Departamento de Matemáticas
Facultad de Ciencias
Universidad Autónoma de Madrid

Problemas de interfase fluida en flujos incompresibles

Francisco Gancedo García

Tesis doctoral dirigida por Diego Córdoba Gazolaz

Madrid, Abril de 2007

Presentación

En esta memoria se consideran dos problemas de interfase fluida. En el capítulo primero tratamos el problema de Muskat, que describe la evolución de dos fluidos incompresibles con diferentes características a través de un medio poroso. En el segundo capítulo presentamos la dinámica de frentes mediante la ecuación quasi-geostrófica, modelando la evolución de un frente de temperatura en la atmósfera.

La dinámica de fluidos en medios porosos se modela mediante la ley experimental de Darcy. Este principio físico, descubierto por Darcy en 1856, proporciona una descripción macroscópica del flujo de manera que la velocidad del fluido es proporcional al gradiente de presión más las fuerzas externas. La ley de Darcy viene dada por una ecuación de momento que involucra la viscosidad y densidad del fluido, la permeabilidad del medio poroso y la aceleración de la gravedad.

El problema de Muskat modela, haciendo uso de la ley de Darcy, la evolución de la interfase dada por dos fluidos en un medio poroso con diferentes viscosidades y densidades. El problema fue propuesto por Muskat en 1934 en un estudio sobre la invasión de agua en petróleo a través de medios porosos.

Un problema físico diferente es la evolución de un fluido bidimensional en una celda de Hele–Shaw. La celda de Hele–Shaw fue inventada por Hele–Shaw en 1898 y consiste en dos láminas paralelas de cristal. Estas láminas están situadas suficientemente próximas de manera que el fluido que se introduce entre éstas sólo se mueve en dos dimensiones esencialmente. De esta forma, la evolución del flujo es tal que la velocidad media es proporcional al gradiente de presión más las fuerzas externas. La dinámica depende de la distancia entre las láminas, la viscosidad y, si la celda de Hele–Shaw no está en posición horizontal, de la densidad del fluido y la gravedad.

El problema de interfase fluida en una celda de Hele–Shaw modela la evolución de la frontera libre dada por dos fluidos con diferentes viscosidades y densidades en una celda de Hele–Shaw. Este problema fue considerado en 1958 por Saffman y Taylor en un estudio de la interfase entre agua y petróleo. Propusieron el trabajo para modelar el problema de Muskat en dimensión dos, ya que aunque los fenómenos físicos son diferentes, se vuelven matemáticamente análogos si la permeabilidad del medio es proporcional al cuadrado de la distancia entre las láminas.

El problema de Muskat y el flujo de interfase en celda de Hele–Shaw ha sido ampliamente estudiado. Estos problemas de frontera libre pueden ser modelados usando la condición de Laplace-Young, dando la presión con un salto de discontinuidad a lo largo de la interfase que es igual a la curvatura local multiplicada por la tensión superficial. Sin tensión superficial las presiones de los fluidos son iguales sobre la interfase. Con tensión superficial, en el caso bidimensional, los problemas tienen soluciones clásicas. Sin tensión superficial, en el caso de fluidos de igual densidad pero diferentes viscosidades (o una celda de Hele–Shaw horizontal), el problema está mal propuesto si el fluido de mayor densidad se contrae. Por otro lado, existen soluciones para todo tiempo cuando el dato inicial es pequeño y el fluido de mayor densidad se expande. La relación entre la densidad del fluido y la viscosidad para que el problema esté bien propuesto es conocida, así como estimaciones de energía de la interfase suponiendo propiedades geométricas en la interfase localmente en tiempo.

La ecuación quasi-geostrófica bidimensional (QG) es un sistema importante en dinámica de fluidos geofísicos. En geofísica, la evolución de fluidos atmosféricos y oceánicos se modelan considerando la importancia de la fuerza de Coriolis en la dinámica. Concretamente, QG proporciona soluciones particulares de la evolución de temperatura de un sistema quasi-geostrófico general para números pequeños de Rossby y Ekman. En estos sistemas se considera estratificación uniforme, vorticidad potencial y rotación rápida.

Desde un punto de vista matemático, el principal interés de QG yace en las fuertes analogías con la ecuación de Euler tridimensional. Estas analogías fueron primero introducidas por Constantin, Majda y Tabak en un trabajo donde presentaron esta ecuación como posible modelo en frontogénesis. Este término técnico se usa en el estudio de formación y evolución de grandes frentes de frío y calor, y la mezcla entre ellos. Desde entonces, esta ecuación ha sido una fuente de inspiración para la ecuación de Euler, de tal manera que los principales resultados para QG se pueden extender para Euler.

En esta tesis estudiaremos la evolución de un frente mediante QG. Consideraremos un dato inicial con una temperatura que toma dos valores constantes en dominios complementarios y cuya evolución es descrita por QG. Este caso es similar al problema 2-D vortex patch, en que la vorticidad viene dada por la función característica de un dominio que evoluciona mediante la ecuación de vorticidad de Euler bidimensional.

Sin embargo nuestro problema es más singular que 2-D vortex patch. En éste, la velocidad es más regular ya que viene dada por la ley de Biot-Savart. Para el problema de frentes en QG, la velocidad está al mismo nivel que la temperatura, ya que la relación entre ellas se da mediante transformadas de Riesz.

La evolución de frentes por QG fue considerado primero por Rodrigo. Él dio la velocidad del frente en la dirección normal y encontró la ecuación de evolución en términos de la función que representa el frente. También probó existencia local y unicidad para un frente periódico e infinitamente diferenciable usando la iteración de Nash-Moser.

Un modelo a caballo entre 2-D vortex patch y frentes en QG fue presentado por Córdoba, Fontelos, Mancho y Rodrigo como el modelo α-patch. Este sistema proporciona una familia de ecuaciones de contorno que dependen de un parámetro α con $0 < \alpha \leq 1$ de tal forma que tendiendo α a cero se obtiene 2-D vortex patch, mientras que el caso $\alpha = 1$ se corresponde con frentes en QG. Probaron existencia local para un frente infinitamente diferenciable y presentaron evidencia de singularidades en tiempo finito. Más específicamente, dieron datos iniciales en los que numéricamente se observaba que la curvatura explotaba debido a que dos patches colapsaban en un punto de una forma autosimiliar.

Conclusiones

En el primer capítulo de esta memoria se estudia el problema de Muskat en dos y tres dimensiones. Consideraremos el caso de dos fluidos incompresibles de igual viscosidad y diferentes densidades. Este caso modela por ejemplo la dinámica de regiones húmedas y secas en un medio poroso. Debido a la forma particular de la vorticidad en este caso, conseguimos reescribir la evolución de la interfase en términos de una función, y se evita que la interfase colapse y por tanto que se produzca una singularidad en el fluido. Mostramos que la conservación de masa se satisface si la ecuación de evolución se cumple y probamos que cuando el fluido más denso está debajo del menos denso (caso estable), el problema está bien propuesto. Cuando el fluido menos denso está debajo del más denso, probamos que el problema está mal propuesto. Damos soluciones globales del caso estable para datos iniciales próximos a cero, y acotamos el máximo y el mínimo de las soluciones para todo tiempo. Finalmente analizamos la ecuación de conservación de masa en medios porosos para un dato inicial regular en dos dimensiones, probando explosión en tiempo finito para dato inicial de energía infinita, y dando criterios de explosión con energía finita.

En el segundo capítulo de este escrito presentamos una prueba de existencia local para un frente evolucionando mediante QG y el modelo α-patch para un contorno cerrado en espacios de Sobolev. Estos resultados se obtienen consiguiendo control local de la evolución de una cantidad que indica cuando el contorno es inyectivo y parametrizado con velocidad positiva. Notar que ésto es crucial ya que los operadores involucrados en la ecuación de contorno están mal definidos en otro caso. Para QG se necesita cambiar la velocidad en la dirección tangencial sobre el contorno para conseguir una cancelación extra, ya que en este caso el operador involucrado en la evolución pierde dos derivadas. Sin embargo este cambio no modifica la geometría del contorno, ya que la velocidad tangencial solo mueve las partículas sobre la curva. En el caso $0 < \alpha < 1$ probamos unicidad de solución.

Abstract

In this dissertation we consider two problems of fluid interface. In the first chapter we treat the Muskat problem, which describes the dynamics of two incompressible fluids with different characteristics, flowing through a porous medium. In the second chapter we present the dynamics of sharp fronts for the quasi-geostrophic equation, where the evolution of a front of temperature through the atmosphere is modelled.

The evolution of fluids in porous media is described using the experimental Darcy's law. This physical principle, first noted by Darcy in 1856, provides a macroscopic description of a flow where the velocity of the fluid is proportional to the pressure gradient and the external forces. Darcy's law is given by a momentum equation involving the viscosity and density of the fluid, the permeability of the medium, and the acceleration due to gravity.

The Muskat problem models, by Darcy's law, the evolution of an interface between two fluids in a porous medium with different viscosities and densities. The problem was proposed by Muskat (1934) in a study about the encroachment of water into oil in a porous medium.

A different physical phenomena is the evolution of a two-dimensional fluid in a Hele-Shaw cell. The Hele-Shaw cell was invented by Hele-Shaw in 1898, and consists of two parallel sheets of glass. The plates are set close enough together, so that the fluid placed between them essentially only moves in two directions. In this configuration, the mean velocity of the fluid is proportional to the pressure gradient and the external forces. The dynamics depend on the distance between the plates, the viscosity and, if the Hele-Shaw cell is not horizontal, the fluid density and gravity.

The two-phase Hele-Shaw flow models the free boundary problem given by two fluids in a Hele-Shaw cell with different viscosities and densities. This was first considered in 1958 by Saffman and Taylor, where the interface between water and oil was studied. They proposed the problem as a model of the Muskat problem in two dimensions, as the different physical phenomena become mathematically analogous when the permeability of the medium is proportional to the square of the distance between the plates.

The Muskat problem and the two-phase Hele-shaw flow have been extensively considered. These free boundary problems can be modelled with a surface tension, so that the pressure is given by a jump discontinuity across the interface that is equal to the local curvature times the surface tension. This is known as the Laplace-Young boundary condition for the pressure function. With surface tension, and in the two-dimensional case, the problems have classical solutions. Without surface tension, the pressures of the fluids are equal on the free boundary. In the case of fluids of the same density but with different viscosities (or a horizontal Hele-

Shaw cell), the problem is ill-posed when the higher-viscosity fluid contracts. On the other hand, there exist global-in-time solutions when the initial data is near planar and the higher-viscosity fluid expands. For the problem to be well-posed, in the two-dimensional case, the required relation between the density and the viscosity of the fluids is known, together with the energy estimates (of the interface) when local-in-time geometric properties of the interface are assumed.

In this dissertation we study the Muskat problem in two and three dimensions. We consider the case with two incompressible fluids of equal viscosities, but with different densities. This problem models, for example, the dynamics of moist and dry regions in a porous medium. Due to the particular form of the vorticity in this case, we can rewrite the evolution equation of the free boundary in terms of a function, and avoid a kind of singularity in the fluid when the interface collapses. We show that the conservation of mass is satisfied if this evolution equation is fulfilled and we prove that when the denser fluid is below the less dense fluid (stable case), the problem is well-posed. When the less dense fluid is below the denser fluid, we prove that the problem is ill-posed. We show global solutions of the stable case for near planar initial data, and bound the maximum and the minimum of the solutions for all time. Finally, we analyze the conservation of mass equation in the porous medium for a regular initial density in two dimensions, proving blow-up for initial data with infinite energy, and giving blow-up criteria for solutions with finite energy.

The two dimensional quasi-geostrophic system (QG) is an important equation in geophysical fluid dynamics. In geophysics, the evolution of atmospheric and oceanic flows are modelled considering the importance of the Coriolis force in the dynamics of the fluids. Specifically, the QG equations give particular solutions of the evolution of the temperature from a general quasi-geostrophic system for small Rossby and Ekman numbers. In these systems uniform stratification and potential vorticity are considered together with fast rotation.

From a mathematical point of view, the main interest of the QG equation lies in the strong analogies with the three dimensional Euler equation. These analogies were first introduced by Constatin, Majda and Tabak, where this equation was presented as a possible model in frontogenesis. This technical term is used in the study of the formation and evolution of strong fronts of hot and cold air, and the mixture between them. Since then, this equation has been a source of inspiration for the three dimensional Euler equation, in such a way that the main results for QG can be extended to the Euler equation.

In this dissertation we study the evolution of a sharp front for the QG equation. That is to say, we consider an initial temperature that takes two constant values in complementary domains, and the evolution is given by the QG equation. This is similar to the 2-D vortex patch problem, in which the vorticity is given by the characteristic function of a domain, and the evolution is governed by the 2-D Euler equation in its vorticity form.

However, our problem is more singular than the 2-D vortex patch problem. In the 2-D vortex patch problem, the velocity is more regular due to fact that is given by the Biot-Savart law. In the QG sharp front problem, the velocity is only as regular as the temperature, as the relationship between them is given by Riesz transforms.

The evolution of a sharp front for the QG equation was first considered by Rodrigo. He gave the velocity of the front in the normal direction and found a closed system only in terms of the function that represents the front. He also proved local-existence and uniqueness for a periodic and infinitely differentiable front using the Nash-Moser iteration.

A model on the borderline between the 2-D vortex patch problem and the QG sharp front was presented by Córdoba, Fontelos, Mancho and Rodrigo as the α-patch model. This system provides a family of contour dynamics equation, depending on a parameter α for $0 < \alpha \leq 1$ in such a way that, letting α tend to 0, the 2-D vortex patch problem is obtained, while the case $\alpha = 1$ corresponds to a sharp front in QG. They proved local-existence for a periodic infinitely differentiable front and presented evidence of singularities in finite time. More specifically, they gave initial data in which the curvature blows up in numerical simulations due to two patches collapsing in a point in a self-similar way.

In this thesis we present a proof of the local-existence for a front convected by the QG equation and the α-model for a closed contour in Sobolev spaces. These results are obtained by getting local control of the evolution of a quantity which indicates when the contour is one-to-one and parameterized with a positive velocity. We note that this is crucial due to the fact that the operators involved in the equations are ill-defined otherwise. For QG we need to change the velocity in the tangential direction on the contour in order to get an extra cancellation, as in this case the operator involved in the evolution loses two derivatives. This change does not alter the geometry of the contour, as the tangential velocity only moves the particles on the curve. For $0 < \alpha < 1$ we show uniqueness.

Agradecimientos

En estas líneas me gustaría ser capaz de expresar mi más sincera gratitud a todas las personas que me han ayudado y han hecho posible esta tesis a lo largo de estos cuatro años de duro trabajo.

En primer lugar quisiera agradecer a mi director Diego Córdoba por transmitirme su pasión por las matemáticas y haberme motivado durante todo este tiempo. Este trabajo es el resultado de su consejo y apoyo en los momentos difíciles. Quiero resaltar su forma de trabajo y el magnífico ambiente laboral que crea, además de darle las gracias por obligarme a ir al fútbol los jueves.

También quiero mostrar mi gratitud a Antonio Córdoba por la incalculable ayuda y simpatía que me ha brindado cuando me he presentado en su despacho en busca de consejo. Los cursos de mecánica de fluidos que impartió me abrieron las puertas de un mundo fascinante.

Me gustaría agradecer a Charles Feffereman, Juan José L. Velázquez y a Luis Vega por mostrar interés en mi trabajo y darme la oportunidad de hablar con ellos de matemáticas. También reconocer a Juan José L. Velázquez la ardua tarea de ser revisor de esta tesis.

Dar las gracias a mi colaborador Rafael Orive, mis compañeros del CSIC y de la Universidad Autónoma de Madrid, por el buen ambiente que respiré con ellos y la interesante plática en las comidas. En especial a todos mis compañeros de despacho, por soportarme y resolverme mis dudas laborales.

Quisiera dar las gracias al gran número de buenísimos profesores de matemáticas que he tenido en mis años académicos por enseñarme la belleza de esta ciencia.

Agradecer a mis padres y mis hermanas por respaldarme siempre en todas las decisiones difíciles que he tomado y apoyarme incondicionalmente. También a mis titos y primos madrileños por hacerme sentir como en el pueblo en mis primeros días por Madrid.

Dar las gracias a mis amigos "madrileños" Dani y Keith, por quitarme presión en los momentos difíciles, ayudarme con el inglés y hacerme un poquito más internacional. También a mis compis de piso y amigos Karim y Juan Carlos; su inestimable compañía durante este tiempo me ha ayudado muchísimo.

Gracias a todos mis amigos de Sevilla; siempre han estado cuando les he necesitado. También a los amigos del pueblo; que me perdonen no haber ido tanto durante los últimos años.

Finalmente, me gustaría agradecer sobremanera a mi pareja María, por apoyarme en todo momento en mi aventura madrileña y ser tan fuerte durante estos cuatro años en los que hemos estado viviendo en ciudades diferentes. Sin ella este trabajo no hubiese sido posible.

A mi familia

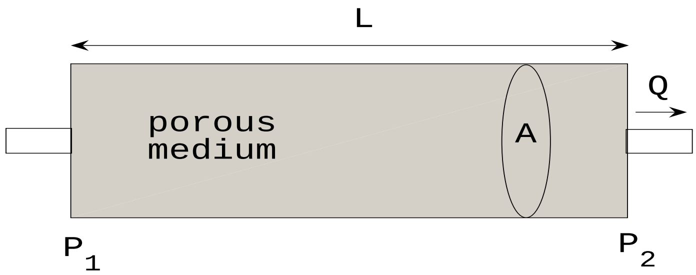
Contents

Chapter 1

Contour dynamics of incompressible fluid in porous media.

1.1 Introduction

The evolution of a fluid in a porous medium is an important topic in fluid mechanics (see [3] and [4]) encountered in engineering, physics and mathematics. This phenomena was described by H. Darcy in 1856 [24] while studying the fountains of the city of Dijon, France. Darcy formulated a principle based on the results of experiments on the flow of water through vertical homogeneous sand filters (see [3] for more details). For a horizontal flow, or a flow in absence of gravity,



the conclusions of Darcy give the following relation between the total discharge Q (volume per time), the cross-sectional area A, the dynamic viscosity of the fluid μ, the pressure drop $P_1 - P_2$, and the length of the pressure drop L:

$$\mu \frac{Q}{A} = -\kappa \frac{P_1 - P_2}{L}.$$

In this formula κ is the permeability of the porous medium, which measures the ability of the medium to transmit a fluid (see [4] Table 1.1 to find permeabilities of several isotropic porous media). In modern notation, Darcy's law, considered with the force of gravity, for a 3-D fluid, is given by the momentum equation

$$\frac{\mu}{\kappa} v = -\nabla p - (0, 0, \mathrm{g}\,\rho),$$

where v is the incompressible velocity, p is the pressure, μ is the dynamic viscosity, κ is the permeability of the isotropic medium, ρ is the liquid density, and g is the acceleration due to gravity. Darcy's law has been determined by the results of many experiments, and has been deduced from the Stokes equation using homogenization [47].

In 1934, Muskat [34] treated the movements of ground water and its interaction with oil-bearing sands where the contact of the inmiscible fluids creates a moveable interface. He was interested in how this could affect the production of oil. Muskat studied the problem using Darcy's law in each region and pointed out the continuity of the normal velocity and the pressure on the interface, and the differences between the viscosities and the densities. The study of the dynamics of the interface between fluids with different viscosities and densities through a porous medium has henceforth been known as the Muskat problem.

A interesting problem of fluid mechanics is the motion of a 2-D flow in a Hele–Shaw cell. This physical phenomena, studied by Hele–Shaw in 1898 (see [29] and [30]), consists of the dynamics of a fluid trapped between two fixed parallel plates, that are close enough together, so that the fluid essentially only moves in two directions. Considering plates at a distance b apart, sufficiently small so that the first component of the velocity V can be safely assumed to be zero, the equation is derived (see [28]) easily from the Stokes equations

$$\rho(V \cdot \nabla V) = -\nabla p + \mu \Delta V - (0, 0, \mathrm{g}\,\rho), \quad \operatorname{div} V = 0,$$

which can be reduced to the system

$$\begin{aligned} 0 &= -\partial_{x_1} p, \\ \rho(V_2 \partial_{x_2} + V_3 \partial_{x_3}) V_2 &= -\partial_{x_2} p + \mu \Delta V_2, \\ \rho(V_2 \partial_{x_2} + V_3 \partial_{x_3}) V_3 &= -\partial_{x_3} p + \mu \Delta V_3 - \mathrm{g}\,\rho. \end{aligned}$$

As b is sufficiently small, we can also assume that the derivatives of V_2 and V_3 in the directions x_2 and x_3 are negligible compared to the derivatives in x_1, to obtain the system

$$\begin{aligned} \partial_{x_1} p &= 0, \\ \partial_{x_2} p &= \mu \partial_{x_1}^2 V_2, \\ \mathrm{g}\,\rho + \partial_{x_3} p &= \mu \partial_{x_1}^2 V_3. \end{aligned}$$

We also assume that the velocity of the fluid reaches its maximum in the middle of the cell, and is zero at the plates given by the points $x_1 = 0$ and $x_1 = b$ (see figure 1.1), so that the previous equation become

$$\mu V_2 = \frac{1}{2}(x_1^2 - b\, x_1) \partial_{x_2} p, \quad \mu V_3 = \frac{1}{2}(x_1^2 - b\, x_1)(\partial_{x_3} p + \mathrm{g}\,\rho).$$

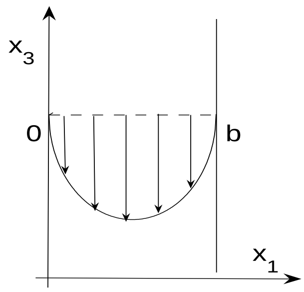


Figure 1.1: Gap in a Hele–Shaw cell.

The integral means of these quantities over the distance between the plates, give the Hele–Shaw equation as follows:

$$\frac{12\mu}{b^2}v = -\nabla p - (0, \mathrm{g}\,\rho),$$

for a mean velocity v depending on two spatial variables.

The evolution of fluid in a porous medium becomes mathematically analogous to that of a Hele–Shaw cell, if we suppress one of the variables in the horizontal plane and identify the permeability of the medium κ with the constant $b^2/12$. This was considered by Saffman and Taylor in 1958 [41] in a study of the dynamics of the interface between two fluids with different viscosities and densities in a Hele–Shaw cell. Saffman and Taylor treated the interface between water and oil, giving a two-dimensional model of the problem proposed by Muskat. Thus, the free boundary problem given by two fluids with different densities and viscosities is known as the two-phase Hele-Shaw flow.

A lot of information can be found in the literature about the Muskat problem and the two-phase Hele-Shaw flow (see [12] and [31] and the references therein). These free boundary problems are modelled using the Laplace–Young condition, so that the pressures of the fluids across the interface are different as follows:

$$p^1 - p^2 = \sigma k,$$

with σ the surface tension, and k the local curvature of the free boundary. With surface tension, and in the two dimensional case, it has been proven that the problems have classical solutions (see [26]). Without surface tension, $\sigma = 0$, the pressures of the fluids are equal on the interface. In this case, Siegel, Caflisch and Howison [42] proved ill-posedness in an unstable 2-D case, namely when the higher-viscosity fluid contracts, and they show global-in-time existence for small initial data in the stable case when the higher-viscosity fluid expands. The results rely on the assumption that the densities of the fluids are equal and the Atwood number

$$A_\mu = \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2}$$

is nonzero, where μ_1 and μ_2 are the viscosities of the fluids. In the same year, Ambrose [1] treated the 2-D problem with an initial data fulfilling

$$(\rho_2 - \rho_1)\mathrm{g}\, cos(\theta(\alpha, 0)) + 2A_\mu U(\alpha, 0) > 0,$$

and assuming the geometric condition,

$$\frac{(x(\alpha,t) - x(\alpha',t))^2 + (y(\alpha,t) - y(\alpha',t))^2}{(\alpha - \alpha')^2} > 0, \tag{1.1}$$

locally in time, where the curve $(x(\alpha,t), y(\alpha,t))$ is the interface, ρ_1 and ρ_2 are the densities of the fluids, θ is the angle that the tangent to the curve forms with the horizontal, and U is the normal velocity (given by the Birkhoff-Rott integral).

We consider the case $A_\mu = 0$, so that the interface is between fluids of different densities. This models, for example, moist and dry regions in a porous medium. The same equation was considered by Dombre, Pumir and Siggia [25], but they treated the interface dynamics for convection in porous media, where the density is replaced by temperature.

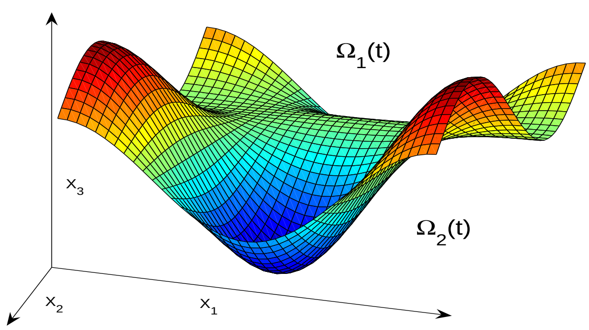
While the work of Ambrose is based on the arclength and the tangent angle formulation used by Hou, Lowengrub and Shelley [31], due to the particular form of the vorticity in the case $A_\mu = 0$, we are able to parameterize the curve in the two-dimensional problem to obtain the condition (1.1) for any time (see equation (1.16)), and to avoid a kind of singularity in the fluid when the interface collapses. We also prove this in the three-dimensional case.

The free boundary problems given by fluids with different densities have been widely considered. We highlight the classical paper of Taylor [48], and the works of Wu [49] and [50]. In the works of Wu, the full water wave problem is solved, where the water has positive density and the air has zero density.

In order to simplify the notation, we let $\mu/k = 12\mu/b^2 = 1$ and $\mathrm{g} = 1$. Thus, the 3-D system is written as

$$v(x_1, x_2, x_3, t) = -\nabla p(x_1, x_2, x_3, t) - (0, 0, \rho(x_1, x_2, x_3, t)), \tag{1.2}$$

where $(x_1, x_2, x_3) \in \mathbb{R}^3$ are the spatial variables and $t \geq 0$ denotes the time.



Here ρ is defined by

$$\rho(x_1, x_2, x_3, t) = \begin{cases} \rho_1 & \text{in} \quad \Omega_1(t), \\ \rho_2 & \text{in} \quad \Omega_2(t), \end{cases}$$

where ρ_1 and ρ_2 are positive constants such that $\rho_1 \neq \rho_2$.

We show in section 2 that in this case it is not necessary to assume any condition on the pressure along the interface to obtain the contour equation. Furthermore, we illustrate below that the solutions to this model are weak solutions to the following conservation of mass equation:

$$\frac{D\rho}{Dt} = \rho_t + v \cdot \nabla \rho = 0, \tag{1.3}$$

where $\operatorname{div} v = 0$.

The chapter is organized as follows. In section 1.2 we derive the contour equation. In section 1.3, we show that the equation fulfills the conservation of mass equation. In section 1.4 we prove local existence and uniqueness in the stable case. In section 1.5 we get a family of global solutions to the 2-D stable case with small initial data. In section 1.6 we prove ill-posedness for the 3-D unstable case, using the results of section 1.5. In section 1.7 we show that the solutions in the stable case remain bounded for any time. Finally, in section 1.8 we analyze the conservation of mass equation for regular initial data.

1.2 The Contour Equation

We consider the equation with $(x_1, x_2, x_3) \in \mathbb{R}^3$, and the fluid with different densities. That is ρ is represented by

$$\rho(x_1, x_2, x_3, t) = \begin{cases} \rho_1, & \{x_3 > f(x_1, x_2, t)\} \\ \rho_2, & \{x_3 < f(x_1, x_2, t)\}, \end{cases} \tag{1.4}$$

where f is the interface. Using Darcy's Law (1.2) we get

$$\operatorname{curl}\operatorname{curl} v = (-\partial_{x_1}\partial_{x_3}\rho, -\partial_{x_2}\partial_{x_3}\rho, (\partial^2_{x_1} + \partial^2_{x_2})\rho),$$

and since $\operatorname{div} v = 0$, we have $\operatorname{curl}\operatorname{curl} v = -\Delta v$. Therefore, taking the inverse of the Laplacian, it follows that

$$v = (\partial_{x_1}\Delta^{-1}\partial_{x_3}\rho, \partial_{x_2}\Delta^{-1}\partial_{x_3}\rho, -(\partial^2_{x_1} + \partial^2_{x_2})\Delta^{-1}\rho). \tag{1.5}$$

The integral operators $\partial_{x_1}\Delta^{-1}$ and $\partial_{x_2}\Delta^{-1}$ are given by the kernels

$$K_1(x_1, x_2, x_3) = \frac{1}{4\pi}\frac{x_1}{(x_1^2 + x_2^2 + x_3^2)^{3/2}}, \quad K_2(x_1, x_2, x_3) = \frac{1}{4\pi}\frac{x_2}{(x_1^2 + x_2^2 + x_3^2)^{3/2}},$$

respectively, thus the velocity can be expressed by

$$v = (K_1 * \partial_{x_3}\rho, K_2 * \partial_{x_3}\rho, -K_1 * \partial_{x_1}\rho - K_2 * \partial_{x_2}\rho). \tag{1.6}$$

Since ρ satisfies (1.4) we have

$$\nabla\rho = (\rho_2 - \rho_1)(\partial_{x_1} f(x_1, x_2, t), \partial_{x_2} f(x_1, x_2, t), -1)\delta(x_3 - f(x_1, x_2, t)), \tag{1.7}$$

where δ is the Dirac distribution. Using (1.6) we obtain

$$v(x_1,x_2,x_3,t)=-\frac{\rho_2-\rho_1}{4\pi}PV\int_{\mathbb{R}^2}\frac{(y_1,y_2,\nabla f(x-y,t)\cdot y)}{[|y|^2+(x_3-f(x-y,t))^2]^{3/2}}dy, \tag{1.8}$$

where we note that $x=(x_1,x_2)$, $y=(y_1,y_2)$,

$$\nabla f(x-y,t)\,\cdot\,y=\partial_{x_1}f(x-y,t)\,y_1+\partial_{x_2}f(x-y,t)\,y_2,$$

and PV indicates a principal value (see [44]). In (1.8) $x_3\neq f(x,t)$, so that the principal value is taken at infinity. When x_3 approaches $f(x,t)$ in the normal direction, we get a discontinuity in the velocity due to the fact that the vorticity is concentrated on the interface. Thus, for $\varepsilon>0$ we define

$$v^1(x,f(x,t),t)=\lim_{\varepsilon\to 0}v(x_1-\varepsilon\partial_{x_1}f(x,t),x_2-\varepsilon\partial_{x_2}f(x,t),f(x,t)+\varepsilon,t),$$

and

$$v^2(x,f(x,t),t)=\lim_{\varepsilon\to 0}v(x_1+\varepsilon\partial_{x_1}f(x,t),x_2+\varepsilon\partial_{x_2}f(x,t),f(x,t)-\varepsilon,t).$$

It follows that

$$\begin{aligned}v^1(x,f(x,t),t)=&-\frac{\rho_2-\rho_1}{4\pi}PV\int_{\mathbb{R}^2}\frac{(y_1,y_2,\nabla f(x-y,t)\cdot y)}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}}dy\\&+\frac{\rho_2-\rho_1}{2}\frac{\partial_{x_1}f(x,t)(1,0,\partial_{x_1}f(x,t))}{1+(\partial_{x_1}f(x,t))^2+(\partial_{x_2}f(x,t))^2}\\&+\frac{\rho_2-\rho_1}{2}\frac{\partial_{x_2}f(x,t)(0,1,\partial_{x_2}f(x,t))}{1+(\partial_{x_1}f(x,t))^2+(\partial_{x_2}f(x,t))^2},\end{aligned} \tag{1.9}$$

$$\begin{aligned}v^2(x,f(x,t),t)=&-\frac{\rho_2-\rho_1}{4\pi}PV\int_{\mathbb{R}^2}\frac{(y_1,y_2,\nabla f(x-y,t)\cdot y)}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}}dy\\&-\frac{\rho_2-\rho_1}{2}\frac{\partial_{x_1}f(x,t)(1,0,\partial_{x_1}f(x,t))}{1+(\partial_{x_1}f(x,t))^2+(\partial_{x_2}f(x,t))^2}\\&-\frac{\rho_2-\rho_1}{2}\frac{\partial_{x_2}f(x,t)(0,1,\partial_{x_2}f(x,t))}{1+(\partial_{x_1}f(x,t))^2+(\partial_{x_2}f(x,t))^2}.\end{aligned} \tag{1.10}$$

The velocity in the tangential directions only moves the particles on the surface $f(x,t)$; i.e., if we rewrite the velocity in the tangential directions, we only make a change on the parametrization and do not alter the shape of the interface. Thus, it follows that

$$v(x,f(x,t),t)=-\frac{\rho_2-\rho_1}{4\pi}PV\int_{\mathbb{R}^2}\frac{(y_1,y_2,\nabla f(x-y,t)\cdot y)}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}}dy, \tag{1.11}$$

due to the fact that the terms

$$\pm\frac{\rho_2-\rho_1}{2}\frac{\partial_{x_1}f(x,t)(1,0,\partial_{x_1}f(x,t))}{1+(\partial_{x_1}f(x,t))^2+(\partial_{x_2}f(x,t))^2},$$

$$\pm\frac{\rho_2-\rho_1}{2}\frac{\partial_{x_2}f(x,t)(0,1,\partial_{x_2}f(x,t))}{1+(\partial_{x_1}f(x,t))^2+(\partial_{x_2}f(x,t))^2},$$

are in the tangential directions. Moreover, if we add the following tangential terms to (1.11)

$$\frac{\rho_2-\rho_1}{4\pi}PV\int_{\mathbb{R}^2}\frac{y_1}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}}dy(1,0,\partial_{x_1}f(x,t)),$$
$$\frac{\rho_2-\rho_1}{4\pi}PV\int_{\mathbb{R}^2}\frac{y_2}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}}dy(0,1,\partial_{x_2}f(x,t)),$$

we obtain

$$v(x,f(x,t),t)=\frac{\rho_2-\rho_1}{4\pi}(0,0,PV\int_{\mathbb{R}^2}\frac{(\nabla f(x,t)-\nabla f(x-y,t))\cdot y}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}}dy). \tag{1.12}$$

In this way, the velocity only moves the particles in the x_3 direction, thus we have the contour equation given by

$$\begin{aligned} f_t(x,t) &= \frac{\rho_2-\rho_1}{4\pi}PV\int_{\mathbb{R}^2}\frac{(\nabla f(x,t)-\nabla f(x-y,t))\cdot y}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}}dy,\\ f(x,0) &= f_0(x). \end{aligned} \tag{1.13}$$

In the periodic case, we can obtain an equivalent equation to (1.13) due to the fact that the integral operators $\partial_{x_1}\Delta^{-1}$ and $\partial_{x_2}\Delta^{-1}$ can be represented by the kernels

$$K_1^p(x_1,x_2,x_3)=\frac{1}{4\pi}(\frac{x_1}{(x_1^2+x_2^2+x_3^2)^{3/2}}L(x_1,x_2,x_3)+M(x_1,x_2,x_3)),$$

$$K_2^p(x_1,x_2,x_3)=\frac{1}{4\pi}(\frac{x_2}{(x_1^2+x_2^2+x_3^2)^{3/2}}L(x_1,x_2,x_3)+M(x_1,x_2,x_3)),$$

respectively, where $(x_1,x_2,x_3)\in\mathbb{T}^2\times\mathbb{R}$, $\mathbb{T}^2=[-\pi,\pi]^2$, and $L,M\in C^\infty(\mathbb{T}^2\times\mathbb{R})$ (see [45] for the kernel of the Riesz potentials on the n-torus). Adding an appropiate function to the singular part of K_1^p and K_2^p, we can choose

$$\begin{aligned} &L\in C_c^\infty(\mathbb{T}^2\times\mathbb{R}),\quad L\geq 0,\quad \text{supp } L\subset\{x_1^2+x_2^2+x_3^2\leq 4\},\\ &L=1 \text{ in } \{x_1^2+x_2^2+x_3^2\leq 1\}\quad\text{and}\quad L(-x_1,-x_2,-x_3)=L(x_1,x_2,x_3). \end{aligned} \tag{1.14}$$

The function M belongs to $C_b^\infty(\mathbb{T}^2\times\mathbb{R})$ and $M(0,0,0)=0$. As before, the velocity can be expressed by

$$v=(K_1^p*\partial_{x_3}\rho,K_2^p*\partial_{x_3}\rho,-K_1^p*\partial_{x_1}\rho-K_2^p*\partial_{x_2}\rho),$$

and due to (1.7) it follows (suppressing the dependence on t)

$$\begin{aligned} v(x_1,x_2,x_3)=&-\frac{\rho_2-\rho_1}{4\pi}PV\int_{\mathbb{T}^2}\frac{(y_1,y_2,\nabla f(x-y)\cdot y)}{[|y|^2+(x_3-f(x-y))^2]^{3/2}}L(y,x_3-f(x-y))dy\\ &-\frac{\rho_2-\rho_1}{4\pi}\int_{\mathbb{T}^2}(1,1,\nabla f(x-y)\cdot(1,1))M(y,x_3-f(x-y))dy, \end{aligned}$$

if $x_3 \neq f(x)$. Adding a suitable term in the tangential directions we obtain

$$\begin{aligned} v(x,f(x)) =& \frac{\rho_2-\rho_1}{4\pi}(0,0,\int_{\mathbb{T}^2}\frac{(\nabla f(x)-\nabla f(x-y))\cdot y}{[|y|^2+(f(x)-f(x-y))^2]^{3/2}}L(y,f(x)-f(x-y))dy \\ &+\int_{\mathbb{T}^2}(\nabla f(x)-\nabla f(x-y))\cdot(1,1))M(y,f(x)-f(x-y))dy). \end{aligned}$$

Finally we have the contour equation in the periodic case given by

$$\begin{aligned} f_t(x,t) =& \frac{\rho_2-\rho_1}{4\pi}\int_{\mathbb{T}^2}\frac{(\nabla f(x,t)-\nabla f(x-y,t))\cdot y}{[|y|^2+(f(x,t)-f(x-y,t))^2]^{3/2}}L(y,f(x,t)-f(x-y,t))dy \\ &+\frac{\rho_2-\rho_1}{4\pi}\int_{\mathbb{T}^2}(\nabla f(x,t)-\nabla f(x-y,t))\cdot(1,1))M(y,f(x,t)-f(x-y,t))dy, \\ f(x,0) =& f_0(x). \end{aligned} \tag{1.15}$$

We use both formulations throughout the paper. Suppose that the function $f(x)$ only depends on x_1 in equation (1.13). Then the contour equation in the 2-D case (with a 1-D interface) follows

$$\begin{aligned} f_t(x,t) &= \frac{\rho_2-\rho_1}{2\pi}PV\int_{\mathbb{R}}\frac{(\partial_x f(x,t)-\partial_x f(x-\alpha,t))\alpha}{\alpha^2+(f(x,t)-f(x-\alpha,t))^2}d\alpha, \\ f(x,0) &= f_0(x); \quad x\in\mathbb{R}. \end{aligned} \tag{1.16}$$

This equation can be obtained in a similar way to (1.13) using the stream function for two dimensional fluids [6]. Performing a two-dimensional analysis using the stream function, we obtain an equivalent equation to (1.16) in the two dimensional periodic case as follows:

$$\begin{aligned} f_t(x,t) =& \frac{\rho_2-\rho_1}{2\pi}\int_{\mathbb{T}}\frac{(\partial_x f(x,t)-\partial_x f(x-\alpha,t))\alpha}{\alpha^2+(f(x,t)-f(x-\alpha,t))^2}P(\alpha,f(x,t)-f(x-\alpha,t))d\alpha \\ &+\frac{\rho_2-\rho_1}{2\pi}\int_{\mathbb{T}}(\partial_x f(x,t)-\partial_x f(x-\alpha,t))Q(\alpha,f(x,t)-f(x-\alpha,t))d\alpha, \\ f(x,0) =& f_0(x), \end{aligned} \tag{1.17}$$

with

$$P(x_1,x_2)\in C_c^\infty(\mathbb{T}\times\mathbb{R}), \quad P\geq 0, \quad \text{supp } P\subset\{x_1^2+x_2^2\leq 4\},$$

$$P=1 \text{ in } \{x_1^2+x_2^2\leq 1\} \quad \text{and} \quad P(-x_1,-x_2)=P(x_1,x_2).$$

The function $Q(x_1,x_2)$ belongs to $C_b^\infty(\mathbb{T}\times\mathbb{R})$ and $Q(0,0)=0$.

If we consider the linearized equation of the motion, we obtain a dissipative equation when $\rho_1<\rho_2$ (the larger density fluid is below the fluid with the smaller density) and an unstable equation when $\rho_1>\rho_2$. The unstable linearized equation presents an instability similar to Kelvin-Helmholtz's (see [7]). As usual, the Riesz transforms in $\mathbb{R}^2$ (see [44]) are defined by

$$R_1 f(x)=\frac{1}{2\pi}PV\int_{\mathbb{R}^2}\frac{y_1}{|y|^3}f(x-y)dy,$$

$$R_2 f(x) = \frac{1}{2\pi} PV \int_{\mathbb{R}^2} \frac{y_2}{|y|^3} f(x-y)dy,$$

and the operator $\Lambda^s f$ is defined by the Fourier transform $\widehat{\Lambda^s f}(\xi) = |\xi|^s \widehat{f}(\xi)$. Suppose that $f(x)$ is uniformly small and we can neglect the terms of order greater than one in (1.13), then it reduces to the following linear equation:

$$\begin{aligned} f_t &= \frac{\rho_1 - \rho_2}{2}(R_1 \partial_{x_1} f + R_2 \partial_{x_2} f) = \frac{\rho_1 - \rho_2}{2}\Lambda f, \\ f(x,0) &= f_0(x). \end{aligned} \tag{1.18}$$

Applying the Fourier transform we get

$$\hat{f}(\xi) = \hat{f}_0(\xi) e^{\frac{\rho_1 - \rho_2}{2}|\xi| t},$$

and therefore (1.18) is a dissipative equation when $\rho_1 < \rho_2$ and an ill-posed problem in the case $\rho_1 > \rho_2$ with general initial data in the Schwartz class. We need analytic initial data in order to get a well-posed problem for $\rho_1 > \rho_2$.

1.3 The conservation of mass equation

We show that if ρ is defined by (1.4) and $f(x,t)$ is convected by the velocity (1.12) then ρ is a weak solution of the conservation of mass equation (1.3) and conversely. From now on, Ω is equal to $\mathbb{R}^2$ or $\mathbb{T}^2$ and $\widetilde{x} = (x_1, x_2, x_3)$.

Definition 1.3.1 *The density ρ is a weak solution of the conservation of mass equation if for any $\varphi \in C^\infty(\Omega \times \mathbb{R} \times (0,T))$, φ with compact support and periodic in (x_1, x_2) in the periodic case, we have*

$$\int_0^T \int_\Omega \int_\mathbb{R} (\rho(\widetilde{x},t)\partial_t \varphi(\widetilde{x},t) + v(\widetilde{x},t)\rho(\widetilde{x},t)\nabla\varphi(\widetilde{x},t)) d\widetilde{x} dt = 0, \tag{1.19}$$

where the incompressible velocity v is given by Darcy's law.

Proposition 1.3.2 *If $f(x,t)$ satisfies* (1.13) *and $\rho(\widetilde{x},t)$ is defined by* (1.4)*, then ρ is a weak solution to the conservation of mass equation. Furthermore, if ρ is a weak solution to the conservation of mass equation given by* (1.4)*, then $f(x,t)$ satisfies* (1.13).

Proof: Let ρ be a weak solution to (1.3) defined by (1.4). Integrating by parts we have

$$\begin{aligned} I &= \int_0^T \int_\Omega \int_\mathbb{R} \rho \partial_t \varphi d\widetilde{x} dt = \rho_1 \int_0^T \int_{\{x_3 > f\}} \partial_t \varphi d\widetilde{x} dt + \rho_2 \int_0^T \int_{\{x_3 < f\}} \partial_t \varphi d\widetilde{x} dt \\ &= (\rho_1 - \rho_2) \int_0^T \int_\Omega \varphi(x, f(x,t), t) \partial_t f(x,t) dx dt. \end{aligned}$$

On the other hand, due to (1.9) and (1.10) we obtain

$$
\begin{aligned}
J &= \int_0^T \int_\Omega \int_{\mathbb{R}} \rho v \,\nabla \varphi \, d\widetilde{x} dt = \rho_1 \int_0^T \int_{\{x_3>f\}} v \nabla \varphi d\widetilde{x} dt + \rho_2 \int_0^T \int_{\{x_3<f\}} v \nabla \varphi d\widetilde{x} dt \\
&= \int_0^T \int_\Omega \varphi(x, f(x,t), t)(\rho_1 v^1(x, f(x,t), t) - \rho_2 v^2(x, f(x,t), t)) \cdot (\partial_{x_1} f(x,t), \partial_{x_2} f(x,t), -1) dx dt \\
&= (\rho_1 - \rho_2) \int_0^T \int_\Omega \varphi(x, f(x,t), t) v(x, f(x,t), t) \cdot (\partial_{x_1} f(x,t), \partial_{x_2} f(x,t), -1) dx dt,
\end{aligned}
$$

where $v(x, f(x,t), t)$ is given by (1.11). We get

$$
J = \frac{(\rho_1 - \rho_2)^2}{4\pi} \int_0^T \int_\Omega \varphi(x, f(x,t), t) PV \int_{\mathbb{R}^2} \frac{(\nabla f(x,t) - \nabla f(x-y,t)) \cdot y}{[|y|^2 + (f(x,t) - f(x-y,t))^2]^{3/2}} dy dx dt.
$$

The identity (1.19) implies that $I + J = 0$, thus if we choose $\varphi(\widetilde{x}, t) = \varphi(x, t)$ for $x_3 \in [-\|f\|_{L^\infty}, \|f\|_{L^\infty}]$, it follows that $f(x,t)$ fulfills (1.13).

Following the same arguments it is easy to check that if $f(x,t)$ satisfies (1.13), then ρ is a weak solution given by (1.4). □

Remark 1.3.3 *Note that due to* (1.5)*, the velocity satisfies*

$$
v = (R_1(R_3\rho),\, R_2(R_3\rho),\, -(R_1^2 + R_2^2)(\rho)),
$$

where the operators R_1, R_2 and R_3 are the Riesz transforms in three dimensions (see [44]). Since $\rho \in L^\infty(\Omega \times \mathbb{R})$ then v belongs to BMO (bounded mean oscillation) and therefore v is in $L^2(\Omega \times \mathbb{R})$ locally (see [46] for the definitions and properties of the BMO space).

1.4 Local well-posedness for the stable case

In this section we prove local existence and uniqueness for the stable case using energy estimates. First we study the case $\Omega = \mathbb{R}^2$, later giving the main differences with the periodic domain, and at the end of the section showing the result for a 1-D interface. Denote the Sobolev spaces by H^k, the Hölder spaces by $C^{k,\delta}$ with $0 \le \delta < 1$ the Hölder continuity and the hessian matrix of a function $f(x)$ by $\nabla^2 f(x)$. The norms of H^k and $C^{k,\delta}$ are defined as follows:

$$
\|f\|_{H^k}^2 = \|f\|_{L^2}^2 + \|\Lambda^k f\|_{L^2}^2,
$$

$$
\|f\|_{C^{k,\delta}} = \|f\|_{C^k} + \max_{i+j=k} \max_{x \neq y} \frac{|\partial_{x_1}^i \partial_{x_2}^j f(x) - \partial_{x_1}^i \partial_{x_2}^j f(y)|}{|x-y|^\delta}.
$$

1.4.1 Two dimensional interface

The main theorem of this section is the following.

Theorem 1.4.1 *Let $f_0(x) \in H^k(\mathbb{R}^2)$ for $k \ge 4$ and $\rho_2 > \rho_1$. Then there exists a time $T > 0$ so that there is a unique solution to* (1.13) *in $C^1([0,T]; H^k(\mathbb{R}^2))$ with $f(x,0) = f_0(x)$.*

Proof: We choose $\rho_2 - \rho_1 = 4\pi$ without loss of generality, then

$$\begin{aligned} f_t(x,t) &= PV\int_{\mathbb{R}^2} \frac{(\nabla f(x,t) - \nabla f(x-y,t))\cdot y}{[|y|^2 + (f(x,t) - f(x-y,t))^2]^{3/2}}dy, \\ f(x,0) &= f_0(x). \end{aligned} \tag{1.20}$$

We let $k = 4$; the proof for $k > 4$ being analogous. We apply energy methods (see [6] for more details). We have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|f\|^2_{L^2}(t) &= \int_{\mathbb{R}^2} f(x) PV\int_{\mathbb{R}^2} \frac{(\nabla f(x) - \nabla f(x-y))\cdot y}{[|y|^2 + (f(x) - f(x-y))^2]^{3/2}}dydx \\ &= \int_{\mathbb{R}^2} f(x)\int_{|y|<1} \frac{(\nabla f(x) - \nabla f(x-y))\cdot y}{[|y|^2 + (f(x) - f(x-y))^2]^{3/2}}dydx \\ &\quad + \int_{\mathbb{R}^2} f(x) PV\int_{|y|>1} \frac{\nabla f(x)\cdot y}{[|y|^2 + (f(x) - f(x-y))^2]^{3/2}}dydx \\ &\quad - \int_{\mathbb{R}^2} f(x) PV\int_{|y|>1} \frac{\nabla f(x-y)\cdot y}{[|y|^2 + (f(x) - f(x-y))^2]^{3/2}}dydx \\ &= I_1 + I_2 + I_3. \end{aligned}$$

The identity

$$\partial_{x_i} f(x) - \partial_{x_i} f(x-y) = \int_0^1 \nabla \partial_{x_i} f(x + (s-1)y)\cdot y\, ds,$$

yields

$$\begin{aligned} I_1 &\leq C\int_0^1 ds\int_{|y|<1} |y|^{-1}\int_{\mathbb{R}^2} \frac{|f(x)||\nabla^2 f(x+(s-1)y)|}{[1 + ((f(x) - f(x-y))^2|y|^{-2}]^{3/2}}dxdy \\ &\leq C\int_0^1 ds\int_{|y|<1} |y|^{-1}dy\|f\|_{L^2}\sum_{i+j=2}\|\partial^i_{x_1}\partial^j_{x_2} f\|_{L^2} \leq C\|f\|^2_{H^2}. \end{aligned}$$

Integrating by parts, the I_2 term is written

$$\begin{aligned} I_2 &= \frac{3}{2}\int_{|y|>1}\int_{\mathbb{R}^2} |f(x)|^2 \frac{(f(x) - f(x-y))(\nabla f(x) - \nabla f(x-y))\cdot y}{[|y|^2 + ((f(x) - f(x-y))^2]^{5/2}}dxdy \\ &\leq C\int_{|y|>1} |y|^{-3}\int_{\mathbb{R}^2} |f(x)|^2 \frac{|f(x) - f(x-y)||y|^{-1}|\nabla f(x) - \nabla f(x-y)|}{[1 + ((f(x) - f(x-y))^2|y|^{-2}]^{5/2}}dxdy \\ &\leq C\|f\|_{L^\infty}\|f\|^2_{H^1}. \end{aligned}$$

Integrating by parts in I_3, it follows that

$$\begin{aligned}
I_3 =& \int_{|y|>1}\int_{\mathbb{R}^2} f(x)f(x-y)\frac{|y|^2-2(f(x)-f(x-y))^2}{[|y|^2+(f(x)-f(x-y))^2]^{5/2}}dxdy \\
&+3\int_{|y|>1}\int_{\mathbb{R}^2} f(x)f(x-y)\frac{(f(x)-f(x-y))\nabla f(x-y)\cdot y}{[|y|^2+(f(x)-f(x-y))^2]^{5/2}}dxdy \\
&-\int_{|y|=1}\int_{\mathbb{R}^2} f(x)f(x-y)\frac{|y|^2}{[|y|^2+(f(x)-f(x-y))^2]^{3/2}}dxd\sigma(y) \\
\leq& C(\|f\|_{L^\infty}+1)\|f\|^2_{H^1}.
\end{aligned}$$

Using Sobolev inequalities, we get finally

$$\frac{d}{dt}\|f\|^2_{L^2}(t) \leq C(\|f\|^3_{H^2}(t)+1). \tag{1.21}$$

We consider the quantity

$$\frac{1}{2}\frac{d}{dt}\|\partial^4_{x_1}f\|^2_{L^2}(t) = I_4+I_5+I_6+I_7+I_8,$$

where

$$\begin{aligned}
I_4 &= \int_{\mathbb{R}^2}\partial^4_{x_1}f(x)PV\int_{\mathbb{R}^2}\frac{(\nabla\partial^4_{x_1}f(x)-\nabla\partial^4_{x_1}f(x-y))\cdot y}{[|y|^2+(f(x)-f(x-y))^2]^{3/2}}dydx, \\
I_5 &= 4\int_{\mathbb{R}^2}\partial^4_{x_1}f(x)\int_{\mathbb{R}^2}(\nabla\partial^3_{x_1}f(x)-\nabla\partial^3_{x_1}f(x-y))\cdot y\ \partial_{x_1}A(x,y)dydx, \\
I_6 &= 6\int_{\mathbb{R}^2}\partial^4_{x_1}f(x)\int_{\mathbb{R}^2}(\nabla\partial^2_{x_1}f(x)-\nabla\partial^2_{x_1}f(x-y))\cdot y\ \partial^2_{x_1}A(x,y)dydx, \\
I_7 &= 4\int_{\mathbb{R}^2}\partial^4_{x_1}f(x)\int_{\mathbb{R}^2}(\nabla\partial_{x_1}f(x)-\nabla\partial_{x_1}f(x-y))\cdot y\,\partial^3_{x_1}A(x,y)dydx, \\
I_8 &= \int_{\mathbb{R}^2}\partial^4_{x_1}f(x)\int_{\mathbb{R}^2}(\nabla f(x)-\nabla f(x-y))\cdot y\ \partial^4_{x_1}A(x,y)dydx,
\end{aligned}$$

and

$$A(x,y) = [|y|^2+(f(x)-f(x-y))^2]^{-3/2}.$$

The most singular term is I_4. In order to estimate it, we write

$$\begin{aligned}
I_4 =& \int_{\mathbb{R}^2}\partial^4_{x_1}f(x)PV\int_{\mathbb{R}^2}\frac{\nabla\partial^4_{x_1}f(x)\cdot y}{[|y|^2+(f(x)-f(x-y))^2]^{3/2}}dydx \\
&-\int_{\mathbb{R}^2}\partial^4_{x_1}f(x)PV\int_{\mathbb{R}^2}\frac{\nabla\partial^4_{x_1}f(y)\cdot(x-y)}{[|x-y|^2+(f(x)-f(y))^2]^{3/2}}dydx \\
=& J_1+J_2.
\end{aligned}$$

Integrating by parts,

$$\begin{aligned}J_1 &= \frac{3}{2}\int_{\mathbb{R}^2} |\partial^4_{x_1} f(x)|^2 PV \int_{\mathbb{R}^2} \frac{(f(x)-f(x-y))(\nabla f(x)-\nabla f(x-y))\cdot y}{[|y|^2+((f(x)-f(x-y))^2]^{5/2}} dydx \\ &= \frac{3}{2}\int_{\mathbb{R}^2} |\partial^4_{x_1} f(x)|^2 (\int_{|y|>1} dy + PV\int_{|y|<1} dy) dx \\ &\leq \frac{3}{2}\|f\|_{C^1}\|\partial^4_{x_1} f\|^2_{L^2} + \frac{3}{2} M(f)\|\partial^4_{x_1} f\|^2_{L^2},\end{aligned} \tag{1.22}$$

where

$$M(f) = \max_x \Big| PV\int_{|y|<1} \frac{(f(x)-f(x-y))(\nabla f(x)-\nabla f(x-y))\cdot y}{[|y|^2+((f(x)-f(x-y))^2]^{5/2}} dy \Big|.$$

We estimate this maximum in the following form:

$$\begin{aligned}M(f) \leq & \max_x \Big| \int_{|y|<1} \frac{(f(x)-f(x-y)-\nabla f(x)\cdot y)(\nabla f(x)-\nabla f(x-y))\cdot y}{[|y|^2+((f(x)-f(x-y))^2]^{5/2}} dy \Big| \\ & + \max_x \Big| \int_{|y|<1} \frac{(\nabla f(x)\cdot y)((\nabla f(x)-\nabla f(x-y))\cdot y - y\cdot\nabla^2 f(x)\cdot y)}{[|y|^2+((f(x)-f(x-y))^2]^{5/2}} dy \Big| \\ & + \max_x \Big| \int_{|y|<1} (\nabla f(x)\cdot y)(y\cdot\nabla^2 f(x)\cdot y)(B(x,y)-C(x,y)) dy \Big| \\ & + \max_x \Big| PV\int_{|y|<1} \frac{(\nabla f(x)\cdot y)(y\cdot\nabla^2 f(x)\cdot y)}{[|y|^2+(\nabla f(x)\cdot y)^2]^{5/2}} dy \Big|,\end{aligned} \tag{1.23}$$

where

$$B(x,y) = [|y|^2+((f(x)-f(x-y))^2]^{-5/2}, \qquad C(x,y) = [|y|^2+(\nabla f(x)\cdot y)^2]^{-5/2}.$$

Making the change of variables $y=-z$, we see that the last integral in (1.23) is null. Thus, we can estimate $M(f)$ by

$$\begin{aligned}M(f) \leq & \|f\|^2_{C^2} \max_x \Big| \int_{|y|<1} \frac{|y|^{-1}}{[1+((f(x)-f(x-y))|y|^{-1})^2]^{5/2}} dy \Big| \\ & + \|f\|_{C^1}\|f\|_{C^{2,\delta}} \Big| \int_{|y|<1} |y|^{-2+\delta} dy \Big| + \|f\|^2_{C^1}\|f\|^2_{C^2} \Big| \int_{|y|<1} |y|^{-1} dy \Big| \\ \leq & C(\|f\|^2_{C^2} + \|f\|_{C^1}\|f\|_{C^{2,\delta}} + \|f\|^2_{C^1}\|f\|^2_{C^2}),\end{aligned}$$

with $0<\delta<1$, so that

$$J_1 \leq C(\|f\|^4_{C^{2,\delta}} + 1)\|\partial^4_{x_1} f\|^2_{L^2}. \tag{1.24}$$

In order to estimate J_2, we integrate by parts getting

$$\begin{aligned}J_2 &= \int_{\mathbb{R}^2} \partial^4_{x_1} f(x) PV\int_{\mathbb{R}^2} \frac{\nabla_y(\partial^4_{x_1} f(x) - \partial^4_{x_1} f(y))\cdot(x-y)}{[|x-y|^2+(f(x)-f(y))^2]^{3/2}} dydx \\ &= K_1 + K_2,\end{aligned}$$

with

$$K_1 = -\int_{\mathbb{R}^2} \partial_{x_1}^4 f(x) PV \int_{\mathbb{R}^2} \frac{\partial_{x_1}^4 f(x) - \partial_{x_1}^4 f(y)}{[|x-y|^2 + (f(x)-f(y))^2]^{3/2}} dydx,$$

and

$$K_2 = \int_{\mathbb{R}^2} \partial_{x_1}^4 f(x) \int_{\mathbb{R}^2} (\partial_{x_1}^4 f(x) - \partial_{x_1}^4 f(y)) \frac{3(f(x)-f(y))(f(x)-f(y)-\nabla f(y)\cdot(x-y))}{[|x-y|^2 + (f(x)-f(y))^2]^{5/2}} dydx.$$

Making a change of variables we obtain

$$\begin{aligned} K_1 &= -PV \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{x_1}^4 f(x) \frac{\partial_{x_1}^4 f(x) - \partial_{x_1}^4 f(y)}{[|x-y|^2 + (f(x)-f(y))^2]^{3/2}} dydx \\ &= PV \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{x_1}^4 f(y) \frac{\partial_{x_1}^4 f(x) - \partial_{x_1}^4 f(y)}{[|x-y|^2 + (f(x)-f(y))^2]^{3/2}} dydx \\ &= -\frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\partial_{x_1}^4 f(x) - \partial_{x_1}^4 f(y))^2}{[|x-y|^2 + (f(x)-f(y))^2]^{3/2}} dydx \\ &\leq 0. \end{aligned}$$

Here we observe the main difference with the unstable case in which we obtain the opposite sign. Now we consider

$$K_2 = L_1 + L_2 + L_3,$$

where

$$L_1 = 3 \int_{\mathbb{R}^2} |\partial_{x_1}^4 f(x)|^2 PV \int_{\mathbb{R}^2} \frac{(f(x)-f(y))(f(x)-f(y)-\nabla f(y)\cdot(x-y))}{[|x-y|^2 + (f(x)-f(y))^2]^{5/2}} dydx,$$

$$L_2 = -\frac{3}{4} PV \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{x_1}^4 f(x) \partial_{x_1}^4 f(y) \frac{(f(x)-f(y))(x-y)\cdot(\nabla^2 f(x) + \nabla^2 f(y))\cdot(x-y)}{[|x-y|^2 + (f(x)-f(y))^2]^{5/2}} dydx,$$

$$L_3 = -3 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \partial_{x_1}^4 f(x) \partial_{x_1}^4 f(y) (f(x)-f(y)) D(x,y) dydx,$$

with

$$D(x,y) = \frac{(f(x)-f(y)-\nabla f(y)\cdot(x-y) - \frac{1}{4}(x-y)\cdot(\nabla^2 f(x) + \nabla^2 f(y))\cdot(x-y))}{[|x-y|^2 + (f(x)-f(y))^2]^{5/2}}.$$

The L_1 term can be estimated like J_1 in (1.22) so that

$$L_1 \leq C(1 + \|f\|_{C^{2,\delta}}^4) \|\partial_{x_1}^4 f\|_{L^2}^2.$$

Exchanging x for y we see that $L_2 = 0$. For the last term, it follows that

$$\begin{aligned}
L_3 &\leq C\int_{\mathbb{R}^2}|\partial_{x_1}^4 f(x)|^2\int_{\mathbb{R}^2}|f(x)-f(y)||D(x,y)|dydx \\
&\quad + C\int_{\mathbb{R}^2}|\partial_{x_1}^4 f(y)|^2\int_{\mathbb{R}^2}|f(x)-f(y)||D(x,y)|dxdy \\
&\leq C\int_{\mathbb{R}^2}|\partial_{x_1}^4 f(x)|^2\int_{\mathbb{R}^2}|f(x)-f(x-y)||D(x,x-y)|dydx \\
&\quad + C\int_{\mathbb{R}^2}|\partial_{x_1}^4 f(y)|^2\int_{\mathbb{R}^2}|f(x+y)-f(y)||D(x+y,y)|dxdy \\
&\leq C\int_{\mathbb{R}^2}|\partial_{x_1}^4 f(x)|^2dx(\int_{|y|<1}dy+\int_{|y|>1}dy)+\int_{\mathbb{R}^2}|\partial_{x_1}^4 f(y)|^2dy(\int_{|x|<1}dx+\int_{|x|>1}dx) \\
&\leq C\|f\|_{C^1}\|f\|_{C^{2,\delta}}\|\partial_{x_1}^4 f\|_{L^2}^2.
\end{aligned}$$

Finally,

$$J_2 = K_1 + K_2 \leq K_2 = L_1 + L_2 + L_3 = L_1 + L_3 \leq C(\|f\|_{C^{2,\delta}}^4 + 1)\|\partial_{x_1}^4 f\|_{L^2}^2,$$

and due to (1.24) we obtain

$$I_4 \leq C(\|f\|_{C^{2,\delta}}^4 + 1)\|\partial_{x_1}^4 f\|_{L^2}^2. \tag{1.25}$$

Now we estimate the I_5 integral. We have $I_5 = J_3 + J_4$ where

$$J_3 = 4\int_{\mathbb{R}^2}\partial_{x_1}^4 f(x)\nabla\partial_{x_1}^3 f(x)\cdot PV\int_{\mathbb{R}^2}y\frac{(f(x)-f(x-y))(\partial_{x_1}f(x)-\partial_{x_1}f(x-y))}{[|y|^2+(f(x)-f(x-y))^2]^{5/2}}dydx,$$

and

$$\begin{aligned}
J_4 &= -4PV\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\partial_{x_1}^4 f(x)\nabla\partial_{x_1}^3 f(y)\cdot(x-y)\frac{(f(x)-f(y))(\partial_{x_1}f(x)-\partial_{x_1}f(y))}{[|x-y|^2+(f(x)-f(y))^2]^{5/2}}dydx \\
&= -4PV\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\partial_{x_1}^4 f(x)\partial_{x_2}\partial_{x_1}^3 f(y)(x_2-y_2)\frac{(f(x)-f(y))(\partial_{x_1}f(x)-\partial_{x_1}f(y))}{[|x-y|^2+(f(x)-f(y))^2]^{5/2}}dydx.
\end{aligned}$$

We estimate J_3 similarly to the term J_1 in (1.22), so that

$$J_3 \leq C(1+\|f\|_{C^{2,\delta}}^4)\|f\|_{H^4}^2.$$

We decompose the term $J_4 = K_3 + K_4 + K_5 + K_6$ as follows:

$$K_3 = -4PV\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\partial_{x_1}^4 f(x)\partial_{x_2}\partial_{x_1}^3 f(x-y)y_2E(x,y)dydx,$$

$$K_4 = -4PV\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\partial_{x_1}^4 f(x)\partial_{x_2}\partial_{x_1}^3 f(x-y)y_2F(x,y)dydx,$$

$$K_5 = -4PV\int_{\mathbb{R}^2}\int_{|y|<1}\partial_{x_1}^4 f(x)\partial_{x_2}\partial_{x_1}^3 f(x-y)y_2(\nabla f(x)\cdot y)(\nabla\partial_{x_1}f(x)\cdot y)(B(x,y)-C(x,y))dydx$$

and

$$K_6 = -4PV \int_{\mathbb{R}^2} \int_{|y|<1} \partial^4_{x_1} f(x) \partial_{x_2} \partial^3_{x_1} f(x-y) y_2 \frac{(\nabla f(x)\cdot y)(\nabla \partial_{x_1} f(x)\cdot y)}{[|y|^2 + (\nabla f(x)\cdot y)^2]^{5/2}} dydx,$$

where

$$E(x,y) = \frac{(f(x) - f(x-y) - \nabla f(x)\cdot y)(\partial_{x_1} f(x) - \partial_{x_1} f(x-y))}{[|y|^2 + (f(x)-f(x-y))^2]^{5/2}},$$

and

$$F(x,y) = \frac{(\nabla f(x)\cdot y)(\partial_{x_1} f(x) - \partial_{x_1} f(x-y) - \nabla \partial_{x_1} f(x)\cdot y X_{\{|y|<1\}})}{[|y|^2 + (f(x)-f(x-y))^2]^{5/2}}.$$

The terms K_3, K_4, and K_5 are estimated in the same way as J_1, so that

$$K_3 \leq C(1 + \|f\|^2_{C^2})\|f\|^2_{H^4},$$

$$K_4 \leq C(1 + \|f\|_{C^1}\|f\|_{C^{2,\delta}})\|f\|^2_{H^4},$$

and

$$K_5 \leq C(1 + \|f\|^2_{C^1}\|f\|^2_{C^2})\|f\|^2_{H^4}.$$

We rewrite K_6 and we get

$$K_6 = -4PV \int_{\mathbb{R}^2} \partial^4_{x_1} f(x) S(\partial_{x_2}\partial^3_{x_1} f)(x) dx,$$

with the operator S defined by

$$S(g)(x) = PV \int_{|y|<1} \frac{\Sigma(x,y)}{|y|^2} g(x-y) dy,$$

and

$$\Sigma(x,y) = \frac{y_2}{|y|} \frac{(\nabla f(x)\cdot \frac{y}{|y|})(\nabla \partial_{x_1} f(x)\cdot \frac{y}{|y|})}{[1 + (\nabla f(x)\cdot \frac{y}{|y|})^2]^{5/2}}. \tag{1.26}$$

The function $\Sigma(x,y)$ satisfies

$$\begin{aligned} (i) &\quad \Sigma(x,\lambda y) = \Sigma(x,y), \quad \forall\, \lambda > 0, \\ (ii) &\quad \Sigma(x,-y) = -\Sigma(x,y), \\ (iii) &\quad \sup_x |\Sigma(x,y)| \leq \|\nabla \partial_{x_1} f\|_{L^\infty}, \end{aligned}$$

so that S is a bounded linear map on $L^p(\mathbb{R}^2)$ for $1 < p < \infty$ and $\|S\|_p \leq C\|\nabla \partial_{x_1} f\|_{L^\infty}$ (see [45] and the references therein for more details). Thus, $K_6 \leq C\|f\|_{C^2}\|\partial^4_{x_1} f\|_{L^2}\|\partial_{x_2}\partial^3_{x_1} f\|_{L^2}$. We obtain finally

$$I_5 \leq C(1 + \|f\|^4_{C^{2,\delta}})\|f\|^2_{H^4}.$$

In order to estimate the term I_6 we take

$$\begin{aligned} I_6 &= 6\int_0^1 ds \int_{\mathbb{R}^2} dy \int_{\mathbb{R}^2} \partial_{x_1}^4 f(x)\, y\cdot(\nabla^2\partial_{x_1}^2 f(x+(s-1)y))\cdot y\ \partial_{x_1}^2 A(x,y)dx \\ &\leq \int_0^1 ds\Big(\int_{|y|<1} dy + \int_{|y|>1} dy\Big)\int_{\mathbb{R}^2} |\partial_{x_1}^4 f(x)||\nabla^2\partial_{x_1}^2 f(x+(s-1)y)||\partial_{x_1}^2 A(x,y)||y|^2\, dx \\ &\leq C\Big(\int_{|y|<1} |y|^{-2+\delta}dy + \int_{|y|>1} |y|^{-3}dy\Big)(1+\|f\|_{C^{2,\delta}}^4)\|f\|_{H^4}^2. \end{aligned}$$

The most singular term of I_7 is K_7

$$K_7 = -12\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial_{x_1}^4 f(x)(\nabla\partial_{x_1} f(x) - \nabla\partial_{x_1} f(x-y))\cdot y\ G(x,y)dydx,$$

where

$$G(x,y) = \frac{(f(x)-f(x-y))(\partial_{x_1}^3 f(x) - \partial_{x_1}^3 f(x-y))}{[|y|^2 + (f(x)-f(x-y))^2]^{5/2}}.$$

Due to $|\nabla\partial_{x_1} f(x) - \nabla\partial_{x_1} f(x-y)| \leq \|f\|_{C^{2,\delta}}|y|^\delta$ and writing

$$\partial_{x_1}^3 f(x) - \partial_{x_1}^3 f(x-y) = \int_0^1 \nabla\partial_{x_1}^3 f(x+(s-1)y)\cdot y\, ds,$$

we obtain $K_7 \leq C(\|f\|_{C^{2,\delta}}^2+1)\|f\|_{H^4}^2$ and $I_7 \leq C(1+\|f\|_{C^{2,\delta}}^4)\|f\|_{H^4}^2$. The most singular term of I_8 is K_8

$$K_8 = -12\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \partial_{x_1}^4 f(x)(\partial_{x_1}^4 f(x) - \partial_{x_1}^4 f(x-y))\ H(x,y)dydx,$$

where

$$H(x,y) = \frac{(f(x)-f(x-y))(\nabla f(x) - \nabla f(x-y))\cdot y}{[|y|^2 + (f(x)-f(x-y))^2]^{5/2}}.$$

Then

$$K_8 = -12\int_{\mathbb{R}^2} |\partial_{x_1}^4 f(x)|^2 PV\int_{\mathbb{R}^2} \frac{(f(x)-f(x-y))(\nabla f(x) - \nabla f(x-y))\cdot y}{[|y|^2 + (f(x)-f(x-y))^2]^{5/2}}dydx,$$

is controlled as before. We obtain $K_8 \leq C(1+\|f\|_{C^{2,\delta}}^4)\|\partial_{x_1}^4 f\|_{L^2}^2$ and $I_8 \leq C(1+\|f\|_{C^{2,\delta}}^4)\|f\|_{H^4}^2$. Finally, we have

$$\frac{d}{dt}\|\partial_{x_1}^4 f\|_{L^2}^2(t) \leq C(1+\|f\|_{C^{2,\delta}}^4(t))\|f\|_{H^4}^2(t),$$

and using Sobolev inequalities we get

$$\frac{d}{dt}\|\partial_{x_1}^4 f\|_{L^2}^2(t) \leq C(\|f\|_{H^4}^6(t)+1). \tag{1.27}$$

In a similar way we obtain

$$\frac{d}{dt}\|\partial_{x_2}^4 f\|_{L^2}^2(t) \leq C(\|f\|_{H^4}^6(t)+1), \tag{1.28}$$

and since we can define $\|f\|_{H^4}^2 = \|f\|_{L^2}^2 + \|\partial_{x_1}^4 f\|_{L^2}^2 + \|\partial_{x_2}^4 f\|_{L^2}^2$, due to (1.21), (1.27) and (1.28) it follows that

$$\frac{d}{dt}\|f\|_{H^4}(t) \le C(\|f\|_{H^4}^5(t) + 1).$$

Using Gronwall's inequality we get that the quantity $\|f\|_{H^4}$ is bounded up to a time $T = T(\|f_0\|_{H^4})$. Then, applying energy methods the local existence result follows.

Let the functions $f_1(x,t), f_2(x,t)$ be two solutions of equation (1.13) with $f_1(x,0) = f_2(x,0) = f_0(x)$, and $f = f_1 - f_2$. Then

$$\frac{d}{dt}\|f\|_{L^2}^2(t) = I_9 + I_{10} + I_{11},$$

with

$$I_9 = \int_{\mathbb{R}^2} f(x)\nabla f(x)\cdot PV\int_{\mathbb{R}^2} y[|y|^2 + (f_1(x) - f_1(x-y))^2]^{-3/2}dydx,$$

$$I_{10} = -\int_{\mathbb{R}^2} f(x) PV\int_{\mathbb{R}^2} \nabla f(y,t)\cdot(x-y)[|x-y|^2 + (f_1(x) - f_1(y))^2]^{-3/2}dydx,$$

and

$$I_{11} = \int_{\mathbb{R}^2} f(x) PV\int_{\mathbb{R}^2} (\nabla f_2(x) - \nabla f_2(x-y))\cdot yN(x,y)dydx,$$

with

$$N(x,y) = [|y|^2 + (f_1(x) - f_1(x-y))^2]^{-3/2} - [|y|^2 + (f_2(x) - f_2(x-y))^2]^{-3/2}.$$

Integrating by parts in I_9, we have

$$I_9 \le C(\|f_1\|_{H^4})\|f\|_{L^2}^2,$$

and computing it follows that

$$I_{11} \le C(\|f_1\|_{H^4}, \|f_2\|_{H^4})\|f\|_{L^2}^2.$$

The term

$$\begin{aligned}
I_{10} &= -\int_{\mathbb{R}^2} f(x) PV\int_{\mathbb{R}^2} \nabla_y(f(y) - f(x))\cdot(x-y)[|x-y|^2 + (f_1(x) - f_1(y))^2]^{-3/2}dydx \\
&= -PV\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)(f(x) - f(y))[|x-y|^2 + (f_1(x) - f_1(y))^2]^{-3/2}dydx \\
&\quad + PV\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} f(x)(f(x) - f(y))\frac{3(f_1(x) - f_1(y))(f_1(x) - f_1(y) - \nabla f_1(x)(x-y))}{[|x-y|^2 + (f_1(x) - f_1(y))^2]^{5/2}}dydx.
\end{aligned}$$

Then we have that $I_{10} \le J_5 + J_6$ where

$$J_5 = \int_{\mathbb{R}^2} |f(x)|^2 PV\int_{\mathbb{R}^2} \frac{3(f_1(x) - f_1(y))(f_1(x) - f_1(y) - \nabla f_1(x)(x-y))}{[|x-y|^2 + (f_1(x) - f_1(y))^2]^{5/2}}dydx,$$

and

$$J_6 = -\int_{\mathbb{R}^2} f(x) PV \int_{\mathbb{R}^2} f(x-y) \frac{3(f_1(x)-f_1(x-y))(f_1(x)-f_1(x-y)-\nabla f_1(x)\cdot y)}{[|y|^2+(f_1(x)-f_1(x-y))^2]^{5/2}} dydx.$$

The term J_5 is estimated in the same way as J_1, so that $J_5 \leq C(\|f_1\|_{H^4})\|f\|_{L^2}^2$. The term J_6 can be expressed as $J_6 = K_9 + K_{10}$ with

$$K_9 = -3\int_{\mathbb{R}^2} f(x) \int_{\mathbb{R}^2} f(x-y) \frac{(f_1(x)-f_1(x-y))G(x,y)}{[|y|^2+(f_1(x)-f_1(x-y))^2]^{5/2}} dydx,$$

where the function $G(x,y)$ is given by

$$G(x,y) = f_1(x) - f_1(x-y) - \nabla f_1(x)\cdot y - \frac{1}{4} y\cdot(\nabla^2 f_1(x) + \nabla^2 f_1(x-y))\cdot y.$$

One finds that the principal value K_{10} is null. Therefore, we obtain finally $J_6 \leq C(\|f_1\|_{H^4})\|f\|_{L^2}^2$. Applying Gronwall's inequality we get uniqueness.

1.4.2 Two dimensional periodic interface

In the periodic case we give the theorem of local well-posedness and the differences with $\Omega = \mathbb{R}^2$.

Theorem 1.4.2 *Let $f_0(x) \in H^k(\mathbb{T}^2)$ for $k \geq 4$ and $\rho_2 > \rho_1$. Then there exists a time $T > 0$ so that there is a unique solution to* (1.15) *in $C^1([0,T]; H^k(\mathbb{T}^2))$ with $f(x,0) = f_0(x)$.*

Proof: The argument is similar to theorem 2.4.1 but we must use the properties of the function L in (1.14). The terms with the function M can be estimated easily integrating by parts. We consider without loss of generality $\rho_2 - \rho_1 = 4\pi$. In order to control the evolution of the quantity $\|\partial_{x_1}^4 f\|_{L^2}$, the most singular term is

$$\begin{aligned} I &= \int_{\mathbb{T}^2} \partial_{x_1}^4 f(x) \int_{\mathbb{T}^2} \frac{(\nabla\partial_{x_1}^4 f(x) - \nabla\partial_{x_1}^4 f(x-y))\cdot y}{[|y|^2+(f(x)-f(x-y))^2]^{3/2}} L(y, f(x)-f(x-y)) dydx \\ &= \int_{\mathbb{T}^2} \partial_{x_1}^4 f(x) \nabla\partial_{x_1}^4 f(x) \cdot PV \int_{\mathbb{T}^2} y \frac{L(y, f(x)-f(x-y))}{[|y|^2+(f(x)-f(x-y))^2]^{3/2}} dydx \\ &\quad - \int_{\mathbb{T}^2} \partial_{x_1}^4 f(x) PV \int_{\mathbb{T}^2} \frac{\nabla\partial_{x_1}^4 f(y)\cdot(x-y)}{[|x-y|^2+(f(x)-f(y))^2]^{3/2}} L(x-y, f(x)-f(y)) dydx \\ &= J_1 + J_2. \end{aligned}$$

Integrating by parts

$$\begin{aligned} J_1 &= \frac{3}{2}\int_{\mathbb{T}^2} |\partial_{x_1}^4 f(x)|^2 PV \int_{\mathbb{T}^2} A(x,y) L(y, f(x)-f(x-y)) dydx \\ &\quad - \frac{1}{2}\int_{\mathbb{T}^2} |\partial_{x_1}^4 f(x)|^2 PV \int_{\mathbb{T}^2} \frac{L_{x_3}(y, f(x)-f(x-y))(\nabla f(x)-\nabla f(x-y))\cdot y}{[|y|^2+(f(x)-f(x-y))^2]^{3/2}} dydx \qquad (1.29) \\ &= K_1 + K_2, \end{aligned}$$

where
$$A(x,y)=\frac{(f(x)-f(x-y))(\nabla f(x)-\nabla f(x-y))\cdot y}{[|y|^2+((f(x)-f(x-y))^2]^{5/2}}$$
and
$$L_{x_3}(x_1,x_2,x_3)=\partial_{x_3}L(x_1,x_2,x_3).$$
Due to $|L(x_1,x_2,x_3)-1|\leq C|(x_1,x_2,x_3)|$ we have
$$\begin{aligned}K_1&\leq C(1+\|f\|^4_{C^{2,\delta}})\|f\|^2_{H^4}+C\|\partial^4_{x_1}f\|^2_{L^2}\max_x\left|PV\int_{\mathbb{T}^2}\frac{(\nabla f(x)\cdot y)(y\cdot\nabla^2 f(x)\cdot y)}{[|y|^2+(\nabla f(x)\cdot y))^2]^{5/2}}dy\right|\\&\leq C(1+\|f\|^4_{C^{2,\delta}})\|f\|^2_{H^4}.\end{aligned}$$
Using that $|f(x)-f(x-y)|\leq\|f\|_{C^1}|y|$ and $L_{x_3}=0$ in $\{x_1^2+x_2^2+x_3^2\leq 4\}$, we have that
$$\begin{aligned}K_2&=-\frac{1}{2}\int_{\mathbb{T}^2}|\partial^4_{x_1}f(x)|^2\int_{|y|>\frac{2}{1+\|f\|_{C^1}}}\frac{L_{x_3}(y,f(x)-f(x-y))(\nabla f(x)-\nabla f(x-y))\cdot y}{[|y|^2+(f(x)-f(x-y))^2]^{3/2}}dydx\\&\leq C(\|f\|^4_{C^{2,\delta}}+1)\|f\|^2_{H^4}.\end{aligned}$$
In order to estimate J_2, we integrate by parts getting
$$\begin{aligned}J_2&=-\int_{\mathbb{T}^2}\partial^4_{x_1}f(x)PV\int_{\mathbb{T}^2}\frac{\nabla_y(\partial^4_{x_1}f(y)-\partial^4_{x_1}f(x))\cdot(x-y)}{[|x-y|^2+(f(x)-f(y))^2]^{3/2}}L(x-y,f(x)-f(y))dydx\\&=K_3+K_4+K_5+K_6\end{aligned}$$
with
$$K_3=-\int_{\mathbb{T}^2}\partial^4_{x_1}f(x)PV\int_{\mathbb{T}^2}\frac{\partial^4_{x_1}f(x)-\partial^4_{x_1}f(y)}{[|x-y|^2+(f(x)-f(y))^2]^{3/2}}L(x-y,f(x)-f(y))dydx,$$
$$K_4=\int_{\mathbb{T}^2}\partial^4_{x_1}f(x)\int_{\mathbb{T}^2}(\partial^4_{x_1}f(x)-\partial^4_{x_1}f(y))B(x,y)L(x-y,f(x)-f(y))dydx,$$
$$K_5=\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\partial^4_{x_1}f(x)(\partial^4_{x_1}f(y)-\partial^4_{x_1}f(x))C(x,y)L_{x_3}(x-y,f(x)-f(y))dydx,$$
$$K_6=\int_{\mathbb{T}^2}\int_{\mathbb{T}^2}\partial^4_{x_1}f(x)(\partial^4_{x_1}f(y)-\partial^4_{x_1}f(x))D(x,y)dydx,$$
and
$$B(x,y)=\frac{3(f(x)-f(y))(f(x)-f(y)-\nabla f(y)\cdot(x-y))}{[|x-y|^2+(f(x)-f(y))^2]^{5/2}},$$
$$C(x,y)=-\frac{\nabla f(y)\cdot(x-y)}{[|x-y|^2+(f(x)-f(y))^2]^{3/2}},$$

$$D(x,y) = -\frac{L_{x_1}(x-y, f(x)-f(y))(x_1-y_1) + L_{x_2}(x-y, f(x)-f(y))(x_2-y_2)}{[|x-y|^2 + (f(x)-f(y))^2]^{3/2}},$$

$$L_{x_1}(x_1,x_2,x_3) = \partial_{x_1} L(x_1,x_2,x_3), \qquad L_{x_2}(x_1,x_2,x_3) = \partial_{x_2} L(x_1,x_2,x_3).$$

Exchanging the variables x and y we obtain $K_3 \leq 0$. The terms K_4, K_5 and K_6 can be estimated in a similar way to K_1. Therefore, we obtain

$$\frac{d}{dt}\|\partial_{x_1}^4 f\|_{L^2}(t) \leq C(\|f\|_{H^4}^5(t) + 1),$$

and analogously

$$\frac{d}{dt}\|\partial_{x_2}^4 f\|_{L^2}(t) \leq C(\|f\|_{H^4}^5(t) + 1).$$

This proves local existence. The proof of the uniqueness is similar to the case $\Omega = \mathbb{R}^2$.

1.4.3 One dimensional interface; nonperiodic and periodic

Using equation (1.16) in $\mathbb{R}$ and equation (1.17) in the periodic case we obtain the following theorem.

Theorem 1.4.3 *Let $f_0(x) \in H^k$ for $k \geq 3$ and $\rho_2 > \rho_1$. Then there exists a time $T > 0$ so that there is a unique solution to* (1.16) *in $C^1([0,T]; H^k)$ with $f(x,0) = f_0(x)$.*

The proof is similar to that of theorems 1.4.1 and 1.4.2, but due to fact that the solution is one-dimensional, we can use Sobolev inequalities in such a way that the initial data can belong to H^3 at least.

1.5 Global solution for small initial data

In this section we obtain a family of global solutions for a 1-D interface with small initial data with respect to a fixed norm. Indeed, we can get the result with initial data with the property $\|f_0\|_{H^s} = \infty$ for $s > 3/2$. We consider $x \in \mathbb{R}$ and

$$\|f\|_a = \sum |\hat{f}(k)| e^{a|k|}.$$

For $a > 0$, if $\|f\|_a < \infty$, then the function f can be analytically extended on the strip $|\Im z| < a$. Furthermore

$$\|\partial_x f\|_a \leq C \frac{\|f\|_b}{b-a}, \tag{1.30}$$

for $b > a$. The main result of this section is

Theorem 1.5.1 *Let $f_0(x)$ be a function such that $\int_{\mathbb{T}} f_0(x)\,dx = 0$, $\|\partial_x f_0\|_0 \leq \varepsilon$ for ε small enough and*

$$\|\partial_x^2 f_0\|_{b(t)} \leq \varepsilon e^{b(t)}(1+|b(t)|^{\gamma-1}), \tag{1.31}$$

with $0<\gamma<1$, $b(t) = a-(\rho_2-\rho_1)t/2$, $\rho_2>\rho_1$ and $a \leq (\rho_2-\rho_1)t/2$. Then, there exists a unique solution of (1.16) with $f(x,0) = f_0(x)$ and $\rho_2 > \rho_1$ satisfying

$$\|\partial_x f\|_a(t) \leq C(\varepsilon)\exp((2\sigma a-(\rho_2-\rho_1)t)/4), \tag{1.32}$$

and

$$\|\partial_x^2 f\|_a(t) \leq C(\varepsilon)(1+|\sigma a - \frac{\rho_2-\rho_1}{2}t|^{\gamma-1})\exp((2\sigma a-(\rho_2-\rho_1)t)/4), \tag{1.33}$$

for $a \leq \frac{\rho_2-\rho_1}{2\sigma}t$, $\sigma = 1+\delta$ and $0<\delta<1$.

The condition (1.31) can be satisfied for example if $\|\Lambda^{1+\gamma} f_0\|_0 < \varepsilon$ and $\hat{f}_0(0) = \hat{f}_0(1) = \hat{f}_0(-1) = 0$ since

$$\|\partial_x^2 f_0\|_{b(t)} \leq e^{b(t)}\|\Lambda^{1+\gamma} f_0\|_0 \max_{k\geq 2}|k|^{1-\gamma}e^{b(t)(|k|-1)}.$$

In order to prove the theorem, we use the Cauchy-Kowalewski method (see [35] and [36]) in a similar way as Caflisch and Orellana [8] and Siegel, Caflisch and Howison [42]. We show the proof with $\rho_2-\rho_1 = 2$ without loss of generality. Let $g(x,t)$ and $h(x,t)$ be functions satisfying

$$\begin{aligned} &g_t = -\Lambda g,\\ &g(x,0) = f_0(x),\\ &h_t = -\Lambda h + T(g+h),\\ &h(x,0) = 0, \end{aligned} \tag{1.34}$$

with

$$T(f) = -\pi^{-1}\int_{\mathbb{R}} \frac{\partial_x f(x)-\partial_x f(x-\alpha)}{\alpha}\frac{\left(\frac{f(x)-f(x-\alpha)}{\alpha}\right)^2}{1+\left(\frac{f(x)-f(x-\alpha)}{\alpha}\right)^2}d\alpha. \tag{1.35}$$

Then the function $f(x,t) = g(x,t)+h(x,t)$ is a solution of (1.16). First, we show some properties of the nonlinear operator T.

Lemma 1.5.2 *If $\|\partial_x f\|_a, \|\partial_x g\|_a < 1$ for $a \geq 0$ then*

$$\widehat{T(f)}(0) = 0, \tag{1.36}$$

$$\|\partial_x T(f)\|_a \leq C_1 \|\partial_x^2 f\|_a \|\partial_x f\|_a, \tag{1.37}$$

and

$$\begin{aligned} \|\partial_x T(f) - \partial_x T(g)\|_a \leq\ & C_2(\|\partial_x^2 f\|_a + \|\partial_x^2 g\|_a)\|\partial_x f-\partial_x g\|_a\\ &+ C_2(\|\partial_x f\|_a+\|\partial_x g\|_a)\|\partial_x^2 f-\partial_x^2 g\|_a, \end{aligned} \tag{1.38}$$

with $C_1 = 4(1-\|\partial_x f\|_a^2)^{-2}$ and $C_2 = 4(1-\|\partial_x f\|_a^2)^{-2} + (1-\|\partial_x g\|_a^2)^{-2}$.

Proof of the Lemma: Due to the inequality $|\partial_x f(x)| \le \|\partial_x f\|_a < 1$, and by (1.35), we obtain

$$T(f) = \pi^{-1} \sum_{n\ge 1} (-1)^n \int_{\mathbb{R}} \frac{\partial_x f(x) - \partial_x f(x-\alpha)}{\alpha} \Big(\frac{f(x)-f(x-\alpha)}{\alpha}\Big)^{2n} d\alpha, \tag{1.39}$$

and

$$T(f) = \pi^{-1} \partial_x \sum_{n\ge 1} \frac{(-1)^n}{2n+1} \int_{\mathbb{R}} \Big(\frac{f(x)-f(x-\alpha)}{\alpha}\Big)^{2n+1} d\alpha.$$

Thus $\widehat{T(f)}(0) = 0$. Using (1.39)

$$\begin{aligned}\widehat{T(f)}(k) &= \pi^{-1} \sum_{n\ge 1} (-1)^n \int_{\mathbb{R}} \sum_{k_0,\dots,k_{2n}} \delta(\sum_{j=0}^{2n} k_j, k) i k_0 \prod_{j=0}^{2n} \hat{f}(k_j) \frac{1-e^{-i\alpha k_j}}{\alpha} d\alpha \\ &= \sum_{n\ge 1} (-1)^n \sum_{k_0,\dots,k_{2n}} \delta(\sum_{j=0}^{2n} k_j, k)\, M_n(k_0,...,k_{2n})\, i k_0 \prod_{j=0}^{2n} \hat{f}(k_j),\end{aligned}$$

where

$$M_n(k_0,...,k_{2n}) = \pi^{-1} \int_{\mathbb{R}} \prod_{j=0}^{2n} \frac{1-e^{-i\alpha k_j}}{\alpha} d\alpha. \tag{1.40}$$

We get

$$M_n(k_0,...,k_{2n}) = (-1)^n m_n(k_0,...,k_{2n}) \prod_{j=1}^{2n} k_j,$$

with

$$\begin{aligned} m_n(k_0,...,k_{2n}) &= \pi^{-1} \int_0^1 ds_1 ... \int_0^1 ds_{2n} \int_{\mathbb{R}} \frac{1-e^{-i\alpha k_0}}{\alpha} exp\Big(i\alpha \sum_{j=1}^{2n} (s_j-1) k_j\Big) d\alpha \\ &= \pi^{-1} \int_0^1 ds_1 ... \int_0^1 ds_{2n} PV \int_{\mathbb{R}} exp\Big(i\alpha \sum_{j=1}^{2n} (s_j-1)k_j\Big) \frac{d\alpha}{\alpha} \\ &\quad - \pi^{-1} \int_0^1 ds_1 ... \int_0^1 ds_{2n} PV \int_{\mathbb{R}} exp\Big(-i\alpha k_0 + i\alpha \sum_{j=1}^{2n}(s_j-1)k_j\Big) \frac{d\alpha}{\alpha} \\ &= i \int_0^1 ds_1 ... \int_0^1 ds_{2n} (\text{sing}\, A - \text{sing}\, B),\end{aligned}$$

and

$$A = \sum_{j=1}^{2n} (s_j-1) k_j, \qquad B = -k + \sum_{j=1}^{2n} s_j k_j.$$

It follows that

$$\widehat{T(f)}(k) = \sum_{n\geq 1} \sum_{k_0,...,k_n} \delta(\sum_{j=0}^{2n} k_j, k)\, m_n(k_0,...,k_{2n}) \prod_{j=0}^{2n} k_j \hat{f}(k_j),$$

with $|m_n(k_0,...,k_{2n})| \leq 2$. We have

$$\begin{aligned}\sum_k e^{a|k|}|k||\widehat{T(f)}(k)| &\leq 2\sum_k \sum_{n\geq 1} \sum_{k_0,...,k_n} e^{a|k|}|k|\delta(\sum_{j=0}^{2n} k_j, k) \prod_{j=0}^{2n} |k_j||\hat{f}(k_j)| \\ &\leq 2\sum_{n\geq 1}(2n+1) \sum_{k_0,...,k_n} e^{a|k_0|}|k_0|^2|\hat{f}(k_0)| \prod_{j=1}^{2n} e^{a|k_j|}|k_j||\hat{f}(k_j)|,\end{aligned}$$

and therefore

$$\|\partial_x T(f)\|_a \leq 2\|\partial_x^2 f\|_a \sum_{n\geq 1}(2n+1)\|\partial_x f\|_a^{2n} = 2\|\partial_x^2 f\|_a \frac{3\|\partial_x f\|_a^3 - \|\partial_x f\|_a^4}{(1-\|\partial_x f\|_a^2)^2}.$$

We get (1.37) for $\|\partial_x f\|_a < 1$. In a similar way we obtain (1.38).□

From (1.34) g can be expressed as follows:

$$\hat{g}(k,t) = e^{-|k|t}\hat{f}_0(k),$$

and by the hypothesis of the initial data we have

$$\|\partial_x g\|_a(t) \leq \varepsilon e^{a-t}, \tag{1.41}$$
$$\|\partial_x^2 g\|_a(t) \leq \varepsilon e^{a-t}(1+(t-a)^{\gamma-1}), \tag{1.42}$$

for $t \geq a$. We will prove the existence of h by an induction argument on the iterative equation:

$$\begin{aligned}&\partial_t h^{n+1} = -\Lambda h^{n+1} + T(g+h^n), \\ &h^{n+1}(x,0) = 0, \\ &h^0 = 0,\end{aligned}$$

or

$$\begin{aligned}&\widehat{h^{n+1}}(k,t) = \int_0^t e^{-|k|(t-s)}(T(g+h^n))\widehat{\ }(k,s)ds, \\ &h^0 = 0.\end{aligned}$$

For h^1 we obtain the following estimates:

$$\|\partial_x h^1\|_a(t) \leq \int_0^t \|T(g)\|_{a+s-t}(s)ds = \int_0^{t-a} + \int_{t-a}^t = I_1 + I_2.$$

Using (1.37), (1.41) and (1.42) we get

$$I_1 \le e^{a-t}\int_0^{t-a} e^s \|\partial_x T(g)\|_0(s)ds \le Ce^{a-t}\int_0^{t-a} e^s \|\partial_x^2 g\|_0(s)\|\partial_x g\|_0(s)ds$$
$$\le C\varepsilon^2 e^{a-t}\int_0^{t-a} e^{-s}(1+s^{\gamma-1})ds \le \frac{C\varepsilon^2(1+2\gamma)}{\gamma}e^{a-t}.$$

By (1.41) and (1.42) we have

$$I_2 \le C\int_{t-a}^t \|\partial_x^2 g\|_{a+s-t}(s)\|\partial_x g\|_{a+s-t}(s)ds \le C\varepsilon^2 e^{2(a-t)}a(1+(t-a)^{\gamma-1}) \le \frac{2C\varepsilon^2}{\delta}e^{a-t},$$

due to the inequalities $(a\delta)^{\gamma-1} > (t-a)^{\gamma-1}$ and $ae^{a-t} \le \delta^{-1}$ for $\sigma a < t$. Then

$$\|\partial_x h^1\|_a(t) \le \frac{5C\varepsilon^2}{\delta\gamma}e^{a-t}.$$

Choosing $b = a+s-t+\frac{t-a}{2}$ we have

$$\|\partial_x^2 h^1\|_a(t) \le \int_0^t \|\partial_x^2 T(g)\|_{a+s-t}(s)ds \le \int_0^t \frac{\|\partial_x T(g)\|_b(s)}{b-(a+s-t)}ds \le 2\int_0^t \frac{\|\partial_x T(g)\|_b}{t-a}ds$$
$$\le \Big(\int_0^{\frac{t-a}{2}} + \int_{\frac{t-a}{2}}^t\Big) = I_3 + I_4,$$

where

$$I_3 \le \frac{2Ce^{\frac{a-t}{2}}}{t-a}\int_0^{\frac{t-a}{2}} e^s\|\partial_x^2 g\|_0(s)\|\partial_x g\|_0(s)ds \le \frac{2C\varepsilon^2 e^{\frac{a-t}{2}}}{t-a}\int_0^{\frac{t-a}{2}} e^{-s}(1+s^{\gamma-1})ds$$
$$\le \frac{2C\varepsilon^2}{\gamma}e^{\frac{a-t}{2}}(1+(t-a)^{\gamma-1}),$$

and

$$I_4 \le \frac{2C}{t-a}\int_{\frac{t-a}{2}}^t \|\partial_x^2 g\|_b(s)\|\partial_x g\|_b(s)ds \le \frac{2C\varepsilon^2}{t-a}e^{a-t}(1+(\frac{t-a}{2})^{\gamma-1})(\frac{t}{2}+\frac{a}{2})$$
$$\le \frac{3C\varepsilon^2}{\delta}e^{a-t}(1+(t-a)^{\gamma-1}).$$

Therefore

$$\|\partial_x h^1\|_a(t) \le \frac{5C\varepsilon^2}{\delta\gamma}e^{a-t}, \tag{1.43}$$
$$\|\partial_x^2 h^1\|_a(t) \le \frac{5C\varepsilon^2}{\delta\gamma}e^{\frac{a-t}{2}}(1+(t-a)^{\gamma-1}). \tag{1.44}$$

Define $r^{n+1} = h^{n+1} - h^n$,

$$R_n = \sup_{\substack{0 \le a < \infty \\ \sigma a < t}} \Big(\|\partial_x r^n\|_a + \frac{\|\partial_x^2 r^n\|_a}{1+(t-\sigma a)^{\gamma-1}} \Big) e^{\frac{t-\sigma a}{2}},$$

and

$$M_n = \sup_{\substack{0 \le a < \infty \\ \sigma a < t}} \Big(\|\partial_x h^n\|_a + \frac{\|\partial_x^2 h^n\|_a}{1+(t-\sigma a)^{\gamma-1}} \Big) e^{\frac{t-\sigma a}{2}}.$$

Take $M_1 = R_1 \le \frac{5C\varepsilon^2}{\delta\gamma} \le \frac{\varepsilon_0}{2}$ and suppose that $M_j, R_j \le \frac{\varepsilon_0}{2}$ for any $j = 2, ..., n$, then

$$\|\partial_x r^{n+1}\|_a \le \int_0^t \|\partial_x T(g+h^n) - \partial_x T(g+h^{n-1})\|_{a+s-t}(s)ds = \int_0^{t-a} + \int_{t-a}^t = I_7 + I_8.$$

Using (1.38) we have

$$\begin{aligned} I_7 &\le Ce^{a-t}\int_0^{t-a} e^s(\|\partial_x r^n\|_0(s)(\|\partial_x^2 g + \partial_x^2 h^n\|_0(s) + \|\partial_x^2 g + \partial_x^2 h^{n-1}\|_0(s)) \\ &\quad + Ce^{a-t}\int_0^{t-a} e^s(\|\partial_x^2 r^n\|_0(s)(\|\partial_x g + \partial_x h^n\|_0(s) + \|\partial_x g + \partial_x h^{n-1}\|_0(s))ds \\ &\le 2C\varepsilon_0 R_n e^{a-t}\int_0^{t-a}(1+s^{\gamma-1})ds \le \frac{2C\varepsilon_0}{\gamma} R_n e^{\frac{\sigma a - t}{2}}, \end{aligned}$$

and

$$\begin{aligned} I_8 &\le C\int_{t-a}^t (\|\partial_x r^n\|_{a+s-t}(s)(\|\partial_x^2 g + \partial_x^2 h^n\|_{a+s-t}(s) + \|\partial_x^2 g + \partial_x^2 h^{n-1}\|_{a+s-t}(s))ds \\ &\quad + C\int_{t-a}^t (\|\partial_x^2 r^n\|_{a+s-t}(s)(\|\partial_x g + \partial_x h^n\|_{a+s-t}(s) + \|\partial_x g + \partial_x h^{n-1}\|_{a+s-t}(s))ds \\ &\le 2C\varepsilon_0 R_n \int_{t-a}^t e^{\delta s - \sigma(t-a)}(1+(\sigma(t-a)-\delta s)^{\gamma-1})ds \\ &\le \frac{2C\varepsilon_0}{\delta} R_n \int_{t-\sigma a}^{t-a} e^{-x}(1+x^{\gamma-1})dx \le \frac{6C\varepsilon_0}{\gamma\delta} R_n e^{\sigma a - t}. \end{aligned}$$

We obtain for $b = a + s - t + \frac{\sigma(t-a)-\delta s}{2\sigma}$

$$\begin{aligned} \|\partial_x^2 r^{n+1}\|_a(t) &\le \int_0^t \|\partial_x^2(T(g+h^n) - T(g+h^{n-1}))\|_{a+s-t}(s)ds \\ &\le \int_0^t \frac{\|\partial_x(T(g+h^n) - T(g+h^{n-1})\|_b(s)}{b-(a+s-t)} ds \\ &\le 2\sigma\int_0^t \frac{\|\partial_x(T(g+h^n) - T(g+h^{n-1})\|_b}{\sigma(t-a)-\delta s} \le \Big(\int_0^{\frac{\sigma}{\sigma+1}(t-a)} + \int_{\frac{\sigma}{\sigma+1}(t-a)}^t \Big) = I_9 + I_{10}. \end{aligned}$$

We have $\sigma(t-a)-\delta s > \frac{2\sigma}{\sigma+1}(t-a)$ for $0 \leq s \leq \frac{\sigma}{\sigma+1}(t-a)$ and therefore we obtain

$$
\begin{aligned}
I_9 \leq & \frac{(\sigma+1)C}{t-a}\int_0^{\frac{\sigma}{\sigma+1}(t-a)} e^b(\|\partial_x r^n\|_0(s)(\|\partial_x^2 g+\partial_x^2 h^n\|_0(s)+\|\partial_x^2 g+\partial_x^2 h^{n-1}\|_0(s))ds \\
& + \frac{(\sigma+1)C}{t-a}\int_0^{\frac{\sigma}{\sigma+1}(t-a)} e^b(\|\partial_x^2 r^n\|_0(s)(\|\partial_x g+\partial_x h^n\|_0(s)+\|\partial_x g+\partial_x h^{n-1}\|_0(s))ds \\
\leq & \frac{2(\sigma+1)C\varepsilon_0}{t-a}R_n\int_0^{\frac{\sigma}{\sigma+1}(t-a)} e^{b-s}(1+s^{\gamma-1})ds \leq \frac{4\sigma C\varepsilon_0}{\gamma}R_n e^{\frac{a-t}{2}}(1+(t-a)^{\gamma-1}).
\end{aligned}
$$

Using (1.38) and the induction hypothesis we get

$$
\begin{aligned}
I_{10} \leq & 4\sigma C\varepsilon_0 R_n\int_{\frac{\sigma}{\sigma+1}(t-a)}^t e^{\sigma b-s}\frac{(1+(s-\sigma b)^{\gamma-1})}{\sigma(t-a)-\delta s}ds \\
\leq & 4\sigma C\varepsilon_0 R_n\int_{\frac{\sigma}{\sigma+1}(t-a)}^t e^{\frac{\delta s-\sigma(t-a)}{2}}\frac{1+(\frac{\sigma(t-a)-\delta s}{2})^{\gamma-1}}{\sigma(t-a)-\delta s}ds \\
\leq & \frac{4\sigma C\varepsilon_0}{\delta}R_n\int_{\frac{t-\sigma a}{2}}^{\frac{\sigma}{\sigma+1}(t-a)} e^{-x}(x^{-1}+x^{\gamma-2})dx \leq \frac{8\sigma C\varepsilon_0}{\delta(1-\gamma)}R_n e^{\frac{\sigma a-t}{2}}(1+(\sigma a-t)^{\gamma-1}).
\end{aligned}
$$

Due to the estimates for I_7, I_8, I_9 and I_{10} we obtain

$$
R_{n+1} \leq \frac{C\sigma\varepsilon_0}{\delta\gamma(\gamma-1)}R_n. \tag{1.45}
$$

Choosing ε_0 small enough we get

$$
R_{n+1} \leq \frac{1}{2}R_n \leq ... \leq \frac{1}{2^n}R_1 \leq \frac{\varepsilon_0}{2^{n+1}},
$$

and

$$
M_{n+1} \leq \sum_{j=1}^{n+1} R_{n+1} \leq \varepsilon_0.
$$

Therefore, we obtain the function $h = \lim_{n\to\infty} h^n$ satisfying

$$
\|\partial_x h\|_a(t) \leq \sum_n R_n e^{\frac{\sigma a-t}{2}} \leq \varepsilon_0 e^{\frac{\sigma a-t}{2}}.
$$

Taking $f(x,t) = g(x,t)+h(x,t)$, we get (1.32) for $\rho_2-\rho_1 = 2$.

In order to show uniqueness, we write the equation (1.13) for $\rho_2-\rho_1 = 2$ in the following form:

$$
\begin{aligned}
f_t &= -\Lambda f + T(f), \\
f(x,0) &= f_0(x).
\end{aligned}
$$

Suppose that there exist two solutions f^1 and f^2 with $f^1(x,0) = f^2(x,0)$. Define R by

$$R = \sup_{\substack{0 \le a < \infty \\ \sigma a < t}} \Big(\|\partial_x f^1 - \partial_x f^2\|_a + \frac{\|\partial_x^2 f^1 - \partial_x^2 f^2\|_a}{1 + (t - \sigma a)^{\gamma - 1}} \Big) e^{\frac{t - \sigma a}{2}}.$$

It follows that $R \le \frac{C(\varepsilon)\sigma}{\delta\gamma(\gamma-1)} R$ and for ε small enough it yields $\frac{C(\varepsilon)\sigma}{\delta\gamma(\gamma-1)} < 1$ and therefore $f_1 = f_2$.

1.6 Ill-posedness for the unstable case

Here we show ill-posedness for the unstable case $\rho_1 > \rho_2$. We use the global solution for the 2-D stable case $f(x_1, t)$ satisfying (1.32) with $\|\Lambda^{1+\gamma} f_0\|_0 < C$ and $\|\Lambda^{1+\gamma+\zeta} f_0\|_0 = \infty$ for $\gamma, \zeta > 0$. Making a change of variables, we define $f_\lambda(x_1, t) = \lambda^{-1} f(\lambda x_1, -\lambda t + \lambda^{1/2})$ obtaining $\{f_\lambda\}_{\lambda > 0}$ a family of solutions to the unstable case. Using (1.32) follows

$$\|f_\lambda\|_{H^s}(0) = |\lambda|^{s - \frac{3}{2}} \|f\|_{H^s}(\lambda^{1/2}) \le C|\lambda|^{s - \frac{3}{2}} \|f\|_1(\lambda^{1/2}) \le C|\lambda|^{s - \frac{3}{2}} e^{-\frac{|\rho_2 - \rho_1|}{4}\lambda^{1/2}},$$

and

$$\|f_\lambda\|_{H^s}(\lambda^{-1/2}) = |\lambda|^{s - \frac{3}{2}} \|f\|_{H^s}(0) \ge |\lambda|^{s - \frac{3}{2}} C \sum_k |k|^{1+\gamma+\zeta} |\hat{f}_0(k)| = \infty,$$

for $s > 3/2$ and γ, ζ small enough. We obtain an ill posed problem for $s > 3/2$.

Theorem 1.6.1 *Let $s > 3/2$, then for any $\varepsilon > 0$ there exists a solution f of* (1.16) *with $\rho_1 > \rho_2$ and $0 < \delta < \varepsilon$ such that $\|f\|_{H^s}(0) \le \varepsilon$ and $\|f\|_{H^s}(\delta) = \infty$.*

Remark 1.6.2 *If one considers a solution of the 3-D problem satisfying $f(x_1, x_2, t) = f(x_1, t)$, from the equation* (1.13) *one obtains a solution of* (1.16)*. This shows that solutions of the 2-D case are solutions of the 3-D problem and therefore, using the above theorem, one obtains ill-posedness for the 3-D case with $\rho_1 > \rho_2$.*

1.7 Decay of the L^∞ norm

Here we show that the L^∞ norm of the solution of the system decreases in time. We consider the set Ω equal to $\mathbb{R}^2$ or $\mathbb{T}^2$. The following proposition is the main result of the section.

Proposition 1.7.1 *Let $f_0 \in H^k(\Omega)$ for $k \ge 4$, and $\rho_2 > \rho_1$. Then the unique solution to* (1.13) *satisfies that $\|f\|_{L^\infty}(t) \le \|f_0\|_{L^\infty}$.*

Proof: Using the theorems 1.4.1 and 1.4.2, there exists a time $T > 0$ and a unique solution $f(x,t) \in C^1([0,T]; H^k(\Omega))$ solution of (1.13). In particular $f(x,t) \in C^1([0,T] \times \Omega)$ using Sobolev inequalities. We consider the application $M(t) = \max_x |f(x,t)|$. In the case $\Omega = \mathbb{R}^2$, there always exists a point $x_t \in \mathbb{R}^2$ where $|f(x,t)|$ reaches its maximum due to the fact that $f(\cdot, t) \in H^s$ with $s > 1$, and using the Riemann-Lebesgue lemma $f(x,t)$ tends to 0 when $|x| \to \infty$. Suppose that this point is for $M(t) = f(x_t, t) > 0$. A similar argument can be used for $M(t) = -f(x_t, t) > 0$. By using the H. Rademacher theorem, the function $M(t)$ is differentiable almost everywhere. We calculate the derivative of $M(t)$ in a similar way as in [10, 16]. If we consider a point in which $M(t)$ is differentiable, we have

$$\begin{aligned} M'(t) &= \lim_{h\to 0^+}(M(t+h)-M(t))h^{-1} \\ &= \lim_{h\to 0^+}(f(x_{t+h}, t+h)-f(x_t,t))h^{-1} \\ &= \lim_{h\to 0^+}(f(x_{t+h}, t+h)-f(x_t,t+h))h^{-1} + (f(x_t,t+h)-f(x_t,t))h^{-1}. \end{aligned}$$

Since $f(x, t+h)$ reachs the maximum at $x = x_{t+h}$, it follows that

$$M'(t) \geq \lim_{h\to 0^+}(f(x_t, t+h)-f(x_t,t))h^{-1} = f_t(x_t,t).$$

Computing for $h < 0$, we obtain

$$M'(t) = f_t(x_t, t). \tag{1.46}$$

Using equation (1.13), the fact that $\nabla f(x_t, t) = 0$, and the last identity, we have

$$M'(t) = \frac{\rho_2-\rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{-\nabla f(y,t)\cdot(x_t-y)}{[|x_t-y|^2 + (f(x_t,t)-f(y,t))^2]^{3/2}} dy.$$

Integrating by parts

$$\begin{aligned} M'(t) &= \frac{\rho_2-\rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{\nabla_y(f(x_t,t)-f(y,t))\cdot(x_t-y)/|x_t-y|^3}{[1+(f(x_t,t)-f(y,t)/|x_t-y|)^2]^{3/2}} dy \\ &= -\frac{\rho_2-\rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{f(x_t,t)-f(y,t)}{[1+(f(x_t,t)-f(y,t)/|x_t-y|)^2]^{3/2}} \operatorname{div}_y \frac{x_t-y}{|x_t-y|^3} dy \\ &\quad - \frac{\rho_2-\rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{f(x_t,t)-f(y,t)}{|x_t-y|} \frac{x_t-y}{|x_t-y|^2} \cdot \nabla_y \Big(1+\Big(\frac{f(x_t,t)-f(y,t)}{|x_t-y|}\Big)^2\Big)^{-3/2} dy \\ &= I_1 + I_2. \end{aligned}$$

We have

$$I_2 = -\frac{\rho_2-\rho_1}{4\pi} PV \int_{\mathbb{R}^2} \nabla_y(\ln|x_t-y|)\cdot\nabla_y(G((f(x_t,t)-f(y,t))/|x_t-y|))dy,$$

where

$$G(x) = \frac{x^3}{(1+x^2)^{3/2}}.$$

The identity $\Delta_y(\ln|x_t-y|)/4\pi = \delta(x_t)$, and the following limit:

$$\lim_{y\to x_t} \frac{f(x_t,t)-f(y,t)}{|x_t-y|} = \lim_{y\to x_t} \frac{f(x_t,t)-f(y,t)-\nabla f(x_t,t)\cdot(x_t-y)}{|x_t-y|} = 0,$$

show that integrating by parts in I_2, we obtain

$$I_2 = \frac{\rho_2-\rho_1}{4\pi} PV \int_{\mathbb{R}^2} \Delta_y(\ln|x_t-y|)G((f(x_t,t)-f(y,t))/|x_t-y|))dy = (\rho_2-\rho_1)G(0),$$

and therefore $I_2 = 0$. The I_1 term is equal to

$$I_1 = -\frac{\rho_2 - \rho_1}{4\pi} PV \int_{\mathbb{R}^2} \frac{M(t) - f(y,t)}{[(x_t - y)^2 + (M(t) - f(y,t))^2]^{3/2}} dy \leq 0,$$

so that $M'(t) \leq 0$ for almost every t. Integrating in time we conclude the proof.

1.8 Conservation of mass equation for a regular initial data

The purpose of this section is to study the nonlinear two-dimensional conservation of mass equation (1.3) in a porous media with a regular initial data and the possible formation of singularities. The mass balance equation reads

$$\rho_t + v \cdot \nabla \rho = 0, \tag{1.47}$$

with the incompressible velocity in the system given by Darcy´s law

$$v = -\nabla p - (0, \rho). \tag{1.48}$$

Using the stream function $\psi(x,t)$, we have

$$v = \nabla^{\perp} \psi \equiv \left(-\frac{\partial \psi}{\partial x_2}, \frac{\partial \psi}{\partial x_1} \right), \tag{1.49}$$

and computing the curl in Darcy's law, we get the following equation for ψ:

$$-\Delta \psi = \frac{\partial \rho}{\partial x_1}. \tag{1.50}$$

The solution of this equation is given by

$$\psi(x,t) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x - y| \frac{\partial \rho}{\partial y_1}(y,t) dy, \qquad x \in \mathbb{R}^2. \tag{1.51}$$

Thus, the velocity v can be recovered from ψ by the operator $\nabla^{\perp}$ using two equivalent formulas

$$v(x,t) = \int_{\mathbb{R}^2} K(x-y)\, \nabla^{\perp} \rho(y,t) dy, \tag{1.52}$$

$$v(x,t) = PV \int_{\mathbb{R}^2} H(x-y)\, \rho(y,t) dy - \frac{1}{2} (0, \rho(x)), \tag{1.53}$$

where the kernels $K(\cdot)$ and $H(\cdot)$ are defined by

$$K(x) = -\frac{1}{2\pi} \frac{x_1}{|x|^2} \qquad \text{and} \qquad H(x) = \frac{1}{2\pi} \left(-2\frac{x_1 x_2}{|x|^4}, \frac{x_1^2 - x_2^2}{|x|^4} \right). \tag{1.54}$$

The equation (1.52) is obtained from (1.51) by integrating by parts. The equation (1.53) is the result of taking (1.52) as a limit as $\varepsilon \to 0$ of the integral on $|y| > \varepsilon$ and integrating by parts. Differentiating the equation (1.47), and using (1.52), we obtain the evolution equation only in terms of

$$\nabla^{\perp}\rho \equiv \left(-\frac{\partial\rho}{\partial x_2}, \frac{\partial\rho}{\partial x_1}\right), \tag{1.55}$$

which is given by

$$\frac{D\nabla^{\perp}\rho}{Dt} = (\nabla v)\nabla^{\perp}\rho. \tag{1.56}$$

In the following subsections we analyze the behavior of the solutions of this system. First, we present the existence of singularities in a class of solutions with infinite energy. In the case of solutions with regular initial data and finite energy, we get local well-posedness using the classical particle trajectories method. We illustrate a criterion of global existence solutions via the norm of the bounded mean oscillation space of (1.55). A similar result is known in the three-dimensional Euler equation (3D Euler) [2]. Also, using the geometric structure of the level sets of the density (where ρ is constant) and the nonlinear evolution equations of the gradient of the arc length of the level sets, we establish a no singularities criterium under not very restrictive conditions. This result is comparable to the 3D Euler equations [13] and to the quasi-geostrophic equation (see next chapter and [14] for more details).

1.8.1 Singularities with infinite energy

Let the stream function ψ be defined by

$$\psi(x_1, x_2, t) = x_2 f(x_1, t) + g(x_1, t). \tag{1.57}$$

Note that under this hypothesis the solution of (1.47) has infinite energy. We reduce the equations to another system with respect to the functions f and g. From (1.50) the density, apart from a constant, satisfies

$$\rho(t, x_1, x_2) = -x_2\frac{\partial f}{\partial x_1}(x_1, t) - \frac{\partial g}{\partial x_1}(x_1, t) = -x_2 f_{x_1} - g_{x_1}, \tag{1.58}$$

and, by (1.49), v verifies

$$v(t, x_1, x_2) = \left(-f(x_1, t), x_2\frac{\partial f}{\partial x_1}(x_1, t) + \frac{\partial g}{\partial x_1}(x_1, t)\right) = (-f, x_2 f_{x_1} + g_{x_1}). \tag{1.59}$$

Therefore, the system under the hypothesis (1.57) is equivalent to

$$(f_x)_t = f f_{xx} - (f_x)^2, \tag{1.60}$$

$$(g_x)_t = f g_{xx} - f_x g_x. \tag{1.61}$$

(Here and in the sequel of the section, we denote with subscripts the derivatives with respect to x.) We note the non-linear character of the first equation. Thus, our study of formation of singularities is concentrated in the solutions of (1.60). The function g depends implicitly on f in equation (1.61).

Now, we show that the system (1.60) and (1.61) is local well posed in the Sobolev spaces $H_0^k(0,1)$.

Lemma 1.8.1 *Let* $f^0 = f(x,0)$ *and* $g^0 = g(x,0)$ *satisfy* $f_x^0, g_x^0 \in H_0^k(0,1)$ *with* $k \geq 1$. *Then, there exists* $T > 0$ *such that* $f_x, g_x \in C^1([0,T]; H_0^k(0,1))$ *are the unique solutions to* (1.60)–(1.61).

Proof: By (1.60) and integrating by parts, we have

$$\frac{1}{2}\frac{d}{dt}\|f_x\|_{L^2}^2 = \int_0^1 f_x f f_{xx} - \int_0^1 f_x^3 = -\frac{3}{2}\int_0^1 f_x^3 \leq C\|f_x\|_{L^\infty}\|f_x\|_{L^2}^2 \leq C\|f_x\|_{H_0^1}^3.$$

Analogously,

$$\frac{1}{2}\frac{d}{dt}\|f_{xx}\|_{L^2}^2 = -\int_0^1 f_{xx}^2 f_x - \int_0^1 f_{xx} f f_{xxx} = -\frac{1}{2}\int_0^1 f_{xx}^2 f_x \leq C\|f_x\|_{L^\infty}\|f_{xx}\|_{L^2}^2 \leq C\|f_x\|_{H_0^1}^3.$$

We can repeat for all $k \geq 1$ to obtain

$$\frac{1}{2}\frac{d}{dt}\|f_x\|_{H_0^k}^2 \leq C\|f_x\|_{H_0^k}^3.$$

Integrating in time, we get

$$\|f_x\|_{H_0^k} \leq \frac{\|f_x^0\|_{H_0^k}}{1 - Ct\|f_x^0\|_{H_0^k}}.$$

On the other hand, by (1.60) and integrating by parts, we have for g_x the following inequalities:

$$\frac{1}{2}\frac{d}{dt}\|g_x\|_{L^2}^2 = \int_0^1 g_x g_{xx} f - \int_0^1 g_x^2 f_x = -\frac{3}{2}\int_0^1 g_x^2 f_x \leq \|f_x\|_{L^\infty}\|g_x\|_{H_0^1}^2,$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|g_{xx}\|_{L^2}^2 &= \int_0^1 g_{xx} g_{xxx} f - \int_0^1 g_{xx} g_x f_{xx} = -\frac{1}{2}\int_0^1 g_{xx}^2 f_x - \int_0^1 g_{xx} g_x f_{xx} \\ &\leq \|f_x\|_{L^\infty}\|g_{xx}\|_{L^2}^2 + \|g_{xx}\|_{L^2}\|g_x\|_{L^\infty}\|f_{xx}\|_{L^2} \leq \|f_x\|_{H_0^1}\|g_x\|_{H_0^1}^2.\end{aligned}$$

Thus, we obtain using Gronwall's Lemma

$$\|g_x\|_{H_0^k}^2 \leq \|g_x^0\|_{H_0^k}^2 \exp\left(C\int_0^t \|f_x\|_{H_0^k} ds\right)$$

and we have existence up to a time $T = T(\|f_x^0\|_{H_0^k})$.

In order to prove the uniqueness, let $f_x(x,t) = h_x(x,t) - k_x(x,t)$, with h_x, k_x two solutions of (1.60) with the same initial data f_x^0. Since h_x, k_x satisfy (1.60) and integrating by parts, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f_x\|_{L^2}^2 &= \int_0^1 f_x(hh_{xx} - kk_{xx}) - \int_0^1 f_x(h_x^2 - k_x^2) \\ &= \int_0^1 f_x h f_{xx} + \int_0^1 f_x f k_{xx} - \int_0^1 (f_x)^2(h_x + k_x) \\ &= -\frac{1}{2}\int_0^1 (f_x)^2 h_x + \int_0^1 f_x f k_{xx} - \int_0^1 (f_x)^2(h_x + k_x).\end{aligned}$$

Thus, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|f_x\|_{L^2}^2 &\leq \|k_{xx}\|_{L^2}\|f_x\|_{L^2}\|f\|_{L^\infty} + C(\|h_x\|_{L^\infty} + \|k_x\|_{L^\infty})\|f_x\|_{L^2}^2 \\ &\leq C(\|h_x\|_{H_0^1} + \|k_x\|_{H_0^1})\|f_x\|_{L^2}^2\end{aligned}$$

and using Gronwall's Lemma it follows that $h_x = k_x$. Finally, we conclude the uniqueness of g_x since (1.61) is a linear differential equation .

The following result shows that the solution of (1.60) blows up in finite time under certain conditions on the initial data.

Proposition 1.8.2 *Let f_x be a solution of* (1.60) *with initial data satisfing $f_x^0 \in H_0^2(0,1)$ and $\min_x f_x^0 < 0$. Then, $\|f_x\|_{L^\infty}$ blows up in finite time $T = -1/\min_x f_x^0$.*

Proof: By the local existence result, we have $f_x \in C^1([0,T];H^2) \subset C^1([0,T]\times[0,1])$. We consider the application $m : [0,T] \to \mathbb{R}$ defined by $m(t) = \min_x f_x(x,t) = f_x(x_t,t)$. By Rademacher Theorem, it follows that m is differentiable at almost every point. We calculate the derivative of m as in the previous section. Let s be a point of differentiability of $m(t)$, then, we obtain

$$m'(s) = f_{xt}(x_s,s) \qquad \text{almost everywhere.}$$

We replace x for x_s in (1.60) which yields

$$m'(s) = -f_x^2(x_s,s) = -(m(s))^2,$$

as $f_{xx}(x_s,s) = 0$, and integrating we are done.

Remark 1.8.3 *There are other blow-up results with initial data of lower regularity. In particular, we consider $f_x^0 \in H_0^1$ and assume that*

$$\int_0^1 f_x^0 \leq 0.$$

Thus, by (1.60)*, we have*

$$\frac{d}{dt}\int_0^1 f_x = \int_0^1 f f_{xx} - \int_0^1 (f_x)^2 = -2\int_0^1 (f_x)^2 \geq -2\left(\int_0^1 f_x\right)^2.$$

Defining

$$c(t) = \int f_x,$$

and integrating, we get

$$c(t) \leq \frac{c(0)}{1+2tc(0)}.$$

Then, $c(t)$ blows up for $c(0) < 0$.

In the case $c(0) = 0$, we have $c'(t) < 0$ for all $t > 0$, therefore, $c(t)$ also blows up.

Remark 1.8.4 *Let $x_1 = x_t$ be the point such that*

$$f_x(x_t, t) = \min_x f_x(x, t),$$

and consider

$$x_2 = 1 - \frac{g_x(x_t, t)}{f_x(x_t, t)}.$$

Then, by (1.58), $\rho(x_1, x_2, t) = -f_x(x_t, t)$ blows up in finite time by Proposition 1.8.2. Analogously, v defined in (1.59) *blows up in finite time.*

1.8.2 Analysis with finite energy

We derive a reformulation of the system as an integro-differential equation for the particle trajectories. Given a smooth field $v(x, t)$, the particle trajectories $\Phi(\alpha, t)$ satisfy

$$\frac{d\Phi}{dt}(\alpha, t) = v(\Phi(\alpha, t), t), \qquad \Phi(\alpha, t)|_{t=0} = \alpha. \tag{1.62}$$

The time-dependent map $\Phi(\cdot, t)$ connects the Lagrangian reference frame (with the variable α) to the Eulerian reference frame (with the variable x). It is well known (Section 2.5 in [6]) that the equation (1.56) implies the following formula:

$$\nabla^\perp \rho(\Phi(\alpha, t), t) \;=\; \nabla_\alpha \Phi(\alpha, t)\, \nabla^\perp \rho_0(\alpha),$$

where $\nabla^\perp \rho_0$ is the orthogonal gradient of the initial density. This last equality shows that the orthogonal gradient of the density is stretched by $\nabla_\alpha \Phi(\alpha, t)$ along particle trajectories. We rewrite (1.52) as

$$v(\Phi(\alpha, t), t) = \int_{\mathbb{R}^2} K(\Phi(\alpha, t) - \Phi(\beta, t))\, \nabla_\alpha \Phi(\beta, t)\, \nabla^\perp \rho_0(\beta) d\beta. \tag{1.63}$$

Notice that the velocity is divergence free, and therefore $\det \nabla_\alpha \Phi(\beta) = 1$ (see [6]). We consider (1.62) as an ODE on a Banach space and using the Picard Theorem the local-in-time existence follows. This is proved analogously to the existence and uniqueness of solutions to the Euler equation (see Section 4.1 in [6]). In fact, we consider $\nabla^\perp \rho_0 \in C^\delta(\mathbb{R}^2)$, $\delta \in (0, 1)$. Let $\mathbf{B}$ be the Banach space defined by

$$\mathbf{B} = \left\{\Phi : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{such that } |\Phi(0)| + |\nabla_\alpha \Phi|_0 + |\nabla_\alpha \Phi|_\delta < \infty\right\},$$

where $|\cdot|_0$ is the L^∞-norm and $|\cdot|_\delta$ is the Hölder semi-norm. Define $\mathcal{O}_M$, the open set of $\mathbf{B}$, as

$$O_M = \left\{\Phi \in \mathbf{B} |\ \inf_{\alpha \in \mathbb{R}^2} \det \nabla_\alpha \Phi(\alpha) > \frac{1}{2} \text{ and } |\Phi(0)| + |\nabla_\alpha \Phi|_0 + |\nabla_\alpha \Phi|_\delta < M\right\}.$$

The mapping $v(\Phi)$, defined by (1.63), satisfies the assumptions of the Picard theorem, i.e., v is bounded and locally Lipschitz continuous on O_M. As a consequence, for any $M > 0$ there exists $T(M) > 0$ and a unique solution

$$\Phi \in C^1((-T(M), T(M)); O_M)$$

to the particle trajectories (1.62, 1.63).

Remark 1.8.5 *The equation* (1.47) *conserves the* L^p *norm of* ρ *for* $1 \leq p \leq \infty$*, i.e.,*

$$\|\rho\|_p(t) = \|\rho_0\|_p, \qquad \forall t > 0, \quad 1 \leq p \leq \infty, \tag{1.64}$$

since $\rho(\Phi(\alpha, t), t) = \rho_0(\alpha)$*. Using* (1.53)*, we see that the velocity is obtained from* ρ *by singular integral operators with Calderón-Zygmund kernels (see [44]). Then for* $1 < p < \infty$ *the* L^p *norm of the velocity is bounded for any time* $t > 0$*, and therefore the energy of the system.*

In order to estimate the growth of the Sobolev norms we use the space of functions of bounded mean oscillation.

Theorem 1.8.6 *Let* ρ *be the solution of the conservation of mass equation with initial data* $\rho_0 \in H^s(\mathbb{R}^2)$ *with* $s > 2$*. Then, the following are equivalent:*

(A) The interval $[0, \infty)$ *is the maximal interval of* H^s *existence for* ρ*.*

(B) The quantity

$$\int_0^T \|\nabla\rho\|_{BMO}(t)\, dt < \infty \qquad \forall\, T > 0. \tag{1.65}$$

Proof. We denote the operator Λ^s by $\Lambda^s \equiv (-\Delta)^{s/2}$. Since the fluid is incompressible, we have for $s > 2$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Lambda^s\rho\|_{L^2}^2 &= -\int_{\mathbb{R}^2} \Lambda^s\rho\Lambda^s(v\nabla\rho)\, dx = -\int_{\mathbb{R}^2} \Lambda^s\rho(\Lambda^s(v\nabla\rho) - v\Lambda^s(\nabla\rho))\, dx,\\ &\leq C\|\Lambda^s\rho\|_{L^2}\|\Lambda^s(v\nabla\rho) - v\Lambda^s(\nabla\rho)\|_{L^2}.\end{aligned}$$

Using the following estimate (see [32]):

$$\|\Lambda^s(fg) - f\Lambda^s(g)\|_{L^p} \leq c\left(\|\nabla f\|_{L^\infty}\|\Lambda^{s-1}g\|_{L^p} + \|\Lambda^s f\|_{L^p}\|g\|_{L^\infty}\right) \qquad 1 < p < \infty,$$

we obtain for $p = 2$

$$\frac{1}{2}\frac{d}{dt}\|\Lambda^s\rho\|_{L^2}^2 \leq C(\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\|\Lambda^s\rho\|_{L^2}^2. \tag{1.66}$$

Integrating, we get for any $t \leq T$

$$\|\Lambda^s\rho\|_{L^2} \leq \|\Lambda^s\rho_0\|_{L^2} \exp\left(C\int_0^T (\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\right). \tag{1.67}$$

Now, we use the following inequality given in [33]: Let $f \in W^{s,p}$ with $1 < p < \infty$ and $s > 2/p$, then, there exists a constant C=C(p,s) such that

$$\|f\|_{L^\infty} \leq C(1 + \|f\|_{BMO}(1 + \ln^+\|f\|_{W^{s,p}})), \tag{1.68}$$

where $\ln^+(x) = \max(0, \ln(x))$. Therefore, for $s > 2$ we have

$$\|\nabla\rho\|_{L^\infty} \leq C(1 + \|\nabla\rho\|_{BMO}(1 + \ln^+\|\nabla\rho\|_{H^{s-1}})),$$

and from (1.67), we obtain that

$$\|\nabla\rho\|_{L^\infty} \leq C(1+\ln^+(\|\rho_0\|_{H^s})\|\nabla\rho\|_{BMO}\int_0^T (\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\,dt). \tag{1.69}$$

On the other hand, applying (1.68) for $v \in H^s(\mathbb{R}^2)$, we have

$$\|\nabla v\|_{L^\infty} \leq C(1+\|\nabla v\|_{BMO}(1+\ln^+\|\nabla v\|_{H^{s-1}})).$$

Since v satisfies (1.53) and the singular integrals are bounded operators in BMO (see [46]), we get

$$\|\nabla v\|_{L^\infty} \leq C(1+\|\nabla\rho\|_{BMO}(1+\ln^+\|\nabla\rho\|_{H^{s-1}})),$$

and, using (1.67), we obtain

$$\|\nabla v\|_{L^\infty} \leq C(1+\ln^+(\|\rho_0\|_{H^s})\|\nabla\rho\|_{BMO}\int_0^T (\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\,dt). \tag{1.70}$$

From (1.69) and (1.70), follows:

$$\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty} \leq C(1+\ln^+(\|\rho_0\|_{H^s})\|\nabla\rho\|_{BMO}\int_0^T (\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\,dt).$$

Applying Gronwall's inequality, we have

$$\int_0^T (\|\nabla v\|_{L^\infty} + \|\nabla\rho\|_{L^\infty})\,dt \leq CT\exp\left(\ln^+(\|\rho_0\|_{H^s})\int_0^T \|\nabla\rho\|_{BMO}\,dt\right),$$

and so (A) is a consequence of (B).

Finally, due to the inequality

$$\|\nabla\rho\|_{BMO} \leq \|\nabla\rho\|_{H^1},$$

we conclude that (A) implies (B).

Remark 1.8.7 *Using that*

$$\|\nabla\rho\|_{BMO} \leq C\|\nabla\rho\|_{L^\infty},$$

we get an easier blow-up characterization for checking in numerical simulations.

From equation (1.47) follows that the level sets, $\rho =$ constant, move with the fluid flow. Then $\nabla^\perp\rho$, defined in (1.55), is tangent to these level sets.

For the conservation of mass equation, the infinitesimal length of a level set for ρ is given by $|\nabla^\perp\rho|$ and from (1.56), the evolution equation for this quantity is given by

$$\frac{D|\nabla^\perp\rho|}{Dt} = \mathcal{L}\,|\nabla^\perp\rho|, \tag{1.71}$$

where the factor $\mathcal{L}(x,t)$ is defined by

$$\mathcal{L}(x,t) = \begin{cases} \mathcal{D}\eta \cdot \eta, & \eta \neq 0, \\ 0, & \eta = 0. \end{cases} \tag{1.72}$$

with the direction of $\nabla^{\perp}\rho$ denoted by

$$\eta = \frac{\nabla^{\perp}\rho}{|\nabla^{\perp}\rho|} \tag{1.73}$$

and $\mathcal{D}(x,t)$ is the symmetric part of the deformation matrix defined by

$$\mathcal{D} = (\mathcal{D}_{ij}) = \left[\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)\right]. \tag{1.74}$$

Now, we show a singularity criterium of the conservation of mass equation using the geometric structure of the level sets and mild hypotheses of the solutions. The theorem stated below is analogous to 3D Euler [13] and to 2DQG [14].

Recall that η is the direction field tangent to the level sets of ρ defined (1.73). Analogously to [14], a set Ω is *smoothly directed* if there exists $\delta > 0$ such that

$$\sup_{x\in\overline{\Omega}} \int_0^T \|\nabla\eta(\cdot,t)\|^2_{L^\infty(B_\delta(\Phi(x,t)))} dt \, < \, \infty, \tag{1.75}$$

where

$$B_\delta(x) = \{y \in \mathbb{R}^2 : \quad |x-y| < \delta\}, \qquad \overline{\Omega} = \{x \in \Omega;\ |\nabla\rho_0(x)| \neq 0\},$$

and Φ is the particle trajectories map. We define $\Omega(t) = \Phi(\Omega,t)$ and $\mathcal{O}_T(\Omega)$ the semi-orbit, i.e.,

$$\mathcal{O}_T(\Omega) = \bigcup_{0\le t\le T} \{t\} \times \Omega(t).$$

Theorem 1.8.8 *If Ω is smoothly directed and*

$$\int_0^T \|R_j\rho\|_{L^\infty}(t)dt \, < \, \infty, \quad j = 1,2, \quad \forall T > 0, \tag{1.76}$$

where R_j denotes the Riesz transform in the direction x_j, then

$$\sup_{\mathcal{O}_T(\Omega)} |\nabla\rho(x,t)| < \, \infty.$$

Remark 1.8.9 *Using the Remark 1.8.7, the previous theorem illustrates that finite-time singularities are impossible in smoothly directed sets.*

Proof: We show a similar formula of the level–set stretching factor $\mathcal{L}$ defined in (1.72). We start by computing the full gradient of the velocity v. From formula (1.52)

$$v(x) = \int_{\mathbb{R}^2} K(y)\, \nabla^{\perp}\rho(x-y)dy,$$

we have

$$\nabla v(x) = \int_{\mathbb{R}^2} K(y)\,(\nabla_y \nabla_y^{\perp} \rho)(x-y)dy.$$

Take the integral as a limit as $\epsilon \to 0$ of integrals on $|y| > \epsilon$ and integrate by parts. In this way, we obtain the formula

$$\nabla v(x) = \frac{1}{2\pi} PV \int_{\mathbb{R}^2} \big(\nabla_y \rho(x-y)\big) \otimes \widetilde{y}\,\frac{dy}{|y|^2} \;-\; \frac{1}{2} \begin{pmatrix} 0 & 0 \\ \dfrac{\partial \rho}{\partial x_1}(x) & \dfrac{\partial \rho}{\partial x_2}(x) \end{pmatrix}, \tag{1.77}$$

where $\widetilde{y}$ is the unit vector defined by

$$\widetilde{y} = \left(-\frac{2y_1 y_2}{|y|^2}, \frac{y_1^2 - y_2^2}{|y|^2} \right).$$

By definition of η in (1.73), we have $\eta \cdot \nabla \rho = 0$. Thus, computing we get

$$\mathcal{L}(x) = \frac{1}{2\pi} PV \int_{\mathbb{R}^2} \big(\widetilde{y} \cdot \eta(x)\big)\big(\eta(x-y)\cdot \eta^{\perp}(x)\big)\}|\nabla^{\perp}\rho(x-y)|\frac{dy}{|y|^2}. \tag{1.78}$$

Let be $\phi \in C_c^{\infty}(\mathbb{R})$, $\phi \geq 0$, supp(ϕ) include in $[-1,1]$ and $\phi(s) = 1$ in $s \in [-1/2, 1/2]$. Consider $r > 0$ and decompose

$$\mathcal{L}(x) = I_1 + I_2,$$

with

$$|I_1| \leq \frac{1}{2\pi} \left| \int_{\mathbb{R}^2} (1 - \phi(|y|^2/r^2))\big(\widetilde{y}\cdot \eta(x)\big)\big(\nabla^{\perp}\rho(x-y)\cdot \eta^{\perp}(x)\big)\frac{dy}{|y|^2} \right|.$$

Integrating by parts and using Cauchy-Schwartz inequality, we get

$$I_1 \leq \frac{C}{r^2}\|\rho\|_{L^2} \leq \frac{C}{r^2}\|\rho_0\|_{L^2}.$$

We have for any $|y| < r$

$$|\eta(x-y)\cdot \eta^{\perp}(x)| \leq |y|\|\nabla \eta\|_{L^{\infty}(B_r(x))}.$$

Applying this in the integral I_2, we get

$$|I_2| \leq \int |\nabla^{\perp}\rho(x-y)|\phi(|y|^2/r^2)\frac{dy}{|y|}\|\nabla \eta\|_{L^{\infty}(B_r(x))}.$$

We integrate by parts and decompose

$$\int |\nabla^{\perp}\rho(x-y)|\phi(|y|^2/r^2)\frac{dy}{|y|} = \int \rho(x-y)\nabla^{\perp}\left(\eta(x-y)\phi(|y|^2/r^2)\frac{1}{|y|}\right)dy = J_1 + J_2 + J_3,$$

where

$$\begin{aligned} J_1 &= \int \rho(x-y)\eta(x-y)\nabla^{\perp}(\phi(|y|^2/r^2))\frac{dy}{|y|}, \\ J_2 &= \int \rho(x-y)\nabla^{\perp}(\eta(x-y))\phi(|y|^2/r^2)\frac{dy}{|y|}, \\ J_3 &= \int \rho(x-y)\eta(x-y)\phi(|y|^2/r^2)\frac{(-y_2,y_1)}{|y|^3}dy, \end{aligned}$$

obtaining the following estimates:

$$|J_1| \leq c\|\rho_0\|_{L^\infty} \qquad \text{and} \qquad |J_2| \leq cr\|\rho_0\|_{L^\infty}\|\nabla\eta\|_{L^\infty(B_r(x))}.$$

The J_3 term can be bounded using the identity

$$J_3 = \eta(x)(-R_2(\rho)(x), R_1(\rho)(x)) + J_4,$$

getting the following estimate for J_4 in a similar way:

$$|J_4| \leq r\|\rho_0\|_{L^\infty}\|\nabla\eta\|_{L^\infty(B_r(x))} + r^{-1}\|\rho_0\|_{L^2}.$$

Thus we conclude the following estimate for the factor $\mathcal{L}$:

$$\begin{aligned} |\mathcal{L}(x)| \leq & C\|\nabla\eta\|_{L^\infty(B_r(x))}\max_{j=1,2}|R_j(\rho)| \\ & + C(r\|\nabla\eta\|_{L^\infty(B_r(x))}+1)(\|\nabla\eta\|_{L^\infty(B_r(x))}\|\rho_0\|_\infty + r^{-2}\|\rho_0\|_2). \end{aligned}$$

Using (1.71), we obtain by Gronwall's lemma

$$\sup_{\mathcal{O}_T(\Omega)}|\nabla\rho(x,t)| \leq \sup_{\Omega}|\nabla\rho_0| \exp\big(\sup_{y\in\Omega}\int_0^T M(t)dt\big),$$

where $M(t)$ is defined by

$$\begin{aligned} M(t) =& \|\nabla\eta\|_{L^\infty(B_r(x))}\max_{j=1,2}|R_j(\rho)| \\ & + (r\|\nabla\eta\|_{L^\infty(B_r(x))}+1)(\|\nabla\eta\|_{L^\infty(B_r(x))}\|\rho_0\|_\infty + r^{-2}\|\rho_0\|_2), \end{aligned}$$

with $x = \Phi(y,t)$. This concludes the proof of Theorem 1.8.8.

Remark 1.8.10 *The condition* (1.76) *depending on the Riesz transform is different than that in QG (see [14]). This appears because the integral kernels* (1.54) *in the conservation of mass equation are different to the kernels in QG.*

Now, we present a geometric conserved quantity that relates the curvature of the level sets and the magnitude $|\nabla^{\perp}\rho|$ in a similar way as in [11] (see the references therein for more details). In particular, if we define the curvature of the level sets κ by

$$\kappa(x,t) = (\eta\cdot\nabla\eta)\cdot\eta^{\perp}(x,t), \tag{1.79}$$

where η is the direction of $\nabla^\perp\rho$ (see (1.73)), the following identity is satisfied:

$$\frac{D(\kappa|\nabla^\perp\rho|)}{Dt}=\nabla^\perp\rho\cdot\nabla\beta \tag{1.80}$$

with

$$\beta(x,t)=(\eta\cdot\nabla v)\cdot\eta^\perp(x,t). \tag{1.81}$$

Indeed, we now prove the identity (1.80). Since $\nabla^\perp\rho$ and $|\nabla^\perp\rho|$ satisfies (1.56) and (1.71) respectively, we get

$$\frac{D\eta}{Dt}=(\nabla v)\eta-\mathcal{L}\,\eta.$$

Using (1.77), we obtain

$$\frac{D\eta}{Dt}=\beta\,\eta^\perp,$$

with β defined in (1.81). By the definition of κ (1.79) and the previous formula, we have

$$\frac{D\kappa}{Dt}=((\nabla v\cdot\eta-\mathcal{L}\,\eta)\nabla\eta)\cdot\eta^\perp+(\eta\cdot(\beta\nabla\eta^\perp+\eta^\perp\otimes\nabla\beta-\nabla\eta\nabla v))\cdot\eta^\perp-\beta(\eta\cdot\nabla\eta)\cdot\eta$$

and, simplifying,

$$\frac{D\kappa}{Dt}=(\nabla\beta)\eta-\mathcal{L}\,\kappa.$$

Using this identity and (1.71), (1.80) is satisfied.

Remark 1.8.11 *The integral of the quantity $\kappa|\nabla^\perp\rho|$ over a region given by two different level sets is conserved along the time, i.e.*

$$\frac{d}{dt}\Big(\int_{\Omega(t)}\kappa|\nabla^\perp\rho|\,dx\Big)=0, \tag{1.82}$$

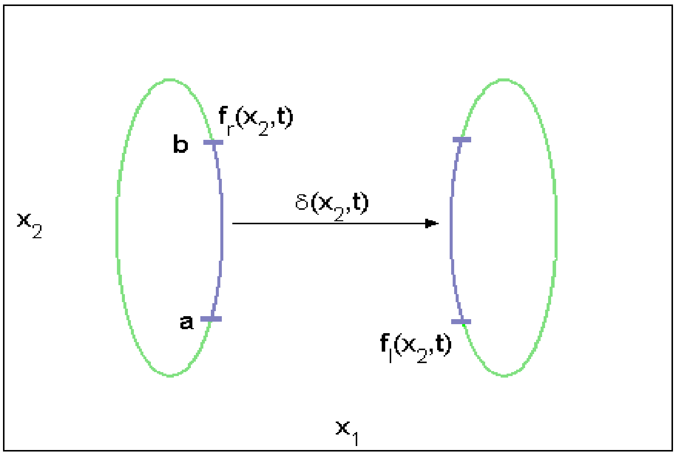
with $\Omega(t)=\{x:\ C_1\le\rho(x,t)\le C_2\}$. This can be shown using the equation (1.80) *and integrating by parts. Thus, in the case that $|\nabla^\perp\rho|$ is large, by* (1.82) *the curvature κ is small if the level sets do not oscillate.*

The numerical experiments (see [23] for example) performed for this equation, show no evidence of level set oscillations. On the contrary, the level sets are flattening where the gradient of ρ is growing. So, if we choose two level sets that are approaching each other we have a scenario as in the following figure, where $\delta=\delta(x_2,t)$ is the distance between the two counters. From previous work (see [18]) it is easy to check that in order for the two graphs f_l, f_r to collapse at time T in any interval $x_2\in[a,b]$, i.e.,

$$\lim_{t\to T_-}[f_r(x_2,t)-f_l(x_2,t)]=0\qquad\forall x_2\in[a,b],$$

it is necessary that

$$\int_0^T\|v\|_\infty(s)ds=\infty. \tag{1.83}$$



No evidence for the quantity $\int_0^T \|v\|_\infty(s)ds$ to blow up in finite time is shown in [23]. Nevertheless, we can obtain an estimate of how close the two graphs approach each other without any assumption on the velocity. It seems reasonable to assume that the minimum and maximum of δ are comparable. This means that there exists a constant $c > 0$ such that

$$\max \delta(x_2, t) \leq c \min \delta(x_2, t) \qquad \forall x_2 \in [a, b]. \tag{1.84}$$

If a curve parameterized by $f(x_2, t) - x_1 = 0$ moves with the fluid, using the Eulerian reference frame, we have

$$f_t(x_2, t) - v^1(f(x_2, t), x_2, t) + f_{x_2}(x_2, t)v^2(f(x_2, t), x_2, t) = 0.$$

The derivative of the stream function with respect to x_2 along the curve is given by

$$(\psi(f(x_2, t), x_2, t))_{x_2} = v^2(f(x_2, t), x_2, t)f_{x_2}(x_2, t) - v^1(f(x_2, t), x_2, t) = -f_t(x_2, t).$$

Then we can obtain an evolution equation for the area $A = A(t)$

$$A(t) = \frac{1}{b-a} \int_a^b [f_r(x_2, t) - f_l(x_2, t)]dx_2,$$

between the two graphs that satisfy (see [19] for more details)

$$\left| \frac{dA}{dt}(t) \right| \leq \frac{C}{b-a} \sup_{a \leq x_2 \leq b} |\psi(f_r(x_2, t), x_2, t) - \psi(f_l(x_2, t), x_2, t)|\,, \tag{1.85}$$

where the stream function satisfies (1.51)

$$\psi(x, t) \;=\; -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x_1 - z_1}{|x - z|^2} \rho(z, t)dz.$$

Using this formula we obtain that for any $x, y \in \mathbb{R}^2$

$$|\psi(x, t) - \psi(y, t)| \leq C(\|\rho_0\|_\infty, \|\rho_0\|_{L^2})|x - y|(1 - \ln^- |x - y|), \tag{1.86}$$

where $\ln^{-}(x) = \min(0, \ln(x))$, due to the following estimates:

$$\begin{aligned}|\psi(x,t) - \psi(y,t)| &= |\frac{1}{2\pi}\int_{\mathbb{R}^2} (\frac{x_1 - z_1}{|x-z|^2} - \frac{y_1 - z_1}{|y-z|^2})\rho(z,t)dz| \\ &\leq |\frac{1}{2\pi}\int_{B_{2r}(x)}| + |\frac{1}{2\pi}\int_{B_2(x) - B_{2r}(x)}| + |\frac{1}{2\pi}\int_{B_2(x)}| \\ &= I_1 + I_2 + I_3,\end{aligned}$$

where $r = |x-y| < 1$ and

$$\begin{aligned}&I_1 \leq C\|\rho_0\|_\infty |x-y|, \\ &I_2 \leq C\|\rho_0\|_\infty |x-y| \int_{2r}^{2} s^{-1} ds \leq C\|\rho_0\|_\infty |x-y|(-\ln|x-y|), \\ &I_3 \leq C\|\rho_0\|_{L^2} |x-y|.\end{aligned}$$

Then, using (1.86) in (1.85) we get that the area $A(t)$ is bounded by

$$A(t) \geq A_0 e^{-Ce^t}.$$

Chapter 2

Sharp fronts for the QG equation.

2.1 Introduction

The 2-D QG equation provides particular solutions to the evolution of the temperature for a general quasi-geostrophic system for atmospheric and oceanic flows. This equation is derived considering small Rossby and Ekman numbers and constant potential vorticity (see [38] and [43] for more details). It reads

$$\frac{D\theta}{Dt} = \theta_t + u \cdot \nabla\theta = 0, \tag{2.1}$$

where $\theta(x,t)$, with $x \in \mathbb{R}^2$, is the temperature of the fluid. The incompressible velocity u is expressed by means of the stream function as follows:

$$u = \nabla^{\perp}\psi = (-\partial_{x_2}\psi, \partial_{x_1}\psi),$$

and the relation between the stream function and the temperature is given by

$$\theta = -(-\Delta)^{1/2}\psi.$$

These identities indicate that the velocity can be recovered form the temperature by the Riesz transform (see [44]) as follows:

$$u = (-R_2\theta, R_1\theta). \tag{2.2}$$

This system has been considered in frontogenesis, where the dynamics of hot and cold fluids are studied together with the formation and the evolution of fronts (see [14], [15], [20], [37]).

From a mathematical point of view, this equation has been presented as a two-dimensional model of the 3-D Euler equation due to their strong analogies. These were first introduced in the literature by Constatin, Majda and Tabak (see [14]). The 3-D Euler equation reads

$$\frac{Dv}{Dt} = -\nabla p, \qquad \operatorname{div} v = 0.$$

For both systems the energy is conserved; that is

$$\|v\|_{L^2}(t) = \|v_0\|_{L^2}, \qquad \|u\|_{L^2}(t) = \|\theta\|_{L^2}(t) = \|\theta_0\|_{L^2},$$

where the last equality follows from the formula (2.2). Identifying the vorticity in the Euler equation with the perpendicular gradient of θ, the following similar evolution equations for these quantities are obtained:

$$\frac{Dw}{Dt} = (\nabla v)w, \qquad \frac{D(\nabla^{\perp}\theta)}{Dt} = (\nabla u)\nabla^{\perp}\theta.$$

These equations show that the integral curves of w and $\nabla^{\perp}\theta$ move with the fluid (see [6] section 1.6). The velocities are given by the following equations:

$$v(x) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \times w(y)dy, \quad u(x) = -\frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1}{|x-y|}\nabla^{\perp}\theta(y)dy,$$

whose kernels are homogeneous of degree the dimension of the space minus one. Finally, the criteria for the formation of singularities are similar in both cases. The fluids blow up in finite time if and only if

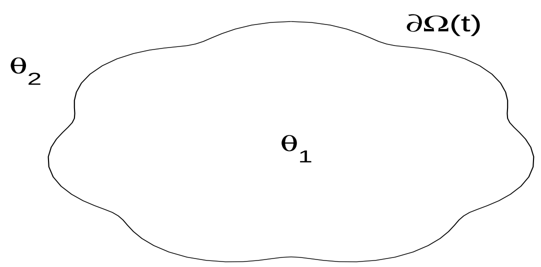
$$\int_0^{T_1} \|w\|_{L^\infty}(t)dt = \infty, \qquad \int_0^{T_2} \|\nabla^{\perp}\theta\|_{L^\infty}(t)dt = \infty,$$

for some $T_1, T_2 > 0$ respectively (see [2] and [14]). In both equations, the formation of singularities for a regular initial data is an open problem (see [14], [17], [18]).

On the other hand, one of the main differences is that there exist global-in-time weak solutions for the QG equation (see [39]), but only a few sparse results are known about weak solutions to the 2-D and 3-D Euler equation in its primitive-variable form.

For the QG equation, we treat the kind of weak solutions for which the temperature takes two different values in complementary domains, modelling the evolution of a sharp front as follows:

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \Omega(t) \\ \theta_2, & \mathbb{R}^2 \setminus \Omega(t). \end{cases} \tag{2.3}$$



We study a problem similar to the 2-D vortex patch problem, where the vorticity of the 2-D Euler equation is given by the characteristic function of a domain,

$$w(x_1,x_2,t)=\begin{cases} w_0, & \Omega(t)\\ 0, & \mathbb{R}^2\smallsetminus\Omega(t),\end{cases}$$

and the regularity of the free boundary of the domain is considered. For this equation the vorticity satisfies

$$\frac{Dw}{Dt}=w_t+u\cdot\nabla w=0 \tag{2.4}$$

in a weak sense, and the velocity is given by the Biot-Savart law or analogously

$$u=\nabla^{\perp}\psi,\quad \text{and}\quad w=\Delta\psi.$$

For this problem the boundary of the domain

$$\partial\Omega(t)=\{x(\gamma,t)=(x_1(\gamma,t),x_2(\gamma,t))\,:\,\gamma\in[-\pi,\pi]\}$$

determines the evolution of the patch by the following contour equation:

$$\begin{aligned} x_t(\gamma,t)&=-\frac{w_0}{2\pi}\int_{-\pi}^{\pi}\ln|x(\gamma,t)-x(\eta,t)|\partial_\gamma x(\eta,t)d\eta,\\ x(\gamma,0)&=x_0(\gamma).\end{aligned} \tag{2.5}$$

Chemin [9] proved global-in-time regularity for the free boundary using paradifferential calculus. A simpler proof can be found in [5] due to Bertozzi and Constantin.

We point out that in the QG equation, the velocity is determined from the temperature by singular integral operators (2.2), making the system more singular than (2.4) (see the contour equation (2.7) for $\alpha=1$ in order to compare QG with (2.5)).

Rodrigo [40] proposed the problem of the evolution of a sharp front for the QG equation. He derived the velocity on the free boundary in the normal direction, and proved local-existence and uniqueness for a periodic C^∞ front, i.e.

$$\theta(x_1,x_2,t)=\begin{cases}\theta_1, & \{f(x_1,t)>x_2\}\\ \theta_2, & \{f(x_1,t)\le x_2\},\end{cases}$$

with $f(x_1,t)$ periodic, using the Nash-Moser iteration.

In this chapter we study a family of contour dynamics equations given by weak solutions of the following system:

$$\begin{aligned} \frac{D\theta}{Dt}&=\theta_t+u\cdot\nabla\theta=0,\\ u=\nabla^{\perp}\psi,&\quad \theta=-(-\Delta)^{1-\alpha/2}\psi,\quad 0<\alpha\le 1,\end{aligned} \tag{2.6}$$

where the active scalar $\theta(x,t)$, with $x\in\mathbb{R}^2$, satisfies (2.3). We notice that the limit case $\alpha=0$ is equivalent to the 2-D vortex patch problem, and $\alpha=1$ corresponds to the sharp front for the QG equation.

This system was introduced by Córdoba, Fontelos, Mancho and Rodrigo in [21], where they prove local-existence for a periodic C^∞ front, and show evidence of singularities in finite time. The singular scenario is due to the point-wise collapse of two patches.

Here we prove local-existence of the system (2.6) when the solution satisfies (2.3), with the boundary $\partial\Omega(t)$ given by the curve

$$\partial\Omega(t) = \{x(\gamma,t) = (x_1(\gamma,t), x_2(\gamma,t)) \,:\, \gamma \in [-\pi,\pi]\},$$

and $x(\gamma,t)$ belongs to a Sobolev space. This system is equivalent to the following contour equation:

$$\begin{aligned} x_t(\gamma,t) &= \frac{\Theta_\alpha}{2\pi}\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|^\alpha} d\eta, \quad 0<\alpha\leq 1,\\ x(\gamma,0) &= x_0(\gamma), \end{aligned} \tag{2.7}$$

where Θ_α depends on α, θ_1 and θ_2. In the case $0<\alpha<1$ we show uniqueness.

It is well-known (see [31] and [40]) that in these kind of contour dynamics equations, the velocity in the tangential direction only moves particles on the boundary. Therefore we do not alter the shape of the contour if we change the tangential component of the velocity; i.e., we are only changing the parametrization. In the most singular case, $\alpha = 1$ or the QG equation, we need to change the velocity in the tangential direction in order to get existence in the Sobolev spaces. We take a tangential velocity in such a way that $|\partial_\gamma x(\gamma,t)|$ satisfies

$$|\partial_\gamma x(\gamma,t)|^2 = A(t),$$

and does not depend on γ. We would like to cite the work of Hou, Lowengrub and Shelley [31] in which this idea was used to study a contour dynamics problem.

We notice that in order to get a nonsingular normal velocity to the curve for $0<\alpha\leq 1$ (see [21] and [40]), we need a one-to-one curve, parameterized in such a way that

$$|\partial_\gamma x(\gamma,t)|^2 > 0.$$

Rigorously, we need that

$$\frac{|x(\gamma,t)-x(\gamma-\eta,t)|}{|\eta|} > 0, \quad \forall\, \gamma,\eta \in [-\pi,\pi], \tag{2.8}$$

so we give initial data satisfying this property, and we prove that this condition is satisfied locally in time. It is evident from the numerical simulations in [21], that one needs to take into account the evolution of this quantity.

2.2 Derivation of the Contour Equation

In this section we deduce the family of contour equations in term of the free boundary $x(\gamma,t)$. We consider the equations given by the system (2.1), with the velocity satisfying

$$u(x,t) = \nabla^\perp\psi(x,t), \tag{2.9}$$

for the stream function it follows:

$$\theta = -(-\Delta)^{1-\alpha/2}\psi, \tag{2.10}$$

and the active scalar fulfills

$$\theta(x_1, x_2, t) = \begin{cases} \theta_1, & \Omega(t) \\ \theta_2, & \mathbb{R}^2 \smallsetminus \Omega(t). \end{cases} \tag{2.11}$$

The boundary of $\Omega(t)$ is given by the curve

$$\partial\Omega(t) = \{x(\gamma, t) = (x_1(\gamma, t), x_2(\gamma, t)) \,:\, \gamma \in [-\pi, \pi] = \mathbb{T}\},$$

where $x(\gamma, t)$ is one-to-one. Due to the identity (2.11), we see that

$$\nabla^{\perp}\theta = (\theta_1 - \theta_2)\, \partial_\gamma x(\gamma, t)\, \delta(x - x(\gamma, t)),$$

where δ is the Dirac distribution. Using (2.9) and (2.10), we have

$$u = -(-\Delta)^{\alpha/2-1}\nabla^{\perp}\theta. \tag{2.12}$$

The integral operators, $-(-\Delta)^{\alpha/2-1}$ are Riesz potentials (see [44]), so that using the last two identities we obtain that

$$u(x, t) = -\frac{\Theta_\alpha}{2\pi}\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma - \eta, t)}{|x - x(\gamma - \eta, t)|^\alpha} d\eta, \tag{2.13}$$

for $x \neq x(\gamma, t)$, and $\Theta_\alpha = (\theta_1 - \theta_2)\Gamma(\alpha/2)/2^{1-\alpha}\Gamma(2 - \alpha/2)$. We notice that for $\alpha = 1$, if $x \to x(\gamma, t)$, then the integral in (2.13) is divergent. As we showed before, we are interested in the normal velocity of the systems. Using the identity (2.13), and taking the limit as follows:

$$u(x, t) \cdot \partial_\gamma^{\perp} x(\gamma, t), \quad x \to x(\gamma, t), \tag{2.14}$$

we obtain

$$u(x(\gamma, t), t) \cdot \partial_\gamma^{\perp} x(\gamma, t) = -\frac{\Theta_\alpha}{2\pi}\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma - \eta, t) \cdot \partial_\gamma^{\perp} x(\gamma, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|^\alpha} d\eta. \tag{2.15}$$

This identity is well defined for $0 < \alpha \leq 1$ and a one-to-one curve $x(\gamma, t)$. Due to the fact that tangential velocity does not change the shape of the boundary, we fix the contour α-patch equations as follows:

$$\begin{aligned} x_t(\gamma, t) &= \frac{\Theta_\alpha}{2\pi}\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma, t) - \partial_\gamma x(\gamma - \eta, t)}{|x(\gamma, t) - x(\gamma - \eta, t)|^\alpha} d\eta, \quad 0 < \alpha \leq 1, \\ x(\gamma, 0) &= x_0(\gamma). \end{aligned} \tag{2.16}$$

Using the equation (2.13), it is easy to check that the velocity in QG presents a logarithmic divergence in the tangential direction on the boundary. Nevertheless it belongs to $L^p(\mathbb{R}^2)$ for $1 < p < \infty$, and to the bounded mean oscillation space (see [46] for the definition of the BMO space). In QG the velocity is given by (2.2), and writing the temperature in the following way:

$$\theta(x,t) = (\theta_1 - \theta_2)X_{\Omega(t)}(x) + \theta_2,$$

we see that

$$u(x,t) = (\theta_1 - \theta_2)(-R_2(X_{\Omega(t)}), R_1(X_{\Omega(t)})).$$

Using that $X_{\Omega(t)} \in L^p(\mathbb{R}^2)$ for $1 \leq p \leq \infty$, we conclude the argument. In particular the energy of the system is conserved due to the fact that

$$\|u\|_{L^2}(t) = |\theta_1 - \theta_2| \|X_{\Omega(t)}\|_{L^2} = |\theta_1 - \theta_2| \, |\Omega(t)|^{1/2},$$

and the area of $\Omega(t)$ is constant in time.

For $0 < \alpha < 1$, the equation (2.12) shows that

$$u = (-\Delta)^{(\alpha-1)/2}(-R_2\theta, R_1\theta),$$

and therefore

$$u = (\theta_1 - \theta_2)(-\Delta)^{(\alpha-1)/2}(-R_2(X_{\Omega(t)}), R_1(X_{\Omega(t)})).$$

Using the inequalities for the Riesz potentials (see [44]) we obtain

$$\|u\|_{L^2}(t) = |\theta_1 - \theta_2| \|(-\Delta)^{\frac{\alpha-1}{2}} X_{\Omega(t)}\|_{L^2} \leq |\theta_1 - \theta_2| \|X_{\Omega(t)}\|_{L^{\frac{2}{2-\alpha}}} = |\theta_1 - \theta_2| \, |\Omega(t)|^{\frac{2-\alpha}{2}}.$$

For the vorticity it reads

$$w = -(-\Delta)^{\alpha/2}\theta \qquad 0 < \alpha \leq 1.$$

Using the following formula for this operator (see [16]):

$$(-\Delta)^{\alpha/2} f(x) = C_\alpha \int_{\mathbb{R}^2} \frac{f(x) - f(y)}{|x-y|^{2+\alpha}} \, dy,$$

we obtain for $x \neq x(\gamma, t)$

$$w(x,t) = -C_\alpha \int_{\mathbb{R}^2} \frac{\theta(x,t) - \theta(y,t)}{|x-y|^{2+\alpha}} \, dy.$$

The equation for θ (2.11) gives a system with finite energy in which the vorticity diverges for $x = x(\gamma, t)$. To find a weak solution of the Euler equation with these properties is an open problem (see [20]).

2.3 Weak solutions for the α-system

In this section we show that if $\theta(x,t)$ is defined by (2.11) and the curve $x(\gamma, t)$ is convected by the normal velocity (2.15), then $\theta(x,t)$ is a weak solution of the system (2.6) and conversely. We give the definition of weak solutions below.

Definition 2.3.1 *The active scalar θ is a weak solution of the α-system if for any function $\varphi \in C_c^\infty(\mathbb{R}^2 \times (0,T))$, we have*

$$\int_0^T \int_{\mathbb{R}^2} \theta(x,t)(\partial_t\varphi(x,t) + u(x,t)\cdot\nabla\varphi(x,t))dxdt = 0, \tag{2.17}$$

where the incompressible velocity u is given by (2.9)*, and the stream function satisfies* (2.10)*.*

Proposition 2.3.2 *If $\theta(x,t)$ is defined by* (2.11)*, and the curve $x(\gamma,t)$ satisfies* (2.8) *and* (2.15)*, then $\theta(x,t)$ is a weak solution of the α-system. Furthermore, if $\theta(x,t)$ is a weak solution of the α-system given by* (2.11)*, and $x(\gamma,t)$ satisfies* (2.8)*, then $x(\gamma,t)$ verifies* (2.15)*.*

Proof: Let $\theta(x,t)$ be a weak solution of the α-system defined by (2.11). Integrating by parts we have

$$I = \int_0^T \int_{\mathbb{R}^2} \theta(x,t)\partial_t\varphi(x,t)dxdt = \theta_1 \int_0^T\int_{\Omega(t)} \partial_t\varphi(x,t)dxdt + \theta_2\int_0^T\int_{\Omega(t)\smallsetminus\mathbb{R}^2}\partial_t\varphi(x,t)dxdt$$
$$= -(\theta_1-\theta_2)\int_0^T\int_{\mathbb{T}}\varphi(x(\gamma,t),t)\,x_t(\gamma,t)\cdot\partial_\gamma^\perp x(\gamma,t)d\gamma dt.$$

On the other hand, we obtain

$$J = \int_0^T\int_{\mathbb{R}^2}\theta\, u\cdot\nabla\varphi\, dxdt = \theta_1\int_0^T\int_\Omega u\cdot\nabla\varphi\,dxdt + \theta_2\int_0^T\int_{\mathbb{R}^2\smallsetminus\Omega} u\cdot\nabla\varphi\,dxdt.$$

Taking

$$\Omega_1^\varepsilon(t) = \{x\in\Omega\,:\, dist(x,\Omega(t))\geq\varepsilon\},$$

and

$$\Omega_2^\varepsilon(t) = \{x\in\mathbb{R}^2\smallsetminus\Omega\,:\, dist(x,\mathbb{R}^2\smallsetminus\Omega(t))\geq\varepsilon\},$$

we have that $J^\varepsilon\to J$ if $\varepsilon\to 0$, where J^ε is given by

$$J^\varepsilon = \theta_1\int_0^T\int_{\Omega_1^\varepsilon(t)} u\cdot\nabla\varphi\,dxdt + \theta_2\int_0^T\int_{\Omega_2^\varepsilon(t)} u\cdot\nabla\varphi\,dxdt.$$

Integrating by parts in J^ε, using that the velocity is divergence free, and taking the limit as in (2.14), we obtain

$$J = (\theta_1-\theta_2)\int_0^T\int_{\mathbb{T}}\varphi(x(\gamma,t),t)u(x(\gamma,t),t)\cdot\partial_\gamma^\perp x(\gamma,t)d\gamma dt$$
$$= -(\theta_1-\theta_2)\frac{\Theta_\alpha}{2\pi}\int_0^T\int_{\mathbb{T}}\varphi(x(\gamma,t),t)\Big(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma-\eta,t)\cdot\partial_\gamma^\perp x(\gamma,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|^\alpha}d\eta\Big)d\gamma dt.$$

We have that $I+J=0$ using (2.17), and it follows that:

$$\int_0^T\int_{\mathbb{T}} f(\gamma,t)\Big(x_t(\gamma,t)\cdot\partial_\gamma^\perp x(\gamma,t) + \frac{\Theta_\alpha}{2\pi}\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma-\eta,t)\cdot\partial_\gamma^\perp x(\gamma,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|^\alpha}d\eta\Big)d\gamma dt = 0,$$

for $f(\gamma,t)$ periodic in γ. We see that (2.15) is satisfied. Following the same arguments it is easy to check that if $x(\gamma,t)$ satisfies (2.15), then θ is a weak solution given by (2.11).

2.4 Local well-posedness for $0 < \alpha < 1$

In this section we prove existence and uniqueness for the contour equation in the cases $0 < \alpha < 1$. We denote the Sobolev spaces by $H^k(\mathbb{T})$, with norms

$$\|x\|^2_{H^k} = \|x\|^2_{L^2} + \|\partial^k_\gamma x\|^2_{L^2},$$

and the spaces $C^k(\mathbb{T})$ with

$$\|x\|_{C^k} = \max_{j\leq k} \|\partial^j_\gamma x\|_{L^\infty}.$$

We require that the curve satisfies

$$\frac{|x(\gamma,t) - x(\gamma-\eta,t)|}{|\eta|} > 0, \quad \forall\, \gamma,\eta \in [-\pi,\pi], \tag{2.18}$$

and we define

$$F(x)(\gamma,\eta,t) = \frac{|\eta|}{|x(\gamma,t) - x(\gamma-\eta,t)|} \quad \forall\, \gamma,\eta \in [-\pi,\pi], \tag{2.19}$$

with

$$F(x)(\gamma,0,t) = \frac{1}{|\partial_\gamma x(\gamma,t)|}.$$

The following theorem is the main result of the section.

Theorem 2.4.1 *Let $x_0(\gamma) \in H^k(\mathbb{T})$ for $k \geq 3$ with $F(x_0)(\gamma,\eta) < \infty$. Then there exists a time $T > 0$ so that there is a unique solution to (2.16) for $0 < \alpha < 1$ in $C^1([0,T];H^k(\mathbb{T}))$, with $x(\gamma,0) = x_0(\gamma)$.*

Proof: We can choose $\Theta_\alpha = 2\pi$ without loss of generality, obtaining the following equation:

$$\begin{aligned} x_t(\gamma,t) &= \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t) - x(\gamma-\eta,t)|^\alpha} d\eta, \quad 0 < \alpha < 1, \\ x(\gamma,0) &= x_0(\gamma). \end{aligned} \tag{2.20}$$

We take $k = 3$, the proof for $k > 3$ being analogous. We use energy estimates (see [6] for more details). We ignore the time dependence to simplify the notation. Considering the quantity

$$\begin{aligned} \int_{\mathbb{T}} x(\gamma)\cdot x_t(\gamma) d\gamma &= \int_{\mathbb{T}}\int_{\mathbb{T}} x(\gamma)\cdot \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\eta)}{|x(\gamma)-x(\eta)|^\alpha} d\eta d\gamma \\ &= -\int_{\mathbb{T}}\int_{\mathbb{T}} x(\eta)\cdot \frac{\partial_\gamma x(\gamma) - \partial_\gamma x(\eta)}{|x(\gamma)-x(\eta)|^\alpha} d\eta d\gamma \\ &= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(x(\gamma)-x(\eta))\cdot(\partial_\gamma x(\gamma) - \partial_\gamma x(\eta))}{|x(\gamma)-x(\eta)|^\alpha} d\eta d\gamma \\ &= \frac{1}{2(2-\alpha)}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma |x(\gamma) - x(\gamma-\eta)|^{2-\alpha} d\gamma d\eta \\ &= 0, \end{aligned} \tag{2.21}$$

we obtain

$$\frac{d}{dt}\|x\|_{L^2}(t) = 0. \tag{2.22}$$

We decompose as follows:

$$\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^3 x_t(\gamma) d\gamma = I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned}
I_1 &= \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot \frac{\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|^\alpha} d\eta d\gamma,\\
I_2 &= 3\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta))\partial_\gamma(|x(\gamma)-x(\gamma-\eta)|^{-\alpha}) d\eta d\gamma,\\
I_3 &= 3\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma-\eta))\partial_\gamma^2(|x(\gamma)-x(\gamma-\eta)|^{-\alpha}) d\eta d\gamma,\\
I_4 &= \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))\partial_\gamma^3(|x(\gamma)-x(\gamma-\eta)|^{-\alpha}) d\eta d\gamma.
\end{aligned}$$

Operating as in (2.21), the term I_1 becomes

$$\begin{aligned}
I_1 &= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} (\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)) \cdot \frac{\partial_\gamma^4 x(\gamma) - \partial_\gamma^4 x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|^\alpha} d\eta d\gamma\\
&= \frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\partial_\gamma|\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^\alpha} d\eta d\gamma\\
&= \frac{\alpha}{4}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)|^2 (x(\gamma)-x(\gamma-\eta)) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+2}} d\eta d\gamma.
\end{aligned}$$

One finds that

$$I_1 \leq \frac{\alpha}{4}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{|\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)|^2 |\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)|}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+1}} d\eta d\gamma,$$

and due to the inequality $|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)||\eta|^{-1} \leq \|x\|_{C^2}$, it follows that:

$$\begin{aligned}
I_1 &\leq \frac{\alpha}{4}\|x\|_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}} |\eta|^{-\alpha}|F(x)(\gamma,\eta)|^{1+\alpha}|\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)|^2 d\eta d\gamma\\
&\leq \frac{1}{2}\|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\int_{\mathbb{T}} |\eta|^{-\alpha}\int_{\mathbb{T}} (|\partial_\gamma^3 x(\gamma)|^2 + |\partial_\gamma^3 x(\gamma-\eta)|^2) d\gamma d\eta\\
&\leq \|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\|\partial_\gamma^3 x\|_{L^2}^2 \int_{\mathbb{T}} |\eta|^{-\alpha} d\eta\\
&\leq C_\alpha \|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\|\partial_\gamma^3 x\|_{L^2}^2.
\end{aligned} \tag{2.23}$$

As before, we have $I_2 = -6I_1$, so that

$$I_2 \le C_\alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \|\partial_\gamma^3 x\|_{L^2}^2. \tag{2.24}$$

In order to estimate the term I_3, we consider $I_3 = J_1 + J_2 + J_3$, where

$$J_1 = -3\alpha \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)) \frac{A(\gamma,\eta)}{|x(\gamma) - x(\gamma-\eta)|^{\alpha+2}} d\eta d\gamma,$$
$$J_2 = -3\alpha \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)) \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma) - x(\gamma-\eta)|^{\alpha+2}} d\eta d\gamma,$$
$$J_3 = 3\alpha(2+\alpha) \int_{\mathbb{T}} \int_{\mathbb{T}} \partial_\gamma^3 x(\gamma) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma - \eta)) \frac{(B(\gamma,\eta))^2}{|x(\gamma) - x(\gamma-\eta)|^{\alpha+4}} d\eta d\gamma,$$

with

$$A(\gamma,\eta) = (x(\gamma) - x(\gamma-\eta)) \cdot (\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma-\eta)),$$

and

$$B(\gamma,\eta) = (x(\gamma) - x(\gamma-\eta)) \cdot (\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)).$$

The identity

$$\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma-\eta) = \eta \int_0^1 \partial_\gamma^3 x(\gamma + (s-1)\eta) ds, \tag{2.25}$$

yields

$$\begin{aligned} J_1 &\le 3 \int_0^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |\eta| \frac{(|\partial_\gamma^2 x(\gamma)| + |\partial_\gamma^2 x(\gamma-\eta)|)|\partial_\gamma^3 x(\gamma)||\partial_\gamma^3 x(\gamma + (s-1)\eta)|}{|x(\gamma) - x(\gamma-\eta)|^{\alpha+1}} d\gamma d\eta ds \\ &\le 3\|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \int_0^1 \int_{\mathbb{T}} |\eta|^{-\alpha} \int_{\mathbb{T}} (|\partial_\gamma^3 x(\gamma)|^2 + |\partial_\gamma^3 x(\gamma + (s-1)\eta)|^2) d\gamma d\eta ds \\ &\le C_\alpha \|F(x)\|_{L^\infty}^{1+\alpha} \|x\|_{C^2} \|\partial_\gamma^3 x\|_{L^2}^2. \end{aligned}$$

Using (2.25), we have for J_2

$$\begin{aligned} J_2 &= -3\alpha \int_0^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |F(x)(\gamma,\eta)|^{2+\alpha} \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)|^2}{\eta} \frac{\partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^3 x(\gamma + (s-1)\eta)}{|\eta|^\alpha} d\gamma d\eta ds \\ &\le 3\|F(x)\|_{L^\infty}^{2+\alpha} \|x\|_{C^2}^2 \int_0^1 \int_{\mathbb{T}} |\eta|^{-\alpha} \int_{\mathbb{T}} (|\partial_\gamma^3 x(\gamma)|^2 + |\partial_\gamma^3 x(\gamma + (s-1)\eta)|^2) d\gamma d\eta ds \\ &\le C_\alpha \|F(x)\|_{L^\infty}^{2+\alpha} \|x\|_{C^2}^2 \|\partial_\gamma^3 x\|_{L^2}^2. \end{aligned}$$

The term J_3 is estimated by

$$\begin{aligned} J_3 &\le 9 \int_0^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |\eta| \frac{|\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta)|^2 |\partial_\gamma^3 x(\gamma)||\partial_\gamma^3 x(\gamma + (s-1)\eta)|}{|x(\gamma) - x(\gamma-\eta)|^{\alpha+2}} d\gamma d\eta ds \\ &\le C_\alpha \|F(x)\|_{L^\infty}^{2+\alpha} \|x\|_{C^2}^2 \|\partial_\gamma^3 x\|_{L^2}^2. \end{aligned}$$

Finally, we obtain

$$I_3 \le C_\alpha(\|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2} + \|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2)\|\partial_\gamma^3 x\|_{L^2}^2. \tag{2.26}$$

We decompose the term $I_4 = J_4 + J_5 + J_6 + J_7 + J_8$ as follows:

$$\begin{aligned}
J_4 &= -\alpha \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\frac{C(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+2}}d\eta d\gamma,\\
J_5 &= -3\alpha\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\frac{D(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+2}}d\eta d\gamma,\\
J_6 &= 5\alpha(\alpha+2)\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\frac{A(\gamma,\eta)B(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+4}}d\eta d\gamma,\\
J_7 &= 5\alpha(\alpha+2)\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\frac{B(\gamma,\eta)|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+4}}d\eta d\gamma,\\
J_8 &= -2\alpha(\alpha+2)(\alpha+4)\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\frac{(B(\gamma,\eta))^3}{|x(\gamma)-x(\gamma-\eta)|^{\alpha+6}}d\eta d\gamma,
\end{aligned}$$

with

$$\begin{aligned}
C(\gamma,\eta) &= (x(\gamma)-x(\gamma-\eta))\cdot(\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)),\\
D(\gamma,\eta) &= (\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\cdot(\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)).
\end{aligned}$$

For the most singular term J_4,

$$\begin{aligned}
J_4 &\le \|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}|\partial_\gamma^3 x(\gamma)||\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)|d\gamma d\eta\\
&\le C_\alpha\|F(x)\|_{L^\infty}^{1+\alpha}\|x\|_{C^2}\|\partial_\gamma^3 x\|_{L^2}^2.
\end{aligned}$$

For J_5, we have

$$\begin{aligned}
J_5 &\le 3\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}|\partial_\gamma^3 x(\gamma)||\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)|d\gamma d\eta\\
&\le C_\alpha\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\|\partial_\gamma^2 x\|_{L^2}\|\partial_\gamma^3 x\|_{L^2}.
\end{aligned}$$

In a similar way, we obtain

$$\begin{aligned}
J_6 &\le 15\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}|\partial_\gamma^3 x(\gamma)||\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)|d\gamma d\eta\\
&\le C_\alpha\|F(x)\|_{L^\infty}^{2+\alpha}\|x\|_{C^2}^2\|\partial_\gamma^2 x\|_{L^2}\|\partial_\gamma^3 x\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
J_7 &\le 15\|F(x)\|_{L^\infty}^{3+\alpha}\|x\|_{C^2}^3\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}|\partial_\gamma^3 x(\gamma)||\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|d\gamma d\eta\\
&\le C_\alpha\|F(x)\|_{L^\infty}^{3+\alpha}\|x\|_{C^2}^3\|\partial_\gamma x\|_{L^2}\|\partial_\gamma^3 x\|_{L^2}.
\end{aligned}$$

For the term J_8, we get

$$\begin{aligned} J_8 &\leq 30\|F(x)\|^{3+\alpha}_{L^\infty}\|x\|^3_{C^2}\int_{\mathbb{T}}|\eta|^{-\alpha}\int_{\mathbb{T}}|\partial^3_\gamma x(\gamma)||\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|d\gamma d\eta \\ &\leq C_\alpha\|F(x)\|^{3+\alpha}_{L^\infty}\|x\|^3_{C^2}\|\partial_\gamma x\|_{L^2}\|\partial^3_\gamma x\|_{L^2}, \end{aligned}$$

so that

$$I_4 \leq C_\alpha(\|F(x)\|^{1+\alpha}_{L^\infty}\|x\|_{C^2}+\|F(x)\|^{2+\alpha}_{L^\infty}\|x\|^2_{C^2}+\|F(x)\|^{3+\alpha}_{L^\infty}\|x\|^3_{C^2})\|x\|^2_{H^3}. \tag{2.27}$$

The inequalities (2.23), (2.24), (2.26) and (2.27) yield

$$\frac{d}{dt}\|\partial^3_\gamma x\|^2_{L^2}(t) \leq C_\alpha\|F(x)\|^{3+\alpha}_{L^\infty}(t)\|x\|^3_{C^2}(t)\|x\|^2_{H^3}(t).$$

Due to the identity $\|x\|^2_{H^3}=\|x\|^2_{L^2}+\|\partial^3_\gamma x\|^2_{L^2}$ and (2.22), we have

$$\frac{d}{dt}\|x\|_{H^3}(t) \leq C_\alpha\|F(x)\|^{3+\alpha}_{L^\infty}(t)\|x\|^3_{C^2}(t)\|x\|_{H^3}(t).$$

Finally, using Sobolev inequalities, we obtain

$$\frac{d}{dt}\|x\|_{H^3}(t) \leq C_\alpha\|F(x)\|^{3+\alpha}_{L^\infty}(t)\|x\|^4_{H^3}(t). \tag{2.28}$$

Notice that if we use energy methods at this point of the proof (see [6] to get the comprehensive argument), we need to regularize the equation (2.20) as follows:

$$\begin{aligned} x^\varepsilon_t(\gamma,t) &= \phi_\varepsilon * \int_{\mathbb{T}}\frac{\partial_\gamma(\phi_\varepsilon * x^\varepsilon(\gamma,t)-\phi_\varepsilon * x^\varepsilon(\gamma-\eta,t))}{|x^\varepsilon(\gamma,t)-x^\varepsilon(\gamma-\eta,t)|^\alpha}d\eta, \\ x^\varepsilon(\gamma,0) &= x_0(\gamma), \end{aligned} \tag{2.29}$$

where ϕ_ε is a regular approximation to the identity. If the inequality (2.18) is satisfied initially, due to the properties of the regular approximations to the identity, we get a Picard system as follows:

$$\begin{aligned} x^\varepsilon_t(\gamma,t) &= G^\varepsilon(x^\varepsilon(\gamma,t)), \\ x^\varepsilon(\gamma,0) &= x_0(\gamma), \end{aligned}$$

where G^ε is Lipschitz. Therefore, for any $\varepsilon>0$, we obtain a time of existence t_ε where (2.18) is fulfilled. In order to have a time of existence for the system (2.29), independent of ε, we need to find energy estimates with bounds independent of ε. Next, by letting $\varepsilon\to 0$, we get solutions of the original equation. In this particular case, we have

$$\frac{d}{dt}\|x^\varepsilon\|_{H^3}(t) \leq C_\alpha\|F(x^\varepsilon)\|^{3+\alpha}_{L^\infty}(t)\|x^\varepsilon\|^4_{H^3}(t),$$

and if we let $\varepsilon \to 0$, it is possible that $\|F(x^\varepsilon)\|_{L^\infty} \to \infty$. In fact, we have an energy estimate that depends on ε, and so the argument fails. We cannot suppose that if the initial data fulfils (2.18), then there exists a time $t > 0$ independent of ε in which (2.18) is satisfied, because just at this moment of the proof we do not have a well-posed system when $\varepsilon \to 0$, as the Lipschitz constant of G^ε goes to infinity when $\varepsilon \to 0$.

In order to solve this problem, we consider the evolution of the quantity $\|F(x)\|_{L^\infty}$. Taking $p > 2$, it follows that:

$$\begin{aligned}\frac{d}{dt}\|F(x)\|^p_{L^p}(t) &= \frac{d}{dt}\int_{\mathbb{T}}\int_{\mathbb{T}}\Big(\frac{|\eta|}{|x(\gamma,t)-x(\gamma-\eta,t)|}\Big)^p d\gamma d\eta \\ &= -p\int_{\mathbb{T}}\int_{\mathbb{T}}|\eta|^p\frac{(x(\gamma,t)-x(\gamma-\eta,t))\cdot(x_t(\gamma,t)-x_t(\gamma-\eta,t))}{|x(\gamma,t)-x(\gamma-\eta,t)|^{p+2}}d\gamma d\eta \\ &\le p\int_{\mathbb{T}}\int_{\mathbb{T}}\Big(\frac{|\eta|}{|x(\gamma,t)-x(\gamma-\eta,t)|}\Big)^{p+1}\frac{|x_t(\gamma,t)-x_t(\gamma-\eta,t)|}{|\eta|}d\gamma d\eta.\end{aligned}$$

We have

$$\begin{aligned}x_t(\gamma)-x_t(\gamma-\eta) &= \int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma)-x(\gamma-\xi)|^\alpha}d\xi - \int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma-\eta)-\partial_\gamma x(\gamma-\eta-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\alpha}d\xi \\ &= \int_{\mathbb{T}}\Big(\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma)-x(\gamma-\xi)|^\alpha} - \frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\alpha}\Big)d\xi \\ &\quad + \int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)+\partial_\gamma x(\gamma-\eta-\xi)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\alpha}d\xi \\ &= I_5 + I_6.\end{aligned}$$

In order to estimate the term I_5, we consider the function $f(a) = a^\alpha$. For $a, b > 0$, we have

$$|a^\alpha - b^\alpha| = \alpha\Big|\int_0^1 (sa+(1-s)b)^{\alpha-1}(a-b)ds\Big| \le \alpha(\min\{a,b\})^{\alpha-1}|a-b|. \tag{2.30}$$

One finds that

$$\begin{aligned}I_5 &\le \int_{\mathbb{T}}\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)|\,\big||x(\gamma)-x(\gamma-\xi)|^\alpha - |x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\alpha\big|}{|x(\gamma)-x(\gamma-\xi)|^\alpha|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\alpha}d\xi \\ &\le \|F(x)\|^{2\alpha}_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}|\xi|^{1-\alpha}\Big|\Big|\frac{x(\gamma)-x(\gamma-\xi)}{\xi}\Big|^\alpha - \Big|\frac{x(\gamma-\eta)-x(\gamma-\eta-\xi)}{\xi}\Big|^\alpha\Big|d\xi.\end{aligned}$$

Using (2.30), we get

$$\begin{aligned}I_5 &\le \alpha\|F(x)\|^{1+\alpha}_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}|\xi|^{1-\alpha}\Big|\Big|\frac{x(\gamma)-x(\gamma-\xi)}{\xi}\Big| - \Big|\frac{x(\gamma-\eta)-x(\gamma-\eta-\xi)}{\xi}\Big|\Big|d\xi \\ &\le \alpha\|F(x)\|^{1+\alpha}_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}|\xi|^{-\alpha}(|x(\gamma)-x(\gamma-\eta)| + |x(\gamma-\xi)-x(\gamma-\eta-\xi)|)d\xi \\ &\le 2\alpha\|F(x)\|^{1+\alpha}_{L^\infty}\|x\|^2_{C^2}|\eta|\int_{\mathbb{T}}|\xi|^{-\alpha}d\xi \\ &\le C_\alpha\|F(x)\|^{1+\alpha}_{L^\infty}\|x\|^2_{C^2}|\eta|.\end{aligned}$$

For I_6, we obtain

$$\begin{aligned} I_6 &\leq \int_{\mathbb{T}} \frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|+|\partial_\gamma x(\gamma-\eta-\xi)-\partial_\gamma x(\gamma-\xi)|}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|^\alpha} d\xi \\ &\leq C_\alpha \|F(x)\|^\alpha_{L^\infty}\|x\|_{C^2}|\eta|. \end{aligned}$$

The last two estimates show that

$$\begin{aligned} \frac{d}{dt}\|F(x)\|^p_{L^p}(t) &\leq pC_\alpha\|x\|^2_{C^2}(t)\|F(x)\|^{1+\alpha}_{L^\infty}(t)\int_{\mathbb{T}^2}(F(x)(\gamma,\eta,t))^{p+1}d\gamma d\eta \\ &\leq pC_\alpha\|x\|^2_{C^2}(t)\|F(x)\|^{2+\alpha}_{L^\infty}(t)\|F(x)\|^p_{L^p}(t), \end{aligned}$$

and therefore

$$\frac{d}{dt}\|F(x)\|_{L^p}(t) \leq C_\alpha\|x\|^2_{C^2}(t)\|F(x)\|^{2+\alpha}_{L^\infty}(t)\|F(x)\|_{L^p}(t).$$

Integrating in time it follows that:

$$\|F(x)\|_{L^p}(t+h) \leq \|F(x)\|_{L^p}(t)\exp\big(C_\alpha\int_t^{t+h}\|x\|^2_{C^2}(s)\|F(x)\|^{2+\alpha}_{L^\infty}(s)ds\big),$$

and taking $p\to\infty$ we obtain

$$\|F(x)\|_{L^\infty}(t+h) \leq \|F(x)\|_{L^\infty}(t)\exp\big(C_\alpha\int_t^{t+h}\|x\|^2_{C^2}(s)\|F(x)\|^{2+\alpha}_{L^\infty}(s)ds\big).$$

In order to estimate the derivative of the quantity $\|F(x)\|_{L^\infty}(t)$, we use the last inequality, so that

$$\begin{aligned} \frac{d}{dt}\|F(x)\|_{L^\infty}(t) &= \lim_{h\to 0}(\|F(x)\|_{L^\infty}(t+h)-\|F(x)\|_{L^\infty}(t))h^{-1} \\ &\leq \|F(x)\|_{L^\infty}(t)\lim_{h\to 0}(\exp\big(C_\alpha\int_t^{t+h}\|x\|^2_{C^2}(s)\|F(x)\|^{2+\alpha}_{L^\infty}(s)ds\big)-1)h^{-1} \\ &\leq C_\alpha\|x\|^2_{C^2}(t)\|F(x)\|^{3+\alpha}_{L^\infty}(t). \end{aligned}$$

Applying Sobolev inequalities we conclude that

$$\frac{d}{dt}\|F(x)\|_{L^\infty}(t) \leq C_\alpha\|x\|^2_{H^3}(t)\|F(x)\|^{3+\alpha}_{L^\infty}(t). \tag{2.31}$$

This estimate does not give a global-in-time bound for $\|F(x)\|_{L^\infty}(t)$ in terms of norms of $x(\gamma,t)$, but adding the estimate (2.31) to (2.28), we have

$$\frac{d}{dt}(\|x\|_{H^3}(t)+\|F(x)\|_{L^\infty}(t)) \leq C_\alpha\|F(x)\|^{3+\alpha}_{L^\infty}(t)\|x\|^4_{H^3}(t),$$

and finally

$$\frac{d}{dt}(\|x\|_{H^3}(t)+\|F(x)\|_{L^\infty}(t)) \leq C_\alpha(\|x\|_{H^3}(t)+\|F(x)\|_{L^\infty}(t))^{7+\alpha}. \tag{2.32}$$

Integrating, we get

$$\|x\|_{H^3}(t)+\|F(x)\|_{L^\infty}(t)\leq \frac{\|x_0\|_{H^3}+\|F(x_0)\|_{L^\infty}}{\big(1-tC_\alpha\big(\|x_0\|_{H^3}+\|F(x_0)\|_{L^\infty}\big)^{6+\alpha}\big)^{\frac{1}{6+\alpha}}},$$

where C_α depends on α. Using the regularized problem (2.29), the same estimate is obtained with x^ε in place of x. Therefore we have found a time of existence independent of ε, and letting $\varepsilon\to 0$, the existence result follows.

Let x and y be two solutions of the equation (2.20) with $x(\gamma,0)=y(\gamma,0)$, and $z=x-y$. We see that

$$\begin{aligned}\int_{\mathbb{T}} z(\gamma)\cdot z_t(\gamma)d\gamma &= \int_{\mathbb{T}}\int_{\mathbb{T}} z(\gamma)\cdot\Big(\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|^\alpha}-\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|y(\gamma)-y(\gamma-\eta)|^\alpha}\Big)d\eta d\gamma\\ &\quad+\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{z(\gamma)\cdot(\partial_\gamma z(\gamma)-\partial_\gamma z(\gamma-\eta))}{|y(\gamma)-y(\gamma-\eta)|^\alpha}d\eta d\gamma\\ &= I_7+I_8.\end{aligned}$$

The term I_7 is estimated using (2.30) by

$$\begin{aligned}I_7 &\leq \int_{\mathbb{T}}\int_{\mathbb{T}}\frac{|z(\gamma)||\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|\big||x(\gamma)-x(\gamma-\eta)|^\alpha-|y(\gamma)-y(\gamma-\eta)|^\alpha\big|}{|x(\gamma)-x(\gamma-\eta)|^\alpha|y(\gamma)-y(\gamma-\eta)|^\alpha}d\eta d\gamma\\ &\leq \|F(x)\|_{L^\infty}^\alpha\|F(y)\|_{L^\infty}^\alpha\|x\|_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}}|\eta|^{1-\alpha}|z(\gamma)|\Big|\Big|\frac{x(\gamma)-x(\gamma-\eta)}{\eta}\Big|^\alpha-\Big|\frac{y(\gamma)-y(\gamma-\eta)}{\eta}\Big|^\alpha\Big|d\eta d\gamma\\ &\leq \|F(x)\|_{L^\infty}\|F(y)\|_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}}|\eta|^{1-\alpha}|z(\gamma)|\Big|\Big|\frac{x(\gamma)-x(\gamma-\eta)}{\eta}\Big|-\Big|\frac{y(\gamma)-y(\gamma-\eta)}{\eta}\Big|\Big|d\eta d\gamma\\ &\leq \|F(x)\|_{L^\infty}\|F(y)\|_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}}|\eta|^{-\alpha}|z(\gamma)||z(\gamma)-z(\gamma-\eta)|d\eta d\gamma\\ &\leq C_\alpha\|F(x)\|_{L^\infty}\|F(y)\|_{L^\infty}\|x\|_{C^2}\|z\|_{L^2}^2.\end{aligned}$$

Integrating by parts in I_8 yields

$$\begin{aligned}I_8 &= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{(z(\gamma)-z(\gamma-\eta))\cdot(\partial_\gamma z(\gamma)-\partial_\gamma z(\gamma-\eta))}{|y(\gamma)-y(\gamma-\eta)|^\alpha}d\eta d\gamma\\ &= \frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{\partial_\gamma(|z(\gamma)-z(\gamma-\eta)|^2)}{|y(\gamma)-y(\gamma-\eta)|^\alpha}d\eta d\gamma\\ &= \frac{\alpha}{4}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{|z(\gamma)-z(\gamma-\eta)|^2(y(\gamma)-y(\gamma-\eta))\cdot(\partial_\gamma y(\gamma)-\partial_\gamma y(\gamma-\eta))}{|y(\gamma)-y(\gamma-\eta)|^{\alpha+2}}d\eta d\gamma\\ &\leq C_\alpha\|F(y)\|_{L^\infty}^{1+\alpha}\|y\|_{C^2}\|z\|_{L^2}^2.\end{aligned}$$

Finally we obtain

$$\frac{d}{dt}\|z\|_{L^2}^2(t)\leq C(\alpha,x,F(x),y,F(y))\|z\|_{L^2}^2(t),$$

and using Gronwall inequality we conclude that $z=0$.

2.5 Existence for $\alpha = 1$; the QG sharp front

In this section we prove existence for the QG sharp front in Sobolev spaces. We give the norm of the Hölder space $C^{k,\frac{1}{2}}(\mathbb{T})$ by

$$\|x\|_{C^{k,\frac{1}{2}}} = \|x\|_{C^k} + \max_{\gamma,\eta\in\mathbb{T}} \frac{|\partial_\gamma^k x(\gamma) - \partial_\gamma^k x(\gamma-\eta)|}{|\eta|^{1/2}}.$$

In the case of $\alpha = 1$, we have the following equation:

$$\begin{aligned} x_t(\gamma,t) &= \frac{\theta_2-\theta_1}{2\pi}\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}d\eta,\\ x(\gamma,0) &= x_0(\gamma). \end{aligned} \tag{2.33}$$

Without lost of generality, we let $\theta_2 - \theta_1 = 2\pi$. This equation loses two derivatives, therefore the technique applied in the last section does not work. Recall that we are trying to solve the QG equation in a weak sense, so we can modify the system (2.33) in the tangential direction without changing the shape of the front, as long as the curve satisfies

$$x_t(\gamma,t)\cdot\partial_\gamma^\perp x(\gamma,t) = -\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma-\eta,t)\cdot\partial_\gamma^\perp x(\gamma,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}d\eta.$$

We showed before that the temperature $\theta(x,t)$ given by (2.11) is a weak solution of the QG equation. We propose to modify the equation (2.33) as follows:

$$\begin{aligned} x_t(\gamma,t) &= \int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}d\eta + \lambda(\gamma,t)\partial_\gamma x(\gamma,t),\\ x(\gamma,0) &= x_0(\gamma). \end{aligned} \tag{2.34}$$

We have introduced the parameter $\lambda(\gamma,t)$ in order to get extra cancellation in such a way that

$$\partial_\gamma x(\gamma,t)\cdot\partial_\gamma^2 x(\gamma,t) = 0. \tag{2.35}$$

Given an initial datum satisfying (2.18), we can reparameterize to obtain $|\partial_\gamma x(\gamma,0)|^2 = 1$, and therefore (2.35) is fulfilled at $t = 0$. We cannot have $|\partial_\gamma x(\gamma,t)|^2 = 1$ for all time, but

$$|\partial_\gamma x(\gamma,t)|^2 = A(t). \tag{2.36}$$

We have

$$\begin{aligned} A'(t) &= 2\partial_\gamma x(\gamma,t)\cdot\partial_\gamma x_t(\gamma,t)\\ &= 2\partial_\gamma x(\gamma,t)\cdot\partial_\gamma\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}d\eta\Big) + 2\partial_\gamma\lambda(\gamma,t)A(t), \end{aligned}$$

so that

$$\partial_\gamma\lambda(\gamma,t) = \frac{A'(t)}{2A(t)} - \frac{1}{A(t)}\partial_\gamma x(\gamma,t)\cdot\partial_\gamma\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}d\eta\Big). \tag{2.37}$$

Because $\lambda(\gamma,t)$ has to be periodic, we obtain

$$\frac{A'(t)}{2A(t)} = \frac{1}{2\pi A(t)}\int_{\mathbb{T}} \partial_\gamma x(\gamma,t)\cdot\partial_\gamma\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}d\eta\Big)d\gamma. \tag{2.38}$$

Using (2.38) in (2.37), and integrating in γ, one gets the following formula:

$$\begin{aligned}\lambda(\gamma,t) = &\frac{\gamma+\pi}{2\pi}\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)}{|\partial_\gamma x(\gamma,t)|^2}\cdot\partial_\gamma\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}d\eta\Big)d\gamma\\ &-\int_{-\pi}^{\gamma} \frac{\partial_\gamma x(\eta,t)}{|\partial_\gamma x(\eta,t)|^2}\cdot\partial_\eta\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\eta,t)-\partial_\gamma x(\eta-\xi,t)}{|x(\eta,t)-x(\eta-\xi,t)|}d\xi\Big)d\eta,\end{aligned} \tag{2.39}$$

taking $\lambda(-\pi,t) = \lambda(\pi,t) = 0$. If we consider solutions of the equation (2.34) with $\lambda(\gamma,t)$ given by (2.39), it is easy to check that

$$\frac{d}{dt}|\partial_\gamma x(\gamma,t)|^2 = \lambda(\gamma,t)\partial_\gamma|\partial_\gamma x(\gamma,t)|^2 + \mu(t)|\partial_\gamma x(\gamma,t)|^2,$$

where

$$\mu(t) = \frac{1}{\pi}\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)}{|\partial_\gamma x(\gamma,t)|^2}\cdot\partial_\gamma\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}d\eta\Big)d\gamma.$$

Solving this linear partial differential equation, if (2.35) is satisfied initially, one finds that the unique solution is given by

$$|\partial_\gamma x(\gamma,t)|^2 = |\partial_\gamma x(\gamma,0)|^2 + \frac{1}{\pi}\int_0^t\int_{\mathbb{T}} \partial_\gamma x(\gamma,s)\cdot\partial_\gamma\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,s)-\partial_\gamma x(\gamma-\eta,s)}{|x(\gamma,s)-x(\gamma-\eta,s)|}d\eta\Big)d\gamma ds.$$

Therefore we obtain (2.36).

The main result of this section is the following theorem.

Theorem 2.5.1 *Let $x_0(\gamma) \in H^k(\mathbb{T})$ for $k \geq 3$ with $F(x_0)(\gamma,\eta) < \infty$. Then there exists a time $T > 0$ so that there is a solution to* (2.34) *in $C^1([0,T];H^k(\mathbb{T}))$ with $x(\gamma,0) = x_0(\gamma)$ and $\lambda(\gamma,t)$ given by* (2.39).

Proof: We let $k = 3$, the proof for $k > 3$ being analogous. We have showed that (2.36) is satisfied if $x(\gamma,t)$ is a solution to (2.34). We can rewrite $\lambda(\gamma,t)$ as follows:

$$\begin{aligned}\lambda(\gamma,t) = &\frac{\gamma+\pi}{2\pi A(t)}\int_{\mathbb{T}} \partial_\gamma x(\gamma,t)\cdot\partial_\gamma\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}d\eta\Big)d\gamma\\ &-\frac{1}{A(t)}\int_{-\pi}^{\gamma} \partial_\gamma x(\eta,t)\cdot\partial_\eta\Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\eta,t)-\partial_\gamma x(\eta-\xi,t)}{|x(\eta,t)-x(\eta-\xi,t)|}d\xi\Big)d\eta.\end{aligned} \tag{2.40}$$

We obtain

$$\begin{aligned}\int_{\mathbb{T}} x(\gamma)\cdot x_t(\gamma)d\gamma &= \int_{\mathbb{T}}\int_{\mathbb{T}} x(\gamma)\cdot\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}d\eta d\gamma + \int_{\mathbb{T}} \lambda(\gamma)x(\gamma)\cdot\partial_\gamma x(\gamma)d\gamma\\ &= I_1 + I_2.\end{aligned}$$

One finds that $I_1 = 0$, since

$$
\begin{aligned}
I_1 &= \int_{\mathbb{T}}\int_{\mathbb{T}} x(\gamma)\cdot\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\eta)}{|x(\gamma)-x(\eta)|}d\eta d\gamma = -\int_{\mathbb{T}}\int_{\mathbb{T}} x(\eta)\cdot\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\eta)}{|x(\gamma)-x(\eta)|}d\eta d\gamma\\
&= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}} \frac{(x(\gamma)-x(\eta))\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\eta))}{|x(\gamma)-x(\eta)|}d\eta d\gamma = \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\gamma |x(\gamma)-x(\gamma-\eta)|d\gamma d\eta\\
&= 0.
\end{aligned}
$$

For the term I_2, one obtains that $I_2 \leq \|\lambda\|_{L^\infty}\|x\|_{L^2}\|\partial_\gamma x\|_{L^2}$, and

$$
\begin{aligned}
\|\lambda\|_{L^\infty} &\leq \frac{2}{A(t)}\int_{\mathbb{T}}|\partial_\gamma x(\gamma)|\Big|\partial_\gamma\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}d\eta\Big|d\gamma\\
&\leq \frac{2}{A(t)}\int_{\mathbb{T}}|\partial_\gamma x(\gamma)|\int_{\mathbb{T}}\frac{|\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta)|}{|x(\gamma)-x(\gamma-\eta)|}d\eta d\gamma\\
&\quad+\frac{2}{A(t)}\int_{\mathbb{T}}|\partial_\gamma x(\gamma)|\int_{\mathbb{T}}\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^2}d\eta d\gamma = J_1+J_2.
\end{aligned}
$$

Due to $1/A(t) \leq \|F(x)\|^2_{L^\infty}(t)$, we have

$$
J_1 \leq 2\|F(x)\|^3_{L^\infty}\int_0^1\int_{\mathbb{T}}\int_{\mathbb{T}}|\partial_\gamma^3 x(\gamma+(s-1)\eta)||\partial_\gamma x(\gamma)|d\gamma d\eta ds \leq 2\|F(x)\|^3_{L^\infty}\|x\|^2_{H^3},
$$

and

$$
J_2 \leq 2\|F(x)\|^4_{L^\infty}\|x\|_{C^1}\int_0^1\int_{\mathbb{T}}\int_{\mathbb{T}}|\partial_\gamma^2 x(\gamma+(s-1)\eta)|^2 d\gamma d\eta ds \leq 2\|F(x)\|^4_{L^\infty}\|x\|^3_{H^3}.
$$

Therefore we obtain that

$$
\frac{d}{dt}\|x\|^2_{L^2}(t) \leq C\|F(x)\|^4_{L^\infty}(t)\|x\|^5_{H^3}(t). \tag{2.41}
$$

We decompose as follows:

$$
\begin{aligned}
\int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^3 x_t(\gamma)d\gamma &= \int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^3\Big(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}d\eta\Big)d\gamma\\
&\quad+\int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^3(\lambda(\gamma)\partial_\gamma x(\gamma))d\gamma\\
&= I_3+I_4.
\end{aligned}
$$

We take $I_3 = J_3+J_4+J_5+J_6$ where

$$
\begin{aligned}
J_3 &= \int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot\frac{\partial_\gamma^4 x(\gamma)-\partial_\gamma^4 x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}d\eta d\gamma,\\
J_4 &= 3\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta))\partial_\gamma(|x(\gamma)-x(\gamma-\eta)|^{-1})d\eta d\gamma,
\end{aligned}
$$

$$J_5 = 3\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta))\partial_\gamma^2(|x(\gamma)-x(\gamma-\eta)|^{-1})d\eta d\gamma,$$

$$J_6 = \int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\partial_\gamma^3(|x(\gamma)-x(\gamma-\eta)|^{-1})d\eta d\gamma.$$

The term J_3 can be written as

$$\begin{aligned}
J_3 &= \frac{1}{2}\int_{\mathbb{T}}\int_{\mathbb{T}}(\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta))\cdot\frac{\partial_\gamma^4 x(\gamma)-\partial_\gamma^4 x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}d\eta d\gamma\\
&= \frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{\partial_\gamma|\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|}d\eta d\gamma\\
&= \frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{|\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)|^2(x(\gamma)-x(\gamma-\eta))\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^3}d\eta d\gamma.
\end{aligned}$$

Defining

$$B(\gamma,\eta) = (x(\gamma)-x(\gamma-\eta))\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)),$$

by (2.35), we see that

$$J_3 = \frac{1}{4}\int_{\mathbb{T}}\int_{\mathbb{T}}|F(x)(\gamma,\eta)|^3|\partial_\gamma^3 x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)|^2\frac{B(\gamma,\eta)\eta^{-2}-\partial_\gamma x(\gamma)\cdot\partial_\gamma^2 x(\gamma)}{|\eta|}d\eta d\gamma.$$

Using

$$\Big|\frac{B(\gamma,\eta)\eta^{-2}-\partial_\gamma x(\gamma)\cdot\partial_\gamma^2 x(\gamma)}{\eta}\Big| \leq 2\|x\|^2_{C^{2,\frac{1}{2}}}|\eta|^{-1/2},$$

we see that

$$\begin{aligned}
J_3 &\leq \|F(x)\|^3_{L^\infty}\|x\|^2_{C^{2,\frac{1}{2}}}\int_{\mathbb{T}}|\eta|^{-1/2}\int_{\mathbb{T}}(|\partial_\gamma^3 x(\gamma)|^2+|\partial_\gamma^3 x(\gamma-\eta)|^2)d\gamma d\eta\\
&\leq C\|F(x)\|^3_{L^\infty}\|x\|^2_{C^{2,\frac{1}{2}}}\|\partial_\gamma^3 x\|^2_{L^2}\\
&\leq C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}.
\end{aligned} \tag{2.42}$$

We have that $J_4 = -6J_3$, which gives

$$J_4 \leq C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}. \tag{2.43}$$

In order to estimate the term J_5, we consider $J_5 = K_1 + K_2 + K_3$, where

$$K_1 = -3\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta))\frac{C(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}d\eta d\gamma,$$

$$K_2 = -3\int_{\mathbb{T}}\int_{\mathbb{T}}\partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma^2 x(\gamma)-\partial_\gamma^2 x(\gamma-\eta))\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^3}d\eta d\gamma,$$

$$K_3 = 9\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma-\eta))\frac{(B(\gamma,\eta))^2}{|x(\gamma)-x(\gamma-\eta)|^5}d\eta d\gamma,$$

and

$$C(\gamma,\eta) = (x(\gamma)-x(\gamma-\eta))\cdot(\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma-\eta)).$$

The inequality

$$|\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma-\eta)||\eta|^{-1/2} \leq \|x\|_{C^{2,\frac{1}{2}}}, \tag{2.44}$$

yields

$$\begin{aligned} K_1 &\leq 3\|F(x)\|^2_{L^\infty}\|x\|_{C^{2,\frac{1}{2}}}\int_0^1\int_{\mathbb{T}}|\eta|^{-1/2}\int_{\mathbb{T}}|\partial_\gamma^3 x(\gamma)||\partial_\gamma^3 x(\gamma+(s-1)\eta)|d\gamma d\eta ds \\ &\leq C\|F(x)\|^2_{L^\infty}\|x\|^3_{H^3}. \end{aligned}$$

As before, we have for K_2 that

$$K_2 \leq C\|F(x)\|^3_{L^\infty}\|x\|^2_{C^2}\|\partial_\gamma^3 x\|^2_{L^2} \leq C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}.$$

The term K_3 is estimated by

$$K_3 \leq C\|F(x)\|^3_{L^\infty}\|x\|^2_{C^2}\|\partial_\gamma^3 x\|^2_{L^2} \leq C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}.$$

Finally, we obtain

$$J_5 \leq C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}. \tag{2.45}$$

Decomposing the term $J_6 = K_4 + K_5 + K_6 + K_7 + K_8$ as

$$K_4 = -\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))\frac{D(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}d\eta d\gamma,$$

$$K_5 = -3\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))\frac{E(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}d\eta d\gamma,$$

$$K_6 = 15\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))\frac{B(\gamma,\eta)C(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^5}d\eta d\gamma,$$

$$K_7 = 15\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))\frac{B(\gamma,\eta)|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^5}d\eta d\gamma,$$

$$K_8 = -30\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot(\partial_\gamma x(\gamma) - \partial_\gamma x(\gamma-\eta))\frac{(B(\gamma,\eta))^3}{|x(\gamma)-x(\gamma-\eta)|^7}d\eta d\gamma,$$

where

$$D(\gamma,\eta) = (x(\gamma)-x(\gamma-\eta))\cdot(\partial_\gamma^3 x(\gamma) - \partial_\gamma^3 x(\gamma-\eta)),$$

$$E(\gamma,\eta) = (\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\cdot(\partial_\gamma^2 x(\gamma) - \partial_\gamma^2 x(\gamma-\eta)),$$

we obtain

$$K_5 \leq 3\|F(x)\|^3_{L^\infty}\|x\|^2_{C^2}\|\partial_\gamma^3 x\|^2_{L^2} \leq 3\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3},$$

$$K_6 \leq 15\|F(x)\|^3_{L^\infty}\|x\|^2_{C^2}\|\partial^3_\gamma x\|^2_{L^2} \leq 15\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3},$$

$$K_7 \leq 15\|F(x)\|^4_{L^\infty}\|x\|^3_{C^2}\|\partial^3_\gamma x\|_{L^2}\|\partial^2_\gamma x\|_{L^2} \leq 15\|F(x)\|^4_{L^\infty}\|x\|^5_{H^3},$$

and

$$K_8 \leq 30\|F(x)\|^4_{L^\infty}\|x\|^3_{C^2}\|\partial^3_\gamma x\|_{L^2}\|\partial^2_\gamma x\|_{L^2} \leq 30\|F(x)\|^4_{L^\infty}\|x\|^5_{H^3}.$$

For the most singular term, we have

$$\begin{aligned}K_4 =& \int_{\mathbb{T}}\int_{\mathbb{T}} \partial^3_\gamma x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\frac{\eta\,\partial_\gamma x(\gamma)\cdot(\partial^3_\gamma x(\gamma)-\partial^3_\gamma x(\gamma-\eta))-D(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}d\eta d\gamma\\ &-\int_{\mathbb{T}}\int_{\mathbb{T}} \partial^3_\gamma x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\frac{\eta\,\partial_\gamma x(\gamma)\cdot(\partial^3_\gamma x(\gamma)-\partial^3_\gamma x(\gamma-\eta))}{|x(\gamma)-x(\gamma-\eta)|^3}d\eta d\gamma\\ =& L_1+L_2,\end{aligned}$$

so that

$$L_1 \leq \|F(x)\|^3_{L^\infty}\|x\|^2_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}}|\partial^3_\gamma x(\gamma)||\partial^3_\gamma x(\gamma)-\partial^3_\gamma x(\gamma-\eta)|d\gamma d\eta \leq C\|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}.$$

Decomposing the L_2 term, we see that

$$\begin{aligned}L_2 =& \int_{\mathbb{T}}\int_{\mathbb{T}} \partial^3_\gamma x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\frac{\eta\,(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\cdot\partial^3_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}d\eta d\gamma\\ &-\int_{\mathbb{T}}\int_{\mathbb{T}} \partial^3_\gamma x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\,\eta\,\frac{\partial_\gamma x(\gamma)\cdot\partial^3_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)\cdot\partial^3_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}d\eta d\gamma\\ =& M_1+M_2.\end{aligned}$$

We estimate the M_1 term as

$$M_1 \leq \|F(x)\|^3_{L^\infty}\|x\|^2_{C^2}\int_{\mathbb{T}}\int_{\mathbb{T}}|\partial^3_\gamma x(\gamma)||\partial^3_\gamma x(\gamma-\eta)|d\gamma d\eta \leq \|F(x)\|^3_{L^\infty}\|x\|^4_{H^3}.$$

Taking the derivative in (2.35), we see that $\partial_\gamma x(\gamma)\cdot\partial^3_\gamma x(\gamma) = -|\partial^2_\gamma x(\gamma)|^2$, and we rewrite

$$M_2 = \int_{\mathbb{T}}\int_{\mathbb{T}} \partial^3_\gamma x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\,\eta\,\frac{|\partial^2_\gamma x(\gamma)|^2-|\partial^2_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^3}d\eta d\gamma.$$

The inequality

$$||\partial^2_\gamma x(\gamma)|^2-|\partial^2_\gamma x(\gamma-\eta)|^2| \leq 2\|x\|_{C^2}|\eta|\int_0^1|\partial^3_\gamma x(\gamma+(s-1)\eta)|ds, \tag{2.46}$$

yields

$$M_2 \leq 2\|F(x)\|_{L^\infty}^3 \|x\|_{C^2}^2 \int_0^1 \int_{\mathbb{T}} \int_{\mathbb{T}} |\partial_\gamma^3 x(\gamma)||\partial_\gamma^3 x(\gamma + (s-1)\eta)| d\gamma d\eta ds \leq C\|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4.$$

Recalling that $K_4 = L_1 + L_2 = L_1 + M_1 + M_2 \leq C\|F(x)\|_{L^\infty}^3 \|x\|_{H^3}^4$, we see that

$$J_6 \leq C\|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^5. \tag{2.47}$$

Due to (2.42), (2.43), (2.45) and (2.47), we obtain

$$I_3 \leq C\|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^5. \tag{2.48}$$

We write $I_4 = J_7 + J_8 + J_9 + J_{10}$, where

$$J_7 = \int_{\mathbb{T}} \lambda(\gamma)\, \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^4 x(\gamma) d\gamma, \qquad J_8 = 3\int_{\mathbb{T}} \partial_\gamma \lambda(\gamma)\, |\partial_\gamma^3 x(\gamma)|^2 d\gamma,$$
$$J_9 = 3\int_{\mathbb{T}} \partial_\gamma^2 \lambda(\gamma)\, \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma^2 x(\gamma) d\gamma, \qquad J_{10} = \int_{\mathbb{T}} \partial_\gamma^3 \lambda(\gamma)\, \partial_\gamma^3 x(\gamma) \cdot \partial_\gamma x(\gamma) d\gamma.$$

Integrating by parts in the term J_7 term, we have

$$J_7 = -\frac{1}{2}\int_{\mathbb{T}} \partial_\gamma \lambda(\gamma) |\partial_\gamma^3 x(\gamma)|^2 d\gamma \leq \frac{1}{2}\|\partial_\gamma \lambda\|_{L^\infty} \|\partial_\gamma^3 x\|_{L^2}^2.$$

Using (2.40), we see that

$$\begin{aligned}\partial_\gamma \lambda(\gamma,t) &= \frac{1}{2\pi A(t)} \int_{\mathbb{T}} \partial_\gamma x(\gamma,t) \cdot \partial_\gamma \Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t) - x(\gamma-\eta,t)|} d\eta \Big) d\gamma \\ &\quad - \frac{1}{A(t)} \partial_\gamma x(\gamma,t) \cdot \partial_\gamma \Big(\int_{\mathbb{T}} \frac{\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t) - x(\gamma-\eta,t)|} d\eta \Big) \\ &= K_9 + K_{10}.\end{aligned} \tag{2.49}$$

The term K_9 is estimated in the same way as J_1 and J_2, so that

$$K_9 \leq \|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^3.$$

We have for K_{10} that

$$\begin{aligned}K_{10} &\leq \frac{\|x\|_{C^2}}{A(t)} \int_{\mathbb{T}} \Big(\frac{|\partial_\gamma^2 x(\gamma,t) - \partial_\gamma^2 x(\gamma-\eta,t)|}{|x(\gamma,t) - x(\gamma-\eta,t)|} + \frac{|\partial_\gamma x(\gamma,t) - \partial_\gamma x(\gamma-\eta,t)|^2}{|x(\gamma,t) - x(\gamma-\eta,t)|^2} \Big) d\eta \\ &\leq 2\|F(x)\|_{L^\infty}^4 \|x\|_{C^{2,\frac{1}{2}}}^3 \int_{\mathbb{T}} |\eta|^{-1/2} d\eta \\ &\leq C\|F(x)\|_{L^\infty}^4 \|x\|_{H^3}^3,\end{aligned}$$

and therefore

$$J_7 \leq C\|F(x)\|^4_{L^\infty}\|x\|^5_{H^3}. \tag{2.50}$$

Due to the identity $J_8 = -6J_7$, one finds that

$$J_8 \leq C\|F(x)\|^4_{L^\infty}\|x\|^5_{H^3}. \tag{2.51}$$

Using

$$\begin{aligned}\partial^2_\gamma\lambda(\gamma,t) = &-\frac{1}{A(t)}\partial^2_\gamma x(\gamma,t)\cdot\partial_\gamma\Big(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}d\eta\Big)\\ &-\frac{1}{A(t)}\partial_\gamma x(\gamma,t)\cdot\partial^2_\gamma\Big(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)}{|x(\gamma,t)-x(\gamma-\eta,t)|}d\eta\Big),\end{aligned}$$

one sees that

$$\begin{aligned}J_9 = &-\frac{1}{A(t)}\int_{\mathbb{T}}\partial^3_\gamma x(\gamma)\cdot\partial^2_\gamma x(\gamma)\ \partial^2_\gamma x(\gamma)\cdot\partial_\gamma\Big(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}d\eta\Big)d\gamma\\ &-\frac{1}{A(t)}\int_{\mathbb{T}}\partial^3_\gamma x(\gamma)\cdot\partial^2_\gamma x(\gamma)\ \partial_\gamma x(\gamma)\cdot\partial^2_\gamma\Big(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}d\eta\Big)d\gamma\\ = &\ L_3 + L_4.\end{aligned}$$

Therefore

$$\begin{aligned}L_3 &\leq \frac{\|x\|^2_{C^2}}{A(t)}\int_{\mathbb{T}}\int_{\mathbb{T}}|\partial^3_\gamma x(\gamma)|\Big(\frac{|\partial^2_\gamma x(\gamma,t)-\partial^2_\gamma x(\gamma-\eta,t)|}{|x(\gamma,t)-x(\gamma-\eta,t)|}+\frac{|\partial_\gamma x(\gamma,t)-\partial_\gamma x(\gamma-\eta,t)|^2}{|x(\gamma,t)-x(\gamma-\eta,t)|^2}\Big)d\eta d\gamma\\ &\leq \|F(x)\|^4_{L^\infty}\|x\|^3_{C^2}\int_0^1\int_{\mathbb{T}}\int_{\mathbb{T}}|\partial^3_\gamma x(\gamma)|(|\partial^3_\gamma x(\gamma+(t-1)\eta)|+|\partial^2_\gamma x(\gamma+(t-1)\eta)|)d\gamma d\eta ds\\ &\leq C\|F(x)\|^4_{L^\infty}\|x\|^5_{H^3}.\end{aligned}$$

Moreover

$$\begin{aligned}L_4 = &-\frac{1}{A(t)}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial^3_\gamma x(\gamma)\cdot\partial^2_\gamma x(\gamma)\ \partial_\gamma x(\gamma)\cdot\frac{\partial^3_\gamma x(\gamma)-\partial^3_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}d\eta d\gamma\\ &+\frac{2}{A(t)}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial^3_\gamma x(\gamma)\cdot\partial^2_\gamma x(\gamma)\ \partial_\gamma x(\gamma)\cdot\frac{(\partial^2_\gamma x(\gamma)-\partial^2_\gamma x(\gamma-\eta))B(\gamma,\eta)}{|x(\gamma)-x(\gamma-\eta)|^3}d\eta d\gamma\\ &-\frac{1}{A(t)}\int_{\mathbb{T}}\int_{\mathbb{T}}\partial^3_\gamma x(\gamma)\cdot\partial^2_\gamma x(\gamma)\ \partial_\gamma x(\gamma)\cdot(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\partial^2_\gamma(|x(\gamma)-x(\gamma-\eta)|^{-1})d\eta d\gamma\\ = &\ M_3+M_4+M_5.\end{aligned}$$

The terms M_4 and M_5 are estimated as before, so that

$$M_4 + M_5 \leq C\|F(x)\|^5_{L^\infty}\|x\|^6_{H^3}.$$

The most singular term is M_3, but we see that

$$\begin{aligned} M_3 = &\frac{1}{A(t)}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^2 x(\gamma)\ \partial_\gamma^3 x(\gamma-\eta)\cdot\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}d\eta d\gamma \\ &-\frac{1}{A(t)}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^2 x(\gamma)\ \frac{\partial_\gamma^3 x(\gamma)\cdot\partial_\gamma x(\gamma)-\partial_\gamma^3 x(\gamma-\eta)\cdot\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}d\eta d\gamma \\ &= N_1+N_2. \end{aligned}$$

We obtain

$$N_1 \leq \|F(x)\|_{L^\infty}^3\|x\|_{C^2}^2\|\partial_\gamma^3 x\|_{L^2}^2 \leq \|F(x)\|_{L^\infty}^3\|x\|_{H^3}^4,$$

and using (2.35)

$$N_2 = \frac{1}{A(t)}\int_{\mathbb{T}}\int_{\mathbb{T}} \partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^2 x(\gamma)\ \frac{|\partial_\gamma^2 x(\gamma)|^2-|\partial_\gamma^2 x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|}d\eta d\gamma.$$

Due to (2.46), we conclude that

$$N_2 \leq 2\|F(x)\|_{L^\infty}^3\|x\|_{C^2}^2\|\partial_\gamma^3 x\|_{L^2}^2 \leq 2\|F(x)\|_{L^\infty}^3\|x\|_{H^3}^4.$$

We have $J_9 = L_3+L_4 = L_3+M_3+M_4+M_5 = L_3+N_1+N_2+M_4+M_5$, so that

$$J_9 \leq \|F(x)\|_{L^\infty}^5\|x\|_{H^3}^6. \tag{2.52}$$

The identity (2.35) yields

$$J_{10} = -\int_{\mathbb{T}}\partial_\gamma^3\lambda(\gamma)\,|\partial_\gamma^2 x(\gamma)|^2 d\gamma = 2\int_{\mathbb{T}}\partial_\gamma^2\lambda(\gamma)\,\partial_\gamma^3 x(\gamma)\cdot\partial_\gamma^2 x(\gamma)d\gamma = \frac{2}{3}J_9,$$

and therefore

$$J_{10} \leq \|F(x)\|_{L^\infty}^5\|x\|_{H^3}^6. \tag{2.53}$$

Due to the inequalities (2.50), (2.51), (2.52), and (2.53), we get

$$I_4 \leq C\|F(x)\|_{L^\infty}^5\|x\|_{H^3}^6.$$

Using (2.48) and the last estimate, we have

$$\frac{d}{dt}\|\partial_\gamma^3 x\|_{L^2}^2(t) \leq C\|F(x)\|_{L^\infty}^5(t)\|x\|_{H^3}^6(t).$$

This inequality and (2.41) bound the evolution of the Sobolev norms of the curve as follows:

$$\frac{d}{dt}\|x\|_{H^3}(t) \leq C\|F(x)\|_{L^\infty}^5(t)\|x\|_{H^3}^5(t). \tag{2.54}$$

We continue the argument considering the evolution of the quantity $\|F(x)\|_{L^\infty}(t)$. Taking $p > 2$, we see that

$$\frac{d}{dt}\|F(x)\|^p_{L^p}(t) \leq p\int_{\mathbb{T}}\int_{\mathbb{T}}\big(\frac{|\eta|}{|x(\gamma,t)-x(\gamma-\eta,t)|}\big)^{p+1}\frac{|x_t(\gamma,t)-x_t(\gamma-\eta,t)|}{|\eta|}d\gamma d\eta.$$

We have

$$\begin{aligned}x_t(\gamma) - x_t(\gamma-\eta) &= \int_{\mathbb{T}}\Big(\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma)-x(\gamma-\xi)|} - \frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|}\Big)d\xi\\ &\quad + \int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)+\partial_\gamma x(\gamma-\eta-\xi)-\partial_\gamma x(\gamma-\xi)}{|x(\gamma-\eta)-x(\gamma-\eta-\xi)|}d\xi\\ &\quad + (\lambda(\gamma)-\lambda(\gamma-\eta))\partial_\gamma x(\gamma) + \lambda(\gamma-\eta)(\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta))\\ &= I_5 + I_6 + I_7 + I_8.\end{aligned}$$

Now,

$$\begin{aligned}I_5 &\leq \int_{\mathbb{T}}\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\xi)|\,\big||x(\gamma)-x(\gamma-\xi)| - |x(\gamma-\eta)-x(\gamma-\eta-\xi)|\big|}{|x(\gamma)-x(\gamma-\xi)||x(\gamma-\eta)-x(\gamma-\eta-\xi)|}d\xi\\ &\leq \|F(x)\|^2_{L^\infty}\|x\|_{C^2}\int_{\mathbb{T}}|\xi|^{-1}|x(\gamma)-x(\gamma-\eta)-(x(\gamma-\xi)-x(\gamma-\eta-\xi))|d\xi\\ &\leq \|F(x)\|^2_{L^\infty}\|x\|_{C^2}|\eta|\int_0^1\int_{\mathbb{T}}\frac{\big|\partial_\gamma x(\gamma+(s-1)\eta)-\partial_\gamma x(\gamma+(s-1)\eta-\xi)\big|}{|\xi|}d\xi ds\\ &\leq 2\pi\|F(x)\|^2_{L^\infty}\|x\|^2_{C^2}|\eta|.\end{aligned}$$

For I_6 we see that

$$\begin{aligned}I_6 &\leq \|F(x)\|_{L^\infty}|\eta|\int_0^1\int_{\mathbb{T}}\frac{\big|\partial^2_\gamma x(\gamma+(s-1)\eta)-\partial^2_\gamma x(\gamma+(s-1)\eta-\xi)\big|}{|\xi|}d\xi ds\\ &\leq \|F(x)\|_{L^\infty}\|x\|_{C^{2,\frac{1}{2}}}|\eta|\int_0^1\int_{\mathbb{T}}|\xi|^{-1/2}d\xi ds\\ &\leq C\|F(x)\|_{L^\infty}\|x\|_{C^{2,\frac{1}{2}}}|\eta|\end{aligned}$$

For I_7, we have

$$\begin{aligned}I_7 &\leq \frac{2\|x\|_{C^2}}{A(t)}|\eta|\max_\gamma|\partial_\gamma x(\gamma)||\partial_\gamma\Big(\int_{\mathbb{T}}\frac{\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)}{|x(\gamma)-x(\gamma-\eta)|}\Big)d\eta|\\ &\leq 2\|F(x)\|^2_{L^\infty}\|x\|^2_{C^2}|\eta|\max_\gamma\Big(\int_{\mathbb{T}}\frac{|\partial^2_\gamma x(\gamma)-\partial^2_\gamma x(\gamma-\eta)|}{|x(\gamma)-x(\gamma-\eta)|}d\eta + \int_{\mathbb{T}}\frac{|\partial_\gamma x(\gamma)-\partial_\gamma x(\gamma-\eta)|^2}{|x(\gamma)-x(\gamma-\eta)|^2}d\eta\Big)\\ &\leq 4\|F(x)\|^4_{L^\infty}\|x\|^4_{H^3}|\eta|.\end{aligned}$$

Estimating $\|\lambda\|_{L^\infty}$ as before, we easily get

$$I_8 \leq \|\lambda\|_{L^\infty}\|x\|_{C^2}|\eta| \leq 4\|F(x)\|^4_{L^\infty}\|x\|^4_{H^3}|\eta|.$$

The last four estimates show that

$$\frac{d}{dt}\|F(x)\|_{L^p}(t) \leq C\|x\|^4_{H^3}(t)\|F(x)\|^5_{L^\infty}(t)\|F(x)\|_{L^p}(t),$$

so that, by integrating in time and taking $p \to \infty$, we obtain

$$\|F(x)\|_{L^\infty}(t+h) \leq \|F(x)\|_{L^\infty}(t)\exp\Big(C\int_t^{t+h}\|x\|^4_{H^3}(s)\|F(x)\|^5_{L^\infty}(s)ds\Big).$$

As in the previous section,

$$\frac{d}{dt}\|F(x)\|_{L^\infty}(t) \leq C\|x\|^4_{H^3}(t)\|F(x)\|^6_{L^\infty}(t),$$

so that, due to (2.54) and the above estimate, we see that

$$\frac{d}{dt}(\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t)) \leq C(\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t))^{10}.$$

Integrating,

$$\|x\|_{H^3}(t) + \|F(x)\|_{L^\infty}(t) \leq \frac{\|x_0\|_{H^3} + \|F(x_0)\|_{L^\infty}}{\big(1 - tC\big(\|x_0\|_{H^3} + \|F(x_0)\|_{L^\infty}\big)^9\big)^{\frac{1}{9}}},$$

where C is a constant.

We have used the equality (2.35) to obtain the a priori estimates. In order to get the solution of (2.34), we have to choose an appropriate regularized problem preserving (2.35). We propose the system

$$\begin{aligned} x^{\varepsilon,\delta}_t(\gamma,t) &= \phi_\varepsilon * \int_{\mathbb{T}} \frac{\partial_\gamma(\phi_\varepsilon * x^{\varepsilon,\delta}(\gamma,t) - \phi_\varepsilon * x^{\varepsilon,\delta}(\gamma-\eta,t))}{|x^{\varepsilon,\delta}(\gamma,t) - x^{\varepsilon,\delta}(\gamma-\eta,t)| + \delta}d\eta + \lambda^{\varepsilon,\delta}(\gamma,t)\partial_\gamma x^{\varepsilon,\delta}(\gamma,t), \\ x^{\varepsilon,\delta}(\gamma,0) &= x_0(\gamma), \end{aligned} \tag{2.55}$$

with

$$\begin{aligned} \lambda^{\varepsilon,\delta}(\gamma,t) = \frac{\gamma+\pi}{2\pi}\int_{\mathbb{T}} &\frac{\partial_\gamma x^{\varepsilon,\delta}(\gamma,t)}{|\partial_\gamma x^{\varepsilon,\delta}(\gamma,t)|^2}\cdot\partial_\gamma\Big(\phi_\varepsilon * \int_{\mathbb{T}} \frac{\partial_\gamma(\phi_\varepsilon * x^{\varepsilon,\delta}(\gamma,t) - \phi_\varepsilon * x^{\varepsilon,\delta}(\gamma-\eta,t))}{|x^{\varepsilon,\delta}(\gamma,t) - x^{\varepsilon,\delta}(\gamma-\eta,t)| + \delta}d\eta\Big)d\gamma \\ &- \int_{-\pi}^{\gamma} \frac{\partial_\gamma x^{\varepsilon,\delta}(\eta,t)}{|\partial_\gamma x^{\varepsilon,\delta}(\eta,t)|^2}\cdot\partial_\eta\Big(\phi_\varepsilon * \int_{\mathbb{T}} \frac{\partial_\gamma(\phi_\varepsilon * x^{\varepsilon,\delta}(\eta,t) - \phi_\varepsilon * x^{\varepsilon,\delta}(\eta-\xi,t))}{|x^{\varepsilon,\delta}(\eta,t) - x^{\varepsilon,\delta}(\eta-\xi,t)| + \delta}d\xi\Big)d\eta. \end{aligned}$$

We can obtain energy estimates for the system (2.55) depending on ε and δ, but without using (2.35), and therefore we obtain existence of (2.55). As long as the solution exists, we have that

$$\partial_\gamma x^{\varepsilon,\delta}(\gamma,t)\cdot\partial_\gamma^2 x^{\varepsilon,\delta}(\gamma,t)=0.$$

Using this property of the solution, we obtain energy estimates that depend only on δ, and taking $\varepsilon\to 0$ we get a solution of the following equation:

$$\begin{aligned}
x_t^\delta(\gamma,t) &= \int_{\mathbb{T}} \frac{\partial_\gamma x^\delta(\gamma,t)-\partial_\gamma x^\delta(\gamma-\eta,t))}{|x^\delta(\gamma,t)-x^\delta(\gamma-\eta,t)|+\delta}d\eta+\lambda^\delta(\gamma,t)\partial_\gamma x^\delta(\gamma,t),\\
x^\delta(\gamma,0) &= x_0(\gamma),
\end{aligned} \tag{2.56}$$

with

$$\begin{aligned}
\lambda^\delta(\gamma,t) = {} & \frac{\gamma+\pi}{2\pi}\int_{\mathbb{T}}\frac{\partial_\gamma x^\delta(\gamma,t)}{|\partial_\gamma x^\delta(\gamma,t)|^2}\cdot\partial_\gamma\Big(\int_{\mathbb{T}}\frac{\partial_\gamma x^\delta(\gamma,t)-\partial_\gamma x^\delta(\gamma-\eta,t)}{|x^\delta(\gamma,t)-x^\delta(\gamma-\eta,t)|+\delta}d\eta\Big)d\gamma\\
&-\int_{-\pi}^{\gamma}\frac{\partial_\gamma x^\delta(\eta,t)}{|\partial_\gamma x^\delta(\eta,t)|^2}\cdot\partial_\eta\Big(\int_{\mathbb{T}}\frac{\partial_\gamma x^\delta(\eta,t)-\partial_\gamma x^\delta(\eta-\xi,t))}{|x^\delta(\eta,t)-x^\delta(\eta-\xi,t)|+\delta}d\xi\Big)d\eta.
\end{aligned}$$

Again we have that the solutions of this system satisfy

$$\partial_\gamma x^\delta(\gamma,t)\cdot\partial_\gamma^2 x^\delta(\gamma,t)=0,$$

and taking advantage of this, we find energy estimates independent of δ. Letting δ tend to 0, we conclude the existence result.

Bibliography

[1] D. Ambrose. Well-posedness of Two-phase Hele-Shaw Flow without Surface Tension. *Euro. Jnl. of Applied Mathematics* 15 597-607, 2004.

[2] J. T. Beale, T. Kato and A. Majda. Remarks on the Breakdown of Smooth Solutions for the 3-D Euler Equations. *Commun. Math. Phys.* 94 (1984), 61–66.

[3] J. Bear, Dynamics of Fluids in Porous Media, *American Elsevier*, New York, 1972.

[4] A. Bejan and D. Nield, Convection in Porous Media, *Springer*, New York, 1992.

[5] A. L. Bertozzi and P. Constantin. Global regularity for vortex patches. *Comm. Math. Phys.* 152 (1): 19–28, 1993.

[6] A. L. Bertozzi and A. J. Majda. Vorticity and the Mathematical Theory of Incompresible Fluid Flow. *Cambridge Press*, 2002.

[7] G. Birkhoff. Helmholtz and Taylor instability. *Hydrodynamics Instability, Proc. Symp. Appl. Math. XII A.M.S.*, 55–76, 1962.

[8] R. Caflisch and O. Orellana. Singular solutions and ill-posedness for the evolution of vortex sheets. *SIAM J. Math. Anal.* 20 (2): 293–307, 1989.

[9] J.Y. Chemin. Persistence of geometric structures in two-dimensional incompressible fluids. *Ann. Sci. Ecole. Norm. Sup.* 26 (4): 517–542, 1993.

[10] A. Constantin and J. Escher. *Wave breaking for nonlinear nonlocal shallow water equations*, Acta Math. 181 (1998), no. 2, 229–243.

[11] P. Constantin. Geometric statistics in turbulence. *SIAM Rev.* 36 (1994), 73–98.

[12] P. Constantin, T.F. Dupont, R.E. Goldstein, L.P. Kadanoff, M.J. Shelley and S.M. Zhou. Droplet breakup in a model of the Hele-Shaw cell. *Physical Review E*, 47, 4169-4181, 1993.

[13] P. Constantin, C. Fefferman and A. Majda. Geometric constraints on potential singularity formulation in the 3-D Euler equations. *Commun. Partial Diff. Eq.* 21 (1996), 559–571.

[14] P. Constantin, A. J. Majda, and E. Tabak. Formation of strong fronts in the 2-D quasigeostrophic thermal active scalar. *Nonlinearity*, 7:1495–1533, 1994.

[15] P. Constantin, Q. Nie, and N. Schorghofer. Nonsingular surface quasi-geostrophic flow. *Phys. Lett. A*, 241, no. 3, 168–172, 1998.

[16] A. Córdoba and D. Córdoba. A maximum principle applied to Quasi-geostrophic equations. *Comm. Math. Phys.* 249 (2004), no. 3, 511–528.

[17] D. Córdoba. Nonexistence of simple hyperbolic blow-up for the quasi-geostrophic equation. *Ann. of Math.*, 148, 1135–1152, 1998.

[18] D. Córdoba and C. Fefferman. Scalars convected by a two-dimensional incompressible flow. *Comm. Pure Appl. Math.* 55 , 255–260, 2002.

[19] D. Córdoba and C. Fefferman. Growth of solutions for QG and 2D Euler equations. *J. Amer. Math. Soc.* 15 (2002), no. 3, 665–670.

[20] D. Córdoba, C. Fefferman and J. L. Rodrigo. Almost sharp fronts for the surface quasi-geostrophic equation. *Proc. Natl. Acad. Sci. USA* 101, 2687–2691, 2004.

[21] D. Córdoba, M. A. Fontelos, A. M. Mancho and J. L. Rodrigo. Evidence of singularities for a family of contour dynamics equations. *Proc. Natl. Acad. Sci. USA* 102, 5949–5952, 2005.

[22] D. Córdoba and F. Gancedo. Contour dynamics of incompressible 3-D fluis in a porous medium with different densities. *To appear in Comm. Math. Phys.*, 2006.

[23] D. Córdoba, F. Gancedo and R. Orive. Analytical behaviour of 2D incompressible flow in porous media. *To appear in J. Math. Phys.*, 2006.

[24] H. Darcy. Les Fontaines Publiques de la Ville de Dijon. *Dalmont*, Paris, 1856.

[25] T. Dombre, A. Pumir and E. Siggia. On the interface dynamics for convection in porous media. *Physica D*, 57 (1992) 311-329.

[26] J. Escher and G. Simonett. Classical solutions for Hele-Shaw models with surface tension. *Adv. Differential Equations*, 2:619-642, 1997.

[27] F. Gancedo. Existence for the α-patch model and the QG sharp front in Sobolev spaces. *Submitted*, 2007.

[28] B. Gustafsson and A. Vasil´ev. Conformal and Potencial Analysis in Hele–Shaw Cells. *Birkhäuser Verlag*, Basel-Boston-Berlin, 2006.

[29] Hele-Shaw. The flow of water. *Nature,* 58, no.1489, 34-36, 520, 1898.

[30] Hele-Shaw. On the motion of a viscous fluid between two parallel plates. *Trans. Royal Inst. Nav. Archit., London* 40, 21, 1898.

[31] T.Y. Hou, J.S. Lowengrub and M.J. Shelley. Removing the Stiffness from Interfacial Flows with Surface Tension. *J. Comput. Phys.*, 114: 312-338, 1994.

[32] T. Kato and G. Ponce. Commutator estimates and the Euler and Navier-Stokes equations. *Comm. Pure Appl. Math.* 41 (1998), 891–907.

[33] H. Kozono and Y. Taniuchi. Limiting case of the Sobolev inequality in BMO, with application to the Euler equations, *Comm. Math. Phys.* 214 (2000), 191–200.

[34] M. Muskat. The flow of homogeneous fluids through porous media. *New York*, 1937.

[35] L. Nirenberg. An abstract form of the nonlinear Cauchy-Kowalewski theorem. *J. Differential Geometry*, 6 561-576, 1972.

[36] T. Nishida. A note on a theorem of Nirenberg. *J. Differential Geometry*, 12 629-633, 1977.

[37] K. Ohkitani, M. Yamada. Inviscid and inviscid-limit behavior of a surface quasi-geostrophic flow. *Phys. Fluids*, 9: 876–882, 1997.

[38] J. Pedlosky. Geophysical fluid dynamics. *Springer-Verlag* New York, 1987.

[39] S. Resnick. Dynamical problems in nonlinear advective partial differential equations. *Ph.D. thesis University of Chicago*, Chicago, 1995.

[40] J.L. Rodrigo. On the Evolution of Sharp Fronts for the Quasi-Geostrophic Equation. *Comm. Pure and Appl. Math.*, 58: 0821-0866, 2005.

[41] P.G. Saffman and Taylor. The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. *Proc. R. Soc. London, Ser. A* 245, 312-329, 1958.

[42] M. Siegel, R. Caflisch and S. Howison. Global Existence, Singular Solutions, and Ill-Posedness for the Muskat Problem. *Comm. Pure and Appl. Math.*, 57: 1374-1411, 2004.

[43] R. Salmon. Geophysical Fluid Dynamics *Oxford University Press.* New York, Oxford, 1998.

[44] E. Stein. Singular Integrals and Differentiability Properties of Function. *Princeton University Press.* Princeton, NJ, 1970.

[45] E. Stein and G. Weiss. Introduction to Fourier Analysis on Euclidean spaces. *Princeton University Press.* Princeton, NJ, 1971.

[46] E. Stein. Harmonic Analysis. *Princeton University Press.* Princeton, NJ, 1993.

[47] L. Tartar. Incomprensible fluid flow in a porous medium - Convergente of the homogenization processs. Apendix in the book *Nonhomogeneous media and vibration theory* by E. Sánchez-Palencia. *Lecture Notes in Physics*, 127. Springer-Verlag, Berlin-New York, 1980.

[48] G. Taylor. The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I. *Proc. Roy. Soc. London. Ser. A.* 201, 192-196 1950.

[49] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 2-D. *Invent. math.* 130, 39-72 1997.

[50] S. Wu. Well-posedness in Sobolev spaces of the full water wave problem in 3-D. *J. Amer. Math. Soc.* 12, 445-495 1999.