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Nonlinear and Nonlocal Models in Fluid Mechanics

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Presentación

Esta memoria está dedicada al estudio de cuatro modelos que provienen de la mecánica de fluidos. Todos ellos son ecuaciones en derivadas parciales no lineales y no locales. Nuestro interés se ha centrado en el análisis de la formación de singularidades en tiempo finito.

El primer capítulo está destinado a introducir las ecuaciones en las que están basados los problemas que se estudian en el resto de la disertación.

En el segundo capítulo analizaremos una ecuación que ha sido previamente estudiada en diferentes contextos, apareciendo como modelo unidimensional de la ecuación quasi-geostrófica superficial (SQG), como modelo del problema del vortex sheet, como aproximación lineal del problema de water waves o como modelo de la dinámica de dislocaciones en sólidos cuando se supone que estas son líneas rectas. Los resultados obtenidos han sido, dependiendo del signo del dato inicial, existencia global y unicidad, ill-posedness, existencia local y formación de singularidades en tiempo finito . La principal idea para la obtención de estos resultados ha sido explotar la relación existente entre este modelo y la ecuación de Burger compleja. De hecho, veremos que podemos entender nuestra ecuación, como la restricción al eje real de la equación de Burger en el semiplano complejo superior. Utilizando una técnica diferente (transformaciones hodógrafas) el fénomeno de blow up había sido probado anteriormente para datos iniciales analíticos y en el caso en el que probaremos que el problema está mal propuesto en espacios menos regulares. El estudio realizado ha sido publicado en [7].

El propósito del tercer capítulo es el análisis del problema del vortex sheet y su relación con el modelo presentado en el primer capítulo. Este problema consiste en el estudio de la evolución de una curva sobre la que se concentra la vorticidad en un fluido ideal en dos dimensiones. Los resultados obtenidos pueden encontrarse en [11].

En el cuarto capítulo estudiaremos soluciones de la ecuación quasi-geostrófica superficial con energía infinita. Esta ecuación tiene un origen geofísico y se obtiene como cierta aproximación del sistema quasi-geostrófico general que considera la dinámica de fluidos atmosféricos sometidos a la fuerza de Coriolis. En concreto, SQG mide la evolución de la temperatura del fluído cuando los números de Rossby y Ekman son pequeños. Un aspecto de gran importancia de esta ecuación, desde el punto de vista matemático, fue señalado por Constantin, Majda y Tabak en un trabajo en la que la propusieron como un modelo escalar y dosdimensional de la Ecuación de Euler tridimensional. Una primera observación para entender la relación entre ambas es que la ecuación para el gradiente perpendicular de la temperatura en SQG tiene la misma estructura que en la ecuación de Euler para la vorticidad. El mismo tipo de soluciones con energía infinita que consideramos, ha sido estudiado en la ecuación de Euler en dos dimensiones. La principal diferencia entre los dos modelos es, que en nuestro caso, la ecuación resultante es no local. Previamente, una generalización de la ecuación obtenida había sido propuesta por De Gregorio como un modelo unidimensional de la ecuación de Euler para la vorticidad en tres dimensiones debido a las similitudes entre sus estructuras: un término cuadrático, formado por el producto de la función y una integral singular de la misma, y un término de transporte en el que las derivadas de la velocidad son integrales singulares de la función. El comportamiento del término cuadrático ya había sido estudiado por Lax, Constantin y Majda, quienes probaron que podía dar lugar a la formación de singularidades. La pregunta que debemos hacernos entonces es si el término convectivo es capaz de cancelarlas. La generalización de De Gregorio consiste en introducir un parámetro en la ecuación que pondere el efecto de los dos términos implicados. En esta memoria mostraremos que para valores negativos del parámetro, el término convectivo no es capaz de impedir la formación de singularidades. Formalmente podemos decir, que en este caso ambos términos trabajan en la misma dirección. Sin embargo, cuando el valor del parámetro es positivo, los términos compiten entre sí y el efecto del término de transporte necesita de un estudio más detallado. Las soluciones con energía infinita de la ecuación SQG son obtenidas cuando el valor del parámetro es igual a uno. Nuestro análisis consistirá en demostrar que bajo ciertas condiciones sobre el dato inicial, que son preservadas por la evolución, los términos son comparables. Aún así, se pueden imponer condiciones más restrictivas sobre la solución que proporcionan un crecimiento cuadrático. Aunque no se ha podido demostrar que dichas condiciones se mantengan a lo largo del tiempo, un estudio numérico muestra que se pueden encontrar datos iniciales que proporcionan soluciones que se comportan de la manera que se les requiere. Otro indicio de la aparición del fenómeno de blow up en el modelo de De Gregorio, es la existencia de soluciones autosimilares para todo los valores del parámetro introducido. El estudio realizado ha sido publicado en [8].

El quinto capítulo está dedicado al estudio de una ecuación que ha sido propuesta como un modelo de la propagación de ondas con una frecuencia linealizada no constante y como una aproximación cuadrática de la dinámica de las ondas sobre una discontinuidad plana de la vorticidad en un fluido ideal en dos dimensiones. El sistema resultante es la ecuación de Burger más un término extra que viene dado por la transformada de Hilbert de la función. Esta transformada de Hilbert es la que introduce el carácter oscilatorio. Nuestro análisis revela que bajo ciertas condiciones sobre el dato inicial, el máximo de la solución crece de forma singular durante el tiempo de existencia. Además presentamos una generalización de la ecuación en la que el término lineal preserva el carácter oscilatorio pero introduce un orden más alto en derivadas. Probaremos, que aún en estas condiciones, las soluciones desarrollan singularidades en tiempo finito. El estudio realizado puede encontrarse en [12].

El sexto capítulo tiene como objetivo el análisis de una ecuación de transporte donde la velocidad viene dada por el vector de las transformadas de Riesz de la función. Este sistema fue estudiado previamente y fue probada la existencia de singularidades. Nosotros probaremos la existencia de soluciones autosimilares, que permiten visualizar como la ausencia de incompresibilidad del flujo interviene en el fenómeno de blow-up. El estudio realizado ha sido publicado en [9].

En el séptimo capítulo estudiaremos soluciones con energía infinita de las ecuaciones que rigen la dinámica de un fluido incompresible en un medio poroso. En este sistema, la velocidad del fluido viene dada por la ley de Darcy, que afirma, que dicha velocidad es proporcional al gradiente de presiones más las fuerzas externas. De modo que las fuerzas no están acelerando el flujo sino produciendo velocidades. Esta suposición pretende modelar el efecto de frenado que sufre el fluido al moverse en un medio poroso. En nuestro caso consideraremos que la única fuerza externa a la que está sometido el fluido es la gravedad. Después de demostrar existencia local para la ecuación resultante, daremos una solución explícita que explota en tiempo finito. El estudio realizado ha sido publicado en [10].

Abstract

This dissertation is devoted to the study of four models which arise in the field of fluid mechanics. All of them are nonlocal and nonlinear partial differential equations. Attention is focused on the analysis of singularity formation.

In the second chapter we shall analyze an equation which has been studied previously in completely different contexts, arising as a one dimensional model of the surface quasi-geostrophic equation (SQG), as a model of the vortex sheet problem, as a quadratic approximation of the water-wave problem, or as model of the dislocation dynamics in solids when the dislocations are supposed to be along straight lines. The obtained results are, depending on the sign of the initial data, global existence and uniqueness, ill-posedness, local existence and singularities. The main idea behind these results is to relate the model and the Burgers complex equation. In fact, we will see that we can understand our equation as the restriction to the real axis of the Burgers equation in the upper half complex plane. By using a different technique (the hodograph transformation) the blow-up phenomena was shown previously for analytic initial data and in the case in which we shall prove ill-posedness for less regular spaces. These results have been published in [7].

The purpose of the third chapter is the analysis of the vortex sheet problem and its relation with the model presented in the first chapter. The vortex sheet problem is concerned with the study of the evolution of a curve where the vorticity accumulates for an ideal fluid in two dimensions. The obtained results can be founded in [11].

In the fourth chapter we shall study solutions with infinite energy for the surface quasi-geostrophic equation. This equation is a model of geophysical origin and is obtained as an aproximation of the general quasi-geostrophic system which considers the dynamics of atmospheric fluids taking into account the Coriolis force. Specifically, SQG measures the evolution of the temperature of the fluid when both the Rossby and Ekman numbers are small. An important aspect of this equation, from a mathematical point of view, was pointed out by Contantin, Majda and Tabak, in a paper where they proposed it as 2D scalar model of the 3D Euler equation. We can understand the relation between both equations by observing that the equation for the perpendicular gradient of the temperature in the SQG equation has the same structure that the equation for the vorticity in the 3D Euler equation. The sort of solutions with infinite energy that we consider, have been studied for the 2D Euler equation. The main difference between the models is that, in our case, the equation is nonlocal. Previously, a generalization of the resulting equation was proposed by De Gregorio as a one dimensional model of the 3D Euler equation for the vorticity, since their structures are similar: one quadratic term, given by the product of the function and a singular integral operator applied to the function, and a transport term in which the derivatives of the velocity are singular integral operators of the function. The behavior of the quadratic term, had been already studied by Lax, Constantin and Majda, who proved that it was able to produce singularities. the question is then whether the convection term can cancel out them. De Gregorio's generalization consists in introducing a parameter in the equation which weights the effect of the both terms. In this dissertation, we will show that, for negative values of the

parameter, the convection term cannot cancel out the singularities. Roughly speaking, in this case, both terms work in the same direction. However, when the parameter is positive, these terms are fighting each other and the effect of the transport term needs a more exhaustive analysis. The infinite energy solutions of the SQG equation are obtained when the parameter is equal to one. Our analysis will consist in showing that, by assuming certain conditions on the initial data, which are preserved along the evolution both terms are comparable. In spite of that, we can impose a more restrictive condition on the solution which provides a quadratic growth. Although it has not been proved that this condition is preserved along the evolution, numerical analysis shows that we can find initial data such that the resulting solution has the required behavior. Another indication about the formation of singularities in the De Gregorio model, is the existence of self-solutions for all values of the introduced parameter. These results have been published in [8].

The fifth chapter is devoted to the study of an equation which has been proposed as a model for waves with constant nonzero linearized frequency and as an effective model describing surface waves on a planar discontinuity in vorticity for a two-dimensional inviscid incompressible fluid. The obtained system is the Burgers equation with an extra term given by the Hilbert transform of the function. This Hilbert transform causes a oscillatory behavior. Our analysis shows that, if the initial data satisfies certain conditions, the maximum of the solution grows in a singular way for the time of existence. In addition, we present a generalization of the equation in which the linear term holds the oscillatory behavior and introduces a higher order in derivatives. We will prove that, also in this case, the solutions develop a singularity in finite time. These results can be founded in [12].

The purpose of the sixth chapter is the analysis of a transport equation whose velocity is given by the Riesz transform of the function. This system was studied previously and the existence of singularities was proven. We shall show the existence of self-similar solutions, which allow us to see how the compressibility influences the blow-up phenomena. These results have been published in [9].

In the seventh chapter we will study infinite energy solutions of the equations for the dynamics of incompresible fluid in a porous medium. In this system, the velocity of the fluid is given by Darcy law, which asserts that this velocity is proportional to the gradient of the pressure plus the external forces. Thus the forces do not accelerate the flow but do produce velocity. These assumptions are intended to model the restraining effect that the fluid experiences when it is moving in a porous medium. In our case, we shall consider that the only external force is gravity. After proving local existence for the obtained equation we will exhibit a explicit solution which blows up in finite time. These results have been published in [10].

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Cuando empiezas la tesis estás convencido de que acabarla es el objetivo, que no importa lo que viene después, que si lo consigues estarás satisfecho. Ahora que termino me doy cuenta de que me he olvidado de aquello, que sólo pienso en los problemas que no he resuelto y en lo que me espera en los próximos años. Bueno, pues uno suele estar equivocado y llevar a la vez algo de razón. Es cierto que aquí no acaba el viaje, pero también es cierto que es un buen momento para valorar el trabajo realizado y, por supuesto, para acordarme de aquellos que me han ayudado a conseguirlo. Además, como es la primera vez que dejo escrito algo parecido, voy a incluir en estos agradecimientos, no sólo a quiénes este trabajo debe una ayuda directa, sino también a quiénes han estado en ocasiones que no recibieron letra impresa a pesar de merecerla.

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"We can forgive a man for making useful thing as long as he does not admite it. The only excuse for making useless thing is that one admires it intensely."

Oscar Wilde

Contents

Chapter 1

Preliminaries

In this chapter we present a brief introduction of the equations which give rise to the problems studied in this dissertation. All of them came from the field of fluid mechanics. The general characteristics and some known results are presented.

1.1 The Surface Quasi-Geostrophic Equation

The Surface Quasi-Geostrophic Equation (SQG) is given by the following expression

$$\begin{aligned} \partial_t \theta + u \cdot \nabla \theta &= 0, \\ \theta(x, 0) &= \theta^0(x), \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} \theta &: \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R} \\ \theta &= \Lambda \Psi \quad \text{and} \\ u &= \nabla^\perp \Psi = (-\partial_{x_2} \Psi, \partial_{x_1} \Psi) \end{aligned}$$

with $\Lambda = (-\Delta)^{1/2}$. Using the Fourier Transform it is easy to check that in the SQG equation the velocity can be written as the perpendicular vector of Riesz transforms of the scalar function

$$u = (R_2 \theta, -R_1 \theta),$$

where (see [65])

$$R\theta(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{(x - y)}{|x - y|^3} \theta(y) dy.$$

Indeed, we have that

$$\hat{u}(\xi) = \left(-\widehat{\partial_{x_2} \Lambda^{-1}} \theta(\xi), \widehat{\partial_{x_1} \Lambda^{-1}} \theta(\xi) \right) = \left(-i \frac{\xi_2}{|\xi|}, i \frac{\xi_1}{|\xi|} \right) \hat{\theta}(\xi) = \left(\widehat{R_2 \theta}(\xi), -\widehat{R_1 \theta}(\xi) \right).$$

The SQG system (1.1) is a model of geophysical origin which was proposed by P. Constantin, A. Majda and E. Tabak [19] as a model of the 3D-Euler equation. Numerical experiments, carried out by those authors, showed evidence of fast growth of the gradient of the active scalar when the geometry of the level sets contain a hyperbolic saddle. Later, further numerical studies were performed in [58] and [20] suggesting a double exponential growth in time. An analytical study in [27] showed that a *simple hyperbolic saddle breakdown* can not occur in finite time. In fact, the angle of the saddle is bounded below by a double exponential in time and a quadruple exponential upper bound was obtained for the growth of the derivatives of the active scalar. Subsequently, this bound was improved, for a formation of a *semi-uniform sharp front* in [28], by a double exponential. In [35], under certain assumptions on the local geometry of the level sets, the same bound is obtained. Recently, there has been different approaches to understand the growth of the derivatives: in [14] an a priori estimate from below for the Sobolev norms is shown, a study of the spectrum of the linearized SQG is performed in [42] and the existence of the unstable eigenvalues is proven, and in [44] they prove that the 0 solution is strongly unstable in H^{11}.

Chapter 2 is devoted to the study of a one dimensional model of the SQG system. In chapter 4 we shall study particular solutions with infinite energy of this equation.

1.2 The vortex-sheet problem

The vortex sheet problem is concerned with a velocity field $v = (v_1, v_2)$ satisfying the incompressible 2D- Euler equations

$$v_t + (v \cdot \nabla)v = -\nabla p, \tag{1.2}$$

$$\nabla \cdot v = 0. \tag{1.3}$$

We will study weak solutions of the system whose vorticity $\omega = \nabla \times v$ is a delta function supported on the curve $z(\alpha, t)$:

$$\omega(x,t) = \varpi(\alpha,t)\delta(x - z(\alpha,t)), \tag{1.4}$$

i.e. ω is a measure defined by

$$< \omega, \eta >= \int \varpi(\alpha,t)\eta(z(\alpha,t))d\alpha,$$

with $\eta(x)$ a test function.

We can assume for the curve the following scenarios:

- Periodicity in the horizontal space variable: $z(\alpha + 2k\pi, t) = z(\alpha, t) + (2k\pi, 0)$.
- A closed contour: $z(\alpha + 2k\pi, t) = z(\alpha, t)$.
- An open contour flat at infinity: $\lim_{\alpha\to\infty} (z(\alpha, t) - (\alpha, 0)) = 0$.

The vortex sheet $z(\alpha, t)$ evolves satisfying the equation,

$$z_t(\alpha,t) = BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t), \tag{1.5}$$

where the Birkhoff-Rott integral on the curve, which comes from Biot-Savart law, is given by

$$BR(z,\varpi)(\alpha,t)=\frac{1}{2\pi}PV\int\frac{(z(\alpha,t)-z(\beta,t))^{\perp}}{|z(\alpha,t)-z(\beta,t)|^2}\varpi(\beta,t)d\beta,$$

and $c(\alpha,t)$ represents the re-parametrization freedom. Then we can close the system using Bernoulli's law with the equation:

$$\varpi_t=\partial_\alpha(c\,\varpi). \tag{1.6}$$

The problem of existence of weak solutions of the Euler equations for a general initial velocity in L^2 is not well understood [51]. There is solution for this problem but the velocity field becomes a Laplace-Young measure (see [37]). Constantin, E and Titi [21] prove a condition of regularity in 3D within the chain of Besov spaces, $v\in L^3([0,T];B_3^{\alpha,\infty})\cap C([0,T];L^2)$ with $\alpha>1/3$, for weak solutions conserving energy (Onsager's conjecture). Nevertheless there are results of non-uniqueness for weaker solutions with zero initial data that becomes nontrivial (see [62] and [63]) even for velocity fields in L^2, i.e. $v(x,t)\in L_c^\infty([0,T];L^2)$ (see [33]). There is also a result of uniqueness for a vorticity in $L^1\cap L^\infty$ due to Yudovich [69].

For the particular case of a vortex sheet there are many papers which consider the case of ϖ with a distinguished sign. We can point out the work of Delort [34] where he prove global existence of weak solutions for initial velocity in L^2_{loc} and vorticity a positive Radon measure. A simpler proof can be found in [49] due to Majda. Existence for a particular case of a Radon measure with non distinguished sign is shown in [48] .

In the chapter 3, our first step will be to deduce the equations of motion of the vortex sheet (1.5) and (1.6) from the weak formulation of the Euler equations. After that, we shall study the case in which the term in the tangential direction is given by $c(\alpha,t)=\frac{1}{2}H(\varpi)(\alpha,t)$, where $H\varpi$ is the Hilbert transform of the function ϖ (see [65]) given by

$$H\varpi(\alpha)=\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{\varpi(\beta)}{\alpha-\beta}d\beta,$$

and in the periodic domains also by

$$H\varpi(\alpha)=\frac{1}{\pi}PV\int_{-\pi}^{\pi}\frac{\varpi(\beta)}{2\tan((\alpha-\beta)/2)}d\beta.$$

This term is just of the same order that the Birkhoff-Rott integral over ϖ for a regular one-to-one curve [26]. In fact, for $z(\alpha,t)=(\alpha,0)$, we have exactly

$$BR(z,\varpi)(\alpha,t)=\frac{1}{2}H(\varpi)(\alpha,t)(0,1).$$

1.3 The vortex-patch problem

The vortex patch problem is concerned with the 2D Euler equation with an initial data given by a vorticity which is the step function

$$w_0(x)=\begin{cases} w_0 & x\in\Omega_0\\ 0 & x\notin\Omega_0\end{cases}, \tag{1.7}$$

where $\Omega_0 \subset \mathbb{R}^2$ is bounded, simply connected and with smooth boundary. Since $w_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ there exists a global solution of the Euler equation in a weak sense (see [50]). In order to study these solutions we shall asume that the vorticity is given by

$$w(x,t) = \begin{cases} w_0 & x \in \Omega(t) \\ 0 & x \notin \Omega(t) \end{cases},$$

where the domain $\Omega(t)$ is transported by the flow. Then, by using Biot-Savart we obtain the following equation for the boundary of $\Omega(t)$

$$\begin{aligned} z_t(\alpha,t) &= -\frac{w_0}{2\pi}\int_0^{2\pi} \log\left(|z(\alpha,t)-z(\beta,t)|\right) z_\alpha(\beta,t)d\beta \\ z(\alpha,0) &= z_0(\alpha). \end{aligned} \tag{1.8}$$

In this situation it can be proven that if $z(\alpha,t)$ is a solution of the system (1.8) such that $z \in C\left([0,T]; C^1\left([0,2\pi)\right)\right)$, then the velocity

$$v(x,t) = -\frac{w_0}{2\pi}\int_0^{2\pi} \log\left(|x-z(\beta,t)|\right) z_\alpha(\beta,t)d\beta$$

is a weak solution of the 2D Euler equation with the initial data (1.7) (see [50] for the details).

In [17] and [3] the authors proved the existence of solutions for all time for the vortex patch system (1.8) with a smooth enough initial curve.

In chapter 5 we will study an equation which was obtained in [53] as a quadratic approximation of this problem in the circle and proposed in [4] as an effective model describing surface wave on a planar discontinuity in vorticity.

1.4 An incompressible fluid in a porous media

The dynamic of an incompressible flow in a porous media is modelled by the following system

$$\rho_t + v\cdot\nabla\rho = 0 \tag{1.9}$$
$$\frac{\mu}{\kappa}v = -\left(\nabla p + g\rho(0,1)\right), \tag{1.10}$$
$$\nabla\cdot v = 0, \tag{1.11}$$

where v is the incompressible velocity, p is the pressure, μ is the dynamic viscosity, κ is the permeability of the isotropic medium, ρ is the liquid density, and g is the acceleration due to gravity. To simplify the notation, we will consider $\frac{\mu}{\kappa} = g = 1$.

The equation (1.10) is Darcy's law which relates the velocity of the fluid with the pressure and gravitational forces in a porous media. Darcy's law has been determined by the results of many experiments, and has been deduced from the Stokes equation using homogenization [67]. The equations (1.9) and (1.11) mean the conservation of the mass and the incompressibility of the flow respectively.

A study of the local existence of the system (1.9), (1.10) and (1.11) can be found in [30].

In chapter 7 we shall study particular solutions with infinite energy of this equation.

Chapter 2

Global Existence, Singularities and Ill-posedness for a Nonlocal Flux

2.1 Introduction.

We consider the following non-local equation:

$$\partial_t f + (fHf)_x = 0 \quad \text{on} \quad \mathbb{R}\times\mathbb{R}^+ \tag{2.1}$$
$$f(x,0) = f_0, \tag{2.2}$$

where Hf is the Hilbert transform of the function f, which is defined by the expression

$$Hf(x) = \frac{1}{\pi}P.V.\int \frac{f(y)}{x-y}dy.$$

One particular feature of equation (2.1) is the relation with the Burgers equation. Applying the Hilbert transform over equation (2.1) yields (for more details see [15]),

$$\partial_t(Hf) + HfHf_x - ff_x = 0 \tag{2.3}$$
$$Hf(x,0) = Hf_0(x). \tag{2.4}$$

Multiplying (2.1) by $-i$, adding (2.3) and defining the complex valued function $z(x,t) = Hf(x,t) - if(x,t)$ we get the equation

$$\partial_t z + zz_x = 0$$

$$z(x,0) = z_0(x) \equiv Hf_0 - if_0,$$

which is known as the inviscid Burgers equation.

In [1] the authors displayed equation (2.1) as a one dimensional model of the motion of a vortex sheet using the ideas of [36]. In the chapter 3, we shall analyze in detail a new relation of this equation with this problem. In [5] it also was proposed as a model of dislocations dynamics in solid.

Another motivation comes from the analogy that equation (2.1) has with the 2D surface quasi-geostrophic equation, which was studied in [15]. Indeed, from the expression (1.1) we can write the SQG system in the following way

$$\theta_t + div\left(\left(R^{\perp}\theta\right)\theta\right) = 0.$$

Since the Riesz transform is a singular integral operator and since, in one dimension, the only operator in this class is the Hilbert transform we can consider the equation (2.1) as a model of the SQG system.

In [54] the authors studied the equation

$$\begin{aligned} f_t + \delta(fHf)_x + (1-\delta)f_x Hf &= 0 && (2.5) \\ f(x,0) &= f_0(x), && (2.6) \end{aligned}$$

and they show formation of singularities for $0 < \delta < 1/3$ and $\delta = 1$. Another proof of the existence of singularities for equation (2.1) can be found in [1] (notice that in this paper the authors take a different sign for the Hilbert transform).

The equation (2.5) is also studied in [15] where the authors showed blow up for $0 < \delta \leq 1$. By an hodograph transformation an explicit solution for $\delta = 1$ is obtained over the torus, with mean zero analytic initial data. In addition they analyzed the equation

$$\begin{aligned} f_t + (fHf)_x &= -\nu H f_x && (2.7) \\ f(x,0) &= f_0(x), && (2.8) \end{aligned}$$

and they showed that the solutions to this equation may also develop singularities with mean zero analytic initial data and with the condition $\nu < ||f_0||_{L^\infty}$. We will study this equation in section 2.4.

The structure of the chapter is the following. In section 2.2 we show, for equation (2.1), global existence for all initial data strictly positive in the class $C^{0,\delta}(\mathbb{R}) \cup L^2(\mathbb{R})$. In section 2.3 we study the case where the initial data have different sign and we prove ill-posedness in Sobolev spaces, $H^s(\mathbb{R})$, with $s > 3/2$. Finally, in section 2.4 we show local existence and blow up in finite time for equation (2.1) when the initial data f_0 is positive and there exists a point $x_0 \in \mathbb{R}$ such that $f_0(x_0) = 0$. In order to obtain the last result we study the equation (2.7) and we show global existence when the sum of the viscosity and the minimum of the initial data is larger than zero. Ill-posedness occurs when this sum is smaller than zero.

Now we will give some comments about the notation.

We will set $H^s(\mathbb{R})$, with $s \in \mathbb{R}$ to the usual Sobolev space,

$$H^s(\mathbb{R}) = \{f \ : \ \hat{f} \text{ is a function and}$$

$$\int_{\mathbb{R}} |\hat{f}(\xi)|^2 (1+\xi^2)^s d\xi < \infty\}.$$

We denote by Λ the operator $(-\Delta)^{\frac{1}{2}}$. This operator can be defined, using the Fourier transform by

$$\widehat{(\Lambda f)}(\xi) = |\xi| \hat{f}(\xi), \tag{2.9}$$

and we will use the representation

$$\Lambda f(x) = \frac{1}{\pi} P.V \int \frac{f(x) - f(y)}{(x-y)^2} dy = H f_x(x). \tag{2.10}$$

We recall the following pointwise inequality (see [22]) for $f \in H^2(\mathbb{R})$.

$$f \Lambda f \geq \frac{1}{2} \Lambda(f^2), \tag{2.11}$$

that will be used in the proofs below.

2.2 Global existence for strictly positive initial data.

In this section we study the equation (2.1) with initial data $f_0(x) > 0$, which imply that the solution will remain strictly positive, $f(x,t) > 0$. The main result is the following:

Theorem 2.2.1 *Let $f_0 \in L^2(\mathbb{R}) \cap C^{0,\delta}(\mathbb{R})$, with $0 < \delta < 1$ and $f_0 > 0$ vanishing at infinity. Then there exists a unique global solution of equation (2.1) in $C^1((0,\infty]; Analytic)$ with $f(x,0) = f_0(x)$.*

Proof: We denote the upper half-plane by

$$M \equiv \{(x,y) \in \mathbb{R}^2 \,:\, y > 0\}$$

and the upper half-plane including the real axis by

$$\overline{M} \equiv \{(x,y) \in \mathbb{R}^2 \,:\, y \geq 0\}.$$

Let

$$P_y(x) \equiv \frac{1}{\pi} \frac{y}{y^2 + x^2} \quad \text{and} \quad R_y(x) = \frac{1}{\pi} \frac{x}{y^2 + x^2},$$

be the Poisson kernel and the conjugate Poisson kernel respectively. Then we denote the convolutions of a function, $f(x)$, with these kernels by

$$Pf(x,y) = (P_y * f)(x,y) \quad \text{and} \quad Rf(x,y) = (R_y * f)(x,y),$$

We recall that with this notation the complex function $g(x,y) = Pf(x,y) + iRf(x,y)$ is analytic on M and

$$\lim_{y \to 0^+} g(x,y) = f(x) + iHf(x). \tag{2.12}$$

Other properties of the Poisson kernel, that we will refer to below, are the following:

- If $f \in L^2$, then

$$Rf(x,y) = PHf(x,y) \quad \text{on} \quad M$$

- If $f \in L^\infty$ and vanishing at infinity, then
$$\lim_{y\to+\infty} Pf(x,y) = 0 \quad \forall x \in \mathbb{R},$$
and
$$\lim_{x\to\pm\infty} Pf(x,y) = 0 \quad \forall y \geq 0.$$
- If $f \in L^\infty$, then $Pf(x,y)$ is a bounded function on $\overline{M}$.

Consider the equation

$$\begin{aligned} \partial_t z(x,t) + z(x,t)z_x(x,t) &= 0 && (2.13)\\ z(x,0) = z_0(x) &\equiv Hf_0(x) - if_0(x), && (2.14)\end{aligned}$$

where $z(x,t) = g(x,t) - if(x,t)$ is a complex valued function and g and $-f$ its the real and imaginary parts respectively. Equation (2.13) written in components can be read

$$\begin{aligned} \partial_t f + (gf)_x &= 0 \quad f(x,0) = f_0(x) && (2.15)\\ \partial_t g + gg_x - ff_x &= 0 \quad g(x,0) = Hf_0(x). && (2.16)\end{aligned}$$

We set the inviscid complex Burgers equation on the upper half plane

$$\begin{aligned} \partial_t Z(w,t) + Z(w,t)Z_w(w,t) &= 0 \quad \text{on } \overline{M} \quad (w = x+iy) && (2.17)\\ Z(w,0) = Z_0(w) &\equiv Rf_0(x,y) - iPf_0(x,y), && (2.18)\end{aligned}$$

where $Z_0(w)$ is an analytic function over M, $\lim_{y\to0^+} Z_0(w) = Hf_0 - if_0$ and $f_0 > 0$.

We introduce the complex trajectories

$$X(w,t) = Z_0(w)t + w, \qquad (2.19)$$

which in components reads

$$\begin{aligned} X_1(x,y,t) &= Rf_0(x,y)t + x && (2.20)\\ X_2(x,y,t) &= -Pf_0(x,y)t + y. && (2.21)\end{aligned}$$

Thus, if $Z_0(w)$ is analytic in w_0 and $\frac{dX}{dw}(w_0,t) \neq 0$ we can define the function

$$Z(\alpha,t) = Z_0(X^{-1}(\alpha,t)),$$

with the property to be analytic on an open neighborhood of $X(w_0,t)$ and $Z(\alpha,0) = Z_0(\alpha)$. Consequently

$$Z(X(w_0,t),t) = Z_0(w_0).$$

Therefore,

$$\begin{aligned} \frac{dZ(X(w_0,t),t)}{dt} &= 0 = \partial_t Z(X,t) + Z_0(w_0)Z_X(X,t)\\ &= \partial_t Z(X,t) + Z(X,t)Z_X(X,t).\end{aligned}$$

Follows immediately that $Z(\alpha,t)$ is a solution of the complex inviscid Burgers equation on a neighborhood of $X(w_0,t)$.

Now we shall show that there exists a suitable analytic inverse function for the problem. First we prove the following lemma:

Lemma 2.2.2 *For all* $(X_1, X_2) \in \overline{M}$ *there exists a unique pair* $(x, y) \in M$ *such that (2.20) and (2.21) holds for all* $t > 0$. *In addition, if* $X(w) \in \overline{M}$ *then* $(1 + t\partial_x Rf_0(x, y)) > 0$.

Proof lemma 2.2.2: By fixing $X_2 \geq 0$ and $t > 0$, for all $x \in \mathbb{R}$, there exist a point $y > 0$ such that the equation (2.21) holds. This is true since $Pf_0(x, y) > 0$ is bounded and we have that $y > X_2$. Now we will prove that this value y is unique by a contradiction argument:

Let us suppose that there exist $y_1 > y_2$ such that

$$
\begin{aligned}
y_1 - X_2 &= Pf_0(x, y_1)t \\
y_2 - X_2 &= Pf_0(x, y_2)t.
\end{aligned}
$$

Dividing both expressions we have

$$
\frac{Pf_0(x, y_1)}{y_1 - X_2} = \frac{Pf_0(x, y_2)}{y_2 - X_2},
$$

which is a contradiction since the product

$$
\frac{y}{y - X_2} \cdot \frac{1}{y^2 + (x - s)^2}
$$

is a decreasing function with respect to y (for $y > X_2$) and $f_0 > 0$. We will denote by $y_{X_2}(x)$ to be the solution of the equation (2.21) with fixed $X_2 \geq 0$, $t > 0$ and $x \in \mathbb{R}$ (the time dependence will be omitted), hence

$$
y_{X_2}(x) - X_2 = tPf_0(x, y_{X_2}(x)) \tag{2.22}
$$

Differentiating implicitly the expression (2.22) with respect to x (fixed X_2) we obtain

$$
\frac{dy_{X_2}(x)}{dx} = \frac{\partial_x Pf_0(x, y_{X_2}(x))}{1 - t\partial_y Pf_0(x, y_{X_2}(x))}. \tag{2.23}
$$

Since $f_0 \in C^{0,\delta} \cap L^2$ follows that $Hf_0 \in L^\infty$ and therefore $Rf_0(x, y)$ is a bounded function over $\overline{M}$. Furthermore

$$
\lim_{x \to \pm\infty} Rf_0(x, y_{X_2}(x)) + x = \pm\infty \quad \forall X_2 \geq 0.
$$

In addition, by differentiating with respect to x the expression

$$
X_1(x, y_{X_2}(x)) = Rf_0(x, y_{X_2}(x))t + x,
$$

and using Cauchy-Riemann equations

$$
\begin{aligned}
\partial_x Pf_0(x, y) &= \partial_y Rf_0(x, y) \\
\partial_y Pf_0(x, y) &= -\partial_x Rf_0(x, y),
\end{aligned}
$$

we obtain from (2.23)

$$
\frac{dX(x, y_{X_2}(x))}{dx} = \frac{(1 + t\partial_x Rf_0(x, y_{X_2}(x)))^2 + (\partial_x Pf_0(x, y_{X_2}(x)))^2}{1 + t\partial_x Rf_0(x, y_{X_2}(x))}.
$$

In the next step we shall prove that if $X_2 \geq 0$ then $0 < \frac{dX(x,y_{X_2}(x))}{dx} < \infty$, which is equivalent to show that

$$1 + t\partial_x Rf_0(x,y) > 0 \;\; \forall (X_1, X_2) \in \overline{M}. \tag{2.24}$$

Suppose that for $t > 0$ we have $1 + t\partial_x Rf_0(x,y) = 0$. Then

$$\partial_x Rf_0(x,y) < 0 \quad \text{and} \quad t = \frac{-1}{\partial_x Rf_0(x,y)}.$$

The time employed by the trajectory $X(x+iy,t)$ to reach the real axis, $X^2 = 0$, is

$$t_r = \frac{y}{Pf_0(x,y)},$$

On the other hand we have

$$-\partial_x Rf_0(x,y) = (\partial_x R_y * f_0)(x) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{-y^2 + (x-s)^2}{(y^2 + (x-s)^2)^2} f_0(s)ds$$

$$< \frac{1}{\pi}\int_{\mathbb{R}} \frac{y^2 + (x-s)^2}{(y^2 + (x-s)^2)^2} f_0(s)ds = \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{(y^2 + (x-s)^2)} f_0(s)ds = \frac{Pf_0(x,y)}{y}.$$

We now observe that on the hypothetical points $(x,y) \in M$ where $\partial_x Rf_0(x,y) < 0$ satisfies

$$\frac{-1}{\partial_x Rf_0(x,y)} > \frac{y}{Pf_0(x,y)}. \tag{2.25}$$

Hence, we have shown that $t_r < t$ and consequently $1 + t\partial_x Rf_0(x,y) > 0 \;\forall (X_1, X_2) \in \overline{M}$. The lemma (2.2.2) is proved. ∎

By the Complex Variable Inverse Function Theorem and by lemma (2.2.2), there exist an analytic inverse function, $X^{-1}(w,t)$ and an open set $O_t \subset \mathbb{C}$, time dependence, such that

$$X^{-1}(\cdot,t) \;:\; O_t \to M,$$

with $\overline{M} \subset O_t \quad \forall t > 0$.

In fact, $Z(w,t) = Z_0(X^{-1}(w,t))$ is an analytic function that satisfies the complex inviscid Burgers equation over $\overline{M}$ and $Z(w,0) = Z_0(w)$. Furthermore $Z(w,t)$ vanishes at infinity $\forall t$ and the restriction $z(x,t) = Z(x,t)$ satisfies (2.13).

Note that the real part of $Z(w,t)$, $\Re Z(w,t)$, is a harmonic function, vanishing at infinity and with an analytic restriction to the real axis, $\Re z(x,t)$. But $P\Re z(x,y,t)$ is a harmonic function with restriction to the real axis equal to $\Re z(x,t)$ and vanishes at infinity. Then $\Re Z(w,t) = P\Re z$ and by unicity of harmonic conjugate we can write

$$z(x,t) = H\Re z(x,t) - i\Re z(x,t).$$

The proof of the existence follows from substituting the previous expression in equations (2.15) and (2.16).

In order to prove uniqueness we consider the weak formulation of equation (2.1)

$$\int_0^T \int_{\mathbb{R}} (f\Phi_t + fHf\Phi_x)\, dxdt = \int_{\mathbb{R}} f_0(x)\Phi(x,0)dx, \tag{2.26}$$

for all $\Phi \in C_c^\infty([0,T);H^s(\mathbb{R}))$, with $s > \frac{3}{2}$.

Let us assume that there exist a solution, $f(x,t)$, of the equation (2.26), such that

$$f \in C\left([0,T); L^2(\mathbb{R}) \cap C^{0,\delta}(\mathbb{R})\right).$$

Then we have that

$$\int_0^T \int_{\mathbb{R}} (fH\Phi_t + fHf(H\Phi_x))\, dxdt = \int_{\mathbb{R}} f_0(x)H\Phi(x,0)dx$$

and therefore

$$\int_0^T \int_{\mathbb{R}} (Hf\Phi_t + H(fHf)\Phi_x)\, dxdt = \int_{\mathbb{R}} Hf_0(x)\Phi(x,0)dx. \tag{2.27}$$

From equations (2.26) and (2.27), the complex function $z(x,t) = Hf(x,t) - if(x,t)$ satisfies

$$\int_0^T \int_{\mathbb{R}} \left(z\Phi_t + \frac{1}{2}z^2\Phi_x\right) dxdt = \int_{\mathbb{R}} z_0(x)\Phi(x,0)dx.$$

Next, we will use Poisson kernel in the following way: we can write

$$\int_0^T \int_{\mathbb{R}} \left(zP_y * \Phi_t + \frac{1}{2}z^2 P_y * \Phi_x\right) dxdt = \int_{\mathbb{R}} z_0(x)P_y * \Phi(x,0)dx,$$

thus

$$\int_0^T \int_{\mathbb{R}} \left(P_y * z\, \Phi_t + \frac{1}{2}P_y * \left(z^2\right)\Phi_x\right) dxdt = \int_{\mathbb{R}} P_y * z_0(x)\Phi(x,0)dx.$$

In addition, we can check that

$$P_y * \left(z^2\right) = \left(P_y * z\right)^2. \tag{2.28}$$

Indeed, both $P_y * \left(z^2\right)$ and $\left(P_y * z\right)^2$ are analytic functions for $y > 0$ with the same restriction to the real axis and with the same decay at infinity. Then they must be equal. Using this identity yields

$$\int_0^T \int_{\mathbb{R}} \left(P_y * z\, \Phi_t + \frac{1}{2}\left(P_y * z\right)^2\Phi_x\right) dxdt = \int_{\mathbb{R}} P_y * z_0(x)\Phi(x,0)dx.$$

The function $Z(x,y,t) \equiv (P_y * z)(x,y,t)$ is an analytic function for $y > 0$, then we can integrate by parts to obtain

$$\int_0^T \int_{\mathbb{R}} (Z\Phi_t - ZZ_x\Phi)\, dxdt = \int_{\mathbb{R}} Z_0(x,y)\Phi(x,0)dx. \tag{2.29}$$

Fixed $\tau \in (0,T)$, let $\{\Phi^\tau_\varepsilon\}_\varepsilon$ be a sequence of smooth test functions, such that $\Phi^\tau_\varepsilon(x,t) = \eta(\frac{t-\tau}{\varepsilon})\phi(x)$, where

$$\eta(t) = \begin{cases} 1 & t < -1 \\ 0 & t > 1 \end{cases}$$

and $\eta(t)$ is decreasing in the interval $(-1,1)$. For $\varepsilon > 0$ small enough, we introduce Φ^τ_ε in the equation (2.29). Taking the limit $\varepsilon \to 0$ yields

$$\int_{\mathbb{R}} Z(\tau)\phi(x)dx = \int_{\mathbb{R}} \left(Z_0(x,y) - \int_0^\tau Z(\tau')Z_x(\tau')d\tau' \right) \phi(x)dx \tag{2.30}$$

Finally, from equation (2.30) and for $y > 0$ we deduce the following equation for the function $Z(w,t)$

$$Z_t + ZZ_w = 0. \tag{2.31}$$

Uniqueness folows from the equation (2.31).

2.3 Ill-posedness for an initial data with different sign.

In this section we analyze the existence of solutions of the equation (2.1) for an initial data, which has positive and negative values. The theorem that we shall prove is the following:

Theorem 2.3.1 *Let $f_0 \in H^s$, with $s > \frac{3}{2}$. Then, if $f_0(x)$ is not C^∞ in a point x_0 where $f_0(x_0) < 0$, there is no solution of equation (2.1), satisfying $f(x,0) = f_0(x)$, in the class $f \in C((0,T), H^s(\mathbb{R})) \cap C^1((0,T), H^{s-1}(\mathbb{R}))$, with $s > \frac{3}{2}$ and $T > 0$. In addition, $f_0 \in C^\infty$ is not sufficient to obtain existence.*

Proof: We will proceed by a contradiction argument.

Let us suppose that there exist a solution of equation (2.1) in the class $C((0,T), H^s(\mathbb{R})) \cap C^1((0,T), H^{s-1}(\mathbb{R}))$ with $f(x,0) = f_0(x)$.

Taking the Hilbert transform on equation (2.1) yields

$$\partial_t Hf + HfHf_x - ff_x = 0.$$

Now we define the complex valued function $z(x,t) = Hf(x,t) - if(x,t)$, that satisfies

$$\partial_t z + zz_x = 0.$$

We set the complex function $Z(x,y,t)$ by the expression (we omit the time dependence):

$$Z(x,y,t) = Rf(x,y) - iPf(x,y) = P(Hf - if)(x,y),$$

so that, $Z(x,y,t) = Z(w,t)$ $(w = x+iy)$ is an analytic function on M. Now we shall prove that this function satisfies the complex Burgers equation on M. In order to do that, we take the time derivative of Z

$$\partial_t Z = \partial_t Pz = P(\partial_t z) = -P(zz_x).$$

From the identity (2.28) we have that $Pz\partial_x Pz = P(zz_x)$. Therefore the analytic function on M, $Z(w,t)$, satisfies the inviscid complex Burgers equation,

$$\begin{aligned} \partial_t Z(w,t) + Z(w,t)Z_w(w,t) &= 0 \quad \text{over } M \\ Z(w,0) &= Z_0(w) \end{aligned}$$

Let us define the complex trajectories

$$\begin{aligned} \frac{dX(w,t)}{dt} &= Z(X(w,t),t) \\ X(w,0) &= w = x+iy, \end{aligned}$$

where we choose w so that $y>0$ and $Pf_0(x,y)<0$. For sufficiently small t, by Picard's theorem, these trajectories exist and $X(w,t)\in M$. Therefore,

$$\frac{dZ(X(w,t),t)}{dt} = \partial_t Z(X,t) + Z_X(X,t)Z(X,t) = 0,$$

and we obtain that $X(w,t) = Z_0(w)t + w$.

Now we take a sequence $w^\varepsilon = x + iy^\varepsilon$ for each x such that $f_0(x)<0$, with $y^\varepsilon>0$, $Pf_0(x,y^\varepsilon)<0$ and $\lim_{\varepsilon\to 0} y^\varepsilon = 0$. Since $X(x,t) = Z_0(x)t + x \in M$, $\forall t>0$, then

$$Z(Z_0(x)t+x,t) = \lim_{\varepsilon\to 0} Z(X(w^\varepsilon,t)) = \lim_{\varepsilon\to 0} Z_0(w^\varepsilon) = Hf_0(x) - if_0(x).$$

Taking one derivative respect to x in both sides of the previous equality we obtain

$$\frac{dZ(X(x,t),t)}{dX} = \frac{\frac{dZ_0(x)}{dx}}{1+t\frac{dZ_0(x)}{dx}}.$$

Taking two derivatives we have the equation

$$\frac{d^2Z(X(x,t),t)}{(dX)^2} = \frac{\frac{d^2Z_0(x)}{dx^2}}{(1+t\frac{dZ_0(x)}{dx})^3}.$$

And for n-th derivatives,

$$\begin{aligned} \frac{d^nZ(X(x,t),t)}{(dX)^n} &= \frac{\frac{d^nZ_0(x)}{dx^n}}{(1+t\frac{dZ_0(x)}{dx})^{n+1}} + \text{lower terms in derivatives} \\ &= \frac{\frac{d^nHf_0(x)}{dx^n} - i\frac{d^nf_0(x)}{dx^n}}{(1+t\frac{dZ_0(x)}{dx})^{n+1}} + \text{lower terms in derivatives} \end{aligned}$$

Indeed, if at x_0 such that $f_0(x_0)<0$ the n-th derivative of f_0 is not continuous we get a contradiction. In addition, if $f_0(x_0)<0$ and $f_0^{(n)}(x_0)=0$ $\forall n$ but f_0 is not constant we have that

$$\frac{dZ(X(x,t),t)}{dX} = \frac{dZ(X(x,t),t)}{dX^1} = \frac{d\Re Z(X^1,X^2,t)}{dX^1} + i\frac{d\Im Z(X^1,X^2,t)}{dX^1}$$

$$= \frac{\frac{dHf_0(x)}{dx} - i\frac{df_0(x)}{dx}}{(1 + t\frac{dHf_0}{dx}) - i\frac{df_0}{dx}}.$$

Therefore

$$\frac{d\Im Z(X^1(x_0), X^2(x_0), t)}{dX^1} = 0.$$

Continuing this process we obtain that all derivatives satisfy

$$\frac{d^n \Im Z(X^1(x_0), X^2(x_0))}{(dX^1)^n} = 0.$$

But $\Im Z(x, y, t)$ is analytic on $(x, y) = (X^1(x_0), X^2(x_0))$, then $\Im Z(x, y, t)$ is constant over the line $y = X^2(x_0)$ and this is a contradiction. Similar result can be obtained if $f_0(x_0) < 0$ and $\frac{d^n Hf(x_0)}{dx^n} = 0 \; \forall n$ but the initial data f_0 is not a constant.

A stronger theorem of ill-posedness can be proven in the case of a strictly negative initial data by using theorem 2.2.1. Indeed we have the following result:

Theorem 2.3.2 *Let f_0 be a non analytic initial data such that $f_0 < 0$. Then, there does not exist a solution of the equation (2.1) in the class $C([0, T]; H^s(\mathbb{R}))$, with $s > \frac{3}{2}$, for any $T > 0$.*

Proof: Let us assume that there exist a solution $f(x, t) \in C([0, T]; H^s(\mathbb{R}))$ for the equation (2.1) with $f_0(x) < 0$ and for some $T > 0$. Then we can check than the solution is strictly negative along the evolution i.e, $f(x, t) < 0$. Indeed, we can define the trajectories

$$\begin{aligned} \frac{dX(x,t)}{dt} &= Hf(X(x,t), t), \\ X(x,0) &= x. \end{aligned} \tag{2.32}$$

Then we have that

$$\frac{df(X(x,t), t)}{dt} = -f(X(x,t), t)Hf_x(X(x,t), t).$$

Integrating in time this expression we obtain the following expression

$$f(X(x,t), t) = f_0(x) \exp\left(\int_0^t Hf_x(X(x,\tau), \tau)d\tau\right). \tag{2.33}$$

In addition, taking a derivative with respect to x in the equation (2.32) and integrating in time yields

$$X_x(x,t) = \exp\left(-\int_0^t Hf_x(X(x,\tau), \tau)d\tau\right).$$

Thus we can invert the trajectory $X(x, t)$ and therefore, since the expression (2.33), we can conclude that $f(x, t) < 0$. But, since changing the time direction in the equation (2.1)is the same as changing the sign of the initial data, we could prove that the initial data is analytic using the theorem 2.2.1. This is a contradiction.

2.4 Local existence and singularities for positive initial data.

The aim of this section is to prove local existence for equation (2.1) with positive initial data. Furthermore, we shall prove blow up in finite time if there exist $x_0 \in \mathbb{R}$ such that the initial data satisfies $f_0(x_0) = 0$.

The argument of the proof requires the introduction of a viscous term. The equation that we shall study is the following

$$\begin{aligned} f_t + (Hff)_x &= -\nu H f_x \qquad (2.34)\\ f(x,0) &= f_0(x), \qquad (2.35) \end{aligned}$$

where $\nu > 0$.

We will divide this study in two subsections. First we analyze the case $f_0 + \nu > 0$ and we will show global existence. In the second part we study the case $f_0 + \nu \geq 0$ and we will show local existence and blow up in finite time.

2.4.1 Global existence for $f_0 + \nu > 0$.

Here we shall prove the following result:

Theorem 2.4.1 *Let be $f_0 \in L^2(\mathbb{R}) \cap C^{0,\delta}(\mathbb{R})$, with $0 < \delta < 1$ and $f_0 + \nu > 0$ vanishing at the infinity. Then there exits a unique global solution of equation (2.34) in $C^1((0,\infty]; Analytic)$ with $f(x,0) = f_0(x)$.*

Proof: This proof is essentially based on the proof of the (2.2.1) which we sketch below. The complex transport equation that we have to consider is the following:

$$\partial_t Z(w,t) + (Z(w,t) - i\nu) Z_w(w,t) = 0$$

$$Z(w,0) = Z_0(w) = Rf_0(x,y) - iPf_0(x,y).$$

In order to obtain global existence of this equation over $\overline{M}$ the only modification respect (2.2.1) is the inequality (2.25). In this case we find that

$$-\partial_x Rf_0(x,y) = \frac{1}{\pi}\int \frac{-y + (x-s)^2}{(y^2 + (x-s)^2)^2} f_0(s)ds$$

$$= \frac{1}{\pi}\int \frac{-y^2 + (x-s)^2}{(y^2 + (x-s)^2)^2}(f_0(s) + \nu)ds < \frac{P(f_0+\nu)(x,y)}{y},$$

where we have used

$$\int \frac{-y^2 + s^2}{(y^2 + s^2)^2} ds = 0.$$

Remark 2.4.2 *Ill-posedness occurs in equation (2.34) if the addition of the minimum of the initial data f_0 plus the viscosity is smaller than zero. In fact, we have the following theorem:*

Theorem 2.4.3 *Let $f_0 \in H^s$, with $s > \frac{3}{2}$. Then, if $f_0(x)$ is not C^∞ in a point x_0 where $f_0(x_0) < -\nu$, there is no solution of equation (2.1), satisfying $f(x,0) = f_0(x)$, in the class $f \in C((0,T), H^s(\mathbb{R})) \cap C^1((0,T), H^{s-1}(\mathbb{R}))$, with $s > \frac{3}{2}$ and $T > 0$. In addition, f_0 analytic in every point where $f_0 < -\nu$ is not sufficient to obtain existence.*

Proof: The proof follows the steps of the theorem (2.3.1).

2.4.2 Local existence in the limit case.

In this subsection we analyze the equation (2.34) with initial data $f_0 \geq -\nu$. We will use energy estimates and the techniques used in the article [15] for the control of the L^∞-norm of the solutions. The theorem that we shall prove is the following:

Theorem 2.4.4 *Let $f_0 \in H^2$ and*

$$m_0 \equiv min_x f_0(x) \leq 0,$$

such that $m_0 + \nu = 0$. Then there exists a time $T > 0$ such that the equation (2.34) has a unique solution in $C([0,T]; H^2(\mathbb{R})) \cap C^1([0,T]; H^1(\mathbb{R}))$ with $f(x,0) = f_0(x)$.

Remark 2.4.5 *In the case $\nu = 0$ this theorem asserts local existence of solutions of equation (2.1) in the class $C([0,T]; H^2(\mathbb{R})) \cap C^1([0,T]; H^1(\mathbb{R}))$ when the initial data $f_0(x) \geq 0$.*

Proof: By theorem (2.4.1) we consider global solutions of the equation

$$f_t + (Hff)_x = -\varepsilon H f_x \tag{2.36}$$

$$f(x,0) = f_0(x) \in H^2 \quad \text{and} \quad \varepsilon + m_0 > 0, \tag{2.37}$$

First we will compute two estimates of the L^∞-norm of f_x and Hf_x which are uniform with respect to $\varepsilon > -m_0$

Lemma 2.4.6 *Let $f_0 \in H^2$, with $m_0 \equiv \min_{x\in\mathbb{R}} f_0(x) \leq 0$ and $m_0 + \varepsilon > 0$. Let f be the solution of equation (2.36) given by the theorem (2.4.1). Then, if we define*

$$m(t) = \min_{x\in\mathbb{R}} f(x,t),$$

we have that

$$m(t) + \varepsilon > 0 \quad \forall t \geq 0.$$

Proof of lemma 2.4.6: From theorem (2.4.1) we know that $f \in C^1([0,\infty) \times \mathbb{R})$, in particular the function $m(t)$ is differentiable almost every t. There always exist a point $x_m \in \mathbb{R}$ (which depends on t) such that $m(t) = f(x_m(t), t)$. Using the same argument as [23] we obtain

$$m'(t) = f_t(x_m(t), t) \quad \text{at almost every } t.$$

Therefore

$$m'(t) = -Hf(x_m(t),t) f_x(x_m(t),t) - Hf_x(x_m,t)(f(x_m(t),t) + \varepsilon)$$

$$= -Hf_x(x_m(t),t)(m(t)+\varepsilon).$$

an by integrating

$$(m(t)+\varepsilon) = (m_0+\varepsilon)\exp(-\int_0^t Hf_x(x_m(\tau),\tau)d\tau).$$

Since

$$Hf_x(x) = \frac{1}{\pi}P.V.\int \frac{f(x)-f(y)}{(x-y)^2}dy,$$

we have that $-Hf_x(x_m(t),t) \geq 0$. Therefore

$$m(t)+\varepsilon \geq m_0+\varepsilon > 0.$$

■

Lemma 2.4.7 *Let $f_0 \in H^2$, with $m_0 \equiv \min_{x\in\mathbb{R}} f_0(x) \leq 0$ and $m_0+\varepsilon > 0$. Let f be the solution of equation (2.36) given by theorem (2.4.1). Then, if we define*

$$\begin{aligned} m(t) &= \min_{x\in\mathbb{R}} f_x(x,t) = f_x(x_m(t),t) \\ M(t) &= \max_{x\in\mathbb{R}} f_x(x,t) = f_x(x_M(t),t) \\ j(t) &= \min_{x\in\mathbb{R}} Hf_x(x,t) = Hf_x(x_j(t),t) \\ J(t) &= \max_{x\in\mathbb{R}} Hf_x(x,t) = f_x(x_J(t),t) \end{aligned}$$

we have that

$$\begin{aligned} m(t) &\geq \frac{m(0)}{(1+j(0)t)^2} \\ M(t) &\leq \frac{M(0)}{(1+j(0)t)^2} \\ j(t) &\geq \frac{j(0)}{1+j(0)t} \\ J(t) &\leq J(0) - \frac{3}{j(0)}\frac{M^2(0)}{(1+j(0)t)^3}. \end{aligned}$$

Therefore

$$\sup_{t\in[0,T]}(||f_x(t)||_{L^\infty} + ||Hf_x(t)||_{L^\infty}) < \infty$$

$$\textit{if} \quad T < T_e \equiv -\frac{1}{j(0)} \quad \forall\varepsilon > -m_0.$$

Proof of lemma 2.4.7: We know that $f_x \in C^1([0,\infty)\times\mathbb{R})$ and $Hf_x \in C^1([0,\infty)\times\mathbb{R})$. Therefore

$$\begin{aligned} m'(t) = \partial_t f_x(x_m(t),t) \quad M'(t) = \partial_t f_x(x_M(t),t) \\ j'(t) = \partial_t Hf_x(x_j(t),t) \quad J'(t) = \partial_t Hf_x(x_J(t),t). \end{aligned}$$

and

$$j'(t) = -j^2(t) + (f_x)^2(x_j(t), t) + (f(x_j(t), t) + \varepsilon) f_{xx}(x_j(t), t).$$

Using the following representation

$$f_{xx}(x) = \frac{1}{\pi} P.V \int \frac{Hf_x(y) - Hf_x(x)}{(x-y)^2} dy,$$

we have that $f_{xx}(x_j(t), t) \geq 0$. Since $j(t)$ is negative this yields

$$j(t) \geq \frac{j(0)}{1 + tj(0)}.$$

Now we shall study the evolution of $m(t)$.

$$m'(t) = -m(t)Hf_x(x_m(t), t) - (f(x_m(t), t) + \varepsilon)Hf_{xx}(x_m(t), t).$$

Since

$$Hf_{xx}(x) = \frac{1}{\pi} \int \frac{f_x(x) - f_x(y)}{(x-y)^2} dy,$$

we have that $Hf_{xx}(x_m(t), t) \leq 0$. We know that $m(t) < 0$ then

$$m'(t) \geq -2m(t)Hf_x(x_m(t), t) \geq -2m(t)j(t),$$

and the following inequality holds

$$m(t) \geq \frac{m(0)}{(1 + j(0)t)^2}.$$

Operating in a similar way we obtain that

$$M(t) \leq \frac{M(0)}{(1 + j(0)t)^2}.$$

Finally we have that

$$J'(t) = -J^2(t) + (f_x)^2(x_J(t), t) + (f(x_J(t), t) + \varepsilon) f_{xx}(x_J(t), t) \leq M^2(t).$$

Therefore

$$J(t) \leq J(0) - \frac{3}{j(0)} \frac{M^2(0)}{(1 + j(0)t)^3}.$$

■

Next, we shall check that the L^2-norm of f is bounded. Multiplying equation (2.34) by f and integrating over $\mathbb{R}$ we have

$$\frac{1}{2}\frac{d||f(t)||^2_{L^2}}{dt} + \int_{\mathbb{R}} (fHf)_x f + \varepsilon \int_{\mathbb{R}} Hf_x f dx$$

$$= \frac{1}{2}\frac{d||f(t)||^2_{L^2}}{dt} - \int_{\mathbb{R}} fHf f_x dx + \varepsilon \int_{\mathbb{R}} Hf_x f dx$$

$$= \frac{1}{2}\frac{d||f(t)||^2_{L^2}}{dt} - \frac{1}{2}\int_{\mathbb{R}} Hf(f^2)_x dx + \varepsilon \int_{\mathbb{R}} Hf_x f dx$$

$$= \frac{1}{2}\frac{d||f(t)||^2_{L^2}}{dt} + \frac{1}{2}\int_{\mathbb{R}} Hf_x f^2 dx + \varepsilon \int_{\mathbb{R}} Hf_x f dx = 0.$$

By the expression (2.10) we can estimate the last two terms of the equality

$$\int_{\mathbb{R}} Hf_x(x) f^2(x) dx = \frac{1}{\pi}\int_{\mathbb{R}} f^2(x) P.V \int \frac{f(x)-f(y)}{(x-y)^2} dy dx$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} P.V. \int \frac{f^2(x)(f(x)-f(y)) + f^2(y)(f(y)-f(x))}{(x-y)^2} dy dx$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} P.V. \int \frac{(f(x)+f(y))(f(x)-f(y))^2}{(x-y)^2} dy dx,$$

and

$$\varepsilon \int_{\mathbb{R}} Hf_x f dx = \frac{\varepsilon}{\pi}\int_{\mathbb{R}} f(x) P.V \int \frac{f(x)-f(y)}{(x-y)^2} dy dx$$

$$= \frac{\varepsilon}{2\pi}\int_{\mathbb{R}} P.V \int \frac{f(x)(f(x)-f(y)) + f(y)(f(y)-f(x))}{(x-y)^2} dy dx$$

$$= \frac{\varepsilon}{\pi}\int_{\mathbb{R}} P.V \int \frac{(f(x)-f(y))^2}{(x-y)^2} dy dx$$

Therefore

$$\frac{1}{2}\int_{\mathbb{R}} Hf_x f^2 dx + \varepsilon \int_{\mathbb{R}} Hf_x f dx$$

$$= \frac{1}{4\pi}\int_{\mathbb{R}} P.V \int \frac{(f(x)+f(y)+2\varepsilon)(f(x)-f(y))^2}{(x-y)^2} dy dx \geq 0,$$

and we can conclude

$$||f(t)||_{L^2} \leq ||f_0||_{L^2}.$$

In addition we have an a priori estimate of L^2-norm of f_{xx}. Taking two derivatives to the equation (2.34), multiplying by f_{xx} and integrating in x we obtain

$$\frac{1}{2}||f_{xx}(t)||_{L^2} = -\int_{\mathbb{R}} (fHf)_{xxx} f_{xx} dx - \varepsilon \int_{\mathbb{R}} Hf_{xxx} f_{xx} dx = 0. \tag{2.38}$$

All the terms of the right side of (2.38), except

$$-\int_{\mathbb{R}} (f+\varepsilon) Hf_{xxx} f_{xx} dx, \tag{2.39}$$

can be controlled in a simple way by

$$C(||f_x(t)||_{L^\infty} + ||Hf_x(t)||_{L^\infty})||f_{xx}||^2_{L^2}.$$

To bound (2.39) we will use the inequality (2.11) as follows

$$-\int_{\mathbb{R}}(f+\varepsilon)Hf_{xxx}f_{xx}dx = -\int_{\mathbb{R}}(f+\varepsilon)\Lambda f_{xx}f_{xx}dx$$

$$\leq \frac{-1}{2}\int_{\mathbb{R}}(f+\varepsilon)\Lambda((f_{xx})^2)dx \leq \frac{1}{2}||Hf_x||_{L^\infty}||f_{xx}||^2_{L^2}.$$

Therefore

$$\frac{d||f_{xx}||^2_{L^2}}{dt} \leq C(||f_x(t)||_{L^\infty} + ||Hf_x(t)||_{L^\infty})||f_{xx}||^2_{L^2}.$$

Finally, integrating in time, we get the estimate

$$||f_{xx}(t)||_{L^2} \leq ||f_{0\,xx}||_{L^2}\exp(C\sup_{t\in[0,T]}(||f_x(t)||_{L^\infty} + ||Hf_x(t)||_{L^\infty})T)$$

Indeed we have that the H^2-norm of f is bounded over $[0,T]$ with $T < T_e$ and this estimate is uniform in ε. The rest of the proof is as follows. We take the approximating problems

$$\begin{aligned} f^\varepsilon_t + (Hf^\varepsilon f^\varepsilon)_x &= -\varepsilon Hf^\varepsilon_x \\ f^\varepsilon(x,0) &= f_0(x)\in H^2 \quad \text{and} \quad \varepsilon + m_0 > 0, \end{aligned}$$

We shall prove that $\{f^\varepsilon\}_\varepsilon$ is a Cauchy sequence in L^2. Let f^{ε_1} and f^{ε_2} be the solutions for the equations

$$\begin{aligned} f^{\varepsilon_1}_t + (Hf^{\varepsilon_1}f^{\varepsilon_1})_x &= -\varepsilon_1 Hf^{\varepsilon_1}_x \\ f^{\varepsilon_2}_t + (Hf^{\varepsilon_2}f^{\varepsilon_2})_x &= -\varepsilon_2 Hf^{\varepsilon_2}_x \\ f^{\varepsilon_1}(x,0) = f^{\varepsilon_1}(x,0) &= f_0(x)\in H^2 \quad \text{and} \quad \varepsilon_{1,\,2} + m_0 > 0. \end{aligned}$$

Then we have

$$\frac{1}{2}\frac{d||f^{\varepsilon_1}-f^{\varepsilon_2}||^2_{L^2}}{dt} = \frac{3}{2}\int_{\mathbb{R}}Hf^{\varepsilon_1}_x(x)(f^{\varepsilon_1}(x)-f^{\varepsilon_2}(x))^2dx$$

$$-\int_{\mathbb{R}}f^{\varepsilon_2}_xH(f^{\varepsilon_1}(x)-f^{\varepsilon_2}(x))(f^{\varepsilon_1}(x)-f^{\varepsilon_2}(x)) + (\varepsilon_1-\varepsilon_2)\int_{\mathbb{R}}Hf^{\varepsilon_1}_x(x)(f^{\varepsilon_1}(x)-f^{\varepsilon_2}(x))dx$$

$$+\int_{\mathbb{R}}(f^{\varepsilon_2}+\varepsilon_2)H(f^{\varepsilon_1}(x)-f^{\varepsilon_2}(x))_x(f^{\varepsilon_1}(x)-f^{\varepsilon_2}(x))dx$$

$$\leq C||f^{\varepsilon_1}-f^{\varepsilon_2}||^2_{L^2} + C|\varepsilon_1-\varepsilon_2|.$$

Therefore, by using the Grönwall inequality we obtain

$$||f^{\varepsilon_1}-f^{\varepsilon_2}||^2_{L^2} \leq C|\varepsilon_1-\varepsilon_2|.$$

Since $\{f^\varepsilon\}_\varepsilon$ is a Cauchy sequence in L^2 and is bounded in H^2, by interpolation, we have that $\{f^\varepsilon\}_\varepsilon$ is a Cauchy sequence in H^s, with $s < 2$.

By taking the limit $\varepsilon \to -m_0^+$ we achieve the conclusion of theorem 2.4.4.

Theorem 2.4.8 *Let $f_0 \in H^3(\mathbb{R})$ and $f_0(x) \geq -\nu$ such that there exist a point x_0 where $f_0(x_0) = -\nu$. Then the solution $f(x,t)$ of the equation (2.34), develops a singularity in finite time in $H^2(\mathbb{R})$.*

Remark 2.4.9 *Theorem (2.4.8) assert blow up in finite time for equation (2.1) when the initial data is positive and there exist a point $x_0 \in \mathbb{R}$ such that $f_0(x_0) = 0$.*

Proof: Adapting the proof of theorem (2.4.4) we can show local existence of a solution,$f \in C([0,T); H^3(\mathbb{R}))$, for the equation (2.34) with a initial data satisfying the requirements of theorem (2.4.8). In addition, we can extend the solution as long as the norm $||Hf_x||_{L^\infty(\mathbb{R})}(t)$ is bounded. Now, we proceed by a contradiction argument. Let us suppose, that $f \in C([0,T); H^3(\mathbb{R}))$, is a solutions of the equation (2.34) for all $T > 0$. We set the trajectory $X(x_0,t)$ by

$$\begin{aligned} \frac{dX(x_0,t)}{dt} &= Hf(X(x_0,t),t) \\ X(x_0,0) &= x_0. \end{aligned}$$

If we evaluate the solution over that trajectory we obtain

$$\frac{df(X(x_0,t),t)}{dt} = \partial_t f(X,t) + \frac{dX}{dt} f_x(X,t)$$

$$= -(f(X(x_0,t),t) + \nu) Hf_x(X(x_0,t),t).$$

Therefore,

$$f(X(x_0,t),t) + \nu = (f(X(x_0,0),0) + \nu) \exp(-\int_0^t Hf_x(X(x_0,\tau),\tau) d\tau) = 0.$$

Evaluating the Hilbert transform of f over that trajectory we obtain,

$$\frac{dHf(X(x_0,t),t)}{dt} = \partial_t Hf(X,t) + HfHf_x(X,t)$$

$$= (f(X(x_0,t),t) + \nu) f_x(X(x_0,t),t) = 0.$$

So that

$$Hf(X(x_0,t),t) = Hf_0(x_0),$$

and

$$X(x_0,t) = Hf_0(x_0)t + x_0.$$

If we evaluate the first derivative of the solution over that trajectory we get

$$\frac{df_x(X(x_0,t),t)}{dt} = \partial_t f_x(X,t) + Hf(X,t) f_{xx}(X,t)$$

$$= -2f_x(X(x_0,t),t) Hf_x(X(x_0,t),t)$$

Since $f_{0x}(x_0) = 0$, by the characteristics of f_0, yields

$$f_x(X(x_0,t),t) = f_x(X(x_0,0),0)\exp(-2\int_0^t Hf_x(X(x_0,\tau),\tau)d\tau) = 0.$$

Finally we evaluate Hf_x over the trajectory and we obtain

$$\begin{aligned}\frac{dHf_x(X(x_0,t),t)}{dt} &= \partial_t Hf_x(X,t) + Hf(X,t)Hf_{xx}(X,t) \\ &= -(Hf_x(X(x_0,t),t))^2,\end{aligned}$$

Which implies

$$Hf_x(X(x_0,t),t) = \frac{Hf_{0\,x}(x_0)}{1 + tHf_{0\,x}(x_0)}.$$

Moreover, we can write

$$Hf_{0\,x}(x_0) = \frac{1}{\pi}P.V\int \frac{f_0(x_0) - f_0(y)}{(x_0 - y)^2}dy = \frac{1}{\pi}P.V\int \frac{m_0 - f_0(y)}{(x_0 - y)^2}dy < 0,$$

if $f_0 \neq 0$. Then we have a contradiction.

Chapter 3

A Naive Parametrization for the Vortex-Sheet Problem.

3.1 Introduction

This chapter is devoted to the vortex sheet problem. In the section 3.2 we shall obtain the classic Birkhoff-Rott system for the motion of the vortex sheet

$$z_t(\alpha,t) = BR(z,\varpi)(\alpha,t) + c(\alpha,t)\partial_\alpha z(\alpha,t) \tag{3.1}$$

$$\varpi_t = \partial_\alpha(c\,\varpi), \tag{3.2}$$

where $c(\alpha,t)$ is a free quantity and the Birkhoff-Rott integral is given by the following expression

$$BR(z,\varpi)(\alpha,t) = \frac{1}{2\pi}PV\int \frac{(z(\alpha,t)-z(\beta,t))^\perp}{|z(\alpha,t)-z(\beta,t)|^2}\varpi(\beta,t)d\beta, \tag{3.3}$$

from the weak formulation of the Euler equation.

In the case of an analytic initial data, a local existence result for system 3.1 and 3.2 is given by Sulem, Sulem, Bardos and Frisch in [66] in the case where the curve is represented by a graph. The first result of ill-posedness in Sobolev spaces for amplitude with a distinguished sign is due to Ebin [41] in a bounded domain. In the same year Duchon and Robert [40] proved global-existence for a peculiar initial data. They consider a particular $c(\alpha,t)$ which gives $z_{1t}(\alpha,t) = 0$ and therefore if one parametrizes initially $z_0(\alpha) = (\alpha, y_0(\alpha))$ the free boundary is given in terms of a function and the equations (3.1) become

$$y_t(\alpha,t) = \frac{1}{2\pi}PV\int \frac{(\alpha-\beta)+(y(\alpha,t)-y(\beta,t))\partial_\alpha y(\alpha,t)}{(\alpha-\beta)^2+(y(\alpha,t)-y(\beta,t))^2}\varpi(\beta,t)d\beta$$

and $c(\alpha,t)$ in equation (1.6) is given by

$$c(\alpha,t) = \frac{1}{2\pi}PV\int \frac{(y(\alpha,t)-y(\beta,t))}{(\alpha-\beta)^2+(y(\alpha,t)-y(\beta,t))^2}\varpi(\beta,t)d\beta.$$

A similar approach is done by Caflish and Orellana [6] to show also global-existence for particular initial data and moreover they give an argument to prove ill-posedness in H^s for

$s > 3/2$. They choose $c(\alpha, t) = 0$ which implies $\varpi(\alpha, t) = \varpi_0(\alpha)$. If $\varpi_0(\alpha)$ has a distinguish sign, the following change of variable is legitimate

$$d\sigma = \varpi_0(\beta) d\beta$$

and equations (3.1) can be written as

$$z_t(\alpha, t) = \frac{1}{2\pi} PV \int \frac{(z(\alpha, t) - z(\beta, t))^{\perp}}{|z(\alpha, t) - z(\beta, t)|^2} d\beta, \tag{3.4}$$

which is the Birkhoff-Rott equation. By taking $z(\alpha, t) = (\alpha + \varepsilon_1(\alpha, t), \varepsilon_2(\alpha, t))$ (or $\varpi(\alpha, t) = 1 + \varepsilon_1(\alpha, t)$ and $y(\alpha, t) = \varepsilon_2(\alpha, t)$ in the parametrization of Duchon and Robert) and linearizing in (3.4) one can obtain

$$\partial_t \varepsilon_1 = -\frac{1}{2}\Lambda(\varepsilon_2), \qquad \partial_t \varepsilon_2 = -\frac{1}{2}\Lambda(\varepsilon_1),$$

where Λ is the operator $\Lambda = (-\Delta)^{\frac{1}{2}}$. Therefore

$$\widehat{\varepsilon}_1(\xi, t) = \frac{\widehat{\varepsilon}_1(\xi, 0) + \widehat{\varepsilon}_2(\xi, 0)}{2} e^{-\pi|\xi|t} + \frac{\widehat{\varepsilon}_1(\xi, 0) - \widehat{\varepsilon}_2(\xi, 0)}{2} e^{\pi|\xi|t},$$

$$\widehat{\varepsilon}_2(\xi, t) = \frac{\widehat{\varepsilon}_1(\xi, 0) + \widehat{\varepsilon}_2(\xi, 0)}{2} e^{-\pi|\xi|t} - \frac{\widehat{\varepsilon}_1(\xi, 0) - \widehat{\varepsilon}_2(\xi, 0)}{2} e^{\pi|\xi|t}.$$

Since the initial data $\varepsilon_1(\xi, 0) = \varepsilon_2(\xi, 0)$ only oscillate the dissipative waves, global existence follows even for non-regular initial data. Applying Fourier techniques to the nonlinear case

$$\partial_t \varepsilon_1 = -\frac{1}{2}\Lambda(\varepsilon_2) + T(\varepsilon_1, \varepsilon_2), \qquad \partial_t \varepsilon_2 = -\frac{1}{2}\Lambda(\varepsilon_1) + S(\varepsilon_1, \varepsilon_2),$$

yields that these particular initial data, small enough, activate only the dissipative waves and control the nonlinear operators T and S obtaining global in time solutions.

The main idea to show ill-posedness of Caflisch and Orellana is to consider the following function

$$s_0(\gamma, t) = \varepsilon(1 - i)[(1 - e^{-t/2 - i\gamma})^{1+\nu} - (1 - e^{-t/2 + i\gamma})^{1+\nu}]$$

which is a solution of the linearization of equation (3.4). For $0 < \nu < 1$, s_0 has a infinite curvature at $\gamma = t = 0$. Then they prove that a function $r(\gamma, t)$ exists such that $z(\gamma, t) = \gamma + s_0 + r$ is an analytic solution of equation (3.4) with infinite curvature at $\gamma = t = 0$. Then they obtain ill-posedness in Sobolev spaces in the Hadamard sense using the following symmetry properties:

If $z(\gamma, t)$ is a solution of (3.4) then so are $z_b(\gamma, t) = \overline{z}(\gamma, -t)$, $z_s(\gamma, t) = z(\gamma, t - t_0)$ and $z_n(\gamma, t) = n^{-1} z(n\gamma, nt)$. A study of the existence of solutions of equation (3.4) in less regular spaces than H^s can be found in [68]. We also quote that the first evidence of singularities with analytic initial data was given by Moore in [55].

We shall study the case in which the term in the tangential direction is given by $c(\alpha, t) = H(\varpi)(\alpha, t)$, where $H\varpi$ is the Hilbert transform of the function ϖ and we shall consider an initial data for the amplitude of the vorticity with mean zero which is preserved by equation (3.2). From Biot-Savart law, at first expansion, the expression at infinity is of the order of $\frac{1}{|x|}\int\varpi$ for a closed curve or near planar at infinity. To obtain a velocity field in L^2 it is necessary to have $\int\varpi = 0$ (for more details see [51]). In the periodic case, $z(\alpha+2\pi k,t) = z(\alpha,t)+(2\pi k,0)$, the following classical identity for complex numbers

$$\frac{1}{\pi}\Big(\frac{1}{z}+\sum_{k\geq 1}\frac{2z}{z^2-(2\pi k)^2}\Big) = \frac{1}{2\pi\tan(z/2)},$$

yields (ignoring the variable t)

$$v(x)=\frac{-1}{4\pi}\int_{-\pi}^{\pi}\varpi(\beta)\Big(\frac{\tanh(\frac{x_2-z_2(\beta)}{2})(1+\tan^2(\frac{x_1-z_1(\beta)}{2}))}{\tan^2(\frac{x_1-z_1(\beta)}{2})+\tanh^2(\frac{x_2-z_2(\beta)}{2})},\frac{\tan(\frac{x_1-z_1(\beta)}{2})(\tanh^2(\frac{x_2-z_2(\beta)}{2})-1)}{\tan^2(\frac{x_1-z_1(\beta)}{2})+\tanh^2(\frac{x_2-z_2(\beta)}{2})}\Big)d\beta,$$

for $x\neq z(\alpha,t)$. Then

$$\lim_{x_2\to\pm\infty} v(x,t) = \mp\frac{1}{4\pi}\int_{-\pi}^{\pi}\varpi(\beta)d\beta(1,0),$$

and to have the same value at infinity it is necessary again mean zero.

In section 3.3 we show that the chosen parametrization provides solutions of the vortex sheet problem. The requirements are the usual for this system: the initial data have to be analytic. With our analysis we do not need to parameterize the interface in terms of a function as in [66], the initial curve has to be one-to-one and with nonzero tangent vector. In the argument we modify the proofs used in the Cauchy-Kowalewski theorems given in [56] and [57] in order to deal with the arc-chord condition.

Finally, in section 3.4, we show ill-posedness for the equation of the amplitude (3.2) for a non-analytic initial data with mean zero by adapting the argument of the section 2.3 to the periodic case.

3.2 The evolution equation

In this section we shall obtain the classic Birkhoff-Rott equations from the weak formulation of the Euler equations for the velocity. For this purpose we use the continuity of the pressure over the vortex sheet. One alternative equation deduction, which does not need to prove this property of the pressure, can be found in [47] where the authors use the weak formulation of the Euler equations for the vorticity in 2D. We consider weak solutions of the system (1.2–1.3): for any smooth functions η and ζ compactly supported on $[0,T)\times\mathbb{R}^2$, i.e. in the space $C_c^\infty([0,T)\times\mathbb{R}^2)$, we have

$$\int_0^T\int_{\mathbb{R}^2}\big(v\cdot(\eta_t+v\cdot\nabla\eta)+p\nabla\cdot\eta\big)dxdt+\int_{\mathbb{R}^2}v_0(x)\cdot\eta(x,0)dx=0 \qquad (3.5)$$

and

$$\int_0^T\int_{\mathbb{R}^2}v\cdot\nabla\zeta dxdt=0, \qquad (3.6)$$

where $v_0(x) = v(x,0)$ is the initial data. We do not require divergence free tests as in the more common weak formulation because in this way we can obtain from (3.5) the continuity of the pressure. Anyway, for vortex sheets both formulation are equivalent.

Let us assume that the vorticity is given by a delta function on the curve $z(\alpha,t)$ multiplied by an amplitude, i.e.

$$\omega(x,t) = \varpi(\alpha,t)\delta(x - z(\alpha,t)), \tag{3.7}$$

where $z(\alpha,t) \in C^{1,\delta}$ splits the plane in two domains $\Omega^j(t)$ $(j = 1,2)$ and $\varpi(\alpha,t) \in C^{1,\delta}$ with $0 < \delta < 1$.

Then by Biot-Savart law we get

$$v(x,t) = \frac{1}{2\pi} PV \int \frac{(x - z(\beta,t))^{\perp}}{|x - z(\beta,t)|^2} \varpi(\beta,t) d\beta \tag{3.8}$$

for $x \neq z(\alpha,t)$. We have

$$\begin{aligned} v^2(z(\alpha,t),t) &= BR(z,\varpi)(\alpha,t) + \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t), \\ v^1(z(\alpha,t),t) &= BR(z,\varpi)(\alpha,t) - \frac{1}{2}\frac{\varpi(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}\partial_\alpha z(\alpha,t), \end{aligned} \tag{3.9}$$

where $v^j(z(\alpha,t),t)$ denotes the limit velocity field obtained approaching the boundary in the normal direction inside Ω^j and $BR(z,\varpi)(\alpha,t)$ is given by (3.3). It is easy to check that the velocity v (3.8) satisfies (3.6) .

Next we shall obtain the equation for the curve $z(\alpha,t)$. We start from equation (3.5) with $\eta(x,0) = 0$, which is

$$\int_0^T \int_{\mathbb{R}^2} [v \cdot (\eta_t + v \cdot \nabla\eta) + p\nabla \cdot \eta]\; dxdt = 0. \tag{3.10}$$

Again we can split the equation (3.10) in the following way

$$\lim_{\varepsilon\to 0}\Big(\int_0^T \int_{\Omega^1_\varepsilon(t)} [v \cdot (\eta_t + v \cdot \nabla\eta) + p\nabla \cdot \eta]\; dxdt +$$

$$\int_0^T \int_{\Omega^2_\varepsilon(t)} [v \cdot (\eta_t + v \cdot \nabla\eta) + p\nabla \cdot \eta]\; dxdt\Big) = 0,$$

where

$$\Omega^1_\varepsilon(t) = \{x \in \Omega^1(t) : \text{dist}\,(x, \partial\Omega^1(t)) \geq \varepsilon\}$$

$$\Omega^2_\varepsilon(t) = \{x \in \Omega^2(t) : \text{dist}\,(x, \partial\Omega^2(t)) \geq \varepsilon\}.$$

We will study the first terms in detail. Integrating by parts we obtain

$$\lim_{\varepsilon\to 0}\int_0^T \int_{\Omega^1_\varepsilon(t)} v \cdot \eta_t \, dxdt = \int_0^T \int (v^1 \cdot \eta)(z_t \cdot \partial^{\perp}_\alpha z) d\alpha dt - \int_0^T \int_{\Omega^1(t)} v_t \cdot \eta \, dxdt.$$

Similarly for the other terms we have

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^1_\varepsilon(t)} v\cdot(v\cdot\nabla)\eta\,dxdt = -\int_0^T\int(v^1\cdot\eta)(v^1\cdot\partial^\perp_\alpha z)\,d\alpha dt - \int_0^T\int_{\Omega^1(t)}\eta\cdot(v\cdot\nabla)v\,dxdt$$

and

$$\lim_{\varepsilon\to 0}\int_0^T\int_{\Omega^1_\varepsilon(t)} p\nabla\cdot\eta,dxdt = \int_0^T\int p^1(\eta\cdot\partial^\perp_\alpha z)d\alpha dt - \int_0^T\int_{\Omega^1}\nabla p\cdot\eta dxdt.$$

Operating in a similar way with the integral over $\Omega^2_\varepsilon(t)$ yields the following equations

$$\int_0^T\int\Big((\partial_t z - BR(z,\varpi))\cdot\partial^\perp_\alpha z\frac{\varpi}{|\partial_\alpha z|^2}(\eta\cdot\partial_\alpha z) + (p^1-p^2)(\eta\cdot\partial^\perp_\alpha z)\Big)\,d\alpha dt = 0 \qquad (3.11)$$

and

$$v_t + (v\cdot\nabla)\cdot v = -\nabla p \quad \text{over } \Omega^1 \text{ and } \Omega^2,$$

where the derivatives of v on $\partial\Omega^1$ and $\partial\Omega^2$ have to be understood like the limits in the normal direction to the curve $z(\alpha,t)$.

By choosing $\eta\cdot\partial_\alpha z = 0$ in the equation (3.11) he have that the pressure is continuos along the vortex sheet. Writing this work we learned of the paper by Shvydkoy [64] who also prove this fact for more general cases in a different way. In addition we obtain the equation

$$(\partial_t z - BR(z,\varpi))\cdot\partial^\perp_\alpha z\frac{\varpi}{|\partial_\alpha z|^2} = 0.$$

Next we close the system giving the evolution equation for the amplitude of the vorticity $\varpi(\alpha,t)$ by means of Bernoulli's law. Using (3.8) for $x\neq z(\alpha,t)$ we get $v(x,t)=\nabla\phi(x,t)$ where

$$\phi(x,t) = \frac{1}{2\pi}PV\int\arctan\Big(\frac{x_2-z_2(\beta,t)}{x_1-z_1(\beta,t)}\Big)\varpi(\beta,t)d\beta.$$

We define

$$\Pi(\alpha,t) = \phi^2(z(\alpha,t),t) - \phi^1(z(\alpha,t),t),$$

where again $\phi^j(z(\alpha,t),t)$ denotes the limit obtained approaching the boundary in the normal direction inside Ω^j. It is clear

$$\begin{aligned}\partial_\alpha\Pi(\alpha,t) &= (\nabla\phi^2(z(\alpha,t),t) - \nabla\phi^1(z(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t)\\ &= (v^2(z(\alpha,t),t) - v^1(z(\alpha,t),t))\cdot\partial_\alpha z(\alpha,t)\\ &= \varpi(\alpha,t).\end{aligned}$$

Now we can check that

$$\begin{aligned}\phi^2(z(\alpha,t),t) &= IT(z,\varpi)(\alpha,t) + \frac12\Pi(\alpha,t)\\ \phi^1(z(\alpha,t),t) &= IT(z,\varpi)(\alpha,t) - \frac12\Pi(\alpha,t),\end{aligned} \qquad (3.12)$$

where

$$IT(z,\varpi)(\alpha,t)=\frac{1}{2\pi}PV\int \arctan\Big(\frac{z_2(\alpha,t)-z_2(\beta,t)}{z_1(\alpha,t)-z_1(\beta,t)}\Big)\varpi(\beta,t)d\beta.$$

Using the Bernoulli's law in (1.2), inside each domain, we have

$$\phi_t(x,t)+\frac{1}{2}|v(x,t)|^2+p(x,t)=0.$$

Taking the limit it follows

$$\phi^j_t(z(\alpha,t),t)+\frac{1}{2}|v^j(z(\alpha,t),t)|^2+p^j(z(\alpha,t),t)=0,$$

and since $p^1(z(\alpha,t),t)=p^2(z(\alpha,t),t)$ we get

$$\phi^2_t(z(\alpha,t),t)-\phi^1_t(z(\alpha,t),t)+\frac{1}{2}|v^2(z(\alpha,t),t)|^2-\frac{1}{2}|v^1(z(\alpha,t),t)|^2=0. \tag{3.13}$$

Then it is clear that $\phi^j_t(z(\alpha,t),t)=\partial_t(\phi^j(z(\alpha,t),t))-z_t(\alpha,t)\cdot\nabla\phi^j(z(\alpha,t),t)$ and using (3.9) together (3.12) in (3.13) we obtain

$$\Pi_t(\alpha,t)=\varpi(\alpha,t)(z_t(\alpha,t)-BR(z,\varpi)(\alpha,t))\cdot\frac{\partial_\alpha z(\alpha,t)}{|\partial_\alpha z(\alpha,t)|^2}. \tag{3.14}$$

Finally it is easy to show that the solutions of the system (3.1) and (3.2) provide weak solutions of the Euler's equation. Given a curve $z(\alpha,t)\in C^{1,\delta}$ and a function $\varpi(\alpha,t)\in C^{1,\delta}$ such that the equations (3.1) and (3.2) are satisfied, we define the velocity $v(x,t)$ by the expression (3.8) and the pressure by

$$p(x,t)=-\phi_t(x,t)-\frac{1}{2}|v(x,t)|^2 \quad \text{over } \Omega^1 \text{ and } \Omega^2,$$

where the potential $\phi(x,t)$ is given by $v=\nabla\phi$. From equation (3.2) we have that the pressure is continuous over the vortex sheet. In order to check that $v(x,t)$ and $p(x,t)$ are weak solutions of Euler's equations we just have to introduce them in the first member of (3.5) and (3.6) and integrate by parts.

3.3 Local-existence for analytic initial data

We have the evolution equation given by

$$\begin{aligned} z_t &= BR(z,\varpi)+H\varpi\partial_\alpha z,\\ \varpi_t &= \partial_\alpha(\varpi H\varpi). \end{aligned} \tag{3.15}$$

In this frame, we consider a scale of Banach spaces $\{X_r\}_{r\geq 0}$ given by periodic real functions that can be extended analytically on the complex strip $B_r=\{\alpha+i\zeta:\alpha\in\mathbb{T},|\zeta|<r\}$ with norm

$$\|f\|_r=\max_{0\leq k\leq 2}\sup_{\alpha+i\zeta\in B_r}|\partial^k_\alpha f(\alpha+i\zeta)|_*+\sup_{\alpha+i\zeta\in B_r,\beta\in\mathbb{T}}\frac{|\partial^2_\alpha f(\alpha+i\zeta)-\partial^2_\alpha f(\alpha+i\zeta-\beta)|_*}{|\beta|^\delta},$$

with $0<\delta<1$ and $|\cdot|_*$ the modulus of a complex number. We then obtain the following theorem.

Theorem 3.3.1 *Let $z^0(\alpha)$ be a curve satisfying the arc-chord condition*

$$\frac{|z^0(\alpha)-z^0(\alpha-\beta)|^2}{|\beta|^2} > \frac{1}{M^2}, \tag{3.16}$$

and $z^0(\alpha), \varpi^0(\alpha) \in X_{r_0}$ for some $r_0 > 0$. Then, there exist a time $T > 0$ and $0 < r < r_0$ so that there is a unique solution to (3.15) in $C([0,T]; X_r)$.

Remark 3.3.2 *In the proof it is easy to check that the tangential term is not harmful to the evolution equation of the curve. In fact, it is the easiest to deal with. Also, with solutions of the system (3.15), by a reparametrization, one could recover solutions of the vortex sheets problem with the more usual choice of c such as the one given by the lagrangian velocities or the one with $c = 0$ (taking $v = \frac{v^1+v^2}{2}$). A similar theorem follows for all these parametrizations.*

It is easy to check that $X_r \subset X_{r'}$ for $r' \le r$ due to the fact that $\|f\|_{r'} \le \|f\|_r$. A simple application of the Cauchy formula gives

$$\|\partial_\alpha f\|_{r'} \le \frac{C}{r-r'}\|f\|_r, \tag{3.17}$$

for $r' < r$.

The equation (3.15) can be extended on B_r as follows:

$$\begin{aligned} z_t(\alpha+i\zeta,t) &= F_1(z(\alpha+i\zeta,t),\varpi(\alpha+i\zeta,t)),\\ \varpi_t(\alpha+i\zeta,t) &= F_2(\varpi(\alpha+i\zeta,t)). \end{aligned} \tag{3.18}$$

with

$$F_1(z,\varpi) = BR(z,\varpi) + H\varpi\partial_\alpha z,$$

and

$$F_2(\varpi) = \partial_\alpha(\varpi H\varpi).$$

Proposition 3.3.3 *Consider $0 \le r' < r$ and the open set O in B_r given by*

$$O = \{z,\varpi \in X_r : \|z\|_r, \|\varpi\|_r < R, \inf_{\alpha+i\zeta\in B_r,\beta\in\mathbb{T}} G(z)(\alpha+i\zeta,\beta) > \frac{1}{R^2}\}, \tag{3.19}$$

with

$$G(z)(\alpha+i\zeta,\beta) = \Big|\frac{(z_1(\alpha+i\zeta)-z_1(\alpha+i\zeta-\beta))^2 + (z_2(\alpha+i\zeta)-z_2(\alpha+i\zeta-\beta))^2}{\beta^2}\Big|_*. \tag{3.20}$$

Then the function $F = (F_1, F_2)$ for $F : O \to X_{r'}$ is a continuous mapping. In addition, there is a constant C_R (depending on R only) such that

$$\|F(z,\varpi)\|_{r'} \le \frac{C_R}{r-r'}\|(z,\varpi)\|_r, \tag{3.21}$$

$$\|F(z^2,\varpi^2) - F(z^1,\varpi^1)\|_{r'} \le \frac{C_R}{r-r'}\|(z^2-z^1,\varpi^2-\varpi^1)\|_r, \tag{3.22}$$

and

$$\sup_{\alpha+i\zeta\in B_r,\beta\in\mathbb{T}} |F_1(z,\varpi)(\alpha+i\zeta)-F_1(z,\varpi)(\alpha+i\zeta-\beta)|_*\leq C_R|\beta|, \tag{3.23}$$

for $z, z^j, \varpi, \varpi^j \in O$.

Using the above proposition we have the proof of theorem 3.3.1.

Proof of Theorem 3.3.1: The argument is analogous as in [56] and [57] (see also [52]). We have to deal with the arc-chord condition so we will point out the main differences. For initial data $z^0, \varpi^0 \in X_{r_0}$ satisfying (3.16), we can find a $0 < r_0' < r_0$ and a constant R_0 such that $\|z^0\|_{r_0'} < R_0$, $\|\varpi^0\|_{r_0'} < R_0$ and

$$\Big|\frac{(z_1^0(\alpha+i\zeta)-z_1^0(\alpha+i\zeta-\beta))^2+(z_2^0(\alpha+i\zeta)-z_2^0(\alpha+i\zeta-\beta))^2}{\beta^2}\Big|_* > \frac{1}{R_0^2}, \tag{3.24}$$

for $\alpha+i\zeta \in B_{r_0'}$. We take $0 < r < r_0'$ and $R_0 < R$ to define the open set O as in (3.19). Therefore we can use the classical method of successive approximations:

$$(z^{n+1}(t),\varpi^{n+1}(t)) = (z^0,\varpi^0)+\int_0^t F(z^n(s),\varpi^n(s))ds,$$

for $F : O \to X_{r'}$ and $0 \leq r' < r$. We assume by induction that

$$\|z^k\|_r(t) < R, \qquad \|\varpi^k\|_r(t) < R \qquad \text{and} \qquad G(z^k)(\alpha+i\zeta,\beta,t) > R^{-2}$$

with $\alpha + i\zeta \in B_r$, $\beta \in \mathbb{T}$ for $k \leq n$ and $0 < t < T$ with $T = \min(T_A, T_{CK})$ and T_{CK} the time obtaining in the proofs in [56] and [57] (see also [52]). Now, we will check that $G(z^{n+1})(\alpha + i\zeta, \beta, t) > R^{-2}$ for $\alpha + i\zeta \in B_r$ and $\beta \in \mathbb{T}$ giving T_A. The rest of the proof follows in the same way as in [56], [57]. The following formula:

$$z^{n+1}(t) = z^0+\int_0^t F_1(z^n(s),\varpi^n(s))ds$$

yields

$$G(z^{n+1})(\alpha+i\zeta,\beta,t)\geq G(z^0)(\alpha+i\zeta,\beta)-I_1-2I_2,$$

for

$$I_1=\int_0^t\Big|\frac{F_1(z^n,\varpi^n)(\alpha+i\zeta,s)-F_1(z^n,\varpi^n)(\alpha+i\zeta-\beta,s)}{\beta}\Big|_*^2 ds$$

and

$$I_2=\Big|\frac{z^0(\alpha+i\zeta)-z^0(\alpha+i\zeta-\beta)}{\beta}\Big|_*\int_0^t\Big|\frac{F_1(z^n,\varpi^n)(\alpha+i\zeta,s)-F_1(z^n,\varpi^n)(\alpha+i\zeta-\beta,s)}{\beta}\Big|_* ds.$$

Using the induction hypothesis and (3.23) it is straightforward to get $I_1 \leq C_R^2 t$. The inequality

$$\Big|\frac{z^0(\alpha+i\zeta)-z^0(\alpha+i\zeta-\beta)}{\beta}\Big|_*\leq\sup_{B_r}|\partial_\alpha z^0(\alpha+i\zeta)|_*<R_0$$

yields $I_2 \leq R_0 C_R t$. Therefore, taking $0 < T_A < (R_0^{-2} - R^{-2})(C_R^2 + 2R_0 C_R)^{-1}$, we obtain $G(z^{n+1})(\alpha + i\zeta, \beta, t) > R^{-2}$. Proof of Proposition 3.3.3: We will show first (3.23). We split as follows

$$F_1(z,\varpi)(\alpha+i\zeta) - F_1(z,\varpi)(\alpha+i\zeta-\beta) = I_1 + I_2 + I_3$$

for

$$I_1 = BR(z,\varpi)(\alpha+i\zeta) - BR(z,\varpi)(\alpha+i\zeta-\beta),$$
$$I_2 = (H(\varpi)(\alpha+i\zeta) - H(\varpi)(\alpha+i\zeta-\beta))\partial_\alpha z(\alpha+i\zeta),$$

and

$$I_3 = H(\varpi)(\alpha+i\zeta)(\partial_\alpha z(\alpha+i\zeta) - \partial_\alpha z(\alpha+i\zeta-\beta)).$$

It is easy to get

$$\sup_{\alpha+i\zeta\in B_r,\beta\in\mathbb{T}} |I_2|_* \leq \sup_{B_r} |\partial_\alpha z(\alpha+i\zeta)|_* \sup_{B_r} |H(\partial_\alpha\varpi)(\alpha+i\zeta)|_* |\beta|,$$

and due to

$$H : C^\delta \to C^\delta, \tag{3.25}$$

(see [65]), yields

$$\sup_{\alpha+i\zeta\in B_r,\beta\in\mathbb{T}} |I_2|_* \leq R^2|\beta|.$$

In a similar fashion it follows:

$$\sup_{\alpha+i\zeta\in B_r,\beta\in\mathbb{T}} |I_3|_* \leq R^2|\beta|.$$

For I_1, a straightforward calculation gives

$$\sup_{\alpha+i\zeta\in B_r,\beta\in\mathbb{T}} |I_1|_* \leq \sup_{B_r} |\partial_\alpha BR(z,\varpi)(\alpha+i\zeta)|_* |\beta|,$$

and it remains to bound $\partial_\alpha BR(z,\varpi)(\alpha+i\zeta)$. For this term we use the following decomposition:

$$\partial_\alpha BR(z,\varpi)(\alpha+i\zeta) = J_1 + J_2 + J_3,$$

with

$$J_1 = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \partial_\alpha\varpi(\gamma-\beta) \frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2} d\beta,$$
$$J_2 = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \varpi(\gamma-\beta) \frac{\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\beta)}{|z(\gamma)-z(\gamma-\beta)|^2} d\beta,$$
$$J_3 = -\frac{1}{\pi} PV \int_{-\pi}^{\pi} \varpi(\gamma-\beta) \frac{(z(\gamma)-z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^4} \big((z(\gamma)-z(\gamma-\beta))\cdot(\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\beta))\big) d\beta,$$

where $\gamma = \alpha + i\zeta$. Here we have to deal with nonlinear singular integral operators given by one-to-one curves. We proceed as in [29] considering the arc-chord condition (see also [26]). we take $J_1 = K_1 + K_2 + K_3$ for

$$K_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \partial_\alpha\varpi(\gamma-\beta) \frac{(z(\gamma)-z(\gamma-\beta)-\partial_\alpha z(\gamma)\beta)^\perp}{|z(\gamma)-z(\gamma-\beta)|^2} d\beta,$$

$$K_2 = \frac{(\partial_\alpha z(\gamma))^\perp}{2\pi}\int_{-\pi}^{\pi}\partial_\alpha\varpi(\gamma-\beta)\Big(\frac{\beta}{|z(\gamma)-z(\gamma-\beta)|^2}-\frac{1}{|\partial_\alpha z(\gamma)|^2\beta}\Big)d\beta,$$

$$K_3 = \frac{(\partial_\alpha z(\gamma))^\perp}{|\partial_\alpha z(\gamma)|^2}\Big(\frac{1}{2\pi}\int_{-\pi}^{\pi}\partial_\alpha\varpi(\gamma-\beta)[\frac{1}{\beta}-\frac{1}{2\tan(\beta/2)}]d\beta+H(\partial_\alpha\varpi)(\gamma)\Big).$$

We rewrite K_1 as follows:

$$K_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\partial_\alpha\varpi(\gamma-\beta)\frac{(z(\gamma)-z(\gamma-\beta)-\partial_\alpha z(\gamma)\beta)^\perp}{\beta^2}\frac{\beta^2}{|z(\gamma)-z(\gamma-\beta)|^2}d\beta,$$

and therefore, using that $z,\varpi\in O$ and the following estimate:

$$\sup_{\gamma\in B_r,\beta\in\mathbb{T}}|z(\gamma)-z(\gamma-\beta)-\partial_\alpha z(\gamma)\beta|_*\leq\sup_{\gamma\in B_r}|\partial_\alpha^2 z(\gamma)|_*|\beta|^2,$$

we obtain $\sup_{B_r}|K_1|_*\leq R^4$. In the integral in K_2 we find

$$\partial_\alpha\varpi(\gamma-\beta)\Big(\frac{(\partial_\alpha z(\gamma)\beta+z(\gamma)-z(\gamma-\beta))\cdot(\partial_\alpha z(\gamma)\beta-(z(\gamma)-z(\gamma-\beta)))}{|z(\gamma)-z(\gamma-\beta)|^2|\partial_\alpha z(\gamma)|^2\beta}\Big).$$

The bound for the infimum of G also gives $\sup_{B_r}||\partial_\alpha z(\gamma)|^{-2}|_*\leq R^2$ in O, so we have $\sup_{B_r}|K_2|_*\leq 2R^8$. The integral in K_3 has a bounded kernel in β and therefore

$$\sup_{B_r}|K_3|_*\leq(C+1)R^4$$

for $C=\max_{\beta\in\mathbb{T}}|\beta^{-1}-(2\tan(\beta/2))^{-1}|$. In J_2 we write $J_2=K_4+K_5+K_6$

$$K_4 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\varpi(\gamma-\beta)-\varpi(\gamma))\frac{(\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\beta))^\perp}{|z(\gamma)-z(\gamma-\beta)|^2}d\beta,$$

$$K_5 = \frac{\varpi(\gamma)}{2\pi}\int_{-\pi}^{\pi}(\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\beta))^\perp\Big(\frac{1}{|z(\gamma)-z(\gamma-\beta)|^2}-\frac{1}{|\partial_\alpha z(\gamma)|^2\beta^2}\Big)d\beta,$$

$$K_6 = \frac{\varpi(\gamma)}{2|\partial_\alpha z(\gamma)|^2}\Big(\frac{1}{\pi}\int_{-\pi}^{\pi}(\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\beta))^\perp[\frac{1}{\beta^2}-\frac{1}{(2\sin(\beta/2))^2}]d\beta+(\Lambda(\partial_\alpha z))^\perp(\gamma)\Big),$$

where $\Lambda=H(\partial_\alpha)$. In K_4 we rewrite

$$K_4 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{(\varpi(\gamma-\beta)-\varpi(\gamma))}{\beta}\frac{(\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\beta))^\perp}{\beta}\frac{\beta^2}{|z(\gamma)-z(\gamma-\beta)|^2}d\beta,$$

and therefore $\sup_{B_r}|K_4|_*\leq R^4$. We take

$$K_5 = \frac{\varpi(\gamma)}{2\pi}\int_{-\pi}^{\pi}\frac{(\partial_\alpha z(\gamma)-\partial_\alpha z(\gamma-\beta))^\perp}{\beta}\Big(\frac{\beta}{|z(\gamma)-z(\gamma-\beta)|^2}-\frac{1}{|\partial_\alpha z(\gamma)|^2\beta}\Big)d\beta,$$

and therefore, as for K_2, we find $\sup_{B_r}|K_5|_*\leq 2R^8$. The first term in K_6 is easy to deal with because the function $\beta^{-2}-(2\sin(\beta/2))^{-2}$ is bounded. For the second term we find

$$\sup_{B_r}|\Lambda(\partial_\alpha z))^\perp|_*=\sup_{B_r}|H(\partial_\alpha^2 z))^\perp|_*\leq CR$$

using (3.25). For the term J_3 we proceed as before to get finally (3.23).

Now we will show how to obtain (3.22). The estimate (3.21) follows in a easier fashion (see also [26]). Here we will use the following estimate:

$$\|fg\|_{C^\delta} \leq \|f\|_{C^\delta}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|g\|_{C^\delta}. \tag{3.26}$$

For F_2 we get $\partial^2_\alpha F_2(\varpi) = \partial^3_\alpha(\varpi H\varpi)$. In the subtraction $\partial^2_\alpha F_2(\varpi^2) - \partial^2_\alpha F_2(\varpi^1)$ we find terms of different order. Here we deal with the terms with greatest order. The rest of the terms can be estimate in a simpler way. We find in the subtraction the term $I_1 = \partial^3_\alpha\varpi^2 H(\varpi^2) - \partial^3_\alpha\varpi^1 H(\varpi^1)$ and we split

$$I_1 = (\partial^3_\alpha\varpi^2 - \partial^3_\alpha\varpi^1)H(\varpi^2) + \partial^3_\alpha\varpi^1(H(\varpi^2) - H(\varpi^1)) = J_1(\gamma) + J_2(\gamma).$$

For $\beta \in \mathbb{T}$ and $\gamma \in B_{r'}$, the inequality (3.26) yields

$$\frac{|J_1(\gamma) - J_1(\gamma-\beta)|_*}{|\beta|^\delta} \leq 2\|\partial_\alpha\varpi^2 - \partial_\alpha\varpi^1\|_{r'}\|\varpi^2\|_{r'},$$

and using (3.17) it follows:

$$\sup_{\gamma\in B_{r'},\beta\in\mathbb{T}} \frac{|J_1(\gamma) - J_1(\gamma-\beta)|_*}{|\beta|^\delta} \leq \frac{2R}{r-r'}\|\varpi^2 - \varpi^1\|_r.$$

Analogously

$$\frac{|J_2(\gamma)-J_2(\gamma-\beta)|_*}{|\beta|^\delta} \leq 2\|\partial_\alpha\varpi^1\|_{r'}\|\varpi^2-\varpi^1\|_{r'} \leq \frac{2\|\varpi^1\|_r}{r-r'}\|\varpi^2-\varpi^1\|_{r'} \leq \frac{2R}{r-r'}\|\varpi^2-\varpi^1\|_r.$$

Also the term I_2, given by $I_2 = \varpi^2 H(\partial^3_\alpha\varpi^2) - \varpi^1 H(\partial^3_\alpha\varpi^1)$, can be decomposed as

$$J_3(\gamma) = (\varpi^2 - \varpi^1)H(\partial^3_\alpha\varpi^2), \quad J_4(\gamma) = \varpi^1 H(\partial^3_\alpha(\varpi^2 - \varpi^1),$$

and as before

$$\begin{aligned}\frac{|J_3(\gamma) - J_3(\gamma-\beta)|_*}{|\beta|^\delta} &\leq 2\|\varpi^2 - \varpi^1\|_{r'}\|H(\partial_\alpha\varpi^2)\|_{r'} \leq 2C\|\varpi^2 - \varpi^1\|_{r'}\|\partial_\alpha\varpi^2\|_{r'} \\ &\leq \frac{2RC}{r-r'}\|\varpi^2 - \varpi^1\|_r.\end{aligned}$$

For J_4 it follows:

$$\frac{|J_4(\gamma) - J_4(\gamma-\beta)|_*}{|\beta|^\delta} \leq 2RC\|\partial_\alpha(\varpi^2 - \varpi^1)\|_{r'} \leq \frac{2RC}{r-r'}\|\varpi^2 - \varpi^1\|_r.$$

Now we consider the operator $F_1(z,\varpi) = BR(z,\varpi) + H\varpi\partial_\alpha z$. The estimates for the second term are as before, we then show the control for the Birkhoff-Rott integral. While the terms of lower order are easier, we consider in $\partial^2_\alpha BR(z,\varpi)$ the most singular:

$$I_3 = \frac{1}{2\pi}PV\int_{-\pi}^{\pi}\varpi(\gamma-\beta)\frac{(\partial^2_\alpha z(\gamma) - \partial^2_\alpha z(\gamma-\beta))^\perp}{|z(\gamma) - z(\gamma-\beta)|^2}d\beta,$$

$$I_4 = \frac{-1}{\pi} PV \int_{-\pi}^{\pi} \varpi(\gamma-\beta) \frac{(z(\gamma)-z(\gamma-\beta))^{\perp}}{|z(\gamma)-z(\gamma-\beta)|^4} (z(\gamma)-z(\gamma-\beta)) \cdot (\partial_\alpha^2 z(\gamma)-\partial_\alpha^2 z(\gamma-\beta)) d\beta,$$

and

$$I_5 = \frac{1}{2\pi} PV \int_{-\pi}^{\pi} \partial_\alpha^2 \varpi(\gamma-\beta) \frac{(z(\gamma)-z(\gamma-\beta))^{\perp}}{|z(\gamma)-z(\gamma-\beta)|^2} d\beta.$$

We take $I_3 = J_5 + J_6 + J_7 + J_8$ with

$$J_5 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\varpi(\gamma-\beta) - \varpi(\gamma)) \frac{(\partial_\alpha^2 z(\gamma) - \partial_\alpha^2 z(\gamma-\beta))^{\perp}}{|z(\gamma)-z(\gamma-\beta)|^2} d\beta,$$

$$J_6 = \frac{\varpi(\gamma)}{2\pi} \int_{-\pi}^{\pi} (\partial_\alpha^2 z(\gamma) - \partial_\alpha^2 z(\gamma-\beta))^{\perp} [\frac{1}{|z(\gamma)-z(\gamma-\beta)|^2} - \frac{1}{|\partial_\alpha z(\gamma)|^2 \beta^2}] d\beta,$$

$$J_7 = \frac{\varpi(\gamma)}{2\pi |\partial_\alpha z(\gamma)|^2} \int_{-\pi}^{\pi} (\partial_\alpha^2 z(\gamma) - \partial_\alpha^2 z(\gamma-\beta))^{\perp} [\frac{1}{\beta^2} - \frac{1}{4\sin^2(\beta/4)}] d\beta,$$

and

$$J_8 = \frac{\varpi(\gamma)}{2|\partial_\alpha z(\gamma)|^2} (\Lambda(\partial_\alpha^2 z)(\gamma))^{\perp}.$$

Then, with this splitting, in $\partial_\alpha^2 BR(z^2, \varpi^2) - \partial_\alpha^2 BR(z^1, \varpi^1)$, one can find the term

$$DJ_8 = \frac{\varpi^2(\gamma)}{2|\partial_\alpha z^2(\gamma)|^2} (\Lambda(\partial_\alpha^2 z^2)(\gamma))^{\perp} - \frac{\varpi^1(\gamma)}{2|\partial_\alpha z^1(\gamma)|^2} (\Lambda(\partial_\alpha^2 z^1)(\gamma))^{\perp}$$

Now, for $h \in \mathbb{T}$ and $\gamma \in B_{r'}$, it follows:

$$\begin{aligned}|DJ_8(\gamma) - DJ_8(\gamma-h)|_* \le C_R(&\|(z^2 - z^1, \varpi^2 - \varpi^1)\|_{r'} |h|^{\delta} (r-r')^{-1} \\ &+ |\Lambda(\partial_\alpha^2 (z^2 - z^1))(\gamma) - \Lambda(\partial_\alpha^2 (z^2 - z^1))(\gamma-h)|_*),\end{aligned}$$

and using (3.25) one finds

$$\begin{aligned}|\Lambda(\partial_\alpha^2 (z^2-z^1))(\gamma) - \Lambda(\partial_\alpha^2 (z^2-z^1))(\gamma-h)|_* &= |H(\partial_\alpha^3 (z^2-z^1))(\gamma) - H(\partial_\alpha^3 (z^2-z^1))(\gamma-h)|_* \\ &\le C\|\partial_\alpha (z^2 - z^1)\|_{r'} |h|^{\delta},\end{aligned}$$

and finally

$$\frac{|DJ_8(\gamma) - DJ_8(\gamma-h)|_*}{|h|^{\delta}} \le \frac{C_R}{r-r'} \|(z^2 - z^1, \varpi^2 - \varpi^1)\|_r.$$

In an analogous way we may define DJ_5 and split it as follows:

$$K_7 = \frac{1}{2\pi} \int_{-\pi}^{\pi} ((\varpi^2 - \varpi^1)(\gamma-\beta) - (\varpi^2 - \varpi^1)(\gamma)) \frac{(\partial_\alpha^2 z^2(\gamma) - \partial_\alpha^2 z^2(\gamma-\beta))^{\perp}}{|z^2(\gamma) - z^2(\gamma-\beta)|^2} d\beta,$$

$$K_8 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\varpi^1(\gamma-\beta) - \varpi^1(\gamma)) \frac{(\partial_\alpha^2 (z^2 - z^1)(\gamma) - \partial_\alpha^2 (z^2 - z^1)(\gamma-\beta))^{\perp}}{|z^2(\gamma) - z^2(\gamma-\beta)|^2} d\beta,$$

and

$$K_9 = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\varpi^1(\gamma-\beta) - \varpi^1(\gamma))(\partial_\alpha^2 z^1(\gamma) - \partial_\alpha^2 z^1(\gamma-\beta))^{\perp} A(\gamma,\beta) d\beta,$$

with $A(\gamma,\beta)=|z^2(\gamma)-z^2(\gamma-\beta)|^{-2}-|z^1(\gamma)-z^1(\gamma-\beta)|^{-2}$. All kernels in the integrals in K_7, K_8 and K_9 have grade 0 so the control of all these terms are analogous. Now we will show in detail the term K_7. We rewrite it as

$$K_7=\frac{1}{2\pi}\int_{-\pi}^{\pi}B(\gamma,\beta)C(\gamma,\beta)D(\gamma,\beta)d\beta,$$

with

$$B(\gamma,\beta)=\frac{(\varpi^2-\varpi^1)(\gamma-\beta)-(\varpi^2-\varpi^1)(\gamma)}{\beta}$$

$$C(\gamma,\beta)=\frac{(\partial_\alpha^2 z^2(\gamma)-\partial_\alpha^2 z^2(\gamma-\beta))^\perp}{\beta},\qquad D(\gamma,\beta)=\frac{\beta^2}{|z^2(\gamma)-z^2(\gamma-\beta)|^2},$$

to get the following splitting

$$K_7(\gamma)-K_7(\gamma-h)=L_1+L_2+L_3,$$

where

$$L_1=\frac{1}{2\pi}\int_{-\pi}^{\pi}(B(\gamma,\beta)-B(\gamma-h,\beta))C(\gamma,\beta)D(\gamma,\beta)d\beta,$$

$$L_2=\frac{1}{2\pi}\int_{-\pi}^{\pi}B(\gamma,\beta)(C(\gamma,\beta)-C(\gamma-h,\beta))D(\gamma,\beta)d\beta,$$

and

$$L_3=\frac{1}{2\pi}\int_{-\pi}^{\pi}B(\gamma,\beta)C(\gamma,\beta)(D(\gamma,\beta)-D(\gamma-h,\beta))d\beta.$$

We take the term B as

$$B(\gamma,\beta)=\int_0^1\partial_\alpha(\varpi^2-\varpi^1)(\gamma-s\beta)\,ds,$$

and therefore

$$|B(\gamma,\beta)-B(\gamma-h,\beta)|_*\le\|\varpi^2-\varpi^1\|_{r'}|h|^\delta.$$

for $\gamma\in B_{r'}$ and $h\in\mathbb{T}$. It yields

$$|L_1|_*\le\|\varpi^2-\varpi^1\|_{r'}|h|^\delta\|\partial_\alpha z^2\|_{r'}R^2\le\frac{R^3}{r-r'}\|\varpi^2-\varpi^1\|_r|h|^\delta.$$

For $C(\gamma,\beta)$ it follows:

$$C(\gamma,\beta)=\int_0^1(\partial_\alpha^3 z^2(\gamma+(s-1)\beta))^\perp\,ds,$$

and analogously one gets

$$|L_2|_*\le\|\varpi^2-\varpi^1\|_{r'}\|\partial_\alpha z^2\|_{r'}|h|^\delta R^2\le\frac{R^3}{r-r'}\|\varpi^2-\varpi^1\|_r|h|^\delta.$$

In L_3 we rewrite the difference $D(\gamma,\beta)-D(\gamma-h,\beta)$ as

$$\frac{\beta^2}{|z^2(\gamma)-z^2(\gamma-\beta)|^2}\frac{\beta^2}{|z^2(\gamma-h)-z^2(\gamma-h-\beta)|^2}E_1(\gamma,h,\beta)\cdot E_2(\gamma,h,\beta),$$

where

$$E_1(\gamma,h,\beta)=\frac{(z^2(\gamma-h)-z^2(\gamma-h-\beta))+(z^2(\gamma)-z^2(\gamma-\beta))}{\beta}$$

$$E_2(\gamma,h,\beta)=\frac{(z^2(\gamma-h)-z^2(\gamma-h-\beta))-(z^2(\gamma)-z^2(\gamma-\beta))}{\beta}$$

As before one can take

$$E_2(\gamma,h,\beta)=\int_0^1\big(\partial_\alpha z^2(\gamma-h+(s-1)\beta)-\partial_\alpha z^2(\gamma+(s-1)\beta)\big)$$

and therefore $|E_2|_*\leq\|z^2\|_{r'}|h|^\delta$. It provides as before

$$|D(\gamma,\beta)-D(\gamma-h,\beta)|_*\leq 2R^6|h|^\delta,$$

and

$$|L_3|_*\leq\|\varpi^2-\varpi^1\|_{r'}\|\partial_\alpha z^2\|_{r'}2R^6|h|^\delta\leq\frac{2R^7}{r-r'}\|\varpi^2-\varpi^1\|_r|h|^\delta.$$

All these estimates for the terms L_j yield

$$\frac{|K_7(\gamma)-K_7(\gamma-h)|_*}{|h|^\delta}\leq\frac{C_R}{r-r'}\|\varpi^2-\varpi^1\|_r,$$

for $\gamma\in B'_r$ and $h\in\mathbb{T}$.

In a similar way it is possible to get the appropriate control for J_6 and J_7. The terms I_4 and I_5 can be estimated as I_3, so that with this argument we finish the proof.

3.4 Ill-posedness for the amplitude equation.

In this section we choose the tangential term $c(\alpha,t)=H\varpi(\alpha,t)$ which gives the following closed equation for the amplitude of the vorticity

$$\varpi_t-(\varpi H\varpi)_\sigma=0, \tag{3.27}$$

$$\varpi(\sigma,0)=\varpi_0(\sigma). \tag{3.28}$$

We shall prove the following theorem:

Theorem 3.4.1 *Let $\varpi_0\in H^s(\mathbb{T})$ with $s>\frac{3}{2}$ and*

$$\int_{\mathbb{T}}\varpi_0=0.$$

Then if there exist a point σ_0 where $\varpi_0(\sigma_0)>0$ and ϖ_0 is not C^∞ in σ_0, there is no solution of equation (3.27) in the class $C([0,T];H^s(\mathbb{T}))$ with $s>\frac{3}{2}$ and $T>0$. In addition, $\varpi_0\in C^\infty$ is not sufficient to obtain existence.

Proof: We will proceed by a contradiction argument.

Let us assume that there exist a solution of equation (3.27) in the class $C([0,T),H^s(\mathbb{T}))$ with $\varpi(\sigma,0)=\varpi_0(\sigma)$.

First we have to note that if the initial data, ϖ_0, is of mean zero the solution ϖ will remain of mean zero.

Now, taking the Hilbert transform on equation (3.27) yields

$$\partial_t H\varpi - (H\varpi H\varpi_\sigma - \varpi\varpi_\sigma) = 0,$$

where we have used the following properties of the Hilbert transform for a periodic function with mean zero:

- $H(H\varpi) = -\varpi$.
- $H(\varpi H\varpi) = \frac{1}{2}((H\varpi)^2 - \varpi^2)$.

We denote the complex valued function $z(\sigma,t) = H\varpi(\sigma,t) - i\varpi(\sigma,t)$ which satisfies

$$\partial_t z - zz_\sigma = 0. \tag{3.29}$$

Take $P_\sigma(u)$ to be the Green's function of the Laplacian for the Dirichlet problem in the unit ball

$$P_\sigma(u) \equiv \frac{1}{2\pi}\frac{1-|u|^2}{|u-\sigma|^2},$$

and $P\varpi(u)$ will be

$$P\varpi(u) \equiv \int_{\partial B(0,1)} P_\tau(u)\varpi(\tau)d\tau.$$

Therefore

$$Z(u) = P(H\varpi - i\varpi)(u) \quad \text{with} \quad u = re^{i\sigma},$$

is an analytic function on the unit ball. Applying P to the equation (2.17-) yields

$$\partial_t Pz = P(zz_\sigma),$$

where we can write the second term in the following way

$$P(zz_\sigma) = Pz(Pz)_\sigma,$$

since both terms have the same restriction to the boundary of the unit ball and both are harmonic.

Thus, we have for $Z(u,t)$ the equation

$$Z_t - ZZ_\sigma = 0 \quad \text{on} \quad u \in \overline{B(0,1)},$$

hence

$$Z_t - iuZZ_u = 0 \quad \text{on} \quad u \in B(0,1), \tag{3.30}$$

$$Z(u,0) = Z_0(u) = P(H\varpi_0 - i\varpi_0)(u). \tag{3.31}$$

We will define the complex trajectories $X(u,t)$ by

$$\frac{dX(u,t)}{dt} = -iX(u,t)Z(X(u,t),t),$$

$$X(u,0) = u, \quad u \in B(0,1).$$

For sufficiently small t, by Picard's Theorem, these trajectories exist and $X(u,t) \in B(0,1)$. Therefore

$$\frac{dZ(X(u,t),t)}{dt} = \partial_t Z(X(u,t),t) - iX(u,t)Z(X(u,t),t)Z_u(X(u,t),t) = 0.$$

Thus, we have

$$Z(X(u,t),t) = Z_0(u),$$

and

$$\frac{dX(u,t)}{dt} = -iX(u,t)Z_0(u).$$

Moreover

$$X(u,t) = ue^{-iZ_0(u)t}.$$

Taking modules in the last expression we obtain

$$R(u,t) = |X(u,t)| = re^{-P\varpi_0(re^{i\sigma})t}.$$

If we consider a point $e^{i\sigma_0} = u_0 \in \partial B(0,1)$ with $\varpi_0(\sigma_0) > 0$, then

$$R(u_0,t) = e^{-w_0(\sigma_0)t} < 1.$$

Hence $X(u_0,t) \in B(0,1)$ for all $t > 0$, and a continuity argument yields

$$Z(X(\sigma_0,t),t) = z_0(\sigma_0) = H\varpi_0(\sigma_0) - i\varpi_0(\sigma_0),$$

where to simplify we denote $X(u_0,t) = X(\sigma_0,t)$. Then we have

$$X(\sigma_0,t) = e^{i(\sigma_0 - z_0(\sigma_0)t)}.$$

Taking a derivative with respect to σ_0 on this equation we find that

$$\frac{dX(\sigma_0,t)}{d\sigma_0} = i(1 - z_{0\,\sigma}(\sigma_0)t)X(\sigma_0,t).$$

With the chain's rule we obtain

$$\frac{dZ}{dX}(X(\sigma_0,t),t)iX(\sigma_0,t) = \frac{dZ}{d\Theta}(X(\sigma_0,t),t) = \frac{z_{0\,\sigma}(\sigma_0)}{(1 - z_{0\,\sigma}(\sigma_0)t)},$$

where

$$X(\sigma_0,t) = R(\sigma_0,t)e^{i\Theta(\sigma_0,t)}.$$

Taking two derivatives

$$\frac{d^2 Z}{d\Theta^2}(X(\sigma_0,t),t) = \frac{z_{0\,\sigma\sigma}(\sigma_0)}{(1 - z_{0\,\sigma}(\sigma_0)t)^3}.$$

For the n-th derivative we have

$$\frac{d^n Z}{d\Theta^n}(X(\sigma_0,t),t) = \frac{\frac{d^n z_0}{d\sigma^n}(\sigma_0)}{(1 - z_{0\,\sigma}(\sigma_0)t)^{n+1}} + \text{“lower terms”}.$$

We observe that $(1 - z_{0\,\sigma}(\sigma_0)t) \neq 0$ for t small enough.

Then if w_0 is not C^∞ in σ_0 this is a contradiction since $Z(u,t)$ is analytic on $X(\sigma_0,t)$ for all $t > 0$.

In addition, if $\varpi_0(\sigma_0) > 0$ and $\frac{d^n \varpi_0}{d\sigma^n}(\sigma_0) = 0\ \forall n$ but ϖ_0 is not constant on any neighborhood of σ_0, we can conclude

$$\frac{d\Im Z}{d\Theta}(X^1(\sigma_0,t),X^2(\sigma_0,t)) = 0.$$

Continuing this process we obtain that all derivatives satisfy

$$\frac{d^n\Im Z}{d\Theta^n}(X^1(\sigma_0,t),X^2(\sigma_0,t)) = 0.$$

The imaginary part $\Im Z(x_1,x_2,t)$ is analytic on $(x_1,x_2) = (X^1(\sigma_0,t),X^2(\sigma_0,t))$ for all $t > 0$, thus $\Im Z(x_1,x_2)$ is constant over the circumference, $R = R(\sigma_0,t)$, and this is a contradiction if ϖ_0 is not constant .

Chapter 4

Infinite Energy Solutions of the Surface Quasi-Geostrophic Equation.

4.1 Introduction.

In this chapter we study the existence of particular solutions with infinite energy of the surface quasi-geostrophic equation (SQG), i.e.

$$\begin{aligned} \partial_t \theta + u \cdot \nabla \theta &= 0, \\ \theta(x,0) &= \theta^0(x), \end{aligned} \tag{4.1}$$

where

$$\theta \quad : \quad \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}$$

$$\theta \quad = \quad \Lambda\Psi \quad \text{and} \tag{4.2}$$

$$u \quad = \quad \nabla^{\perp}\Psi = (-\partial_{x_2}\Psi, \partial_{x_1}\Psi) \tag{4.3}$$

with $\Lambda = (-\Delta)^{1/2}$.

Specifically, we shall analyze the case in which the stream function, Ψ, is given by the expression

$$\Psi(x_1, x_2, t) = -x_2 Hf(x_1, t), \tag{4.4}$$

where H is the Hilbert transform, i.e.

$$Hf(x) = \frac{1}{\pi} P.V. \int \frac{f(y)}{x-y} dy.$$

With the choice (4.4) of the stream function (see below in section 2), the solutions of (4.1) can be written as

$$\theta(x_1, x_2, t) = x_2 f_{x_1}(x_1, t), \tag{4.5}$$

where f_x satisfies the following one dimensional equation

$$\begin{aligned} \partial_t f_x + Hf f_{xx} &= Hf_x f_x, \\ f_x(x,0) &= f_x^0(x). \end{aligned} \tag{4.6}$$

In this way, for an odd initial data, the geometry of the level set of the active scalar contain a hyperbolic saddle in a neighborhood of zero. Nevertheless, the angle of opening of the saddle is not observed to go to zero in time. Similar stagnation-point solutions were considered for 2D Navier-Stokes equation in [16].

We will consider that the unknown quantity is the function f_x and we will define f by the expression

$$f(x) = \int_{-\infty}^{x} f_x(y)dy.$$

Then, if we take f_x with zero mean and with a suitable decay at infinity, we have

$$Hf(x) \equiv H\left(\int_{-\infty}^{x} f_x(y)dy\right) = \frac{1}{\pi}\int_{-\infty}^{\infty} \log(|x-y|) f_x(y)dy.$$

At this point it is important to stress that equation (4.6) is mean preserving. In order to verify this property it is enough to recall the orthogonality character of the Hilbert transform.

A more general version of (4.6) was proposed, in a different context, by H. Okamoto, T. Sakajo and M. Wunsch in [60]

$$\partial_t f_x + aHf f_{xx} = Hf_x f_x, \tag{4.7}$$

where a is real parameter. It was motivated by the work of P. Constantin, P. Lax and A. Majda ([18]) and the work of De Gregorio ([31] and [32]) where the equation is presented as a 1D-model of the 3D vorticity Euler equation.

Indeed, we can write the 3D Euler equation as

$$\partial_t w + (u\cdot\nabla)w = Dw, \tag{4.8}$$

where

$$u(w) = \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{(x-y)}{|x-y|^3} \times w(y)dy$$

and

$$D(w) = \frac{1}{2}(\nabla u + \nabla u^{\top}).$$

Thus, D is a singular integral operator and it is easy to check that equation (4.7) and (4.8) are of the same order. The natural question, behind equation (4.7), is if a transport term (preserving the structure of the Euler equation) can cancel the singularities of the model ($a = 0$) in [18]. See [31], [32] and [59] for a discussion on the role played by the convection term.

In [60] the authors show local existence of classical solutions for (4.7) with $f_x^0 \in H^1(\mathbb{T})$ and they present a blow up criteria: *The local solution of (4.6) can be extended to time T if*

$$\int_0^T ||Hf_x||_{L^\infty} dt < \infty.$$

In addition they carry out a numerical analysis (see also [61]) and conjecture that solutions may blow-up for $-1 \leq a < 1$ and global existence otherwise.

The case $a = -1$ has been proposed as a 1-D model of the SQG equation (see [24]) and as a 1-D model of the vortex sheet problem (see [1] and [54]). For this case, local existence is proven in [54], singularities in finite time are shown for even, compact support and positive classical solutions in [24] and for a more general positive initial data in [25].

The main results of this chapter are organized as follows: In section 4.2 we will show that the solutions of equation (4.6) provide solutions of the SQG equation. In section 4.3 we will construct self-similar solutions for equation (4.11) for any value of the parameter a. The existence of such solutions for the SQG equation has been studied by D. Chae in [13]. He showed that self-similar solutions do not exist with the form

$$\begin{aligned} u(x,t) &= \frac{1}{(T-t)^{\frac{\alpha}{1+\alpha}}} U\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}}\right), \\ \theta(x,t) &= \frac{1}{(T-t)^{\beta}} \Phi\left(\frac{x}{(T-t)^{\frac{1}{1+\alpha}}}\right), \\ \alpha,\ \beta \in \mathbb{R}, & \qquad \alpha \neq -1, \end{aligned}$$

if the profile Φ is in the class $L^{p_1} \cap L^{p_2}$, with $0 < p_1 < p_2 \leq \infty$, and if the profile $U \in C([0,T); C^1(\mathbb{R}^2; \mathbb{R}^2))$ generates a C^1 diffeomorphism from $\mathbb{R}^2$ onto itself. Nevertheless, this theorem can not be applied to solutions with the form (4.4). Finally, in the section 4.4, we will prove blow up for classical solutions of (4.11) with $a < 0$ and in the section 4.5 we will analyze the case $a = 1$.

4.2 SQG solutions with infinite energy.

In order to obtain the evolution equation for the function, $f(x,t)$, we will use the following representation of the operator Λ

$$\Lambda\Psi(x_1,x_2) = \frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{\Psi(x_1,x_2) - \Psi(y_1,y_2)}{|x-y|^3} dy. \tag{4.9}$$

Then, introducing (4.4) in (4.9) we have

$$\Lambda\Psi(x_1,x_2) = -\frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{x_2 Hf(x_1) - y_2 Hf(y_1)}{|x-y|^3} dy$$

$$= -\frac{1}{2\pi} P.V. \int_{\mathbb{R}^2} \frac{x_2\left(Hf(x_1) - Hf(y_1)\right) + \eta Hf(y_1)}{((x_1-y_1)^2 + \eta^2)^{\frac{3}{2}}} dy_1 d\eta$$

$$= -\frac{1}{2\pi} x_2 \, P.V. \int_{\mathbb{R}} Hf(x_1) - Hf(y_1) \left(P.V. \int_{\mathbb{R}} \frac{1}{\left((x_1 - y_1)^2 + \eta^2\right)^{\frac{3}{2}}} d\eta \right) dy_1$$

$$-\frac{1}{2\pi} \, P.V. \int_{\mathbb{R}} Hf(y_1) \left(P.V. \int_{\mathbb{R}} \frac{\eta}{((x_1 - y_1)^2 + \eta^2)^{\frac{3}{2}}} d\eta \right) dy_1$$

$$= -\frac{1}{\pi} x_2 P.V. \int_{\mathbb{R}} \frac{Hf(x_1) - Hf(y_1)}{(x_1 - y_1)^2} dy_1 = -x_2 \partial_{x_1} H(Hf(x_1)) = x_2 \partial_{x_1} f(x_1).$$

Therefore, from (4.2) follows

$$\theta(x_1, x_2, t) = \Lambda\Psi(x_1, x_2, t) = x_2 \partial_{x_1} f(x_1, t). \tag{4.10}$$

Combining (4.10) with (4.3) and with the equation (4.1) yields

$$x_2 \left(\partial_t \left(\partial_{x_1} f \right)(x_1, t) + Hf(x_1, t) \partial^2_{x_1} f(x_1, t) - \partial_{x_1} Hf(x_1, t) \partial_{x_1} f(x_1, t) \right) = 0.$$

Thus, the solutions of (4.6) provides solutions of the equation (4.1) with infinite energy.

4.3 Self-Similar solutions for any a.

The aim of this section is to prove the existence of self-similar solutions of the equation (4.7) but since the lack of regularity of this type of solutions we will work with the equation for f, instead of f_x, which is given by

$$\begin{aligned} \partial_t f + aHf f_x &= (1+a) \int_{-\infty}^{x} Hf_x(y) f_x(y) dy \\ f(x, 0) &= f^0(x). \end{aligned} \tag{4.11}$$

The theorem we will prove is the following:

Theorem 4.3.1 *Let*

$$G(x) = \sqrt{(1 - x^2)_+},$$

where and f_+ is the positive part of the function f. Then, the function

$$f(x, t) = -\frac{1}{t^{(1+a)}} G(t^a x)$$

is a self-similar solution of the equation (4.11).

The proof of this theorem is based on the next lemma:

Lemma 4.3.2 *The Hilbert transform of the function*

$$G(x) = \sqrt{(1 - x^2)_+}$$

is given by the expression

$$HG(x) = \begin{cases} x - \sqrt{x^2 - 1} & \text{if } x > 1 \\ x & \text{if } |x| < 1 \\ x + \sqrt{x^2 - 1} & \text{if } x < -1 \end{cases}$$

Remark 4.3.3 *A more general statement is obtained in [43]. Here we will give a simpler proof for lemma 4.3.2.*

Proof of lemma 4.3.2. Consider the complex function

$$F(z) = \sqrt{1 - z^2} + iz, \quad z = x + iy,$$

where the square root is defined by

$$\sqrt{z} \equiv |z|^{\frac{1}{2}} \exp^{\frac{i}{2}\arg(z)} \quad \text{with} \quad -\pi < arg(z) \leq \pi.$$

Then the following properties of F can be checked:

1. $F(z)$ is an analytic function for $y > 0$.
2. $F(z)$ vanishes at infinity.
3. The restriction of $F(z)$ to the real axis is given by the expression

$$\lim_{y \to 0^+} F(z) = \begin{cases} i(x - \sqrt{x^2 - 1}) & \text{if } x > 1 \\ \sqrt{1 - x^2} + ix & \text{if } |x| < 1 \\ i(x + \sqrt{x^2 - 1}) & \text{if } x < -1 \end{cases}$$

Then, since the restriction of $F(z)$ has to be of the form

$$\lim_{y \to 0^+} F(z) = f(x) + iHf(x),$$

the lemma 4.3.2 is proven. ▪

By introducing the ansatz, $f(x,t) = \frac{1}{t^{(1+a)}}\Phi(t^a x)$, in the equation (4.11) we obtain

$$a\Phi'(\xi)\left(H\Phi + \xi\right) = (1 + a)\int_{-\infty}^{\xi} \Phi'(y)\left(H\Phi'(\eta) + 1\right) d\eta. \tag{4.12}$$

Using lemma 4.3.2 we have that the function, $\Phi(\xi) = -G(\xi)$, satisfies equation (4.12) for any a.

An important consequence of this self-similar solutions is the following corollary:

Corollary 4.3.4 *The function $f(x,t) = \frac{1}{(1-t)^{1+a}}G\left((1-t)^a x\right)$ is a solution of equation (4.11) with the following behavior at time $t = 1$: i) For $a > -1$ there is blow-up i.e. $f(0,t)$ tends to infinity in finite time. ii) For $a = -1$ the solution collapses in a point.*

In order to prove this corollary it is enough to observe that the equation is time translations invariant and that changing the time direction is the same that changing the sign of the initial data.

4.4 Blow up for classical solutions with $a < 0$.

In this section we will present a proof of blow up of classical solutions for the equation (4.11) with $a < 0$. We will say that a solution $f(x,t)$ of equation (4.11) "blows up in finite time" if there exits $0 < T < \infty$ such that either f is not in $C^\infty(\mathbb{R}\times[0,T])$ or $Hf_x(x,t)$ is unbounded on $\mathbb{R}\times[0,T]$.

Theorem 4.4.1 *Let $f_x^0 \in C_c^\infty(\mathbb{R})$ an odd function such that $Hf_x^0(0) > 0$. Then the solution of (4.11) with $a < 0$ blows up in finite time.*

Proof. We will proceed by a contradiction argument. Let us suppose that there exist a solution of (4.11), $f_x \in C^1([0,T], C^\infty(\mathbb{R}))$ for all $T < \infty$ with f_x^0 as in the theorem. Then, f_x satisfies the following properties:

1. $f_x(\cdot,t)$ is odd.
2. $f_x(\cdot,t)$ is of compact support.

The first property is evident. In order to check the second property, we define the trajectories

$$\begin{aligned} \frac{d\,X(x,t)}{dt} &= aHf(X(x,t),t), \\ X(x,0) &= x. \end{aligned}$$

Then the function $f_x(X(x,t),t)$ satisfies the equation

$$\begin{aligned} \frac{d\,f_x(X(x,t),t)}{dt} &= aHf_x(X(x,t),t)f_x(X(x,t),t), \\ f_x(X(x,0),0) &= f_x^0(x) \end{aligned}$$

and therefore

$$f_x(X(x,t),t) = \exp\left(a\int_0^t Hf_x(X(x,\tau),\tau)d\tau\right) f_x^0(x).$$

Taking the Hilbert transform on (4.7) yields

$$\partial_t Hf_x(x,t) + aH(Hff_{xx})(x,t) = \frac{1}{2}\left(Hf_x(x,t)^2 - f_x(x,t)^2\right).$$

By evaluating this equation in $x = 0$ we obtain

$$\frac{d\Lambda f(0,t)}{dt} = \frac{1}{2}(\Lambda f(0,t))^2 - aH(Hff_{xx})(0,t). \tag{4.13}$$

Thus, if we prove that $H(Hff_{xx})(0,t)$ is bigger than 0, we obtain a contradiction since $a < 0$. Therefore, in order to prove theorem (4.4.1) we just have to show the following lemma:

Lemma 4.4.2 *Let $f \in C_c^\infty(\mathbb{R})$ an even function. Then*

$$H(Hff_{xx})(0) \geq 0.$$

Proof of lemma 4.4.2: We will use the Fourier transform:

$$\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{-ikx}dx \quad \text{and} \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(k)e^{ikx}dk.$$

Then, we can write

$$H(Hff_{xx})(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H\widehat{(Hff_{xx})}(k)dk$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} -i\text{sign}\,(k)\int_{-\infty}^{\infty} \widehat{Hf}(k-\eta)\widehat{f_{xx}}(\eta)d\eta dk$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\text{sign}\,(k)|\eta|}{k-\eta}\widehat{\Lambda f}(k-\eta)\widehat{\Lambda f}(\eta)d\eta dk$$

$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\text{sign}\,(\xi+\eta)|\eta|}{\xi}\widehat{\Lambda f}(\xi)\widehat{\Lambda f}(\eta)d\eta d\xi.$$

Since Λf is a real even function, we have that $\widehat{\Lambda f}$ is also a real even function. Therefore we can write the previous expression in the following way

$$H(Hff_{xx})(0) = \frac{2}{(2\pi)^2}\int_{0}^{\infty}\int_{0}^{\infty}(1+\text{sign}\,(\xi-\eta))\frac{\eta}{\xi}\widehat{\Lambda f}(\eta)\widehat{\Lambda f}(\xi)d\eta d\xi$$

$$= \frac{4}{(2\pi)^2}\int_{0}^{\infty}\int_{0}^{\xi}\frac{\eta}{\xi}\widehat{\Lambda f}(\eta)\widehat{\Lambda f}(\xi)d\eta d\xi = \frac{4}{(2\pi)^2}\int_{0}^{\infty}\int_{0}^{1}\alpha\xi\widehat{\Lambda f}(\alpha\xi)\xi\widehat{\Lambda f}(\xi)d\alpha\frac{d\xi}{\xi}$$

Defining the function, $g(x) = x\widehat{\Lambda f}(x)$, and the dilatation $g_\alpha(x) = g(\alpha x)$ we have that

$$H(Hff_{xx})(0) = \frac{4}{(2\pi)^2}\int_{0}^{1}\left(\int_{0}^{\infty} g(\xi)g_\alpha(\xi)\frac{d\xi}{\xi}\right)d\alpha.$$

Now we recall the definition of the Mellin transform:

Definition 4.4.3 *Let g be a real function such that the integral*

$$\int_{0}^{\infty}|g(x)|\frac{dx}{x} < \infty.$$

Then, we define the Mellin transform, $Mg(\lambda)$, of $g(x)$ by the expression

$$Mg(\lambda) = \int_{0}^{\infty} x^{i\lambda}g(x)\frac{dx}{x}.$$

This operator has the following properties:

1. The Mellin transform of a dilatation is given by

$$Mg_\alpha(\lambda) = \alpha^{-i\lambda}Mg(\lambda).$$

2. The Parseval identity

$$\int_0^\infty f(x)g(x)\frac{dx}{x} = \frac{1}{2\pi}\int_{-\infty}^\infty Mf(\lambda)\overline{Mg}(\lambda)d\lambda.$$

Therefore,

$$H(Hff_{xx})(0) = \frac{4}{(2\pi)^3}\int_0^1\int_{-\infty}^\infty \alpha^{-i\lambda}|Mg|^2(\lambda)d\lambda d\alpha.$$

Since $|Mg|(\cdot)$ is an even function we can conclude that

$$H(Hff_{xx})(0) = \frac{4}{(2\pi)^3}\int_{-\infty}^\infty |Mg|^2(\lambda)\left(\Re\int_0^1 \alpha^{-i\lambda}d\alpha\right)d\lambda \tag{4.14}$$

$$= \frac{4}{(2\pi)^3}\int_{-\infty}^\infty \frac{|Mg|^2(\lambda)}{1+\lambda^2}d\lambda \geq 0.$$

The lemma (4.4.2) is proven. ∎

4.5 A study of blow up for classical solutions with a=1.

In this section we shall study the blow up problem for classical solutions of the equation (4.6) which is concerned with the SQG equation as we have shown in the section 4.2.

We choose an initial data satisfying the requirements of theorem 4.4.1. Let us assume that there exists a solution of (4.6), $f_x \in C^1([0,T], C^\infty(\mathbb{R}))$ for all $T < \infty$. Then, from the expressions (4.13) and (4.14) we have that

$$\frac{d\Lambda f(0,t)}{dt} = \frac{1}{2}(\Lambda f(0,t))^2 - \frac{4}{(2\pi)^3}\int_{-\infty}^\infty \frac{|Mg|^2(\lambda)}{1+\lambda^2}d\lambda, \tag{4.15}$$

where

$$Mg(\lambda) = \int_0^\infty \xi^{i\lambda}\widehat{\Lambda f}(\xi)d\xi.$$

In order to compare the terms in the right hand side of equation (4.15) we will use the following lemma:

Lemma 4.5.1 *Let $f \in C_c^\infty(\mathbb{R})$ be an even function. Then*

$$\frac{4}{(2\pi)^3}\int_{-\infty}^\infty \frac{|Mg|^2(\lambda)}{1+\lambda^2}d\lambda = \frac{16}{(2\pi)^3}\int_{-\infty}^\infty \frac{\cosh^2(\frac{\lambda\pi}{2})}{1+\lambda^2}\frac{\lambda\pi}{\sinh(\lambda\pi)}|M(f_x)(\lambda)|^2 d\lambda.$$

Proof of lemma 4.5.1: From the definition of $Mg(\lambda)$ yields

$$Mg(\lambda) = \int_0^\infty \xi^{i\lambda}\widehat{\Lambda f}(\xi)d\xi$$

$$= \lim_{\varepsilon\to 0^+}\int_0^\infty \xi^{i\lambda}\exp(-\varepsilon\xi)\widehat{\Lambda f}(\xi)d\xi = \lim_{\varepsilon\to 0^+} -i\int_0^\infty \xi^{i\lambda}\exp(-\varepsilon\xi)\widehat{f_x}(\xi)d\xi,$$

where we have used the symbol of the Hilbert transform.

Using the Fourier transform formula we obtain

$$\begin{aligned}
Mg(\lambda) &= \lim_{\varepsilon\to 0^+} -i\int_0^\infty \xi^{i\lambda}\exp(-\varepsilon\xi)\int_{-\infty}^{\infty} f_x(x)\exp(-ix\xi)dx(\xi)d\xi \\
&= \lim_{\varepsilon\to 0^+} -i\int_{-\infty}^{\infty} f_x(x)\int_0^\infty \xi^{i\lambda}\exp(-\varepsilon\xi)\exp(-i\xi x)d\xi dx \\
&= \lim_{\varepsilon\to 0^+} -i\Gamma(1+i\lambda)\int_{-\infty}^{\infty}\frac{f_x(x)}{(\varepsilon+ix)^{1+i\lambda}}dx.
\end{aligned}$$

Since f_x is an odd function we have that

$$Mg(\lambda) = -i\Gamma(1+i\lambda)\lim_{\varepsilon\to 0^+}\int_0^\infty \left((\varepsilon+ix)^{-1-i\lambda}-(\varepsilon-ix)^{-1-i\lambda}\right)f_x(x)dx$$

$$\begin{aligned}
&= -i\Gamma(1+i\lambda)\lim_{\varepsilon\to 0^+}\int_0^\infty \left(\exp\left(-(1+i\lambda)\log(\varepsilon+ix)\right)-\exp\left(-(1+i\lambda)\log(\varepsilon-ix)\right)\right)f_x(x)dx \\
&= -i\Gamma(1+i\lambda)\lim_{\varepsilon\to 0^+}\int_0^\infty (\varepsilon^2+x^2)^{\frac{-1-i\lambda}{2}}\left(\exp\left(-(1+i\lambda)i\arg(\varepsilon+ix)\right)\right. \\
&\quad \left.-\exp\left(-(1+i\lambda)i\arg(\varepsilon-ix)\right)\right)f_x(x)dx \\
&= -i\Gamma(1+i\lambda)\int_0^\infty +x-1-i\lambda\left(\exp\left(-(1+i\lambda)i\frac{\pi}{2}\right)-\exp\left((1+i\lambda)i\frac{\pi}{2}\right)\right)f_x(x)dx \\
&\quad 2i\Gamma(1+i\lambda)\cosh\left(\frac{\lambda\pi}{2}\right)\int_0^\infty x^{-i\lambda}f_x(x)\frac{dx}{x}.
\end{aligned}$$

Therefore, the square of the module of $Mg(\lambda)$ is given by the expression

$$|Mg(\lambda)|^2 = 4|\Gamma(1+i\lambda)|^2|M(f_x)(\lambda)|^2.$$

Since the classical formula

$$|\Gamma(1+i\lambda)|^2 = \frac{\lambda\pi}{\sinh(\lambda\pi)},$$

lemma 4.5.1 is proven. ■

Now, we shall observe that the value of $M(f_x)(0)$ is related with the value $\Lambda f(0)$. Indeed

$$M(f_x)(0) = \int_0^\infty f_x(x)\frac{dx}{x} = \frac{1}{2}\int_{-\infty}^{\infty} f_x(x)\frac{dx}{x} = -\frac{\pi}{2}Hf_x(0).$$

Therefore, we can write

$$|M(f_x)(0)|^2 = \frac{\pi^2}{4}\Lambda f(0)^2. \tag{4.16}$$

The following lemma provides a sufficient condition on the function $f_x(x)$ in order to obtain a suitable decay of the function $|M(f_x)(\lambda)|$ with respect to its value at the point $\lambda = 0$.

Lemma 4.5.2 *Let $f \in C_c^\infty([0,\infty))$ be a function such that:*

1. *The value of f at the point $x = 0$ is equal to zero.*

2. *f reaches a global minimum at the point $x_m \in [0, \infty)$.*

3. *f is a convex function in the interval $[0, x_m]$.*

4. *f is an increasing function in the interval $[x_m, \infty)$.*

Then

$$|Mf(\lambda)| \leq \frac{1}{\sqrt{1+\lambda^2}}|Mf(0)|.$$

Proof of lemma 4.5.2: By integrating by parts we obtain

$$Mf(\lambda) = \int_0^\infty x^{i\lambda} f_x(x) \frac{dx}{x} = \frac{1}{1+i\lambda} \int_0^\infty \frac{dx^{1+i\lambda}}{dx} f(x) \frac{dx}{x}$$

$$= -\frac{1}{1+i\lambda} \int_0^\infty x^{1+i\lambda} \frac{d}{dx}\left(\frac{f(x)}{x}\right) dx = \frac{1}{1+i\lambda} \int_0^\infty x^{i\lambda} \left(\frac{f(x)}{x} - f_x(x)\right) dx.$$

Therefore

$$|Mf(\lambda)| = \frac{1}{\sqrt{1+\lambda^2}} \left| \int_0^\infty x^{i\lambda} \left(\frac{f(x)}{x} - f_x(x)\right) dx \right|$$

$$\leq \frac{1}{\sqrt{1+\lambda^2}} \int_0^\infty \left| \frac{f(x)}{x} - f_x(x) \right| dx.$$

Since $f(x)$ is convex in the interval $[0, x_m]$ we have the following inequality

$$f(y) \geq f_x(x)(y - x) + f(x) \quad x, y \in [0, x_m].$$

Thus, taking $y = 0$ yields

$$\frac{f(x)}{x} - f_x(x) \leq 0 \quad x \in [0, x_m].$$

In addition, we know that $f(x) \leq 0$ for all $x \in \mathbb{R}$ and $f_x(x) \geq 0$ for all $x \in [x_m, \infty)$ and we can write

$$|Mf(\lambda)| \leq \frac{1}{\sqrt{1+\lambda^2}} \int_0^\infty \left(f_x(x) - \frac{f(x)}{x}\right) dx$$

$$= \frac{1}{\sqrt{1+\lambda^2}} \int_0^\infty \left(-\frac{f(x)}{x}\right) dx = \frac{1}{\sqrt{1+\lambda^2}}|Mf(0)|.$$

Lemma 4.5.2 is proven. ∎

By applying lemmas 4.5.1 and 4.5.2 together with expression (4.16) we have the following result

Lemma 4.5.3 *Let $f \in C_c^\infty(\mathbb{R})$ be an even function such that:*

1. *f_x reaches a global minimum at the point $x_m \in [0, \infty)$.*

2. *f_x is a convex function in the interval $[0, x_m]$.*

3. *f_x is a increasing function in the interval $[x_m, \infty)$.*

Then

$$\frac{4}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{|Mg|^2(\lambda)}{1+\lambda^2}d\lambda \leq \frac{1}{2\pi}\Lambda f(0)^2\int_{-\infty}^{\infty}\frac{\cosh^2\left(\frac{\lambda\pi}{2}\right)}{(1+\lambda^2)^2}\frac{\lambda\pi}{\sinh(\lambda\pi)}d\lambda = \frac{\beta}{2}\Lambda f(0)^2,$$

where $0<\beta<1$ and

$$Mg(\lambda) = \int_0^{\infty}\xi^{i\lambda}\widehat{\Lambda f}(\xi)d\xi.$$

Finally we summarize section 4.5 in the following corollary:

Corollary 4.5.4 *Let $f_x^0 \in C_c^{\infty}(\mathbb{R})$ an odd function such that:*

1. *f_x^0 reaches a global minimum at the point $x_m \in [0,\infty)$.*
2. *f_x^0 is a convex function in the interval $[0, x_m]$.*
3. *f_x^0 is an increasing function in the interval $[x_m, \infty)$.*

Let us assume that the solution $f(x,t)$ of the equation (4.6) holds the properties 1, 2 and 3 along the evolution. Then $f(x,t)$ blows up in finite time.

Proof. Suppose that there exist a solution of (4.6), $f_x \in C^1([0,T], C^{\infty}(\mathbb{R}))$, for all $T<\infty$ satisfying the requirements of corollary 4.5.4. Then using the expression (4.15) and lemma 4.5.3 we have

$$\frac{d\Lambda f(0)}{dt} \geq \frac{1-\beta}{2}\Lambda f(0)^2,$$

where $\beta<1$, which is a contradiction.

Remark 4.5.5 *It has been checked that the conditions 1, 2 and 3 on the initial data in corollary 4.5.4 do not assure the assumption on the solution $f(x,t)$. However, numerical experiments , carried out by the author, show evidence of the existence of initial data such that the solution to the equation (4.6) holds these conditions along the evolution.*

4.5.1 Numerical Analysis

In order to carry out a numerical analysis of the equation (4.6) we shall study the equation of the trajectories

$$\begin{aligned}\frac{dX(x,t)}{dt} &= Hf(X(x,t),t)\\ X(x,0) &= x.\end{aligned}$$

Using the equation (4.6) we obtain that the trajectories satisfies

$$\begin{aligned}\frac{dX(x,t)}{dt} &= \frac{1}{\pi}\int_{\mathbb{R}}\log\left(|X(x,t)-X(y,t)|\right)\left(X_x(y,t)\right)^2 f_x^0(y)dy \\ X(x,0) &= x.\end{aligned} \tag{4.17}$$

In addition the solution will have the expression

$$f_x(X(x,t),t) = X_x(x,t) f_x^0(x).$$

In the numerical simulations, which are presented below, we have taken as a initial data

$$f_x^0(x) = \begin{cases} -8x(1-x^2)^3 & |x| < 1 \\ 0 & |x| > 1 \end{cases}. \tag{4.18}$$

Thus, we only have to solve the equation (4.17) for $x \in [-1,1]$.

We approximate the solution in a mesh with $N = 401$ nodes and second order splines. In order to compute the integral we have used the MATLAB routine `cumtrapz` with the step size $h = \frac{2}{10(N-1)} = 0.005$. The time integration has been carried out with a fourth order solver `ode45` of MATLAB up to $t = 1$.

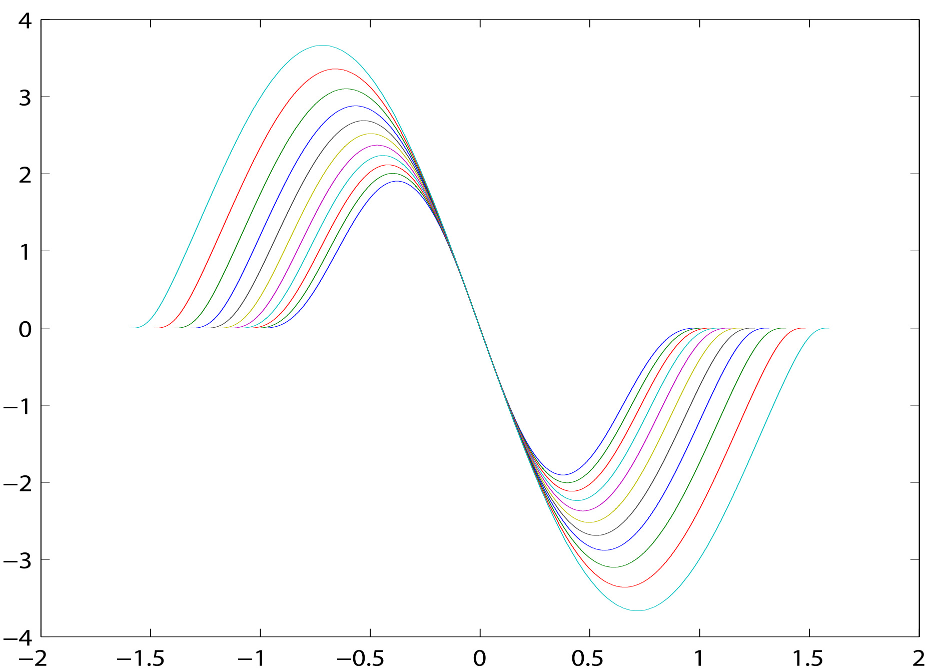

Figure 4.1: The solution $f_x(x,t)$ of equation (4.6) between $t = 0$ and $t = 1$. The initial data is given by the expression (4.18).

Chapter 5

Singularity Formations for a Surface Wave Model.

5.1 Introduction.

In this chapter we shall study the formation of singularities for the equation

$$\begin{aligned} u_t + uu_x &= \Lambda^\alpha H u, \\ u(x,0) &= u_0(x), \end{aligned} \tag{5.1}$$

with $0 \leq \alpha < 1$, where H is the Hilbert transform [65] defined by

$$Hf(x) = \frac{1}{\pi} P.V. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy$$

and $\Lambda^\alpha \equiv (-\Delta)^{\alpha/2}$ is given by the following expression

$$\Lambda^\alpha f(x) = k_\alpha \int_{\mathbb{R}} \frac{f(x)-f(y)}{|x-y|^{1+\alpha}} dy, \qquad k_\alpha = \frac{\Gamma(1+\alpha)\cos((1-\alpha)\pi/2)}{\pi}.$$

The case $\alpha = 0$

$$u_t + uu_x = Hu \tag{5.2}$$

was introduced by J. Marsden and A. Weinstein, in [53], as a second order approximation for the dynamics of a free boundary of a vortex patch (see [17] and [3]). Recently J. Biello and J.K. Hunter, in [4], proposed it as a model for waves with constant nonzero linearized frequency. They gave a dimensional argument to show that it models nonlinear Hamiltonian waves with constant frequency. In addition, an asymptotic equation from (5.2) is derived, describing surface waves on a planar discontinuity in vorticity for a two-dimensional inviscid incompressible fluid. They also carried out numerical analysis showing evidence of singularity formation in finite time. Let us point out that the Hamiltonian structure of the equation (5.1) (in particular for $\alpha = 0$) comes from the representation

$$u_t + \partial_x \left[\frac{\delta \mathcal{H}}{\delta u}\right] = 0, \qquad \text{where} \qquad \mathcal{H}(u) = \int_{\mathbb{R}} \left(\frac{1}{2} u \Lambda^{\alpha-1} u + \frac{1}{6} u^3\right) dx. \tag{5.3}$$

In section 5.2 we show that the linear term in the equation (5.2) is too weak to prevent the singularity formation of the Burgers equation. In fact, we show that, if the L^∞ norm of the initial data is large enough compared with the L^2 norm, the maximum of the solution has a singular behavior during the time of existence. One of the ingredients in the proof is to use the following pointwise inequality

$$u(x)^4 \leq 16||u||^2_{L^2(\mathbb{R})} \int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2} dy, \tag{5.4}$$

(see lemma 5.2.2 below) which can be understood as the local version of the well-known bound

$$||u||^4_{L^4} \leq C||u||^2_{L^2}||\Lambda^{1/2}u||^2_{L^2} = \frac{C}{2\pi}||u||^2_{L^2} \int_{\mathbb{R}}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2} dydx.$$

In the appendix we provide a generalized pointwise inequality ($n-$dimensional) in terms of fractional derivatives.

In section 3 we consider the more general family of equations, with a higher order term in derivatives, given by (5.1). By a different method, we prove that the blow up phenomena still arises. Let us note that, since $\Lambda H u = -u_x$, the case $\alpha = 1$ trivializes. Using the same approach as in section 2, it is possible to obtain blow up for $0 < \alpha < 1/3$. Inspired by the method used in [24], we check the evolution of the following quantity

$$J^p_q u(x) = \int_{\mathbb{R}} w^p_q(x-y)u(y)dy, \quad \text{where} \quad w^p_q(x) = \begin{cases} |x|^{-q}\text{sign}\,(x) & \text{if } |x|<1 \\ |x|^{-p}\text{sign}\,(x) & \text{if } |x|>1 \end{cases},$$

with $0 < q < 1$ and $p > 2$ to find a singular behavior. Let us note that a similar approach was used by H. Dong, D. Du and D. Li (see [39]) to show blow up for the Burgers equation with fractional dissipation in the supercritical case ($0 < \alpha < 1$):

$$u_t + uu_x = -\Lambda^\alpha u. \tag{5.5}$$

A different method to show singularities can be found in [45].

It is well known that the L^p norms of the solutions of equation (5.5) are bounded for all $1 \leq p \leq \infty$. However, to the best to the authors knowledge, two quantities are conserved by equation (5.1). The orthogonality property of the Hilbert transform provides the conservation of the L^2 norm, i.e.

$$||u(\cdot,t)||_{L^2(\mathbb{R})} = ||u_0||_{L^2(\mathbb{R})}. \tag{5.6}$$

Since the equation is given by (5.3), we have that

$$\int_{\mathbb{R}} \left(\frac{1}{3}u^3(x,t) + \left(\Lambda^{\frac{\alpha-1}{2}}u(x,t)\right)^2\right)dx = \int_{\mathbb{R}} \left(\frac{1}{3}u_0^3(x) + \left(\Lambda^{\frac{\alpha-1}{2}}u_0(x)\right)^2\right)dx. \tag{5.7}$$

5.2 Blow up for the Burgers-Hilbert equation.

The purpose of this section is to show finite time formation of singularities for solutions of the equation (5.2). The result we shall prove is the following:

Theorem 5.2.1 *Let $u_0 \in L^2(\mathbb{R}) \cap C^{1+\delta}(\mathbb{R})$, with $0 < \delta < 1$, satisfying the following condition:*

There exists a point $\beta_0 \in \mathbb{R}$ with

$$Hu_0(\beta_0) > 0, \tag{5.8}$$

such that

$$u_0(\beta_0) \geq \left(32\pi ||u_0||^2_{L^2(\mathbb{R})}\right)^{1/3}. \tag{5.9}$$

Then there is a finite time T such that

$$\lim_{t\to T} ||u(\cdot,t)||_{C^{1+\delta}(\mathbb{R})} = \infty,$$

where $u(x,t)$ is the solution to the equation (5.2).

Proof: Let us assume that there exist a solution of the equation (5.2)

$$u(x,t) \in C([0,T), C^{1+\delta}(\mathbb{R}))$$

for all time $T < \infty$ and with u_0 satisfying the hypotheses.

Now, we shall define the trajectories $x(\beta,t)$ by the equation

$$\begin{aligned} \frac{dx(\beta,t)}{dt} &= u(x(\beta,t),t), \\ x(\beta,0) &= \beta. \end{aligned}$$

Considering the evolution of the solution along trajectories, it is easy to get the identity

$$\frac{du(x(\beta,t),t)}{dt} = u_t(x(\beta,t),t) + \frac{dx(\beta,t)}{dt} u_x(x(\beta,t),t) = Hu(x(\beta,t),t),$$

and taking a derivative in time we obtain

$$\begin{aligned} \frac{d^2u(x(\beta,t),t)}{dt^2} &= Hu_t(x(\beta,t),t) + u(x(\beta,t),t)Hu_x(x(\beta,t),t) \\ &= -H(uu_x)(x(\beta,t),t) - u(x(\beta,t),t) + u(x(\beta,t),t)Hu_x(x(\beta,t),t). \end{aligned}$$

Since

$$H(uu_x)(x) = \frac{1}{2}H((u^2)_x) = \frac{1}{2}\Lambda(u^2)(x),$$

we can write

$$\frac{1}{2}\Lambda(u^2)(x) = \frac{1}{2\pi} P.V \int_{\mathbb{R}} \frac{u(x)^2 - u(y)^2}{(x-y)^2} dy = u(x)\Lambda u(x) - \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2} dy,$$

and therefore it follows that

$$\frac{d^2u(x(\beta,t),t)}{dt^2} = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{(u(x(\beta,t),t)) - u(y,t)))^2}{(x(\beta,t)-y)^2} dy - u(x(\beta,t),t). \tag{5.10}$$

In order to continue with the proof we will prove the lemma below (for similar approach see [23]):

Lemma 5.2.2 *Let $u \in L^2(\mathbb{R}) \cap C^{1+\delta}(\mathbb{R})$, for $0 < \delta < 1$. Then*

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2}dy \geq Cu(x)^4,$$

where

$$C = \frac{1}{32\pi E}$$

and

$$E = ||u||^2_{L^2(\mathbb{R})}.$$

Proof of lemma 5.2.2: Let us assume that $u(x) > 0$ (a similar proof holds for $u(x) < 0$). Let Ω be the set

$$\Omega = \{y \in \mathbb{R} \quad : \quad |x-y| < \Delta\},$$

where Δ will be given below. And let Ω^1 and Ω^2 be the subsets

$$\begin{aligned} \Omega^1 &= \{y \in \Omega \, : u(x) - u(y) \geq \frac{u(x)}{2}\}, \\ \Omega^2 &= \{y \in \Omega \, : u(x) - u(y) < \frac{u(x)}{2}\} = \{y \in \Omega \, : u(y) > \frac{u(x)}{2}\}. \end{aligned}$$

Then

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2}dy \geq \frac{u(x)^2}{8\pi\Delta^2}|\Omega^1|.$$

On the other hand

$$E = \int_{\mathbb{R}} u(y)^2 dy \geq \int_{\Omega^2} u(y)^2 dy \geq \frac{u(x)^2}{4}|\Omega^2|,$$

and therefore

$$|\Omega^2| \leq \frac{4E}{u(x)^2}.$$

Since $|\Omega^1| = |\Omega| - |\Omega^2|$ and $|\Omega| = 2\Delta$, we have that

$$\frac{1}{2\pi}\int_{\mathbb{R}} \frac{(u(x)-u(y))^2}{(x-y)^2}dy \geq \frac{u(x)^2}{8\pi\Delta^2}(2\Delta - \frac{4E}{u(x)^2}).$$

We achieve the conclusion of lemma 5.2.2 by taking $\Delta = \dfrac{4E}{u(x)^2}$. ∎

Next, let us define $J(t) = u(x(\beta_0, t), t)$. Thus, applying lemma 5.2.2 to the expression (5.10), we obtain the inequality

$$J_{tt}(t) \geq CJ(t)^4 - J(t). \tag{5.11}$$

Since $Hu_0(\beta_0) > 0$ and $J_t(t) = Hu(x(\beta_0, t), t)$, we obtain that $J_t(t) > 0$ and $J(t) > J(0)$ for $t \in (0, t^*)$ and t^* small enough. Therefore, multiplying (5.11) by $J_t(t)$ we have that

$$\frac{1}{2}(J_t(t)^2)_t \geq \frac{C}{5}(J(t)^5)_t - \frac{1}{2}(J(t)^2)_t, \quad \forall t \in [0, t^*).$$

Integrating this inequality in time from 0 to t we get

$$J_t(t) \geq \Big(J_t(0)^2 + \frac{2C}{5}(J(t)^5 - J(0)^5) - (J(t)^2 - J(0)^2)\Big)^{\frac{1}{2}}, \quad \forall t \in [0, t^*). \tag{5.12}$$

Now, since $CJ(0)^4 - J(0) \geq 0$, by the statements of the theorem we obtain that $J_{tt}(t) > J_{tt}(0) \geq 0$ for $t \in (0, t^*)$. Therefore, $J_t(t)$ is an increasing function $[0, t^*)$. Thus, the inequality (5.12) holds for all time t and we have a contradiction.

Remark 5.2.3 *It is easy to check that there exists a large class of functions satisfying the requirement of the theorem (5.2.1). For example, we can consider the function*

$$\begin{aligned} u_0(x) &= \frac{-ax}{1+(bx)^2}, \\ Hu_0(x) &= \frac{a}{1+(bx)^2}, \end{aligned}$$

where a, $b > 0$. Choosing a and b in a suitable way we can have the norm $||u_0||_{L^2(\mathbb{R})}$ as small as we want and the norm $||u_0||_{L^\infty(\mathbb{R})}$ as large as we want.

Remark 5.2.4 *We note that the requirements (5.8) and (5.9) in theorem 5.2.1 can be replaced by*

$$\begin{aligned} Hu_0(\beta_0) &\geq 0, \\ u_0(\beta_0) &> \Big(32\pi||u_0||^2_{L^2(\mathbb{R})}\Big)^{1/3}, \end{aligned}$$

attaining the same conclusion.

5.3 Blow up for the whole range $0 < \alpha < 1$.

In this section we shall show formation of singularities for the equation (5.1), with $0 < \alpha < 1$. The aim is to prove the following result:

Theorem 5.3.1 *There exist initial data $u_0 \in L^2(\mathbb{R}) \cap C^{1+\delta}(\mathbb{R})$, with $0 < \delta < 1$, and a finite time T, depending on u_0, such that*

$$\lim_{t \to T} ||u(\cdot, t)||_{C^{1+\delta}(\mathbb{R})} = \infty$$

where $u(x,t)$ is the solution to the equation (5.1).

Proof: Let us assume that there exists a solution of the equation (5.1)

$$u(x,t) \in C([0,T), C^{1+\delta}(\mathbb{R}))$$

for all time $T < \infty$. Let $J^p_q u$ be the convolution

$$J^p_q u(x) = \int_{\mathbb{R}} w^p_q(x-y)u(y)dy$$

where

$$w_q^p(x) = \begin{cases} \frac{1}{|x|^q}\text{sign}\,(x) & \text{if } |x| < 1 \\ \frac{1}{|x|^p}\text{sign}\,(x) & \text{if } |x| > 1 \end{cases},$$

with $0 < q < 1$ and $p > 2$. In order to prove theorem 2.2.1 we shall need the following two lemmas.

Lemma 5.3.2 *Let f in $C^{1+\delta}(\mathbb{R}) \cap L^2(\mathbb{R})$ and $0 < \alpha < 1$. Then*

$$\Lambda^\alpha H f(x) = k_\alpha \int_{\mathbb{R}} \frac{f(x) - f(y)}{|x-y|^{1+\alpha}} \text{sign}\,(x-y) dy$$

where

$$k_\alpha = -\frac{\Gamma(1+\alpha)\sin((1+\alpha)\pi/2)}{\pi}.$$

Proof of lemma 5.3.2: Let f be a function on the Schwartz class. The inverse Fourier transform formula yields

$$\Lambda^\alpha H f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} -i\text{sign}\,(k)|k|^\alpha \hat{f}(k) \exp(ikx) dk.$$

We will understand the above identity as the following limit

$$\begin{aligned} \Lambda^\alpha H f(x) &= \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\mathbb{R}} -i\text{sign}\,(k)|k|^\alpha \exp(-\varepsilon|k|) \exp(ikx) \Big(\int_{\mathbb{R}} f(y) \exp(-iky) dy \Big) dk \\ &= \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{\mathbb{R}} f(y) \Big(\int_0^\infty k^\alpha \exp(-\varepsilon k) \sin(k(x-y)) dk \Big) dy. \end{aligned}$$

Next, we can compute that

$$\begin{aligned} \Lambda^\alpha H f(x) &= \lim_{\varepsilon \to 0^+} \frac{\Gamma(1+\alpha)}{\pi} \int_{\mathbb{R}} \frac{f(y)}{(\varepsilon^2 + (x-y)^2)^{(1+\alpha)/2}} \sin\Big((1+\alpha)\arctan\Big(\frac{x-y}{\varepsilon}\Big)\Big) dy \\ &= -\lim_{\varepsilon \to 0^+} \frac{\Gamma(1+\alpha)}{\pi} \int_{\mathbb{R}} \frac{f(x) - f(y)}{(\varepsilon^2 + (x-y)^2)^{(1+\alpha)/2}} \sin\Big((1+\alpha)\arctan\Big(\frac{x-y}{\varepsilon}\Big)\Big) dy \\ &= -\frac{\Gamma(1+\alpha)\sin((1+\alpha)\pi/2)}{\pi} \int_{\mathbb{R}} \frac{f(x) - f(y)}{|x-y|^{1+\alpha}} \text{sign}\,(x-y) dy. \end{aligned}$$

We achieve the conclusion of lemma 5.3.2 by the classical density argument. ▪

Lemma 5.3.3 *Let $I_q^p(x)$ be the integral*

$$I_q^p(x) = \int_{\mathbb{R}} \frac{w_q^p(x) - w_q^p(y)}{|x-y|^{1+\alpha}} \text{sign}\,(x-y) dy$$

where $0 < q < 1$ and $p > 2$. Then

$$|I_q^p(x)| \le \begin{cases} \frac{K^1}{|x|^{q+\alpha}} & \text{if } 0 < |x| < \frac{1}{2} \\ \frac{K^2}{|x|^{2+\alpha}} & \text{if } 2 < |x| < \infty \\ K^3 & \text{if } \frac{1}{2} \le |x| \le 2 \end{cases}$$

where K^1, K^2 and K^3 are universal constants depending on q and p.

Proof of lemma 5.3.3: Since the function $I_q^p(x)$ is even, we can assume that $x>0$. The constant values of K^1 and K^2 can be different along the estimates below.

First, let us consider the case $0<x<1/2$. We split as follows

$$I_q^p(x)=\int_{|y|<1}dy+\int_{|y|>1}dy=I_1(x)+I_2(x).$$

It yields

$$\begin{aligned}I_1(x)&=\int_{|y|<1}\frac{\frac{1}{x^q}-\operatorname{sign}(y)\frac{1}{|y|^q}}{|x-y|^{1+\alpha}}\operatorname{sign}(x-y)dy\\&=\int_0^1\Big(\frac{\frac{1}{x^q}-\frac{1}{y^q}}{|x-y|^{1+\alpha}}\operatorname{sign}(x-y)+\frac{\frac{1}{x^q}+\frac{1}{y^q}}{|x+y|^{1+\alpha}}\Big)dy,\end{aligned}$$

and a change of variables allow us to split further

$$\begin{aligned}I_1(x)&=\frac{1}{x^{q+\alpha}}\int_0^{\frac{1}{x}}\Big(\frac{1-\frac{1}{\eta^q}}{|1-\eta|^{1+\alpha}}\operatorname{sign}(1-\eta)+\frac{1+\frac{1}{\eta^q}}{|1+\eta|^{1+\alpha}}\Big)d\eta\\&=\frac{1}{x^{q+\alpha}}\Big(\int_0^1+\int_1^{\frac{1}{x}}\Big)=\frac{1}{x^{q+\alpha}}(F_1(x)+F_2(x)).\end{aligned}$$

For $F_1(x)$ we find the bound

$$|F_1(x)|\le\int_0^1\Big|\frac{1-\frac{1}{\eta^q}}{|1-\eta|^{1+\alpha}}\Big|d\eta+\int_0^1\Big|\frac{1+\frac{1}{\eta^q}}{|1+\eta|^{1+\alpha}}\Big|d\eta\le K_1.$$

On the other hand

$$F_2(x)=\int_1^{\frac{1}{x}}\Big(\frac{\frac{1}{\eta^q}-1}{|1-\eta|^{1+\alpha}}+\frac{\frac{1}{\eta^q}+1}{|1+\eta|^{1+\alpha}}\Big)d\eta=\int_1^{\frac{3}{2}}+\int_{\frac{3}{2}}^{\frac{1}{x}}=j_1(x)+j_2(x).$$

For $j_1(x)$ it is easy to obtain

$$|j_1(x)|\le\int_1^{\frac{3}{2}}\Big|\frac{\frac{1}{\eta^q}-1}{|1-\eta|^{1+\alpha}}\Big|d\eta+\int_1^{\frac{3}{2}}\Big|\frac{\frac{1}{\eta^q}+1}{|1+\eta|^{1+\alpha}}\Big|d\eta\le K_1.$$

For $j_2(x)$ we decompose as follows

$$j_2(x)=\int_{\frac{3}{2}}^{\frac{1}{x}}\frac{1}{\eta^q}\Big(\frac{1}{|1-\eta|^{1+\alpha}}+\frac{1}{|1+\eta|^{1+\alpha}}\Big)d\eta+\int_{\frac{3}{2}}^{\frac{1}{x}}\Big(\frac{1}{|1-\eta|^{1+\alpha}}-\frac{1}{|1+\eta|^{1+\alpha}}\Big)d\eta.$$

Thus, since $0<q<1$ and

$$\Big|\frac{1}{|1-\eta|^{1+\alpha}}+\frac{1}{|1+\eta|^{1+\alpha}}\Big|\le\frac{K^1}{\eta}$$

we have that

$$|j_2(x)| \leq K^1 \int_{\frac{3}{2}}^{\infty} \frac{1}{\eta^{q+1}} d\eta + \int_{\frac{3}{2}}^{\infty} \Big| \frac{1}{|1-\eta|^{1+\alpha}} - \frac{1}{|1+\eta|^{1+\alpha}} \Big| d\eta \leq K^1.$$

Let us continue with I_2 which can be written in the form

$$I_2(x) = \int_{|y|>1} \frac{\frac{1}{x^q} - \operatorname{sign}(y)\frac{1}{|y|^p}}{|x-y|^{1+\alpha}} \operatorname{sign}(x-y) dy = \int_1^{\infty} \Big(- \frac{\frac{1}{x^q} - \frac{1}{|y|^p}}{|x-y|^{1+\alpha}} + \frac{\frac{1}{x^q} + \frac{1}{|y|^p}}{|x+y|^{1+\alpha}} \Big) dy$$
$$= \frac{1}{x^{q+\alpha}} \int_{\frac{1}{x}}^{\infty} \Big(\frac{\frac{x^{q-p}}{\eta^p} - 1}{|1-\eta|^{1+\alpha}} + \frac{1 + \frac{x^{q-p}}{\eta^p}}{|1-\eta|^{1+\alpha}} \Big) d\eta.$$

The following decomposition

$$I_2(x) = \frac{1}{x^{q+\alpha}} \int_{\frac{1}{x}}^{\infty} \Big(\frac{1}{|1+\eta|^{1+\alpha}} - \frac{1}{|1-\eta|^{1+\alpha}} \Big) d\eta$$
$$+ \frac{1}{x^{p+\alpha}} \int_{\frac{1}{x}}^{\infty} \frac{1}{\eta^p} \Big(\frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}} \Big) d\eta$$

yields

$$|I_2(x)| \leq \frac{K^1}{x^{q+\alpha}} + \frac{1}{x^{p+\alpha}} \int_{\frac{1}{x}}^{\infty} \frac{1}{\eta^p} \Big| \frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}} \Big| d\eta$$
$$\leq \frac{K^1}{x^{q+\alpha}} + \frac{K^1}{x^{p+\alpha}} \int_{\frac{1}{x}}^{\infty} \frac{1}{\eta^{p+1}} d\eta \leq K^1 \Big(\frac{1}{|x|^{\alpha+q}} + \frac{1}{|x|^{\alpha}} \Big) \leq \frac{K_1}{x^{q+\alpha}}.$$

Next, we consider the case $2 < x < \infty$ taking

$$I_q^p(x) = \int_{\mathbb{R}} \frac{\frac{1}{x^p} - w(y)}{|x-y|^{1+\alpha}} \operatorname{sign}(x-y) dy = \int_{|y|<1} dy + \int_{|y|>1} dy = J_1(x) + J_2(x).$$

For $J_2(x)$ we have that

$$J_2(x) = \int_{|y|>1} \frac{\frac{1}{x^p} - \operatorname{sign}(y)\frac{1}{|y|^p}}{|x-y|^{1+\alpha}} \operatorname{sign}(x-y) dy$$
$$= \int_1^{\infty} \Big(\frac{\frac{1}{x^p} - \frac{1}{|y|^p}}{|x-y|^{1+\alpha}} \operatorname{sign}(x-y) + \frac{\frac{1}{x^p} + \frac{1}{|y|^p}}{|x+y|^{1+\alpha}} \Big) dy$$

and a change of variables provides

$$J_2(x) = \frac{1}{x^{p+\alpha}} \int_{\frac{1}{x}}^{\infty} \Big(\frac{1 - \frac{1}{\eta^p}}{|1-\eta|^{1+\alpha}} \operatorname{sign}(1-\eta) + \frac{1 + \frac{1}{\eta^p}}{|1+\eta|^{1+\alpha}} \Big) d\eta$$
$$= \frac{1}{x^{\alpha+p}} \Big(\int_{\frac{1}{x}}^{1} + \int_1^{\infty} \Big) = \frac{1}{x^{\alpha+p}} (H_1(x) + H_2(x)).$$

For $H_2(x)$ one could bound as follow

$$|H_2(x)| \leq \int_1^\infty \Big|\frac{1-\frac{1}{\eta^p}}{|1-\eta|^{1+\alpha}}\Big| d\eta + \int_1^\infty \Big|\frac{1+\frac{1}{\eta^p}}{|1+\eta|^{1+\alpha}}\Big| d\eta \leq K^2.$$

On the other hand, in $H_1(x)$ we split further

$$H_1(x) = \int_{\frac{1}{x}}^1 \Big(\frac{1-\frac{1}{\eta^p}}{|1-\eta|^{1+\alpha}} + \frac{1+\frac{1}{\eta^p}}{|1+\eta|^{1+\alpha}}\Big) d\eta = \int_{\frac{1}{x}}^{\frac{2}{3}} d\eta + \int_{\frac{2}{3}}^1 d\eta = h_1(x) + h_2(x).$$

The term $h_2(x)$ is bounded by

$$|h_2(x)| \leq \int_{\frac{2}{3}}^1 \Big|\frac{1-\frac{1}{\eta^p}}{|1-\eta|^{1+\alpha}}\Big| d\eta + \int_{\frac{2}{3}}^1 \Big|\frac{1+\frac{1}{\eta^p}}{|1+\eta|^{1+\alpha}}\Big| d\eta \leq K^2.$$

We reorganize $h_1(x)$ so that

$$h_1(x) = \int_{\frac{1}{x}}^{\frac{2}{3}} \Big(\frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}}\Big) d\eta + \int_{\frac{1}{x}}^{\frac{2}{3}} \frac{1}{\eta^p} \Big(\frac{1}{|1-\eta|^{1+\alpha}} - \frac{1}{|1+\eta|^{1+\alpha}}\Big) d\eta.$$

Since $p > 2$ and

$$\Big|\frac{1}{|1-\eta|^{1+\alpha}} - \frac{1}{|1+\eta|^{1+\alpha}}\Big| \leq K^2 \eta \qquad \text{for} \quad \eta \in [0, 2/3],$$

we obtain that

$$|h_1(x)| \leq \int_0^{\frac{2}{3}} \Big|\frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}}\Big| d\eta + K^2 \int_{\frac{1}{x}}^{\frac{2}{3}} \frac{1}{\eta^{p-1}} d\eta \leq K^2 (1 + x^{p-2}).$$

Therefore

$$|J_2(x)| \leq K^2 \Big(\frac{1}{x^{p+\alpha}} + \frac{1}{x^{2+\alpha}}\Big) \leq \frac{K^2}{x^{2+\alpha}}.$$

Next, we deal with J_1 given by

$$J_1(x) = \int_{|y|<1} \frac{\frac{1}{x^p} - \text{sign}\,(y) \frac{1}{|y|^q}}{|x-y|^{1+\alpha}} dy = \int_0^1 \left(\frac{\frac{1}{x^p} - \frac{1}{|y|^q}}{|x-y|^{1+\alpha}} + \frac{\frac{1}{x^p} + \frac{1}{|y|^q}}{|x+y|^{1+\alpha}} \right) dy$$
$$= \frac{1}{x^{p+\alpha}} \int_0^{\frac{1}{x}} \left(\frac{1 - \frac{x^{p-q}}{\eta^q}}{|1-\eta|^{1+\alpha}} + \frac{1 + \frac{x^{p-q}}{\eta^q}}{|1+\eta|^{1+\alpha}} \right) d\eta.$$

Hence

$$J_1(x) = \frac{1}{x^{p+\alpha}} \int_0^{\frac{1}{x}} \Big(\frac{1}{|1-\eta|^{1+\alpha}} + \frac{1}{|1+\eta|^{1+\alpha}}\Big) d\eta + \frac{1}{x^{q+\alpha}} \int_0^{\frac{1}{x}} \frac{1}{\eta^q} \Big(\frac{1}{|1+\eta|^{1+\alpha}} - \frac{1}{|1-\eta|^{1+\alpha}}\Big) d\eta.$$

Since $p>2$ and

$$\Big|\frac{1}{|1+\eta|^{1+\alpha}}-\frac{1}{|1-\eta|^{1+\alpha}}\Big|\leq K^2\eta, \qquad \text{for} \quad \eta\in[0,1/2],$$

we obtain that

$$|J_1(x)|\leq\frac{K^2}{x^{p+\alpha}}+\frac{K^2}{x^{q+\alpha}}\int_0^{\frac{1}{x}}\eta^{1-q}d\eta\leq K^2\Big(\frac{1}{x^{p+\alpha}}+\frac{1}{x^{2+\alpha}}\Big)\leq\frac{K^2}{x^{2+\alpha}}.$$

The bound for $1/2<x<2$ is obvious, which allow us to conclude the proof. ▪

In order to prove theorem 2.2.1, we shall study the evolution of $J(t)=J_q^p u(x_b(t),t)$, where x_b is the trajectory $x_b(t)=x(0,t)$. Hence

$$\frac{dJ(t)}{dt}=-\frac{1}{2}J_q^p((u^2)_x)(x_b(t),t)+J_q^p\Lambda^\alpha Hu(x_b(t),t)+u(x_b(t),t)(\partial_x J_q^p u)(x_b(t),t).$$

We can write

$$J_q^p((u^2)_x)=\int_{\mathbb{R}}(u(x)^2-u(y)^2)W_q^p(x-y)dy$$

and

$$\partial_x(J_q^p u)(x)=\int_{\mathbb{R}}(u(x)-u(y))W_q^p(x-y)dy$$

where

$$W_q^p=\begin{cases}\frac{q}{|x|^{q+1}} & \text{if } |x|<1\\ \frac{p}{|x|^{p+1}} & \text{if } |x|>1\end{cases}.$$

Then, it is easy to check that

$$-\frac{1}{2}J_q^p((u^2)_x)(x)+u(x)(\partial_x J_q^p u)(x)=\frac{1}{2}\int_{\mathbb{R}}(u(x)-u(y))^2W_q^p(x-y)dy,$$

and therefore

$$\frac{dJ(t)}{dt}=\frac{1}{2}\int_{\mathbb{R}}(u(x_b(t))-u(y))^2W_q^p(x_b(t)-y)dy+J_q^p\Lambda^\alpha Hu(x_b(t),t). \tag{5.13}$$

Using lemma (5.3.2), the linear term becomes

$$J_q^p\Lambda^\alpha Hu(x)=k_\alpha\int_{\mathbb{R}}w_q^p(x-y)\int_{\mathbb{R}}\frac{u(y)-u(s)}{|y-s|^{1+\alpha}}\text{sign}\,(y-s)dsdy,$$

and a wise use of the principal value provides

$$\begin{aligned}J_q^p\Lambda^\alpha Hu(x)&=k_\alpha\int_{\mathbb{R}}w_q^p(x-y)P.V.\int_{\mathbb{R}}\frac{-u(s)}{|y-s|^{1+\alpha}}\text{sign}\,(y-s)dsdy\\&=k_\alpha\int_{\mathbb{R}}w_q^p(x-y)P.V.\int_{\mathbb{R}}\frac{u(x)-u(s)}{|y-s|^{1+\alpha}}\text{sign}\,(y-s)dsdy\\&=k_\alpha\int_{\mathbb{R}}(u(x)-u(s))P.V.\int_{\mathbb{R}}\frac{w_q^p(x-y)}{|y-s|^{1+\alpha}}\text{sign}\,(y-s)dyds\\&=k_\alpha\int_{\mathbb{R}}(u(x)-u(s))\int_{\mathbb{R}}\frac{w_q^p(x-s)-w_q^p(x-y)}{|y-s|^{1+\alpha}}\text{sign}\,(s-y)dyds\end{aligned}$$

to find finally

$$J_q^p \Lambda^\alpha H u(x) = k_\alpha \int_{\mathbb{R}} (u(x) - u(s)) I_q^p(x-s) ds.$$

Therefore

$$\begin{aligned} |J_q^p \Lambda^\alpha H u(x)| &\leq |k_\alpha| \int_{\mathbb{R}} |u(x) - u(y)||I_q^p(x-y)| dy \\ &\leq |k_\alpha| \left(\int_{\mathbb{R}} (u(x)-u(y))^2 W_q^p(x-y) dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{I_q^p(x)^2}{W_q^p(x)} dx \right)^{\frac{1}{2}}. \end{aligned}$$

Since

$$\frac{I_q^p(x)^2}{W_q^p(x)} \leq \begin{cases} \frac{C}{|x|^{2\alpha+q-1}} & \text{when } |x| \to 0 \\ \frac{C}{|x|^{3+2\alpha-p}} & \text{when } |x| \to \infty \end{cases},$$

by taking $2 < p < 2 + 2\alpha$ and $q < 2(1-\alpha)$, we obtain that

$$\begin{aligned} |J_q^p \Lambda^\alpha H u(x)| &\leq C(p,q) \left(\int_{\mathbb{R}} (u(x)-u(y))^2 W_q^p(y) dy \right)^{\frac{1}{2}} \\ &\leq \frac{1}{4} \int_{\mathbb{R}} (u(x)-u(y))^2 W_q^p(x-y) dy + C. \end{aligned}$$

This inequality in the equation (5.13) yields

$$\frac{dJ(t)}{dt} \geq \frac{1}{4} \int_{\mathbb{R}} (u(x_b(t)) - u(y))^2 W_q^p(x_b(t)-y) dy - C(p,q)$$

On the other hand

$$\begin{aligned} J(t) = \int_{\mathbb{R}} u(y) w_q^p(x_b(t)-y) dy &= \int_{\mathbb{R}} (u(y) - u(x_b(t))) w_q^p(x_b(t)-y) dy \\ \leq \left(\int_{\mathbb{R}} (u(x_b(t)) - u(y))^2 W_q^p(x_b(t)-y) dy \right)^{\frac{1}{2}} &\left(\int_{\mathbb{R}} \frac{w_q^p(x)^2}{W_q^p(x)} dx \right)^{\frac{1}{2}}. \end{aligned}$$

The following bound

$$\frac{w_q^p(x)^2}{W_q^p(x)} \leq \begin{cases} \frac{C}{|x|^{q-1}} & \text{when } |x| \to 0 \\ \frac{C}{|x|^{p-1}} & \text{when } |x| \to \infty \end{cases}$$

allows us to obtain, for $2 < p < 2 + 2\alpha$ and $0 < q < 1$,

$$\int_{\mathbb{R}} (u(x_b(t)) - u(y))^2 W_q^p(x_b(t)-y) dy \geq c(q,p) J(t)^2.$$

Therefore we obtain a quadratic evolution equation

$$\frac{dJ(t)}{dt} \geq c(q,p) J(t)^2 - C(q,p)$$

and by taking $c(q,p)J(0)^2 - C(q,p) > 0$, we find a contradiction for the mere fact that

$$J(t) \leq C(q,p) ||u||_{L^\infty}.$$

5.4 Appendix

In this section we generalize the pointwise inequality (5.4) envolving the nonlocal operator $2f\Lambda^\alpha f - \Lambda^\alpha(f^2)$. Some simple applications to Gagliardo-Nirenberg-Sobolev inequalities are also shown.

Lemma 5.4.1 *Consider a function $f : \mathbb{R}^n \to \mathbb{R}$ in the Schwartz class, $0 < \alpha < 2$ and $0 < p < \infty$. Then*

$$|f(x)|^{2+\frac{\alpha p}{n}} \leq C(\alpha,p,n)||f||^{\frac{\alpha p}{n}}_{L^p(\mathbb{R}^n)}(2f(x)\Lambda^\alpha f(x) - \Lambda^\alpha(f^2)(x)) \tag{5.14}$$

for any $x \in \mathbb{R}^n$.

Proof of lemma 5.4.1: The formula for the operator Λ^α in $\mathbb{R}^n$

$$\Lambda^\alpha f(x) = k_{\alpha,n}\int_{\mathbb{R}^n}\frac{f(x)-f(y)}{|x-y|^{n+\alpha}}dy$$

and $0 < \alpha < 2$, allows us to find

$$2f(x)\Lambda^\alpha f(x) - \Lambda^\alpha(f^2)(x) = k_{\alpha,n}\int_{\mathbb{R}^n}\frac{(f(y)-f(x))^2}{|x-y|^{n+\alpha}}dy.$$

We consider $f(x) > 0$, being the case $f(x) < 0$ analogous. Let Ω, Ω^1 and Ω^2 be the sets

$$\begin{aligned}\Omega &= \{y \in \mathbb{R} : |x-y| < \Delta\},\\ \Omega^1 &= \{y \in \Omega : f(x)-f(y) \geq f(x)/2\},\\ \Omega^2 &= \{y \in \Omega : f(x)-f(y) < f(x)/2\} = \{y \in \Omega : f(y) > f(x)/2\}.\end{aligned}$$

Then

$$2f(x)\Lambda^\alpha f(x) - \Lambda^\alpha(f^2)(x) \geq k_{\alpha,n}\int_{\Omega}\frac{(f(y)-f(x))^2}{|x-y|^{n+\alpha}}dy \geq k_{\alpha,n}\frac{f(x)^2}{4\Delta^{n+\alpha}}|\Omega^1|. \tag{5.15}$$

On the other hand

$$||f||^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n}|f(y)|^p dy \geq \frac{f(x)^p}{2^p}|\Omega^2|,$$

therefore

$$2f(x)\Lambda^\alpha f(x) - \Lambda^\alpha(f^2)(x) \geq k_{\alpha,n}\frac{f(x)^2}{4\Delta^{n+\alpha}}(|\Omega| - |\Omega^2|) \geq k_{\alpha,n}\frac{f(x)^2}{4\Delta^{n+\alpha}}(c_n\Delta^n - \frac{2^p||f||^p_{L^p(\mathbb{R}^n)}}{f(x)^p}),$$

where $c_n = 2\pi^{n/2}/(n\Gamma(n/2))$. By choosing

$$\Delta^n = \frac{(n+\alpha)2^p||f||^p_{L^p(\mathbb{R}^n)}}{\alpha c_n f(x)^p}$$

we obtain the desired inequality. ▪

Remark 5.4.2 *Inequality* (5.14) *allows us to get easily the following Gagliardo-Nirenberg-Sobolev estimate:*

$$||f||^{2+\frac{\alpha p}{n}}_{L^{2+\frac{\alpha p}{n}}} \leq 2C(\alpha,p,n)||f||^{\frac{\alpha p}{n}}_{L^p(\mathbb{R}^n)}||\Lambda^{\frac{\alpha}{2}}f||^2_{L^2(\mathbb{R}^n)}$$

for $0 < \alpha < 2$ and $0 < p < \infty$.

Chapter 6

Self-Similar Solutions for a Transport Equation With Non-Local Flux.

In this chapter we shall construct self-similar solutions of the transport equation

$$\theta_t + R\theta \cdot \nabla\theta = 0 \quad \text{on} \quad \mathbb{R}^N \times \mathbb{R}^+, \tag{6.1}$$

$$\theta(x,0) = \theta_0(x), \tag{6.2}$$

where $\theta : \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}$, $N \geq 2$, $R\theta = (R_1\theta\,,\,...\,,\,R_N\theta)$ and $R_i\theta$ are the Riesz transform of θ in the i-th direction, i.e.

$$R_i\theta(x) = \Gamma\left(\frac{N+1}{2}\right)\pi^{-\frac{N+1}{2}} P.V.\int_{\mathbb{R}^N} \frac{x_i - y_i}{|x-y|^{N+1}}\theta(y)dy, \quad 1 \leq i \leq N. \tag{6.3}$$

The equation (6.1) was studied in [2] and the authors showed blow-up in finite time for all positive initial data. For a simple proof of the formation of singularities with radial initial data see [38] and for the viscous case see [46].

The technique used in this chapter to construct self-similar solutions of the form

$$\theta(x,t) = Nk(N)\Big(\Big(1 - \Big(\frac{|x|}{t}\Big)^2\Big)_+\Big)^{\frac{1}{2}} \quad \in C^{\frac{1}{2}}(\mathbb{R}^N), \tag{6.4}$$

is based on a result of [43] where the author shows that the function, $\theta(x,1)$ is such that $\Lambda\theta(x,1) = N$ in the unit ball (see section (6.1)).

In section (6.2) we will construct self-similar solutions of the equation

$$\theta_t + (\theta H\theta)_x = 0 \quad \text{in}\ \mathbb{R} \times \mathbb{R}^+, \tag{6.5}$$

$$\theta(x,0) = \theta_0(x), \tag{6.6}$$

which was studied in the chapter 2. Although the same techniques and lemma 4.3.2 can be applied to construct these solutions, we shall use the results of the chapter 2.

Next we shall comment briefly the notation: the spaces $W^{k,p}$ are the classical Sobolev space (k derivatives in L^p).

6.1 Riesz transport equation.

6.1.1 Self-Similar Solutions.

From the scaling invariance of equation (6.1), $\theta(x,t) \to \theta(\lambda x, \lambda t)$, with $\lambda > 0$, we will consider a self-similar function with the following form

$$\theta(x,t) = \Phi(x/t) = \Phi(\xi), \tag{6.7}$$

where $\xi = x/t$. The equalities

$$\begin{aligned} \partial_t\theta(x,t) &= \partial_t\Phi(x/t) = -\frac{\xi}{t}\nabla\Phi(\xi) \\ R\theta(x,t) &= R\Phi(\xi) \\ \nabla\theta(x,t) &= \nabla\big(\Phi(x/t)\big) = \frac{1}{t}\nabla\Phi(\xi) \end{aligned}$$

yields, from equation (6.1),

$$\nabla\Phi(\xi)\cdot\big(R\Phi(\xi)-\xi\big) = 0. \tag{6.8}$$

Now we shall show the existence of a solution of equation (6.8) by means of the following lemma.

Lemma 6.1.1 *The function*

$$v(\xi) = Nk(N)\big((1-|\xi|^2)_+\big)^{\frac{1}{2}} \quad \in C^{\frac{1}{2}}(\mathbb{R}^N), \tag{6.9}$$

where $k(N) = \Gamma(N/2)\left(2^{1/2}\Gamma(3/2)\Gamma((2N+1)/2)\right)^{-1}$ *and* f_+ *is the positive part of the function* f*, satisfies the equalities:*

$$Rv(\xi) = \xi \quad \textit{if } |\xi| < 1,$$

and

$$\nabla v(\xi) = 0 \quad \textit{if } |\xi| > 1.$$

Proof of lemma 6.9: From [43] we know that $v(\xi)$ satisfies the following properties:

1. $\Lambda v(\xi) = N$ if $|\xi| < 1$.
2. $\Lambda v(\xi) \in L^1(R^N)$.
3. Λv is radial.

Since

$$Rv = -\nabla\big(\Lambda^{-1}v\big) \equiv \nabla\Psi, \tag{6.10}$$

$$\nabla\cdot Rv = \Lambda v, \tag{6.11}$$

we have that $\Delta\Psi = \Lambda v$ and therefore Ψ is a radial function with $\Delta\Psi(\xi) = N$ if $|\xi| < 1$. This implies the following expression for Ψ,

$$\Psi(\xi) = \frac{|\xi|^2}{2} + a_0 \quad \text{if } |\xi| < 1,$$

where a_0 is constant. By using (6.10) we obtain

$$Rv(\xi) = \frac{\xi}{|\xi|}\frac{\partial}{\partial|\xi|}\Psi(\xi) = \xi \qquad \text{if } |\xi| < 1. \tag{6.12}$$

▪

Thus, the function

$$\theta(x,t) = Nk(N)\Big(\Big(1 - \Big(\frac{|x|}{t}\Big)^2\Big)_+\Big)^{\frac{1}{2}} \quad \in C^{\frac{1}{2}}(\mathbb{R}^N) \tag{6.13}$$

is a self-similar solution of equation (6.1) (almost everywhere).

Remark 6.1.2 *We can check that the functions $\theta^T(x,t) = -\theta(x,(T-t))$, with $0 < T < \infty$ are solutions with an initial data $\theta^T(x,0) = -\theta(x,T)$ which collapse in a point in finite time T.*

6.1.2 Formal weak solutions and non-uniqueness.

In this section we shall check that the previous functions are solutions of the equation (6.1) in the weak sense that we define below. In addition we will be able to show non-uniqueness.

Definition 6.1.3 *The function $\theta(x,t)$ is a weak solutions of equation (6.1) if*

$$\theta \in C\big((0,T), L^q(\mathbb{R}^{\mathbb{N}})\big) \cap C\big((0,T), W^{1,p}(\mathbb{R}^{\mathbb{N}})\big) \quad \textit{with } 1 \le q < \infty \textit{ and } 1 \le p < 2,$$

$$\partial_t\theta \in W^{1,p}(\mathbb{R}^N) \quad \forall t > 0 \quad \textit{with } 1 \le p < 2,$$

$$\int_{\mathbb{R}^N} \big(\theta(x,t)_t + R\theta(x,t)\cdot\nabla\theta(x,t)\big)\phi(x,t)\,dx = 0 \quad \forall t \in (0,T)\forall\phi \in C_c^\infty((0,T)\times\mathbb{R}^N),$$

and

$$\lim_{t\to 0^+}\theta(x,t) = \theta_0(x) \quad \textit{in } L^q(\mathbb{R}^N)$$

Theorem 6.1.4 *(Non-Uniqueness). The function*

$$\Phi(x,t) = Nk(N)\Big(\Big(1 - \Big(\frac{|x|}{t}\Big)^2\Big)_+\Big)^{\frac{1}{2}}$$

is a global weak solution of the equation (6.1) in the sense of the definition (6.1.3) with zero initial data.

Proof: Given a function $\phi(x,t) \in C_c^\infty((0,\infty)\times\mathbb{R}^N)$ and a fixed time $t>0$ we have that

$$\int_{\mathbb{R}^N} \big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx$$

$$= \int_{|x|<t} \big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx$$

$$= \int_{\varepsilon<|x|<t} \big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx$$

$$+ \int_{|x|<\varepsilon} \big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx,$$

where $0<\varepsilon<t$. The second term on the right hand side of the last expression is equal to zero. In addition we have the following identities,

$$\nabla\Phi(x,t) = \begin{cases} 0 & |x|>t \\ Nk(N)\frac{\frac{x}{t^2}}{(1-\frac{|x|^2}{t^2})^{1/2}} & |x|<t \end{cases} \tag{6.14}$$

$$\partial_t\Phi(x,t) = \begin{cases} 0 & |x|>t \\ Nk(N)\frac{-\frac{|x|^2}{t^3}}{(1-\frac{|x|^2}{t^2})^{1/2}} & |x|<t \end{cases} \tag{6.15}$$

Thus, if $p<2$, we obtain,

$$||\nabla\Phi(\cdot,t)||_{L^p(\mathbb{R}^N)} = Nk(N)\left(\int_{|x|<t} \frac{\frac{|x|^p}{t^{2p}}}{(1-\frac{|x|^2}{t^2})^{p/2}}dx\right)^{\frac{1}{p}} \tag{6.16}$$

$$= Nk(N)t^{\frac{N}{p}-p}\left(\int_{|x|<t} \frac{|x|^p}{(1-|x|^2)^{p/2}}dx\right)^{\frac{1}{p}}$$

$$= C(N)t^{\frac{N}{p}-p}\left(\int_0^1 \frac{r^{N-1+p}}{(1-r^2)^{p/2}}dr\right)^{\frac{1}{p}} = C(N,p)t^{\frac{N}{p}-p},$$

$$||\partial_t\Phi(\cdot,t)||_{L^1(\mathbb{R}^N)} = Nk(N)\left(\int_{|x|<t} \frac{\frac{|x|^{2p}}{t^{3p}}}{(1-\frac{|x|^2}{t^2})^{p/2}}dx\right)^{\frac{1}{p}}$$

$$= Nk(N)t^{\frac{N}{p}-p}\left(\int_{|x|<1} \frac{|x|^{2p}}{(1-|x|^2)^{p/2}}dx\right)^{\frac{1}{p}}$$

$$= C(N)t^{\frac{N}{p}-p}\left(\int_0^1 \frac{r^{2p+N-1}}{(1-r^2)^{p/2}}dr\right)^{\frac{1}{p}} = C(N,p)t^{\frac{N}{p}-p}$$

Therefore,

$$\int_{\mathbb{R}^N} \partial_t \Phi(x,t)\phi(x,t)dx \leq ||\partial_t \Phi(\cdot,t)||_{L^1(\mathbb{R}^{\mathbb{N}})}||\phi(\cdot,t)||_{L^\infty(\mathbb{R}^N)} = C(N,1)t^{N-1}||\phi(\cdot,t)||_{L^\infty(\mathbb{R}^N)}$$

and

$$\int_{\mathbb{R}^N} R\Phi(x,t)\cdot\nabla\Phi(x,t)\phi(x,t)dx \leq ||R\Phi(\cdot,t)||_{L^q(\mathbb{R}^N)}||\nabla\Phi(\cdot,t)||_{L^p(\mathbb{R}^N)}||\phi(\cdot,t)||_{L^\infty(\mathbb{R}^N)} \leq$$

$$C(N,q,p)t^{N-p}||\phi(\cdot,t)||_{L^\infty(\mathbb{R}^N)},$$

where $1 < p < 2$, $1/p + 1/q = 1$ and $t > 0$. Then, we can conclude that

$$\lim_{\varepsilon\to t}\int_{\varepsilon<|x|<t}\big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx = 0 \quad \forall t>0,$$

and

$$\int_{\mathbb{R}^N}\big(\Phi(x,t)_t + R\Phi(x,t)\cdot\nabla\Phi(x,t)\big)\phi(x,t)\,dx = 0 \quad \forall t>0 \quad \forall\phi\in C_c^\infty((0,\infty)\times\mathbb{R}^{\mathbb{N}}).$$

In addition is easy to check that

$$\lim_{t\to 0^+}\Phi(x,t) = 0 \quad \text{in } L^p(\mathbb{R}^{\mathbb{N}}) \text{ with } 1\leq p<\infty.$$

6.2 Self-similar solutions for the equation 2.1

In this section we will construct self-similar solutions for the equation

$$\theta_t + (\theta H\theta)_x = 0 \quad \text{in } \mathbb{R}\times\mathbb{R}^+, \tag{6.17}$$

$$\theta(x,0) = \theta_0(x), \tag{6.18}$$

where $\theta:\mathbb{R}\to\mathbb{R}$ and $H\theta$ is the Hilbert transform of the function θ.

We sketch the main features of the equation (6.17) in the following lemma.

Lemma 6.2.1 *Let $Z(w,t)$ be a complex function, $Z: M\to\mathbb{C}$, where $M = \{w = x+iy : y>0\}$ such that*

$$Z_t + ZZ_w = 0 \quad \text{on } M \tag{6.19}$$

$$Z(w,0) = R\theta_0(x,y) - iP\theta_0(x,y). \tag{6.20}$$

$P\theta(x,y)$ is the convolution with the Poisson kernel and $R\theta(x,y)$ is the convolution with the harmonic conjugate Poisson kernel, i.e.

$$P\theta(x,y) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{y}{y^2+(x-s)^2}\theta(s)ds \qquad R\theta(x,y) = \frac{1}{\pi}\int_{\mathbb{R}}\frac{x-s}{y^2+(x-s)^2}\theta(s)ds. \tag{6.21}$$

Then, if $Z(w,t)$ is analytic on M and vanishing at infinity

$$\theta(x,t) = -\Im\left(Z(w,t)|_{y=0}\right) \tag{6.22}$$

is a solution of equation (6.17), with $\theta(x,0)=\theta_0(x)$ on the points where θ and $H\theta$ are differentiable.

Proof of lemma 6.2.1: If $Z(w,t)$ satisfies the statements of lemma (6.2.1) we can write it in the following way

$$Z(w,t) = R\theta(x,y;t)) - iP\theta(x,y;t) \tag{6.23}$$

where $\theta(x,t) = -\Im\left(Z(w,t)|_{y=0}\right)$. In addition we know that

$$Z_t + ZZ_x = 0 \quad \text{on } M,$$

and from (6.23) follows $Z(w,t)|_{y=0} = H\theta(x,t) - i\theta(x,t)$. By taking the limit $y \to 0^+$ in equation (6.23) we have the desired result. ▪

Next we shall use the previous lemma to prove the following theorem.

Theorem 6.2.2 *The function*

$$\theta(x,t) = \frac{1}{\sqrt{t\pi}}\left(\left(1 - \frac{\pi x^2}{4t}\right)_+\right)^{\frac{1}{2}} \in C^{\frac{1}{2}}(\mathbb{R})$$

is a self-similar solution (at least in a weak sense) of equation (6.17) with the initial data $\theta_0 = \delta_0$*, where* δ_0 *is the Dirac Delta.*

Proof: By the lemma (6.2.1), we have to study the solutions of the equation,

$$Z_t + ZZ_w = 0 \quad \text{on } M \tag{6.24}$$

$$Z(w,0) = \frac{1}{\pi}\frac{x}{x^2+y^2} - i\frac{1}{\pi}\frac{y}{x^2+y^2}, \tag{6.25}$$

A standard argument yields that the solution is constant along the following complex trajectories

$$X^1(x,y,t) = \frac{1}{\pi}\frac{x}{x^2+y^2}t + x \tag{6.26}$$

$$X^2(x,y,t) = -\frac{1}{\pi}\frac{y}{x^2+y^2}t + y. \tag{6.27}$$

Thus

$$Z(X^1(x,y,t), X^2(x,y,t), t) = Z_0(x,y),$$

and one can check that the solution, $Z(w,t)$, satisfies the requirements of the lemma (6.2.1). In addition

$$\theta(X^1,t) = -\Im\left(Z(X^1,X^2,t)|_{X^2=0}\right) = P\theta_0(x,y,t)|_{X^2=0} = \frac{y}{\pi t}|_{X^2=0}.$$

The function

$$y = \sqrt{\pi t}\left(\left(1 - \frac{\pi x^2}{t}\right)_+\right)^{\frac{1}{2}}$$

satisfies equation (6.27) with $X^2 = 0$ and by the equation (6.26) we have that

$$X^1 = \begin{cases} 2x & |x| < \sqrt{t/\pi} \\ \frac{t}{\pi x} + x & |x| > \sqrt{t/\pi} \end{cases}$$

Furthermore we can conclude that,

$$\theta(x,t) = \frac{1}{\sqrt{t\pi}} \left(\left(1 - \frac{\pi x^2}{4t} \right)_+ \right)^{\frac{1}{2}}.$$

Remark 6.2.3 *This solution was obtained in [5] and by using the techniques of section (6.1). In fact they constructed self-similar solutions for the equation*

$$u_t + \Lambda^\alpha u u_x = 0 \quad on\ \mathbb{R} \times \mathbb{R}^+, \tag{6.28}$$

$$u(x,0) = H(x), \tag{6.29}$$

where $H(x)$ is the Heaviside function and $0 < \alpha \leq 2$.

Chapter 7

Infinite Energy Solutions for an Incompressible flow in a Porous Medium

7.1 Introduction

In this chapter we will study solutions with infinite energy for a incompressible flow in a Porous Medium. We are concerned with the following system

$$\begin{aligned} \rho_t + v \cdot \nabla \rho &= 0 && (7.1) \\ v &= -\left(\nabla p + \rho(0,1)\right), && (7.2) \\ \nabla \cdot v &= 0, && (7.3) \end{aligned}$$

For a divergence free velocity field there exists a stream function ψ, in the two dimensional case, such that

$$v = \nabla^{\perp}\psi = (-\partial_{x_2}\psi, \partial_{x_1}\psi).$$

We shall choose a stream function of the form

$$\psi(x_1, x_2, t) = -x_2 f(x_1, t).$$

Taking the rotational over the equation (7.2) we obtain

$$\nabla \times v = \partial_{x_2} v_1 - \partial_{x_1} v_2 = -\Delta\psi = x_2 \partial^2_{x_1} f = \partial_{x_1}\rho.$$

Therefore, the function ρ has the following expression

$$\rho(x_1, x_2, t) = x_2 \partial_{x_1} f(x_1, t) + \hat{g}(x_2, t)$$

where we choose

$$\hat{g}(x_2, t) = \frac{1}{\pi} x_2 \int_0^t ||\partial_{x_1} f(\tau)||^2_{L^2(-\pi,\pi)} d\tau.$$

Substituting the expression in (7.1), without diffusion in the two dimensional case, we obtain

$$\partial_t f_x = -\partial_t g - f f_{xx} + (f_x)^2 + g f_x, \tag{7.4}$$

(here and in the following section, we denote with subscript the derivatives with respect to x) where g satisfies

$$g(t) = \frac{1}{\pi}\int_0^t ||f_x(\tau)||^2_{L^2(-\pi,\pi)} d\tau \tag{7.5}$$

and we define f as

$$f(x,t) = \int_{-\pi}^{x} f_x(x',t)dx'. \tag{7.6}$$

Notice that the difference between this system and the one obtained in [30] is that this has the property of conserving the mean zero value of the initial data. Indeed, integrating equation (7.4) over the interval $[-\pi,\pi)$ and imposing periodical condition on f_x, we have

$$\partial_t \int_{-\pi}^{\pi} f_x(x',t)dx' = (-f_x(\pi,t) + g(t))\int_{-\pi}^{\pi} f_x(x')dx'.$$

Therefore, if f_x is a solution of equation (7.4) and

$$\int_{-\pi}^{\pi} f_x(x',0)dx' = 0,$$

we obtain

$$\int_{-\pi}^{\pi} f_x(x',t)dx' = 0, \qquad \forall t > 0.$$

7.2 Existence.

In this section we prove the following theorem.

Theorem 7.2.1 *Let $\varphi_0 \in H^2(\mathbb{T})$ with mean zero value and*

$$M_0 = \max_{x\in\mathbb{T}} \varphi_0(x).$$

Then, there exist a solution $f(x,t)$ of the equation (7.4) with initial datum $f_x(x,0) = \varphi_0(x)$ such that

$$f_x \in C([0,T), H^2(\mathbb{T})),$$

with $T = M_0^{-1}$.

In order to prove this theorem, first, we add other diffusion term to the equation (7.4). Thus, we have the following system

$$\begin{cases} \partial_t f_x = -\partial_t g - f f_{xx} + (f_x)^2 + g f_x + \nu(||f_{xx}||^2_{L^2} + g^2) f_{xxx}, \\ f_x(x,0) = \varphi_0(x), \end{cases} \tag{7.7}$$

where g satisfies (7.5) and $\nu > 0$. In the next lemma, we prove the global existence of the solutions of (7.7).

Lemma 7.2.2 *Let $\varphi_0 \in H^3(\mathbb{T})$ with mean zero value and $\nu > 0$. Then, there exists a function $f(x,t)$ defined by (7.6) where f_x is solution of the equation (7.7) such that*

$$f_x \in C([0,\infty), H^3(\mathbb{T})).$$

Proof of lemma 7.2.2: We note that if φ_0 has mean zero value then f has mean zero value. Multiplying the equation (7.7) by f_x and integrating over the interval $[-\pi,\pi)$, we obtain

$$\frac{1}{2}\frac{d}{dt}||f_x||_{L^2}^2 = \frac{3}{2}\int_{-\pi}^{\pi}(f_x)^3 + g||f_x||_{L^2}^2 - \nu(||f_{xx}||_{L^2}^2 + g^2)||f_{xx}||_{L^2}^2.$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}||f_x||_{L^2}^2 \leq \frac{3}{2}||f_x||_{L^\infty}||f_x||_{L^2}^2 + g||f_x||_{L^2}^2 - \nu(||f_{xx}||_{L^2}^2 + g^2)||f_{xx}||_{L^2}^2.$$

Using Gagliardo-Nirenberg and Poincaré inequalities, we have

$$||f_x||_{L^\infty} \leq C||f_x||_{L^2}^{\frac{1}{2}}||f_{xx}||_{L^2}^{\frac{1}{2}} \leq C||f_{xx}||_{L^2}.$$

Hence,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||f_x||_{L^2}^2 \leq\ & C(||f_{xx}||_{L^2}^2||f_x||_{L^2} + g||f_x||_{L^2}||f_{xx}||_{L^2}) \\ & -\nu||f_{xx}||_{L^2}^4 - \nu g^2||f_{xx}||_{L^2}^2.\end{aligned}$$

And using the Young's inequality we obtain

$$\frac{1}{2}\frac{d}{dt}||f_x||_{L^2}^2 + \frac{\nu}{4}(||f_{xx}||_{L^2}^4 + g^2||f_{xx}||_{L^2}^2) \leq C_\nu||f_x||_{L^2}^2.$$

Therefore,

$$||f_x||_{L^2} \leq ||\varphi_0||_{L^2}\exp(C_\nu t), \tag{7.8}$$

and

$$\int_0^T ||f_{xx}||_{L^2}^4 dt \leq C(||\varphi_0||_{L^2}, \nu, T), \qquad \forall T > 0. \tag{7.9}$$

Taking a derivative over equation (7.7), multiplying by f_{xx} and integrating over the interval $[-\pi,\pi)$ yield

$$\frac{1}{2}\frac{d}{dt}||f_{xx}||_{L^2}^2 + \nu(||f_{xx}||_{L^2}^2 + g^2)||f_{xxx}||_{L^2}^2 = \frac{3}{2}\int_{-\pi}^{\pi} f_x(f_{xx})^2 dx + g||f_{xx}||_{L^2}^2.$$

Hence,

$$\frac{1}{2}\frac{d}{dt}||f_{xx}||_{L^2}^2 \leq \frac{3}{2}||f_x||_{L^\infty}||f_{xx}||_{L^2}^2 + g||f_{xx}||_{L^2}^2 \leq C(||f_{xx}||_{L^2}^3 + g||f_{xx}||_{L^2}^2).$$

Integrating between 0 and T we obtain

$$\begin{aligned}||f_{xx}||_{L^2}^2 &\leq ||\varphi_{0,x}||_{L^2}^2 + C(\int_0^T ||f_{xx}||_{L^2}^3 dt + \int_0^T g||f_{xx}||_{L^2}^2 dt) \\ &\leq ||\varphi_{0,xx}||_{L^2}^2 + C(T)\int_0^T ||f_{xx}||_{L^2}^4,\end{aligned}$$

and we can conclude that $||f_{xx}||_{L^2}$ is bounded for all $T < \infty$.

Finally, we estimate $||f_{xxx}||_{L^2}$ and $||f_{xxxx}||_{L^2}$. Taking two derivatives on the equation (7.7), multiplying by f_{xxx} and integrating over the interval $[-\pi, \pi]$ yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||f_{xxx}||^2_{L^2} + \nu(||f_{xx}||^2_{L^2} + g^2)||f_{xxxx}||^2_{L^2} &= \frac{1}{2}\int_{-\pi}^{\pi} f_x(f_{xxx})^2 + g||f_{xxx}||^2_{L^2}\\ &\leq C(T)||f_{xxx}||^2_{L^2}.\end{aligned}$$

Applying Gronwall inequality we have that $||f_{xxx}||_{L^2}$ is bounded for all $T < \infty$. We obtain that $||f_{xxxx}||_{L^2}$ is bounded in a similar form and we conclude the proof. ▪

In order to prove theorem 7.2.1 we show some estimates, independent of ν, of the global solutions of the equation (7.7) by the lemma 7.2.2, which allows us to obtain the local existence for the equation (7.4). Next, we prove the following lemma.

Lemma 7.2.3 *Let f_x be a global solution of the equation (7.7) with initial data φ_0 and $M(t)$ the maximum of f_x. Then*

$$M(t) + g(t) \leq \frac{M(0)}{1 - M(0)t}. \tag{7.10}$$

Proof of lemma 7.2.3 In this proof we use the techniques of article [30] for the control of the maximum of the solution of equation (7.4). Let $x_M(t)$ denote the point where f_x reaches the maximum, then

$$\begin{aligned}(M+g)_t &= M^2 + gM + \nu(||f_{xx}||^2 + g^2)f_{xx}(x_M(t), t)\\ &\leq M^2 + gM \leq (M+g)^2.\end{aligned}$$

Since $g(0) = 0$ we obtain

$$M + g \leq \frac{M(0)}{1 - M(0)t},$$

and the proof is finished. ▪

Proof of theorem 7.2.1. Multiplying the equation (7.7) by f_x and integrating over the interval $[-\pi, \pi)$ yields

$$\frac{1}{2}\frac{d}{dt}||f_x||^2_{L^2} \leq (M+g)||f_x||^2_{L^2}.$$

Applying lemma 7.2.3 and Gronwall inequality we have that $||f_x||_{L^2}$ is bounded for all $T < M(0)^{-1}$. In a similar way, we can obtain that $||f_{xx}||_{L^2}$ and $||f_{xxx}||_{L^2}$ are bounded for all $T < M(0)^{-1}$ independently of ν .

To finish the proof, we consider a sequence of solutions $\{f^\epsilon\}_{\epsilon>0}$ of the equations

$$\begin{cases} \partial_t f^\varepsilon_x = -g^\varepsilon_t - f^\varepsilon f^\varepsilon_{xx} + (f^\varepsilon_x)^2 + g^\varepsilon f^\varepsilon_x + \varepsilon(||f^\varepsilon_{xx}||_{L^2} + (g^\varepsilon)^2)f^\varepsilon_{xx}\\ f^\varepsilon_x(x,0) = \varphi^\varepsilon_0(x), \end{cases} \tag{7.11}$$

where $\{\varphi^\varepsilon_0\}_\varepsilon$ is a sequence in $H^3(\mathbb{T})$ such that

$$\lim_{\varepsilon\to 0}\varphi^\varepsilon_0 = \varphi_0 \in H^2,$$

$$M^\varepsilon(0) \equiv \max_{x\in\mathbb{T}} f^\varepsilon_{x0}(x) \le M(0) \equiv \max_{x\in\mathbb{T}} \varphi_0(x),$$

and

$$||\varphi_0^\varepsilon||_{H^2(\mathbb{T})} \le ||\varphi_0||_{H^2(\mathbb{T})}.$$

The above estimates provide that

$$||f_x^\varepsilon||_{H^2(\mathbb{T})} \quad \text{is bounded} \quad \forall T < M(0)^{-1} \quad \text{uniformly in } \varepsilon.$$

Using the Rellich's theorem we conclude the proof taking the limit $\varepsilon \to 0$. ∎

Remark 7.2.4 *We shall consider the equation*

$$\partial_t f_x = -\partial_t g - f f_{xx} + (f_x)^2 + g f_x - \nu \Lambda^\alpha f_x, \tag{7.12}$$

which is (7.4) with an extra dissipating term. For this system we have a local existence result similar to theorem 7.2.1. Moreover, we can construct solutions that blow-up in finite time for $\alpha = 1,\, 2$ *and* $\nu \ge 0$ *(see below) which show the existence of singularities for PM with infinite energy.*

7.3 Blow up.

Next, we show that there exist a particular solution of the equation (7.12), with $\nu \ge 0$ and $\alpha = 1,\, 2$, which blows up in finite time.

We consider the following ansatz of (7.4) and (7.12)

$$f_x(x,t) = r(t)\cos(x), \tag{7.13}$$

then r satisfies

$$\frac{dr(t)}{dt} = r(t)\int_0^t r^2(\tau)d\tau + \nu r. \tag{7.14}$$

We define the function

$$\beta(t) = \int_0^t r^2(\tau)d\tau. \tag{7.15}$$

Multiplying the equation (7.14) by $r(t)$ we have that β satisfies

$$\frac{d^2\beta(t)}{dt^2} = 2\beta(t)\frac{d\beta(t)}{dt} + 2\nu\frac{d\beta}{dt}.$$

Integrating with respect to the variable t yields

$$\beta'(t) - \beta'(0) = \beta^2(t) - \beta^2(0) + 2\nu(\beta - \beta(0)).$$

Since $\beta(0) = 0$ and $\beta'(0) = r^2(0)$ we obtain

$$\frac{\beta'(t)}{r(0)^2 + 2\nu\beta + \beta(t)^2} = 1.$$

If we choose $r(0)^2 > \nu^2$ it follows

$$\beta(t) = \sqrt{r(0)^2 - \nu^2} \tan\big(\sqrt{r(0)^2 - \nu^2}\, t + \arctan(\frac{\nu}{\sqrt{r(0)^2 - \nu^2}})\big) - \nu.$$

Therefore, the function

$$f_x(x,t) = r(t)\cos(x), \tag{7.16}$$

is a solution of equation (7.4) and (7.12) which blows up at time

$$t = \frac{\big(\frac{\pi}{2} - \arctan(\frac{\nu}{\sqrt{r(0)^2-\nu^2}})\big)}{\sqrt{r(0)^2 - \nu^2}}.$$

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