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PROPIEDADES CUALITATIVAS DE ESQUEMAS NUMÉRICOS DE APROXIMACIÓN DE ECUACIONES DE DIFUSIÓN Y DE DISPERSIÓN

Memoria para optar al título de Doctor en Ciencias Matemáticas

presentada por

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Resumen

Esta memoria tiene como objeto el estudio de diversos esquemas numéricos para ecuaciones del calor, de Schrödinger y de ondas. Nuestro principal objetivo es describir el comportamiento de las soluciones de discretizaciones numéricas clásicas por diferencias finitas prestando especial atención a sus propiedades cualitativas como decaimiento, dispersión, propagación, etc.

Para la ecuación del calor demostramos que las soluciones del método semi-discreto de diferencias finitas estándar reproducen exactamente el decaimiento de las soluciones continuas. Para probar este hecho se demuestran estimaciones del núcleo de convolución discreto en variable Fourier. Este resultado es útil posteriormente en el estudio de las aproximaciones viscosas de la ecuación de Schrödinger. También obtenemos una expansión completa de las soluciones discretas, usando los momentos del dato inicial, semejante a la bien conocida en el caso continuo.

En referencia a la semi-discretización clásica conservativa por diferencias finitas de la ecuación de Schrödinger probamos en primer lugar que no se tienen propiedades dispersivas independientes del parámetro de la discretización. Lo hacemos construyendo paquetes de ondas concentrados en los puntos del espectro donde el símbolo del laplaciano discreto anula todas sus derivadas de segundo orden. Se trata por tanto de un fenómeno debido a la presencia de altas frecuencias espurias.

Para remediar este hecho introducimos tres métodos numéricos: filtrado de los datos iniciales en variable Fourier; viscosidad numérica; precondicionamiento bimalla. Para cada uno de estos tres esquemas probamos estimaciones dispersivas y de ganancia de regularidad espacial local, uniformes en los parámetros de discretización. Los métodos empleados se basan en las estimaciones previas obtenidas para la ecuación del calor y estimaciones clásicas para integrales oscilatorias. Gracias a estos resultados obtenemos desigualdades de tipo Strichartz para los modelos numéricos. Esto nos permite abordar problemas no lineales para datos iniciales en el espacio L^2, sin hipótesis adicionales de regularidad. Probamos la convergencia para no linealidades que no se pueden abordar por métodos de energía y que, incluso en el caso continuo, exigen estimaciones de tipo Strichartz.

Analizamos también esquemas totalmente discretos para la ecuación de Schrödinger unidimensional. Obtenemos condiciones necesarias y suficientes para garantizar que las mismas propiedades analizadas en el caso semi-direcreto se cumplen con independencia de los parámetros de las discretización. Usando una aproximación de Euler implícito para el semigrupo lineal

introducimos un esquema numérico convergente para la ecuación no lineal bajo las mismas propiedades de regularidad del caso anterior.

En el caso del problema de Cauchy para la ecuación de ondas multidimensional introducimos un esquema semi-discreto en diferencias finitas. Probamos que para datos iniciales en el espacio de Besov discreto $\dot{B}^{d-1/2}_{1,1}(h\mathbb{Z}^d)$, las soluciones decaen en norma $l^\infty(h\mathbb{Z}^d)$ como $t^{-1/2}$ uniformemente con respecto al paso del mallado, a diferencia de las soluciones continuas que para datos iniciales en $\dot{B}^{(d+1)/2}_{1,1}(\mathbb{R}^d)$ decaen como $t^{-(d-1)/2}$. Sobre la base de este resultado de decaimiento, a pesar de la falta de homogeneidad del símbolo de la ecuación semi-discreta, utilizando una descomposición de tipo Paley-Littlewood, conseguimos probar desigualdades de tipo Strichartz en una clase de espacios que no cubre por completo la del modelo continuo, dado que la tasa de decaimiento en norma L^∞ es distinta. Sin embargo, en tres dimensiones espaciales, las estimaciones obtenidas son suficientes para probar que el esquema numérico en diferencias finitas, para la ecuación de ondas semi-lineal con exponente subcrítico, tiene soluciones uniformemente acotadas en uno de los espacios donde la ecuación continua está también bien puesta.

Los resultados obtenidos son analizados no sólo en el contexto de la aproximación numérica de ondas continuas, sino también en el contexto de la ecuación de ondas en retículos, donde la cuestión de la uniformidad con respecto al tamaño del retículo se obvia.

Finalmente, consideramos el esquema conservativo semi-discreto clásico en diferencias finitas para la ecuación de ondas en un cuadrado, y estudiamos la observabilidad frontera desde dos lados consecutivos del mismo, motivado en el control de vibraciones. Consideramos una clase de datos iniciales obtenidos por un método de filtrado bimalla. A partir de resultados conocidos de observabilidad uniforme para datos filtrados en Fourier, obtenemos el mismo resultado en esta clase bimalla. La demostración utiliza una descomposición espectral diádica introducida en el contexto del control de la ecuaciones de Schrödinger y de ondas. Este resultado es novedoso puesto que extiende a varias dimensiones resultados que solo se conocían en una dimensión espacial. Este método que desarrollamos permite abordar el mismo tipo de problemas para una clase más amplia de ecuaciones.

Capítulo 1

Introducción

En esta memoria analizamos propiedades cualitativas de esquemas numéricos de aproximación de ecuaciones de difusión y dispersión. Concretamente estudiamos cómo las discretizaciones numéricas de las ecuaciones en derivadas parciales que describen estos procesos afectan a las propiedades bien conocidas en los modelos continuos, como por ejemplo la propagación de energía, estimaciones de decaimiento de soluciones, propiedades dispersivas, etc.

El análisis numérico clásico, basado en los resultados fundamentales de P. Lax, reduce la prueba de convergencia de un esquema numérico a probar resultados de consistencia y estabilidad. Sin embargo, este análisis tiene sus limitaciones a la hora de abordar problemas no lineales, de control, problemas inversos, etc.

En particular, en el marco no lineal no existe una teoría general que permita trasladar al ámbito numérico los resultados conocidos en el marco de las EDP's. En efecto, las EDP's no lineales han sufrido un espectacular desarrollo en las últimas décadas y muchos de los resultados finos de existencia y unicidad conocidos escapan a la teoría clásica y necesitan métodos profundos relacionados con la geometría y el análisis de Fourier, entre otros. El diseño de métodos numéricos convergentes en esos casos es un problema principalmente abierto.

Esta memoria puede situarse en este ámbito. Más concretamente, nos proponemos hacer un estudio exhaustivo de las propiedades de dispersión de los esquemas numéricos para la ecuaciones de Schrödinger y ondas que, además de tener importancia en sí mismas, tienen una gran relevancia a la hora de abordar problemas no lineales que no se pueden tratar por métodos de energía. Dado que las propiedades dispersivas de los modelos continuos se usan de manera crucial para probar la existencia y la unicidad de sus soluciones en los problemas no lineales, la demostración de la convergencia de los esquemas numéricos en el contexto no lineal no puede ser probada si dichas propiedades no se verifican a nivel numérico.

En esta memoria presentamos resultados sobre tres temas:

1. Comportamiento asintótico para la aproximación semi-discreta de la ecuación del calor.

2. Estimaciones dispersivas para aproximaciones numéricas de las ecuaciones de Schrödinger y ondas.

   En concreto analizamos los siguientes esquemas numéricos:

   a) Esquema clásico semi-discreto conservativo para la ecuación de Schrödinger, y además dos métodos de filtrado:

      i. Filtración de los datos iniciales en variable Fourier.

      ii. Precondicionamiento bimalla.

   b) Un esquema viscoso de aproximación para la ecuación de Schrödinger.

   c) Esquemas totalmente discretos para la ecuación de Schrödinger unidimensional.

   d) Esquema conservativo clásico semi-discreto para la ecuación de ondas.

3. Observabilidad frontera uniforme de un método bimalla para la ecuación de ondas.

A continuación describimos brevemente los aspectos más relevantes de los problemas estudiados, los resultados obtenidos y los métodos que hemos desarrollado.

1. Ecuación del calor

Consideramos en el primer lugar el problema homogéneo de valores iniciales para la ecuación del calor:

$$\begin{cases} u_t - \Delta u = 0, & x \in \mathbb{R}^d,\ t > 0, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.0.1}$$

Utilizando la transformada de Fourier es fácil ver que las soluciones de (1.0.1) verifican

$$u(t) = G(t,\cdot) * \varphi$$

donde $G(t,x)$ es la solución fundamental del problema (1.0.1):

$$G(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}},\ t > 0.$$

Usando la fórmula explícita de la solución fundamental se puede demostrar que

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \leq c(p,q,d) t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|\varphi\|_{L^p(\mathbb{R}^d)},\ t > 0, \tag{1.0.2}$$

para todo $1 \leq p \leq q \leq \infty$.

En [43] se complementa el resultado de decaimiento (1.0.2), obteniéndose un desarrollo asintótico completo de la forma

$$u(t,\cdot) \sim \sum_{|\alpha|\leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int f(x) x^\alpha dx \right) D^\alpha G(t,\cdot). \tag{1.0.3}$$

En este contexto, en el Capítulo 2 de la memoria consideramos esquemas semi-discretos en diferencias finitas y analizamos si sus soluciones tienen propiedades de decaimiento semejantes al caso continuo, uniformes con respecto al paso del mallado. Como veremos en el Capítulo 3, estas propiedades tienen gran importancia en el análisis de los esquemas numéricos disipativos para la aproximación de la ecuación de Schrödinger.

En un primer paso, mediante técnicas de análisis de Fourier, obtenemos el decaimiento temporal para las soluciones fundamentales de la ecuación semi-discreta considerada y estimaciones de decaimiento $l^p(h\mathbb{Z}^d) - l^q(h\mathbb{Z}^d)$ de las soluciones, uniformes con respecto al paso del mallado. El decaimiento temporal obtenido es exactamente el mismo que en el caso continuo. El siguiente paso de nuestro análisis es identificar el perfil espacial de las soluciones semi-discretas para tiempos grandes. En el Teorema 2.4.1 obtenemos, para datos iniciales en espacios con peso, un desarrollo asintótico completo de las soluciones análogo a (1.0.3).

2. Ecuación de Schrödinger

Los Capítulos 3 y 4 de la memoria están dedicados al estudio de la ecuación de Schrödinger.

Consideramos en primer lugar el problema homogéneo de valores iniciales asociado a la ecuación lineal de Schrödinger

$$\begin{cases} iu_t + \Delta u = 0, \ x \in \mathbb{R}^d, \ t \neq 0, \\ u(0,x) = \varphi(x), \ x \in \mathbb{R}^d, \end{cases} \tag{1.0.4}$$

y denotamos su solución por $S(t)\varphi = e^{it\Delta}\varphi$.

Como sucede con la ecuación del calor, en la ecuación de Schrödinger se puede hallar una fórmula explícita para las soluciones. Utilizando la transformada de Fourier es fácil ver que la solución de (1.0.4) viene representada en forma integral como:

$$S(t)\varphi(x) = \frac{\exp(\frac{i|\cdot|^2}{4t})}{(4\pi it)^{d/2}} * \varphi(x) = (4\pi it)^{-d/2} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} \varphi(y) dy. \tag{1.0.5}$$

Por la identidad de Plancherel, $S(t)$ define una isometría en $L^2(\mathbb{R}^d)$ para cada valor real de t, es decir

$$\|S(t)\varphi\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)}. \tag{1.0.6}$$

Tenemos así la llamada *ley de conservación de la masa*. Además, como consecuencia del comportamiento del núcleo de convolución se tiene la *estimación dispersiva*

$$\|S(t)\varphi\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(4\pi|t|)^{d/2}}\|\varphi\|_{L^1(\mathbb{R}^d)}, \quad t \neq 0. \tag{1.0.7}$$

Esta propiedad es la clave de todas las estimaciones que presentamos a continuación.

Interpolando (1.0.6) y (1.0.7) se obtienen las siguientes desigualdades

$$\|S(t)\varphi\|_{L^p(\mathbb{R}^d)} \leq c(p)|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{L^{p'}(\mathbb{R}^d)}, \, t \neq 0, \tag{1.0.8}$$

donde $2 \leq p \leq \infty$.

Strichartz [121] utilizó estas estimaciones para resolver problemas de valores iniciales asociados a la ecuación lineal de Schrödinger no homogénea

$$\begin{cases} iu_t + \Delta u = F(x,t), \, x \in \mathbb{R}^d, \, t \neq 0, \\ u(0,x) = \varphi(x), \, x \in \mathbb{R}^d. \end{cases} \tag{1.0.9}$$

Concretamente probó que, si el dato inicial φ pertenece a $L^2(\mathbb{R}^d)$ y el término no homogéneo $F(x,t)$ está en el espacio $L^{2(d+2)/(d+4)}(\mathbb{R}^{d+1})$, entonces la solución del problema (1.0.9) está en $L^{2(d+2)/d}(\mathbb{R}^{d+1})$.

Usando la fórmula de Duhamel, Strichartz escribió la solución de (1.0.9) como

$$u(t) = S(t)\varphi - i\int_0^t S(t-s)F(\cdot,s)ds$$

y probó las siguientes estimaciones para los dos términos de la misma:

$$\|S(t)\varphi\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^{d+1})} \leq c\|\varphi\|_{L^2(\mathbb{R}^d)} \tag{1.0.10}$$

y

$$\left\|\int_0^t S(t-s)F(\cdot,s)ds\right\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^{d+1})} \leq c\|F\|_{L^{\frac{2(d+2)}{d+4}}(\mathbb{R}^{d+1})}. \tag{1.0.11}$$

La última acotación se consigue aplicando la desigualdad integral de Minkowski, las estimaciones (1.0.8) y el teorema de Hardy-Littlewood-Sobolev.

En 1985, J. Ginibre y G. Velo [50] necesitaron generalizar los resultados de Strichartz (1.0.10) y (1.0.11) para poder demostrar que el problema de Cauchy no lineal con datos iniciales en $H^1(\mathbb{R}^d)$

$$\begin{cases} iu_t + \Delta u = F(u), \, x \in \mathbb{R}^d, \, t \neq 0, \\ u(0,x) = \varphi(x), \, x \in \mathbb{R}^d, \end{cases} \tag{1.0.12}$$

tiene solución única. Para ello utilizaron el principio de la aplicación contractiva, probando que, bajo ciertas condiciones sobre la no linealidad, F, el operador

$$\Phi_\varphi(u) = S(t)\varphi - i\int_0^t S(t-s)F(u(\cdot,s))ds,$$

es una contracción en un espacio que exige a la solución determinadas propiedades de integrabilidad en las variables temporal y espacial. Esto deriva en la necesidad de probar que la solución de la ecuación homogénea, $S(t)\varphi$, tiene propiedades de integrabilidad de la forma

$$\|S(t)\varphi\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \leq c\|\varphi\|_{L^2(\mathbb{R}^d)} \tag{1.0.13}$$

para ciertos valores de q y de r, generalizando así el resultado de Strichartz (1.0.10).

Estas propiedades han sido utilizadas para probar la existencia y la unicidad de las soluciones para problemas no lineales en los que el término no lineal no es localmente Lipschitz en el espacio de la energía. En concreto para datos iniciales en espacio $L^2(\mathbb{R}^d)$ y término no lineal $F(u) = |u|^{p-1}u$, $p < 1 + 4/d$, Tsutsumi [132], usando de manera crucial las propiedades de integrabilidad (1.0.13), probó en [132] la existencia global de las soluciones. El caso crítico $p = 1 + 4/d$ ha sido analizado por Cazenave y Weissler en [28].

En esta memoria nos proponemos construir esquemas numéricos convergentes para la ecuación de Scrödinger no lineal con datos iniciales en el espacio $L^2(\mathbb{R}^d)$. El análisis numérico clásico garantiza la convergencia de las aproximaciones numéricas por diferencias finitas de la ecuación (1.0.12) en la clase de no linealidades, F, que sean localmente Lipschitz en el espacio $L^2(\mathbb{R}^d)$. Sin embargo, la ecuación (1.0.12) está bien puesta también para una clase de no linealidades que no satisfacen esta propiedad.

Para motivar más nuestro trabajo y la necesidad de probar propiedades dispersivas para los modelos numéricos, vamos a suponer que hemos probado la existencia global de la soluciones en el espacio $L^\infty(\mathbb{R},\ l^2(h\mathbb{Z}^d))$ para el problema no lineal

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = 2|u^h|^2 u^h, \quad t > 0, \\ u^h(0) = \varphi^h. \end{cases} \tag{1.0.14}$$

El vector u^h es una aproximación de la solución en el nodo $x_{\mathbf{j}} = \mathbf{j}h$, y Δ_h es la aproximación clásica de segundo orden por diferencias finitas del operador Δ:

$$(\Delta_h u^h)_{\mathbf{j}} = h^{-2}\sum_{k=1}^{d}(u^h_{\mathbf{j}+e_k} + u^h_{\mathbf{j}-e_k} - 2u^h_{\mathbf{j}}).$$

La acotación uniforme en el espacio $L^\infty(\mathbb{R},\ l^2(h\mathbb{Z}^d))$ de $\{u^h\}_{h>0}$ no asegura su convergencia a la solución continua de la ecuación de Schrödinger no lineal. Para probar la convergencia hace falta demostrar su compacidad y que permanecen acotadas en los espacios donde se puede probar la unicidad de las soluciones del problema continuo: $L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R},\ L^r(\mathbb{R}^d))$. Estas dos cuestiones exigen que se cumplan a nivel numérico técnicas semejantes a las desarrolladas en el ámbito continuo y que, en gran medida, reposen en la utilización fina de técnicas de análisis armónico.

La ecuación (1.0.14) no es integrable. Sin embargo, en el caso unidimensional, existe una discretización de la ecuación (1.0.12) con término no lineal $F(u) = 2|u|^2u$ que si lo es y tiene soluciones explícitas. Este esquema ha sido propuesto por Ablowitz y Ladik en [1]:

$$i\partial_t u_n^h + \Delta_h u^h = |u_n^h|^2(u_{n+1}^h + u_{n-1}^h). \tag{1.0.15}$$

Esta ecuación tiene soluciones explícitas (véase [2]) que no permanecen uniformemente acotadas en el espacio $L^1_{loc}(\mathbb{R}, L^r(\mathbb{R}))$ para ningún $r > 2$, cuando $h \to 0$. Vemos por tanto que, para datos iniciales generales en el espacio $L^2(\mathbb{R})$, no se puede esperar que las soluciones de la ecuación (1.0.15) tengan acotaciones uniformes en algún espacio auxiliar $L^q_{loc}(\mathbb{R}, L^r(\mathbb{R}))$. Conviene subrayar que esto es sin embargo compatible con la convergencia del esquema (1.0.15) para datos iniciales muy regulares [2].

El hecho de que el esquema (1.0.15) no posea propiedades de integrabilidad uniformes hace pensar que tampoco se tengan en (1.0.14) y ni siquiera en el esquema lineal subyacente.

A continuación detallamos los resultados principales obtenidos es este contexto, adaptando al nivel semi-discreto y completamente discreto los métodos e ideas principales de la teoría continua de la ecuación de Schrödinger no lineal.

2.1. Análisis de las propiedades dispersivas para aproximaciones semi-discretas de la ecuación de Schrödinger.

En primer lugar, consideramos el esquema semi-discreto conservativo clásico en diferencias finitas para la ecuación lineal de Schrödinger.

A diferencia de lo que sucede con la ecuación del calor, para la ecuación de Schrödinger las estimaciones dependen del parámetro de la malla. Primero probamos que el decaimiento $l^{q_0}(h\mathbb{Z}^d) - l^q(h\mathbb{Z}^d)$ no puede ser uniforme con respecto al paso del mallado h, para ningún par (q_0, q) con $q > q_0$. También, demostramos que no hay ninguna propiedad de integrabilidad en espacios del tipo $L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}^d))$ que sea uniforme en h. Estas propiedades se pierden ya que el símbolo del esquema, $p_h(\xi) = 4/h^2 \sum_{k=1}^d \sin^2(\xi_k h/2)$, anula todas sus derivadas de segundo orden. Esto ocurre en los puntos $(\pm\pi/2h, \dots, \pm\pi/2h)$, cosa que no acontece al símbolo continuo $|\xi|^2$. Por el contrario, en el caso continuo el hecho de que todas las derivadas de orden dos del símbolo $|\xi|^2$ no se anulen implica, mediante el Lema de Van der Corput, la estimación $L^1(\mathbb{R}^d) - L^\infty(\mathbb{R}^d)$ para el semigrupo lineal. Para probar la falta de propiedades dispersivas uniformes, construimos ejemplos de soluciones concentradas en variable Fourier en los puntos mencionados anteriormente.

Una vez entendidas las peculiaridades patológicas del modelo semi-discreto, introducimos varios remedios (filtración de los datos iniciales, métodos numéricos con viscosidad numérica añadida y métodos bimalla) que restablecen las propiedades dispersivas.

a. **Filtrado de los datos iniciales.**

Como hemos mencionado, las propiedades de decaimiento dejan de ser uniformes por la presencia de soluciones espurias concentradas en las frecuencias $(\pm\pi/2h)^d$. Por esto consideramos datos iniciales para los cuales el soporte de la transformada de Fourier no contenga los puntos $(\pm\pi/2h,\ldots,\pm\pi/2h)$ y probamos en el Teorema 3.3.1 el decaimiento $l^1(h\mathbb{Z}^d)-l^\infty(h\mathbb{Z}^d)$ de las soluciones, uniformemente con respecto al parámetro de discretización h. Una vez probada esta propiedad, usamos los argumentos de Keel y Tao [74] para conseguir estimaciones de tipo Strichartz para el modelo semi-discreto.

b. **Viscosidad numérica.**

A continuación, con objeto de evitar el filtrado del dato inicial en variable Fourier, introducimos un esquema que contiene un término de viscosidad numérica añadida:

$$i\frac{du^h}{dt}+\Delta_h u^h = ia(h)\mathrm{sgn}(t)\Delta_h u^h, \tag{1.0.16}$$

donde $a(h)$ es una función positiva que tiende a cero cuando h tiende a cero.

Este esquema puede ser entendido como una combinación de la aproximación conservativa de Schrödinger

$$iu_t^h+\Delta_h u^h=0$$

y de la ecuación de calor semi-discreta a una escala adecuada

$$u_t^h=a(h)\Delta_h u^h.$$

Para las bajas frecuencias, el comportamiento de las soluciones del problema (1.0.16) está dado por la ecuación conservativa. Por el contrario, en la zona de altas frecuencias, el efecto disipativo de la ecuación del calor se activa y anula los efectos introducidos por las altas frecuencias espurias en la aproximación conservativa de la ecuación de Schrödinger.

De este modo probamos que para cualquier $\alpha>d/2$ existe una elección de la función $a(h)$, para la cual las soluciones del esquema considerado satisfacen

$$\|u^h(t)\|_{l^\infty(h\mathbb{Z}^d)}\leq c(d,\alpha)\left[\frac{1}{|t|^{d/2}}+\frac{1}{|t|^{\alpha}}\right]\|\varphi^h\|_{l^1(h\mathbb{Z}^d)}, \tag{1.0.17}$$

uniformemente con respecto al parámetro de discretización $h > 0$.

Esta estimación implica estimaciones espacio-temporales de tipo Strichartz. Sin embargo, dado el comportamiento diferente del semigrupo lineal cerca de $t = 0$, $t^{-\alpha}$, y de $t = \infty$, $t^{-d/2}$, las estimaciones tipo Strichartz obtenidas en el Teorema 3.4.2 se cumplen en espacios $L^q(\mathbb{R}, l^r(h\mathbb{Z}^d)) + L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))$. La prueba sigue, en general, las líneas clásicas para este tipo de estimaciones [121], [74], [25].

A continuación, introducimos un esquema numérico para la ecuación de Schrödinger con término no lineal $|u|^p u$, $p \leq 4/d$, basado en la aproximación anterior del semigrupo lineal. Las soluciones semi-discretas pertenecen al espacio de energía $L^\infty(\mathbb{R}, l^2(h\mathbb{Z}^d))$ y además a espacios $L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^d))$, permaneciendo uniformemente acotadas con respecto a $h > 0$.

Por último, estudiamos la convergencia del método. Las soluciones numéricas obtenidas pertenecen a espacios de funciones integrables, que no permiten usar ningún tipo de argumento de compacidad para probar la convergencia de los términos no lineales. Con el objeto de obtener la compacidad requerida analizamos propiedades de integrabilidad de las soluciones del tipo $L^2_{loc}(\mathbb{R}, H^s_{loc}(\mathbb{R}^d))$.

La Sección 3.5.3 está dedicada al estudio de la regularidad local del semigrupo lineal usando las estimaciones de Kenig, Ponce y Vega [75] en la zona de bajas frecuencias y un argumento de energía para las altas. En el caso no homogéneo, usamos las técnicas de Constantin y Saut [36] que, aún siendo menos finas que las de Christ y Kiselev [33], se adaptan mejor a nuestro caso. El semigrupo generado por el esquema considerado no es conservativo, por lo que las técnicas de [33], que permiten obtener la regularidad local para la parte no homogénea a partir de las estimaciones sobre el semigrupo lineal, no se pueden aplicar.

c. **Precondicionamiento bimalla.**

Estudiamos la convergencia del algoritmo bimalla, introducido por R. Glowinski en [52] en el contexto del control de la ecuación de ondas, para la aproximación semi-discreta conservativa de la ecuación de Schrödinger.

A diferencia de la implementación propuesta en [53] y [97] (mallados con razón 1/2) consideramos dos mallados G^{4h} y G^h con la proporción de mallas $1/4$. El método consiste en resolver la ecuación semi-discreta sobre el mallado fino G^h, pero solamente para datos iniciales lentos obtenidos por interpolación del mallado grueso G^{4h}.

La demostración pone de manifiesto que la proporción $1/4$ de las mallas es importante para garantizar la existencia de propiedades dispersivas uniformes con respecto a h, al anular las peculiaridades singulares del símbolo semi-discreto en los puntos $(\pm\pi/2h)^d$ descritas anteriormente.

Esto permite probar propiedades de decaimiento y de tipo Strichartz uniformes, en la clase de los datos iniciales obtenidos por el método bimalla.

A la luz de estos resultados, introducimos un esquema numérico convergente para la ecuación de Schrödinger no lineal con nolinealidad $|u|^p u$, $p \leq 4/d$. La aproximación del término no lineal se hace de tal manera que se garantiza la conservación de la norma $l^2(h\mathbb{Z}^d)$ de las soluciones, y por tanto su existencia global, a la vez que dicha aproximación proporciona una nolinealidad adaptada al algoritmo bimalla.

La convergencia del método se demuestra en el Teorema 3.6.4. De nuevo la dificultad se presenta en el paso al límite en el término no lineal como en el apartado anterior. Para resolverla probamos, usando los argumentos de [75] y de [33], que las soluciones obtenidas permanecen acotadas en el espacio $L^2_{loc}(\mathbb{R}, H^{1/2}_{loc}(\mathbb{R}^d))$ y por tanto la convergencia de las soluciones.

2.2. Análisis de las propiedades dispersivas para esquemas totalmente discretos de la ecuación de Schrödinger en una dimensión espacial.

A continuación detallamos los resultados obtenidos en el Capítulo 4. Primero consideramos esquemas totalmente discretos para la ecuación de Schrödinger lineal, con número de Courant fijo $\lambda = k/h^2$, siendo k y h los pasos de discretización temporal y espacial respectivamente. Para estos esquemas, en primer lugar, analizamos la propiedad de decaimiento $l^1(h\mathbb{Z})-l^\infty(h\mathbb{Z})$ de las soluciones. Mediante un análisis fino de Fourier, obtenemos condiciones necesarias y suficientes para garantizar que la propiedad anterior se cumple uniformemente con respecto a los parámetros de discretización.

También analizamos la propiedad de regularidad local de las soluciones. Como hemos mencionada anteriormente, esta propiedad tiene una gran importancia a la hora de probar la convergencia del método. Obtenemos condiciones necesarias y suficientes para garantizar que la propiedad de regularidad local sea uniforme en los parámetros de discretización.

A la luz de estos resultados, consideramos un esquema numérico de aproximación para el semigrupo de Schrödinger lineal que satisface la propiedad de decaimiento $l^1(h\mathbb{Z}) - l^\infty(h\mathbb{Z})$

uniforme con respecto al parámetro de la discretización h. Usando la aproximación anterior para el semigrupo lineal introducimos un esquema numérico para la ecuación de Schrödinger no homogénea y probamos (Teorema 4.5.1) estimaciones $l^q(k\mathbb{Z}, l^r(h\mathbb{Z}))$ discretas semejantes a las de Strichartz en el caso continuo usando técnicas semejantes a los de Keel y Tao [74].

Aplicamos los resultados obtenidos al análisis cuidadoso de los dos siguientes esquemas: el de Euler implícito y el de Crank-Nicolson.

Como veremos, al hacer una discretización temporal por el método de Euler implícito se introduce una viscosidad numérica adicional que permite probar que el modelo considerado tiene propiedades dispersivas semejantes a las del modelo continuo, y uniformes en los pasos de discretización. En el caso de Crank-Nicolson no se agrega viscosidad artificial, y el esquema resulta conservativo. En este caso aparecen las mismas patologías encontradas en el caso del esquema semi-discreto conservativo y por tanto, no se verifica ninguna propiedad de dispersividad uniforme.

A continuación introducimos un esquema numérico para la ecuación de Schrödinger no lineal basado en la aproximación de la ecuación lineal por el método de Euler implícito. Usando las propiedades de tipo Strichartz analizadas anteriormente, conseguimos probar que las soluciones discretas están uniformemente acotadas en $L^\infty(\mathbb{R}, L^2(\mathbb{R})) \cap L^q_{loc}(\mathbb{R}, L^r(\mathbb{R}))$, espacio donde la ecuación de Schrödinger no lineal está bien puesta. Los mismos argumentos utilizados en el caso semi-discreto permiten probar resultados de compacidad de las soluciones y por tanto su convergencia a las soluciones continuas.

Finalmente, hacemos un análisis de la posible aplicación del método bimalla para el esquema de Crank-Nicolson. Usando propiedades finas de teoría de números, y en concreto de polinomios ciclotómicos, probamos que no existe ningún algoritmo de tipo bimalla para el que el esquema clásico de Crank-Nicolson tenga la propiedad de decaimiento $l^1(h\mathbb{Z}) - l^\infty(h\mathbb{Z})$ uniforme en los parámetros de discretización.

3. Ecuación de ondas.

El Capítulo 5 de la memoria está dedicada al estudio de la ecuación de ondas.

Consideramos el problema de valores iniciales

$$\begin{cases} u_{tt} - \Delta u = F,\ x \in \mathbb{R}^d,\ t > 0 \\ u_{|t=0} = u_0, \partial_t u_{|t=0} = u_1, \end{cases} \tag{1.0.18}$$

con $d \geq 2$.

Las estimaciones de Strichartz más simples existentes son de la forma:

$$\|u\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \leq C(\|f\|_{\dot{H}^s(\mathbb{R}^d)} + \|g\|_{\dot{H}^{s-1}(\mathbb{R}^d)} + \|F\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(\mathbb{R}^d))})$$

para algunos valores de $s, q, r, \tilde{q}, \tilde{r}$.

Este tipo de estimaciones han sido utilizadas en [50] para probar que el problema de Cauchy con término no lineal $F(u) = |u|^{p-1}u$, $p < 1 + 4/(d-2)$, tiene una única solución en el espacio de energía $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. El caso crítico $p = (d+2)/(d-2)$ ha sido analizado con éxito en [57] y [110].

Introducimos el esquema semi-discreto en diferencias finitas para la ecuación de ondas (1.0.18) y analizamos si las propiedades dispersivas del modelo continuo se mantienen en esta aproximación numérica. También analizamos la relación con la ecuación de ondas sobre retículos, que recientemente ha sido objeto de estudio en la literatura física [31], [29], [30].

Probamos que para datos iniciales es el espacio de Besov discreto $\dot{B}^{d-1/2}_{1,1}(h\mathbb{Z}^d)$, las soluciones semidiscretas decaen en norma $l^\infty(h\mathbb{Z}^d)$ como $t^{-1/2}$, uniformemente con respecto al paso del mallado, a diferencia de las continuas que para datos iniciales en $\dot{B}^{(d+1)/2}_{1,1}(\mathbb{R}^d)$ decaen como $t^{-(d-1)/2}$.

Este resultado tiene que ver con los teoremas de restricción sobre superficies de Stein y Tomas [127]. En nuestro caso, el cono d-dimensional, $\tau = |\xi|$, se reemplaza por una variedad $\tau = p_1(\xi) = 2(\sum_{k=1}^d \sin^2(\xi_k/2))^{1/2}$ que tiene, al menos, una curvatura principal no nula en cada punto distinto de cero. Por tanto, los resultados obtenidos son diferentes del caso continuo, donde las propiedades geométricas del símbolo involucrado, $|\xi|$, que tiene $d-1$ curvaturas principales no nulas, juegan un papel importante a la hora de obtener el decaimiento de las soluciones.

Tras el estudio de estas propiedades de decaimiento, en el Teorema 5.1.2, probamos estimaciones de tipo Strichartz para las soluciones semi-discretas. A diferencia de los resultados para el caso continuo, en el caso semi-discreto, las soluciones pertenecen a espacios del tipo $L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))$, con (q,r) un par 1/2-admisible:

$$\frac{1}{q} \leq \frac{1}{2}\left(\frac{1}{2} - \frac{1}{r}\right),\ q \geq 2,\ r \geq 2.$$

En el caso continuo los pares (q,r) para los que se tienen las estimaciones son los denominados $(d-1)/2$-admisibles:

$$\frac{1}{q} \leq \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right),\ q \geq 2,\ r \geq 2.$$

En vista de las restricciones sobre los pares (q,r), las estimaciones para el problema semi-discreto son uniformes en h en una clase de espacios aun más restringida.

Este hecho está relacionado con las distintas tasas de decaimiento del problema continuo y semi-discreto. La prueba de estas estimaciones para el método numérico sigue las líneas del caso continuo: se usan estimaciones del operador de evolución semi-discreto $\exp(it\sqrt{\Delta_1})$ sobre datos iniciales localizados en variable Fourier y una descomposición diádica tipo Paley-Littlewood. Sin embargo, aparecen nuevas dificultades a causa de la falta de homogeneidad del símbolo semi-discreto $p_1(\xi)$.

A la luz de estos resultados introducimos un esquema numérico para la ecuación de ondas semi-lineal en dimensión tres con exponente subcrítico $p < 5$ y probamos la existencia global de las soluciones. Las soluciones semi-discretas, además de permanecer uniformemente acotadas en el espacio de energía, tienen cotas uniformes en uno de los espacios auxiliares introducidos por Ginibre y Velo [49], espacio en el que la unicidad de las soluciones continuas se puede probar en el caso subcrítico. Uno de los aspectos novedosos de esta memoria es probar estimaciones de tipo Strichartz para las soluciones de esquemas numéricos hasta ahora sólo analizados en el espacio de la energía.

4. El problema de control para la ecuación de ondas

El problema de controlabilidad de un sistema de evolución, (la ecuación de ondas, por ejemplo), puede expresarse en un marco general de la siguiente forma: consideramos un sistema a controlar, sobre el cual podemos actuar mediante un mecanismo dado a través de un subconjunto de la frontera, en una parte interior del sistema o de cualquier otro modo. Dado un tiempo $T > 0$, el problema consiste en conducir el sistema desde un estado inicial arbitrario a un estado final fijado previamente.

El método HUM (Hilbert Uniqueness Method) introducido por J. L. Lions en [86], es una herramienta sistemática que permite el estudio de problemas de control en el marco general multidimensional y para una amplia gama de ecuaciones. Este método se basa en el hecho de que el problema de control para un sistema de evolución es equivalente a ciertas estimaciones a priori, llamadas desigualdades de observabilidad, para el sistema adjunto homogéneo correspondiente. Estas desigualdades pueden ser demostradas utilizando técnicas de multiplicadores (Komornik en [77]), métodos de series no-armónicas de Fourier (Lions en [86]), análisis micro-local (Bardos et al. en [6], Burq y Gerard en [18]), desigualdades de Ingham (Lions en [86] y Young en [139]), entre otras. El lector interesado puede consultar los artículos recopilatorios [144], [141] y [143].

En el contexto de la aproximación numérica del control de la ecuación de ondas, un modo natural de proceder sería aproximar el operador de ondas por sucesiones de operadores semi-discretos y obtener el control como el límite de las sucesiones de controles de las ecuaciones aproximadas. Sin embargo, la interacción de las ondas con los mallados discretos produce fenómenos de dispersion numérica y oscilaciones de las de altas frecuencias espurias [129], [135]. Como consecuencia de este hecho, la velocidad de propagación de las ondas numéricas puede converger a cero cuando la longitud de onda de las soluciones numéricas es del orden del tamaño del mallado discreto y este último tiende a cero. Esto tiene consecuencias negativas respecto a los problemas de controlabilidad, y en particular, los controles de los modelos discretos pueden divergir. Este fenómeno ha sido observado por Glowinski et al. en [53], [55] y [56] en conexión con la controlabilidad exacta de la ecuación de ondas y la implementación numérica del método HUM.

En general, para obtener una secuencia de controles aproximados que converjan al control continuo, hace falta probar una desigualdad de observabilidad, uniforme en el parámetro de la discretización, para los problemas adjuntos semi-discretos en una clase de datos iniciales filtrados por algún procedimiento. Para esto se han introducido algunas técnicas: regularización Tychonoff [53], filtración de las altas frecuencias [65], [142], [145], elementos finitos mixtos [54], [22], [23], algoritmos bimalla [97], [88]. Este último método ha sido introducido por Glowinski [55] y consiste en considerar dos mallados, uno fino y otro grueso, e interpolar los datos iniciales del problema numérico adjunto en el mallado fino desde el mallado grueso.

En esta memoria (Capítulo 6) consideramos el esquema conservativo semi-discreto clásico en diferencias finitas para la ecuación de ondas en un cuadrado y estudiamos el problema de observabilidad frontera desde dos lados consecutivos del mismo. Probamos desigualdades de observabilidad uniformes con respecto al parámetro de discretización en la clase de datos precondicionados o filtrados a través de un algoritmo bimalla. Las técnicas utilizadas en esta memoria permiten obtener los mismos resultados en cualquier dimensión espacial. Este resultado, así como el método de demostración, es novedoso y completa los resultados previamente conocidos en una dimensión espacial [97] y [88].

La demostración que desarrollamos consiste en usar las desigualdades de observabilidad para soluciones filtradas en variable Fourier, probadas en [142], junto con una descomposición espectral diádica introducida en [78] y [17] en el contexto del control de la ecuaciones de Schrödinger y de ondas. Un aspecto importante de este método es su posible aplicación en un contexto más general (dominios generales, ecuaciones de Schrödinger y de placas,...). El análisis espectral de las funciones que se obtienen mediante un proceso bimalla juega un papel esencial en la prueba con el objeto de mostrar que la energía de las altas frecuencias se puede acotar en función las bajas.

El algoritmo bimalla utilizado involucra las mallas G^h y G^{4h} y la desigualdad de observabilidad se prueba para cualquier tiempo $T > 4$, uniformemente en el paso de la discretización. La elección de las dos mallas de razón $1/4$ se hace por razones técnicas y cabe esperar que el mismo resultado sea cierto para las mallas G^h y G^{2h}.

También hacemos un análisis heurístico del valor del tiempo óptimo de observabilidad que cabría esperarse, estudiando el tiempo mínimo para que todos los rayos de la óptica geométrica discreta entren en la zona de observabilidad. Este análisis permite conjeturar que las estimaciones obtenidas en esta memoria son mejorables siendo el tiempo mínimo previsible $T > 2\sqrt{2}/\cos(\pi/8)$. En el caso continuo, el análisis basado en los rayos de la óptica geométrica ha sido usado con éxito en [6], [17], [78].

Notaciones

Sea $\{x_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$, una discretización uniforme de paso h del espacio $\mathbb{R}^d$. La solución aproximada de una EDP en el punto $(t, x_{\mathbf{j}})$ va ser denotada por $u_{\mathbf{j}}(t)$.

Para las diferencias finitas, en el caso unidimensional, usaremos la siguiente notación estándar para las diferencias progresivas y regresivas:

$$(d_h^+ u)_j = \frac{u_{j+1} - u_j}{h},\ (d_h^- u)_j = \frac{u_j - u_{j-1}}{h},\ j \in \mathbb{Z}.$$

También definimos

$$(d_h^2 u)_j := (d_h^+ d_h^- u)_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2},$$

que corresponde a la aproximación por diferencias finitas del operador de derivación de segundo orden.

En dimensión $d \geq 2$ introducimos operadores semejantes aplicando los anteriores en cada dimensión espacial:

$$(\nabla_h^+ u)_{\mathbf{j}} = (d_h^+ u_{j_1}, ..., d_h^+ u_{j_d}),\ \mathbf{j} = (j_1, ..., j_d)$$

y

$$(\nabla_h^- u)_{\mathbf{j}} = (d_h^- u_{j_1}, ..., d_h^- u_{j_d}),\ \mathbf{j} = (j_1, ..., j_d).$$

Para la aproximación del operador Δ utilizamos la notación Δ_h:

$$(\Delta_h u)_{\mathbf{j}} = h^{-2} \sum_{k=1}^{d} (u_{\mathbf{j}+\mathbf{e}_k} + u_{\mathbf{j}-\mathbf{e}_k} - 2u_{\mathbf{j}}),$$

donde $\{\mathbf{e}_k\}_{k=1}^d$ es la base canónica del $\mathbb{R}^d$:

$$\mathbf{e}_1 = (1, 0, ..., 0),\ \mathbf{e}_2 = (0, 1, 0, ..., 0), ...,\ \mathbf{e}_n = (0, ..., 0, 1)$$

Los mismos operadores van a ser definidos para funciones de argumento continuo $x \in \mathbb{R}^d$, $u : \mathbb{R}^d \to \mathbb{C}^d$. En dimensión uno usamos $d_h^{\pm}$ para los operadores en diferencias

$$(d_h^+ u)(x) = \frac{u(x+h) - u(x)}{h},\ (d_h^- u)(x) = \frac{u(x) - u(x-h)}{h},\ x \in \mathbb{R}.$$

La notación $d_h^2 u$ significa entonces:

$$(d_h^2 u)(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2},\ x \in \mathbb{R}.$$

En dimensión $d \geq 2$ los operadores discretos $\nabla_h^{\pm}$ and Δ_h introducidos antes se extienden a operadores continuos:

$$(\nabla_h^+ u)(x) = (d_h^+ u(x_1), ..., d_h^+ u(x_d)),\ x = (x_1, ..., x_d) \in \mathbb{R}^d,$$

$$(\nabla_h^- u)(x) = (d_h^- u(x_1), ..., d_h^- u(x_d),\ x = (x_1, ..., x_d) \in \mathbb{R}^d$$

y

$$(\Delta_h u)(x) = h^{-2} \sum_{k=1}^{d} (u(x + h\mathbf{e}_k) + u(x - h\mathbf{e}_k) - 2u(x)),\ x \in \mathbb{R}^d.$$

También usaremos los espacios discretos $l^p(h\mathbb{Z}^d), 1 \leq p < \infty$, definidos por

$$l^p(h\mathbb{Z}^d) = \left\{ \{\varphi_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d} : \|\varphi\|_{l^p(h\mathbb{Z}^d)} := \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} |u_{\mathbf{j}}|^p \right)^{1/p} < \infty \right\}.$$

En particular, para $p = \infty$ consideramos el espacio $l^\infty(h\mathbb{Z})$ de secuencias uniformemente acotadas:

$$l^\infty(h\mathbb{Z}^d) = \left\{ \{\varphi_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d} : \|\varphi\|_{l^\infty(h\mathbb{Z}^d)} := \sup_{\mathbf{j}\in\mathbb{Z}^d} |u_{\mathbf{j}}| < \infty \right\}.$$

Los espacios con peso $l^p(h\mathbb{Z}^d, |x|^m)$, $1 \leq m < \infty$ and $1 \leq p \leq \infty$, son aquellos espacios de secuencias φ para cual $\{|\mathbf{j}h|^m \varphi_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$ pertenece al $l^p(h\mathbb{Z}^d)$:

$$l^p(h\mathbb{Z}^d, |x|^m) = \left\{ \{\varphi_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d} : \|\varphi\|_{l^p(h\mathbb{Z}^d, |x|^m)} := \left(h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} |\mathbf{j}h|^{mp} |u_{\mathbf{j}}|^p \right)^{1/p} < \infty \right\},\ 1 \leq p < \infty$$

y

$$l^\infty(h\mathbb{Z}^d, |x|^m) = \left\{ \{\varphi_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d} : \|\varphi\|_{l^\infty(h\mathbb{Z}^d, |x|^m)} := \sup_{\mathbf{j}\in\mathbb{Z}^d} |\mathbf{j}h|^m |u_{\mathbf{j}}| < \infty \right\}.$$

Para dos funciones reales f y g usamos la notación $f \lesssim g$ para enfatizar la existencia de una constante C tal que $f \leq Cg$.

Abstract

This thesis analyzes various numerical schemes for the heat, Schrödinger and wave equations. Our main goal is to describe the behaviour of the solutions of the classical finite difference approximations, focusing on their qualitative properties: decay rates, dispersion, propagation, etc.

For the heat equation, we show that the solutions of the standard finite difference semi-discrete scheme have the same decay rates as the continuous ones. To prove this fact we estimate the discrete convolution kernel in the Fourier variable. This result is helpful for the later analysis of the viscous schemes for the Schrödinger equation. Using the moments of the initial data, we obtain a complete expansion of the discrete solutions similar to the well known one of the continuous case.

With respect to the conservative finite difference semi-discretization of the Schrödinger equation, we show that there is no dispersive property of its solutions that is independent of the mesh size. We prove this property by constructing wave packets concentrated at the points of the spectrum, where the symbol of the discrete laplacian has all its second order derivatives equal to zero. Therefore this refers to a phenomenon which is due to the presence of spurious high-frequencies.

To recover the dispersive properties of the solutions at the discrete level, we introduce three numerical methods: filtering the initial data in the Fourier variable; numerical viscosity; two-grid preconditioner. For each of them we prove dispersive estimates and the local space smoothing effect, uniform with respect to the mesh size. The methods we employ are based on the previous work for the heat equation and classical estimates for oscillatory integrals. In view of these results we obtain Strichartz-like estimates for the numerical models. These estimates allow us to treat nonlinear problems with L^2-initial data, without additional hypotheses of regularity. We prove the convergence of the proposed methods for nonlinearities that cannot be handled by energy arguments and which even in the continuous case require Strichartz estimates.

We also analyze fully discrete schemes for the one dimensional Schrödinger equation. Necessary and sufficient conditions are given to guarantee that the properties discussed above hold independently of the mesh sizes. Under the same hypothesis of regularity as in the semi-discrete case, and by using the backward Euler scheme to approximate the linear semigroup, we introduce a convergent numerical scheme for the nonlinear problem.

For the multidimensional wave equation we analyze the finite difference semidiscrete scheme. In contrast with the continuous case, where initial data in Besov's space $\dot{B}_{1,1}^{(d+1)/2}(\mathbb{R}^d)$ guarantee that the solutions decay as $t^{-(d-1)/2}$, for the semidiscrete problem we prove that initial data in the discrete Besov's space $\dot{B}_{1,1}^{d-1/2}(h\mathbb{Z}^d)$ implies that the solutions decay as $t^{-1/2}$ uniformly with respect to the mesh size. In view of this result, despite the inhomogeneity of the symbol introduced by the scheme, we prove Strichartz-like estimates by using a Paley-Littlewood decomposition. These estimates hold for a class of spaces smaller than in the continuous case, since the L^∞-decay rates are different. However, in dimension three, these estimates are sufficient to prove that the solutions of the finite difference scheme for the subcritical semilinear wave equation are uniformly bounded in one of the spaces, where the continuous equation is well-posed.

These results are analyzed not only in the context of the numerical approximations of the wave equation, but also for the wave equation on lattices, where we get rid of the mesh size.

Finally, we consider the finite difference semidiscrete approximation of the wave equation in a square, and analyze the boundary observability from two consecutive sides of itself, a problem that occurs in the control of vibrations. We consider the class of initial data obtained by a two-grid filtering. Using the known results of uniform boundary observability for Fourier filtered data, we obtain the same result in this class of two-grid data. The proof uses a spectral dyadic decomposition introduced in the context of the control of the Schrödinger and wave equations. This result is new since it extends to greater dimensions results that were known only in the one-dimensional case. The method employed here allows us to study the same type of problems for a large class of equations.

Chapter 1

Introduction

In this thesis we analyze some qualitative properties of the numerical approximation schemes for diffusion and dispersive equations. More precisely, we study how the numerical discretizations of the partial differential equations that describe these process affect the well-known properties of the continuous models as, for example, the propagation of energy, decay rates of the solutions, dispersion properties, etc.

Classical numerical analysis based on the fundamental work of P. Lax reduces the convergence of a numerical scheme to the proof of its consistency and stability. However, this analysis has its limitations when we want to approach nonlinear problems, control, inverse problems, etc.

In particular, there is no general theory that allows known results in nonlinear PDE's, to be translated to the numerical approach. In fact, the nonlinear PDE's have undergone a great development in the last decades. Many deep results about the existence and uniqueness of their solutions do not belong to classical theory, and need deep results related to the geometry and the Fourier analysis, among others. The construction of convergent numerical methods in these cases is largely an open problem.

This thesis can be located in this setting. More precisely, our aim is to thoroughly study the dispersion properties of the numerical schemes for Schrödinger and wave equations, properties which are not only important in themselves, but also deal with the approximation of nonlinear problems which cannot be treated by using energy methods. Indeed, since the proof of the well-posedness of the nonlinear equations in the continuous framework requires a subtle use of the dispersion properties, the proof of the convergence of the numerical scheme in the nonlinear context is hopeless if these dispersion properties are not verified at the numerical level.

In this thesis we present results on the following three subjects:

1. Asymptotic behaviour of the semidiscrete scheme for the heat equation

2. Dispersive estimates for numerical approximations of the Schrödinger and wave equations.

   More precisely we analyze the following numerical schemes:

   a) The semidiscrete conservative scheme for the Schrödinger equation and additionally two filtering methods:

      i. Fourier filtering of the initial data

      ii. Two-grid preconditioner

   b) A viscous numerical approximation for the Schrödinger equation

   c) Fully discrete schemes for the Schrödinger equation

   d) Conservative semidiscrete scheme for the wave equation

3. Uniform boundary observability of a two-grid method for the wave equation.

In the following we briefly describe the most important aspects of the studied problems, the obtained results and the methods we have developed.

1. Heat equation

Let us consider the initial value problem

$$\begin{cases} u_t - \Delta u = 0, & x \in \mathbb{R}^d,\ t > 0, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^d. \end{cases} \tag{1.0.1}$$

By means of the Fourier transform it is easy to see that, at least formally, the solutions of (1.0.1) verify

$$u(t) = G(t,\cdot) * \varphi$$

where $G(t,x)$ is the fundamental solution of problem (1.0.1):

$$G(t,x) = \frac{1}{(4\pi t)^{d/2}} e^{-\frac{|x|^2}{4t}},\ t > 0.$$

The explicit formula of $G(t,\cdot)$ gives us that

$$\|u(t)\|_{L^q(\mathbb{R}^d)} \leq c(p,q,d)t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})}\|\varphi\|_{L^p(\mathbb{R}^d)},\ t>0, \tag{1.0.2}$$

for all $1 \leq p \leq q \leq \infty$.

In [43] the authors obtain a complete asymptotic expansion of the solutions:

$$u(t,\cdot) \sim \sum_{|\alpha|\leq k} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\int f(x)x^\alpha dx\right) D^\alpha G(t,\cdot) \tag{1.0.3}$$

for initial data φ belonging to weighted $L^p(\mathbb{R}^d)$-spaces.

In this context, in Chapter 2 of the thesis we consider the finite difference semidiscrete scheme for equation (1.0.1) and we analyze whether its solutions have decay properties similar to the continuous case, uniform with respect to the mesh size. As we will see in Chapter 3, these properties have great importance in the analysis of dissipative schemes for the approximation of the Schrödinger equation.

Firstly by means of the semidiscrete Fourier transform, we obtain the long time behaviour of the semidiscrete kernel and uniform $l^p(h\mathbb{Z}^d) - l^q(h\mathbb{Z}^d)$ estimates (with respect to the mesh size) of the solutions. The next step in our analysis is to identify the space shape of the solutions for large time. For initial data in the weighted $l^1(h\mathbb{Z}^d)$-space, in Theorem 2.4.1 we obtain a complete asymptotic expansion of the solutions similar to (1.0.3).

2. Schrödinger equation

Chapter 3 and Chapter 4 of this thesis are devoted to the study of numerical schemes for the Schrödinger equation.

Let us consider the initial value problem for the homogenous Schrödinger equation:

$$\begin{cases} iu_t + \Delta u = 0,\ x\in\mathbb{R}^d,\ t\neq 0, \\ u(0,x) = \varphi(x),\ x\in\mathbb{R}^d \end{cases} \tag{1.0.4}$$

and denote its solution by $S(t)\varphi = e^{it\Delta}\varphi$.

As it happens for the heat equation, for the Schrödinger equation it is also possible to find the integral representation of its solutions. Using the Fourier transform it is easy to see that the solutions of (1.0.4) are represented in integral form as:

$$S(t)\varphi(x) = \frac{\exp(i|\cdot|^2/4t)}{(4\pi i t)^{d/2}} * \varphi(x) = (4\pi i t)^{-d/2}\int_{\mathbb{R}^d} e^{i|x-y|^2/4t}\varphi(y)dy. \tag{1.0.5}$$

By Plancherel's identity, $S(t)$ defines for any real t an isometry in $L^2(\mathbb{R}^d)$, i.e.:

$$\|S(t)\varphi\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)}. \tag{1.0.6}$$

Thus we have the so-called *mass conservation law.* In addition, as a result of the behaviour of the convolution kernel we have the *dispersive estimate*:

$$\|S(t)\varphi\|_{L^\infty(\mathbb{R}^d)} \leq \frac{1}{(4\pi|t|)^{d/2}}\|\varphi\|_{L^1(\mathbb{R}^d)}, \quad t \neq 0. \tag{1.0.7}$$

This property is the key point for all the estimates we will analyze in what follow.

Interpolating (1.0.6) and (1.0.7) we obtain the following inequalities

$$\|S(t)\varphi\|_{L^p(\mathbb{R}^d)} \leq c(p)|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{L^{p'}(\mathbb{R}^d)}, \; t \neq 0, \tag{1.0.8}$$

where $2 \leq p \leq \infty$.

Strichartz [121] used these estimates to solve the nonhomogeneous linear Schrödinger equation

$$\begin{cases} iu_t + \Delta u = F(x,t), \; x \in \mathbb{R}^d, \; t \neq 0, \\ u(0,x) = \varphi(x), \; x \in \mathbb{R}^d. \end{cases} \tag{1.0.9}$$

More precisely, for initial datum φ belonging to $L^2(\mathbb{R}^d)$ and the nonhomogeneous term $F(x,t)$ in the space $L^{2(d+2)/(d+4)}(\mathbb{R}^{d+1})$ he has proven that the solution of equation (1.0.9) belongs to $L^{2(d+2)/d}(\mathbb{R}^{d+1})$.

Using Duhamel's formula, Strichartz has written the solutions of (1.0.9) as

$$u(t) = S(t)\varphi - i\int_0^t S(t-s)F(\cdot,s)ds$$

and proved the following estimates:

$$\|S(t)\varphi\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^{d+1})} \leq c\|\varphi\|_{L^2(\mathbb{R}^d)} \tag{1.0.10}$$

and

$$\left\|\int_0^t S(t-s)F(\cdot,s)ds\right\|_{L^{\frac{2(d+2)}{d}}(\mathbb{R}^{d+1})} \leq c\|F\|_{L^{\frac{2(d+2)}{d+4}}(\mathbb{R}^{d+1})}. \tag{1.0.11}$$

The last estimate is a consequence of Minkowsky's inequality, estimates (1.0.8) and Hardy-Littlewood-Sobolev's inequality.

In 1985, J. Ginibre y G. Velo needed to generalize the results of Strichartz (1.0.10) and (1.0.11) to prove that the nonlinear problem

$$\begin{cases} iu_t + \Delta u = F(u), \; x \in \mathbb{R}^d, \; t \neq 0, \\ u(0,x) = \varphi(x), \; x \in \mathbb{R}^d. \end{cases} \tag{1.0.12}$$

is well posed in the case of $H^1(\mathbb{R}^d)$-initial data. They used the Banach fix point theorem, proving that, under certain conditions on F, the operator

$$\Phi_\varphi(u) = S(t)\varphi - i\int_0^t S(t-s)F(u(\cdot,s))ds,$$

is a contraction in a space which requires space-time integrability properties of solutions. To do it they needed to extended the results of Strichartz, by proving that the linear semigroup $S(t)\varphi$ satisfies

$$\|S(t)\varphi\|_{L^q(\mathbb{R},\,L^r(\mathbb{R}^d))} \leq c(q,r,d)\|\varphi\|_{L^2(\mathbb{R}^d)} \tag{1.0.13}$$

for some q and r, not necessarily equals.

These properties have been used to prove the well-posedness of nonlinear problems with nonlinearities that are not locally Lipschitz in the energy space. For $L^2(\mathbb{R}^d)$-initial data and nonlinearity $F(u)=|u|^{p-1}u$, $p<1+4/d$, Tsutsumi [132], using in essential manner estimates (1.0.13), has proven the global well-posedness of solutions. The critical case $p=1+4/d$ has been later analyzed by Cazenave and Weissler [28].

In this thesis our propose is to construct convergent numerical schemes for the nonlinear Shrödinger equation with $L^2(\mathbb{R}^d)$-initial data. The classical numerical analysis guarantees the convergence of the numerical schemes for equation (1.0.12) for nonlinearities that are locally Lipschitz in the space $L^2(\mathbb{R}^d)$. However, equation (1.0.12) is also well posed for a class of nonlinearities that does not satisfy this requirement and thus the numerical approximation of these equations cannot be treated with classical tools.

To give a reason to our work and to explain the necessity of analyzing these dispersive properties for numerical models, let us consider the nonlinear problem

$$\begin{cases} i\dfrac{du^h}{dt}+\Delta_h u^h = 2|u^h|^2u^h, \quad t>0,\\ \qquad\qquad u^h(0)=\varphi^h. \end{cases} \tag{1.0.14}$$

Here u^h is an approximation of the solution at the node $x_{\mathbf{j}}=\mathbf{j}h$, and Δ_h is the classical second order finite difference approximation of Δ:

$$(\Delta_h u^h)_{\mathbf{j}} = h^{-2}\sum_{k=1}^{d}(u^h_{\mathbf{j}+e_k}+u^h_{\mathbf{j}-e_k}-2u^h_{\mathbf{j}}).$$

A classical fix point guarantee that equation (1.0.14) has a global solutions $u^h\in L^\infty(\mathbb{R},l^2(h\mathbb{Z}^d))$. The uniform boundedness of $\{u^h\}_{h>0}$ in $L^\infty(\mathbb{R},\,l^2(h\mathbb{Z}^d))$ does not imply the convergence towards the solution of the NSE. To guarantee the convergence we need to prove their compactness and that remain uniformly bounded in the spaces where the continous problem is well-posed. This two difficulties require that in the numerical framework, techniques similar to the ones developed in the continuous case have to be achieved. The development of such tools suggests a delicate use of harmonic analysis.

Unfortunately equation (1.0.14) is not integrable. However, in the one-dimensional case, an alternative integrable type of discretization of the NSE with nonlinearity $F(u)=2|u|^2u$

has been proposed in [1] and is accordingly often referred to as the Ablowitz-Ladik NSE:

$$i\partial_t u_n^h + \Delta_h u^h = |u_n^h|^2 (u_{n+1}^h + u_{n-1}^h). \quad (1.0.15)$$

This equation has explicit solutions (see [2]) and it is posible to prove that for any $r > 2$, they are not uniformly bounded in the space $L^1_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^d))$ as $h \to 0$. Thus for general L^2-initial data we cannot expect that the solutions of scheme (1.0.15) will stay uniformly bounded in some auxiliary space $L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^d))$. We point out that this is compatible with the convergence of the numerical scheme for smooth initial data (see [2]). The fact that scheme (1.0.15) has no uniform integrability properties it makes think that neither we have them in (1.0.14) nor in the underlying linear scheme.

In the following we make precise the results we have obtained in this context, adapting the main methods and ideas of the continous theory of the nonlinear Schrödinger equation to the semidiscrete and fully discrete framework.

2.1. Analysis of the dispersive estimates for semidiscrete approximations of the Schrödinger equation.

We first consider the conservative finite difference semidiscrete scheme for the linear Schrödinger equation.

In contrast with what happens in the case of the heat equation, for the Schrödinger equation the estimates depend on the mesh size. For any pairs (q_0, q), $q > q_0 \geq 1$ we prove that the estimate $l^{q_0}(h\mathbb{Z}^d) - l^q(h\mathbb{Z}^d)$ is not uniform with respect to the mesh size h. We also prove that there is no integrability property of the type $L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^d))$ uniform with respect to h. These properties do not hold since the symbol of the scheme $p_h(\xi) = 4/h^2 \sum_{k=1}^d \sin^2(\xi_k h/2)$, vanishes all its second order derivatives at the points $(\pm\pi/2h, \ldots, \pm\pi/2h)$ of the spectrum. By contrary, in the continuous case the fact that all the second order derivatives of the symbol $|\xi|^2$ do not vanish implies, by means of the Van der Corput Lemma, the $L^1(\mathbb{R}^d) - L^\infty(\mathbb{R}^d)$ estimate for the linear semigroup. To prove the lack of any uniform dispersive properties, we construct solutions that are concentrated in the Fourier variable at the points mentioned before.

Once we have understood the pathologies of the semidiscrete model, we introduce several remedies (filtration of the initial data, numerical viscosity and a two-grid method) that reestablish the dispersive properties.

a. **Filtering the initial data.**

As we have mentioned before, the decay properties are not uniform by the presence of spurious solutions concentrated at the frequencies $(\pm\pi/2h)^d$. To avoid the spurious effects introduced at these points we consider initial data supported in the Fourier variable far away from the points $(\pm\pi/2h, \dots, \pm\pi/2h)$. We obtain in Theorem 3.3.1 the uniform $l^1(h\mathbb{Z}^d) - l^\infty(h\mathbb{Z}^d)$ decay of the solutions. Once this property is proved, using the arguments of Keel and Tao [74] we obtain Strichartz-like estimates for the semidiscrete solutions.

b. **Numerical viscosity.**

To avoid the use of the Fourier filtering we introduce a numerical scheme that contains a numerical viscosity term:

$$i\frac{du^h}{dt} + \Delta_h u^h = ia(h)\mathrm{sgn}(t)\Delta_h u^h, \tag{1.0.16}$$

where $a(h)$ is a positive function which tends to zero as h goes to zero.

We remark that the proposed scheme is a combination of the conservative approximation of the Schrödinger equation and a semidiscretization of the heat equation in the appropriate time-scale.

On the low frequency component, the behaviour of the solutions of (1.0.16) is given by the conservative scheme. On the contrary, on the high frequencies the dissipative effect of the heat equation makes presence and vanishes the spurious effects introduced by the conservative scheme for the Schrödinger equation.

In this way we prove that for any $\alpha > d/2$ there exists an election of the function $a(h)$, such that all the solutions of scheme (1.0.16) satisfy

$$\|u^h(t)\|_{l^\infty(h\mathbb{Z}^d)} \leq c(d,\alpha)\left[\frac{1}{|t|^{d/2}} + \frac{1}{|t|^\alpha}\right]\|\varphi^h\|_{l^1(h\mathbb{Z}^d)}, \tag{1.0.17}$$

uniformly with respect to the mesh size $h > 0$.

This estimate implies more general Strichartz-like estimates. However, given the different behaviour of the linear semigroup near to $t = 0$, $t^{-\alpha/2}$, and to $t = \infty$, $t^{-d/2}$, the Strichartz-like estimates obtained in Theorem 3.4.2, hold in spaces of the type $L^q(\mathbb{R}, l^r(h\mathbb{Z}^d)) + L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))$. The proof follows, in general, the classic lines for this type of estimates [121], [74], [25].

Next, we introduce a numerical scheme for the Schrödinger equation with nonlinearity $|u|^p u$, $p \leq 4/d$, based on the above approximation of the linear semigroup. The semidiscrete solutions belong to the energy space $L^\infty(\mathbb{R}, l^2(h\mathbb{Z}^d))$ and in addition are uniformly bounded in the spaces $L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^d))$.

Finally, we study the convergence of the method. The numerical solutions belong to spaces of integrable functions, spaces that do not allow us to use any compactness argument to prove the convergence of the nonlinear terms. In order to obtain the required compactness we analyze whether the solutions remain bounded in some space $L^2_{loc}(\mathbb{R}, H^s_{loc}(\mathbb{R}^d))$ with $s > 0$.

Section 3.5.3 is devoted to the study of the local smoothing effect of the linear discrete semigroup. This is proved by using the ideas of Kenig, Ponce and Vega [75] for the low frequencies and an energy argument for the high ones. In the nonhomogeneous case, we use the techniques of Constantin and Saut [36]. We emphasize that the semigroup generated by the considered scheme is not conservative. Therefore, the techniques of Christ y Kiselev [33] that show the smoothing effect of the nonhomogeneous term from the previous estimates for the linear semigroup, cannot be applied. The techniques of Constantin and Saut are less fine than those of Christ and Kiselev but better adapted in our case.

c. **Two-grid preconditioner.**

We study the convergence of the two-grid algorithm, introduced by R. Glowinski in [52] in the context of the control of the wave equation, for the conservative semidiscrete finite difference approximation of the Schrödinger equation.

Unlike the proposed implementation in [53] and [97] (grids with quotient 1/2) we consider two grids G^{4h} and G^h with quotient of the meshes of 1/4. The method consists of solving the semidiscrete equation on the fine grid one G^h, but only for slow initial data obtained as interpolation from the grid G^{4h}.

The proof emphasizes that the proportion 1/4 of the meshes is important to guarantee the existence of uniform dispersive properties, by vanishing the pathologies of the semidiscrete symbol at the points $(\pm\pi/2h)^d$ previously described.

This allows us to obtain uniform decay properties and Strichartz type estimates in the class of initial data obtained by the two-grid method.

In view of these results we introduce a numerical scheme based on the two-grid method for the nonlinear Schrödinger equation with nonlinearity $|u|^p u$, $p \leq 4/d$. The approximation of the nonlinear term is such that guarantees the conservation of the $l^2(h\mathbb{Z}^d)$-norm of the solutions and then their global existence.

The convergence of the method is proved in Theorem 3.6.4. Again, like in the case of the viscous scheme the difficulty appears in the passage to the limit in the nonlinear term. Using the previous ideas of [75] and [33] we prove that the approximated solutions are uniformly bounded in the space $L^2_{loc}(\mathbb{R},\, H^{1/2}_{loc}(\mathbb{R}^d))$ and then the convergence of the scheme.

2.2. Analysis of the dispersive properties for fully discrete schemes in the one dimensional case.

We now describe the results obtained in Chapter 4. We consider fully discrete schemes with fixed Courant's number $\lambda = k/h^2$, where k and h are the time-step, respectively space-step discretization. Firstly we analyze the $l^1(h\mathbb{Z}) - l^\infty(h\mathbb{Z})$ decay of the solutions. Through a careful Fourier analysis, we obtain necessary and sufficient conditions to guarantee that the previous property is fulfilled independently of the mesh parameters.

We also analyze the local smoothing property of the solutions. As we say before, this property has great importance in the future proof of the convergence of the numerical schemes for nonlinear problems. We obtain necessary and sufficient conditions to guarantee that this property holds uniformly on the mesh sizes.

In view of these results, we consider a numerical scheme for the linear Schrödinger equation that uniform $l^1(h\mathbb{Z}) - l^\infty(h\mathbb{Z})$ decay property and use it to introduce a numerical scheme for the nonhomogeneous Schrödinger equation. For both schemes, using similar techniques to those of [74] and [98], we prove discrete $l^q(k\mathbb{Z}, l^r(h\mathbb{Z}))$-estimates similar to the Strichartz ones in the continuous case.

We exemplify the obtained results by two numerical schemes: the backward Euler and Crank-Nicolson. Doing a backward Euler discretization, an additional numerical viscosity is introduced and thus the scheme has uniform dispersive properties similar to the continuous model. The Crank-Nicolson scheme is conservative and the same pathologies as in the semidiscrete case occur, i.e. there are no uniform dispersive estimates for its solutions.

Next, we introduce a numerical scheme for the nonlinear Schrödinger equation based in the backward Euler approximation of the linear Schrödinger semigroup. Using the Strichartz-like estimates proved before, we obtain that the discrete solutions are uniformly bounded in the space $L^\infty(\mathbb{R}, L^2(\mathbb{R})) \cap L^q_{loc}(\mathbb{R}, l^r(\mathbb{R}))$, space where the nonlinear Schrödinger equation is well-posed. The same arguments used in the semidiscrete case allow us to prove the compactness of the solutions and thus their convergence toward the continuous ones.

Finally, we analyze the possible application of the two-grid method for the Crank-Nicolson scheme. Using fine properties of number theory, in particular cyclotomic polynomials, we prove that any two-grid algorithm applied to the Crank-Nicolson scheme is not sufficient to recover the uniform $l^1(h\mathbb{Z}) - l^\infty(h\mathbb{Z})$ estimate of the solutions.

3. Wave equation

Chapter 5 of this thesis is devoted to the study of the wave equation.

Let us consider the initial value problem

$$\begin{cases} u_{tt} - \Delta u = F,\, x \in \mathbb{R}^d,\, t > 0 \\ u_{|t=0} = u_0, \partial_t u_{|t=0} = u_1, \end{cases} \tag{1.0.18}$$

with $d \geq 2$.

The simplest Strichartz estimates are as follows:

$$\|u\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \leq C(\|f\|_{\dot{H}^s(\mathbb{R}^d)} + \|g\|_{\dot{H}^{s-1}(\mathbb{R}^d)} + \|F\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(\mathbb{R}^d))}) \tag{1.0.19}$$

for some $s, q, r, \tilde{q}$ and $\tilde{r}$.

This type of estimates has been used in [50] to prove that the Cauchy problem with nonlinearity $F(u) = |u|^{p-1}u$, $p < 1 + 4/(d-2)$, has a unique solution in the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. The critical case $p = (d+2)/(d-2)$ has been analyzed in [57] and [110].

We continue our analysis considering the finite difference semidiscrete scheme for the wave equation (1.0.18) and we analyze whether dispersive properties similar to (1.0.19), hold for this numerical scheme. We also analyze these properties in the context of the wave equation on lattices, that has lately had a great importance in physics [31], [29], [30].

In contrast with the continuous case where for initial data in the Besov's space $\dot{B}^{(d+1)/2}_{1,1}(\mathbb{R}^d)$ the solutions decay as $t^{-(d-1)/2}$, for the semidiscrete problem we prove that for initial data in the discrete Besov's space $\dot{B}^{d-1/2}_{1,1}(h\mathbb{Z}^d)$ the discrete solutions decay as $t^{-1/2}$ uniformly with respect to the mesh size. This result is connected with the restriction theorems of Stein and Tomas [127]. In our case, the d-dimensional cone $\tau = |\xi|$ is replaced by a manifold $\tau = p_1(\xi) = 2(\sum_{k=1}^{d} \sin^2(\xi_k/2))^{1/2}$ which has, at least, one non-vanishing principal curvature at each point far away from zero. Therefore, the results are different from the continuous case, where the geometrical properties of the symbol $|\xi|$, that has $d-1$ non-vanishing principal curvatures at each point far away from zero plays a key role in proving the decay of the localized solutions.

After the study of these decay properties, in Theorem 5.1.2 we prove Strichartz-like estimates for the semidiscrete solutions. In the semidiscrete case the solutions belong to the spaces $L^q(\mathbb{R}, l^r(h\mathbb{Z}^d))$, with (q,r) an 1/2-admissible pair

$$\frac{1}{q} \leq \frac{1}{2}\left(\frac{1}{2} - \frac{1}{r}\right), \; q \geq 2, \, r \geq 2,$$

in contrast with the continuous case where the solutions belong to $L^q(\mathbb{R}, L^r(\mathbb{R}^d))$, (q,r) being a $(d-1)/2$-admissible pair:

$$\frac{1}{q} \leq \frac{d-1}{2}\left(\frac{1}{2} - \frac{1}{r}\right), \; q \geq 2, \, r \geq 2.$$

This fact is related to the different decay rates of the two problems. The proof of these estimates for the numerical scheme follows the lines of the continuous case: estimates for the operator $\exp(it\sqrt{-\Delta_1})$ acting on initial data localized in the Fourier space and a Paley-Littlewood dyadic decomposition. However, new difficulties caused by the lack of homogeneity of the semidiscrete symbol $p_1(\xi)$ appear.

Thanks to these results we introduce a numerical scheme for the three dimensional semilinear wave equation with subcritical nonlinearity $|u|^{p-1}u$, $p < 5$, and prove the global existence of the approximate solutions. Besides being uniformly bounded in the energy space, the semidiscrete solutions are also uniformly bounded in one of the auxiliary spaces introduced by Ginibre y Velo [49], space where the uniqueness of the continuous solutions can be proved. One of the novel aspects of this thesis is the proof of Strichartz estimates for the solutions of the finite difference scheme for the wave equation that up to now have been analyzed only by energy estimates.

4. The observability problem for the wave equation

The problem of controllability of an evolution system, (the wave equation for example), can be expressed in the following form: we consider a system to control, on which we can act by means of a mechanism given through a subset of the border, in an inner part of the system or in any other way. Given a time $T > 0$, the problem of the controllability consists in studying the possibility of leading the system from an arbitrary initial state to a fixed final state.

The HUM method (Hilbert Uniqueness Method) introduced by J. L. Lions in [86], is a systematic tool that allows the study of the control problems within the general framework and for an ample range of equations. This method is based on the fact that the control problem for an evolution system is equivalent to certain a priori estimates, named observability inequalities, for the corresponding homogenous attached system. These inequalities can be proved using multipliers technique (Komornik in [77]), no-harmonic series (Lions in [86]), microlocal analysis (Bardos et al. in [6], Burq y Gerard in [18]), Ingham inequalities (Lions in [86] and Young in [139]), among others. For more on this topic we refer to the survey articles [144], [141] and [143].

In the context of numerical approximation a natural way is to approximate the wave operator by semidiscrete ones and obtain the control as the limit of the continuous controls of the approximate problems. However, numerical waves yield dispersion phenomena and spurious high frequency oscillations [129, 135]. As a consequence, the group velocity of these nonphysical waves converges to zero when the wavelength of the numerical solutions is of the order of the mesh size and the latter tends to zero. This has important consequences for the approximation of the control, in particular the controls of the discrete models may diverge. This phenomena has been observed by Glowinski et al. [53, 55, 56] in the context of the exact controlability of the wave equation and the numerical implementation of the HUM method.

In general to obtain a convergent approximation of the continuos control is necessary to prove a uniform observability inequality for the adjoint semidiscrete problems in some class of filtered initial data. For this reason several techniques have been introduced as possible remedies to the high frequencies spurious oscillations: Tychonoff regularization [53], filtering of the high frequencies [65, 142, 145], mixed finite elements [54, 22, 23], two-grid algorithm [97, 88]. The last method was proposed by Glowinski [55] and consists in using a coarse and a fine grid, and interpolating the initial data from the coarse grid to the fine one.

In this thesis (Chapter 6) we consider the classical finite difference scheme for the wave equation in a square and study the problem of border observability from two consecutive sides of itself. We prove uniform with respect to the mesh size h observability inequalities in the class of data filtrated through a two-grid algorithm. This result as well as its proof is new and complete the previous work in the one-dimensional case [97, 88].

The proof consists in using the observability inequalities for Fourier filtered solutions obtained in [142], together with a dyadic spectral decomposition introduced in [78] and [17] in the context of the controllability of the Schrödinger and wave equations. An important aspect is the possible application of this method to a more general context (general domains, Schrödinger and plate equations,...). The spectral analysis of the functions obtained by a two-grid method plays a key role in the proof, by allowing us to estimate the energy of the high frequencies in terms of the low ones.

The two-grid method that we propose involves the grids G^h and G^{4h} and the observability inequality holds for any time $T > 4$ uniformly with respect to the mesh size. The election of the two grids with ratio of the meshes $1/4$, is due to technical reasons and we expect that the result is certain for the grids G^h y G^{2h}.

Also we make a heuristic analysis of the value of the optimal time of observability. We analyze the time needed by all the rays of the geometric optics to reach the observability area, analysis that has been used successfully in [6], [17], [78]. This analysis allows us to conjecture that the expected value is $T > 2\sqrt{2}/\cos(\pi/8)$ and the estimates obtained in this memory could be improved.

Notations

Let $\{x_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$, $x_{\mathbf{j}} = h\mathbf{j}$ be a uniform discretization of $\mathbb{R}^d$ with mesh size h. The approximate solution to a PDE at $(t, x_{\mathbf{j}})$ will be denoted by $u_{\mathbf{j}}(t)$.

In discussing one-dimensional finite difference schemes, we will use the following standard notation for forward and backward discrete derivatives:

$$(d_h^+ u)_j = \frac{u_{j+1} - u_j}{h},\ (d_h^- u)_j = \frac{u_j - u_{j-1}}{h},\ j \in \mathbb{Z}.$$

We also define

$$(d_h^2 u)_j := (d_h^+ d_h^- u)_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2},$$

which corresponds to the most common finite difference discretization of the second derivative operator.

In dimensions $d \geq 2$, we introduce similar operators by applying the above ones in each space direction:

$$(\nabla_h^+ u)_{\mathbf{j}} = (d_h^+ u_{j_1}, ..., d_h^+ u_{j_d}),\ \mathbf{j} = (j_1, ..., j_d)$$

and

$$(\nabla_h^- u)_{\mathbf{j}} = (d_h^- u_{j_1}, ..., d_h^- u_{j_d}),\ \mathbf{j} = (j_1, ..., j_d).$$

We will use the notation Δ_h to the approximation of the second order operator Δ:

$$(\Delta_h u)_{\mathbf{j}} = h^{-2} \sum_{k=1}^{d} (u_{\mathbf{j}+\mathbf{e}_k} + u_{\mathbf{j}-\mathbf{e}_k} - 2u_{\mathbf{j}}),$$

where $\{\mathbf{e}_k\}_{k=1}^d$ is the canonical basis of $\mathbb{R}^d$:

$$\mathbf{e}_1 = (1, 0, ..., 0),\ \mathbf{e}_2 = (0, 1, 0, ..., 0), ...,\ \mathbf{e}_n = (0, ..., 0, 1).$$

The same operators will be defined for functions of continuous argument $x \in \mathbb{R}^d$, $u : \mathbb{R}^d \to \mathbb{C}^d$. In dimension one we use $d_h^{\pm}$ to denote the difference operators

$$(d_h^+ u)(x) = \frac{u(x+h) - u(x)}{h}, \; (d_h^- u)(x) = \frac{u(x) - u(x-h)}{h}, \; x \in \mathbb{R}.$$

The notation $d_h^2 u$ means

$$(d_h^2 u)(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}, \; x \in \mathbb{R}.$$

In dimension $d \geq 2$ the discrete operators $\nabla_h^{\pm}$ and Δ_h introduced above extend to continuous ones as below:

$$(\nabla_h^+ u)(x) = (d_h^+ u(x_1), ..., d_h^+ u(x_d)), \; x = (x_1, ..., x_d) \in \mathbb{R}^d,$$

$$(\nabla_h^- u)(x) = (d_h^- u(x_1), ..., d_h^- u(x_d), \; x = (x_1, ..., x_d) \in \mathbb{R}^d$$

and

$$(\Delta_h u)(x) = h^{-2} \sum_{k=1}^{d} (u(x + h\mathbf{e}_k) + u(x - h\mathbf{e}_k) - 2u(x)), \; x \in \mathbb{R}^d.$$

We also make use of the spaces $l^p(h\mathbb{Z}^d), 1 \leq p < \infty$ defined by

$$l^p(h\mathbb{Z}^d) = \left\{ \{\varphi_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{Z}^d} : \|\varphi\|_{l^p(h\mathbb{Z}^d)} := \left(h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} |u_{\mathbf{j}}|^p \right)^{1/p} < \infty \right\}.$$

For $p = \infty$ we denote $l^\infty(h\mathbb{Z})$ the space of uniformly bounded sequences:

$$l^\infty(h\mathbb{Z}^d) = \left\{ \{\varphi_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{Z}^d} : \|\varphi\|_{l^\infty(h\mathbb{Z}^d)} := \sup_{\mathbf{j} \in \mathbb{Z}^d} |u_{\mathbf{j}}| < \infty \right\}.$$

The weighted spaces $l^p(h\mathbb{Z}^d, |x|^m)$, $1 \leq m < \infty$ and $1 \leq p \leq \infty$, are those spaces of sequences φ such that $\{|\mathbf{j}h|^m \varphi_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{Z}^d}$ belongs to $l^p(h\mathbb{Z}^d)$:

$$l^p(h\mathbb{Z}^d, |x|^m) = \left\{ \{\varphi_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{Z}^d} : \|\varphi\|_{l^p(h\mathbb{Z}^d, |x|^m)} := \left(h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} |\mathbf{j}h|^{mp} |u_{\mathbf{j}}|^p \right)^{1/p} < \infty \right\}$$

and

$$l^\infty(h\mathbb{Z}^d, |x|^m) = \left\{ \{\varphi_{\mathbf{j}}\}_{\mathbf{j} \in \mathbb{Z}^d} : \|\varphi\|_{l^\infty(h\mathbb{Z}^d, |x|^m)} := \sup_{\mathbf{j} \in \mathbb{Z}^d} |\mathbf{j}h|^m |u_{\mathbf{j}}| < \infty \right\}.$$

For real functions f and g we use the notation $f \lesssim g$ to mean the existence of a constant C such that $f \leq Cg$.

Chapter 2

Preliminaries on the Heat Equation

2.1. Introduction

Let us consider the linear heat equation in the whole space

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^d \times (0,\infty) \\ u(0,x) = \varphi(x) & \text{on } \mathbb{R}^d. \end{cases} \tag{2.1.1}$$

The heat or diffusion equation describes in typical applications the evolution in time of the density u of some quantity such as temperature or chemical concentration in uniform materials; see e.g. Evans [45]. The well-posedness of the problem (2.1.1) is by now a textbook result. We refer to [45] and [26] for classical results.

In this chapter we analyze by means of Fourier techniques the semi-discretizations of the linear heat equation (2.1.1). First we summarize the properties of the heat equation that we will analyze in the semidiscrete setting. These properties concern the long-time behaviour and the spatial shape for large time t of the solutions.

In the Fourier space, equation (2.1.1) becomes

$$\begin{cases} \widehat{u}_t &= -|\xi|^2\widehat{u} \quad \text{for } t>0, \\ \widehat{u} &= \widehat{\varphi} \quad\quad\;\; \text{for } t=0. \end{cases} \tag{2.1.2}$$

Hence

$$\widehat{u}(t) = e^{-|\xi|^2 t}\widehat{\varphi}.$$

Consequently $u(t) = \left(e^{-|\xi|^2 t}\widehat{\varphi}\right)^{\vee}$ and therefore

$$u(t) = G(t,\cdot) * \varphi, \tag{2.1.3}$$

where

$$G(t,x) = (4\pi t)^{-d/2} e^{-\frac{|x|^2}{4t}} = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\xi} e^{-|\xi|^2 t} d\xi \tag{2.1.4}$$

is the fundamental solution or the *heat Kernel* (the solution of (2.1.1) with $u(0) = \delta$, the Dirac delta). The explicit expression of the fundamental solution allows us to obtain information about the decay rates of the solutions of (2.1.1). The fundamental solution $G(t,\cdot)$ satisfies for any $1 \le p \le \infty$ (cf. [45], Ch.2, p.46 and [26], p.44):

$$\|G(t,\cdot)\|_{L^p(\mathbb{R}^d)} \le c(p) t^{-d/2(1-1/p)}, \ \forall\, t>0.$$

Using the representation of the solutions as the convolution between the fundamental solution and the initial data (2.1.3) one can obtain the following asymptotic properties of the solutions:

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \leq c(p,q,d)t^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}\|\varphi\|_{L^q(\mathbb{R}^d)},\ t>0, \tag{2.1.5}$$

for all $1 \leq q \leq p \leq \infty$ and some positive constants $c(p,q,d)$. The same result is obtained in [140] by means of energy estimates, i.e. differentiating the quantity $\|u(t)\|^p_{L^p(\mathbb{R}^d)}$ with respect to the time variable and using Sobolev inequalities and interpolation. This argument was introduced by Veron [134] in the context of semilinear parabolic PDE's.

A finer analysis is given in [43] where the authors study how the mass of the solution is distributed as $t \to \infty$ and obtain the following result:

Theorem 2.1.1. *Let $1 \leq q < \frac{d}{d-1}$, $k \in \mathbb{N}$ and $q \leq p \leq \infty$. There exists a positive constant $C = C(p,k,q,d)$ such that*

$$\left\| u(t,\cdot) - \sum_{|\alpha|\leq k} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\int f(x)x^\alpha dx\right) D^\alpha G(t,\cdot)\right\|_{L^p(\mathbb{R}^d)} \leq Ct^{-\frac{d}{2}(\frac{k+1}{d}+\frac{1}{q}-\frac{1}{p})}\| |x|^{k+1}\varphi\|_{L^q(\mathbb{R}^d)} \tag{2.1.6}$$

for all $\varphi \in L^1(\mathbb{R}^d, 1+|x|^k)$ with $|x|^{k+1}\varphi(x) \in L^q(\mathbb{R}^d)$.

For $k=0$, this Theorem essentially says that for large time t the solution of (2.1.1) is close to the product between the mass of the solution and the fundamental solution. From (2.1.6) one can obtain the first k terms in the asymptotic expansion of u. We point out that for $k=0$ the same result can be obtained by scaling arguments.

Let us introduce the semidiscrete finite difference approximation of the Heat Equation (2.1.1):

$$\begin{cases} \dfrac{du^h}{dt} &=& \Delta_h u^h,\ t>0, \\ u^h(0) &=& \varphi^h. \end{cases} \tag{2.1.7}$$

Here u^h stands for the infinite unknown vector $\{u^h_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$, $u_{\mathbf{j}}(t)$ being the approximation of the solution at the node $x_{\mathbf{j}} = \mathbf{j}h$, and Δ_h is the classical second order finite difference approximation of Δ:

$$(\Delta_h u^h)_{\mathbf{j}} = h^{-2}\sum_{k=1}^{d}(u^h_{\mathbf{j}+\mathbf{e}_k} + u^h_{\mathbf{j}-\mathbf{e}_k} - 2u^h_{\mathbf{j}}).$$

This scheme satisfies the classical properties of consistency and stability which imply L^2-convergence. In fact stability holds because the discrete l^2-norm does not increase under the flow (2.1.7):

$$\frac{d}{dt}\left(h^d\sum_{\mathbf{j}\in\mathbb{Z}^d}|u^h_{\mathbf{j}}(t)|^2\right) \leq 0.$$

In the following we are concerned with finer properties of the semidiscrete solution: the long time behavior or, more precisely, the spatial shape for large time. Of course we are interested in obtaining estimates on the norms of the semidiscrete solutions uniformly with respect to the mesh size h.

The main tool in our work is the semidiscrete Fourier transform (SDFT). Appendix A contains the basic properties of this transform which will be used along this chapter. For a fine analysis of the semidiscrete transform one can look into the book [58]. In the context of the numerical approximations of PDE good references are [128] and [131].

By means of SDFT we compute the solutions of (2.1.7) in a similar way as in the continuous case, writing them as a convolution of a fundamental solution (i.e. with initial datum $(\delta_0^h)_j = (1/h)\delta_{0j}$, δ_{ij} being the Knoneker's symbol) with the initial data. This allows us to obtain decay rates of the solution in different $l^q - l^p$ norms. As we shall see, all the estimates are uniform with respect to the mesh size h.

In the case of transport equations the l^p-stability has been studied by Brenner and Thomée [14] and Threfethen [130] using similar techniques based on Fourier analysis.

Concerning the fundamental solutions of equation (2.1.7), we prove that they are related to the modified Bessel function (see [99] for a survey on special functions).

Finally we introduce the moments of order $k \geq 0$ of a discrete function and obtain the first m terms, $m \geq 1$, in the asymptotic expansion of the semidiscrete solution in different norms. In contrast with Theorem 2.1.1 our result is valid only for the initial data in the weighted space $l^1(h\mathbb{Z}^d, |x|^{m+1})$. This is due to technical reasons and we expect these results to hold for initial data in $l^p(h\mathbb{Z}^d, |x|^{m+1})$, $p > 1$.

2.2. Long time behaviour of the solutions

In this section we explain how to obtain the decay rates for the solutions of the semidiscrete heat equation (2.1.7). The main result is the following Theorem:

Theorem 2.2.1. *Let $1 \leq q \leq p \leq \infty$. Then there is a positive constant $c(p,q)$ such that all the solutions of (2.1.7) satisfy*

$$\|u^h(t)\|_{l^p(h\mathbb{Z}^d)} \leq c(p,q)t^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}\|\varphi^h\|_{l^q(h\mathbb{Z}^d)} \tag{2.2.1}$$

for all $t > 0$, uniformly on $h > 0$.

This kind of estimates has been analyzed by Davies [41] in a more general setting of abstract heat-like equations. The decay properties (2.2.1) still hold for continuous time Markov chains and to the second order difference operators associated with random walks. In that case, the above estimates are obtained by energy methods, which reduce the $l^q - l^p$ decay of heat operators to logarithmic inequalities.

Here we give a proof which relies on the use of SDFT.

We first observe that the parameter h can be neglected in (2.2.1). A scaling argument shows that $u^h(t) = u^1(t/h^2)$. This reduces all the norm estimates (2.2.1) to the case $h = 1$. Then let us consider the case $h = 1$ and study the heat equation on the lattice $\mathbb{Z}^d$.

We are in the setting of [41]. We present the general results as they are stated in [41] and apply them to our case: *the heat equation on the lattice $\mathbb{Z}^d$.*

Let us consider Ω a locally compact, second countable Hausdorff space and dx a Borel measure on Ω. Also let H be a positive operator on $L^2(\Omega, dx)$, i.e.

$$\langle Hf, f\rangle \geq 0,\ \forall\ f \in L^2(\Omega, dx),$$

and the quadratic form

$$Q(f) = \langle H^{1/2}f, H^{1/2}f\rangle.$$

Under additional hypothesis on H, e^{-tH} is positivity preserving and a contraction on $L^p(\Omega, dx)$ for all $1 \leq p \leq \infty$ and $t \geq 0$. To obtain the positivity preserving property it is sufficient to assume that the quadratic form Q satisfies (Th. 1.3.2 ii, p.13, [41])

$$Q(|u|) \leq Q(u), \; u \in D(H^{1/2}). \tag{2.2.2}$$

The $L^p(\Omega, dx)$ contraction property holds if the quadratic form Q satisfies

$$Q(g) \leq Q(f) \tag{2.2.3}$$

for all $f, g \in D(H^{1/2})$ satisfying $|g(x)| \leq |f(x)|$ for all $x \in \Omega$ and $|g(x) - g(y)| \leq |f(x) - f(y)|$ for all $x, y \in \Omega$.

In our case we choose $\Omega = \mathbb{Z}^d$, $H = -\Delta_1$ and dx the counting measure. The bilinear form $Q(f)$ is then defined as

$$Q(f) = \sum_{k=1}^{d} \sum_{\mathbf{j} \in \mathbb{Z}^d} |f_{\mathbf{j}+\mathbf{e}_k} - f_{\mathbf{j}}|^2.$$

In this case it is easy to check that Q satisfies (2.2.2) and (2.2.3). Thus the operator e^{-tH} is positivity preserving and satisfies the contraction property

$$\|e^{tH} f\|_{l^p(\mathbb{Z}^d)} \leq \|f\|_{l^p(\mathbb{Z}^d)}, \; t > 0.$$

In the terminology of [41], e^{-tH} is called ultracontractive if e^{-tH} is bounded from $L^2(\Omega, dx)$ to $L^\infty(\Omega, dx)$ for all $t > 0$. The author proves the equivalence between the ultracontractivity of the operator e^{-tH} and the existence of Sobolev inequalities in the measure space (Ω, dx). For example a bound of the form

$$\|e^{-tH} f\|_{L^\infty(\Omega,dx)} \leq c_1 t^{-\mu/4} \|f\|_{L^2(\Omega,dx)}, \; \forall \; t > 0, \; \forall \; f \in L^2(\Omega, dx) \tag{2.2.4}$$

is equivalent, in the case $\mu > 2$, to the inequality

$$\|f\|^2_{L^{2\mu/(\mu-2)}(\Omega,dx)} \leq c_2 Q(f), \; \forall \; f \in D(H^{1/2}). \tag{2.2.5}$$

In our case for any $d \geq 3$ a bound of the form

$$\|e^{t\Delta_1}\|_{l^\infty(\mathbb{Z}^d)} \leq c_1 t^{-d/4} \|f\|_2, \; \forall \; t > 0, \; \forall \; f \in L^2(\mathbb{Z}^d)$$

is equivalent to the Sobolev inequality

$$\|f\|_{L^{2d/(d-2)}(\mathbb{Z}^d)} \leq c_2 \|\nabla_1^+ f\|_{l^2(\mathbb{Z}^d)},$$

which holds for any function $f \in l^2(\mathbb{Z}^d)$.

According [41] there is another approach to ultracontractive estimates which is even more direct. Unlike the above condition (2.2.5) it does not require $\mu > 2$. More precisely, the ultracontractivity property (2.2.4) is equivalent with the following inequality:

$$\|f\|^{2+4/\mu}_{L^2(\Omega,dx)} \leq c_2 Q(f) \|f\|^{4/\mu}_{L^1(\Omega,dx)},$$

for some constant $c_2 < \infty$ and all $0 \leq f \in D(H^{1/2}) \cap L^1(\Omega, dx)$.

For $\Omega = \mathbb{Z}^d$ and $\mu = d$, this inequality reads:

$$\|f\|_{l^2(\mathbb{Z}^d)}^{2+4/d} \leq c_2 \|\nabla_1^+ f\|_{l^2(\mathbb{Z}^d)}^2 \|f\|_{l^1(\mathbb{Z}^d)}^{4/d},$$

which holds for all $f \in l^1(\mathbb{Z}^d)$. This analysis shows that the semigroup generated by the numerical scheme (2.1.7) in the case $h = 1$ is ultracontractive:

$$\|e^{t\Delta} f\|_{l^\infty(\mathbb{Z}^d)} \leq c_3 t^{-d/4} \|f\|_{l^2(\mathbb{Z}^d)}.$$

By duality we get

$$\|e^{t\Delta} f\|_{l^2(\mathbb{Z}^d)} \leq c_4 t^{-d/4} \|f\|_{l^1(\mathbb{Z}^d)}.$$

These results, the $l^p(\mathbb{Z}^d)$-contraction property and classical interpolation show that for any $1 \leq q \leq p \leq \infty$ the following decay rate holds for some constant $c(p,q)$, uniformly on $h > 0$:

$$\|e^{t\Delta_1} f\|_{l^p(\mathbb{Z}^d)} \leq c(p,q) t^{-d/2(1/q-1/p)} \|f\|_{l^q(\mathbb{Z}^d)}.$$

2.3. Fourier analysis of the semidiscrete scheme

The previous analysis essentially reduces (by energy methods, see [41], Ch. 2) the long time behaviour of the solutions of (2.1.7) to classical Sobolev inequalities. In this section we obtain the same results by Fourier analysis techniques. This analysis will be used in the following chapters to study the long time behaviour of the approximated solutions for the linear Schrödinger equation.

In this section we make use of the SDFT to analyze the decay rates of solutions of (2.1.7). Taking the SDFT in (2.1.7) we obtain that $\widehat{u}^h$ satisfies the following ODE with ξ as a parameter:

$$\begin{cases} \dfrac{d\widehat{u}^h}{dt}(t,\xi) = -\dfrac{4}{h^2}\displaystyle\sum_{k=1}^d \sin^2\left(\dfrac{\xi_k h}{2}\right)\widehat{u}^h(t,\xi), & t>0, \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d, \\ \widehat{u}^h(0,\xi) = \widehat{\varphi}^h(\xi), & \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d. \end{cases}$$

In the Fourier space the solution $\widehat{u}^h$ can be written as

$$\widehat{u}^h(t,\xi) = e^{-tp_h(\xi)}\widehat{\varphi}^h(\xi),\ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d,\ t > 0, \tag{2.3.1}$$

where the symbol $p_h : [-\pi/h, \pi/h]^d \to \mathbb{R}$ is defined by

$$p_h(\xi) = \frac{4}{h^2}\sum_{k=1}^d \sin^2\left(\frac{\xi_k h}{2}\right). \tag{2.3.2}$$

Observe that the new symbol differs from the continuous one: $|\xi|^2$. The two symbols are comparable on the set $[-\pi/h, \pi/h]^d$, $p_h(\xi) \sim |\xi|^2$ in the sense that there exist two positive constants c_1 and c_2, independent of the mesh size h, such that

$$c_1|\xi|^2 \leq p_h(\xi) \leq c_2|\xi|^2, \forall\, \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d.$$

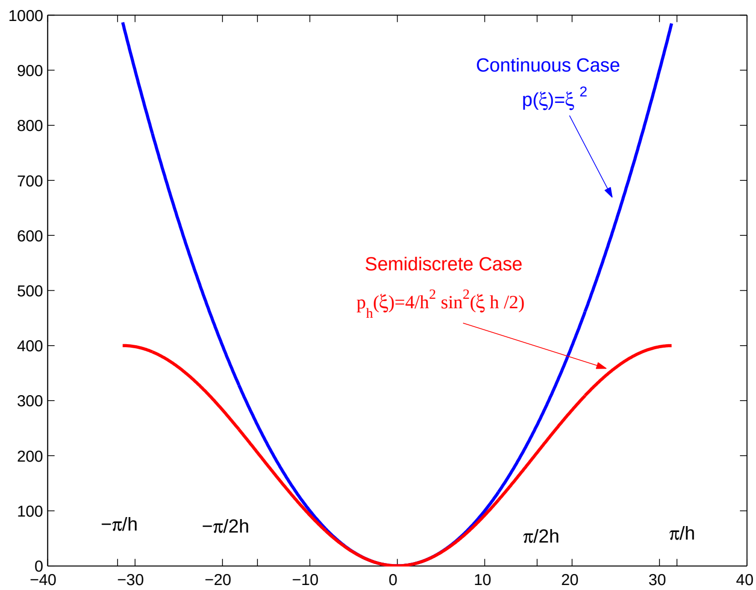

Figure 2.1: The two symbols in dimension one

By (2.3.1) we obtain the explicit solution of equation (2.1.7) as convolution between the fundamental solution $K_t^{d,h}$ and the initial datum:

$$u^h(t) = K_t^{d,h} * \varphi^h. \tag{2.3.3}$$

Here, the fundamental solution is given by the inverse SDFT of the function $e^{-tp_h(\xi)}$:

$$(K_t^{d,h})_{\mathbf{j}} = \frac{1}{(2\pi)^d} \int_{[-\pi/h,\pi/h]^d} e^{-tp_h(\xi)} e^{i\mathbf{j}\cdot\xi h} d\xi,\ \mathbf{j} \in \mathbb{Z}^d. \tag{2.3.4}$$

We point out that for any $\mathbf{j} = (j_1, j_2, ..., j_d) \in \mathbb{Z}^d$ the fundamental solution $K_t^{d,h}$ can be written as the product of one-dimensional kernels $K_t^{1,h}$ as follows

$$(K_t^{d,h})_{\mathbf{j}} = \prod_{k=1}^{d} (K_t^{1,h})_{j_k}. \tag{2.3.5}$$

This separation of variable property helps us to derive $l^p(h\mathbb{Z}^d)$-estimates for $K_t^{d,h}$ in terms of the one-dimensional kernel $K_t^{1,h}$. Using Fourier analysis techniques we obtain in the following theorem the decay rates of the kernel $K_t^{1,h}$.

Theorem 2.3.1. *Let $p \in [1, \infty]$. Then there is a positive constant $c(p)$ such that*

$$\|K_t^{d,h}\|_{l^p(h\mathbb{Z}^d)} \leq c(p) t^{-\frac{d}{2}(1-\frac{1}{p})} \tag{2.3.6}$$

holds for all positive time t, uniformly on $h > 0$.

Proof of Theorem 2.2.1. The proof consists in writing the solution of equation (2.1.7) in convolution form $u^h(t) = K_t^{d,h} * \varphi^h$ and to apply Young's inequality:

$$\|u^h(t)\|_{l^p(h\mathbb{Z}^d)} \leq \|K_t^{d,h}\|_{l^r(h\mathbb{Z}^d)} \|\varphi^h\|_{l^q(h\mathbb{Z}^d)}, \tag{2.3.7}$$

where $1/p = 1/r + 1/q - 1$. Theorem 2.3.1 shows that

$$\|K_t^{d,h}\|_{l^r(h\mathbb{Z}^d)} \le c(r)t^{-\frac{d}{2}(1-\frac{1}{r})} \tag{2.3.8}$$

for all $t > 0$ and $h > 0$. Hence by (2.3.7) and (2.3.8) we obtain

$$\|u^h(t)\|_{l^p(h\mathbb{Z}^d)} \le c(p,q)t^{-\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}\|\varphi^h\|_{l^q(h\mathbb{Z}^d)}$$

and this finishes the proof.

□

Therefore, our next task is to prove Theorem 2.3.1.

2.3.1. Kernel estimates

In this section we study the long time behavior of the fundamental solutions of equation (2.1.7). We will reduce the proof of (2.3.6) to the one-dimensional case and $h = 1$.

The analysis uses the properties of band limited functions. The band-limited functions are those with Fourier transform supported in some cube $[-a, a] = \prod_{k=1}^{d}[-a_k, a_k]$. In our case $a = (\pi, \dots, \pi) \in \mathbb{R}^d$. The relation between the norms of band limited functions and the norms of its Fourier series has been analyzed in [102] (see also [117], Chapter 4, p. 99, for a different approach).

Proof of Theorem 2.3.1

First we remark that it is sufficient to consider the cases $p = 1$ and $p = \infty$, since the other cases follow by the Hölder inequality. In the following, using separation of variables we reduce the proof to the one-dimensional case.

As we said before the solution of the d-dimensional case can be written as the product of 1-dimensional kernels:

$$(K_t^{d,h})_{\mathbf{j}} = \prod_{k=1}^{d}(K_t^{1,h})_{j_k},\ \mathbf{j} = (j_1, j_2, \dots, j_d) \in \mathbb{Z}^d.$$

This is easily obtained from (2.3.4) by writing $\mathbf{j} = (j_1, \dots, j_d)$ and $\xi = (\xi_1, \dots, \xi_d)$. As a consequence

$$\|K_t^{d,h}\|_{l^p(h\mathbb{Z}^d)} = (\|K_t^{1,h}\|_{l^p(h\mathbb{Z})})^d.$$

Then it is sufficient to consider the one-dimensional case. To do that, we prove that $K_t^{1,h}$, given by

$$(K_t^{1,h}) = \frac{1}{2\pi}\int_{-\pi/h}^{\pi/h} e^{-\frac{4t}{h^2}\sin^2\frac{\xi h}{2}} e^{ijh\xi} d\xi,\ j \in \mathbb{Z}, \tag{2.3.9}$$

satisfies

$$\|K_t^{1,h}\|_{l^1(h\mathbb{Z})} \le c_1 \quad \text{and} \quad \|K_t^{1,h}\|_{l^\infty(h\mathbb{Z})} \le c_2 t^{-1/2} \quad \text{for all} \quad t > 0, \tag{2.3.10}$$

for some positive constants c_1 and c_2.
We point out that a change of variables in (2.3.9) implies

$$(K_t^{1,h})_j = \frac{1}{h}(K_{t/h^2}^{1,1})_j,\ j \in \mathbb{Z}.$$

The scaling argument is as follows

$$\|K_t^{1,h}\|_{l^1(h\mathbb{Z})} = h\|K_t^{1,h}\|_{l^1(\mathbb{Z})} = \left\|K_{t/h^2}^{1,1}\right\|_{l^1(\mathbb{Z})}.$$

Thus, inequalities (2.3.10) hold uniformly on $h > 0$, provided that

$$\|K_t^{1,1}\|_{l^1(\mathbb{Z})} \leq c_1 \quad \text{and} \quad \|K_t^{1,1}\|_{l^\infty(h\mathbb{Z})} \leq c_2 t^{-1/2} \tag{2.3.11}$$

for all $t > 0$ and some positive constants c_1 and c_2. The decay for $p = \infty$ easily follows by the rough estimate:

$$\begin{aligned} |(K_t^{1,1})_j| &= \frac{1}{2\pi}\left|\int_{-\pi}^{\pi} e^{-4t\sin^2\frac{\xi}{2}} e^{ij\xi} d\xi\right| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-4t\sin^2\frac{\xi}{2}} d\xi \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-4t\left(\frac{2\xi}{\pi}\right)^2} d\xi = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-\frac{16t\xi^2}{\pi^2}} d\xi \leq c_2 t^{-1/2}. \end{aligned}$$

The case $p = 1$ is more tricky and for that we make use of a band-limited interpolator to reduce the estimates of the discrete $l^1(\mathbb{Z})$-norm to the continuous $L^1(\mathbb{R})$-one.

Let us define the function $K_*^1(t, x) : \mathbb{R} \to \mathbb{R}$ by

$$K_*^1(t, x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-4t\sin^2\frac{\xi}{2}} e^{ix\xi} d\xi.$$

Observe that $K_*^1(t, j) = (K_t^1)_j$ for all $j \in \mathbb{Z}$. Also its Fourier transform in the x variable $\widehat{K}_t^1$, is supported on $[-\pi, \pi]$ and

$$\widehat{K}_*^1(t, \xi) = \widehat{K}_t^{1,1}(\xi), \quad \forall\, \xi \in [-\pi, \pi].$$

Using the properties of the band limited interpolator $K_*^1(t, \cdot)$ (see Appendix A) we get

$$\|K_t^{1,1}\|_{l^1(\mathbb{Z})} \leq c\|K_*^1(t, \cdot)\|_{L^1(\mathbb{R})},$$

so it is sufficient to prove that

$$\|K_*^1(t, \cdot)\|_{L^1(\mathbb{R})} \leq c_1. \tag{2.3.12}$$

We recall the Carlson-Beurling inequality (cf. [14], [9]):

$$\|\widehat{a}\|_{L^1(\mathbb{R})} \leq (2\|a\|_{L^2(\mathbb{R})}\|a'\|_{L^2(\mathbb{R})})^{1/2} \tag{2.3.13}$$

which holds for all functions $a \in H^1(\mathbb{R})$.

This inequality applied to $K_*^1(t)$ yields

$$\|K_*^1(t)\|_{L^1(\mathbb{R})} \leq \left(2\|\widehat{K}_*^1(t)\|_{L^2_\xi(\mathbb{R})}\|\partial_\xi \widehat{K}_*^1(t)\|_{L^2_\xi(\mathbb{R})}\right)^{1/2}. \tag{2.3.14}$$

The explicit expression of $\widehat{K}_*^1$:

$$\widehat{K}_*^1(t, \xi) = \frac{1}{2\pi} e^{-4t\sin^2\frac{\xi}{2}} \chi_{[-\pi,\pi]}$$

gives us

$$\partial_\xi(\widehat{K}_*^1(t, \cdot))(\xi) = -\frac{t\sin\xi}{\pi} e^{-4t\sin^2\frac{\xi}{2}} \chi_{(-\pi,\pi)}.$$

These identities allow us to estimate the L^2-norm of $\widehat{K}_*^{1,1}(t,\cdot)$ and its derivative. In fact

$$\begin{aligned}\|\widehat{K}_*^1(t,\cdot)\|_{L^2(\mathbb{R})} &\lesssim \left(\int_{-\pi}^{\pi} e^{-8t\sin^2\frac{\xi}{2}}d\xi\right)^{1/2} \lesssim \left(\int_{-\pi}^{\pi} e^{-8t(\frac{2}{\pi}\xi)^2}d\xi\right)^{1/2}\\ &\lesssim t^{-1/4}\end{aligned}$$

and

$$\begin{aligned}\|\partial_\xi\widehat{K}_*^1(t)\|_{L^2(\mathbb{R})} &\lesssim \left(\int_{-\pi}^{\pi} t^2\sin^2\xi e^{-8t\sin^2\frac{\xi}{2}}d\xi\right)^{1/2} \lesssim \left(\int_{-\pi}^{\pi} t^2\xi^2 e^{-8t(\frac{2}{\pi}\xi)^2}d\xi\right)^{1/2}\\ &\lesssim \left(\int_{-\pi}^{\pi} t\eta e^{-\eta^2}\frac{d\eta}{t^{1/2}}\right)^{1/2} \lesssim t^{1/4}.\end{aligned}$$

These inequalities together with (2.3.14) imply (2.3.12). Then by Hölder's Inequality

$$\begin{aligned}\|K_t^{1,1}\|_{l^p(h\mathbb{Z})} &\leq \|K_t^{1,1}\|_{l^1(h\mathbb{Z})}^{1/p}\|K_t^{1,1}\|_{l^\infty(h\mathbb{Z})}^{1-1/p} \leq c_1^{1/p}(c_2t^{-1/2})^{1-1/p}\\ &\leq c(p)t^{-\frac{1}{2}\left(1-\frac{1}{p}\right)},\end{aligned}$$

which finishes the proof. □

2.3.2. Relation between $K_t^{d,h}$ and the Modified Bessel Function

In this section we relate the fundamental solution of the semidiscrete heat equation with the modified Bessel function. The modified Bessel function $I_\nu(z)$ is defined for all values of ν and z, other than $z=0$, by the series

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{s=0}^{\infty}\frac{(z^2/4)^s}{s!\Gamma(\nu+s+1)}.$$

We recall a few properties of this function (cf. [99], Ch.2, p. 60 and Ch. 7, p. 251):

Theorem 2.3.2. *([99], Ch. 2 and 7) The modified Bessel function $I_\nu(x)$ has the following properties:*

1. *For all real numbers ν,x, $I_\nu(x)$ satisfies*

$$I_\nu(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\sigma}\cos(\nu\sigma)d\sigma - \frac{\sin\nu\pi}{\pi}\int_0^\infty e^{-x\cosh t-\nu t}dt, \tag{2.3.15}$$

2. *For all integers ν and for all $x\in\mathbb{R}$*

$$I_\nu(x) = \frac{1}{\pi}\int_0^\pi e^{x\cos\sigma}\cos(\nu\sigma)d\sigma, \tag{2.3.16}$$

3. *For all integers ν and $x\in\mathbb{R}$*

$$I_\nu(x) = I_{-\nu}(x), \tag{2.3.17}$$

4. *If $x>0$ is fixed, $I_\nu(x)$ is positive and decreasing for $\nu\in[0,\infty)$*

The following Theorem gives the relation between the kernel $K_t^{d,h}$ and the modified Bessel function.

Theorem 2.3.3. *For any $t \in \mathbb{R}$ and $\mathbf{j} = (j_1, j_2, \dots, j_d) \in \mathbb{R}^d$ the following holds*

$$(K_t^{d,h})_{\mathbf{j}} = \left(\frac{\exp(-\frac{2t}{h^2})}{\pi h} \right)^d \prod_{k=1}^{d} I_{j_k} \left(\frac{2t}{h^2} \right). \tag{2.3.18}$$

Remark 2.3.1. *The main ingredient in the proof of this identity is property (2.3.16) of the modified Bessel functions.*

Theorem 2.3.3 allows us to use the well known properties of the modified Bessel function to derive the basic properties of the semidiscrete kernel $K_t^{d,h}$.

Theorem 2.3.4. *Let $t > 0$ and $h > 0$. Then*
i) For any $\mathbf{j} \in \mathbb{Z}^d$, the kernel $(K_t^{d,h})_{\mathbf{j}}$ is positive.
ii) The map $\mathbb{Z} \ni j \mapsto (K_t^{1,h})_j$ is increasing for $j \leq 0$ and decreasing for $j \geq 0$.
iii) For any $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{Z}^d$ and $\mathbf{b} = (b_1, b_2, \dots, b_d) \in \mathbb{Z}^d$ with

$$|a_1| \leq |b_1|, |a_2| \leq |b_2|, \dots, |a_d| \leq |b_d|$$

the following holds

$$(K_t^{d,h})_{\mathbf{a}} \geq (K_t^{d,h})_{\mathbf{b}}. \tag{2.3.19}$$

Remark 2.3.2. *The positivity of the kernel $K_t^{d,h}$ can be obtained from the work of Davies [41] explained in Section 2.2 (see conditions (2.2.2) and (2.2.3) and their applications to heat equation on the lattice $\mathbb{Z}^d$). Choosing the discrete Dirac delta δ_0 as initial datum in (2.1.7), the positivity preserving property of the heat operator $e^{-t\Delta_h}$ (the maximum principle) gives us the positivity of the fundamental solution $K_t^{d,h}$.*

Proof of Theorem 2.3.3. First we analyze the one-dimensional case. With a simple change of variables we get for any $j \in \mathbb{Z}$

$$\begin{aligned}(K_t^h)_j =& \frac{1}{2\pi h} \int_{-\pi}^{\pi} e^{-\frac{4t}{h^2}\sin^2(\frac{\sigma}{2})} e^{ij\sigma} d\sigma = \frac{1}{\pi h} \int_0^{\pi} e^{-\frac{4t}{h^2}\sin^2(\frac{\sigma}{2})} \cos(j\sigma) d\sigma \\ =& \frac{\exp(-\frac{2t}{h^2})}{\pi h} \int_0^{\pi} e^{\frac{2t}{h^2}\cos(\sigma)} \cos(j\sigma) d\sigma = \frac{\exp(-\frac{2t}{h^2})}{\pi h} I_j \left(\frac{2t}{h^2} \right).\end{aligned} \tag{2.3.20}$$

Using property (2.3.5) of the d-dimensional kernel we obtain

$$(K_t^{d,h})_{\mathbf{j}} = \prod_{k=1}^{d} (K_t^{1,h})_{j_k} = \left(\frac{\exp(-\frac{2t}{h^2})}{\pi h} \right)^d \prod_{k=1}^{d} I_{j_k} \left(\frac{2t}{h^2} \right).$$

□

Proof of Theorem 2.3.4. In view of (2.3.18), Theorem 2.3.2 gives us the positivity of the kernel. Also property #3 of Theorem 2.3.2 gives us the second statement of Theorem 2.3.4.

We remark that in order to prove

$$(K_t^{d,h})_{\mathbf{a}} \leq (K_t^{d,h})_{\mathbf{b}}$$

it is sufficient to show that $|a_k| \le |b_k|$, $k = 1, .., d$ implies

$$I_{a_k}\left(\frac{2t}{h^2}\right) \le I_{b_k}\left(\frac{2t}{h^2}\right).$$

The case when both a_k and b_k have the same sign easily follows from (2.3.17) and the monotonicity property #4 of Theorem 2.3.2. Suppose that $a_k \le 0 \le b_k$. Then $0 \le -a_k \le b_k$ and

$$I_{a_k}\left(\frac{2t}{h^2}\right) = I_{-a_k}\left(\frac{2t}{h^2}\right) \le I_{b_k}\left(\frac{2t}{h^2}\right)$$

holds as a simple consequence of (2.3.16) and (2.3.17). □

2.4. Asymptotic expansion of $u^h(t)$

In this section we obtain the asymptotic expansion as $t \to \infty$ for the solution $u^h(t)$ of equation (2.1.7). For that we define the moments of the discrete function φ^h by

$$M_\alpha \varphi \stackrel{def}{=} h^d \sum_{\mathbf{j} \in \mathbb{Z}^d} \frac{(\mathbf{j}h)^\alpha}{\alpha!} \varphi^h_{\mathbf{j}}. \tag{2.4.1}$$

We introduce the band-limited interpolator of the semidiscrete Kernel $K_t^{d,h}$ by

$$K_*^{d,h}(t,x) = \sum_{\mathbf{n} \in \mathbb{Z}^d} (K_t^{d,h})_{\mathbf{n}} \Psi^h_{\mathbf{n}}(x)$$

where $\Psi^h_{\mathbf{n}}(x) = \Psi\left(\frac{x - h\mathbf{n}}{h}\right)$ with

$$\Psi(x) = \prod_{k=1}^{d} \frac{\sin(\pi x_k)}{\pi x_k},\ x = (x_1, \ldots, x_d) \in \mathbb{R}^d. \tag{2.4.2}$$

The main result of this section is contained in the following Theorem:

Theorem 2.4.1. *Let $m \in \mathbb{N}$ and $p \ge 1$. There exists a positive constant $c(m,p)$ such that*

$$\left\| u^h(t) - \sum_{|\alpha| \le m} \frac{(-1)^{|\alpha|}}{\alpha!} (M_\alpha \varphi^h) \partial^\alpha K_*^{d,h}(t, \cdot h) \right\|_{l^p(h\mathbb{Z}^d)} \le c(m,p) t^{-\frac{m+1}{2}} t^{-\frac{d}{2}(1-\frac{1}{p})} \|\varphi^h\|_{l^1(h\mathbb{Z}^d, |x|^{m+1})} \tag{2.4.3}$$

for all $\varphi^h \in l^1(h\mathbb{Z}^d, |x|^{m+1})$ and $t > 0$, uniformly in $h > 0$.

Remark 2.4.1. *The condition on φ^h to belong to $l^1(h\mathbb{Z}^d, |x|^{m+1})$ could be merely technical. Similar results can be expected for initial data in $l^q(h\mathbb{Z}^d, |x|^{m+1})$ with a different decay rate.*

Remark 2.4.2. *Similar results can be obtained differently. Indeed, one could consider the interpolator Iu^h of u^h and then decompose $I\varphi^h$ on the Dirac basis as in [43]. In this way one could obtain a decomposition, similar to (2.4.3), but involving the continuous moments of $I\varphi^h$ instead of the discrete ones.*

In order to prove this Theorem we will make use of the following Lemma, which gives estimates on the functions $K_*^{d,h}$ and their derivatives. These estimates hold uniformly with respect to the mesh size h.

Lemma 2.4.1. *Let $\alpha = (\alpha_1, \dots, \alpha_d)$ be a multi-index and $1 \le p \le \infty$. Then there is a positive constant $c(\alpha, p)$ such that*

$$\|\partial^\alpha K_*^{d,h}(t,\cdot)\|_{L^p(\mathbb{R}^d)} \le c(\alpha,p) t^{-\frac{|\alpha|}{2}-\frac{d}{2}(1-\frac{1}{p})} \tag{2.4.4}$$

for all $t > 0$ and $h > 0$.

We postpone the proof of this lemma and proceed with the proof of Theorem 2.4.1.

Proof of Theorem 2.4.1. First a scaling argument reduces the proof to the case $h = 1$. We will consider the cases $p = 1$ and $p = \infty$, the other ones being a consequence of the interpolation between these two. The solution $u^1(t)$ of equation (2.1.7) is given by convolution of the fundamental solutions $K_t^{d,1}$ with the initial datum φ^1:

$$u^1_{\mathbf{j}}(t) = (K_t^{d,1} * \varphi^1)_{\mathbf{j}} = \sum_{\mathbf{n}\in\mathbb{Z}^d} (K_t^{d,1})_{\mathbf{j}-\mathbf{n}} \varphi^1_{\mathbf{n}}. \tag{2.4.5}$$

Using the fact that the band-limited interpolator $K_*^{d,1}$ satisfies

$$K_*^{d,1}(t,\mathbf{j}) = (K_t^{d,1})_{\mathbf{j}} \tag{2.4.6}$$

we obtain that

$$u^1_{\mathbf{j}}(t) = \sum_{\mathbf{n}\in\mathbb{Z}^d} K_*^{d,1}(t,\mathbf{j}-\mathbf{n})\varphi^1_{\mathbf{n}}. \tag{2.4.7}$$

Let us introduce the sequence $\{a_{\mathbf{j}}(t)\}_{\mathbf{j}\in\mathbb{Z}^d}$ as follows

$$\begin{aligned}
a_{\mathbf{j}}(t) &\stackrel{not}{=} \left(u^1(t) - \sum_{|\alpha|\le m} \frac{(-1)^{|\alpha|}}{\alpha!} (M_\alpha \varphi^1) \partial^\alpha K_*^{d,1}(t,\cdot) \right)_{\mathbf{j}} \\
&= u^1_{\mathbf{j}}(t) - \sum_{|\alpha|\le m} \frac{(-1)^{|\alpha|}}{\alpha!} (M_\alpha \varphi^1) \partial^\alpha K_*^{d,1}(t,\mathbf{j}) \\
&= \sum_{\mathbf{n}\in\mathbb{Z}^d} K_*^{d,1}(t,\mathbf{j}-\mathbf{n})\varphi^1_{\mathbf{n}} - \sum_{|\alpha|\le m} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\sum_{\mathbf{n}\in\mathbb{Z}^d} \mathbf{n}^\alpha \varphi^1_{\mathbf{n}} \right) (\partial^\alpha K_*^{d,1})(t,\mathbf{j}) \\
&= \sum_{\mathbf{n}\in\mathbb{Z}^d} \varphi^1_{\mathbf{n}} \left[K_*^{d,1}(t,\mathbf{j}-\mathbf{n}) - \sum_{|\alpha|\le m} \frac{(-\mathbf{n})^\alpha}{\alpha!} (\partial^\alpha K_*^{d,1})(t,\mathbf{j}) \right].
\end{aligned}$$

The sequence $\{a_{\mathbf{j}}(t)\}_{\mathbf{j}\in\mathbb{Z}^d}$ is exactly the one involved in the right side hand of (2.4.3). Thus it remains to prove that

$$\sup_{\mathbf{j}\in\mathbb{Z}^d} |a_{\mathbf{j}}(t)| \lesssim t^{-\frac{m+1}{2}-\frac{d}{2}} \|\varphi^1\|_{l^1(\mathbb{Z}^d,|x|^{m+1})} \tag{2.4.8}$$

and

$$\sum_{\mathbf{j}\in\mathbb{Z}^d} |a_{\mathbf{j}}(t)| \lesssim t^{-\frac{m+1}{2}} \|\varphi^1\|_{l^1(\mathbb{Z}^d,|x|^{m+1})}. \tag{2.4.9}$$

The first case corresponds to $p=\infty$ and the second one to $p=1$.

The Taylor formula

$$f(x) - \sum_{|\alpha|\le m} \frac{D^\alpha f(x_0)}{\alpha!}(x-x_0)^\alpha = \int_0^1 \frac{(1-s)^m}{m!} \sum_{|\alpha|=m+1} (D^\alpha f)(x_0+s(x-x_0))(x-x_0)^\alpha ds$$

with $f(\cdot)=K_*^{d,1}(t,\cdot)$, $x=\mathbf{j}-\mathbf{n}$ and $x_0=\mathbf{j}$ yields

$$K_*^{d,1}(t,\mathbf{j}-\mathbf{n}) - \sum_{|\alpha|\le m} \frac{(-\mathbf{n})^\alpha}{\alpha!}(\partial^\alpha K_*^{d,1})(t,\mathbf{j}) = \int_0^1 \frac{(1-s)^m}{m!} \sum_{|\alpha|=m+1} (D^\alpha K_*^{d,1})(t,\mathbf{j}-s\mathbf{n})(-\mathbf{n})^\alpha ds.$$

As a consequence, for any $\mathbf{j}\in\mathbb{Z}^d$, the sequence $a_{\mathbf{j}}(t)$ satisfies

$$|a_{\mathbf{j}}(t)| \lesssim \sum_{\mathbf{n}\in\mathbb{Z}^d} |\varphi^1_{\mathbf{n}}||\mathbf{n}^\alpha| \sum_{|\alpha|=m+1} \int_0^1 |(D^\alpha K_*^{d,1})(t,\mathbf{j}-s\mathbf{n})| ds.$$

Observe that we can compare $|\mathbf{n}^\alpha|$ with $|\mathbf{n}|^{|\alpha|}$ as follows

$$|\mathbf{n}^\alpha| = |n_1^{\alpha_1}\dots n_d^{\alpha_d}| \le (n_1^2+\dots+n_d^2)^{\frac{\alpha_1+\dots+\alpha_d}{2}} = |\mathbf{n}|^{|\alpha|}.$$

This allows us to obtain the following inequality :

$$\begin{aligned} |a_{\mathbf{j}}(t)| &\lesssim \sum_{\mathbf{n}\in\mathbb{Z}^d} |\varphi^1_{\mathbf{n}}||\mathbf{n}|^{m+1} \sum_{|\alpha|=m+1} \int_0^1 |(D^\alpha K_*^{d,1})(t,\mathbf{j}-s\mathbf{n})| ds \\ &\overset{not}{=} \sum_{\mathbf{n}\in\mathbb{Z}^d} |\varphi^1_{\mathbf{n}}||\mathbf{n}|^{m+1} \sum_{|\alpha|=m+1} b^\alpha_{\mathbf{j},\mathbf{n}}(t). \end{aligned} \tag{2.4.10}$$

In order to prove inequality (2.4.8) it is sufficient to show that

$$b^\alpha_{\mathbf{j},\mathbf{n}}(t) \le t^{-\frac{m+1}{2}-\frac{d}{2}}$$

for all multi-indexes α with $|\alpha|=m+1$, $\mathbf{j},\mathbf{n}\in\mathbb{Z}^d$ and $t>0$. By Lemma 2.4.1 we get

$$b_{\mathbf{j},\mathbf{n}} \le \|D^\alpha K_*^{d,1}\|_{L^\infty(\mathbb{R}^d)} \lesssim t^{-\frac{|\alpha|}{2}-\frac{d}{2}} = t^{-\frac{m+1}{2}-\frac{d}{2}},$$

which finishes proof in the first case.

Let us now consider the case $p=1$. We sum over $\mathbf{j}\in\mathbb{Z}^d$ on (2.4.10) and obtain

$$\begin{aligned} \sum_{\mathbf{j}\in\mathbb{Z}^d} |a_{\mathbf{j}}(t)| &\lesssim \sum_{\mathbf{j}\in\mathbb{Z}^d}\sum_{\mathbf{n}\in\mathbb{Z}^d} |\varphi^1_{\mathbf{n}}||\mathbf{n}|^{m+1} \sum_{|\alpha|=m+1} b^\alpha_{\mathbf{j},\mathbf{n}}(t) \\ &= \sum_{\mathbf{n}\in\mathbb{Z}^d} |\varphi^1_{\mathbf{n}}||\mathbf{n}|^{m+1} \sum_{|\alpha|=m+1}\sum_{\mathbf{j}\in\mathbb{Z}^d} b^\alpha_{\mathbf{j},\mathbf{n}}(t). \end{aligned}$$

Observe that it is sufficient to show that

$$\sum_{\mathbf{j}\in\mathbb{Z}^d} b^{\alpha}_{\mathbf{j},\mathbf{n}}(t) \lesssim t^{-\frac{m+1}{2}}$$

for all $\mathbf{n} \in \mathbb{Z}^d$ and for all multi-indexes α with $|\alpha| = m+1$. Using the separation of variables we get for all $\mathbf{j} = (j_1, ..., j_d)$ and $\mathbf{n} = (n_1, ..., n_d)$:

$$b^{\alpha}_{\mathbf{j},\mathbf{n}}(t) = \int_0^1 \prod_{k=1}^{d} |\partial^{\alpha_k} K_*^{1,1}(j_k - sn_k)| ds$$

and

$$\begin{aligned}\sum_{\mathbf{j}\in\mathbb{Z}^d} b^{\alpha}_{\mathbf{j},\mathbf{n}}(t) &= \int_0^1 \prod_{k=1}^{d} \left(\sum_{j_k\in\mathbb{Z}} |\partial^{\alpha_k} K_*^{1,1}(j_k - sn_k)|\right) ds \\ &\leq \sup_{s\in\mathbb{R}} \prod_{k=1}^{d} \left(\sum_{j_k\in\mathbb{Z}} |\partial^{\alpha_k} K_*^{1,1}(j_k - s)|\right).\end{aligned}$$

We prove that each term in the last product is dominated by $t^{-\alpha_k/2}$ and consequently the product will be controlled by $t^{-|\alpha|/2}$. The key point is to show that

$$\sum_{j\in\mathbb{Z}} |\partial^{\alpha_k} K_*^{1,1}(j - s)| \lesssim \int_{\mathbb{R}} |\partial^{\alpha_k} K_*^{1,1}(x - s)| dx = \int_{\mathbb{R}} |\partial^{\alpha_k} K_*^{1,1}(x)| dx \qquad (2.4.11)$$

and finally apply Lemma 2.4.1 to conclude the proof.

In Appendix A we show that for any band limited function f:

$$\sum_{j\in\mathbb{Z}} |f(j)| \lesssim \int_{\mathbb{R}} |f(x)| dx.$$

Clearly the new function $g_s(x) = f(x - s)$, with $s \in \mathbb{R}$ fixed, satisfies $\widehat{g}_s(\xi) = e^{is\xi}\widehat{f}(\xi)$, its Fourier transform being supported in the same band as f. Then

$$\sum_{j\in\mathbb{Z}} |f(j - s)| = \sum_{j\in\mathbb{Z}} |g_s(j)| \lesssim \int_{\mathbb{R}} |g(x)| dx = \int_{\mathbb{R}} |f(x - s)| dx.$$

Applying this argument to the function $f = \partial^{\alpha_k} K_*^{1,1}$ we obtain (2.4.11) and the proof is finished.

□

Proof of Lemma 2.4.1. First observe, by scaling, that

$$K_*^{d,h}(t, x) = K_*^{d,1}\left(\frac{t}{h^2}, \frac{x}{h}\right).$$

This reduces the proof of the Lemma to the case $h = 1$. More than that, for any $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$, using separation of variables we get

$$K_*^{d,1}(t, x) = \prod_{k=1}^{d} K_*^{1,1}(t, x_k)$$

and

$$(\partial^\alpha K_*^{d,1})(t,x) = \prod_{k=1}^{d} (\partial^{\alpha_h} K_*^{1,1})(t,x_k).$$

Taking L^p-norms in the last identity we obtain

$$\|\partial^\alpha K_*^{d,1}(t,\cdot)\|_{L^p(\mathbb{R}^d)} = \prod_{k=1}^{d} \|\partial^{\alpha_k} K_*^{1,1}(t,\cdot)\|_{L^p(\mathbb{R})},$$

reducing the proof to the one-dimensional case. It remains to show that

$$\|\partial^m K_*^{1,1}(t,\cdot)\|_{L^p(\mathbb{R})} \lesssim t^{-\frac{m+1}{2}-\frac{1}{2p}} \tag{2.4.12}$$

for all $m \in \mathbb{N}$ and $t > 0$. By Hölder's inequality it is sufficient to consider (2.4.12) with $p \in \{1, \infty\}$. These cases follow by the same argument as in the proof of Theorem 2.3.1 by using the triangle inequality for $p = \infty$ and the Carlson-Beurling inequality (2.3.13) to the function $x \mapsto (\partial^m K_*^{1,1})(t,x)$ for $p = 1$. □

Chapter 3

Semidiscrete Schemes for the Schrödinger Equation

3.1. Introduction

Let us consider the linear (LSE) and the nonlinear (NLS) Schrödinger Equations

$$\begin{cases} iu_t + \Delta u = 0, \; x \in \mathbb{R}^d, \; t \neq 0, \\ u(0,x) = \varphi(x), \; x \in \mathbb{R}^d, \end{cases} \tag{3.1.1}$$

and

$$\begin{cases} iu_t + \Delta u = |u|^p u, \; x \in \mathbb{R}^d, \; t > 0, \\ u(0,x) = \varphi(x), \; x \in \mathbb{R}^d, \end{cases} \tag{3.1.2}$$

respectively.

Here, for simplicity we choose the semilinear problem (3.1.2). All the results presented in this Chapter can be extended to more general nonlinearities $f(u)$ (see [25], Ch. 4.6, p. 109, for L^2-solutions).

The linear equation is solved by $u(x,t) = S(t)\varphi$, where the free Schrödinger operator $S(t) = e^{it\Delta}$ is given by

$$S(t)\varphi(x) = (4\pi i t)^{-d/2} \int_{\mathbb{R}^d} e^{i|x-y|^2/4t} \varphi(y) dy. \tag{3.1.3}$$

and defines a unitary transformation group in $L^2(\mathbb{R}^d)$.

The nonlinear initial value problem being considered in the entire space $\mathbb{R}^d$, the problem can be conveniently rewritten in the integral form

$$u(t) = S(t)\varphi + i \int_0^t S(t-s)|u(s)|^p u(s) ds. \tag{3.1.4}$$

The existence of a solution for small enough t (local existence) is proved by a fixed point method for (3.1.4), using that, as a result of the dispersion properties of the linear operator, this equation defines a contraction in a suitable Banach space of functions for small enough t. Existence for all time (global existence) holds in the case where the local solutions can be continued for all t, by means of a priori estimates for the norms of the solutions in the corresponding spaces.

The estimates discussed in this section play an important role in the proof of well-posedness of the NLS equation in $H^1(\mathbb{R}^d)$ or $L^2(\mathbb{R}^d)$. They are in fact very general and extend to cases where the operator $i\Delta$ is replaced by any skew-Hermitian operator for which the L^∞-norm of the kernel behaves like $t^{-d/2}$.

The linear semigroup has two important properties, the conservation of the L^2-norm

$$\|u(t)\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)} \tag{3.1.5}$$

and a dispersive estimate:

$$|S(t)\varphi(x)| = |u(t,x)| \leq \frac{1}{(4\pi|t|)^{d/2}}\|\varphi\|_{L^1(\mathbb{R}^d)},\ x \in \mathbb{R}^d,\ t \neq 0. \tag{3.1.6}$$

The first one follows immediately from Plancherel's Theorem and the second one by (3.1.3). Interpolation between these two estimates immediately yields the $L^{p'}$ - L^p estimate

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \leq c(p)|t|^{-d\left(\frac{1}{2}-\frac{1}{p}\right)}\|u_0\|_{L^{p'}(\mathbb{R}^d)} \tag{3.1.7}$$

for $1/p + 1/p' = 1$ and $2 \leq p \leq \infty$.

The Space-Time Estimate

$$\|u\|_{L^{2+4/d}(\mathbb{R},\, L^{2+4/d}(\mathbb{R}^d))} \leq c\|u_0\|_{L^2(\mathbb{R}^d)}, \tag{3.1.8}$$

due to Segal and Strichartz' [121], is deeper. It tells us two important informations. It says that the solutions decay in some sense as t becomes large and that they gain a little bit of spatial integrability for $t > 0$. Since the Schrödinger equation is invariant under the scaling $x \to \lambda x$, $t \to \lambda^2 t$, a simple scaling argument shows that the exponent $2 + 4/d$ in the estimate is the unique possible.

Inequality (3.1.8) was generalized by Ginibre and Velo [50]. They proved an estimate more general than (3.1.8), the *Mixed Space-Time Estimate*

$$\|S(\cdot)\varphi\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \leq C(q,r)\|\varphi\|_{L^2(\mathbb{R})} \tag{3.1.9}$$

for the so-called admissible pairs: $2 \leq r < 2/(d-2)$ and

$$\frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right),\ (q,r,d) \neq (2,\infty,2). \tag{3.1.10}$$

The end-point case $q = 2, r = 2d/(d-2)$ has been finally achieved in [74].

The extension to the inhomogeneous linear Schrödinger equation is due to Yajima [138] and Cazenave and Weissler [27]. Improvements of the Strichartz inequalities are presented in a series of papers by Bourgain [12], Merle and Vega [92] and Moyua, Vargas, and Vega [96].

These properties are not only relevant for a better understanding of the dynamics of the linear system but also to derive well-posedness results for inhomogeneous [121] and nonlinear Schrödinger equations. The nonlinear problem with initial data in $L^2(\mathbb{R}^d)$ has been first analyzed by Tsutsumi [132]. The author proved the global existence of the solutions of (3.1.2) in the subcritical case $p < 4/d$. Also, Cazenave and Weissler [28] prove the local existence in the critical case $p = 4/d$. For H^1-solutions the existence was proved by Baillon, Cazenave and Figueira [4], Lin and Strauss [83], Ginibre and Velo [47] [48], Cazenave [24], and in a more general context by Kato [71] [72].

Typically the dispersive estimates are used when the energy methods fail to provide well posedness of the nonlinear problems. Consequently one has to introduce auxiliary spaces. It is well known by now that for $p < 4/d$ NSE with $L^2(\mathbb{R}^d)$ initial data is locally well posed in $L^\infty(\mathbb{R},\ L^2(\mathbb{R}^d) \cap L^q_{loc}(\mathbb{R},\ L^r(\mathbb{R}^d))$, where (q,r) satisfies (3.1.10).

The Schrödinger equation has another remarkable property: the gain of one half space derivative in $L^2_{x,t}$ (cf. [112], [35], [36] and [75]):

$$\sup_{x_0,R} \frac{1}{R} \int_{B(x_0,R)} \int_{-\infty}^{\infty} |D_x^{1/2} e^{it\Delta}\varphi|^2 dt dx \leq C\|\varphi\|^2_{L^2(\mathbb{R}^d)}. \tag{3.1.11}$$

It has played a crucial role in the study of the nonlinear Schrödinger equation with nonlinearities involving derivatives (see [76]). Also, this type of local smoothing effect has been used to prove the existence a.e. of $\lim_{t\to 0} u(x,t)$ for solutions of the Schrödinger equation with $H^s(\mathbb{R}^d)$, $s > 1/2$, initial data [112], [133].

For other deep results on the Schrödinger equations we refer to [123], [25] and the bibliography in the end of this Thesis.

The goal of this chapter is to develop numerical schemes for the nonlinear Schrödinger equations. In what follows we construct convergent schemes for NSE with low regularity initial data, in $L^2(\mathbb{R}^d)$. We first introduce numerical schemes for LSE. We will analyze whether these numerical approximation schemes have the same dispersive properties, uniformly with respect to the mesh-size h, as in the case of the continuous Schrödinger equation (3.1.1). In particular we analyze whether the decay rate (3.1.6) holds for the solutions of the numerical scheme, uniformly in h. The study of these dispersion properties of the numerical scheme in the linear framework is relevant also for proving their convergence in the nonlinear context. Indeed, since the proof of the well-posedness of the nonlinear Schrödinger equation in the continuous framework requires a subtle use of the dispersion properties, the proof of the convergence of the numerical scheme in the nonlinear context is hopeless if these dispersion properties are not verified at the numerical level.

To explain the necessity of analyzing these properties let us suppose that we proved the global existence of solutions in the space $L^\infty(\mathbb{R},\ l^2(h\mathbb{Z}^d))$ for the nonlinear problem

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = 2|u^h|^2 u^h, \quad t > 0, \\ \qquad\qquad u^h(0) = \varphi^h. \end{cases} \tag{3.1.12}$$

Here u^h stands for the infinite unknown vector $\{u^h_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$, $u_{\mathbf{j}}(t)$ being the approximation of the solution at the node $x_{\mathbf{j}} = \mathbf{j}h$, and Δ_h is the classical second order finite difference approximation of Δ:

$$(\Delta_h u^h)_{\mathbf{j}} = h^{-2} \sum_{k=1}^{d} (u^h_{\mathbf{j}+e_k} + u^h_{\mathbf{j}-e_k} - 2u^h_{\mathbf{j}}).$$

The uniform boundedness of $\{u^h\}_{h>0}$ in $L^\infty(\mathbb{R},\ l^2(h\mathbb{Z}^d))$ does not allow us to prove its convergence towards the solution of the NSE. We recall that, as explained above, in order to prove the well-posedness of NSE we have to introduce an auxiliary space $L^q_{loc}(\mathbb{R},\ L^r(\mathbb{R}^d))$ with suitable q and r. One then needs to analyze whether the solutions of (3.1.12) belong to this auxiliary space $L^q_{loc}(\mathbb{R},\ l^r(h\mathbb{Z}^d))$.

In what follows we will present an approximation of the cubic NSE where we can explicitly compute its solutions. This allows us to analyze whether its solutions remain uniformly bounded in any auxiliary space introduced above.

Let us consider the cubic one-dimensional NSE:

$$iu_t + u_{xx} = 2|u|^2 u. \tag{3.1.13}$$

It is well known that this equation is integrable. There is a class of solutions, the solitons, which have the explicit form:

$$u = (2\Lambda)^{1/2} \operatorname{sech}(\Lambda^{1/2}(x - ct - x_0)) \exp\left(i\left(\frac{c}{2}x + (\Lambda - \frac{c^2}{4})t\right)\right),$$

where Λ is the frequency of the wave, x_0 the initial position of its center and c its speed. Oftentimes the physical applications where the NSE equation arises impose either an explicitly discrete setting. Therefore it is relevant to study discrete forms of the NSE equation as well. The most direct such example is of the form

$$i\partial_t u_n^h + \Delta_h u^h = 2|u_n^h|^2 u_n^h,$$

where n is the index of the spatial lattice and Δ_h is the classical second order finite difference approximation of d^2/dx^2. However, this discrete equation is not integrable. An alternative integrable type of discretization of the NSE with nonlinearity $2|u|^2u$ was proposed in [1] and is accordingly often referred to as the Ablowitz-Ladik NSE of the form:

$$i\partial_t u_n^h + \Delta_h u^h = |u_n^h|^2(u_{n+1}^h + u_{n-1}^h). \tag{3.1.14}$$

Equation (3.1.14) also has explicit standing as well as travelling soliton solutions. We remark that any solution u^h satisfies

$$u_n^h(t) = \frac{1}{h}\, u_n^1\left(\frac{t}{h^2}\right),\ n \in \mathbb{Z},\ t \geq 0.$$

In the case $h = 1$, the explicit solutions of (3.1.14) (cf. [2], p. 84) take the form

$$u_n^1(t) = A\exp(i(an - bt)) \operatorname{sech}(cn - dt)$$

for suitable constants A, a, b, c, d. We will not make precise the constants (for the explicit values we refer to [2]).

This type of solutions u^h are not uniformly bounded in any auxiliary space $L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}))$. A simple rescaling arguments shows that

$$\frac{\|u^h\|_{L^q([0,T],l^r(h\mathbb{Z}))}}{\|u^h(0)\|_{l^2(h\mathbb{Z})}} = h^{\frac{1}{r}+\frac{2}{q}-\frac{1}{2}} \frac{\|u^1\|_{L^q([0,T/h^2],l^r(\mathbb{Z}))}}{\|u^1(0)\|_{l^2(\mathbb{Z})}}.$$

Observe that for any $t > 0$, the behaviour of the $l^r(\mathbb{Z})$-norm is given by:

$$\|u^1(t)\|_{l^r(\mathbb{Z})} \sim \left(\int_{\mathbb{R}} \operatorname{sech}^r(cx - dt)dx\right)^{1/r} = \left(\int_{\mathbb{R}} \operatorname{sech}^r(cx)dx\right)^{1/r}.$$

Thus, for all $T > 0$ and $h > 0$ the solution u^1 satisfies

$$\|u^1\|_{L^q([0,T/h^2],l^r(\mathbb{Z}))} \sim (Th^{-2})^{1/q}.$$

Consequently for any $r > 2$ the solution on the lattice $h\mathbb{Z}$ satisfies:

$$\frac{\|u^h\|_{L^q([0,T],l^r(h\mathbb{Z}))}}{\|u^h(0)\|_{l^2(h\mathbb{Z})}} \sim h^{\frac{1}{r}-\frac{1}{2}} \to \infty,\ h \to 0.$$

This means that, except for the case $r = 2$, the solutions will not be uniformly bonded in any auxiliary space $L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}))$. In the case $r = 2$, the solution belongs to $L^\infty(\mathbb{R},\, l^2(h\mathbb{Z}))$ and then to all $L^q_{loc}(\mathbb{R},\, l^2(h\mathbb{Z}))$ spaces.

However, this does not imply the existence of $\varphi \in L^2(\mathbb{R})$ and $\varphi^h \in l^2(h\mathbb{Z})$ such that $\varphi^h \to \varphi$ in $L^2(\mathbb{R})$ and $\|\varphi^h\|_{L^q((0,T),l^r(h\mathbb{Z}))} \to \infty$. The existence of such example remains an open problem.

The above example shows the existence of numerical schemes with solutions that are not uniformly bounded in any space $L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}))$, $r > 2$. Thus, in general, one cannot expect that the solutions of a numerical scheme for NSE will have a limit in $L^q_{loc}(\mathbb{R},\, l^r(\mathbb{R}))$. This motivates us to follow, at the semidiscrete level, the main steps of the theory of the well posedness of NSE and analyze whether we can derive similar dispersive properties for the linear part of the numerical scheme.

To better illustrate the problems we shall address, let us first consider the conservative semidiscrete numerical scheme

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = 0, & t > 0, \\ \quad u^h(0) = \varphi^h. \end{cases} \tag{3.1.15}$$

First, the blow-up of the solutions of the nonlinear problem (3.1.14) implies that there are no uniform dispersive estimates for the linear semigroup generated by scheme (3.1.15). If there exists any dispersive estimate similar to the ones in (3.1.9), the nonlinear problem will admit solutions which will remains bounded in some auxiliary space $L^q_{loc}(\mathbb{R},\, l^r(h\mathbb{Z}^d))$ which is not the case. Thus we conclude that there are no dispersive estimates for the linear problem.

In this chapter we first construct explicit examples of solutions for the conservative scheme (3.1.15) which fails to have uniform dispersive properties. We then introduce two numerical schemes for which the estimates are uniform. The first one uses an artificial numerical viscosity term and the second one involves a two-grid algorithm to precondition the initial data. Both approximation schemes of the linear semigroup converge and have uniform dispersion properties. This allows us to build two convergent numerical schemes for the NSE in the class of $L^2(\mathbb{R}^d)$ initial data. Also in the case of the conservative scheme (3.1.1) we prove that a convenient filtering of the initial data allows to recover the dispersive properties of the continuous model.

This Chapter is organized as follows. In Section 3.2 we consider the conservative approximations of the LSE. We prove that this scheme does not gain any uniform integrability or local smoothing of the solutions with respect to the initial data. Afterwards, in Section 3.3 we propose a frequency filtering of initial data which will recover both integrability and local smoothing of the continuous model.

In Section 3.4 we introduce a numerical scheme containing a numerical viscosity term of the form $ia(h)\Delta_h u$. We prove that choosing a convenient $a(h)$ we are able to recover the properties mentioned above. Schemes with higher order dissipative terms are also analyzed. We will introduce an approximation of NSE based in the approximation of LSE introduced before. We prove the well-posedness of the nonlinear semidiscrete scheme and the convergence of its solutions towards the solutions of NSE.

Section 3.6 is dedicated to a two-grid preconditioner. We analyze the action of the linear semigroup $\exp(it\Delta_h)$ on the subspace V^h of $l^2(h\mathbb{Z}^d)$ generated by the two-grid method. Once we obtain Strichartz-like estimates in this subspace we apply them to approximate the NSE.

3.2. A conservative scheme

In this section we analyze the conservative scheme (3.1.15). This scheme satisfies the classical properties of consistency and stability which imply L^2-convergence. In fact stability holds because the discrete l^2-norm is conserved under the flow (3.1.15):

$$\frac{d}{dt}\left(h^d \sum_{j\in\mathbb{Z}^d} |u_{\mathbf{j}}^h(t)|^2\right) = 0. \tag{3.2.1}$$

We make use of the semidiscrete Fourier transform (SDFT) in the analysis of the properties of our schemes. To do that we apply SDFT to equations (3.1.15). We obtain the relation between the solution at the time t and the initial data. This is usually done in the study of the stability of numerical schemes.

Taking SDFT in (3.1.15) we obtain that $\widehat{u}^h$ satisfies the following ODE:

$$\begin{cases} i\dfrac{d\widehat{u}^h}{dt}(t,\xi) + \dfrac{4}{h^2}\displaystyle\sum_{k=1}^{d} \sin^2\left(\frac{\xi_k h}{2}\right)\widehat{u}^h(t,\xi) = 0, & t>0,\ \xi\in\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^d, \\ \widehat{u}^h(0,\xi) = \widehat{\varphi}^h(\xi), & \xi\in\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^d. \end{cases}$$

In the Fourier space the solution $\widehat{u}^h$ can be written as

$$\widehat{u}^h(t,\xi) = e^{-itp_h(\xi)}\widehat{\varphi}^h(\xi),\ \xi\in\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^d, \tag{3.2.2}$$

where the function $p_h : [-\pi/h, \pi/h]^d \to \mathbb{R}$ is defined by

$$p_h(\xi) = \frac{4}{h^2}\sum_{k=1}^{d} \sin^2\left(\frac{\xi_k h}{2}\right). \tag{3.2.3}$$

Observe that the new symbol is different from the continuous one: $|\xi|^2$. In the one-dimensional case, the symbol $p_h(\xi)$ changes convexity at the points $\xi = \pm\pi/2h$ (see Figure 3.1) and has critical points also at $\xi = \pm\pi/h$, two properties that the continuous symbol does not have.

In dimension d, the same can be said in terms of the number of nonvanishing principal curvatures of the symbol and its gradient. Observe that at the points $\xi = (\pm\pi/2h, \dots, \pm\pi/2h)$ all the eigenvalues of the hessian matrix $H_{p_h} = (\partial_{ij}p_h)_{ij}$ vanish.

Also at the points $\xi = (\pm\pi/h, \dots, \pm\pi/h)$ the gradient of the symbol vanishes. As we will see, these pathologies affect the dispersive properties of the semidiscrete scheme. The gradient of the two symbols (continuous and discrete ones) are plotted in Figure 3.4 and Figure 3.5.

The first pathology $H_p((\pm\pi/2h, \dots, \pm\pi/2h)) = 0$ shows that there are no uniform estimates similar to (3.1.6) at the discrete level. Consequently, solutions of the semidiscrete system do not have the $L^q(l^r(h\mathbb{Z}^d))$ integrability property of the continuous Schrödinger

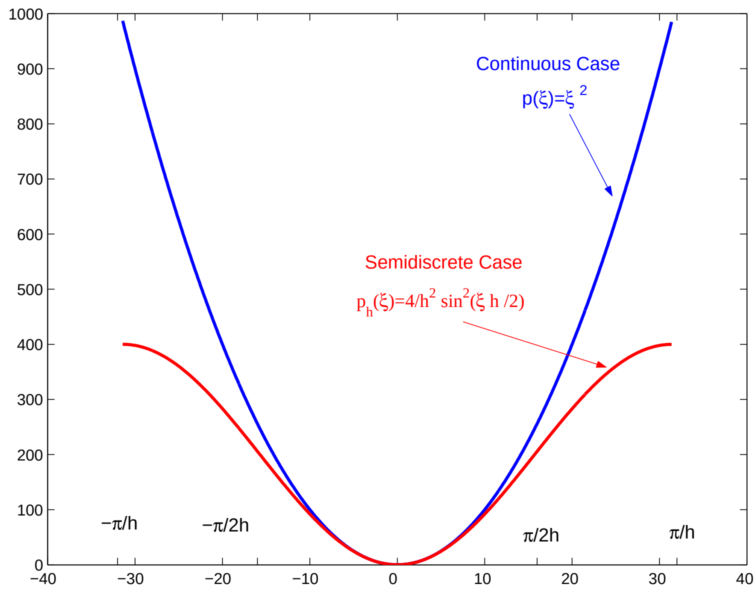

Figure 3.1: The two symbols in dimension one

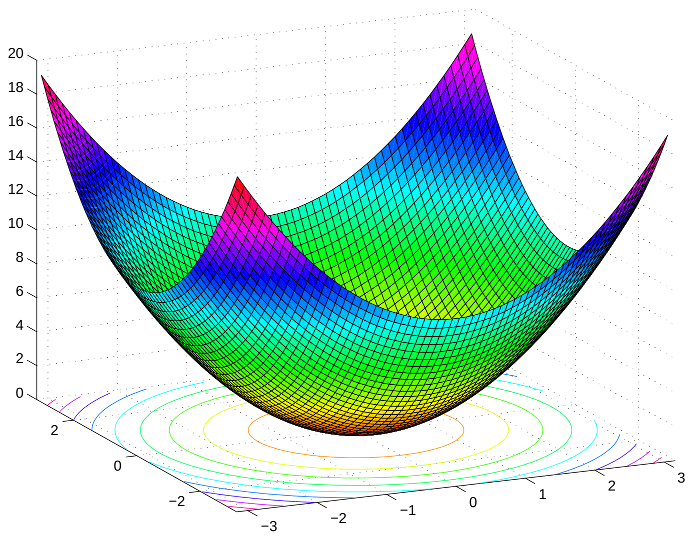

Figure 3.2: The continuous symbol $|\xi|^2$ in dimension two

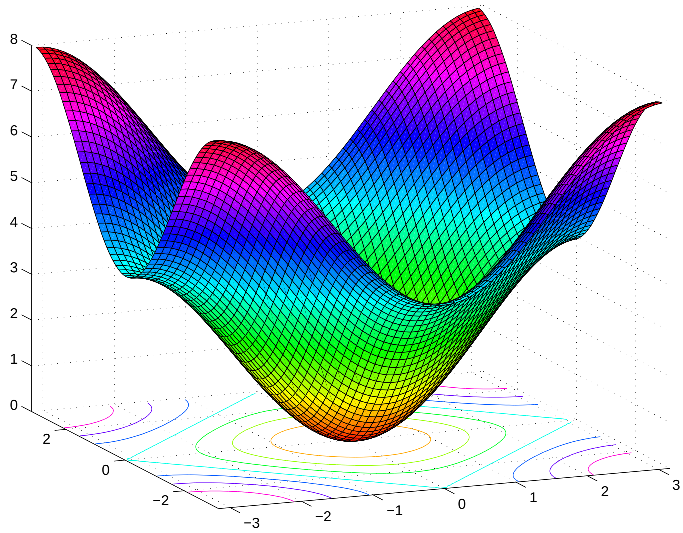

Figure 3.3: The discrete symbol $p_1(\xi)$ in dimension two

equation. This condition is necessary in order to prove the convergence of the semidiscrete solutions towards the continuous one in the nonlinear case. We recall that the uniqueness of the nonlinear Schrödinger equation can not be established in the energy space $L^2(\mathbb{R}^d)$.

The second pathology does not allow us to prove the convergence of the scheme for the NSE. In order to pass to the limit in the nonlinear term we have to use a compactness argument on the discrete solutions u^h. To obtain compactness, we need a small gain of regularity of the solution with respect to the energy space.

We remark that for any time t, the solution is given by a vector $\{u_{\mathbf{j}}(t)\}_{\mathbf{j}\in\mathbb{Z}^d}$. So, we will not make explicit the mesh size h in the notations unless it is necessary. Also the semigroup $S^h\varphi$ satisfies

$$S^h(t)\varphi = S^1\left(\frac{t}{h^2}\right)\varphi \tag{3.2.4}$$

for all time t and mesh size $h>0$. This fact is a consequence of the properties of the SDFT:

$$\begin{aligned}(S^h(t)\varphi)_{\mathbf{j}} &= \int_{[-\pi/h,\pi/h]^d} e^{-itp_h(\xi)} e^{ih\xi\cdot\mathbf{j}}\mathcal{F}_h(\varphi)(\xi)d\xi \\ &= \frac{1}{h^d}\int_{[-\pi,\pi]^d} e^{-it/h^2 p_1(\xi)} e^{i\xi\cdot\mathbf{j}}\mathcal{F}_h(\varphi)\left(\frac{\xi}{h}\right)d\xi \\ &= \int_{[-\pi,\pi]^d} e^{-it/h^2 p_1(\xi)} e^{i\xi\cdot\mathbf{j}}\mathcal{F}_1(\varphi)(\xi)d\xi = (S^1(t/h^2)\varphi)_{\mathbf{j}}.\end{aligned}$$

This reduces all the estimates to the case $h=1$. To fix ideas let us consider the one-dimensional case.

A useful tool to study the decay properties of solutions to dispersive equations is the classical Van der Corput Lemma:

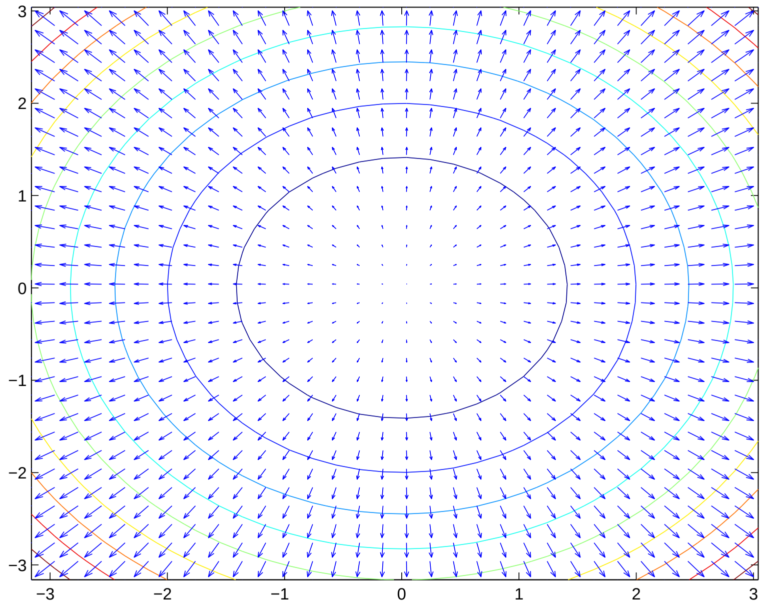

Figure 3.4: The gradient of the continuous symbol

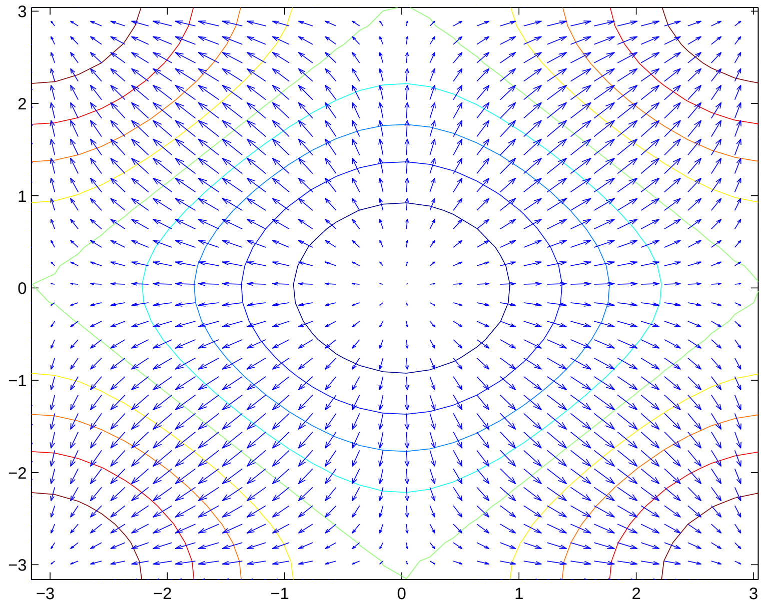

Figure 3.5: The gradient of the discrete symbol

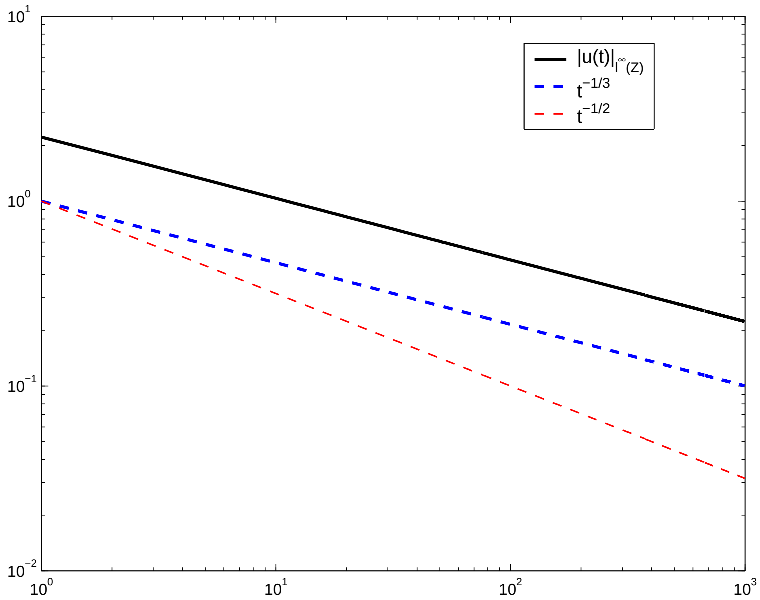

Figure 3.6: Log-log plot of the time evolution of the l^∞ norm of u^1 with initial datum δ_0.

Lemma 3.2.1. *(Van der Corput, Prop. 2, Ch. 8, p. 332, [118]) Suppose ψ is real-valued and smooth in (a, b), and that $|\psi^{(k)}(x)| \geq 1$ for all $x \in (a, b)$. Then*

$$\left| \int_a^b e^{i\lambda\psi(x)} dx \right| \leq c_k \lambda^{-1/k}$$

holds when:

(i) $k \geq 2$, or

(ii) $k = 1$ and $\psi'(x)$ is monotonic.

The bound c_k is independent of ψ and λ.

Essentially it says that

$$\left| \int_a^b e^{it\psi(\xi)} d\xi \right| \lesssim t^{-1/k}$$

provided that ψ is real valued and smooth in (a, b) satisfying $|\partial^k \psi(x)| \geq 1$ for all $x \in (a, b)$. In the continuous case, i.e., with $\psi(\xi) = \xi^2$, using that the second derivative of the symbol is identically two ($\psi''(\xi) = 2$), one easily obtains (3.1.6). However, in the semidiscrete case the symbol of the semidiscrete approximation $p_1(\xi)$ satisfies

$$|\partial^2 p_1(\xi)| + |\partial^3 p_1(\xi)| \geq c$$

for some positive constant c, a property that the second derivative does not satisfy. This implies that for any t

$$\|u^1(t)\|_{l^\infty(\mathbb{Z})} \lesssim \left(\frac{1}{t^{1/2}} + \frac{1}{t^{1/3}} \right) \|u^1(0)\|_{l^1(\mathbb{Z})}. \tag{3.2.5}$$

This estimate was also obtained in [116] for the semidiscrete Schrödinger equation on the lattice $\mathbb{Z}$. But here we are interested on the behavior of the system as the mesh-size h tends to zero.

The decay estimate (3.2.5) contains two terms. The first one, $t^{-1/2}$, is of the order of that of the continuous Schrödinger equation. The second term, $t^{-1/3}$, is due to the discretization scheme and, more precisely, to the behavior of the semidiscrete symbol at the frequencies $\pm\pi/2$.

A scaling argument implies that

$$\frac{\|u^h(t)\|_{l^\infty(h\mathbb{Z})}}{\|u^h(0)\|_{l^1(h\mathbb{Z})}} \lesssim \frac{1}{t^{1/2}} + \frac{1}{(th)^{1/3}},$$

an estimate which fails to be uniform with respect to the mesh size h.

As we have seen, the $l^\infty(\mathbb{Z})$ norm of the discrete solution $u^1(t)$ behaves as $t^{-1/3}$ as $t \to \infty$. This is illustrated in Figure 3.6 by choosing the discrete Dirac delta, δ_0, as initial datum. That is $u(0)_j = \delta_{0j}$ where δ is the Kronecker symbol.

More generally one can prove that there is no gain of integrability similar to (3.1.9), uniformly with respect to the mesh size h. The same occurs in what concerns the gain of the local smoothing property (3.1.11). The last pathology is due to the fact that, in contrast with the continuous case, the symbol $p_h(\xi)$ has critical points also at $\pm\pi/h$. These negative results are summarized in the following Section.

The results plotted in Figure 3.6 use the techniques given by [67] and [68], based on stationary phase method, to compute highly-oscillatory integrals.

3.2.1. Lack of Strichartz estimates

In this section we prove that there is no gain of integrability or local smoothing of the solutions of the considered semidiscrete scheme, uniformly with respect to the mesh size.

Theorem 3.2.1. *Let $T > 0$, $r_0 \geq 1$ and $r > r_0$. Then*

$$\sup_{h>0,\, \varphi\in l^r(h\mathbb{Z}^d)} \frac{\|S^h(T)\varphi\|_{l^r(h\mathbb{Z}^d)}}{\|\varphi\|_{l^{r_0}(h\mathbb{Z}^d)}} = \infty \tag{3.2.6}$$

and

$$\sup_{h>0,\, \varphi\in l^r(h\mathbb{Z}^d)} \frac{\|S^h(\cdot)\varphi\|_{L^1((0,T),l^r(h\mathbb{Z}^d))}}{\|\varphi\|_{l^{r_0}(h\mathbb{Z}^d)}} = \infty. \tag{3.2.7}$$

Remark 3.2.1. *Let I^h be an interpolator, piecewise constant or linear. For any fixed $T > 0$, the uniform boundedness principle guarantees the existence of a function $\varphi \in L^2(\mathbb{R}^d)$ and a sequence φ^h such that $I^h\varphi^h \to \varphi$ in $L^2(\mathbb{R}^d)$ such that the corresponding solutions u^h of (3.1.15) satisfy*

$$\|I^h u^h\|_{L^1((0,T),\, L^r(\mathbb{R}^d))} \to \infty.$$

This guarantees the existence of an initial datum φ and approximations φ^h such that the solutions of (3.1.15) do not remain uniformly bounded in any auxiliary space $L^q_{loc}(\mathbb{R},\, L^r(\mathbb{R}^d))$.

The main steps of the proof are as follows: first, we reduce the proof to the one-dimensional case by means of separation of variables. Also we get rid of the parameter h by scaling.

In contrast with the discrete case, the continuous L^p-norms are better adapted to scaling arguments. Thus we will replace the discrete norms by continuous ones by using a band-limited interpolator. We will introduce the operator $S_1 : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined as:

$$(S_1(t)\varphi)(x) = \int_{-\pi}^{\pi} e^{itp_1(\xi)} e^{ix\xi} \widehat{\varphi}(\xi). \tag{3.2.8}$$

and will prove the following Lemma, which will be the key point of the proof.

Lemma 3.2.2. *For large enough τ there exists a function φ_τ with its Fourier transform supported in $[-\pi, \pi]$ such that*

$$\|S_1(s)\varphi_\tau\|_{L^q(\mathbb{R})} \gtrsim \tau^{-\frac{1}{3}\left(\frac{1}{q_0}-\frac{1}{q}\right)} \|\varphi_\tau\|_{L^{q_0}(\mathbb{R})} \tag{3.2.9}$$

for all $|s| \le \tau$.

We postpone the proof of Lemma 3.2.2 and proceed with the above steps.

Proof of Theorem 3.2.1. **Step 1. Reduction to the one-dimensional case.**

Let us denote by $S^{1,h}$ the linear semigroup generated by the equation (3.1.15) in the one-dimensional case. Let us choose a sequence $\{\psi_j\}_{j\in\mathbb{Z}}$ and set

$$\varphi_{\mathbf{j}} = \psi_{j_1}\psi_{j_2}\dots\psi_{j_d}, \quad \mathbf{j} = (j_1, j_2, \dots, j_d).$$

Then, for any t and $\mathbf{j} = (j_1, j_2, \dots, j_d)$

$$(S^h(t)\varphi)_{\mathbf{j}} = (S^{1,h}(t)\psi)_{j_1}(S^{1,h}(t)\psi)_{j_2}\dots(S^{1,h}(t)\psi)_{j_d}.$$

Also for any $q \ge 1$:

$$\|S^h(t)\varphi\|_{l^q(h\mathbb{Z}^d)} = \|S^{1,h}(t)\psi\|^d_{l^q(h\mathbb{Z})}.$$

As a consequence

$$\sup_{h>0,\varphi\in l^q(h\mathbb{Z}^d)} \frac{\|S^h(T)\varphi\|_{l^q(h\mathbb{Z}^d)}}{\|\varphi\|_{l^{r_0}(h\mathbb{Z}^d)}} \ge \left(\sup_{h>0,\psi\in l^q(h\mathbb{Z})} \frac{\|S^{1,h}(T)\psi\|_{l^q(h\mathbb{Z})}}{\|\psi\|_{l^{q_0}(h\mathbb{Z})}} \right)^d \tag{3.2.10}$$

and

$$\sup_{h>0,\varphi\in l^q(h\mathbb{Z}^d)} \frac{\|S^h(\cdot)\varphi\|_{L^1((0,T),l^q(h\mathbb{Z}^d))}}{\|\varphi\|_{l^{q_0}(h\mathbb{Z}^d)}} \ge T^{1-d} \left(\sup_{h>0,\psi\in l^q(h\mathbb{Z})} \frac{\|S^h(\cdot)\psi\|_{L^1((0,T),l^q(h\mathbb{Z}))}}{\|\psi\|_{l^{q_0}(h\mathbb{Z})}} \right)^d. \tag{3.2.11}$$

Inequalities (3.2.10) and (3.2.11) reduce the proof of (3.2.6) and (3.2.7) to the one-dimensional case. In the sequel we consider the one-dimensional case, denoting by S^h the semigroup generated by the equation (3.1.15).

In order to simplify the presentation we rescale all the $l^q(h\mathbb{Z})$ norms. We observe that

$$\frac{\|S^h(T)\varphi\|_{l^q(h\mathbb{Z})}}{\|\varphi\|_{l^{q_0}(h\mathbb{Z})}} = h^{\frac{1}{q}-\frac{1}{q_0}} \frac{\|S^1(T/h^2)\varphi\|_{l^q(\mathbb{Z})}}{\|\varphi\|_{l^{q_0}(\mathbb{Z})}} \tag{3.2.12}$$

and

$$\frac{\|S^h(\cdot)\varphi\|_{L^1((0,T),l^q(h\mathbb{Z}))}}{\|\varphi\|_{l^{q_0}(h\mathbb{Z})}} = h^{2+\frac{1}{q}-\frac{1}{q_0}}\frac{\|S^1(\cdot)\varphi\|_{L^1((0,T/h^2),l^q(\mathbb{Z}))}}{\|\varphi\|_{l^{q_0}(\mathbb{Z})}}. \tag{3.2.13}$$

Step 2. Replacing the discrete norm by the continuous one.

For each $t\in\mathbb{R}$ the operator $S_1(t)\varphi$ has the Fourier transform supported in $[-\pi,\pi]$. Thus $S_1(t)\varphi$ is a continuous function in the x variable and its discrete norms make sense. More than that, for each $j\in\mathbb{Z}$ the semigroups $S^1(t)\varphi$ and $S_1(t)\varphi$ are related by:

$$(S^1(t)\varphi)_j = (S_1(t)\varphi)(j).$$

In the identity above the discrete or continuous character of φ does not matter as long as the two Fourier transforms (discrete and continuous one) are identical on $(-\pi,\pi)$. We claim that there are two constants c and C such that

$$c\|S^1(t)\varphi\|_{l^q(\mathbb{Z})} \le \|S_1(t)\varphi\|_{L^q(\mathbb{R})} \le C\|S^1(t)\varphi\|_{l^q(\mathbb{Z})}.$$

The results of [101] and [90] on band-limited functions, i.e. functions with compactly supported Fourier transform, allow us to compare the continuous and discrete norms of these functions. More precisely (see Theorem A.0.3) there are two constants A and B such that

$$A\|f\|^q_{l^q(\mathbb{Z})} \le \|f\|^q_{L^q(\mathbb{R})} \le B\|f\|^q_{l^q(\mathbb{Z})}$$

for all functions f with its Fourier transform supported in $[-\pi,\pi]$ and $q>1$. Also for $q=1$, the first inequality holds:

$$A\|f\|_{l^1(\mathbb{Z})} \le \|f\|_{L^1(\mathbb{R})}.$$

For $q=1$, the second inequality is not true in the whole class of functions with support on $[-\pi,\pi]$.

Applying this result to the function $S_1(\cdot)\varphi$ we replace in (3.2.12) and (3.2.13) the discrete norms by continuous ones :

$$\sup_{\varphi\in l^{q_0}(\mathbb{Z})}\frac{\|S^1(T/h^2)\varphi\|_{l^q(\mathbb{Z})}}{\|\varphi\|_{l^{q_0}(\mathbb{Z})}} \ge \frac{B^{-1/q}}{A^{-1/q_0}}\sup_{\operatorname{supp}\widehat{\varphi}\subset[-\pi,\pi]}\frac{\|S_1(T/h^2)\varphi\|_{l^q(\mathbb{R})}}{\|\varphi\|_{l^{q_0}(\mathbb{R})}} \tag{3.2.14}$$

and

$$\sup_{\varphi\in l^{q_0}(\mathbb{Z})}\frac{\|S^1(\cdot)\varphi\|_{L^1((0,T/h^2),l^q(\mathbb{Z}))}}{\|\varphi\|_{l^{q_0}(\mathbb{Z})}} \ge \frac{B^{-1/q}}{A^{-1/q_0}}\sup_{\operatorname{supp}\widehat{\varphi}\subset[-\pi,\pi]}\frac{\|S_1(\cdot)\varphi\|_{L^1((0,T/h^2),\,L^q(\mathbb{R}))}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}}. \tag{3.2.15}$$

Let us fix $T>0$ (the other case being similar) and set $\tau=T/h^2$. This means that $h\sim\tau^{-1/2}$. We replace T/h^2 by τ in both (3.2.14) and (3.2.15). In view of (3.2.12), (3.2.13), (3.2.14) and (3.2.15) the proof of the Theorem is reduced to the following estimates:

$$\tau^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{q_0}\right)}\sup_{\operatorname{supp}\widehat{\varphi}\subset[-\pi,\pi]}\frac{\|S_1(\tau)\varphi\|_{l^q(\mathbb{R})}}{\|\varphi\|_{l^{q_0}(\mathbb{R})}} \to\infty,\quad \tau\to\infty \tag{3.2.16}$$

and

$$\tau^{-1-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{q_0}\right)}\sup_{\operatorname{supp}\widehat{\varphi}\subset[-\pi,\pi]}\frac{\|S_1(\cdot)\varphi\|_{L^1((0,\tau),\,L^q(\mathbb{R}))}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}} \to\infty,\quad \tau\to\infty. \tag{3.2.17}$$

Step 3. Proof of (3.2.16) and (3.2.17).
At this point we make use of Lemma 3.2.2. By (3.2.9) the limit (3.2.16) easily follows:

$$\tau^{-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{q_0}\right)} \sup_{\operatorname{supp}\widehat{\varphi}\subset[-\pi,\pi]} \frac{\|S_1(\tau)\varphi\|_{l^q(\mathbb{R})}}{\|\varphi\|_{l^{q_0}(\mathbb{R})}} \gtrsim \tau^{\frac{1}{6}\left(\frac{1}{q_0}-\frac{1}{q}\right)} \to \infty, \quad \tau\to\infty. \tag{3.2.18}$$

The second one, (3.2.17) can be obtained in a similar way

$$\begin{aligned} \tau^{-1-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{q_0}\right)} \sup_{\operatorname{supp}\widehat{\varphi}\subset[-\pi,\pi]} \frac{\|S_1(\cdot)\varphi\|_{L^1((0,\tau),\,L^q(\mathbb{R}))}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}} &\gtrsim \tau^{-1-\frac{1}{2}\left(\frac{1}{q}-\frac{1}{q_0}\right)} \int_0^\tau \tau^{-\frac{1}{3}\left(\frac{1}{q_0}-\frac{1}{q}\right)} \\ &\gtrsim \tau^{\frac{1}{2}\left(\frac{1}{q_0}-\frac{1}{q}\right)} \tau^{-\frac{1}{3}\left(\frac{1}{q_0}-\frac{1}{q}\right)} \\ &\gtrsim \tau^{\frac{1}{6}\left(\frac{1}{q_0}-\frac{1}{q}\right)} \to \infty, \quad \tau\to\infty. \end{aligned}$$

Both estimates (3.2.18) and (3.2.17) finish the proof of Theorem 3.2.1.

□

Proof of Lemma 3.2.2. Our proof is similar to the one used in [46] to bound from below the magnitude of a family of oscillatory integrals. The key point is to apply the mean value theorem to functions with their Fourier's transform concentrated near the points where Van der Corput's Lemma 3.2.1 fails to provide good results. Next we prove inequality (3.2.9).

The mean value theorem, applied to the function $\Psi(\xi) = -sp_1(\xi) + x\xi$ on the interval $[a,b]\subset[-\pi,\pi]$, gives

$$\left|\int_a^b e^{i\Psi(\xi)}\widehat{\varphi}(\xi)d\xi\right| \geq \left(1-|b-a|\sup_{\xi\in[a,b]}|\Psi'(\xi)|\right)\int_a^b \widehat{\varphi}(\xi)d\xi, \tag{3.2.19}$$

provided that $\widehat{\varphi}$ is nonnegative on $[a,b]$. Taylor's formula applied to the function Ψ' at the point $\pi/2$ shows

$$\begin{aligned} |\Psi'(\xi)-\Psi'(\pi/2)-(\xi-\pi/2)\Psi''(\pi/2)| &\leq \frac{|\xi-\pi/2|^2}{2}\sup_{\eta\in[a,b]}|\Psi^{(3)}(\eta)| \\ &\leq \frac{|\xi-\pi/2|^2 s}{2}\sup_{\eta\in[a,b]}|p_1^{(3)}(\eta)| \\ &\leq 2s|\xi-\pi/2|^2 \end{aligned}$$

for all $\xi\in[a,b]$. Taking into account that $\Psi''(\pi/2)=0$, the last inequality implies

$$|\Psi'(\xi)| \leq |x-sp_1'(\pi/2)| + 2s|\xi-\pi/2|^2,\ \xi\in[a,b]. \tag{3.2.20}$$

Let $\epsilon>0$ be a small positive number that we shall fix below, and $\widehat{\varphi}_\epsilon\geq 0$ be a function supported on the set $\{\xi : |\xi-\pi/2|\leq\epsilon\}$. Choosing $a=\pi/2-\epsilon$ and $b=\pi/2+\epsilon$, by (3.2.8) and (3.2.19) we get

$$|(S_1(s)\varphi_\epsilon)(x)| \geq \left(1-2\epsilon\sup_{\xi\in[\pi/2-\epsilon,\pi/2+\epsilon]}|\Psi'(\xi)|\right)\int_{\pi/2-\epsilon}^{\pi/2+\epsilon}\widehat{\varphi}_\epsilon(\xi)d\xi. \tag{3.2.21}$$

Inequality (3.2.20) shows that

$$|\Psi'(\xi)| \leq \frac{\epsilon^{-1}}{4}$$

as long as

$$|x - sp_1'(\pi/2)| \leq \frac{\epsilon^{-1}}{8} \text{ and } |s| \leq \frac{\epsilon^{-3}}{8}. \tag{3.2.22}$$

Thus by (3.2.21)

$$|(S_1(s)\varphi_\epsilon)(x)| \geq \frac{1}{2}\int_{\pi/2-\epsilon}^{\pi/2+\epsilon} \widehat{\varphi}_\epsilon(\xi)d\xi$$

for all x and s satisfying (3.2.22). Integrating the last inequality on the set $\{x : x - sp_1'(\pi/2) = \epsilon^{-1}/8\}$, we see that $S_1(s)\varphi_\epsilon$ satisfies

$$\begin{aligned} \|S_1(s)\varphi_\epsilon\|_{L^q(\mathbb{R})} &\geq \left|\left\{x : |x - sp_1'(\pi/2)| \leq \frac{\epsilon^{-1}}{8}\right\}\right|^{1/q_0} \left(\int_{\pi/2-\epsilon}^{\pi/2+\epsilon} \widehat{\varphi}_\epsilon(\xi)d\xi\right) \\ &\geq c(q)\epsilon^{-\frac{1}{q}} \int_{\pi/2-\epsilon}^{\pi/2+\epsilon} \widehat{\varphi}_\epsilon(\xi)d\xi, \end{aligned} \tag{3.2.23}$$

for some positive constant $c(q)$.

Observe that (3.2.23) does not imply directly (3.2.9). We have to choose a particular φ_ϵ in order to compare the right hand sides of (3.2.23) and (3.2.9). Let us now be more precise about the choice of φ_ϵ. Let φ be such that its Fourier transform $\widehat{\varphi}$ has compact support in $(-1, 1)$ and $\widehat{\varphi}(\xi) > 1$ on $(-1/2, 1/2)$. Set φ_ϵ in the following manner:

$$\widehat{\varphi}_\epsilon(\xi) = \epsilon^{-1}\widehat{\varphi}\left(\epsilon^{-1}\left(\xi - \frac{\pi}{2}\right)\right).$$

Clearly, this implies that the mass of $\widehat{\varphi}_\epsilon$ does not depend on ϵ. Also, the conditions imposed on φ guarantee that:

$$\widehat{\varphi}_\epsilon(\xi) \geq \epsilon^{-1},\ \xi \in \left(\frac{\pi}{2} - \frac{\epsilon}{2}, \frac{\pi}{2} + \frac{\epsilon}{2}\right)$$

and consequently, the right hand side of (3.2.23) satisfies

$$\int_{\pi/2-\epsilon}^{\pi/2+\epsilon} \widehat{\varphi}_\epsilon(\xi)d\xi \geq \epsilon^{-1}\int_{\pi/2-\epsilon/2}^{\pi/2+\epsilon/2} d\xi = 1. \tag{3.2.24}$$

Also, classical properties of the Fourier transform guarantee that

$$\varphi_\epsilon(x) = \exp\left(\frac{i\pi x}{2}\right)\varphi(\epsilon x)$$

and its L^{q_0}-norm behaves as ϵ^{-1/q_0}:

$$\|\varphi_\epsilon\|_{L^{q_0}(\mathbb{R})} = \epsilon^{-1/q_0}\|\varphi\|_{L^{q_0}(\mathbb{R})}. \tag{3.2.25}$$

Finally let us choose $\tau = \epsilon^{-3}/8$. Thus, in view of (3.2.23), (3.2.24) and (3.2.25), for any $|s| \leq \tau$:

$$\frac{\|S_1(s)\varphi_\epsilon\|_{L^q(\mathbb{R})}}{\|\varphi_\epsilon\|_{L^{q_0}(\mathbb{R})}} \gtrsim \epsilon^{\frac{1}{q_0}-\frac{1}{q}} \gtrsim \tau^{\frac{1}{3}\left(\frac{1}{q}-\frac{1}{q_0}\right)}.$$

This proves (3.2.9) and finishes the proof.

□

3.2.2. Lack of local smoothing effect

In order to analyze the local smoothing effect at the discrete level we introduce the discrete fractional derivatives on the lattice $h\mathbb{Z}^d$. We define for any $s \geq 0$, the fractional derivative $D_h^s u$ at the scale h as:

$$(D_h^s u)_{\mathbf{j}} = \int_{[-\pi/h,\pi/h]^d} p_h^{s/2}(\xi) e^{i\mathbf{j}\cdot\xi h} \mathcal{F}_h(u)(\xi) d\xi, \ \mathbf{j} \in \mathbb{Z}^d$$

where $\mathcal{F}_h(u)$ is the SDFT of the sequence $\{u_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$ at the scale h and $p_h(\xi)$ is as in (3.2.3).

The main result is the following theorem:

Theorem 3.2.2. *Let be $T > 0$, $s > 0$ and $q \geq 1$. Then*

$$\sup_{h>0,\varphi\in l^q(h\mathbb{Z}^d)} \frac{h^d \sum_{|\mathbf{j}|h\leq 1} |(D_h^s S^h(T)\varphi)_{\mathbf{j}}|^2}{\|\varphi\|^2_{l^q(h\mathbb{Z}^d)}} = \infty \tag{3.2.26}$$

and

$$\sup_{h>0,\varphi\in l^q(h\mathbb{Z}^d)} \frac{h^d \sum_{|\mathbf{j}|h\leq 1} \int_0^T |(D_h^s S^h(t)\varphi)_{\mathbf{j}}|^2 dt}{\|\varphi\|^2_{l^q(h\mathbb{Z}^d)}} = \infty. \tag{3.2.27}$$

Remark 3.2.2. *In contrast with the proof of Theorem 3.2.1 we are not able to reduce it to the one-dimensional case. This is due to the extra factor $p_h^{s/2}(\xi)$ which does not allow to use the separation of variables argument.*

The proof consists in reducing (3.2.26) and (3.2.27) to the case $h = 1$. Afterwards the main ingredient will be the following Lemma.

Lemma 3.2.3. *Let $s > 0$, $c_0 > 0$ and $q > 1$. There is a constant $C(s, q, c_0)$ such that for large enough τ there exists a function $\varphi = \varphi_\tau$ such that*

$$|(D^s S^1(t)\varphi)_{\mathbf{j}}| \geq C(s, q, c_0)\tau^{-d/2q} \|\varphi\|_{l^q(\mathbb{Z}^d)} \tag{3.2.28}$$

for all $|t| \leq \tau$, $|\mathbf{j}| \leq c_0\tau^{1/2}$.

This Lemma is similar to the one given in the previous Section. We postpone the proof of Lemma 3.2.3 and proceed with the proof of Theorem 3.2.2.

Proof of Theorem 3.2.2. As before we rescale all $l^p(h\mathbb{Z}^d)$-norms and reduce the proof to the case $h = 1$. By the definition of the discrete fractional derivatives D_h^s we have:

$$\begin{aligned}
(D_h^s S^h(t)\varphi)_{\mathbf{j}} &= \int_{[-\pi/h,\pi/h]^d} p_h^{s/2}(\xi) e^{-itp_h(\xi)} \mathcal{F}_h(\varphi)(\xi) e^{i\mathbf{j}\xi h} d\xi \\
&= \frac{1}{h^s} \int_{[-\pi,\pi]^d} p_1^{s/2}(\xi) e^{-\frac{it}{h^2}p_1(\xi)} \mathcal{F}_1(\varphi)(\xi) e^{i\mathbf{j}\xi h} d\xi \\
&= \frac{1}{h^s} \left(D_1^s S^1\left(\frac{t}{h^2}\right)\varphi \right)_{\mathbf{j}}.
\end{aligned}$$

Therefore estimates (3.2.26) and (3.2.27) are equivalent to

$$\sup_{\varphi\in l^q(\mathbb{Z}^d)} \frac{h^{d-2s}\sum\limits_{|\mathbf{j}|\le 1/h} |(D_1^s S^h(T/h^2)\varphi)_{\mathbf{j}}|^2}{h^{\frac{2d}{q}}\|\varphi\|^2_{l^q(\mathbb{Z}^d)}} \to \infty, \quad h\to 0 \tag{3.2.29}$$

and

$$\sup_{\varphi\in l^q(\mathbb{Z}^d)} \frac{h^{d+2-2s}\sum\limits_{|\mathbf{j}|h\le 1} \int_0^{T/h^2} |(D_1^s S^1(t)\varphi)_{\mathbf{j}}|^2 dt}{h^{\frac{2d}{q}}\|\varphi\|^2_{l^q(\mathbb{Z}^d)}} \to \infty, \quad h\to 0. \tag{3.2.30}$$

We will show that each of the above terms are of the order of h^{-2s} for h small enough.

To prove (3.2.29) and (3.2.30) we make use of Lemma 3.2.3. First, let us choose in (3.2.28): $\tau = T/h^2$. Thus there exists a function φ such that

$$\begin{aligned} h^{d-2s}\sum_{|\mathbf{j}|\le 1/h} \left|\left(D_1^s S^1\left(\frac{T}{h^2}\right)\varphi\right)_{\mathbf{j}}\right|^2 &\ge Ch^{d-2s}\sum_{|\mathbf{j}|\le 1/h}\left(\frac{T}{h^2}\right)^{-d/q}\|\varphi\|_{l^q(\mathbb{Z}^d)} \\ &\ge Ch^{-2s}\left(\frac{T}{h^2}\right)^{-d/q}\|\varphi\|_{l^q(\mathbb{Z}^d)}. \end{aligned}$$

This implies

$$\sup_{\varphi\in l^q(\mathbb{Z}^d)} \frac{h^{d-2s}\sum\limits_{|\mathbf{j}|\le 1/h} |(D_1^s S^h(T/h^2)\varphi)_{\mathbf{j}}|^2}{h^{\frac{2d}{q}}\|\varphi\|^2_{l^q(\mathbb{Z}^d)}} \gtrsim h^{-2s} \to \infty, \quad h\to 0,$$

which proves (3.2.29). With the same τ and φ as above, the following holds:

$$\begin{aligned} \frac{h^{d+2-2s}\sum\limits_{|\mathbf{j}|\le 1/h}\int\limits_0^{T/h^2} |(D_1^s S^1(t)\varphi)_{\mathbf{j}}|^2 dt}{h^{\frac{2d}{q}}\|\varphi\|_{l^q(\mathbb{Z}^d)^2}} &\ge \frac{Th^{-2s}\inf\limits_{t\in[0,T/h^2],|\mathbf{j}|\le 1/h}|(D_1^s S^1(t))_{\mathbf{j}}|^2}{h^{\frac{2d}{q}}\|\varphi\|^2_{l^q(\mathbb{Z}^d)}} \\ &\ge \frac{Th^{-2s}(T/h^2)^{-\frac{d}{q}}\|\varphi\|^2_{l^q(\mathbb{Z}^d)}}{h^{\frac{2d}{q}}\|\varphi\|_{l^q(\mathbb{Z}^d)}} \gtrsim h^{-2s}. \end{aligned}$$

This proves (3.2.30) and finishes the proof. □

Proof of Lemma 3.2.3. The definition of the fractional derivative D_1^s is as follows

$$(D_1^s S^1(t)\varphi)_{\mathbf{j}} = \int_{[-\pi,\pi]^d} p_1^{s/2}(\xi)e^{-itp_1(\xi)}e^{i\mathbf{j}\xi}\widehat{\varphi}(\xi)d\xi. \tag{3.2.31}$$

We employ the letter C to denote any constant that can be explicitly computed in terms of known quantities. The exact value of C may change from one line to another.

The Mean Value Theorem, applied to the function $\Psi(\xi) = -tp_1(\xi) + \mathbf{j}\xi$ on the set $\Omega \subset [-\pi, \pi]^d$ yields:

$$\left| \int_\Omega e^{i\Psi(\xi)} p_1^{s/2}(\xi)\widehat{\varphi}(\xi)d\xi \right| \geq \left(1 - \text{diam}(\Omega) \sup_{\xi\in\Omega} |\nabla\Psi(\xi)|\right) \int_\Omega p_1^{s/2}(\xi)\widehat{\varphi}(\xi)d\xi,$$

provided that $\widehat{\varphi}$ is nonnegative on Ω. With the notation $\pi^d = (\pi, \dots, \pi) \in \mathbb{R}^d$, the gradient $\nabla\Psi$ satisfies

$$\begin{aligned} \nabla\Psi(\xi) &\sim \mathbf{j} - t\nabla p_1(\pi^d) + tO(|\xi - \pi^d|) \\ &\sim \mathbf{j} - tO(|\xi - \pi^d|), \quad \xi \sim \pi^d. \end{aligned}$$

Let ϵ be a small positive number that we shall fix below and $\widehat{\varphi}_\epsilon$ a function supported on the set

$$\Omega_\epsilon = \{\xi : |\xi - \pi^d| \leq C\epsilon\} \cap [-\pi, \pi]^d.$$

Then, choosing possibly a smaller constant C in the definition of Ω_ϵ,

$$\text{diam}(\Omega_\epsilon) \sup_{\xi\in\Omega_\epsilon} |\nabla\Psi(\xi)| \leq \frac{1}{2}$$

as long as

$$|\mathbf{j}| \leq C\epsilon^{-1} \quad \text{and} \quad |t| \leq \epsilon^{-2}. \tag{3.2.32}$$

This implies that the discrete derivative D_1^s of $S^1(t)\varphi$ satisfies

$$|(D_1^s S^1(t)\varphi_\epsilon)_{\mathbf{j}}| = \left| \int_\Omega e^{i\Psi(\xi)} p_1^{s/2}(\xi)\widehat{\varphi}_\epsilon(\xi)d\xi \right| \geq \frac{1}{2} \int_{\Omega_\epsilon} p_1^{s/2}(\xi)\widehat{\varphi}_\epsilon(\xi)d\xi \geq c_1 \int_{\Omega_\epsilon} \widehat{\varphi}_\epsilon(\xi)d\xi \tag{3.2.33}$$

for all $\mathbf{j}$ and t satisfying (3.2.32).

In what follows we make precise the function φ_ϵ and the dependence between ϵ and τ. With this choice, (3.2.33) will imply (3.2.28). As we will see $\epsilon = \tau^{-1/2}$ suffices for a *convenient* choice of φ_ϵ.

Let us be more precise about the function $\widehat{\varphi}_\epsilon$. We choose a function φ such that its SDFT $\widehat{\varphi}$ has compact support on the set (see Figure 3.7)

$$\Omega^1 = B_1(0) \cap \{\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d : -\pi \leq \xi_i \leq 0\}$$

and $\widehat{\varphi}(\xi) > 1$ on

$$\Omega^2 = B_{1/2}(0) \cap \{\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d : -\pi \leq \xi_i \leq 0\}.$$

Let us set φ_ϵ in the following manner:

$$\widehat{\varphi}_\epsilon(\xi) = \epsilon^{-d}\widehat{\varphi}(\epsilon^{-1}(\xi - \pi^d)).$$

The condition imposed on φ guaranties that

$$\widehat{\varphi}_\epsilon(\xi) \geq \epsilon^{-d}, \ \xi \in \Omega_\epsilon^2 = \left\{\xi \in [-\pi, \pi]^d : |\xi - \pi^d| \leq \frac{\epsilon}{2}\right\}.$$

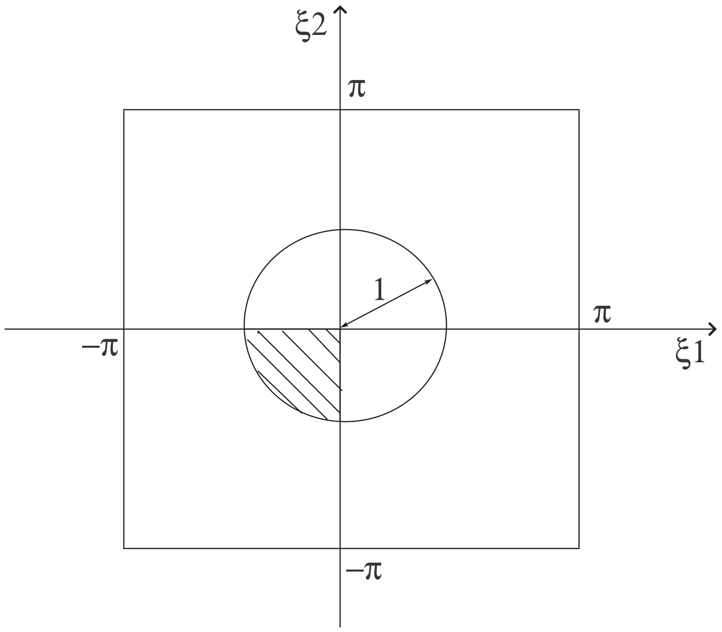

Figure 3.7: The support of $\widehat{\varphi}$ in dimension two

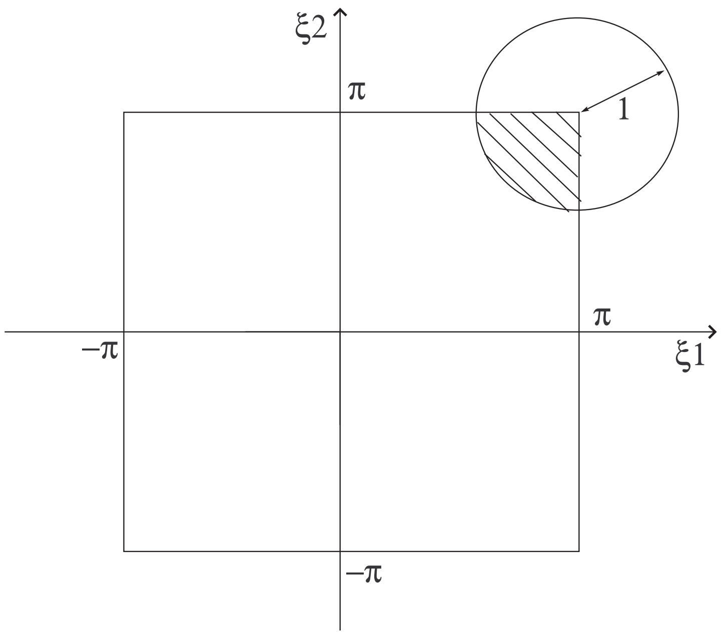

Figure 3.8: The support of $\widehat{\varphi}(\xi - \pi^2)$

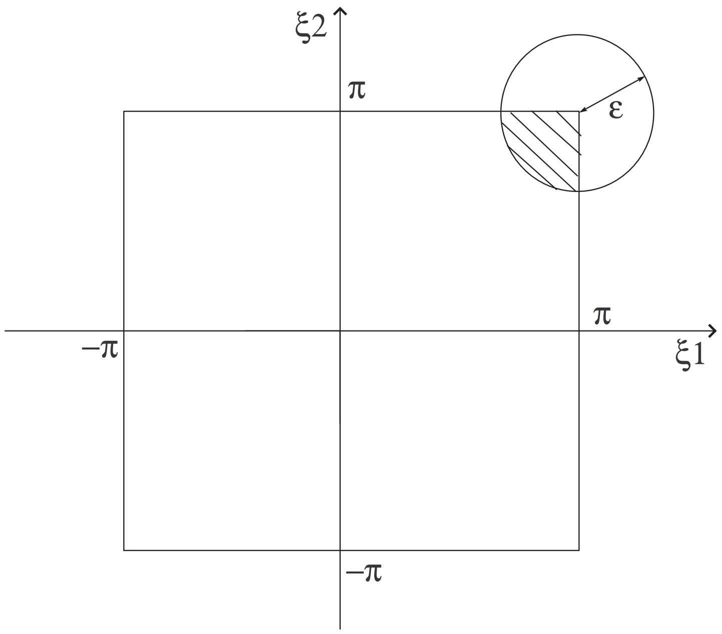

Figure 3.9: The support of $\widehat{\varphi}_\epsilon$

Consequently, the right hand side of (3.2.33) satisfies

$$\int_{\Omega_\epsilon} \widehat{\varphi}_\epsilon(\xi)d\xi \geq \int_{\Omega_\epsilon^2} \epsilon^{-d} d\xi = \epsilon^{-d}|\Omega_\epsilon^2| \geq C.$$

We claim that the discrete function φ_ϵ has an $l^q(\mathbb{Z}^d)$-norm smaller than $C(q)\epsilon^{-d/q}$, for some positive constant $C(q)$. Therefore for all $\mathbf{j}$ and t satisfying (3.2.32)

$$\frac{|(D_1^s S^1(t)\varphi_\epsilon)_\mathbf{j}|}{\|\varphi_\epsilon\|_{l^q(\mathbb{Z}^d)}} \geq C\epsilon^{\frac{d}{q}}.$$

Finally we choose $\epsilon = \tau^{-1/2}$ and finish the proof of Lemma 3.2.3.

It remains to prove that the $l^q(\mathbb{Z}^d)$-norm of the function φ_ϵ is smaller than $C(q)\epsilon^{-d/q}$. The trick is the same as in the proof of Theorem 3.2.1: we change the continuous and the discrete norms. The advantage of the continuous norm comes from the fact that the norms are easily computed when we rescale the involved function. We choose the band-limited interpolator $\varphi^* : \mathbb{R}^d \to \mathbb{C}$:

$$\widehat{\varphi^*}(\xi) = \widehat{\varphi}(\xi)\mathbf{1}_{[-\pi,\pi]^d}(\xi).$$

Also we introduce the family of functions $\{\varphi_\epsilon^*\}_{\epsilon>0}$ verifying

$$\widehat{\varphi^*}_\epsilon(\xi) = \widehat{\varphi}_\epsilon(\xi)\mathbf{1}_{[-\pi,\pi]^d}(\xi).$$

Thus for all $\mathbf{j} \in \mathbb{Z}^d$, $\varphi_\epsilon^*(\mathbf{j}) = (\varphi_\epsilon)_\mathbf{j}$ and

$$\|\varphi_\epsilon^*(\mathbf{j})\|_{l^q(\mathbb{Z}^d)} = \|\varphi_\epsilon\|_{l^q(\mathbb{Z}^d)}.$$

The same arguments as in the proof of Theorem 3.2.1 show the existence of a constant C such that

$$\|\varphi_\epsilon\|_{l^q(\mathbb{Z}^d)} \leq C^{1/q}\|\varphi_\epsilon^*\|_{L^q(\mathbb{R}^d)}.$$

It remains to prove that

$$\|\varphi_\epsilon^*\|_{L^q(\mathbb{R}^d)} \leq C\epsilon^{-d/q}. \tag{3.2.34}$$

Using the definition of φ_ϵ^* we get: $\widehat{\varphi_\epsilon^*}(\xi) = \epsilon^{-d}\varphi^*(\epsilon(\xi - \pi^d))$ and $\varphi_\epsilon^*(x) = \exp(i\pi^d \cdot x)\varphi^*(\epsilon x)$. Thus its $L^q(\mathbb{R}^d)$-norm satisfies

$$\|\varphi_\epsilon^*\|_{L^q(\mathbb{R}^d)} = \epsilon^{-\epsilon/q}\|\varphi^*\|_{L^q(\mathbb{R}^d)}$$

which proves (3.2.34). □

3.3. Filtered initial data

As we have seen in the previous section the conservative scheme does not reflect the dispersive properties of the LSE. In this section we prove that a filtration of the initial data in the Fourier space will recover the dispersive properties specified in the introduction. The key point to recover the decay rates (3.1.6) at the discrete level is to choose initial data with their SDFT supported far from the pathological points $a \in \{(\pm\pi/2h)^d\}$. For example, in the one-dimensional case, choosing φ^h such that the support of $\widehat{\varphi}^h$ belongs to $(-(\pi/2-\epsilon)/h, (\pi/2-\epsilon)/h)$, shows that

$$\|e^{it\Delta_h}\varphi^h\|_{l^\infty(\mathbb{Z})} \leq \frac{c(\epsilon)}{|t|^{1/2}}\|\varphi^h\|_{l^1(h\mathbb{Z})}.$$

Let us consider $h = 1$ and $\widehat{\varphi}^1 = \mathbf{1}_{(-\pi/4,\pi/4)}$. In contrast with the results presented in Figure 3.6, we can see in Figure 3.10 that the long time behaviour of the solution is $t^{-1/2}$ rather than $t^{-1/3}$. This is due to the fact that there is no influence of the bad frequencies $\pm\pi/2$. Their influence has been observed in Figure 3.6.

For any positive ϵ and h we define the set of all the points inside the cube $[-\pi/h, \pi/h]^d$ which have distance at least ϵ from all the points $a = (\pm\pi/2h, \dots, \pm\pi/2h) \in \mathbb{R}^d$:

$$\Omega_\epsilon^h = \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d \setminus \bigcup_{a\in(\pm\pi/2h)^d} B_\epsilon(a).$$

Let us define the class of function $\mathcal{I}_\epsilon^h \subset l^2(h\mathbb{Z}^2)$, with their SDFT supported on Ω_ϵ^h:

$$\mathcal{I}_\epsilon^h = \{\varphi \in l^2(h\mathbb{Z}^d) : \text{ supp } \widehat{\varphi} \subset \Omega_\epsilon^h\}.$$

Theorem 3.3.1. *Let be $\epsilon > 0$ and $p \geq 2$. There exists a positive constant $C(\epsilon, p)$ such that*

$$\|S^h(t)\varphi\|_{l^p(h\mathbb{Z}^d)} \leq \frac{C(\epsilon, d)}{|t|^{\frac{d}{2}\left(1-\frac{2}{p}\right)}}\|\varphi\|_{l^{p'}(h\mathbb{Z}^d)}, \ t > 0 \tag{3.3.1}$$

holds for all $\varphi \in l^{p'}(h\mathbb{Z}^d) \cap \mathcal{I}_\epsilon^h$, uniformly on $h > 0$.

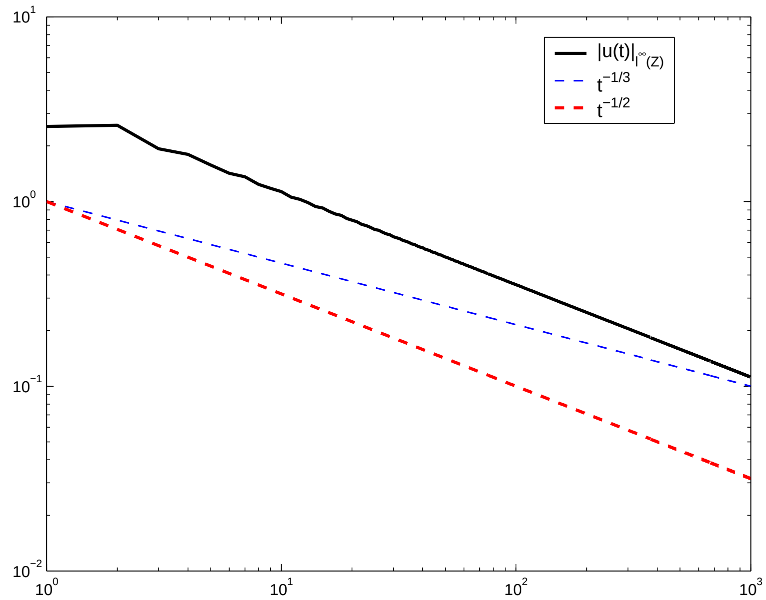

Figure 3.10: Log-log plot of the time evolution of the l^∞ norm of $u^1(t)$ for initial datum $\widehat{\varphi}^1 = \mathbf{1}_{(-\pi/4,\pi/4)}$

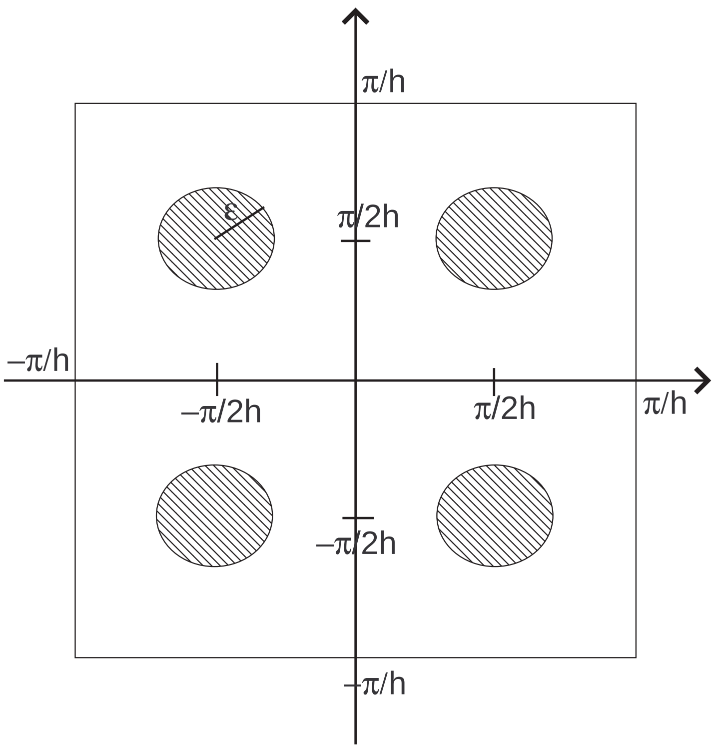

Figure 3.11: The set Ω_ϵ^h is the complement of the dashed area

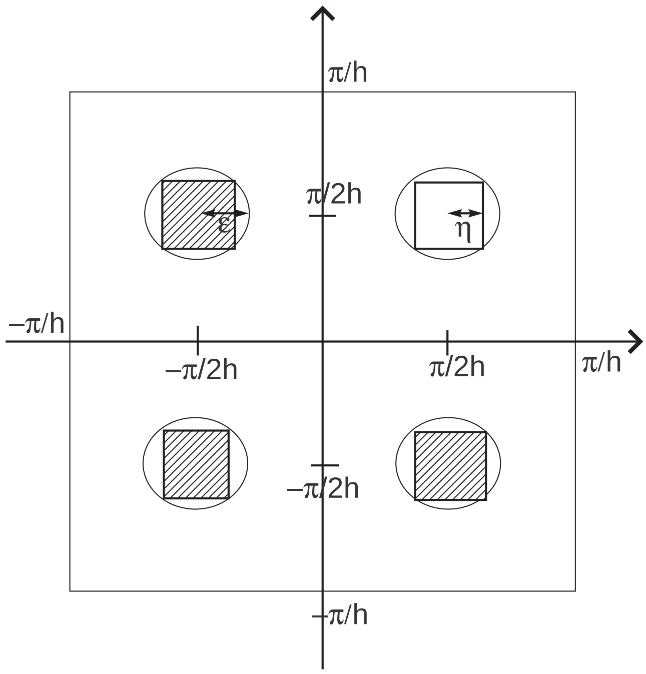

Figure 3.12: The cubes $\mathcal{C}^h_\eta$ form the dashed area

Remark 3.3.1. *This result says that (3.3.1) holds for all functions φ with its SDFT supported far from the points where the Hessian matrix H_{p_h} vanishes identically (the centers of the dashes areas in Figure 3.11).*

Proof. Let φ be such that $\widehat{\varphi}$ identically vanishes in $\cup_{a=(\pm\pi/2h)^d} B_\epsilon(a)$. Let us choose $\eta = \eta(\epsilon)$ such that for each $a = (a_1, \ldots, a_d) = (\pm\pi/2h)^d$ the d-dimensional cube $\mathcal{C}^h_a(\eta) = \prod_{k=1}^d [a_k - \eta, a_k + \eta]$ is contained in the ball $B_a(\epsilon)$ (see Figure 3.12). We denote by $\mathcal{C}^h_\eta$ the union of all these cubes and Λ^h_η its complement. We introduce the cubes $\mathcal{C}^h_a(\eta)$ to take advantage of the separation of variables.

Clearly $\widehat{\varphi}$ identically vanishes on the sets $\mathcal{C}^h_\eta$. Using the Fourier representation of the solutions

$$\mathcal{F}_h(S^h(t)\varphi)(\xi) = e^{-itp_h(\xi)}(\mathcal{F}_h\varphi)(\xi)\mathbf{1}_{\Lambda^h_\eta}(\xi),$$

we find that $S^h(t)\varphi$ satisfies

$$S^h(t)\varphi = K^{h,\eta}(t) * \varphi,$$

where

$$K^{h,\eta}(t, \mathbf{j}) = \int_{\Lambda^h_\eta} e^{-itp_h(\xi)} e^{ih\mathbf{j}\xi} d\xi, \ \mathbf{j} \in \mathbb{Z}^d.$$

Let us introduce the operator $T^h_\eta(t) : l^2(h\mathbb{Z}^d) \to l^2(h\mathbb{Z}^d)$ defined by

$$T^h_\eta(t)\varphi = K^{h,\eta}(t) * \varphi. \tag{3.3.2}$$

Obviously, for any function φ with $\widehat{\varphi}$ supported in Ω^h_ϵ the two operators act identically

$$T^h_\eta(t)\varphi = S^h(t)\varphi.$$

Thus it is sufficient to show that the new operator $T^h_\eta(t)$ satisfies

$$\|T^h_\eta(t)\varphi\|_{l^p(h\mathbb{Z}^d)} \leq \frac{C(\eta,d)}{|t|^{\frac{d}{2}\left(1-\frac{2}{p}\right)}}\|\varphi\|_{l^{p'}(h\mathbb{Z}^d)} \tag{3.3.3}$$

for all $\varphi \in l^{p'}(h\mathbb{Z}^d)$. We shall consider only the cases $p = 2$ and $p = \infty$, since the other ones follow by interpolation:

$$\|T^h_\eta(t)\varphi\|_{l^2(h\mathbb{Z}^d)} \leq C(d,\eta)\|\varphi\|_{l^2(h\mathbb{Z}^d)}$$

and

$$\|T^h_\eta(t)\varphi\|_{l^\infty(h\mathbb{Z}^d)} \leq \frac{C(d,\eta)}{|t|^{1/2}}\|\varphi\|_{l^1(h\mathbb{Z}^d)}. \tag{3.3.4}$$

The case $p = 2$ easily follows by Plancherel's identity. Let us consider the case $p = \infty$. We claim that the kernel $K^{h,\eta}(t)$ satisfies

$$|K^{h,\eta}(t,\mathbf{j})| \leq \frac{C(d,\eta)}{|t|^{1/2}},\ t\neq 0,\ \mathbf{j}\in\mathbb{Z}^d. \tag{3.3.5}$$

Thus (3.3.2) and Young's inequality shows that T^h_η verifies (3.3.3).

In the following we look to (3.3.5). We use separation of variables on the set Λ^h_η to reduce it to the one-dimensional case. Observe that for any $\mathbf{j} = (j_1,\ldots,j_d) \in \mathbb{Z}^d$, $K^{h,\eta}_t$ satisfies:

$$K^{h,\eta}_t(t,\mathbf{j}) = \prod_{k=1}^d \left(\int_{-\pi/h}^{(-\pi/2-\eta)/h} + \int_{(-\pi/2+\eta)h}^{(\pi/2-\eta)/h} + \int_{(\pi/2+\eta)/h}^{\pi/h} e^{-i\frac{4t}{h^2}\sin^2(\frac{\xi_k h}{2})}e^{ij_k\xi_k h}d\xi_k\right). \tag{3.3.6}$$

Then it suffices to show that each term in the above product satisfies

$$\left|\int_{-\pi/h}^{(-\pi/2-\eta)/h} + \int_{(-\pi/2+\eta)h}^{(\pi/2-\eta)/h} + \int_{(\pi/2+\eta)/h}^{\pi/h} e^{-i\frac{4t}{h^2}\sin^2(\frac{\xi_k h}{2})}e^{ij_k\xi_k h}d\xi_k\right| \leq \frac{c(\eta)}{|t|^{1/2}},\ t>0,\ j_k\in\mathbb{Z}.$$

Applying Van der Corput's Lemma 3.2.1 to each of the above integrals I_l, $l = 1,2,3$ we get

$$\left|\int_{I_l} e^{-4it\sin^2(\frac{\xi_k}{2})}e^{ij_k\xi_k}d\xi_k\right| \leq \left(|t|\inf_{\xi\in I_l}\left|\left(\sin^2\left(\frac{\xi}{2}\right)\right)''\right|\right)^{-1/2}.$$

This proves (3.3.6) and finishes the proof. □

3.3.1. Strichartz estimates in the class of filtered data

We say that (q,r) is an α-admissible pair (cf. [25], [74]) if

$$\frac{1}{q} = \alpha\left(\frac{1}{2}-\frac{1}{r}\right), \tag{3.3.7}$$

where r satisfies

$$\begin{cases} 2\leq r<\frac{2d}{d-2}, & d\geq 3,\\ 2\leq r<\infty, & d=1,2. \end{cases}$$

For continuous Schrödinger equations $\alpha = d/2$.

There is a large literature concerning the integrability properties of the Schrödinger semigroup. One of the results which reduces the computations is due to Keel and Tao, [74]. We state the original result

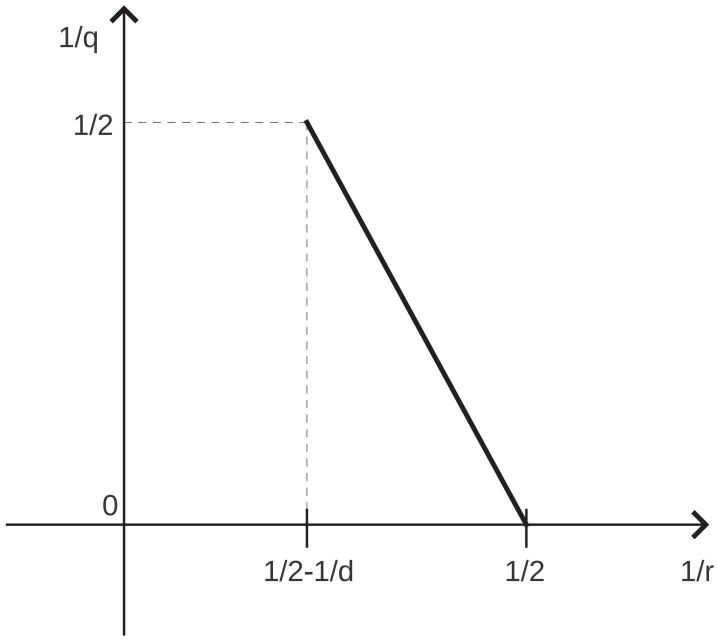

Figure 3.13: $d/2$-Admissible set in dimension $d \geq 3$.

Proposition 3.3.1. *([74], Theorem 1.2) Let H be a Hilbert space, (X, dx) be a measure space and $U(t) : H \to L^2(X)$ be a one parameter family of mappings, which obey the energy estimate*

$$\|U(t)f\|_{L^2(X)} \leq C\|f\|_H \tag{3.3.8}$$

and the decay estimate

$$\|U(t)U(s)^*g\|_{L^\infty(X)} \leq C|t-s|^{-\sigma}\|g\|_{L^1(X)} \tag{3.3.9}$$

for some $\sigma > 0$. Then

$$\|U(t)f\|_{L^q(\mathbb{R},\,L^r(X))} \leq C\|f\|_{L^2(X)},$$
$$\left\|\int_{\mathbb{R}}(U(s))^*F(s,\cdot))ds\right\|_{L^2(X)} \leq C\|F\|_{L^{q'}(\mathbb{R},\,L^{r'}(X))},$$
$$\left\|\int_0^t U(t)(U(s))^*F(s)ds\right\|_{L^q(\mathbb{R},\,L^r(X))} \leq C\|F\|_{L^{\tilde{q}'}(\mathbb{R},\,L^{\tilde{r}'}(X))}$$

for all (q, r) and $(\tilde{q}, \tilde{r})$, σ-admissible pairs.

Our goal is to apply this result to the operator $T^h_\eta(t)$. This yields space-time integrability properties similar to the continuous one. The two operators $T^h_\eta(t)$ and $S^h(t)$ identically acts on the class of filtered data $\mathcal{I}^h_\epsilon$. Thus we obtain the same estimates for the operator $S^h(t)$.

Our main result is the following:

Theorem 3.3.2. *Let $\epsilon > 0$ and (q, r), $(\tilde{q}, \tilde{r})$ two admissible-pairs.*
i) There exists a positive constant $C(d, r, \epsilon)$ such that

$$\|S^h(\cdot)\varphi\|_{L^q(\mathbb{R},\,l^r(h\mathbb{Z}^d))} \leq C(d, r, \epsilon)\|\varphi\|_{l^2(h\mathbb{Z}^d)} \tag{3.3.10}$$

holds for all functions $\varphi \in \mathcal{I}_\epsilon^h$ *and for all* $h > 0$.
ii) There exists a positive constant $C(d,r,\epsilon)$ *such that*

$$\left\| \int_{\mathbb{R}} S^h(s)^* f(s) ds \right\|_{l^2(h\mathbb{Z}^d)} \leq C(d,r,\epsilon) \|f\|_{L^{q'}(\mathbb{R},\, l^{r'}(h\mathbb{Z}^d)))} \tag{3.3.11}$$

holds for all functions f *with* $f(t) \in \mathcal{I}_\epsilon^h$ *and for all* $h > 0$.
iii) There exists a positive constant $C(d,r,\tilde{r},\epsilon)$ *such that*

$$\left\| \int_0^t S^h(t-s) f(s) ds \right\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \leq C(d,r,\tilde{r},\epsilon) \|f\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))} \tag{3.3.12}$$

holds for all functions f *with* $f(t) \in \mathcal{I}_\epsilon^h$ *and for all* $h > 0$.

Proof of Theorem 3.3.2. In Proposition 3.3.1 we consider the space $X = h\mathbb{Z}^d$, the counting measure dx and $U(t) = T_\eta^h(t)$. Estimate (3.3.3) shows that the hypothesis (3.3.8) and (3.3.9) are verified. Thus $T_\eta^h(t)$ satisfies

$$\|T_\eta^h(\cdot)\varphi\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \leq C(d,r,\eta)\|\varphi\|_{l^2(h\mathbb{Z}^d)},$$

$$\left\| \int_{\mathbb{R}} T_\eta^h(s)^* f(s) ds \right\|_{l^2(h\mathbb{Z}^d)} \leq C(d,r,\eta) \|f\|_{L^{q'}(\mathbb{R},\, l^{r'}(h\mathbb{Z}^d)))}$$

and

$$\left\| \int_0^t T_\eta^h(t-s) f(s) ds \right\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \leq C(d,\eta,r,\tilde{r}) \|f\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))}.$$

Using that the two operators T_η^h and $S^h(t)$ act identically on I_ϵ^h we get a similar result for $S^h(t)$. This proves (3.3.10), (3.3.11) and (3.3.12) and finishes the proof. □

3.3.2. Local smoothing effect

As we pointed before the second pathology of the semidiscrete symbol is that its gradient vanishes at the points $(\pm\pi/h)^d$. Filtering these critical points allows us to recover the local smoothing effect of the continuous model. As we have we seen in Section 3.2.2 it cannot be uniform with respect to the mesh size h. This property is relevant in the analysis of the convergence of our models for the nonlinear problems. Here we will state the result without proof.

For a positive ϵ, let us define the set A_ϵ^h of all points situated at a distance at least ϵ from the points $(\pm\pi/h)^d$:

$$A_\epsilon^h = \left\{ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d ; \left| \xi - \left(\pm\frac{\pi}{h}\right)^d \right| \geq \epsilon \right\}.$$

Observe (see Figure 3.14) that A_ϵ^h contains all the point of $[-\pi/h, \pi/h]^d$ situated at a distance at least ϵ from the corners $(\pm\pi/h)^d$. Exactly at these points the gradient of the symbol $p_h(\xi)$ vanishes. On the set A_ϵ^h the symbol $p_h(\xi)$ has no critical points far from the origin. A similar argument as in [75] shows that the linear semigroup $S^h(t)$ gains 1/2-space derivative in $L^2_{t,x}$ with respect to the initial datum.

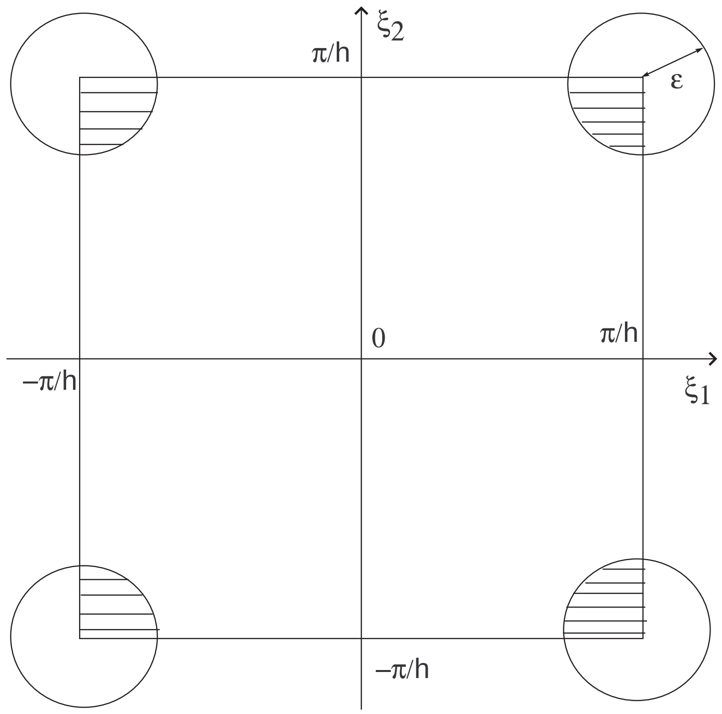

Figure 3.14: The set A^h_ϵ is the complement of the dashed area.

Theorem 3.3.3. *Let $\epsilon > 0$. There exists a positive constant $C(\epsilon, d, T)$ such that*

$$\sup_{R>0} h^d \sum_{|\mathbf{j}h|\le R} \int_{-\infty}^{\infty} |(D_h^{1/2} e^{it\Delta_h}\varphi)_{\mathbf{j}}|^2 dt \le C(q,d,T)\|\varphi\|^2_{l^2(h\mathbb{Z}^d)}$$

holds for all $\varphi \in l^2(h\mathbb{Z}^d)$, uniformly on $h > 0$.

3.4. A dissipative scheme

In the previous section we have analyzed the Fourier filtering method for the conservative scheme (3.1.15). Another possible remedy is to introduce a scheme containing numerical viscosity term in order to compensate the artificial numerical dispersion.

We propose the following viscous semidiscretization of (3.1.1):

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = ia(h)\mathrm{sgn}(t)\Delta_h u^h, \ t \neq 0, \\ u^h(0) = \varphi^h, \end{cases} \tag{3.4.1}$$

where $a(h)$ is a positive function which tends to 0 as h tends to 0. We remark that the proposed scheme is a combination of the conservative approximation of the Schrödinger equation and a semidiscretization of the heat equation in a suitable time-scale. More precisely, the scheme

$$\frac{du^h}{dt} = a(h)\Delta_h u^h, \ t > 0,$$

which is underlined in (3.4.1) may be viewed as a discretization of

$$u_t = a(h)\Delta u,\ t > 0,$$

which is, indeed, a heat equation in the appropriate time-scale. The scheme (3.4.1) generates a semigroup $S_+^h(t)$, for $t > 0$. Similarly one may define $S_-^h(t)$, for $t < 0$. In the sequel we denote by $S^h(t)$ the two operators. Using classical semigroup theory we deduce that

$$S^h(\cdot) \in C(\mathbb{R}, l^2(h\mathbb{Z}^d)) \cap C^\omega(\mathbb{R}\backslash\{0\}, l^2(h\mathbb{Z}^d)).$$

In this section we will obtain norm decay estimates for the operator $S^h(t)$. We first analyze the $l^1(h\mathbb{Z}^d) - l^\infty(h\mathbb{Z}^d)$ decay of $S^h(t)$. In contrast with the continuous case where $\|u(t)\|_{L^\infty(\mathbb{R})} \lesssim t^{-1/2}$ for all $t \neq 0$, the behaviour of the l^∞-norm of the solutions will be different when $t \to 0$ and when $t \to \infty$. The low frequency component of the solution u^h gives the behaviour for large time t, similar to the continuous one $t^{-1/2}$. For $t \sim 0$, the behaviour is given by the high frequency component.

Once the $l^{p'}(h\mathbb{Z}^d) - l^p(h\mathbb{Z}^d)$ analysis will been done, we will give in Section 3.4.1 Strichartz-like estimates for the linear operator $S^h(t)$.

In Section 3.4.2 we introduce a higher-order dissipative scheme by replacing the dissipative term Δ_h by a higher order one Δ_h^m, $m \geq 2$. In this case the dissipative term is strong enough to recover the $l^\infty(h\mathbb{Z}^d)$ behaviour of solutions for small time as well: $t^{-1/2}$. In view of Proposition 3.3.1 we will obtain Strichartz-like estimates for $S^h(t)$ similar to the continuous case.

Finally, we give an application to a nonlinear problem. We will consider a numerical scheme based on the dissipative scheme (3.4.1). The same can be done in the case of the scheme with a higher order dissipative term.

The main result for the approximation scheme (3.4.1) is given by the following Theorem.

Theorem 3.4.1. *Let $p \in [2, \infty]$, $\alpha > d/2$ and $a(h)$ be a positive function such that*

$$\inf_{h>0} \frac{a(h)}{h^{2-\frac{d}{\alpha}}} > 0. \tag{3.4.2}$$

Then $S^h(t)$ maps continuously $l^{p'}(h\mathbb{Z}^d)$ to $l^p(h\mathbb{Z}^d)$ and there exist positive constants $c(d,p,\alpha)$ such that

$$\|S^h(t)(S^h(s))^*\varphi\|_{l^p(h\mathbb{Z}^d)} \leq c(d,p,\alpha)\left[\frac{1}{|t-s|^{\frac{d}{2}(1-\frac{2}{p})}} + \frac{1}{|t-s|^{\alpha(1-\frac{2}{p})}}\right]\|\varphi\|_{l^{p'}(h\mathbb{Z}^d)} \tag{3.4.3}$$

holds for all $t \neq s$, $\varphi \in l^{p'}(h\mathbb{Z})$ and $h > 0$.

Remark 3.4.1. *For $s = 0$ we obtain that $S^h(t)$ satisfies*

$$\|S^h(t)\|_{l^p(h\mathbb{Z})} \leq c(d,p,\alpha)\left[\frac{1}{|t|^{\frac{d}{2}(1-\frac{2}{p})}} + \frac{1}{|t|^{\alpha(1-\frac{2}{p})}}\right]\|\varphi\|_{l^{p'}(h\mathbb{Z}^d)}. \tag{3.4.4}$$

The decay of $S^h(t)$ for large time t is the same as in the continuous case. However, for more general space-time estimates, the behaviour at $t = 0$ is important. According the (3.4.4) the behaviour at $t \sim 0$ is more singular since $\alpha > d/2$. As we will see in the proof the first term in the estimate (3.4.4) given by the low-frequencies and the second one by the high-frequencies.

Remark 3.4.2. *The condition imposed to $a(h)$ in (3.4.2) guarantees that the high frequency component of the fundamental solutions of (3.4.1) behaves in $l^\infty(h\mathbb{Z}^d)$-norm as $|t|^{-\alpha}$.*

Observe that the operator $S^h(t)$ satisfies the semigroup property only restricted to the sets $\{t : t \geq 0\}$, $\{t : t \leq 0\}$ and not on the whole line $\mathbb{R}$. Then for any t and s having the same sign:

$$S^h(t)(S^h(s))^* = S^h(t)S^h(-s) \neq S^h(t-s),\ ts > 0.$$

Thus we cannot reduce directly (3.4.3) to the case $s = 0$. However, the properties of the semidiscrete heat operator $e^{|t|\Delta_h}$ will guarantee that for any $t, s \in \mathbb{R}$:

$$\|S^h(t)(S^h(s)^*)\varphi\|_{l^p(h\mathbb{Z}^d)} \leq \|S^h(t-s)\varphi\|_{l^p(h\mathbb{Z}^d)}, \tag{3.4.5}$$

and this allows reducing (3.4.3) to the case $s = 0$. Observe that, according to (3.4.5), $S^h(t)(S^h(s))^*$ is more dissipative than $S^h(t-s)$.

The proof is divided in several steps. First we prove that it is sufficient to analyze the case $s = 0$. In this case we write the solutions as the convolution of the fundamental solutions $K^h(t)$ with the initial data. This reduces the proof of (3.4.3) to estimates on $K^h(t)$. The fundamental solutions will be split in two frequency components: low and high frequencies : $K^{h,1}$ and $K^{h,2}$ respectively. The first one behaves as in the conservative case. For $K^{h,2}$ we use the dissipation effects introduced by our scheme.

Proof of Theorem 3.4.1. In order to simplify the notation we omit the index h for the solutions of equation (3.4.1). We write the equation (3.4.1) in the Fourier variable:

$$\begin{cases} i\dfrac{d\widehat{u}}{dt}(t,\xi) - p_h(\xi)\widehat{u}(t,\xi) = -ia(h)\,\mathrm{sgn}(t)p_h(\xi)\widehat{u}(t,\xi), \quad t \neq 0,\ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d, \\ \\ \widehat{u}(0,\xi) = \widehat{\varphi}(\xi),\ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d. \end{cases}$$

Then for all $t \in \mathbb{R}$ and $\xi \in [-\pi/h, \pi/h]^d$ the solution $u(t) = S^h(t)\varphi$ satisfies

$$\mathcal{F}_h(S^h(t)\varphi)(\xi) = \exp(-itp_h(\xi) - |t|a(h)p_h(\xi))\mathcal{F}_h\varphi(\xi) \tag{3.4.6}$$

and

$$(S^h(t)\varphi)_{\mathbf{j}} = (e^{it\Delta_h}e^{|t|a(h)\Delta_h}\varphi)_{\mathbf{j}} = \int_{[-\frac{\pi}{h},\frac{\pi}{h}]^d} e^{-itp_h(\xi)}e^{-|t|a(h)p_h(\xi)}e^{i\mathbf{j}\cdot\xi h}\widehat{\varphi}(\xi)d\xi,\ \mathbf{j} \in \mathbb{Z}^d. \tag{3.4.7}$$

Observe that for any $t \in \mathbb{R}$ the operator $S^h(t)$ satisfies

$$(S^h(t))^* = S^h(-t) \quad \text{and} \quad S^h(-t)\varphi = \overline{S^h(t)\overline{\varphi}}.$$

Step 1. Reduction to the case $s = 0$.

We will reduce the proof of (3.4.3) to the case $s = 0$. By (3.4.6) and (3.4.7) we get for any $t, s \in \mathbb{R}$:

$$\begin{aligned} S^h(t)(S^h(s))^* &= S^h(t)S^h(-s) = e^{it\Delta_h}e^{|t|a(h)\Delta_h}e^{-is\Delta_h}e^{|s|a(h)\Delta_h} \\ &= e^{i(t-s)\Delta_h}e^{a(h)|t-s|\Delta_h}e^{(|t|+|s|-|t-s|)a(h)\Delta_h} \\ &= S^h(t-s)e^{(|t|+|s|-|t-s|)a(h)\Delta_h}. \end{aligned}$$

The operator $\exp((|t|+|s|-|t-s|)a(h)\Delta_h)$ is exactly the semidiscrete heat operator studied in Chapter 2 at the time $(|t|+|s|-|t-s|)a(h)$. Let us assume that (3.4.3) holds for $s=0$, i.e.:

$$\|S^h(t)\varphi\|_{l^p(h\mathbb{Z}^d)} \leq c(d,p,\alpha)\left[\frac{1}{|t|^{\frac{d}{2}(1-\frac{2}{p})}}+\frac{1}{|t|^{\alpha(1-\frac{2}{p})}}\right]\|\varphi\|_{l^{p'}(h\mathbb{Z}^d)},\ t\neq 0. \tag{3.4.8}$$

The properties of the semidiscrete heat equation (see Chapter 2) give us that the operator $\exp(|t|\Delta_h)$ is uniformly stable in any $l^q(h\mathbb{Z}^d)$-norm, $q\geq 1$:

$$\|\exp(|t|\Delta_h)\varphi\|_{l^q(h\mathbb{Z}^d)}\leq \|\varphi\|_{l^q(h\mathbb{Z}^d)},\ t\in\mathbb{R}.$$

Thus, for any $t\neq s$ the operator $S^h(t)S^h(s)^*$ satisfies:

$$\begin{aligned}
\|S^h(t)(S^h(s))^*\varphi\|_{l^p(h\mathbb{Z}^d)} &= \|S^h(t-s)e^{(|t|+|s|-|t-s|)a(h)\Delta_h}\varphi\|_{l^p(h\mathbb{Z}^d)}\\
&\leq c(d,p,\alpha)\left[\frac{1}{|t-s|^{\frac{d}{2}(1-\frac{2}{p})}}+\frac{1}{|t-s|^{\alpha(1-\frac{2}{p})}}\right]\|e^{(|t|+|s|-|t-s|)a(h)\Delta_h}\varphi\|_{l^{p'}(h\mathbb{Z}^d)}\\
&\leq c(d,p,\alpha)\left[\frac{1}{|t-s|^{\frac{d}{2}(1-\frac{2}{p})}}+\frac{1}{|t-s|^{\alpha(1-\frac{2}{p})}}\right]\|\varphi\|_{l^{p'}(h\mathbb{Z}^d)}.
\end{aligned}$$

Step 2. The case $s=0$.

We will prove (3.4.8) for $p=2$ and $p=\infty$. The other cases follow by interpolation. The case $p=2$ is a simple consequence of Plancherel's identity. Let us now analyze the case $p=\infty$. At any time t we write the $S^h(t)$ as the a discrete convolution operator:

$$S^h(t)\varphi = K^h(t)*\varphi, \tag{3.4.9}$$

where the kernel $K^h(t)$ is given by

$$K^h(t,\mathbf{j})=\int_{[-\frac{\pi}{h},\frac{\pi}{h}]^d} e^{-itp_h(\xi)}e^{-|t|a(h)p_h(\xi)}e^{i\mathbf{j}\cdot\xi h}d\xi,\ \mathbf{j}\in\mathbb{Z}^d.$$

Young's inequality applied to (3.4.9) reduces the proof to the following estimate on $K^h(t)$:

$$|K^h(t,\mathbf{j})|\leq c(d,p,\alpha)\left[\frac{1}{|t|^{\frac{d}{2}}}+\frac{1}{|t|^{\alpha}}\right],\ t\neq 0,\ \mathbf{j}\in\mathbb{Z}^d. \tag{3.4.10}$$

With the notation

$$\Omega_h=\left[-\frac{\pi}{h},\frac{\pi}{h}\right]^d\setminus\left[-\frac{\pi}{4h},\frac{\pi}{,}4h\right]^d.$$

we split the kernel $K^h(t,\cdot)$ in two parts $K^{h,1}(t,\cdot)+K^{h,2}(t,\cdot)$ where

$$K^{h,1}(t,\mathbf{j})=\int_{[-\frac{\pi}{4h},\frac{\pi}{4h}]^d} e^{-itp_h(\xi)}e^{-|t|a(h)p_h(\xi)}e^{i\mathbf{j}\cdot\xi h}d\xi,\ \mathbf{j}\in\mathbb{Z}^d$$

and

$$K^{h,2}(t,\mathbf{j})=\int_{\Omega_h} e^{-itp_h(\xi)}e^{-|t|a(h)p_h(\xi)}e^{i\mathbf{j}\cdot\xi h}d\xi,\ \mathbf{j}\in\mathbb{Z}^d.$$

The kernel $K^{h,1}(t)$ will behave as the conservative kernel when applied to initial data belonging to the *low frequency* domain $[-\pi/4h, \pi/4h]^d$. Since the Hessian matrix $H_{p_h}(\xi) = (\partial_{ij}p_h(\xi))_{i,j=1}^d$ has no vanishing components on the diagonal and thus its rank is always d, the same is true in any cube $[-(\pi/2-\epsilon)/h, (\pi/2-\epsilon)/h]^d$. In other words, no artificial viscosity is needed in this low frequency range.

To estimate the second kernel $K^{h,2}(t)$ we use in an essential way the dissipative effect introduced by the term $\exp(-|t|p_h(\xi))$ far from the origin. These two kernels give the terms $|t|^{-d/2}$ and $|t|^{-\alpha}$ respectively in (3.4.10).

The kernel $K^{h,2}$ satisfies for all $t\neq 0$ and $\mathbf{j}\in\mathbb{Z}^d$ the following rough estimate:

$$\begin{aligned}
|K^{h,2}(t,\mathbf{j})| &\leq \int_{\Omega_h} e^{-|t|a(h)p_h(\xi)}d\xi \leq \int_{\Omega_h}\exp\left(-4d\sin^2(\frac{\pi}{8})|t|\frac{a(h)}{h^2}\right)d\xi\\
&\leq \frac{c(d)}{h^d}\exp\left(-4d\sin^2\left(\frac{\pi}{8}\right)|t|\frac{a(h)}{h^2}\right)\leq \frac{c(\alpha,d)}{h^d}\left(\frac{h^2}{|t|a(h)}\right)^{\alpha}\\
&\leq \frac{c(\alpha,d)}{|t|^{\alpha}}\left(\frac{h^{2-\frac{d}{\alpha}}}{a(h)}\right)^{\alpha}\leq \frac{c(\alpha,d)}{|t|^{\alpha}}\left[\inf_{h>0}\frac{a(h)}{h^{2-d/\alpha}}\right]^{-\alpha}\\
&\leq \frac{c(\alpha,d)}{|t|^{\alpha}}.
\end{aligned}$$

This gives us the second term in the right hand side of (3.4.10).

Going back to $K^{1,h}$, it is convenient to rewrite it as a convolution:

$$K^{h,2}(t,\cdot)=K^{h,3}(t,\cdot)*H^h(ta(h),\cdot)$$

where $K^{h,3}(t,\cdot)$ is the conservative semidiscrete kernel restricted to the set $[-\pi/4h,\pi/4h]^d$:

$$K^{h,3}(t,\mathbf{j})=\int_{[-\frac{\pi}{4h},\frac{\pi}{4h}]^d} e^{-itp_h(\xi)}e^{i\mathbf{j}\cdot\xi h}d\xi,\ t\in\mathbb{R},\ \mathbf{j}\in\mathbb{Z}^d \tag{3.4.11}$$

and $H^h(t,\cdot)$ is the semidiscrete heat kernel at the time $|t|$:

$$H^h(t,\mathbf{j})=\int_{[-\frac{\pi}{h},\frac{\pi}{h}]^d} e^{-|t|p_h(\xi)}e^{i\mathbf{j}\cdot\xi h}d\xi,\ t\in\mathbb{R},\ \mathbf{j}\in\mathbb{Z}^d.$$

As we proved in Chapter 2 the semidiscrete heat kernel satisfies

$$\|H^h(t)\|_{l^1(h\mathbb{Z}^d)}\leq C(d),\ t\in\mathbb{R}.$$

Thus, the estimates on $K^{h,2}(t,\cdot)$ are reduced to the ones on $K^{h,3}(t,\cdot)$:

$$\|K^{h,2}(t)\|_{l^\infty(h\mathbb{Z}^d)}\leq \|K^{h,3}(t)\|_{l^\infty(h\mathbb{Z}^d)}\|H^h(ta(h))\|_{l^1(h\mathbb{Z}^d)}\leq C(d)\|K^{h,3}(t)\|_{l^\infty(h\mathbb{Z}^d)}. \tag{3.4.12}$$

Remark that $K^{h,3}$ corresponds to filtered initial data involving only frequencies on the set $[-\pi/4h,\pi/4h]^d$. As we have proved in Section 3.3 in this case we have the same decay as in the continuous case:

$$\|K^{h,3}(t)\|_{L^\infty(h\mathbb{Z}^d)}\leq \frac{C(d)}{|t|^{d/2}},\ t\neq 0. \tag{3.4.13}$$

The proof is now complete. □

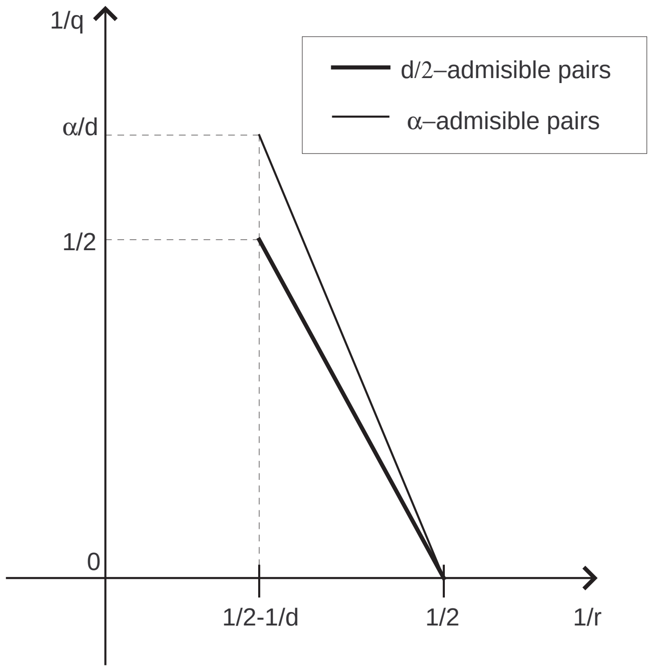

Figure 3.15: $d/2$ and α-admissible pairs in dimension $d \geq 3$.

3.4.1. Strichartz like Estimates

In this Section we derive more general estimates on the linear operator $S^h(t)$. The estimates are different from the ones obtained in the continuous case. Observe that the behaviour of the semigroup as $t \to \infty$ and $t \to 0$ is different. As a consequence, the estimates will not be in the same spaces as in the continuous case. This is the reason why the estimates in Theorem 3.4.2 hold in spaces of the form $L^q(\mathbb{R}, l^r(h\mathbb{Z}^d)) + L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))$. A careful analysis of the Hardy-Littlewood-Sobolev (HLS) inequality (cf. [117], p. 119, Ch. V.1, Theorem 1) shows that we have to consider both spaces. More precisely the term $|t-s|^{-d/2}$ in the $l^\infty(h\mathbb{Z}^d)$-norm of $S(t)S(s)^*$ gives us estimates in the space $L^q(\mathbb{R}, l^r(h\mathbb{Z}^d))$ with (q,r) an 1/2-admissible pair. The second one, $|t-s|^{-\alpha}$, provides estimates in the space $L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))$ with (q_1, r) an α-admissible pair. Obviously these estimates provide local estimates in the space $L^{q_1}(I, l^r(h\mathbb{Z}^d))$. This fails on unbounded time intervals, where the L^q-norms cannot be compared. The local in time estimate is a consequence of the fact that $\alpha > d/2$ and $q > q_1$. We recall that the Strichartz estimates are used to prove the local existence of the nonlinear problem. So the local version of them suffices to prove the local well possed of the nonlinear problem. In fact in Section 3.5 we use the local results of Corollary 3.4.1 and not the global ones of Theorem 3.4.2.

Theorem 3.4.2. *Let $\alpha \in (d/2, d]$ and $a(h)$ be satisfying (3.4.2). Also let us consider (q,r), $(\tilde{q}, \tilde{r})$, 1/2-admissible pairs and (q_1, r), $(\tilde{q}_1, \tilde{r})$, α-admissible pairs. Then*
i) There exists a positive constant $C(d,\alpha,r)$ such that

$$\|S^h(\cdot)\varphi\|_{L^q(\mathbb{R}, l^r(h\mathbb{Z}^d)) + L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))} \leq C(d,\alpha,r)\|\varphi\|_{l^2(h\mathbb{Z}^d)} \tag{3.4.14}$$

holds for all $\varphi \in l^2(h\mathbb{Z}^d)$, uniformly on $h > 0$.

ii) There exists a positive constant $C(d,\alpha,r)$ such that

$$\left\|\int_{\mathbb{R}} S^h(s)^* f(s)ds\right\|_{l^2(h\mathbb{Z}^d)} \leq C(d,\alpha,r)\|f\|_{L^{q'}(\mathbb{R},l^{r'}(h\mathbb{Z}^d))\cap L^{q_1'}(\mathbb{R},l^{r'}(h\mathbb{Z}^d))} \tag{3.4.15}$$

holds for all $f\in L^{q'}(\mathbb{R},\, l^{r'}(h\mathbb{Z}^d))\cap L^{q_1'}(\mathbb{R},\, l^{r'}(h\mathbb{Z}^d))$, uniformly on $h>0$.
iii) There exists a positive constant $C(d,\alpha,r,r_1)$ such that

$$\left\|\int_0^t S^h(t-s)f(s)ds\right\|_{L^q(\mathbb{R},l^r(h\mathbb{Z}^d))+L^{q_1}(\mathbb{R},l^r(h\mathbb{Z}^d))} \leq C(d,\alpha,r,\tilde{r})\|f\|_{L^{\tilde{q}'}(\mathbb{R},l^{\tilde{r}'}(h\mathbb{Z}^d))\cap L^{\tilde{q}_1'}(\mathbb{R},l^{\tilde{r}'}(h\mathbb{Z}^d))}. \tag{3.4.16}$$

holds for all $f\in L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))\cap L^{\tilde{q}_1'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))$, uniformly on $h>0$.

The following Corollary represents a simple consequence of the above Theorem. It uses only the definition of the sum space and Hölder's inequality.

Corollary 3.4.1. *Let $I\subset\mathbb{R}$ be a bounded interval, (q,r), $(\tilde{q},\tilde{r})$, $1/2$-admissible pairs and (q_1,r), $(\tilde{q}_1,\tilde{r})$, α-admissible pairs. Then*

i) There exists a positive constant $C(d,\alpha,r)$ such that

$$\|S^h(t)\varphi\|_{L^{q_1}(I,l^r(h\mathbb{Z}^d))} \leq C(d,\alpha,r)(|I|^{\frac{q-q_1}{q}}+1)\|\varphi\|_{l^2(h\mathbb{Z}^d)}. \tag{3.4.17}$$

ii) There exists a positive constant $C(d,\alpha,r,r_1)$ such that

$$\left\|\int_{s<t} S^h(t-s)f(s)ds\right\|_{L^{q_1}(I,l^r(h\mathbb{Z}^d))} \leq C(d,\alpha,r,\tilde{r})(|I|^{\frac{q-q_1}{q}}+1)(|I|^{\frac{\tilde{q}_1'-\tilde{q}'}{\tilde{q}_1'}}+1)\|f\|_{L^{\tilde{q}_1'}(I,l^{\tilde{r}'}(h\mathbb{Z}^d))}. \tag{3.4.18}$$

Remark 3.4.3. *All the involved constants occur from the HLS inequality and interpolation between the involved spaces. As proved in [82] (see also [117] and [84]) the HLS inequality for Rietz potentials writes*

$$\||x|^{-2/q}f\|_{L^q(\mathbb{R})} \leq \pi^{1/q}\frac{\Gamma(\frac{1}{2}-\frac{1}{q})}{\Gamma(1-\frac{1}{q})}\left(\frac{\Gamma(\frac{1}{2})}{\Gamma(1)}\right)^{-1+\frac{2}{q}}\|f\|_{L^{q'}(\mathbb{R})}, \tag{3.4.19}$$

where the constants involved are optimal. In our case

$$\frac{1}{q}=\alpha\left(\frac{1}{2}-\frac{1}{r}\right),\qquad r\in\left[2,\frac{2d}{d-2}\right).$$

This implies that for any $\alpha\in(d/2,d]$ the following holds

$$0\leq\frac{1}{q}<d\left(\frac{1}{2}-\frac{d-2}{2d}\right)=1.$$

Thus, all the above constants remain bounded as $\alpha\to d/2$.

The proof is divided in three steps. First we show that (3.4.15) implies (3.4.14). Thus it is sufficient to analyze (3.4.15). The proof of this estimate is by now standard (see [24], [74]), but we have to take into account the different behaviour of $S^h(t)$ for $t\sim 0$ and $t\sim\infty$. The last step contains the proof of (3.4.16).

Proof. **Step I. (3.4.15) implies (3.4.14).**
We reduce the proof of estimate (3.4.14) to the second one (3.4.15). By duality (cf. [7], p. 32, Ch. 2, Th. 2.7.1):

$$\|S^h(\cdot)\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))+L^{q_1}(\mathbb{R},\, l^r(h\mathbb{Z}^d))} = \sup_{\|\psi\|_{L^{q'}(\mathbb{R},\, l^{r'}(h\mathbb{Z}^d))\cap L^{q_1'}(\mathbb{R},\, l^{r'}(h\mathbb{Z}^d))}\leq 1} \langle\langle S^h(\cdot)\varphi, \psi\rangle\rangle,$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ is the inner product on $L^2(\mathbb{R},\, l^2(h\mathbb{Z}^d))$. The last term can be estimated as follows:

$$\begin{aligned}
|\langle\langle S^h(\cdot)\varphi, \psi\rangle\rangle| &= \left|\int_{\mathbb{R}} \langle S^h(s)\varphi, \psi(s)\rangle ds\right| = \left|\int_{\mathbb{R}} \langle \varphi, S^h(s)^*\psi(s)\rangle ds\right| \\
&= \left|\left\langle \varphi, \int_{\mathbb{R}} S^h(s)^*\psi(s) ds\right\rangle\right| \\
&\leq \|\varphi\|_{l^2(h\mathbb{Z}^d)} \left\|\int_{\mathbb{R}} S^h(s)^*\psi(s) ds\right\|_{l^2(h\mathbb{Z}^d)}.
\end{aligned}$$

Thus, it suffices to prove (3.4.15).

Step II. Proof of (3.4.15).
By duality, (3.4.15) turns to be equivalent to the bilinear estimate

$$\left|\left\langle \int_{\mathbb{R}} S^h(t)^* f(t) dt, \int_{\mathbb{R}} S^h(s)^*\psi(s) ds\right\rangle\right| \lesssim \|f\|_{L^{q'}(l^{r'})\cap L^{q_1'}(l^{r'})} \|\psi\|_{L^{q'}(l^{r'})\cap L^{q_1'}(l^{r'})}.$$

We prove that the following holds:

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}} |\langle S^h(t)^* f(t), S^h(s)^*\psi(s)\rangle| dt ds \leq \\
&\qquad \leq (\|f\|_{L^{q'}(l^{r'}(h\mathbb{Z}^d))} + \|f\|_{L^{q_1'}(l^{r'}(h\mathbb{Z}^d))})(\|\psi\|_{L^{q'}(l^{r'}(h\mathbb{Z}^d))} + \|\psi\|_{L^{q_1'}(l^{r'}(h\mathbb{Z}^d))}).
\end{aligned}$$

Using estimate (3.4.3) on $S^h(t)S^h(s)^*$ we get

$$\begin{aligned}
|\langle S^h(t)^* f(t), S^h(s)^*\psi(s)\rangle| &= |\langle f(t), S^h(t)S^h(s)^* g(s)\rangle| \\
&\leq \|f(t)\|_{l^{r'}(h\mathbb{Z}^d)} \|S^h(t)S^h(s)^* g(s)\|_{l^r(h\mathbb{Z}^d)} \\
&\leq \|f(t)\|_{l^{r'}(h\mathbb{Z}^d)} \|g(s)\|_{l^{r'}(h\mathbb{Z}^d)} \left(\frac{1}{|t-s|^{2/q}} + \frac{1}{|t-s|^{2/q_1}}\right).
\end{aligned}$$

Integrating both in t and s we obtain

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}} |\langle S^h(t)^* f(t), S^h(s)^*\psi(s)\rangle| ds dt &\leq \int_{\mathbb{R}}\int_{\mathbb{R}} \|f(t)\|_{l^{r'}} \|g(s)\|_{l^{r'}} \left(\frac{1}{|t-s|^{2/q}} + \frac{1}{|t-s|^{2/q_1}}\right) ds dt \\
&\lesssim \int_{\mathbb{R}_t} \|f(t)\|_{l^{r'}(h\mathbb{Z}^d)} \int_{|t-s|<1} \frac{\|g(s)\|_{l^{r'}(h\mathbb{Z}^d)} ds}{|t-s|^{2/q_1}} dt \\
&\quad + \int_{\mathbb{R}_t} \|f(t)\|_{l^{r'}(h\mathbb{Z}^d)} \int_{|t-s|>1} \frac{\|g(s)\|_{l^{r'}(h\mathbb{Z}^d)} ds}{|t-s|^{2/q}} dt.
\end{aligned}$$

Let us denote by Γ_1 and Γ_2 the operators

$$(\Gamma_1\psi)(t) = \int_{|t-s|<1} \frac{\psi(s)ds}{|t-s|^{2/q_1}}$$

and

$$(\Gamma_2\psi)(t) = \int_{|t-s|>1} \frac{\psi(s)ds}{|t-s|^{2/q}}.$$

Applying Holder's inequality we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |\langle S^h(t)^* f(t), S^h(s)^*\psi(s)\rangle| dsdt \leq$$
$$\leq \|f(t)\|_{L^{q_1'}(l^{r'})} \|\Gamma_1(\|g\|_{l^{r'}})\|_{L^{q_1}(\mathbb{R})} + \|f(t)\|_{L^{q'}(l^{r'})} \|\Gamma_2(\|g\|_{l^{r'}})\|_{L^{q}(\mathbb{R})}.$$

So, it suffices to prove that the two operators Γ_1, Γ_2 satisfy

$$\|\Gamma_1\psi\|_{L^{q_1}(\mathbb{R})} \leq \|\psi\|_{L^{q_1'}(\mathbb{R})} \quad \text{and} \quad \|\Gamma_2\psi\|_{L^{q}(\mathbb{R})} \leq \|\psi\|_{L^{q'}(\mathbb{R})}. \tag{3.4.20}$$

We have that the operators Γ_1, Γ_2 are given by

$$\Gamma_1\psi = K_1 * \psi \quad \text{and} \quad \Gamma_2\psi = K_2 * \psi$$

where

$$K_1(s) = \begin{cases} \frac{1}{|s|^{2/q_1}} & \text{if } |s|<1, \\ 0 & \text{if } |s|>1 \end{cases} \quad \text{and} \quad K_2(s) = \begin{cases} 0 & \text{if } |s|<1, \\ \frac{1}{|s|^{2/q}} & \text{if } |s|>1. \end{cases}$$

Thus both estimates (3.4.20) are consequences of the HLS inequality (cf. [117], p. 119, Ch. V.1, Theorem 1) applied to each of the two kernels K_1 and K_2.

Step III. Proof on (3.4.16).
We recall that the norm of the sum space $L^q(\mathbb{R}, l^r(h\mathbb{Z}^d)) + L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))$ is given by

$$\|a\| = \inf_{a=a_0+a_1} \left(\|a_0\|_{L^q(\mathbb{R}, l^r(h\mathbb{Z}^d))} + \|a_1\|_{L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))} \right).$$

Let us denote by T_1 and T_2 the operators

$$(T_1 f)(t) = \int_{s<t-1} S^h(t-s) f(s) ds$$

and

$$(T_2 f)(t) = \int_{t-1<s<t} S^h(t-s) f(s) ds.$$

Clearly,

$$\int_{s<t} S^h(t-s) f(s) ds = (T_1 f)(t) + (T_2 f)(t).$$

Using the definition of the sum space $L^q(\mathbb{R}, l^r(h\mathbb{Z}^d)) + L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))$ we have

$$\left\| \int_{s<t} S^h(t-s) f(s) ds \right\|_{L^q(\mathbb{R}, l^r(h\mathbb{Z}^d)) + L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))} \leq \|T_1 f\|_{L^q(\mathbb{R}, l^r(h\mathbb{Z}^d))} + \|T_2 f\|_{L^{q_1}(\mathbb{R}, l^r(h\mathbb{Z}^d))}.$$

Thus, it is sufficient to prove that

$$\|T_1 f\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \le C(d,r,\tilde{r})\|f\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))}$$

and

$$\|T_2 f\|_{L^{q_1}(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \le C(d,r,\tilde{r})\|f\|_{L^{\tilde{q}_1'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))}. \tag{3.4.21}$$

We proceed with the proof of the the last estimate, the proof of the first one being similar. The operator T_2 being linear, the proof of (3.4.21) is reduced to the cases $(\tilde{q}_1,\tilde{r})=(\infty,2)$, $(q_1,r)=(\infty,2)$ and $(q_1,r)=(\tilde{q}_1,\tilde{r})$. The other cases are a consequence of an interpolation between these cases (cf. [7], [74] and [25]). By duality

$$\|T_2 f\|_{L^{q_1}(\mathbb{R},\, l^r(h\mathbb{Z}^d)} = \sup_{\|g\|_{L^{q_1'}(\mathbb{R},\, l^{r'}(h\mathbb{Z}^d))}\le 1} \langle\langle T_2 f, g\rangle\rangle.$$

In all the analyzed cases we use the following property of the operator $T_2 f$:

$$\begin{aligned}
\langle\langle T_2 f, g\rangle\rangle &= \int_{\mathbb{R}_t}\left\langle \int_{t-1}^{t} S^h(t-s)f(s)ds, g(t)\right\rangle dt\\
&= \int_{\mathbb{R}_t}\int_{t-1}^{t}\left\langle S^h(t-s)f(s), g(t)\right\rangle dsdt\\
&= \int_{\mathbb{R}_t}\int_{t-1}^{t}\left\langle f(s), S^h(t-s)^* g(t)\right\rangle dsdt\\
&= \int_{\mathbb{R}_s}\left\langle f(s), \int_{s}^{s+1} S^h(t-s)^* g(t)dt\right\rangle ds.
\end{aligned}$$

Case I: $(\tilde{q}_1,\tilde{r})=(\infty,2)$**.** Applying Cauchy's inequality in the space variable we obtain:

$$\begin{aligned}
\langle\langle T_2 f, g\rangle\rangle &\le \int_{\mathbb{R}_s}\|f(s)\|_{l^2(h\mathbb{Z}^d)}\left\|\int_{s}^{s+1} S^h(t-s)^* g(t)dt\right\|_{l^2(h\mathbb{Z}^d)} ds\\
&\le \|f\|_{L^1(\mathbb{R},\, l^2(h\mathbb{Z}^d))}\sup_{s\in\mathbb{R}}\left\|\int_{s}^{s+1} S^h(t-s)^* g(t)dt\right\|_{l^2(h\mathbb{Z}^d)}\\
&\le \|f\|_{L^1(\mathbb{R},\, l^2(h\mathbb{Z}^d))}\sup_{s\in\mathbb{R}}\left\|\int_{0}^{1} S^h(t)^* g(t+s)dt\right\|_{l^2(h\mathbb{Z}^d)}.
\end{aligned}$$

The arguments used in Step II give us

$$\left\|\int_0^1 S^h(t)^* g(t+s)dt\right\|_{l^2(h\mathbb{Z}^d)} \le \|g(\cdot+s)\|_{L^{q_1'}(\mathbb{R},\, l^{r'}(h\mathbb{Z}^d))} \le 1.$$

This shows that

$$\langle\langle T_2 f, g\rangle\rangle \le \|f\|_{L^1(\mathbb{R},\, l^2(h\mathbb{Z}^d))}.$$

and finishes the proof of the first case.

Case II: $(q,r)=(\infty,2)$**.** With the same notations as above

$$\begin{aligned}\langle\langle T_2 f, g\rangle\rangle &= \int_{\mathbb{R}_t}\left\langle \int_{t-1}^t S^h(t-s)f(s)ds, g(t)\right\rangle dt\\ &\leq \sup_{t\in\mathbb{R}}\left\|\int_{t-1}^t S^h(t-s)f(s)ds\right\|_{l^2(h\mathbb{Z}^d)}\|g\|_{L^1(\mathbb{R},\,l^2(h\mathbb{Z}^d))}.\end{aligned}$$

It remains to prove that for any α-admissible pair $(\tilde{q},\tilde{r})$ the following holds:

$$\left\|\int_{t-1}^t S^h(t-s)f(s)ds\right\|_{L^\infty(\mathbb{R},\,l^2(h\mathbb{Z}^d))}\leq \|f\|_{L^{\tilde{q}'}(\mathbb{R},\,l^{\tilde{r}'}(h\mathbb{Z}^d))}.$$

To do that, we write

$$\int_{t-1}^t S^h(t-s)f(s)ds=\int_{-1}^0 S^h(-s)f(t+s)ds=\int_{-1}^0 S^h(s)^*f(t+s)ds$$

and apply the same arguments as in Step II to the function $f(\cdot+t)$. This implies that

$$\left\|\int_{-1}^0 S^h(s)^*f(t+s)ds\right\|\leq \|f(\cdot+t)\|_{L^{\tilde{q}'}(\mathbb{R},\,l^{\tilde{r}'}(h\mathbb{Z}^d))}=\|f\|_{L^{\tilde{q}'}(\mathbb{R},\,l^{\tilde{r}'}(h\mathbb{Z}^d))}$$

and finishes the second case.

Case III: $(q,r)=(\tilde{q},\tilde{r})$. Observe that T_2f satisfies

$$\|T_2f(t)\|_{l^r(h\mathbb{Z}^d)} \leq \int_{t-1}^t \|S^h(t-s)f(s)\|_{l^r(h\mathbb{Z}^d)}ds\leq \int_{t-1}^t\frac{\|f(s)\|_{l^{r'}(h\mathbb{Z}^d)}}{1+|t-s|^{2/q_1}}ds.$$

The same arguments as in Step II show that

$$\|T_2f\|_{L^{q_1}(\mathbb{R},\,l^r(h\mathbb{Z}^d)}\leq\|f\|_{L^{\tilde{q}_1'}(\mathbb{R},\,l^{\tilde{r}_1'}(h\mathbb{Z}^d))}.$$

This ends the proof. □

Proof of Corollary 3.5.18. First we remark that in Theorem 3.4.2 we can replace the whole line $\mathbb{R}$ by any finite interval I. The two estimates (3.4.17) and (3.4.18) are consequences of properties of the spaces involved in Theorem 3.4.2. Using that $q>q_1$ the definition of these spaces gives us for any finite interval $I\subset\mathbb{R}$ that

$$\|f\|_{L^{q_1}(I)}\leq \max\{1,|I|^{(q-q_1)/q}\}\|f\|_{L^{q_1}(I)+L^q(I)}$$

and

$$\|f\|_{L^{q_1}(I)\cap L^q(I)}\leq (1+|I|^{(q-q_1)/q})\|f\|_{L^q(I)}.$$

Applying these inequalities to the estimates obtained in Theorem 3.4.2 we obtain the desired result. □

3.4.2. A higher order dissipative scheme

Let us consider the following scheme

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h = i(-1)^{m-1}a(h)\Delta_h^m u^h, & t>0, \\ i\dfrac{du^h}{dt} + \Delta_h u^h = -i(-1)^{m-1}a(h)\Delta_h^m u^h, & t<0, \\ u^h(0) = \varphi^h. \end{cases} \tag{3.4.22}$$

In contrast with the scheme introduced before, the term Δ_h is replaced by a higher order one Δ_h^m. This introduces more dissipation in our scheme. Observe that for high frequencies the contribution of the term Δ_h^m is of order $1/h^{2m}$, which is greater than $1/h^2$, introduced by the previous scheme.

In the Fourier space the solution of (3.4.22), $u^h(t) = S^h(t)\varphi^h$, satisfies

$$\mathcal{F}_h(S^h(t)\varphi)(\xi) = \exp(-itp_h(\xi) - |t|a(h)p_h^m(\xi))\mathcal{F}_h\varphi(\xi),\ \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right].$$

The following Theorem shows that in the case $m \geq 2$ we can recover the same behaviour of the solutions as in the continuous case.

Theorem 3.4.3. *Let $m \geq 2$, $p \in [2, \infty]$ and $a(h)$ be a positive function such that*

$$\inf_{h>0} \frac{a(h)}{h^{2m-2}} > 0. \tag{3.4.23}$$

Then $S^h(t)$ maps continuously $l^{p'}(h\mathbb{Z}^d)$ to $l^p(h\mathbb{Z}^d)$ and there exist positive constants $c(d,p,m,a)$ such that

$$\|S^h(t)(S^h(s))^*\varphi\|_{l^p(h\mathbb{Z}^d)} \leq \frac{c(d,p,m,a)}{|t-s|^{\frac{d}{2}(1-\frac{2}{p})}}\|\varphi\|_{l^{p'}(h\mathbb{Z}^d)} \tag{3.4.24}$$

holds for all $t \neq s$, $\varphi \in l^{p'}(h\mathbb{Z})$ and $h > 0$.

Remark 3.4.4. *For $m = 1$ there is no function $a(h) \to 0$ satisfying (3.4.23). Thus, as we have seen in Theorem 3.4.1, new conditions on the function a have to be imposed to guarantee that the l^p-norm behaviour of solutions is uniform on h.*

Remark 3.4.5. *In contrast with the scheme (3.4.1) proposed before, in this case the behaviour of the solutions is the same for $t \sim 0$ and $t \sim \infty$.*

The estimate of the low frequencies is reduced to estimates on $K^{h,3}(t, \mathbf{j})$ defined in (3.4.11) and the result obtained in this case is the same.

The extra term $t^{-\alpha(1-2/p)}$ in Theorem 3.4.1 is given by the high-frequency estimates. In the present case the high frequency estimates of the kernel

$$K^h(t, \mathbf{j}) = \int_{[-\frac{\pi}{h}, \frac{\pi}{h}]^d} e^{-itp_h(\xi)}e^{-|t|a(h)p_h^m(\xi)}e^{i\mathbf{j}\cdot\xi h}d\xi,$$

give a better result. To illustrate this fact we observe that, with the notation

$$\Omega_h = \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d \setminus \left[-\frac{\pi}{4h}, \frac{\pi}{,}4h\right]^d$$

we obtain

$$\begin{aligned}
\left|\int_{\Omega_h} e^{-itp_h(\xi)} e^{-|t|a(h)p_h^m(\xi)} e^{i\mathbf{j}\cdot\xi h} d\xi\right| &\leq \int_{\Omega_h} e^{-|t|a(h)p_h^m(\xi)} d\xi \leq \int_{\Omega_h} e^{-\frac{|t|a(h)}{h^{2m}}(d\sin^2\frac{\pi}{8})^m} d\xi \\
&\leq \frac{c(d)}{h^d} e^{-\frac{|t|a(h)}{h^{2m}}(d\sin^2\frac{\pi}{8})^m} \leq \frac{c(m,d)}{h^d}\left(\frac{h^{2m}}{|t|a(h)}\right)^{d/2} \\
&= \frac{c(m,d)}{|t|^{d/2}}\left(\frac{h^{2m-2}}{a(h)}\right)^{d/2} \leq \frac{c(m,d)}{|t|^{d/2}}\left(\inf_{h>0}\frac{a(h)}{h^{2m-2}}\right)^{-d/2} \\
&\leq \frac{c(m,d,a)}{|t|^{d/2}}.
\end{aligned}$$

This shows that the low and high frequency components of $K^h(t)$ have the same behaviour.

Once (3.4.24) is proved, we can apply Proposition 3.3.1 to obtain Strichartz-like estimates for the solutions of (3.4.22).

Theorem 3.4.4. *Let $a(h)$ be satisfying (3.4.23) and (q,r), $(\tilde{q},\tilde{r})$ two 1/2-admissible pairs. Then*
i) There exists a positive constant $C(d,r,m,a)$ such that

$$\|S^h(\cdot)\varphi\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \leq C(d,r,m,a)\|\varphi\|_{l^2(h\mathbb{Z}^d)}$$

holds for all $\varphi^h \in l^2(h\mathbb{Z}^d)$ uniformly on $h>0$.
ii) There exists a positive constant $C(d,r,m,a)$ such that

$$\left\|\int_{\mathbb{R}} S^h(s)^* f(s) ds\right\|_{l^2(h\mathbb{Z}^d)} \leq C(d,r,m,a)\|f\|_{L^{q'}(\mathbb{R},\, l^{r'}(h\mathbb{Z}^d))}$$

holds for all $f \in L^{q'}(\mathbb{R},\, l^{r'}(h\mathbb{Z}^d))$, uniformly on $h>0$.
iii) There exists a positive constant $C(d,\alpha,r,m,a)$ such that

$$\left\|\int_{s<t} S^h(t-s) f(s) ds\right\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \leq C(d,\alpha,r,m,a)\|f\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))}.$$

holds for all $f \in L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))$, uniformly on $h>0$.

3.5. Application to a nonlinear problem

We concentrate on the semilinear NSE equation in $\mathbb{R}^d$:

$$\begin{cases} iu_t + \Delta u = |u|^p u, \ t>0, \\ u(0,x) = \varphi(x), \ x \in \mathbb{R}^d, \end{cases} \tag{3.5.1}$$

the case when nonlinearity is given by $f(u) = -|u|^p u$ being the same. In fact, the key point in the global existence of the solutions is that the L^2-scalar product $(f(u),u)$ is a real number. All the results extend to more general nonlinearities $f(u)$ (see [25], Ch. 4.6, p. 109, for L^2-solutions).

The first result concerning the L^2 solution is the following

Theorem 3.5.1. *(Global existence in $L^2(\mathbb{R}^d)$, Tsutsumi, [132]). For $0 \le p < 4/d$ and $\varphi \in L^2(\mathbb{R}^d)$, there exists a unique solution u in $C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R}, L^{p+2}(\mathbb{R}^d))$ with $q = 4(p+1)/pd$ that satisfies the L^2-norm conservation property and depends continuously on the initial condition in $L^2(\mathbb{R}^d)$.*

Local existence is proved by applying a fixed point argument to the integral formulation. Global existence holds because of the $L^2(\mathbb{R}^d)$-conservation property which excludes finite-time blow-up. In order to introduce a numerical approximation of equation (3.5.1) it is convenient to introduce the definition of the weak solution of equation (3.5.1). The solution obtained by the semigroup method and the weak solutions being the same (cf. [5] and [81], Ch. II) we introduce an approximation of equation (3.5.1) and prove its convergence.

Definition 3.5.1. *We say that u is a weak solution of (3.5.1) if*
i) $u \in C(\mathbb{R}, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R}, L^{p+2}(\mathbb{R}^d))$
ii) $u(0) = \varphi$ a.e. and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} u(-i\psi_t + \Delta\psi)dxdt = \int_{\mathbb{R}} \int_{\mathbb{R}^d} |u|^p u\psi dxdt \tag{3.5.2}$$

for all $\psi \in \mathcal{D}(\mathbb{R}, H^2(\mathbb{R}^d))$, where p and q are as in the statement of Theorem 3.5.1.

In this section we consider the following approximation of the nonlinear problem (3.5.1):

$$\begin{cases} i\dfrac{du^h}{dt} + \Delta_h u^h &= ia(h)\Delta_h u^h + |u^h|^p u^h, \quad t > 0, \\ \qquad u^h(0) &= \varphi^h, \\ i\dfrac{du^h}{dt} + \Delta_h u^h &= -ia(h)\Delta_h u^h + |u^h|^p u^h, \quad t < 0, \end{cases} \tag{3.5.3}$$

with $0 < p < 4/d$ and $a(h) = h^{2-d/\alpha(h)}$ such that $\alpha(h) \downarrow d/2$ and $a(h) \to 0$ as $h \downarrow 0$. The critical case $p = 4/d$ will be analyzed in Section 3.5.5.

The main result in the subcritical case $p < 4/d$ is the following:

Theorem 3.5.2. *Let $p \in (0, 4/d)$ and $\alpha(h) \in (d/2, 2/p)$. Set*

$$\frac{1}{q(h)} = \alpha(h)\left(\frac{1}{2} - \frac{1}{p+2}\right)$$

so that $(q(h), p+2)$ is an $\alpha(h)$-admissible pair. Then for every $\varphi^h \in l^2(h\mathbb{Z}^d)$, there exists a unique global solution

$$u^h \in C(\mathbb{R}, l^2(h\mathbb{Z}^d)) \cap L^{q(h)}_{loc}(\mathbb{R}, l^{p+2}(h\mathbb{Z}^d))$$

of the problem (3.5.3). Moreover, u^h satisfies

$$\|u^h\|_{L^\infty(\mathbb{R}, l^2(h\mathbb{Z}^d))} \le \|\varphi^h\|_{l^2(h\mathbb{Z}^d)} \tag{3.5.4}$$

and for any finite interval $I \subset \mathbb{R}$

$$\|u^h\|_{L^{q(h)}(I, l^{p+2}(h\mathbb{Z}^d))} \le c(I)\|\varphi^h\|_{l^2(h\mathbb{Z}^d)} \tag{3.5.5}$$

where the above constant is independent of h.

Remark 3.5.1. *The restriction imposed on α: $\alpha(h) < 2/p$ guarantees that $q(h) > p+2$. The condition $q(h) > p+2$ is essential in the proof of the local existence. It is always satisfied in the subcritical case $p < 4/d$ and allows us to apply Banach fix point theorem for small time T. In the critical case $p = 4/d$, this condition is not fulfilled and additional hypotheses on the initial data have to be imposed (see Section 3.5.5).*

The content of this section can be summarized as follows. First we prove the global existence and uniqueness of solutions of (3.5.3). The next step is devoted to prove the convergence of the method. The fact that we work in the $L^2(\mathbb{R}^d)$ space does not allow us to pass to the limit in the nonlinear term. To do it we need to use a compactness argument that needs to assure that the solutions gain some regularity with respect to space of initial data. We will prove that our solutions u^h gain a fractional space-derivative in L^2_{xt}. We will first analyze the smoothing effect of the linear semigroup $S^h(t)$ to later extend it to the inhomogeneous case.

Once the local smoothing effect is proved we will prove the convergence of the semidiscrete solutions towards the continuous one. Finally, we analyze the critical case $p = 4/d$.

3.5.1. Global existence of solutions

First we establish an a priori estimate on the $l^2(h\mathbb{Z}^d)$-norm of solutions. Afterwards using the Banach's Fix Point Theorem we prove the local existence of solutions. These arguments are standard.

Step I. A priori estimates for $\|u^h\|_{l^2(h\mathbb{Z}^d)}$.
Multiplying (3.5.3) by $\overline{u}^h_{\mathbf{j}}$ and summing in $\mathbf{j}$ we get

$$i\sum_{\mathbf{j}\in\mathbb{Z}^d}\frac{du^h_{\mathbf{j}}}{dt}\overline{u}^h_{\mathbf{j}} + \sum_{\mathbf{j}\in\mathbb{Z}^d}(\Delta_h u^h)_{\mathbf{j}}\overline{u}^h_{\mathbf{j}} = ia(h)\sum_{\mathbf{j}\in\mathbb{Z}^d}(\Delta_h u^h)_{\mathbf{j}}\overline{u}^h_{\mathbf{j}} + \sum_{\mathbf{j}\in\mathbb{Z}^d}|u^h_{\mathbf{j}}|^{p+2}.$$

We take the imaginary part in the above identity and obtain :

$$\begin{aligned}
\Re\sum_{\mathbf{j}\in\mathbb{Z}^d}\frac{du^h_{\mathbf{j}}}{dt}\overline{u}^h_{\mathbf{j}} &= \frac{a(h)}{h^2}\sum_{\mathbf{j}\in\mathbb{Z}^d}\sum_{k=1}^{d}\Big(u^h_{\mathbf{j}+\mathbf{e}_k} - 2u^h_{\mathbf{j}} + u_{\mathbf{j}-\mathbf{e}_k}\Big)\overline{u}^h_{\mathbf{j}} \\
&\leq \frac{a(h)}{h^2}\sum_{\mathbf{j}\in\mathbb{Z}^d}\sum_{k=1}^{d}\Big(|u^h_{\mathbf{j}+\mathbf{e}_k}||u^h_{\mathbf{j}}| + |u^h_{\mathbf{j}-\mathbf{e}_k}||u^h_{\mathbf{j}}| - 2|u^h_{\mathbf{j}}|^2\Big) \\
&\leq \frac{a(h)}{h^2}\sum_{\mathbf{j}\in\mathbb{Z}^d}\sum_{k=1}^{d}\left(\frac{|u^h_{\mathbf{j}+\mathbf{e}_k}|^2 + |u^h_{\mathbf{j}-\mathbf{e}_k}|^2}{2} - |u^h_{\mathbf{j}}|^2\right) = 0.
\end{aligned}$$

This implies that

$$\frac{d}{dt}\|u^h(t)\|^2_{l^2(h\mathbb{Z}^d)} = \sum_{\mathbf{j}\in\mathbb{Z}^d}\Big(u^h_{\mathbf{j}}\frac{d\overline{u}^h_{\mathbf{j}}}{dt} + \frac{du^h_{\mathbf{j}}}{dt}\overline{u}^h_{\mathbf{j}}\Big) = 2\Re\sum_{\mathbf{j}\in\mathbb{Z}^d}\frac{du^h_{\mathbf{j}}}{dt}\overline{u}^h_{\mathbf{j}} \leq 0 \tag{3.5.6}$$

and proves (3.5.4).

Step II. Local existence of solutions.
We now proceed to prove the local existence of solutions. Fix $T < 1$, $M > 0$ and set

$$E_h = \{u \in L^\infty((-T,T),\, l^2(h\mathbb{Z}^d)) \cap L^{q(h)}((-T,T),\, l^{p+2}(h\mathbb{Z}^d)),$$
$$\|u\|_{L^\infty((-T,T),\, l^2(h\mathbb{Z}^d))} + \|u\|_{L^{q(h)}((-T,T),\, l^{p+2}(h\mathbb{Z}^d))} \le M\}.$$

It follows that E_h is a complete metric space when equipped with the distance

$$d(u,v) = \|u-v\|_{L^\infty((-T,T),\, l^2(h\mathbb{Z}^d))} + \|u-v\|_{L^{q(h)}((-T,T),\, l^{p+2}(h\mathbb{Z}^d))}.$$

We set

$$\mathcal{G}^h(u)(t) = i\int_0^t S^h(t-s)|u|^p u(s)ds$$

and

$$\mathcal{H}^h(u)(t) = S^h(t)\varphi^h + \mathcal{G}^h(u)(t).$$

We use the Strichartz-like estimates given by Corollary 3.4.1 to prove that for small enough T, independent of h, $\mathcal{H}^h(u)$ is a contraction on E_h.

We claim the existence of a constant $c(p)$, independent of $h > 0$, such that for any $u, v \in E_h$, the function $\mathcal{G}^h$ satisfies

$$\|\mathcal{G}^h(u) - \mathcal{G}^h(v)\|_{L^\infty((-T,T),\, l^2(h\mathbb{Z}^d))} \le c(p)T^{\frac{q(h)-p-2}{q(h)}} M^p d(u,v). \tag{3.5.7}$$

and

$$\|\mathcal{G}^h(u) - \mathcal{G}^h(v)\|_{L^{q(h)}((-T,T),\, l^{p+2}(h\mathbb{Z}^d))} \le c(p)T^{\frac{q(h)-p-2}{q(h)}} M^p d(u,v). \tag{3.5.8}$$

We deduce from Strichartz's estimates (3.4.17) and (3.4.18) that for every $u \in E_h$,

$$\begin{aligned}
\|\mathcal{H}^h(u)\|_{L^\infty((-T,T),\, l^2(h\mathbb{Z}^d))} &+ \|\mathcal{H}^h(u)\|_{L^{q(h)}((-T,T),\, l^{p+2}(h\mathbb{Z}^d))} \le \\
&\le \|S^h\varphi^h\|_{L^\infty((-T,T),\, l^2(h\mathbb{Z}^d))} + \|S^h\varphi^h\|_{L^{q(h)}((-T,T),\, l^{p+2}(h\mathbb{Z}^d))} \\
&\quad + \|\mathcal{G}^h(u)\|_{L^\infty((-T,T),\, l^2(h\mathbb{Z}^d))} + \|\mathcal{G}^h(u)\varphi^h\|_{L^{q(h)}((-T,T),\, l^{p+2}(h\mathbb{Z}^d))} \\
&\le \|\varphi^h\|_{l^2(h\mathbb{Z}^d)} + c(p)\|\varphi^h\|_{l^2(h\mathbb{Z}^d)} \\
&\quad + c(p)T^{\frac{q(h)-(p+2)}{q(h)}} M^{p+1} + c(p)T^{\frac{q(h)-(p+2)}{q(h)}} M^{p+1} \\
&\le c(p)\|\varphi^h\|_{l^2(h\mathbb{Z}^d)} + c(p)T^{\frac{q(h)-(p+2)}{q(h)}} M^{p+1}.
\end{aligned}$$

Choosing $M = 2c(p)\|\varphi^h\|_{l^2(h\mathbb{Z}^d)}$, we see that if T is sufficiently small (depending on $\|\varphi^h\|_{l^2(h\mathbb{Z}^d)}$) $\mathcal{H}^h(u) \in E_h$ for all $u \in E_h$. Moreover choosing T smaller (but still depending on $\|\varphi^h\|_{l^2(h\mathbb{Z}^d)}$) we obtain

$$\begin{aligned}
d(\mathcal{H}^h(u), \mathcal{H}^h(v)) &= \|\mathcal{H}^h(u) - \mathcal{H}^h(v)\|_{L^\infty((-T,T),\, l^2(h\mathbb{Z}^d))} + \|\mathcal{H}^h(u) - \mathcal{H}^h(v)\|_{L^{q(h)}((-T,T),\, l^{p+2}(h\mathbb{Z}^d))} \\
&= \|\mathcal{G}^h(u) - \mathcal{G}^h(v)\|_{L^\infty((-T,T),\, l^2(h\mathbb{Z}^d))} + \|\mathcal{G}^h(u) - \mathcal{G}^h(v)\|_{L^{q(h)}((-T,T),\, l^{p+2}(h\mathbb{Z}^d))} \\
&\le c(p)T^{\frac{q(h)-p-2}{q(h)}} M^p d(u,v) \le \frac{1}{2} d(u,v)
\end{aligned}$$

for all $u, v \in E_h$. Fixing $T = T_0$ sufficiently small we get that $\mathcal{H}^h$ has a unique fixed point $u \in E_h$. This proves local existence. We have to point out that T_0 depends only on p and

$\|\varphi^h\|_{l^2(h\mathbb{Z}^d)}$ and it is independent of h. The a priori estimate (3.5.4) excludes the blow-up alternative and proves the global existence of the solution.

It remains to prove (3.5.7) and (3.5.8). For this we make use of the following inequalities in the $l^q(h\mathbb{Z}^d)$ space. Note that

$$\||u|^p u\|_{l^{(p+2)'}(h\mathbb{Z}^d)} = \||u|^p u\|_{l^{(p+2)/(p+1)}(h\mathbb{Z}^d)} = \|u\|^{p+1}_{l^{p+2}(h\mathbb{Z}^d)}.$$

To simplify the presentation let us denote $I=(-T,T)$. Applying Hölder's inequality in time:

$$\begin{aligned}
\||u|^p u\|_{L^{q(h)'}(I,\,l^{(p+2)'}(h\mathbb{Z}^d))} &= \left(\int_I \||u|^p u\|^{q(h)'}_{l^{(p+2)'}(h\mathbb{Z}^d)}\right)^{1/q(h)'} = \left(\int_I \|u\|^{(p+1)q(h)'}_{l^{p+2}(h\mathbb{Z}^d)}\right)^{1/q(h)'}\\
&= \|u\|^{p+1}_{L^{(p+1)q(h)'}(I,\,l^{p+2}(h\mathbb{Z}^d))}\\
&\le \|1\|^{p+1}_{L^{\frac{(p+1)q(h)}{q(h)-p-2}}(I)} \|u\|^{p+1}_{L^{q(h)}(I,\,l^{p+2}(h\mathbb{Z}^d))}\\
&\le T^{\frac{q(h)-(p+2)}{q(h)}} M^{p+1}.
\end{aligned}$$

Using the inequality

$$\||u|^p u - |v|^p v\|_{l^{\frac{p+2}{p+1}}(h\mathbb{Z}^d)} \le c(p)(\|u\|^p_{l^{p+2}(h\mathbb{Z}^d)} + \|v\|^p_{l^{p+2}(h\mathbb{Z}^d)})\|u-v\|_{l^{p+2}(h\mathbb{Z}^d)}$$

and Hölder's inequality in time we get

$$\begin{aligned}
\||u|^p u - |v|^p v\|_{L^{q(h)'}(I,\,l^{(p+2)'}(h\mathbb{Z}^d))} &\le c(p) \left\|(\|u\|^p_{l^{p+2}(h\mathbb{Z}^d)} + \|v\|^p_{l^{p+2}(h\mathbb{Z}^d)})\|u-v\|_{l^{p+2}(h\mathbb{Z}^d)}\right\|_{L^{q(h)'}(I)}\\
&\le c(p) \left\|\|u\|^p_{l^{p+2}(h\mathbb{Z}^d)} + \|v\|^p_{l^{p+2}(h\mathbb{Z}^d)}\right\|_{L^{\frac{q(h)}{q(h)-2}}(I)} \|u-v\|_{L^{q(h)}(I,\,l^{p+2}(h\mathbb{Z}^d))}\\
&\le c(p) \left(\|u\|^p_{L^{\frac{pq(h)}{q(h)-2}}(I,\,l^{p+2}(h\mathbb{Z}^d))} + \|v\|^p_{L^{\frac{pq(h)}{q(h)-2}}(I,\,l^{p+2}(h\mathbb{Z}^d))}\right) \|u-v\|_{L^{q(h)}(I,\,l^{p+2}(h\mathbb{Z}^d))}\\
&\le c(p)\|1\|^p_{L^{\frac{pq(h)}{q(h)-p-2}}(I)} \left(\|u\|^p_{L^{q(h)}(I,\,l^{p+2}(h\mathbb{Z}^d))} + \|v\|^p_{L^{q(h)}(I,\,l^{p+2}(h\mathbb{Z}^d))}\right) d(u,v)\\
&\le c(p) T^{\frac{q(h)-p-2}{q(h)}} M^p d(u,v), \qquad (3.5.9)
\end{aligned}$$

where we used that

$$\frac{q(h)-p-2}{pq(h)} + \frac{1}{q(h)} = \frac{q(h)-2}{pq(h)}.$$

Applying (3.5.9) and Strichartz's estimate (3.4.18), we see that $\mathcal{G}^h(u)$ satisfies

$$\begin{aligned}
\|\mathcal{G}^h(u) - \mathcal{G}^h(v)\|_{L^\infty(I,\,l^2(h\mathbb{Z}^d)} &\le c(p)\||u|^p u - |v|^p v\|_{L^{q(h)'}(I,\,l^{(p+2)'}(h\mathbb{Z}^d))}\\
&\le c(p) T^{\frac{q(h)-p-2}{q(h)}} M^p d(u,v).
\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{G}^h(u)-\mathcal{G}^h(v)\|_{L^{q(h)}(I,\,l^{p+2}(h\mathbb{Z}^d)} &\leq c(p)\||u|^pu-|v|^pv\|_{L^{q(h)'}(I,\,l^{(p+2)'}(h\mathbb{Z}^d))}\\ &\leq c(p)T^{\frac{q(h)-p-2}{q(h)}}M^pd(u,v).\end{aligned}$$

The proof is now complete.

3.5.2. Uniqueness

We first note that uniqueness is a local property, so that we need only to establish it on possibly small time intervals. Suppose now that $u, v \in C([0,T],\, l^2(h\mathbb{Z}^d))\cap L^{q(h)}((0,T),\, l^{p+2}(h\mathbb{Z}^d))$ are any two solutions of (3.5.3), then $u = v$ on $(0,\theta)$ for $0<\theta\leq T$ sufficiently small. We may then define $0<\theta^*\leq T$ by

$$\theta^*=\sup\{0<\theta<T;\ u=v \,\text{on}\ (0,\theta)\}.$$

It follows that $u=v$ on $[0,\theta^*]$. If $\theta^*=T$, uniqueness follows, so we assume by contradiction that $\theta^*<T$. We see that $u_1(\cdot)=u(\theta^*+\cdot)$ and $v_1(\cdot)=v(\theta^*+\cdot)$ are two solutions of (3.5.3) with φ replaced by $u(\theta^*)=v(\theta^*)$ on the interval $(0,T-\theta^*)$. By uniqueness for small time, we deduce that $u_1=v_1$ on some interval $[0,\epsilon]$ with $0<\epsilon\leq T-\theta^*$. This means that $u=v$ on $[0,\theta^*+\epsilon]$, contradicting the definition of θ.

We now show uniqueness for small time. The proof of the existence shows that

$$\begin{aligned}\|\mathcal{G}(u)-\mathcal{G}(v)\|_{L^{q(h)}((0,T),\,l^{p+2}(h\mathbb{Z}^d)} &\leq c(p)\||u|^pu-|v|^pv\|_{L^{q(h)'}((0,T),\,l^{(p+2)'}(h\mathbb{Z}^d))}\\ &\leq T^{\frac{q(h)-p-2}{q(h)}}(\|u\|^p_{L^{q(h)}((0,T),\,l^{p+2}(h\mathbb{Z}^d))}+\|v\|^p_{L^{q(h)}((0,T),\,l^{p+2}(h\mathbb{Z}^d))})\|u-v\|_{L^{q(h)}((0,T),\,l^{p+2}(h\mathbb{Z}^d))}\\ &\leq c(p)(\|\varphi\|_{l^2(h\mathbb{Z}^d)})T^{\frac{q(h)-p-2}{q(h)}}\|u-v\|_{L^{q(h)}(0,T),\,l^{p+2}(h\mathbb{Z}^d))}.\end{aligned}$$

Since $\mathcal{G}(u)-\mathcal{G}(v)=u-v$, we deduce that if T is sufficiently small, then

$$\|u-v\|_{L^{q(h)}((0,T),\,l^{p+2}(h\mathbb{Z}^d))}\leq\frac{1}{2}\|u-v\|_{L^{q(h)}((0,T),\,l^{p+2}(h\mathbb{Z}^d))},$$

i.e. $u=v$ on $[0,T]$.

3.5.3. Smoothing effect of the discrete operator $S^h(t)$

In the following, let us consider the piecewise linear interpolator Iu^h. In the Fourier space it reads

$$\widehat{Iu^h}(\xi)=\prod_{k=1}^{d}\left|\frac{e^{i\xi_kh}-1}{\xi_kh}\right|^2\widehat{u^h}(\xi),\quad \xi\in\mathbb{R}^d. \tag{3.5.10}$$

In the following Theorem we prove the local smoothing property of $I^hS^h(t)$. We use a piecewise lineal interpolator instead of a piecewise constant one to avoid some technical difficulties. To be more precise, in the one-dimensional case, the piecewise constant interpolator does not belong to $H^{1/2}_{loc}(\mathbb{R}^d)$, having less regularity than the continuous Schrödinger semigroup.

The following holds:

Theorem 3.5.3. *Let $\alpha \in (d/2, d]$ and $\chi \in C_c^\infty(\mathbb{R}^d)$. Then*
1.) For all $\varphi \in l^2(h\mathbb{Z}^d)$ the following holds

$$\int_I \int_{\mathbb{R}^d} \chi^2 |(I-\Delta)^{1/4} I S^h(t)\varphi|^2 ds dt \leq C(I,\chi)\|\varphi\|_{l^2(h\mathbb{Z}^d)} \tag{3.5.11}$$

for all $\varphi \in l^2(h\mathbb{Z}^d)$,uniformly on h.
2.) For all $f \in L^1(I, l^2(h\mathbb{Z}^d))$ the following holds

$$\int_I \int_{\mathbb{R}^d} \chi^2 \left| (I-\Delta)^{1/4} I \left(\int_0^t S^h(t-s) f(s) ds \right) \right|^2 dx dt \leq C(I,\chi) \|f\|^2_{L^1(I, l^2(h\mathbb{Z}^d))} \tag{3.5.12}$$

uniformly on $h > 0$. 3.) Let (q, r) be an α-admissible pair. Then there is a positive constant $s(r)$ such that

$$\int_I \int_{\mathbb{R}^d} \chi^2 \left| (I-\Delta)^{s(r)} I \left(\int_0^t S^h(t-s) f(s) ds \right) \right|^2 dx dt \leq C(I,\chi) \|f\|^2_{L^{q'}(I, l^{r'}(h\mathbb{Z}^d))} \tag{3.5.13}$$

for all $f \in L^{q'}(I, l^{r'}(h\mathbb{Z}^d))$, uniformly on $h > 0$.

Remark 3.5.2. *In the continuous case, the last estimate holds for $s(r) = 1/4$. In that case, the homogenous case have been proved by Kenig, Ponce and Vega [75]. The inhomogeneous case is reduced to the homogenous one using the results of Christ and Kiselev [33] and Strichartz estimates.*

Remark 3.5.3. *In our case the arguments of [33] can not be applied. The key point in their proof is that the Schrödinger semigroup satisfies $S(t-s) = S(t)S(s)^*$ for all reals t and s, identity which does not hold in our case. We recall that for t and s positive the operator $S^h(t)S^h(s)^*$ is more dissipative than $S^h(t-s)$.*

Remark 3.5.4. *Estimate (3.5.13) follows by interpolation of (3.5.12) and the Strichartz estimate (3.4.18) applied to a suitable α-admissible pair (q_1, r_1).*

More precisely, by (3.4.18) we obtain for any α-admissible pair (q_1, r_1) that

$$\int_I \int_{\mathbb{R}^d} \chi^2 \left| I \left(\int_0^t S^h(t-s) f(s) ds \right) \right|^2 dx dt \leq C(I,\chi) \|f\|^2_{L^{q_1'}(I, l^{r_1'}(h\mathbb{Z}^d))}. \tag{3.5.14}$$

Let us choose an α-admissible pair (q, r). Using that our estimates does not involve the endpoint $r = 2d/(d-2)$ we can choose an r_1 satisfying $r < r_1 < 2d/(d-2)$. An example can be

$$r_1 = \frac{1}{2}\left(r + \frac{2d}{d-2} \right). \tag{3.5.15}$$

An interpolation between (3.5.12) and (3.5.14) gives us the existence of a positive constant $s(r)$ such that (3.5.13) is satisfied.

Proof. First we consider the homogenous case. Once the estimates for this case are proved we use them to prove the inhomogeneous ones (3.5.12).

Step I. Proof of the homogenous estimate. We take advantage of the dissipative term. Let us write Iu^h as

$$Iu^h = I_a u^h + I_b u^h,$$

where

$$I_b u^h = \int_{|\xi|\leq \pi/2h} \widehat{Iu^h} e^{ix\xi} d\xi.$$

We will prove that, for any $R>0$, the two terms satisfy the following inequalities

$$\int_{|x|<R} \int_{-\infty}^{\infty} |D^{1/2} I_b S^h(t)\varphi|^2 dt dx \leq C(R) \|\varphi\|^2_{l^2(h\mathbb{Z}^d)}$$

and

$$\int_{\mathbb{R}^d} \int_{-\infty}^{\infty} |D^{d/2\alpha} I_a S^h(t)\varphi|^2 dt dx \leq C(R) \|\varphi\|^2_{l^2(h\mathbb{Z}^d)}.$$

These inequalities give estimates on the $H^s_{loc}(\mathbb{R}^d)$-norm of $I_a u^h$ and $I_b u^h$:

$$\int_I \int_{\mathbb{R}^d} \chi^2 |(I-\Delta)^{1/4} I_b S^h(t)\varphi|^2 dx dt \leq C(I,\chi) \|\varphi\|^2_{l^2(h\mathbb{Z}^d)}$$

and

$$\int_I \int_{\mathbb{R}^d} \chi^2 |(I-\Delta)^{d/4\alpha} I_b S^h(t)\varphi|^2 dx dt \leq C(I,\chi) \|\varphi\|^2_{l^2(h\mathbb{Z}^d)}.$$

Finally taking into account that $\alpha \leq d$ we obtain

$$\int_I \int_{\mathbb{R}^d} \chi^2 |(I-\Delta)^{1/4} I S^h(t)\varphi|^2 dx dt \leq C(I,\chi) \|\varphi\|^2_{l^2(h\mathbb{Z}^d)}$$

which shows that $IS^h(t)\varphi$ belongs to the space $L^2_{loc}(I, H^{1/2}_{loc}(\mathbb{R}^d))$.

Case a). Estimates on $I_b u^h$. By definition

$$(I_b S^h(t)\varphi)(x) = \int_{|\xi|\leq \pi/4h} e^{-itp_h(\xi)} e^{-|t|a(h)p_h(\xi)} e^{ix\xi} \widehat{I\varphi}(\xi) d\xi.$$

We reduce the estimates on $I_b S^h(t)\varphi$ to those on $J_b\varphi$, where $J_b\varphi$ is defined by

$$(J_b\varphi)(t,x) = \int_{|\xi|\leq \pi/4h} e^{-itp_h(\xi)} e^{ix\xi} \widehat{I\varphi}(\xi) d\xi.$$

Classical properties of Poisson's integrals (Th. 1, p. 62, Ch. III, [117]) give us :

$$|I_b(S^h(t)\varphi)(x)| \leq \sup_{s\geq 0} \left| \int_{|\xi|\leq \pi/4h} e^{-itp_h(\xi)} e^{-sp_h(\xi)} e^{ix\xi} \widehat{I\varphi}(\xi) d\xi \right| = \Psi(t) \qquad (3.5.16)$$

where the function Ψ satisfies $\|\Psi\|_{L^2(\mathbb{R}_t)} \leq \|J_b\varphi\|_{L^2(\mathbb{R}_t)}$. It remains to prove that $J_b\varphi$ satisfies

$$\int_{|x|<R}\int_{-\infty}^{\infty} |D^{1/2}J_b\varphi(t,x)|^2 dtdx \lesssim C(R)\|I\varphi\|_{L^2(\mathbb{R}^d)}. \tag{3.5.17}$$

To prove the last inequality we make use of the following result

Lemma 3.5.1. *(Th. 4.1, [75]) Let Ω be an open set in $\mathbb{R}^d$, and ψ be a $C^1(\Omega)$ function such that $\nabla\psi(\xi) \neq 0$ for any $\xi \in \Omega$. Assume that there is $N \in \mathbb{N}$ such that for any $\overline{\xi} \in \mathbb{R}^{n-1}$ and $r \in \mathbb{R}$ the equations*

$$\psi(\xi_1, \ldots, \xi_k, x, \xi_{k+1}, \ldots, \xi_{n-1}) = r$$
$$\overline{\xi} = (\xi_1, \ldots, \xi_{n-1}),\ k = 0, \ldots, n-1$$

have at most N solutions. For $f \in \mathcal{S}(\mathbb{R}^d)$ define

$$W(t)f(x) = \int_{\Omega} e^{i(t\psi(\xi)+x\xi)}\widehat{f}(\xi)d\xi;$$

then for $d \geq 1$

$$\int_{|x|\leq R}\int_{-\infty}^{\infty} |W(t)f(x)|^2 dtdx \leq cRN\int_{\Omega}\frac{|\widehat{f}(\xi)|^2}{|\nabla\psi(\xi)|}d\xi$$

where c is independent of R and N.

Applying this result with $W = J_b$ we obtain

$$\int_{|x|<R}\int_{-\infty}^{\infty} |J_b\varphi(t,x)|^2 dtdx \leq CR\int_{|\xi|\leq\pi/4h}\frac{|\widehat{Iu^h}|^2}{|\nabla p_h(\xi)|}d\xi \lesssim R\int_{|\xi|\leq\pi/4h}\frac{|\widehat{Iu^h}|^2}{|\xi|}d\xi$$

which proves inequality (3.5.17).

Case b). Estimates on $I_a u^h$. The definition of $I_a u^h$ gives us

$$\begin{aligned}
\int_{\mathbb{R}^d} |D^{d/2\alpha}I_aS^h(t)\varphi|^2dx &= \int_{|\xi|\geq\frac{\pi}{4h}} |\xi|^{d/\alpha}|\widehat{IS^h(t)}\varphi|^2d\xi \\
&\leq h^{2-d/\alpha}\int_{|\xi|\geq\frac{\pi}{4h}} |\xi|^2|\widehat{IS^h(t)}\varphi|^2d\xi \\
&\leq a(h)\int_{\mathbb{R}^d} |\xi|^2|\widehat{IS^h(t)}\varphi|^2d\xi \\
&= a(h)\|\nabla(IS^h(t)\varphi)\|^2_{L^2(\mathbb{R}^d)} = a(h)\|\nabla S^h(t)\varphi\|^2_{l^2(h\mathbb{Z}^d)} \\
&= a(h)\int_{[-\pi/h,\pi/h]^d} p_h(\xi)e^{-2ta(h)p_h(\xi)}|\widehat{\varphi}(\xi)|^2d\xi.
\end{aligned}$$

Integrating the last inequality on time we get

$$\begin{aligned}
\int_{\mathbb{R}}\int_{\mathbb{R}^d} |D^{d/2\alpha}I_aS^h(t)\varphi|^2dxdt &\leq a(h)\int_{[-\pi/h,\pi/h]^d} p_h(\xi)|\widehat{\varphi}(\xi)|^2\int_{\mathbb{R}} e^{-2ta(h)p_h(\xi)}dtd\xi \\
&\leq \int_{[-\pi/h,\pi/h]^d} |\widehat{\varphi}(\xi)|^2d\xi = \|\varphi\|^2_{l^2(h\mathbb{Z}^d)}.
\end{aligned}$$

Step II. Estimates of the Inhomogeneous Term. Let us denote

$$\Psi_f = \int_0^t IS^h(t-s)f(s)ds.$$

Without loss of generality we consider $I=[0,T]$. For any $\chi \in C_c^\infty(\mathbb{R}^d)$ we have

$$\begin{aligned} \|\chi\Psi_f(t)\|_{H^{1/2}(\mathbb{R}^d)} &= \left\| \int_0^t \chi IS^h(t-s)f(s)ds \right\|_{H^{1/2}(\mathbb{R}^d)} \\ &\leq \int_0^t \|\chi IS^h(t-s)f(s)\|_{H^{1/2}(\mathbb{R}^d)}ds = \int_0^t g(t,s)ds. \end{aligned}$$

Integrating on time we obtain

$$\begin{aligned} \|\chi\Psi_f\|_{L^2(0,T),H^{1/2}(\mathbb{R}^d)} &= \left\| \int_0^T \mathbf{1}_{(0,t)}(s)g(t,s)ds \right\|_{L_t^2(0,T)} = \left\| \int_0^T \mathbf{1}_{(s,T)}(t)g(t,s)ds \right\|_{L_t^2(0,T)} \\ &\leq \int_0^T \|\mathbf{1}_{(s,T)}(t)g(t,s)\|_{L_t^2(0,T)}ds. \end{aligned}$$

Using (3.5.11) on the homogenous term we have

$$\begin{aligned} \|\mathbf{1}_{(s,T)}(t)g(t,s)\|^2_{L_t^2(0,T)} &= \int_s^T |g(t,s)|^2 dt = \int_s^T \|\chi IS^h(t-s)f(s)\|^2_{H^{1/2}(\mathbb{R}^d)}dt \\ &\leq C(T,\chi)\|f(s)\|^2_{L^2(\mathbb{R}^d)}. \end{aligned}$$

Integrating on $t \in (0,T)$ the last inequality we obtain

$$\|\chi\Psi_f\|_{L^2((0,T),H^{1/2}(\mathbb{R}^d)} \leq C(T,\chi)\|f\|_{L^1((0,T),\,L^2(\mathbb{R}^d)}.$$

□

3.5.4. Convergence of the method

Let us consider the piecewise constant interpolator Eu^h. This choice is motivated by the fact that it commutes with the nonlinearity. Let $\varphi \in L^2(\mathbb{R}^d)$ and φ^h such that $E\varphi^h \to \varphi$ in $L^2(\mathbb{R}^d)$. Clearly $\|E\varphi^h\|_{L^2(\mathbb{R}^d)} \leq C(\|\varphi\|_{L^2(\mathbb{R}^d)})$. Then the interpolator Eu^h satisfies:

Proposition 3.5.1. *Let $I \subset \mathbb{R}$ a finite interval. There exists a constant $C(I, \|\varphi\|_{L^2(\mathbb{R}^d)})$ such that*

$$\|Eu^h\|_{L^\infty(\mathbb{R},\,L^2(\mathbb{R}^d))} \leq C,\ \ \|Eu^h\|_{L^{q(h)}(I,\,L^{p+2}(\mathbb{R}^d))} \leq C(I, \|\varphi\|_{L^2(\mathbb{R}^d)}), \tag{3.5.18}$$

and

$$\||Eu^h|^p Eu^h\|_{L^{q'}(I,\,L^{(p+2)'}(\mathbb{R}^d))} \leq C(I, \|\varphi\|_{L^2(\mathbb{R}^d)}). \tag{3.5.19}$$

Moreover, Eu^h satisfies

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} Eu^h(i\psi_t + \Delta_h\psi)dxdt = \int_{\mathbb{R}}\int_{\mathbb{R}^d} |Eu^h|^p Eu^h \psi dxdt + a(h)\int_{\mathbb{R}}\int_{\mathbb{R}^d} Eu^h \Delta^h\psi dxdt \tag{3.5.20}$$

for all $\psi \in C_c^\infty(\mathbb{R}^{d+1})$.

These uniform estimates and the regularity property proved in the previous section give us the following result on the convergence of the scheme:

Theorem 3.5.4. *The sequence Eu^h satisfies*

$$Eu^h \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)),\ Eu^h \rightharpoonup u \text{ in } L^s_{loc}(\mathbb{R}, L^{p+2}(\mathbb{R}^d)), \forall\, s<q,$$

$$Eu^h \to u \text{ in } L^2_{loc}(\mathbb{R}\times\mathbb{R}^d),\ |Eu^h|^p|Eu^h| \rightharpoonup |u|^p u \text{ in } L^{q'}_{loc}(\mathbb{R}, L^{(p+2)'}(\mathbb{R}^d))$$

where u is the unique weak solution of (NSE).

Proof. The first three convergences are consequences of the uniform estimates in Proposition 3.5.1. We prove the existence of a function v such that Eu^h converges a.e. to v. This allows us to pass to the limit in the nonlinear term. This is the most difficult part of the proof. To do that we first prove that Iu^h converges strongly in $L^2_{loc}(\mathbb{R}^{d+1})$ and a.e. to a function v. We may transfer the strong convergence property of Iu^h to Eu^h by proving that $Iu^h - Eu^h$ tends to zero in $L^2_{loc}(\mathbb{R}^{d+1})$.

The results of [34] (Th. 3.1.5, p. 122) and [103] (Th. 3.4.1, p. 88), give us

$$\begin{aligned}\int_{\mathbb{R}^d} |Iu^h(t)-Eu^h(t)|^2dx &\le h^2\|\nabla_h u^h(t)\|^2_{l^2(h\mathbb{Z}^d)} = h^2\int_{[-\pi/h,\pi/h]} p_h(\xi)|\widehat{u}^h(t,\xi)|^2d\xi\\ &= h^2\int_{[-\pi/h,\pi/h]^d} p_h(\xi)e^{-2|t|p_h(\xi)a(h)}|\widehat{\varphi}^h(\xi)|^2d\xi.\end{aligned}$$

Integrating on time and using that $\alpha(h)\to 1/2$ we obtain

$$\begin{aligned}\int_{\mathbb{R}}\int_{\mathbb{R}^d} |Iu^h(t)-Eu^h(t)|^2dx &\le h^2\int_{\mathbb{R}}\int_{[-\pi/h,\pi/h]^d} p_h(\xi)e^{-2|t|p_h(\xi)a(h)}|\widehat{\varphi}^h(\xi)|^2d\xi\\ &= \frac{h^2}{a(h)}\int_{[-\pi/h,\pi/h]^d}|\widehat{\varphi}^h(\xi)|^2d\xi\\ &= h^{1/\alpha(h)}\|\varphi^h\|^2_{l^2(h\mathbb{Z}^d)}\to 0.\end{aligned}$$

We proceed with the proof of the strong convergence of Iu^h. Let us consider a bounded interval $I\subset\mathbb{R}$ and a bounded domain $\Omega\subset\mathbb{R}^d$. Theorem 3.5.3 gives us the existence of a positive s such that

$$\|Iu^h\|_{L^2(I,H^s(\Omega))}\le C(I,\Omega,\|\varphi\|_{L^2(\mathbb{R}^d)}).$$

We also have the uniform boundeness of its time derivative $\frac{d}{dt}(Iu^h)$:

$$\begin{aligned}\left\|\frac{dIu^h}{dt}\right\|_{L^1(I,H^{-2}(\mathbb{R}^d))} &\le \|\Delta_h Iu^h\|_{L^1(I,H^{-2}(\mathbb{R}^d))}+\|I(|u^h|^pu^h)\|_{L^1(I,H^{-2}(\mathbb{R}^d))}\\ &\le \|Iu^h\|_{L^1(I,L^2(\mathbb{R}^d))}+\|I(|u^h|^pu^h)\|_{L^1(I,L^{(p+2)'}(\mathbb{R}^d))}\\ &\le C(I,\|\varphi^h\|_{l^2(h\mathbb{Z}^d)})\le C(I,\|\varphi\|_{L^2(\mathbb{R}^d)}).\end{aligned}$$

The compactness results of [111] provide the existence of a function v such that $Iu^h\to v_1$ in $L^2(I\times\Omega)$. By a diagonal process we get $Iu^h\to v_1$ in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$. The strong convergence $Iu^h-Eu^h\to 0$ in $L^2_{loc}(\mathbb{R}^{d+1})$ shows that $v_1=v$ and $Eu^h\to v$ in $L^2_{loc}(\mathbb{R}\times\mathbb{R}^d)$. Moreover, up to a subsequence

$$Eu^h\to v \text{ a.e. on compact sets.} \tag{3.5.21}$$

Proposition 3.5.1 gives us that $v \in L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$ and $Eu^h \overset{*}{\rightharpoonup} v$ in $L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))$. Let us choose an $s < q$. For h sufficiently small, $s \leq q(h) < q$. Estimates (3.5.18) and Hölder's inequality show that Eu^h remains bounded in $L^s(I, L^{p+2}(\mathbb{R}^d))$. This implies that $v \in L^s(I, L^{p+2}(\mathbb{R}^d))$, $Eu^h \rightharpoonup v$ in $L^s(I, L^{p+2}(\mathbb{R}^d))$ and

$$\|v\|_{L^s(I,\, L^{p+2}(\mathbb{R}^d))} \leq \liminf_h \|Eu^h\|_{L^{q(h)}(I,\, L^{p+2}(\mathbb{R}^d))} \leq C(I, \|\varphi\|_{L^2(\mathbb{Z}^d)}).$$

Fatou's Lemma shows that $v \in L^q(\mathbb{R}, L^{p+2}(\mathbb{R}^d))$.

In the following we show how to pass to the limit in the nonlinear term. The a.e. convergence on compact sets $Eu^h \to v$ implies

$$|Eu^h|^p Eu^h \to |v|^p v \quad \text{a.e..}$$

Strauss's Lemma (see [120] and [25], Ch. 1.2, Prop. 1.2.1) shows that

$$|Eu^h|^p Eu^h \rightharpoonup |v|^p v \quad \text{in } L^{q'}(\mathbb{R}, L^{p+2}(\mathbb{R}^d)). \tag{3.5.22}$$

It remains to prove that v satisfies (3.5.2). It is sufficient to prove that

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} Eu^h \Delta_h \psi dx dt \to \int_{\mathbb{R}}\int_{\mathbb{R}^d} v \Delta \psi dx dt$$

for all $\psi \in C_c^\infty(\mathbb{R}^{d+1})$. This is a consequence of the strong convergence (3.5.21) and the weak convergence

$$\Delta_h \psi \rightharpoonup \Delta \psi.$$

Finally using that v belongs to $L^q_{loc}(\mathbb{R}, L^{p+2}) \hookrightarrow L^q(\mathbb{R}, H^{-2}(\mathbb{R}^d))$ we obtain (3.5.2) for all $\psi \in C_c^\infty(\mathbb{R}, H^2(\mathbb{R}))$. □

3.5.5. The critical case $p = 4/d$.

Our method works similarly in the critical case $p = 4/d$ for small initial data. It suffices to modify the approximation scheme by taking a nonlinear term of the form $|u^h|^{2/\alpha(h)} u^h$ in the semidiscrete equation (3.5.3) with $a(h) = h^{2-d/\alpha(h)}$ and $\alpha(h) \downarrow d/2$, $a(h) \downarrow 0$, so that, asymptotically, it approximates the critical nonlinearity of the continuous Schrödinger equation. In this way the critical continuous exponent $p = 4/d$ is approximated by semidiscrete critical problems.

The critical semidiscrete problem presents the same difficulties as the continuous one. Thus, the initial datum needs to be assumed to be small. But the smallness condition is independent of the mesh-size $h > 0$. More precisely, the following holds.

Theorem 3.5.5. *Let $\alpha(h) > d/2$ and $p(h) = 2/\alpha(h)$. There exists a constant ϵ, independent of h, such that for all $\|\varphi^h\|_{l^2(h\mathbb{Z}^d)} < \epsilon$, the semidiscrete critical equation has a unique global solution*

$$u^h \in C(\mathbb{R}, l^2(h\mathbb{Z}^d)) \cap L^{p(h)+2}_{loc}(\mathbb{R}, l^{p(h)+2}(h\mathbb{Z}^d)).$$

Moreover $u^h \in L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^d))$ for all $\alpha(h)$-admissible pairs (q, r) and

$$\|u^h\|_{L^q(I, l^r(h\mathbb{Z}^d))} \leq C(q, I)\|\varphi^h\|_{l^2(h\mathbb{Z}^d)}$$

for all finite interval I.

Observe that, in particular, $(d+2/\alpha(h), 4/d+2)$ is an $\alpha(h)$-admissible pair. This allows us to bound the solutions u^h in a space $L^s_{loc}(\mathbb{R}, L^{4/d+2}(\mathbb{R}))$ with $s < 4/d+2$. With the same notation as in the subcritical case the following convergence result holds.

Theorem 3.5.6. *When $p = 4/d$ and under the smallness assumption on the initial datum u_0, the sequence Eu^h satisfies*

$$Eu^h \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)),$$

$$Eu^h \rightharpoonup u \text{ in } L^s_{loc}(\mathbb{R}, L^{4/d+2}(\mathbb{R}^d)), \ \forall\, s < 4/d+2,$$

$$Eu^h \to u \text{ in } L^2_{loc}(\mathbb{R}\times\mathbb{R}^d),$$

$$|Eu^h|^{p(h)}|Eu^h| \rightharpoonup |u|^{4/d}u \text{ in } L^{(4/d+2)'}_{loc}(\mathbb{R}, L^{(4/d+2)'}(\mathbb{R}^d))$$

where u is the unique weak solution of critical (NSE).

3.6. A two-grid algorithm

To compensate the lack of dispersion proved in Section 3.2 we propose a two-grid algorithm (inspired by [52]) and that, to some extent, acts as a filter for those unwanted high frequency components.

The method is roughly as follows. We consider two meshes: the coarse one of size $4h$, $h > 0$, $4h\mathbb{Z}^d$, and the finer one, $h\mathbb{Z}^d$, of size $h > 0$. The method relies basically on solving the finite-difference semi-discretization (3.1.15) on the fine mesh $h\mathbb{Z}^d$, but only for slow data, interpolated from the coarse grid $4h\mathbb{Z}^d$. As we shall see, the $1/4$ ratio between the two meshes is important to guarantee the convergence of the method. This particular structure of the data cancels the two pathologies of the discrete symbol mentioned above. Indeed, a careful Fourier analysis of those initial data (we refer to [136] for the theory of multi-grid methods) shows that their discrete Fourier transform vanishes quadratically in each variable at the points $\xi = (\pm\pi/2h)^d$ and $\xi = (\pm\pi/h)^d$. As we shall see, this suffices to recover the dispersive properties of the continuous model.

Once we get the discrete version of the dispersive properties we are able to apply it to a semi-discretization of the NLS with nonlinearity $f(u) = |u|^p u$. The nonlinear term is approximated in a such way that allows to apply the dispersive estimates of the linear semigroup. We recall that such estimates are valid only in a subspace of $\mathbb{C}^{h\mathbb{Z}^n}$ of data interpolated from the coarse grid. In the subcritical case we prove the global existence of the solutions for initial data in $l^2(h\mathbb{Z}^n)$. We also consider the critical case $p = 4/d$ for small initial data.

3.6.1. Dispersive estimates in the class of slowly oscillating sequences

We introduce the space of the slowly oscillating sequences (SOS). The SOS on the fine grid $h\mathbb{Z}^d$ are those which are obtained from the coarse grid $4h\mathbb{Z}^d$ by an interpolation process. Any function defined on the lattice $h\mathbb{Z}^d$ can be viewed as a function on the lattice $\mathbb{Z}^d$. This is the way we will proceed in the definition of the projection operator $\widetilde{\Pi}$ and its adjoint.

Let us consider the multilinear interpolator I acting on the coarse grid $4\mathbb{Z}^d$. We define the operator $\widetilde{\Pi} : l^2(4\mathbb{Z}^d) \to l^2(\mathbb{Z}^d)$ by

$$(\widetilde{\Pi} f)_{\mathbf{j}} = (If)_{\mathbf{j}}, \quad \mathbf{j} \in \mathbb{Z}^d \tag{3.6.1}$$

and its adjoint $\widetilde{\Pi}^* : l^2(\mathbb{Z}^d) \to l^2(4\mathbb{Z}^d)$:

$$(\widetilde{\Pi} f, g)_{l^2(\mathbb{Z}^d)} = (f, \widetilde{\Pi}^* g)_{l^2(4\mathbb{Z}^d)}. \tag{3.6.2}$$

We now define the space V_4 (subspace of $l^2(\mathbb{Z}^d)$) of slowly oscillating sequences as the image of the operator $\widetilde{\Pi}$:

$$V_4 = \{\widetilde{\Pi}\psi,\ \psi : 4\mathbb{Z}^d \to \mathbb{C}\}.$$

In dimension one, the explicit expressions of the two interpolators $\widetilde{\Pi}$ and $\widetilde{\Pi}^*$ are

$$(\widetilde{\Pi} f)_{4j+r} = \frac{4-r}{4} f_{4j} + \frac{r}{4} f_{4j+4},\ j \in \mathbb{Z},\ r \in \mathbb{Z}_4 = \{0,1,2,3\},$$

and

$$(\widetilde{\Pi}^* g)_{4j} = \sum_{r=0}^{3} \frac{4-r}{4} g_{4j+r} + \frac{r}{4} g_{4j-4+r},\ j \in \mathbb{Z}.$$

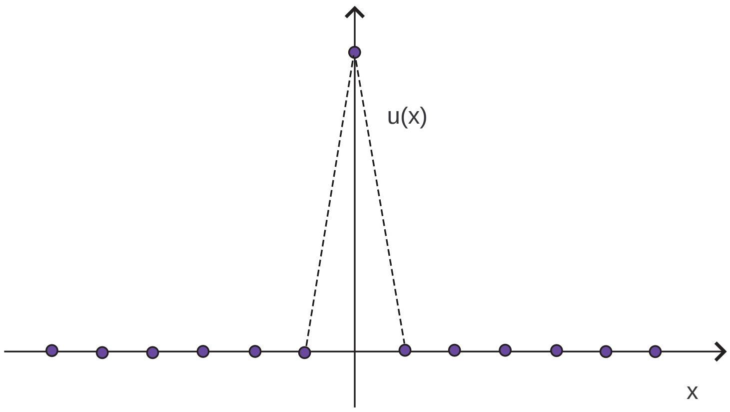

Figure 3.16: $u^1(0) = \delta_0$

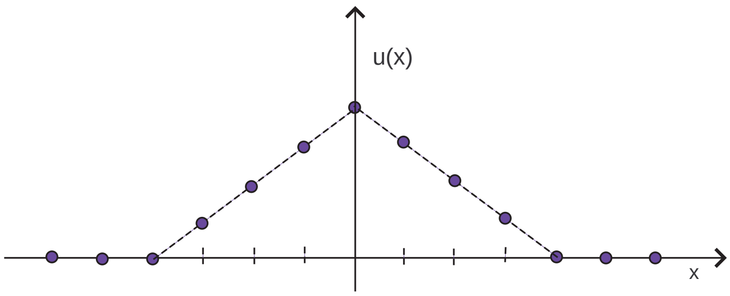

Figure 3.17: $u^1(0) = E\delta_0$

In the general case the explicit expressions are more complicated. However these operators have a simple and useful representation in the Fourier space. The same occurs when we consider the multilineal interpolator I. In the physical space the number of terms in its representation is of order 4^d which difficult an explicit formula. By contrast, in the Fourier space, its representation reads:

$$\widehat{Iu}(\xi) = \prod_{k=1}^{d} \left| \frac{e^{i\xi_k} - 1}{\xi_k} \right|^2 \widehat{u}(\xi).$$

The definition of the operator $\widetilde{\Pi}$ and its adjoint $\widetilde{\Pi}^*$ gives us

$$\|f\|_{l^p(4\mathbb{Z}^d)} \leq \|\widetilde{\Pi} f\|_{l^p(\mathbb{Z}^d)} \leq \|f\|_{l^p(4\mathbb{Z}^d)}, \ f \in l^p(4\mathbb{Z}^d)$$

and

$$\|\widetilde{\Pi}^* g\|_{l^p(4\mathbb{Z}^d)} \leq \|g\|_{l^p(\mathbb{Z}^d)}, \ g \in l^p(\mathbb{Z}^d).$$

Recall that in Section 3.2, we proved that there is no gain (uniformly in h) of integrability of the linear semigroup $e^{it\Delta_h}$. The same happened with the local smoothing effect. However, there are subspaces of $l^2(h\mathbb{Z}^d)$ where the linear semigroup has appropriate decay properties, uniformly on $h > 0$. The main results concerning the gain of integrability are the following.

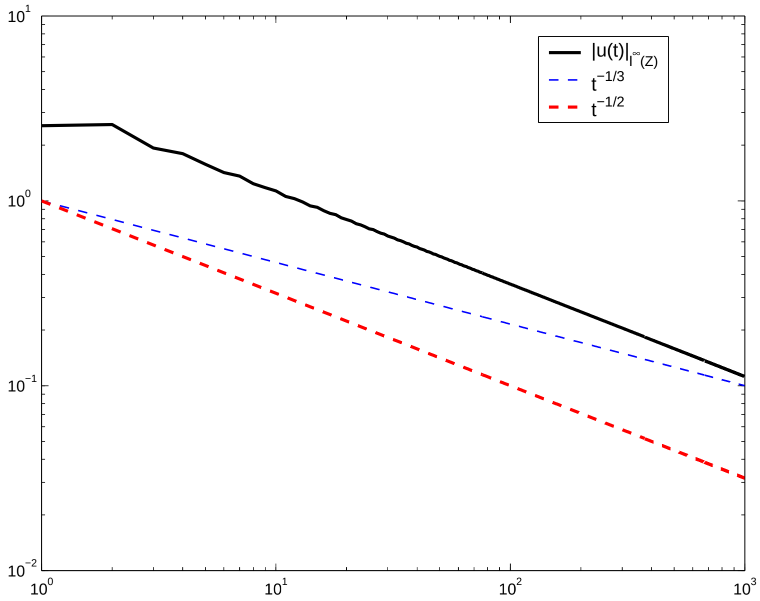

Figure 3.18: Log-log plot of the time evolution of the l^∞ norm of $u^1(t)$

Theorem 3.6.1. *Let $p \geq 2$ and (q, r), $(\tilde{q}, \tilde{r})$ two 1/2-admissible pairs. The following properties hold*

i) There exists a positive constant $C(d,p)$ such that

$$\|e^{it\Delta_h}\widetilde{\Pi}\varphi\|_{l^p(h\mathbb{Z}^d)} \leq C(d,p)|t|^{-d(\frac{1}{2}-\frac{1}{p})}\|\widetilde{\Pi}\varphi\|_{l^{p'}(h\mathbb{Z}^d)} \tag{3.6.3}$$

for all $\varphi \in l^{p'}(4h\mathbb{Z}^d)$, $h > 0$ and $t \neq 0$.

ii) For every $\varphi \in l^2(4h\mathbb{Z}^d)$, the function $t \to e^{it\Delta_h}\widetilde{\Pi}\varphi$ belongs to $L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d)) \cap C(\mathbb{R},\, l^2(h\mathbb{Z}^d))$. Furthermore, there exists a positive constant $C(d,r)$ such that

$$\|e^{it\Delta_h}\widetilde{\Pi}\varphi\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \leq C(d,r)\|\widetilde{\Pi}\varphi\|_{l^2(h\mathbb{Z}^d)} \tag{3.6.4}$$

uniformly on $h > 0$.

iii) There exists a positive constant $C(d,r)$ such that

$$\left\|\int_{-\infty}^{\infty} e^{-is\Delta_h}\widetilde{\Pi}f(s)ds\right\|_{l^2(h\mathbb{Z}^d)} \leq C(d,r)\|\widetilde{\Pi}f\|_{L^{q'}(\mathbb{R}, l^{r'}(h\mathbb{Z}^d))} \tag{3.6.5}$$

for all $f \in L^{q'}(\mathbb{R},\, l^{\tilde{r}'}(4h\mathbb{Z}^d))$, uniformly in $h > 0$.

iv) There exists a positive constant $C(d,r,\tilde{r})$ such that

$$\left\|\int_{s<t} e^{i(t-s)\Delta_h}\widetilde{\Pi}f(s)ds\right\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \leq C(d,r,\tilde{r})\|\widetilde{\Pi}f\|_{L^{\tilde{q}'}(\mathbb{R}, l^{\tilde{r}'}(h\mathbb{Z}^d))} \tag{3.6.6}$$

for all $f \in L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(4h\mathbb{Z}^d))$, uniformly in $h > 0$.

The results given by Theorem 3.6.1 i) are plotted in Figure 3.18. We choose an initial datum as in Figure 3.17, obtained by interpolation of the Dirac delta: $\Pi u(0) = \delta_0$ (see Figure 3.16). The $l^\infty(\mathbf{Z})$-norm of the solution $u^1(t)$ for the two-grid algorithm behaves like

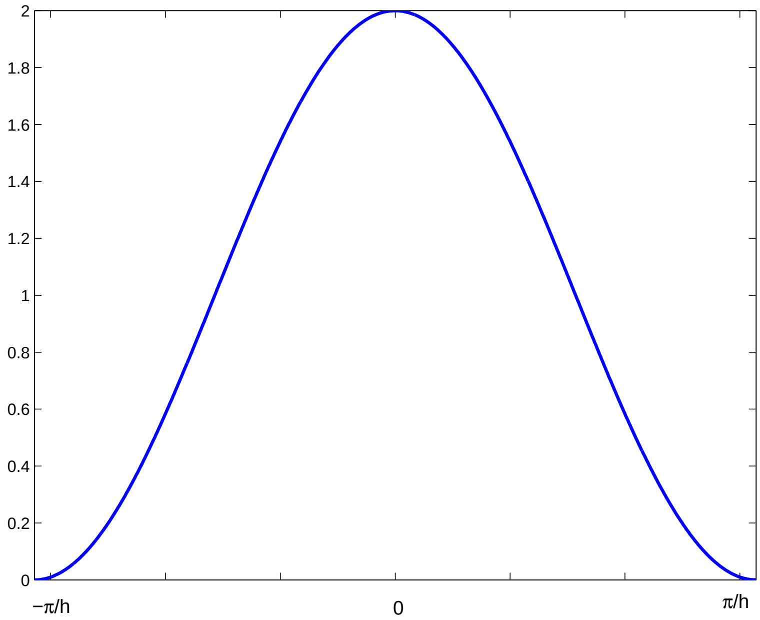

Figure 3.19: The multiplicative factor $2\cos^2(\xi h/2)$
for the two-grid method with mesh ratio 1/2

$t^{-1/2}$ as $t \to \infty$, with the decay rate predicted above, while the solutions of the conservative scheme, without the two-grid filtering, decay like $t^{-1/3}$.

Concerning the SDFT of SOS we have the following result:

Lemma 3.6.1. *Let $\psi \in l^2(4h\mathbb{Z}^d)$. Then for all $\xi \in [-\pi/h, \pi/h]^d$*

$$\widehat{\tilde{\Pi}\psi}(\xi) = 4^d \widehat{\psi}(\xi) \prod_{k=1}^{d} \cos^2(\xi_k h) \cos^2\left(\frac{\xi_k h}{2}\right).$$

Remark 3.6.1. *A simpler construction may be done by interpolating $2h\mathbb{Z}^d$ sequences. We then get for all $\psi \in l^2(2h\mathbb{Z}^d)$ and $\xi \in [-\pi/h, \pi/h]^d$*

$$\widehat{\tilde{\Pi}\psi}(\xi) = 2^d \widehat{\psi}(\xi) \prod_{k=1}^{d} \cos^2\left(\frac{\xi_k h}{2}\right)$$

This cancels the spurious numerical solutions at the frequencies $\{\pm\pi/h\}^d$, but not at $\{\pm\pi/2h\}^d$. In this case, as we proved in Section 3.2, the Strichartz estimates fail to be uniform on h. Thus we rather choose $1/4$ as the ratio between the grids for the two-grid algoorithm.

Proof. We consider the one-dimensional case. The general case follows by applying the one-

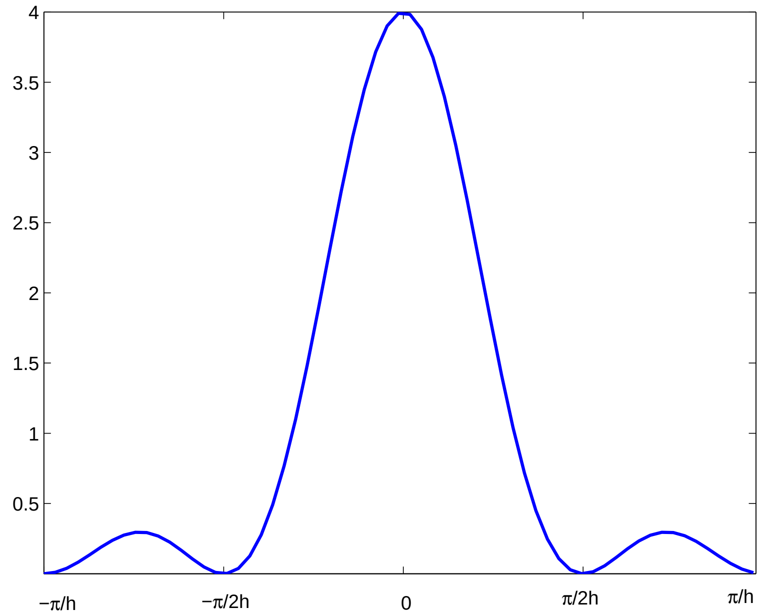

Figure 3.20: The multiplicative factor $4\cos^2(\xi h)\cos^2\left(\frac{\xi h}{2}\right)$ for the two-grid method with mesh ratio 1/4

dimensional argument in each variable. In this case:

$$\begin{aligned}
(\widehat{\widetilde{\Pi}\psi})(\xi) &= \sum_{j\in\mathbb{Z}}\sum_{r=0}^{3} e^{i(4j+r)h\xi}\left(\frac{4-r}{4}\psi_{4j}+\frac{r}{4}\psi_{4j+4}\right)\\
&= \sum_{j\in\mathbb{Z}} e^{i4jh\xi}\psi_{4j}\left(\sum_{r=0}^{3}\frac{4-r}{4}e^{ir\xi h}+\frac{r}{4}e^{i(r-4)\xi h}\right)\\
&= 4\cos^2(\xi h)\cos^2\left(\frac{\xi h}{2}\right)\sum_{j\in\mathbb{Z}} e^{4ij\xi h}\psi_{4j}\\
&= 4\cos^2(\xi h)\cos^2\left(\frac{\xi h}{2}\right)\widehat{\psi}(\xi).
\end{aligned}$$

□

As we have seen in the above Lemma, the operator $\widetilde{\Pi}$ acts in each variable as a multiplicative factor in the Fourier space. This factor vanishes quadratically in each variable at the points $\{\pm\pi/2h\}^d$ and $\{\pm\pi/h\}^d$.

In the following we introduce a more general class of operators and give an extension of Theorem 3.6.1. As we will see the multiplier introduced by the operator $\widetilde{\Pi}$ is too strong. In fact we need only that the multiplier vanishes in each variable with order 1/4 at the points $\{\pm\pi/2h\}^d$ and 1/2 at the points $\{\pm\pi/h\}^d$.

Let us define the family of weighted operators $A^h_\alpha(t): l^2(h\mathbb{Z}^d)\to l^2(h\mathbb{Z}^d)$ by

$$(\widehat{A^h_\alpha(t)f})(\xi)=e^{-itp_h(\xi)}|g(\xi h)|^\alpha\widehat{f}(\xi),\ \xi\in\left[-\frac{\pi}{h},\frac{\pi}{h}\right],$$

where

$$g(\xi)=\prod_{k=1}^{d}\cos(\xi_k)\cos\left(\frac{\xi_k}{2}\right).$$

The results of Theorem 3.6.1 are consequences of the following one.

Theorem 3.6.2. *Let $\alpha \geq 1/4$, $p \geq 2$ and (q,r), $(\tilde{q},\tilde{r})$ two admissible pairs. Then the following estimates are uniform with respect to $h>0$:*

i) There is a positive constant $c(\alpha,p)$ such that

$$\|A_\alpha^h(t)A_\alpha^h(s)^*\varphi\|_{l^p(h\mathbb{Z}^d)}\leq c(\alpha,p)|t-s|^{-\frac{d}{2}(\frac{1}{p'}-\frac{1}{p})}\|\varphi\|_{l^{p'}(h\mathbb{Z}^d)}, \tag{3.6.7}$$

holds for all $t\neq s$ and $\varphi\in l^{p'}(h\mathbb{Z}^d)$.

ii) There is a positive constant $c(\alpha,r)$ such that

$$\|A_\alpha^h(\cdot)\varphi\|_{L^q(\mathbb{R},\,l^r(h\mathbb{Z}^d))}\leq c(\alpha,r)\|\varphi\|_{l^2(h\mathbb{Z}^d)}, \tag{3.6.8}$$

holds for all $\varphi\in l^2(h\mathbb{Z}^d)$.

iii) There is a positive constant $c(\alpha,r)$ such that

$$\left\|\int_{-\infty}^{\infty}A_\alpha^h(s)^*f(s)ds\right\|_{l^2(h\mathbb{Z}^d)}\leq c(\alpha,r)\|f\|_{L^{q'}(\mathbb{R},\,l^{r'}(h\mathbb{Z}^d))}, \tag{3.6.9}$$

holds for all $f\in L^{q'}(\mathbb{R},\,l^{r'}(h\mathbb{Z}^d))$.

iv) There is a positive constant $c(\alpha,r,\tilde{r})$ such that

$$\left\|\int_{s<t}A_\alpha^h(t)A_\alpha^h(s)^*F(s)ds\right\|_{L^q(\mathbb{R},\,l^r(h\mathbb{Z}^d))}\leq c(\alpha,r,\tilde{r})\|F(s)\|_{L^{q'}(\mathbb{R},\,l^{r'}(h\mathbb{Z}^d))} \tag{3.6.10}$$

holds for all $f\in L^{\tilde{q}'}(\mathbb{R},\,l^{\tilde{r}'}(h\mathbb{Z}^d))$.

Remark 3.6.2. *In all the above inequalities we assume $\alpha\geq 1/4$. However, in order to obtain*

$$\|A_\alpha^h(t)\varphi\|_{l^p(h\mathbb{Z}^d)}\leq C(\alpha,p)|t|^{-\frac{d}{2}(\frac{1}{p'}-\frac{1}{p})}\|\varphi\|_{l^{p'}(h\mathbb{Z}^d)},\ t\neq 0, \tag{3.6.11}$$

we have to assume $\alpha\geq 1/2$. This is a consequence of the fact that the contribution of the multiplicative factor g in $A_\alpha^h(t)A_\alpha^h(s)^*$ is twice that in $A_\alpha^h(t-s)$.

We postpone the proof of Theorem 3.6.2 and show how Theorem 3.6.1 immediately follows once Theorem 3.6.2 will be proved.

Proof of Theorem 3.6.1. We observe that $e^{it\Delta_h}\widetilde{\Pi}\varphi=A_2^h(t)\varphi$. Then (3.6.7) and (3.6.8) imply (3.6.3) and (3.6.4). Remark that

$$\int_{\mathbb{R}}e^{-is\Delta_h}\widetilde{\Pi}F(s)ds=\int_{\mathbb{R}}(A_2^h(s))^*F(s)ds$$

and

$$\int_{s<t} e^{i(t-s)\Delta_h} \widetilde{\Pi} F(s) ds = \int_{s<t} A_1^h(t)(A_1^h(s))^* F(s) ds.$$

Then (3.6.9) and (3.6.10) imply (3.6.5) respectively (3.6.6). □

Proof of Theorem 3.6.2. A scaling argument reduces all the estimates to the case $h = 1$. We reduce the proof to the conditions of Keel and Tao [74] given in Proposition 3.3.1. These conditions say that it is sufficient to show that $A_\alpha^1(t)$ maps $l^2(\mathbb{Z}^d)$ to $l^2(\mathbb{Z}^d)$ and $A_\alpha^1(t)A_\alpha^1(s)^*$ maps $l^1(\mathbb{Z}^d)$ to $l^\infty(\mathbb{Z}^d)$ with an appropriate norm decay. More precisely we have to check (3.6.7) for $p = 1$ and $p = 2$.

The case $p = 2$ follows by Plancherel's identity. For that we remark that

$$(\widehat{A_\alpha^1(t)\psi})(\xi) = e^{-itp_1(\xi)} \widehat{\psi}(\xi) |g(\xi)|^\alpha$$

and obviously

$$\|A_\alpha^1(t)\psi\|_{l^2(\mathbb{Z}^d)} = \|\widehat{A_\alpha^1(t)\psi}\|_{L^2([-\pi,\pi]^d)} \lesssim \|\widehat{\psi}\|_{L^2([-\pi,\pi]^d)} = \|\psi\|_{l^2(\mathbb{Z}^d)}.$$

It remains to prove (3.6.7) for $p = 1$, i.e.

$$\|A_\alpha^1(t)A_\alpha^1(s)^*\psi\|_{l^\infty(\mathbb{Z}^d)} \leq c(\alpha, d)|t-s|^{-d/2}\|\psi\|_{l^1(\mathbb{Z}^d)}. \tag{3.6.12}$$

Let us first analyze the operator $A_\alpha^1(t)$. We claim that for any $\alpha \geq 1/2$ the following holds

$$\|A_\alpha^1(t)\psi\|_{l^\infty(\mathbb{Z}^d)} \leq c(\alpha, d)|t|^{-d/2}\|\psi\|_{l^1(\mathbb{Z}^d)}.$$

We write $A_\alpha^1(t)$ as a convolution $A_\alpha^1(t)\psi = K_\alpha^t * \psi$ where

$$\widehat{K_\alpha^t}(\xi) = e^{-itp_1(\xi)}|g(\xi)|^\alpha.$$

Thus, it is sufficient to prove that for any $\alpha \geq 1/2$

$$\|K_\alpha^t\|_{l^\infty}(\mathbb{Z}^d) \leq c(\alpha, d)|t|^{-d/2}.$$

We observe that K_α^t can be written by separation of variables as

$$\widehat{K_\alpha^t}(\xi) = \prod_{k=1}^d e^{-4it\sin^2(\frac{\xi_k}{2})} \left|\cos(\xi_k)\cos\left(\frac{\xi_k}{2}\right)\right|^\alpha = \prod_{j=1}^d \widehat{K_{1,\alpha}^t}(\xi_j).$$

It remains to prove that $\|K_{1,\alpha}^t\|_{l^\infty(\mathbb{Z})} \leq c(\alpha)|t|^{-1/2}$. We make use of the following result:

Lemma 3.6.2. *(Corollary 2.9, [75]) Let $(a, b) \subset \mathbb{R}$ and $\psi \in C^3(a, b)$ be such that ψ'' has a finite number of changes of monotonicity. Then*

$$\left|\int_a^b e^{i(t\psi(\xi)-x\xi)}|\psi''(\xi)|^{1/2}\phi(\xi)d\xi\right| \leq c_\psi|t|^{-1/2}\left\{\|\phi\|_{L^\infty(a,b)} + \int_a^b |\phi'(\xi)|d\xi\right\}.$$

holds for all real numbers x and t.

Remark that $(4\sin^2(\xi/2))'' = 2\cos(\xi)$. Applying the above Lemma with $\psi(\xi) = -4\sin^2(\xi/2)$ we obtain for any $\alpha \geq 1/2$:

$$
\begin{aligned}
\|K_1^t\|_{l^\infty(\mathbb{Z})} &\lesssim |t|^{-1/2}\left(\left\| |\cos(\xi)|^{\alpha-1/2}\left|\cos\left(\frac{\xi}{2}\right)\right|^{\alpha}\right\|_{L^\infty([-\pi,\pi])} \right.\\
&\quad \left. + \int_{-\pi}^{\pi}\left|\left(|\cos(\xi)|^{\alpha-1/2}\left|\cos\left(\frac{\xi}{2}\right)\right|^{\alpha}\right)'\right| d\xi \right)\\
&\leq c(\alpha)|t|^{-1/2}.
\end{aligned}
$$

In the following we prove that (3.6.12) holds for any $\alpha \geq 1/4$. Observe that the operator $A^1_\alpha(t)$ satisfies $A^1_\alpha(t)^* = A^1_\alpha(-t)$ for all real t. As a consequence we obtain

$$
\begin{aligned}
\|A^1_\alpha(t)A^1_\alpha(s)^*\psi\|_{l^\infty(\mathbb{Z}^d)} &= \|A^1_\alpha(t)A^1_\alpha(-s)^*\psi\|_{l^\infty(\mathbb{Z}^d)}\\
&= \|A^1_{2\alpha}(t-s)\psi\|_{l^\infty(\mathbb{Z}^d)} \lesssim |t-s|^{-d/2}\|\psi\|_{l^1(\mathbb{Z}^d)},
\end{aligned}
$$

for all $t \neq s$ and $\psi \in l^1(\mathbb{Z}^d)$.

We fall into the hypothesis of [74] (Theorem 1.2, p. 956). Thus for all admissible pairs (q,r) and $(\tilde{q},\tilde{r})$ we get all the desired estimates on A^1_α.

□

3.6.2. A conservative approximation of the NSE

We concentrate on the semilinear NSE equation in $\mathbb{R}^d$:

$$iu_t + \Delta u = |u|^p u,\ x \in \mathbb{R}^d,\ t \in \mathbb{R}; \qquad u(0,x) = \varphi(x),\ x \in \mathbb{R}^d,$$

the case when nonlinearity is given by $f(u) = -|u|^p u$ being the same. In fact, the key point in the global existence of the solutions is that the L^2-scalar product $(f(u), u)$ is a real number. All the results extend to more general nonlinearities $f(u)$ (see [25], Ch. 4.6, p. 109, for L^2-solutions).

We consider the following semi-discretization

$$i\frac{du^h}{dt} + \Delta_h u^h = \widetilde{\Pi} f(\widetilde{\Pi}^* u^h),\ t \in \mathbb{R}; \qquad u^h(0) = \widetilde{\Pi}\varphi^h, \tag{3.6.13}$$

where $f(u) = |u|^p u$. In order to prove the global well-posedness of (3.6.13), it is sufficient to guarantee the conservation of the $l^2(h\mathbb{Z}^d)$ norm of solutions, a property that the solutions of NSE satisfy. This is why we choose $\widetilde{\Pi} f(\widetilde{\Pi}^* u^h)$ as an approximation of the nonlinear term $f(u)$. The following holds:

Theorem 3.6.3. *Let $p \in (0, 4/d)$ and $q = 4(p+2)/dp$. Then for all $h > 0$ and for every $\varphi^h \in l^2(4h\mathbb{Z}^d)$, there exists a unique global solution*

$$u^h \in C(\mathbb{R},\, l^2(h\mathbb{Z}^d)) \cap L^q_{loc}(\mathbb{R},\, l^{p+2}(h\mathbb{Z}^d)) \tag{3.6.14}$$

of (3.6.13) which satisfies

$$\|u^h\|_{L^\infty(\mathbb{R},\, l^2(h\mathbb{Z}^d))} \leq \|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)} \text{ and } \|u^h\|_{L^q(I, l^{p+2}(h\mathbb{Z}^d))} \leq c(I)\|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)} \tag{3.6.15}$$

for all finite interval I, where the above constants are independent of h.

Remark 3.6.3. *The choice of the approximation of the nonlinear term is motivated by the following identity:*

$$(\widetilde{\Pi} f(\widetilde{\Pi}^* u^h), u^h)_{l^2(h\mathbb{Z}^d)} = (f(\widetilde{\Pi}^* u^h), \widetilde{\Pi}^* u^h)_{l^2(4h\mathbb{Z}^d)} \in \mathbb{R}. \tag{3.6.16}$$

This allows us to prove the conservation of the $l^2(h\mathbb{Z}^d)$-norm of the solutions.

Proof of Theorem 3.6.3. The local existence and uniqueness are consequences of the Strichartz-like estimates and of a fixed point argument in the space $L^\infty((-T,T), l^2(h\mathbb{Z}^d)) \cap L^q((-T,T), l^{p+2}(h\mathbb{Z}^d))$ where T have to be assumed small. Identity (3.6.16) proves the global existence of the solution. □

In the sequel we consider the piecewise constant interpolator E. We use a piecewise constant interpolator because it commutes with the nonlinear term $f(u)$ in the sense that $Ef(u^h) = f(Eu^h)$. This will be useful in order to transfer the pointwise convergence of solutions $Eu^h(x) \to u(x)$ to the nonlinear term. We choose $(\varphi^h_{\mathbf{j}})_{\mathbf{j}\in\mathbb{Z}^d}$, an approximation of the initial datum $\varphi \in L^2(\mathbb{R}^d)$, such that $E\widetilde{\Pi}\varphi^h$ converges strongly to φ in $L^2(\mathbb{R}^d)$. Thus, in particular,

$$\|E\widetilde{\Pi}\varphi^h\|_{L^2(\mathbb{R}^d)} \le C(\|\varphi\|_{L^2(\mathbb{R}^d)}). \tag{3.6.17}$$

The main convergence result is the following:

Theorem 3.6.4. *Let be p and q as in Theorem 3.6.3 and u^h be the unique solution of (3.6.13). Then the sequence Eu^h satisfies*

$$Eu^h \overset{\star}{\rightharpoonup} u \ \text{in} \ L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)), \ Eu^h \rightharpoonup u \ \text{in} \ L^q_{loc}(\mathbb{R}, L^{p+2}(\mathbb{R}^d)), \tag{3.6.18}$$

$$Eu^h \to u \ \text{in} \ L^2_{loc}(\mathbb{R}^{d+1}), \ E\widetilde{\Pi} f(\widetilde{\Pi}^* u^h) \rightharpoonup |u|^p u \ \text{in} \ L^{q'}_{loc}(\mathbb{R}, L^{(p+2)'}(\mathbb{R}^d)) \tag{3.6.19}$$

where u is the unique solution of NSE.

Our method works similarly in the critical case $p = 4/d$ for small initial data. The initial datum needs to be assumed to be small, but the smallness condition is independent of the mesh-size $h > 0$. More precisely, the following holds.

Theorem 3.6.5. *There exists a constant ϵ, independent of h, such that for all initial data $\|\varphi^h\|_{l^2(h\mathbb{Z}^d)} < \epsilon$, the semidiscrete critical equation (3.6.13) with $p = 4/d$ has a unique global solution*

$$u^h \in C(\mathbb{R}, l^2(h\mathbb{Z}^d)) \cap L^{4/d+2}_{loc}(\mathbb{R}, l^{4/d+2}(h\mathbb{Z}^d)).$$

Moreover $u^h \in L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^d))$ for all 1/2-admissible pairs (q,r) and

$$\|u^h\|_{L^q(I, l^r(h\mathbb{Z}^d)} \le C(q,I)\|\varphi^h\|_{l^2(h\mathbb{Z}^d)}$$

for all finite intervals I, uniformly on h.

With the same notation as in the subcritical case the following convergence result holds.

Theorem 3.6.6. *Let $p = 4/d$. Under the smallness assumption of Theorem 3.6.5, the sequence Eu^h satisfies*

$$Eu^h \overset{\star}{\rightharpoonup} u \text{ in } L^\infty(\mathbb{R}, L^2(\mathbb{R}^d)), \; Eu^h \rightharpoonup u \text{ in } L^{4/d+2}_{loc}(\mathbb{R}, L^{4/d+2}(\mathbb{R}^d))$$

and

$$Eu^h \to u \text{ in } L^2_{loc}(\mathbb{R}\times\mathbb{R}^d), \; E\widetilde{\Pi}(f(\widetilde{\Pi}^* u^h)) \rightharpoonup |u|^{4/d}u \text{ in } L^{(4/d+2)'}_{loc}(\mathbb{R}, L^{(4/d+2)'}(\mathbb{R}^d))$$

where u is the unique weak solution of the critical NSE with $p = 4/d$.

The main difficulty in the proof of Theorem 3.6.4 is the strong convergence $Eu^h \to u$ in $L^2_{loc}(\mathbb{R}^{d+1})$. Once it is obtained, the second convergence in (3.6.19) easily follows. Without the strong convergence of Eu^h towards u we are not able to pass to the limits in the nonlinear term. Another difficulty comes from the fact that the interpolator E has no compact support in the Fourier space. To simplify the proof we consider a band-limited interpolator I_* (cf. [135], Ch. II) and prove the compactness for $I_* u^h$. Once this is obtained we transfer the L^2-strong convergence of $I_* u^h$ to Eu^h. This is consequence of the following property of the piecewise constant interpolator Eu^h (cf. [34], [103]):

$$\|Eu^h - I_* u^h\|_{L^2(\Omega)} \le h\|I_* u^h\|_{H^1(\Omega)}, \tag{3.6.20}$$

which holds for all $\Omega \subset \mathbb{R}^d$.

We will prove that $I_* u^h$ is uniformly bounded in $L^2_{loc}(\mathbb{R}, H^{1/2}_{loc}(\mathbb{R}^d))$. Also we will obtain estimates on the $L^2_{loc}(\mathbb{R}, H^1_{loc}(\mathbb{R}^d))$-norm. The last ones are not uniform on h but give sufficient information to ensure that $Eu^h - I_* u^h$ strongly converges to zero in $L^2_{loc}(\mathbb{R}^{d+1})$.

The following proposition gives uniform bounds on Eu^h. These estimates are consequence of (3.6.15) on the solutions of equation (3.6.13).

Proposition 3.6.1. *Let I be a finite interval and Ω a bounded set of $\mathbb{R}^d$. Then*

$$\|Eu^h\|_{L^\infty(\mathbb{R}, L^2(\mathbb{R}^d))} \le C(\|\varphi\|_{L^2(\mathbb{R}^d)}), \; \|Eu^h\|_{L^q(I, L^{p+2}(\mathbb{R}^d))} \le C(I, \|\varphi\|_{L^2(\mathbb{R}^d)}), \tag{3.6.21}$$

and

$$\|E\widetilde{\Pi}f(u^h)\|_{L^{q'}(I, L^{(p+2)'}(\mathbb{R}^d))} \le C(I, \|\varphi\|_{L^2(\mathbb{R}^d)}), \tag{3.6.22}$$

hold uniformly for all $h > 0$. Moreover, Eu^h verifies

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} Eu^h(-i\psi_t + \Delta^h\psi)dxdt = \int_{\mathbb{R}}\int_{\mathbb{R}^d} E\widetilde{\Pi}f(\widetilde{\Pi}^* u^h)\psi dxdt \tag{3.6.23}$$

for all $\psi \in C_c^\infty(\mathbb{R}^{d+1})$.

In order to obtain the strong convergence of Eu^h in $L^2_{loc}(\mathbb{R}^{d+1})$ we have to prove regularity results for the band limited interpolator $I_* u^h$. This interpolator satisfies :

Lemma 3.6.3. *Let be $s \geq 1/2$, $I \subset \mathbb{R}$ a bounded interval and $\chi \in C_c^\infty(\mathbb{R}^d)$. Then there is a constant $C(I,\chi)$ such that*

$$\|\chi I_*(e^{it\Delta_h}\widetilde{\Pi}\varphi^h)\|_{L^2(I,H^s(\mathbb{R}^d))} \leq \frac{C(I,\chi)}{h^{s-1/2}}\|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)} \tag{3.6.24}$$

holds for all functions $\varphi^h \in l^r(4h\mathbb{Z}^d)$. Moreover for any $1/2$-admissible pair (q,r)

$$\left\|\chi I_*\left(\int_0^t e^{i(t-\tau)\Delta_h}\widetilde{\Pi}f^h(\tau)d\tau\right)\right\|_{L^2(I,H^s(\mathbb{R}^d))} \leq \frac{C(I,\chi)}{h^{s-1/2}}\|\widetilde{\Pi}f^h\|_{L^{q'}(I,l^{r'}(h\mathbb{Z}^d))} \tag{3.6.25}$$

for all $f^h \in L^{q'}(I, l^{r'}(4h\mathbb{Z}^d))$.

We postpone the proof of Lemma 3.6.3 and proceed to prove Theorem 3.6.4.

Proof of Theorem 3.5.4. **Step i). Weak convergence.**
By (3.6.21) there is a subsequence of Eu^h and a function $v \in L^\infty(\mathbb{R},\, L^2(\mathbb{R}^d))$ such that

$$Eu^h \overset{\star}{\rightharpoonup} v \text{ in } \; L^\infty(\mathbb{R},\, L^2(\mathbb{R}^d)).$$

Using that $\|Eu^h\|_{L^q(I,\, L^{p+2}(\mathbb{R}^d))} \leq c(I)$ there is a subsequence Eu^h and a function $v_1 \in L^q(I,\, L^{p+2}(\mathbb{R}^d))$ such that

$$Eu^h \rightharpoonup v_1 \text{ in } \; L^q(I,\, L^{p+2}(\mathbb{R}^d)). \tag{3.6.26}$$

Step ii). Continuity of v in $L^2(\mathbb{R}^d)$.
Using the uniqueness of the limit in the sense of distributions we can identify a.e. v_1 with v. To prove that $v \in C(\mathbb{R},\, L^2(\mathbb{R}^d))$ it is sufficient to prove the continuity at $t=0$. We remark that for any positive $0 \leq t \leq T$:

$$\begin{aligned}\|u^h(t) - S^h(t)\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)} &\leq \left\|\int_0^t S^h(t-s)f(\widetilde{\Pi}^* u^h)ds\right\|_{L^\infty([0,T],l^2(\mathbb{Z}^d))}\\ &\leq \||u^h|^{p+1}\|_{L^{q'}([0,T],\, l^{(p+2)'}(h\mathbb{Z}^d))} \leq T^\alpha \|u^h\|^\beta_{L^q(\mathbb{R},\, l^{p+2}(h\mathbb{Z}^d))} \leq CT^\alpha\end{aligned}$$

for some positive α, β and C independent of h. Using the weak convergence $Eu^h(t) - ES^h(t)\widetilde{\Pi}\varphi^h \rightharpoonup v(t) - \varphi$ in $L^2(\mathbb{R}^d)$ we get

$$\|v(t)-\varphi\|_{L^2(\mathbb{R}^d)} \leq \liminf \|Eu^h(t) - ES^h(t)\widetilde{\Pi}\varphi^h\|_{L^2(\mathbb{R}^d)} \leq T^\alpha$$

which prove that $v(t) \to \varphi$ in $L^2(\mathbb{R}^d)$ as $t \to 0$.

The case $p = 4/d$ is more tricky. Let us consider $\varphi \in L^2(\mathbb{R}^d)$ and $E\widetilde{\Pi}\varphi^h \to 0$ in $L^2(\mathbb{R}^d)$. First we prove that for every $\epsilon > 0$ there exist $T_\epsilon > 0$ and $h_\epsilon > 0$ such that

$$\|S^h(t)\widetilde{\Pi}\varphi^h\|_{L^{2+4/d}([0,T],l^{2+4/d}(h\mathbb{Z}^d))} < \epsilon \tag{3.6.27}$$

for all all $0 < T < T_\epsilon$ and $h < h_\epsilon$.

The same argument as in the subcritical case shows that for any $T>0$

$$\|u^h(t)-S^h(t)\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)}\lesssim \|u^h\|^{4/d}_{L^{2+4/d}([0,T],l^{2+4/d}(h\mathbb{Z}^d)}. \tag{3.6.28}$$

The proof of the existence shows the existence of a positive time T_0 such that for all $T<T_0$

$$\|u^h\|^{4/d}_{L^{2+4/d}([0,T],l^{2+4/d}(h\mathbb{Z}^d)}\lesssim \|S^h(t)\varphi^h\|_{L^{2+4/d}([0,T],l^{2+4/d}(h\mathbb{Z}^d)}. \tag{3.6.29}$$

Putting together (3.6.27), (3.6.28 and (3.6.29) we obtain that for any positive ϵ there exist $T_\epsilon>0$ and $h_\epsilon>0$ such that

$$\|u^h(t)-S^h(t)\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)}\le\epsilon \tag{3.6.30}$$

for all $T<T_\epsilon$ and $h<h_\epsilon$. Keeping ϵ fixed, we obtain by getting $h\to 0$ that

$$\|v(t)-S(t)\varphi\|_{L^2(\mathbb{R}^d)}\le\epsilon \tag{3.6.31}$$

for all $t<T_\epsilon$, which prove the continuity of u at $t=0$.

It remains to prove (3.6.27). The main difficulty is to prove that the time obtained in (3.6.27) does not depend by h. For the moment let us assume the existence of a sequence $\tilde{\varphi}^h$ such that $\|\varphi^h-\tilde{\varphi}^h\|\le\epsilon/2$ and for some $s>d(1/2-1/(2+4/d))$ the following holds $\|\tilde{\varphi}^h\|_{\hbar^s(h\mathbb{Z}^d)}<c(\epsilon)$. Thus

$$\begin{aligned}\|S^h(t)\widetilde{\Pi}\varphi^h\|_{L^{2+4/d}([0,T],l^{2+4/d}(h\mathbb{Z}^d)} &\le \|S^h(t)(\widetilde{\Pi}\varphi^h-\widetilde{\Pi}\tilde{\varphi}^h)\|_{L^{2+4/d}([0,T],l^{2+4/d}(h\mathbb{Z}^d))}\\ &\quad+\|S^h(t)\widetilde{\Pi}\tilde{\varphi}^h\|_{L^{2+4/d}([0,T],l^{2+4/d}(h\mathbb{Z}^d))}\\ &\le \|\varphi^h-\tilde{\varphi}^h\|_{l^2(h\mathbb{Z}^d)}+T^{1/(2+4/d)}\|S^h(t)\widetilde{\Pi}\tilde{\varphi}^h\|_{L^\infty([0,T],\hbar^s(h\mathbb{Z}^d))}\\ &\le \frac{\epsilon}{2}+T^{1/(2+4/d)}\|\tilde{\varphi}^h\|_{\hbar^s(h\mathbb{Z}^d)}\le\epsilon\end{aligned}$$

for all $T<T_\epsilon$, where $T_\epsilon^{1/(2+4/d)}<\epsilon/2$.

It remains to prove the assumptions on $\tilde{\varphi}^h$. Let us choose $\tilde{\varphi}\in H^s(\mathbb{R}^d)$, $d/(d+2)<s\le 1$ such that $\|\varphi-\tilde{\varphi}\|_{L^2(\mathbb{R}^d)}\le\epsilon/4$. For this new function we choose an approximation $\tilde{\varphi}^h$ such that $I\tilde{\varphi}^h\to\tilde{\varphi}$ in $H^s(\mathbb{R}^d)$, I being the multi-linear interpolator. Thus, there exists an h_ϵ such that for all $h<h_\epsilon$ the following hold

$$\begin{aligned}\|\varphi^h-\tilde{\varphi}^h\|_{l^2(h\mathbb{Z}^d)} &\lesssim \|I\varphi^h-\varphi\|_{L^2(\mathbb{R}^d)}+\|\varphi-\tilde{\varphi}\|_{L^2(\mathbb{R}^d)}+\|\tilde{\varphi}-I\tilde{\varphi}^h\|_{L^2(\mathbb{R}^d)}\\ &\le \frac{\epsilon}{8}+\frac{\epsilon}{4}+\frac{\epsilon}{8}=\frac{\epsilon}{2}\end{aligned}$$

and

$$\|\tilde{\varphi}^h\|_{\hbar^s(h\mathbb{Z}^d)}\lesssim\|I\tilde{\varphi}^h\|_{H^s(\mathbb{R}^d)}\le C(\tilde{\varphi}).$$

Step iii). Strong Convergence of I_*u^h.

Using Lemma 3.6.3 with $s = 1/2$ we obtain that I_*u^h satisfies

$$\begin{aligned}
\|\chi I_* u^h\|_{L^2(I,H^{1/2}(\mathbb{R}^d))} &\leq \|\chi I_*(e^{it\Delta_h}\widetilde{\Pi}\varphi^h)\|_{L^2(I,H^{1/2}(\mathbb{R}^d))} \\
&\quad + \left\|\chi I_*\left(\int_0^t e^{i(t-s)\Delta_h}\widetilde{\Pi} f(\widetilde{\Pi}^* u^h)ds\right)\right\|_{L^2(I,H^{1/2}(\mathbb{R}^d))} \\
&\leq C(I,\chi)(\|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)} + \|\widetilde{\Pi} f(\widetilde{\Pi}^* u^h)\|_{L^{q'}(I,l^{(p+2)'}(h\mathbb{Z}^d))}) \\
&= C(I,\chi)(\|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)} + \||u^h|^{p+1}\|_{L^{q'}(I,l^{(p+2)'}(h\mathbb{Z}^d))}) \\
&\leq C(I,\chi)(\|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)} + \|u^h\|^{p+1}_{L^q(I,l^{p+2}(h\mathbb{Z}^d))}) \\
&\leq C(I,\chi,\|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)}) \leq C(I,\chi,\|\varphi\|_{L^2(\mathbb{R}^d)}).
\end{aligned}$$

Let I be a finite interval and $\Omega \subset \mathbb{R}^d$ bounded. We use the arguments of Simon ([111], Corollary 4, p. 85). For that it is sufficient to remark that Iu^h satisfies

$$\|I_* u^h\|_{L^2(I,H^{1/2}(\Omega))} \leq C(I,\Omega,\|\varphi\|_{L^2(\mathbb{R}^d)})$$

and

$$\left\|\frac{dI_* u^h}{dt}\right\|_{L^1(I,H^{-2}(\Omega)} \leq C(I,\Omega,\|\varphi\|_{L^2(\mathbb{R}^d)}).$$

Using the embeddings

$$H^s(\Omega) \underset{comp}{\hookrightarrow} L^2(\Omega) \hookrightarrow H^{-2}(\Omega)$$

we obtain the existence of a function v_1 such that $I_*u^h \to v_1$ in $L^2(I \times \Omega)$. By a diagonal process we get that $I_*u^h \to v_1$ in $L^2_{loc}(\mathbb{R} \times \mathbb{R}^2)$.

Step iv). Transferring the strong convergence from I_*u^h to Eu^h.
Classical properties of the interpolator Eu^h (see [34], [103]) give us

$$\int_\Omega |Eu^h - I_*u^h|^2 dx \leq h^2 \|I_*u^h\|^2_{H^1(\Omega)}.$$

Applying Lemma 3.6.3 with $s = 1$ we obtain for any $\chi \in C_c^\infty(\mathbb{R}^d)$

$$\begin{aligned}
\int_I \int_{\mathbb{R}^d} \chi^2 |Eu^h - I_*u^h|^2 dxdt &\leq h^2 \int_I \int_{\mathbb{R}^d} \chi^2 |(I-\Delta)^{1/2} I_* u^h|^2 dxdt \\
&\leq hC(I,\|\widetilde{\Pi}\varphi^h\|^2_{l^2(h\mathbb{Z}^d)}) \to 0,\ h \to 0.
\end{aligned}$$

This shows that $Eu^h - I_*u^h \to 0$ in $L^2_{loc}(\mathbb{R} \times \mathbb{R}^d)$. Using the strong convergence of I_*u^h towards v_1, we obtain that $v_1 = v$ and

$$Eu^h \to v \text{ in } L^2_{loc}(\mathbb{R} \times \mathbb{R}^d).$$

Let $\Gamma \subset \mathbb{Z}^d$ be a finite set. Thus for any $s \in \Gamma$ we have $Eu^h(\cdot + sh) \to v$ in $L^2_{loc}(\mathbb{R} \times \mathbb{R}^d)$ and

$$Eu^h(\cdot + sh) \to v \text{ a.e. on compact sets.}$$

The operators $\widetilde{\Pi}$ and $\widetilde{\Pi}^*$ involve only a finite number of translations. This shows that

$$E\widetilde{\Pi}f(\widetilde{\Pi}^*u^h) \to |v|^p v \text{ a.e. on compact sets.}$$

By Strauss's Lemma we obtain that

$$E\widetilde{\Pi}f(\widetilde{\Pi}^*u^h) \rightharpoonup |v|^p v \text{ in } L^{q'}(I,\, L^{(p+2)'}(\mathbb{R}^d)). \tag{3.6.32}$$

Step v) Passing to the limit in (3.6.23).
It remains to prove that v is the weak solution of NSE. This means that $v \in C(\mathbb{R},\, L^2(\mathbb{R}^d)) \cap L^q_{loc}(\mathbb{R},\, L^{p+2}(\mathbb{R}^d))$ and

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} u(-i\psi_t + \Delta\psi)dxdt = \int_{\mathbb{R}}\int_{\mathbb{R}^d} |u|^p u\psi dxdt \tag{3.6.33}$$

for all $\psi \in C_c^\infty(\mathbb{R},\, H^2(\mathbb{R}^d))$. Using that v belongs to $L^q_{loc}(\mathbb{R},\, L^{p+2}(\mathbb{R}^d)) \hookrightarrow L^q_{loc}(\mathbb{R},\, H^{-2}(\mathbb{R}^d))$ it is sufficient to prove (3.6.33) for all $\psi \in C_c^\infty(\mathbb{R}^{d+1})$. By (3.6.26) and (3.6.32)

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} Eu^h\psi_t dxdt \to \int_{\mathbb{R}}\int_{\mathbb{R}^d} v\psi_t dxdt$$

and

$$\int_{\mathbb{R}}\int_{\mathbb{R}^d} \widetilde{\Pi}f(\widetilde{\Pi}^*u^h)\psi dxdt \to \int_{\mathbb{R}}\int_{\mathbb{R}^d} |v|^p v\psi dxdt$$

for all $\psi \in C_c^\infty(\mathbb{R}^{d+1})$. The strong convergence $Eu^h \to v$ on compact sets and the weak convergence $\Delta_h\psi \rightharpoonup \Delta\psi$ imply

$$\int_{\mathbb{R}}\int_{\mathbb{R}^2} Eu^h\Delta_h\psi dxdt \to \int_{\mathbb{R}}\int_{\mathbb{R}^2} v\Delta\psi dxdt.$$

This finishes the proof. □

Proof of Lemma 3.6.3. **Step I. Regularity of the homogenous term.**
To prove (3.6.24) it is sufficient to show for any $R > 0$ the existence of a positive constant $C(I, R)$ such that

$$\int_I\int_{|x|<R} |D^s I_*(e^{it\Delta_h}\widetilde{\Pi}\varphi^h)|^2 dxdt \leq \frac{C(R, I)}{h^{2s-1}}\int_{[-\pi/h,\pi/h]^d} |\widehat{\varphi}^h(\xi)|^2 d\xi.$$

In view of the above estimate and using the properties of pseudodifferential operators we have

$$\begin{aligned}
&\|\chi I_*(e^{it\Delta_h}\widetilde{\Pi}\varphi^h)\|_{L^2(I,H^s(\mathbb{R}^d))} \leq \\
&\qquad \leq \|\chi D^s I_*(e^{it\Delta_h}\widetilde{\Pi}\varphi^h)\|_{L^2(I,\, L^2(\mathbb{R}^d))} + C(I,\chi)\|I_*(e^{it\Delta_h}\widetilde{\Pi}\varphi^h)\|_{L^2(I,\, L^2(\mathbb{R}^d))} \\
&\qquad \leq C(I,\chi)\|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)}\left(\frac{1}{h^{s-1/2}} + 1\right) \leq \frac{C(I,\chi)}{h^{s-1/2}}\|\widetilde{\Pi}\varphi^h\|_{l^2(h\mathbb{Z}^d)}.
\end{aligned}$$

Let us consider $\psi^h \in l^2(h\mathbb{Z}^d)$. Applying the results of [75] (see Lemma 3.5.1) to the function $I_*(e^{it\Delta_h}\psi^h)$ we obtain

$$\begin{aligned}
\int_I\int_{|x|<R}|D^sI_*(e^{it\Delta_h}\psi^h)|^2dxdt &\leq C(I,R)\int_{[-\pi/h,\pi/h]^d}|\xi|^{2s}\frac{|\widehat{I_*\psi^h}(\xi)|^2}{|\nabla p_h(\xi)|}d\xi\\
&\leq C(I,R)h^{1-2s}\int_{[-\pi/h,\pi/h]^d}|\xi|\frac{|\widehat{I_*^h\psi^h}(\xi)|^2}{|\nabla p_h(\xi)|}d\xi\\
&\leq C(I,R)h^{1-2s}\int_{[-\pi/h,\pi/h]^d}\frac{(\sum_{j=1}^d\xi_k^2)^{1/2}|\widehat{\psi^h}(\xi)|^2}{(\sum_{j=1}^d\sin^2(\xi_jh)/h^2)^{1/2}}d\xi\\
&\lesssim C(I,R)h^{1-2s}\int_{[-\pi/h,\pi/h]^d}\frac{|\widehat{\psi}^h|^2}{\prod_{j=1}^d|\cos(\xi_jh/2)|}d\xi. \qquad (3.6.34)
\end{aligned}$$

Now, we apply this inequality with $\psi^h=\widetilde{\Pi}\varphi^h$ to obtain

$$\begin{aligned}
\int_I\int_{|x|<R}|D^sI_*(e^{it\Delta_h}\widetilde{\Pi}\varphi^h)|^2dxdt &\leq C(I,R)h^{1-2s}\int_{[-\pi/h,\pi/h]^d}\frac{|\widehat{\widetilde{\Pi}\varphi^h}(\xi)|^2}{\prod_{j=1}^d|\cos(\xi_jh/2)|}d\xi\\
&\leq\frac{C(I,R)}{h^{2s-1}}\int_{[-\pi/h,\pi/h]^d}|\widehat{\varphi}^h(\xi)|^2\prod_{j=1}^d|\cos(\xi_jh/2)|^3d\xi\leq\\
&\leq\frac{C(I,R)}{h^{2s-1}}\int_{[-\pi/h,\pi/h]^d}|\widehat{\varphi}^h(\xi)|^2d\xi.
\end{aligned}$$

Step II. Regularity of the inhomogeneous part

In the following we prove that

$$\left\|\chi I_*\left(\int_0^te^{i(t-\tau)\Delta_h}\widetilde{\Pi}f^h(\tau)d\tau\right)\right\|_{L^2(I,H^s(\mathbb{R}^d))}\leq C(I,\chi)\|\widetilde{\Pi}f^h\|_{L^{q'}(I,l^{r'}(h\mathbb{Z}^d))}.$$

The estimates on the nonhomogeneous term will be reduced to the homogenous ones by using the argument of Christ and Kiselev [33] (see also [20], [115] in the context of PDE). A simplified version, useful in PDE application is given in [115] :

Lemma 3.6.4. *Let X and Y be Banach spaces and assume that $K(t,s)$ is a continuous function taking its values in $B(X,Y)$, the space of bounded linear mappings from X to Y. Suppose that $-\infty\leq a<b\leq\infty$ and set*

$$Tf(t)=\int_a^bK(t,s)f(s)ds,\quad Wf(t)=\int_a^tK(t,s)f(s)ds.$$

Assume that $1\leq p<q\leq\infty$ and

$$\|Tf\|_{L^q([a,b],Y)}\leq\|f\|_{L^p([a,b],X)}.$$

Then

$$\|Wf\|_{L^q([a,b],Y)}\leq\|f\|_{L^p([a,b],X)}.$$

In view of the above lemma it is sufficient to prove that

$$\left\| \chi I_*^h \left(\int_{-T}^{T} e^{i(t-\tau)\Delta_h} \widetilde{\Pi} f^h(\tau) d\tau \right) \right\|_{L^2((-T,T),H^s(\mathbb{R}^d))} \leq C(T,\chi) \|\widetilde{\Pi} f^h\|_{L^{q'}((-T,T),l^{r'}(h\mathbb{Z}^d))}$$

for any $T>0$. Consider the operator $A : l^2(h\mathbb{Z}^d) \to L^2((-T,T), l^2(h\mathbb{Z}^d))$ defined by

$$(A\varphi^h)(t) = e^{it\Delta_h}\varphi^h.$$

Its adjoint $A^* : L^2((-T,T), l^2(h\mathbb{Z}^d)) \to l^2(h\mathbb{Z}^d)$ is given by

$$A^* g^h = \int_{-T}^{T} e^{-is\Delta_h} g^h(s) ds.$$

Then the operator $AA^* : L^2((-T,T), l^2(h\mathbb{Z}^d)) \to L^2((-T,T), l^2(h\mathbb{Z}^d))$ verifies

$$(AA^* g^h)(t) = \int_{-T}^{T} e^{i(t-s)\Delta_h} g^h(s) ds.$$

It remains to prove that

$$\|\chi I_* AA^* \widetilde{\Pi} f^h\|_{L^2((-T,T),H^s(\mathbb{R}^d))} \leq C(T,\chi) \|\widetilde{\Pi} f^h\|_{L^{q'}((-T,T),l^{r'}(h\mathbb{Z}^d))}$$

holds for all $f^h \in L^{q'}((-T,T), l^{r'}(4h\mathbb{Z}^d))$

Using (3.6.34) on $e^{it\Delta_h}$ we get

$$\|\chi I_* AA^* \widetilde{\Pi} f^h\|_{L^2((-T,T),H^s(\Omega))} \leq C(T,\Omega) \left\| \frac{\widehat{A^* \widetilde{\Pi} f^h}}{\prod_{k=1}^d |\cos(\frac{\xi_k h}{2})|^{1/2}} \right\|_{L^2((-\pi/h,\pi/h)^d)}.$$

Hence, it is sufficient to prove that

$$\left\| \frac{(\widehat{A^* \widetilde{\Pi} f^h})}{\prod_{k=1}^d |\cos(\frac{\xi_k h}{2})|^{1/2}} \right\|_{L^2((-\pi/h,\pi/h)^2)} \leq C(T,\Omega) \|\widetilde{\Pi} f^h\|_{L^{q'}((-T,T),l^{r'}(h\mathbb{Z}^d))}.$$

Explicit computations shows that $A^* \widetilde{\Pi} f^h$ satisfies

$$\begin{aligned} \frac{(\widehat{A^* \widetilde{\Pi} f^h})(\xi)}{\prod_{k=1}^d |\cos(\frac{\xi_k h}{2}) \cos(\xi_k h)|^{1/2}} &= \int_{-T}^{T} e^{isp_h(\xi)} \left| \cos\left(\frac{\xi_k h}{2}\right) \cos(\xi_k h) \right|^{3/2} \widehat{\Pi f^h(s)} ds \\ &= \left(\int_{-T}^{T} A_{3/2}^h(s)^* \Pi f^h(s) ds \right)\widehat{\ }(\xi), \end{aligned}$$

where Πf^h extends f^h by zero on $h\mathbb{Z}^d \setminus 4h\mathbb{Z}^d$. Applying Theorem 3.6.2 with $\alpha = 3/2$ we get

$$\left\| \int_{-T}^{T} A_{3/2}^h(s)^* \Pi f^h(s) ds \right\|_{l^2(h\mathbb{Z}^d)} \lesssim \|\Pi f^h\|_{L^{q'}((-T,T),l^{r'}(h\mathbb{Z}^d))} \lesssim \|\widetilde{\Pi} f^h\|_{L^{q'}((-T,T),l^{r'}(h\mathbb{Z}^d))}.$$

This finishes the proof. □

Chapter 4

Fully Discrete Schemes for the Schrödinger Equation

4.1. Introduction

In this chapter we present some results on the qualitative properties of the fully discrete schemes for the one-dimensional linear Schrödinger equation and their consequences in the context of nonlinear problems.

More precisely, we analyze whether these numerical approximation schemes have the same dispersive properties, uniformly with respect to the mesh-size h, as in the case of the continuous Schrödinger equation (3.1.1). In particular we analyze whether the decay rate (3.1.6) holds for the solutions of the numerical scheme, uniformly in h. The study of these dispersion properties of the numerical scheme in the linear framework is relevant also for proving the convergence in the nonlinear context. Indeed, since the proof of the well-posedness of the nonlinear Schrödinger equations in the continuous framework requires a delicate use of the dispersion properties, the proof of the convergence of the numerical scheme in the nonlinear context is hopeless if these dispersion properties are not verified at the numerical level.

In the context of the one-dimensional KdV-equation, Nixon in [98] analyzes the backward Euler scheme for the linear semigroup. The author obtains space-time estimates for the discrete solutions and apply these results to obtain an approximation for a nonlinear problem. Here we consider a general two-level scheme for the one-dimensional Schrödinger equation and give necessary and sufficient conditions to guarantee the existence of uniform dispersive properties.

In order to introduce the finite-difference approximation of the LSE, it will be necessary to first introduce some notations. The space $\mathbb{R} \times \mathbb{R}$ will be replaced by the lattice $\mathbb{Z} \times \mathbb{Z}$, and instead of functions $u(t, \cdot)$ depending on $t \in \mathbb{R}$, consideration will be given to sequences $U^n = (U_j^n)_{j\in\mathbb{Z}}$ for $n \in \mathbb{Z}$. For a mesh size $h > 0$ and a time step $k > 0$, U_j^n is supposed to approximate $u(nk, jh)$; $u(t, x)$ being a solution of the LSE. In the sequel we shall assume that

$$\lambda = \frac{k}{h^2} \tag{4.1.1}$$

is kept constant as $h, k \to 0$, and we shall consider the two-level, constant coefficient, difference scheme:

$$A_{1,\lambda}U^{n+1} = A_{2,\lambda}U^n, n \geq 0 \tag{4.1.2}$$

and

$$\overline{A}_{1,\lambda}U^{n-1} = \overline{A}_{2,\lambda}U^{n}, n \leq 0. \tag{4.1.3}$$

The operators $A_{l,\lambda}$ and $\overline{A}_{l,\lambda}$, $l = 1, 2$, are defined by

$$A_{l,\lambda} = \sum_{\gamma \in F} a_{l,\gamma}(\lambda)\tau^{\gamma}, \quad \overline{A}_{l,\lambda} = \sum_{\gamma \in F} \overline{a}_{l,\gamma}(\lambda)\tau^{\gamma}, l = 1, 2,\ F \subset \mathbb{Z}, \text{finite set},$$

$$(\tau^{\gamma}U)_j = U_{j+\gamma},\ \text{for}\, U = (U_j)_{j\in\mathbb{Z}},\ \gamma \in \mathbb{Z};$$

so that, explicitly,

$$\begin{cases} \displaystyle\sum_{\gamma \in F} a_{1,\gamma}(\lambda)U^{n+1}_{j+\gamma} = \sum_{\gamma \in F} a_{2,\gamma}(\lambda)U^{n}_{j+\gamma}, & n \geq 0, \quad j \in \mathbb{Z}, \\ \displaystyle\sum_{\gamma \in F} \overline{a}_{1,\gamma}(\lambda)U^{n-1}_{j+\gamma} = \sum_{\gamma \in F} \overline{a}_{2,\gamma}(\lambda)U^{n}_{j+\gamma}, & n \leq 0, \quad j \in \mathbb{Z}. \end{cases} \tag{4.1.4}$$

The choice of $\overline{A}_{1,\lambda}$ and $\overline{A}_{2,\lambda}$ is motivated by the fact that once we introduce the scheme (4.1.2) to approximate LSE for $t \geq 0$, we automatically have an approximation of LSE for $t \leq 0$ given by (4.1.3).

We will be more precise on the type of estimates we are looking for. Let us consider $T \neq 0$, $h \to 0$ and $n \in \mathbb{Z}$ such that $nk \to T$. We establish necessary and sufficient conditions on the operators $A_{1,\lambda}$ and $A_{2,\lambda}$ in order to guarantee that

$$\|U^n\|_{l^q(h\mathbb{Z})} \leq C(T, \lambda, q, q_0)\|U^0\|_{l^{q_0}(h\mathbb{Z})} \tag{4.1.5}$$

for some $q_0 < q$ with $C(T, \lambda, q, q_0)$ independent of k and h. Such estimates will guarantee that the solution gains integrability with respect to the initial data and that the integrability property is uniform with respect to the mesh size. Once such requirements on the scheme are imposed we prove more general estimates of the type:

$$\|U\|_{l^q(k\mathbb{Z},\, l^r(h\mathbb{Z}))} \leq C(q, r, \lambda)\|U^0\|_{l^2(h\mathbb{Z})}, \tag{4.1.6}$$

uniformly on k and h, related by $k/h^2 = \lambda$. In Section 4.5 we consider approximations of the inhomogeneous Schrödinger equation and obtain similar estimates for that problem. The estimates obtained in Section 4.5 allow us to introduce a scheme for the Nonlinear Schrödinger Equation in Section 4.7 and to prove its convergence to the continuous one.

Also the local smoothing property will be analyzed. To do it we introduce the discrete fractional derivatives on the lattice $h\mathbb{Z}$. For that, we define for any $s < 1$, the fractional derivative $D^s_h U$ at the scale h as:

$$(D^s_h U)_j = \int_{-\pi/h}^{\pi/h} \left|\frac{e^{i\xi h} - 1}{h}\right|^s e^{ij\xi h}\mathcal{F}_h(U)(\xi)d\xi,\ j \in \mathbb{Z}^d.$$

where $\mathcal{F}_h(U)$ is the semidiscrete Fourier transform at the scale h of the sequence U (see Appendix A).

We will obtain necessary and sufficient conditions in order to guarantee that our scheme satisfies

$$k \sum_{|n|k\leq 1} \left[h \sum_{|j|h\leq 1} |(D^s_h U^n)_j|^2 \right] \leq C(s, \lambda) \left[h \sum_{j\in\mathbb{Z}} |U^0_j|^2 \right] \tag{4.1.7}$$

for some constant $s > 0$ and $C(s, \lambda)$, independent of h and k. In fact, once (4.1.7) is satisfied for some $s > 0$ the above sums can be taken in all finite intervals $|n|k \leq T$ and $|j|h \leq R$.

Such kind of estimates on the discrete solution give us sufficient conditions to prove the convergence of the scheme considered in Section 4.7 towards the solution of the nonlinear Schrödinger equation. Without such an estimate, despite the global well-posedness of the discrete problem in the spaces $l^\infty(k\mathbb{Z},\, l^2(h\mathbb{Z})) \cap l^q_{loc}(k\mathbb{Z},\, l^r(h\mathbb{Z}))$, one cannot pass to the limit in the nonlinear term. Of course in the case of linear problems the condition (4.1.7) is not necessary, the L^2-stability being sufficient to prove the convergence of the scheme.

Finally we concentrate on two schemes: backward Euler and Crank-Nicolson. The first one introduces dissipation and consequently has similar properties to the continuous one. The second one is conservative and has no local integrability property or local smoothing effect, uniform with respect to the mesh size h. We also prove that there is no two-grid algorithm allowing to recover the gain of integrability of the scheme.

4.2. Fully discrete schemes

In this section we give necessary and sufficient conditions in order to guarantee that the properties presented in the previous section are verified. Most of the dispersive properties of the continuous Schrödinger equation are studied by means of the Fourier transform. It is then natural to consider similar tools at the discrete level. We make use of the semidiscrete Fourier transform in the analysis of the properties of our schemes. To do that we will apply SDFT to equations (4.1.4). We obtain the relation between the solution at the time step n and the initial data. This is usually done in the study of stability of numerical schemes.

The properties of the semidiscrete Fourier transform (see Appendix A) give us, for all $n \geq 0$:

$$\begin{aligned}
\mathcal{F}_1\left(\sum_{\gamma\in F} a_{1,\gamma}(\lambda) U^{n+1}_{j+\gamma}\right)(\xi) &= \sum_{j\in\mathbb{Z}} e^{-ij\xi}\left(\sum_{\gamma\in F} a_{1,\gamma}(\lambda) U^{n+1}_{j+\gamma}\right) \\
&= \sum_{\gamma\in F} a_{1,\gamma}(\lambda)\left(\sum_{j\in\mathbb{Z}} e^{-ij\xi} U^{n+1}_{j+\gamma}\right) \\
&= \sum_{\gamma\in F} a_{1,\gamma}(\lambda) e^{i\gamma\xi}\left(\sum_{j\in\mathbb{Z}} e^{-i(j+\gamma)\xi} U^{n+1}_{j+\gamma}\right) \\
&= \left(\sum_{\gamma\in F} a_{1,\gamma}(\lambda) e^{i\gamma\xi}\right)\mathcal{F}_1(U^{n+1})(\xi) \\
&= P_{1,\lambda}(\xi)\mathcal{F}_1(U^{n+1})(\xi).
\end{aligned}$$

In view of this property, equations (4.1.4) can be written in the Fourier space as:

$$\begin{cases}
P_{1,\lambda}(\xi)\widehat{U}^{n+1}(\xi) = P_{2,\lambda}(\xi)\widehat{U}^{n}(\xi), \quad n \geq 0, \quad \xi \in [-\pi, \pi]\,, \\
\overline{P_{1,\lambda}(\xi)}\widehat{U}^{n-1}(\xi) = \overline{P_{2,\lambda}(\xi)}\widehat{U}^{n}(\xi), \quad n \leq 0, \quad \xi \in [-\pi, \pi]\,.
\end{cases}$$

In general

$$\widehat{U}^n(\xi) = \begin{cases} \left(\frac{P_{2,\lambda}(\xi)}{P_{1,\lambda}(\xi)}\right)^n \widehat{U}^0(\xi), & n \geq 0,\, \xi \in [-\pi, \pi]\,, \\ \left(\frac{\overline{P_{2,\lambda}(\xi)}}{\overline{P_{1,\lambda}(\xi)}}\right)^{|n|} \widehat{U}^0(\xi), & n \leq 0,\, \xi \in [-\pi, \pi]\,, \end{cases} \tag{4.2.1}$$

for some 2π-periodic functions $P_{1,\lambda}, P_{2,\lambda}$. Observe that in order to define U^{n+1} in terms of U^n (or the converse) a natural condition is to impose that both symbols $P_{1,\lambda}$ have no roots on $\xi \in [-\pi, \pi]$. Also we point out that LSE is time reversible, so it is natural to consider schemes which allows us to construct, for example, U^0 from U^1. Thus we also impose that $P_{1,\lambda}$ has no roots in $[-\pi, \pi]$. This always happens in practice as we can see in the two examples contained in the next section.

Our results will be expressed in terms of the symbol a_λ, defined as the quotient of the trigonometric polynomials $P_{1,\lambda}$ and $P_{2,\lambda}$:

$$a_\lambda(\xi) = \frac{P_{2,\lambda}(\xi)}{P_{1,\lambda}(\xi)},\, \xi \in [-\pi, \pi]\,.$$

From now on we write the symbol a_λ in the polar form

$$a_\lambda(\xi) = m_\lambda(\xi) e^{i\psi_\lambda(\xi)},\, \xi \in [-\pi, \pi],$$

where

$$m_\lambda(\xi) = \sqrt{\Re(a_\lambda(\xi))^2 + \Im(a_\lambda(\xi))^2},\, \psi_\lambda(\xi) = \arctan\left(\frac{\Im(a_\lambda(\xi))}{\Re(a_\lambda(\xi))}\right),\, \xi \in [-\pi, \pi],$$

$\Re$ and $\Im$ being the real, respectively the imaginary part.

A Taylor expansion shows that in order to be consistent, the scheme introduced above has to satisfy:

$$\begin{cases} \sum_{\gamma \in F} a_{1,\gamma} = \sum_{\gamma \in F} a_{2,\gamma},\; \sum_{\gamma \in F} \gamma a_{1,\gamma} = \sum_{\gamma \in F} \gamma a_{2,\gamma}, \\ \lambda \sum_{\gamma \in F} a_{1,\gamma} = \frac{i}{2} \sum_{\gamma \in F} \gamma^2 \left[a_{1,\gamma} - a_{2,\gamma}\right]. \end{cases}$$

In terms of $P_{1,\lambda}$ and $P_{2,\lambda}$, these conditions are equivalently with

$$\begin{cases} P_{1,\lambda}(0) = P_{2,\lambda}(0),\; P'_{1,\lambda}(0) = P'_{2,\lambda}(0), \\ \lambda P_{1,\lambda}(0) = \frac{i}{2}\left[P''_{1,\lambda}(0) - P''_{2,\lambda}(0)\right]. \end{cases}$$

This guarantees that

$$a_\lambda(\xi) \sim a_\lambda(0) + \xi a'_\lambda(0) + \frac{\xi^2}{2} a''_\lambda(0) = 1 - i\lambda\xi^2,\, \xi \sim 0, \tag{4.2.2}$$

where the values of a_λ and its first two derivatives in $\xi = 0$ are obtained by those of $P_{l,\lambda}, l = 1, 2$.

In order to guarantee the L^2-stability of the scheme we have to assume that

$$|a_\lambda(\xi)| \leq 1$$

for all $\xi \in [-\pi, \pi]$. It follows, since a_λ is analytic, that one of the following conditions is satisfied, namely :

$$|a_\lambda(\xi)| \equiv 1,\ \xi \in [-\pi, \pi] \tag{4.2.3}$$

or

$$|a_\lambda(\xi)| < 1 \tag{4.2.4}$$

for all but a finite number of points ξ_k, $k = \overline{1, N}$ in $[-\pi, \pi]$. The first case corresponds to a conservative scheme; the second one to a dissipative scheme.

The consistency condition (4.2.2), $a_\lambda(\xi) \sim 1 - i\lambda\xi^2$ as $\xi \sim 0$, excludes the case $\Re a_\lambda \equiv 0$. Then $\Re a_\lambda$ has a finite number of roots in $[-\pi, \pi]$ and ψ'_λ is defined except for a finite number of points. Using that ψ'_λ satisfies

$$\psi'_\lambda = \frac{\Im(a_\lambda)'\Re(a_\lambda) - \Re(a_\lambda)'\Im(a_\lambda)}{\Im(a_\lambda)^2 + \Re(a_\lambda)^2}$$

and that $\Im(a_\lambda)$ and $\Re(a_\lambda)$ are C^1 functions, we obtain that ψ'_λ is defined at all points in $[-\pi, \pi]$. In the above calculus of ψ'_λ we have used that the symbol $a_\lambda(\xi)$ has no roots in the whole interval $[-\pi, \pi]$.

Observe that (4.2.1) allows us to write the solution at any step $n \in \mathbb{Z}$ in terms of the initial datum U^0:

$$\widehat{U^n}(\xi) = m_\lambda^{|n|}(\xi) e^{in\psi_\lambda(\xi)} \widehat{U^0}(\xi),\ n \in \mathbb{Z},\ \xi \in [-\pi, \pi].$$

Concerning the gain of integrability of our scheme we will show that the following condition is necessary and sufficient:

$$m_\lambda(\xi_0) = 1 \quad \Rightarrow \quad |\psi''_\lambda(\xi_0)| > 0 \ \text{ or } \ m''_\lambda(\xi_0) \neq 0. \tag{4.2.5}$$

The analicity of m_λ guarantees that the set of points ξ_0 where $m_\lambda(\xi_0) = 1$ is the whole interval $[-\pi, \pi]$ or a finite set of points Λ. The consistency of the scheme guarantees that $m_\lambda = 1$ at least at one point, $\xi = 0$. This shows that the set Λ is nonempty.

The case when m_λ is identically one corresponds to a conservative scheme and the other one to a dissipative scheme. As we have seen in the semidiscrete case, the strict convexity of the symbol ψ_λ plays a key role.

Regarding the local gain of smoothness we prove that (4.1.7) holds if and only if the symbol a_λ satisfies

$$\xi_0 \neq 0,\ \psi'_\lambda(\xi_0) = 0 \quad \Rightarrow \quad m_\lambda(\xi_0) < 1. \tag{4.2.6}$$

The above condition ensures that the dissipative effect occurs at the points (different from zero) where the first derivative of the symbol vanishes. This will avoid the spurious effects introduced by the scheme and will recover the properties of the continuous model. In the continuous case the derivative of the symbol, $\psi'(\xi) = 2\xi$, has no roots except for the point $\xi = 0$.

The main results are given by the following theorems :

Theorem 4.2.1. *Let us assume that a_λ satisfies (4.2.5). Then for any $q \geq 2$ there is a positive constant $C(q, \lambda)$ such that*

$$\|U^n\|_{l^q(h\mathbb{Z})} \leq C(q, \lambda)(|n|k)^{\frac{1}{2}\left(\frac{1}{q}-\frac{1}{q'}\right)}\|U^0\|_{l^{q'}(h\mathbb{Z})} \tag{4.2.7}$$

holds for all $n \neq 0$, $h, k > 0$.

Remark 4.2.1. *This estimate is similar to the $L^1 - L^\infty$ decay of the continuous Schrödinger semigroup. Choosing a positive time T, $n \in \mathbb{N}$ and $k_n \to 0$ such that $nk_n \to T$ one can obtain in the limit exactly the estimates of the continuous case (3.1.6).*

Remark 4.2.2. *With the same notations as in the one-dimensional case the same result holds if the following holds :*

$$m_\lambda(\xi_0) = 1 \Rightarrow \mathrm{rank}(H_{\psi_\lambda}(\xi_0)) = d \text{ or } \xi H_{m_\lambda}(\xi_0)\xi^t < 0, \forall\, \xi \in \mathbb{R}^d,$$

where H_{m_λ} is the hessian matrix. However, we do not know if the condition is necessary.

Theorem 4.2.2. *Let $q > q_0 \geq 1$. Assume that a_λ does not satisfy (4.2.5). Then for any $T > 0$*

$$\lim_{\substack{h\to 0\\ nk\to T}} \sup_{U^0\in l^{q_0}(h\mathbb{Z})} \frac{\|U^n\|_{l^q(h\mathbb{Z})}}{\|U^0\|_{l^{q_0}(h\mathbb{Z})}} = \infty. \tag{4.2.8}$$

Remark 4.2.3. *A similar argument as in the proof of the above Theorem allows us to prove the lack of uniform Strichartz-like estimates: For any $T > 0$, $q > q_0 \geq 1$*

$$\lim_{\substack{h\to 0\\ nk\to T}} \sup_{U^0\in l^{q_0}(h\mathbb{Z})} \frac{k\sum_{nk\leq T}\|U^n\|_{l^q(h\mathbb{Z})}}{\|U^0\|_{l^{q_0}(h\mathbb{Z})}} = \infty.$$

The above quotient can be understood as the quotient between the $l^1(l^q)$-norm of the solution and the l^{q_0}-norm of the initial datum. For a complete proof of the above result one has to refine the proof of (4.2.8) as in Chapter 3, Section 3.2.1.

Theorem 4.2.3. *There is a positive s and a constant $C(s, \lambda)$ such that (4.1.7) holds for all $U^0 \in l^2(h\mathbb{Z})$ and for all $h > 0$ if and only if condition (4.2.6) is satisfied.*

In that case $s = 1/2$ and

$$\sup_{j\in\mathbb{Z}}\left[k\sum_{n\in\mathbb{Z}}|D_h^{1/2}U^n|^2\right] \leq C(\lambda)h\sum_{j\in\mathbb{Z}}|U_j^0|^2 \tag{4.2.9}$$

holds for all $U^0 \in l^2(h\mathbb{Z})$ and all h and k satisfying (4.1.1).

Remark 4.2.4. *In more than one dimension, the same result holds if one impose that the following condition on the symbol a_λ:*

$$\xi_0 \neq 0,\ \nabla\psi_\lambda(\xi_0) = 0 \quad \Rightarrow \quad m_\lambda(\xi_0) < 1.$$

Once these results are obtained, following Tao [74], and writing $U^n = S(n)U^0$, we obtain Strichartz-like estimates for the operators $\{S(n)\}_{n\in\mathbb{Z}}$. These results are contained in Section 4.5.

4.3. Two examples

We present two particular fully discrete schemes for LSE. One is the backward Euler scheme:

$$\begin{cases} i\dfrac{U_j^{n+1}-U_j^n}{k}+\dfrac{U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}}{h^2}=0,\ n\geq 0,\ j\in\mathbb{Z},\\[2ex] i\dfrac{U_j^{n}-U_j^{n-1}}{k}+\dfrac{U_{j+1}^{n-1}-2U_j^{n-1}+U_{j-1}^{n-1}}{h^2}=0,\ n\leq 0,\ j\in\mathbb{Z},\end{cases} \tag{4.3.1}$$

and the second one is the Crank-Nicolson scheme:

$$\begin{cases} i\dfrac{U_j^{n+1}-U_j^n}{k}+\dfrac{U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}}{2h^2}+\dfrac{U_{j+1}^{n}-2U_j^{n}+U_{j-1}^{n}}{2h^2}=0,\ n\geq 0,\ j\in\mathbb{Z},\\[2ex] i\dfrac{U_j^{n}-U_j^{n-1}}{k}+\dfrac{U_{j+1}^{n-1}-2U_j^{n-1}+U_{j-1}^{n-1}}{2h^2}+\dfrac{U_{j+1}^{n}-2U_j^{n}+U_{j-1}^{n}}{2h^2}=0,\ n\leq 0,\ j\in\mathbb{Z}.\end{cases} \tag{4.3.2}$$

The Fourier analysis of the backward Euler scheme gives us

$$\begin{cases} i\widehat{U}^{n+1}-i\widehat{U}^n+\lambda\widehat{U}^{n+1}(e^{i\xi}+e^{-i\xi}-2)=0, & n\geq 0,\\[1ex] i\widehat{U}^{n}-i\widehat{U}^{n-1}+\lambda\widehat{U}^{n-1}(e^{i\xi}+e^{-i\xi}-2)=0, & n\leq 0.\end{cases}$$

With the above notations

$$\begin{cases} \widehat{U}^{n+1}(\xi)(i-4\lambda\sin^2\dfrac{\xi}{2})=i\widehat{U}^n(\xi), & n\geq 0,\\[2ex] \widehat{U}^{n-1}(\xi)(-i-4\lambda\sin^2\dfrac{\xi}{2})=-i\widehat{U}^n(\xi), & n\leq 0,\end{cases} \tag{4.3.3}$$

and

$$a_\lambda(\xi)=\frac{\exp(-i\arctan(4\lambda\sin^2\frac{\xi}{2}))}{\Big(1+16\lambda^2\sin^4\frac{\xi}{2}\Big)^{1/2}}. \tag{4.3.4}$$

In general

$$\widehat{U}^n(\xi)=\frac{\exp(-in\arctan(4\lambda\sin^2\frac{\xi}{2}))}{\Big(1+16\lambda^2\sin^4\frac{\xi}{2}\Big)^{|n|/2}}\widehat{U}^0(\xi),\ n\in\mathbb{Z},\ \xi\in[-\pi,\pi].$$

Thus the modulus of the symbol $a_\lambda(\xi)$ is smaller than one, except at the origin. Explicit computations show that $\psi_\lambda(\xi)=-\arctan(4\lambda\sin^2\frac{\xi}{2})$ satisfies (4.2.5) and (4.2.6). Let us choose $\lambda=1$. In Figure 4.1 we can see that at some point ξ_0 the symbol ψ_1 changes its convexity. Also the first derivative vanishes at $\xi=\pm\pi$ (see Figure 4.3). However, these pathologies are compensated by the dissipative character of the multiplier m_1 (see Figure 4.2) which is strictly less than one outside the origin.

We recall that a scheme is dissipative of order s if there exists $C>0$ such that, for $\xi\in[-\pi,\pi]$,

$$|a_\lambda(\xi)|\leq 1-C|\xi|^s. \tag{4.3.5}$$

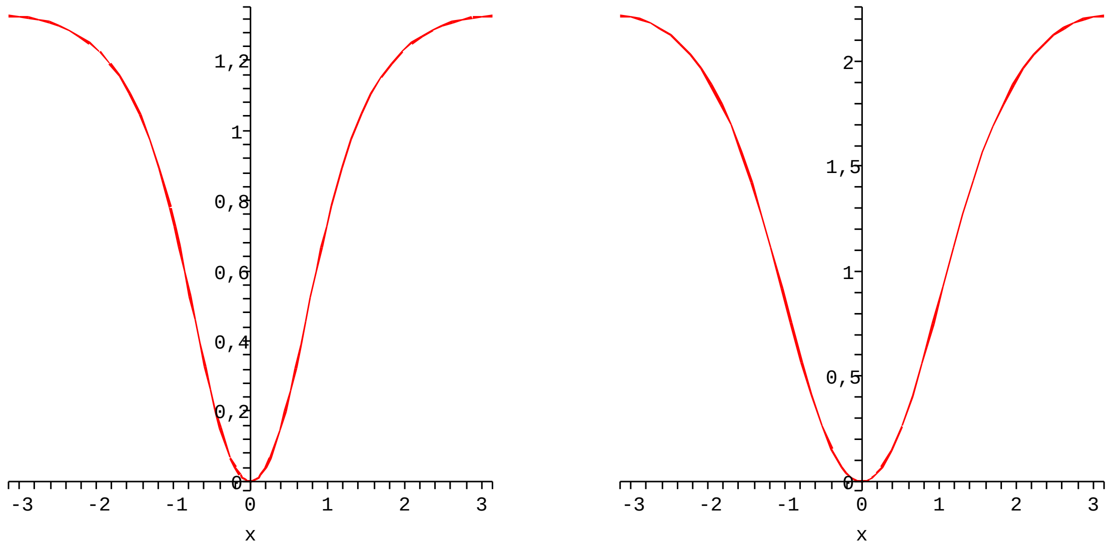

Figure 4.1: The symbol ψ_1 for backward Euler and CN Scheme

In the case of backward Euler scheme the symbol a_λ satisfies for $\xi \sim 0$:

$$a_\lambda(\xi) = \frac{1}{1 + 4i\lambda \sin^2 \frac{\xi}{2}} \sim 1 - i\lambda\xi^2 - \lambda^2\xi^4 + O(|\xi|^6)$$

and

$$|a_\lambda(\xi)| \sim 1 - \lambda^2\xi^4 + O(|\xi|^6). \tag{4.3.6}$$

Choosing possibly a smaller constant C in (4.3.5) we obtain that the scheme is dissipative of order 4.

In the case of the Crank-Nicolson scheme, the Fourier analysis shows that

$$\begin{cases} i\widehat{U}^{n+1}(\xi) - i\widehat{U}^n(\xi) + \widehat{U}^{n+1}(\xi)\frac{\lambda}{2}(e^{i\xi} + e^{-i\xi} - 2) + \widehat{U}^n(\xi)\frac{\lambda}{2}(e^{i\xi} + e^{-i\xi} - 2) = 0, & n \geq 0, \\ i\widehat{U}^n(\xi) - i\widehat{U}^{n-1}(\xi) + \widehat{U}^{n-1}(\xi)\frac{\lambda}{2}(e^{i\xi} + e^{-i\xi} - 2) + \widehat{U}^n(\xi)\frac{\lambda}{2}(e^{i\xi} + e^{-i\xi} - 2) = 0, & n \leq 0, \end{cases}$$

and

$$\begin{cases} \widehat{U}^{n+1}\left(i - 2\lambda \sin^2 \frac{\xi}{2}\right) = \widehat{U}^n\left(i + 2\lambda \sin^2 \frac{\xi}{2}\right), & n \geq 0, \\ \widehat{U}^{n-1}\left(-i - 2\lambda \sin^2 \frac{\xi}{2}\right) = \widehat{U}^n\left(-i + 2\lambda \sin^2 \frac{\xi}{2}\right), & n \leq 0, \end{cases} \tag{4.3.7}$$

The symbol a_λ is given by

$$a_\lambda(\xi) = \frac{1 + 2i\lambda \sin^2 \frac{\xi}{2}}{1 - 2i\lambda \sin^2 \frac{\xi}{2}} = \exp\left(2i \arctan\left(2\lambda \sin^2 \frac{\xi}{2}\right)\right) \overset{not}{=} \exp(i\psi_\lambda(\xi)). \tag{4.3.8}$$

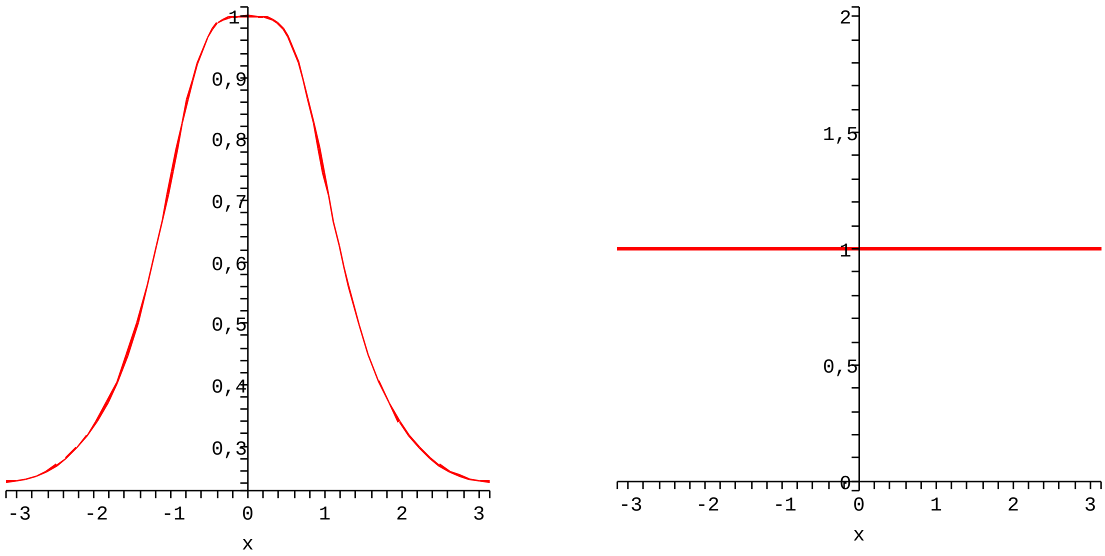

Figure 4.2: The symbol m_1 for backward Euler and CN Scheme

In general

$$\widehat{U}^n = \exp\left(2in\arctan\left(2\lambda\sin^2\frac{\xi}{2}\right)\right)\widehat{U}^0,\ n\in\mathbb{Z},\ \xi\in[-\pi,\pi].$$

This scheme is a conservative one and the dissipative effect does not occur.

Explicit computations show that the derivative of the function ψ_λ is given by

$$\psi'_\lambda(\xi) = \frac{8\lambda\sin\xi}{1+4\lambda^2\sin^4\frac{\xi}{2}}$$

and the scheme fails to have property (4.2.6) at the point $\xi = \pi$. Also explicit computations show that $\psi''_\lambda(0)\psi''_\lambda(\frac{\pi}{2}) < 0$. This suffices to show hat the scheme does not satisfy (4.2.5).

These pathologies are similar to the ones of the semidiscrete conservative scheme introduced in Chapter 3, Section 3.2. As we have seen in that Chapter additional techniques have to be introduced to cancel these spurious effects: filtering, numerical viscosity or a two-grid preconditioner.

We point out that any filtration of initial data which excludes the end points $\xi = \pm\pi/h$ ($\pm\pi$ if one lock at the mesh size $h = 1$) will guarantee the local smoothing property (4.1.7).

Regarding the $l^1 - l^\infty$ norm decay, Figure 4.4 shows the existence of two points $\pm\xi_0 \in [-\pi,\pi]$ where the second derivative of ψ_1 vanishes. Any filtration of initial data which excludes the two points will recover the right decay property of solutions and then the Strichartz-like estimates for $S_1(n)$.

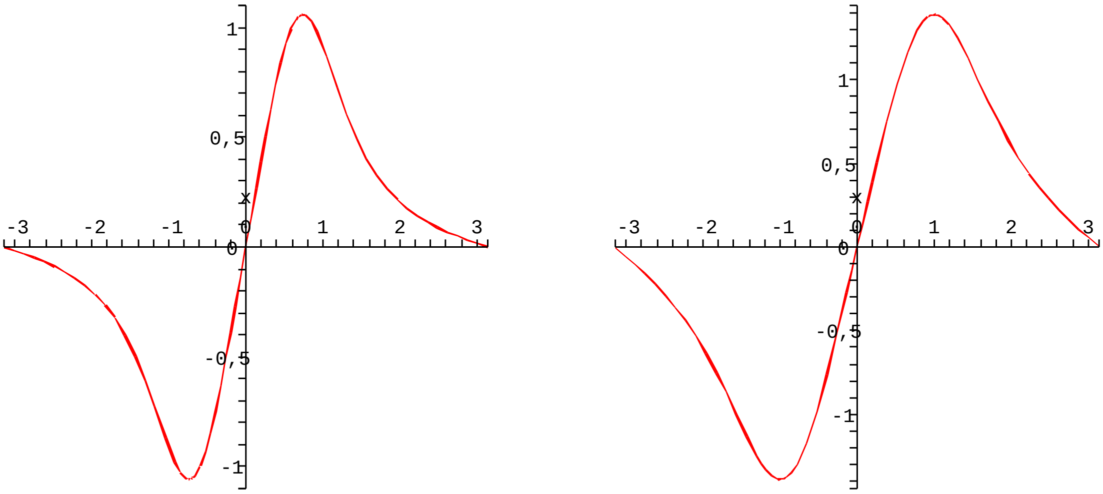

Figure 4.3: The first derivative of the symbol ψ_1 for backward Euler and CN Scheme

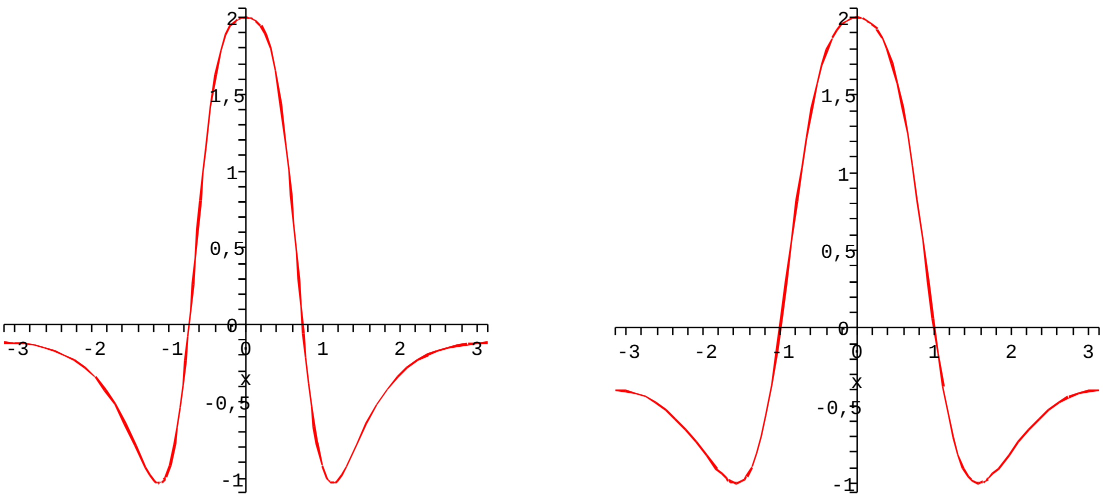

Figure 4.4: The second derivative of the symbol ψ_1 for backward Euler and CN Scheme

4.4. Main results

This section is devoted to the proofs of the Theorems given in the first section. The methods we use are similar to those in Chapter 3. In all the proofs we rescale all the estimates, reducing them to $h = 1$.

4.4.1. Proof of Theorem 4.2.1

We will consider the cases $q = 2$ and $q = \infty$. The case $q = 2$ easily follows by the stability of the scheme. Let us assume that the following inequality is proved

$$\|U^n\|_{l^\infty(h\mathbb{Z})} \lesssim (|n|k)^{-\frac{1}{2}} \|U^0\|_{l^1(h\mathbb{Z})},\ \forall\, n \neq 0. \tag{4.4.1}$$

Then by interpolation between $q = 2$ and $q = \infty$ we obtain the desired result (4.2.7).

Let us consider the case $q = \infty$. We rescale inequality (4.4.1), reducing the proof to the case $h = 1$. This is just a re-normalization of the $l^p(h\mathbb{Z})$ norms:

$$\frac{|nk|^{1/2}\|U^n\|_{l^\infty(h\mathbb{Z})}}{\|U^0\|_{l^1(h\mathbb{Z})}} = \frac{|nk|^{1/2}\|U^n\|_{l^\infty(\mathbb{Z})}}{h\|U^0\|_{l^1(\mathbb{Z})}} = \lambda^{1/2}|n|^{1/2}\frac{\|U^n\|_{l^\infty(\mathbb{Z})}}{\|U^0\|_{l^1(\mathbb{Z})}}. \tag{4.4.2}$$

Now we prove that the right hand side remains bounded as long as n varies in $\mathbb{Z}\backslash\{0\}$.

We follow the ideas of the continuous case. We write the solution U^n as the discrete convolution of some kernel K^n with the initial datum U^0 and we estimate the $l^\infty(\mathbb{Z})$-norm of the kernel K^n. Finally, Young's inequality gives us (4.4.1) and finishes the proof.

Using the representation of $\widehat{U}^n$ in Fourier space:

$$\widehat{U}^n(\xi) = m_\lambda^{|n|}(\xi)e^{in\psi_\lambda(\xi)}\widehat{U}^0(\xi),\ \ \xi \in [-\pi, \pi],$$

we obtain that

$$U^n = K_\lambda^n * U^0,$$

where the kernel K_λ^n is given by

$$K_{\lambda,j}^n = \frac{1}{2\pi}\int_{-\pi}^{\pi} m_\lambda^{|n|}(\xi)e^{in\psi_\lambda(\xi)}e^{ij\xi}d\xi,\ \ n \in \mathbb{Z},\ \ j \in \mathbb{Z}. \tag{4.4.3}$$

Young's inequality shows that

$$\|U^n\|_{l^\infty(\mathbb{Z})} \le \|K_\lambda^n\|_{l^\infty(\mathbb{Z})}\|U^0\|_{l^1(\mathbb{Z})},$$

so, it is sufficient to show that the kernel K_λ^n satisfies

$$\sup_{j\in\mathbb{Z}} |K_{\lambda,j}^n| \lesssim \frac{1}{|n|^{1/2}},\ \ n \in \mathbb{Z},\ n \neq 0. \tag{4.4.4}$$

In fact once we prove this estimate the following one also holds:

$$\sup_{j\in\mathbb{Z}} |K_{\lambda,j}^n| \lesssim \frac{1}{1+|n|^{1/2}},\ n \in \mathbb{Z}, \tag{4.4.5}$$

the kernel being uniformly bounded by 2π. The last estimate will be useful to establish more general estimates for U^n in Section 4.5.

Now, we write the symbol a_λ in polar form as $a_\lambda(\xi) = m_\lambda(\xi)e^{i\psi_\lambda(\xi)}$. As we said before $m_\lambda \equiv 1$ or $m_\lambda < 1$ except a finite set of points Λ. We will consider two cases, the conservative and the dissipative one.

Case 1. The conservative case: $m_\lambda \equiv 1$.
In this case the kernel K^n_λ is given by

$$K^n_{\lambda,j} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{in\psi_\lambda(\xi)}e^{ij\xi}d\xi.$$

Using that $|\psi''_\lambda(\xi)|$ is bounded below by a positive constant, we obtain by Van der Corput's Lemma 3.2.1 (cf. [118], Proposition 2, Ch. VIII. §1, p. 332) that

$$|K^n_{\lambda,j}| \lesssim \frac{1}{|n|^{1/2}}.$$

Case 2. The dissipative case: $m_\lambda(\xi) < 1$ except on the set Λ.
In this case we will not look at the convexity of ψ_λ at the points where m_λ has modulus smaller than one. The dissipation introduced by m_λ will be sufficient to guarantee (4.4.4).

Let us suppose the existence of a positive constant δ such that $|m_\lambda(\xi)| \le 1-\delta$ for all ξ in some interval $[a,b]$. Clearly, the component of K^n_λ coresponding to the interval $[a,b]$ satisfies the rough estimates

$$\left|\frac{1}{2\pi}\int_a^b m_\lambda^{|n|}(\xi)e^{in\psi_\lambda(\xi)}e^{ij\xi}d\xi\right| \le (1-\delta)^{|n|} \lesssim \frac{c(\delta)}{|n|^{1/2}}$$

for all integers $n \neq 0$.

Using the fact that the set Λ is finite then its points are isolated and the proof is reduced to the case when Λ has a single point, namely, ξ_0, where $m_\lambda(\xi_0) = 1$.

The condition $\psi''_\lambda(\xi_0) \neq 0$ is similar to the one of conservative case. However, the fact that m_λ is not identically one in a neighborhood of some point ξ_0 is important. The fact that $m''_\lambda(\xi_0) \neq 0$ guarantees that its contribution to the kernel K^n_λ in a neighborhood $(\xi_0-\epsilon, \xi_0+\epsilon)$ has the order of magnitude $|n|^{-1/2}$.

Case 2a): $\psi''(\xi_0) > 0$.
Let us choose $\delta > 0$. The continuity of m_λ and ψ_λ shows the existence of $\epsilon > 0$ such that

$$m_\lambda(\xi) \le (1-\delta),\ \xi \in [-\pi,\pi]\backslash(\xi_0-\epsilon,\xi_0+\epsilon)$$

and

$$|\psi_\lambda(\xi)| \ge \frac{c}{2} > 0,\ \xi \in (\xi_0-\epsilon,\xi_0+\epsilon),$$

for some positive constant c.

With ϵ as above we split the kernel K^n in two parts :

$$K^n_{\lambda,j} = \frac{1}{2\pi}\int_{\xi_0-\epsilon}^{\xi_0+\epsilon} e^{in\psi_\lambda(\xi)}m_\lambda^{|n|}(\xi)e^{ij\xi}d\xi + \int_{[-\pi,\pi]\backslash(\xi_0-\epsilon,\xi_0+\epsilon)} e^{in\psi_\lambda(\xi)}m_\lambda^{|n|}(\xi)e^{ij\xi}d\xi$$

and estimate each of the two terms. For the first one we will use the strict convexity of the function ψ_λ on $(\xi_0-\epsilon,\xi_0+\epsilon)$. For the second term we make use of the dissipative effect introduced by m_λ on $[-\pi,\pi]\backslash(\xi_0-\epsilon,\xi_0+\epsilon)$.

The kernel K_λ^n satisfies

$$\begin{aligned}|K^n_{\lambda,j}| &\lesssim \left|\int_{\xi_0-\epsilon}^{\xi_0+\epsilon} e^{in\psi_\lambda(\xi)}m_\lambda^{|n|}(\xi)e^{ij\xi}d\xi\right| + \int_{[-\pi,\pi]\backslash(\xi_0-\epsilon,\xi_0+\epsilon)} m_\lambda^{|n|}(\xi)d\xi\\ &\lesssim \left|\int_{\xi_0-\epsilon}^{\xi_0+\epsilon} e^{in\psi_\lambda(\xi)}m_\lambda^{|n|}(\xi)e^{ij\xi}d\xi\right| + (1-\delta)^{|n|}\\ &\lesssim \left|\int_{\xi_0-\epsilon}^{\xi_0+\epsilon} e^{in\psi_\lambda(\xi)}m_\lambda^{|n|}(\xi)e^{ij\xi}d\xi\right| + \frac{1}{|n|^{1/2}}.\end{aligned}$$

Therefore it is sufficient to prove that the first term is dominated by $|n|^{-1/2}$. Applying Van der Corput's Lemma 3.2.1 we get

$$\left|\int_{\xi_0-\epsilon}^{\xi_0+\epsilon} e^{in\psi_\lambda(\xi)}m_\lambda^{|n|}(\xi)e^{ij\xi}d\xi\right| \leq \frac{1}{\left(|n|\inf\limits_{\xi\in(\xi_0-\epsilon,\xi_0+\epsilon)}|\psi''_\lambda(\xi)|\right)^{1/2}}\left[\|m_\lambda^{|n|}\|_{L^\infty} + \int_{\xi_0-\epsilon}^{\xi_0+\epsilon}|(m_\lambda^{|n|})'|d\xi\right].$$

The function m_λ is analytic and therefore the derivative of the function $m_\lambda^{|n|}$ is given by:

$$(m_\lambda^{|n|})'(\xi) = |n|m_\lambda^{|n|-1}(\xi)m'_\lambda(\xi),$$

and changes sign a finite number of times. This implies that

$$\int_{\xi_0-\epsilon}^{\xi_0+\epsilon}|(m_\lambda^{|n|})'|d\xi \leq \|m_\lambda^{|n|}\|_{L^\infty(\xi_0-\epsilon,\xi_0+\epsilon)}$$

and

$$|K^n_{\lambda,j}| \lesssim \frac{1}{|n|^{1/2}}\|m_\lambda^{|n|}\|_{L^\infty} \lesssim \frac{1}{|n|^{1/2}}.$$

Case 2b): $m''_\lambda(\xi_0)\neq 0$.
In this case we do not use any assumption on ψ_λ. The conditions on m_λ are sufficient to guarantee the right decay of the kernel K_λ^n. We remark that m_λ has a maximum point at ξ_0 : $m_\lambda(\xi)\leq 1 = m_\lambda(\xi_0)$. Then

$$m'_\lambda(\xi_0)=0,\ m''_\lambda(\xi_0)<0$$

and

$$m_\lambda(\xi) = 1 + (\xi-\xi_0)^2 m''_\lambda(\xi_0) + O(|\xi-\xi_0|^3),\ \xi\sim\xi_0.$$

Let us choose $0<\delta<1$. Then there exists $\epsilon>0$ such that

$$m_\lambda(\xi)\leq 1-\delta \quad \text{for all} \quad \xi\in[-\pi,\pi]\backslash(\xi_0-\epsilon,\xi_0+\epsilon)$$

and

$$|m_\lambda(\xi)| \leq 1 + \frac{(\xi-\xi_0)^2 m''(\xi_0)}{2} \quad \text{for all} \quad \xi\in(\xi_0-\epsilon,\xi_0+\epsilon).$$

Then the kernel K^n_λ satisfies

$$\begin{aligned}
|K^n_{\lambda,j}| &\leq \int_{-\pi}^{\pi} m_\lambda^{|n|}(\xi)d\xi = \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} m_\lambda^{|n|}(\xi)d\xi + \int_{[-\pi,\pi]\setminus(\xi_0-\epsilon,\xi_0+\epsilon)} m_\lambda^{|n|}(\xi)d\xi \\
&\leq \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} \left(1+\frac{(\xi-\xi_0)^2 m''_\lambda(\xi_0)}{2}\right)^{|n|} d\xi + 2\pi(1-\delta)^{|n|} \\
&\lesssim \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} \exp\left(|n|\frac{(\xi-\xi_0)^2 m''_\lambda(\xi_0)}{2}\right) d\xi + \frac{1}{|n|^{1/2}} \\
&\lesssim \frac{1}{(|nm''_\lambda(\xi_0)|)^{1/2}} + \frac{1}{|n|^{1/2}} \lesssim \frac{1}{|n|^{1/2}}.
\end{aligned}$$

This ends the proof.

4.4.2. Proof of Theorem 4.2.2

First, we rescale the quotient $\|U^n\|_{l^q(h\mathbb{Z})}/\|U^0\|_{l^{q_0}(h\mathbb{Z})}$:

$$\frac{\|U^n\|_{l^q(h\mathbb{Z})}}{\|U^0\|_{l^{q_0}(h\mathbb{Z})}} = h^{\frac{1}{q}-\frac{1}{q_0}} \frac{\|U^n\|_{l^q(\mathbb{Z})}}{\|U^0\|_{l^{q_0}(\mathbb{Z})}} \sim k^{\frac{1}{2}\left(\frac{1}{q}-\frac{1}{q_0}\right)} \frac{\|U^n\|_{l^q(\mathbb{Z})}}{\|U^0\|_{l^{q_0}(\mathbb{Z})}}.$$

Let us consider $T>0$, the case when $T<0$ being similar. In order to prove (4.2.8) we choose $nk \to T$. Then $n \sim k^{-1}$ and the proof is reduced to the following

$$\lim_{\substack{h\to 0\\ nk\to T}} n^{\frac{1}{2}\left(\frac{1}{q_0}-\frac{1}{q}\right)} \sup_{U^0\in l^{q_0}(h\mathbb{Z})} \frac{\|U^n\|_{l^q(\mathbb{Z})}}{\|U^0\|_{l^{q_0}(\mathbb{Z})}} = \infty. \tag{4.4.6}$$

Let us assume that (4.2.5) is not satisfied. Then there is a point $\xi_0 \in [-\pi,\pi]$ such that $m_\lambda(\xi_0)=1$, $\psi''_\lambda(\xi_0)=0$ and $m''_\lambda(\xi_0)=0$. Using that m_λ has a maximum at $\xi=\xi_0$ we get $m'_\lambda(\xi_0)=0$ and

$$m_\lambda(\xi) = 1 + O((\xi-\xi_0)^3), \xi \sim \xi_0.$$

To prove (4.4.6) we choose initial data U^0 with their SDFT, $\widehat{U^0}$, concentrated at the point ξ_0. We show that for n large enough the following holds

$$\sup_{U^0\in l^{q_0}(\mathbb{Z})} \frac{\|U^n\|_{l^q(\mathbb{Z})}}{\|U^0\|_{l^{q_0}(\mathbb{Z})}} \gtrsim n^{-\frac{1}{3}\left(\frac{1}{q_0}-\frac{1}{q}\right)}. \tag{4.4.7}$$

This obviously implies (4.4.6).

We use the same techniques as in the semi-discrete case: extend the operators to continuous ones, apply the norm equivalence and prove blow-up for the continuous operators. We introduce the operators S^n_λ defined as:

$$(S^n_\lambda\varphi)(x) = \int_{-\pi}^{\pi} m^n_\lambda(\xi) e^{in\psi_\lambda(\xi)} e^{ix\xi} \widehat{\varphi}(\xi) d\xi.$$

Using the relation between the norms of the discrete functions and their band-limited interpolator (see Appendix A) we obtain

$$\sup_{U^0 \in l^{q_0}(\mathbb{Z})} \frac{\|U^n\|_{l^q(\mathbb{Z})}}{\|U^0\|_{l^{q_0}(\mathbb{Z})}} \geq \sup_{\operatorname{supp} \widehat{\varphi} \subset [-\pi,\pi]} \frac{\|S^n_\lambda \varphi\|_{L^q(\mathbb{R})}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}}. \tag{4.4.8}$$

We claim that for n large enough

$$\sup_{\operatorname{supp} \widehat{\varphi} \subset [-\pi,\pi]} \frac{\|S^n_\lambda \varphi\|_{L^q(\mathbb{R})}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}} \gtrsim n^{-\frac{1}{3}\left(\frac{1}{q_0}-\frac{1}{q}\right)}. \tag{4.4.9}$$

Then, in view of (4.4.8) we obtain (4.4.7), and then (4.4.6) which finishes the proof.

In the following we prove (4.4.9). To do that, we will show that for n sufficiently large there exists φ with $\operatorname{supp} \widehat{\varphi} \in [-\pi, \pi]$ such that

$$\frac{\|S^n_\lambda \varphi\|_{L^q(\mathbb{R})}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}} \gtrsim n^{-\frac{1}{3}\left(\frac{1}{q_0}-\frac{1}{q}\right)}. \tag{4.4.10}$$

We use similar techniques as in the semidiscrete case.

The Mean Value Theorem, applied to the function $\Psi(\xi) = n\psi_\lambda(\xi) + x\xi$ on the interval $[a, b] \subset [-\pi, \pi]$, implies that

$$\left|\int_a^b e^{i\Psi(\xi)} m^n_\lambda(\xi)\widehat{\varphi}(\xi)d\xi\right| \geq \left(1 - |b-a| \sup_{\xi\in[a,b]} |\Psi'(\xi)|\right) \int_a^b m^n_\lambda(\xi)\widehat{\varphi}(\xi)d\xi$$

provided that $\widehat{\varphi}$ is nonnegative on $[a, b]$.

We will choose suitable a and b to guarantee that the term $(1 - |b-a| \sup_{\xi\in[a,b]} |\Psi'(\xi)|)$ is grater than 1/2. This allows us to obtain a lower bound for $S^n_\lambda \varphi$ in terms of $\int_a^b m^n_\lambda(\xi)\widehat{\varphi}(\xi)d\xi$.

Observe that

$$\begin{aligned} \Psi'(\xi) &\sim x + n\psi'_\lambda(\xi_0) + n(\xi-\xi_0)\psi''_\lambda(\xi_0) + nO((\xi-\xi_0)^2) \\ &\sim x + n\psi'_\lambda(\xi_0) + nO((\xi-\xi_0)^2), \quad \xi \sim \xi_0. \end{aligned}$$

Let $\epsilon \sim n^{-1/2}$ be a small positive number. Let us choose $a = \xi_0 - \epsilon$, $b = \xi_0 + \epsilon$ and φ_ϵ supported in (a, b). Then for any $\xi \in (a, b)$ the function $\Psi'(\xi)$ satisfies

$$|\Psi'(\xi)| \leq C\epsilon^{-1}$$

as long as

$$|x + n\psi'_\lambda(\xi_0)| \leq c_1\epsilon^{-1} \text{ and } n = c_2\epsilon^{-3}. \tag{4.4.11}$$

Taking into account that $|b - a| = 2\epsilon$ we obtain, choosing eventually c_1 and c_2 smaller, (see Chapter 3, Theorem 3.2.1 for a precise choice of the above constants) that

$$|b-a| \sup_{\xi\in[a,b]} |\Psi'(\xi)| \leq \frac{1}{2}.$$

This implies that

$$|(S^n_\lambda \varphi_\epsilon)(x)| \gtrsim \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} m^n_\lambda(\xi)\widehat{\varphi}_\epsilon(\xi)d\xi \tag{4.4.12}$$

for all x and n satisfying (4.4.11). Integrating (4.4.12) on the set $\{x : |x + n\psi'_\lambda(\xi_0)| \le c_1\epsilon^{-1}\}$ we obtain, for all $n = c_2\epsilon^{-3}$:

$$\|S_\lambda^n \varphi_\epsilon\|_{L^q(\mathbb{R})} \gtrsim \epsilon^{-\frac{1}{q}} \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} m_\lambda^n(\xi)\widehat{\varphi}_\epsilon(\xi)d\xi. \tag{4.4.13}$$

Let us be more precise about φ_ϵ. Choose a function φ such that its Fourier transform $\widehat{\varphi}$ has compact support in $(-1,1)$ and $\widehat{\varphi} > 1$ on $(-1/2, 1/2)$. Take φ_ϵ as:

$$\widehat{\varphi}_\epsilon(\xi) = \epsilon^{-1}\widehat{\varphi}\left(\epsilon^{-1}(\xi - \xi_0)\right).$$

Classical properties of the Fourier transform guarantee that the L^{q_0}-norm of φ_ϵ behaves as ϵ^{-1/q_0} and

$$\int_{\xi_0-\epsilon}^{\xi_0+\epsilon} m_\lambda^n(\xi)\widehat{\varphi}_\epsilon(\xi)d\xi \gtrsim \epsilon^{-1}\int_{\xi_0-\epsilon/2}^{\xi_0+\epsilon/2} m_\lambda^n(\xi)d\xi. \tag{4.4.14}$$

Thus (4.4.13) and (4.4.14) imply that

$$\frac{\|S_\lambda^n\varphi_\epsilon\|_{L^q(\mathbb{R})}}{\|\varphi_\epsilon\|_{L^{q_0}(\mathbb{R})}} \gtrsim \epsilon^{\frac{1}{q_0}-\frac{1}{q}}\epsilon^{-1}\int_{\xi_0-\epsilon/2}^{\xi_0+\epsilon/2} m_\lambda^n(\xi)d\xi \gtrsim n^{-\frac{1}{3}\left(\frac{1}{q_0}-\frac{1}{q}\right)}\epsilon^{-1}\int_{\xi_0-\epsilon/2}^{\xi_0+\epsilon/2} m_\lambda^n(\xi)d\xi. \tag{4.4.15}$$

Therefore, to prove (4.4.9) it is sufficient to show that the last term is bounded below by a positive constant, independent of ϵ, and then of n. This is the point where the second derivative of m_λ becomes important.

Choosing possibly ϵ smaller we obtain the existence of a negative constant c such that

$$m_\lambda(\xi) \sim 1 + c(\xi-\xi_0)^3, \text{ for all } \xi \in \left(\xi_0, \xi_0 - \frac{\epsilon}{2}\right)$$

and

$$\begin{aligned}
\epsilon^{-1}\int_{\xi_0}^{\xi_0+\frac{\epsilon}{2}} m_\lambda^n(\xi)d\xi &\gtrsim \epsilon^{-1}\int_{\xi_0}^{\xi_0+\frac{\epsilon}{2}} \left(1+c(\xi-\xi_0)^3\right)^n d\xi \\
&\gtrsim \epsilon^{-1}\int_{\xi_0}^{\xi_0+\frac{\epsilon}{2}} e^{cn(\xi-\xi_0)^3}d\xi \gtrsim \epsilon^{-1}\int_0^{\frac{\epsilon}{2}} e^{cn\xi^3}d\xi \\
&\gtrsim \frac{1}{\epsilon n^{1/3}}\int_0^{\frac{\epsilon n^{1/3}}{2}} e^{c\xi^3}d\xi.
\end{aligned}$$

Using that $n \sim \epsilon^{-3}$, the last term is bounded below by a positive constant C_1.

Then inequality (4.4.15) gives us

$$\sup_{\operatorname{supp}\widehat{\varphi}\subset[-\pi,\pi]} \frac{\|S_\lambda^n\varphi\|_{L^q(\mathbb{R})}}{\|\varphi\|_{L^{q_0}(\mathbb{R})}} \gtrsim n^{-\frac{1}{3}\left(\frac{1}{q_0}-\frac{1}{q}\right)},$$

which finishes the proof.

4.4.3. Proof of Theorem 4.2.3

Step 1: The conditions are necessary.

First we rescale all the terms of (4.1.7) as in the proof of Theorem 4.2.1. We apply the same ideas as in the previous proof choosing initial data U^0 with SDFT concentrated at ξ_0, one of the points where (4.2.6) fails.

The definition of D_h^s gives us for all $h > 0$ and $j \in \mathbb{Z}$:

$$\begin{aligned}(D_h^s U^n)_j &= \int_{-\pi/h}^{\pi/h} \left|\frac{e^{i\xi h}-1}{h}\right|^s e^{ij\xi h}\mathcal{F}_h(U^n)(\xi)d\xi \\ &= h\int_{-\pi/h}^{\pi/h} \left|\frac{e^{i\xi h}-1}{h}\right|^s e^{ij\xi h}\mathcal{F}_1(U^n)(h\xi)d\xi \\ &= \frac{1}{h^{2s}}\int_{-\pi}^{\pi} |e^{i\xi}-1|^s e^{ij\xi}\mathcal{F}_1(U^n)(\xi)d\xi = \frac{1}{h^{2s}}(D_1^s U^n)_j. \end{aligned} \tag{4.4.16}$$

Using that $k \sim h^2$ we get

$$\frac{k\sum\limits_{|n|k\leq 1} h\sum\limits_{|j|h\leq 1} |(D_h^s U^n)_j|^2}{h\sum\limits_{j\in\mathbb{Z}} |U_j^0|^2} \simeq h^{2-2s}\frac{\sum\limits_{|n|h^2\leq 1}\sum\limits_{|j|h\leq 1} |(D_1^s U^n)_j|^2}{\sum\limits_{j\in\mathbb{Z}} |U_j^0|^2}.$$

Thus, it is sufficient to prove that

$$\limsup_{h\to 0, U^0\in l^2(\mathbb{Z})} h^{2-2s}\frac{\sum\limits_{|n|h^2\leq 1}\sum\limits_{|j|h\leq 1} |(D_1^s U^n)_j|^2}{\sum_{j\in\mathbb{Z}} |U_j^0|^2} = \infty. \tag{4.4.17}$$

The key point is the following: for small enough h there exists initial data U^0 such that

$$|(D_1^s U^n)_j| \gtrsim h\sum_{j\in\mathbb{Z}} |U_j^0|^2 \tag{4.4.18}$$

holds for all $|j| \leq 1/h$ and $|n| \leq 1/h^2$.

Therefore

$$\sup_{U^0\in l^2(\mathbb{Z})} h^{2-2s}\frac{\sum\limits_{|n|h^2\leq 1}\sum\limits_{|j|h\leq 1} |(D_1^s U^n)_j|^2}{\sum\limits_{j\in\mathbb{Z}} |U_j^0|^2} \quad\gtrsim\quad h^{-2s}$$

and then we obtain (4.4.17), which finishes the proof.

In the following we prove (4.4.18). Let us consider $\xi_0 \neq 0$ such that

$$\psi'_\lambda(\xi_0) = 0 \text{ and } m_\lambda(\xi_0) = 1.$$

Let us choose ϵ positive, $\epsilon \sim h$, and a function $\widehat{\varphi}$ supported in $(-1,1)$ with $\widehat{\varphi} > 1$ on $(-1/2, 1/2)$. We set

$$\widehat{U_\epsilon^0} = \epsilon^{-1}\widehat{\varphi}\left(\epsilon^{-1}(\xi-\xi_0)\right). \tag{4.4.19}$$

The Plancherel identity gives us

$$\sum_{j\in\mathbb{Z}} |(U^0_\epsilon)_j|^2 \simeq \epsilon^{-1} \simeq h^{-1}.$$

In view of (4.4.19) it is sufficient to show that for the above choice of U^0 the following holds

$$|(D^s_1 U^n)_j| \gtrsim 1$$

for all $|j|h \leq 1$ and $|n|h^2 \leq 1$.

By the definition of the discrete derivative D^s_1 we obtain

$$(D^s_1 U^n_\epsilon)_j = \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} |e^{i\xi}-1|^s m_\lambda^{|n|}(\xi) e^{ij\xi+n\psi_\lambda(\xi)} \widehat{U^0_\epsilon}(\xi) d\xi.$$

The mean value theorem applied to the function $\Psi(\xi) = ij\xi + n\psi_\lambda(\xi)$ on the interval $[\xi_0 - \epsilon, \xi_0+\epsilon]$ gives us

$$|(D^s_1 U^n_\epsilon)_j| \geq \left(1 - 2\epsilon \sup_{\xi\in[\xi_0-\epsilon,\xi_0+\epsilon]} |\Psi'(\xi)|\right) \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} \left|\sin\frac{\xi}{2}\right|^s m_\lambda^{|n|}(\xi)\widehat{U^0_\epsilon}(\xi) d\xi.$$

The fact that ψ'_λ vanishes at the point ξ_0 implies that $\psi'_\lambda(\xi) = O(\xi-\xi_0)$ as $\xi \sim \xi_0$. Then the function Ψ satisfies

$$\Psi'(\xi) = j + n\psi'_\lambda(\xi) = j + nO(\xi-\xi_0) = O(\epsilon^{-1}) \tag{4.4.20}$$

as long as $j = O(\epsilon^{-1})$ and $n = O(\epsilon^{-2})$.

Using that $\epsilon \sim h$ and $|\xi - \xi_0| = O(\epsilon)$ we obtain for all $|j|h \leq 1$ and $|n|k \leq 1$ that the derivative $(D^s_1 U^n_\epsilon)$ satisfies

$$|(D^s_1 U^n_\epsilon)_j|^2 \gtrsim \left(\int_{\xi_0-\epsilon}^{\xi_0+\epsilon} \left|\sin\frac{\xi}{2}\right|^s m_\lambda^{|n|}(\xi)\widehat{U^0_\epsilon}(\xi)\right)^2 \gtrsim \left|\sin\frac{\xi_0}{2}\right|^{2s} \left(\epsilon^{-1}\int_{\xi_0-\epsilon/2}^{\xi_0+\epsilon/2} m_\lambda^{|n|}(\xi)\right)^2.$$

Hence it is sufficient to prove that the last term remains bounded from below as long as $\epsilon \sim h$ and $|n| = O(h^{-2})$. The fact that $m_\lambda(\xi_0) = 1$ implies that m_λ has a maximum at ξ_0 and consequently $m'_\lambda(\xi_0) = 0$. Then

$$m_\lambda(\xi) \sim 1 + O((\xi-\xi_0)^2),\ \xi \sim \xi_0.$$

Choosing possibly ϵ smaller we obtain the existence of a negative constant C such that

$$m_\lambda(\xi) \geq 1 + C(\xi-\xi_0)^2$$

for all $\xi \in (\xi_0 - \epsilon/2, \xi_0 + \epsilon/2)$. Then

$$\begin{aligned}
\epsilon^{-1}\int_{\xi_0-\epsilon/2}^{\xi_0+\epsilon/2} m_\lambda^{|n|}(\xi) &\gtrsim \epsilon^{-1}\int_{\xi_0-\epsilon/2}^{\xi_0+\epsilon/2} (1+C(\xi-\xi_0)^2)^{|n|} d\xi \\
&\gtrsim \epsilon^{-1}\int_{\xi_0-\epsilon/2}^{\xi_0+\epsilon/2} \exp(|n|C(\xi-\xi_0)^2) d\xi \\
&\gtrsim \frac{1}{\epsilon|n|^{1/2}} \int_0^{\frac{\epsilon|n|^{1/2}}{2}} \exp(C\xi^2) d\xi.
\end{aligned}$$

Using that $|n| = O(\epsilon^{-2})$ we get $\epsilon|n|^{1/2} = O(1)$ and the last integral is bounded below. This finishes the proof.

Step II: The conditions are sufficient.
We prove that under condition (4.2.6) the following holds:

$$\sup_{j\in\mathbb{Z}} k\sum_{n\in\mathbb{Z}} |(D_h^{1/2}U^n)_j|^2 \lesssim h\sum_{j\in\mathbb{Z}} |U_j^0|^2.$$

Using the relation (4.4.16) between the discrete derivatives $D_h^{1/2}$ and $D_1^{1/2}$ we obtain

$$\frac{k\sum\limits_{n\in\mathbb{Z}} |(D_h^{1/2}U^n)_j|^2}{h\sum\limits_{j\in\mathbb{Z}} |U_j^0|^2} = \frac{kh^{-1}\sum\limits_{n\in\mathbb{Z}} |(D_1^{1/2}U^n)_j|^2}{h\sum\limits_{j\in\mathbb{Z}} |U_j^0|^2} = \frac{\lambda\sum\limits_{n\in\mathbb{Z}} |(D_1^{1/2}U^n)_j|^2}{\sum\limits_{j\in\mathbb{Z}} |U_j^0|^2}.$$

Now we prove that under condition (4.2.6) the last term is uniformly bounded above :

$$\sup_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |(D_1^{1/2}U^n)_j|^2 \lesssim \sum_{j\in\mathbb{Z}} |U_j^0|^2. \tag{4.4.21}$$

The definition of $D_1^{1/2}U^n$

$$(D_1^{1/2}U^n)_j = \int_{-\pi}^{\pi} \left|\sin\frac{\xi}{2}\right| e^{ij\xi}e^{in\psi_\lambda(\xi)}m_\lambda^{|n|}(\xi)\widehat{U^0}(\xi)d\xi,$$

and Plancherel's identity

$$\sum_{j\in\mathbb{Z}} |U_j^0|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{U^0}(\xi)|^2 d\xi,$$

show that (4.4.21) is equivalent with

$$\sup_{j\in\mathbb{Z}} \sum_{n\in\mathbb{Z}} |U_j^n|^2 \lesssim \int_{-\pi}^{\pi} \frac{|\widehat{U^0}(\xi)|^2}{|\sin(\xi/2)|}d\xi. \tag{4.4.22}$$

Now the proof uses similar techniques as in [75].

Case I. ψ_λ' has no roots except at the point $\xi_0 = 0$.
In this case ψ_λ is one to one on each interval $(-\pi, 0)$ and $(0, \pi)$. We can assume also, without loss of generality that ψ_λ' is positive on each interval. This allows us to make the change of variables $\xi \to \psi_\lambda^{-1}(\xi)$ on each of the above intervals. In fact we prove that for any $j \in \mathbb{Z}$ the following estimates hold:

$$\sum_{n\in\mathbb{Z}} \left|\int_0^{\pi} e^{ij\xi}e^{in\psi_\lambda(\xi)}m_\lambda^{|n|}(\xi)\widehat{U^0}(\xi)d\xi\right|^2 \lesssim \int_0^{\pi} \frac{|\widehat{U^0}(\xi)|^2}{|\psi_\lambda'(\xi)|}d\xi \tag{4.4.23}$$

and

$$\sum_{n\in\mathbb{Z}} \left|\int_{-\pi}^{0} e^{ij\xi}e^{in\psi_\lambda(\xi)}m_\lambda^{|n|}(\xi)\widehat{U^0}(\xi)d\xi\right|^2 \lesssim \int_{-\pi}^{0} \frac{|\widehat{U^0}(\xi)|^2}{|\psi_\lambda'(\xi)|}d\xi. \tag{4.4.24}$$

Recall that the consistency of the scheme guarantees that $\psi_\lambda(\xi)\sim 1+i\xi^2$ when $\xi\sim 0$. Thus $\psi'_\lambda(\xi)\sim 2i\xi$ for $\xi\sim 0$.

Once we prove (4.4.23) and (4.4.24), using that the symbol $\psi'_\lambda(\xi)$ behaves as $2i\xi$ when $\xi\sim 0$ and has no roots far from the point $\xi=0$, we can replace $|\psi'_\lambda(\xi)|$ by $\sin(\xi/2)$ in the above estimates.

We will prove the first inequality (4.4.23), the second one being similar. First we make the change of variables $\xi\to\psi_\lambda^{-1}(\xi)$ to get

$$\int_0^\pi e^{ij\xi}e^{in\psi_\lambda(\xi)}m_\lambda^{|n|}(\xi)\widehat{U^0}(\xi)d\xi=\int_{\psi_\lambda(0)}^{\psi_\lambda(\pi)}e^{in\xi}e^{ij\psi_\lambda^{-1}(\xi)}m_\lambda^{|n|}(\psi_\lambda^{-1}(\xi))\widehat{U^0}(\psi_\lambda^{-1}(\xi))(\psi_\lambda^{-1})'(\xi)d\xi.$$

Then, inequality (4.4.23) is equivalent to the following one :

$$\sum_{n\in\mathbb{Z}}\left|\int_{\psi_\lambda(0)}^{\psi_\lambda(\pi)}e^{in\xi}e^{ij\psi_\lambda^{-1}(\xi)}m_\lambda^{|n|}(\psi_\lambda^{-1}(\xi))\widehat{U^0}(\psi_\lambda^{-1}(\xi))(\psi_\lambda^{-1})'(\xi)d\xi\right|^2\leq\int_0^\pi\frac{|\widehat{U^0}(\xi)|^2}{|\psi'_\lambda(\xi)|}d\xi. \quad (4.4.25)$$

Each term in the above sum is similar to the Fourier coefficients of the function

$$e^{ij\psi_\lambda^{-1}(\xi)}\widehat{U^0}(\psi_\lambda^{-1}(\xi))(\psi_\lambda^{-1})'(\xi)$$

except for the weight term $m_\lambda^{|n|}(\psi_\lambda^{-1}(\xi))$. Using that the involved extra term is always smaller than one, we can apply Lemma B.0.2 (see Appendix B) to obtain

$$\begin{aligned}\sum_{n\in\mathbb{Z}}&\left|\int_{\psi_\lambda(0)}^{\psi_\lambda(\pi)}e^{in\xi}e^{ij\psi_\lambda^{-1}(\xi)}m_\lambda^{|n|}(\psi_\lambda^{-1}(\xi))\widehat{U^0}(\psi_\lambda^{-1}(\xi))(\psi_\lambda^{-1})'(\xi)d\xi\right|^2\lesssim\\ &\lesssim\int_{\psi_\lambda(0)}^{\psi_\lambda(\pi)}|\widehat{U^0}(\psi_\lambda^{-1}(\xi))|^2|(\psi_\lambda^{-1})'(\xi)|^2d\xi\\ &=\int_{\psi_\lambda(0)}^{\psi_\lambda(\pi)}|\widehat{U^0}(\psi_\lambda^{-1}(\xi))|^2\frac{d\xi}{|\psi'_\lambda(\psi_\lambda^{-1}(\xi))|^2}\\ &=\int_0^\pi\frac{|\widehat{U^0}(\xi)|^2}{|\psi'_\lambda(\xi)|}d\xi.\end{aligned}$$

This shows (4.4.23) and finishes the proof the first case.

Case II. ψ'_λ has roots other than $\xi_0=0$.
Using that ψ'_λ has a finite number of roots we can assume that there is only one. Let us consider a point $\xi_0\neq 0$ such that

$$\psi'_\lambda(\xi_0)=0 \text{ and } m_\lambda(\xi_0)<1.$$

Far from the point ξ_0 we use the same argument as in the first case. Close to the point ξ_0 we will take into account the dissipation introduced by m_λ.

Let us fix $\delta>0$. We choose $\epsilon>0$ such that

$$m_\lambda(\xi)\leq 1-\delta,\ \forall\,\xi\in(\xi_0-\epsilon,\xi_0+\epsilon).$$

Without loss of generality we can assume that $\xi_0 > \epsilon > 0$. We write the solution U^n as $U^n = V^n + W^n$ where

$$W_j^n = \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} e^{ij\xi} e^{in\psi_\lambda(\xi)} m_\lambda^{|n|}(\xi)\widehat{U^0}(\xi)d\xi. \tag{4.4.26}$$

Both terms V^n and W^n satisfy an estimate similar to (4.4.22). For V^n we use the same argument as in the previous case. In the case of W^n we use the dissipative character of $m_\lambda(\xi)$ near the point ξ_0. On $[-\pi,\pi]\backslash(\xi_0-\epsilon,\xi_0+\epsilon)$ the function $\psi_\lambda'(\xi)$ has a single root $\xi = 0$. Then as in the first case:

$$\sum_{n\in\mathbb{Z}} |V_j^n|^2 \lesssim \int_{[-\pi,\pi]\backslash(\xi_0-\epsilon,\xi_0+\epsilon)} \frac{|\widehat{U^0}(\xi)|^2}{|\psi_\lambda'(\xi)|} d\xi \lesssim \int_{[-\pi,\pi]\backslash(\xi_0-\epsilon,\xi_0+\epsilon)} \frac{|\widehat{U^0}(\xi)|^2}{|\sin(\xi/2)|} d\xi. \tag{4.4.27}$$

Applying Cauchy's inequality to (4.4.26), we obtain that W^h satisfies

$$|W_j^n|^2 \le \left[\int_{\xi_0-\epsilon}^{\xi_0+\epsilon} m_\lambda^{|n|}(\xi)|\widehat{U^0}(\xi)|\right]^2 \lesssim (1-\delta)^{2|n|} \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} |\widehat{U^0}(\xi)|^2.$$

We sum the above inequality over $n \in \mathbb{Z}$ and get

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |W_j^n|^2 &\lesssim \left[\sum_{n\ge 0}(1-\delta)^{2n}\right] \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} |\widehat{U^0}(\xi)|^2 \\ &\lesssim \frac{1}{\delta}\int_{\xi_0-\epsilon}^{\xi_0+\epsilon} |\widehat{U^0}(\xi)|^2 \lesssim \int_{\xi_0-\epsilon}^{\xi_0+\epsilon} |\widehat{U^0}(\xi)|^2. \end{aligned}\tag{4.4.28}$$

Putting together (4.4.27) and (4.4.28) and using that $\xi_0 \neq 0$ we finally obtain

$$\sum_{n\in\mathbb{Z}} |U_j^n|^2 \lesssim \int_{-\pi}^{\pi} \frac{|\widehat{U^0}(\xi)|^2}{|\sin(\xi/2)|} d\xi,$$

which finishes the proof.

4.5. Strichartz-like estimates

Let us consider the following approximation of LSE:

$$U^{n+1} = A_\lambda U^n,\ n \ge 0.$$

We denote by $S_\lambda(n)U^0 = U^n$ the solution at step n.

We consider now a scheme that satisfies (4.2.7). Then for all $n \neq 0$:

$$\|S_\lambda(n)U^0\|_{l^\infty(h\mathbb{Z})} \lesssim \frac{1}{(k|n|)^{1/2}} \|U^0\|_{l^1(h\mathbb{Z})}. \tag{4.5.1}$$

We consider the inhomogeneous problem:

$$\begin{cases} U^{n+1} = A_\lambda U^n + kf(n+1), \quad n \ge 0, \\ U^0 = 0. \end{cases} \tag{4.5.2}$$

The explicit solution of this difference equation is given by the discretized version of Duhamel's Principle, defined by

$$(\Lambda f)(n,\cdot) \overset{not}{=} k\sum_{j=0}^{n} A_\lambda^{n-j} f(j,\cdot), \tag{4.5.3}$$

with the convention $f(0)\equiv 0$. It is convenient to write Λf in a semigroup formulation :

$$(\Lambda f)(n,\cdot) = k\sum_{j=0}^{n} S_\lambda(n-j) f(j,\cdot). \tag{4.5.4}$$

With these notations we prove the following result, that can be considered as the discrete version of Keel and Tao [74].

Theorem 4.5.1. *Let (q,r) and $(\tilde q,\tilde r)$ be two $1/2$-admissible pairs. Then*
i) There exists a positive constant $C(\lambda,q,r)$ such that

$$\|S_\lambda(\cdot)U^0\|_{l^q(k\mathbb{N},\, l^r(h\mathbb{Z}))} \le C(\lambda,q,r)\|U^0\|_{l^2(h\mathbb{Z})} \tag{4.5.5}$$

for all $U^0\in l^2(h\mathbb{Z})$, uniformly on $h>0$.
ii) There exists a positive constant $C(\lambda,q,r)$ such that

$$\left\|\sum_{n\in\mathbb{N}} S_\lambda(n)^* f(n)\right\|_{l^2(h\mathbb{Z})} \le C(\lambda,q,r)\|f\|_{l^{q'}(k\mathbb{N},\, l^{r'}(h\mathbb{Z}))} \tag{4.5.6}$$

for all $f\in l^{q'}(k\mathbb{N},\, l^{r'}(h\mathbb{Z}))$, uniformly on $h>0$.
iii) There exists a positive constant $C(\lambda,q,r,\tilde q,\tilde r)$ such that

$$\|\Lambda f\|_{l^q(k\mathbb{N},\, l^r(h\mathbb{Z}))} \le C(\lambda,q,r,\tilde q,\tilde r)\|f\|_{l^{\tilde q'}(k\mathbb{N},\, l^{\tilde r'}(h\mathbb{Z}))} \tag{4.5.7}$$

for all $f\in l^{q'}(k\mathbb{N},\, l^{r'}(h\mathbb{Z}))$, uniformly on $h>0$.

The above results require estimates for the operators $S_\lambda(n)S_\lambda(m)^*$, with $m,n\in\mathbb{N}$. We remark that $S_\lambda(n)$ satisfies the semigroup condition $S_\lambda(m+n)=S_\lambda(n)S_\lambda(m)$ for all $m,n\in\mathbb{N}$.

With the notation used in Section 4.1, let us write $S_\lambda(n)$, $n\ge 0$, in the Fourier variable:

$$\mathcal{F}_1(S_\lambda(n)\varphi)(\xi) = e^{in\psi_\lambda(\xi)} m_\lambda^n(\xi)\mathcal{F}_1(\varphi)(\xi),\ \xi\in[-\pi,\pi].$$

Thus its adjoint is given by

$$\mathcal{F}_1(S_\lambda(n)^*\varphi)(\xi) = e^{-in\psi_\lambda(\xi)} m_\lambda^n\mathcal{F}_1(\varphi)(\xi),\ \xi\in[-\pi,\pi].$$

We point out that, by a scaling argument, the estimates of the operators $S_\lambda(n), n\in\mathbb{Z}$ in $l^p(h\mathbb{Z})$-spaces are reduced to estimates on the spaces $l^p(\mathbb{Z})$. We give the following result on these spaces. Of course one can obtain a similar result on the normalized spaces.

Lemma 4.5.1. *Let $q\ge 2$. There is a positive constant $C(\lambda)$ such that*

$$\|S_\lambda(n)S_\lambda(m)^*\varphi\|_{l^q(\mathbb{Z})} \le \frac{C(\lambda)}{1+|n-m|^{\frac12(\frac1{q'}-\frac1q)}}\|\varphi\|_{l^{q'}(\mathbb{Z})} \tag{4.5.8}$$

holds for all nonnegative integers n and m and $\varphi\in l^{q'}(\mathbb{Z})$.

The same result can be obtained in the normalized spaces $l^p(h\mathbb{Z})$ in the case $n \neq m$.

Lemma 4.5.2. *Let $q \geq 2$. Then for all nonnegative integers $n \neq m$*

$$\|S_\lambda(n)S_\lambda(m)^*\varphi\|_{l^q(h\mathbb{Z})} \leq \frac{C(\lambda)}{(k|n-m|)^{\frac{1}{2}(\frac{1}{q'}-\frac{1}{q})}}\|\varphi\|_{l^{q'}(h\mathbb{Z})} \tag{4.5.9}$$

holds for all $\varphi \in l^{q'}(\mathbb{Z})$, uniformly on k and h with $k/h^2 = \lambda$.

We postpone the proof of Lemma 4.5.1 to Section 4.6 and proceed with the proof of Theorem 4.5.1.

Proof of Theorem 4.5.1. First, remark that when all the inequalities are rescaled, the proof is reduced to the case $h = 1$:

$$\begin{aligned}\frac{\|S_\lambda(\cdot)U^0\|_{l^q(k\mathbb{Z},\, l^r(h\mathbb{Z}))}}{\|U^0\|_{l^2(h\mathbb{Z})}} &= \frac{k^{1/q}h^{1/r}}{h^{1/2}}\frac{\|S_\lambda(\cdot)U^0\|_{l^q(\mathbb{Z},\, l^r(\mathbb{Z}))}}{\|U^0\|_{l^2(\mathbb{Z})}} = \frac{(\lambda h^2)^{1/q}h^{1/r}}{h^{1/2}}\frac{\|S_\lambda(\cdot)U^0\|_{l^q(\mathbb{Z},\, l^r(\mathbb{Z}))}}{\|U^0\|_{l^2(\mathbb{Z})}}\\ &= \lambda^{1/q}h^{1/r-1/2+2/q}\frac{\|S_\lambda(\cdot)U^0\|_{l^q(\mathbb{Z},\, l^r(\mathbb{Z}))}}{\|U^0\|_{l^2(\mathbb{Z})}} = \lambda^{1/q}\frac{\|S_\lambda(\cdot)U^0\|_{l^q(\mathbb{Z},\, l^r(\mathbb{Z}))}}{\|U^0\|_{l^2(\mathbb{Z})}}.\end{aligned}$$

The other estimates (4.5.6) and (4.5.7) are rescaled in a similar manner. Then we shall consider the case $h = 1$ and prove estimates with constants depending only on $\lambda, q, r, \tilde{q}, \tilde{r}$. In the sequel we denote by Λ^1 the operator Λ defined in (4.5.4) corresponding to the case $k = \lambda, h = 1$.

The main ingredient used in the proof is the estimate of $S_\lambda(n)S_\lambda(m)^*$ in (4.5.8).

Step I. Estimate (4.5.6) implies (4.5.5).

Using similar arguments as in the classical Strichartz-estimates ([74]) we will reduce (4.5.5) to (4.5.6). By duality

$$\|S_\lambda(\cdot)U^0\|_{l^q(\mathbb{N},\, l^r(\mathbb{Z}))} = \sup_{\|\psi\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}\leq 1} \langle\langle S_\lambda(\cdot)U^0, \psi\rangle\rangle$$

where $\langle\langle\cdot,\cdot\rangle\rangle$ is the duality product between $l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))$ and $l^q(\mathbb{N},\, l^r(\mathbb{Z}))$. The last term can be estimated in the following way:

$$\begin{aligned}|\langle\langle S_\lambda(\cdot)U^0, \psi\rangle\rangle| &= \left|\sum_{n\in\mathbb{N}}\langle S_\lambda(n)U^0, \psi(n)\rangle\right| = \left|\sum_{n\in\mathbb{N}}\langle U^0, S_\lambda(n)^*\psi(n)\rangle\right|\\ &= \left|\left\langle U^0, \sum_{n\in\mathbb{N}} S_\lambda(n)^*\psi(n)\right\rangle\right| \leq \|U^0\|_{l^2(\mathbb{Z})}\left\|\sum_{n\in\mathbb{N}} S_\lambda(n)^*\psi(n)\right\|_{l^2(\mathbb{Z})},\end{aligned}$$

reducing the proof to (4.5.6), namely

$$\left\|\sum_{n\in\mathbb{N}} S_\lambda(n)^*f(n)\right\|_{l^2(\mathbb{Z})} \lesssim \|f\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}.$$

Step II. Proof of estimate (4.5.6).
By the TT^* method, inequality (4.5.6) turns out to be equivalent to the bilinear estimate

$$\left|\left\langle \sum_{n\in\mathbb{N}} S_\lambda(n)^* f(n), \sum_{m\in\mathbb{N}} S_\lambda(m)^* g(m)\right\rangle\right| \lesssim \|f\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}\|g\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}.$$

It is equivalent to

$$\left|\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \langle S_\lambda(n)^* f(n), S_\lambda(m)^* g(m)\rangle\right| \lesssim \|f\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}\|g\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}. \tag{4.5.10}$$

Let us prove that

$$\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} |\langle S_\lambda(n)^* f(n), S_\lambda(m)^* g(m)\rangle| \lesssim \|f\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}\|g\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}.$$

and then (4.5.10).

In view of estimate (4.5.8) on $S_\lambda(n)S_\lambda(m)^*$ we have

$$\begin{aligned}
|\langle S_\lambda(n)^* f(n), S_\lambda(m)^* g(m)\rangle| &= |\langle f(n), S_\lambda(n)S_\lambda(m)^* g(m)\rangle| \\
&\le \|f(n)\|_{l^{r'}(\mathbb{Z})}\|S_\lambda(n)S_\lambda(m)^* g(m)\|_{l^r(\mathbb{Z})} \\
&\lesssim \|f(n)\|_{l^{r'}(\mathbb{Z})}\frac{\|g(m)\|_{l^{r'}(\mathbb{Z})}}{1+|n-m|^{2/q}}.
\end{aligned}$$

Therefore, we obtain by the discrete Riesz potential inequality (see [98])

Lemma 4.5.3. *Let be $0<\alpha<1$ and k a sequence such that*

$$|k(n)| \le \frac{1}{1+|n|^{1-\alpha}}. \tag{4.5.11}$$

Then the operator $\mathcal{T}$ defined by $\mathcal{T}(f) = f * k$ maps continuously $l^p(\mathbb{Z})$ into $l^q(\mathbb{Z})$ for any p and q satisfying

$$1 < p < q < \infty \text{ and } \frac{1}{q} = \frac{1}{p} - \alpha. \tag{4.5.12}$$

In view of the above lemma we obtain:

$$\begin{aligned}
\sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} |\langle S_\lambda(n)^* f(n), S_\lambda(m)^* g(m)\rangle| &\lesssim \sum_{n\in\mathbb{N}}\sum_{m\in\mathbb{N}} \|f(n)\|_{l^{r'}(\mathbb{Z})}\frac{\|g(m)\|_{l^{r'}(\mathbb{Z})}}{1+|n-m|^{2/q}} \\
&\lesssim \|f\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}\left\|\left(\sum_m \frac{\|g(m)\|_{l^{r'}(\mathbb{Z})}}{1+|n-m|^{2/q}}\right)\right\|_{l^q(\mathbb{Z})} \\
&\lesssim \|f\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}\|g\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}.
\end{aligned}$$

Step III. Proof of (4.5.7).
We consider the cases $(\tilde{q},\tilde{r}) = (\infty, 2)$, $(q,r) = (\infty, 2)$ and $(\tilde{q},\tilde{r}) = (q,r)$, since the other cases follow by interpolation. By duality

$$\|\Lambda^1 f\|_{l^q(\mathbb{N}),\, l^r(\mathbb{Z})} = \sup_{\|g\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))}\le 1} \left\langle\left\langle \Lambda^1 f, g\right\rangle\right\rangle.$$

Let us choose a function g such that $g \in l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))$. The definition of Λ^1 gives us

$$\begin{aligned}\langle\langle \Lambda^1 f, g\rangle\rangle &= \sum_{n\geq 0}\left\langle \sum_{j=0}^{n} S_\lambda(n-j)f(j), g(n)\right\rangle = \sum_{n\geq 0}\sum_{j=0}^{n}\langle S_\lambda(n-j)f(j), g(n)\rangle \\ &= \sum_{n\geq 0}\sum_{j=0}^{n}\langle f(j), S_\lambda(n-j)^* g(n)\rangle = \sum_{j\geq 0}\sum_{n\geq j}\langle f(j), S_\lambda(n-j)^* g(n)\rangle \\ &= \sum_{j\geq 0}\left\langle f(j), \sum_{n\geq j} S_\lambda(n-j)^* g(n)\right\rangle. \qquad (4.5.13)\end{aligned}$$

Case 1: $(\tilde{q},\tilde{r}) = (\infty, 2)$**.**

The Cauchy inequality applied to (4.5.13) shows that

$$\begin{aligned}\langle\langle \Lambda^1 f, g\rangle\rangle &\leq \sum_{j\geq 0}\|f_j\|_{l^2(\mathbb{Z})}\left\|\sum_{n\geq j} S_\lambda(n-j)^* g(n)\right\|_{l^2(\mathbb{Z})} \\ &\leq \|f\|_{l^1(\mathbb{N},\, l^2(\mathbb{Z}))}\sup_{j\geq 0}\left\|\sum_{n\geq j} S_\lambda(n-j)^* g(n)\right\|_{l^2(\mathbb{Z})} \\ &\leq \|f\|_{l^1(\mathbb{N},\, l^2(\mathbb{Z}))}\sup_{j\geq 0}\left\|\sum_{m\geq 0} S_\lambda(m)^* g(m+j)\right\|_{l^2(\mathbb{Z})}.\end{aligned}$$

Applying estimate (4.5.6) to the function $g(\cdot + j)$ we get

$$\left\|\sum_{m\geq 0} S_\lambda(m)^* g(m+j)\right\|_{l^2(\mathbb{Z})} \leq \|g(\cdot + j)\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))} \leq \|g\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))},$$

which finally proves that

$$\langle\langle \Lambda^1 f, g\rangle\rangle \leq \|f\|_{l^1(\mathbb{N},\, l^2(\mathbb{Z}))}.$$

Case II: $(q, r) = (\infty, 2)$**.**

With the same notations as above

$$\begin{aligned}\langle\langle \Lambda^1 f, g\rangle\rangle &= \sum_{n\geq 0}\left\langle \sum_{j=0}^{n} S_\lambda(n-j)f(j), g(n)\right\rangle \\ &\lesssim \left\|\sum_{0\leq j\leq n} S_\lambda(n-j)f(j)\right\|_{l^\infty(\mathbb{N},\, l^2(\mathbb{Z}))}\|g\|_{l^1(\mathbb{N},\, l^2(\mathbb{Z}))}.\end{aligned}$$

It remains to prove that for any admissible pair $(\tilde{q},\tilde{r})$

$$\left\|\sum_{0\leq j\leq n} S_\lambda(n-j)f(j)\right\|_{l^\infty(\mathbb{N},\, l^2(\mathbb{Z}))} \lesssim \|f\|_{l^{\tilde{q}'}(\mathbb{N},\, l^{\tilde{r}'}(\mathbb{Z}))}.$$

To do that, we write

$$\sum_{0\leq j\leq n} S_\lambda(n-j)f(j) = \sum_{0\leq j\leq n} S_\lambda(j-n)^* f(j) = \sum_{k=-n}^{0} S_\lambda(k)^* f(n+k)$$

and apply estimate (4.5.6) to the function $f(\cdot + n)$.

Case III: $(q,r) = (\tilde{q},\tilde{r})$**.**

Observe that Λ^1 satisfies the rough estimate

$$\|(\Lambda^1 f)(n)\|_{l^r(\mathbb{Z})} \leq \sum_{j=0}^{n} \|S_\lambda(n-j)f(j)\|_{l^r(\mathbb{Z})} \leq \sum_{0\leq j\leq n} \frac{\|f(j)\|_{l^{r'}(\mathbb{Z})}}{1+|n-j|^{2/q}}.$$

The same arguments as in Step II, based on the discrete Riesz's potential inequality (4.5.3), show that

$$\|(\Lambda^1 f)(n)\|_{l^q(\mathbb{N},\, l^r(\mathbb{Z}))} \lesssim \|f\|_{l^{q'}(\mathbb{N},\, l^{r'}(\mathbb{Z}))},$$

which finishes the proof of the last case and that of Theorem 4.5.1. □

4.6. Estimates of $S_\lambda(n)S_\lambda(m)^*$

Proof of Lemma 4.5.1. Let us first consider the case $n=m$. In that case, using the fact that the spaces $l^q(\mathbb{Z})$ are nested:

$$l^{q'}(\mathbb{Z}) \hookrightarrow l^2(\mathbb{Z}) \hookrightarrow l^q(\mathbb{Z}),$$

we obtain by Plancherel's identity

$$\|S_\lambda(n)S_\lambda(n)^*\varphi\|_{l^q(\mathbb{Z})} \leq \|S_\lambda(n)S_\lambda(n)^*\varphi\|_{l^2(\mathbb{Z})} \lesssim \|\varphi\|_{l^2(\mathbb{Z})} \leq \|\varphi\|_{l^{q'}(\mathbb{Z})}. \tag{4.6.1}$$

In the general case, i.e. $n\neq m$, we prove that the operator $S_\lambda(n)S_\lambda(m)^*$ satisfies

$$\|S_\lambda(n)S_\lambda(m)^*\varphi\|_{l^q(\mathbb{Z})} \lesssim |n-m|^{-\frac{1}{2}(\frac{1}{q'}-\frac{1}{q})}\|\varphi\|_{l^{q'}(\mathbb{Z})}. \tag{4.6.2}$$

Both (4.6.1) and (4.6.2) imply (4.5.8). We will prove estimate (4.6.2).

We consider the cases $q=2$:

$$\|S_\lambda(n)S_\lambda(m)^*\varphi\|_{l^2(\mathbb{Z})} \lesssim \|\varphi\|_{l^2(\mathbb{Z})},$$

and $q=\infty$:

$$\|S_\lambda(n)S_\lambda(m)^*\varphi\|_{l^\infty(\mathbb{Z})} \lesssim |n-m|^{-1/2}\|\varphi\|_{l^1(\mathbb{Z})}.$$

The other cases follow by interpolation. The first estimate easily follows by Plancherel's identity. For the second one we use similar arguments as in the first section, based on Van der Corput's Lemma 3.2.1. Observe that the operator $S_\lambda(n)S_\lambda(m)^*$ satisfies :

$$\mathcal{F}_1(S_\lambda(n)S_\lambda(m)^*\varphi)(\xi) = m_\lambda^{|n|+|m|}(\xi)e^{i\psi_\lambda(\xi)(n-m)}\mathcal{F}_1(\varphi).$$

Then

$$S_\lambda(n)S_\lambda(m)^*\varphi = K_\lambda^{n,m} * \varphi, \tag{4.6.3}$$

where

$$(K_\lambda^{n,m})_j = \int_{-\pi}^{\pi} e^{ij\xi} m_\lambda^{|n|+|m|}(\xi) e^{i(n-m)\psi_\lambda(\xi)} d\xi.$$

By Young's Inequality, (4.6.3) implies

$$\|S_\lambda(n)S_\lambda(m)^*\varphi\|_{l^\infty(\mathbb{Z})} \leq \|K_\lambda^{n,m}\|_{l^\infty(\mathbb{Z})}\|\varphi\|_{l^1(\mathbb{Z})}.$$

It is sufficient to show that the kernel $K^{n,m}$ satisfies for all $n \neq m$:

$$|K_{\lambda,j}^{n,m}| \lesssim |n-m|^{1/2}.$$

Applying the same argument as in the proof of Theorem 4.2.1, based on Van der Corput's Lemma 3.2.1, we obtain the desired estimates on $K_\lambda^{n,m}$. □

4.7. Application to a nonlinear problem

In this section we consider a numerical scheme for the semilinear NSE equation in $\mathbb{R}$ with repulsive power law nonlinearity :

$$\begin{cases} iu_t + u_{xx} &= |u|^p u, \quad x \in \mathbb{R}, t > 0, \\ u(0,x) &= \varphi(x), \quad x \in \mathbb{R}, \end{cases} \tag{4.7.1}$$

the case when nonlinearity is given by $f(u) = -|u|^p u$ being the same. In fact, the key point in the global existence of the solutions is that the L^2-scalar product $(f(u), u)$ is a real number. All the results extend to more general nonlinearities $f(u)$ (see [25], Ch. 4.6, p. 109, for L^2-solutions).

With the notation $f(x) = |x|^p x$ the scheme is given by

$$i\frac{U_j^{n+1} - U_j^n}{k} + \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_j^{n+1}}{h^2} = f(U_j^{n+1}), \quad n \geq 0,\, j \in \mathbb{Z}, \tag{4.7.2}$$

$U^0 \in l^2(h\mathbb{Z})$ being an approximation of the initial datum φ and h and k such that k/h^2 remains constant. With the notations of the first section we can write the above equation as

$$A_{1,\lambda}U^{n+1} = A_{2,\lambda}U^n + kf(U^{n+1}),\, n \geq 0.$$

Then the solution at step $n+1$ is the solution of the following equation

$$U^{n+1} = (A_{1,\lambda}^{-1}A_{2,\lambda})U^n + kA_{1,\lambda}^{-1}f(U^{n+1}).$$

Using the discrete Duhamel's principle, U^{n+1} is given by:

$$U^{n+1} = (A_{1,\lambda}^{-1}A_{2,\lambda})^n U^0 + k\sum_{j=1}^{n}(A_{1,\lambda}^{-1}A_{2,\lambda})^{n-j}A_{1,\lambda}^{-1}f(U^j).$$

We point out that all the expressions above make sense. First we prove a priori that U^n belongs to $l^2(h\mathbb{Z})$ if $U^0 \in l^2(h\mathbb{Z})$. The fact that $A_{1,\lambda}$ is a continuous operator on $l^2(h\mathbb{Z})$ implies that all the above terms make sense.

The main result is the following:

Theorem 4.7.1. *Let k and h be such that the Courant number k/h^2 is kept constant. Also let be $p \in [0,4)$ and $U^0 \in l^2(h\mathbb{Z})$. Then there is a unique solution of equation (4.7.2) which satisfies:*

$$\|U^n\|_{l^2(h\mathbb{Z})} \leq \|U^0\|_{l^2(h\mathbb{Z})} \tag{4.7.3}$$

for all $n \geq 0$.

Moreover for all $T > 0$ there is a constant $C(T)$ such that

$$\|U\|_{l^q(nk\leq T, l^r(h\mathbb{Z}))} \leq C(T)\|U^0\|_{l^2(h\mathbb{Z})} \tag{4.7.4}$$

uniformly on h.

Remark 4.7.1. *The case $p = 4$ has to be treated as in the semidiscrete case. In this case smallness assumptions on the initial data are required.*

Remark 4.7.2. *In the case of the backward Euler scheme, there is no need to add numerical viscosity or a two-grid preconditioner as for the semidiscrete case. The scheme itself introduces viscosity as we have shown in (4.3.6).*

Proof of Theorem 4.7.1. The proof consists in applying the Banach fix point Theorem in a ball of $l^q(nk \leq T, l^r(h\mathbb{Z})) \cap L^\infty(nk \leq T, l^2(h\mathbb{Z}))$ and make use of the Strichartz-like estimates proved in Theorem 4.5.1. Observe that the nonlinear term $f(U)$ is composed with the operator $(A_{1,\lambda})^{-1}$. In order to apply the Banach fix point Theorem we have to prove that the operator $(A_{1,\lambda})^{-1}$ is continuous from $l^s(h\mathbb{Z})$ to $l^s(h\mathbb{Z})$ for any $s \in [1,2]$, with a norm independent of $h > 0$. Observe that it is sufficient to prove that

$$\|(A_{1,\lambda})^{-1}f\|_{l^s(\mathbb{Z})} \leq c(\lambda)\|f\|_{l^s(\mathbb{Z})}$$

for all $f \in l^s(\mathbb{Z})$. Using the kernel representation of $(A_{1,\lambda})^{-1}f$ as

$$(A_{1,\lambda})^{-1}f = K_\lambda * f,$$

where K_λ is given by

$$\mathcal{F}_1(K_\lambda)(\xi) = \frac{1}{1+\lambda\sin^2(\frac{\xi}{2})},$$

it is sufficient to show that $\|K_\lambda\|_{l^1(\mathbb{Z})} \leq c(\lambda)$. Using the same arguments an in Chapter 2 we have

$$\|K_\lambda\|_{l^1(\mathbb{Z})} \leq \big(\|\mathcal{F}_1(K_\lambda)\|_{L^2(\mathcal{T}^1)}\|(\mathcal{F}_1(K_\lambda))'\|_{L^2(\mathcal{T}^1)}\big)^{1/2} = c(\lambda).$$

This allows us to prove the local existence of the solution of equation (4.7.2) and estimate (4.7.4).

In order to guarantee the global existence of the solution we have to obtain a priori estimates of the l^2-norm of the solution. Multiplying equation (4.7.2) by $\overline{U}_j^{n+1}$ we get for all $n \geq 0$ and $j \in \mathbb{Z}$:

$$i|U_j^{n+1}|^2 - iU_j^n\overline{U}_j^{n+1} + \lambda(U_{j+1}^{n+1} - 2U_j^n + U_{j-1}^{n+1})\overline{U}_j^{n+1} = kf(U_j^{n+1})\overline{U}_j^{n+1}.$$

Summing up on $j \in \mathbb{Z}$ and taking the imaginary part we obtain

$$\sum_{j\in\mathbb{Z}} |U_j^{n+1}|^2 \leq \sum_{j\in\mathbb{Z}} |U_j^{n+1} U_j^n|,$$

which guarantees that the l^2-norm of U^n is bounded above by the l^2-norm of the initial datum. This guarantees the l^2-stability of the scheme and the global existence of a solution $(U^n)_{n\geq 0}$.
□

4.8. Convergence of the method

In the sequel we introduce the interpolator $I^h U$, piecewise linear in time and piecewise constant in space:

$$(I^h U)(t,x) = U_j^n \frac{(n+1)k - t}{k} + U_j^{n+1} \frac{t - nk}{k},$$

for all $t \in [nk, (n+1)k)$, $x \in [jh, (j+1)h)$ with $n, j \in \mathbb{Z}, n \geq 0$. The following theorem gives uniform estimates on $I^h U$ and its convergence to the weak solution of the nonlinear Schrodinger equation (4.7.1).

Theorem 4.8.1. *Let $p < 4$ and k and h be such that the Courant number k/h^2 is a fixed constant. Then the interpolator $I^h U$ satisfies*

$$\|I^h U\|_{L^\infty([0,\infty),L^2(\mathbb{R}))} \lesssim \|(I^h U)(0)\|_{L^2(\mathbb{R})}.$$

and for all $T > 0$, there is a positive constant $C(T)$ such that

$$\|I^h U\|_{L^q([0,T),L^r(\mathbb{R}))} \leq C(T) \|(I^h U)(0)\|_{L^2(\mathbb{R})}.$$

Moreover

$$I^h U \overset{*}{\rightharpoonup} u \quad in \quad L^\infty([0,\infty), L^2(\mathbb{R})), \tag{4.8.1}$$

$$I^h U \rightharpoonup u \quad in \quad L^q_{loc}([0,\infty), L^r(\mathbb{R})), \tag{4.8.2}$$

$$I^h U \to u \quad a.e.\ on\ compact\ sets\ of\ [0,\infty) \times \mathbb{R}, \tag{4.8.3}$$

where u is the unique weak solution of the NSE.

Proof. The first two estimates are a consequence of (4.7.3) and (4.7.4). Thus, obviously (4.8.1) and (4.8.2) hold. The limit (4.8.3) is a consequence of the local smoothing property of the discrete operator S_λ. For a complete proof see Chapter 3. All the above properties show the convergence of $I^h u$ towards the unique solution u of the NSE. □

4.9. A finer analysis of the Crank-Nicolson scheme

In this section we analyze whether the two-grid pre-conditioner, introduced by Glowinsky in [52] recovers the dispersive properties (4.2.7) of the Crank-Nicolson scheme. As we proved in [63] it is sufficient to show that the multiplicative factor $m(\xi)$ introduced by the two-grid algorithm vanishes at the points where the second derivative of the symbol $\psi(\xi)$ vanishes. On the other hand the condition is necessary. If not, we take initial data with SDFT concentrated at the point ξ_0 where the second derivative of ψ vanishes, and we obtain a similar result as in (4.2.8). This is a consequence of the fact that the multiplicative factor $m(\xi)$ behaves as a nonzero constant close to ξ_0. Next we prove that for any Courant number $\lambda = k/h^2 \in \mathbb{Q}$, there is no two-grid pre-conditioner guaranteeing the dispersive properties (4.2.7). We prove that any two-grid algorithm introduces a multiplicative factor which vanishes only at points of the form $2\pi\mathbb{Q}$. On the other hand, for any rational Courant number λ, the second derivative of the symbol introduced by the Crank-Nicolson scheme vanishes at some points which do not belong to the set $2\pi\mathbb{Q}$.

As we proved in Section 4.3, the symbol introduced by the Crank Nicolson scheme is given by

$$a_\lambda(\xi) = \exp\left(i\psi_\lambda(\xi)\right)$$

where the function ψ_λ is given by

$$\psi_\lambda(\xi) = \arctan\left(\frac{\lambda}{2}\sin^2\frac{\xi}{2}\right).$$

Its first derivatives are given by

$$\psi_\lambda'(\xi) = \frac{2\lambda\sin\xi}{1+\frac{\lambda^2}{4}\sin^4\frac{\xi}{2}}, \quad \xi\in[-\pi,\pi]$$

and

$$\psi''(\xi) = \frac{2\lambda\left(\cos\xi - \frac{\lambda^2}{4}(\frac{1-\cos\xi}{2})^2(2+\cos\xi)\right)}{(1+\frac{\lambda^2}{4}\sin^4\frac{\xi}{2})^2}, \quad \xi\in[-\pi,\pi].$$

We prove that for any $\lambda\in\mathbb{Q}$, the function

$$Q(\xi) = \cos\xi - \frac{\lambda^2}{4}\left(\frac{1-\cos\xi}{2}\right)^2(2+\cos\xi) \tag{4.9.1}$$

has at least one root which does not belong to $2\pi\mathbb{Q}$. It is clear that the above function has a root in $(0,\pi/2)$: $Q(0)Q(\pi/2) < 0$. Let us suppose that there is $\lambda\in\mathbb{Q}$ such that the function Q in (4.9.1) has a root of the form $2m\pi/n$ with $m,n\in\mathbb{Z}$, $(m,n)=1$. We write $\cos\xi = (e^{i\xi}+e^{-i\xi})/2$ and $\lambda/8=\mu$ in equation (4.9.1). This gives us

$$e^{i\xi}+e^{-i\xi}-\mu(2-e^{-i\xi}-e^{i\xi})^2(4+e^{i\xi}+e^{-i\xi}) = 0.$$

Then the polynomial $P_\mu(x)$, defined by

$$P_\mu(x) = x^4+x^2-\mu(x^2-2x+1)^2(x^2+4x+1)$$

admits a root of the form $x=\exp(2i\pi m/n)$, with $(m,n)=1$. This implies that $P_\mu(x)$ is divisible by some cyclotomic polynomial associated with the root $2\pi m/n$. Using the fact that the degree of the cyclotomic polynomial of order n, Q_n, is $\varphi(n)$, where $\varphi(n)$ is the Euler indicator function, we obtain that n satisfies

$$\varphi(n) \leq 6.$$

The only possible values of n are $\{1, 2, 3, 5, 6, 7, 9, 12\}$. Then in order to obtain a contradiction we have to prove that none of the polynomials Q_n divide the polynomial P_μ :

$$\begin{array}{ll} Q_1 = x - 1, & Q_2 = x + 1, \\ Q_3 = x^2 + x + 1, & Q_5 = x^4 + x^3 + x^2 + x + 1, \\ Q_6 = x^2 - x + 1, & Q_7 = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1, \\ Q_9 = x^6 + x^3 + 1, & Q_{12} = x^4 - x^2 + 1. \end{array}$$

Explicit calculations show that

$$P_\mu \equiv 2 \mod Q_1, \quad P_\mu \equiv 2 + 2^5\mu \mod Q_2$$

and

$$P_\mu \equiv -1 - 27\mu^2 \mod Q_3, \quad P_\mu \equiv -1 + 5\mu^2 \mod Q_6$$

which exclude the cases Q_1, Q_2, Q_3, Q_6. In the case of Q_5 we get

$$P_\mu \equiv -x^3(25\mu^2 + 1) + ... \mod Q_5$$

which proves that $Q_5 \not| P_\mu$. Similar computations show that

$$P_\mu \equiv x^4(9\mu^2 + 1) + ... \mod Q_9, \quad P_\mu \equiv -15\mu^2x^3 \mod Q_{12}.$$

It remains to study the case Q_7. Using that both polynomials have the same degree, P_μ equals Q_7 multiplied by a constant. Using the fact that the coefficient of x in P_μ vanishes, we also exclude this case.

This analysis shows that for any rational λ, the Crank-Nicolson scheme introduces a symbol ψ_λ that has a second derivative vanishing at some point $\xi_0 \notin 2\pi\mathbb{Q}$. As we will prove in Section 4.10, the two grid algorithm cannot control the effects introduced by the scheme at the point $\xi_0 \notin 2\pi\mathbb{Q}$. This implies that a two-grid algorithm, at any scale n, cannot provide uniform Strichartz-like estimates for the Crank-Nicolson scheme.

For the local smoothing properties, a simple two-grid pre-conditioner, at the scale $n = 2$, allows us to recover that property. The essential point is that the first derivative of ψ_λ vanishes at the point $\pm\pi$ and a two grid algorithm with the quotient of the meshes $1/2$ will vanish the effects introduced by the scheme, at these points.

4.10. The two-grid algorithm

In this subsection we consider a general two-grid algorithm, based in the ones introduced by Glowinsky ([52]), Negreanu and Zuazua ([97]) and Zuazua and the author ([63]). To be more precise, we consider a function defined on the grid $\mathbb{Z}$, obtained as an interpolation of functions defined on the coarse grid $n\mathbb{Z}$, with $n \geq 2$. In [97], $n = 2$, and in [63] $n = 4$. We obtain the relation between the SDTF of the initial function on the coarse grid and the new one on the fine grid.

Lemma 4.10.1. *Let $n \geq 2$ and $\{V(kn)\}_{k\in\mathbb{Z}}$ a function defined on the coarse grid $n\mathbb{Z}$. Then the new function, $\{U(k)\}_{k\in\mathbb{Z}}$, defined by*

$$U(kn+j) = \frac{(n-j)V(kn) + jV((k+1)n)}{n},\ k \in \mathbb{Z},\ j = \overline{0, n-1},$$

satisfies

$$\mathcal{F}_1(U)(\xi) = \frac{e^{i(n-1)\xi}\mathcal{F}_1(\Pi V)(\xi)}{n}\left(\sum_{k=0}^{n-1} e^{ik\xi}\right)^2,\ \xi \in [-\pi, \pi],$$

where $(\Pi V)(nk+j) = V(nk)\delta_{0j}$, $k \in \mathbb{Z}$, $j = \overline{0, n-1}$.

Remark 4.10.1. *We point out that the multiplicative factor $\sum_{k=0}^{n-1} e^{ik\xi}$ vanishes only at the points $\xi_k = 2k\pi/n$ with $k = \overline{1, n-1}$.*

Proof. By the definition of SDFT we have

$$\begin{aligned}
\mathcal{F}_1(U)(\xi) &= \sum_{j\in\mathbb{Z}} e^{-ij\xi}U_j = \sum_{k\in\mathbb{Z}}\sum_{j=0}^{n-1} e^{-i(kn+j)\xi}\frac{(n-j)V(kn) + jV((k+1)n)}{n} \\
&= \frac{1}{n}\sum_{k\in\mathbb{Z}}\sum_{j=0}^{n-1}(n-j)e^{-ikn\xi}e^{-ij\xi}V(kn) \\
&\quad + \frac{1}{n}\sum_{k\in\mathbb{Z}}\sum_{j=0}^{n-1} je^{-i(k+1)n}e^{i(n-j)\xi}V((k+1)n) \\
&= \frac{1}{n}\left[\sum_{j=0}^{n-1}(n-j)e^{-ij\xi} + je^{i(n-j)\xi}\right]\left(\sum_{k\in\mathbb{Z}} e^{-ikn\xi}V(kn)\right) \\
&= \frac{e^{i(n-1)\xi}}{n}\left(\sum_{j=0}^{n-1}(n-j)e^{-i(n-1+j)\xi} + \sum_{j=1}^{n-1} je^{-i(j-1)\xi}\right)\mathcal{F}_1(\Pi V).
\end{aligned}$$

Using the polynomial identity

$$Q_n(x) = (1 + x + ... + x^{n-1})^2 = \sum_{j=0}^{n-1}(n-j)x^{n-1+j} + \sum_{j=1}^{n-1} jx^{j-1}$$

we obtain that

$$\mathcal{F}_1(U)(\xi) = \frac{e^{i(n-1)\xi}(Q_n(e^{-i\xi}))^2}{n}\mathcal{F}_1(\Pi V).$$

which finishes the proof.

□

Chapter 5

The Wave Equation on Lattices

5.1. Introduction

Let us consider the d-dimensional linear wave equation (LWE) in the whole space :

$$\begin{cases} u_{tt} - \Delta u = F, \text{ on } \mathbb{R}^{1+d} \\ \\ u_{|t=0} = u_0, \partial_t u_{|t=0} = u_1. \end{cases} \tag{5.1.1}$$

We assume $d \geq 2$ throughout. The wave equation models the propagation of different kinds of waves (for example light waves) in homogenous media. Nonlinear models of conservative type arise in quantum mechanics. Other perturbations of the wave equation appear in the study of vibrating systems. This model has been an object of very intensive investigation.

Regarding the existence and uniqueness (and more generally the well-posedness) of such modes, we mention some classical results. Ginibre and Velo [49] (see also [74]) proved the time space integrability properties of the linear semigroup generated by the wave equation. Those estimates are called Strichartz estimates since the pioneering work [121]. They guarantee that, in addition to the energy estimate, a gain of space-time integrability occurs. The most simple estimate takes the form

$$\|u\|_{L^q_t(\mathbb{R},\, L^r(\mathbb{R}^d))} \leq C(\|f\|_{\dot{H}^s(\mathbb{R}^d)} + \|g\|_{\dot{H}^{s-1}(\mathbb{R}^d)} + \|F\|_{L^{\tilde{q}'}_t(\mathbb{R},\, L^{\tilde{r}'}(\mathbb{R}^d))}) \tag{5.1.2}$$

for suitable values of $s, q, r, \tilde{q}, \tilde{r}$.

The pairs (q, r) and $(\tilde{q}, \tilde{r})$ are the so-called $(n-1)/2$ wave-admissible pairs. We recall that an α-wave admissible pair is such that (cf. [74])

$$\begin{cases} 2 \leq q \leq \infty,\ 2 \leq r < \infty, \\ \\ \dfrac{2}{q} \leq \left(1 - \dfrac{2}{r}\right)\alpha. \end{cases}$$

A scaling argument on (5.1.2) implies

$$\frac{d}{2} - s = \frac{d}{r} + \frac{1}{q} \tag{5.1.3}$$

and the so-called *gap condition*

$$\frac{d}{r} + \frac{1}{q} = \frac{d}{\tilde{r}'} + \frac{1}{\tilde{q}'} - 2. \tag{5.1.4}$$

These properties are not only relevant for a better understanding of the dynamics of the linear system but also to derive well-possedness results for nonlinear wave equations.

Let us consider the nonlinear problem

$$\begin{cases} u_{tt} - \Delta u + |u|^{p-1}u = 0 & \text{on} \quad \mathbb{R}^{1+d}, \\ u_{|t=0} = u_0, \partial_t u_{|t=0} = u_1 & \text{in} \quad \mathbb{R}^d. \end{cases} \tag{5.1.5}$$

This problem has been studied mostly by compactness or contraction methods. The existence of a weak global solution, satisfying (5.1.5) in the distributional sense, with no upper bound on p has been proved by [85]. Also their uniqueness for $p \leq 1+4d/(d+1)(d-2)$ has been proved in [51]. In this case, it even suffices to assume that the initial data $u_0, u_1 \in L^2_{loc}(\mathbb{R}^d)$ with $u_0 \in L^{p+1}_{loc}(\mathbb{R}^d)$ and the distributional derivative $\nabla u_0 \in L^2_{loc}(\mathbb{R}^d)$ (see [122] for more references).

Jörgens [70] proved the existence of unique global strongly continuous solutions in the energy space under the assumption $p \leq 1 + 2/(d-2)$.

In order to improve the range of admisible exponents, more sophisticated tools were developed, based in particular, on the $L^p - L^q$-estimates for the wave operator by Strichartz [121]; see also Brenner [13]. Ginibre and Velo [49] proved the uniqueness of weak solutions and the existence and uniqueness of global strongly continuous solutions in the energy space $u \in C(\mathbb{R}, H^1(\mathbb{R}^d)) \cap C^1(\mathbb{R}, L^2(\mathbb{R}^d))$, under the assumption $p < 1 + 4/(d-2)$. For the critical case $p = 1 + 4/(d-2)$ we refer to [106], [109] and [110].

Difference equations and differential-difference equations, have recently raised a lot of interest in the physics literature. This is due, in part, to the fact that they constitute a natural way to approach numerically real physical situations, but also because there are many models based on such kinds of equations, for instance applications to dissipative systems, to nuclear physics [31], and to the study of phonons [29] and magnons [30]. However, some work has already been developed concerning the symmetries of linear difference equations on geometric lattices (q-lattices).

The so called nonlinear lattices are a class of nonlinear dynamical systems (sometimes also allowing solitonic solutions). These are systems of (a great number of) coupled ordinary differential equations. They arise naturally in the spatial discretization of nonlinear partial differential equations, which is required for their numerical integration, and in a variety of physical phenomena including the dynamic description of solids, wave propagation in periodic media, etc. The latter are examples of systems ranging from molecular crystals to interacting biological species that are modeled directly in terms of nonlinear lattices, meaning that we are considering a spatial grid, instead of continuous space coordinates. The development of analytic tools for these systems, and the relation between their properties and those of their continuous counterparts, are a source of interesting problems.

Let us first consider the difference scheme

$$\begin{cases} \dfrac{d^2u^h}{dt^2} - \Delta_h u^h = F^h, & t > 0, \\ u^h_{\mathbf{j}}(0) = u^h_{0,\mathbf{j}},\, u^h_t(0) = u^h_{1,\mathbf{j}}, & \mathbf{j} \in \mathbb{Z}^d. \end{cases} \tag{5.1.6}$$

Here u^h stands for the infinite vector unknown $\{u^h_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d}$, $u^h_{\mathbf{j}}(t)$ being the approximation of the solution at the node $x_{\mathbf{j}} = h\mathbf{j}$, and Δ_h the classical second order finite difference approximation of Δ_x^2:

$$(\Delta_h u^h)_{\mathbf{j}} = h^{-2}\sum_{k=1}^{d}(u^h_{\mathbf{j}+he_k} + u^h_{\mathbf{j}-he_k} - 2u^h_{\mathbf{j}}).$$

This scheme can be viewed as an approximation of the continuous model and also as a wave equation on the lattice $h\mathbb{Z}^d$. Our main purpose in this paper is to analyze whether the numerical approximation scheme (5.1.6) has the same dispersive properties (5.1.2), uniformly with respect to the mesh-size h. From a numerical point of view it is important that such estimates are uniform with respect to the mesh size. If one looks only for the properties of the discrete wave equation on the lattice $h\mathbb{Z}^d$ without taking care of the uniformity on h, the same results are obtained in a large class of integrability spaces.

Lions [85] (Ch. 1, p.9) proved the existence of a weak solution of (5.1.5) using Faedo-Galerkin's method. He introduced an approximation $\{(u_n, u_{n,t})\}_{n\geq 0}$ of (5.1.5), uniformly bounded in the space $L^\infty((0,T); H^1(\mathbb{R}^3)\bigoplus L^2(\mathbb{R}^3))$ for all $T>0$. These estimates allow to obtain in the limit a solution of (5.1.5): $u\in L^\infty_{loc}(\mathbb{R}, H^1(\mathbb{R}^3))$ with $u_t \in L^\infty_{loc}(\mathbb{R}, L^2(\mathbb{R}^3))$. In that case the uniqueness can be established only for $p\leq 3$ as proved by [70], [85].

In this chapter we introduce a numerical scheme for (5.1.5). The analysis of our scheme will provide an approximation u^h which belongs to the above mentioned spaces. More than that, u^h will belong to some auxiliary space $L^q_{loc}(\mathbb{R}, L^r(\mathbb{R}^3))$ where the uniqueness of the solutions of (5.1.5) can be established by the arguments of [49] for $p<5$.

This well-posedness result may not be proved simply as a consequence of energy estimates and the dispersive properties of the LWE play a key role. In order to obtain results similar to (5.1.2) we need to introduce the corresponding $\dot{H}^s$-norms at the discrete level. We define the $\dot{\hbar}^s(h\mathbb{Z}^d)$-spaces as:

$$\dot{\hbar}^s(h\mathbb{Z}^d) = \left\{ \{u_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^d} : \|u\|_{\dot{\hbar}^s(h\mathbb{Z}^d)} = \left(\int_{[-\pi/h,\pi/h]^d} |p_h(\xi)|^{2s}|\widehat{u}(\xi)|^2 d\xi\right)^{1/2} < \infty \right\},$$

where

$$\widehat{u}(\xi) = h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} e^{i\mathbf{j}\cdot\xi h} u_{\mathbf{j}}$$

is the semidiscrete Fourier's transform of the function u.

We first consider the wave equation on the lattice $\mathbb{Z}^d$. Once we establish dispersive properties similar to (5.1.2) on the lattice $\mathbb{Z}^d$, by a scaling argument we obtain the same results on the lattice $h\mathbb{Z}^d$. In order to obtain uniform estimates with respect to the mesh size h we have to assume additional hypotheses on the involved spaces.

The main result says that a dispersive estimate, similar to (5.1.2), holds for the semidiscretization (5.1.6).

Theorem 5.1.1. *Let $h=1$, (q,r) and $(\tilde{q},\tilde{r})$ be two $1/2$-wave admissible pairs. Assume that*

$$\frac{d}{r} + \frac{1}{q} \leq \frac{d}{2} - s \tag{5.1.7}$$

and

$$\frac{d}{2} - s + 2 \leq \frac{d}{\tilde{r}'} + \frac{1}{\tilde{q}'}. \tag{5.1.8}$$

Then any solution u^1 *of (5.1.6) satisfies*

$$\|u^1\|_{L^q(\mathbb{R},\, l^r(\mathbb{Z}^d))} + \|u^1\|_{C(\mathbb{R},\, \dot{\hbar}^s(\mathbb{Z}^d))} + \|u^1_t\|_{C(\mathbb{R},\, \dot{\hbar}^{s-1}(\mathbb{Z}^d))} \lesssim \|u^1_0\|_{\dot{\hbar}^s(\mathbb{Z}^d)} + \|u^1_1\|_{\dot{\hbar}^{s-1}(\mathbb{Z}^d)} + \|F^1\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(\mathbb{Z}^d))}. \tag{5.1.9}$$

Remark 5.1.1. *In contrast with the continuous case, we have to assume that* (q, r) *and* $(\tilde{q}, \tilde{r})$ *are* $1/2$*-admissible pairs. In the continuous case the hessian of the symbol* $|\xi|$ *has rank* $d-1$ *at any point* $\xi \neq 0$*. In the semidiscrete case the rank of* $H_{p_1^{1/2}}$ *is at least one for all* $\xi \neq 0$*. Moreover,* $\operatorname{rank}(H_{p_1^{1/2}}(\pi/2, \dots, \pi/2)) = 1$.

Remark 5.1.2. *In contrast with the continuous case, in this case the Fourier analysis involves only the frequencies in the range* $[-\pi, \pi]^d$*. This is why we only need to assume the restrictions (5.1.7) and (5.1.8). In the continuous case, the analysis in the whole range of frequencies* $\mathbb{R}^d$ *requires equalities in (5.1.7) and (5.1.8).*

Remark 5.1.3. *In the case of the Schrödinger equation the symbol is given by* $|\xi|^2$ *and its Hessian matrix satisfies* $H_{|\xi|^2} = 2I_d$*. Its rank equals* d *at all points* ξ*. Its semidiscrete counterpart* $H_{p_1} = O_d$ *at the points* $\xi = (\pm\pi/2)^d$*. This is why we introduced various filtering methods in Chapter 3.*

Concerning the wave equation on the lattice $h\mathbb{Z}^d$, by a rescaling argument a similar result can be stated:

Theorem 5.1.2. *Let* $h > 0$*,* (q, r) *and* $(\tilde{q}, \tilde{r})$ *be as in the theorem above. Then*

$$\|u^h\|_{L^q(\mathbb{R},\, l^r(h\mathbb{Z}^d))} \lesssim h^{\frac{d}{r}+\frac{1}{q}+s-\frac{d}{2}}(\|u^h_0\|_{\dot{\hbar}^s(h\mathbb{Z}^d)} + \|u^h_1\|_{\dot{\hbar}^{s-1}(h\mathbb{Z}^d)}) + h^{\frac{d}{r}+\frac{1}{q}+2-\frac{d}{\tilde{r}'}-\frac{1}{\tilde{q}'}}\|F^h\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))}. \tag{5.1.10}$$

and

$$\|u^h\|_{C(\mathbb{R},\, \dot{\hbar}^s(h\mathbb{Z}^d))} + \|u^h_t\|_{C(\mathbb{R},\, \dot{\hbar}^{s-1}(h\mathbb{Z}^d))} \lesssim \|u^h_0\|_{\dot{\hbar}^s(h\mathbb{Z}^d)} + \|u^h_1\|_{\dot{\hbar}^{s-1}(h\mathbb{Z}^d)} + h^{\frac{d}{2}-s+2-\frac{d}{\tilde{r}'}-\frac{1}{\tilde{q}'}}\|F^h\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(h\mathbb{Z}^d))}. \tag{5.1.11}$$

Remark 5.1.4. *From a numerical point of view, it is convenient that all the estimates be uniform with respect to the mesh size* h*. Thus we have to impose that all the exponents of* h *in (5.1.10) and (5.1.11) are positive:*

$$\begin{cases} \dfrac{d}{r} + \dfrac{1}{q} + s - \dfrac{d}{2} \geq 0, \\[2ex] \dfrac{d}{r} + \dfrac{1}{q} + 2 - \dfrac{d}{\tilde{r}'} - \dfrac{1}{\tilde{q}'} \geq 0, \\[2ex] \dfrac{d}{2} - s + 2 - \dfrac{d}{\tilde{r}'} - \dfrac{1}{\tilde{q}'} \geq 0. \end{cases}$$

In view of (5.1.7) and (5.1.8):

$$\frac{d}{r} + \frac{1}{q} = \frac{d}{2} - s = \frac{d}{\tilde{r}'} + \frac{1}{\tilde{q}'} - 2. \tag{5.1.12}$$

On the lattice $h\mathbb{Z}^d$, the Fourier analysis involves the range of frequencies $[-\pi/h, \pi/h]^d$. As h goes to zero, this range is each time larger and in the limit one recovers the whole Fourier space $\mathbb{R}^d$. Then it is natural to have similar restrictions as in the continuous case.

Remark 5.1.5. *Let $\eta \in C_c^\infty(\mathbb{R}^d)$ be such that $\eta = 1$ on $B_1(0)$ and $\eta = 0$ outside $B_2(0)$ and define $\beta(\xi) = \eta(\xi) - \eta(2\xi)$, $\rho_j(\xi) = \beta(\xi/2^j)$, $j \in \mathbb{Z}$. Then any $s \in \mathbb{R}$ we define the homogenous Besov norm by letting:*

$$\|\varphi\|_{\dot{B}^s_{1,1}(\mathbb{R}^d)} = \sum_{j=-\infty}^{\infty} 2^{js} \|(\rho_j \widehat{\varphi})^\vee\|_{L^1(\mathbb{R}^d)}.$$

In the continuous case all the above estimates comes from the energy estimate

$$\|e^{it\sqrt{-\Delta}}\varphi\|_{L^2(\mathbb{R}^d)} = \|\varphi\|_{L^2(\mathbb{R}^d)}$$

and the pointwise estimate

$$\|e^{it\sqrt{-\Delta}}\varphi\|_{L^\infty(\mathbb{R}^d)} \le |t|^{-(d-1)/2} \|\varphi\|_{\dot{B}^{\frac{d+1}{2}}_{1,1}(\mathbb{R}^d)}. \tag{5.1.13}$$

For any φ with its Fourier transform having compact support far away from zero the following inequality also holds:

$$\|e^{it\sqrt{-\Delta}}\varphi\|_{L^\infty(\mathbb{R}^d)} \le \frac{C\|\varphi\|_{L^1(\mathbb{R}^d)}}{(1+|t|)^{(d-1)/2}},$$

where the constant C depend by the support of $\widehat{\varphi}$.

We sketch the proof of (5.1.13). Let us choose a smooth function γ supported away from zero such that $\beta\gamma = \beta$ and set $\gamma_j = \gamma(\xi/2^j)$. Then

$$\varphi = \sum_{j \in \mathbb{Z}} (\gamma_j \rho_j \widehat{\varphi})^\vee$$

and remains to prove that

$$\|e^{it\sqrt{-\Delta}}(\gamma_j \rho_j \widehat{\varphi})^\vee\|_{L^\infty(\mathbb{R}^d)} \le |t|^{(d-1)/2} 2^{j(d+1)/2} \|(\rho_j \widehat{\varphi})^\vee\|_{L^\infty(\mathbb{R}^d)}.$$

In view of Young's inequality it is sufficient to prove that

$$\sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{it|\xi|} e^{ix\xi} \gamma\left(\frac{\xi}{2^j}\right) d\xi \right| \le |t|^{-(d-1)/2} 2^{j(d+1)/2},$$

or by a change of variables:

$$\sup_{x \in \mathbb{R}^d} \left| \int_{\mathbb{R}^d} e^{i\tau|\eta|} e^{ix\eta} \gamma(\eta) d\eta \right| \le |\tau|^{-(d-1)/2}$$

where $\tau = 2^j t$. The last estimate follows by applying classical restriction results (see Lemma 5.3.2).

By means of semidiscrete Fourier transform we also could define the discrete Besov norms:

$$\|\varphi\|_{\dot{B}^s_{1,1}(h\mathbb{Z}^d)} = \sum_{j=-\infty}^{\infty} 2^{js} \|(\rho_j \widehat{\varphi})^\vee\|_{l^1(h\mathbb{Z}^d)}.$$

Then the following also holds:

$$\|e^{it\sqrt{-\Delta}}\varphi\|_{l^\infty(h\mathbb{Z}^d)} \leq |t|^{-1/2}\|\varphi\|_{\dot{B}_{1,1}^{d-\frac{1}{2}}(h\mathbb{Z}^d)}. \tag{5.1.14}$$

The decay rate is different from the continuous case and it holds uniformly for φ in $\dot{B}_{1,1}^{d-\frac{1}{2}}(h\mathbb{Z}^d)$. Estimate (5.1.14) is a consequence of the following result that is obtained in the same manner as estimate (5.3.8) in the proof of Lemma (5.3.1) (see Section 5.3):

Lemma 5.1.1. *There exists a constant C such that*

$$\sup_{0\leq\alpha\leq\pi}\left|\int_{\xi\sim 1}\exp\left(i\tau\Big(\alpha^{-2}\sum_{k=1}^{d}\sin^2(\frac{\alpha\xi_k}{2})\Big)^{1/2}\right)e^{ix\xi}d\xi\right| \leq \frac{C}{(1+|\tau|)^{1/2}}$$

holds for all $\tau\in\mathbb{R}$.

The difference with the continuous case comes from the fact that for j such that $\operatorname{supp}(\rho_{\mathrm{j}})$ *contains the points $(\pm\pi/2h)$ the decay rate $|t|^{-1/2}$ in (5.1.14) cannot be improved.*

In fact (5.1.14) is reduced, by using similar arguments an in the continuous case, to the following uniform (on h) estimate:

$$\sup_{x\in\mathbb{R}^d}\left|\int_{\xi\sim 2^j} e^{itp_1(\xi h)/h}e^{ix\xi}d\xi\right| \lesssim \frac{2^{j(d-1/2)}}{|t|^{1/2}},\ \forall\, j,\ 2^j\leq\pi/h.$$

The last estimate is equivalent with the following one:

$$\left|\int_{\xi\sim 1}\exp\left(i\frac{t}{h}p_1\left(\frac{2^j\xi}{h}\right)\right)\exp(i2^jx\xi)d\xi\right| \lesssim \frac{2^{-j/2}}{|t|^{1/2}}$$

which follows from Lemma 5.1.1 applied to $\alpha = 2^j/h$ and $\tau = t/2^j$.

5.2. Proof of the main result

Let us consider the wave equation on the lattice $h\mathbb{Z}^d$:

$$\begin{cases} \dfrac{d^2u^h(t)}{dt^2} - \Delta_h u^h(t) = F^h(t), \quad t>0, \\[2ex] u^h_{\mathbf{j}}(0) = u^h_{0,\mathbf{j}}, (\partial_t u)_{\mathbf{j}}(0) = u^h_{1,\mathbf{j}}, \quad \mathbf{j}\in\mathbb{Z}^d. \end{cases} \tag{5.2.1}$$

Let us define $v(t) = u(th)$. Then v is the solution of the wave equation on the lattice $\mathbb{Z}^d$:

$$\begin{cases} \dfrac{d^2v(t)}{dt^2} - \Delta_1 v(t) = h^2F^h(th), \quad t>0, \\[2ex] v_{\mathbf{j}}(0) = u^h_{0,\mathbf{j}},\ (\partial_t v)_{\mathbf{j}}(0) = hu^h_{1,\mathbf{j}}, \quad \mathbf{j}\in\mathbb{Z}^d. \end{cases}$$

Using the results on the lattice $\mathbb{Z}^d$ we can rescale all the norms in (5.1.10) and (5.1.11) to obtain the same results on the lattice $h\mathbb{Z}^d$. This reduces all the estimates to the case of the lattice $\mathbb{Z}^d$.

Proof of Theorem 5.1.1. In the following we concentrate on the wave equation on the lattice $\mathbb{Z}^d$. To simplify the presentation we get rid of the parameter h in our notation, unless it is necessary.

We consider the following equation

$$\begin{cases} \dfrac{d^2u(t)}{dt^2} - \Delta_1 u(t) = F(t), & t > 0, \\ \\ u_{\mathbf{j}}(0) = u_{0,\mathbf{j}},\ (\partial_t u)_{\mathbf{j}}(0) = u_{1,\mathbf{j}}, & \mathbf{j} \in \mathbb{Z}^d. \end{cases} \tag{5.2.2}$$

and prove (5.1.9).

The main ingredient of the proof is the semidiscrete Fourier transform. Applying the semidiscrete Fourier Transform to (5.2.2) we obtain

$$\begin{cases} \widehat{u}_{tt}(\xi) + 4\left(\displaystyle\sum_{k=1}^{d} \sin^2 \frac{\xi_k}{2}\right)\widehat{u}(\xi) = \widehat{F}(t,\xi), & t > 0,\ \xi \in [-\pi,\pi]^d, \\ \\ \widehat{u}(0,\xi) = \widehat{u}_0(\xi),\ \widehat{u}_t(0,\xi) = \widehat{u}_1(\xi), & \xi \in [-\pi,\pi]^d. \end{cases}$$

Solving this ODE we find the explicit expression of the solution u:

$$\begin{aligned} u_{\mathbf{j}}(t) = \int_{[-\pi,\pi]^d} \frac{1}{2}\left(\widehat{u_0}(\xi) - i\frac{\widehat{u_1}(\xi)}{q(\xi)}\right) e^{itq(\xi)} e^{i\mathbf{j}\cdot\xi} d\xi + \int_{[-\pi,\pi]^d} \frac{1}{2}\left(\widehat{u_0}(\xi) + i\frac{\widehat{u_1}(\xi)}{q(\xi)}\right) e^{-itq(\xi)} e^{i\mathbf{j}\cdot\xi} d\xi \\ + \int_0^t \int_{[-\pi,\pi]^d} \frac{e^{i(t-s)q(\xi)} - e^{-i(t-s)q(\xi)}}{q(\xi)} \widehat{F}(s,\xi) e^{i\mathbf{j}\cdot\xi} d\xi \end{aligned}$$

where $q(\xi) = p_1^{1/2}(\xi) = 2(\sum_{k=1}^{d} \sin^2(\xi_k/2))^{1/2}$ and $\mathbf{j} \in \mathbb{Z}^d$.

Using that $\|(\widehat{g}q)^\vee\|_{\dot{h}^s(\mathbb{Z}^d)} = \|g\|_{\dot{h}^{s-1}(\mathbb{Z}^d)}$ it is sufficient to consider the case $g \equiv 0$. In this case

$$\begin{aligned} u_{\mathbf{j}}(t) = \frac{1}{2}\int_{[-\pi,\pi]^d} \widehat{u_0}(\xi) e^{itq(\xi)} e^{i\mathbf{j}\cdot\xi} d\xi + \frac{1}{2}\int_{[-\pi,\pi]^d} \widehat{u_0}(\xi) e^{-itq(\xi)} e^{i\mathbf{j}\cdot\xi} d\xi \\ + \int_0^t \int_{[-\pi,\pi]^d} \frac{e^{i(t-s)q(\xi)} - e^{-i(t-s)q(\xi)}}{q(\xi)} \widehat{F}(s,\xi) e^{i\mathbf{j}\cdot\xi} d\xi. \end{aligned}$$

We will prove that

$$\|u\|_{L^q(\mathbb{R},\, l^r(\mathbb{Z}^d))} + \|u\|_{C(\mathbb{R},\, \dot{h}^s(\mathbb{Z}^d))} \lesssim \|u_0\|_{\dot{h}^s(\mathbb{Z}^d)} + \|F\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(\mathbb{Z}^d))}.$$

In fact we prove that the operators $e^{\pm it\sqrt{-\Delta_1}}$, defined by

$$(e^{\pm it\sqrt{-\Delta_1}} f)_{\mathbf{j}} = \int_{[-\pi,\pi]^d} e^{\pm itq(\xi)} e^{i\mathbf{j}\xi} \widehat{f}(\xi) d\xi, \tag{5.2.3}$$

satisfy

$$\|e^{\pm it\sqrt{-\Delta_1}} f\|_{L^q(\mathbb{R},\, l^r(\mathbb{Z}^d))} + \|e^{\pm it\sqrt{-\Delta_1}} f\|_{C(\mathbb{R},\, \dot{h}^s(\mathbb{Z}^d))} \lesssim \|f\|_{\dot{h}^s(\mathbb{Z}^d)} \tag{5.2.4}$$

and

$$\begin{aligned} \left\| \int_{s<t} \frac{e^{\pm i(t-s)\sqrt{-\Delta_1}}}{\sqrt{-\Delta_1}} F(s) ds \right\|_{L^q(\mathbb{R},\, l^r(\mathbb{Z}^d))} + \left\| \int_{s<t} \frac{e^{\pm i(t-s)\sqrt{-\Delta_1}}}{\sqrt{-\Delta_1}} F(s) ds \right\|_{C(\mathbb{R},\, \dot{h}^s(\mathbb{Z}^d))} \\ \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(\mathbb{Z}^d))}. \end{aligned} \tag{5.2.5}$$

Step I. Estimates on $\|e^{\pm it\sqrt{-\Delta_1}}f\|_{C(\mathbb{R},\,\dot{h}(\mathbb{Z}^d))}$.

The continuity of $e^{\pm it\sqrt{-\Delta_1}}$ in $\dot{h}^s(\mathbb{Z}^d)$ easily follows by Plancherel's Theorem. Let us set $G_\pm F$ as

$$G_\pm F(t) = \int_{s<t} \frac{e^{\pm i(t-s)\sqrt{-\Delta_1}}}{\sqrt{-\Delta_1}} F(s)ds. \tag{5.2.6}$$

Assume for the moment that the following holds:

$$\|G_\pm F(t)\|_{L^q(\mathbb{R},\,l^r(\mathbb{Z}^d))} + \|G_\pm F(t)\|_{L^\infty(\mathbb{R},\,\dot{h}^s(\mathbb{Z}^d))} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R},\,l^{\tilde{r}'}(\mathbb{Z}^d))}. \tag{5.2.7}$$

To show that $G_\pm F$ is continuous in $\dot{h}^s(\mathbb{Z}^d)$ one can use the identity

$$(G_\pm F)(t+\epsilon) = e^{i\epsilon\sqrt{-\Delta_1}} G_\pm F(t) + G_\pm(\chi_{[t,t+\epsilon]}F)(t),$$

the continuity of $e^{i\epsilon t\sqrt{-\Delta_1}}$ on $\dot{h}^s(\mathbb{Z}^d)$ and the fact that

$$\|\chi_{[t,t+\epsilon]}F\|_{L^{\tilde{q}'}(\mathbb{R},\,l^{\tilde{r}'}(\mathbb{Z}^d))} \to 0, \text{ as } \epsilon \to 0.$$

Step II. Proof of (5.2.7).

Observe that $\|e^{\pm it\sqrt{-\Delta_1}}f\|_{L^q(\mathbb{R},\,l^r(\mathbb{Z}^d))} \lesssim \|f\|_{\dot{h}^s(\mathbb{Z}^d)}$ implies by duality

$$\left\| \int_{\mathbb{R}} e^{\mp is\sqrt{-\Delta_1}} F(s)ds \right\|_{\dot{h}^{-s}(\mathbb{Z}^d)} \leq \|F\|_{L^{q'}(\mathbb{R},\,l^{r'}(\mathbb{Z}^d))} \tag{5.2.8}$$

for all q, r and s satisfying (5.1.7).

In the following we prove that the estimates on the homogenous part imply the same estimates for the inhomogeneous part. This is a consequence of the argument of Christ and Kiselev. A simplified version, useful in PDE applications, is given in [115] :

Lemma 5.2.1. *Let X and Y be Banach spaces and assume that $K(t,s)$ is a continuous function taking its values in $B(X,Y)$, the space of bounded linear mappings from X to Y. Suppose that $-\infty \leq a < b \leq \infty$ and set*

$$Tf(t) = \int_a^b K(t,s)f(s)ds, \quad Wf(t) = \int_a^t K(t,s)f(s)ds.$$

Assume that $1 \leq p < q \leq \infty$ and

$$\|Tf\|_{L^q([a,b],Y)} \leq \|f\|_{L^p([a,b],X)}.$$

Then

$$\|Wf\|_{L^q([a,b],Y)} \leq \|f\|_{L^p([a,b],X)}.$$

In view of the above Lemma, (5.2.7) turns to be equivalent to

$$\left\| \int_{\mathbb{R}} \frac{e^{\pm i(t-s)\sqrt{-\Delta_1}}}{\sqrt{-\Delta_1}} F(s)ds \right\|_{L^q(\mathbb{R},\,l^r(\mathbb{Z}^d))} + \left\| \int_{\mathbb{R}} \frac{e^{\pm i(t-s)\sqrt{-\Delta_1}}}{\sqrt{-\Delta_1}} F(s)ds \right\|_{L^\infty(\mathbb{R},\,\dot{h}^s(\mathbb{Z}^d))} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R},\,l^{\tilde{r}'}(\mathbb{Z}^d))}.$$

We will prove that each of the above left hand terms is upper bounded by $\|F\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(\mathbb{Z}^d))}$. Remark that the second term satisfies:

$$\begin{aligned}
\left\|\int_{\mathbb{R}} \frac{e^{\pm i(t-s)\sqrt{-\Delta_1}}}{\sqrt{-\Delta_1}} F(s)ds\right\|_{L^\infty(\mathbb{R},\, \dot{\hbar}^s(\mathbb{Z}^d))} &= \left\|e^{\pm it\sqrt{-\Delta_1}} \int_{\mathbb{R}} \frac{e^{\mp is\sqrt{-\Delta_1}}}{\sqrt{-\Delta_1}} F(s)ds\right\|_{\dot{\hbar}^s(\mathbb{Z}^d)} \\
&= \left\|\int_{\mathbb{R}} \frac{e^{\mp is\sqrt{-\Delta_1}}}{\sqrt{-\Delta_1}} F(s)ds\right\|_{\dot{\hbar}^s(\mathbb{Z}^d)} \\
&= \left\|\int_{\mathbb{R}} e^{\mp is\sqrt{-\Delta_1}} F(s)ds\right\|_{\dot{\hbar}^{s-1}(\mathbb{Z}^d)}.
\end{aligned}$$

Applying (5.2.8) we get

$$\left\|\int_{\mathbb{R}} e^{\mp is\sqrt{-\Delta_1}} F(s)ds\right\|_{\dot{\hbar}^{s-1}(\mathbb{Z}^d)} \lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(\mathbb{Z}^d))}$$

provided that

$$\frac{d}{\tilde{r}} + \frac{1}{\tilde{q}} \leq \frac{d}{2} - (1-s) \tag{5.2.9}$$

or, equivalently, (5.1.8).

For the first term we have

$$\begin{aligned}
\left\|\int_{\mathbb{R}} \frac{e^{\pm i(t-s)\sqrt{-\Delta_1}}}{\sqrt{-\Delta_1}} F(s)ds\right\|_{L^q(\mathbb{R},\, l^r(\mathbb{Z}^d))} &= \left\|e^{\pm it\sqrt{-\Delta_1}} \int_{\mathbb{R}} \frac{e^{\mp is\sqrt{-\Delta_1}}}{\sqrt{-\Delta_1}} F(s)ds\right\|_{L^q(\mathbb{R},\, l^r(\mathbb{Z}^d))} \\
&\lesssim \left\|\int_{\mathbb{R}} \frac{e^{\mp is\sqrt{-\Delta_1}}}{\sqrt{-\Delta_1}} F(s)ds\right\|_{\dot{\hbar}^s(\mathbb{Z}^d)} \\
&= \left\|\int_{\mathbb{R}} e^{\mp is\sqrt{-\Delta_1}} F(s)ds\right\|_{\dot{\hbar}^{s-1}(\mathbb{Z}^d)} \\
&\lesssim \|F\|_{L^{\tilde{q}'}(\mathbb{R},\, l^{\tilde{r}'}(\mathbb{Z}^d))}
\end{aligned}$$

for all $q, r, s, \tilde{q}, \tilde{r}$ which satisfy (5.1.7) and (5.1.8).

Step III. Estimates on $\|e^{\pm it\sqrt{-\Delta_1}} f\|_{L^q(\mathbb{R},\, l^r(\mathbb{Z}^d))}$.

We will prove that

$$\|e^{\pm it\sqrt{-\Delta_1}} f\|_{L^q(\mathbb{R},\, l^r(\mathbb{Z}^d))} \lesssim \|f\|_{\dot{\hbar}^s(\mathbb{Z}^d)}. \tag{5.2.10}$$

We consider the case $e^{it\sqrt{\Delta_1}}$, the second one being similar. Let us introduce the operator

$$(Tf)(t,x) = \int_{[-\pi,\pi]^d} e^{-itq(\xi)} e^{ix\cdot\xi} \widehat{f}(\xi) d\xi, t \in \mathbb{R}, x \in \mathbb{R}^d.$$

In view of (5.2.3), $(Tf)(t)$ is the band-limited interpolator of $e^{-it\sqrt{-\Delta_1}} f$. Similar arguments to the ones in [101] and [90] guarantee that:

$$\|e^{\pm it\sqrt{-\Delta_1}} f\|_{L^q(\mathbb{R},\, l^r(\mathbb{Z}^d))}\| \lesssim \|Tf\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))}$$

and

$$\|(Tf)(0)\|_{\dot{H}^s(\mathbb{R}^d)} \lesssim \|f\|_{\dot{h}^s(\mathbb{Z}^d)}.$$

Thus (5.2.10) is reduced to the following estimate on the operator T:

$$\|Tf\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^d)}. \tag{5.2.11}$$

We first prove (5.2.11) for frequency-localized functions f, and then obtain the general case by using Paley-Littlewood theory. The main difficulty is given by the lack of homogeneity of the symbol $q(\xi)$. Remark that, in the continuous case, the symbol associated with the wave equation is given by $|\xi|$ which is one degree homogenous. In order to avoid this lack of homogeneity we define the following class of symbols

$$p^j(\xi) = 2^j q\left(\frac{\xi}{2^j}\right),\ j \geq 0,\ \xi \in [-\pi,\pi]^d.$$

We point out that in the continuous case $p^j(\xi) = |\xi|$, for all $j \in \mathbb{Z}$.

We fix a radial cutoff function $\beta \in C_c^\infty([-\pi,\pi]^d)$ supported away from zero, and consider the *truncated cone operators*

$$(T_\beta^j f)(t,x) = \int_{[-\pi,\pi]^d} e^{ix\cdot\xi} e^{itp^j(\xi)} \beta(\xi)\widehat{f}(\xi) d\xi,\ j \geq 0. \tag{5.2.12}$$

Remark that

$$(T_\beta^j)(t,x) = (U_\beta^j(t)f)(x),$$

where

$$\widehat{U^j(t)f}(\xi) = e^{itp^j(\xi)}\beta(\xi)\widehat{f}(\xi)\chi_{[-\pi,\pi]^d}.$$

Using the results of Section 5.3, Corollary 5.3.1 i), on the operators $U^j(t)$ we obtain the existence of a constant $C(\beta)$, independent of j, such that

$$\|T_\beta^j f\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \leq C(\beta)\|f\|_{L^2(\mathbb{R}^d)} \tag{5.2.13}$$

for all $j \geq 0$.

The last estimate allows us to use a Paley-Littlewood decomposition. For that let us choose a radial bump function γ such that

$$\begin{cases} \gamma(\xi) \equiv 1, & |\xi| \leq \frac{1}{2}, \\ \operatorname{supp}\gamma \subset \{|\xi| \leq 1\}, & \\ \gamma(\xi) \equiv 0, & |\xi| \geq 1. \end{cases}$$

Set $\beta(\xi) = \gamma(\xi) - \gamma(2\xi)$. Then $\beta(\xi) \equiv 0$ if $|\xi| \leq \frac{1}{4}$ or $|\xi| \geq 1$ and $\operatorname{supp}\beta \subset \{1/4 \leq |\xi| \leq 1\}$. Also, for any $j \geq 0$,

$$\operatorname{supp}\beta(2^j\xi) \subset \left\{\frac{1}{2^{j+2}} \leq |\xi| \leq \frac{1}{2^j}\right\}$$

and

$$\sum_{j\geq 0}\beta(2^j\xi) = \sum_{j\geq 0}(\gamma(2^j\xi) - \gamma(2^{j+1}\xi)) = \gamma(\xi).$$

We split the operator T as

$$Tf(t) = T_1f(t) + T_2f(t)$$

where

$$\widehat{T_1f(t)}(\xi) = \widehat{Tf(t)}(\xi)\chi_{(|\xi|\leq 1/2)} \text{ and } \widehat{T_2f(t)}(\xi) = \widehat{Tf(t)}(\xi)(1-\chi_{(|\xi|\leq 1/2)}).$$

It is sufficient to prove (5.2.11) for T_1 and T_2.

In the case of T_2, using that $\widehat{f}(1-\chi_{(|\xi|\leq 1/2)})$ is localized far from zero, inequality (5.2.13) gives us

$$\|T_2f\|_{L^q(\mathbb{R}),L^r(\mathbb{R}^d)} \lesssim \|(\widehat{f}(1-\chi_{|\xi|\leq 1/2})\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^d)}.$$

Step. IV. $\|T_1f\|_{L^q(\mathbb{R},\,L^r(\mathbb{R}^d))} \lesssim \|f\|_{\dot{H}^s(\mathbb{R}^d)}$.

We choose a dyadic decomposition generated by the function β. Define the frequency projections Δ_j, $j\geq 0$, by

$$\widehat{\Delta_j f}(\xi) = \beta(2^j\xi)\widehat{f}(\xi).$$

We claim that for all $j \geq 0$ and for any function g the following holds

$$\|T_1(\Delta_j g)\|_{L^q(\mathbb{R},\,L^r(\mathbb{R}^d))} \lesssim 2^{-js}\|g\|_{L^2(\mathbb{R}^d)}. \tag{5.2.14}$$

We postpone its proof until the end of the proof.

The definition of the projectors Δ_j guarantee that

$$\sum_{j\geq 0}\widehat{\Delta_j f}(\xi) = \widehat{f}(\xi)\gamma(\xi) = \widehat{f}(\xi), \text{ for all } |\xi| \leq \frac{1}{2}.$$

Remark that

$$\begin{aligned}
\widehat{T_1(t)f}(\xi) &= e^{itq(\xi)}\widehat{f}(\xi)\chi_{(|\xi|\leq 1/2)} = e^{itq(\xi)}\chi_{(|\xi|\leq 1/2)}\left(\sum_{j\geq 0}\widehat{\Delta_j f}(\xi)\right) \\
&= \sum_{j\geq 0} e^{itq(\xi)}\widehat{\Delta_j f}(\xi)\chi_{(|\xi|\leq 1/2)} = \sum_{j\geq 0}(T_1(t)\Delta_j f)\widehat{\ }(\xi)
\end{aligned}$$

and

$$\begin{aligned}
(T_1(t)\Delta_j f)\widehat{\ }(\xi) &= e^{itq(\xi)}\chi_{(|\xi|\leq 1/2)}\widehat{\Delta_j f}(\xi) = e^{itq\xi)}\chi_{(|\xi|\leq 1/2)}\beta(2^j\xi)\widehat{f}(\xi) \\
&= e^{itq(\xi)}\chi_{(|\xi|\leq 1/2)}\beta(2^j\xi)\sum_{k\geq 0}\widehat{\Delta_k f}(\xi) = e^{itq(\xi)}\chi_{(|\xi|\leq 1/2)}\beta(2^j\xi)\sum_{k\geq 0}\beta(2^k\xi)\widehat{f}(\xi) \\
&= e^{itq(\xi)}\chi_{(|\xi|\leq 1/2)}\sum_{|k-j|\leq 2}\beta(2^j\xi)\beta(2^k\xi)\widehat{f}(\xi) = \sum_{|k-j|\leq 2}(T_1(t)\Delta_j\Delta_k f)\widehat{\ }(\xi).
\end{aligned}$$

Using that $T_1 f = \sum_{j\geq 0}\Delta_j(T_1 f)$ and classical estimates for the Hardy-Litlewood decomposition we obtain:

$$\begin{aligned}
\|T_1f\|_{L^q(\mathbb{R},\,L^r(\mathbb{R}^d))} &\lesssim \left\|\sqrt{\sum_{j\geq 0}\|\Delta_j(T_1f)\|^2_{L^r(\mathbb{R}^d)}}\right\|_{L^q(\mathbb{R})} = \left\|\sqrt{\sum_{j\geq 0}\|T_1(\Delta_jf)\|^2_{L^r(\mathbb{R}^d)}}\right\|_{L^q(\mathbb{R})} \\
&\lesssim \sqrt{\sum_{j\geq 0}\|T_1(\Delta_jf)\|^2_{L^q(\mathbb{R},\,L^r(\mathbb{R}^d))}},
\end{aligned}$$

where the last inequality follows by Minkowski's integral inequality since $q \geq 2$.

Finally by (5.2.14) we get

$$
\begin{aligned}
\|T_1 f\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} &\leq \sqrt{\sum_{j\geq 0} \|T_1(\Delta_j f)\|^2_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))}} \\
&= \sqrt{\sum_{j\geq 0}\sum_{|k-j|\leq 2} \|T_1(\Delta_k f)\|^2_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))}} \\
&\lesssim \sqrt{\sum_{j\geq 0}\sum_{|j-k|\leq 2} 2^{-2js}\|\Delta_k f\|^2_{L^2(\mathbb{R}^d)}} \\
&\lesssim \sqrt{\sum_{j\geq 0} \|\Delta_j f\|^2_{\dot{H}^s(\mathbb{R}^d)}} \\
&\lesssim \|(\widehat{f}\chi_{(|\xi|\leq 1/2)})^\vee\|_{\dot{H}^s(\mathbb{R}^d)} \leq \|f\|_{\dot{H}^s(\mathbb{R}^d)}.
\end{aligned}
$$

which finishes the proof.

Step V. Proof of (5.2.14).

We distinguish two cases $j = 0$ and $j \geq 1$. The difference occurs from the fact that for all $j \geq 1$, the support of the function $\beta(2^j\xi)$ is contained in $\{\xi : |\xi| \leq 1/2\}$.

In the case $j = 0$ we get

$$
\begin{aligned}
(T_1\Delta_0 g)(t,x) &= \int_{[-\pi,\pi]^d} e^{itq(\xi)}e^{ix\xi}\beta(\xi)\chi_{(|\xi|\leq 1/2)}\widehat{g}(\xi)d\xi \\
&= T^0_\beta[(\widehat{g}\chi_{(|\xi|\leq 1/2)})^\vee](t,x)
\end{aligned}
$$

and by (5.2.13)

$$
\begin{aligned}
\|T_1\Delta_0 g\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} &= \|T^0_\beta[(\widehat{g}\chi_{(|\xi|\leq 1/2)})^\vee]\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \\
&\lesssim \|(\widehat{g}\chi_{(|\xi|\leq 1/2)})^\vee\|_{L^2(\mathbb{R}^d)} \lesssim \|g\|_{L^2(\mathbb{R}^d)}.
\end{aligned}
$$

For $j \geq 1$ we get

$$
\begin{aligned}
(T_1\Delta_j g)(t,x) &= \int_{[-\pi,\pi]^d} e^{itq(\xi)}e^{ix\xi}\widehat{\Delta_j g}(\xi)\chi_{(|\xi|\leq 1/2)}d\xi \\
&= \int_{[-\pi,\pi]^d} e^{itq(\xi)}e^{ix\xi}\beta(2^j\xi)\widehat{g}(\xi)\chi_{(|\xi|\leq 1/2)}d\xi \\
&= \int_{1/2^{j+2}\leq|\xi|\leq 1/2^j} e^{itq(\xi)}e^{ix\xi}\beta(2^j\xi)\widehat{g}(\xi)d\xi \\
&= \frac{1}{2^{jd}}\int_{1/4\leq|\eta|\leq 1} e^{itq(\eta/2^j)}e^{ix\eta/2^j}\beta(\eta)\widehat{g}\left(\frac{\eta}{2^j}\right)d\eta \\
&= \frac{1}{2^{jd}}\int_{1/4\leq|\eta|\leq 1} e^{i\frac{t}{2^j}2^j q(\eta/2^j)}e^{ix\eta/2^j}\beta(\eta)\widehat{g}\left(\frac{\eta}{2^j}\right)d\eta \\
&= T^j_\beta[g(2^j\cdot)]\left(\frac{t}{2^j},\frac{x}{2^j}\right)
\end{aligned}
$$

and

$$\begin{aligned}
\|T_1\Delta_j g\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} &= \left\|T^j_\beta[g(2^j\cdot)]\left(\frac{\cdot}{2^j},\frac{\cdot}{2^j}\right)\right\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \\
&= 2^{j(\frac{d}{r}+\frac{1}{q})}\|T^j_\beta[g(2^j\cdot)]\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \\
&\leq c(\beta)2^{j(\frac{d}{r}+\frac{1}{q})}\|g(2^j\cdot)\|_{L^2(\mathbb{R}^d)} \\
&= c(\beta)2^{j(\frac{d}{r}+\frac{1}{q}-\frac{d}{2})}\|g\|_{L^2(\mathbb{R}^d)} = 2^{-js}\|g\|_{L^2(\mathbb{R}^d)}.
\end{aligned}$$

□

5.3. A uniform estimate for truncated operators

Lemma 5.3.1. *Let β be a smooth function supported in a open set far from zero. Let us define the family of operators U^j as*

$$\widehat{U^j(t)f} = e^{itp^j(\xi)}\beta(\xi)\widehat{f}(\xi)\chi_{[-\pi,\pi]^d}. \tag{5.3.1}$$

Then there is a positive constant $c = c(d,\beta)$ such that

i)

$$\|U^j(t)f\|_{L^2(\mathbb{R}^d)} \leq c\|f\|_{L^2(\mathbb{R}^d)} \tag{5.3.2}$$

for all $t \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^d)$, uniformly on $j \geq 0$;

ii)

$$\|U^j(t)(U^j(s))^*f\|_{L^\infty(\mathbb{R}^d)} \leq \frac{c}{(1+|t-s|)^{1/2}}\|f\|_{L^1(\mathbb{R}^d)} \tag{5.3.3}$$

for all $t \neq s$ and $f \in L^1(\mathbb{R})$ uniformly on $j \geq 0$.

As a consequence of this result we can apply the Strichatz estimates of [74] to obtain::

Corollary 5.3.1. *Let (q,r) and $(\tilde{q},\tilde{r})$ be two $1/2$-admissible pairs. Then there is a positive constant $c = c(d,\beta,q,r,\tilde{q},\tilde{r})$ such that*

i)

$$\|U^j(t)f\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \leq c\|f\|_{L^2(\mathbb{R}^d)}, \tag{5.3.4}$$

ii)

$$\left\|\int_{\mathbb{R}}(U^j(s))^*F(s)ds\right\|_{L^2(\mathbb{R}^d)} \leq c\|F\|_{L^{q'}(\mathbb{R},\, L^{r'}(\mathbb{R}^d))} \tag{5.3.5}$$

iii)

$$\left\|\int_{s<t}U^j(t)(U^j(s))^*F(s)ds\right\|_{L^q(\mathbb{R},\, L^r(\mathbb{R}^d))} \leq c\|F\|_{L^{\tilde{q}'}(\mathbb{R},\, L^{\tilde{r}'}(\mathbb{R}^d))} \tag{5.3.6}$$

uniformly on $j \geq 0$;

Proof of Lemma 5.3.1. Observe that $U^j(t)$ is a convolution operator $U^j(t)f = K^j(t,\cdot) * f$ where

$$K^j(t,x) = \int_{[-\pi,\pi]^d} e^{ix\xi}e^{itp^j(\xi)}\beta(\xi)d\xi. \tag{5.3.7}$$

The estimate (5.3.2) is just an energy estimate and is a trivial consequence of Plancherel's theorem:

$$\|K^j(t,\cdot) * f\|_{L^2(\mathbb{R}^d)} = \|\widehat{K^j}(t)\widehat{f}\|_{L^2(\mathbb{R}^d)} \le \|\beta\|_{L^\infty(\mathbb{R}^d)}\|f\|_{L^2(\mathbb{R}^d)} \le c\|f\|_{L^2(\mathbb{R}^d)}.$$

Inequality (5.3.3) is called the dispersive inequality. To prove it, note by Young's inequality,

$$\|K^j(t,\cdot) * f\|_{L^\infty(\mathbb{R}^d)} \le \|K^j(t,\cdot)\|_{L^\infty(\mathbb{R}^d)}\|f\|_{L^1(\mathbb{R}^d)},$$

so it is sufficient to prove that

$$|K^j(t,x)| \le \frac{c}{(1+|t|)^{1/2}} \tag{5.3.8}$$

holds uniformly on $\mathbb{R}^{1+d}$ and on $j \ge 0$.

We make use of the following lemma (Lemma 3, [13], see also [87], [60]):

Lemma 5.3.2. *Let P be real, $C^\infty(\mathbb{R}^d)$ in a neighborhood of the support of $v \in C_0^\infty(\mathbb{R}^d)$. Assume that the rank of $H_p(y) = (\partial^2 P(y)/\partial y_k \partial y_l)$ is at least ρ on the support of v. Then for some integer M,*

$$\|\mathcal{F}^{-1}(e^{itP}v)\|_{L^\infty(\mathbb{R}^d)} \le C(1+|t|)^{-\frac{1}{2}\rho} \sum_{|\alpha|\le M} \|D^\alpha v\|_{L^1(\mathbb{R}^d)}. \tag{5.3.9}$$

Here C depends on bounds of finitely many derivatives of P on supp(v) *and on a lower bound of the maximum of the absolute values of the minors of order ρ of H_p on* supp(v)*, and on* supp(v)*.*

We claim that rank$(H_q(\xi)) \ge 1$, $\xi \ne 0$. Thus rank$(H_{p^j}) \ge 1$. We postpone its proof.

Applying the above lemma with $P = p^j$ and $v = \beta$ we obtain the existence of a constant c_j such that (5.3.8) holds for some constant c_j. Remains to prove that all these constant are uniformly bounded.

Using that for a fixed multi-index α, $(\partial^\alpha p^j) \to \partial^\alpha(|\xi|)$ uniformly in the support of ψ we obtain that all the derivatives, up to a finite order N_1, converge uniformly. Then there exists an index j_0 and a positive constant C such that all the derivatives $\partial^\alpha p^j$ satisfy

$$\|\partial^\alpha p^j\|_{L^\infty(\operatorname{supp}\psi)} \le C$$

for all $j \ge j_0$ and $|\alpha| \le N_1$. The functions p^j being smooth, the above inequality also holds for a finite number of indices $j \le j_0$.

The uniform convergence of the partial derivatives of p^j gives us

$$H_{p^j}(\xi) \to H_{|\xi|}(\xi)$$

uniformly on the support of ψ. This means that

$$\mu_{kj}(\xi) \to \mu_k(\xi)$$

where $\mu_k(\xi)$ is the k-eigenvalue of $H_{|\xi|}(\xi)$. In consequence there exists an index j_0' such that

$$|\mu_{kj}(\xi)| \ge \frac{|\mu_k(\xi)|}{2}$$

for all $j \geq j_0'$ and ξ in support of ψ. We remark that H_{p^1} has the rank at least one at each point $\overline{x}$ in the support of ψ. Consequently the same holds for all $H_{p_j}(\overline{x})$ with $j = \overline{0, j_1}$. Applying Lemma 5.3.2 for each $j \leq j_1$ one can obtain (5.3.9) with the constants $C_0, C_1, ..., C_{j_1}$. Choosing $C^1 = \max\{C, C_0, ..., C_{j_1}\}$ we finish the proof.

In the following we show that $\text{rank}(H_q(\xi)) \geq 1$ for all $\xi \neq 0$. Explicit calculus shows that

$$(\partial^2_{\xi_i\xi_j} q)(\xi) = \frac{1}{q^3(\xi)} \begin{cases} \cos(\xi_i) q^2(\xi) - \sin^2(\xi_i), & i = j \\ \\ -\sin(\xi_i)\sin(\xi_j), & i \neq j. \end{cases}$$

This shows that at $\xi = (\pi/2, \dots, \pi/2)$ the following holds

$$(\partial^2_{\xi_i\xi_j} q)(\xi) = \frac{-1}{q((\pi/2)^d)} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ & & \dots & \\ 1 & 1 & \dots & 1 \end{pmatrix}.$$

Hence the are points where $\text{rank}(H_q) = 1$. It remains to prove that there are no points ξ where $\text{rank}(H_q(\xi)) = 0$. In this case $\sin(\xi_i)\sin(\xi_j) = 0$ for all $1 \leq i \neq j \leq d$.

Let us consider the case when for all $1 \leq i \leq d$, $\sin(\xi_i) = 0$. Thus $H_q = 1/q^2 I_d$ and $\text{rank}(H_d) = d$. It remains to analyze the case when at least one of $\sin(\xi_i)$, $1 \leq i \leq d$ does not vanish. Without restricting the generality, we can assume that $\sin(\xi_1) \neq 0$. Thus $\sin(\xi_i) = 0$ for all $2 \leq i \leq d$. Hence

$$H_q(\xi) = \frac{1}{q^3(\xi)} \begin{pmatrix} \cos(\xi_1) q^2(\xi) - \sin^2(\xi_1) & 0 & \dots & 0 \\ 0 & q^2(\xi) & \dots & 0 \\ 0 & 0 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & q^2(\xi) \end{pmatrix}$$

and $\text{rank}(H_q) \geq d - 1$. □

5.4. An application to a nonlinear problem

Let us consider the following nonlinear problem.

$$\begin{cases} \dfrac{d^2 u^h}{dt^2} - \Delta_h u^h + |u^h|^{p-1} u^h = 0, & t > 0, \\ \\ u^h_{\mathbf{j}}(0) = f^h_{\mathbf{j}}, (u^h_{\mathbf{j}})_t(0) = g^h_{\mathbf{j}}, & \mathbf{j} \in \mathbb{Z}^3 \end{cases} \tag{5.4.1}$$

We have the following well posedness result :

Theorem 5.4.1. *Let $h > 0$, $1 < p < 5$ and (q, r) be a 1/2-admissible pair with $3/r + 1/q = 1/2$. Then for any initial data $(f^h, g^h) \in \hbar^1(h\mathbb{Z}^3) \times l^2(h\mathbb{Z}^3)$ there is a unique solution u^h of the equation (5.4.1) in the class*

$$u^h \in C(\mathbb{R}, \hbar^1(h\mathbb{Z}^3)) \cap C^1(\mathbb{R}, l^2(h\mathbb{Z}^3)) \cap L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^3)).$$

More than that, for any finite interval I of $\mathbb{R}$

$$\|u^h\|_{L^q(I,l^r(h\mathbb{Z}^3))} \leq C(I)\|(f^h,g^h)\|_{\hbar^1(h\mathbb{Z}^3)\times l^2(h\mathbb{Z}^3)} \tag{5.4.2}$$

and

$$\|u^h\|_{L^\infty(I,\hbar^1(h\mathbb{Z}^3))} + \|u^h_t\|_{L^\infty(I,l^2(h\mathbb{Z}^3))} \leq C(I,\|f^h\|_{\hbar^1(h\mathbb{Z}^3)},\|g^h\|_{l^2(h\mathbb{Z}^3)}) \tag{5.4.3}$$

hold uniformly on $h>0$.

Remark 5.4.1. *In the case $p=5$, the same results can be obtained by imposing smallness conditions on the initial data.*

Remark 5.4.2. *By the gap condition we have*

$$\frac{3}{r}+\frac{1}{q}=\frac{1}{2}=\frac{3}{\tilde{r}'}+\frac{1}{\tilde{q}'}-2.$$

Then $(\tilde{q},\tilde{r})$ satisfies

$$\frac{3}{\tilde{r}'}+\frac{1}{\tilde{q}'}=\frac{5}{2} \quad or \quad \frac{3}{\tilde{r}}+\frac{1}{\tilde{q}}=\frac{3}{2} \tag{5.4.4}$$

and the $1/2$-admissibility condition

$$\frac{2}{\tilde{q}}+\frac{1}{\tilde{r}}\leq\frac{1}{2},\ \tilde{q}\geq 2, \tilde{r}\geq 2. \tag{5.4.5}$$

The unique $\tilde{r}$ which satisfies (5.4.4) and (5.4.5) is $\tilde{r}=2$, thus $\tilde{q}=\infty$.

Remark 5.4.3. *The convergence of the scheme follows the same steps as in [85] (Theorem 1.1, Ch.1, p. 13) with the difference that the limit point u will belong to the space $L^q_{loc}(\mathbb{R}, L^r(\mathbb{R}^3))$.*

Proof. Let us write $v^h=u^h_t$. Then the problem (5.4.1) is written in an equivalent form as:

$$\begin{cases} \begin{pmatrix} u^h \\ v^h \end{pmatrix}_t = \begin{pmatrix} 0 & I \\ -\Delta_h & 0 \end{pmatrix}\begin{pmatrix} u^h \\ v^h \end{pmatrix} + \begin{pmatrix} 0 \\ |u^h|^{p-1}u^h \end{pmatrix}, \ t\neq 0, \\ \begin{pmatrix} u^h \\ v^h \end{pmatrix}(0) = \begin{pmatrix} f^h \\ g^h \end{pmatrix}. \end{cases}$$

The operator $A_h = \begin{pmatrix} 0 & I \\ -\Delta_h & 0 \end{pmatrix}$ is bounded on $\hbar^1(h\mathbb{Z}^3)\times l^2(h\mathbb{Z}^3)$. Thus there exists $T_h>0$ and a unique solution u^h of (5.4.1) which satisfies

$$u^h\in C((-T_h,T_h),\hbar^1(h\mathbb{Z}^3))\cap C^1((-T_h,T_h),l^2(h\mathbb{Z}^3)).$$

The blow-up alternative provides that either $T_h=\infty$ or

$$\lim_{t\nearrow T_h}\|u^h(t)\|_{\hbar^1(h\mathbb{Z}^3)}+\|u^h_t(t)\|_{l^2(h\mathbb{Z}^3)}=\infty.$$

Let us consider the perturbed energy

$$F(t)=\frac{h}{2}\left(\sum_{\mathbf{j}\in\mathbb{Z}^3}|u^h_{\mathbf{j}}|^2+\sum_{\mathbf{j}\in\mathbb{Z}^n}|(\nabla u^h)_{\mathbf{j}}|^2+\sum_{\mathbf{j}\in\mathbb{Z}^3}|(u^h_j)_t|^2\right)+\frac{h}{p+1}\sum_{\mathbf{j}\in\mathbb{Z}^n}|u_j|^{p+1}.$$

Then

$$\frac{dF}{dt} = h \sum_{\mathbf{j}\in\mathbb{Z}^3} u^h_{\mathbf{j}} v^h_{\mathbf{j},t} \leq \frac{h}{2} \sum_{\mathbf{j}\in\mathbb{Z}} (|u^h_{\mathbf{j}}|^2 + |u^h_{\mathbf{j},t}|^2) \leq F(t) \tag{5.4.6}$$

so

$$F(t) \leq F(0)e^t.$$

This implies $T_h = \infty$ and global existence of the solutions.

In the following we prove the estimates (5.4.2) and (5.4.2). Let us choose $T > 0$. Using that $p < 5$ we get

$$\begin{aligned}
\|u^h\|^2_{L^\infty((-T,T),\hbar^1(h\mathbb{Z}^3))} &\leq e^T F(0) \leq C(T)\left(\|f^h\|_{\hbar^1(h\mathbb{Z}^3)} + \|g^h\|_{l^2(h\mathbb{Z}^3)} + \frac{\|f^h\|^{p+1}_{l^{p+1}(h\mathbb{Z}^3)}}{p+1}\right)\\
&\leq C(T)\left(\|f^h\|_{\hbar^1(h\mathbb{Z}^3)} + \|g^h\|_{l^2(h\mathbb{Z}^3)} + \frac{\|f^h\|^{p+1}_{\hbar^1(h\mathbb{Z}^3)}}{p+1}\right)\\
&\leq C(I, \|f^h\|_{\hbar^1(h\mathbb{Z}^3)}, \|g^h\|_{l^2(h\mathbb{Z}^3)}).
\end{aligned}$$

Moreover, for any $2 \leq s \leq 6$ we have

$$\|u^h\|_{L^\infty((-T,T),l^s(h\mathbb{Z}^d)} \leq C(I, \|f^h\|_{\hbar^1(h\mathbb{Z}^3)}, \|g^h\|_{l^2(h\mathbb{Z}^3)}). \tag{5.4.7}$$

This shows that for $p \leq 3$ and (q, r) a 1/2-admisible pair

$$\begin{aligned}
\|u^h\|_{L^q((-T,T),l^r(h\mathbb{Z}^3)} &\leq \|f^h\|_{\hbar^1(h\mathbb{Z}^3)} + \|g^h\|_{l^2(h\mathbb{Z}^3)} + \||u^h|^p\|_{L^1((-T,T),l^2(h\mathbb{Z}^3))}\\
&\leq \|f^h\|_{\hbar^1(h\mathbb{Z}^3)} + \|g^h\|_{l^2(h\mathbb{Z}^3)} + T\|u^h\|^p_{L^\infty((-T,T),l^{2p}(h\mathbb{Z}^3))}\\
&\leq C(I, \|f^h\|_{\hbar^1(h\mathbb{Z}^3)}, \|g^h\|_{l^2(h\mathbb{Z}^3)}).
\end{aligned}$$

It remains to analyze the case $3 < p < 5$. We remark that $|u|^{p-1}u$ maps $L^{2p/(p-3)}([0,T], l^{2p}(h\mathbb{Z}^3))$ to $L^1([0,T], l^2(h\mathbb{Z}^3))$ with a norm independent of h. Also the pairs $(2p/(p-3), 2p)$ and $(\infty, 2)$ are 1/2-admissible pairs which satisfy the conditions (5.1.3) and (5.1.4). The dispersive properties given by Theorem 5.1.1 and a fixed point argument in the space $L^{2p/(p-3)}([0,T], l^{2p}(h\mathbb{Z}^3))$ give the existence of a unique local solution u^h which satisfies $\|v^h\|_{L^{2p/p-3}([0,T],l^{2p}(h\mathbb{Z}^3))} \leq C\|(f^h, g^h)\|_{\hbar^1(h\mathbb{Z}^3)\times l^2(h\mathbb{Z}^3)}$. The energy estimate (5.4.6) proves the global existence of the solution v^h and the estimation (5.4.2) with $(q, r) = (2p/(p-3), 2p)$.

Remains to prove that $u^h \equiv v^h$. Using that $u^h \in C(\mathbb{R}, \hbar^1(h\mathbb{Z}^3))$ and that $\hbar^1(h\mathbb{Z}^3) \hookrightarrow l^2(h\mathbb{Z}^3) \hookrightarrow l^{2p}(h\mathbb{Z}^3)$ we obtain that u^h belongs to $L^{2p/(p-3)}_{loc}(\mathbb{R}, l^{2p}(h\mathbb{Z}^3))$. By uniqueness we get $u^h \equiv v^h$. Once we obtain that the solution belongs toone of the spaces $L^q_{loc}(\mathbb{R}, l^r(h\mathbb{Z}^3))$, we apply again the dispersive properties of Theorem (5.1.1) to obtain that the solution belongs to all the other spaces and satisfies (5.4.2).

□

Chapter 6

Uniform Boundary Observability. A Two-Grid Method.

6.1. Introduction

Let Ω be the square $\Omega = (0,1) \times (0,1)$ of $\mathbb{R}^2$ and consider the wave equation with Dirichlet boundary conditions:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } Q = \Omega \times (0,T), \\ u(0) = 0 & \text{on } \Sigma = \partial\Omega \times (0,T), \\ u(x,0) = u^0(x), \ u_t(x,0) = u^1(x) & \text{in } Q = \Omega. \end{cases} \tag{6.1.1}$$

Given $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ system (6.1.1) admits a unique solution

$$u \in C([0,T], H_0^1(\Omega)) \cap C^1([0,T], L^2(\Omega)).$$

Moreover, the energy

$$\mathcal{E}(t) = \frac{1}{2} \int_\Omega [|u_t(x,t)|^2 + |\nabla u(x,t)|^2] dx \tag{6.1.2}$$

remains constant, i.e.

$$\mathcal{E}(t) = \mathcal{E}(0), \ \forall \ 0 < t < T. \tag{6.1.3}$$

Let Γ_0 denote a subset of the boundary of Ω constituted by two consecutive sides, for instance,

$$\Gamma_0 = \{(x_1, 1) : \ x_1 \in (0,1)\} \cup \{(1, x_2) : \ x_2 \in (0,1)\}. \tag{6.1.4}$$

It is by now well known (see [86]) that for $T > 2\sqrt{2}$ there exists $C(T) > 0$ such that

$$\mathcal{E}(0) \leq C(T) \int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \tag{6.1.5}$$

holds for every finite-energy solution of (6.1.1). The condition imposed on T is due to the fact that the velocity of the waves is one and then any perturbation of the initial data will take some time in order to arrive at the observation zone.

In (6.1.5), n denotes the unit normal to Ω, $\partial \cdot / \partial n$ the normal derivative and $d\sigma$ the surface measure.

Let us consider the finite-difference semi-discretization of (6.1.1). Given $N \in \mathbb{N}$ we set $h = 1/(N+1)$.

We denote by u_{jk} the approximation of (6.1.1) at the point $x_{jk} = (jh, kh)$. The finite-difference semi-discretization of (6.1.1) is as follows:

$$\begin{cases} u''_{j,k} - \dfrac{u_{j+1,k} + u_{j-1,k} - 2u_{jk}}{h^2} - \dfrac{u_{j,k+1} + u_{j,k-1} - 2u_{j,k}}{h^2} = 0, \\ \quad 0 < t < T, \ j = 0, ..., N; \ k = 0, ..., N, \\ u_{j,k} = 0, \ 0 < t < T, \ j = 0, ..., N+1; \ k = 0, ..., N+1, \\ u_{j,k}(0) = u^0_{j,k}, \ u'_{j,k}(0) = u^1_{j,k}, \ j = 0, ..., N+1; \ k = 0, ..., N+1. \end{cases} \tag{6.1.6}$$

In (6.1.6), the first equation provides a 5-point approximation of the wave equation. The second equation takes into account the homogenous Dirichlet boundary conditions. The last one provides the initial conditions guaranteeing the uniqueness of the solution. System (6.1.6) is a coupled system of N^2 linear ordinary differential equations of second order.

Let us now introduce the *discrete energy* associated with system (6.1.6):

$$\mathcal{E}_h(t) = \frac{h^2}{2} \sum_{j,k=0}^{N} \left[|u'_{jk}(t)|^2 + \left| \frac{u_{j+1,k}(t) - u_{jk}(t)}{h} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{jk}(t)}{h} \right|^2 \right]. \tag{6.1.7}$$

It is easy to see that the energy remains constant in time, i.e.

$$\mathcal{E}_h(t) = \mathcal{E}_h(0), \ \forall \ 0 < t < T \tag{6.1.8}$$

for every solution of (6.1.6).

We now observe that the discrete version of the energy observed on the boundary is given by:

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \sim \int_0^T \left[h \sum_{j=1}^{N} \left| \frac{u_{jN}}{h} \right|^2 + h \sum_{k=1}^{N} \left| \frac{u_{Nk}}{h} \right|^2 \right] dt. \tag{6.1.9}$$

In the following for any $j = 1, ..., N$ and $k = 1, ..., N$, we denote

$$(\partial^h_n \overline{u})_{j,N+1} := \frac{u_{jN}}{h}, \ (\partial^h_n \overline{u})_{N+1,k} := \frac{u_{Nk}}{h}.$$

We introduce the discrete boundary Γ_h as the set of grid points belonging to Γ_0:

$$\Gamma_h = \{(jh, N+1), \ j = 1, ..., N\} \cup \{(N+1, kh), \ k = 1, ..., N\}.$$

Also, in order to simplify the presentation, we introduce the following notation

$$\int_{\Gamma_h} |\partial^h_n \overline{u}|^2 d\Gamma_h := h \sum_{j=1}^{N} \left| \frac{u_{jN}}{h} \right|^2 + h \sum_{k=1}^{N} \left| \frac{u_{Nk}}{h} \right|^2. \tag{6.1.10}$$

The discrete version of (6.1.5) is then an inequality of the form

$$\mathcal{E}_h(0) \le C_h(T) \int_0^T \int_{\Gamma_h} |\partial^h_n \overline{u}|^2 d\Gamma_h dt. \tag{6.1.11}$$

For all $T > 0$ and $h > 0$ there exists a constant $C_h(T)$ such that (6.1.11) holds for all the solutions of equation (6.1.1). As it was proved in [142], for all $T > 0$ the best constant $C_h(T)$ necessarily blows-up as $h \to 0$. This is due to the fact that spurious high frequency oscillations are present in the semi-discrete system (6.1.1). This phenomenon was already observed by R. Glowinski et al in [53], [55] and [56], in connection with the exact boundary controllability of the wave equation and the numerical implementation of the so-called HUM method.

Several techniques have been introduced as possible remedies to the high frequencies spurious oscillations: Tychonoff regularization [53], filtering of the high frequencies [65], [142], [145], mixed finite elements [54], [22], [23], two-grid algorithm [97], [88]. The last method was proposed by Glowinski [55] and consists in using a coarse and a fine grid, and interpolating the initial data in the adjoint problem (6.1.6) from the coarse grid to the fine one. This method eliminates the short wave-length component of the initial conditions u^0, u^1 of the wave equation by defining them on a coarse grid of twice the step size, $2h$.

In general the two grid algorithm reduces the oscillations in the initial data. In Figure 6.1 we choose the function $g(x_1, x_2) = \sin^2(20x_1) + 4\sin^2(20x_2)$ and restrict it on the fine grid $G^{1/52}$. We choose only the grid points from $G^{1/26}$ and make a linear interpolation of them. As we can see in Figure 6.2 the new function has less oscillations. Also choosing only the nodes of $G^{1/13}$ we obtain a better result, plotted in Figure 6.2.

In the one dimensional case, the two-grid method was analyzed by Negreanu and Zuazua in [97] with a discrete multiplier approach. The authors also proved the convergence of the method as $h \to 0$ for $T > 4$. In a recent work, Mehrenberer and Loreti [88], used a fine extension of Ingham's inequality to improve the time of observability $T > 2$. However as far as we know there is no proof in the 2-dimensional case. The main goal of this Chapter is to give the first complete proof of the uniform observability inequality in the multi-dimensional case.

In contrast with the strategy adopted in [97] where the authors consider the ratio between the size of the grids 1/2, we choose the quotient to be 1/4. This is done for merely technical reasons and one may expect the same result should hold when the ratio of the grids is 1/2.

This idea of considering the quotient of the grids to be 1/4 has been used successfully in [63] when proving dispersive estimates for conservative semi-discrete approximation schemes of the Schrödinger equation. When diminishing the ratio between grids, the filtering that the two-grid algorithm introduces concentrates the solutions of the numerical problem on lower and lower frequencies for which the velocity of propagation becomes closer and closer to that of the continuous wave equation.

The two-grid algorithm which we analyze is the following: Let N be such that $N \equiv 3 \pmod 4$ and $h = 1/(N+1)$. We introduce a coarse grid (see Figure 6.5 for $N = 11$)

$$G^{4h} : x_{\mathbf{j}},\ x_{\mathbf{j}} = 4h\mathbf{j},\ \mathbf{j} \in \left[0, \frac{N+1}{4}\right]^2 \cap \mathbb{Z}^2$$

and a fine one (see Figure 6.4 for $N = 11$) :

$$G^h : y_{\mathbf{j}},\ y_{\mathbf{j}} = \mathbf{j}h,\ \mathbf{j} \in [0, N+1]^2 \cap \mathbb{Z}^2.$$

We consider the space V^h of all functions $\{\varphi_{\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^2_{N+2}}$ defined on the fine grid G^h as a linear interpolation of the functions $\{\varphi_{4\mathbf{j}}\}_{\mathbf{j}\in\mathbb{Z}^2_{(N+1)/4+1}}$ defined on the coarse grid.

In this paper, by using a different approach than the one in [97], [88], we prove that (6.1.11) holds uniformly for all $T > 4$, in the class of initial data $V^h \times V^h$. The new method

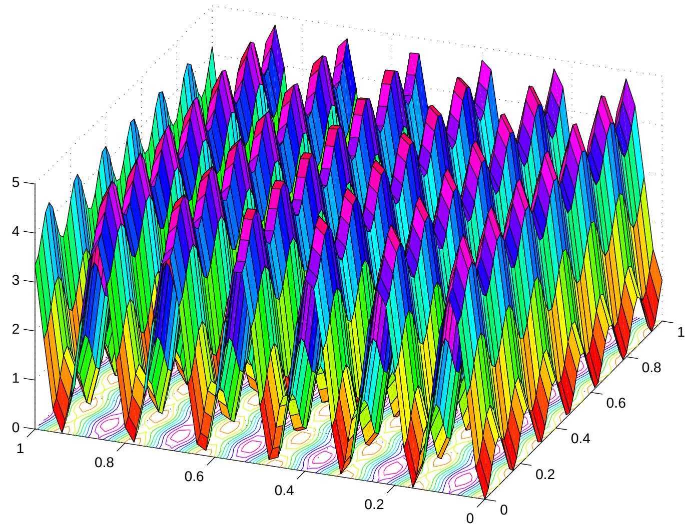

Figure 6.1: The function $\sin^2(20x_1) + 4\sin^2(20x_2)$ restricted on $G^{1/52}$

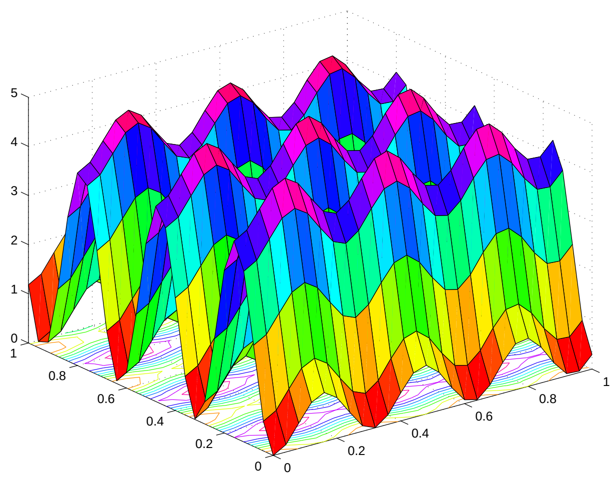

Figure 6.2: Interpolation from $G^{1/26}$.

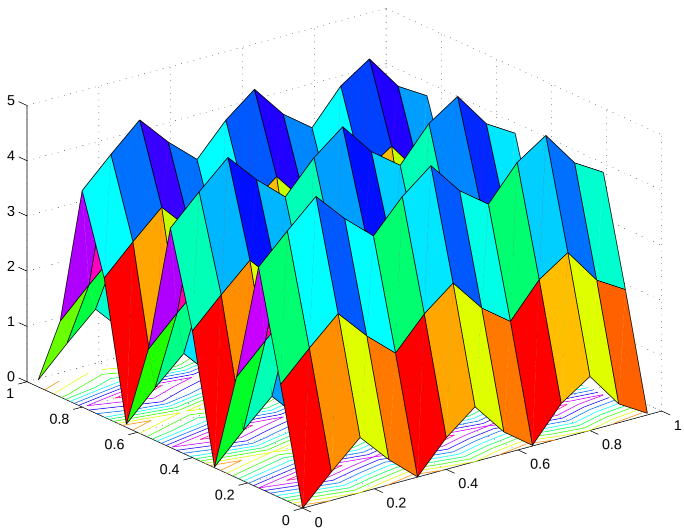

Figure 6.3: Interpolation from $G^{1/13}$.

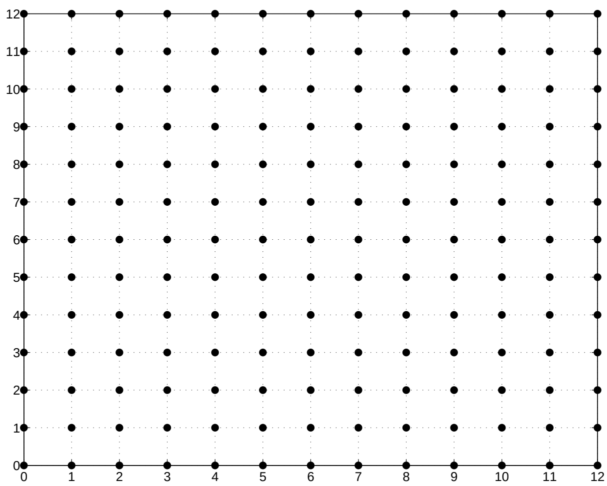

Figure 6.4: The fine grid G^h; N=11

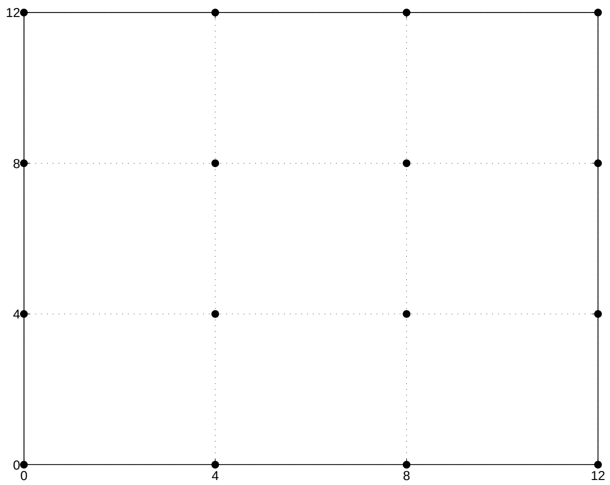

Figure 6.5: The coarse grid G^{4h}; N=11

consists in using the already well known observability inequality for a class of *low frequency* data and a time spectral decomposition of the solutions.

In the sequel when we focus on the two grid method associated with t G^h and G^{4h} we implicitly assume that $N \equiv 3 \pmod 4$.

This Chapter is organized as follows: in Section 6.2 we specify what the problem consists of and we state the main result in Theorem 6.2.1. The uniform observability for the solutions with initial data in $V^h \times V^h$ is a consequence of Theorem 6.2.1 and is given by Theorem 6.2.2. In Section 6.3 we make an analysis of the bicharacterisc rays and comment on what the optimal for observability should be. Section 6.4 contains a Fourier analysis of the functions belonging to V^h. The final Section is dedicated to the proof of Theorem 6.2.1.

6.2. Main results

To make our statements precise, let us consider the eigenvalue problem associated to (6.1.6):

$$\begin{cases} -\dfrac{\varphi_{j+1,k}+\varphi_{j-1,k}-2\varphi_{jk}}{h^2}-\dfrac{\varphi_{j,k+1}+\varphi_{j,k-1}-2\varphi_{jk}}{h^2}=\lambda\varphi_{jk} \\ \quad j=1,...,n;\ k=1,...,N, \\ \varphi_{jk}=0,\ j=0,...,N+1;\ k=0,...,N+1. \end{cases} \tag{6.2.1}$$

System (6.2.1) admits N^2 eigenvalues. We denote $\Lambda_N := [1,N]^2 \cap \mathbb{Z}^2$.

The eigenvalues and eigenvectors of system (6.2.1) (cf. [66]) are

$$\lambda_{\mathbf{k}}(h)=\frac{4}{h^2}\left[\sin^2\left(\frac{k_1\pi h}{2}\right)+\sin^2\left(\frac{k_2\pi h}{2}\right)\right],\ \mathbf{k}=(k_1,k_2)\in\Lambda_N$$

and $\{\overline{\varphi}^{\mathbf{k}}\}_{\mathbf{k}\in\Lambda_N}$:

$$\overline{\varphi}^{\mathbf{k}}_{\mathbf{j}}=\sin(j_1k_1\pi h)\sin(j_2k_2\pi h),\ \mathbf{k}=(k_1,k_2)\in\Lambda_N,\ \mathbf{j}=(j_1,j_2)\in\Lambda_N.$$

The vectors $\{\overline{\varphi}^{\mathbf{k}}\}_{\mathbf{k}\in\Lambda_N}$ form a basis for the functions defined on G^h and vanishing on its boundary. Any real function defined on the grid G^h admits the Fourier expansion:

$$\overline{\phi} = \sum_{\mathbf{k}\in\Lambda_N} \widehat{\phi}(\mathbf{k})\overline{\varphi}^{\mathbf{k}}, \ \widehat{\phi}(\mathbf{k}) \in \mathbb{R}.$$

We define the Hilbert spaces $\hbar^s, \ s \geq 0$ by:

$$\hbar^s = \{\overline{\phi} : \|\overline{\phi}\|^2_{\hbar^s} = \sum_{\mathbf{k}\in\Lambda_N} \lambda^s_{\mathbf{k}}(h)|\widehat{\phi}(\mathbf{k})|^2 < \infty\}. \tag{6.2.2}$$

We point out that in the physical space the following identities hold:

$$\|\overline{\phi}\|^2_{\hbar^0} = h \sum_{\mathbf{j}\in\Lambda_N} |\phi_{\mathbf{j}}|^2,$$

$$\|\overline{\phi}\|^2_{\hbar^1} = h \sum_{j,k=0}^{N} \left(\left|\frac{\phi_{j+1,k} - \phi_{j,k}}{h}\right|^2 + \left|\frac{\phi_{j,k+1} - \phi_{j,k}}{h}\right|^2 \right).$$

In view of (6.1.7), the energy of the solution of the semi-discrete system (6.1.6) can be written in the following way:

$$\mathcal{E}_h(\overline{u}) = \frac{1}{2}(\|\overline{u}^0\|^2_{\hbar^1} + \|\overline{u}^1\|^2_{\hbar^0}). \tag{6.2.3}$$

The solution of system (6.1.6) in Fourier series is given by

$$\begin{aligned}
\overline{u}(t) &= \frac{1}{2}\sum_{\mathbf{j}\in\Lambda_N} \left[e^{it\sqrt{\lambda_{\mathbf{j}}(h)}} \left(\widehat{u}^0_{\mathbf{j}} + \frac{\widehat{u}^1_{\mathbf{j}}}{i\sqrt{\lambda_{\mathbf{j}}(h)}} \right) + e^{-it\sqrt{\lambda_{\mathbf{j}}(h)}} \left(\widehat{u}^0_{\mathbf{j}} - \frac{\widehat{u}^1_{\mathbf{j}}}{i\sqrt{\lambda_{\mathbf{j}}(h)}} \right) \right] \overline{\varphi}^{\mathbf{j}} \\
&= \frac{1}{2}\sum_{\mathbf{j}\in\Lambda_N} \left[e^{it\omega_{\mathbf{j}}(h)}\widehat{u}_{\mathbf{j}+} + e^{-it\omega_{\mathbf{j}}(h)}\widehat{u}_{\mathbf{j}-} \right] \overline{\varphi}^{\mathbf{j}},
\end{aligned}$$

where $\{\widehat{u}^0_{\mathbf{j}}\}_{\mathbf{j}\in\Lambda_N}$ and $\{\widehat{u}^1_{\mathbf{j}}\}_{\mathbf{j}\in\Lambda_N}$ are the coefficients of the initial data $(\overline{u}^0, \overline{u}^1)$ in the basis $\{\overline{\varphi}^{\mathbf{j}}\}_{\mathbf{j}\in\Lambda_N}$, $\omega_{\mathbf{j}}(h) = \sqrt{\lambda_{\mathbf{j}}(h)}$ and

$$\widehat{u}_{\mathbf{j}\pm} = \widehat{u}^0_{\mathbf{j}} \pm \frac{\widehat{u}^1_{\mathbf{j}}}{i\sqrt{\lambda_{\mathbf{j}}(h)}}.$$

Using the above notations, the energy is given by

$$\begin{aligned}
\mathcal{E}_h(\overline{u}) &= \frac{1}{2}\sum_{\mathbf{j}\in\Lambda_N} \omega^2_{\mathbf{j}}(h) \left(|\widehat{u}_{\mathbf{j}+}e^{it\omega_{\mathbf{j}}(h)} - \widehat{u}_{\mathbf{j}-}e^{-it\omega_{\mathbf{j}}(h)}|^2 + |\widehat{u}_{\mathbf{j}+}e^{it\omega_{\mathbf{j}}(h)} + \widehat{u}_{\mathbf{j}-}e^{-it\omega_{\mathbf{j}}(h)}|^2 \right) \\
&= \sum_{\mathbf{j}\in\Lambda_N} \omega^2_{\mathbf{j}}(h)(|\widehat{u}_{\mathbf{j}+}|^2 + |\widehat{u}_{\mathbf{j}-}|^2).
\end{aligned}$$

As proved in [142], the blow-up of the observability constant in (6.1.11) is due to solutions of (6.1.6) of the form $u = e^{it\sqrt{\lambda}}\overline{\varphi}$, λ being a sufficiently large eigenvalue of (6.2.1) and $\overline{\varphi}$ the corresponding eigenfunction. The high frequency eigenfunctions of the system (6.2.1) are such that the energy concentrated on the boundary is asymptotically smaller than the total energy. In fact the observability constant blows up exponentially. This was proved by Micu in [93] in the 1-d case using the explicit expression of a biorthogonal sequence of functions to the underlying time complex-exponentials.

We introduce two classes of solutions of (6.1.6) in which the high frequencies have been truncated or filtered. More precisely, for any $0 < \gamma \leq 2\sqrt{2}$ and $0 < \eta \leq 1$ we set

$$I_h(\gamma) = \left\{ \overline{u}(t) = \sum_{\omega_{\mathbf{j}}(h) \leq \gamma/h} \left[e^{it\omega_{\mathbf{j}}(h)} \widehat{u}_{\mathbf{j}+} + e^{-it\omega_{\mathbf{j}}(h)} \widehat{u}_{\mathbf{j}-} \right] \overline{\varphi}^{\mathbf{j}} \text{ with } \widehat{u}_{\mathbf{j}+}, \widehat{u}_{\mathbf{j}-} \in \mathbb{C} \right\} \tag{6.2.4}$$

and

$$J_h(\eta) = \left\{ \overline{u}(t) = \sum_{\|\mathbf{j}\|_\infty \leq \eta(N+1)} \left[e^{it\omega_{\mathbf{j}}(h)} \widehat{u}_{\mathbf{j}+} + e^{-it\omega_{\mathbf{j}}(h)} \widehat{u}_{\mathbf{j}-} \right] \overline{\varphi}^{\mathbf{j}} \text{ with } \widehat{u}_{\mathbf{j}+}, \widehat{u}_{\mathbf{j}-} \in \mathbb{C} \right\}. \tag{6.2.5}$$

For any solution $\overline{u}$ of equation (6.1.6) we denote by $\Pi^h_\gamma \overline{u}$ and $\Upsilon^h_\eta \overline{u}$, its projection on the space $I_h(\gamma)$ respectively $J_h(\eta)$. Problem (6.1.6) being linear $\Pi^h_\gamma \overline{u}$ and $\Upsilon^h_\eta \overline{u}$ are also solutions of (6.1.6) with the corresponding initial data.

If we look on the frequency domain $[0, N+1]^2$, the first class introduces a filtering along the level curves of the function $2/h(\sin^2(\xi_1 h\pi/2) + \sin^2(\xi_2 h\pi/2))^{1/2}$. The second one consists in filtering the range of indices $\mathbf{j}$ to the square of length side $\eta(N+1)$. Observe that in dimension one there exists a one-to-one correspondence between the two classes. In dimension two, excepting the case $\gamma = 2\sqrt{2}$, $\eta = 1$, the situation is different. Both classes can be easily compared with each other. For any $0 < \gamma < 2\sqrt{2}$ there exist two parameters η_1, η_2 such that

$$J_h(\eta_1) \subsetneq I_h(\gamma) \subsetneq J_h(\eta_2).$$

The converse also holds: for any $0 < \eta \leq 1$ there exist two constants γ_1 and γ_2 such that

$$I_h(\gamma_1) \subsetneq J_h(\eta) \subsetneq I_h(\gamma_2).$$

The first class $I_h(\gamma)$ is helpful when using arguments involving the time variable t. It has been intensively used, in connection with the so-called semi-classical analysis, for control problems ([78], [17], [79]) and the dispersive properties of PDE's ([21], [19]). The second one is better adapted to the two-grid methods. In fact we will prove that the total energy of a solution with initial data in the space V^h is bounded above by the energy of its projection on the space $J_h(1/4)$:

$$\mathcal{E}_h(\overline{u}) \leq 4\mathcal{E}_h(\Upsilon^h_{1/4}\overline{u}). \tag{6.2.6}$$

A similar result holds for the case of a two-grid method involving the grids G^h and G^{2h}. In that case the energy satisfies $\mathcal{E}_h(\overline{u}) \leq 2\mathcal{E}_h(\Upsilon^h_{1/2}\overline{u})$.

The uniform observability in the class $I_h(\gamma)$ has been analyzed in [142] by the multipliers technique. In that article it is shown that for any $0 < \gamma < 2$ and

$$T > T(\gamma) = \frac{8\sqrt{2}}{4 - \gamma^2} \tag{6.2.7}$$

there exists $C(\gamma, T) > 0$ such that

$$\mathcal{E}_h(\overline{u}) \leq C(\gamma, T) \int_0^T \int_{\Gamma_h} |\partial^h_n \overline{u}(t)|^2 d\Gamma dt \tag{6.2.8}$$

holds for every solution $\overline{u}$ of (6.1.6) in the class $I_h(\gamma)$ and $h > 0$. More than that for $\gamma = 2$ and $T > 0$ there is no constant C such that (6.2.8) holds for all solutions $\overline{u}$ of (6.1.6), uniformly on h:

$$\sup_{\overline{u}\in I_h(2)} \frac{\mathcal{E}_h(\overline{u})}{\displaystyle\int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}(t)|^2 d\Gamma dt} \to \infty,\ h \to 0.$$

This is a consequence of the presence of the frequencies near the points $(\pi/h, 0)$, $(0, \pi/h)$ (see green area in Figure 6.7) with group velocity of order h that spend a time of order $1/h$ to reach the boundary.

The observability result (6.2.8) will be systematically used along the paper.

The main result is given by Theorem 6.2.1. The observability inequality for the two-grid method is a consequence of it and will be stated in Theorem 6.2.2.

We state a general result, the observability for our two-grid class being a consequence of it.

Theorem 6.2.1. *Let $\overline{u}$ be a solution of (6.1.6) and $\gamma > 0$ such that*

$$\mathcal{E}_h(\overline{u}) \le C\mathcal{E}_h(\Pi_\gamma^h \overline{u}). \tag{6.2.9}$$

holds for some constant C, independent of h. Let us assume the existence of a time $T(\gamma)$ such that for all $T > T(\gamma)$ there exists a constant $C(T)$, independent of h, such that

$$\mathcal{E}_h(\overline{v}) \le C(\gamma, T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{v}(t)|^2 d\Gamma dt \tag{6.2.10}$$

for all $\overline{v} \in I_h(\gamma)$. Then for all $T > T(\gamma)$ there exists a constant $C_1(T, \gamma)$, independent of h, such that

$$\mathcal{E}_h(\overline{u}) \le C_1(T, \gamma) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt \tag{6.2.11}$$

Remark 6.2.1. *A two grid method with mesh-ratio of the grids $1/2^k q$, $k \ge 1$, q odd, implies*

$$\mathcal{E}_h(\overline{u}) \le C\mathcal{E}_h(\Upsilon_{1/2^k}^h \overline{u}).$$

Remark 6.2.2. *A two-grid algorithm involving G^h and G^{2h} implies $\mathcal{E}_h(\overline{u}) \le 2\mathcal{E}_h(\Upsilon_{1/2}^h \overline{u}) \le 2\mathcal{E}_h(\Pi_2^h \overline{u})$ for all solutions $\overline{u}$ obtained by this method. As we can see in Figure 6.7, the smallest γ such that $I_h(\gamma)$ contains all the frequencies $\omega_{\mathbf{j}}(h)$, $\|\mathbf{j}\|_\infty \le (N+1)/2$ (blue area in Figure 6.6) is $\gamma = 2$ (red area in Figure 6.7). Unfortunately, as we pointed before, inequality (6.2.10) does not hold in the class $I_h(2)$. This is why we choose the ratio between the fine and coarse grid in the two-grid method to be $1/4$. This will guarantee that the two hypotheses (6.2.9) and (6.2.10) are verified.*

Remark 6.2.3. *In the above Theorem we use that the so-called "direct inequality" holds. In fact (see [142]) for any $T > 0$ and $h > 0$ there exists a constant $C(T)$, independent of h, such that*

$$\int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt \le C(T)\mathcal{E}_h(\overline{u}). \tag{6.2.12}$$

for all solutions $\overline{u}$ of the semidiscrete system (6.1.6).

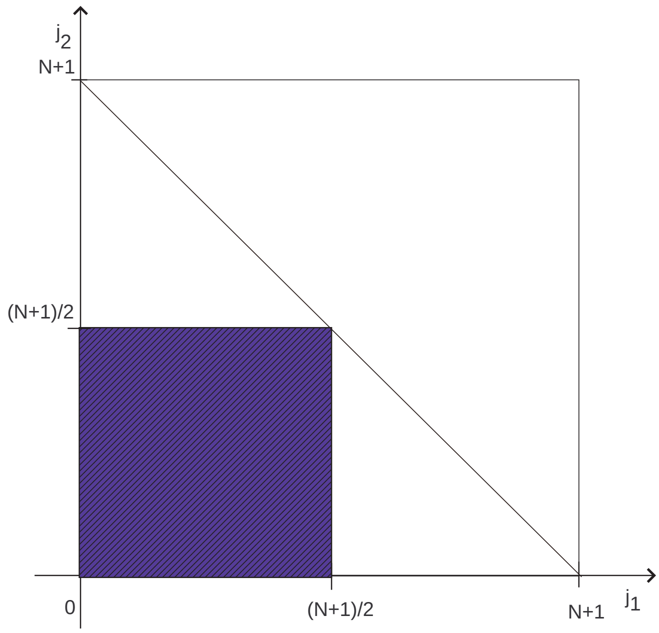

Figure 6.6: The blue area represents the frequencies involved in $J_h(1/2)$.

In the case of a two-grid method involving the grids G^{4h} and G^h, the total energy of solutions is controlled (see the proof of Theorem 6.2.2) by the energy associated with 1/16-th of its spectrum (blue area in Figure 6.8):

$$\mathcal{E}_h(\overline{u}) \leq 4 \sum_{\mathbf{j}\in\Lambda_{(N+1)/4}} \omega_{\mathbf{j}}^2(h)(|\widehat{u}_{\mathbf{j}+}|^2 + |\widehat{u}_{\mathbf{j}-}|^2) = 4\mathcal{E}_h(\Upsilon^h_{1/4}\overline{u}). \tag{6.2.13}$$

Clearly any $\omega_{\mathbf{j}}(h)$ with $\|\mathbf{j}\|_\infty \leq (N+1)/4$ satisfies

$$\omega_{\mathbf{j}}(h) \leq \left(\frac{8}{h^2}\sin^2\left(\frac{\pi}{8}\right)\right)^{1/2} \leq \frac{2\sqrt{2}\sin(\pi/8)}{h}.$$

This implies that the total energy of the solution is bounded above by the energy of its projection on the space $I_h(2\sqrt{2}\sin(\pi/8))$:

$$\mathcal{E}_h(\overline{u}) \leq 4\mathcal{E}_h(\Upsilon^h_{1/4}\overline{u}) \leq 4\mathcal{E}_h(\Pi^h_{2\sqrt{2}\sin(\pi/8)}\overline{u}).$$

In view of Theorem 6.2.1 we have the following result:

Theorem 6.2.2. *Let $T > 4$. There exists a constant $C(T)$ such that*

$$\mathcal{E}_h(\overline{u}) \leq C(T)\int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt$$

holds for all solutions of (6.1.6) with $(\overline{u}^0, \overline{u}^1) \in V^h \times V^h$, uniformly on $h > 0$, V^h being the class of the two-grid data obtained with ratio 1/4.

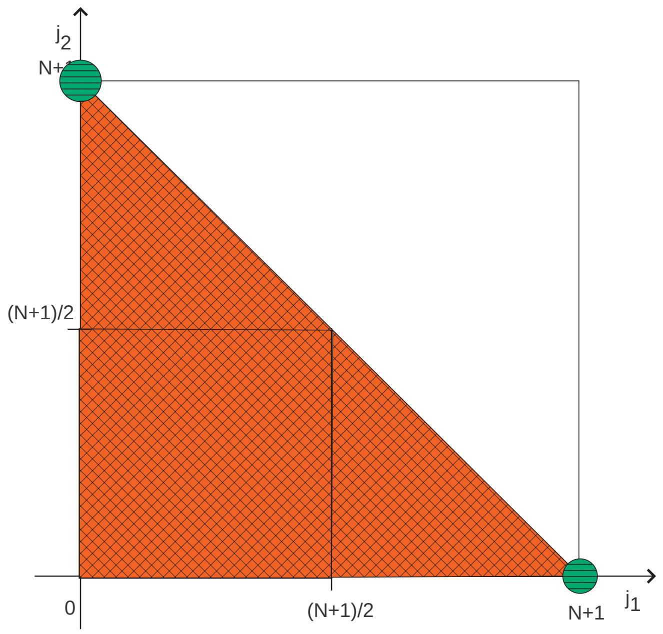

Figure 6.7: The red area represents the frequencies involved in $I_h(2)$. The green area corresponds to frequencies with group velocity of order h.

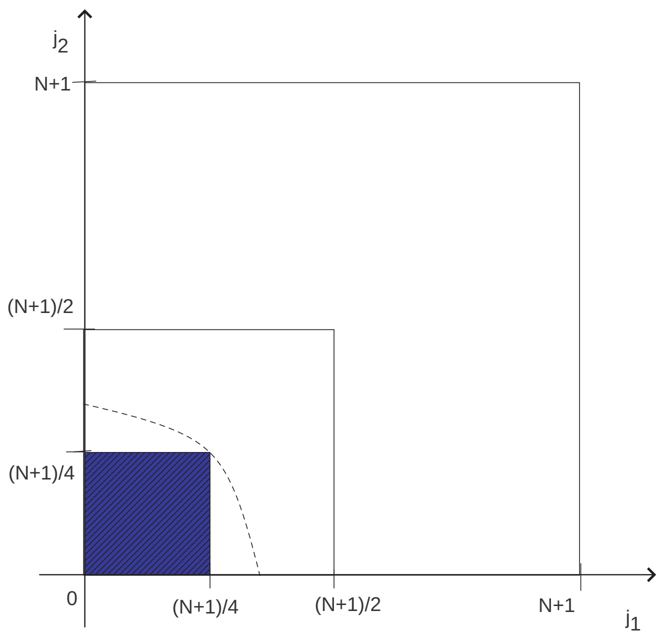

Figure 6.8: The blue area represents the frequencies $\omega_{\mathbf{j}}(h)$, $\mathbf{j} \in \Lambda_{(N+1)/4}$

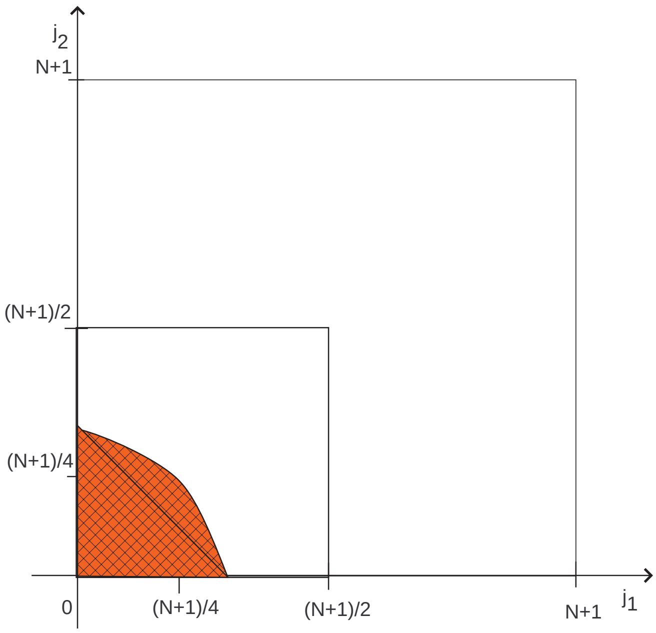

Figure 6.9: The red area represents the frequencies involved in $I_h(2\sqrt{2}\sin(\pi/8))$

Remark 6.2.4. *The time $T > 4$ is given by the observability time obtained in [142] for the class of solutions belonging to $I_h(2\sqrt{2}\sin(\pi/8))$. We recall that in view of (6.2.7) the observability time for the above class of solutions is given by:*

$$T\left(2\sqrt{2}\sin\left(\frac{\pi}{8}\right)\right) = \frac{2\sqrt{2}}{1-2\sin^2\left(\frac{\pi}{8}\right)} = \frac{2\sqrt{2}}{\cos\left(\frac{\pi}{4}\right)} = 4.$$

The time $T_0 = 4$ is not the optimal one. Its optimality depends on the optimality time $T(2\sqrt{2}\sin(\pi/8))$ in the low frequency class $I_h(2\sqrt{2}\sin(\pi/8))$. The analysis of the geometrical rays will allow us to conjecture that the minimal time should be

$$T_0 = \frac{2\sqrt{2}}{\cos(\pi/8)}.$$

Remark 6.2.5. *Any improvement of the time $T(\gamma)$ for (6.2.8) in the class $I_h(\gamma)$ would improve the time we have obtained. The observability time T_0 obtained in [142] is given in terms of the largest eigenvalue occuring in the Fourier representation of $\overline{u}$. The microlocal analysis ([78], [17], [6]) shows that, at least in the continuous case (see [129], [89], [94] and [3] for a semidiscrete case), the optimal time is that in which all the geometrical rays touch the observation subset of the boundary. In dimension one the modes associated with larger eigenvalues have smaller group velocity as we can see in Figure 6.10. This does not remain true for a dimension greater than one. This can be seen in Figure 6.12 and Figure 6.13 where we plot the level curves of the wave numbers and their group velocity.*

Remark 6.2.6. *We are not able to prove the observability inequality directly from (6.2.13). In our proof we introduce artificial modes associated with frequencies which belong to*

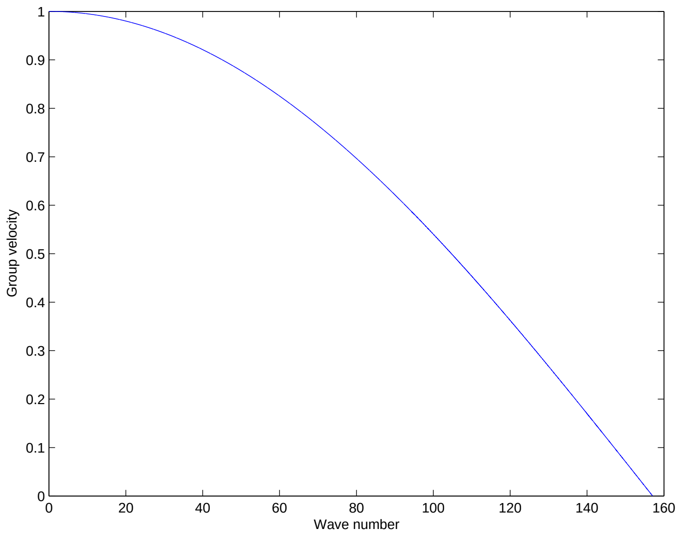

Figure 6.10: Group velocity in dimension one, $h = 1/50$

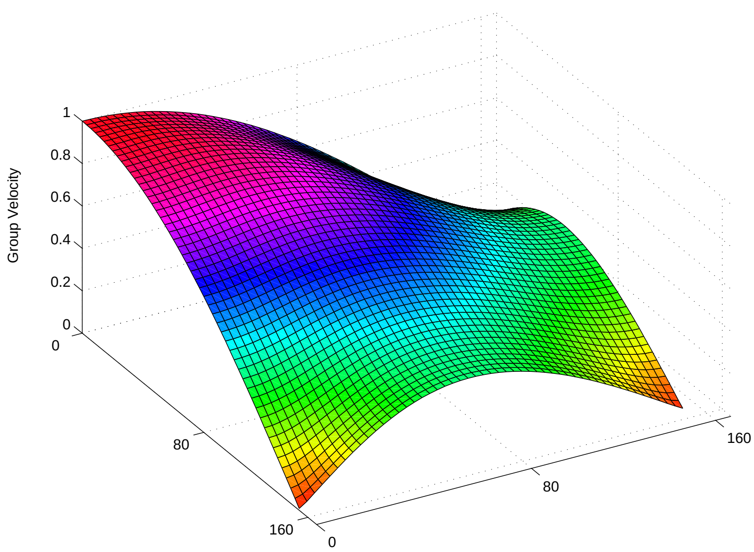

Figure 6.11: Group velocity in dimension two, $h = 1/50$

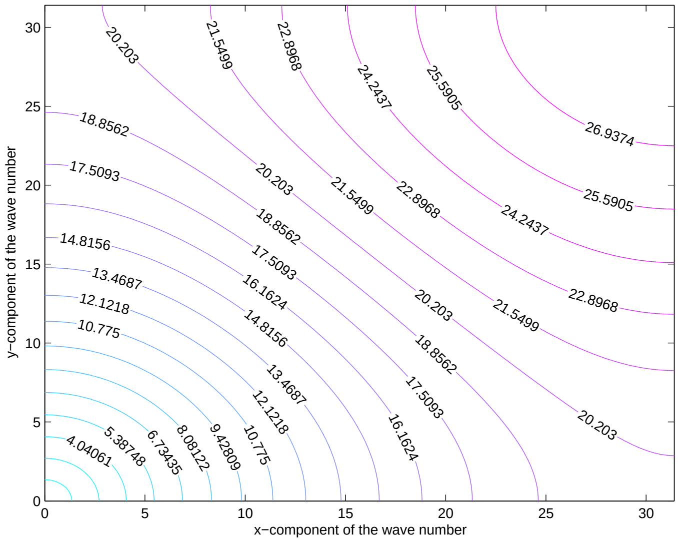

Figure 6.12: Level curves of the wave numbers, $h = 1/10$

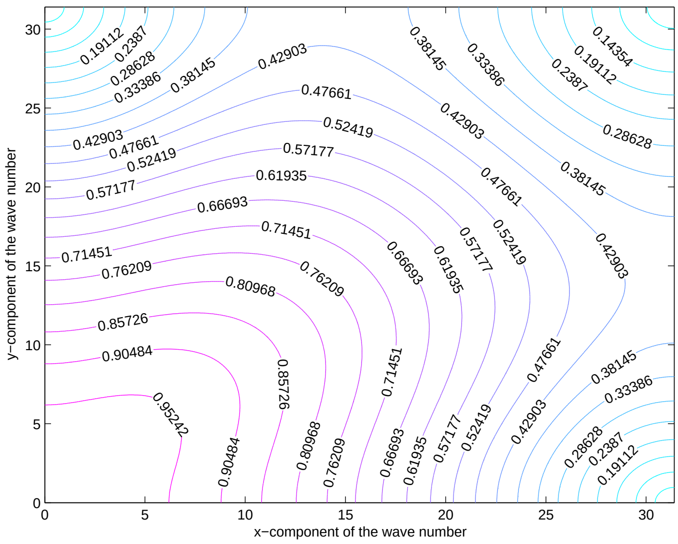

Figure 6.13: Level curves of the group velocity, $h = 1/10$

$I_h(2\sqrt{2}\sin(\pi/8))$ *but not to* $J_h(1/4)$. *However, as we will see in Section 6.3 the minimal time should be given by the unique common point of the two sets* $(\xi_1, \xi_2) = (\pi/4h, \pi/4h)$*:*

$$T_0 = \frac{2\sqrt{2}}{\cos(\pi/8)}.$$

Proof of Theorem 6.2.2. As we said before $J_h(1/4) \subset I_h(2\sqrt{2}\sin(\pi/8))$. This implies that

$$\mathcal{E}_h(\Upsilon^h_{1/4}\overline{u}) \leq \mathcal{E}_h(\Pi^h_{2\sqrt{2}\sin(\pi/8)}\overline{u}).$$

To apply Theorem 6.2.1 with $\gamma = 2\sqrt{2}\sin(\pi/8)$ it remains to prove that $\mathcal{E}_h(\overline{u}) \leq 4\mathcal{E}_h(\Upsilon^h_{1/4}\overline{u})$. The conservation of energy implies that

$$\mathcal{E}_h(\overline{u}) = \|\overline{u}^0\|^2_{\hbar^1} + \|\overline{u}^1\|^2_{\hbar^0}$$

and

$$\mathcal{E}_h(\Upsilon^h_{1/4}\overline{u}) = \|\Upsilon^h_{1/4}\overline{u}^0\|^2_{\hbar^1} + \|\Upsilon^h_{1/4}\overline{u}^1\|^2_{\hbar^0}.$$

We make use of the following Lemma, which will be proved in Section 6.4.

Lemma 6.2.1. *For any* $\overline{v} \in V^h$ *the following holds:*

$$\|\overline{v}\|_{\hbar^s} \leq 2^{(s+1)/2}\|\Upsilon^h_{1/4}\overline{v}\|_{\hbar^s}, \; 0 \leq s \leq 2. \tag{6.2.14}$$

Applying this Lemma to $\overline{u}^0 \in V^h$ and $\overline{u}^1 \in V^h$ we get

$$\|\overline{u}^0\|_{\hbar^1} \leq 2\|\Upsilon^h_{1/4}\overline{u}^0\|_{\hbar^1} \qquad \text{and} \qquad \|\overline{u}^1\|_{\hbar^0} \leq 2\|\Upsilon^h_{1/4}\overline{u}^1\|_{h^0}.$$

This proves that

$$\mathcal{E}_h(\overline{u}) \leq 4\mathcal{E}_h(\Upsilon^h_{1/4}\overline{u})$$

and finishes the proof.

□

6.3. The rays of geometric optics

Let us first define bicharacteristic rays. Consider the wave equation $u_{tt} - \Delta u = 0$. Bicharacteristic rays solve the Hamiltonian system

$$\begin{cases} x'(s) = -\xi; & t'(s) = \tau \\ \xi'(s) = |\xi|^2; & \tau'(s) = 0. \end{cases} \tag{6.3.1}$$

These rays describe the microlocal propagation of energy. The projection of the bicharacteristic rays in the (x, t) variables are the rays of geometric optics that play a fundamental role in the analysis of the observation and control properties through the geometrical control condition (GCC). As time evolves, the rays move in the physical space according to the solutions of (6.3.1). Moreover, the direction in the Fourier space (ξ, τ) in which the energy of solutions is concentrated as they propagate is given precisely by the projection of the bicharacteristic ray in the (ξ, τ) variables. This Hamiltonian system describes the dynamics of rays in the interior of the domain where the equation is satisfied. When rays reach the boundary they are reflected according to the laws of geometric optics.

In the continuous case, the observability inequality holds if and only if the GCC is satisfied (see Bardos, Lebeau, and Rauch [6] and Burq [17].)

In the case of semi-discrete wave equations with periodic boundary conditions the microlocal analysis has been used in [89] to prove the internal observability A similar approach has been used for the propagation of the local energy on infinite harmonic lattices [94].We also refer to [129] for the numerical waves and their group velocity.

As far as we know there is no microlocal approach for the uniform controllability of the semidiscrete problem (6.1.6), i.e. boundary controlability . This technique should provide the optimal time for boundary observability.

In the following we discuss the bicharacteristic rays associated with solutions of wavelength h.

The symbol of the semidiscrete system for the solutions of wavelength h is

$$p(\tau,\xi)=\tau^2-4\left(\sin^2\left(\frac{\xi_1}{2}\right)+\sin^2\left(\frac{\xi_2}{2}\right)\right).$$

The bicharacteristic rays are then defined as follows:

$$\begin{cases} x_j'(s)=-\sin(\xi_j),\ j=1,2,\\ t'(s)=\tau,\\ \xi_j'(s)=0,\ j=1,2,\\ \tau'(s)=0. \end{cases} \tag{6.3.2}$$

The projections into the physical space are:

$$x_j(t)=-\frac{\sin(\xi_j)}{\tau}t+x_{j,0},\ j=1,2. \tag{6.3.3}$$

It is interesting to note that the rays are straight lines, as for the constant coefficient wave equation, since the coefficients of the equation and the numerical discretization are both constant. We see, however, that in (6.3.2) both the direction and the velocity of propagation change with respect to those of the continuous wave equation. In particular, the velocity of propagation of the rays is given by

$$|\dot{x}(t)|=\frac{1}{2}\left(\frac{\sin^2(\xi_1)+\sin^2(\xi_2)}{\sin^2\frac{\xi_1}{2}+\sin^2\frac{\xi_2}{2}}\right)^{1/2}. \tag{6.3.4}$$

In the following we will obtain the minimal time needed by all the rays associated with the frequencies (ξ_1,ξ_2) belonging to $F_1=[0,\pi/4]^2$, respectively $F_2=\{\xi:\sin^2(\xi_1/2)+\sin^2(\xi_2/2)\leq 2\sin^2(\pi/8)\}$, to touch the observability zone Γ_0. These two sets of frequencies arise in our proof of observability of the two-grid algorithm.

Let us consider a straight line with direction $(\sin(\xi_1),\sin(\xi_2))$ starting from some point (x_0,y_0) situated inside the square $[0,1]\times[0,1]$. Wherever such a ray starts its velocity is the same, given by (6.3.4). Thus we have to find the maximal length of such a ray before it touches Γ_0. Typically, this ray will touch the boundary of the square $[0,1]\times[0,1]$ at the points A,B,C,D (see Figure 6.14). Its length is the same as the new one $ABC'D'$ obtained by reflection of the boundary.

It remains to find the largest straight line which starts from some point inside the square $[0,1]\times[0,1]$ and finishes on the boundary of the large square $[-1,1]\times[-1,1]$. We will prove

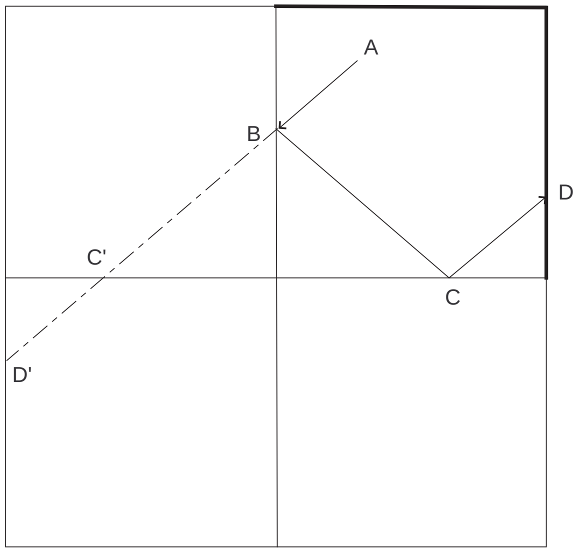

Figure 6.14: The propagation of a ray.

that the largest line of this type has the length

$$L(\xi_1,\xi_2) = 4\left(\frac{\sin^2(\xi_1)+\sin^2(\xi_2)}{\max\{\sin^2(\xi_1),\sin^2(\xi_2)\}}\right)^{1/2}. \quad (6.3.5)$$

Thus, the needed time $T(\xi_1,\xi_2)$ for such a ray to reach the observability area Γ_0 will be:

$$T(\xi_1,\xi_2) = 4\left(\frac{\sin^2(\xi_1/2)+\sin^2(\xi_2/2)}{\max\{\sin^2(\xi_1),\sin^2(\xi_2)\}}\right)^{1/2}.$$

In view of this analysis, the observability time within a class of solutions coresponding to a certain range of frequencies (ξ_1,ξ_2) should be the maximum of $T(\xi_1,\xi_2)$ on that range. Obviously, for that maximum to be finite one needs to exclude the points (ξ_1,ξ_2) in with both $\sin(\xi_1)$ and $\sin(\xi_2)$ vanish.

In Figure 6.15 we can see that when (ξ_1,ξ_2) is close to some of the points $(0,\pi)$, (π,π) or $(\pi,0)$, time blows-up. However, for $(\xi_1,\xi_2)\in[0,\pi/4]^2$, the time $T(\xi_1,\xi_2)$ achieves its maximum at the point $(\pi/4,\pi/4)$.

Now we maximize $T(\xi_1,\xi_2)$ along the sets F_1 and F_2 defined before. In both cases

$$T(\xi_1,\xi_2)\le 4\sqrt{2}\left(\frac{\max\{\sin^2(\xi_1/2),\sin^2(\xi_2/2)\}}{\max\{\sin^2(\xi_1),\sin^2(\xi_2)\}}\right)^{1/2}\le\frac{4\sqrt{2}}{2\cos(\max\{\xi_1,\xi_2\}/2)}\le\frac{2\sqrt{2}}{\cos(\pi/8)}$$

with equality for $\xi_1=\xi_2=\pi/4$.

The time needed for all the rays to touch the boundary is the same in the two classes F_1 and F_2. In view of this property we do not lose the optimality of the time T_0 by passing from $J_h(1/4)$ to $I_h(2\sqrt{2}\sin(\pi/8))$.

In the following we prove (6.3.5). Let (x_0,y_0) be a point inside the square $[0,1]\times[0,1]$, i.e. $0\le x_1,y_1\le 1$. The straight line passing by this point in the direction $(\sin(\xi_1),\sin(\xi_2))$ has the following equation

$$x-x_0=\alpha\sin(\xi_1),\ y-y_0=\alpha\sin(\xi_2),\ \alpha\in\mathbb{R}.$$

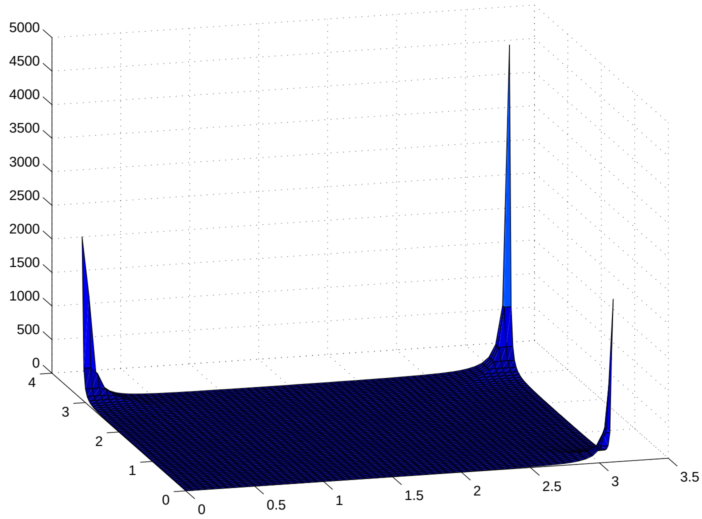

Figure 6.15: The time $T(\xi_1, \xi_2)$ for $(\xi_1, \xi_2) \in [0, \pi]^2$.

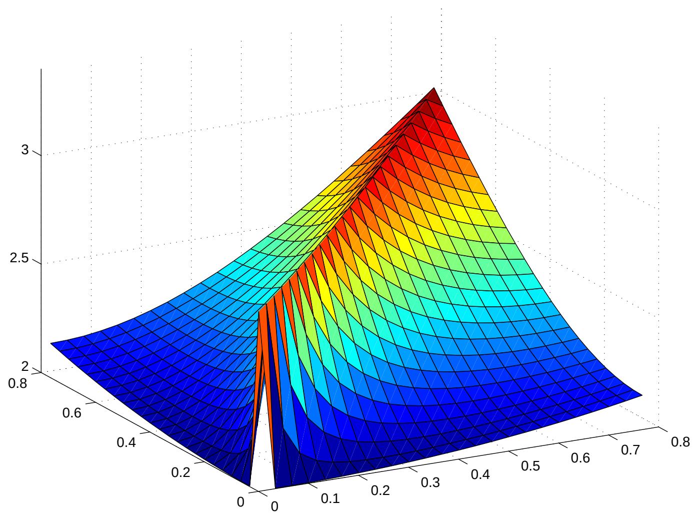

Figure 6.16: The time $T(\xi_1, \xi_2)$ for $(\xi_1, \xi_2) \in [0, \pi/4]^2$.

Consider the point (x_1, y_1) where this line touches the boundary of the square $[-1,1]\times[-1,1]$. Thus

$$x_1 = \pm 1 \quad \text{and} \quad -1 \leq y_1 \leq 1$$

or

$$y_1 = \pm 1 \quad \text{and} \quad -1 \leq y_2 \leq 1.$$

The length of the segment ending at (x_0, y_0) and (x_1, y_1) is given by

$$L^2(x_0, y_0, \xi_1, \xi_2) = (x_1 - x_0)^2 + (y - y_0)^2 = \alpha^2(\sin^2(\xi_1) + \sin^2(\xi_2)).$$

It remains to prove that

$$\alpha^2 \leq \frac{4}{\max\{\sin^2(\xi_1), \sin^2(\xi_2)\}}.$$

Observe that α satisfies

$$\alpha^2 = \frac{|x_1 - x_0|^2}{\sin^2(\xi_1)} \leq \frac{4}{\sin^2(\xi_1)} \quad \text{and} \quad \alpha^2 = \frac{|y_1 - y_0|^2}{\sin^2(\xi_2)} \leq \frac{4}{\sin^2(\xi_2)}.$$

Thus

$$\alpha^2 \leq \frac{4}{\max\{\sin^2(\xi_1), \sin^2(\xi_2)\}}$$

and then (6.3.5).

6.4. Spectral analysis of the V^h-Functions.

In this Section we analyze the h^s-norms of the functions belonging to V^h, i.e. the space of functions defined on the fine grid as a linear interpolation of the functions defined on the coarse one, and we prove Lemma 6.2.1.

Fix M a positive integer and $h = 1/4M$. Let us consider a function $\overline{v} \in V^h$ which admits the following representation in the basis $\{\overline{\varphi}^{\mathbf{k}}\}_{\mathbf{k}\in\Lambda_{4M-1}}$:

$$\overline{v} = \sum_{k\in\Lambda_{4M-1}} \widehat{v}(\mathbf{k})\overline{\varphi}^{\mathbf{k}}.$$

For each $\overline{v}$ we define its projection $\Upsilon^h_{1/4}\overline{v}$ on the space generated by the eigenfunctions $\{\overline{\varphi}^{\mathbf{k}}\}_{\mathbf{k}\in\Lambda_M}$ as

$$\Upsilon^h_{1/4}\overline{v} = \sum_{\mathbf{k}\in\Lambda_M} \widehat{v}(\mathbf{k})\overline{\varphi}^{\mathbf{k}}.$$

Clearly for each positive s the norm of the projection satisfies $\|\Upsilon^h_{1/4}\overline{v}\|_{\hbar^s} \leq \|\overline{v}\|_{\hbar^s}$. In Lemma 6.2.1 we analyze whether the converse inequality

$$\|\overline{v}\|_{\hbar^s} \leq C\|\Upsilon^h_{1/4}\overline{v}\|_{\hbar^s} \tag{6.4.1}$$

holds in the space V^h. A fine analysis of the Fourier coefficients $\widehat{v}(\mathbf{k})$ shows that (6.4.1) holds for all $0 \leq s \leq 2$ with a constant which does not depend on $h > 0$.

The following Lemma gives a description of the coefficients $\widehat{v}(\mathbf{k})$.

Lemma 6.4.1. *Let $\overline{v} \in V^h$. Then for any $\mathbf{k} = (k_1, k_2) \in \Lambda_{4M-1}$ the $\mathbf{k}$-th Fourier coefficient satisfies*

$$\widehat{v}(\mathbf{k}) = 4 \prod_{l=1}^{2} \cos^2\left(\frac{k_l \pi h}{2}\right) \cos^2(k_l \pi h) \sum_{\mathbf{j} \in \Lambda_{M-1}} v_{4\mathbf{j}} \overline{\varphi}_{\mathbf{k}}^{4\mathbf{j}}. \tag{6.4.2}$$

Remark 6.4.1. *For any $\mathbf{k} = (k_1, k_2)$ with at least one of its components belonging to the set $\{M, 2M, 3M\}$, the $\mathbf{k}$-th coefficient $\widehat{v}(\mathbf{k})$ vanishes. This comes from the fact that for any $\mathbf{k} = (k_1, k_2)$ and $\mathbf{j} = (j_1, j_2)$ the eigenfunction $\overline{\varphi}^{\mathbf{j}}$ satisfies $\overline{\varphi}^{4\mathbf{j}}(\mathbf{k}) = \sin(k_1 j_1 \pi/M) \sin(k_2 j_2 \pi/M)$.*

Proof. Firstly we analyze the one-dimensional case. The result extends to the 2d-case by iterating the same argument in each direction.

For each $1 \le k \le 4M - 1$ the coefficient $\widehat{v}(k)$ is given by

$$\widehat{v}(k) = (\overline{v}, \overline{\sin(k\pi x)}) = \frac{h}{2\pi} \sum_{j=1}^{4M-1} v_j \sin(kj\pi h).$$

In order to simplify the computations we first use that the coefficients v_{2j+1}, corresponding to odd indexes, satisfy $v_{2j+1} = (v_{2j} + v_{2j+2})/2$. Secondly we make use of the same property for v_{4j+2}, which verifies $v_{4j+2} = (v_{4j} + v_{4j+4})/2$.

Using that the function $\overline{v}$ satisfies $v_{2j+1} = (v_{2j} + v_{2j+2})/2$, $j = 0, ..., 2M - 1$, we obtain

$$\begin{aligned}
\sum_{j=1}^{4M-1} v_j \sin(kj\pi h) &= \sum_{j=1}^{2M-1} v_{2j} \sin(2kj\pi h) + \sum_{j=0}^{2M-1} v_{2j+1} \sin((2j+1)k\pi h) \\
&= \sum_{j=1}^{2M-1} v_{2j} \sin(2kj\pi h) + \sum_{j=0}^{2M-1} \frac{v_{2j} + v_{2j+2}}{2} \sin((2j+1)k\pi h) \\
&= \sum_{j=1}^{2M-1} v_{2j} \left(\sin(2kj\pi h) + \frac{\sin((2j+1)k\pi h) + \sin((2j-1)k\pi h)}{2} \right) \\
&= 2\cos^2\left(\frac{k\pi h}{2}\right) \sum_{j=1}^{2M-1} v_{2j} \sin(2kj\pi h).
\end{aligned}$$

In a similar way:

$$\begin{aligned}
\sum_{j=1}^{2M-1} v_{2j} \sin(2kj\pi h) &= \sum_{j=1}^{M-1} v_{4j} \sin(4kj\pi h) + \sum_{j=1}^{M-1} v_{4j+2} \sin((4j+2)k\pi h) \\
&= \sum_{j=1}^{M-1} v_{4j} \sin(4kj\pi h) + \sum_{j=1}^{M-1} \frac{v_{4j} + v_{4j+2}}{2} \sin((4j+2)k\pi h) \\
&= \sum_{j=1}^{M-1} v_{4j} \left(\sin(4kj\pi h) + \frac{\sin((4j+2)k\pi h) + \sin((4j-2)k\pi h)}{2} \right) \\
&= 2\cos^2(k\pi h) \sum_{j=1}^{M-1} v_{4j} \sin(4kj\pi h).
\end{aligned}$$

Thus, the coefficient $\widehat{v}(k)$ satisfies

$$\widehat{v}(k) = \sum_{j=1}^{4M-1} v_j \sin(kj\pi h) = 4\cos^2\left(\frac{k\pi h}{2}\right)\cos^2(k\pi h) \sum_{j=1}^{M-1} v_{4j}\sin(4kj\pi h),$$

which proves (6.4.2) in the one-dimensional case. Applying the same argument in each space direction we obtain (6.4.2) in the two-dimensional case. □

In view of these results we proceed to proving Lemma 6.2.1. Firstly we consider the $1-d$ case and extend it to the $2-d$ case. The same argument works in any space dimension.

Proof of Lemma 6.2.1. Let us choose an integer M such that $N+1 = 4M$.

Step I. The one-dimensional case

By Lemma 6.4.1, for any $k = 1, ..., 4M-1$, the k-th Fourier coefficient $\widehat{v}(k)$ is given by:

$$\widehat{v}(k) = 4\cos^2\left(\frac{k\pi h}{2}\right)\cos^2(k\pi h)\, g(k) := a(k)g(k)$$

with $g(k)$ defined by

$$g(k) := \sum_{j=0}^{M-1} v_{4j}\sin(4kj\pi h).$$

Firstly we show that for any $k = 1, ..., 4M-1$ the coefficients $a(k)$ satisfy

$$a_k\lambda_k(h) = a_{2M+k}\lambda_{2M+k}(h) = a_{2M-k}\lambda_{2M-k}(h) = a_{4M-k}\lambda_{4M-k}(h) \qquad (6.4.3)$$

where $\lambda_k(h) = 4/h^2\sin^2(k\pi h/2)$ are the eigenvalues of the one-dimensional counterpart of the system (6.2.1).

Indeed, for any $k = 1, ..., 4M-1$ the coefficients $a(k)$ verify:

$$\begin{aligned} h^2 a_k\lambda_k(h) &= 16\cos^2\left(\frac{k\pi h}{2}\right)\cos^2(k\pi h)\sin^2\left(\frac{k\pi h}{2}\right) \\ &= \sin^2(2k\pi h) = \sin^2\left(\frac{k\pi}{2M}\right). \end{aligned}$$

Using the periodicity of the function $x \mapsto \sin^2(x)$ we obtain (6.4.3).

Moreover the function g satisfies

$$g(k) = g(2M+k) = -g(2M-k) = -g(4M-k),\ k = 1, ..., M \qquad (6.4.4)$$

and

$$g(0) = g(M) = g(2M) = g(3M) = 0.$$

Then, by definition of the norm in $\hbar^s$:

$$\begin{aligned}
\|\overline{v}\|^2_{\hbar^s} &= \sum_{k=1}^{4M-1} a^2(k)g^2(k)\lambda_k^s(h) \\
&= \sum_{k=1}^{M-1} a^2(k)g^2(k)\lambda_k^s(h) + \sum_{k=1}^{M-1} a^2(2M-k)g^2(2M-k)\lambda_{2M-k}^s(h) \\
&\quad + \sum_{k=1}^{M-1} a^2(2M+k)g^2(2M+k)\lambda_{2M+k}^s(h) \\
&\quad + \sum_{k=1}^{M-1} a^2(4M-k)g^2(4M-k)\lambda_{4M-k}^s(h)
\end{aligned}$$

and

$$\|\Upsilon^h_{1/4}\overline{v}\|^2_{\hbar^s} = \sum_{k=1}^{M-1} a^2(k)g^2(k)\lambda_k^s(h).$$

We point out that for any $k = 1, ..., M-1$ and $j = M+1, ..., 4M-1$ the following holds:

$$\lambda_k(h) \le \frac{4}{h^2}\sin^2\left(\frac{\pi}{8}\right) \le \lambda_j(h).$$

Also for any $s \in [0,2]$, $k = 1, ..., M-1$ and $j \in \{2M-k, 2M+k, 4M-k\}$ we get

$$\begin{aligned}
a^2(k)\lambda_k^s(h) &= \lambda_k^{s-2}(h)a^2(k)\lambda_k^2(h) = \lambda_k^{s-2}(h)a^2(j)\lambda_j^2(h) \\
&= \left(\frac{\lambda_j(h)}{\lambda_k(h)}\right)^{2-s} a^2(j)\lambda_j^s(h) \\
&\ge a^2(j)\lambda_j^s(h).
\end{aligned}$$

In view of (6.4.4) the $\hbar^s$-norm of $\overline{v}$ satisfies:

$$\begin{aligned}
\|\overline{v}\|^2_{\hbar^s} &= \sum_{k=1}^{M-1} g^2(k)\left[a^2(k)\lambda_k^s(h) + a^2(2M-k)\lambda_{2M-k}^s(h)\right. \\
&\qquad\qquad \left.+a^2(2M+k)\lambda_{2M+k}^s(h) + a^2(4M-k)\lambda_{4M-k}^s(h)\right] \\
&\le 4\sum_{k=1}^{M-1} g^2(k)a^2(k)\lambda_k^s(h) = 4\|\Upsilon^h_{1/4}\overline{v}\|^2_{\hbar^s}.
\end{aligned}$$

which finishes the proof of the $1-d$ case.

Step II. Reduction to the one-dimensional case.

We reduce the two-dimensional case to the one-dimensional one. The function $\overline{v}$ admits a representation in the Fourier space as:

$$\overline{v}(x,y) = \sum_{j,k=1}^{4M-1} a_{jk}\overline{\varphi}^j(x)\overline{\varphi}^k(y),\ x = j_1h,\ y = k_1h,\ j_1, k_1 \in \Lambda_{4M-1},$$

where $\overline{\varphi}^j(x) = \sin(j\pi x)$, $j \in \Lambda_{4M-1}$.

Thus its projection $\Upsilon^h_{1/4}\overline{u}$ satisfies

$$(\Upsilon^h_{1/4}\overline{v})(x,y) = \sum_{j,k=1}^{M-1} a_{jk}\overline{\varphi}^j(x)\overline{\varphi}^k(y).$$

We shall prove that for each fixed x, the one dimensional $\hbar^s$-norm in the variable y satisfies

$$\|\overline{v}(x,\cdot)\|^2_{\hbar^s} \leq 4\|\Upsilon^h_{1/4}\overline{v}(x,\cdot)\|^2_{\hbar^s}. \tag{6.4.5}$$

A similar argument will guarantee that

$$\|\overline{v}(\cdot,y)\|^2_{\hbar^s} \leq 4\|\Upsilon^h_{1/4}\overline{v}(\cdot,y)\|^2_{\hbar^s}. \tag{6.4.6}$$

Taking into account that

$$\lambda^s_{jk} = (\lambda_j + \lambda_k)^s \leq 2^{s-1}(\lambda^s_j + \lambda^s_k),$$

(6.4.5) and (6.4.6) give us

$$\|\overline{v}\|_{\hbar^s} \leq 2^{(s+1)/2}\|\Upsilon^h_{1/4}\overline{v}\|_{\hbar^s}.$$

To prove (6.4.5) we write $\overline{v}(x,\cdot)$ as

$$\overline{v}(x,y) = \sum_{k=1}^{4M-1} a_k(x)\overline{\varphi}^k(y).$$

For each x fixed, the function $\overline{v}(x,\cdot)$ is obtained by a one-dimensional interpolation of the two-grid type. Then, applying Step I we obtain (6.4.5) and finish the proof.

□

6.5. Proof of the main result

This section is devoted to the proof of Theorem 6.2.1 and of some technical Lemmas. We introduce a time-spectral decomposition of the solution $\overline{u}$. In the following we make precise the time projectors P_k which give us the a time-spectral decomposition of $\overline{u}$. These are essentially the ones introduced in [78].

Firstly, a similar argument as in the proof of (6.2.8) (see [142] for more details) shows the existence of two positive constants $\delta(T,\gamma)$ and $\epsilon(T,\gamma,\delta)$ for any $T > T(\gamma)$ such that

$$\mathcal{E}_h(\overline{v}) \leq C(T,\gamma,\epsilon,\delta)\int_{2\delta}^{T-2\delta}\int_{\Gamma_h}|\partial^h_n\overline{v}|^2 d\Gamma_h dt \tag{6.5.1}$$

for all $\overline{v} \in I_h(\gamma+\epsilon)$. More precisely, for any given $T > T(\gamma)$ we can choose positive $\delta = \delta(T,\delta)$ such that $0 < \delta < (T - T(\gamma))/4$. Thus we can choose an $\epsilon = \epsilon(T,\gamma,\delta)$ such that (6.5.1) still holds.

With ϵ verifying (6.5.1) let us choose positive constants a, b, c and μ with $b > a$ satisfying

$$1 < c < \frac{b-\mu}{a+\mu} < \frac{b}{a} < \frac{\gamma+\epsilon}{\gamma}. \tag{6.5.2}$$

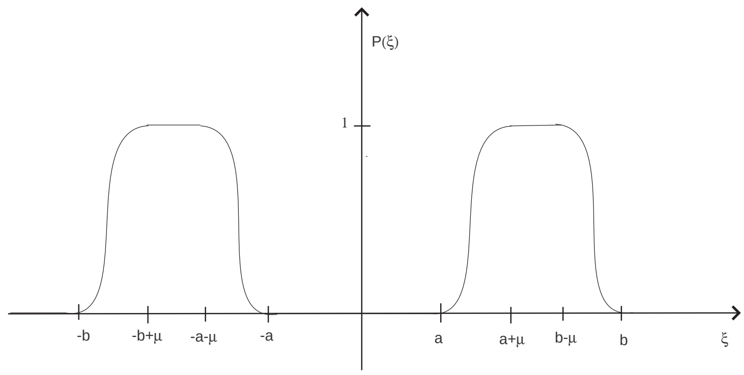

Figure 6.17: An example of a function $P(\xi)$

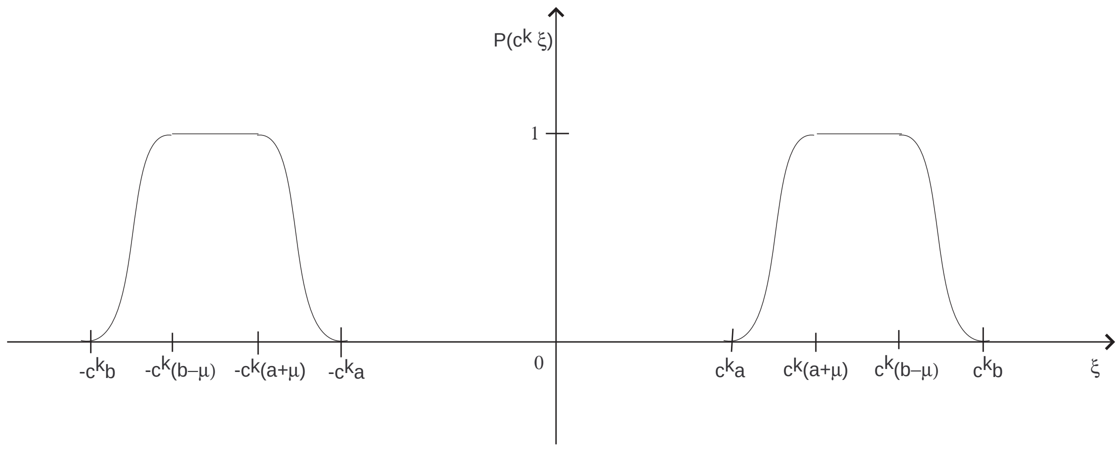

Figure 6.18: The function $P(c^{-k}\xi)$

Also let $F \in C_c^\infty(\mathbb{R})$ be supported in (a, b), $0 \le F \le 1$ such that $F \equiv 1$ in $[a + \mu, b - \mu]$. Set $P(\tau) = F(\tau) + F(-\tau)$ (see Figure 6.17 for an example of such a function). For any function $f \in L^1(\mathbb{R})$ and $k \ge 0$ we consider the projector $P_k f$ defined by

$$(P_k f)(t) = \int_{\mathbb{R}_\tau} \int_{\mathbb{R}_s} P(c^{-k}\tau) f(s) e^{i(t-s)\tau} ds d\tau. \tag{6.5.3}$$

The Fourier transform of $\overline{u}$, in the t variable, reads

$$\widehat{\overline{u}}(\tau) = \sum_{\mathbf{j}\in\mathbb{Z}^2} [\delta(\tau - \omega_{\mathbf{j}}(h))\widehat{u}_+(\mathbf{j}) + \delta(\tau + \omega_{\mathbf{j}}(h))\widehat{u}_-(\mathbf{j})] \overline{\varphi}^{\mathbf{j}}. \tag{6.5.4}$$

Therefore, the projector $P_k\overline{u}$ satisfies

$$P_k\overline{u}(t) = \sum_{\mathbf{j}\in\mathbb{Z}^2} F(c^{-k}\omega_{\mathbf{j}}(h)) \left[e^{it\omega_{\mathbf{j}}(h)}\widehat{u}_+(\mathbf{j}) + e^{-it\omega_{\mathbf{j}}(h)}\widehat{u}_-(\mathbf{j})\right] \overline{\varphi}^{\mathbf{j}} \tag{6.5.5}$$

and its energy is given by

$$\mathcal{E}_h(P_k\overline{u}) = \sum_{\mathbf{j}\in\mathbb{Z}^2} F^2(c^{-k}\omega_{\mathbf{j}}(h))\omega_{\mathbf{j}}^2(h)(|\widehat{u}_{\mathbf{j}+}|^2 + |\widehat{u}_{\mathbf{j}-}|^2). \tag{6.5.6}$$

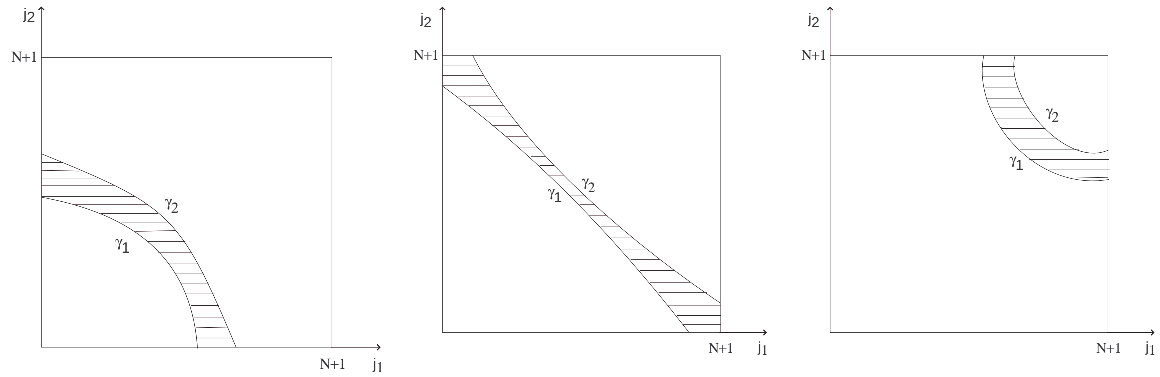

Figure 6.19: The dashed area corresponds to $\mathbf{j} = (j_1, j_2)$ such that $F(c^{-k}\omega_{\mathbf{j}}) \neq 0$. The level curves of the function $\frac{2}{h}(\sin^2 \frac{\xi_1 \pi h}{2} + \sin^2 \frac{\xi_2 \pi h}{2})^{1/2}$ corresponding to $c^k a$ and $c^k b$ are denoted by γ_1 and γ_2.

We briefly sketch the main steps of the proof. Inequalities (6.2.9) and (6.2.10) show that

$$\mathcal{E}_h(\overline{u}) \leq C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \Pi_\gamma^h \overline{u}|^2 d\Gamma_h dt.$$

Unfortunately, the right side term cannot be estimated directly in terms of the energy of the solution $\overline{u}$ measured at the boundary Γ_h: $\int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt$. To avoid this difficulty, we use the spectral-time decomposition introduced above. We will choose two positive integers k_0 and k_h, $k_0 \leq k_h$, k_0 independent of h, such that $\{P_k \overline{u}\}_{k=k_0}^{k_h}$ covers, except possibly for a finite number, all the frequencies occurring in $\Pi_\gamma^h \overline{u}$. The term containing a finite number of frequencies constitutes a lower order term which will be absorbed by a compactness-uniqueness argument.

Firstly we prove that

$$\mathcal{E}_h(\Pi_\gamma^h \overline{u}) \leq \sum_{k=k_0}^{k_h} \mathcal{E}_h(P_k \overline{u}) + LOT \tag{6.5.7}$$

where LOT is a lower order term.

Next we use that each projection $P_k \overline{u}$, $k_0 \leq k \leq k_h$ belongs to the class $I_h(\gamma + \epsilon)$ and consequently, according to (6.5.1), satisfies the observability inequality:

$$\mathcal{E}_h(P_k \overline{u}) \leq C(T, \gamma, \delta, \epsilon) \int_\delta^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \overline{u}|^2 d\Gamma_h dt. \tag{6.5.8}$$

Thus, combining (6.5.7) and (6.5.8),

$$\mathcal{E}_h(\Pi_\gamma^h \overline{u}) \leq C(T, \gamma, \delta, \epsilon) \sum_{k=k_0}^{k_h} \int_\delta^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \overline{u}|^2 d\Gamma_h dt + LOT.$$

Using the ideas of [78] and [17] we will obtain estimates of the above sum in terms of $\int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt$. More precisely, for any solution $\overline{u}$ of equation (6.1.6) we will obtain:

$$\sum_{k=k_0}^{k_h} \int_\delta^{T-\delta} \int_{\Gamma_h} |\partial_n^h P_k \overline{u}|^2 d\Gamma_h dt \leq 2 \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt + \frac{C(\epsilon, \delta, T)}{c^{2k_0}} \mathcal{E}_h(\overline{u}),$$

for some constant $C(\epsilon,\delta,T)$, independent of k_0. We remark that the right hand term contains the whole solution $\overline{u}$ and not only one projection of it.

Summing up all the above estimates we obtain

$$\mathcal{E}_h(\overline{u}) \leq C\mathcal{E}_h(\Pi_\gamma^h\overline{u}) \leq C(T,\gamma,\delta,\epsilon)\int_0^T\int_{\Gamma_h}|\partial_n^h\overline{u}|^2 d\Gamma_h dt + \frac{C(\epsilon,\delta,T)}{c^{2k_0}}\mathcal{E}_h(\overline{u}) + LOT.$$

Choosing h small and k_0 sufficiently large, but still independent of h, the energy term from the right side will be absorbed and then

$$\mathcal{E}_h(\overline{u}) \leq C(T,\gamma,\delta,\epsilon)\int_0^T\int_{\Gamma_h}|\partial_n^h\overline{u}|^2 d\Gamma_h dt + LOT.$$

Finally, classical arguments of compactness-uniqueness (see [65] and [142] in the semi-discrete settings) allow us to get rid of the lower order term.

We remark that the high frequency component of $\overline{u}$, i.e. the part of the solution $\overline{u}$ orthogonal to $\Pi_\gamma^h\overline{u}$, occurs only in the hypothesis (6.2.9). The rest of the arguments still work for all solutions $\overline{u}$ of equation (6.1.6).

In the following we give the details of the proofs of the above steps.

Proof of Theorem 6.2.1. **Step I. Upper bounds of $\mathcal{E}_h(\Pi_\gamma^h\overline{u})$ in terms of the energy of projections $\mathcal{E}_h(P_k\overline{u})$.**

The condition $1 < c < b/a$ imposed in (6.5.2) shows that

$$\bigcup_{k\geq 0}(ac^k, bc^k) = (a,\infty). \tag{6.5.9}$$

This means that any frequency $\omega_{\mathbf{j}}(h) \geq a$ occurs in at least one of the projections $P_k\overline{u}$, $k \geq 0$.

Let us now consider k_h such that $c^{k_h}(a+\mu) \leq \gamma/h < c^{k_h+1}(a+\mu)$. This choice is always possible for the small parameter h. In fact for any h verifying $a+\mu < \gamma/h$ there exists such a k_h. We make use of the second condition imposed in (6.5.2): $1 < c < (b-\mu)/(a+\mu)$. The election of k_h shows that for all $0 \leq k \leq k_h$

$$a+\mu \leq c^k(a+\mu) \leq c^{k_h}(a+\mu) \leq \gamma/h \leq c^{k_h+1}(a+\mu) \leq c^{k_h}(b-\mu).$$

Then any frequency $\omega_{\mathbf{j}}(h)$ belonging to $[a+\mu,\gamma/h]$ is contained in at least one interval of the form $[c^k(a+\mu), c^k(b-\mu)]$. Using that F is identically one on $[a+\mu, b-\mu]$ we get $F(c^{-k}\omega_{\mathbf{j}}(h)) = 1$ and consequently

$$1 \leq \sum_{k=0}^{k_h} F(c^{-k}\omega_{\mathbf{j}}(h))^2,\ \forall\,\omega_{\mathbf{j}}(h) \in \left[a+\mu, \frac{\gamma}{h}\right]. \tag{6.5.10}$$

In view of (6.5.6) and (6.5.10) the energy of the low frequency component $\Pi_\gamma^h\overline{u}$ of the solution $\overline{u}$, a low order term being excepted, can be bounded from above by the energy of all the projections $(P_k\overline{u})_{k=1}^{k_h}$:

$$\begin{aligned}
\mathcal{E}_h(\Pi_\gamma^h\overline{u}) &= \sum_{\omega_{\mathbf{j}}(h)\leq\gamma/h} \omega_{\mathbf{j}}^2(h)\left(|\widehat{u}_{\mathbf{j}+}|^2+\widehat{|}u_{\mathbf{j}-}|^2\right)\\
&= \sum_{\omega_{\mathbf{j}}(h)<a+\mu} \omega_{\mathbf{j}}^2(h)\left(|\widehat{u}_{\mathbf{j}+}|^2+|\widehat{u}_{\mathbf{j}-}|^2\right)+\sum_{a+\mu\leq\omega_{\mathbf{j}}(h)\leq\gamma/h} \omega_{\mathbf{j}}^2(h)\left(|\widehat{u}_{\mathbf{j}+}|^2+|\widehat{u}_{\mathbf{j}-}|^2\right)\\
&\leq (a+\mu)^2\sum_{\omega_{\mathbf{j}}(h)<a+\mu}\left(|\widehat{u}_{\mathbf{j}+}|^2+|\widehat{u}_{\mathbf{j}-}|^2\right)\\
&\quad+\sum_{a+\mu\leq\omega_{\mathbf{j}}(h)\leq\gamma/h}\sum_{k=0}^{k_h} F^2(c^{-k}\omega_{\mathbf{j}}(h))\omega_{\mathbf{j}}^2(h)\left(|\widehat{u}_{\mathbf{j}+}|^2+|\widehat{u}_{\mathbf{j}-}|^2\right)\\
&\leq C(a,\mu)\sum_{\omega_{\mathbf{j}}(h)<a+\mu}\left(|\widehat{u}_{\mathbf{j}+}|^2+|\widehat{u}_{\mathbf{j}-}|^2\right)+\sum_{k=0}^{k_h}\sum_{\mathbf{j}\in\mathbb{Z}^d} F^2(c^{-k}\omega_{\mathbf{j}}(h))\omega_{\mathbf{j}}^2(h)\left(|\widehat{u}_{\mathbf{j}+}|^2+|\widehat{u}_{\mathbf{j}-}|^2\right)\\
&= C(a,\mu)\sum_{\omega_{\mathbf{j}}(h)<a+\mu}\left(|\widehat{u}_{\mathbf{j}+}|^2+|\widehat{u}_{\mathbf{j}-}|^2\right)+\sum_{k=0}^{k_h}\mathcal{E}_h(P_k\overline{u}).
\end{aligned}$$

Let be $k_0 \leq k_h$, positive, independent of h, which will be specified later. A similar argument as above implies the existence of a positive constant $C(k_0, a, \mu)$ such that

$$\mathcal{E}_h(\Pi_\gamma^h\overline{u}) \leq \sum_{k=k_0}^{k_h}\mathcal{E}_h(P_k\overline{u}) + C(k_0,a,\mu)\sum_{\omega_{\mathbf{j}}(h)<(a+\mu)c^{k_0}}\left[|\widehat{u}_{\mathbf{j}+}|^2+|\widehat{u}_{\mathbf{j}-}|^2\right]. \tag{6.5.11}$$

Step II. Observability inequalities for the projections $(P_k\overline{u})_{k\geq k_0}^{k_h}$.

The next step is to apply the observability inequality (6.5.1) to each projection $P_k\overline{u}$, $k_0 \leq k \leq k_h$. To do that we have to prove that each of them belongs to the class where (6.5.1) holds:

$$P_k\overline{u}(t) \in I_h(\gamma+\epsilon),\ k_0 \leq k \leq k_h.$$

We remark that the projector $P_k\overline{u}(t)$ contains only the frequencies $\omega_{\mathbf{j}}(h) \in (c^k a, c^k b)$. For a given $k < k_h$ any frequency $\omega_{\mathbf{j}}(h)$ involved in the decomposition (6.5.5) of $P_k\overline{u}$ satisfies

$$\omega_{\mathbf{j}}(h) \leq c^k b \leq c^{k_h} b c^{-1} < c^{k_h}(a+\mu) \leq \frac{\gamma}{h}.$$

This shows that $P_k\overline{u} \in I_h(\gamma)$ for any $0 \leq k \leq k_h$. In the case $k = k_h$, the following holds:

$$\begin{aligned}
c^{k_h}a < c^{k_h}(a+\mu) &\leq \frac{\gamma}{h} < c^{k_h+1}(a+\mu) < c^{k_h}c(a+\mu) < c^{k_h}(b-\mu)\\
&\leq c^{k_h}b < \frac{\gamma}{ah}b < \frac{\gamma+\epsilon}{h},
\end{aligned}$$

which shows that $P_{k_h}\overline{u}(t) \in I_h(\gamma+\epsilon)$.

Now we apply inequality (6.5.1) to each projection $P_k\overline{u}$:

$$\mathcal{E}_h(P_k\overline{u}) \leq C(T,\delta,\epsilon,\gamma)\int_{2\delta}^{T-2\delta}\int_{\Gamma_h}|\partial_n^h(P_k\overline{u})|^2 d\Gamma_h dt,\ k_0 \leq k \leq k_h. \tag{6.5.12}$$

Using (6.5.11) and the above inequalities we obtain that

$$\begin{aligned}\mathcal{E}_h(\Pi^h_\gamma\overline{u}) \quad &\leq \quad C(T,\gamma,\delta,\epsilon)\sum_{k=k_0}^{k_h}\int_{2\delta}^{T-2\delta}\int_{\Gamma_h}|\partial^h_n(P_k\overline{u})|^2d\Gamma_h dt \\ &\quad + C(k_0,a,\mu)\sum_{\omega_{\mathbf{j}}(h)<(a+\mu)c^{k_0}}\left[|\widehat{u}_{\mathbf{j}+}|^2+|\widehat{u}_{\mathbf{j}-}|^2\right]. \end{aligned} \tag{6.5.13}$$

Step III. Boundary Observability of the Solution $\overline{u}$.

The argument we will use is by now classical and has been successfully applied in the context of the semiclassical reduction of the boundary controllability of continuous Schrödinger and wave equations: [78], [17], [79].

Let us denote $\psi = \mathbf{1}_{[0,T]}$ and let be $\varphi \in C_c^\infty(0,T)$ such that $0 \leq \varphi \leq 1$ and

$$\varphi|_{[2\delta,T-2\delta]} \equiv 1, \text{ supp } \varphi \subset (\delta, T-\delta). \tag{6.5.14}$$

We claim the existence of a constant $C(T,\varphi,\psi,\delta)$ such that

$$\begin{aligned}\int_{\mathbb{R}}\varphi^2(t)\int_{\Gamma_h}|\partial^h_n(P_k\overline{u})|^2d\Gamma_h dt \quad &\leq \quad 2\int_{\mathbb{R}}\varphi^2(t)\int_{\Gamma_h}|P_k(\psi\partial^h_n\overline{u})|^2d\Gamma_h dt \\ &\quad + \frac{C(T,\varphi,\psi,\delta)}{c^{2k}}\mathcal{E}_h(\overline{u}) \end{aligned} \tag{6.5.15}$$

for all $k \geq 0$. Also we will prove that the following inequality holds for some constant C independent of h

$$\sum_{k=k_0}^{k_h}\int_{\mathbb{R}}\varphi^2(t)\int_{\Gamma_h}|P_k(\psi\partial^h_n\overline{u})|^2d\Gamma_h dt \leq C\int_0^T\int_{\Gamma_h}|\partial^h_n\overline{u}|^2d\Gamma_h dt. \tag{6.5.16}$$

We postpone the proofs of (6.5.15) and (6.5.16) and proceed to the last step of the proof. We point out that the last two inequalities hold for all solutions of the semi-discrete wave equation (6.1.6), without imposing any additional hypotheses (there is no need of filtering or using any two-grid algorithm).

Taking the sum in (6.5.15) we get

$$\begin{aligned}\sum_{k=k_0}^{k_h}\int_{\mathbb{R}}\varphi^2(t)\int_{\Gamma_h}|\partial^h_n(P_k\overline{u})|^2d\Gamma_h dt \quad &\leq \quad 2\sum_{k=k_0}^{k_h}\int_{\mathbb{R}}\varphi^2(t)\int_{\Gamma_h}|P_k(\psi\partial^h_n\overline{u})|^2d\Gamma_h dt \\ &\quad + C(T,\varphi,\psi,\delta)\mathcal{E}_h(\overline{u})\sum_{k\geq k_0}\frac{1}{c^{2k}} \\ &\leq \quad 2\sum_{k=k_0}^{k_h}\int_{\mathbb{R}}\varphi^2(t)\int_{\Gamma_h}|P_k(\psi\partial^h_n\overline{u})|^2d\Gamma_h dt \\ &\quad + \frac{C(T,\varphi,\psi,\delta)}{c^{2k_0}}\mathcal{E}_h(\overline{u}). \end{aligned} \tag{6.5.17}$$

Putting together the energy estimate $\mathcal{E}_h(\overline{u}) \leq C\mathcal{E}_h(\Pi^h_\gamma(\overline{u}))$, (6.5.13) and (6.5.17) we obtain the existence of a constant $C(T) = C(T, \gamma, \delta, \epsilon, \varphi, \psi)$ such that

$$\begin{aligned} \mathcal{E}_h(\overline{u}) &\leq C(k_0, a, \mu) \sum_{\omega_{\mathbf{j}}(h) \leq c^{k_0}(a+\mu)} \left[|\widehat{u}_{\mathbf{j}+}|^2 + |\widehat{u}_{\mathbf{j}-}|^2 \right] \\ &\quad + C(T) \sum_{k=k_0}^{k_h} \int_{\mathbb{R}} \varphi^2(t) \int_{\Gamma_h} |P_k(\psi \partial_n^h \overline{u})|^2 d\Gamma_h dt \\ &\quad + \frac{C(T, \varphi, \psi, \delta)}{c^{2k_0}} \mathcal{E}_h(\overline{u}). \end{aligned}$$

Choosing a $k_0 = k_0(T)$ verifying

$$\frac{C(T, \varphi, \psi, \delta)}{c^{2k_0}} \leq \frac{1}{2}$$

we obtain

$$\begin{aligned} \frac{1}{2} \mathcal{E}_h(\overline{u}) &\leq C(k_0, a, \mu) \sum_{\omega_{\mathbf{j}}(h) \leq c^{k_0}(a+\mu)} \left[|\widehat{u}_{\mathbf{j}+}|^2 + |\widehat{u}_{\mathbf{j}-}|^2 \right] \\ &\quad + C(T) \sum_{k=k_0}^{k_h} \int_{\mathbb{R}} \varphi^2(t) \int_{\Gamma_h} |P_k(\psi \partial_n^h \overline{u})|^2 d\Gamma_h dt. \end{aligned}$$

By (6.5.16) we obtain

$$\frac{1}{2} \mathcal{E}_h(\overline{u}) \leq C(T) \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}|^2 d\Gamma_h dt + C(T) \sum_{\omega_{\mathbf{j}}(h) \leq c^{k_0}(a+\mu)} \left[|\widehat{u}_{\mathbf{j}+}|^2 + |\widehat{u}_{\mathbf{j}-}|^2 \right], \tag{6.5.18}$$

which is exactly the desired inequality (6.2.11), except for the last term. Using a compactness-uniqueness argument as in [78], [17], [142] we obtain (6.2.11).

It remains to prove (6.5.15) and (6.5.16).

Step IV. Proof of (6.5.15). The main ingredient of the proof is the following lemma:

Lemma 6.5.1. *([78], [17]) Let $\varphi \in C_0^\infty(0,T)$, $\psi \in L^\infty(\mathbb{R})$ be such that $\psi \equiv 1$ on $(0,T)$ and $(P_k)_{k \geq 0}$ be defined as above. There exists a constant $C = C(T, \varphi, \psi, F)$ such that*

$$\int_{\mathbb{R}} \|\varphi(t) P_k(a)(t)\|_{l^2(\Gamma_h)}^2 dt \leq 2 \int_{\mathbb{R}} \|\varphi(t) P_k(\psi a)(t)\|_{l^2(\Gamma_h)}^2 dt + C c^{-2k} \sup_{l \in \mathbb{Z}} \|a\|_{L^2((lT,(l+1)T),\, l^2(\Gamma_h))}^2 \tag{6.5.19}$$

holds for all $a \in L^2_{loc}(\mathbb{R}, l^2(\Gamma_h))$ and for all $k \geq 0$.

Let us choose $a(t) = (\partial_n^h \overline{u})(t)$ in the above Lemma. Using the definitions of the discrete normal derivative ∂_n^h and the operator P_k it is easy to see that they commute:

$$\partial_n^h(P_k \overline{u}) = P_k(\partial_n^h \overline{u}).$$

Thus, Lemma 6.5.1 yields

$$\begin{aligned} \int_{\mathbb{R}} \varphi^2(t) \int_{\Gamma_h} |\partial_n^h(P_k \overline{u})|^2 d\Gamma_h dt &= \int_{\mathbb{R}} \varphi^2(t) \int_{\Gamma_h} |P_k(\partial_n^h \overline{u})(t)|^2 d\Gamma_h dt \\ &\leq 2 \int_{\mathbb{R}} \varphi^2(t) \|P_k(\psi \partial_n^h \overline{u})(t)\|_{l^2(\Gamma_h)}^2 dt \\ &\quad + C c^{-2k} \sup_{l \in \mathbb{Z}} \|\partial_n^h \overline{u}\|_{L^2(lT,(l+1)T)}^2. \end{aligned} \tag{6.5.20}$$

At this point we apply the so-called *"direct inequality"* (6.2.12), which holds for all solutions $\overline{u}$ of (6.1.6). Thus, a translation in time in (6.2.12) together with the conservation of energy shows

$$\sup_{l\in\mathbb{Z}} \int_{lT}^{(l+1)T} \int_{\Gamma_h} |\partial_n^h \overline{u}(t)|^2 d\Gamma_h dt \leq C(T)\mathcal{E}_h(\overline{u}). \tag{6.5.21}$$

We apply (6.5.20) in (6.5.21) obtaining

$$\begin{aligned} \int_{\mathbb{R}} \varphi^2(t) \|\partial_n^h (P_k \overline{u})(t)\|^2_{l^2(\Gamma_h)} dt \quad &\leq \quad 2\int_{\mathbb{R}} \varphi^2(t) \|P_k(\psi \partial_n^h \overline{u})(t)\|^2_{l^2(\Gamma_h)} dt \\ &\quad + C(T,\varphi,\psi,\delta) c^{-2k} \mathcal{E}_h(\overline{u}). \end{aligned} \tag{6.5.22}$$

which proves (6.5.15).

Step V. Proof of (6.5.16). Observe that any real number τ belongs either to a finite number of intervals of the form $(\pm ac^k, \pm bc^k)$ or to none of them. Then there is a positive constant C such that

$$\sup_{\tau\in\mathbb{R}} \sum_{k\geq 0} P^2(c^{-k}\tau) \leq C. \tag{6.5.23}$$

Applying Plancherel's identity in time we obtain

$$\begin{aligned} \sum_{k=k_0}^{k_h} \int_{\mathbb{R}} \varphi^2(t) \int_{\Gamma_h} |P_k(\psi \partial_n^h \overline{u})(t)|^2 d\Gamma_h dt &= \sum_{k=k_0}^{k_h} \int_{\mathbb{R}} \varphi^2(t) \|P_k(\psi \partial_n^h \overline{u})(t)\|^2_{l^2(\Gamma_h)} dt \\ &\leq \|\varphi\|^2_{L^\infty(\mathbb{R})} \sum_{k\geq 0} \int_{\mathbb{R}} \|P_k(\psi \partial_n^h \overline{u})(t)\|^2_{l^2(\Gamma_h)} dt \\ &= \|\varphi\|^2_{L^\infty(\mathbb{R})} \sum_{k\geq 0} \int_{\mathbb{R}} P^2(c^{-k}\tau) \|\widehat{(\psi \partial_n^h \overline{u})}(\tau)\|^2_{l^2(\Gamma_h)} d\tau \\ &\leq \|\varphi\|^2_{L^\infty(\mathbb{R})} \sup_{\tau\in\mathbb{R}} \sum_{k\geq 0} P^2(c^{-k}\tau) \int_{\mathbb{R}} \|\widehat{(\psi \partial_n^h \overline{u})}(\tau)\|^2_{l^2(\Gamma_h)} d\tau \\ &\leq C\|\varphi\|^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \|\psi(t) \partial_n^h \overline{u}(t)\|^2_{l^2(\Gamma_h)} dt \\ &= C\|\varphi\|^2_{L^\infty(\mathbb{R})} \int_{\mathbb{R}} \psi^2(t) \|\partial_n^h \overline{u}(t)\|^2_{l^2(\Gamma_h)} dt \\ &= C\|\varphi\|^2_{L^\infty(\mathbb{R})} \int_0^T \int_{\Gamma_h} |\partial_n^h \overline{u}(t)|^2 d\Gamma_h dt. \end{aligned}$$

□

6.6. Proof of lemma 6.5.1

In what follows, for reasons of completeness, we prove Lemma 6.5.1. For any $l \in \mathbb{Z}$ and $a \in L^2_{loc}(\mathbb{R}, l^2(\Gamma_h))$ we denote $I_l = [lT, (l+1)T)$ and $a_l = 1_{I_l} a$. Lemma 6.5.1 will follow from the following one:

Lemma 6.6.1. *([78], [17]) There exists a positive constant $C = C(P)$ such that for all $\varphi \in C_0^\infty(\mathbb{R})$ and $l \in \mathbb{Z}$ with $dist(I_l, \text{supp}(\varphi)) \geq \delta > 0$ the following holds:*

$$\sup_{t\in\mathbb{R}} \|\varphi(t)P_k(a_l)\|_{l^2(\Gamma_h)} \leq Cc^{-k}\delta^{-2}T^{1/2}\|\varphi\|_{L^\infty(\mathbb{R})} \sup_{l\in\mathbb{Z}} \|a_l\|_{L^2(\mathbb{R}\times\Gamma_h)}, \tag{6.6.1}$$

uniformly in $h > 0$.

Proof. The definition of the projector P_k and integration by parts give us

$$\begin{aligned}\varphi(t)P_k(a_l)(t) &= \int_{\mathbb{R}_\tau}\int_{\mathbb{R}_s} e^{i\tau(t-s)}P(c^{-k}\tau)\varphi(t)a_l(s)dsd\tau \\ &= \int_{\mathbb{R}_\tau}\int_{\mathbb{R}_s} e^{i\tau(t-s)}i^2\partial_\tau^2[P(c^{-k})\tau]\frac{\varphi(t)a_l(s)}{(t-s)^2}dsd\tau.\end{aligned}$$

Thus, for any t in the support of φ and $\mathbf{j} \in \Gamma_h$:

$$\begin{aligned}|\varphi(t)P_k(a_l)(t,\mathbf{j})| &\leq \int_{\mathbb{R}_\tau}|\partial_\tau^2[P(c^{-k}\tau)]|d\tau\int_{\mathbb{R}_s}\frac{|\varphi(t)||a_l(s,\mathbf{j})|}{(t-s)^2}ds \\ &\leq c^{-2k}\|\varphi\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}_\tau}|(\partial_\tau^2P)(c^{-k}\tau)|d\tau\int_{I_l}\frac{|a_l(s,\mathbf{j})|}{(t-s)^2}ds \\ &\leq c^{-k}\delta^{-2}\|\varphi\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}_\tau}|(\partial_\tau^2P)(\tau)|d\tau\int_{I_l}|a_l(s,\mathbf{j})|ds\end{aligned}$$

and

$$|\varphi(t)P_k(a_l)(t,\mathbf{j})|^2 \leq \left(c^{-k}\delta^{-2}\|\varphi\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}_\tau}|(\partial_\tau^2P)(\tau)|d\tau\right)^2 T\int_{I_l}|a_l(s,\mathbf{j})|^2ds. \tag{6.6.2}$$

Making the sum on $\mathbf{j} \in \Gamma_h$ yields

$$h\sum_{\mathbf{j}\in\Gamma_h}|\varphi(t)P_k(a_l)(t,\mathbf{j})|^2 \leq \left(c^{-k}\delta^{-2}\|\varphi\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}_\tau}|(\partial_\tau^2P)(\tau)|d\tau\right)^2 T\int_{I_l}h\sum_{\mathbf{j}\in\Gamma_h}|a_l(s,\mathbf{j})|^2ds$$

and

$$\sup_{t\in\mathbb{R}}\|\varphi(t)P_k(a_l)(t)\|_{l^2(\Gamma_h)} \leq c^{-k}\delta^{-2}\|\varphi\|_{L^\infty(\mathbb{R})}T^{1/2}\int_{\mathbb{R}_\tau}|(\partial_\tau^2P)(\tau)|d\tau\ \sup_{l\in\mathbb{Z}}\|a_l\|_{L^2(\mathbb{R},\,l^2(\Gamma_h))}.$$

□

Proof of Lemma 6.5.1. Using Lemma 6.6.1 we will prove the existence of a positive constant $C = C(T, \varphi, \psi, P)$ such that

$$\sup_{t\in\mathbb{R}}\|\varphi(t)(P_k(a) - P_k(\psi a))(t)\|_{l^2(\Gamma_h)} \leq Cc^{-k}\sup_{l\in\mathbb{Z}}\|a_l\|_{L^2(\mathbb{R},\,l^2(\Gamma_h))}. \tag{6.6.3}$$

Then, (6.5.19) will be a consequence of Cauchy's inequality:

$$\begin{aligned}\int_{\mathbb{R}}\|\varphi(t)P_k(a)(t)\|_{l^2(\Gamma_h)}^2dt &\leq 2\int_{\mathbb{R}}\|\varphi(t)P_k(\psi a)(t)\|_{l^2(\Gamma_h)}^2dt + 2\int_{\mathbb{R}}\|\varphi(t)P_k(a-\psi a)(t)\|_{l^2(\Gamma_h)}^2dt \\ &\leq 2\int_{\mathbb{R}}\|\varphi(t)P_k(\psi a)(t)\|_{l^2(\Gamma_h)}^2dt + 2T\sup_{t\in\mathbb{R}}\|\varphi(t)(P_k(a-\psi a))(t)\|_{l^2(\Gamma_h)}^2 \\ &\leq 2\int_{\mathbb{R}}\|\varphi(t)P_k(\psi a)(t)\|_{l^2(\Gamma_h)}^2dt + Cc^{-k}\sup_{l\in\mathbb{Z}}\|a_l\|_{L^2(\mathbb{R},\,l^2(\Gamma_h))}^2.\end{aligned}$$

In the following we prove (6.6.3). Observe that on I_0, $a \equiv a\psi$. This yields to the following decomposition of the difference $P_k(a) - P_k(\psi a)$:

$$P_k(a) - P_k(\psi a) = \sum_{|l|\geq 1} P_k(a_l) - P_k((\psi a)_l) = \sum_{|l|\geq 1} P_k(a_l - (\psi a)_l) = \sum_{|l|\geq 1} P_k(b_l), \tag{6.6.4}$$

with $b_l = a_l - (\psi a)_l$. Let us choose an $\delta > 0$ such that φ is supported on $(\delta, T-\delta)$. Thus for all $|l| \geq 2$, the function b_l satisfies $\text{dist}(\text{supp}(\varphi), I_l) \geq T(|l|-1)$. Also, for $|l| = 1$: $\text{dist}(\text{supp}(\varphi), I_l) \geq \delta$. Lemma 6.6.1 shows the existence of a constant $C = C(T, \varphi, \psi, P)$ such that

$$\sup_{t\in\mathbb{R}} \|\varphi(t)P_k(b_l)(t)\|_{l^2(\Gamma_h)} \leq Cc^{-k} \sup_{l\in\mathbb{Z}} \|b_l\|_{L^2(\mathbb{R},\, l^2(\Gamma_h))} \begin{cases} \frac{1}{(|l|-1)^2}, & |l| \geq 2, \\ \frac{1}{\delta^2}, & |l| = 1. \end{cases} \tag{6.6.5}$$

Finally, (6.6.4) and (6.6.5) give for any $t \in \mathbb{R}$

$$\begin{aligned} \|\varphi(t)[P_k(a) - P_k(\psi a)]\|_{l^2(\Gamma_h)} &\leq \sum_{|l|\geq 1} \|\varphi(t)P_k(b_l)\|_{l^2(\Gamma_h)} \\ &\leq Cc^{-k} \sup_{l\in\mathbb{Z}} \|b_l\|_{L^2(\mathbb{R},\, l^2(\Gamma_h))} \left[\sum_{|l|\geq 2} \frac{1}{(|l|-1)^2} + \frac{2}{\delta^2}\right] \\ &\leq Cc^{-k} \sup_{l\in\mathbb{Z}} \|b_l\|_{L^2(\mathbb{R},\, l^2(\Gamma_h))} \\ &\leq Cc^{-k} \sup_{l\in\mathbb{Z}} \|a\|_{L^2(\mathbb{R},\, l^2(\Gamma_h))}. \end{aligned}$$

□

Capítulo 7

Conclusiones y Problemas Abiertos

En esta memoria hemos obtenido resultados sobre tres temas:

1. Comportamiento asintótico para una aproximación de la ecuación del calor.
2. Estimaciones dispersivas para aproximaciones numéricas de la ecuaciones de Schrödinger y ondas.
3. Observabilidad frontera uniforme de un método bimalla para la ecuación de ondas en un cuadrado.

Para la ecuación del calor demostramos que las soluciones del método de diferencias finitas estándar reproducen exactamente el decaimiento de las soluciones continuas. También, usando los momentos de los datos iniciales, obtenemos una expansión completa de las soluciones discretas, semejante a la bien conocida en el caso continuo.

En referencia a la semi-discretización clásica conservativa por diferencias finitas de la ecuación de Schrödinger, probamos la falta de propiedades asintóticas independientes del parámetro de la discretización. Para remediar este hecho introducimos tres métodos numéricos: filtrado de los datos iniciales en variable Fourier; viscosidad numérica; precondicionamiento bimalla, y probamos estimaciones dispersivas uniformes para cada uno de los métodos analizados. Gracias a estos resultados obtenemos desigualdades de tipo Strichartz para los modelos numéricos, y los aplicamos a las aproximaciones de problemas no lineales. Los resultados obtenidos permiten probar la convergencia para no linealidades que no se pueden abordar por métodos de energía y que, incluso en el caso continuo, exigen estimaciones de tipo Strichartz. El mismo análisis se ha hecho también para esquemas totalmente discretos.

Para la ecuación de ondas introducimos un esquema semi-discreto en diferencias finitas y obtenemos desigualdades de tipo Strichartz para su soluciones. Usando estas propiedades introducimos esquemas convergentes para problemas no lineales. Estos resultados son analizados no sólo en el contexto numérico sino también en el contexto de la ecuación de ondas en retículos, donde no hace falta preocuparse de la uniformidad con respecto al tamaño del retículo.

El último capítulo de esta memoria trata el problema de observabilidad frontera para las aproximaciones de la ecuación de ondas en un cuadrado utilizando métodos bimalla. La demostración consiste en usar las desigualdades de observabilidad para soluciones filtradas junto con una descomposición espectral diádica. Este resultado constituye la primera demostración de la observabilidad uniforme para el método bimalla en varias dimensiones espaciales.

A continuación presentamos algunos problemas abiertos relacionados con los temas y problemas abordados en esta tesis.

- **Propiedades asintóticas de aproximaciones numéricas para ecuaciones de convección-difusión.**

  Las estimaciones obtenidas para el decaimiento de las aproximaciones de la ecuación del calor permiten, junto con un método de energía, tratar el mismo tipo de propiedades para las aproximaciones de la ecuación de convección-difusión con convección no lineal ($|u|^{q-1}u$) con exponente $q > 1 + 1/d$.

  Uno de los problemas abiertos es analizar el caso $q = 1 + 1/d$. En el caso continuo, Escobedo y Zuazua [44] demuestran que el comportamiento asintótico de las soluciones está dado por una familia uniparamétrica de soluciones auto-semejantes. La demostración dada por los autores se basa en el principio de invariancia de La Salle aplicado para la ecuación escrita en variables auto-semejantes. La mayor dificultad a la hora de aplicar este argumento a nivel semi-discreto es encontrar una manera de escribir la ecuación semi-discreta en variables auto-semejantes. Esto sugiere introducir esquemas con mallas variables.

  En un futuro nos proponemos estudiar qué sucede con el decaimiento de las soluciones semidiscretas cuando consideramos mallados no uniformes tanto en el caso lineal como no lineal.

- **Propiedades asintóticas de aproximaciones numéricas para ecuaciones de difusión no locales.**

En un trabajo reciente, Chasseigne, Chaves y Rossi [32] han considerado el problema

$$u_t = \int_{\mathbb{R}^n} J(x-y)(u(y,t)-u(x,t))dy$$

y han obtenido resultados sobre el comportamiento asintótico de las soluciones bajo algunas condiciones sobre la función J. Este tipo de ecuaciones tiene aplicaciones en física y biología. Por ejemplo, u puede ser la densidad de una población y J la distribución de probabilidad de que un individuo salte de un lugar y a otro x.

Como las técnicas usadas por los autores en el caso continuo utilizan el comportamiento de la transformada de Fourier de la función J en la proximidad del origen, esperamos que lo mismo se pueda hacer a nivel discreto. Las técnicas que vamos a usar se basan en la transformada semi-discreta de Fourier y sus propiedades. Nuestra experiencia acumulada en los trabajos de esta tesis nos dice que las diferencias entre el modelo continuo y el discreto aparecen en el análisis de altas frecuencias y no en las bajas. Sin embargo, es necesario un análisis cuidadoso debido a las grandes diferencias entre los modelos continuos y los discretos.

- **Esquemas numéricos para la ecuación de Schrödinger con coeficientes variables.**

  Una vez que hemos entendido cuáles son las propiedades del método numérico que garantizan las propiedades dispersivas para las ecuaciones de Schrödinger con coeficientes constantes, el siguiente paso sería estudiar las mismas propiedades para ecuaciones con coeficientes variables. En el caso continuo, Staffilani y Tataru [115] han demostrado estimaciones de tipo Strichartz para ecuaciones con coeficientes C^2. Los autores usan la transformación FBI (ver [124], [125] y [126]) para construir una paramétrica microlocal para la ecuación de Schrödinger.

  Tal y como hemos probado en esta memoria los esquemas clásicos en diferencias finitas no verifican las propiedades de dispersión del modelo continuo. El análisis de las ecuaciones con coeficientes variables introduce nuevas e importantes dificultades, tanto en lo que respecta a la prueba de contraejemplos como en los resultados positivos mediante argumentos bimalla o métodos viscosos. Para afrontar estas dificultades es preciso adaptar la transformada FBI al marco semi-discreto o totalmente discreto del problema.

- **Esquemas totalmente discretos para la ecuación de Schrödinger.**

  En el Capítulo 4 hemos analizado esquemas totalmente discretos para la ecuación de Schrödinger involucrando dos pasos temporales. Sin embargo, hay esquemas numéricos que involucran más pasos temporales, como por ejemplo

$$i\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{(\Delta x)^2}=0,\ n\geq 1,\ j\in\mathbf{Z}, \tag{7.0.1}$$

o los introducidos en [69], [91] y [10].

Usando las técnicas desarrolladas en el Capítulo 4 se pueden analizar condiciones necesarias y suficientes para garantizar que el decaimiento l^1-l^∞ de las soluciones discretas sea uniforme con respecto al paso del mallado. Como hemos visto éste es el punto clave a la hora de obtener estimaciones espacio-temporales más generales tipo Strichartz.

También se pueden analizar las mismas propiedades en esquemas numéricos donde los pasos espacial Δx y temporal Δt verifican la condición de Courant-Friedrichs-Levy pero el cociente $\Delta t/(\Delta x)^2$ no se mantiene constante.

Otro problema abierto interesante es extender las técnicas de esta tesis para tratar problemas de la ecuación de Schrödinger en varias dimensiones espaciales. No es claro todavía si las condiciones dadas en el Capitulo 3 (véase las Remark 4.2.2 y Remark 4.2.4) para el caso d-dimensional son necesarias.

En el caso de varias dimensiones espaciales también se pueden utilizar mallas uniformes (pero distintas) en cada dirección espacial, por ejemplo, h en variable x y una potencia de h en la variable y y ver si con estas ideas se mejora en algo los resultados obtenidos.

- **Métodos de descomposición (Splitting).**

En Besse, Bidégaray y Descombes [8], (ver también Sanz-Serna y Calvo [108], Descombes y Schatzman [42]) los autores consideran la ecuación de Schrödinger no lineal con dato inicial en $H^2(\mathbb{R}^2)$ y termino no lineal $|u|^2u$, e introducen un método de descomposición para aproximar la solución. De manera más precisa, la ecuación de Schrödinger no lineal se descompone en el flujo X^t generado por la ecuación de Schrödinger lineal

$$\begin{cases} v_t-i\Delta v=0,\ x\in\mathbb{R}^2,\ t>0,\\ v(0,x)=v_0(x),\ x\in\mathbb{R}^2,\end{cases} \tag{7.0.2}$$

y el flujo Y^t de la ecuación diferencial

$$\begin{cases} w_t-i|w|^2w=0,\ x\in\mathbb{R}^2,\ t>0,\\ w(0,x)=w_0(x),\ x\in\mathbb{R}^2.\end{cases} \tag{7.0.3}$$

El flujo de NSE combinando puede aproximarse por los dos flujos X^t y Y^t usando los métodos clásicos de descomposición, la fórmula de Lie, $Z_L^t=X^tY^t$, o la fórmula de Strang, $Z_S^t=X^{t/2}Y^tX^{t/2}$.

En [8] la convergencia del método se prueba para datos iniciales en $H^2(\mathbb{R}^2)$. Conviene observar que la no linealidad $|u|^2u$ es localmente Lipschitz en $H^2(\mathbb{R}^2)$. En consecuencia, este tipo de no linealidades se pueden tratar por métodos de energía sin usar desigualdades de tipo Strichartz.

Sin embargo para datos iniciales en el espacio L^2 la no linealidad $|u|^2u$ no es localmente Lipschitz y por tanto la convergencia del método no se puede probar usando solamente métodos de energía. En este sentido se deberían desarrollar ideas parecidas a las esta memoria para probar la convergencia del método.

Otro posible problema es reemplazar las ecuaciones (7.0.2), (7.0.3), continuas en la variable x por unas discretas y analizar la convergencia de las soluciones para datos iniciales en $L^2(\mathbb{R}^2)$.

Como hemos visto en el Capítulo 3, la aproximación de (7.0.2) por diferencias finitas no tiene las propiedades dispersivas del modelo continuo. Es natural, por tanto, considerar uno de los remedios propuestos en el Capítulo 3, viscosidad numérica o precondicionamiento bimalla. La convergencia del método es un problema abierto por la falta de propiedades de dispersión de la EDO (7.0.3) y sus semi-discretizaciones.

- **Aproximaciones de la ecuación de Schrödinger en medios periódicos**.

  En el caso continuo, Bourgain [11] considera la ecuación de Scrödinger no lineal con condiciones de contorno periódicas y demuestra la existencia y la unicidad de las soluciones. El punto clave son las estimaciones sobre el semigrupo de Schrödinger. El autor obtiene para datos iniciales en L^2 que el semigrupo lineal pertenece al espacio $L^4_tL^4_x$. Como consecuencia consigue resultados sobre la ecuación no lineal.

  A la hora de diseñar un esquema numérico para la ecuación no lineal hemos obtenido resultados parciales que no han sido incluidos en esta memoria. Para la discretización conservativa hemos probado la falta de la propiedad de dispersión clásica $L^4(l^4)$ algo que el modelo continuo sí cumple. Es un problema abierto determinar si hay algún par de exponentes (q,r) para los cuales la solución discreta pertenece al espacio $L^p(l^q)$. Sin embargo sabemos que usando esquemas con viscosidad numérica, la propiedad de integrabilidad $L^4(l^4)$ se puede recuperar. También queda por analizar si los métodos bimalla pueden recuperar las propiedades de integrabilidad espacio-temporal.

- **Esquemas totalmente discretos para la ecuación de ondas**

  El mismo análisis que hemos hecho en el Capítulo 4 para esquemas totalmente discretos para la ecuación de Schrödinger se puede hacer en el contexto de la ecuación de ondas en varias dimensiones espaciales. Constituye un problema abierto dar condiciones necesarias y suficientes para garantizar las propiedades de decaimiento uniforme l^∞ de las soluciones que ha sido analizado en el Capítulo 5 para esquemas semi-discretos.

- **Análisis microlocal para observabilidad frontera.**

  En el caso de la observabilidad interna de la ecuación de ondas, Macía [89] obtiene mediante técnicas microlocales condiciones necesarias y suficientes para garantizar dicha propiedad de observabilidad en dominios sin frontera. Sin embargo en caso de dominios con frontera aparecen varias dificultades, inducidas por la propagación de los rayos geométricos en el contacto con el borde. Este tipo de dificultades aparecen también en el marco continuo y bajo varias hipótesis de regularidad sobre la frontera han sido analizadas en [6], [17] [79]. Unos de los problemas abiertos es intentar utilizar estos métodos microlocales para probar las desigualdades de observabilidad en la frontera tanto para soluciones filtradas como para el método bimalla.

  Cabe mencionar también que el tiempo de observabilidad $T > 4$ obtenido en el Capítulo 6 no es óptimo. Las técnicas microlocales dan resultados mucho más precisos que los basados en multiplicadores [65], [97], [145] o series no-armónicas [88] y uno puede esperar que aplicando las técnicas microlocales se pueda obtener el tiempo óptimo.

- **Métodos bimalla**

  En el Capítulo 6, hemos introducido un algoritmo bimalla para probar la observabilidad frontera para el esquema semi-discreto en diferencias finitas para la ecuación de ondas. El argumento usado reduce la observabilidad uniforme en el caso del precondicionamiento bimalla a la observabilidad para una clase de soluciones filtradas.

  La prueba de la observabilidad para soluciones filtradas ha sido objeto de varios trabajos, no sólo para la ecuación de ondas [65], [142], sino para la ecuación de Schrödinger [89] y la ecuación de vigas [80] entre otras. Por tanto cabe esperar que el mismo método se puede aplicar para la controlabilidad uniforme de otras ecuaciones.

  En general, para el algoritmo bimalla se demuestra que con un cociente de los mallados conveniente $1/2^k$, se reduce la demostración de observabilidad para una clase de soluciones más filtradas cuanto mayor es k. Para un k suficientemente grande nos situamos en la clase de soluciones filtradas que han sido estudiadas anteriormente para clases mucho más generales de ecuaciones.

  Otra clase importante de problemas abiertos que se destaca en esta memoria es la relativa a la aplicación del método bimalla, tanto en el análisis de propiedades dispersivas como para estudiar observabilidad uniforme en mallados generales. En estos casos el análisis de Fourier de los datos obtenidos por el método bimalla no se puede aplicar y nuevas técnicas basadas en el análisis en el espacio físico tienen que ser desarrolladas.

- **Condiciones espectrales para el control y/o observabilidad**

  En unos trabajos recientes, Tusnack [104], Miller [95], véase también Russell y Weiss [107], se da una condición espectral que garantiza la observabilidad de sistemas infinito dimensionales. Este tipo de condiciones generalizan el test de Hautus para sistemas finito dimensionales a sistemas infinito dimensionales. También, Tucsnak et al. [105] dan una condición espectral suficiente para garantizar la estabilidad interna de una aproximación numérica de la ecuación de placas. Los autores introducen un esquema basado en viscosidad numérica.

  Sería interesante ver si estos métodos espectrales pueden garantizar resultados de observabilidad y controlabilidad (interna o frontera) para métodos numéricos basados en precondicionamiento bimalla. En particular se puede plantear dar una condición espectral suficiente para probar las desigualdades de observabilidad para el método bimalla.

- **Propiedades dispersivas de esquemas en mallados generales.**

  Todo el análisis presentado en esta memoria esta basado en técnicas de Fourier. Este tipo de análisis está bien adaptado a problemas numéricos donde las mallas involucradas son de tipo retículo $h\mathbb{Z}^d$. Sin embargo, en muchos problemas que interesan en ingeniería se plantean problemas en mallados irregulares. En este caso el análisis de Fourier parece no poderse aplicar.

  Un primer paso para entender las dificultades presentes en mallas no-regulares es considerar un esquema numérico en diferencias finitas para la ecuación de Schrödinger unidimensional donde la malla considerada sea $3h\mathbb{Z} \cup (3\mathbb{Z}+1)h$. Este método, que rompe la simetría del esquema clásico en diferencias finitas, es un método intermedio entre mallados uniformes y mallados totalmente no estructurados y en el cual el análisis de Fourier todavía se puede aplicar. Una vez entendidas las dificultades del caso anterior se puede internar abordar problemas en mallados estructurados como los del Figura 7.1.

  Mucho más ambicioso sería analizar cuándo las propiedades de decaimiento de las soluciones de ecuación de Schrödinger se mantienen en una aproximación por elementos finitos con triangulaciones generales no estructuradas. Es claro que en este caso el análisis de Fourier no se puede aplicar y se tienen que desarrollar nuevas técnicas para abordar las propiedades de decaimiento de las soluciones de las aproximaciones numéricas.

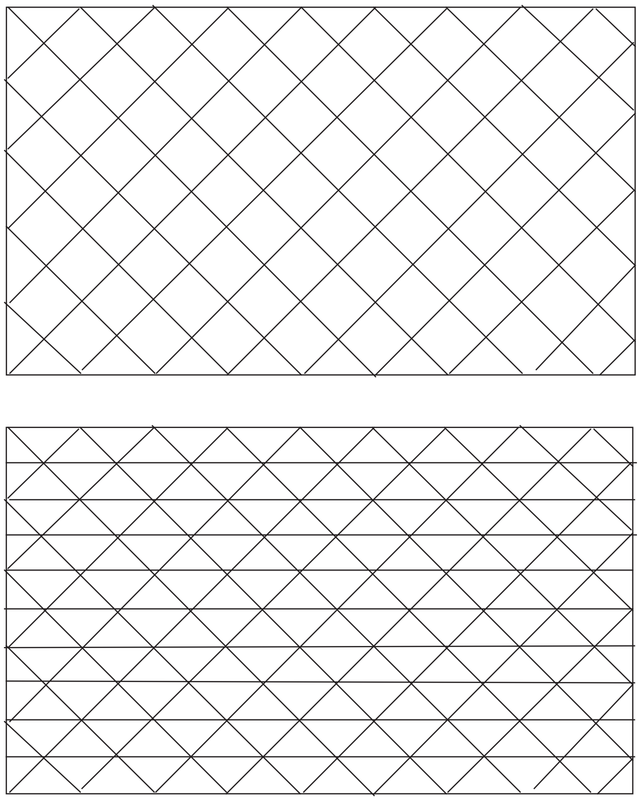

Figura 7.1: Mallas con simetría

- **Ecuaciones sobre redes de cuerdas.**

  En vista de los trabajos de Dager y Zuazua [38], [39], [37], [40] sobre el control de ondas en redes de cuerdas nos planteamos analizar las propiedades dispersivas de las ecuaciones de tipo Schrödinger sobre este tipo de estructuras.

  Un primer ejemplo sería considerar la ecuación $iu_t + u_{xx} = 0$ sobre una estructura de semi-rectas como en la Figura 7.2 y analizar las propiedades dispersivas de esta ecuación. También se puede analizar la ecuación semi-discreta $iu_h + \Delta_h u = 0$ y sus variantes bimalla y viscosa sobre la estructura dada en la Figura 7.3.

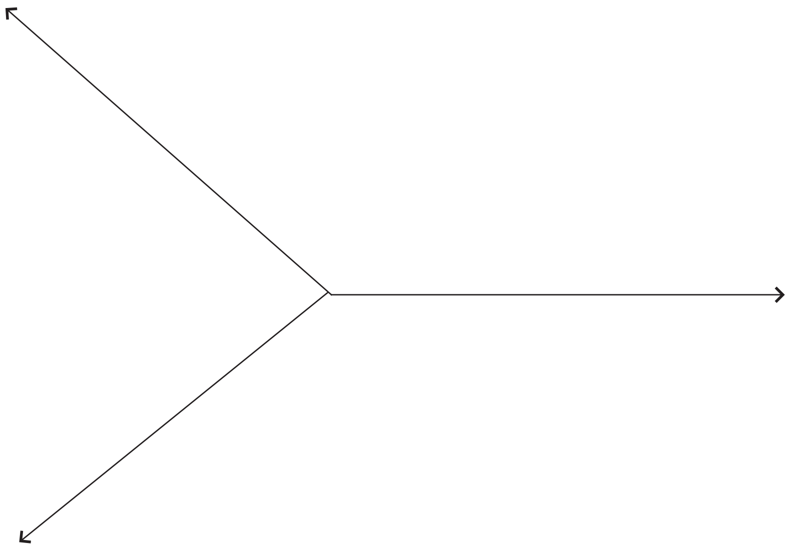

Figura 7.2: Estructura continua tipo árbol

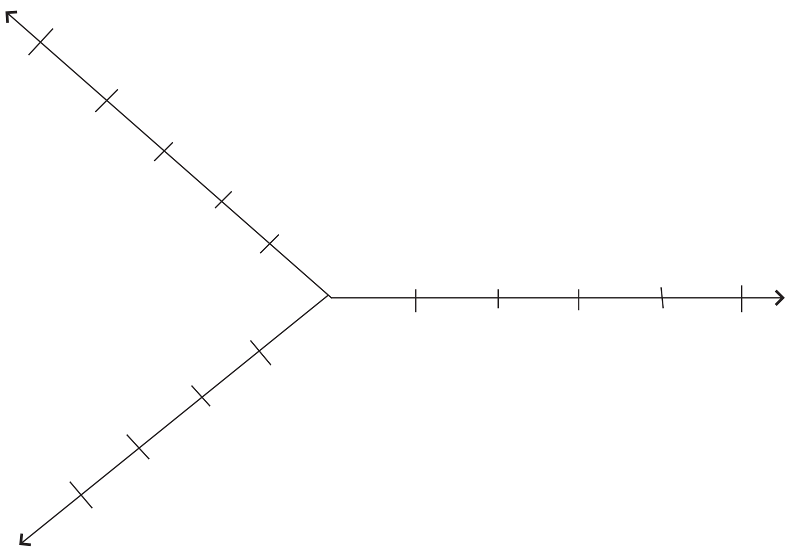

Figura 7.3: Estructura discreta tipo árbol

Chapter 7

Conclusions and Open Problems

In this thesis we have analyzed the following problems:

1. Asymptotic behaviour of a numerical scheme for the heat equation,

2. Dispersive properties of numerical schemes for the Schrödinger and wave equations,

3. Uniform boundary observability of a two-grid method for the wave equation in a square.

For the heat equation we prove that the solutions of the finite difference scheme have the same decay rates as the continuous ones. Using the moments of the initial data, we obtain a complete expansion of the discrete solutions, similar to the well-known continuous case.

With respect to the conservative finite difference semi-discretization of the Schrödinger equation, we prove the lack of any dispersive property independent of the mesh size. To recover the dispersive properties of the solutions we introduce three numerical methods: filtering the initial data in the Fourier variable; numerical viscosity; two-grid preconditioner. For each of them we prove uniform dispersive properties similar to those of the continuous case. Thanks to these results we obtain Strichartz-like estimates for the numerical models and apply them to approximate nonlinear problems. The results obtained allow us to prove the convergence of the numerical methods for nonlinearities which cannot be treated by energy arguments and even in the continuous case require Strichartz estimates. The same analysis has also been done in the context of fully discrete schemes.

For the wave equation we introduce a semidiscrete scheme and obtain Strichartz-like estimates for its solutions. We apply these estimates to approximate a nonlinear wave equation. These results are not only analyzed in the numerical context but also for the wave equation on lattices, where there is no need for uniformity with respect to the size of the lattice.

The last result of this thesis deals with the uniform boundary observability for the finite difference approximation of the wave equation by using a two-grid method. The proof is based on observability inequalities for filtered solutions together with a dyadic spectral decomposition. This is the first proof, in dimensions greater than one, of the uniform observability for the two-grid method.

In the following we present some open problems related to the subjects and problems treated in this thesis.

- **Asymptotic properties of numerical schemes for convection-diffusion equations**

  The decay rates obtained for the semidiscrete approximation of the heat equation, together with an energy argument, allow us to obtain the long time behaviour of the solutions of a semidiscrete approximation for the convection-diffusion equations with nonlinear convection ($|u|^{q-1}u$), $q > 1 + 1/d$.

  One of the open problems is the study of the case $q = 1 + 1/d$. In the continuous case, Escobedo and Zuazua [44] proved that the asymptotic behavior of the solutions is given by a family of self-similar solutions. The proof given by the authors is based on La Salle's invariance principle applied for the equation written in self-similar variables. The great difficulty to apply this argument at the semidiscrete level is to find a way to write the semidiscrete equation in self-similar variables. This suggests to introduce numerical schemes with mesh size varying in time and study the long time behaviour of their solutions.

- **Asymptotic properties of numerical schemes for nonlocal diffusion equations**

  In a recent work, Chasseigne, Chaves y Rossi [32] have considered the equation

  $$u_t = \int_{\mathbb{R}^d} J(x-y)(u(y,t) - u(x,t))dy$$

  and they obtained the asymptotic behaviour of the solutions under some assumption on function J. This kind of equations have applications in physics and biology. For example, u can be the density of the population and J the probability distribution that a person goes from y to x.

  Taking into account that the techniques used by the authors use the behaviour of the Fourier transform of J near zero, we think the same analysis can be done at the discrete level. The analysis we have done here shows that the possible differences between the two models, continuos and discrete, occur at the high frequencies. However, taking into account the grand differences between the two models, a careful analysis have to be done.

- **Numerical schemes for variable coefficients Schrödinger equations**

  Once we have understood which are the properties of the numerical schemes for the linear Schrödinger equation that guarantee the existence of uniform dispersive properties, the next step will be to analyze the same properties for equations with variable coefficients. In the continuous case, Staffilani y Tataru [115] have proved Strichartz estimates for equations with C^2-coefficients. The authors use the FBI transform (see [124], [125] y [126]) to construct a microlocal parametrix for the considered Schrödinger equation.

  As we have seen in this thesis the conservative finite difference scheme has no dispersive properties similar to the continuous model. Doing the same analysis for equations with variable coefficients introduces new and important difficulties, both in obtaining counterexamples as positive results. In order to face these difficulties it is precise to adapt the FBI transform to the semidiscrete or fully discrete problems and to follow at the discrete level the techniques introduced in the continuous case.

- **Fully discrete schemes for the Schrödinger equations.**

  In Chapter 4 we have analyzed two-level schemes for the Schrödinger equations. However, there are multilevel schemes as for example:

$$i\frac{u_j^{n+1}-u_j^{n-1}}{2\Delta t}+\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{(\Delta x)^2}=0,\ n\geq 1,\ j\in\mathbb{Z}, \tag{7.0.1}$$

  or the ones introduced in [69], [91] and [10].

  Using similar techniques as in Chapter 4, we expect to be able to characterize the numerical schemes that guarantee the uniform l^1-l^∞ decay of the solutions. As we have seen, this is the key point in obtaining more general space-time estimates of Strichartz type.

  Another open problem is to extend the techniques of this thesis to the multi-dimensional case. We still do not know whether the conditions given in Chapter 3 (see Remark 4.2.2 and Remark 4.2.4) are necessary. In the d-dimensional case, $d\geq 2$, it is possible to further analyze the use of meshes with different size in each direction and to see whether the results can be improved.

- **Splitting Methods.**

In Besse, Bidégaray y Descombes [8], (see also Sanz-Serna and Calvo [108], Descombres and Schatzman [42]) the authors consider the NSE with initial data in $H^2(\mathbb{R}^2)$ and nonlinearity $|u|^2u$. A time splitting method is used in order to approximate the solution. More precisely, the nonlinear Schrödinger equation is split into the flow X^t generated by the linear Schrödinger equation

$$\begin{cases} v_t - i\Delta v = 0,\ x \in \mathbb{R}^2,\ t > 0, \\ v(0,x) = v_0(x),\ x \in \mathbb{R}^2, \end{cases} \tag{7.0.2}$$

and the flow Y^t for the differential equation

$$\begin{cases} w_t - i|w|^2 w = 0,\ x \in \mathbb{R}^2,\ t > 0, \\ w(0,x) = w_0(x),\ x \in \mathbb{R}^2. \end{cases} \tag{7.0.3}$$

One can then approximate the flow of NSE by combining the two flows X^t and Y^t using some of the classical splitting methods: Lie's formula $Z_L^t = X^tY^t$ or Strang's formula $Z_S^t = X^{t/2}Y^tX^{t/2}$.

In [8] the convergence of these methods is proved for initial data in $H^2(\mathbb{R}^2)$. Note however that the nonlinearity $|u|^2u$ is locally Lipschitz in $H^2(\mathbb{R}^2)$. Consequently this nonlinearity in this functional setting can be dealt with by means of classical energy methods, without using the Strichartz type estimate.

For L^2 initial data the nonlinear term $|u|^2u$ is not locally Lipschitz and then the convergence of the method can not be guaranteed using only energy estimates. Thus, one has to use similar methods to the ones of this thesis to prove the convergence of the method.

Another possible open problem consists in replacing equations (7.0.2) and (7.0.3) continuous in the x-variable by discrete ones and analyze the convergence of the solutions for L^2-initial data.

As we have seen in Chapter 3, the conservative approximation of (7.0.2) does not keep the dispersive properties of the continuous model. Thus it seems natural to consider one of the remedies proposed in that chapter: numerical viscosity or a two-grid method. The convergence of the method is an open problem for the lack of dispersion property of ODE (7.0.3) and its semi-discretizations.

- **Approximations of the Schrödinger equation with periodic boundary conditions.**

In the continuous case Bourgain [11] considers the nonlinear Scrödinger equation with periodic boundary conditions and proved its well-posedness. The key points are the estimates on the lineal semigroup. For initial data in L^2, the author proves that the linear semigroup belongs to $L^4_t L^4_x$. As a consequence he obtains well-posedness results for the nonlinear equation.

For this type of problem we introduced some numerical schemes for the linear problem and we have obtained partial results that have not been included in this thesis. For the conservative semidiscrete approximation, a similar l^2_x-$L^4_t l^4_x$ estimate fails to be uniform with respect to the mesh size Δx. It is an open problem to establish what is the complete range of (q, r) (if any) for which the estimates l^2_x-$L^q_t l^r_x$ are uniform with respect to the mesh size. Thus it is natural to consider schemes with numerical viscosity or involving a two-grid algorithm. In the first case we are able to prove that the solutions remain uniformly bounded in the space $L^4_t l^4_x$. It remains to study whether the two-grid method recovers this property.

- **Fully discrete schemes for the wave equation**

  The same analysis we have done in Chapter 4 for the Schrödinger equation can be done in the context of the wave equation. It is an open problem to establish necessary and sufficient conditions which guarantee the existence of uniform Strichartz estimates for the solutions of fully discrete schemes for the wave equation.

- **Microlocal analysis and boundary observability.**

  In the case of internal observability, using microlocal tools, Macía [89] obtained conditions which guarantee the uniform observability on domains without boundary. However, for domains with boundary new difficulties occur given by the propagation of the rays when they touch the boundary. These difficulties also occur in the continuous case and have been analyzed in [6], [17], [79]. As far we know, there is no proof of the uniform boundary observability for semi-discretizations of the wave equation using these techniques.

  We also mention that the observability time obtained in Chapter 6, $T > 4$, is not optimal. The microlocal techniques give more precise results than the ones based on multipliers [65], [97], [145] or non-harmonic series [88]. One can expect that applying microlocal techniques it will be possible to obtain the optimal time for the two-grid method.

- **Two-grid Methods**

In Chapter 6, we introduced a two-grid algorithm to prove the boundary observability for a semidiscrete scheme for the wave equation. The argument we used, reduces the uniform observability of the two-grid method to the observability inequality in a class of Fourier filtered solutions. In general, as k increases, a two grid method with quotient of the meshes $1/2^k$ reduces the proof of the observability inequality to a class of solutions each time more filtered.

The proof of the observability for filtered solutions has been the object of several works, not only for the waves equation [65], [142], but also for Schrödinger equation [89] and beam equation [80] among others. Thus the method we used here can be applied to other types of equations both for internal and boundary observability.

Another important class of open problems related with this thesis consists in the application of the two-grid method for general meshes. In these cases the Fourier analysis cannot be applied and new techniques must be developed.

- **Spectral conditions for observability.**

  In recent works Tusnack [104], Miller [95] (see also Russell y Weiss [107]), the authors give a spectral condition which guarantees the observability for infinite dimensional systems. This type of conditions generalize the Hautus test for finite dimensional systems to infinitely dimensional ones.

  It would be interesting to see if these spectral methods can be adapted in order to guarantee uniform observability results for numerical methods based on the two-grid method.

- **Dispersive properties of schemes in general grids**

  All the analysis we have done here is based on Fourier analysis. This type of analysis is well adapted to numerical problems where the meshes are lattices. Nevertheless, in many engineering applications the same problems are studied on asymmetrical meshes. In these cases the Fourier analysis cannot be used.

  A first step to understand the difficulties occurring in asymmetrical grids is to consider finite difference schemes for the one-dimensional Schrödinger equation on the grid $3h\mathbb{Z}\cup(3\mathbb{Z}+1)h$. This method breaks the symmetry of the classical scheme and represents a intermediate step between the symmetrical grids and the unstructured ones. For this method the Fourier analysis is yet useful.

  Once we understand the difficulties of the previous case the same problems can be analyzed on the structured meshes as those of Figure 7.1.

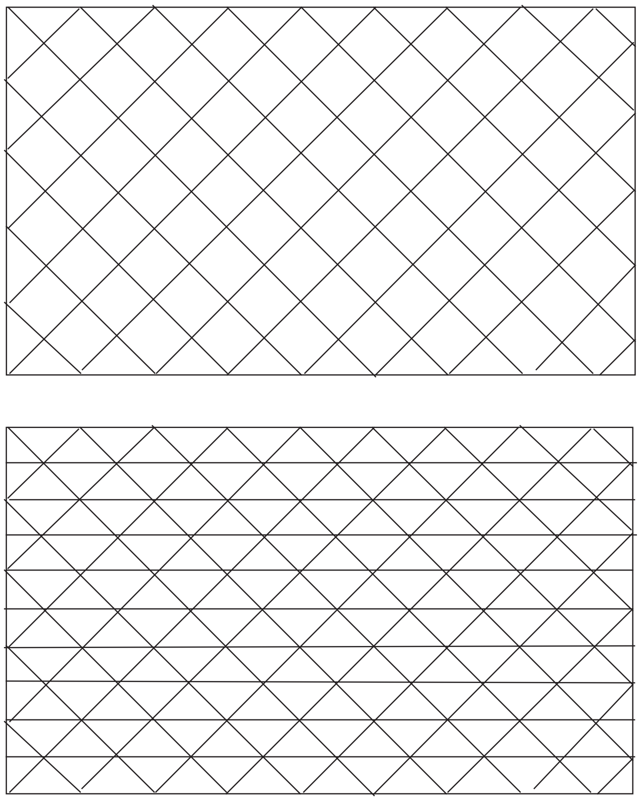

Figure 7.1: Meshes with symmetry

More ambitious would be to analyze where the decay properties of the Schrödinger equation are maintained in approximations by finite elements. In this case it is clear that the Fourier analysis cannot be applied and new techniques have to be developed.

- **Equation on networks**

  In view of the works of Dager and Zuazua [38], [39], [37], [40] on the control of the wave equation on networks we propose to analyze the dispersive properties of the Schrödinger equations on this type of structures.

  A first example would be to consider the equation $iu_t + u_{xx} = 0$ on a network as in Figure 7.2 and analyze the dispersive properties of this equation. Also the discrete version $iu_h + \Delta_h u = 0$ and its viscous and two-grid alternatives can be studied on the network given by Figure 7.3.

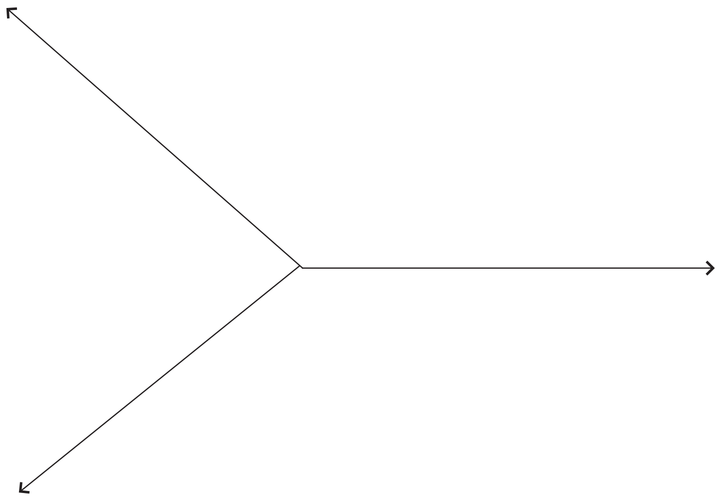

Figure 7.2: A continuous tree

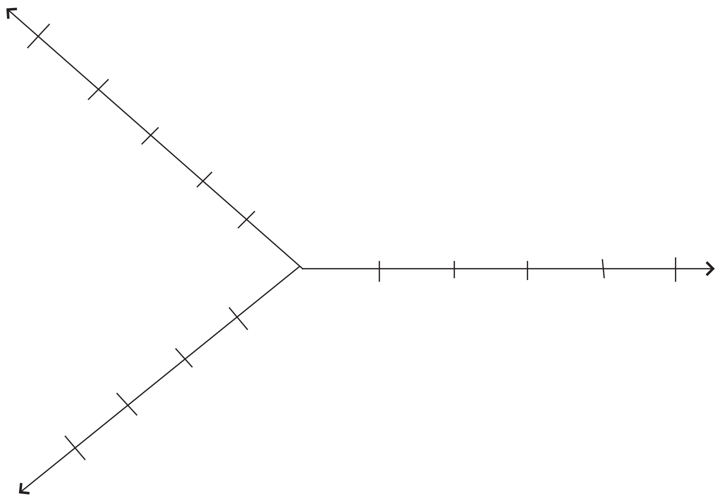

Figure 7.3: A discrete tree

Appendix A

The Semidiscrete Fourier Transform and the sinc Function

In this Appendix we present some classical results on the semidiscrete Fourier transform and band-limited sinc function interpolation. The main results we present here are collected from [128] and [131].

The Fourier transform of a continuous function $u(x)$, $x \in \mathbb{R}^d$ is the function $\widehat{u}(\xi)$ defined by

$$\widehat{u}(\xi) = \int_{\mathbb{R}^d} e^{i\xi \cdot x} u(x) dx, \ \xi \in \mathbb{R}^d. \tag{A.0.1}$$

The number $\widehat{u}(\xi)$ can be interpreted as the amplitude density of u at the wavenumber ξ, and this process of decomposing a function into its constituent waves is called Fourier analysis. Conversely, we can reconstruct u from $\widehat{u}$ by the inverse Fourier transform:

$$u(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} \widehat{u}(\xi) d\xi, \ x \in \mathbb{R}^d. \tag{A.0.2}$$

The variable x is the physical variable, and ξ is the Fourier variable or wavenumber.

We want to consider x ranging over $h\mathbb{Z}^d$ rather than $\mathbb{R}^d$. Precise analogues of the Fourier transform and its inverse exist for this case. The crucial point is that, because the spatial domain is discrete, the wavenumber ξ will no longer range over all of $\mathbb{R}^d$. Instead, the appropriate wavenumber domain is a bounded interval of length $2\pi/h$ in each direction, and one suitable choice is $[-\pi/h, \pi/h]^d$:

$$\begin{array}{lccc} \text{Physical space:} & \text{discrete,} & \text{unbounded :} & x \in h\mathbb{Z}^d \\ & \updownarrow & \updownarrow & \\ \text{Fourier space:} & \text{bounded,} & \text{continuous:} & \xi \in \left[-\frac{\pi}{h}, \frac{\pi}{h}\right]^d. \end{array}$$

The reason for these connections is the phenomenon known as *aliasing*. Two complex exponentials $f(x) = \exp(ik_1x)$ and $g(x) = \exp(ik_2x)$ are unequal over $\mathbb{R}$ if $k_1 \neq k_2$. If we restrict f and g to $h\mathbb{Z}$, however, they take values $f_j = \exp(ik_1x_j)$ and $g_j = \exp(ik_2x_j)$, and if $k_1 - k_2$ is an integer multiple of $2\pi/h$, then $f_j = g_j$ for each j. It follows that for any complex exponential $\exp(ikx)$, there are infinitely many other complex exponentials that match it on the grid $h\mathbb{Z}$-aliases of ξ. Consequently it suffices to measure wavenumbers for the grid in an interval of length $2\pi/h$, and for reasons of symmetry, we choose the interval $[-\pi/h, \pi/h]$.

We now introduce the semidiscrete Fourier transform and state the main result (cf. [128], Ch.2, p. 94):

Theorem A.0.1. *If $v \in l^2(h\mathbb{Z}^d)$, then the semidiscrete Fourier transform*

$$\widehat{v}(\xi) = (\mathcal{F}_h v)(\xi) = h^d \sum_{\mathbf{j}\in\mathbb{Z}^d} e^{-i\xi\cdot\mathbf{j}h} v_{\mathbf{j}},\ \xi \in [-\pi/h, \pi/h]^d \tag{A.0.3}$$

belongs to $L^2((-\pi/h, \pi/h)^d)$, and v can be recovered from $\widehat{v}$ by the inverse semidiscrete Fourier transform

$$v_{\mathbf{j}} = (\mathcal{F}_h^{-1}\widehat{v})(x) = \frac{1}{(2\pi)^d}\int_{[-\pi/h,\pi/h]^d} e^{i\xi\cdot x_{\mathbf{j}}}\widehat{v}(\xi)d\xi,\ \mathbf{j} \in \mathbb{Z}^d. \tag{A.0.4}$$

The $l^2(h\mathbb{Z}^d)$-norm of v and the L^2-norm of $\widehat{v}$ are related by Parseval's equality,

$$\|\widehat{v}\|_{L^2((-\pi/h,\pi/h)^d)} = (2\pi)^{d/2}\|v\|_{l^2(h\mathbb{Z}^d)}. \tag{A.0.5}$$

If $u \in l^2(h\mathbb{Z}^d)$ and $v \in l^1(h\mathbb{Z}^d)$, then $u * v \in l^2(h\mathbb{Z}^d)$, and $\widehat{u * v}$ satisfies

$$\widehat{u * v}(\xi) = \widehat{u}(\xi)\widehat{v}(\xi). \tag{A.0.6}$$

Note that (A.0.3) approximates (A.0.1) by a trapezoid rule, and (A.0.4) approximates (A.0.2) by truncating $\mathbb{R}^d$ to $[-\pi/h, \pi/h]^d$. As $h \to 0$, the two pairs of formulas converge.

We now introduce the band-limited interpolator of a discrete function. Consider the discrete set $\{u_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$ and the continuous function derived from it by the relation

$$u^*(x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} u_{\mathbf{n}}\Psi\left(\frac{x - h\mathbf{n}}{h}\right) \tag{A.0.7}$$

where the function $\Psi : \mathbb{R}^d \to \mathbb{R}$ is defined by

$$\Psi(x) = \prod_{k=1}^{d}\frac{\sin(\pi x_k)}{\pi x_k},\ x = (x_1, ..., x_d) \in \mathbb{R}^d. \tag{A.0.8}$$

We may define the set of basis functions

$$\Psi_{\mathbf{n}}(x) = \Psi\left(\frac{x - h\mathbf{n}}{h}\right)$$

and rewrite (A.0.7) as

$$u^*(x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} u_{\mathbf{n}}\Psi_{\mathbf{n}}(x).$$

Since

$$\Psi_{\mathbf{n}}(h\mathbf{m}) = \begin{cases} 1 & \text{when } \mathbf{m} = \mathbf{n}, \\ 0 & \text{when } \mathbf{m} \neq \mathbf{n}, \end{cases}$$

it follows that

$$u^*(h\mathbf{n}) = u_n.$$

Hence $u^*(x)$ is an interpolator between the discrete points $\{h\mathbf{n}; u_\mathbf{n}\}$. The function $\Psi_0(x)$ is called the sinc *function* or Whittaker's cardinal function, Shannon, and Nyquist [137], [59], [100]. For much more about sinc function and associated numerical methods, see [119].

It has the interesting Fourier transform :

$$\widehat{\Psi}_0(\xi) = \begin{cases} h^d, & \xi \in [-\pi/h, \pi/h]^d, \\ 0, & \text{elsewhere.} \end{cases}$$

Since $\Psi_\mathbf{n}(x)$ is obtained by shifting $\Psi_0(x)$ by $h\mathbf{n}$, we also have

$$\widehat{\Psi}_\mathbf{n}(\xi) = e^{-ih\xi\cdot\mathbf{n}}\widehat{\Psi}_0(\xi),$$

which results from a direct application of the shifting rule for Fourier transforms.

We now return to the function $u^*(x)$. Its Fourier transform is easily found:

$$\widehat{u}^*(\xi) = \widehat{\Psi}_0(\xi) \sum_{\mathbf{n}\in\mathbb{Z}^d} u_\mathbf{n} e^{-ih\xi\cdot\mathbf{n}}, \ \xi \in \mathbb{R}^d;$$

that is

$$\widehat{u}^*(\xi) = \begin{cases} \widehat{u}(\xi), & \xi \in (-\pi/h, \pi/h)^d, \\ 0, & \text{elsewhere.} \end{cases}$$

If we apply Parseval's equality to the continuous function $u^*(x)$ and to the sequence $\{u_\mathbf{n}\}_{\mathbf{n}\in\mathbb{Z}^d}$ we easily obtain the equality of the L^2-norms :

$$\|u^*\|_{L^2(\mathbb{R}^d)} = \|u\|_{l^2(h\mathbb{Z}^d)}.$$

More generally we can prove that

Theorem A.0.2. *Let u be a sequence and u^* its Shannon interpolator. There is a positive constant A such that*
i) For any $p > 0$

$$\|u\|_{l^p(h\mathbb{Z}^d)} \leq A^{d/p}\|u^*\|_{L^p(\mathbb{R}^d)}; \tag{A.0.9}$$

ii) For any $p > 1$

$$\|u^*\|_{L^p(\mathbb{R}^d)} \leq A^{d/p}\|u\|_{l^p(h\mathbb{Z}^d)}; \tag{A.0.10}$$

iii)

$$\frac{1}{A}\|\nabla_h u\|_{l^2(h\mathbb{Z}^d)} \leq \|\nabla u^*\|_{L^2(\mathbb{R})} \leq A\|\nabla_h u\|_{l^2(h\mathbb{Z}^d)}. \tag{A.0.11}$$

This theorem is a consequence of a result of Plancherel and Polya [1] ([101], [102], p. 157) concerning the Fourier series.

1

Theorem A.0.3. *Soit $f(x_1, x_2, ..., x_d)$ une function définie et intégrable dans le domaine $-\pi \leq x_\nu \leq \pi$, $\nu = 1, 2, ..., d$. Soit*

$$\sum_{m_1}\sum_{m_2}...\sum_{m_d} c_{m_1 m_2 ... m_d} e^{-i(m_1x_1+m_2x_2+...+m_dx_d)} \sim f(x_1, x_2, ..., x_d) \tag{A.0.12}$$

sa série de Fourier et $F(z_1, z_2, ..., z_d)$ la function entière de type exponentiel définie par

$$F(z_1, z_2, ..., z_d) = \int_\pi^\pi \int_\pi^\pi ... \int_\pi^\pi f(t_1, t_2, ..., t_d)e^{-i(t_1z_1+t_2z_2+...+t_dz_d)}dt_1dt_2...dt_d. \tag{A.0.13}$$

Proof of Theorem A.0.2. First we reduce the general case to the case $h = 1$. This reduction is a simple consequence of a scaling argument. To avoid the possible confusions let us denote

$$u^{*,h}(x) = \sum_{\mathbf{n}\in\mathbb{Z}^d} u_{\mathbf{n}}\Psi\left(\frac{x-h\mathbf{n}}{h}\right),\ h > 0. \tag{A.0.16}$$

We remark that for any $x \in \mathbb{R}^d$ the following holds:

$$u^{*,h}(x) = u^{*,1}\left(\frac{x}{h}\right).$$

As a consequence

$$\|u\|_{l^p(h\mathbb{Z}^d)} = h^{d/p}\|u\|_{l^p(\mathbb{Z}^d)} \tag{A.0.17}$$

and

$$\begin{aligned}\|u^{*,h}\|_{L^p(\mathbb{R}^d)} &= \left(\int_{\mathbb{R}^d} |u^{*,h}(x)|^p dx\right)^{1/p} = \left(\int_{\mathbb{R}^d} \left|u^{*,1}\left(\frac{x}{h}\right)\right|^p dx\right)^{1/p}\\ &= \left(h^d \int_{\mathbb{R}^d} |u^{*,1}(x)|dx\right)^{1/p} = h^{d/p}\|u^{*,1}\|_{L^p(\mathbb{R}^d)}.\end{aligned}$$

These results reduce the general case to the case $h = 1$. Let us take $f = \widehat{u}$ in Theorem A.0.3, the discrete Fourier transform of the sequence $\{u_{\mathbf{n}}\}_{\mathbf{n}\in\mathbb{Z}^d}$. Then by the definition of the inverse Fourier transforms (discrete and continuous) :

$$u_{\mathbf{n}} = \int_{[-\pi,\pi]^d} e^{i\mathbf{n}\cdot\xi}\widehat{u}(\xi)d\xi,\ \mathbf{n} \in \mathbb{Z}^d$$

and

$$u^{*,1}(x) = \int_{[-\pi,\pi]^d} e^{ix\cdot\xi}\widehat{u}(\xi)d\xi,\ x \in \mathbb{R}^d.$$

Clearly by (A.0.14) and (A.0.15) we obtain the desired results (A.0.9), (A.0.10) respectively.

For the last inequality we recall that

$$\|\nabla_1 u\|^2_{l^2(\mathbb{Z}^d)} = \sum_{\mathbf{j}\in\mathbb{Z}^d}\sum_{k=1}^d |u_{\mathbf{j}+\mathbf{e}_k} - u_{\mathbf{j}}|^2 = \int_{[-\pi,\pi]^d} |\widehat{u}(\xi)|^2 \left(\sum_{k=1}^d |e^{i\xi_k} - 1|^2\right) d\xi.$$

By the definition of $u^{*,1}$ we have

$$\|\nabla u^{*,1}\|^2_{L^2(\mathbb{R})} = \int_{[-\pi,\pi]^d} |\widehat{u}(\xi)|^2 \left(\sum_{k=1}^d |\xi_k|^2\right) d\xi.$$

Il existe, si $p > 0$, une constante A ne dépendant de p telle que

$$\sum_{m_1}\sum_{m_2}...\sum_{m_d} |c_{m_1 m_2 ... m_d}|^p < A^d \int_{\mathbb{R}}\int_{\mathbb{R}}...\int_{\mathbb{R}} |F(x_1, x_2, ..., x_d)|^p dx_1 dx_2 ... dx_d \tag{A.0.14}$$

et, si $p > 1$, une constante B ne dépendant de p telle que

$$\int_{\mathbb{R}}\int_{\mathbb{R}}...\int_{\mathbb{R}} |F(x_1, x_2, ..., x_d)|^p dx_1 dx_2 ... dx_d < B^d \sum_{m_1}\sum_{m_2}...\sum_{m_d} |c_{m_1 m_2 ... m_d}|^p. \tag{A.0.15}$$

It remains to prove that

$$\int_{[-\pi,\pi]^d} |\widehat{u}(\xi)|^2 \left(\sum_{k=1}^{d} |\xi_k|^2 \right) d\xi \simeq \int_{[-\pi,\pi]^d} |\widehat{u}(\xi)|^2 \left(\sum_{k=1}^{d} |e^{i\xi_k} - 1|^2 \right) d\xi.$$

This is a consequence of the following inequality

$$\xi^2 \geq |e^{i\xi} - 1|^2 = 4 \sin^2 \frac{\xi}{2} \geq 4 \left(\frac{2}{\pi} \xi \right)^2 \geq \frac{16}{\pi^2} \xi^2,$$

for all $\xi \in [-\pi, \pi]$. The proof is now complete. □

Appendix B

A Result on Fourier Series

Lemma B.0.2. *Let* $m : [-\pi, \pi]$ *be a continuous function satisfying*

$$0 \leq m(\xi) \leq 1, \ \xi \in [-\pi, \pi].$$

Then the following holds for any function $f \in L^2(\mathbb{T}^1)$

$$\sum_{n\in\mathbb{Z}} \left| \int_{-\pi}^{\pi} e^{in\xi} m^{|n|}(\xi) f(\xi) d\xi \right|^2 \lesssim \int_{-\pi}^{\pi} |f(\xi)|^2 d\xi, \tag{B.0.1}$$

where $\mathbb{T}^1$ *is the one-dimensional torus.*

Proof. Let us define the linear operator

$$(Tf)_n = \int_{-\pi}^{\pi} e^{in\xi} m^{|n|}(\xi) f(\xi) d\xi.$$

Inequality (B.0.1) means that T maps continuously $L^2(\mathbb{T})$ to $l^2(\mathbb{Z})$. To prove the continuity of T is equivalent to prove that its formal adjoint T^* maps continuously $l^2(\mathbb{Z})$ to $L^2(\mathbb{T}^1)$. By the definition of the formal adjoint T^* :

$$\langle Tf, g \rangle_{l^2(\mathbb{Z})} = \langle f, T^* g \rangle_{L^2((-\pi,\pi))},$$

we get

$$\sum_{n\in\mathbb{Z}} \overline{g_n} \int_{-\pi}^{\pi} e^{in\xi} m^{|n|}(\xi) f(\xi) d\xi = \int_{-\pi}^{\pi} f(\xi) \overline{(T^* g)(\xi)} d\xi.$$

Then the operator T^* is given by

$$(T^* g)(\xi) = \sum_{n\in\mathbb{Z}} e^{-in\xi} m^{|n|}(\xi) g_n.$$

In the following we prove that T^* maps continuously $l^2(\mathbb{Z})$ to $L^2(\mathbb{T}^1)$. The key point is the following pointwise estimate on T^*:

$$|(T^* g)(\xi)| \leq \sup_{0\leq r\leq 1} \left| \sum_{n\in\mathbb{Z}} e^{in\xi} r^{|n|} g_n \right| \ \ \forall \ \xi \in [-\pi, \pi].$$

Classical results on harmonic analysis (cf. [73], p.76) show that for any $0 \leq r \leq 1$

$$\left|\sum_{n\in\mathbb{Z}} e^{in\xi} r^{|n|} g_n\right| \leq M_{g^\vee}(\xi),$$

where

$$g^\vee(\xi) = \sum_{n\in\mathbb{Z}} e^{in\xi} g_n$$

and M_f is the maximal function of f, defined by

$$M_f(t) = \sup_{0<s\leq\pi} \left|\frac{1}{2s}\int_{t-s}^{t+s} f(\tau)d\tau\right|.$$

For further reading on estimates involving maximal functions of elements of $L^2(\mathbb{T})$ see [73]. Using the properties of the maximal function $M_{g^\vee}$ (cf. [73], p.88) we get

$$\|M_{g^\vee}\|_{L^2(\mathbb{T})} \leq \|g^\vee\|_{L^2(\mathbb{T})} = \|g\|_{l^2(\mathbb{Z})},$$

which in fact proves that T^* maps continuously $L^2(\mathbb{T})$ to $l^2(\mathbb{Z})$:

$$\|T^* g\|_{L^2((-\pi,\pi))} \leq \|g\|_{l^2(\mathbb{Z})}.$$

□

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