d_0 \approx 1 \, h^{-1}\text{Mpc} \) (in the fully isolated case, we have \( d_0 \approx 0.8 \, h^{-1}\text{Mpc} \)). Hence, we suggest that pairs that are not completely isolated trace the cosmological expansion almost as well as the fully isolated ones, with the added benefit of much better statistics.

### 3.6.4.5 Varying Redshift

We illustrate the redshift dependence of the peculiar velocity for isolated pairs in the top panel of Fig. 3.6.6, where we plot \( v_{12}(d) \) for the four redshifts \( z = 0, 0.5085, 0.989, 1.504 \). It is remarkable that for separations \( d \gtrsim 0.2 \, h^{-1}\text{Mpc} \) the peculiar velocity depends only slightly on redshift. The scale \( d_0 \), defined as the separation where \( v_{12} \) vanishes, ranges from 0.6 \( h^{-1}\text{Mpc} \) at \( z = 0 \) to 0.9 \( h^{-1}\text{Mpc} \) at \( z = 1.504 \). This is a small variation if we consider that at \( z = 1.504 \) the Universe was at one third of its current age.

![Figure 3.6.6](image_url)

**Figure 3.6.6:** Upper panel: variation of the average pairwise velocity of isolated galaxy pairs with redshift as a function of separation. The isolation radius is taken to be \( 4 \, h^{-1}\text{Mpc} \) for each member of the pair. Lower panel: range in pairwise velocity for each separation bin over range in the static solutions at each separation bin. Note that the y axis of this panel is in logarithmic scale. The error bars shown are the propagated one-sigma errors from the upper panel.

In the bottom panel of Fig. 3.6.6 we plot the ratio between the range in \( v_{12} \) and the range
in $Hd$ at a given separation. For separations $1 < d < 4 \, h^{-1}\text{Mpc}$, the change of peculiar velocity with redshift is only 10% of the change in Hubble flow. This implies a weak dependence of $v_{12}$ on cosmological expansion on those scales.

### 3.6.4.6 Cosmological implications

Fig. 3.6.3 shows that for $0.4 < d < 4 \, h^{-1}\text{Mpc}$ isolated galaxy pairs at $z = 0$ are nearly comoving with the Hubble flow. Thus, they move with the cosmological expansion and, for this cosmology and epoch, only need a small RSD correction. Using different redshift slices as a way to test different cosmological expansion rates, in the lower panel of Fig. 3.6.6, we identify a second regime for $1 < d < 4 \, h^{-1}\text{Mpc}$ where the peculiar velocity $v_{12}(z)$ depends only slightly on the cosmological expansion. This suggests the intriguing possibility of isolated pairs behaving in the same way on those scales regardless of the assumed cosmology. In particular, the lower panel of Fig. 3.6.6 shows that the variation in RSD model is less than the variation in expansion rates. Thus we conclude that there is cosmological signal to be extracted here.

We refer to the intersection of these regimes, where we have almost comoving pairs with small redshift evolution, as a cosmological regime, since we might be able to use these pairs as cosmological tracers. Measuring galaxy pairs in the cosmological regime would still induce a systematic error due to the fact that peculiar velocities are non-zero. As the correction is of the 10% level – see lower panel of Fig. 3.6.3 – and we might suppose to be able to model this at the same level, we would have a 1% systematic correction to contend with. This claim is little more than a speculation at this stage; in order to falsify or confirm it, one needs to model isolated pairs in detail and to analyze N-body simulations with different underlying cosmologies.

Pairs isolated within a radius of $4 \, h^{-1}\text{Mpc}$ are rare objects – see Fig. 3.6.7 – and this might result in significant statistical error when dealing with observations. In Sections 3.6.4.3 and 3.6.4.4 we found that one can drastically increase the number of pairs while keeping the RSD correction small by either reducing the isolation radius to $2 \, h^{-1}\text{Mpc}$ or allowing up to 10 galaxies to be neighbours of the pair.

### 3.6.5 Varying galaxy properties

The main result of the previous section, illustrated in Fig. 3.6.3 and Fig. 3.6.6, is that there is a regime where isolated galaxy pairs may be used as tracers of expansion with correction for RSD that is weaker than the signal to be measured. Such a finding relies on the
ability of measuring the redshift of galaxies regardless of their mass or luminosity. This is clearly not the case when dealing with actual galaxy surveys, whose flux sensitivity is limited. To model such a selection bias, we need to investigate ways of selecting galaxies from simulations which mimic the selection process of surveys. In this section we address this issue by forming subsamples where galaxies are selected according to subhalo mass, stellar mass and magnitude. We also study the redshift dependence of our results by analysing 4 different redshifts: $z = 0, 0.5085, 0.989, 1.504$.

### Method

We select galaxy subsamples from the semi-analytic models by applying cuts on galaxy properties. We then study the dynamics of each subsample by applying the same analyses of Sections 3.6.3 and 3.6.4. Initially we look at the number $n_p$ of dark matter particles of the subhalo the galaxy is in. For reference, this is the $n_p$ field of the Guo2010 database in the Millennium simulation servers. We consider $n_p = 100, 500, 1000, 5000$ corresponding to masses $m = 8.6 \times 10^{10}, 4.3 \times 10^{11}, 8.6 \times 10^{11}, 4.3 \times 10^{12} \, h^{-1} M_\odot$. Note...
Figure 3.6.7: Number of galaxy pairs for different mass cuts as a function of separation at redshift $z = 0.989$. In the right panel we omitted the $m > 8.6 \times 10^{11} \, h^{-1} M_{\odot}$ curve for the sake of readability.

that, where no cut is made, the number of particles in each subhalo is always $n_p > 20$, corresponding to $1.72 \times 10^{10} \, h^{-1} M_{\odot}$, since this is the threshold that defines a bound subhalo according to [160].

We choose the dark matter mass as our main cut because it is directly related to the pairwise velocity dynamics, which is the main subject of this paper. In order to make a more direct link with observations, we also study the pairwise velocity statistics when varying the rest-frame $r$-band magnitude and stellar mass of galaxies. More precisely, this is the rest-frame total absolute magnitude in the SDSS $r$-band, corresponding to the $r_{\text{mag}}$ field in the Guo2010 database of the Garching mirror and to the $r_{\text{SDSS}}$ field in the Font2008 database of the Durham mirror.

The limits on $r$-band magnitude (hereafter $r$-mag) and stellar mass (hereafter $m_*$) are chosen such that, for a given $n_p$ cut, the corresponding $r$-mag and $m_*$ cuts yield the same number of surviving galaxies. Table 3.6.1 reports the values of the limits used, together with the resulting fraction of surviving galaxies at each redshift. We apply these cuts to the data sets before running the pair-finder algorithms. Thus, a pair that is isolated within its subsample may not be isolated when considering the full catalog, i.e. our isolation criterion is sample dependent. This implementation is in line with an analysis of an actual galaxy survey, limited by these cuts.

In Fig. 3.6.7 we show how many galaxy pairs we find after imposing the cuts given in Table 3.6.1. The number of unconstrained pairs (left panel) decreases monotonically with increasing mass cut. Note that the drop-off in the number of pairs at large separations is due to the size of the individual boxes we consider and has no physical meaning. For
isolated pairs (right panel), the number density increases as we increase the mass cut, as a higher mass cut results in a sparser distribution of galaxies where it is easier to find isolated pairs. Only for our most stringent mass cut, that is for \( m > 4.3 \times 10^{12} \, \text{h}^{-1} \, \text{M}_\odot \), do we see a slight decrease in the density of isolated pairs due to the small number of high mass galaxies.

\[
\begin{align*}
\text{No mass cut} & \quad \text{Mass} > 8.6 \times 10^{10} \, \text{Msun}/h \\
\text{Mass} > 4.3 \times 10^{11} \, \text{Msun}/h & \\
\text{Mass} > 8.6 \times 10^{11} \, \text{Msun}/h & \\
\text{Mass} > 4.3 \times 10^{12} \, \text{Msun}/h & \\
\end{align*}
\]

\( (a) \text{ Redshift } z=0 \)

\( (b) \text{ Redshift } z=0.5085 \)

\( (c) \text{ Redshift } z=0.989 \)

\( (d) \text{ Redshift } z=1.504 \)

**Figure 3.6.8:** Variation of the average pairwise velocity of all galaxy pairs with redshift for different mass cuts (number of dark matter particles) as a function of separation.
3.6.5.2 Results

3.6.5.3 All galaxy pairs

In Fig. 3.6.8 we present the average pairwise velocity $v_{12}$ for non-isolated pairs above different subhalo mass thresholds. The four panels show the same selection procedure at different redshifts. The mass cuts range from $1.72 \times 10^{10} h^{-1} M_\odot$ up to $4.3 \times 10^{12} h^{-1} M_\odot$ as tabulated in Table 3.6.1. Note that the lowest mass cut corresponds to the smallest subhalo in the [160] semianalytic model, consisting of 20 dark matter particles.

The imposition of a mass cut has a significant impact on non-linear scales. Independent of redshift, Fig. 3.6.8 shows that massive galaxy pairs experience an infall regime for separations smaller than $d \lesssim 3 h^{-1}$ Mpc. While in such a regime, peculiar velocity increases with mass, with the most massive galaxies ranging from $v_{12} \simeq 330 \text{ km/s}$ at $z = 0$ and $v_{12} \simeq 600 \text{ km/s}$ at $z = 1.504$. Lower mass galaxies, on the other hand, seem to follow the static solution up to higher separations, especially at low redshift. Our interpretation for such behaviour is that galaxies in high mass pairs are more affected at small separations by their mutual attraction than the underlying density field.

For separations $d > 10 h^{-1}$ Mpc, the velocity curves at each redshift seem to converge to a common asymptote. In Sec. 3.6.3, we have shown that this limit is correctly predicted by linear theory – see the agreement between the green curve and $v_{12}$ in Fig. 3.6.2. This means that, even though the non-linear dynamics of the different mass limit pairs differs, their behaviour at large separations seems to be predicted by the same linear theory.

A remarkable feature of Fig. 3.6.8 is the different redshift dependence of the various $v_{12}$ curves. The pairwise velocity of the heaviest galaxies (cyan triangles) greatly increases with redshift, while in the uncut case (red circles) it decreases. The intermediate curves seem to experience smaller variations.

3.6.5.4 Isolated pairs

Having analysed the dynamics of non-isolated pairs with varying subhalo mass, we now do the same for pairs isolated within a $4 h^{-1}$ Mpc radius. In Fig. 3.6.9 we show the average pairwise velocity $v_{12}$ for different mass cuts, with the redshift varying from panel to panel. This is the same setup as in Fig. 3.6.8; note, however, that here we only plot separations up to $4 h^{-1}$ Mpc.

All curves in Fig. 3.6.9, regardless of redshift, present the same features found in the
Figure 3.6.9: Variation of the average pairwise velocity of isolated galaxy pairs with redshift for different mass cuts (number of dark matter particles) as a function of separation. The isolation radius is taken to be \(4 \, h^{-1} \text{Mpc} \) for each member of the pair.

uncut sample shown in Fig. 3.6.3 and explained in Sec. 3.6.4. Namely, we see a virialised region on the smallest scales, an infalling regime on intermediate scales, and a roughly logarithmic growth due to the void effect on the largest separations analysed. The main difference from the uncut case is the separations at which these different regimes hold. Most importantly, we can see that the logarithmic growth of \(v_{12}\) begins at larger separations for higher masses. This is intuitive, since we expect the mutual attraction to be stronger in heavier pairs, thus overcoming the void effect even when the galaxies are closer to the edge of the void.

As a result of this stronger mutual attraction, the peculiar velocity contribution increases
as we consider heavier pairs. This means that when it comes to isolated galaxy pairs, low-mass pairs trace the cosmological expansion better than high-mass ones. More quantitatively, the scale $d_0$ where the peculiar velocity vanishes is reached at larger separations for massive pairs. At $z = 0.989$, $d_0$ ranges from 0.8 $h^{-1}$Mpc in the uncut case to almost 4 $h^{-1}$Mpc for pairs with $m > 4.3 \times 10^{11} \, h^{-1} M_{\odot}$. For higher masses, $v_{12}$ does not even cross the zero line.

To illustrate the redshift dependence of the peculiar velocity in more detail, in Fig. 3.6.11 we plot $v_{12}$ for a given mass-cut at four different redshifts, with the mass-cuts varying across the panels. Increasing the mass-cut makes the redshift evolution of $v_{12}$ more evident. As a result, for $m > 4.3 \times 10^{11} \, h^{-1} M_{\odot}$, we cannot identify a cosmological regime where the pairs are comoving and have a redshift independent peculiar velocity, where “independence” here means that the evolution is significantly less than the change in expansion rate.

A closer look at Fig. 3.6.9 shows a pattern in the different mass cuts. We notice that, as we increase the mass cut, the absolute value of the pairwise velocity increases and the minimum shifts to the right. This suggests that applying a mass dependent scaling to both separation and velocity may stack the curves. This is shown in Fig. 3.6.10, where we have
Figure 3.6.11: Variation of the average pairwise velocity of isolated galaxy pairs with redshift for different mass cuts (number of dark matter particles) as a function of separation. The isolation radius is taken to be 4 \( h^{-1} \) Mpc for each member of the pair. Note that panel Fig. 3.6.11a is equivalent to the upper panel of Fig. 3.6.6.

have taken as an example the plot at redshift \( z = 0.989 \) (Fig. 3.6.9c) and scaled both the x and y axis by a factor \((M_{\text{ref}}/M)^{1/3}\) where \(M_{\text{ref}}\) is the minimum mass of a subhalo: \(M_{\text{ref}} = 1.72 \times 10^{10} M_{\odot}/h\). The physical motivation for this scaling is to have the same orbital period for all curves in the Keplerian regime \(i.e.\) on small scales. Indeed we see on this plot that such scaling collapses the curves especially in the infalling and virialised regions.
We now make a more direct link with observations and study the dependence of $v_{12}$ on $r$-band absolute magnitude and stellar mass.

In the left panel of Fig. 3.6.12, we show the average pairwise velocity $v_{12}$ of all galaxy pairs at $z = 0.989$ for the $r$-mag cuts given in Table 3.6.1. Although the curves retain their qualitative shape, there are two major differences with respect to the mass-cut sample in Fig. 3.6.8c. Firstly, the $r$-mag selected pairs have smaller average velocities. Secondly, the velocity minima are all approximately aligned at the same scale of $r_{\text{min}} \sim 2 \, h^{-1}\text{Mpc}$, while for the subhalo mass cuts the different velocity curves have their minima at different

3.6.5 Magnitude & stellar mass

Figure 3.6.12: Average pairwise velocity of galaxy pairs at $z = 0.989$ varying $r$-band magnitude.

Figure 3.6.13: Average pairwise velocity of galaxy pairs at $z = 0.989$ varying stellar mass.
separations. These two differences are also seen when we apply the cuts in stellar mass (Fig. 3.6.13a).

The velocity differences can be explained by the fact that, although the subsamples chosen based on limits in $r$-band magnitude and stellar mass preserve the number density of galaxies selected, these are not the same galaxies as the ones selected by the subhalo mass cuts. In particular, most massive galaxies do not necessarily coincide with the most luminous ones. In general, dark matter haloes trace the velocity of galaxies more directly than stellar mass or $r$-band magnitude. Cuts based on stellar mass or luminosity add an additional dispersion, affecting the position of the minima with respect to subhalo mass cuts.

The right panels of Fig. 3.6.12 and Fig. 3.6.13 show the average pairwise velocity $v_{12}$ for pairs with an isolation radius of $4 \, h^{-1}\text{Mpc}$ for cuts in $r$-band magnitude and stellar mass respectively. These plots should be compared with the corresponding cuts in subhalo mass at redshift $z = 0.989$ (Fig. 3.6.9c). Even though we again appreciate that the pairwise velocities in the $r$-mag and stellar mass cut plots are smaller, the general dynamics shown on the plots are the same. It is worth noting that the almost comoving regime mentioned in Sec. 3.6.4.6 for each curve remains unchanged both for the $r$-band magnitude and the stellar mass cuts. It is clear that the effects of galaxy selection (be it subhalo mass, stellar mass or $r$-band magnitude) play an important role in the behaviour of pairwise velocities for isolated galaxy pairs.
The results presented in the previous sections were based on the semi-analytic model of \[160\]. To check the robustness of these results, we also compute the average peculiar velocities \(v_{12}\) for the semi-analytic model in \[161\]. This catalog is an improvement over the one presented in \[164\] to better match the colours of satellite galaxies observed in the SDSS sample. In order to do this, the main modification introduced in \[161\] is the stripping of hot gaseous haloes of satellite galaxies into the GALFORM semi-analytic model for galaxy formation, while \[160\] concentrates more on the independence of satellite galaxies from the FOF group. Both semi-analytic models have similar galaxy luminosity functions that fit the data well.

In Fig. 3.6.14 we show the average pairwise velocity \(v_{12}\) as a function of separation for all the galaxy pairs (left panel) and for isolated pairs with an isolation radius of \(4 \, h^{-1}\) Mpc (right panel). Each curve corresponds to one of the \(r\)-mag cuts in Table 3.6.1, and should be compared with the matching curve for the \[160\] catalog in Fig. 3.6.12. Note that we omitted to plot \(v_{12}\) for our most stringent cut of \(r\)-mag \(< -22.86\) because of poor statistics. In general, we found that \[161\] has significantly less bright galaxies with \(r\)-mag \(< -22.17\) than \[160\], as can be seen by the large error bars in Fig. 3.6.14.

A direct comparison between Figures 3.6.14 and 3.6.12 shows that galaxy pairs have very similar dynamics regardless of the semi-analytic model used. Not only do we see almost the same \(v_{12}\) range, but also the almost comoving regime introduced in Sec. 3.6.4.6 is found approximately in the same range. Such findings suggest that the parameters of the semi-analytic model used do not significantly affect the average pairwise velocities of galaxies.

### 3.7 The Homogeneity Scale of the Universe

One of the main assumptions of the ΛCDM theory of cosmology is that the Universe is homogeneous and isotropic on large scales, and hence can be described by the FLRW metric. ‘Homogeneous’ means that its statistical properties (such as density) are translationally invariant; ‘isotropic’ means it should be rotationally invariant. The Universe clearly deviates from this on small scales, where galaxies are clustered, but on large enough scales (\(\gtrsim 100 h^{-1}\) Mpc in ΛCDM), the distribution of matter is assumed to be ‘statistically homogeneous’ *i. e.* the small-scale inhomogeneities can be considered as perturbations, which have a statistical distribution that is independent of position. However, this is merely an assumption, and it is important for it to be accurately verified by observation.
Over the last decade there has been a debate in the literature as to whether the Universe really is homogeneous, or whether it has a fractal-like structure extending to large scales. It is important to resolve this contention if we are to be justified in assuming the FLRW metric.

In fact, although ΛCDM is based on the assumption of large-scale homogeneity and an FLRW metric, inflation predicts that the primordial density power spectrum is close to scale-invariant. This means that a certain level of density fluctuations is predicted on all scales. In the standard model, the scalar index $n_s$, which quantifies the scale-dependence of the primordial power spectrum, is close to 0.96 [16], while a scale-invariant power spectrum has $n_s = 1$. In this case, these density fluctuations induce fluctuations in the metric, $\delta \Phi$, which are virtually independent of scale, and are of the order of $\delta \Phi / c^2 \sim 10^{-5}$ [165]. Since these perturbations are small, the FLRW metric is still valid, but it means that we expect the Universe to have a gradual approach to large-scale homogeneity rather than a sudden transition, since our observations are performed on the light cone.

Homogeneity is required by several important statistical probes of cosmology, such as the galaxy power spectrum and $n$-point correlation functions, in order for them to be meaningful. Applying these to a galaxy sample below the scale of homogeneity would be problematic, since if a distribution has no transition to homogeneity, it does not have a defined mean density, which is required to calculate and interpret these statistics. It is also not possible to model its cosmic variance, so the error in the measurements would be ill-defined, making it impossible to relate these statistics to a theoretical model. It is therefore important to quantify the scale on which the Universe becomes close enough to homogeneous to justify their use.

Large-scale homogeneity is already well supported by a number of different observations. In particular, the high degree of isotropy of the CMB gives very strong support for large-scale homogeneity in the early Universe, at redshift $z \sim 1100$. The isotropy of the CMB also indicates the Universe has remained homogeneous, since there are no significant ISW effects distorting our view of the isotropic CMB [166].

However, these measurements of high-redshift isotropy do not necessarily imply homogeneity of the present Universe. If around every point the Universe is isotropic, then this implies the Universe is homogeneous; so if we accept the Copernican principle that our location is non-special, then the observed isotropy should imply homogeneity. Note that the Copernican principle is not incompatible with an inhomogeneous Universe. It assumes only that our location is non-special, not that every location is the same. However most of these measurements (except the ISW effect) only tell us about the high-redshift
Universe. We know that it has evolved to a clustered distribution since then, and it is possible that it could also have become anisotropic. It is also possible for the matter distribution to be homogeneous while the galaxy distribution is not, since the galaxy distribution is biased relative to the matter field – although since galaxy bias is known to be linear on large scales \([167]\), this seems unlikely.

A way to look at homogeneity is by using galaxy surveys. In order to do this, it is useful to employ a fractal analysis. Fractal dimensions can be used to quantify clustering; they quantify the scaling of different moments of galaxy counts in spheres, which in turn are related to the \(n\)-point correlation functions. The most commonly used is the correlation dimension \(D_2(r)\), which quantifies the scaling of the 2-point correlation function, and is based on the counts-in-spheres, which scale as \(N(< r) \sim r^{D_2}\). One can also consider the more general dimensions \(D_q\), where \(q\) are different moments of the counts-in-spheres \([168]\).

This idea has lately been explored by the WiggleZ Dark Energy Survey \([169]\), which has enough volume to conduct such a homogeneity analysis \([170]\). They use the correlation dimension mentioned above \(D_2(r)\) to estimate the homogeneity scale and find it to be between \(60 \lesssim R_H \lesssim 90\) Mpc for a redshift range between \(0.2 < z < 0.8\). Defining a scale of homogeneity is not trivial, since the approach to homogeneity is gradual. Some authors consider this scale to be reached when the data become consistent with homogeneity within \(1\)\(\sigma\) (see, for example \([168]\)). This method, however, presents several disadvantages: it depends on the size of the errors and hence, on the survey area, and also on the bin spacing. A larger survey will have smaller error bars, measuring a larger homogeneity scale. In \([170]\) they use a different technique, by which they fit a polynomial curve through their data points and find where this curve intercepts chosen values close to homogeneity (usually taken to be within \(1\)\%).

### 3.7.1 Defining the fractal dimension

Let us assume we have a distribution of points in 3 dimensions. We can define the quantity \(C_2(r)\) as

\[
C_2(r) = \frac{1}{N} \sum_{n=0}^{N} nP(n; r, N),
\]

where \(N\) is the total number of particles in our distribution and \(P(n; r, N)\) is the normalised probability of finding \(n\) out of \(N\) points as neighbours within a sphere of radius \(r\) centred around each of the points of the distribution. One can define the quantity \(D_2(r)\),
known as the fractal or correlation dimension, as
\[ D_2(r) \equiv \frac{d \log C_2(r)}{d \log r}. \] (3.39)

For a three dimensional homogeneous distribution of points, we should have that \( V(r) \propto r^{D_2} \) with \( D_2 = 3 \). However, in a real distribution of particles that experience a gravitational force between them, there exists clustering on small scales and hence one should expect the quantity \( D_2 \) to approach 3 only on large-scales.

Eq. (3.39) can be expressed in terms of the 3 dimensional 2-point correlation function in real space \( \xi(r) \) as
\[ D_2(r) = 3 \frac{1 + \bar{\xi}(r)}{1 + \bar{\xi}(r)}, \] (3.40)
where \( \bar{\xi}(r) \) is the volume averaged three dimensional 2-point correlation function defined as
\[ \bar{\xi}(r) \equiv \frac{3}{r^3} \int_0^r r' \xi(r') \, dr'. \] (3.41)

The relationship in Eq. (3.40) is easy to derive considering the definition of the 2-point correlation function: the excess probability of finding two objects separated by a given distance \( r \).

To apply this to the data, one needs to define an estimator for \( C_2(r) \). In the case of [170], the estimator used was built to take into account the edge effects one might expect from a galaxy survey. In order to do this, they take spheres around all the particles within the survey and weigh their particle content by the number of random particles one would expect inside those same spheres. Thus, their estimator for \( C_2(r) \) is defined as
\[ \mathcal{N}(< r) \equiv \frac{1}{N_p} \sum_{i=1}^{N_p} \frac{n_{iDD}^D(r)}{N_R} \sum_{j=1}^{N_R} \rho_j n_{ij}^{DR}(r), \] (3.42)
where \( n_{iDD}^D(r) \) is the number of data points inside the sphere of radius \( r \) centred on the \( i \)-th data object, \( N_R \) is the number of random catalogues used, \( \rho_j \) is the ratio of data to random objects in the \( j \)-th catalogue and \( n_{ij}^{DR}(r) \) is the number of object from the \( j \)-th random catalogue inside a sphere of radius \( r \) centred on the \( i \)-th data particle. Once \( \mathcal{N}(< r) \)
is estimated this way, the fractal dimension is calculated as

$$D_2(r) = 3 + \frac{d \log N(< r)}{d \log r}.$$  (3.43)

The main drawback of this method is that this weighing intrinsically assumes homogeneity (by assuming a mean number density). Also, it is computationally costly (since the same operation must be performed on the data catalogue and on a number of random catalogues).

In the case of $[168]$, the estimator for $C_2(r)$ is calculated by just taking into account the spheres that fit within the survey volume. Thus, the estimator in this case is defined as

$$C_2(r) \equiv \frac{1}{N_s(r)} \sum_{i=1}^{N_p} n_i(r),$$  (3.44)

where $N_s(r)$ is the number of spheres of radius $r$ fitting inside the survey mask, $N_p$ is the number of objects in the survey and $n_i(r)$ is the number of objects inside the sphere of radius $r$ centred on the $i$-th object. Once we have $C_2(r)$, the fractal dimension is estimated by taking the logarithmic derivative as in Eq. (3.39). The drawback of this estimator is that significant parts of the survey will be unused and we will not be able to probe large scales (this is not a problem with the estimator used by $[170]$).

These estimators work nicely when we have a galaxy survey with sufficient volume and a good resolution i.e. a wide spectroscopic survey like WiggleZ. However, if one has a wide photometric survey such as DES $[171]$, one cannot use the fractal dimension as described in Eq. (3.39) since we do not have precise information on the position of objects on the radial direction. In order to use the fractal dimension as a study of homogeneity for such surveys, it is convenient to define the correlation dimension on the sphere, $S^2$. Two main differences arise here: on the one hand $S^2$ is a 2-dimensional manifold, and on the other hand, the volume of a spherical-cap of radius $\theta$ (in radians) grows like $V(\theta) \propto (1 - \cos \theta)$. A logarithmic derivative with respect to $\theta$ will not capture the approach to homogeneity in a simple manner independent of the angular radius $\theta$. For this reason it is useful to define the fractal dimension on $S^2$ as

$$H(\theta) \equiv \frac{d \log N(< \theta)}{d \log V(\theta)},$$  (3.45)

where $N(< \theta)$ is the number of particles within a sphere of angular radius $\theta$. The quantity
$H(\theta)$ should approach 1 as we reach homogeneity.

Again, this definition of the fractal dimension can also be expressed as a function of the correlation function. In this case, it is related to the angular correlation function as

$$H(\theta) = \frac{1 + \bar{w}(\theta)}{1 + \bar{\bar{w}}(\theta)}, \tag{3.46}$$

where $\bar{w}(\theta)$ is the volume averaged angular correlation function defined as

$$\bar{w}(\theta) \equiv \frac{1}{1 - \cos \theta} \int_{\cos \theta}^{1} w(\theta) \, \mathrm{d} \cos \theta. \tag{3.47}$$

For a redshift bin of width $z_{0} < z < z_{f}$ the angular correlation function is given by

$$w(\theta) = \int_{z_{0}}^{z_{f}} dz_{1} \varphi(z_{1}) \int_{z_{0}}^{z_{f}} dz_{2} \varphi(z_{2}) \xi(r(z_{1}, z_{2}, \cos \theta), \mu(z_{1}, z_{2}, \cos \theta)), \tag{3.48}$$

where the relative radial separation and transverse angle are related to $z_{1}$, $z_{2}$ and $\cos \theta$ via

$$r = \left[ \chi^{2}(z_{1}) + \chi^{2}(z_{2}) - 2 \chi(z_{1})\chi(z_{2}) \cos \theta \right]^{1/2},$$

$$\mu = \frac{|\chi^{2}(z_{1}) - \chi^{2}(z_{2})|}{[(\chi^{2}(z_{1}) + \chi^{2}(z_{2}))^{2} - 4\chi^{2}(z_{1})\chi^{2}(z_{2}) \cos^{2} \theta]^{1/2}}. \tag{3.49}$$

### 3.7.2 Results

Here we present the results obtained by [170] for the WiggleZ survey for the correlation dimension in a three dimensional survey and some theoretical plots for the fractal dimension on the sphere, a work in preparation which we plan to apply to future DES data.

#### 3.7.2.1 The Fractal Dimension for the WiggleZ Data

In Fig. 3.7.2 we see the correlation dimension $D_{2}$ for the four redshift bins used in the WiggleZ analysis. The black dots represent the WiggleZ combined data while the red and blue lines represent a polynomial fit and the prediction from $\Lambda$CDM with a best-fitting bias parameter, respectively.

We see in Fig. 3.7.2 that, for each redshift bin, the correlation or fractal dimension starts below 3 on small scales due to the clustering of structure and then grows until it asymp-
Figure 3.7.1: Illustration of the method of defining the homogeneity scale, $R_H$, for the $D_2(r)$ measurement. A model-independent polynomial (red curve) is fit to the data (black data points). Then, the point where this intercepts a chosen value close to homogeneity, e.g., 1 per cent from homogeneity, $D_2 = 2.97$ (dotted grey line) is found. This gives $R_H$. The uncertainty in $R_H$ is found from the root mean square variance of 100 lognormal realisations (pink curves). Figure taken from [170].

In [170] they consider that homogeneity is reached when the polynomial fit crosses the line corresponding with $2.97$ (i.e., 1% from homogeneity). To get the error on the scale of homogeneity obtained this way, they compare with 100 lognormal catalogues (see Fig. 3.7.1).

An analysis of homogeneity can also help us distinguish the clustering of matter on large scales with fractal models. In order to do this, in [170] they generate fractal distributions using the $\beta$-model, a simple self-similar cascading model [172]. This method starts with a cube of side $L_0$ and splits it up into $M$ smaller cubes of side $L_0/n$ (they take $n = 2$, so $M = 8$). Each subcube is then assigned a probability $p$ of surviving to the next iteration. This is repeated for a certain number of iterations $k$, and the resultant set of survived points is taken as the final distribution. In the limit of an infinite number of iterations, this produces a monofractal with a correlation dimension given by

$$D_2 = \lim_{k \to \infty} \frac{\log(pM)^k}{\log n^k} = \frac{\log pM}{\log n}. \quad (3.50)$$

To use these models to test their homogeneity analysis they first choose a range of $D_2$ values and for each generate 100 fractal distributions with boxsize $(L_0 \, h^{-1}\text{Gpc})^3$, where $L_0$ is the length of the longest side of the WiggleZ 15-hr $0.5 < z < 0.7$ selection func-
Figure 3.7.2: The $D_2(r)$ measurements for the combined WiggleZ data in each of the four redshift slices are shown as black error bars. A $\Lambda$CDM model with best-fitting bias $b^2$ is shown in blue. A 5th-degree polynomial fit to the data is shown in red. The red errorbar and label show the homogeneity scale $R_H$ measured by the intercept of the polynomial fit with 2.97 (1 per cent away from homogeneity), with the error given by lognormal realisations. This scale is consistent with the $\Lambda$CDM intercept with 2.97, labelled in blue. Figure taken from [170].

...
Figure 3.7.3: Fractal model comparisons of the measured correlation dimension from the WiggleZ 15-hr region, 0.5 < z < 0.7 redshift slice. The WiggleZ data (black error bars) and ΛCDM model (blue curve) are compared with several different β-models with different fractal dimension (D₂ = 2.7, 2.8, 2.9 and 2.95), which have been sampled with the 15-hr selection function and analysed in the same way as the WiggleZ data (coloured error bars). The uncertainties shown are the error-in-the-mean of 100 fractal realisations. The input fractal dimensions are shown as dotted lines with corresponding colours. The best-fit D₂(r) values fit over the range r = [80, 300] h⁻¹Mpc are shown as solid lines. Figure taken from [170].

variance for a box of a particular volume. To quantify how well they can exclude fractal models with the WiggleZ data, they fit a line of constant D₂ to each set of fractal data, over the range [80, 300] h⁻¹Mpc (shown in Fig. 3.7.3). This gives the best-fit D₂ value one would expect to measure for each fractal distribution over this range, taking into account bias from the selection function. They then find the formal probability of these values fitting the WiggleZ data. Doing this, they discover that they can exclude a fractal dimension of D₂(r) = [2.9, 2.95, 2.97] at the [19, 6, 4]-σ level. In other words, they can exclude fractal distributions following β-models with dimension D₂(r) < 2.97 at over 99.99% confidence on scales from 80 to 300 h⁻¹Mpc.

There are several effects that can affect the homogeneity scale. One if these is the galaxy bias, since it is degenerate with the amplitude of the correlation function, σ₈(z), as the correlation function depends on a combination of these, ξ(r, z) ∝ b²σ₈(z)². In order to quantify this effect, in [170] they study how the homogeneity scale R_H changes as a function of the combination b²σ₈(z)². This is shown in Fig. 3.7.4. They see that the model R_H - b²σ₈(z)² curves are monotonically increasing. Since it is expected that σ₈(z) in ΛCDM
grows over time due to the growth of structure, it is not surprising to expect the homogeneity scale to increase over time for galaxies with fixed bias.

![Figure 3.7.4: Homogeneity scale $R_h$ as a function of $b^2\sigma_8(z)^2$, as predicted by the ΛCDM model, for different thresholds approaching homogeneity (10, 1 and 0.1 per cent from homogeneity, coloured curves from bottom to top), for $D_2(r)$. The corresponding WiggleZ results are shown as error bars of corresponding colour (the errors are found using lognormal realisations). The $b^2\sigma_8(z)^2$ values of the data increase with redshift slice (so the data points from left to right are from low to high redshift). Not all redshift slices have measurements at 0.1 per cent (blue) since the data do not reach this value in those slices. Figure taken from [170].](image)

### 3.7.2.2 The fractal dimension on the sphere

Here we present preliminary results of the theoretical fractal dimension on the sphere as described by Eq. (3.46). In order to do this, we take a flat ΛCDM matter power spectrum with the Planck best-fit parameters [16]. Since we want to apply this technique to DES, we take a selection function $n(z)$ compatible with the mock catalogues tailored for such a survey (see Fig. 3.7.5). When taking into account redshift space distortions, we use the Kaiser formula [173] and hence expand the matter power spectrum up to the fourth multipole.

In Fig. 3.7.6 we plot the angular fractal dimension for the same redshift bins as in the WiggleZ analysis (see Fig. 3.7.2). We can see that, also in the angular case, the fractal dimension starts below homogeneity (1 in this case, recall Eq. (3.46)) for small scales (angles) and then approaches homogeneity as we go to larger angles. We can also see from
Figure 3.7.5: The selection function $n(z)$ as a function of redshift $z$ for a DES-like survey.

<table>
<thead>
<tr>
<th>$z$ bin</th>
<th>$\theta_H$ (deg)</th>
<th>equivalent $R_H$ (Mpc)</th>
<th>$R_H$ from the WiggleZ survey (Mpc)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.1 &lt; z &lt; 0.3$</td>
<td>0.3</td>
<td>4.5</td>
<td>100</td>
</tr>
<tr>
<td>$0.3 &lt; z &lt; 0.5$</td>
<td>0.45</td>
<td>11.3</td>
<td>100</td>
</tr>
<tr>
<td>$0.5 &lt; z &lt; 0.7$</td>
<td>0.7</td>
<td>27.5</td>
<td>115</td>
</tr>
<tr>
<td>$0.7 &lt; z &lt; 0.9$</td>
<td>1.7</td>
<td>85.3</td>
<td>106</td>
</tr>
</tbody>
</table>

Table 3.7.1: Approximate angular homogeneity scale and the corresponding separation in Mpc for the different redshift bins analysed in Fig. 3.7.6. On the last column we see the equivalent homogeneity scale from the WiggleZ survey (see Fig. 3.7.2).

Fig. 3.7.6 that, as we increase in redshift, homogeneity is achieved on smaller scales. This seems logical since, at earlier times, there was less clustering in the Universe, meaning it was more homogeneous. We have also plotted as a dotted line the 1% deviation from homogeneity to compare with Fig. 3.7.2. We can do a rough estimate of what the homogeneity scale is in this case. In Table 3.7.1 we can see the angle at which $H(\theta)$ reaches the 0.99 mark, $\theta_H$, and the separation it corresponds to in Mpc. As we can see from Table 3.7.1, the homogeneity scale derived from the angular fractal dimension $H(\theta)$, $\theta_H$, is very different from the homogeneity scale found using the estimator described in Sec. 3.7.1 for the WiggleZ survey. This is mainly due to projection effects. When we calculate the
fractal dimension on the sphere, we take all the galaxies inside the redshift bin and project them on a spherical surface, meaning that our result is a much more homogeneous distribution and, hence, the scale of homogeneity is reached on smaller scales. It should also be noted that, in the case of the angular fractal dimension, the 0.99 line does not necessarily correspond to the 2.97 line in the WiggleZ analysis, since the methods to determine homogeneity are very different. If we take smaller, quasi-infinite redshift bins, we can see how the angular fractal dimension depends on redshift bin width. In Fig. 3.7.7 we see that, just by decreasing the bin width from 0.2 to 0.1, the homogeneity scale goes up by about a factor of 10, showing how the projection effect homogenises the distribution.

We can also explore how the angular fractal dimension changes with varying bias or error in photometric redshift. In Fig. 3.7.8 we plot again the angular fractal dimension $H(\theta)$ for a redshift bin $0.5 < z < 0.6$ for different biases and photo-z errors. Upon inspection of Fig. 3.7.8 we see that, if we increase the bias, the homogeneity scale is reached only on larger scales. Physically, this makes sense since increasing the bias means that the galaxies cluster more than the underlying dark matter distribution. This means that the galaxy distribution is only homogeneous on larger scales. The opposite effect happens when one in-
Figure 3.7.7: The angular homogeneity scale $H(\theta)$ as a function of the angle $\theta$ for different redshift bins using redshift space distortions and a bias of $b = 1$ with a redshift bin width of $\Delta z = 0.1$. The line below homogeneity by $1\%$ is also shown (dotted line).

(a) The angular fractal dimension $H(\theta)$ as a function of the angle $\theta$ for different values of the galaxy bias.

(b) The angular fractal dimension $H(\theta)$ as a function of the angle $\theta$ for different values of the photometric error.

Figure 3.7.8: The angular homogeneity scale $H(\theta)$ as a function of the angle $\theta$ for different values of the galaxy bias and the photometric error. The line below homogeneity by $1\%$ is also shown (dotted line).

creases the photometric redshift uncertainty. This is also expected since a larger $\sigma_z$ means that it is more likely that galaxies from other redshift bins are pulled into the bin under
consideration. Thus, when we project on the sphere we will have an intrinsically more dense and homogeneous sample.

It is important to note that this kind of analysis of homogeneity is a consistency check and one should not expect to be able to use it to constrain cosmological parameters. This is so since, as also stated in [170], when applying this technique to data, one needs to convert the measured redshifts into distances. This means that, if one assumes an FLRW metric to do so (which is intrinsically homogeneous), one should expect to obtain a correlation dimension that approaches homogeneity. It should be noted that, in the case of the angular fractal dimension, this problem does not appear, since angles are measured directly from the data, without the need to assume a model or a metric.

With this analysis one can exclude fractal models as a description of the galaxy distribution in our universe. In the future, we would like to explore different fractal models from the $\beta$-model explored in the WiggleZ analysis. Another interesting question we wish to address in our future research is what would happen if, for a three dimensional homogeneous sample (a $\Lambda$CDM N-body simulation, for example), one were to convert redshifts into distances employing an inhomogeneous metric, such as the LTB metric. Perhaps this will give us further insight into how the fractal dimension depends on such model-dependent assumptions.
“We are all interested in the future, for that is where you and I are going to spend the rest of our lives.”

– Woody Allen

4

Future observations

With current and future surveys, the flow of data in cosmology has become impressively huge (for example, the billion galaxies that the future Euclid space mission plans to detect [174]). It is thus necessary to use and develop statistical tools to analyse this amount of data. It is not only vital to develop methods to analyse current data, but also those that produce forecasts for future surveys. Forecasts can tell us how well a survey will be able to measure a set of parameters given its characteristics. This is useful when planning a future experiment, since some probes will prefer a larger area, while others a larger depth. Some probes will need to measure the shapes of galaxies very accurately (for example, for weak lensing), while others will just need the positions of galaxies in the sky (for example, the BAO scale). For these reasons, it is important to perform forecasts having in mind what one wants to measure with each survey. In Sec. 4.1 we will discuss the different statistical methods used in cosmology, putting particular emphasis on the Fisher matrix analysis that we used in [1]. In Sec. 4.2 we will give details of the future Euclid survey and its specifications and objectives. Finally, in Sec. 4.3, we will see different forecasts for Euclid.
4.1 Statistical methods in cosmology

4.1.1 The likelihood function

Let us suppose that a random variable $x$ has a probability distribution function (PDF) $f(x; \theta)$ that depends on an \textit{a priori} unknown parameter $\theta$. Such a probability is known as the \textit{conditional probability} of having the data $x$ given the theoretical parameter $\theta$. If we repeat the experiment such that we obtain a whole data set, \textit{i.e.} $x_1, x_2, \ldots$, then the law of combined probability tells us that the probability of obtaining $x_1$ in the interval $dx_1$ around $x_1, x_2$ in the interval $dx_2$ around $x_2$ and so forth is given by

$$f(x_i; \theta)dx_i \equiv \prod_i f_i(x_i; \theta)dx_i,$$  \hspace{1cm} (4.1)

if the measures are independent of each other, although this is not always the case as there might be systematic correlations between. For each different theoretical parameter $\theta$, this multivariate PDF will assume different values. It is always possible to define a best $\theta$ such that $\prod_i f(x_i; \theta)$ is maximal. If we generate random variables whose distribution is given by $f(x; \theta)$, the most likely outcome for $x$ is the value that maximises $f(x; \theta)$. On the other hand, if we have a particular outcome $x$, then the best value of $\theta$ is such as to maximise the occurrence of that $x$. This is actually a general statement, we do not need to have independent random variables or a single parameter. We can define the best $\theta_i$ as those that maximise the joint function $f(x_1, x_2, \ldots, x_n; \theta_1, \theta_2, \ldots, \theta_m)$. We will denote this function as $f(x_i, \theta_j)$ from now on.

The maximum likelihood method consists on finding the parameters that maximise the likelihood function $f(x_i; \theta_j)$, \textit{i.e.}:

$$\frac{\partial f(x_i; \theta_j)}{\partial \theta_j} = 0, \quad \text{for} \quad j = 1\ldots m.$$  \hspace{1cm} (4.2)

Let us denote the solution to this system as $\hat{\theta}_j$. They are functions of the data $x_i$ and are therefore random variables themselves. The \textit{frequentist} approach would try to determine the full distribution of the solutions $\hat{\theta}_j$ knowing the distribution of $x_i$. One problem with this approach is that it is not always possible to derive the distribution of $\hat{\theta}_j$ analytically and it is computationally demanding to find them numerically through simulated datasets. However, the main problem is that the frequentist approach does not take into account what we already know about the theoretical parameters, for example, information from
previous experiments. This, however, is solved by the Bayesian approach in which, instead of looking for the probability \( f(x_i; \theta_j) \) of having the data \( x_i \) given the model \( \theta_j \), we estimate the probability \( L(\theta_j; x_i) \) of finding the model given the data. These two probabilities are related by the Bayes’ theorem:

\[
P(T; D) = \frac{P(D; T)P(T)}{P(D)},
\]

where the known data \( x_i \) are denoted here as \( D \) and the unknown theoretical parameters with \( T \). Here \( P(D; T) \) is the conditional probability of having the data given the theory, while \( P(T) \) and \( P(D) \) are the probabilities of having the theory and the data, respectively. Finally, \( P(T; D) \) is the conditional probability of having the theory given the data. Applying Bayes’ theorem to our PDFs \( f(x_i; \theta_j) \) and \( L(\theta_j; x_i) \) it follows that

\[
L(\theta_j; x_i) = \frac{f(x_i; \theta_j)p(\theta_j)}{g(x_i)},
\]

where \( p(\theta_j) \) is called the prior probability for the parameters \( \theta_j \), while \( g(x_i) \) is the PDF of the data \( x_i \). The function \( L(\theta_j; x_i) \) is known as the posterior or likelihood. This is the function we are looking for: the probability distribution of the parameters \( \theta_j \) given the data \( x_i \) and that we have some prior knowledge about the parameters themselves. Since this function is a PDF, it needs to be normalised, giving

\[
\int L(\theta_j; x: i)d^n\theta_j = 1 = \int f(x_i; \theta_j)p(\theta_j)d^n\theta_j, \quad (4.5)
\]

and therefore

\[
E(x_i) \equiv \int f(x_i; \theta_j)d^n\theta_j = g(x_i). \quad (4.6)
\]

The left hand side of Eq. (4.6) is what is commonly named the evidence. Generally, we do not know the probability distribution of the theoretical parameters, i. e. whether a model is more probable than another. However, what we do know are the results from previous experiments. For example, if we know that \( \Omega^0_m < 0.1 \) has been ruled out by experiments, we can use this information in the prior by saying \( p(\Omega^0_m < 0.1) = 0 \). If we do not have such a strict bound but instead believe that, for example \( \Omega^0_m = 0.24 \pm 0.02 \), then we could use as the prior \( p(\Omega^0_m) \) a Gaussian distribution with mean 0.024 and standard deviation 0.02. One can also exclude unphysical values with the prior. It is relevant to note the
importance of priors. Clearly, from Eq. (4.4), the result depends on the prior we choose. It is also important to note that priors are unavoidable. Even if a prior is not consciously chosen, the way we manage the statistical problem at hand always implies some sort of prior. No prior on one parameter means assuming a “flat” prior, i.e. \( p(\theta) = 1 \) in the domain where \( \theta \) is defined and \( p(\theta) = 0 \) outside. Even choosing a theory and its parameters is already a strong prior.

Because of the presence of the prior in Eq. (4.4), when we maximise the likelihood \( L(\theta_j; x_i) \), the resulting parameters \( \hat{\theta}_j \) will be different than those obtained maximising \( f(x_i; \theta_j) \). The equation to solve becomes

\[
\frac{\partial L(\theta_j; x_i)}{\partial \theta_j} = 0, \quad j = 1...n. 
\]  

(4.7)

4.1.2 Confidence regions

A confidence region of parameters \( R(a) \) is a region of constant \( L(\theta_j; x_i) \) in which

\[
\int_{R(a)} L(\theta_j; x_i) d^n \theta_j = a. 
\]  

(4.8)

To find \( R \), one evaluates

\[
\int \hat{L}(L_i) d^n \theta = a_i, 
\]  

(4.9)

where \( \hat{L}(L_i) = L \) if \( L > L_i \) and 0 elsewhere, i.e. the volume lying within the curve of height \( L_i \) smaller than the peak of \( L \). To find the \( a_i \) one can use a trial and error approach or interpolate over a grid. The usual values of \( a \) are 0.683 (1\( \sigma \)), 0.854 (2\( \sigma \)) and 0.997 (3\( \sigma \)). The value \( L_i \) that corresponds to \( a_i \) is the level at which we have to cut \( L \) to find the region \( R(a_i) \).

Sometimes, we are not interested in obtaining the solution to Eq. (4.7) for all the theoretical parameters. This is done by integrating out the parameters we are not interested in (process known as marginalisation). In this way, one can marginalise a multidimensional problem down to a two or one parameter likelihood (for example, to quote final confidence regions). Sometimes, instead of marginalising over a parameter, one wants to fix its value. One reason to do this is because one may be interested in the result for particular values of the parameter. The result will again depend on the the value of the fixed parameter. When this value corresponds to the maximum likelihood estimator, the posterior is said to the maximised with respect to that parameter.
4.1.3 Model selection

When we select a model to test, we also select the free functions that characterise the model and its parameterisations. If we change the model, how can we decide which model is better? The frequentist approach to answer this question is to evaluate the goodness of fit: once we have found the best fit parameters for models A and B, we calculate the $\chi^2$ statistic of the model prediction with respect to data and choose the one with the best $\chi^2$. It is important to note that this does not necessarily mean the model with the lowest $\chi^2$, since the $\chi^2$ statistic depends on the degrees of freedom and the number of parameters. If model B has a poorly constrained parameter, this would not help in the fit but would still count as an extra parameter, penalising the model for it. Even if both models had the same $\chi^2$ statistic, if, say, model B has 3 parameters and A only 2, model B would be penalised for it.

To overcome this dilemma, we can use another method of model selection known as evidence or marginal likelihood. We have already defined the evidence in Eq. (4.6). For a model $M$, the evidence is given by

$$E(x_i; M) = \int f(x_i; \theta^M_j) p(\theta^M_j) d^n \theta^M_j,$$

(4.10)

where $\theta^M_j$ are the $n$ parameters that describe the model $M$. If we want to compare two models, $M_1$ and $M_2$, we calculate what is known as the Bayes’ ratio $B_{12}$

$$B_{12} = \frac{E(x_i; M_1)}{E(x_i; M_2)} = \frac{\int f(x_i; \theta^M_{M_1}) p(\theta^M_{M_1}) d^n \theta^M_{M_1}}{\int f(x_i; \theta^M_{M_2}) p(\theta^M_{M_2}) d^n \theta^M_{M_2}}.$$

(4.11)

If $B_{12} > 1$, then model $M_1$ is favoured, while if $B_{12} < 1$, then model $M_2$ is preferred. This, however, is not strong enough to discard one model or the other. Large or small values should incline us towards one of the two models, but there is no absolute statistic to associate to any level. The scale most used in the literature is known as the Jeffrey scale by which if $| \ln B_{12} | < 1$, there is no evidence or inconclusive evidence in favour of any of the models, whereas if $| \ln B_{12} | > 1$, there is weak evidence; if $| \ln B_{12} | > 2.5$, there is moderate evidence, while, finally, $| \ln B_{12} | > 5$ means strong evidence. We can also weigh the models by assigning a model prior $p(M_j)$ and using the Bayes’ theorem again to write

$$L(M; x_i) \propto E(x_i; M)p(M),$$

(4.12)
and evaluate the ratio
\[
\frac{L(M_1; x_i)}{L(M_2; x_i)} = B_{12} \frac{p(M_1)}{p(M_2)}.
\] (4.13)

Generally, it is assumed that \( p(M_1) = p(M_2) \). This is so when the priors come from data from previous experiments, but it is a strong assumption when comparing different models such as general relativity and scalar-tensor theories. In such a case, it is hard to decide if assuming the same priors for both models does not penalise one of them.

Let us see how this type of model selection does not penalise a model if a parameter is poorly constrained. If parameter \( \theta_k \) is poorly constrained, then the likelihood \( f(x_i; \theta_j) \) is practically independent of \( \theta_k \), which means that \( f \) remains almost constant when varying \( \theta_k \). If the prior is factorisable (which is often the case if the systematic errors do not mix parameters), then
\[
p(\theta_j) = \prod_j p_j(\theta_j),
\] (4.14)

which, in turn, makes the evidence integral Eq. (4.6) become
\[
\int f(x_i; \theta_j)p(\theta_j) d^n \theta_j = \int f(x_i; \theta_j) \prod_{j=1}^{n-1} p(\theta_j)p_k(\theta_k) d^{n-1} \theta_j d\theta_k.
\] (4.15)

Since priors are PDFs, it holds that \( \int p_k(\theta_k) d\theta_k = 1 \) and hence the evidence is finally
\[
E(x_i) = \int f(x_i; \theta_j) \prod_{j=1}^{n-1} p(\theta_j) d^{n-1} \theta_j;
\] (4.16)

which is independent on \( \theta_k \). This means that the evidence correctly discards poorly constrained parameters and does not penalise models for introducing them.

### 4.1.4 The Fisher matrix

As we have seen in the previous section, the likelihood method is a powerful tool to analyse parameter space. However, this can be computationally expensive, especially if there is a large number of parameters involved. We need to evaluate \( L(\theta_j; x_i) \) for every \( \theta_j \). If there are, say 10 parameters, we would need to perform \( 10^{10} \) evaluations, computationally impossible in a reasonable amount of time. A possible solution is to use the Monte Carlo approach, in which, instead of evaluating across the whole parameter grid, one uses
a random number generator to explore the parameter space. The number of evaluations using this technique increases with the number of parameters instead of exponentially, like the full likelihood method described earlier. However, this might still mean a lot of calculations, which can make the process very time consuming.

There is a faster method that does not require a computation over the full or partial parameter grid. This is known as the Fisher (or information) matrix method. The idea is to approximate the full likelihood with a multivariate Gaussian distribution:

$$L \sim N \exp \left[ -\frac{1}{2} (\theta - \hat{\theta}) F_{ij} (\theta - \hat{\theta}) \right],$$  \tag{4.17}

where $F_{ij}$ is the Fisher matrix or the inverse of the correlation matrix. It is important to note that in Eq. (4.17) the likelihood is assumed to be a Gaussian function of the parameters and not of the data. One can argue that Eq. (4.17) is a coarse approximation and should not be used, however we can hope that it is a reasonable one, if we have a large enough sample, the central limit theorem ensures us that the probability distribution function of a random variable becomes Gaussian centred about their true value, at least near the peak of the distribution, since around a local maximum, every smooth function can be approximated as a quadratic one. For this reason we expect this approximation to work better when $\theta_i$ is close to $\hat{\theta}_i$.

Expanding Eq. (4.17) near its peak (i. e. near the maximum likelihood estimators, $\hat{\theta}_i$)

$$\ln L(\theta_i) \approx \ln L(\hat{\theta}_i) + \frac{1}{2} \left. \frac{\partial^2 \ln L(\theta_i)}{\partial \theta_i \partial \theta_j} \right|_{ML} (\theta_i - \hat{\theta}_i) (\theta_j - \hat{\theta}_j).$$  \tag{4.18}

Comparing Eq. (4.17) and Eq. (4.18), the Fisher matrix is defined as the expected value of the matrix $-\partial^2 \ln L / \partial \theta_i \partial \theta_j$ averaged over the data distribution:

$$F_{ij} \equiv -\left\langle \frac{\partial^2 \ln L(\theta_i)}{\partial \theta_i \partial \theta_j} \right\rangle = -\int \frac{\partial^2 \ln L(\theta_i)}{\partial \theta_i \partial \theta_j} L(x_i; \theta_j) d^n x_i.$$  \tag{4.19}

It seems that we still need to calculate the maximum likelihood estimators $\hat{\theta}_i$ to calculate the Fisher matrix. This can be done with some techniques like the Newton-Raphson method which is based on an iterative process of guessing the maximum likelihood estimators. However, the most useful application of the Fisher matrix and the one we used it for in [11] is to produce forecasts for future experiments. This is done by considering a fiducial model (i. e. a set of fixed parameters) as the maximum likelihood estimators.
These values are usually taken as the best fit parameters of some previous experiment or set of experiments. In this way, we can find the confidence regions around this particular parameter set. If we change the fiducial parameter values, our results will also change. It is important to note that, with this approach, $F_{ij}$ is not necessarily diagonal even if the original correlation function was. Taking fiducial values as the maximum likelihood estimators alters this.

What we have developed so far is a way to propagate errors from the observational errors $\sigma_i$ to the cosmological parameters. The errors $\sigma_i$ are, in turn, based on the expected performance of the experiment. Since the Fisher matrix is the covariance matrix, we can use it to extract errors on the parameters. Let us look at some theorems to understand this better.

- **Theorems:**

  - **Unbiased estimators.** If we have a parameter set $\theta_i$ whose true values are given by $\theta_i^0$, we say that $\theta_i$ will be unbiased if $\langle \theta_i \rangle = \theta_i^0$ with smallest possible errors: $\Delta \theta_i = \sqrt{\langle \theta_i^2 \rangle - \langle \theta_i \rangle^2}$. For any unbiased estimator, it holds that $\Delta \theta_i \geq \sqrt{F_{ii}^{-1}}$.

  - **Best unbiased estimator.** If there exists a best unbiased estimator, it is the maximum likelihood estimator. It also holds that the maximum likelihood estimator is asymptotically the best unbiased estimator.

  - **The Cramér-Rao inequality.** This inequality gives a lower bound on errors. If all the parameters are known, it gives the minimum error bar, while if not all of them are known, then $\Delta \theta_i \geq \sqrt{F_{ii}^{-1}}$. For a large enough dataset, the maximum likelihood estimator is that for which $\Delta \theta_i = \sqrt{F_{ii}^{-1}}$.

In Appendix C we can see the details of how to calculate the Fisher matrix in the case of a Gaussian likelihood.

We can also use the Fisher matrix method to find the errors on other variables, granted the parameters depend on them. Say we have parameters $\theta_i$ that depend on parameters $y_i$ and we wish to know the errors of $y_i$. This is easily done by multiplying the elements of the Fisher matrix $F_{ij}$ with the Jacobian of the transformation i.e.:

$$
\tilde{F}_{lm}(y_i) = \frac{\partial \theta_k}{\partial y_l} F_{kp} \frac{\partial \theta_p}{\partial y_m}.
$$

(4.20)
Thus, we can get the errors on the parameters $y_i$ simply by doing $\Delta y_i = \sqrt{F_{ii}^{-1}}$.

Let us now look into how one can marginalise or maximise parameters within the Fisher matrix formalism. Maximising with respect to a parameter means to fix one of the parameters to its maximum likelihood estimator. This means setting the difference $\theta_i - \hat{\theta}_i = 0$, therefore all the rows and columns in the Fisher matrix related to the $i$-th parameter vanish. In practice, this means that if we want to maximise our likelihood with respect to a parameter, we just need to remove the rows and columns of the Fisher matrix of the maximised parameters. In the case of marginalisation, if we wish to marginalise the $j$-th parameter, we must remove the $j$-th row and column from the inverse of the Fisher matrix $F^{-1}$. If we wish to marginalise over several parameters, we just remove the corresponding rows and columns from the inverse matrix. The resulting inverse Fisher matrix contains in its diagonal the fully marginalised $1\sigma$ errors of the remaining parameters.

It is often useful to reduce the Fisher matrix to a $2 \times 2$ matrix for two parameters to plot the resulting two-dimensional confidence regions, defined as the regions of constant likelihood that contain a fraction of the total likelihood volume. Since we have approximated from the start our likelihood with a Gaussian function, the resulting confidence regions will be elliptical in the plane of the two remaining parameters. The semiaxes of this elliptic region will be defined by the eigenvectors of the reduced $2 \times 2$ Fisher matrix, while the ratio of the semiaxes will be given by the square root of the eigenvalues ratio. The length of the semiaxes depends on the level of confidence required. If we take the semiaxes length along the $i$-th eigenvector equal to $\sqrt{\lambda_i}$, where $\lambda_i$ is the $i$-th eigenvalue, we are finding the $1\sigma$ region, but because we are in two dimensions, this level does not contain the usual $68.3\%$ of the probability but rather less than $40\%$. Instead, we find by integrating a two-dimensional Gaussian that the one-dimensional $1\sigma$ region corresponding to $68.3\%$ of the probability content is found for semiaxes which are roughly $1.51$ times the eigenvalues. Regions at $95.4\%$ and $99.7\%$ correspond to semiaxes $2.49$ and $3.44$ times the eigenvalues respectively. The area of the $68.3\%$ ellipses is $\pi ab$, with $a$ and $b$ the semiaxes length, that is $1.51$ times the eigenvalues $i. e.$ the area is $(1.51)^2 \pi (\det F)^{-1/2}$. Since an experiment is more constraining if the confidence region is smaller, it is useful to define what is known as the figure of merit (FOM) \[177\]

\[ FOM = \sqrt{\det F}. \tag{4.21} \]

The FOM naturally depends on how many parameters have been marginalised. Every parameter marginalisation increases the amount of uncertainty with respect to a maximised likelihood and therefore decreases the available information and the FOM of the final set
of parameters.

Even though calculations involving the Fisher matrix seem rather easy, in practice some problems can arise. The main problem arises when the Fisher matrix is singular i.e. its determinant is 0 and it cannot be inverted. A singular Fisher matrix appears when rows or columns are not linearly independent. This can, however, be used to our advantage. If the Fisher matrix is singular, then it means that there is a linear combination of two or more parameters hidden somewhere in the likelihood. Therefore, we can substitute the linear combination with a new parameter, e.g. \( \tilde{\theta} = a\theta_1 + b\theta_2 \), and remove the singularity by restricting ourselves to \( \tilde{\theta} \) instead of the original pair of parameters. In this case, we say that there is a degeneracy between parameters \( \theta_1 \) and \( \theta_2 \).

It is also simple to add priors to the Fisher matrix (apart from the obvious choice of fiducial model). If the priors is the outcome of another experiment and we have the Fisher matrix of that experiment (let us call it \( F^P_{ij} \)), and both experiments have the same maximum likelihood estimators or the same fiducial model, the new Fisher matrix is given by

\[
F_{ij}^{\text{tot}} = F_{ij} + F^P_{ij}.
\]  

In summing these two matrices, one must be careful to have the same number of parameters and in the same order. If an experiment does not contain information on a parameter, we just set the rows and columns corresponding to that parameter to zero. In this way, the two Fisher matrices are rendered of the same rank by filling the one with less parameters with zeros in the correct position.

Let us now summarise the properties of the Fisher matrices in 5 points:

1. To transform to different parameter spaces, we apply the Jacobian transformation to the left and right of the Fisher matrix.

2. To maximise over a set of parameters, we remove the rows and columns related to those parameters from the Fisher matrix.

3. To marginalise over a set of parameters, we remove the rows and columns related to those parameters from the inverse of the Fisher matrix.

4. To combine Fisher matrices from independent experiments with the same fiducial model, we sum the corresponding Fisher matrices, ensuring the same order of parameters and, if necessary, inserting rows and columns of zeros for unconstrained parameters.
5. The elliptic confidence regions of the reduced $2 \times 2$ marginalised matrix have semi-axes lengths equal to the square root of the eigenvalues of the inverse Fisher matrix, while the semi-axes are oriented along the corresponding eigenvectors. The area of the ellipse is proportional to the square root of the determinant of the inverse of the Fisher matrix. The determinant of the Fisher matrix is the indicator of the performance of an experiment in constraining a set of parameters or, in other words, a figure of merit.

On a final note, it is important to remember that the Fisher matrix analysis gives the best possible error estimates. If the fiducial values of the parameters are chosen away from the peak of the distribution, the resulting errors will grow. A discussion on the comparison of the marginalised errors and the figure of merit obtained using the Fisher matrix versus an MCMC analysis can be found in [178].

### 4.1.4.1 Complete galaxy power spectrum

There are two ways we can get information about dark energy parameters by looking at the galaxy power spectrum: first, by marginalising over the main observables: the Hubble parameter, the angular diameter distance, the growth factor and the redshift space distortion parameter and projecting onto the dark energy parameter space; and secondly, by taking the full information of the observed galaxy power spectrum and deducing from the latter the parameters of interest. Let us look into the case of using the full galaxy power spectrum within the Fisher matrix formalism.

Following [92, 179, 180] we write the observed galaxy power spectrum as:

$$P_{\text{obs}}(z; k, \mu) = \frac{D_A^2(z)H(z)}{D_A^2(z)H_r(z)} G^2(z)b(z)^2 (1 + \beta \mu^2)^2 P_{0r}(k) + P_{\text{shot}}(z), \quad (4.23)$$

where the subscript $r$ refers to the values assumed for the fiducial cosmological model, i.e. the model at which we evaluate the Fisher matrix, $b(z)$ is the galaxy bias factor and $G(z)$ is the growth factor of matter perturbations. Also, here $P_{\text{shot}}$ is the shot noise due to discreteness in the survey, $\mu$ is the direction cosine within the survey and $P_{0r}$ is the present matter power spectrum for the fiducial (reference) cosmology. For the linear matter power spectrum today we adopt the CAMB output [181].

The wavenumber $k$ is also to be transformed between the fiducial cosmology and the general one (see [92, 179, 180] for more details).

The distortion induced by redshift can be expressed in terms of the $\beta(z)$ factor which is
related to the bias factor via:

$$\beta(z) = \frac{\Omega_m(z)^{\gamma}}{b} = \frac{f(z)}{b}. \quad (4.24)$$

The \((1 + \mu^2 \beta)^2\) factor accounts only for linear distortions in redshift space and it should be considered as a first approximation.

The galaxy over-densities are assumed to trace the underlying matter distribution through a fraction called bias factor, \(b(z, k)\). This quantity could be arbitrary, it could even depend on both time and scale, see \([182–184]\). Usually it is assumed that the bias on large scales is independent on scale \([185]\), hence in the matter power spectrum, this term appears as a multiplicative factor which modulates the overall amplitude of the galaxy power spectrum. We will assume here a Gaussian linear bias, with redshift dependence \(b(z) = \sqrt{1 + z}\), because it provides a good fit to the \(H_a\) line galaxies in the near-infrared, which are the target of a Euclid-like survey, see \([186]\).

The factor \(G(z)\) is related to the growth rate by \(f(a) = \frac{d \log G}{d \log a} - 1\) and is usually assumed to be independent of scale: the late-time change in the expansion rate affects all scales equally. However, the growth factor may depend on the scale \(k\), for instance allowing perturbations also in the dark energy sector. In the last case, the existence of the dark energy sound horizon would introduce a \(k\) dependence on the growth factor, see for instance \([39, 40, 187]\). However, here we assume a scale independent \(G(z)\), since in \([1]\) we have neglected the dark energy perturbations.

The total galaxy power spectrum including the errors on the redshift can be written as \([179]\)

$$P(z, k) = P_{\text{obs}}(z, k)e^{k^2 \mu^2 \sigma_r^2} \quad (4.25)$$

where \(\sigma_r = \delta z/H(z)\) is the absolute error on the measurement of the distance and \(\delta z\) is the absolute error on redshift.

The Fisher matrix provides a useful method for evaluating the marginalised errors on cosmological parameters. Assuming the likelihood function to be Gaussian, the Fisher matrix for the matter power spectrum is defined as \([188, 189]\)

$$F_{ij} = 2\pi \int_{k_{\text{min}}}^{k_{\text{max}}} \frac{\partial \log P(k)}{\partial \theta_i} \frac{\partial \log P(k)}{\partial \theta_j} \cdot V_{\text{eff}} \cdot \frac{k^2}{8\pi^3} \cdot dk, \quad (4.26)$$

where the \(\theta\)'s are the parameters shown in Table 4.1.1; the derivatives are evaluated at the parameter values of the fiducial model and \(V_{\text{eff}}\) is the effective volume of the survey, given
\begin{table}
\centering
\begin{tabular}{|c|c|c|c|}
\hline
Parameters & P (k) & mP(k) & WL \\
\hline 1 & total matter density & $\Omega_m h^2$ & $\Omega_m h^2$ & $\Omega_m h^2$ \\
2 & total baryon density & $\Omega_b h^2$ & $\Omega_b h^2$ & $\Omega_b h^2$ \\
3 & optical thickness & $\tau$ & $\tau$ & $\tau$ \\
4 & spectral index & $n_s$ & $n_s$ & $n_s$ \\
5 & matter density today & $\Omega_m$ & $\Omega_m$ & $\Omega_m$ \\
6 & equation of state parameter & $w_0$ & $w_0$ & $w_0$ \\
7 & equation of state parameter & $w_0$ & $w_0$ & $w_0$ \\
8 & rms fluctuations & $\sigma_8$ & $\sigma_8$ & $\sigma_8$ \\
\hline
\end{tabular}
\caption{Cosmological parameters for the complete and marginalised (mP(k)) galaxy power spectrum and weak lensing.}
\end{table}

by:

$$
V_{\text{eff}} = \int d^3 r \left( \frac{n(r)}{n(r)} P(k, \mu) + 1 \right)^2 = \left( \frac{\bar{n} P(k, \mu)}{\bar{n} P(k, \mu) + 1} \right)^2 V_{\text{survey}}.
$$

Here $\mu = k \cdot \hat{r}/k$, $\hat{r}$ is the unit vector along the line of sight, $k$ is the wave vector and the last equality holds for an average comoving number density $\bar{n}$. The highest frequency, $k_{\text{max}}(z)$, is evaluated at $z$ of the corresponding bin and it is chosen so as to avoid non-linearity problems both in the spectrum and in the bias; we choose values from $0.11 \ h/\text{Mpc}$ for small $z$ bins to $0.25 \ h/\text{Mpc}$ for the highest redshift bins. For a detailed calculation of the Fisher matrix for the galaxy power spectrum, see Appendix C.

**4.1.4.2 Marginalised galaxy power spectrum**

This approach consists on evaluating the Fisher matrix $F_{ij}$ first for the main observables shown in Table 4.1.1 and then project into the dark energy parameters, $w_0$, $w_a$ and $\gamma$ as we
have seen in Sec. 4.1.4. Since we want to propagate the errors to the cosmological parameters above, we need to change parameter space. This will be done taking the inverse of the Fisher matrix $F^{-1}_{ij}$ and then extracting a submatrix, called $F^{-1}_{mn}$ containing only the rows and columns with the parameters that depend on $w_0, w_a$ and $\gamma$, namely $H(z), D_A(z), G(z)$ and $\beta(z)$. We then contract the inverse of the submatrix with the new set of parameters; the new Fisher matrix will be given by

$$S_{ij} = \frac{\partial P_m}{\partial q_i} F_{mn} \frac{\partial P_n}{\partial q_i}.$$  \hspace{1cm} (4.28)

The square root of the diagonal elements of the inverse of the matrix $S_{ij}$ gives the errors on the parameters $w_0, w_a$ and $\gamma(z)$.

4.1.4.3 Weak Lensing

Following [190, 191], the lensing potential is $\Phi_L = \Psi + \Phi$, which describes the deviation of light rays by gravitational sources. As previously mentioned, we limit ourselves to scalar perturbations at linear order in the Newtonian gauge: Eq. (1.26). Since we are considering dark energy only at background level, the gravitational potential is simply given by Eq. (1.26). As $\Phi = \Psi$, it follows that the weak lensing potential is:

$$k^2 \Phi_L = \frac{3H_0^2 \Omega_m}{2a} \Delta_m .$$  \hspace{1cm} (4.29)

The convergence weak lensing power spectrum (which, in the linear regime, is equal to the ellipticity power spectrum) is a linear function of the matter power spectrum convoluted with the lensing properties of space. For a $\Lambda$CDM cosmology it can be written as [192]

$$P_{ij}(\ell) = H_0^4 \int_0^\infty \frac{dz}{H(z)} W_i(z) W_j(z) P_{nl} \left[ P_l \left( \frac{H_0 \ell}{r(z)}, z \right) \right]$$  \hspace{1cm} (4.30)

where the $W_i$'s are the window functions, $P_{nl} [P_l (k, z)]$ is the non linear power spectrum at redshift $z$ obtained correcting the linear matter power spectrum $P_l (k, z)$, see [38] for more details.

The Fisher matrix for weak lensing is given by:

$$F_{\alpha\beta} = f_{\text{sky}} \sum_{\ell} \frac{(2\ell + 1) \Delta \ell}{2} \partial (P_{ij})_{,\alpha} C^{-1}_{jk} \partial (P_{kn})_{,\beta} C^{-1}_{mi}$$  \hspace{1cm} (4.31)
where the partial derivatives represent $\frac{\partial}{\partial \theta_a}$, the corresponding cosmological parameters $\theta_a$ are shown in Table 4.1.1 and

$$C_{jk} = P_{jk} + \delta_{jk} \frac{\langle \gamma_{\text{int}}^{1/2} \rangle}{n_j}$$

where $f_{\text{sky}}$ is the fraction of the sky available to the survey under consideration, $\gamma_{\text{int}}$ is the rms intrinsic shear (here we assume $\langle \gamma_{\text{int}}^{1/2} \rangle = 0.22$ [193]) and $n_j$ is the number of galaxies per steradians belonging to the $i$-th bin.

### 4.2 The Euclid Survey

In the past decade, cosmology has become a data-driven science. After the huge ongoing success of the CMB [12, 114], we have now opened the door to exploiting the data coming from present and future galaxy surveys (for a possibly incomplete list, see [45–47, 171, 194–198]). Since our work is centred on producing forecasts for the future Euclid survey, we shall discuss the survey and its forecasts in this section. If the reader is interested in forecasts for surveys such as the Dark Energy Survey (DES), he can find the details in [199].

Euclid is an ESA selected M-class mission that was selected in October 2011, adopted in June 2012 and will be launched around 2020. It is a wide-field space imager whose main goal is to map the geometry and evolution of the dark universe with unprecedented accuracy and precision in order to place stringent constraints on dark energy, dark matter, gravity and cosmic initial conditions. This will be done via research of dark energy and dark matter through the study of galaxies and clusters of galaxies with a galaxy clustering and a weak lensing probe.

The following are the key topics that will be addressed by Euclid:

1. Dynamical dark energy: is the dark energy equation of state constant or does it, on the other, hand vary with time?

2. Modification of gravity: is the late time acceleration a manifestation of a breakdown of Einstein’s theory of GR, or maybe a failure of the cosmological assumptions of homogeneity and isotropy?

3. Dark matter: what constitutes dark matter? What are the neutrino mass scale and the number of relativistic degrees of freedom in the Universe?
4. Initial conditions: what is the primordial power spectrum of density perturbations and are they described by a Gaussian probability distribution?

In order to address these questions, Euclid will measure the shape and spectra of galaxies over the best 15000 deg$^2$ of the extragalactic sky in the visible and near infrared, out to redshift about $z \sim 2$, thus covering the period over which dark energy took over and started accelerating the Universe. The payload baseline comprises two wide field instruments (> 0.5 deg): a visible instrument (VIS), and a near infrared photometric and spectroscopic instrument (NISP P and NISP S) [200]. The visible channel is used to measure the shapes of galaxies for weak lensing, with a platescale of 0.1 arcsec in a wide visible red band (R+I+Z, 0.55 to 0.92 μm). The near infrared photometric channel provides three near infrared bands (Y, J and H, spanning 1.1 to 2.0 μm) with a platescale of 0.3 arcsec. The baseline for the near infrared spectroscopic channel operates in the wavelength range 1.0 to 2.0 μm in slitless mode at a spectral resolution $R = 250$ to 1 arcsec diameter source, with 0.3 arcsec per pixel.

In order to achieve its goals, the wide survey must ensure that the following parameters are properly sampled by both the imaging and the spectroscopic probes: firstly, a high density of objects for a high enough signal to noise ratio and, secondly, a sufficiently large volume to minimise cosmic variance. The pattern on the sky covered by Euclid must sample all relevant angular scales that probe interesting signatures of dark energy, both in the correlation function and angular power spectrum (up to $\ell \sim 5000$). The requirement for redshift depth from $z = 0.7$ to $z = 2.0$ translates into requirements on the weak lensing galaxy probe both in the visible and the near infrared.

The optimum parameters for Euclid have been set to: a joint 15000 deg$^2$ survey of the extragalactic sky, an average of 30 galaxies per arcmin for imaging channels used in the weak lensing probe and an average of 3500 galaxies per square degree for the spectroscopic channel used for the galaxy clustering probe. A summary of parameters for the Euclid survey can be seen in Fig. 4.2.1 and the expected galaxy number densities can be seen in Fig. 4.2.2 and in Table 4.3.3 in Sec. 4.3.

4.3 Forecasts for Euclid

In order to produce meaningful forecasts for Euclid, it is necessary to look at parameters that will help us answer the 4 questions envisioned by Euclid and mentioned in Sec. 4.2.
**Euclid Mission Summary**

**Main Scientific Objectives**

- Understand the nature of Dark Energy and Dark Matter by:
  - Reach a dark energy FoM > 400 using only weak lensing and galaxy clustering; this roughly corresponds to 1 sigma errors on $w_p$ and $w_a$ of 0.02 and 0.1, respectively.
  - Measure $\Omega_m$, the exponent of the growth factor, with a 1 sigma precision of < 0.02, sufficient to distinguish General Relativity and a wide range of modified-gravity theories.
  - Test the Cold Dark Matter paradigm for hierarchical structure formation, and measure the sum of the neutrino masses with a 1 sigma precision better than 0.03eV.
  - Constrain $n_s$, the spectral index of primordial power spectrum, to percent accuracy when combined with Planck, and to probe inflation models by measuring the non-Gaussianity of initial conditions parameterised by $f_{NL}$ to a 1 sigma precision of ~2.

### SURVEYS

<table>
<thead>
<tr>
<th>Area (deg²)</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>15,000 (required) 20,000 (goal)</td>
<td>Step and stare with 4 dither pointings per step.</td>
</tr>
<tr>
<td>40</td>
<td>In at least 2 patches of &gt; 10 deg² 2 magnitudes deeper than wide survey</td>
</tr>
</tbody>
</table>

### PAYLOAD

- **Telescope**: 1.2 m Korsch, 3 mirror anastigmat, f=24.5 m
- **Instrument**: VIS - NIR
- **Field-of-View**: 0.787×0.709 deg² 0.763×0.722 deg²
- **Capability**:
  - Visual Imaging
  - NIR Imaging Photometry
  - NIR Spectroscopy
- **Sensitivity**:
  - 24.5 mag 10σ extended source
  - 24 mag 5σ point source
  - 24 mag 5σ point source
  - 24 mag 5σ point source
  - $3 \times 10^{-16}$ erg cm⁻² s⁻¹ unresolved line flux
- **Detector Technology**:
  - 36 arrays 4k×4k CCD
  - 16 arrays 2k×2k NIR sensitive HgCdTe detectors
- **Pixel Size**:
  - 0.1 arcsec
  - 0.3 arcsec
  - 0.3 arcsec
  - R=250

### SPACECRAFT

- **Launcher**: Soyuz ST-2.1 B from Kourou
- **Orbit**: Large Sun-Earth Lagrange point 2 (SEL2), free insertion orbit
- **Pointing**:
  - 25 mas relative pointing error over one dither duration
  - 30 arcsec absolute pointing error
- **Observation mode**: Step and stare, 4 dither frames per field, VIS and NISP common FoV = 0.54 deg²
- **Lifetime**: 7 years
- **Operations**: 4 hours per day contact, more than one ground station to cope with seasonal visibility variations;
- **Communications**: maximum science data rate of 850 Gbit/day downlink in K band (26GHz), steerable HGA

### Budgets and Performance

<table>
<thead>
<tr>
<th>Industry</th>
<th>TAS</th>
<th>Astrrium</th>
<th>TAS</th>
<th>Astrrium</th>
</tr>
</thead>
<tbody>
<tr>
<td>Payload Module</td>
<td>897</td>
<td>696</td>
<td>410</td>
<td>496</td>
</tr>
<tr>
<td>Service Module</td>
<td>786</td>
<td>835</td>
<td>647</td>
<td>692</td>
</tr>
<tr>
<td>Propellant</td>
<td>148</td>
<td>232</td>
<td>65</td>
<td>108</td>
</tr>
<tr>
<td>Total (including margin)</td>
<td>2160</td>
<td>1368</td>
<td>1690</td>
<td></td>
</tr>
</tbody>
</table>

**Figure 4.2.1**: Euclid survey specifications. Figure taken from [174].
These parameters and their parameterisations can be seen in Table 4.3.1.

An important question addressed when designing the Euclid experiment [174] was how well Euclid should measure \( w(a) \). If one considers two models, say the cosmological constant and one in which the equation of state parameter varies with redshift and can be parameterised linearly, then, for small variations of \( w \) with \( z \) the cosmological constant model would be favoured because of Occam’s razor. For large deviations from \( w \neq -1 \) the more complex two-parameter model would be preferred instead. This is quantified with the Bayesian evidence described in Sec. 4.1.3. If the data is consistent with the cosmological constant model with a figure of merit \( FOM > 400 \) (e.g. \( \Delta w_p \sim 0.01 \) and \( \Delta w_a \sim 0.1 \)) then LCDM would be favoured with odds of more than 100 : 1, which is considered “decisive” evidence.

The fiducial model used to produce forecasts for Euclid is the LCDM model with parameter values \( \Omega_m = 0.25 \), \( \Omega_{\Lambda} = 0.75 \), \( \Omega_b = 0.0045 \), \( \sigma_8 = 0.8 \), \( n_s = 1.0 \) and \( h = 0.7 \). This model assumes GR and Gaussian initial conditions and a cosmological constant (i.e. \( w(a) = -1 \)) and dark matter components. Let us now look at how well Euclid will constrain the the parameters described on Table 4.3.1. The forecasts quoted here are the ones appearing in Euclid’s definition study report [174]. In Table 4.3.2 we can see the forecasted errors on the science goals parameters.
<table>
<thead>
<tr>
<th>Goal</th>
<th>Parameter</th>
<th>Parameter details</th>
</tr>
</thead>
<tbody>
<tr>
<td>Dark energy</td>
<td>$w(a)$</td>
<td>The dark energy equation of state, parameterised as $w(a) = w_p + (a_p - a)w_a$, where $a_p$ is the “pivot” scale factor such that the statistical errors $\Delta w_p$ and $\Delta w_a$ are not correlated. Detecting $w(a) \neq -1$ at any redshift would demonstrate that dark energy is not a cosmological constant, but a dynamical field or a void (see Fig. 2.4.1c).</td>
</tr>
<tr>
<td>Modified gravity</td>
<td>$\gamma(z)$</td>
<td>The growth index quantifies the amount of clustering in our universe as a function of redshift. It is directly sensitive to the nature of gravity. It is parameterised as $\gamma$ or as $\gamma(a) = \gamma_0 + (1 - a)\gamma_a$. A detection of $\gamma \neq 0.55$ would indicate a deviation from GR (see Fig. 2.4.1d).</td>
</tr>
<tr>
<td>Dark matter</td>
<td>$m_\nu$</td>
<td>The total neutrino mass, damping of structure growth on small scales. The larger the mass, the larger the damping observed in the matter power spectrum.</td>
</tr>
<tr>
<td>Initial conditions</td>
<td>$f_{NL}$</td>
<td>The standard cosmological model assumes an initial Gaussian random field of perturbations which seed the growth of large scale structure. A detection of non-Gaussianity would imply a deviation from the vanilla single field inflationary model. The $f_{NL}$ parameter quantifies the amplitude of such an effect. Euclid hopes to improve the current bounds [16].</td>
</tr>
</tbody>
</table>

Table 4.3.1: Determining parameters for the study of the scientific goals of the Euclid survey.
Table 4.3.2: Summary of the forecasted constraints for Euclid. The Euclid primary constraints include combined constraints from weak lensing tomography and galaxy clustering. The Euclid all constraints include the primary ones plus the galaxy clusters and ISW probes. The current constraints come from [16, 204, 205] while the improvement factor compares the current constraints with the Euclid + Planck case.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Modified gravity</th>
<th>Dark matter</th>
<th>Initial conditions</th>
<th>Dark energy</th>
</tr>
</thead>
<tbody>
<tr>
<td>Euclid primary</td>
<td>0.010</td>
<td>0.027 eV</td>
<td>5.5</td>
<td>0.015</td>
</tr>
<tr>
<td>Euclid all</td>
<td>0.009</td>
<td>0.020 eV</td>
<td>2.0</td>
<td>0.013</td>
</tr>
<tr>
<td>Euclid + Planck</td>
<td>0.007</td>
<td>0.019 eV</td>
<td>2.0</td>
<td>0.007</td>
</tr>
<tr>
<td>Current</td>
<td>0.200</td>
<td>0.580 eV</td>
<td>5.0</td>
<td>0.100</td>
</tr>
<tr>
<td>Improvement factor</td>
<td>30</td>
<td>30</td>
<td>2.5</td>
<td>&gt;10</td>
</tr>
</tbody>
</table>

4.3.1 Constraints on dark energy

Table 4.3.2 and Fig. 4.3.1 show that the Euclid primary probes alone will determine the dark energy equation of state with a FOM > 400. In combination with galaxy clusters and ISW, Euclid will exceed the requirement of FOM > 400 by a factor of 3 and, in combination with Planck results, by a factor of 10, improving the current constraints by a factor of over 100. These constraints will allow to distinguish between some models of dark energy to be tested: for example, freezing models in which $w$ tends to $-1$ at low redshifts, thawing models where $w$ deviates from $-1$ at low redshifts (see Fig. 2.1.1) and phantom models where $w$ is less than $-1$ at any redshift.

As we have said before, a deviation from $w = -1$ at any $z$ would indicate that dark energy is not the cosmological constant. Expressing constraints in the $(w_p, w_a)$ plane in Fig. 4.3.1a shows how the redshift evolution of $w$ can be constrained in the redshift range (0, 2) to percent accuracy. Fig. 4.3.1b shows that the Euclid primary probes can constrain $w(z)$ to percent accuracy at around $z = 0.5$, which could provide evidence on its own of a departure from ΛCDM. In combination with the secondary probes and the data from Planck, the redshift dependence can be constrained even further, as can be seen in Fig. 4.3.1b.
2. Scientific Objectives

Constraints on Dark Energy: Table 2.2 and Figure 2.4 show that the Euclid primary probes can constrain the dark energy equation of state with a 68% confidence level around $w(z) = -0.55$. Figure 2.5(a) shows the cosmological constant model and the contours on both panels are the 1σ confidence regions. Figure taken from [174].

4.3.2 Constraints on modified gravity

In Fig. 4.3.2 we can see how Euclid will constrain the growth index $\gamma$ and the growth rate $f(z)$. In this figure, $\gamma$ is considered as a constant parameter (see following sections for a more detailed study on the constraints Euclid can provide for $\gamma(z)$). As can be seen in Fig. 4.3.2, the growth index parameter will be constrained to 0.01. To test modified gravity theories, however, the growth index is not enough and one must consider a second parameter (for example, the gravitational slip). Forecasts using two parameters have been considered in [206] which have shown that a Euclid-like survey can measure these parameters to high precision.

We can see in Fig. 4.3.2 that Euclid will be able to distinguish between different dark energy models such as coupled dark energy models with a constant or time-dependent coupling or a braneworld scenario such as DGP. As we will see in the following section, using the growth index we will be able to distinguish also between other models such as $f(R)$ theories or LTB.
Constraints on Initial Conditions: As shown in Figure 2.5, Euclid will constrain the shape of the primordial power spectrum parameterised by the spectral index $n_s$ to percent accuracy when combined with Planck results. If the assumption of a Gaussian random field is relaxed then Euclid can constrain the amplitude of the non-Gaussianity $f_{NL}$ through 3-point statistics of the weak lensing and galaxy clustering signals and through the correlation function of clusters of galaxies. We find agreement with previous results (e.g. Fedeli et al., 2011), where the combination of the galaxy power spectrum with the cluster-galaxy cross spectrum can decrease the error on the determination of $f_{NL}$ by up to a factor of 2 relative to either probe individually. Through the combination of lensing, galaxy clustering and clusters we find that Euclid can constrain $\Delta f_{NL} \approx 2$, competitive and possibly superior to future CMB experiments. In fact, if the simplest inflationary scenario holds, Euclid is expected to detect a non-Gaussian signal due to large-scale corrections needed in the Poisson equation from general relativistic effects, while no such imprint should be detectable in the CMB. Here the unique combination of the two primary cosmological probes again enables the discrimination among models for the origin of cosmological structures.

To conclude, we have presented the primary science goals of Euclid, and shown that these laudable objectives can be met by the experiment that we present. Euclid provides a major step forward, reducing the uncertainties of a number of key cosmological parameters by impressive factors. It will either confirm the concordance model with unprecedented accuracy, or else lead the way to exciting alterations of it, signalling the need for a revision of fundamental physics.

Figure 2.5: In the left panel we show the parameter space constraints on the $J$ parameter describing the growth factor and the scalar spectral index. Green is lensing, blue galaxy clustering, orange includes the primary and secondary Euclid probes and red is combined with Planck. These errors are marginalised over all other parameters. Right panel: Predicted Euclid measurements of the growth rate of structure $f(z)$ using redshift-space distortions alone. The cyan (shaded) area gives the expected $1\sigma$ error, with the red points illustrating a corresponding simulated observation. Current state-of-the-art measurements by the SDSS (filled pentagons), 2dF (filled square, Hawkins et al., 2003) and WiggleZ (open hexagons, Blake et al. 2011) are also shown. The lines show predictions for $f(z)$ by the concordance model and by three alternative models in which DE couples with DM (Di Porto & Amendola, 2007) or gravity is generalised to a 5-dimensional brane-world (DGP, Dvali et al., 2000).

2.4 Legacy science

The design of Euclid is driven by our desire to study some of the most fundamental problems in cosmology, but the survey that is needed to achieve these goals will provide a dataset that will be of immense value for astrophysics as well: it will be important for understanding the formation and evolution of structures in the Universe at all scales, from galaxy clusters to brown dwarfs. The Euclid wide survey required to achieve the cosmological goals (see Section 3) will image 15,000 deg$^2$ of extra-galactic sky in the optical with a spatial resolution approaching that of HST, and to a depth in the near-IR at which only an area 1000 times smaller can feasibly be surveyed from the ground.

Figure 4.3.2: Euclid constraints for the growth index and the growth rate. In the left panel we can see the parameter $\gamma$ versus the scalar spectral index. In this case the fiducial values are $\gamma = 0.55$ and $n_s = 0.95$. As in Fig. 4.3.1, green corresponds to the forecasts from the lensing probe, blue to the galaxy clustering probe, orange to all the Euclid probes combined and red to Euclid plus Planck. In the right panel we can see the forecasted errors on the growth rate $f(z)$ using redshift-space distortions alone. The red dots correspond to a simulated observation with the predicted $1\sigma$ errors shown in the shaded area. Current observations are also shown: filled pentagons represent SDSS data, filled squares are 2dF, while open hexagons correspond to WiggleZ. The lines shown correspond to the predictions of different theoretical cosmological models. Figure taken from [174].
Table 4.3.3: Expected galaxy number density in each redshift bin for the Euclid survey in units of \((h/\text{Mpc})^3\) for the optimistic (middle column) and realistic (last column) cases. Let us notice that the number densities depend on the fiducial cosmology adopted in the computation of the survey volume, needed for the conversion from the galaxy numbers \(dN/dz\) to \(n(z)\).

<table>
<thead>
<tr>
<th>(z)</th>
<th>(n_1(z) \times 10^{-3})</th>
<th>(n_2(z) \times 10^{-3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5 − 0.7</td>
<td>4.69</td>
<td>3.56</td>
</tr>
<tr>
<td>0.7 − 0.9</td>
<td>3.33</td>
<td>2.42</td>
</tr>
<tr>
<td>0.9 − 1.1</td>
<td>2.57</td>
<td>1.81</td>
</tr>
<tr>
<td>1.1 − 1.3</td>
<td>2.1</td>
<td>1.44</td>
</tr>
<tr>
<td>1.3 − 1.5</td>
<td>1.52</td>
<td>0.90</td>
</tr>
<tr>
<td>1.5 − 1.7</td>
<td>0.92</td>
<td>0.55</td>
</tr>
<tr>
<td>1.7 − 1.9</td>
<td>0.54</td>
<td>0.29</td>
</tr>
<tr>
<td>1.9 − 2.1</td>
<td>0.31</td>
<td>0.15</td>
</tr>
</tbody>
</table>

4.3.3 Forecasts using the growth index

Here we look in more detail how measuring the growth index with Euclid will help us discern between different theoretical dark energy models. This section is based in [1] and the forecasts are based on the specifications of the Euclid survey according to the Yellow Book (Assessment Phase Study Report), see [207], although they do not differ much from the results cited above for the Red Book specifications in [174].

In order to provide forecasts for the growth index we make use of the Fisher matrix formalism for the three different observables introduced in Sec. 4.1.4: complete and marginalised galaxy power spectrum and weak lensing.

Galaxy redshift survey specifications: for the galaxy power spectrum cases we consider a spectroscopic survey from \(z = 0.5 − 2.1\) divided in equally spaced bins of width \(\Delta z = 0.2\) and with a covering area of 20000 deg\(^2\); the galaxy number densities in each bin are shown in Table 4.3.3 with an efficiency of 50%. For all cases we assume that the error on the measured redshift is spectroscopic, \(\delta z = 0.001 (1 + z)\).

While modelling the redshift survey, we choose two different cases according to the galaxy number density [130]:

- optimistic case; this corresponds to the middle column in Table 4.3.3.
- realistic case; this corresponds to the last column in Table 4.3.3.

Weak lensing survey specifications: for the WL survey we consider a photometric
survey characterized by the sky fraction \( f_{\text{sky}} = \frac{1}{2} \), that is, a covering area of 20,000 deg\(^2\); an overall radial distribution \( n(z) = z^2 \exp\left[-(z/z_0)^{1.5}\right] \) with \( z_0 = z_{\text{mean}}/1.412 \) and mean redshift \( z_{\text{mean}} = 0.9 \), the number density is \( d = 35 \) galaxies per arcmin\(^2\). Moreover, we consider the range \( 10 < \ell < 10,000 \) and we extend our survey up to \( z_{\text{max}} = 3 \) divided in 5 bins each containing the same number of galaxies. For the non linear correction we use the halo fit model by Smith et al. \([208]\). We assume the error on the measured redshift is photometric, \( \delta z = 0.05 \ (1 + z) \).

**Fiducial model:** our fiducial model corresponds to the \( \Lambda \)CDM, WMAP-7yr best-fit parameters \([114]\): \( \Omega_{m,0} h^2 = 0.134 \), \( \Omega_b h^2 = 0.022 \), \( n_s = 0.96 \), \( \tau = 0.085 \), \( h = 0.7 \), \( \Omega_{m,0} = 0.275 \) and \( \Omega_K = 0 \). For the dark energy parameters we choose \( w_0 = -1 \) and \( w_a = 0 \).

We can now derive the sensitivity of the parameters introduced above. More specifically, we consider separately three different cases: in Case 1 the growth index has been chosen to be a free parameter but independent in each redshift bin; in Case 2 we consider the growth index as a free parameter but equal for all redshift bins; in Case 3 the growth index depends directly on the equation of state parameters \((w_0 \text{ and } w_a)\), using Eq. (1.40) as the analytic solution.

**Case 1:**
We consider the growth index \( \gamma \) as a free parameter and independent for each redshift bin in order to map its variation over time; its value is chosen to be 0.545. The errors are shown in Fig. 4.3.3 for both galaxy power spectrum cases; in Table 4.3.4 are reported their 1\( \sigma \) errors. Overall, the errors on the growth index are of about 0.02 and 0.03 for the complete and marginalised galaxy power spectrum, respectively. In Fig. 4.3.4 the errors for the WL case are shown. It is worth noticing that after \( z \sim 1 \) the errors on \( \gamma \) are basically unchanged. Furthermore, we also plot the analytic expression for the growth index Eq. (1.40) as a comparison. However, the difference between the full analytic expression for the growth index and its asymptotic value \( \gamma = 0.545 \) is too small to be detected even for a half sky survey like Euclid for both the spectroscopic and the photometric cases.

**Case 2:**
Figure 4.3.3: Marginalised errors for the growth index \( \gamma \) at different redshift bins for the complete (left panel) and marginalised (right panel) galaxy power spectrum. The blue dashed and the red errors bars refer to the optimistic and realistic case, respectively. The black long dashed line is the growth index given by the Eq. (1.40).

Figure 4.3.4: Marginalised errors for the growth index \( \gamma \) for different redshift bins for the WL convergence spectrum for a Euclid-like survey.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( \sigma_{\gamma}^{\text{real}} )</th>
<th>( \sigma_{\gamma}^{\text{opt}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.0305</td>
<td>0.0296</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0231</td>
<td>0.0219</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0194</td>
<td>0.0183</td>
</tr>
<tr>
<td>1.2</td>
<td>0.0159</td>
<td>0.0147</td>
</tr>
<tr>
<td>1.4</td>
<td>0.0153</td>
<td>0.0139</td>
</tr>
<tr>
<td>1.6</td>
<td>0.0162</td>
<td>0.0141</td>
</tr>
<tr>
<td>1.8</td>
<td>0.0197</td>
<td>0.0153</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0281</td>
<td>0.0185</td>
</tr>
</tbody>
</table>

Table 4.3.4: Here are listed the 1\( \sigma \) errors for the growth index \( \gamma \) at different redshifts for the \( P(k) \) methods.

<table>
<thead>
<tr>
<th>( z )</th>
<th>( \sigma_{\gamma}^{\text{real}} )</th>
<th>( \sigma_{\gamma}^{\text{opt}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.0311</td>
<td>0.0286</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0286</td>
<td>0.0256</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0284</td>
<td>0.0250</td>
</tr>
<tr>
<td>1.2</td>
<td>0.0308</td>
<td>0.0268</td>
</tr>
<tr>
<td>1.4</td>
<td>0.0337</td>
<td>0.0292</td>
</tr>
<tr>
<td>1.6</td>
<td>0.0367</td>
<td>0.0318</td>
</tr>
<tr>
<td>1.8</td>
<td>0.0397</td>
<td>0.0342</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0427</td>
<td>0.0366</td>
</tr>
</tbody>
</table>

Table 4.3.5: Here are listed the 1\( \sigma \) errors for the growth index \( \gamma \) at different redshifts for the WL method.
For this case we chose the growth index to be $[130, 131]$

$$\gamma(z) = \gamma_0 + \gamma_a(1 - a) = \gamma_0 + \gamma_a \frac{z}{1 + z}$$

(4.33)

in order to compare the alternative theories introduced in the previous sections. Our reference cosmology is always $\Lambda$CDM with $\gamma_0 = 0.55620$ and $\gamma_a = 0.0182537$. We choose these values according to the Taylor expansion of Eq. (1.40) for the $\Lambda$CDM cosmology. In Fig. 4.3.5 the 1σ confidence regions for the parameters $w_0 - w_a$ (left panel) and $\gamma_0 - \gamma_a$ (right panel) for the galaxy power spectrum cases are shown.

For the WL case we found $\gamma_a$ to be degenerate with $\gamma_0$ indicating probably that a weak lensing experiment is insensitive to the variation of the growth index with redshift; this can also be seen in Fig. 4.3.4 where the errors of the growth index are basically the same for all the redshift bins above $z \simeq 1$. In Table 4.3.6 we report the errors for the parameters introduced above for all the three surveys.

We reach a sensitivity of about $\sigma_{w_0} = 0.018$, $\sigma_{w_0} = 0.068$ and $\sigma_{w_0} = 0.122$ for the galaxy power spectrum cases and WL survey, respectively. These errors are sufficiently small to rule out independently most of the models we considered in this paper, see Fig. 4.3.5 where we also plot the expected values of the equation of state parameters $w_0 - w_a$ and the growth index $\gamma_0 - \gamma_a$ for the alternative cosmological models: quintessence model (yellow box), DGP (blue diamond), LTB (black triangle) and $f(R)$ model (brown inverted triangle). Only a measurement of the equation of state parameter is able to rule out $w$CDM, DGP and LTB model; however, $f(R)$ is almost indistinguishable from the reference cosmology and none of the surveys assumed here are able to rule it out. Fortunately, it is also possible to measure the growth index $\gamma_0$; in this case we reach a sensitivity of about 0.02, 0.092 and 0.075 for the galaxy power spectrum cases and WL survey, respectively. On the other hand, measuring only $\gamma_0 - \gamma_a$, the model that could be clearly ruled out is DGP, while LTB could still be a viable theory. However, since we are dealing with two different surveys we can consider measuring one set of parameters for each experiment; for example, we can measure $w_0 - w_a$ with the WL survey and $\gamma_0 - \gamma_a$ with a $P(k)$ experiment. In this case the models considered in this work would be excluded at least at the 2σ level. With respect to $f(R)$ models, note that the Taylor expansion of $\gamma(a)$ does not work appropriately on the whole redshift range explored by the survey. Therefore, we have left this class of models out of Fig. 4.3.5. However, inspecting Fig. 4.3.7 one can easily conclude that models $f(R)$ would also be ruled out by the measurements of a
Figure 4.3.5: Confidence level for $w_0$, $w_a$ (left panel) and $\gamma_0$, $\gamma_a$ (right panel). The blue and light blue areas are for the galaxy power spectrum case, optimistic and realistic case respectively. The red dashed and solid lines are for the $mP(k)$ case, optimistic and realistic case respectively. The light green shaded area is for the WL case. These two plots correspond to case 2 of our analysis, taking $\gamma = \gamma_0 + \gamma_a(1 - a)$. Note that the vertical axis has been broken on the left panel plot to include the point corresponding to the LTB model. The points corresponding to the different models are the $w_0$, $w_a$ and $\gamma_0$, $\gamma_a$ pairs calculated for each model.

### Table 4.3.6

<table>
<thead>
<tr>
<th></th>
<th>$P(k)$</th>
<th></th>
<th>$mP(k)$</th>
<th></th>
<th>WL</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>real.</td>
<td>opt.</td>
<td>real.</td>
<td>opt.</td>
<td></td>
</tr>
<tr>
<td>$\sigma_{w_0}$</td>
<td>0.021</td>
<td>0.018</td>
<td>0.076</td>
<td>0.068</td>
<td>0.122</td>
</tr>
<tr>
<td>$\sigma_{w_a}$</td>
<td>0.051</td>
<td>0.041</td>
<td>0.375</td>
<td>0.324</td>
<td>0.524</td>
</tr>
<tr>
<td>$\sigma_{\gamma_0}$</td>
<td>0.022</td>
<td>0.020</td>
<td>0.102</td>
<td>0.092</td>
<td>0.075</td>
</tr>
<tr>
<td>$\sigma_{\gamma_a}$</td>
<td>0.120</td>
<td>0.116</td>
<td>0.339</td>
<td>0.296</td>
<td></td>
</tr>
</tbody>
</table>

Table 4.3.6: Here are listed the 1σ errors for $w_0$, $w_a$, $\gamma_0$ and $\gamma_a$ for the $P(k)$, $mP(k)$ and WL cases.
Table 4.3.7: Here are listed the 1σ errors for $w_0$ and $w_a$ for the $P(k)$, $mP(k)$ and WL cases.

<table>
<thead>
<tr>
<th></th>
<th>$P(k)$</th>
<th></th>
<th>$mP(k)$</th>
<th></th>
<th>$WL$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>real.</td>
<td>opt.</td>
<td>real.</td>
<td>opt.</td>
<td></td>
</tr>
<tr>
<td>$\sigma_{w_0}$</td>
<td>0.0052</td>
<td>0.0047</td>
<td>0.0063</td>
<td>0.0056</td>
<td>0.072</td>
</tr>
<tr>
<td>$\sigma_{w_a}$</td>
<td>0.0155</td>
<td>0.0135</td>
<td>0.0281</td>
<td>0.025</td>
<td>0.106</td>
</tr>
</tbody>
</table>

Euclid-like survey at the many $\sigma$ level.

**Case 3:**

In this case, the growth index $\gamma$ depends on the cosmological parameters $w_0 - w_a$ according to Eq. (1.40); the derivatives of the growth factor are given by:

$$\frac{\partial \log G}{\partial q_i} = - \int \left[ \frac{\partial \gamma}{\partial q_i} \log \Omega_m(z) + \gamma \frac{\partial \log \Omega(z)}{\partial q_i} \right] \Omega(z)^{\gamma} \frac{dz}{1 + z}$$

(4.34)

which has an extra term $\partial \gamma/\partial q_i$; our new set of parameters becomes now $q_i = \{w_0, w_a\}$.

In Fig. 4.3.6 we plot the 1σ confidence regions for $w_0 - w_a$ for all the three experiments; in the same figures are also plotted the values of the equation of state parameters $w_0 - w_a$ for the alternative cosmological models: quintessence model (yellow box), DGP (blue diamond) and LTB model (black triangle). We have not plotted the $f_S(R)$ model here because the Ansatz for $\gamma(a)$ is not valid in the whole range of redshifts.

However, in order to be able to include in the same graphic the other cosmological models we had to multiply the confidence regions by a factor of 5 for the galaxy power spectrum cases and by a factor of 3 the WL case; in the right panel of Fig. 4.3.6 we zoomed into the confidence region to preserve the proportions of the errors obtained with the three different methods. Giving a direct dependence to the growth index $\gamma$ on the equation of state parameters, the errors on $w_0 - w_a$ are reduced by a factor 4 for the $P(k)$ case, by a factor 10 for $mP(k)$ and by about a factor 2 for WL. In this case we reach an extreme sensitivity of about $0.5 - 1.0\%$ for $w_0$ and of about $2 - 3\%$ for $w_a$, however still not sufficiently good to rule out completely the $f(R)$ model studied.

In Fig. 4.3.7 we can see the evolution of the growth index with redshift for four different dark energy models: wCDM, DGP, $f(R)$ and LTB, and the 1σ errors one would expect to obtain from a Euclid-like survey with a fiducial $\Lambda$CDM cosmology. We find that using the galaxy power spectrum, one can reach (2%, 5%) errors in $(w_0, w_a)$ and (4%, 12%) errors...
Figure 4.3.6: Confidence level for $w_0$ and $w_a$ for all the three cases with the growth index given by Eq. (4.0). The red solid line and blue dashed line refer to the optimistic and realistic case, respectively. Note that the vertical axis has been broken on the left panel plot to include the point corresponding to the LTB model. Note that the right figure contains a scaled version of the left figure. For details see the text. The points corresponding to the different models are the $w_0$, $w_a$ and $\gamma_0$, $\gamma_a$ pairs calculated for each model.

in $(\gamma_0, \gamma_a)$, while using WL we get errors at least twice as large. These estimates allow us to differentiate easily between DGP, $f(R)$ Starobinsky model and $\Lambda$CDM, see Fig. 4.3.7, while it would be more difficult to distinguish the latter from a variable $w$CDM or LTB models using only the growth index.
Figure 4.3.7: The redshift evolution of the growth index compared with the expected errors from a Euclid-like survey. Note that DGP and the studied $f(R)$ models can in principle be ruled out at many sigma.
“Don’t adventures ever have an end? I suppose not. Someone else always has to carry on the story.”
– J. R. R. Tolkien

5

Conclusions

With the turn of the millennium, we have entered a new era in which cosmological observations have improved to the point that we can now start to define a Standard Model of Cosmology based on the CDM paradigm in addition to some sort of vacuum energy responsible for the observed dimming of distant supernovae. The nature of either dark matter (DM) or dark energy (DE) remains a mystery, in spite of the improved determinations of their contribution to the total energy density of the Universe.

While the nature of dark matter seems less uncertain (most cosmologists are in favour of a particle physics origin), that of dark energy is still unexplored territory. In the last decade there has been a plethora of proposals to account for the observed acceleration of the Universe. All these proposals fall into four main categories: i) the inclusion of some extra field (scalar, vector or tensor), coupled or not to the rest of matter, like in quintessence, chameleon, vector dark energy or massive gravity; ii) the extension of GR by inclusion of higher order terms in the Einsten-Hilbert action, like $f(R)$ theories, Gauss-Bonnet terms, etc.; iii) the modification of gravity on large scales by introduction of extra dimensions, like in the Dvali-Gabadadze-Porrati model, Kaluza-Klein gravity, etc.; iv) the reinterpretation in terms of a nontrivial spatial geometry, like in large-void inhomogeneous LTB models.

All of these proposals have very specific predictions for the background evolution of cosmological spacetimes, and most of them can be well fitted to the present observations.
with just a few phenomenological parameters: the equation of state, the speed of sound, the coupling between DM and DE, bulk viscosity, etc. However, in order to discriminate between the different alternatives it has been realised that one has to go beyond the background evolution and start to consider also the theory of linear cosmological perturbations and parameterise their evolution in terms of the growth function and growth index, as well as the shift parameter, amongst others.

At the moment, the main observables used to constrain the dark sector are the cepheids and supernovae magnitudes for the determination of the expansion rate as a function of redshift, the power spectrum of matter and the baryon acoustic oscillation (BAO) scale for the determination of the matter content as a function of redshift, together with the cosmic microwave background for the determination of global spatial curvature and the asymptotic values of cosmological parameters, and the weak lensing shear power spectrum, or the ISW-galaxy cross-correlation, for consistency of the whole scenario. Most of these measurements are rather preliminary and suggest a detection of DE at the 2 to 3 sigma level. However, a significant improvement is expected in the present decade thanks to future surveys such as DES, BOSS and Euclid.

During this thesis, we have studied the prospects that a survey like Euclid would have in distinguishing between the four main classes of DE models. In the process, we find exact and approximate solutions for the growth index in terms of simple functions, for $\Lambda$CDM models with a constant and variable equation of state parameter. We then propose a simple parameterisation of the growth index that fits well the recent history, except for extreme models like $f(R)$. We note that, while the background parameters $H(z)$, $\Omega_M(z)$ and $w(z)$ seem to be rather similar to those of $\Lambda$CDM, the growth index can differ significantly for most classes of models.

The parameterisation of the density contrast with the growth index was first introduced by Peebles in 1980. We knew that the rate of growth of structures should be a function of the matter density. Several parameterisation attempts including a power law expansion or simply the square root of the matter density parameter did not quite fit the data. Finally, the $\gamma$ parameterisation was the most widely accepted one; however, we will have to wait for direct measurements of the growth index, which will only be possible with the next generation of experiments. It is therefore tantalising to explore the possibilities of distinguishing between different DE models with a better determination of the growth index than what we have at present. This is the reason why we study a survey like Euclid, which will allow us to obtain information not only about the matter distribution (power spectrum $P(k)$ of perturbations) but also about the weak lensing (WL) spectrum.
We provide exact solutions to the cosmological matter perturbation equation in a homogeneous FLRW universe with a vacuum energy that can be parameterised with a variable equation of state parameter $w(a) = w_0 + w_a(1 - a)$. We compute the growth index $\gamma = \log f(a)/\log \Omega_m(a)$, and its redshift dependence, using the exact solutions in terms of Legendre polynomials and show that it can be parameterised as $\gamma(a) = \gamma_0 + \gamma_a(1 - a)$ for most cases, see Fig. 2.4.2. We then compare four different types of dark energy models: wCDM, DGP, $f(R)$ and a LTB-large-void model, which have very different behaviours at $z \sim 1$, see Fig. 4.3.7. This allows us to study the possibility to differentiate between various alternatives using full sky deep surveys like Euclid, which will measure both photometric and spectroscopic redshifts for several million galaxies up to redshift $z = 2$. We do a Fisher matrix analysis for the prospects of differentiating among the different DE models in terms of the growth index, taken as a given function of redshift or with a principal component analysis, with a value for each redshift bin, see also [209, 210] for a similar analysis for DGP and $f(R)$ models. We use as observables the complete and marginalised power spectrum of galaxies $P(k)$ and the Weak Lensing (WL) power spectrum. We find that using $P(k)$ one can reach (2%, 5%) errors in $(w_0, w_a)$ and (4%, 12%) errors in $(\gamma_0, \gamma_a)$, while using WL we get errors at least twice as large. These estimates allow us to differentiate easily between a DGP, a $f(R)$ Starobinsky model and ΛCDM, see Fig. 4.3.7, while it would be more difficult to distinguish the latter from a variable wCDM or LTB models using only the growth index.

It is also important to devise new observables and consistency checks that will allow us to further research the nature and origin of dark energy. For this purpose we have investigated the pairwise velocities of galaxies having in mind using them as cosmological tracers by means of an Alcock Paczynski (AP) style test, as recently proposed by [154]. We have analysed the dynamics of such objects within the semi-analytic models of [160] and [161] applied to the Millennium simulation [156], and studied the dependence of their relative velocity on local density, redshift, mass of the hosting subhaloes, $r$-band magnitude and stellar mass.

We have first analysed the dynamics of all galaxy pairs at redshift $z = 0$ (see Fig. 3.6.2). We have found that, on scales $d > 10$ h$^{-1}$ Mpc, the peculiar velocity is correctly predicted by linear theory [162, 163]. On the other hand, for separations $d < 3$ h$^{-1}$ Mpc, the pairs are decoupled from the Hubble flow and close to the static solution. We argue that pairs in this regime cannot be used as cosmological tracers (see Appendix B).

Being interested in investigating the claims by [154], we have studied the dynamics of galaxy pairs that are isolated within a radius of 4 h$^{-1}$ Mpc. At $z = 0$, isolated galaxy pairs
are almost comoving already for separations of $0.4 < d < 4 \, h^{-1}\text{Mpc}$ and only need up to 20% RSD correction (see Fig. 3.6.3). By analysing redshift slices up to $z = 1.504$, we have found that the peculiar velocities are only weakly dependent on the cosmological expansion ($< 10\%$ variation) for separations of $1 < d < 4 \, h^{-1}\text{Mpc}$ (see Fig. 3.6.6). Since expansion is the main property characterising a cosmological model, we might assume that in this regime the dynamics of isolated pairs are independent of the underlying cosmology. Hence, we argue that isolated pairs in this regime could possibly be used as cosmological tracers with minimal RSD corrections.

Imposing an isolation criterion of $4 \, h^{-1}\text{Mpc}$, as done in Ref. [154], greatly reduces the number of pairs (see Fig. 3.6.7). We have found that one can drastically increase the statistics while keeping the RSD corrections small by either reducing the isolation radius to $2 \, h^{-1}\text{Mpc}$ or allowing up to 10 galaxies to be neighbours of the pair. When dealing with observations, these adjustments may be helpful to reduce possibly large statistical uncertainties.

As galaxy surveys are flux limited, we have studied the feasibility of a measurement by varying the following properties of galaxy pairs: mass of the subhaloes that host the galaxies, $r$-band absolute magnitude in the rest-frame, and stellar mass. Low-mass pairs appear to be the best cosmological tracers, as RSD corrections increase with mass. More precisely, a nearly comoving regime is reached in our analysis only for subhalo masses of $m \lesssim 4.3 \times 10^{11} \, h^{-1}\text{M}_{\odot}$, corresponding to $r$-band magnitudes of $r_{\text{mag}} \gtrsim -21.27$ and stellar masses of $m_{\star} \lesssim 1.59 \times 10^{10} \, h^{-1}\text{M}_{\odot}$ (see, respectively, Figures 3.6.9, 3.6.12 and 3.6.13). We have also found that the peculiar velocities of galaxy pairs become more redshift-dependent as we increase the subhalo mass (see Fig. 3.6.6). Therefore, we suggest that isolated pairs may not be adequate as cosmological tracers if their mass or luminosity is above the given thresholds.

Marinoni and Buzzi [154] selected isolated galaxy pairs from the DEEP2 galaxy survey [211, 212] with comoving transverse separation $r_{\perp}$ in the range $20 \, \text{kpc}/h – 0.7 \, h^{-1}\text{Mpc}$. DEEP2 galaxies are known to reside in dark matter haloes of approximately $10^{12} \, \text{M}_{\odot}/h$ [212, 213]. The results of our analysis imply that such galaxy pairs do require corrections for evolution and a cosmology dependent RSD component, which is significant with respect to the evolution being measured.

Indeed, the primary concern for observational studies is the extent to which RSD “corrections” need to be modelled. Cosmological measurements from Baryon Acoustic Oscillation and RSD measurements on large-scales are reaching a precision at the 2-5% level [147, 214, 215], and it is therefore reasonable to suppose that this is also the level to which
we need to understand RSD corrections in order to make a useful contribution to the field from small-scale measurements. We have investigated whether selection based on local density can reduce the modelling burden, and have found that low-mass, isolated galaxy pairs are preferred. However, even for these galaxies, the corrections depend on sample properties, and would need to be recalculated for each cosmological model to be tested: the only currently available way to do this is via numerical simulations. We conclude that observations of close-pairs of galaxies do show promise for AP-style cosmological measurements, particularly for low mass, isolated galaxies. However, it is likely that modelling limitations will continue to be the limiting factor in the foreseeable future.

We have also explored the large-scale transition to homogeneity in a distribution of galaxies measured by a photometric survey (i.e., one in which we only have a precise angular position for the galaxies). We have seen that, by defining the angular fractal dimension as in Eq. (3.45), we obtain a correlation dimension that behaves as the three dimensional one in the sense that, on small scales, $H(\theta)$ lies below 1, while for large scales, it asymptotically reaches homogeneity (1 in the case of the angular fractal dimension $H(\theta)$ instead of 3, as is the case in the three dimensional analysis of $D_2(r)$).

We have seen that, even if the same behaviour is exhibited by both $H(\theta)$ and $D_2(r)$, there are projection effects present in the angular analysis that are not in the three dimensional case. These can be seen explicitly in Table 3.7.1, where homogeneity in the angular case is reached at much smaller scales than in the 3D case for equally-sized redshift bins. We see that if we reduce the bin size, the homogeneity scale is met on larger scales, more compatible with the 3D results from the WiggleZ survey. Aside from the projection effects, the angular analysis is intrinsically different, so it is not clear if comparing the homogeneity scale defined as when the fractal dimension is within 1% of 1, to the three dimensional homogeneity scale defined the same way, is compatible.

We have also looked into how the angular fractal dimension depends on bias and the photometric redshift uncertainty. We see that, for increasing bias, homogeneity is reached on larger scales. This is logical since, for larger biases, galaxies are more clustered. In the case of the photometric redshift uncertainty, for a larger error in redshift, homogeneity is reached on smaller scales. This is also consistent, since a bigger uncertainty means that more galaxies from other redshift bins than the one under consideration are taken into account, homogenising the sample.

It is important to note that, when using this homogeneity analysis on real data, if redshifts are translated into distances using an intrinsically homogeneous metric, the result will be homogeneous. This is not the case when dealing with the angular fractal dimen-
sion, since we measure angles directly and will not be biased towards homogeneity since no model-dependent conversion is needed. However, we understand this homogeneity analysis as a consistency check rather than a way to constrain cosmological parameters, since it contains the same information as the correlation function but can be harder to extract from the data. With the angular fractal dimension framework we have developed, we can apply it to photometric surveys such as DES and confirm the WiggleZ results.
Appendices
Small scale approximation

In this appendix we present the small scale solution found to the perturbation equation for the matter density contrast. The master equation for the perturbations in the small scale regime (i.e.: large wave number $k$) Eq. (1.27) can be solved by changing the independent variable from time to the scale factor,

$$a^2 \delta''(a) + (3 - \epsilon(a)) a \delta'(a) - \frac{3}{2} \Omega_m(a) \delta(a) = 0.$$  \hspace{1cm} (A.1)

where $\epsilon$ is given by

$$\epsilon(a) = -\frac{aH'}{H} = \frac{3}{2} \left[ 1 + w(a) \left( 1 - \Omega_m(a) \right) \right].$$

Furthermore, by noting that the density contrast for pure matter grows like $\delta(a) \propto a$, one can write the general function

$$\delta(a) = a \cdot G(a),$$  \hspace{1cm} (A.2)

with which the master equation becomes

$$a^2 G''(a) + \left( 5 - \epsilon(a) \right) a G'(a) + \left( 3 - \epsilon(a) - \frac{3}{2} \Omega_m(a) \right) G(a) = 0.$$  \hspace{1cm} (A.3)
Now we make a change of variables,

\[ u = -\frac{\Omega_{DE}}{\Omega_m} a^{-3w}, \quad \implies \quad a \frac{d}{da} = -3w u \frac{d}{du}, \quad \Omega_m(a) = \frac{1}{1 - u}, \quad (A.4) \]

and thus the master equation becomes

\[ u(1 - u) G''(u) + \left[ 1 - \frac{5}{6w} - \left( \frac{3}{2} - \frac{5}{6w} \right) u \right] G'(u) - \frac{1 - w}{6w^2} G(u) = 0, \quad (A.5) \]

which has the form of a Hypergeometric equation with constant coefficients \( a = (w - 1)/2w, \beta = -1/3w, \gamma = 1 - 5/6w, \) and thus the exact solutions are written in terms of two independent constants, \( C_1 \) and \( C_2, \)

\[ \delta(a) = C_1 a \, _2F_1 \left( \frac{w - 1}{2w}, \frac{-1}{3w}, 1 - \frac{5}{6w}; -\frac{\Omega_{DE}}{\Omega_m} a^{-3w} \right) \quad (A.6) \]
\[ + C_2 a^{1 + \frac{5}{6w}} \, _2F_1 \left( \frac{1}{2w}, \frac{1}{2} + \frac{1}{3w}, 1 + \frac{5}{6w}; -\frac{\Omega_{DE}}{\Omega_m} a^{-3w} \right). \quad (A.7) \]

The first term corresponds to the growing mode solution and the second one to the decaying mode. When describing late time solutions we will always take the growing mode solution; furthermore, the integration constant \( C_1 \) is not a problem, as we are interested in the evolution of the growth rate \( f(a) \) which is the ratio of the matter density and its derivative.

There are a number of way that one can play with the solution above in order to simplify the Hypergeometric function and even to drop the Hypergeometric functions. Here we use one relation which seems to be the easiest among all the others; we can notice that the third coefficient \( \gamma \) of the Hypergeometric function can be written as:

\[ \gamma = \frac{1}{2} + a + \beta \quad (A.8) \]

being \( a \) and \( \beta \) the first and second coefficient of the Hypergeometric function. In this case the solution for the matter density contrast can be written in terms of Legendre polynomials. However, being our goal to find an exact solution for the growth rate and growth index, we find easier to first evaluate these terms using the Hypergeometric functions and
then to simplify the result. The growth rate is defined as

\[
f(a) = \frac{a \delta'(a)}{\delta(a)} = 1 + a \frac{a \Omega_m'(a)}{\gamma \Omega_m^2(a)} F_1 \left[ a + 1, \beta + 1, \gamma + 1, 1 - \Omega_m^{-1}(a) \right],
\]

using Eqs. (15.4.12) and (15.4.21) of [216] we have:

\[
f(a) = 1 + 6w\alpha\beta \sqrt{1 - \Omega_m(a)} \left( \frac{P_{\beta-a-\frac{1}{2}}^{-\beta-a}}{P_{\beta-a-\frac{1}{2}}^{-2\beta-a}} \right) \left[ \frac{1}{\sqrt{\Omega_m(a)}} \right],
\]

being \( P_n^m(x) \) the Legendre polynomial. This last equation can be further simplified making use of the recurrence relations of the Legendre polynomials, see Eqs. (8.5.3) and (8.5.5) of [216], then we find

\[
f(a) = \Omega_m^{1/2}(a) \left( \frac{P_{5/6w}^{5/6w} \left[ \Omega_m^{-1/2}(a) \right]}{P_{-1/6w}^{5/6w} \left[ \Omega_m^{-1/2}(a) \right]} \right).
\]

We can then express the growth index \( \gamma(a) \) in terms of the Legendre polynomials, and we find:

\[
\gamma(a) = \frac{1}{2} + \frac{1}{\ln \Omega_m(a)} \ln \left[ \frac{P_{5/6w}^{5/6w} \left[ \Omega_m^{-1/2}(a) \right]}{P_{-1/6w}^{5/6w} \left[ \Omega_m^{-1/2}(a) \right]} \right].
\]

Note that the growth index parameter is close to 0.5 \pm 0.1, depending on the range of values of the equation of state parameter and the matter content of the Universe.

### A.1 Varying equation of state parameter

We can extend our discussion also to a varying dark energy equation of state parameter \( w(a) \). It is fair to be said that in the last case there is no exact analytic solution for the matter density contrast as it was for the case in which \( w \) is constant; the main problem here is that there is no a direct transformation between the scale factor \( a \) and the new variable \( u \).

Here we assume the equation of state parameter to be:

\[
w(a) = w_0 + w_a (1 - a),
\]

\( w_0 \) and \( w_a \) being the equation of state parameter at the present and at the time of decoupling, respectively. Under this assumption, the growth index parameter is given by:

\[
\gamma(a) = \frac{1}{2} + \frac{1}{\ln \left( \frac{P_{5/6w}^{5/6w} \left[ \Omega_m^{-1/2}(a) \right]}{P_{-1/6w}^{5/6w} \left[ \Omega_m^{-1/2}(a) \right]} \right)}.
\]
for which the matter density parameter can be integrated
\[
\Omega_m(a) = \left( 1 + \frac{\Omega_{DE}}{\Omega_m} a^{-3(w_0+w_a)} e^{3w_a(a-1)} \right)^{-1},
\] (A.14)

The master equation for a varying equation of state parameter still looks the same as Eq. (A.3), except that now \(w\) is a function of the scale factor. In order to integrate out we use again the variable
\[
u = -\frac{1 - \Omega_m}{\Omega_m} a^{-3\hat{w}(a)} \quad \text{with} \quad \hat{w}(a) = \frac{1}{\ln a} \int_1^a \frac{w(a')}{a'} \, da'.
\] (A.15)

However, we can make the approximation that the \(w(a)\) is slowly varying with time and integrate out the master equation. The approximation is not so rude. To see this, we need to have another look at the master equation for matter perturbations.

In this case the dark matter density contrast reads
\[
\delta(a) = a \, {}_1F_1 \left[ a, \beta, \frac{1}{2} + a + \beta; 1 - \Omega_m^{-1}(a) \right]
\] (A.16)

with parameters
\[
a = \frac{w(a) - 1}{2w(a)} - \frac{w_a}{6w^2(a)},
\] (A.17)
\[
\beta = -\frac{1}{3w(a)} + \frac{w_a}{6w^2(a)},
\] (A.18)

which gives a density growth function
\[
f(a) = \Omega_m^{1/2}(a) \frac{P^{5/6w(a)+w_a/6w^2(a)}}{P^{5/6w(a)-w_a/6w^2(a)}} \left[ \Omega_m^{-1/2}(a) \right].
\] (A.19)

Comparison of the numerical solution for the density growth function with the approximate expression Eq. (A.19) shows accordance within less than 0.1% for a very wide range of values of \(-2 < w_a < 2\).
Observing bound systems

B.1 Gravitationally bound systems

Consider applying the AP effect for bound systems, such as shown by the triangle OBC in Fig. B.1.1. Here we cannot relate $\Delta z$ to the proper distance using the Hubble parameter, or if we force this, we have to consider a peculiar velocity that cancels the expansion. We can write the observed redshift width

$$\Delta z^0 = \Delta z + \frac{dv_\parallel}{c} (1 + z).$$  \hfill (B.1)

If $v_\parallel$ is the orbital motion only $v_\parallel = v_{\text{orb}}$, then $\Delta z = 0$ as photons from both galaxies are subject to the same cosmological expansion.

We can calculate a variable with the units of distance as in [154]

$$\Delta x^0 = \Delta x + \frac{dv_\parallel}{H(z)} (1 + z).$$  \hfill (B.2)

But using the arguments above $\Delta x = 0$ and, as $dv_\parallel$ is independent of $H(z)$, using $H(z)$ to translate to distance does not provide any extra cosmological information. Hence any information, even if from apparent orientation of pairs, is independent of $H(z)$. 

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We now consider bound systems that have broken free from cosmological expansion. In general, the Hubble expansion velocity can be defined for any pair of particles in the Universe. With this definition, a static body has peculiar velocities that oppose and balance the expansion. To see this more clearly, consider an N-body simulation with a comoving coordinate system. In this coordinate system, a body that has a constant proper size would appear to be collapsing. In interpreting this system from an N-body simulation, one might consider this infalling velocity as a peculiar velocity, although in effect this is simply balancing the cosmological expansion. One therefore sees that there is a general interplay between the expansion rate and the peculiar velocities, which must be included in any interpretation of data.

Figure B.1.1: This figure shows schematically two collapsed regions. Objects \( A \) and \( B \), which are not gravitationally bound, will have different cosmological redshifts given by

\[
z_{A,B} = H_0 r_{A,B}^\text{com} + \frac{\vec{v}_{A,B} \cdot \vec{r}_{A,B}}{c},
\]

while gravitationally bound objects \( B \) and \( C \), shown in the second collapsed region, will have a line of sight component of their velocity that exactly cancels the expansion so that particles within that region will all have the same cosmological redshift.

Interpreting this in terms of local curvature, Fig. B.1.1 shows two collapsed regions being observed. In the standard interpretation, objects \( B \) and \( C \), which are in a collapsed system, have peculiar velocities that cancel any cosmological redshift between them. The infall peculiar velocity must therefore be

\[
v_{\text{pec}} = -H(z) r_{BC}. \tag{B.3}
\]

A light ray sent from \( B \) to \( C \), will experience a Doppler shift due to the motion of the objects towards each other in addition to the cosmological redshift. Assuming that the light ray
is emitted at a wavelength $\lambda_{em}$, the Doppler shift changes this wavelength to $\lambda_{dop}$ and the observed wavelength at $B \lambda_{obs}$. The change in the wavelength due to the Doppler shift, assuming velocities much smaller than the speed of light, is

$$\lambda_{dop} = \frac{\lambda_{em}}{1 + \frac{H(z)d}{c}}. \quad (B.4)$$

Due to the cosmological redshifting, the light ray is then observed at a wavelength of

$$\lambda_{obs} = \left(1 + \frac{H(z)d}{c}\right)\lambda_{dop}, \quad (B.5)$$

where we can substitute Eq. (B.4) and obtain

$$\lambda_{obs} = \lambda_{em}. \quad (B.6)$$

This shows that if one treats the redshift difference between two objects as including a cosmological expansion component, one cannot assume that the peculiar velocity is independent of cosmology: for bound systems their combined effect is zero. In an alternative and equally valid interpretation, $B$ and $C$ live in a flat Minkowski space-time, which does not lead to a cosmological redshift due to cosmological expansion. Photons only start to experience a cosmological redshift once they are free from the bound system, and subject to cosmological expansion: photons from $B$ and $C$ traveling to $O$ experience the same cosmological redshift. The line of sight radial velocity distribution is independent of $H(z)$ due to decoupling from cosmological expansion and hence, whatever the true expansion rate $H(z)$, we should expect the same velocities from isolated bound systems, which are simply all behaving as if they were in Minkowski space-time.

### B.2 Cosmological geometry on small scales

Structure in the Universe is clearly inhomogeneous on small scales, with maps of galaxy positions revealing a wealth of voids, clusters, walls and filaments [217, 218]. Rather than treat this structure as forming within a homogeneous and isotropic background, we should instead consider that space-time itself has a locally varying curvature. A discussion of how the evolution of mass should be considered is provided by [219], and further insight can be obtained from the spherical top-hat collapse model, which we now consider.
Birkhoff’s theorem states that a spherically symmetric gravitational field in empty space is static and is always described by the Schwarzschild metric \[ NJ^L \]. In this simplified picture, the behaviour of an homogeneous sphere of uniform density can itself be modelled using the same equations as the Universe as a whole, albeit with a different curvature that depends on the initial density. For a ΛCDM cosmology, the dynamical expansion of these spheres and the background is specified by the Friedmann equation

\[
\left( \frac{da_p}{d(H_0 t)} \right)^2 = \frac{\Omega_M}{a_p} + \varepsilon_p + \Omega_\Lambda a_p^2,
\]

where \( H_0, \Omega_M \) and \( \Omega_\Lambda \) are constants, given by their standard background values. The initial curvature \( \varepsilon_p \), which is a function of the perturbed initial over-density, alters the derived evolution of \( a_p \) (see, for example, \[ NJ^L \] for more details).

Based on the premise that “mass tells space how to curve”, we should consider that this equation reveals the behaviour of space-time within patches of the Universe. This simplified picture shows that not every part of the Universe behaves in the same way, and that evolution is dependent on density. Patches with significant over-density behave as closed universes, while under-dense regions behave as open universes.

The evolution of the scale factor for each of these patches as calculated using the spherical top-hat collapse model is shown in Fig. B.2.1a. This assumes a flat ΛCDM background cosmological model with \( \Omega_m = 0.25 \) and \( h = 0.7 \). The y-axis gives the scale factor of the perturbation, while the x-axis shows time.

Along with the local variation in the scale factor, we should also expect local variation in the Hubble parameter. In Fig. B.2.1b we plot this local Hubble parameter as a function of time for the same models as in Fig. B.2.1a. From this plot it is clear that collapsing regions (bottom four, red lines) have local Hubble parameters which are vastly different from the background (black, dashed line). However, even regions which are over-dense but not collapsing (blue lines) can have a significantly different local Hubble parameter.

The inhomogeneous nature of realistic perturbations means that, in general, they cannot collapse to singularities, but instead stabilise at finite size, meaning that the Hubble parameter does not asymptote to negative infinity. Energy considerations can then be used to determine the final radius and density of the virialised perturbation (e.g. for ΛCDM cosmologies see \[ 43 \], while for more general dark energy models see \[ 221 \] and references therein). For these virialised perturbations (if assumed to be close to spherical and
Figure B.2.1: This figure shows the evolution of the scale factor and the Hubble parameter depending on the local density of the patch of universe under consideration. The orange (top two) lines represent under-dense regions, the black, dashed line shows the background evolution, the two blue lines show that the effect of dark energy on small over-densities which have not collapsed sufficiently by the onset of dark energy domination is to keep them from collapsing. The bottom four red lines show the evolution of closed mini-universes, where collapse to a singularity is predicted within a fixed time.

homogeneous), Birkhoff’s theorem suggests that they evolve in a way that remains independent of the background (except for dark energy models where the dark energy density depends on the background scale only). We should therefore assume no further cosmological evolution in the scale associated with these regions.
The Fisher matrix

C.1 Gaussian likelihood function

Let us see, as an example, how to calculate the Fisher matrix elements for a Gaussian likelihood. Let us consider the following likelihood function:

\[
L(\theta_i; x_j) = \frac{1}{\sqrt{\det C}} \frac{1}{(2\pi)^{n/2}} \exp \left[ -\frac{1}{2} (x - \mu(\theta_i))^t C^{-1}(\theta_i) (x - \mu(\theta_i)) \right],
\]

where \(x\) is the vector with the data, \(\mu\) is the mean vector and \(C\) is the covariance matrix. Both \(\mu\) and \(C\) depend on the model parameters. The covariance matrix is defined as \(C = \langle (x - \mu)^t(x - \mu) \rangle\). Let us define \(D \equiv (x - \mu)^t(x - \mu)\). Denoting \(\mathcal{L} = -\ln L\), then

\[
2\mathcal{L} = \ln(\det C) + n \ln(2\pi) + (x - \mu)C^{-1}(x - \mu)^t
\]

\[
= \ln(\det C) + n \ln(2\pi) + \text{Tr}(C^{-1}D)
\]

\[
= \text{Tr}(\ln C) + n \ln(2\pi) + \text{Tr}(C^{-1}D)
\]

\[
= n \ln(2\pi) + \text{Tr}(\ln C + C^{-1}D),
\]
where we have used trace theorems and $\det A = \exp(\text{Tr}(\ln A))$. Differentiating Eq. (C.2) with respect to $\theta_i$

$$2 \frac{\partial L}{\partial \theta_i} = \text{Tr} \left[ \frac{\partial (\ln C)}{\partial \theta_i} + \frac{\partial (C^{-1}D)}{\partial \theta_i} \right]$$

$$= \text{Tr} \left[ C^{-1} \frac{\partial C}{\partial \theta_i} - C^{-1} \frac{\partial C}{\partial \theta_i} C^{-1}D + C^{-1} \frac{\partial D}{\partial \theta_i} \right], \quad (C.3)$$

where we have used the facts that $\frac{\partial (\ln C)}{\partial \theta_i} = C^{-1} \frac{\partial C}{\partial \theta_i}$ and $\frac{\partial (C^{-1})}{\partial \theta_i} = -C^{-1}(\frac{\partial C}{\partial \theta_i})C^{-1}$. Differentiating again with respect to $\theta_j$:

$$2 \frac{\partial^2 L}{\partial \theta_i \partial \theta_j} = \text{Tr} \left[ C^{-1} \frac{\partial C}{\partial \theta_i} C^{-1} \frac{\partial C}{\partial \theta_j} + C^{-1} \frac{\partial^2 D}{\partial \theta_i \partial \theta_j} \right], \quad (C.4)$$

This means that the Fisher matrix is given by

$$F_{ij} = \left\langle \frac{\partial^2 L}{\partial \theta_i \partial \theta_j} \right\rangle = \frac{1}{2} \text{Tr} \left[ C^{-1} \frac{\partial C}{\partial \theta_i} C^{-1} \frac{\partial C}{\partial \theta_j} + C^{-1} \frac{\partial^2 D}{\partial \theta_i \partial \theta_j} \right]$$

$$= \frac{1}{2} \left[ C_{i}^{-1}m \frac{\partial C_{mn}}{\partial \theta_i} C_{nk}^{-1} \frac{\partial C_{kl}}{\partial \theta_j} + C_{im}^{-1} \frac{\partial \mu_i}{\partial \theta_i} \frac{\partial \mu_m}{\partial \theta_j} \right]. \quad (C.5)$$

### C.2 The Fisher matrix for the galaxy power spectrum

Let us suppose we have a galaxy survey that measures the galaxy power spectrum $P(k)$ for a density of galaxies $n(z)$ distributed in redshift bins $[z, z + \Delta z]$, so we have

$$\Delta_k^2 \equiv \langle \delta_k^* \delta_k \rangle = P(k, z) + \frac{1}{n(z)}, \quad (C.6)$$

where the term $1/n$ accounts for the Poisson shot noise coming from measuring a discrete number of galaxies. Since the average galaxy density is estimated from the survey itself, we have by construction $\langle \delta(x) \rangle = 0$, and therefore $\langle \delta_k \rangle = 0$ for any $k$. The coefficients $\delta_k$ are complex variables in which both real and imaginary parts obey the same Gaussian statistics. This means that we can calculate the Fisher matrix for the real part and the whole Fisher matrix will be the sum of two identical Fisher matrices \textit{i.e.} twice the result for the real part. However, when we count the total number of independent modes, we have to remember that only half of them are statistically relevant, since $\delta_k^* = \delta_{-k}$, so we should

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divide the total result by two.

If we assume the galaxy distribution to be well described by a Gaussian function, then we can write the likelihood as

\[ L = \frac{1}{(2\pi)^{m/2} \prod_i \Delta_i} \exp \left[ -\frac{1}{2} \sum_i \frac{\delta_i^2}{\Delta_i^2} \right], \tag{C.7} \]

where \( \Delta_i = \Delta_k \) and \( \delta_i = \text{Re} \delta_k \) assuming that the measures at every \( k_i \) are statistically independent. Now we compute \( \mathcal{L} = -\ln \mathcal{L} \), giving

\[ \mathcal{L} = \frac{m}{2} \ln (2\pi) + \sum_i \ln (\Delta_i) + \frac{1}{2} \sum_i \frac{\delta_i^2}{\Delta_i^2}. \tag{C.8} \]

Now, let us differentiate with respect to the \( l \)-th parameter. To simplify notation, let us denote \( \partial A/\partial \theta_l = A_{,l} \).

\[ \mathcal{L}_{,l} = \sum_i \frac{\Delta_{i,l}}{\Delta_i} - \sum_i \frac{\delta_i^2}{\Delta_i^3} \Delta_{i,l}. \tag{C.9} \]

Differentiating again with respect to a second parameter, say \( \theta_m \), we get

\[ \mathcal{L}_{,lm} = \sum_i \frac{\Delta_{i,lm}}{\Delta_i} \Delta_i - \sum_i \frac{\delta_i^2}{\Delta_i^3} \Delta_{i,lm} - \sum_i \frac{\Delta_{i,lm} \Delta_i^3 - 3 \Delta_{i,l} \Delta_{i,m} \Delta_i^2}{\Delta_i^6} \]. \tag{C.10} \]

The Fisher matrix elements will be given by \( F_{lm} = \langle \mathcal{L}_{,lm} \rangle \), which leaves us with

\[ F_{lm} = \sum_i \left[ \frac{\Delta_{i,lm}}{\Delta_i} \frac{\Delta_{i,m}}{\Delta_i^2} - \frac{\delta_i^2}{\Delta_i^3} \left( \frac{\Delta_{i,lm}}{\Delta_i} \frac{\Delta_{i,m}}{\Delta_i^4} \right) \right]. \tag{C.11} \]

Now, since \( \langle \delta_i^2 \rangle = \Delta_i^2 \), then

\[ F_{lm} = 2 \sum_i \left[ \frac{\Delta_{i,l} \Delta_{i,m}}{\Delta_i^2} \right]. \tag{C.12} \]

From Eq. \( \text{(C.6)} \), we have that \( \Delta_i = \sqrt{P(k, z) + 1/n(z)} \). now, forgetting about the sum.
over the modes for the time being (it will be retaken later), we have that

$$\Delta_{i,k} = \frac{1}{2} \frac{P(k, z)}{\sqrt{P(k, z) + \frac{1}{n(z)}}},$$  \hspace{1cm} \text{(C.13)}$$

which means that the elements of the Fisher matrix in terms of the power spectrum are

$$F_{lm} = \frac{1}{2} \frac{\partial \ln P(k, z)}{\partial \theta_l} \frac{\partial \ln P(k, z)}{\partial \theta_m} \frac{P^2(k, z)n^2(z)}{[1 + n(z)P(k, z)]^2}. $$  \hspace{1cm} \text{(C.14)}$$

Now let’s retake the sums in the $k$ modes. We can approximate the sum with an integral over $k$, however, one must be careful to count how many modes fall into a bin defined by the interval $[k, k + dk]$ and the cosine interval $d\mu$. This translates into a Fourier volume of $2\pi k^2 dk d\mu$. The number of modes available to us is limited by two factors: the size of the survey and the shot noise. Modes larger than the survey volume cannot be measured and shot modes sampled only by a few galaxies cannot be reliable. To take into account these two shortcomings, we discretise the Fourier space into cells of volume $V_{\text{cell}} = (2\pi)^3 / V_{\text{survey}}$, which means that we have $2\pi k^2 dk d\mu / V_{\text{cell}} = (2\pi)^{-2} V_{\text{survey}} k^2 dk d\mu$ modes inside the survey volume. Putting all of this together in our expression for the Fisher matrix, we finally obtain

$$F_{lm} = \frac{1}{8\pi^2} \int_{-1}^{1} d\mu \int_{k_{\text{min}}}^{k_{\text{max}}} \frac{\partial \ln P(k, z)}{\partial \theta_l} \frac{\partial \ln P(k, z)}{\partial \theta_m} \left[ \frac{n(z)P(k, z)}{1 + n(z)P(k, z)} \right]^2 k^2 dk V_{\text{survey}}. $$  \hspace{1cm} \text{(C.15)}$$

The factor

$$\left[ \frac{n(z)P(k, z)}{1 + n(z)P(k, z)} \right]^2 V_{\text{survey}}$$  \hspace{1cm} \text{(C.16)}$$

can be seen as an effective survey volume. When $nP \gg 1$, the sampling is good enough to derive all the cosmological information that can be extracted from the survey at hand and there is no need for more sources. For $nP \ll 1$, the effective volume is severely reduced. It is important to note that what we calculated here is the Fisher matrix for one redshift bin. If the data is subdivided in several of these bins, (assuming they are independent), one can sum the Fisher matrices for each bin.
References


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