Discrete gauge symmetries in string theory

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Abstract

In particle physics model building discrete symmetries are often invoked for phenomenological reasons, like forbidding unwanted or dangerous couplings, or explaining and reproducing the matrices of masses and mixings of fermions in flavour physics. While these discrete symmetries do their phenomenological job, they are poorly motivated at a fundamental level in the context of field theory. Moreover, there are arguments suggesting that discrete global symmetries are expected to be violated in consistent theories of quantum gravity; therefore, any exact symmetry of the theory should be gauge. This motivates the study of discrete gauge symmetries in string theory, which is one of the most promising candidates for a complete theory of quantum gravity. The goal of this thesis is to study different mechanisms that give rise to discrete gauge symmetries in string theory, specially in compactifications to four dimensions, although we also explore several generalisations, like compactifications to a number of dimensions different from four, or supercritical string theories.

Resumen

A la hora de construir modelos de física de partículas a menudo se emplean simetrías discretas por razones fenomenológicas, como prohibir acoplos indeseados o peligrosos, o explicar y reproducir las matrices de masa y de mezcla de los fermiones en física del sabor. Si bien cumplen con su labor fenomenológica, desde el punto de vista de teoría de campos, estas simetrías discretas no están suficientemente motivadas a un nivel fundamental. Además, hay indicios de que en una teoría cuántica que incluye también gravedad las simetrías globales (continuas o discretas) son violadas, por lo que las simetrías exactas de la naturaleza deberían ser simetrías locales o gauge. Esto motiva el estudio de las simetrías gauge, en especial las discretas, en Teoría de Cuerdas, la cual es una de las propuestas más prometedoras para una teoría completa de gravedad cuántica. El objetivo de esta tesis es estudiar diversos mecanismos que llevan a la aparición de simentrías gauge discretas en teoría de cuerdas, principalmente en compactificaciones a cuatro dimensiones, aunque también se exploran diversas generalizaciones, como compactificaciones a dimensiones diferentes de cuatro, o en teoría de cuerdas supercríticas.

Esta tesis está basada en los siguientes artículos:

1. *Discrete Gauge Symmetries in D-brane Models*, M.B.-G., Á. M. Uranga, L. E. Ibáñez and P. Soler, JHEP 1112 (2011) 113. [arXiv: hep-th/1106.4169]

2. *Non-Abelian discrete gauge symmetries in 4d string models*, M.B.-G., P. G. Cámara, F. Marchesano, D. Regalado and Á. M. Uranga, JHEP 1209 (2012) 059. [arXiv: hep-th/1206.2383]

3. *Z_p charged branes in flux compactifications*, M.B.-G., P. G. Cámara, F. Marchesano and Á. M. Uranga, JHEP 1304 (2013) 183. [arXiv:hep-th/1211.5317]

4. *Discrete gauge symmetries from (closed string) tachyon condensation*, M.B.-G., M. Montero, A. Retolaza and Á. M. Uranga, JHEP 1311 (2013) 144. [arXiv:hep-th/1305.6788]

5. *Antisymmetric tensor Z_p gauge symmetries in field theory and string theory*, M.B.-G., G. Ramírez and Á. M. Uranga, JHEP 1401 (2014) 059. [arXiv:hep-th/1310.5582]

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Motivación

El Modelo Estándar de física de partículas es uno de los mayores éxitos de la física del siglo pasado. Describe las partículas elementales y sus interacciones fuertes y electrodébiles en un amplio rango de energías.

El Modelo Estándar es una teoría cuántica de campos basada en un grupo gauge

$$G_{SM} = SU(3) \times SU(2)_L \times U(1)_Y, \tag{1.1}$$

donde $SU(3)$ describe las interacciones fuertes a través de la Cromodinámica Cuántica, y $SU(2)_L \times U(1)_Y$ describe las interacciones electrodébiles.

Los campos de materia forman tres generaciones de quarks y leptones, descritos por espinores de Weyl de dos componentes, con estructurea electrodébil dada por

$$Q_L^i = \begin{pmatrix} U_L^i \\ D_L^i \end{pmatrix}, \quad U_R^i, \quad D_R^i, \quad L^i = \begin{pmatrix} \nu_L^i \\ E_L^i \end{pmatrix}, \quad E_R^i; \quad i = 1,2,3. \tag{1.2}$$

Además, los quarks en Q_L transforman como tripletes de color, mientras que U_R, D_R transforman como tripletes conjugados. los números cuánticos gauge de los fermiones del Modelo Estándar se muestran en la tabla 1.1

Field	$SU(3)$	$SU(2)_L$	$U(1)_Y$
Q_L^i	$\mathbf{3}$	$\mathbf{2}$	1/6
U_R^i	$\bar{\mathbf{3}}$	$\mathbf{1}$	2/3
D_R^i	$\bar{\mathbf{3}}$	$\mathbf{1}$	-1/3
L^i	$\mathbf{1}$	$\mathbf{2}$	-1/2
E_R^i	$\mathbf{1}$	$\mathbf{1}$	-1
H	$\mathbf{1}$	$\mathbf{2}$	1/2

Table 1.1: Números cuánticos gauge de los quarks, leptones y escalar de Higgs del Modelo Estándar.

El modelo también incluye un campo escalar complejo H que transforma como un doblete de $SU(2)_L$, llamado el campo de Higgs, cuyo valor esperado en el vacío rompe el

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Motivation

The Standard Model of particle physics is one of the greatest successes of last century physics. It describes elementary particles and their strong and electroweak interactions in a large range of energies.

The Standard Model is a quantum field theory based on a gauge group

$$G_{SM} = SU(3) \times SU(2)_L \times U(1)_Y, \tag{2.1}$$

with $SU(3)$ describing strong interactions via Quantum Chromodynamics, and $SU(2)_L \times U(1)_Y$ describing electroweak interactions.

The matter fields form three generations of quarks and leptons, described as Weyl 2-component spinors, with the electroweak structure

$$Q_L^i = \begin{pmatrix} U_L^i \\ D_L^i \end{pmatrix}, \quad U_R^i, \quad D_R^i, \quad L^i = \begin{pmatrix} \nu_L^i \\ E_L^i \end{pmatrix}, \quad E_R^i; \quad i = 1, 2, 3. \tag{2.2}$$

In addition, quarks in Q_L transform as colour triplets, while U_R, D_R transform as conjugate triplets. The gauge quantum numbers of the Standard Model fermions are shown in table 2.1

Field	$SU(3)$	$SU(2)_L$	$U(1)_Y$
Q_L^i	$\mathbf{3}$	$\mathbf{2}$	1/6
U_R^i	$\bar{\mathbf{3}}$	$\mathbf{1}$	2/3
D_R^i	$\bar{\mathbf{3}}$	$\mathbf{1}$	-1/3
L^i	$\mathbf{1}$	$\mathbf{2}$	-1/2
E_R^i	$\mathbf{1}$	$\mathbf{1}$	-1
H	$\mathbf{1}$	$\mathbf{2}$	1/2

Table 2.1: Gauge quantum numbers of Standard Model quarks, leptons and the Higgs scalars.

The model also includes a complex scalar field H transforming as a $SU(2)_L$ doublet, called the Higgs field, whose vacuum expectation value breaks the gauge group

$$\langle H \rangle = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix} \quad \Rightarrow \quad G_{SM} \to SU(3) \times U(1)_{EM}. \tag{2.3}$$

This vacuum expectation value generates masses for the $W^{\pm}$ and Z vector bosons and, at the same time, produces the quark and lepton masses through the Yukawa couplings

$$\mathcal{L}_{\text{Yuk}} = Y_U^{ij}\overline{Q}_l^i U_R^j H^* + Y_D^{ij}\overline{Q}_L^i D_R^j H + Y_L^{ij}\overline{L}^i E_R^j H + \text{h.c.} \tag{2.4}$$

These interactions are the most general consistent with gauge invariance and renormalizability, and accidentally are invariant under the global symmetries related to the baryon number B and the three generation lepton numbers L_i.

The electric charge is given by

$$Q_{EM} = T_3 + Q_Y, \tag{2.5}$$

where $T_3 = \text{diag}\left(\frac{1}{2}, -\frac{1}{2}\right)$ is an $SU(2)_L$ generator.

The validity of the Standard Model has been confirmed to great precision in a lot of experiments, with one of the most recent successes being the discovery of a scalar particle at the LHC which is compatible with the Standard Model Higgs Particle.

Despite all of its success, the Standard Model is far from being complete.

- **Gravity.** One of the most clear hints indicating that the Standard Model cannot be the ultimate theory is that it does not include gravitational interactions, due to the difficulties to reconcile them with Quantum Mechanics. Gravity implies that the Standard Model should be considered as an effective theory, with a cutoff at most at the Planck scale

$$M_p = \frac{1}{\sqrt{8\pi} G_N^{1/2}} = 1.2 \times 10^{19} \text{ GeV}, \tag{2.6}$$

  where G_N is Newton's constant.

- **Neutrino masses.** Another evidence which indicates that the Standard Model is not complete is the masses of the neutrinos. It is a well established experimental fact that neutrinos oscillate (see e.g. [1, 2]). The idea of neutrino oscillations was developed back in the 1960's [3, 4, 5]; in particular, [4, 5] showed that flavour oscillations arise if neutrinos are massive and mixed.

  On the other hand, in the Standard Model described above, neutrinos are massless. So one needs to extend the Standard Model to incorporate neutrino masses. The usual extra ingredients are right-handed neutrinos and/or a high scale of lepton number violation.

  One possibility is to produce neutrino masses from Dirac mass terms, in analogy with quark and charged lepton masses, with Yukawa couplings $Y_\nu^{ij}\overline{L}^i \nu_R^j H^*$. This mechanism preserves lepton number, but does not explain the smallness of observed neutrino masses, or their essentially left-handed character.

  If one accepts violation of lepton number at a relatively high scale M, one can get effective Majorana mass terms for left-handed neutrinos at lower scales. In particular, the dimension five Weinberg operator

$$\frac{h_{ij}}{M}\nu_L^i \nu_L^j HH + \text{h.c.}, \tag{2.7}$$

produces left-handed neutrino Majorana masses of order

$$M_L^{ij} = \frac{h_{ij}\langle H\rangle^2}{M} \tag{2.8}$$

upon electroweak symmetry breaking. For $h_{ij} \simeq 1$, and $M \simeq 10^{14} - 10^{15}$ GeV, we get neutrino masses of order $M_L \simeq 0.01 - 0.1$ eV, consistent with neutrino oscillation experiments.

An implementation of the above idea is the seesaw mechanism [6, 7, 8]. The theory includes right-handed neutrinos with a large Majorana mass M_R, and Yukawa couplings for left-handed neutrinos, leading to Dirac masses m_D. Diagonalization of the neutrino mass matrix leads to a predominantly left-handed neutrino eigenstate with very small mass $\simeq m_D^2/M_R$, and a predominantly right-handed eigenstate with very large mass $\simeq M_R$.

- **Electroweak hierarchy problem.** The electroweak hierarchy problem is the question of the origin of the hierarchy between the electroweak and the Plank scale. The electroweak scale is fixed by the Standard Model Higgs vev v, which is related to the squared mass of the Higgs μ^2 by $\mu^2 = 2\lambda v^2$, with λ being the quartic coupling of the Higgs. The squared mass of the Higgs, which according to the experiments is $\mu \simeq 126$ GeV, receives huge quadratically divergent quantum corrections that take it to the cutoff scale of the theory,

$$\delta\mu^2 \simeq \frac{\alpha}{4\pi}\Lambda^2_{cutoff}. \tag{2.9}$$

For a theory with a finite physical cutoff, this correction is finite and generally huge. One could choose the naked value of μ^2 such that the renormalised value is of order of $(126 \text{ GeV})^2$, but for $\Lambda_{cutoff} = M_p$ this implies a fine-tuning with a precision of 10^{-34}.

Therefore, we need to solve the question of maintaing a light Higgs with $\mu \simeq 126$ GeV to trigger electroweak symmetry breaking, given that its natural value would be an ultraviolet scale of the theory.

A possible solution to the problem is low energy supersymmetry. The main idea is to extend the Standard Model to include one extra partner for each SM field, related by $\mathcal{N} = 1$ supersymmetry, a symmetry relating bosons and fermions (see e.g. [9] for a review).The interactions of these new particles precisely cancel the quadratically divergent corrections to the Higgs mass, stabilising the hierarchy. In addition, this approach is perturbative and predictive, and it is possible to test it at the LHC. Furthermore, supersymmetry is a key ingredient in string theory compactifications.

The minimal extension to the Standard Model that realizes supersymmetry is the Minimal Supersymmetric Standard Model (MSSM), see e.g. chapter 2 of [10] for a review. The spectrum includes a supersymmetric partner for all the Standard Model fields, but it needs two Higgs doublets instead of one. The most general superpotential consistent with gauge invariance and leading to dimension four operators is given in 2.11.

One can easily construct SUSY versions of grand unified theories (see below for a short review on grand unified theories) with a unification scale of $M_{GUT} \sim 10^{16}$ GeV. If in addition to gauge coupling unification, one also wants realistic mass spectra of the Higgs and SUSY particles, this can be done in the framework of the MSSM coupled to the minimal supergravity model which relates many unknown quantities of the MSSM in terms of a few basic parameters at the unification scale; this is the so-called Constrained MSSM (CMSSM), see [11] for an early review and [12] for the current status of the CMSSM after the first run of the LHC.

There are also non-minimal supersymmetric extensions, the simplest one being the Next-to-Minimal Supersymmetric Standard Model (NMSSM), which adds an additional singlet chiral superfield to the MSSM (see [13] for a review).

Other solutions that have been proposed to solve the problem are: Dynamical generation by strongly coupled gauge sectors, e.g. technicolor [14, 15, 16, 17, 18]; lowering the fundamental gravity scale, e.g. with extra dimensions, see [19, 20, 9] for reviews; or even an anthropic approach [21, 22, 23].

- **Grand Unified Theories.** Grand unified theories (GUTs) propose that there is an underlying gauge group G_{GUT}, assumed to be simple in most cases, which is broken spontaneously down to the Standard Model group at very high energies (see [24, 25, 26] for some introduction to the subject). The SM gauge factors are unified into a single gauge force, and the SM matter fields are unified into multiplets of G_{GUT}.

  A consequence of the unification of the SM gauge groups into a simple group is the unification of gauge coupling constants into a single one. Evidence for such unification comes from the evolution of the SM gauge couplings to high energy with the renormalization group equations. Assuming no further relevant degrees of freedom at intermediate scales, the three couplings tend to join around 10^{15} GeV into a single coupling α_{GUT}; if one only considers the matter content of the Standard Model the joining is only in qualitative agreement with the experiment [27], but it is enhanced sharply if one considers supersymmetric versions.

  The different GUTs are classified in terms of the chosen G_{GUT}, which has to be at least of rank four, contain the SM group and admit complex representations to accommodate a chiral fermion spectrum. Usual choices for G_{GUT} are $SU(5)$ [28, 29], $SO(10)$ [30, 31], or E_6 [32].

- **Dark matter** It is an established fact that there is dark matter present in the universe (see e.g. [33] for recent precise measurements of the Planck satellite). It represents the 23% of the universe, while only 5% consists of ordinary atomic matter and the remainder 72% is dark energy. Identifying the nature of this dark matter is one of the open problems of modern physics.

  A particular candidate for dark matter, which is being searched for very actively, is a Weakly Interacting Massive Particle, or WIMP. In addition to the feeling of gravity, these particles undergo weak interactions, but do not take part in strong or electromagnetic interactions. The expected WIMP mass ranges from 1 GeV to 10 TeV. To make an idea of how weakly interacting the WIMPs are, in [34] the authors estimated the interaction rate of WIMPs with a standard 70 kg human body; from the billions of WIMPs that pass through every second, the average number of interactions ranges between approximately one per month for 60 GeV WIMPs and one per minute for 10 GeV WIMPs.

The relic density of WIMPs is given by

$$\Omega_\chi h^2 \sim (3 \times 10^{-26}\text{cm}^3/\text{sec})/\langle\sigma \text{v}\rangle_{\text{ann}}, \tag{2.10}$$

where Ω_χ is the fractional contribution of WIMPs to the energy density of the Universe. An annihilation cross section $\langle\sigma v\rangle_{\text{ann}}$ of weak interaction strength automatically gives the right answer, near the value measured by WMAP [35]. This coincidence is known as the “WIMP miracle”, and it is the reason for WIMPs to be taken so seriously as dark matter candidates.

One of the best WIMP candidates is motivated by supersymmetry: the lightest neutralino in the MSSM and its extensions [36]. However, other WIMP candidates arise in a variety of theories beyond the Standard Model, like Kaluza-Klein excitations of Standard Model fields which appear in models of universal extra dimensions [37], a light scalar particle with a mass between 1 and 100 MeV [38, 39], heavy fourth generation neutrinos [40, 41], etc. (see [42, 43] for reviews).

In addition to WIMPs, there many other candidates for dark matter. For instance, axions [44], axinos [45, 46, 47], sterile neutrinos [48], Kaluza-Klein gravitons [49, 50, 51]... (see [52] for a review of different dark matter candidates).

Most of the models consider only one species. However, there are also studies that attempt to extend the idea of Standard Model flavour to the dark sector, and consider scenarios where dark matter is composed of multiples species [53, 54, 55, 56] or unflavoured dark matter which is stabilised by requiring dark matter interactions with the Standard Model fields to obey flavour conservation [57, 58] (see [59] for a more detailed review on flavoured dark matter).

For more detailed reviews on the subject and the status of direct search experiments see e.g. [60, 61].

- **Others.** Cosmological constant problem [62, 63, 64], strong CP problem and axions [65, 66, 67, 68]...

One of the most promising candidates for a quantum theory that also includes gravity is string theory (see [69, 70, 71, 72, 10] for some classical and more recent texts on string theory), since besides being mathematically consistent, the graviton arises in a natural way. In particular, there is a branch of string theory, called string phenomenology (see [10] for a recent review of the field), that attempts to construct realistic models of particle physics, including the different issues mentioned above. Within the framework of string theory one can easily construct models with the gauge group and spectrum of the Standard Model [73], Yukawa couplings [74, 75, 76, 77], neutrino masses [78, 79], supersymmetric extensions of the Standard Model [80, 81, 82, 83], grand unified theories [84, 85, 86, 87, 88, 89, 90, 91], etc.

However, let us focus in the low energy theories for a bit longer. Many theories of physics beyond the Standard Model make use of discrete symmetries for phenomenological reasons. For instance:

- Discrete symmetries are used in flavour physics (see [92] for a review in the subject) to solve (at least partially) what is called the flavour puzzle, i.e. the origin of fermions masses and mixings, their hierarchies, and the difference of the flavour parameters in the neutrino sector.

Let us consider the issue of neutrino masses and mixings. Before 2012, experimental results determining the mixing angles and mass square differences [93, 94, 95, 96] suggested that some special mixing patterns like Tri-Bimaximal (TB) [97, 98, 99, 100, 101] or Golden Ratio (GR) [102, 103, 104, 105] were good first approximations[1]. The specific mixing patterns in leading order can be reproduced by a broken discrete flavour symmetry based, for instance, on the A_4 group for TB (for a review see [106] and references therein) and the A_5 group for GR (see e.g. [103, 104, 105]). This changed after 2012, when new experimental results showed that $\theta_{13} \approx 0.15$ and $\sin^2\theta_{23}$ is no longer maximal [107, 108, 109], which rule out the previous models at leading order. However, this does not mean that discrete symmetries are no longer useful; for instance, they can be combined with CP symmetry to obtain patterns which are in agreement with the experimental results [110].

For more general implementations of discrete non-Abelian symmetries in the lepton sector see e.g. [111, 112, 113, 114, 115] or [116, 117, 118, 119, 120] for reviews.

- In supersymmetric extensions of the Standard Model, discrete symmetries are invoked to get rid of unwanted operators. Let us focus on the Minimal Supersymmetric Standard Model (MSSM), where discrete symmetries are unavoidable in order to explain the observed proton stability.

  Indeed, a crucial difference between the non-supersymmetric Standard Model and the MSSM is that in the latter the most general dimension four effective Lagrangian respects neither baryon- nor lepton-number conservation. The most general superpotential consistent with gauge invariance and leading to dimension four operators has the structure

$$\begin{aligned} W_{\rm MSSM} = & \, Y_U^{ij} Q_i U_j H_u + Y_D^{ij} Q_i D_j H_d + Y_L^{ij} L_i E_j H_d + \mu H_u H_d \\ & + \lambda^{ijk} U_i D_j D_k + \lambda^{ijk'} Q_i D_j L_k + \lambda^{ijk''} L_i L_j E_k + \mu_R^i L_i H_u, \end{aligned} \tag{2.11}$$

  where we use a standard notation for quark, lepton and Higgs superfields.

  The first line in (2.11) contains the usual Yukawa couplings and the μ-term, and respects both baryon- and lepton-number; the UDD terms in the second line violate baryon-number, whereas the rest violate lepton-number in one unit. If all the terms in the second line were present and unsuppressed, the proton would decay with a lifetime of minutes.

  The simplest solution to avoid this problem is to assume some discrete symmetry. For example, R-parity, which forbids all the couplings in the second line; or baryon triality B_3, first introduced in [121], which forbids the first term in the second line, which violates baryon-number, while allows for the lepton-number violating terms to be present. Although in this case the proton is still unstable and can decay, the proton lifetime is long enough to be consistent with the experimental bounds.

  It was mentioned earlier while discussing dark matter candidates that one of the best WIMP candidates was motivated by supersymmetry. Let us elaborate this point a bit more. For R-parity to forbid all the couplings in the second line in

[1] These models assume $\theta_{13} = 0$ and $\sin^2\theta_{23} = 1/2$ at leading order.

(2.11) the charges of the fields under it are such that Standard Model fields are even and their supersymmetric partners are odd. Since R-charge has to be conserved, a supersymmetric particle has to decay necessarily into another supersymmetric particle. This decay continues until no other supersymmetric particles with lighter mass exist in the spectrum. This particle is called the 'lightest supersymmetric particle' (LSP), and it is the one thought to be the dark matter candidate. Depending on the model, the LSP can be a neutralino, a gravitino, etc.

All the discrete symmetries mentioned above are introduced because of their phenomenological relevance but, although they do their phenomenological job, their fundamental origin remains obscure. Therefore, one should investigate the nature of discrete symmetries at a fundamental level.

There are diverse arguments suggesting that global symmetries, either continuous or discrete, are violated by quantum gravitational effects, and hence cannot exist in any consistent quantum theory including gravity (see [122, 123, 124] for early viewpoints, and e.g. [125, 126] and references therein for more recent discussions).

In one hand, there are the usual arguments of black hole evaporation. Let us consider a black hole with a given amount of charge under a global symmetry. When it evaporates, it emits the same number of particles with a given charge and its opposite, in such a way that the total emitted charge is zero. This process violates the global charge in as many units as the black hole had initially.

This does not happen if the symmetry is a gauge symmetry. One way to see this is to recall that an object charged under a gauge symmetry generates a gauge field; in the case of black hole evaporation, this gauge field favours the emission of particles with charge of a given sign, in such a way that the total emitted charge is not zero, and is of the same sign that the charge the black hole had initially. Equivalently, the fact that there is a gauge field means that the charge can be measured at infinite distance from the black hole; this, together with the existence of charge conservation equations, implies that the total gauge charge must be conserved, and therefore, it cannot be violated.

On the other hand, there are microscopic arguments in string theory [122]. The basic idea is that a continuous symmetry will lead to a worldsheet current which should serve as a vertex operator for a gauge boson.

Therefore, any symmetry in a consistent theory of quantum gravity, no matter whether it is continuous or discrete, should have a gauge nature and be respected by such corrections [127, 128, 129]. So one should study discrete gauge symmetries both in field theory (see e.g. [130, 121, 131] for the Abelian case and [132, 133, 134, 135, 136, 137] for the non-Abelian case) and in string theory (see e.g. [138, 139, 140, 141, 142, 143, 144, 145, 146, 147, 148, 149]).

2.1 Outline of the thesis

The goal of this thesis is to show how discrete gauge symmetries can be realized in the context of string theory, and their embedding into the broken continuous gauge symmetries.

In chapter 3 we present a review of discrete gauge symmetries in 4d field theory. In section 3.1 we consider the case of an Abelian discrete symmetry, what the anomaly cancellation conditions are and, since it will be useful in some of the string theory constructions in later chapters, we review the possible anomaly-free discrete symmetries of the MSSM. In section 3.2 we analyze one of the possible ways to study the realization of discrete symmetries in string theory, the BF couplings. Section 3.3 introduces a different but equivalent way to study discrete symmetries, by considering them as arising from isometries of the moduli space of the scalars in the theory, and is particularly useful to study non-Abelian discrete symmetries.

In chapter 4 we apply the BF formulation of section 3.2 to models with intersecting D-branes. First we review how this models are constructed, and how the BF couplings arise in them. Subsequently we study two different classes of intersecting D-brane models (sections 4.3 and 4.4); in both cases, after presenting the general characteristics of the model, we consider specific cases giving rise to both the Standard Model and its supersymmetric extension given by the MSSM, and analyze what the possible discrete gauge symmetries that can be realized are. We also study D-brane instantons and how they preserve the discrete gauge symmetries in section 4.5. Finally, we comment on $\mathbb{Z}_2$ discrete gauge symmetries associated to the discrete K-theory charge cancellation conditions, and suggest the intriguing possibility of identifying it with R-parity in explicit constructions in section 4.6.

In chapter 5 we study the discrete gauge symmetries that arise in compacfications with torsion cycles, making use of the ideas developed in section 3.3, both in the Abelian (section 5.1) and non-Abelian case (sections 5.2 and 5.3 for a concrete example and the general case, respectively).

Chapter 6 presents another type of string theory compactifications where the BF formulation is useful to study the possible discrete gauge symmetries: string theory compactifications in the presence of background fluxes. After a general review of how the presence of background fluxes leads to 4d BF couplings (section 6.1), we study the appearance of discrete gauge symmetries when there is only one kind of flux, both for the Abelian (section 6.2) and the non-Abelian case (section 6.3). Section 6.4 studies the cases where several kind of background fluxes are present at the same time, and how one can deal with the inconsistencies that arise when doing things naively. In section 6.5 we turn to $\mathbb{Z}_p$ valued domain walls, and their relation to string duality symmetries relating vacua with different flux vacua. Section 6.6 provides a brief discussion of flux-induced discrete gauge symmetries arising from Kaluza-Klein $U(1)$'s in compactifications with isometries.

Chapter 7 provides an analysis of discrete gauge symmetries coming from discrete isometries of the compactification space rather than isometries of the moduli space of the scalars of the theory. In section 7.1 we consider the case of compactifications on a twisted torus to illustrate the main ideas; these ideas, together with the formalism of section 3.3, are applied to a more realistic construction based on magnetized D-branes in section sec:magnetized. We also study non-perturbative instanton effects, and how they manage to preserve the non-Abelian discrete gauge symmetry in section 7.4.

In chapter 8 we generalize the analysis to arbitrary number of dimensions (instead of considering just compactifications to 4 dimensions like in the previous chapters) and discrete symmetries arising as remnants of broken continuous gauge symmetries carried by general antisymmetric tensor fields, rather that by standard 1-forms. In sections 8.1 and

8.2 we present a generalization of the BF formulation of section 3.2 to D dimensions and arbitrary p-forms. In the rest of the chapter we consider some string theory constructions that make use of these ideas; in particular, section 8.3 provides an analysis of discrete symmetries arising in compactifications to 6d in the presence of background fluxes, similar to the 4d analysis in section 6.2, while section 8.4 considers the case of compactifications with torsion cycles and section 8.5 studies the possibility of realising non-Abelian discrete symmetries.

Chapter 9 presents a completely different, albeit related, topic. In the previous chapters it was easy to see what the continuous group that was broken to the discrete subgroup was; however, this embedding is not always possible. In this chapter we show that it is possible if we consider extra space-time dimensions, using the supercritical string theories that are explained in section A; the critical theory is obtained by closed string tachyon condensation. After a simple warmup exercise (section 9.2), we present two different ways to realize the embedding of discrete Abelian gauge symmetries, dubbed "quenched rotations" (section 9.3) and "quenched translations" (section 9.4). Finally, in section 9.5 we apply the previous ideas to the non-Abelian case.

Finally, in chapter 10 we present the conclusion and some final remarks.

3

Discrete gauge symmetries in 4d field theory

3.1 4d Abelian discrete gauge symmetries in field theory

3.1.1 Generalities

In field theory, discrete gauge symmetries appear when a continuous discrete gauge symmetry is broken to a discrete subgroup by some mechanism. The most simple example is a $U(1)$ gauge symmetry Higgsed down to a $\mathbb{Z}_k$ subgroup when a scalar field with charge k under the $U(1)$ acquires a non-zero vev.

The basic action for a $\mathbb{Z}_k$ discrete gauge symmetry is

$$\int_{4d} \frac{1}{2} \left(d\phi + kA_1\right) \wedge *_{4d} \left(d\phi + kA_1\right) = \int_{4d} \frac{1}{2} \left|d\phi + kA_1\right|^2 . \tag{3.1}$$

where the gauge field A_1 is normalised such that the minimum electric charge is 1, and ϕ is a scalar field (considered to be the phase of the Higgs field with charge k under the $U(1)$) with a periodic identification

$$\phi \simeq \phi + 1. \tag{3.2}$$

This action is invariant under a gauge transformation of the form

$$A_1 \longrightarrow A_1 - d\lambda, \tag{3.3a}$$

$$\phi \longrightarrow \phi + k\lambda. \tag{3.3b}$$

The action (3.1) can be written in terms of the four dimensional dual fields to ϕ and A_1, given by B_2 and V_1, respectively, as [126]

$$\int_{4d} \frac{1}{2} \left(dV_1 + kB_2\right) \wedge *_{4d} \left(dV_1 + kB_2\right) = \int_{4d} \frac{1}{2} \left|dV_1 + kB_2\right|^2 \tag{3.4}$$

where

$$dB_2 = *_{4d} d\phi, \tag{3.5a}$$

$$dV_1 = *_{4d} dA_1. \tag{3.5b}$$

This action is invariant under a gauge transformation of the form

$$B_2 \longrightarrow B_2 + d\Lambda_1, \tag{3.6a}$$

$$V_1 \longrightarrow V_1 - k\Lambda_1. \tag{3.6b}$$

It is also invariant under

$$V_1 \longrightarrow V_1 + d\Xi_0 \tag{3.7}$$

for any 0-form Ξ_0. This dual description makes manifest an *emergent* $\mathbb{Z}_k$ discrete gauge symmetry[1].

Theories with discrete gauge symmetries have sets of (possibly massive) charged particle states, which often provide a practical way to identify the discrete gauge symmetry in a given theory. In the case of the above $\mathbb{Z}_k$ theory, charge n particles with worldline C are described as insertions of the line operators

$$\mathcal{O}_{\text{particle}} \sim e^{2\pi i n \int_C A_1}. \tag{3.8}$$

Their charge is conserved modulo k, since there are gauge invariant 'instanton' vertices which create/annihilate sets of particles with total charge k,

$$e^{-2\pi i\phi} e^{2\pi i n \int_C A_1} = e^{-2\pi i\phi}\, \mathcal{O}_{\text{particle(s)}}, \tag{3.9}$$

describing an insertion $e^{-2\pi i\phi}$ at a point P, out of which a charge k set of particles emerges along a worldline C (i.e. $\partial C = P$). In many realizations, the above operators are induced in the 4d action by effects $e^{S_{\text{inst}}}$, on-perturbative in some suitable coupling, with $S_{\text{inst}} = 2\pi i\phi + \ldots$ linear in the gauged axion. The overall $U(1)$ charge of $\mathcal{O}_{\text{particle(s)}}$ is thus compensated by shifts of S_{inst}.

In addition, the theory contains $\mathbb{Z}_k$-charged strings, described as the insertion of operators along a worldsheet Σ

$$\mathcal{O}_{\text{string}} \sim e^{-2\pi i p \int_\Sigma B_2}, \tag{3.10}$$

where B_2 is the 2-form dual to ϕ and p is defined modulo k. String charge is also conserved modulo k, since there are operators describing strings of total charge k on worldsheets Σ ending along a junction line L ($\partial\Sigma = L$)

$$e^{-2\pi i \int_L V_1} e^{2\pi i k \int_\Sigma B_2}. \tag{3.11}$$

A charge n particle defined by (3.8) suffers a $\mathbb{Z}_k$ discrete gauge transformation, $n \to n + np$, when moved around the charge p string (3.10), i.e. its wavefunction picks up an Aharonov-Bohm phase $e^{2\pi i \frac{np}{k}}$. Conversely, a charge p string looped around a charge n particle picks up a phase $e^{2\pi i \frac{np}{k}}$. In more abstract terms, the amplitude associated to a charge p string on a worldsheet Σ and a charge n particle on a worldline C contains an Aharonov-Bohm phase

$$\exp\left[2\pi i \frac{np}{k} L(\Sigma, C)\right], \tag{3.12}$$

where $L(\Sigma, C)$ is the so-called linking number of Σ and C.

These ingredients have a natural yet more involved generalization to the non-Abelian case [132, 133, 134, 135, 136] (see [137] for a review).

Whether the $\mathbb{Z}_k$-charged particles and strings play the role of fundamental objects or the associated codimension-2 topological defect depends on what theory we are considering, the 'electric' theory (3.1) or the 'magnetic' theory (3.4). In the 'electric' case, charged

[1] An important point, not manifest in the examples in [126], is that the emergent discrete symmetry may differ from the original one. This is illustrated explicitly in the next section.

particles are the fundamental objects and charged strings are the topological defects, while in the 'magnetic' case, charged strings are the fundamental objects and charged particles are the topological defects.

Since in 4d the most natural thing is to think of particles as the fundamental objects, in the rest of this thesis we will consider this to be the case and make all the 4d analysis taking this fact into account.

3.1.2 Discrete gauge symmetries and anomaly cancellation

Given a gauge theory with a continuous gauge group, anomaly cancellation conditions strongly constrain the chiral fermion content of the theory. As shown in [130] there are analogous constraint for discrete gauge symmetries. In this section we will review the main results of [130].

Consider a $\mathbb{Z}_N$ discrete gauge symmetry like the one in the previous section. Under this symmetry, the light fermions ψ_i in the theory transform as

$$\psi_i \quad \longrightarrow \quad \exp\left(2\pi i \frac{q_i}{N}\right)\psi_i. \tag{3.13}$$

Let us consider now the original $U(1)$ theory from which the $\mathbb{Z}_N$ arises after the symmetry breaking; the original $U(1)$ charges of the fermions are necessarily of the form

$$q_i + m_i N, \qquad q_i, m_i \in \mathbb{Z}\,. \tag{3.14}$$

These are not the only fermions that contribute to the anomalies. One should also take into account the contribution to the anomaly coming from fermions which became massive when the symmetry broke. Let us denote by Q_j the charge of such fermions[2]. There are two types of masses fermions may acquire:

1. Pairs of different Weyl fermions Ψ_1^j and Ψ_2^j combine to get a mass. In this case, their charges must obey

$$Q_1^j + Q_2^j = p_j, N, \qquad p_j \in \mathbb{Z}\,. \tag{3.15}$$

2. One fermion (singlet with respect to the rest of the gauge interactions) acquires a Majorana mass. The charge of such a fermion χ has to obey

$$Q_\chi^j = \frac{1}{2}p_j', \qquad p_j' \in \mathbb{Z}\,. \tag{3.16}$$

Cancellation of the $U(1)^3$ anomaly requires that the $\mathbb{Z}_N$ charges q_j of the massless fermions in the theory should verify

$$\sum_i (q_i)^3 = mN + \eta n\frac{N^3}{8}, \qquad m, n \in \mathbb{Z}, \tag{3.17}$$

where $\eta = 1, 0$ for N =even, odd.

[2] It is convenient to scale the $U(1)$ coupling so that the charges q_i and Q_j are integers

Cancellation of the $\mathbb{Z}_N$-graviton-graviton anomaly requires that the q_i's verify

$$\sum_i q_i = pN + \eta q \frac{N}{2}, \qquad p, q \in \mathbb{Z}, \tag{3.18}$$

where $\eta = 1, 0$ for N =even, odd.

In the case of mixed $\mathbb{Z}_N$ and non-Abelian gauge anomalies (e.g. $\mathbb{Z}_N - SU(M) - SU(M)$) one obtains

$$\sum_i T_i(q_i) = \frac{1}{2} rN, \qquad r \in \mathbb{Z}, \tag{3.19}$$

where T_j is the quadratic $SU(M)$ Casimir corresponding to each given representation, with the normalization being such that the Casimir of an M-plet is $\frac{1}{2}$.

3.1.3 Discrete gauge symmetries in the MSSM

The MSSM and models based on it are common extensions of the standard model, both in field theory and string theory (see e.g. [74, 75, 80, 81, 82, 79, 77, 83] for some examples in string theory). In this section we will review the possible anomaly free discrete gauge symmetries that one can obtain in the MSSM, following [130, 121].

In [121] the possible $\mathbb{Z}_N$ generation independent discrete symmetries of the MSSM where classified in terms of the three generators R, L, A, given in table 3.1, where (Q, U, D, L, E, N_R, H_u, H_d) are the MSSM quark, lepton and Higgs superfields in standard notation.

	Q	U	D	E	L	N_R	H_u	H_d
R	0	-1	1	0	1	-1	1	-1
L	0	0	0	-1	1	1	0	0
A	0	0	-1	-1	0	1	0	1

Table 3.1: Generation independent generators of discrete Z_N gauge symmetries in the MSSM

Defining

$$R_N = e^{i2\pi R/N}, \tag{3.20a}$$
$$L_N = e^{i2\pi L/N}, \tag{3.20b}$$
$$A_N = e^{i2\pi A/N}, \tag{3.20c}$$

a $\mathbb{Z}_N$ generator may be written as

$$g_N = R_N^m \times A_N^n \times L_N^p, \quad m, n, p = 0, 1, \ldots, N-1 \tag{3.21}$$

This is the most general $\mathbb{Z}_N$ symmetry allowing for the presence of all standard Yukawas QUH_u, QDH_d, LEH_d and, in the presence of right-handed neutrinos, LN_RH_u.

Further but equivalent discrete symmetries can be obtained by multiplying by some power of a discrete subgroup of the hypercharge generator $e^{i2\pi(6Y)/N}$, where $6Y$ is used to make hypercharges integer.

As discussed in [130, 121], the mixed $\mathbb{Z}_N \times SU(3)^2$, $\mathbb{Z}_N \times SU(2)^2$ and mixed gravitational anomaly constraints yield

$$\begin{aligned} nN_g &= 0 \mod N & (3.22)\\ (n+p)N_g - nN_d &= 0 \mod N & (3.23)\\ -N_g(5n+p-m)+2nN_D &= \eta\frac{N}{2} \mod N & (3.24)\end{aligned}$$

where N_g is the number of generations, N_D is the number of Higgs sets and $\eta = 0, 1$ for n =odd,even. In the presence of right-handed neutrinos, the mixed gravitational anomaly gets simplified to

$$-4nN_g + 2nN_D = \eta\frac{N}{2} \mod N \tag{3.25}$$

One necessary condition for the above symmetries to be discrete gauge symmetries is anomaly cancellation. It is clear from the previous equations that, in the presence of right-handed neutrinos, all R_N are anomaly free. In particular, R_2, which corresponds to the usual R-parity is anomaly free.

	H_uH_d	UDD	QDL	LLE	LH_u	LLH_uH_u	$QQQL$	$UUDE$
R_2		x	x	x	x			
R_3L_3		x					x	x
L_3			x	x	x	x	x	x
$R_3L_3^2$		x	x	x	x		x	x
$R_2 \times R_3L_3$		x	x	x	x		x	x

Table 3.2: Operators forbidden by the anomaly-free Z_2 and Z_3 symmetries

In the physical case $N_g = 3$, in addition to R_2, which is anomaly free even in the absence of right-handed neutrinos, there are three anomaly-free $\mathbb{Z}_3$: L_3, R_3L_r and $R_3L_3^2$. The symmetry $B_3 = R_3L_3$, usually called baryon triality, was introduced in [121] and it allows for dimension 4 operators violating lepton number, but not violating baryon number, so the proton is sufficiently stable.

As shown in [150], there are additional $\mathbb{Z}_9$ and $\mathbb{Z}_{18}$ anomaly free discrete symmetries involving the A_N generators, as well as several $\mathbb{Z}_6$ involving just R_6 and L_6. However, imposing the purely abelian condition of [121] and the absence of massive fractionally charged states singles out R-parity R_2 and baryon triality B_3.

Table 3.2 displays the phenomenologically interesting couplings allowed or forbidden by these discrete symmetries. The $\mathbb{Z}_6$ obtained by multiplying R_2 and B_3 is called hexality, and forbids all dangerous couplings but allows for a μ-term and the Weinberg operator LLH_uH_u (and hence left-handed and right-handed neutrino Majorana masses).

3.2 Discrete gauge symmetries and the BF formulation

3.2.1 The BF coupling

Recall that the 4d action for a $\mathbb{Z}_k$ gauge discrete symmetry is

$$\mathcal{S} = \int_{4d} \frac{1}{2}\left(d\phi + kA_1\right) \wedge *_{4d}\left(d\phi + kA_1\right). \tag{3.26}$$

It contains the terms

$$\mathcal{S} \supset \frac{k}{2}\int_{4d}(d\phi \wedge *_{4d}A_1 + A_1 \wedge *_{4d}d\phi) = k\int_{4d} A_1 \wedge *_{4d}d\phi, \tag{3.27}$$

where the equality follows from the fact that $\alpha \wedge *\beta = \beta \wedge *\alpha$. If we define $dB_2 = *_{4d}d\phi$, then (3.27) can be rewritten as

$$\mathcal{S} \supset k\int_{4d} A_1 \wedge dB_2. \tag{3.28}$$

Integrating (3.28) by parts, we get

$$\mathcal{S} \supset k\int_{4d} dA_1 \wedge B_2 = k\int_{4d} B_2 \wedge F_2, \tag{3.29}$$

where $F_2 = dA_1$.

Therefore, the order of the $\mathbb{Z}_k$ discrete symmetry is encoded into the coefficient of the BF coupling. This kind of couplings are ubiquitous in string theory. Hence, a way of studying the possible gauge discrete symmetries of some 4d string theory model is to analyze the coefficients of its BF couplings. Concrete examples can be found in chapter 4 for intersecting D-brane models and in chapter 6 for compactifications to 4d in the presence of background fluxes.

3.2.2 Generalization of BF

The previous argument can easily be generalized to the case where we have multiple fields of each type.

Consider a single scalar field ϕ and several 1-form gauge fields A_1^k in 4 dimensions. The action is given by

$$\int_{4d}\sum_{k=1}^{n}\left|d\phi + q_k A_1^k\right|^2. \tag{3.30}$$

This is gauge invariant under

$$A_1^k \longrightarrow A_1^k - d\lambda^k, \tag{3.31a}$$

$$\phi \longrightarrow \phi + \sum_k q_k\lambda^k. \tag{3.31b}$$

The corresponding BF coupling is

$$\int_{4d}\sum_{k=1}^{n} q_k B_2 \wedge F_2^k \tag{3.32}$$

where $dB_2 = *_{4d}d\phi$ and $F_2^k = dA_1^k$.

The remnant discrete gauge symmetry is not manifest by inspection. Naively, it may seem that each $U(1)_k$ factor leaves an unbroken $\mathbb{Z}_{q_k}$. This is however not correct, since the different $U(1)$ factors couple simultaneously to a *single* 2-form field. Indeed, there is only one broken $U(1)$, given by a linear combination of the $U(1)_k$, while an unbroken $U(1)^{n-1}$ remains.

Let $\vec{q} = (q_1, \ldots, q_n)$ be the charge vector of ϕ under the $U(1)_k$, $k = 1, \ldots, n$ and let Q_k be the generator of $U(1)_k$ for $k = 1, \ldots, n$. The unbroken $U(1)^{n-1}$ is generated by Q_a, $a = 1, \ldots, n$, which are given by linear combinations

$$A_1 = \sum_{k=1}^{n} c_a^k Q_k, \qquad c_a^k \in \mathbb{Z}, \tag{3.33}$$

with $\vec{c}_a \cdot \vec{p} = 0$. The only broken $U(1)$ linear combination is the one orthogonal to all the massless $U(1)$'s, namely it is given by

$$Q = \sum_{k=1}^{n} \frac{q_k}{q} Q_k, \tag{3.34}$$

where the factor $q = \gcd(q_k)$ is included in order to keep the normalization such that minimal charge is 1. Its BF couplings are

$$\sum_{k=1}^{n} \frac{(q_k)^2}{q} B_2 \wedge F_2. \tag{3.35}$$

The symmetry is therefore $\mathbb{Z}_r$ with $r = \sum_k \frac{(q_k)^2}{q}$. The $\mathbb{Z}_r$ structure follows from the structure of charged particle states, which are created by operators

$$\exp(-2\pi i \phi) \exp\left(i \int_C \sum_k q_k A_1^k \right). \tag{3.36}$$

This violates Q_k charge conservation in q_k units, and hence $Q = \sum_k \frac{q_k}{q} Q_k$ in $r = \sum_k \frac{(q_k)^2}{q}$ units.

The other possibility is to consider a single 1-form field made massive by coupling to several scalar fields ϕ^k in 4 dimensions. The action is given by

$$\int_{4d} \sum_{k=1}^{m} \left| d\phi^k + q_k A_1 \right|^2. \tag{3.37}$$

This is gauge invariant under

$$A_1 \longrightarrow A_1 - d\lambda, \tag{3.38a}$$
$$\phi^k \longrightarrow \phi^k + q_k \lambda. \tag{3.38b}$$

The corresponding BF coupling is

$$\int_{4d} \sum_{k=1}^{m} q_k B_2^k \wedge F_2, \tag{3.39}$$

where $dB_2^k = *_{4d} d\phi^k$ and $F_2 = dA_1$.

The potential A_1 actually eats only one linear combination of the fields ϕ^k, while the orthogonal linear combinations remain as massless scalar fields. Denoting $q = \gcd(q_j)$, the massive gauge symmetry leaves a remnant $\mathbb{Z}_q$ gauge symmetry[3]. This follows from the structure of $\mathbb{Z}_q$-charged particles, whose number can be violated by operators

$$\exp\left(-2\pi i\phi^k\right)\exp\left(2\pi i\int_C q_k A_1\right). \tag{3.40}$$

Each such vertex creates q_k particles, so by Bezout's lemma, there exists a set of vertices which (minimally) violates their number in q units, making the particles $\mathbb{Z}_q$-valued.

In terms of the dual fields, the action for the theory dual to (3.30) is

$$\int_{4d}\sum_{k=1}^{m}\left|dV_1^k + q_k B_2\right|^2, \tag{3.41}$$

where $dB_2 = *_{4d}d\phi$ and $dV_1^k = *_{4d}dA_1^k$, and it is gauge invariant under

$$B_2 \longrightarrow B_2 - d\Lambda_1, \tag{3.42a}$$
$$V_1^k \longrightarrow V_1^k + q_k\Lambda_1, \tag{3.42b}$$

in addition to the gauge transformations

$$V_1^k \longrightarrow V_1^k + d\Xi_0^k. \tag{3.43}$$

On the other hand, the action for the theory dual to (3.37) is

$$\int_{4d}\left|dV_1 + \sum_{k=1}^{n} q_k B_2^k\right|^2, \tag{3.44}$$

were $dB_2^k = *_{4d}d\phi^k$ and $dV_1 = *_{4d}dA_1$, and it is gauge invariant under

$$B_2^k \longrightarrow B_2^k - d\Lambda_1^k, \tag{3.45a}$$
$$V_1 \longrightarrow V_1 + \sum_k q_k\Lambda_1^k, \tag{3.45b}$$

in addition to the gauge transformation

$$V_1 \longrightarrow V_1 + d\Xi_0. \tag{3.46}$$

Applying the same analysis to (3.41) and (3.44), it is easy to see that the emergent gauge symmetry associated to the original $\mathbb{Z}_q$ is a $\mathbb{Z}_r$, and vice versa. Hence, the discrete part of the emergent gauge group in the dual description is different from the original one; this is a novel feature as compared with the system in [126] and in section 3.1.

The fact that the original $\mathbb{Z}_q$ (resp. $\mathbb{Z}_r$) and the emergent $\mathbb{Z}_r$ (resp. $\mathbb{Z}_q$) gauge symmetries are different is not in contradiction with charge quantization of the dual charged objects, i.e. the $\mathbb{Z}_q$ (resp. $\mathbb{Z}_r$) strings and the $\mathbb{Z}_r$ (resp. Z_q) particles, because of the presence of additional charges under the additional continuous gauge symmetries in the system.

[3] Each of the scalars ϕ^k wants to be eaten by the potential A_1 and break the $U(1)$ into a $\mathbb{Z}_{q_k}$ subgroup; therefore, the $\mathbb{Z}_q$ discrete symmetry can be seen as a compromise solution, since it is the biggest subgroup all of them can break the $U(1)$ into.

To finish the analysis, let us consider the case with several fields of each kind. The 4d action is given by

$$\int_{4d} \sum_{l=1}^{m} \left| d\phi^l + \sum_{k=1}^{n} p_{lk} A_1^k \right|^2 . \tag{3.47}$$

This is gauge invariant under

$$A_1^k \longrightarrow A_1^k - d\lambda^k, \tag{3.48a}$$

$$\phi^l \longrightarrow \phi^l + \sum_{k=1}^{n} p_{lk} \lambda^k. \tag{3.48b}$$

The corresponding BF couplings are

$$\int_{4d} \sum_{l=1}^{m} \sum_{k=1}^{n} p_{kl} B_2^l \wedge F_2^k, \tag{3.49}$$

where $dB_2^l = *_{4d} d\phi^l$ and $F_2^k = dA_1^k$.

Let Q_k be the generator of $U(1)_k$, $k = 1, \dots, n$, and consider a linear combination $Q = \sum_k c^k Q_k$ such that $\gcd(c_k) = 1$. Then the BF couplings for the field strength F_2 of the $U(1)$ gauge symmetry generated by Q are

$$\int_{4d} \sum_{l=1}^{m} \left(\sum_{k=1}^{n} p_{lk} c^k \right) B_2^l \wedge F_2 = \int_{4d} \sum_{l=1}^{m} q_l B_2^l \wedge F_2 \tag{3.50}$$

where $q_l = \sum_k p_{lk} c^k$. Hence, the $U(1)$ generated by Q is broken to a Z_q subgroup where $q = \gcd(q_l)$.

In general, it is not immediate to identify the surviving discrete gauge symmetry in the case (3.47). In the literature this is usually done by 'trial and error', by scanning through different integral linear combinations of the $U(1)$ generators and looking at the greatest common divisor of their couplings, as explained above. This is the approach taken in [140], and used in chapter 4. A more systematic way of describing the surviving gauge symmetries is presented in section 3.3. A generalization of this analysis to an arbitrary number of dimensions can be found in chapter 8.

3.3 Discrete gauge symmetries from isometries of the moduli space of scalars

In this chapter we will consider a different approach to the analysis of 4d discrete gauge symmetries in field theory. Instead of looking at the BF couplings of the theory, we will show that the possible discrete gauge symmetries can be inferred from the isometries of the moduli space of the scalars in the theory charged under the continuous groups that will be broken to a discrete subgroup.

3.3.1 One Abelian discrete gauge symmetry

Let us recall that the basic action for a $\mathbb{Z}_k$ discrete gauge symmetry is

$$\int d^4x\,(\partial_\mu\phi - kA_\mu)^2\,, \tag{3.51}$$

where the gauge field A_1 is normalised such that the minimum electric charge is 1, and ϕ is a scalar field (henceforth dubbed 'axion') with a periodic identification

$$\phi \simeq \phi + 1. \tag{3.52}$$

In the previous chapter we showed that we could dualise the above Lagrangian in terms of a 2-form and a (magnetic) gauge potential, and study the discrete gauge symmetries using the BF formulation. However, in this chapter, we will stick to the axion formulation.

Although this form is largely inspired by considering ϕ to be the phase of a Higgs field with charge k under a broken $U(1)$ gauge group, for the moment we will regard it just as a scalar, whose moduli space (locally given by $\mathbb{R}$) has a continuous isometry

$$\phi \longrightarrow \phi + \epsilon. \tag{3.53}$$

The action (3.51 describes the gauging of this isometry by a $U(1)$,

$$A_\mu \;\longrightarrow\; A_\mu + \partial_\mu\lambda, \tag{3.54a}$$
$$\phi \;\longrightarrow\; \phi + k\lambda. \tag{3.54b}$$

Before taking into account the periodicity (3.52), the value of k could be removed by rescaling ϕ, and would not be relevant. The integer k is thus properly interpreted as the winding number in the map between the $\mathbb{S}^1$ of $U(1)$ gauge transformations $e^{2\pi i\alpha}$ (with $\alpha \simeq \alpha + 1$ due to charge quantization), and the $\mathbb{S}^1$ parametrized by the axion ϕ. The fact that k is integer is a compatibility condition of the gauging by the $U(1)$ with the pre-existing discrete equivalence (3.52).

The gauging directly implements the field identification $\phi \simeq \phi + k$. On the other hand the discrete equivalence (3.52) corresponds to a 'fractional' $1/k\; U(1)$ gauge transformation, namely a $\mathbb{Z}_k$ gauge transformation.

This perspective displays the close relation of the discrete gauge symmetry with the underlying field identification in the scalar manifold. More precisely, the discrete gauge symmetry is the group of field identifications in the scalar manifold modulo those already accounted for by the gauging. This intuition is the key to the non-Abelian generalization in the coming sections.

3.3.2 The multiple Abelian case

Before moving onto the non-Abelian case, let us sharpen our intuitions in a slightly more involved (yet Abelian) situation. Consider a theory with several U(1) gauge symmetries, labelled with an index α, and several axions ϕ^a, $a = 1, \ldots, N$. The generalization of equation (3.51) is

$$\mathcal{L} \supset \sum_\alpha \left(\partial_\mu \phi^a - k_\alpha{}^a A_\mu^\alpha\right) \left(\partial_\nu \phi^b - k_\alpha{}^b A_\nu^\alpha\right) \eta^{\mu\nu} \delta_{ab} \tag{3.55}$$

with integer $k_\alpha{}^a \in \mathbb{Z}$. We take $U(1)$ generators normalized such that charges are integer and axions have integer periodicity.

There is a systematic closed description of the surviving discrete gauge symmetry based on our earlier intuitions. For that aim, we consider the space spanned by the scalars ϕ^a. This is a torus $\mathbb{T}^N$ which we regard as $\mathbb{R}^N/\Gamma$, with Γ the lattice of translations defined by vectors of integer entries

$$\Gamma = \{(r_1, \ldots, r_N) | r_a \in \mathbb{Z}\}. \tag{3.56}$$

The Lagrangian (3.55) implies that $U(1)_\alpha$ gauge transformations act as translations in $\mathbb{R}^N$ along the vectors $\vec{k}_\alpha$

$$A_\alpha \longrightarrow A_\alpha + d\lambda_\alpha, \tag{3.57a}$$

$$\phi^a \longrightarrow \phi^a + \sum_\alpha k_\alpha{}^a \lambda_\alpha. \tag{3.57b}$$

For simplicity, we focus on the case where the number of axions and $U(1)$ gauge symmetries is equal.[4] Finite $U(1)$ gauge transformations leaving all charged fields invariant (i.e. gauge parameter $\lambda_\alpha = 1$) act as discrete translations in $\mathbb{R}^N$ by the integer vectors $\vec{k}_\alpha$, and therefore span a sublattice $\hat{\Gamma} \subset \Gamma$,

$$\hat{\Gamma} = \langle \vec{k}_1, \ldots \vec{k}_N \rangle_{\mathbb{Z}} = \left\{ \sum_\alpha c_\alpha \vec{k}_\alpha | c_\alpha \in \mathbb{Z} \right\}. \tag{3.58}$$

Following our previous discussion for the single Abelian case, the discrete gauge symmetry is given by the set of identifications in the space of scalars modulo those implemented by the finite $U(1)$ gauge symmetries, namely by the quotient

$$\mathbf{P} = \frac{\Gamma}{\hat{\Gamma}}. \tag{3.59}$$

As we will now see these intuitions generalize to the non-Abelian case as well.

3.3.3 Non-Abelian discrete gauge symmetries and gaugings

While the construction introduced above describes the well-known case of Abelian discrete gauge symmetries, it admits a natural generalization to the non-Abelian case. In the non-Abelian version instead of a single field we will have a whole set of scalars (dubbed 'non-Abelian axions') which span a manifold with non-commuting isometries. This more general construction can also be regarded as a procedure to construct a Lagrangian formulation for (at least certain) non-Abelian discrete gauge theories.

[4] Generalization is straightforward. If the number n of $U(1)$'s, is smaller than the number N of scalars, we restrict to those scalars which actually shift: we consider the $\mathbb{R}^n \subset \mathbb{R}^N$ given by real linear combinations of the vectors $\vec{k}_\alpha$ (assumed linearly independent for simplicity), and the sublattice $\Gamma_n \subset \Gamma$ lying in this $\mathbb{R}^n$, and proceed as above with n playing the role of N. For $\vec{k}_\alpha$ not linearly independent, we just eliminate the decoupled linear combinations of $U(1)$'s, and restart. Similarly if the number of $U(1)$ gauge symmetries is larger than the number of scalars to start with.

3.3.3.1 The scalar manifold

Let $\mathcal{M}$ be the moduli space of N scalars ϕ^a, endowed with a metric $G_{ab}(\phi)$ with a set of (in general non-Abelian) continuous isometries with Killing vector fields $X_A = X_A^b \partial_b$. Under infinitesimal space-time independent isometry transformations the scalars transform as

$$\phi^b \longrightarrow \phi^b + \epsilon^A X_A^b, \tag{3.60}$$

and their kinetic term

$$\int d^4x\, G(\vec{\phi})_{ab}\, \partial_\mu \phi^a \partial^\mu \phi^b \tag{3.61}$$

is invariant provided that $(\mathcal{L}_{X_A} G)_{ab} = 0$. The Killing vector fields satisfy a Lie algebra

$$[X_A, X_B] = f_{AB}{}^C X_C \tag{3.62}$$

with $f_{AB}{}^C$ the structure constants and $[\ ,\ ]$ the Lie Bracket.

Given the 4d Lagrangian (3.61) it is easy to guess how to implement a gauging analogous to equation (3.55), see equation (3.71) below. Before doing that it is however useful to consider the scalar manifold $\mathcal{M}$ and try to understand which kind of metrics $G_{ab}(\phi)$ one may obtain in the case where all the fields ϕ^a are axions. This will allow in particular to rewrite (3.61) in a simpler form (namely equation (3.67) below) which we will use extensively when reproducing non-Abelian discrete gauge symmetries from string theory setups.

In order to characterize the metric G_{ab} it is useful to describe the manifold $\mathcal{M}$ in the language of group theory, as follows. Note that each Killing vector field describes a flow within $\mathcal{M}$, and so there is a natural action of the Lie group of isometries Iso($\mathcal{M}$) on the scalar manifold $\mathcal{M}$. We may then consider that Iso($\mathcal{M}$) acts transitively on $\mathcal{M}$,[5] and so identify $\mathcal{M}$ with the coset $K_p \backslash \text{Iso}(\mathcal{M})$, with K_p the stabilizer or little group of an arbitrary point $p \in \mathcal{M}$. Therefore we may apply the usual procedure (see for instance appendix A.4 of [151]) for building a Riemannian metric $G_{ab}(\phi)$ for $\mathcal{M}$ in terms of the elements of Iso($\mathcal{M}$) and K_p.

In general, the quotient $K_p \backslash \text{Iso}(\mathcal{M})$ will not be a Lie group itself: for this it is necessary that K_p is a normal subgroup of Iso($\mathcal{M}$). However, if $\mathcal{M}$ parametrizes the vevs of *only* axion-like scalars, the choice of $\mathcal{M}$ as a Lie group is quite natural. Indeed, for an 'axionic manifold' $\mathcal{M}$ the number of independent shift symmetries at any point should equal the dimension of $\mathcal{M}$. This is automatically satisfied if $\mathcal{M}$ is a Lie group, since in this case we can identify each axion with an element of the Lie algebra of the group $\mathcal{M}$, while the continuous shift symmetry corresponds to the one-parameter subgroup generated by such Lie algebra element. Hence, in the following we will consider the case where our axionic manifold $\mathcal{M}$ is a Lie group.[6]

In the case that $\mathcal{M}$ is a Lie group we can systematically build an affine representation of $\mathcal{M}$ acting on the plane $\mathbb{R}^{N+1}$, with $N = \dim \mathcal{M}$. For this construction, familiar from

[5] If not, we may take the orbit $\mathcal{O}_p$ created when Iso($\mathcal{M}$) acts on a point $p \in \mathcal{M}$, and then restrict the initial set of scalars ϕ^a to those that span $\mathcal{O}_p$.

[6] In general we would expect that a coset $\mathcal{M}$ that is not a Lie group but is nevertheless a parallelizable manifold could also qualify as an axionic manifold. We are nevertheless unaware of any example of this kind arising from a string compactification, and so this possibility will not be analyzed here.

the description of twisted tori geometries, we first consider the affine plane $\mathbb{R}^{N+1}$ described by vectors

$$\vec{v} = \begin{pmatrix} \vec{\phi} \\ 1 \end{pmatrix}, \tag{3.63}$$

as well as a vector $\vec{\epsilon} \in \mathbb{R}^N$ that parametrizes an element of the Lie algebra of $\mathcal{M}$. Second, we consider the adjoint representation of Lie($\mathcal{M}$), given by $(\mathrm{ad}_{\vec{\epsilon}})_b{}^c = \epsilon^a f_{ab}{}^c$, and construct the matrices

$$\mathfrak{g}(\vec{\epsilon}) = \begin{pmatrix} \frac{1}{2}\,\mathrm{ad}_{\vec{\epsilon}} & \vec{\epsilon} \\ 0 & 0 \end{pmatrix}, \tag{3.64}$$

which provide a faithful $(N+1)$-dimensional linear representation of Lie$(\mathcal{M}) \subset \mathfrak{iso}(\mathcal{M})$. Taking the exponential map, we obtain

$$g(\vec{\epsilon}) = \begin{pmatrix} e^{\frac{1}{2}\,\mathrm{ad}_{\vec{\epsilon}}} & 2\,\mathrm{ad}_{\vec{\epsilon}}^{-1}(e^{\frac{1}{2}\,\mathrm{ad}_{\vec{\epsilon}}} - \mathbb{I}_{n\times n})\,\vec{\epsilon} \\ 0 & 1 \end{pmatrix}, \tag{3.65}$$

where ϵ^a now parametrize arbitrarily large translations in $\mathcal{M}$. Finally, we can build an explicit expression for the metric $G_{ab}(\phi)$ in terms of the right-invariant 1-forms η^a, which are defined as

$$(dg \cdot g^{-1})(\vec{\phi}) = \eta^a(\vec{\phi})\, t_a, \tag{3.66}$$

with t_a the generators of Lie($\mathcal{M}$). We then obtain that the metric for $\mathcal{M}$ is such that

$$\int d^4x\, G_{ab}(\vec{\phi})\, \partial^\mu \phi^a \partial_\mu \phi^b = \int d^4x\ \mathcal{P}_{ab}\, \eta^a \cdot \eta^b, \tag{3.67}$$

where $\mathcal{P}_{ab}$ is the metric in the tangent space of $\mathcal{M}$, and so independent of ϕ, while $\eta^a \cdot \eta^b \equiv \eta^{\mu\nu}\eta^a_\mu \eta^b_\nu$ with $\eta^{\mu\nu}$ the 4d Minkowski metric. Notice that this expression is automatically invariant under continuous right-translations by group elements $g(\vec{\phi}) \to g(\vec{\phi})g(\vec{\epsilon})$, and so it indeed respects the axionic shift symmetries.

A particularly relevant case to forthcoming applications is when Lie($\mathcal{M}$) is a 2-step nilpotent algebra (see [152] for a recent review). In this case we have that $e^{\frac{1}{2}\,\mathrm{ad}_{\vec{\epsilon}}} = 1 + \frac{1}{2}\,\mathrm{ad}_{\vec{\epsilon}}$ and so equation (3.65) reduces to

$$g(\vec{\epsilon}) = \begin{pmatrix} 1 + \frac{1}{2}\,\mathrm{ad}_{\vec{\epsilon}} & \vec{\epsilon} \\ 0 & 1 \end{pmatrix}. \tag{3.68}$$

Then, applying equation (3.66) we obtain

$$\eta^a_\mu = \partial_\mu \phi^a + \frac{1}{2} f_{bc}{}^a \phi^b\, \partial_\mu \phi^c, \tag{3.69}$$

yielding a particularly simple expression for the right-invariant forms η^a and hence for the metric in (3.67).

Since the above construction is general it is important to note that, unless Lie($\mathcal{M}$) is semi-simple, $\mathcal{M}$ will be a non-compact manifold which is unsuitable to describe the moduli space of axionic-like scalars. We may however make this moduli space compact by taking its quotient by a lattice $\Gamma \subset \mathcal{M}$. This is in fact something quite common in string theory, where moduli spaces are quotients of the form $\tilde{\mathcal{M}} = \mathcal{M}/\Gamma$, with Γ a discrete subgroup of Iso($\mathcal{M}$) that takes into account the dualities of the theory. A well-known example is the 10d axio-dilaton coupling τ of type IIB theory, whose moduli space is not $\mathcal{M}_\tau = SO(2)\backslash SL(2,\mathbb{R})$ but rather $\tilde{\mathcal{M}}_\tau = SO(2)\backslash SL(2,\mathbb{R})/SL(2,\mathbb{Z})$ once S-duality has been taken into account.[7]

Going back to the general case, if Γ is cocompact, namely if there is a subset $X \subset \mathcal{M}$ such that the image of X under the action of Γ covers the entire $\mathcal{M}$, then $\tilde{\mathcal{M}} = \mathcal{M}/\Gamma$ is compact. Finding such a lattice is in general a complicate task and its existence is not guaranteed. However, if $\mathcal{M}$ is a nilpotent Lie group it is enough to require that the structure constants are integer in some particular basis and that they satisfy $f_{ab}{}^a = 0$ [153]. For the time being we will assume that such cocompact Γ exists, but ignore its effect until subsection 3.3.3.3.

3.3.3.2 The gauging

Let us now write a 4d Lagrangian describing a set of non-commuting U(1) gauge symmetries that gauge some of the isometries of $\mathcal{M}$, ignoring the effect of the discrete lattice Γ. To describe such gauging, instead of (3.60), we consider infinitesimal space-time dependent isometry transformations

$$\phi^b \to \phi^b + \epsilon^A(x) X^b_A, \tag{3.70}$$

where x represents the set of 4d coordinates. Invariance of the action under local transformations becomes manifest once we introduce the corresponding set of gauge fields (see e.g. [154]). We have the generalization of (3.55)

$$\int d^4x\, G_{ab}(\phi) \left(\partial_\mu\phi^a - k_\alpha{}^a A^\alpha_\mu\right) \left(\partial_\nu\phi^b - k_\beta{}^b A^\beta_\nu\right) \eta^{\mu\nu}, \tag{3.71}$$

where the set of vector fields $\{k_\alpha\}$ is similar to the above $\{X_A\}$, but not necessarily identical due to relative normalizations to be discussed in the next subsections. In order for this action to be invariant under the infinitesimal isometry transformations (3.70), covariant derivatives have to transform as

$$\partial_\mu\phi^a - k_\alpha{}^a A^\alpha_\mu \;\longrightarrow\; (\delta^a_b + \epsilon^A\partial_b X^a_A)(\partial_\mu\phi^b - k_\alpha{}^b A^\alpha_\mu), \tag{3.72}$$

which means that the gauge fields A^α_μ transform as

$$k_\alpha{}^a A^\alpha_\mu \longrightarrow k_\alpha{}^a A^\alpha_\mu + X^a_C\partial_\mu\epsilon^C + f^C{}_{AB}\, X^a_C (X^{-1})^A_b A^\beta_\mu k_\beta{}^b \epsilon^B. \tag{3.73}$$

As in the previous section, let us focus on the case where $\mathcal{M}$ is a Lie group. For notational simplicity, we will assume that all the right isometries of $\mathcal{M}$ are gauged. It is easy to see that the right-invariant 1-forms are now given by

$$(Dg \cdot g^{-1})(\vec{\phi}) = \eta^a(\vec{\phi}) t_a, \tag{3.74}$$

[7] As a slightly more involved example, we may reconsider the multiple Abelian case in subsection 3.3.2. Before taking the quotient by the lattice (3.56), the scalar manifold is $\mathcal{M} = \mathbb{R}^N$ and its isometry group is the Euclidean group, $\mathrm{Iso}(\mathcal{M}) = \mathbb{R}^N \rtimes O(N)$. Since the action of $\mathrm{Iso}(\mathcal{M})$ on $\mathcal{M}$ is transitive and the little group of any point of $\mathcal{M}$ is $O(N)$, $\mathcal{M}$ can be identified with the quotient $O(N)\backslash\mathrm{Iso}(\mathcal{M})$, which is nothing but the group of translations in $\mathcal{M}$. Finally, this space is made compact by taking the quotient $\tilde{\mathcal{M}} = \mathcal{M}/\Gamma$, with Γ a group of discrete translations.

with

$$Dg = dg - t_a k^a_\alpha A^\alpha_\mu, \tag{3.75}$$

and so are built by performing the replacement $dg \to Dg$ everywhere. In terms of these new 1-forms the action is still given by

$$\int d^4x\, \mathcal{P}_{ab}\eta^a \cdot \eta^b. \tag{3.76}$$

As before, for the particular case of 2-step nilpotent groups things simplify and these right-invariant 1-forms read

$$\eta^a_\mu = \partial_\mu\phi^a - k_\alpha{}^a A^\alpha_\mu + \frac{1}{2} f_{bc}{}^a \phi^b(\partial_\mu\phi^c - k_\beta{}^c A^\beta_\mu), \tag{3.77}$$

and so under a space-time dependent right-translation $g(\vec{\phi}) \to g(\vec{\phi})g(\vec{\epsilon})$ gauge fields transform as

$$k_\alpha{}^a A^\alpha_\mu t_a \longrightarrow k_\alpha{}^a A^\alpha_\mu t_a g(\vec{\epsilon}) + g(\vec{\phi})\partial_\mu g(\vec{\epsilon}). \tag{3.78}$$

3.3.3.3 A simple example

The above construction provides the Lagrangian for a massive non-Abelian gauge symmetry, but it still does not reveal potential residual discrete gauge symmetry. To proceed further and make the discussion concrete, we introduce here an example of scalar manifold $\mathcal{M}$ and lattice Γ whose gauging leads to a non-Abelian discrete symmetry group. The example is constructed using the Heisenberg group, $\mathcal{M} = \mathcal{H}_3(\mathbb{R})$, and will be realized in several physical systems in coming chapters. In the next subsection we then extend the discussion to the general case.

Thus, we consider the 3-dimensional Heisenberg group as generated by matrices of the form

$$g(\vec{\epsilon}) = \begin{pmatrix} 1 & 0 & 0 & \epsilon^1 \\ 0 & 1 & 0 & \epsilon^2 \\ -\frac{M}{2}\epsilon^2 & \frac{M}{2}\epsilon^1 & 1 & \epsilon^3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.79}$$

with M an integer. The associated Lie algebra is

$$[t_1, t_2] = M t_3, \tag{3.80}$$

where t_1, t_2 and t_3 are the elements of the algebra that generate the 1-dimensional subgroups parametrized by ϵ^1, ϵ^2 and ϵ^3. The right-invariant 1-forms are given by equation (3.69), which in this particular case corresponds to

$$\eta^1_\mu = \partial_\mu\phi^1, \tag{3.81a}$$

$$\eta^2_\mu = \partial_\mu\phi^2, \tag{3.81b}$$

$$\eta^3_\mu = \partial_\mu\phi^3 + \frac{M}{2}(\phi^1\partial_\mu\phi^2 - \phi^2\partial_\mu\phi^1), \tag{3.81c}$$

in terms of which the metric of $\mathcal{M}$ is given by the r.h.s. of

$$\int d^4x\, G_{ab}(\vec{\phi})\, \partial^\mu\phi^a\partial_\mu\phi^b = \int d^4x\ \mathcal{P}_{ab}\, \eta^a \cdot \eta^b. \tag{3.82}$$

Since $\mathcal{M}$ is non-compact, we take our axionic moduli space to be given by the compact coset $\tilde{\mathcal{M}} = \mathcal{H}_3(\mathbb{R})/\Gamma$ where, for concreteness, we take the cocompact lattice $\Gamma \subset \mathcal{H}_3(\mathbb{R})$ to be generated by $(\phi^1, \phi^2, \phi^3) = (n_1, n_2, n_3)$ with $n_i \in \mathbb{Z}$, namely by the discrete transformations

$$\Gamma(1,0,0): \quad \phi^1 \to \phi^1 + 1, \quad \phi^3 \to \phi^3 - \frac{M}{2}\phi^2, \tag{3.83a}$$

$$\Gamma(0,1,0): \quad \phi^2 \to \phi^2 + 1, \quad \phi^3 \to \phi^3 + \frac{M}{2}\phi^1, \tag{3.83b}$$

$$\Gamma(0,0,1): \quad \phi^3 \to \phi^3 + 1. \tag{3.83c}$$

We can gauge the right isometries of $\tilde{\mathcal{M}}$ following the general procedure described in the previous subsection. Thus, we introduce a set of U(1) gauge bosons A^α_μ, $\alpha = 1, 2, 3$, and replace the right-invariant 1-forms (3.81) by their gauged counterparts eq. (3.77), which in this particular case read

$$\eta^1_\mu = \partial_\mu \phi^1 - k_1 A^1_\mu, \tag{3.84a}$$

$$\eta^2_\mu = \partial_\mu \phi^2 - k_2 A^2_\mu, \tag{3.84b}$$

$$\eta^3_\mu = \partial_\mu \phi^3 - k_3 A^3_\mu + \frac{M}{2}\left[\phi^1(\partial_\mu\phi^2 - k_2 A^2_\mu) - \phi^2(\partial_\mu \phi^1 - k_1 A^1_\mu)\right], \tag{3.84c}$$

with $k_\alpha \in \mathbb{Z}$, $\alpha = 1, 2, 3$.

After the gauging, U(1) gauge transformations of the gauge bosons A^α_μ

$$A^1_\mu \longrightarrow A^1_\mu + \partial_\mu \lambda^1, \tag{3.85a}$$

$$A^2_\mu \longrightarrow A^2_\mu + \partial_\mu \lambda^2, \tag{3.85b}$$

$$A^3_\mu \longrightarrow A^3_\mu + \partial_\mu \lambda^3 + \frac{M k_1 k_2}{2k_3}\left(\lambda^2 A^1_\mu - \lambda^1 A^2_\mu\right) + \frac{M}{2k_3}(k_2\phi^1\partial_\mu\lambda^2 - k_1\phi^2\partial_\mu\lambda^1), \tag{3.85c}$$

induce non-trivial shifts on the scalars

$$\phi^1 \longrightarrow \phi^1 + k_1\lambda^1, \tag{3.85d}$$

$$\phi^2 \longrightarrow \phi^2 + k_2\lambda^2, \tag{3.85e}$$

$$\phi^3 \longrightarrow \phi^3 + \frac{M}{2}(k_2\phi^1\lambda^2 - k_1\phi^2\lambda^1) + k_3\lambda^3. \tag{3.85f}$$

Compatibility of these transformations with (3.83) then leads to a set of non-commuting $\mathbb{Z}_{k_\alpha}$ discrete gauge symmetries. Indeed, the gauge symmetry is given by the set of identifications (3.83) modulo these finite gauge transformations, in analogy with the Abelian case. For instance, for $k_1 = k_2 = k_3 = k \in \mathbb{Z}$ and $M = 1$ we have that the discrete gauge symmetry is given by $\mathbf{P} = (\mathbb{Z}_k \times \mathbb{Z}_k) \rtimes \mathbb{Z}_k$, with generators $\tilde{T}_1$, $\tilde{T}_2$ and $\tilde{T}_3$ satisfying

$$\tilde{T}_1^k = \tilde{T}_2^k = \tilde{T}_3^k = 1, \tag{3.86a}$$

$$\tilde{T}_1\tilde{T}_2 = \tilde{T}_3\tilde{T}_2\tilde{T}_1. \tag{3.86b}$$

For $k = 2$ this is isomorphic to the dihedral group, $\mathbf{P} \simeq \text{Dih}_4$, whereas for $k = 3$ the discrete symmetry group is $\mathbf{P} \simeq \Delta(27)$

3.3.3.4 The discrete gauge symmetry

To obtain the non-Abelian discrete gauge symmetry group in the above example we have closely followed a similar reasoning to the one that we used for Abelian discrete gauge symmetries. Indeed, we have seen that gauge transformations span a lattice $\hat{\Gamma} \subset \tilde{\mathcal{M}}$ and in order to gauge the left isometries of $\tilde{\mathcal{M}}$ it is enough to specify such a lattice. As in the Abelian case, the discrete gauge symmetry arises when we take into account the group Γ of scalar field identifications; namely when we specify the periodicities of the isometries generated by X_A and compare them with those of the gauge transformations (3.73), generated by k_α. Thus, once Γ is taken into account, a non-trivial compatibility condition for the gauging arises.

The discrete gauge symmetry of the theory is

$$\mathbf{P} = \frac{\Gamma}{\hat{\Gamma}}. \tag{3.87}$$

Fields charged under the original U(1) symmetries end up in some representation of this discrete gauge symmetry (whether they are massless fields or not).

3.4 A comment on discrete gauge symmetries and instantons

One point we have not emphasised yet is that the continuous symmetries which are broken to the discrete ones behave as exact global symmetries at the perturbative level. However, they are violated by non-perturbative effects, in particular D-brane instantons [155, 78, 156] (see [157, 158, 10] for reviews. The existence of a gauged discrete subgroup implies that it will be preserved by any such non-perturbative effect; we will illustrate this in section 4.5 in the case of intersecting brane models for Abelian symmetries, and in section 7.4 for non-Abelian symmetries in the context of magnetised brane models.

One may think that, for practical purposes, instanton effects may be negligible, and discrete gauge symmetries are irrelevant, since they are just part of the perturbatively exact global symmetries. However, in many SM-like D-brane models, instanton effects are often invoked to generate phenomenologically interesting (but perturbatively forbidden) couplings, see e.g. [159, 160, 161, 77], and so must be non-negligible. Hence it is relevant to ensure that other instantons do not induce dangerous coupling.

4

Discrete gauge symmetries in intersecting D-brane models

Intersecting D-brane models are one of the possible ways that realistic models of particle physics. It consists of several stacks of D-branes, which contain the gauge group and gauge bosons in their worldvolume, and matter fields arising in the intersections of two such stacks. They have been extensively studied in the literature, see e.g. [162, 163, 164, 73, 165, 166, 167, 168, 169, 170, 74, 171, 172, 76, 140, 149].

In this chapter we present an analysis of the possible discrete gauge symmetries one can realize in intersecting D-branes models, using the BF formalism we explained in section 3.2. We first review the construction of intersecting D-brane models, and how the BF couplings arise in them. Subsequently, we consider two kinds of models, which differ in the way the $SU(2)_L$ group of the Standard Model is generated, and study what the possible discrete gauge symmetries in those theories are. We also study D-brane instantons, and check that all instantons preserve the discrete gauge symmetries. Finally, we comment on $\mathbb{Z}_2$ discrete gauge symmetries associated to the discrete K-theory charge cancellation conditions, and suggest the intriguing possibility of identifying it with R-parity in explicit constructions.

4.1 Introduction to intersecting brane models

4.1.1 Intersecting D6-branes in flat 10d space

4.1.1.1 Local geometry and spectrum

The basic configuration of intersecting D-branes leading to chiral 4d fermions at their intersection corresponds to two stacks of D6-branes in flat 10d, intersecting over a 4d subspace of their volumes. Consider flat 10d space, decomposed as $\mathbb{M}_4 \times \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2$, abd two stacks of D6-branes spanning $\mathbb{M}_4$ times a line in each of the three 2-planes. The local geometry is fully defined by the three angles θ_i which define the rotation between the two stacks of D6-branes. As discussed below, the chiral fermions are localized at the intersection of teh brane volumes.

The appearance of chirality can be understood from the fact that the geometry of the two D6-branes introduces a preferred orientation in the transverse 6d space; the rotation from the first to the second D6-brane defines an orientation by a 6d version of the 'right-hand rule'.

Consider a stack of N_1 coincident D6-branes (denoted D$6'_1$s) intersecting a second stack of N_2 D6-branes (denoted D$6'_2$). The open string spectrum is the following:

- $\mathbf{6}_1\mathbf{6}_1$: Strings stretching among the D6_1-branes produce 7d $U(N_1)$ gauge bosons, three real adjoint scalars and fermion superpartners, propagating over the 7d world-volume of the D6_1-branes.
- $\mathbf{6}_2\mathbf{6}_2$: Strings stretching among the D6_2-branes produce 7d $U(N_2)$ gauge bosons, three real adjoint scalars and fermion superpartners, propagating over the 7d world-volume of the D6_2-branes.
- $\mathbf{6}_1\mathbf{6}_2$+$\mathbf{6}_2\mathbf{6}_1$ Open strings between both kinds of D6-branes are naturalluy localized at their intersection, to minimize their stretching, and lead to 4d chiral fermions in the bi-fundamental representation $(N_1, \bar{N}_2)$ of $U(N_1)\times U(N_2)$, as well as scalar fields in the same representation. For a detailed computation, see the next section.

Chirality of the sector of open strings stretching between the two D6-branes is consistent with the fact that any continuous motion of the branes (preserving gauge symmetry) maintains the existence of an intersection; this corresponds to the fact that chiral particles at the intersection do not become massive upon deforming their effective action in a continuous fashion.

4.1.1.2 Open strings at D6-brane intersections

Consider open strings stretching between the D6_1 and D6_2-branes. the boundary conditions for the coordinates along $\mathbb{M}_4$ are of the NN kind and lead to the oscillators α_n^μ, ψ_{n+r}^μ. For the directions where the branes form non-trivial angles, like in the (x^4, x^5) two-plane, we have boundary conditions

$$\begin{aligned} \partial_\sigma X^4|_{\sigma=0} = 0 \qquad & (\cos\theta_1 \partial_\sigma X^4 + \sin\theta_1 \partial_\sigma X^5)|_{\sigma=l} = 0 \\ \partial_t X^5|_{\sigma=0} = 0 \qquad & (-\sin\theta_1 \partial_t X^4 + \cos\theta_1 \partial_t X^5)|_{\sigma=l} = 0 \end{aligned} \tag{4.1}$$

and similarly for the two remaining two-planes with angles θ_2, θ_3. Defining complex coordinates $Z^i = X^{2i+2} + iX^{2i+3}$, $i = 1, 2, 3$, the boundary conditions read

$$\begin{aligned} \partial_\sigma(\mathrm{Re}Z^i)|_{\sigma=0} = 0 \qquad & \partial_t(\mathrm{Im}Z^i)|_{\sigma=0} = 0 \\ \partial_\sigma[\mathrm{Re}(e^{i\theta_i}Z^i)]|_{\sigma=l} = 0 \qquad & \partial_t[\mathrm{Im}(e^{i\theta_i}Z^i)]|_{\sigma=l} = 0 \end{aligned} \tag{4.2}$$

Using mode expansion for these coordinates, these boundary conditions shift the oscillator modding by an amount $\pm\nu_i = \pm\theta/\pi$. The oscillator operators, which are now associated to complex coordinates, are $\alpha^i_{n-\nu_i}$, $\bar{\alpha}^{\bar{i}}_{n+\nu_i}$, $\psi^i_{n+r-\nu_i}$, $\bar{\psi}^{\bar{i}}_{n+r+\nu_i}$, with $r = \frac{1}{2}, 0$ in the NS, R sectors. Also, the center of mass degrees of freedom for the bosonic coordinates are frozen in these directions, so that the open strings are localized at the D6-brane intersection.

The states are localized at the 4d intersection and transform in the bifundamental $(N_1, \bar{N}_2)_{1,-1}$ of the $U(N_1) \times U(N_2)$ gauge factor, with the subindices denoting the $U(1)_1 \times U(1)_2$ charges. The antiparticles arise from the D6_2-D6_1 open string sector.

The light spectrum contains in the NS sector a set of light scalars with masses depending on the θ_i (so they can be massive, massless or tachyonic), and in the R sector a massless 4d chiral fermion. The chirality of the 4d fermion is due to the GSO projections, and its handedness is determined by the orientation of the intersection. The masses of the scalars are given by

$$\begin{aligned} \alpha' \frac{M^2}{2} &= \frac{1}{2\pi}(\theta_1 + \theta_2 - \theta_3); \quad \frac{1}{2\pi}(\theta_1 - \theta_2 + \theta_3) \\ & \quad \frac{1}{2\pi}(-\theta_1 + \theta_2 + \theta_3); \quad 1 - \frac{1}{2\pi}(\theta_1 + \theta_2 + \theta_3) \end{aligned} \tag{4.3}$$

where $\theta_i \in [-\pi, \pi]$ for $i = 1, 2, 3$.

In the generic case, there is no supersymmetry invariant under the two stacks of branes, and the open string configuration is non-supersymmetric. However, if $\theta_1 \pm \theta_2 \pm \theta_3 = 0$ for some choice of signs, one of the scalars becomes massless, and the configuration is 4d $\mathcal{N}= 1$ supersymmetric. In the case that one of the angles vanishes, e.g. $\theta_1 = 0$, and $\theta_2 \pm \theta_3 = 0$, the system preserves 4d $\mathcal{N}= 2$ supersymmetry, while for $\theta_1 = \theta_2 = \theta_3 = 0$ we have 4d $\mathcal{N}=4$ supersymmetry.

4.1.2 Four-dimensional models

In the configuration we studied in the previous section, we obtained 4d chiral fermions; however, gauge interactions remain 7d and gravity interactions remain 10d, so one needs to consider compactification of space time.

The general kind of configurations that will be considered are type IIA string theory on a space time of the form $\mathbb{M}_4 \times \mathbb{X}_6$, where $\mathbb{X}_6$ is a compact Calabi-Yau manifold $\mathbb{X}_6$, so that we have supersymmetry in the closed sector. In addition we introduce stacks (labelled by an index a) of N_a D6_a-branes spanning 4d Minkowski and wrapped on 3-cycles Π_a in $\mathbb{X}_6$. Introduction of orientifold planes will be discussed in a later section.

Each D6-brane stack leads to a 4d gauge factor, while intersection between D6-brane stacks lead to 4d charged chiral fermions. A novelty in compact models is that generically two 3-cycles in a 6d space intersect several times, leading to replication of charged chiral fermions. This is a natural mechanism to explain/reproduce the appearance of replicated generations of chiral fermions in Nature.

4.1.2.1 Toroidal models

Many features of general compactifications with intersecting D-branes can be illustrated in the simpler setup of toroidal compactifications.

Consider type IIA compactified on a factorized torus $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$, and stacks of N_a D6_a-branes spanning 4d spacetime and wrapping a 1-cycle (n_a^i, m_a^i) in the i^{th} 2-torus; namely, the D6_a-brane wraps n_a^i times along the horizontal direction and m_a^i times along the vertical direction in the i^{th} two-torus. The 3-cycles Π_a are the product of three 1-cycles in the three 2-tori of $\mathbb{T}^6$. Note that for each i, the integers (n^i, m^i) must be coprime; otherwise, the system describes r D-branes with wrapping numbers $(n/r, m/r)$, where $r = \gcd(n, m)$. There exists more general non-factorizable 3-cycles, that for simplicity will not be considered here.

Near each intersection of D6-brane stacks, the configuration reduces to the intersecting D6-branes in 10d flat spacetime. However, in this case, the angles are derived quantities. For a rectangular $\mathbb{T}^2$ of radii R_1, R_2 along the horizontal and vertical directions, the angle between the 1-cycles $(1, 0)$ and (n_a^i, m_a^i) is given by

$$\tan\theta_a^i = \frac{m_a^i R_2}{n_a^i R_1} \tag{4.4}$$

The intersection number is given by the product of the number of intersections in each 2-torus, and reads

$$I_{ab} = (n_a^1 m_b^1 - m_a^1 n_b^1) \times (n_a^2 m_b^2 - m_a^2 n_b^2) \times (n_a^3 m_b^3 - m_a^3 n_b^3) \tag{4.5}$$

It is useful to introduce the 3-homology class $[\Pi_a]$ of the 3-cycle Π_a. $[\Pi_a]$ can be thought of as a vector of RR charges of the corresponding D6-brane. The 1-homology class of an (n, m) 1-cycle in a 2-torus is $n[a] + m[b]$, with $[a]$, $[b]$ the basic homology cycles in $\mathbb{T}^2$. For a 3-cycle with wrapping numbers we have

$$[\Pi_a] = \otimes_{i=1}^3 (n_a^i [a_i] + m_a^i [b_i]) \tag{4.6}$$

The intersection number (4.5) is the homological intersection number, denoted $I_{ab} = [\Pi_a]\cdot[\Pi_b]$. This result follows easily from $[a_i]\cdot[b_j] = \delta_{ij}$ and the linearity and antisymmetry of the intersection pairing.

The multiplicities N_a and the intersection numbers I_{ab} are sufficient to compute the gauge symmetry and chiral matter content of the 4d compactification. The closed string sector is just a toroidal compactification, and produces 4d $\mathcal{N}=8$ supergravity (this can be reduced to 4d $\mathcal{N}=1$ in more general orbifold or CY compactifications). There are also different open string sectors:

- $\mathbf{6}_a\mathbf{6}_a$ Strings stretched among D6-branes in the a^{th} stack produce 4d $U(N_a)$ gauge bosons, 6 real adjoint scalars and 4 adjoint Majorana fermions, filling out a vector multiplet of the 4d $\mathcal{N}=4$ supersymmetry preserved by the corresponding brane.
- $\mathbf{6}_a\mathbf{6}_b + \mathbf{6}_b\mathbf{6}_a$ Strings streched between the a^{th} and b^{th} stack lead to I_{ab} replicated left-handed chiral fermions in the bi-fundamental representation $(N_a, \bar{N}_b)$. Negative intersecion numbers indicate a positive number of right-handed chiral fermions. Additional light scalars may be present, with masses (4.3) in terms of angles fixed by the wrapping numbers and the $\mathbb{T}^2$ moduli.

String theories with open string sectors must satisfy the condition of RR tadpole cancellation, which amounts to requiring the total RR charge of D-branes to vanish. In our setup, the vector of RR charges is encoded in the D6-brane 3-cycle homology class, so the condition reads

$$[\Pi_{tot}] \equiv \sum_a N_a [\Pi_a] = 0 \tag{4.7}$$

In the toroidal setup, this condition becomes

$$\sum_a N_a n_a^1 n_a^2 n_a^3 = 0,$$
$$\sum_a N_a n_a^1 n_a^2 m_a^3 = 0, \quad \text{and permutations}$$
$$\sum_a N_a n_a^1 m_a^2 m_a^3 = 0, \quad \text{and permutations}$$
$$\sum_a N_a m_a^1 m_a^2 m_a^3 = 0, \tag{4.8}$$

Since the total D6-brane charge adds up to zero, models without O6-planes implicitly contain brane and antibrane charges; they are not symmetric, and may have potential instabilities.

Cancellation of RR tadpoles in the underlying string configuration implies cancellation of anomalies in the 4d effective theory.

- **Cubic non-abelian anomalies**
  The $SU(N_a)^3$ cubic anomaly is proportional to the number of fundamental minus antifundamental representations of $SU(N_a)$, hence it is proportional to $\sum_b I_{ab} N_b$. This vanishes as a consequence of RR tadpole cancellation: taking the intersection of $[\Pi_{tot}]$ with any $[\Pi_a]$ one gets

$$0 = [\Pi_a] \cdot [\Pi_{tot}] = [\Pi_a] \cdot \sum_b N_b [\Pi_b] = \sum_b N_b I_{ab} \tag{4.9}$$

- **Mixed anomalies**
  The $U(1)_a - SU(N_b)^2$ mixed anomalies cancel involving a 4d Green-Schwarz mechanism mediated by closed string RR fields. The triangle diagrams for $U(1)_a - SU(N_b)^2$ give a contribution which, even after using RR tadpole cancellation, is non-zero and proportional to

$$A_{ab} \simeq N_a I_{ab} \tag{4.10}$$

  The theory also contains contributions from Green-Schwarz diagrams, where the gauge boson of $U(1)_a$ mixes with a 2-form which subsequently couples to two gauge bosons of $SU(N_b)$. The couplings arise from the KK reduction of the D6-brane couplings $N_a \int_{D6_a} C_5 \wedge \mathrm{Tr} F_a$ and $\int_{D6_b} C_3 \wedge \mathrm{Tr} F_b^2$ in the Chern-Simons action. Let $\{\alpha_k\}$ be a basis of 3-cycles and $\{\beta^k\}$ its dual basis, i.e. $\alpha_k \cdot \beta^l = \delta_k^l$. We define the KK reduced 4d 2-forms ands scalar fields

$$B_2^k = \int_{\alpha_k} C_5, \quad a_l = \int_{\beta^l} C_3, \quad \text{with } \partial_\mu B_{\nu\rho}^k = -\delta^{kl} \epsilon_{\mu\nu\rho\sigma} \partial^\sigma a_l \tag{4.11}$$

  where the 4d duality relation follows from the 10d duality between C_5 and C_3. The KK reduced couplings read

$$N_a \int_{D6_a} C_5 \wedge \mathrm{Tr} F_a \quad \to \quad N_a Q_{ak} \int_{4d} B_2^k \mathrm{Tr} F_a$$
$$\int_{D6_b} C_3 \wedge \mathrm{Tr} F_b^2 \quad \to \quad q_B^l \int_{4d} a_l \mathrm{Tr} F_b^2 \tag{4.12}$$

with $Q_{ak} = [\Pi_a] \cdot [\alpha_k]$, $q_b^l = [\Pi_b] \cdot [\beta^l]$. One can check that the total amplitude is proportional to

$$A_{ab}^{GS} \simeq -I_{ab} N_a \tag{4.13}$$

leading to a cancellation between both kinds of contributions.

- **Mixed gravitational anomalies**
  Mixed gravitational triangle anomalies cancel automatically since the sum of $U(1)_a$ charges is $N_a \sum_b I_{ab} N_b$, which vanishes from (4.7).

4.1.2.2 Generalization beyond torus

One may take any compact 6-manifold as internal space; for instance, a Calabi-Yau threefold $\mathbb{X}_6$, which would lead to 4d $\mathcal{N}=2$ supersymmetry in the closed string sector. One should pick a set of 3-cycles Π_a on which we wrap N_a D6$_a$-branes making sure they satisfy the RR tadpole condition $\sum_a N_a[\Pi_a] = 0$. If one is interested in preserving supersymmetry, the 3-cycles should be special lagrangian of $\mathbb{X}_6$, which are defined by the conditions

$$J|_\Pi = 0, \quad \mathrm{Im}(e^{-i\phi}\Omega_3)|_\Pi = 0 \quad \text{for some fixed } \phi \tag{4.14}$$

where J and Ω_3 are the Calabi-Yau Kähler 2-form and holomorphic 3-form, and $|_\Pi$ denotes the restriction to the 3-cycle.

The final open string spectrum, in the case of supersymmetric wrapped D6-branes, arises in two kinds of sectors:

- $\mathbf{6}_a\mathbf{6}_a$ Leads to $U(N_a)$ vector multiplets of the 4d $\mathcal{N}=1$ supersymmetry preserved by the D6$_a$brane. In addition, there may be $b_1(\Pi_a)$ chiral multipletes in the adjoint, where $b_1(\Pi_a)$ is the first Betti number of Π_a.

- $\mathbf{6}_a\mathbf{6}_b+\mathbf{6}_b\mathbf{6}_a$ It produces I_{ab} chiral fermions in the representation $(N_a, \bar{N}_b)$ (plus light scalars, with masses determined by the relative angles $(\theta_i)_{ab}$, and which become massless for supersymmetry preserving intersections). Here $I_{ab} = [\Pi_a] \cdot [\Pi_b]$ is the topological intersection number of 3-cycles.

Since the chiral spectrum involves only purely topological data of the configuration, the discussion of RR tadpole cancellation and anomaly cancellation can be borrowed directly from the previous section.

4.1.3 Orientifold compactifications with intersecting D6-branes

RR tadpole cancellation implies that models without O6-planes are necessarily non-supersymmetric. A putative fully supersymmetric configuration of D6-branes would be, as a whole, a BPS state of type IIA on $\mathbb{X}_6$. For a BPS state, the tension is proportional to the RR charge, and since the latter vanishes due to tadpole cancellation, so must the former; hence, the only supersymmetric configuration is type IIA on $\mathbb{X}_6$, with no D6-branes at all. The way out of this impasse is to introduce O6-planes, which have negative RR charge and tension, and preserve the same supersymmetry as the D6-branes.

Consider type IIA theory on a CY $\mathbb{X}_6$ and mod out the configuration by $\Omega\mathcal{R}(-1)^{F_L}$, where Ω is worldsheet orientation reversal, $\mathcal{R}$ is an antiholomorphic $\mathbb{Z}_2$ symmetry on $\mathbb{X}_6$, acting as $(z_1, z_2, z_3) \to (\bar{z}_1, \bar{z}_2, \bar{z}_3)$ on local complex coordinates. The fixed points of $\mathcal{R}$ form an orientifold plane, the O6-plane, namely a subspace of spacetime where the orientation of the string can flip. The $(-1)^{F_l}$ operation, where F_L is the number of left-moving spacetime fermion number, is needed for the orientifold action to square to the identity operator.

The models also include N_a D6$_a$-branes wrapped on 3-cycles Π_a, and their image D6$_{a'}$-branes on 3-cycles denoted by $\Pi_{a'}$. The D6-branes preserve the 4d $\mathcal{N}= 1$ supersymmetry of the model if they preserve a common supersymmetry with the O6-planes, i.e. if their local relative anbles with the O6-planes satisfy

$$\theta_1 + \theta_2 + \theta_3 = 0 \tag{4.15}$$

The RR tadpole cancellation conditions include contributions from D6-branes, image D6′-branes and O6-planes (with -4 units of D6-brane charge) and read

$$\sum_a N_a[\Pi_a] + \sum_a N_a[\Pi_{a'}] - 4[\Pi_{O6}] = 0 \tag{4.16}$$

In models with no D6-brane coinciding with its image D6′-brane, the light spectrum in the different sector is:

- $\mathbf{6}_a\mathbf{6}_a$+$\mathbf{6}_{a'}\mathbf{6}_{a'}$ Contains $U(N_a)$ gauge bosons (plus possible additional adjoint fields).
- $\mathbf{6}_a\mathbf{6}_b$+$\mathbf{6}_b\mathbf{6}_a$+$\mathbf{6}_{a'}\mathbf{6}_{b'}$+$\mathbf{6}_{b'}\mathbf{6}_{a'}$ Gives I_{ab} chiral fermions in the representation $(N_a, \bar{N}_b)$, plus light (possibly massless) scalars.
- $\mathbf{6}_a\mathbf{6}_{b'}$+$\mathbf{6}_{b'}\mathbf{6}_a$+$\mathbf{6}_{a'}\mathbf{6}_b$+$\mathbf{6}_b\mathbf{6}_{a'}$ Gives $I_{ab'}$ chiral fermions in the representation (N_a, N_b), plus light (possibly massless) scalars.
- $\mathbf{6}_a\mathbf{6}_{a'}$+$\mathbf{6}_{a'}\mathbf{6}_a$ Contains $n_{sym,a}$ 4d chiral fermions in the representation $\square\square_a$ and $n_{asym,a}$ in the $\boxminus_a$, with

$$n_{sym,a} = \frac{1}{2}(i_{aa'} - I_{a,O6}), \quad n_{asym,a} = \frac{1}{2}(i_{aa'} + I_{a,O6}) \tag{4.17}$$

  where $I_{a,O6} = [\Pi_a] \cdot [\Pi_{O6}]$ is the number of aa' intersections on top of O6-planes. In addition, there are light (possibly massless scalars).

The RR tadpole condition (4.16) guarantees the cancellation of 4d anomalies of the new chiral spectrum. Anomalous and non-anomalous $U(1)'$s may acquire masses from their couplings to RR 2-form fields. The condition for a $U(1)$ to remain massless is

$$\sum_a N_a(Q_{ak} - Q_{a'k})c_a = 0 \quad \text{for all } k \tag{4.18}$$

In the orientifold case, there are in general mixed gravitational triangular anomalies, which cancel via Green-Schwarz contributions arising from both D6-brane and O6-plane worldvolume couplings.

4.1.3.1 Toroidal orientifold models

We focus on factorized tori $\mathbb{T}^6 = \mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$, parametrized by $z_i = x^i + iy^i$. We take the orientifold action $\Omega\,\mathcal{R}(-1)^{F_L}$ with $\mathcal{R}$: $z_i \to \bar{z}_i$, i.e. $y^i \to -y^i$, leaving x^i invariant.

There are two possible kinds of 2-tori compatible with this symmetry: rectangular tori, defined by the periodicities $z_i \sim z_i + R_1$, $z_i \sim z_i + iR_2$; and tilted tori, defined by the periodicities $z_i \sim z_i + R_1$, $z_i \sim z_i + R_1 + iR_2/2$. The orientofold action on (n,m)-cycles is different in each case, with $(n,m) \to (n,-m)$ for rectangular tori, and $(n,m) \to (n,-m-n)$ for tilted tori. In the latter case, one usually defines $\tilde{m} = m + n/2$, so that the orientifold acts as $(n,\tilde{m}) \to (n,-\tilde{m})$. For simplicity, in this section only geometries with three rectangular 2-tori will be considered.

For a geometry with three rectangular 2-tori, there are O6-planes spanning the 3-cycle parametrized by x^i and $y^i = 0, iR_2/2$. There are thus 8 O6-planes along the 3-cycle with wrapping numbers $(n^i, m^i) = (1,0)$, so their total homology class is $[\Pi_{O6}] = 8[a_1][a_2][a_3]$. The model contains stacks of N_A D6$_a$-branes with wrapping numbers (n_a^i, m_a^i) and images with wrapping numbers $(n_a^i, -m_a^i)$. The RR tadpole conditions (4.16) read

$$\begin{aligned} \sum_a N_a n_a^1 n_a^2 n_a^3 &= 16 \\ \sum_a N_a n_a^1 m_a^2 m_a^3 &= 0, \text{ and permutations} \end{aligned} \tag{4.19}$$

The number of conditions is halved with respect to the number of conditions in the un-orientifolded case (4.8) because the orientifold projection removes half of the components of the RR fields. Equivalently, because branes and their orientifold images cancel each other's contributions, and thus the conditions are automatically fulfilled.

The spectrum of the orientifold theory is given by the general CY result above, with intersection numbers given by (4.5).

In the presence of orientifold planes, there are additional discrete $\mathbb{Z}_2$-valued charges of non-BPS branes, classified by K-theory. In compact models, global consistency requires the cancellation of these discrete charges, leading to additional conditions. In the toroidal orientifold case, there are conditions requiring the cancellation of $\mathbb{Z}_2$-valued charges of D6-branes along $(1,0) \times (1,0) \times (0,1)$ and permutations; namely

$$\sum_a N_a m_a^1 n_a^2 n_a^3 \in 2\,\mathbb{Z}, \quad \text{and permutations} \tag{4.20}$$

4.2 Discrete gauge symmetries from BF couplings

4.2.1 General analysis

Consider type IIA compactified on a Calabi-Yau $\mathbb{X}_6$, with an orientifold action $\Omega\,\mathcal{R}(-1)^{F_L}$. Here $\mathcal{R}$ is an antiholomorphic involution of $\mathbb{X}_6$, acting as $z_i \to -z_i$ on local complex coordinates, so it introduces O6-planes. The compactification also contains stacks of N_A D6$_A$-branes wrapped on 3-cyces Π_A (along with their orientifold images $\Pi_{A'}$). We do not impose the supersymmetry conditions at this level since the analysis is essentially topological, and it holds even in non-supersymmetric models.

Let us introduce a basis of 3-cycles $\{\alpha_k\}$, $\{\beta_k\}$, even and odd under the geometric action $\mathcal{R}$, with $k = 1, \ldots, h_{2,1} + 1$, and for simplicity, such that $\alpha_k \cdot \beta_l = \delta_{kl}$.[1] ‘ We expand the wrapped cycles in this basis as

$$\Pi_A = \sum_k \left(r_A^k \alpha_k + s_A^k \beta_k \right), \quad (4.21a)$$

$$\Pi_{A'} = \sum_k \left(r_a^k \alpha_k - s_A^k \beta_k \right). \quad (4.21b)$$

The RR tadpole cancellation conditions are

$$\sum_A N_A [\Pi_A] + \sum_{A'} N_A [\Pi_{A'}] - 4[\Pi_{O6}] = 0, \quad (4.22)$$

where $[\Pi_{O6}]$ denotes the total homology class of the O6-planes (with the -4 from their RR charge, assumed to be negative).[2]

The gauge group is given by $\Pi_A U(N_A)$, and the spectrum of chiral matter is given by

$$\sum_{AB} I_{AB}\,(\square_A, \overline{\square}_B) + \sum_{AB'} (\square_A, \square_B) + \sum_A \left(\frac{I_{AA'} + I_{A,\mathrm{O6}}}{2} \,\begin{smallmatrix}\square\\[-2pt]\square\end{smallmatrix}_A + \frac{I_{AA'} - I_{A,\mathrm{O6}}}{2} \,\square\!\square_A \right) \quad (4.23)$$

where $I_{AB} = [\Pi_A] \cdot [\Pi_B]$, $I_{AB'} = [\Pi_A] \cdot [\Pi_{B'}]$ and $I_{A,\mathrm{O6}} = [\Pi_A] \cdot [\Pi_{\mathrm{O6}}]$ are the relevant intersection numbers giving the multiplicities.

The RR 5- and 3-form are intrinsically even and odd under the orientifold action, respectively; therefore, the Kaluza-Klein reduction leads to the following basis of RR 2-forms and their dual RR scalars

$$B_k = \int_{\beta_k} C_5, \quad (4.24a)$$

$$a_k = \int_{\alpha_k} C_3, \quad (4.24b)$$

with $dB_k = *_{4d} da_k$.

The Kaluza-Klein reduction of the D6-brane Chern-Simons action leads to the following BF coupling

$$S_{BF_A} = \frac{1}{2} \left(\int_{\Pi_A} C_5 \wedge \mathrm{tr}\, F_A - \int_{\Pi_{A'}} C_5 \wedge \mathrm{tr}\, F_A \right) = \sum_k N_A s_A^k B_K \wedge F_A, \quad (4.25)$$

where the factor of $1/2$ is due to the orientifold action, and the relative minus sign of the orientifold image contributions arises because $F_{A'} = -F_A$. Also, the factor of N_A arises from the $U(1)_A$ trace quantization.

As discussed previously in chapter 3, the factor of N_A implies the appearance of a Z_{N_A} discrete gauge symmetry. This corresponds to the general fact that the actual gauge

[1] There is an alternative class of orientifold actions, satisfying $\alpha_k \cdot \beta_l = 2\delta_{kl}$, which leads to very similar physical results, but requires a careful tracking of factors of 2. See section 4.2.3 for more details.

[2] In addition, there are certain discrete K-theory charge cancellation conditions [173], which actually play an interesting role, discussed in section 4.6.

group on a stack of N D-branes is $[SU(N) \times U(1)]/\mathbb{Z}_N$, with the $\mathbb{Z}_N$ corresponding to the centre of $SU(N)$, i.e. the N-ality. Namely, the group element diag$(\alpha, \ldots, \alpha)$ with $\alpha = e^{2\pi i/N}$ can be regarded as belonging to $SU(N)$ or to the diagonal $U(1)$; the quotient by $\mathbb{Z}_N$ implies that the two possibilities should be regarded as completely equivalent. The charges of fields under this $\mathbb{Z}_N$ are given by their N-ality, and so this $\mathbb{Z}_N$ does not imply any selection rule beyond $SU(N)$ gauge invariance; therefore, it is not very interesting by itself.

It follows from the structure of the BF couplings (4.25) that an additional $\mathbb{Z}_n$ discrete gauge symmetry appears whenever the coefficients s_A^k are multiples of n, for all k; more precisely, when $n = gcd(s_A^k)$.

In general, one may be interested in discrete subgroups of $U(1)$ linear combinations of the form

$$Q = \sum_A c_A Q_A. \tag{4.26}$$

In order to properly identify the discrete gauge symmetry from the BF coupling, the quantization is fixed such that $c_A \in \mathbb{Z}$, andgcd$(c_A) = 1$.

The BF couplings read

$$S_{BF} = \left(\sum_A c_A N_A s_A^k \right) B_k \wedge F, \tag{4.27}$$

where F is the field strength associated to the Q generator. So there is a $\mathbb{Z}_n$ gauge symmetry if $(\sum_A c_A N_A s_A^k) \in n\,\mathbb{Z}$ for all k. This condition can be rewritten as

$$\sum_A c_A N_A [\Pi_A] \cdot [\alpha_k] = 0 \mod n, \quad \forall k. \tag{4.28}$$

Although it has been derived for the case where $[\alpha_k] \cdot [\beta_l] = \delta_{kl}$, this expression for the condition is also valid in cases where $[\alpha_k] \cdot [\beta_l] = 2\delta_{kl}$ for some subset of the k's, see section 4.2.3.

Under a $U(1)_Q$ gauge transformation, the shift of the scalars a_k, dual to the 2-forms B_k, is given by

$$A_\mu \longrightarrow A_\mu + \partial_\mu \lambda, \tag{4.29a}$$

$$a_k \longrightarrow a_k + \sum_A c_A N_A s_A^k \lambda. \tag{4.29b}$$

In our quantizations, fields in the fundamental of $SU(N_A)$ have $U(1)_A$ charges $q_A = 1$, while fields in the two-index symmetric or antisymmetric tensor representation have $q_A = 2$ (and the opposite charge for the conjugate representations). For a field with charges q_A under the $U(1)_A$, its charge under the $\mathbb{Z}_n$ is given by $\sum_A c_A q_A \mod n$.

4.2.2 Toroidal orbifolds

In this section we particularise the above general analysis to the case of toroidal orientifolds. This is also valid for orbifolds thereof, as long as the relevant D6-branes do not wrap twisted cycles.

Consider a $\mathbb{T}^6$, taken factorable for simplicity, with each $(\mathbb{T}^2)^i$ parametrised by x^i, y^i, $i = 1, 2, 3$, and denote $[a_i]$, $[b_i]$ the 1-cycles along its two independent 1-cycles, with $[a_i] \cdot [b_j] = \delta_{ij}$. The orientifold acts as $x^i \to x^i$, $y^i \to -y^i$, and we take the action on the 1-cycles to be $[a_i] \to [a_i]$, $[b_i] \to -[b_i]$ (other tilted actions are also possible, see next section).

The basis of even and odd 3-cycles are

$$[\alpha_0] = [a_1][a_2][a_3], \qquad [\beta_0] = [b_1][b_2][b_3], \tag{4.30a}$$
$$[\alpha_1] = [a_1][b_2][b_3], \qquad [\beta_1] = [b_1][a_2][a_3], \tag{4.30b}$$
$$[\alpha_2] = [b_1][a_2][b_3], \qquad [\beta_2] = [a_1][b_2][a_3], \tag{4.30c}$$
$$[\alpha_3] = [b_1][b_2][a_3], \qquad [\beta_3] = [a_1][a_2][b_3]. \tag{4.30d}$$

The coefficients s_A^k are given by

$$s_A^0 = m_A^1 m_A^2 m_A^3, \tag{4.31a}$$
$$s_A^1 = m_A^n m_A^2 n_A^3, \tag{4.31b}$$
$$s_A^2 = n_A^1 m_A^2 n_A^3, \tag{4.31c}$$
$$s_A^3 = n_A^1 n_A^2 m_A^3, \tag{4.31d}$$

where (n^i, m^i) denote the wrapping numbers on the i-th torus with coordinates (x^i, y^i).

4.2.3 Tilted orientifolds

Let us introduce a basis of 3-cycles $\{\tilde{\alpha}_k\}$, $\{\tilde{\beta}_k\}$, satisfying $\tilde{\alpha}_k \cdot \tilde{\beta}_l = \delta_{kl}$, before the orientifold projection. Earlier in this chapter we have focused on the situation where $\tilde{\alpha} \to \tilde{\alpha}_k$ and $\tilde{\beta} \to -\tilde{\beta}_k$, which were denoted α_k and β. However, the orientifold projections is also compatible with other possibilities, e.g. in which of a sunsets of k's we have $\tilde{\alpha}_k \to \tilde{\alpha}_k - \tilde{\beta}_k$, $\tilde{\beta}_k \to -\tilde{\beta}_k$, or in which for a subset we have $\tilde{\alpha}_k \to \tilde{\alpha}_k$, $\tilde{\beta}_k \to -\tilde{\beta}_k + \tilde{\alpha}_k$. Since the latter turns into the former by renaming $\tilde{\alpha}' = 2\tilde{\alpha} - \tilde{\beta}$, $\tilde{\beta}' = \tilde{\alpha}$, we will focus on the action $\tilde{\alpha} \to \tilde{\alpha} - \tilde{\beta}$, $\tilde{\beta} \to \tilde{\beta}$ for concreteness. Also, for simplicity, we will assume that this happens for all k.

This kind of situation is familiar in compactifications with tilted $\mathbb{T}^2$'s, so we dub them 'tilted orientifolds'.

The cycles with definite parity under the orientifold action are given by $\alpha_k = 2\tilde{\alpha}_k - \tilde{\beta}_k$, $\beta_k = \tilde{\beta}_k$, and $\alpha_k \cdot \beta_l = 2\delta kl$.

The 3-cycles wrapped by the D6$_A$-branes and their images are

$$[\Pi_A] = r_A^k \tilde{\alpha}_k + s_A^k \tilde{\beta}_k = \frac{1}{2} r_A^k \alpha_k + \tilde{s}_A^k \beta_k, \tag{4.32a}$$
$$[\Pi_{A'}] = r_A^k \tilde{\alpha}_k - s_A^k \tilde{\beta}_k = \frac{1}{2} r_A^k \alpha_k - \tilde{s}_A^k \beta_k, \tag{4.32b}$$

where we have introduced $\tilde{s}_A^k = s_a^k + \frac{1}{2} r_A^k$.

We define the RR 2-forms and scalars as

$$B_k = \int_{\beta_k} C_5, \tag{4.33a}$$

$$a_k = \int_{\alpha_k} C_3. \tag{4.33b}$$

Notice that since $\alpha_k = 2\tilde{\alpha}_k - \tilde{\beta}$, there is a factor of 2 in the duality relation.

The BF coupling read

$$N_A \sum_k \tilde{s}_A^k B_k \wedge F_A. \tag{4.34}$$

Considering a $U(1)$ linear combination $Q = \sum_A c_A Q_A$, under a $U(1)$ gauge transformation, the shift in the RR scalar is

$$A_\mu \longrightarrow A_\mu + \partial_\mu \lambda, \tag{4.35a}$$

$$a_k \longrightarrow a_k + 2\sum_A c_A N_A \tilde{s}_A^k \lambda, \tag{4.35b}$$

where the factor of 2 (related to the one mentioned above) arises because $\alpha_l = 2\tilde{\alpha}_k - \tilde{\beta}$; this ensures the coefficient to be integer, even though the $\tilde{s}$ can be $\frac{1}{2} \mod \mathbb{Z}$.

Noting that $[\Pi_A]\cdot[\alpha_n] = -2\tilde{s}_A^k$, the condition for a $\mathbb{Z}_n$ subgroup to remain as a discrete gauge symmetry is given by the expression (4.28).

4.3 The $Sp(2)$ class

4.3.1 Generalities

In this class of models there are four stacks of D-branes, denoted a (*baryonic*), b (*left*), c (*right*) and d (*leptonic*). They have $N_1 = 3$, $N_b = 1$, $N_c = 1$, $N_d = 1$, but the stack b is taken coincident with its orientifold image, so that the initial gauge group is $U(3)_a \times Sp(2)_b \times U(1)_c \times U(1)_d$.

The chiral fermion content reproduces the SM quark and leptons if the D6-brane intersection numbers are given by[3]

$$I_{ab} = I_{ab^*} = 3, \tag{4.36a}$$

$$I_{ac} = I_{ac^*} = -3, \tag{4.36b}$$

$$I_{db} = I_{db^*} = -3, \tag{4.36c}$$

$$I_{cd} = -I_{dc^*} = -3, \tag{4.36d}$$

with the remaining intersections vanishing. As usual, negative intersection numbers denote positive multiplicities of the conjugate representation. The spectrum of chiral fermions is showed in table 4.1. It corresponds to the three SM quark-lepton generations. In addition there are three right-handed neutrinos N_R, whose presence is generic in this kind of constructions.

[3]Here and in what follows, we use the notation A^* for the orientifold image of the branes A.

Intersection	Matter fields		Q_q	Q_c	Q_d	Y
$(ab), (ab^*)$	Q_L	$3(3,2)$	1	0	0	1/6
(a,c)	U_R	$3(\bar{3},1)$	-1	1	0	-2/3
(ac^*)	D_R	$3(\bar{3},1)$	-1	-1	0	1/3
$(bd), (bd^*)$	L	$3(1,2)$	0	0	-1	-1/2
(cd)	E_R	$3(1,1)$	0	-1	1	1
(cd^*)	N_R	$3(1,1)$	0	1	1	0

Table 4.1: Standard model spectrum and $U(1)$ charges in the realization in terms of D6-branes with intersection numbers (4.36).

At the intersections there are also complex scalars with the same charge as the chiral fermions [73]; in supersymmetric realizations, some of these scalars are massless and complete the matter chiral multiplets, while in non-supersymmetric realizations they are generally massive (their possible tachyonic character can be avoided by a judicious choice of the complex structure moduli in concrete examples, see [73] for the toroidal case).

One linear combination of the three $U(1)$'s,

$$Y = \frac{1}{6}\left(Q_a - 3Q_c + 3Q_d\right), \tag{4.37}$$

corresponds to the hypercharge generator; it is anomaly free, and should be required to be massless, namely its BF coupling should vanish. In the language of section 4.2.1, we have

$$s_1^k - s_c^k + s_d^k = 0 \quad \text{for all } k, \tag{4.38}$$

where we have accounted for a factor of $N_a = 3$ in the s_a^k term, and have recalled that $N_c = N_d = 1$. Another one, $(3Q_a - Q_d)$ is anomalous, with the anomaly cancelled by the Green-Schwarz mechanism, and becomes massive. The remaining orthogonal combination Y' is anomaly free, and will become massive or not depending on the structure of the BF couplings in the given model.

One can identify the generators discussed in section 3.1.3 as $R = -Q_c$, $L = Q_d$ and $Q_a = 3B$, with B the baryon number. There is no analogue of the A generator in this class of models due to the absence of a $U(1)_b$ associated to the electroweak group.

Depending on the structure of the BF couplings in the model, it is possible to realize the following discrete symmetries:

- **R_N symmetries**

  Since $R = -Q_c$, a R_N symmetry will appear if $s_c^k \in N\mathbb{Z}$ for all k in the model. In particular, the standard R-parity will appear if $s_c^k \in 2\mathbb{Z}$ for all k.

- **L_n symmetries**

  Since $L = Q_d$, a L_N symmetry will appear if $s_d^k \in N\mathbb{Z}$ for all k.

- **Baryon triality**

  This symmetry is realized as a combination $B_3 = R_3 L_3$. This requires the condition $s_c^k + s_d^k \in 3\,\mathbb{Z}$ for all k. It follows from (4.38) that this is equivalent to the condition $s_a^k \in 3\,\mathbb{Z}$ for all k.

  An equivalent derivation is that B_3 can be related to the baryon number B by

  $$B_3 = \frac{2}{3}Y - \frac{1}{3}B. \tag{4.39}$$

  In any SM-like D-brane model, baryon number is realized as $U(1)_a$, and hence B_3 arises from its $\mathbb{Z}_9$ subgroup. Due to the additional multiplicity of $N_a = 3$, this only requires $s_a^k \in 3\,\mathbb{Z}$ for all k in the model.

- Other combinations may be studied analogously.

4.3.2 Non-supersymmetric example

Consider the class of non-supersymmetric SM-like models constructed in [78], based on a toroidal orientifold like the ones described in section 4.2.2. Consider a set of SM branes with wrapping numbers as shown in table 4.2. Here N_g, n_a^2, m_a^3, n_c^1, n_d^2, m_d^3 are integers. The brane b is mapped to itself under the orientifold action, so that the corresponding gauge group is $Sp(2)$, identified with $SU(2)_l$. It is easy to check that these wrapping numbers give rise to the spectrum of a SM with N_g quark/lepton generations.

N_i	(n^1, m^1)	$(n^2.m^2)$	(n^3, m^3)
$N_a = 3$	$(1, 0)$	$(n_a^2, 1)$	(N_g, m_a^3)
$N_b = 1$	$(0, 1)$	$(1, 0)$	$(0, -1)$
$N_c = 1$	$(n_c^1, 1)$	$(1, 0)$	$(0, 1)$
$N_d = 1$	$(1, 0)$	$(n_d^2, -N_g)$	$(1, m_d^3)$

Table 4.2: D6-brane wrapping numbers giving rise to a SM spectrum.

The hypercharge remains massless as long as

$$n_c^1 = n_a^2 m_a^3 + n_d^2 m_d^3. \tag{4.40}$$

The other two linear combinations are generically massive.

RR tadpoles cancel in this model if

$$3m_a^3 = N_g m_d^3. \tag{4.41}$$

In addition one should add $(3n_a^2 N_g + n_d^2 - 16)$ D6-branes (or antibranes, depending on the sign) along the orientifold plane. They have no intersection with the rest of the branes and do not modify the discussion in any way.

The non-vanishing BF couplings are

$$F^a \wedge 3\left(N_g B_2^2 + n_a^2 m_a^3 B_2^3\right), \tag{4.42a}$$
$$F^c \wedge n_c^1 B_2^3, \tag{4.42b}$$
$$F^d \wedge \left(-N_g B_2^2 + n_d^2 m_d^3 B_2^3\right), \tag{4.42c}$$

where we have denoted B_2^p, $p = 0, 1, 2, 3$ the RR 2-forms.

This structure contains some of the discrete gauge symmetries discussed above:

- Baryon triality is quite generic. The $\mathbb{Z}_9$ required for matter parity appears automatically for the physical case $N_g = 3$ as long as $n_a^2 m_a^3 \in 3\,\mathbb{Z}$. More generally, a Z_{N_g} discrete baryon symmetry will be present if $n_a^2 m_a^3 \in N_g\,\mathbb{Z}$.
- Since $R = -Q_c$, the R_N discrete symmetries (including R-parity) are naturally generated with $N = n_c^1$.
- Since $L = Q_d$, a L_{N_g} discrete symmetry appears whenever $n_d^2 m_d^3 \in N_g\,\mathbb{Z}$.
- The symmetry $R_3 L_3^2$ is a $\mathbb{Z}_3$ subgroup of the $U(1)$ generated by $Q_c + Q_d$; therefore, it is realized as a discrete gauge symmetry whenever $n_c^1 + n_d^2 m_d^3 = 3$. This is still compatible with (4.40); for instance, $n_c^1 = 1$, $n_d^2 m_d^3 = 2$, $n_a^2 m_a^3 = -1$.

Some of these symmetries may be realized simultaneously, generating a larger discrete gauge symmetry group. For example, hexality, being a product of R_2 and B_3, will appear for $n_c^1 = 2$ and $n_a^2 m_a^3 \in 3/Z$. These conditions are still compatible with (4.40).

4.3.3 Supersymmetric example

Consider the MSSM-like model in [74] realized in an orientifold of $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ as in [80]. The wrapping numbers (n_α^i, m_α^i) of the different MSSM D6$_\alpha$-branes on the different 2-tori are shown in table 4.3 (ignoring the additional branes required for RR tadpole cancellation), and the resulting spectrum and charge assignments are shown in table 4.4. This corresponds to the intersection numbers (4.36) with a trivial relabelling $d \leftrightarrow d^*$. Note that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold truncates the gauge group on $2N_A$ D6$_A$-branes to $U(N_A)$.

N_α	(n^1, m^1)	$(n^2.m^2)$	(n^3, m^3)
$N_a = 6$	$(1, 0)$	$(N_g, 1)$	$(N_g, -1)$
$N_b = 2$	$(0, 1)$	$(1, 0)$	$(0, -1)$
$N_c = 2$	$(0, 1)$	$(0, -1)$	$(1, 0)$
$N_d = 2$	$(1, 0)$	$(N_g, 1)$	$(N_g, -1)$

Table 4.3: D6-brane wrapping numbers giving rise to a $SU(3) \times SU(2) \times SU(2) \times U(1)_{B-L}$ extension of the MSSM with N_g quark-lepton generations. The $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold truncates the gauge group on $2N_A$ D6$_A$-branes to $U(N_A)$.

Sector	Matter fields	$SU(3)\times SU(2)_L\times SU(2)_R$	Q_a	Q_d	Q_{B-L}
(ab)	Q_L	$3(3,2,1)$	1	0	1/3
(ac)	Q_R	$3(\bar{3},2,1)$	-1	0	-1/3
(db)	L_L	$3(1,2,1)$	0	-1	-1
(dc)	L_R	$3(1,1,2)$	0	1	1
(bc)	H	$(1,2,2)$	0	0	0

Table 4.4: Left-right MSSM spectrum and $U(1)$ charges obtained from table 4.3, for the particular choice $N_g = 3$. The $B-L$ generator is defined as $Q_{B-L} = \frac{1}{3}Q_a + Q_d$.

The BF couplings are

$$F^a \quad\wedge\quad 3N_g\left(B_2^2 - B_2^3\right), \tag{4.43a}$$
$$F^d \quad\wedge\quad N_g\left(B_2^2 - B_2^3\right). \tag{4.43b}$$

In this model $U(1)_{B-L}$ remains as a continuous gauge symmetry, generated by $\frac{1}{3}Q_a + Q_d$. Using a hypercharge shift, this means that Q_c has no BF couplings. Therefore, it does not make sense to discuss discrete R_N symmetries which are contained in a continuous symmetry.

On the other hand, the realization of B_3 as a discrete gauge symmetry is automatic for the physical case with $N_g = 3$. Besides realising baryon triality in a nice and simple way in an explicit MSSM-like D-brane model, this example shows an interesting link between this symmetry and the number of generations. Note that, since $U(1)_{B-L}$ is also a symmetry of the massless spectrum, $R_3L_3^2$ also remains as a discrete gauge symmetry.

4.4 The $U(2)$ class

4.4.1 Generalities

In this class of models the electroweak gauge group $SU(2)_L$ is contained in a $U(2)_b$ factor. There are four stacks of D-branes, denoted a (*baryonic*), b (*left*), c (*right*) and d (*leptonic*). They have $N_1 = 3$, $N_b = 2$, $N_c = 1$, $N_d = 1$, and the initial gauge group is $U(3)_a \times U(2)_b \times U(1)_c \times U(1)_d$.

The main difference with respect to the $Sp(2)$ class is that now there is an extra $U(1)$ gauge boson; although this continuous symmetry is anomalous, it could lead to new anomaly-free discrete $\mathbb{Z}_N$ symmetries. The assignments of Q_b are not family independent. This follows from the structure of intersection numbers required to reproduce the (MS)SM matter content, where the intersection numbers with the b branes and those with their orientifold image b^* (or the intersection numbers of the branes b with other branes or their images) are different; for instance, in the concrete examples that will be studied, $I_{ab} = 1$ but $I_{ab^*} = 2$. Also, it should be noted that this class of models does not have a generalization to arbitrary number of generations N_g, since the latter is related to the number of colours by anomaly cancellation [73].

As in the previous class of models, the hyper charge generator is given by

$$Y = \frac{1}{6}(Q_a - 3Q_c + 3Q_d). \tag{4.44}$$

It is anomaly free, and should be required to be massless, i.e. its BF coupling should vanish.

Let us now identify the discrete symmetries. The symmetry R_N is associated to the generator $-Q_c$, while L_N is associated to the generator Q_d. The presence of $U(1)_b$ allows the realization of an axial symmetry, given by a generation-dependent version of the A_N symmetry in section 3.1.3. $U(1)_b$ also forbids the Yukawa couplings for some of the quark families. Since the A_N symmetry in section 3.1.3 was constructed to preserve Yukawa couplings, we will try to realize a discrete symmetry which will correspond to A_N for those families with Yukawa couplings.

For instance, in the explicit realization shown in section 4.4.2, the generator of the A_N symmetry is given by

$$\tilde{A} = \frac{1}{2}\left(Q_a + Q_b + Q_c + Q_d\right). \tag{4.45}$$

The above linear combination has non-integer coefficients, contrary to our quantization (4.26), so some clarification is needed here.

The quantization of equation (4.45) follows from the fact that any SM field arises from a string with both endpoints on the branes a, b, c or d, so its charge under $Q_a+Q_b+Q_c+Q_d$ is even. The factor of 1/2 in (4.45)brings back the quantization to minimum unit charge. However, one should note that other possible (potentially massive) states in the full theory, arising from strings stretching between the SM and hidden branes, would have fractional charge assignments under $\tilde{A}$. Namely, taking into account all fields in the string model, we should normalise the combination as $Q_a+Q_b+Q_c+Q_d$, according to (4.26). Nevertheless, a $\mathbb{Z}_{2N}$ subgroup acts only as a $\mathbb{Z}_N$ symmetry in the SM fields, identified with the generator (4.45).

To study the appearance of diverse discrete gauge symmetries, we turn to concrete explicit realizations of the model, in the toroidal setup for simplicity.

4.4.2 Non-supersymmetric example

A large number of three generation toriodal non-SUSY SM-like models were constructed in [73]. In those models, the intersection numbers are given by

$$I_{ab} = 1, \qquad I_{ab*} = 2, \tag{4.46a}$$
$$I_{ac} = -3, \qquad I_{ac*} = -3, \tag{4.46b}$$
$$I_{bd} = 0, \qquad I_{bd*} = -3, \tag{4.46c}$$
$$I_{cd} = -3, \qquad I_{cd*} = 3. \tag{4.46d}$$

The wrapping numbers of the SM D6-branes in this family of models are given in table 4.5. The models are parametrised by a phase $\epsilon = \pm 1$, four integers $n_a^2, n_b^1, n_c^1, n_d^2$ and a

N_A	(n_A^1, m_A^1)	(n_A^2, m_A^2)	(n_A^3, m_A^3)
$N_a = 3$	$(\frac{1}{\beta^1}, 0)$	$(n_a^2, \epsilon\beta^2)$	$(\frac{1}{\rho}, \frac{1}{2})$
$N_b = 2$	$(n_b^1, -\epsilon\beta^1)$	$(\frac{1}{\beta^2}, 0)$	$(1, \frac{3}{2}\rho)$
$N_c = 1$	$(n_c^1, 3\rho\epsilon\beta^1)$	$(\frac{1}{\beta^2}, 0)$	$(0, 1)$
$N_d = 1$	$(\frac{1}{\beta^1}, 0)$	$(n_d^2, -\epsilon\frac{\beta^2}{\rho})$	$(1, \frac{3}{2}\rho)$

Table 4.5: D6-brane wrapping numbers giving rise to a SM spectrum for the model with intersection numbers (4.46).

Intersection	Matter fields		Q_a	Q_b	Q_c	Q_d	Y
(ab)	Q_L	$(3, 2)$	1	-1	0	0	1/6
(ab^*)	q_L	$2(3, 2)$	1	1	0	0	1/6
(ac)	U_R	$3(\bar{3}, 1)$	-1	0	1	0	-2/3
(ac^*)	D_R	$3(\bar{3}, 1)$	-1	0	-1	0	1/3
(bd^*)	L	$3(1, 2)$	0	-1	0	-1	-1/2
(cd)	E_R	$3(1, 1)$	0	0	-1	1	1
(cd^*)	N_R	$3(1, 1)$	0	0	1	1	0

Table 4.6: Standard model spectrum and U(1) charges for the model with intersection numbers (4.46).

parameter $\rho = 1, 1/3$. In addition, $\beta^i = 1, 1/2$ depending on whether the corresponding tori are tilted or not; the third torus is tilted for the whole class. The massless chiral spectrum is shown in table 4.6.

Since there are tilted tori, the computation of the conditions for discrete gauge symmetries requires the results from section 4.2.3 (note that in table 4.5 the labels m_A^i for tilted tori actually denote the corresponding tilded quantities of section 4.2.3).

These models have up to four $U(1)$ gauge fields, but generically three of hem acquire Stückelberg masses due to the BF coupling. The masslessness of the hypercharge, generated by (4.44), requires the condition

$$n_c^1 = \frac{\beta^2}{2\beta^1} \left(n_a^2 + 3\rho n_d^2 \right) . \tag{4.47}$$

Two of the three remaining $U(1)$'s are anomalous and massive, and the third one is anomaly free and generically massive, although it may become massless for some choices of wrapping numbers.

The relevant BF couplings are

$$F^a \wedge 3\left(\frac{1}{\rho}B_2^2 + n_a^2\frac{B_2^3}{2}\right), \tag{4.48a}$$

$$F^b \wedge 2\left(-B_2^1 + 3\rho n_b^1\frac{B_2^3}{2}\right), \tag{4.48b}$$

$$F^c \wedge 2n_c^1\frac{B_2^3}{2}, \tag{4.48c}$$

$$F^d \wedge \left(-\frac{1}{\rho}B_2^2 + 3\rho n_d^2\frac{B_2^3}{2}\right), \tag{4.48d}$$

where we have taken $\beta^1 = \beta^2 = \epsilon = 1$ to simplify the expressions, since no new interesting possibilities appear by relaxing those conditions. The factor 1/2 multiplying B_2^3 arises because of the tilting of the third torus; on the other hand, this tilting simultaneously leads to a factor of 2 in the actual shift of the RR scalar dual a_3, as compared with the coefficient of the $F^A B_2^3$ coupling.

The set of discrete gauge symmetries that can be realized is quite analogous to the previous class of models, $Sp(2)$, but now the symmetries cannot be generalized beyond $N_g = 3$:

- Barion triality is obtained for $\rho = 1/3$ if $n_a^2 \in 3/Z$.
- R_N discrete symmetries with N even are naturally generated with $N = 2n_c^1$. In particular, R-parity is automatically implemented in all models in this class.
- The L_3 symmetry appears whenever $\rho = 1/3$ and $n_d^2 \in 3\,\mathbb{Z}$.
- Hexality arises if $n_c^1 = 1$, $\rho = 1/3$ and $n_a^2 \in 3\,\mathbb{Z}$. These conditions are still consistent with (4.47).

The combination $U(1)_{\tilde{A}}$ in (4.45), including the factor 1/2, has coupling $F^{\tilde{A}} \wedge (-B_2^1 + \ldots)$. This means that there is no discrete gauge $\tilde{A}_N$ symmetry that can be realized. This is in fact expected, since such symmetries are anomalous for $N < 9$. Still, it might be possible that such symmetries participate in some anomaly free combination, although none were found in a preliminary search.

Note that there is a seemingly new $\mathbb{Z}_2$ symmetry coming from $U(1)_b$. However, it is just the centre of the $SU(2)_L$ group, and as already discussed in section 4.2.1, does not lead to any useful discrete gauge symmetry.

4.4.3 Supersymmetric example

Consider the MSSM-like models in [167], realized in an orientifold of $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ in [174, 77]. In those models, the intersection numbers are given by

$$I_{ab} = 1, \qquad I_{ab*} = 2, \tag{4.49a}$$

$$I_{ac} = -3, \qquad I_{ac*} = -3, \tag{4.49b}$$

$$I_{bd} = -1, \qquad I_{bd*} = 2, \tag{4.49c}$$

$$I_{cd} = 3, \qquad I_{cd*} = -3. \tag{4.49d}$$

The wrapping numbers are shown in table 4.7 and the massless spectrum and $U(1)$ charges in table 4.8. It is easy to find additional branes so that all RR tadpoles cancel [174]. Note that the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold truncates the gauge group on $2N_A$ D6-branes to $U(N_A)$.

N_A	(n_A^1, m_A^1)	(n_A^2, m_A^2)	(n_A^3, m_A^3)
$N_a = 6$	$(1,0)$	$(3,1)$	$(3,-\frac{1}{2})$
$N_b = 4$	$(1,1)$	$(1,0)$	$(1,-\frac{1}{2})$
$N_c = 2$	$(0,1)$	$(0,-1)$	$(2,0)$
$N_d = 2$	$(1,0)$	$(3,1)$	$(3,-\frac{1}{2})$

Table 4.7: D6-brane wrapping numbers realising the intersection numbers in (4.49) and giving rise to (a SUSY version of) the SM spectrum in table 4.8. The $\mathbb{Z}_2 \times Z_2$ orbifold truncates the gauge group on $2N_A$ D6-branes to $U(N_A)$.

Intersection	Matter fields		Q_a	Q_b	Q_c	Q_d	Y
(ab)	Q_L	$(3,2)$	1	-1	0	0	1/6
(ab^*)	q_L	$2(3,2)$	1	1	0	0	1/6
(ac)	U_R	$3(\bar{3},1)$	-1	0	1	0	-2/3
(ac^*)	D_R	$3(\bar{3},1)$	-1	0	-1	0	1/3
(bd)	L	$(1,2)$	0	-1	0	1	-1/2
(bd^*)	l	$2(1,2)$	0	1	0	1	-1/2
(cd)	N_R	$3(1,1)$	0	0	1	-1	0
(cd^*)	E_R	$3(1,1)$	0	0	-1	-1	1
(bc)	H_d	$(1,2)$	0	-1	1	0	-1/2
(bc^*)	H_u	$(1,2)$	0	-1	-1	0	1/2

Table 4.8: Chiral spectrum spectrum and U(1) charges for the model with intersection numbers (4.49).

In this example, there are two massive and two massless $U(1)$'s, including hypercharge and $B-L$.

The relevant BF couplings are

$$F^a \quad \wedge \quad 9\left(B_2^2 - \frac{B_2^3}{2}\right), \tag{4.50a}$$

$$F^b \quad \wedge \quad 2\left(B_2^1 - \frac{B_2^3}{2}\right), \tag{4.50b}$$

$$F^d \quad \wedge \quad 3\left(B_2^2 - \frac{B_2^3}{2}\right). \tag{4.50c}$$

Again, the third $\mathbb{T}^2$ is tilted, so the coefficients of the $F^A B_2^3$ coupling receives and additional factor of 2 upon dualization to a shift of the dual RR scalar; this effectively removes the factors 1/2 accompanying B_2^3.

In this example, baryon triality B_3 is automatic, and so is L_3. Also, no new non-trivial discrete gauge symmetries arise from the presence of a $U(1)_b$ gauge symmetry.

4.5 Discrete gauge symmetries and D-brane instanton effects

Type IIA compactifications have non-perturbative effects from D2-brane instantons on 3-cycles. Let us denote Π_{inst} the 3-cycle wrapped by the instanton (and probably its orientifold image, if it wraps a 3-cycle invariant under $\mathcal{R}$). Such Π_{inst} can be expanded in terms of the 3-cycles $\{\alpha_k\}$ as

$$[\Pi_{\text{inst.}}] = \sum_k r_{inst}^k \alpha_k. \tag{4.51}$$

In supersymmetric models, there are certain conditions for such instantons to contribute to the superpotential; instantons not satisfying them contribute to other higher dimensional operators, and are often neglected. However, here we are interested in showing that *all* instantons respect the discrete gauge symmetries; hence we must not restrict to super potential generating instantons, and not even to BPS instantons. We must consider instantons in the most general possible class.

The non-perturbative contribution of the instanton to the 4d effective action contains a piece

$$e^{-S_{cl}} = e^{-\frac{V}{g_s}+ia}, \tag{4.52}$$

where

$$a = \int_{\Pi_{\text{inst.}}} C_3 = \sum_k r_{\text{inst.}}^k a_k. \tag{4.53}$$

Under a $U(1)$ gauge transformation (4.29), the instanton exponential rotates by a phase

$$\sum_k r_{\text{inst.}}^k \sum_A c_a N_A s_A^k \lambda. \tag{4.54}$$

As described in [155, 78, 156], this phase rotation is cancelled by the insertion, in the complete instanton amplitude, of 4d fields charged under the $U(1)$ symmetry. This effectively leads to operators whose appearance was forbidden in perturbation theory. Now in the presence of a discrete $\mathbb{Z}_n$ symmetry, namely when the quantities $(\sum_A c_a N_A s_A^k)$ are multiples of n for all k, the instanton exponential shift is a multiple of n, so the non-perturbative effects preserve the $\mathbb{Z}_n$ discrete subgroup. Conversely, the set of charged operators required to cancel the phase rotation of $e^{-S_{cl}}$ have $U(1)$ charges adding up to multiples of n.

It is interesting to provide an alternative microscopic view of the argument. The phase shift (4.54) of the instanton exponent may be written as

$$\sum_A c_A N_A \sum_k r^k_{\text{inst.}} s^k_A = -\sum_A c_A N_A [\Pi_A]\cdot[\Pi_{\text{inst.}}] \equiv -[\Pi_Q]\cdot[\Pi_{\text{inst.}}], \tag{4.55}$$

where in the first equality we have used

$$[\Pi_A]\cdot[\Pi_{inst}] = \sum_k s^k_A r^l_{\text{inst.}}[\beta_k]\cdot[\alpha_l] = -\sum_k r^k_{\text{inst.}} s^k_A, \tag{4.56}$$

and in the second we have defined

$$[\Pi_Q] = \sum_A c_A N_A [\Pi_A]. \tag{4.57}$$

The $\mathbb{Z}_n$ discrete gauge symmetry implies that the intersection number of any instanton with the homology class associated to the $U(1)$ is multiple of n, as follows from (4.28). This intersection number determines the number of instanton fermion modes charged under $U(1)$, and therefore the amount of $U(1)$ charge violation.

Let us finally remark on a complementary mechanism, already mentioned in [79], to ensure that instantons preserve discrete (presumably gauge) $\mathbb{Z}_2$ symmetries. In models where all instantons mapped to themselves under the orientifold action experience an Sp type projection (i.e. $\gamma_\Omega^2 = -1$ for open string with both endpoints on the instanton D-brane), the instanton class expands in the basis α_k as a linear combination with *even* coefficients; in other words, the minimal instanton has worldvolume gauge group $USp(2)$, and arises from two D-brane instantons on the covering space. Hence, the violation of *any* $U(1)$ symmetry by instantons automatically preserves a $\mathbb{Z}_2$ subgroup. A milder version guaranteeing a $\mathbb{Z}_2$ subgroup of *some* $U(1)$, is that any instanton intersecting the class $\Pi_Q]$ of the $U(1)$ and invariant under the orientifold, is of $USp(2)$ type. In the sext subsection we develop the realization of such $\mathbb{Z}_2$ symmetries in a few examples, including a realization of R-parity in an SM-like D-brane construction.

4.5.1 $\mathbb{Z}_2$ symmetries and R-parity from $SP(2)$ instantons

As we mentioned above, it is possible to construct models in which a $\mathbb{Z}_2$ subgroup of *each* $U(1)$ is automatically preserved by instantons. This happens in compactificaitons for which all D-brane instantons mapped to themselves under the orientifold action (invariant instantons for short) have USp worldvolume symmetry (rather than SO). More specifically, a $U91$) inn the model with associated homology charge $[\Pi_Q]$ is violated by a D2-brane instanton (with Chan-Paton multiplicity k) on $[\Pi_{\text{inst.}}]$ by an amount

$$k[\Pi_Q]\cdot([\Pi_{\text{inst.}}] - [\Pi'_{\text{inst.}}]) \in 2\,\mathbb{Z} \quad \text{for non-invariant instantons,} \tag{4.58a}$$

$$k[\Pi_Q]\cdot[\Pi_{\text{inst.}}] \in 2\,\mathbb{Z} \quad \text{for invariant instantons,} \tag{4.58b}$$

where the first line corresponds to an even quantity due to the contributions of branes and images, while the second is an even quantity due to the USp character assumed for invariant instantons.

An example of compactification realizing this mechanism is the $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orientifold with positively charge O6-planes. In the usual choice with negative charge for all four kinds of O6-planes, invariant D6-branes have USp-type orientifold projection, while invariant instantons have SO-type projections; this is reversed in the case of positively charged O6-planes. These models are necessarily non-supersymmetric, because RR-tadpole cancellation must be achieved by the introduction of anti-D6-branes; however, a more general CY orientifold compactification may allow the realization of this mechanism in a supersymmetric fashion.

These $\mathbb{Z}_2$ symmetries are actual gauge discrete symmetries of the theory. Recall from chapter 3 that the Lagrangian description for a $\mathbb{Z}_n$ gauge theory contains

$$\frac{1}{2}(da - nA) \wedge *(da - nA), \tag{4.59}$$

where the order of the symmetry is given by n, if the scalar a has a periodicity 2π, and charges under A are integer.

In the above contest, in which all invariant instantons are of USp-type, instanton numbers are effectively truncated to be even. The periodicity of the scalar a is halved to π, so we must introduce a scalar $a' = 2a$, and write (4.59) as

$$\frac{1}{2}(da - nA) \wedge *(da - nA) = \frac{1}{8}(da' - 2nA) \wedge *(da' - 2nA). \tag{4.60}$$

The latter expression shows that the actual gauge symmetry is $\mathbb{Z}_{2n}$. So, even for $n = 1$ there is a $\mathbb{Z}_2$ discrete gauge symmetry associated to the restriction in the available instanton numbers; models with only USp-type instanton provide a microscopic implementation of the phenomenon in [175] (see also [176]).

Obtaining a $\mathbb{Z}_2$ subgroup of every single $U(1)$ on the model may not be necessarily appealing from a phenomenological point of view. For instance, they may prevent some instantons from generating phenomenologically interesting couplings. So, it may be better to consider models where a $\mathbb{Z}_2$ subgroup of some $U(1)$ is preserved, because all invariant instantons violating it have USp projection, whereas others, not violating the $U(1)$, may have $O(1)$ projections.

As an example, we consider a version of the model in table 4.3, embedded in a $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orientifold; this requires doubling the number of D6-branes in each stack to $2N$ to generate an $U(N)$ symmetry. We make the usual choice of discrete torsion corresponding to Hodge numbers $(h_{1,1}, h_{2,1}) = (51, 3)$. As shown in [177], this choice requires having an even number of negatively charged O6-planes, among the four kinds present in the model. Instead of choosing all O6-planes to have negative charge (which is the usual choice), we choose negatively charged O6-planes along $[a_1][a_2][a_3]$, $[a_1][b_2][b_3]$, and positively charged O6-planes along $[b_1][a_2][a_3]$ and $[b_1][b_2][b_3]$. This does not modify the appearance of the SM spectrum from the visible branes, since the orientifold signs only enter in the multiplicities of two-index tensor representations in AA' sectors, which are massless in the model, and would only change the set of hidden branes to cancel the tadpole, which is ignored for simplicity.

Consider the $U(1)_c$ gauge factor, which is associated to the R_N generators. The only invariant instanton intersection $[\Pi_c]$ is that wrapped on $[b_1][b_2][a_3]$, and it has Sp orientifold projection, due to the choice of O6-plane charge. Hence a $\mathbb{Z}_2$ subgroup of $U(1)_c$, which corresponds to R-parity, is automatically preserved by all D-brane instanton effects in the model.

4.6 K-theory $\mathbb{Z}_2$ and R-parity

The K-theory constraints in orientifold models force some combinations of quantities to be even [173]. Interestingly enough, these quantities arise as the coefficients of the BF couplings in the model. Hence, in certain classes of construction, the K-theory constraints imply the existence of an anomaly-free $\mathbb{Z}_2$ discrete gauge symmetry, which we denote K_2. In this section we will describe the conditions for its existent and its interplay with the massless $U(1)$ possibly present in the model.

Consider an orientifold with D6$_A$-branes on a general Calabi-Yau orientifold, with basis $\{\alpha_k\}$, $\{\beta_k\}$ of even and odd cycles, and assume for simplicity that $\alpha_k \cdot \beta_l = \delta_{kl}$. The K-theory constraints in the model have the structure

$$\sum_A N_A c_{Ak_1} \in 2\,\mathbb{Z}, \tag{4.61}$$

for all k_1 in a subset of the odd cycles. This condition is not necessarily imposed to all odd cycles. For instance, in orientifolds of $\mathbb{T}^6$ the K-theory constraints are

$$\sum_a N_a m_A^1 n_a^2 n_a^3 \in 2\,\mathbb{Z}, \tag{4.62a}$$

$$\sum_a N_a n_A^1 m_a^2 n_a^3 \in 2\,\mathbb{Z}, \tag{4.62b}$$

$$\sum_a N_a n_A^1 n_a^2 m_a^3 \in 2\,\mathbb{Z}, \tag{4.62c}$$

whereas there is no constraint on the combination $\sum_A N_A m_A^1 m_a^2 m_a^3$. So k_1 labels the odd cycles α_1, α_2, α_3 but not α_0.

Also, not all branes contribute, i.e. some c_{Ak} may be zero. For instance, the branes b, c, in the model in table 4.3 have no contribution to the K-theory constraints. Labelling with A_1 those branes for which $c_{A_1k_1} \neq 0$, the K-theory constraints read

$$\sum_{A_1} N_{A_1} c_{A_1k_1} \in 2\,\mathbb{Z}\,. \tag{4.63}$$

The BF couplings have a structure

$$\sum_A \sum_k N_A c_{Ak} B_k \wedge F_A = \sum_{A_1} \sum_{k_1} N_A c_{A_1k_1} B_{k_1} \wedge F_A + \sum_A \sum_{k_2} N_A c_{Ak_2} B_{k_2} \wedge F_A \tag{4.64}$$

where k runs over all odd cycles, and k_1, k_2 label those with or without associated K-theory charge cancellation constraint. Note that in the first term, there are contributions only from branes with label A_1, i.e. participating in (4.63).

Now assume that $c_{A_1 k_2} = 0$ for all A_1. Even if this condition sounds strong, holds even in the simplest semi-realistic intersecting D6-brane models in $\mathbb{T}^6$ orientifolds studied in sections 4.3 and 4.4.

Under these assumptions, the diagonal combination of the $U(1)'$s contributing to the K-theory charges

$$Q_k = \sum_{A_1} Q_{A_1} \tag{4.65}$$

has BF coupling

$$\sum_{A_1} N_{A_1} c_{A_1 k_1} B_{k_1} \wedge F_{A_1} \tag{4.66}$$

The K-theory constraint (4.63) implies the existence of a $\mathbb{Z}_2$ discrete gauge symmetry K_2.

Since many models have massless $U(1)'$s, one must ensure that K_2 is not just a subgroup of these. We write the massless $U(1)$ generator as

$$Q = \sum_{A} r_a Q_A \tag{4.67}$$

with $r_A \in \mathbb{Z}$ and $\gcd(r_A) = 1$, so that charges are integer with minimal charge one. If $r_A = odd$ for all A, then $Q = Q_K \text{mod} 2$, and K_2 is just a subgroup of the massless $U(1)$. Even if this seems non-generic, occurs in many SM-like D-brane models, where the hypercharge generator is tipically of the form

$$6Y = Q_a - 3Q_c + 3Q_d \tag{4.68}$$

Since its coefficients are odd, in models where the branes a, c, d contribute to the K-theory charges, the symmetry K_2 is just a subgroup of hypercharge. This is the case in all the models studied in sections 4.3 and 4.4.

There are several possible ways to relax the constraints on the BF cowlings of c. For instance, the hypercharge combination may involve extra ‘hidden’ $U(1)$ generators; this however invokes symmetries beyond the visible MSSM-like sector. Another possibility is to relax the condition $c_{A_1 k_2} = 0$ to $c_{A_1 k_2} \in 2\mathbb{Z}$, and still have a $\mathbb{Z}_2$ symmetry from (4.65); however this exploits additional even-ness requirements, beyond the genuinely K-theoretical one. Since both possibilities are beyond the intended scope of this section, we will not pursue them.

5

Torsion p-forms and discrete gauge symmetries

In this chapter we will present an explicit string theory example of the ideas described in section 3.3.

We will show that non-Abelian discrete gauge symmetries can arise from compactifications of p-form fields along torsion homology classes with non-trivial relations, extending the observation for 5d theories in [138] (see also [178, 179]).

We will perform the dimensional reduction from 10d, and find a 4d Lagrangian realising non-Abelian discrete gauge symmetries, typically discrete Heisenberg groups, in terms of gaugings of non-Abelian axions.

This generalizes the relation between torsion homology and discrete symmetries observed in the Abelian case in [141].

5.1 Abelian discrete gauge symmetries and torsion homology

Before describing the non-Abelian case let us review the relation of Abelian discrete gauge symmetries to torsion classes [141]. As mentioned in section 3.1, a practical way to identify discrete gauge symmetries is to tag a set of $\mathbb{Z}_k$ charged particles and $\mathbb{Z}_k$ charged strings inducing relative holonomies on each other via the Aharonov-Bohm phase (3.12), which we reproduce here for convenience

$$\exp\left[2\pi i \frac{np}{k} L(\Sigma, C)\right], \tag{5.1}$$

In string theory compactifications, we thus search for dynamical objects in the higher-dimensional theory that lead to Aharonov-Bohm strings and particles in the 4d effective theory.

5.1.1 Aharonov-Bohm strings and particles from torsion

A simple way to obtain Aharonov-Bohm strings and particles in type II vacua is to consider D-branes or NS-branes wrapped on p-cycles of the compactification manifold, the inequivalent possibilities being classified in terms of homology. In general, the homology group of a D-dimensional manifold $\mathbb{X}_D$ consists of a free part, given by $b_p \equiv \dim H_r(X_D, \mathbb{R})$ copies of $\mathbb{Z}$, and a torsion part, given by a set of finite $\mathbb{Z}_k$ groups,

$$H_p(\mathbb{X}_D, \mathbb{Z}) = H_p^{\text{free}}(\mathbb{X}_D, \mathbb{Z}) \oplus \text{Tor}\, H_p(X_D, \mathbb{Z}) = \mathbb{Z}^{b_p} \oplus (\mathbb{Z}_{k_1} \oplus \ldots \oplus \mathbb{Z}_{k_n}). \tag{5.2}$$

It has been argued in [141] that 4d Aharonov-Bohm strings and particles arising from a compactification in $\mathbb{X}_D$ are associated to the torsion part of the corresponding homology lattice. This is based on the observation that if we wrap a p-brane on a torsion p-cycle π_p^{tor} and a dual $(D-p)$-brane on a torsion $(D-p-1)$-cycle π_{D-p-1}^{tor} then we will have a 4d particle and string, respectively, that induce fractional holonomies on each other proportional to the torsion linking number $L([\pi_p^{\text{tor}}], [\pi_{D-p-1}^{\text{tor}}])$ in the internal dimensions. Such torsion linking number is one of the main topological invariants that can be defined for the torsion homology classes of $\mathbb{X}_D$, and it univocally relates torsion classes of p-cycles to torsion classes of $(D-p-1)$-cycles, such that

$$\text{Tor}\, H_p(X_D, \mathbb{Z}) \simeq \text{Tor}\, H_{D-p-1}(X_D, \mathbb{Z}). \tag{5.3}$$

Let us be more specific and consider M-theory compactified on a manifold $\mathbb{X}_7$ with G_2 holonomy. Gauge symmetries in the 4d effective theory arise from the M-theory 3-form A_3 and are classified by elements of $H_2(\mathbb{X}_7, \mathbb{Z})$. On the one hand, elements belonging to the free part of $H_2(\mathbb{X}_7, \mathbb{Z})$ are in one-to-one correspondence with harmonic 2-forms in $\mathbb{X}_7$ so, upon expanding A_3 in such 2-forms, we obtain standard $U(1)$ gauge symmetries in the 4d effective theory. On the other hand, elements that belong to $\text{Tor}\, H_2(\mathbb{X}_7, \mathbb{Z})$ must correspond to discrete $\mathbb{Z}_{k_i}$ gauge symmetries.[1] This can be seen from the fact that M2-branes wrapping torsion 2-cycles lead to Aharanov-Bohm particles in 4d, whereas M5-branes wrapping the dual torsion 4-cycles (which exist because of eq.(5.3)) lead to 4d Aharanov-Bohm strings.

Indeed, let us consider an M2-brane wrapping a $\mathbb{Z}_k$ torsion 2-cycle π_2^{tor} and with 4d worldline C, as well as a 4d string with worldsheet Σ that arises from an M5-brane wrapping a $\mathbb{Z}_k$ torsion 4-cycle π_4^{tor} of $\mathbb{X}_7$. Following [141], one can see that the holonomy that these two objects induce on each other is given by

$$\frac{1}{2\pi i}\log\,[\text{hol}(\Sigma, C)] \overset{\text{mod}\,1}{=} \int_{C\times\pi_2^{\text{tor}}} A_3 = \frac{1}{k}\int_{D\times k\pi_2^{\text{tor}}} F_4 = \frac{1}{k}\int_{D\times S_3} \delta_5, \tag{5.4a}$$

$$\overset{\text{mod}\,1}{=} \int_{\Sigma\times\pi_4^{\text{tor}}} A_6 = \frac{1}{k}\int_{B\times k\pi_4^{\text{tor}}} F_7 = \frac{1}{k}\int_{B\times S_5} \delta_8. \tag{5.4b}$$

The upper chain of equalities represent the Aharanov-Bohm effect that a 4d string creates on a 4d particle circling around it with a path $C = \partial D$. Indeed, the M5-brane that becomes a 4d string will create a flux F_4 via backreaction, and we should integrate the corresponding potential A_3 on the M2-brane worldvolume $C \times \pi_2^{\text{tor}}$ to compute the induced holonomy on the 4d particle. The computation is then carried by applying Stokes' theorem and by noticing that because π_2^{tor} is k-torsion there is a 3-chain S_3 such that $\partial S_3 = k\pi_2^{\text{tor}}$, and that $dF_4 = \delta_5$ with δ_5 a bump 5-form transverse to the M5-brane worldvolume $\Sigma \times \pi_4^{\text{tor}}$. Similarly, the lower chain represents the holonomy created by the 4d particle on a 4d string surrounding it with $\Sigma = \partial B$, with now $\partial S_5 = k\pi_4^{\text{tor}}$ and $dF_7 = \delta_8$. Notice that the

[1] If the manifold has discrete isometries, there can be in addition discrete gauge symmetries coming from the metric, see chapter 7.

integral of a bump function like δ_5 or δ_8 is always an integer, and so we end up with a fractional holonomy of the form exp $(2\pi i\ell/k)$ with $\ell \in \mathbb{Z}$. One can see that the integer ℓ in equations (5.4a) and (5.4b) is the same integer mod k, since both quantities in the r.h.s. are the definition of the torsion linking number $L([\pi_2^{\rm tor}],[\pi_4^{\rm tor}])$ multiplied by the 4d linking number $L(\Sigma, C)$ of equation (5.1).

To summarize, one finds that the Aharanov-Bohm phase that an M2-brane and an M5-brane wrapped on torsion cycles create on each other is given by

$$\exp\left[\,2\pi i\, L([\pi_2^{\rm tor}],[\pi_4^{\rm tor}])\cdot L(\Sigma, C)\right] \tag{5.5}$$

Comparing with eq.(5.1), we can identify $L([\pi_2^{\rm tor}],[\pi_4^{\rm tor}]) = np/k$, and so the charges n and p of the 4d objects correspond in the higher dimensional M-theory picture to choose torsion cycles with appropriate linking numbers.

This M-theory picture allows to reinterpret the Abelian discrete gauge symmetries that arise in type IIA compactifications with intersecting D6-branes [140]. Indeed, if the G_2 manifold $\mathbb{X}_7$ admits a weakly coupled type IIA limit with D6-branes, some of the $U(1)$ symmetries classified by $H_2(\mathbb{X}_7,\mathbb{Z})$ are downlifted to $U(1)$ symmetries localized at D6-branes. Massless 4d particles charged under such $U(1)$'s, which in type IIA are open strings at the D6-brane intersections, correspond to M2-branes wrapping collapsed 2-cycles of $\mathbb{X}_7$. The $U(1)$ gauge symmetries that in M-theory are related to $H_2^{\rm free}(\mathbb{X}_7,\mathbb{Z})$ become in type IIA D6-brane $U(1)$ symmetries without any axion coupling, while those discrete gauge symmetries related to Tor $H_2(\mathbb{X}_7,\mathbb{Z})$ become D6-brane U(1)'s broken to $\mathbb{Z}_k$ through axion couplings. Consequently, massless 4d particles are charged under the unbroken $U(1)$'s if they are M2-branes wrapped on non-torsional 2-cycles, while particles that only have $\mathbb{Z}_k$ charges correspond to M2-branes wrapping collapsed torsional 2-cycles of $\mathbb{X}_7$.

This M-theory perspective provides also a geometrization of the instanton contribution structure (3.9), as follows. Consider a set of particles ψ_i with $\mathbb{Z}_k$ charges, namely a set of M2-branes wrapping torsion 2-cycles D_i; whenever the total homology charge of the combination is zero (in particular, the torsion classes add up to a trivial class), there exists a 3-chain S connecting them ($\partial S = \sum_i D_i$). An M2-brane wrapped on S would describe an instanton effect on the 4d theory, but it contains open holes. A completely consistent instanton can be obtained by glueing M2-branes on D_i, emerging from the instanton from the 4d perspective. This is precisely the dressed instanton structure (3.9) with $\mathcal{O} = \prod_i \psi_i$. Also, this is the M-theory picture of a D2-brane instanton with insertions of 4d charged matter multiplets, observed in [155, 78, 156].

5.1.2 Torsion and dimensional reduction

Interestingly, this geometrical picture that relates torsion to discrete gauge symmetries can also be made manifest by means of dimensional reduction [141]. For this, we need to associate to each generator of Tor $H_p(\mathbb{X}_D,\mathbb{Z})$ a differential p-form which is also an eigenform of the Laplacian, just like we do when we associate harmonic p-forms to the generators of $H_p^{\rm free}(\mathbb{X}_D,\mathbb{Z})$. In the case of torsion groups, however, these eigenforms must have a non-zero eigenvalue and in order to reproduce the topological information of Tor $H_p(\mathbb{X}_D,\mathbb{Z})$ we must consider non-closed p-forms satisfying specific relations. More precisely, given the generators of Tor $H_p(\mathbb{X}_D,\mathbb{Z})$ and Tor $H_{D-p-1}(\mathbb{X}_D,\mathbb{Z})$ we consider non-closed p- and $(D-p-1)$-forms ω_α and α^β such that [141]

$$d\omega_\alpha = k_\alpha{}^\beta \beta_\beta, \tag{5.6a}$$
$$d\alpha^\beta = (-1)^{D-p} k^\beta{}_\alpha \tilde{\omega}^\alpha, \tag{5.6b}$$

where β_β and $\tilde{\omega}^\alpha$ are exact eigenforms of the Laplacian which are trivial in de Rham cohomology but represent non-trivial elements of $H^{p+1}(\mathbb{X}_D,\mathbb{Z})$ and $H^{D-p}(\mathbb{X}_D,\mathbb{Z})$, respectively. Moreover, $k_\alpha{}^\beta \in \mathbb{Z}$ must be given by

$$L([\pi^{\rm tor}_{p,\alpha}],[\pi^{{\rm tor},\beta}_{D-p-1}]) = (k^{-1})_\alpha{}^\beta, \tag{5.7}$$

so that it contains the topological information of the torsion cycles that these eigenforms are related to. Finally, the integral of these forms satisfy

$$\int_{\mathbb{X}_D} \alpha^\rho \wedge \beta_\sigma = \int_{\mathbb{X}_D} \tilde{\omega}^\rho \wedge \omega_\sigma = \delta^\rho_\sigma \tag{5.8}$$

Including this set of non-harmonic eigenforms when performing the dimensional reduction allows to reproduce the 4d Lagrangian for Abelian discrete gauge symmetries, and in particular displays the gauging structure discussed in section 3.3.1. Indeed, taking again the above example of M-theory on 7-manifolds, for each torsion 2-cycle we need to consider an exact 3-form α_3 and a non-closed 2-form ω_2, with $d\omega_2 = k\beta_3$ and $k \in \mathbb{Z}$. Expanding the M-theory 3-form A_3 in such eigenforms we obtain

$$A_3 = \phi(x^\mu) \wedge \beta_3 + A_1(x^\mu) \wedge \omega_2 + \dots \tag{5.9}$$

namely, a 4d $U(1)$ gauge boson A_1 and a 4d scalar ϕ. One can check that the gauge transformation (3.54) shifts A_3 by the exact 3-form $d(\lambda\omega_2)$ and so it indeed leaves any 4d quantity invariant. In particular we have that

$$dA_3 = (d\phi - kA_1) \wedge \beta_3 + dA_1(x^\mu) \wedge \omega_2 + \dots \tag{5.10}$$

and so the 4d Lagrangian (3.51) arises from the dimensional reduction of the 11d kinetic term $dA_3 \wedge *_{11} dA_3$. These observations will be exploited and generalized in the next subsections in the context of type IIB compactifications, in order to reproduce via dimensional reduction the 4d Lagrangian of non-Abelian discrete gauge symmetries.

5.2 Non-Abelian discrete symmetries from torsion homology

Torsion classes have appeared in an example in [138] as a source of discrete non-Abelian gauge symmetries in 5d in the AdS/CFT setup (see also [178, 179]). In this subsection we further explore and generalize this realization in the 4d setup, unveiling that the key to non-Abelianity lies in the existence of wedge (or cup) product relations among torsion classes. The corresponding dimensional reduction allows an elegant derivation of a general class of 4d theories with non-Abelian discrete gauge symmetry.

5.2.1 Non-Abelian strings and the Hanany-Witten effect

In order to describe the link between non-Abelian discrete gauge symmetries and torsion let us consider the class of models given by type IIB compactifications to 6d. In a generic 6d manifold there are two independent torsion classes, corresponding to torsion 1-cycles (and 4-cycles) and torsion 2-cycles (and 3-cycles)

$$\mathrm{Tor}\, H_1(\mathbb{X}_6, \mathbb{Z}) \simeq \mathrm{Tor}\, H_4(\mathbb{X}_6, \mathbb{Z}), \tag{5.11a}$$

$$\mathrm{Tor}\, H_2(\mathbb{X}_6, \mathbb{Z}) \simeq \mathrm{Tor}\, H_3(\mathbb{X}_6, \mathbb{Z}). \tag{5.11b}$$

The first class actually describes two different kinds of discrete gauge symmetries: one of them associated to spontaneously broken $U(1)$ symmetries that result from reducing the RR 2-form C_2 and the other to the spontaneously broken $U(1)$'s that result from reducing the NSNS 2-form B_2. In the latter case the 4d particles and strings charged under the discrete gauge symmetry arise from fundamental strings wrapping torsion 1-cycles and NSNS 5-branes wrapping torsion 4-cycles, respectively, while in the former case they arise from D1 and D5-branes. On the other hand, the second class in (5.11) describes discrete gauge symmetries associated to the RR 4-form C_4, with charged particles and strings arising from D3-branes wrapping torsion 3-cycles and 2-cycles, respectively.

As emphasized above, in compactifications with torsion classes the key to non-Abelianity is encoded in the existence of relations between torsion elements. Let us be more specific and consider the simple case where the torsion groups of $\mathbb{X}_6$ are given by

$$\mathrm{Tor}\, H_1(\mathbb{X}_6, \mathbb{Z}) = \mathrm{Tor}\, H_4(\mathbb{X}_6, \mathbb{Z}) = \mathbb{Z}_k, \tag{5.12a}$$

$$\mathrm{Tor}\, H_2(\mathbb{X}_6, \mathbb{Z}) = \mathrm{Tor}\, H_3(\mathbb{X}_6, \mathbb{Z}) = \mathbb{Z}_{k'}\,. \tag{5.12b}$$

In general $k \neq k'$ although their precise relation is not relevant for our momentary purposes. Naively, considering general (p, q)-strings and 5-branes, the torsion 1-cycles would seem to produce a $\mathbb{Z}_k \times \mathbb{Z}_k$ symmetry, while also considering D3-branes in torsion cycles would add an extra $\mathbb{Z}_{k'}$ factor. This mere Abelian structure is however promoted to a non-Abelian one if the corresponding classes have non-trivial relations. Indeed, if the torsion 4-cycles dual to the 1-cycles intersect non-trivially along a torsion 2-cycle, there is a non-trivial Hanany-Witten effect [180] between the 4d strings obtained from NS5 and D5-branes wrapping the torsion 4-cycles. Crossing the strings in 4d leads to the creation of D3-branes wrapped on the torsion 2-cycle at the intersection of the 4-cycles, namely the creation of a 4d string associated to the RR 4-form. This 4d string creation effect is associated to non-Abelian discrete symmetry groups [127, 132, 133, 134, 135, 137]. At the level of the gauge holonomies that result from moving around the 4d strings, we have the non-Abelian relation

$$\tilde{T}_1 \tilde{T}_2 = \tilde{T}_3 \tilde{T}_2 \tilde{T}_1 \tag{5.13}$$

among the generators $\tilde{T}_1$, $\tilde{T}_2$ of the two $\mathbb{Z}_k$'s and the generator $\tilde{T}_3$ of $\mathbb{Z}_{k'}$. This defines a finite Heisenberg group (c.f. equation (3.86)). The same result can be obtained by working out the non-Abelian transformations undergone by particles moving around 4d strings, again by invoking the Hanany-Witten effect [138].

5.2.2 Dimensional reduction and four-dimensional effective action

Just like in the Abelian case, this microscopic description of a non-Abelian discrete gauge symmetry should have a macroscopic counterpart via dimensional reduction. Indeed, we will show below how a 4d effective Lagrangian reproducing such non-Abelian symmetries can be obtained by following the same procedure as in the Abelian case. Again, in order to perform the dimensional reduction we need to consider a set of non-harmonic forms satisfying (5.6), together with certain relations among them which are necessary for the non-Abelian pattern to emerge, and are equivalent to the topological conditions which allow for the Hanany-Witten effect. For simplicity, we will consider here the simple case where the torsion classes of $\mathbb{X}_6$ are given by (5.12). The more general case can be worked out in a similar way, as it is explicitly done in section 5.3.

More precisely, we consider a set of non-harmonic Laplacian eigenforms in $\mathbb{X}_6$

$$d\gamma_1 = k\rho_2, \tag{5.14a}$$
$$d\alpha_3 = k'\tilde{\omega}_4, \tag{5.14b}$$
$$d\tilde{\rho}_4 = k\zeta_5, \tag{5.14c}$$
$$d\omega_2 = k'\beta_3, \tag{5.14d}$$

with ρ_2, $\tilde{\omega}_4$, ζ_5 and β_3 representing the generators of the torsion cohomology Poincaré dual to (5.12), and such that

$$\int_{\mathbb{X}_6} \gamma_1 \wedge \zeta_5 = \int_{\mathbb{X}_6} \rho_2 \wedge \tilde{\rho}_4 = \int_{\mathbb{X}_6} \alpha_3 \wedge \beta_3 = \int_{\mathbb{X}_6} \omega_2 \wedge \tilde{\omega}_4 = 1 \tag{5.15}$$

In these expressions k^{-1} and k'^{-1} are the torsion linking numbers between dual p- and $(5-p)$-cycles, with $p = 1, 3$ respectively, and encode the monodromies which are felt by an electric (magnetic) charge when moved in a closed loop around its dual magnetic (electric) source. The fact that these torsion cycles have a non-trivial intersection pattern as described above is expressed in terms of these dual forms as

$$\rho_2 \wedge \rho_2 = M\tilde{\omega}_4, \tag{5.16}$$

with $M \in \mathbb{Z}$, which can be integrated to[2]

$$\rho_2 \wedge \gamma_1 = M'\,\alpha_3 \qquad M' \in \mathbb{Z} \;\text{ such that }\; k\,M = k'\,M' \tag{5.17}$$

Let us then perform dimensional reduction of the type IIB supergravity action, taking into account the relations that we have introduced above. The relevant part of the action written in the 10d Einstein frame is

$$S_{10\mathrm{d}} = \frac{1}{4\kappa_{10}^2}\int d^{10}x \left[(-G_E)^{1/2}\left(-\mathcal{M}_{ij}\; dB_2^i \cdot dB_2^j - \frac{1}{2}(F_5)^2\right) + \frac{\epsilon_{ij}}{2} dC_4 \wedge B_2^i \wedge dB_2^j\right] \tag{5.18}$$

where $B_2^1 \equiv B_2$ and $B_2^2 \equiv C_2$ are respectively the NSNS and RR 2-form potentials, $F_5 = dC_4 - C_2 \wedge dB_2$, the matrix $\mathcal{M}_{ij}$ denotes the SO(2)\SL(2,$\mathbb{Z}$) coset metric

$$\mathcal{M}_{ij} = \frac{1}{\mathrm{Im}\,\tau}\begin{pmatrix} |\tau|^2 & -\mathrm{Re}\;\tau \\ -\mathrm{Re}\;\tau & 1\end{pmatrix} \tag{5.19}$$

[2]In principle, instead of (5.17) one could have chosen the more general condition

$$\rho_2 \wedge \gamma_1 = M'\alpha_3 + M''\beta_3 \qquad M', M'' \in \mathbb{Z}$$

This choice however, corresponds to gauging also the magnetic degrees of freedom and it will not be explored here.

and $\tau = C_0 + ie^{-\phi}$ the complex axio-dilaton.

In order to dimensionally reduce this action, we expand the NSNS and RR 2-forms and the RR 4-form potentials as[3]

$$B_2^i = b^i \rho_2 + A_1^i \wedge \gamma_1, \qquad i = 1, 2 \tag{5.20a}$$
$$C_4 = b^3 \tilde{\omega}_4 + A_1^3 \wedge \alpha_3 + V_1^3 \wedge \beta_3 + c_2 \wedge \omega_2, \tag{5.20b}$$

obtaining several 4d vectors and scalars. The corresponding 10d field-strengths read

$$dB_2^i = \eta^i \wedge \rho_2 + dA_1^i \wedge \gamma_1, \qquad i = 1, 2 \tag{5.21a}$$
$$F_5 = \eta^3 \wedge \tilde{\omega}_4 - F_2^3 \wedge \alpha_3 + \tilde{F}_2^3 \wedge \beta_3 + dc_2 \wedge \omega_2, \tag{5.21b}$$

where we have introduced the following 4d 1-form potentials

$$\eta_\mu^i \equiv \partial_\mu b^i - k A_\mu^i, \tag{5.22a}$$
$$\eta_\mu^3 \equiv \partial_\mu b^3 - k' A_\mu^3 - M b^2 \eta_\mu^1 \tag{5.22b}$$

and field-strengths

$$k' F_2^3 \equiv d\eta^3 - \frac{\epsilon_{ij}}{2} M \eta^i \wedge \eta^j, \tag{5.23a}$$
$$\tilde{F}_2^3 \equiv dV_1^3 + k' c_2, \tag{5.23b}$$

and we have made use of the relations (5.16) and (5.17).

Substituting these expansions into equation (5.18) we get (up to total derivatives and in 4d Planck mass units)

$$\begin{aligned} S_{4\text{d}} =& \frac{1}{4}\int d^4x \Big[(-g)^{1/2} \Big(- \mathcal{M}_{ij}\, \mathcal{N}\, dA_1^i \cdot dA_1^j - \mathcal{M}_{ij}\, \mathcal{T}\, \eta^i \cdot \eta^j - \frac{\mathcal{R}}{2} (F_2^3)^2 \\ &+ \mathcal{Q}\, F_2^3 \cdot \tilde{F}_2^3 + \frac{\mathcal{S}}{2} (\tilde{F}_2^3)^2 - \frac{\mathcal{G}}{2} (dc_2)^2 - \frac{\mathcal{G}^{-1}}{2} (\eta^3)^2 \Big) - \eta^0 \wedge dc_2 - \tilde{F}_2^3 \wedge F_2^3 \Big], \end{aligned} \tag{5.24}$$

where we have defined[4]

$$\mathcal{N} \equiv \int_{X_6} \gamma_1 \wedge *_6 \gamma_1, \tag{5.25a}$$
$$\mathcal{T} \equiv \int_{X_6} \rho_2 \wedge *_6 \rho_2, \tag{5.25b}$$
$$\mathcal{Q} \equiv \int_{X_6} \alpha_3 \wedge *_6 \beta_3, \tag{5.25c}$$
$$\mathcal{R} \equiv \int_{X_6} \alpha_3 \wedge *_6 \alpha_3, \tag{5.25d}$$
$$\mathcal{S} \equiv \int_{X_6} \beta_3 \wedge *_6 \beta_3, \tag{5.25e}$$
$$\mathcal{G} \equiv \int_{X_6} \omega_2 \wedge *_6 \omega_2. \tag{5.25f}$$

[3]This expansion is the most general one if we assume an underlying orientifold structure, according to which γ_1 and ρ_2 must be odd and ω_2, $\tilde{\omega}_4$, α_3 and β_3 even forms under the orientifold action. We also ignore 4d 2-forms resulting from B_2^i, as they do not play any role in what follows.

[4]Note that idempotency of the hodge operator imply the non-trivial relation $\mathcal{R}\,\mathcal{S} + \mathcal{Q}^2 = -1$, so these quantities are not all independent.

Since we have not yet imposed the self-duality condition of the RR 5-form field-strength, $F_5 = *_{10}F_5$, the 4d effective action (5.24) contains redundant degrees of freedom. Making use of

$$\tilde{F}_2^3 = -F_2^3\,\mathcal{Q}\,\mathcal{S}^{-1} - *_4\,F_2^3\,\mathcal{S}^{-1}, \tag{5.26a}$$
$$dc_2 = \mathcal{G}^{-1}\,*_4\eta^3, \tag{5.26b}$$

we finally obtain

$$\begin{aligned} S_{4\mathrm{d}} =& \frac{1}{4}\int d^4x\Big[(-g)^{1/2}\Big(- \mathcal{M}_{ij}\,\mathcal{T}\,\eta^i\cdot\eta^j - \mathcal{G}^{-1}(\eta^3)^2 \\ & - \mathcal{M}_{ij}\,\mathcal{N}\,dA_1^i\cdot dA_1^j + \mathcal{S}^{-1}(F_2^3)^2\Big) + \mathcal{Q}\,\mathcal{S}^{-1}\,F_2^3\wedge F_2^3\Big] \end{aligned} \tag{5.27}$$

From the first line of this equation and comparing to equation (3.76) we observe that the 4d axion-like scalars in this setup span a gauged scalar manifold with tangent space metric

$$\mathcal{P}_{ab} = -\frac{1}{4}\begin{pmatrix}\mathcal{G}^{-1} & 0\\ 0 & \mathcal{T}\,\mathcal{M}_{ij}\end{pmatrix}\,, \tag{5.28}$$

right-invariant 1-forms given by the equations (3.84) upon the following identifications

$$\phi^1 = b^1, \qquad \phi^2 = b^2, \qquad \phi^3 = b^3 - \frac{M}{2}b^1b^2, \tag{5.29a}$$
$$k_1 = k_2 = k, \qquad k_3 = k', \tag{5.29b}$$
$$A^1_\mu\big|_{\mathrm{sec.3.3.3.3}} = A^1_\mu,\quad A^2_\mu\big|_{\mathrm{sec.3.3.3.3}} = A^2_\mu,\quad A^3_\mu\big|_{\mathrm{sec.3.3.3.3}} = A^3_\mu - \frac{M'}{2}(b^1A^2_\mu + b^2A^1_\mu), \tag{5.29c}$$

and structure constants of the Heisenberg algebra $\mathcal{H}_3$. The example based on the Heisenberg manifold $\tilde{\mathcal{M}} = \mathcal{H}_3(\mathbb{R})/\Gamma$ discussed in section 3.3.3.3 is thus physically realized in a large class of type IIB compactifications with torsional homology.

5.2.3 Non-Abelian discrete gauge symmetries

As the 4d effective action (5.27) is identical to the one analyzed in section 3.3.3.3, the discrete gauge symmetries that one obtains from it can be directly extracted from the discussion therein. It is however illustrative to reproduce the previous 4d discussion from a 10d perspective. In the present context, the shift symmetries of the scalars b^1, b^2, and b^3 are inherited from the 10d gauge transformations of B_2, C_2 and C_4. Indeed, at the perturbative level we have that the 10d field strengths (5.21a) dB_2, dC_2 and F_5 are invariant under any of the following shifts

$$B_2 \longrightarrow B_2 + \epsilon^1\rho_2,, \tag{5.30a}$$
$$C_2 \longrightarrow C_2 + \epsilon^2\rho_2, \tag{5.30b}$$
$$C_4 \longrightarrow C_4 + \epsilon^2\rho_2\wedge B_2 + \epsilon^3\tilde{\omega}_4, \tag{5.30c}$$

with $\epsilon^{1,2,3}\in\mathbb{R}$. Hence, they are symmetries of the Lagrangian (5.18). Upon dimensional reduction they become isometries of this axionic manifold, which at this level can be thought to be $\mathcal{H}_3$. On the other hand, one should impose the discrete identifications

$$C_4 \longrightarrow C_4 + \tilde{\omega}_4, \tag{5.31a}$$
$$B_2 \longrightarrow B_2 + \rho_2 \tag{5.31b}$$
$$C_2 \longrightarrow C_2 + \rho_2, \qquad C_4 \longrightarrow C_4 + \rho_2 \wedge B_2, \tag{5.31c}$$

which in 4d become the discrete transformations

$$b^1 \longrightarrow b^1 + 1, \tag{5.32a}$$
$$b^2 \longrightarrow b^2 + 1, \qquad b^3 \longrightarrow b^3 + Mb^1 \tag{5.32b}$$
$$b^3 \longrightarrow b^3 + 1, \tag{5.32c}$$

in agreement with equations (3.83) once we make use of the identifications (5.29). These symmetries generate a non-Abelian discrete group Γ, so that the final axionic manifold is $\tilde{\mathcal{M}} = \mathcal{H}_3/\Gamma$. The corresponding algebra generators satisfy equation (3.80) and the symplectic $Sp(2,\mathbb{Z}) \simeq SL(2,\mathbb{Z})$ global structure of this algebra is in this context inherited from the $SL(2,\mathbb{Z})$ invariance of the 10d action.

Because of the torsion, the discrete shifts of B_2, C_2 and C_4 above not only imply the discrete transformations (5.32), but also discrete transformations of the 4d massive gauge vectors A^i that must occur simultaneously with them

$$A^1_\mu \longrightarrow A^1_\mu + \partial_\mu \lambda^1, \tag{5.33a}$$
$$A^2_\mu \longrightarrow A^2_\mu + \partial_\mu \lambda^2, \qquad A^3_\mu \longrightarrow A^3_\mu + M'k\lambda^2 A^1_\mu + M'b^1\partial_\mu\lambda^2, \tag{5.33b}$$
$$A^3_\mu \longrightarrow A^3_\mu + \partial_\mu \lambda^3. \tag{5.33c}$$

That is, we find that the discrete shifts of the scalars are gauged to

$$b^1 \longrightarrow b^1 + k\lambda^1, \tag{5.33d}$$
$$b^2 \longrightarrow b^2 + k\lambda^2, \qquad b^3 \longrightarrow b^3 + Mkb^1\lambda^2, \tag{5.33e}$$
$$b^3 \longrightarrow b^3 + k'\lambda^3. \tag{5.33f}$$

Compatibility with the discrete transformations (5.32) leads to a sublattice $\hat{\Gamma} \subset \Gamma$, as in equations (3.85).

As already discussed, the gauge symmetries of the action (5.27) for non-vanishing k and k' are then given by the quotient $\mathbf{P} = \Gamma/\hat{\Gamma}$. It is insightful to work out the transformation of charged fields under such discrete gauge group. For that aim, consider a 4d charged particle $\psi(x)$ with integer charges q_I under A^I_1, $I = 1,2,3$. From a 10d perspective this corresponds to a bound state of q_0 D3-branes wrapping the torsion 3-cycle above, and q_1 fundamental strings and q_2 D1-branes wrapping the torsion 1-cycle. The 4d covariant derivative is given by

$$D\psi(x) = \left[d + iq_I \hat{A}^I_1\right]\psi(x) \tag{5.34}$$

with $\hat{A}^i_1 = k^{-1}\eta^i$, $i = 1,2$, and $\hat{A}^3_1 = k'^{-1}\eta^3$. In general, under a discrete gauge transformation (5.33) the field $\psi(x)$ will transform with a holonomy phase and a charge redefinition.

Indeed, acting on (5.34) with (5.33) we obtain the following transformation properties under the action of **P**

$$\tilde{T}_1 : \qquad \psi(x) \longrightarrow \exp\left[2\pi i k^{-1} q_1\right] \psi(x), \tag{5.35a}$$

$$\tilde{T}_2 : \qquad \psi(x) \longrightarrow \exp\left[2\pi i k^{-1} q_2\right] \mathcal{U}\, \psi(x), \tag{5.35b}$$

$$\tilde{T}_3 : \qquad \psi(x) \longrightarrow \exp\left[2\pi i k'^{-1} q_3\right] \psi(x), \tag{5.35c}$$

where $\mathcal{U}$ is the charge redefinition

$$\mathcal{U} : \qquad \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & M' \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} \tag{5.36}$$

The above monodromies can also be derived from a higher dimensional point of view, by simply performing the discrete shifts (5.32) on the Chern-Simons actions of the corresponding type IIB p-branes, and reading the induced charges before and after the shift. Note that a particle with charge q_I is indistinguishable from a particle with charge $q_i + kn^i$ (or $q_3 + k'n^3$ for the case $I = 3$) for $n^i \in \mathbb{Z}$, and therefore represent the same physical state, in agreement with the underlying discrete symmetry discussed above. Moreover, due to the non-Abelian structure the basis of charge eigenstates of $\tilde{T}_1$ and $\tilde{T}_2$ are not compatible with each other, and the two types of charges cannot be simultaneously measured.

5.2.4 A simple example revisited

In order to illustrate the usefulness of the above results let us consider the simple setup of [138], consisting on a set of N fractional D3-branes at a $\mathbb{C}^3/\mathbb{Z}_3$ singularity on type II string theory. In the large N limit this setup backreacts to string theory on $AdS_5 \times \mathbb{S}^5/\mathbb{Z}_3$, dual to a certain supersymmetric $\mathrm{SU}(N)^3$ gauge theory with bifundamental matter. The SCFT has a $\Delta(27)$ discrete symmetry, which can be obtained from torsion homology in the 5d AdS dual, as described in [138]. Alternatively, we can make use of the results of the previous subsection for torsion p-forms in order to make explicit the non-Abelian discrete gauge symmetry directly from dimensional reduction of the backreacted setup.

Indeed, in this case the torsion homology of $\mathbb{S}^5/\mathbb{Z}_3 \times \mathbb{S}^1$ corresponds to equations (5.12) with $k = k' = 3$ and $M = M' = 1$ in equation (5.16). Charged particles in the 4d theory are thus labeled by three fractional charges $\frac{1}{N}(q_1, q_2, q_3)$, with q_I defined mod 3.

In particular, the three types of bifundamental fields described in [138] correspond to states $\psi_r(x)$, $r = 1, 2, 3$ with $(q_1, q_2, q_3) = (r-1, 0, 1)$. These are 4d particles which result from wrapping a D3-brane in the torsion 3-cycle and 0, 1 or 2 fundamental strings in the torsion 1-cycle. From our previous results, we obtain that the three generators of discrete symmetries act in these states as

$$\tilde{T}_1 : \qquad (\psi_1, \psi_2, \psi_3) \longrightarrow (\psi_1, \xi\psi_2, \xi^2\psi_3), \tag{5.37a}$$

$$\tilde{T}_2 : \qquad (\psi_1, \psi_2, \psi_3) \longrightarrow (\psi_2, \psi_3, \psi_1), \tag{5.37b}$$

$$\tilde{T}_3 : \qquad (\psi_1, \psi_2, \psi_3) \longrightarrow (\xi\psi_1, \xi\psi_2, \xi\psi_3), \tag{5.37c}$$

with $\xi = e^{2\pi i/3N}$, in complete agreement with the results of [138].

5.3 Non-Abelian discrete symmetries from torsion forms: general case

We have explored in section 5.2 the structure of non-Abelian discrete gauge symmetries from the perspective of dimensional reduction, for the simplest case with torsion groups (5.12). In this section we perform dimensional reduction of the type IIB action for the more general case with arbitrary torsion homology groups (5.11). Thus, we introduce a set of non-harmonic eigenforms associated to the generators of the torsion homology groups with

$$d\gamma_1^\alpha = k^\alpha{}_\beta \rho_2^\beta, \tag{5.38a}$$
$$d\tilde{\rho}_{4,\beta} = k^\alpha{}_\beta \zeta_{5,\alpha}, \tag{5.38b}$$
$$d\alpha_3^\alpha = k'^\alpha{}_\beta \tilde{\omega}_4^\beta, \tag{5.38c}$$
$$d\omega_{2,\beta} = k'^\alpha{}_\beta \beta_{3,\alpha}, \tag{5.38d}$$

and

$$\int_{X_6} \gamma_1^\alpha \wedge \zeta_{5,\beta} = \int_{X_6} \rho_2^\alpha \wedge \tilde{\rho}_{4,\beta} = \int_{X_6} \alpha_3^\alpha \wedge \beta_{3,\beta} = \int_{X_6} \omega_{2,\beta} \wedge \tilde{\omega}_4^\alpha = \delta_\beta^\alpha \tag{5.39}$$

In these expressions k^{-1} and k'^{-1} are the linking matrices between dual p- and $(5-p)$-cycles, with $p = 1, 3$ respectively.

We recast the torsion cycle intersection pattern in terms of these dual forms as

$$\gamma_1^\alpha \wedge \gamma_1^\beta = 0, \tag{5.40a}$$
$$\rho_2^\alpha \wedge \gamma_1^\beta = \mathcal{A}^{\alpha\beta}{}_\gamma\, \alpha_3^\gamma, \tag{5.40b}$$
$$\rho_2^\alpha \wedge \rho_2^\beta = \mathcal{K}^{\alpha\beta}{}_\gamma\, \tilde{\omega}_4^\gamma, \tag{5.40c}$$

where consistency with the exterior derivative requires

$$\mathcal{A}^{\alpha[\beta}{}_\gamma\, k^{\delta]}{}_\alpha = 0, \tag{5.41a}$$
$$k^\alpha{}_\beta\, \mathcal{K}^{\delta\beta}{}_\gamma = k'^\beta{}_\gamma\, \mathcal{A}^{\delta\alpha}{}_\beta\, . \tag{5.41b}$$

We proceed now to perform dimensional reduction of the type IIB supergravity action (5.18), taking into account these relations. Following the same reasoning than in section 5.2, we expand the NSNS and RR 2-forms and the RR 4-form as

$$B_2^i = b_\alpha^i \rho_2^\alpha + A_{1,\alpha}^i \wedge \gamma_1^\alpha, \qquad i = 1, 2, \tag{5.42a}$$
$$C_4 = b_\alpha^3 \tilde{\omega}_4^\alpha + A_{1,\alpha}^3 \wedge \alpha_3^\alpha + V_1^{3,\alpha} \wedge \beta_{3,\alpha} + c_2^\alpha \wedge \omega_{2,\alpha} \tag{5.42b}$$

The corresponding 10d field-strengths read

$$dB_2^i = \eta_\alpha^i \wedge \rho_2^\alpha + dA_{1,\alpha}^i \wedge \gamma_1^\alpha, \qquad i = 1, 2, \tag{5.43a}$$
$$F_5 = \eta_\beta^3 \wedge \tilde{\omega}_4^\gamma - F_{2,\alpha}^3 \wedge \alpha_3^\alpha + \tilde{F}_2^{3,\alpha} \wedge \beta_{3,\alpha} + dc_2^\alpha \wedge \omega_{2,\alpha}, \tag{5.43b}$$

where now

$$\eta^i_\alpha \equiv db^i_\alpha - k^\beta{}_\alpha A^i_{1,\beta}, \tag{5.44a}$$

$$\eta^3_\alpha \equiv db^3_\alpha - k'^\beta{}_\alpha A^3_{1,\beta} - \mathcal{K}^{\gamma\rho}{}_\alpha\, b^2_\gamma \eta^1_\rho, \tag{5.44b}$$

and

$$k'^\alpha{}_\beta F^3_{2,\alpha} \equiv d\eta^3_\beta - \frac{\epsilon_{ij}}{2}\,\mathcal{K}^{\rho\alpha}{}_\beta\, \eta^i_\alpha \wedge \eta^j_\rho, \tag{5.45a}$$

$$\tilde{F}^{3,\alpha}_2 \equiv dV^{3,\alpha}_1 + k'^\alpha{}_\beta c^\beta_2. \tag{5.45b}$$

Substituting into equation (5.18) and making use of the relations (5.41) we get

$$\begin{aligned} S_{4\text{d}} =& \frac{1}{4}\int d^4x\Big[(-g)^{1/2}\Big(- \mathcal{M}_{ij}\,\mathcal{N}^{\alpha\beta}\, dA^i_{1,\alpha}\cdot dA^j_{1,\beta} - \mathcal{M}_{ij}\,\mathcal{T}^{\alpha\beta}\,\eta^i_\alpha\cdot\eta^j_\beta \\ & - \frac{\mathcal{R}^{\alpha\beta}}{2} F^3_{2,\alpha}\cdot F^3_{2,\beta} + \mathcal{Q}^\alpha{}_\beta\, F^3_{2,\alpha}\cdot \tilde{F}^{3,\beta}_2 + \frac{\mathcal{S}_{\alpha\beta}}{2}\tilde{F}^{3,\alpha}_2\cdot\tilde{F}^{3,\beta}_2 - \frac{\mathcal{G}_{\alpha\beta}}{2} dc^\alpha_2\cdot dc^\beta_2 \\ & - \frac{(\mathcal{G}^{-1})^{\alpha\beta}}{2}\eta^3_\alpha\cdot\eta^3_\beta\Big) - \eta^3_\alpha\wedge dc^\alpha_2 - \tilde{F}^{3,\alpha}_2\wedge F^3_{2,\alpha}\Big] \end{aligned} \tag{5.46}$$

where

$$\mathcal{N}^{\alpha\beta} \equiv \int_{X_6} \gamma^\alpha_1 \wedge *_6\gamma^\beta_1, \tag{5.47a}$$

$$\mathcal{T}^{\alpha\beta} \equiv \int_{X_6} \rho^\alpha_2 \wedge *_6\rho^\beta_2, \tag{5.47b}$$

$$\mathcal{Q}^\alpha{}_\beta \equiv \int_{X_6} \alpha^\alpha_3 \wedge *_6\beta_{3,\beta}, \tag{5.47c}$$

$$\mathcal{R}^{\alpha\beta} \equiv \int_{X_6} \alpha^\alpha_3 \wedge *_6\alpha^\beta_3, \tag{5.47d}$$

$$\mathcal{S}_{\alpha\beta} \equiv \int_{X_6} \beta_{3,\alpha} \wedge *_6\beta_{3,\beta}, \tag{5.47e}$$

$$\mathcal{Q}_{\alpha\beta} \equiv \int_{X_6} \omega_{2,\alpha} \wedge *_6\omega_{2,\beta}, \tag{5.47f}$$

and

$$\mathcal{R}^{\alpha\beta}\,\mathcal{S}_{\beta\gamma} + \mathcal{Q}^\alpha{}_\beta\,\mathcal{Q}^\beta{}_\gamma = -\delta^\alpha_\gamma, \tag{5.48a}$$

$$\mathcal{S}_{\alpha\beta}\,\mathcal{Q}^\beta{}_\gamma - \mathcal{Q}^\beta{}_\alpha\,\mathcal{S}_{\beta\gamma} = 0. \tag{5.48b}$$

The self-duality condition of the RR 5-form field-strength, $F_5 = *_{10}F_5$, implies

$$\tilde{F}^{3,\alpha}_2 = -F^3_{2,\beta}\,\mathcal{Q}^\beta{}_\gamma(\mathcal{S}^{-1})^{\gamma\alpha} - *_4F^3_{2,\beta}(\mathcal{S}^{-1})^{\beta\alpha}, \tag{5.49a}$$

$$dc^\alpha_2 = (\mathcal{G}^{-1})^{\alpha\beta} *_4 \eta^3_\beta, \tag{5.49b}$$

so we finally obtain

$$\begin{aligned} S_{4\text{d}} =& \frac{1}{4}\int d^4x\big[(-g)^{1/2}\big(- \mathcal{M}_{ij}\,\mathcal{T}^{\alpha\beta}\,\eta^i_\alpha\cdot\eta^j_\beta - (\mathcal{G}^{-1})^{\alpha\beta}\eta^3_\alpha\cdot\eta^3_\beta \\ & - \mathcal{M}_{ij}\,\mathcal{N}^{\alpha\beta}\,F^i_{2,\alpha}\cdot F^j_{2,\beta} + (\mathcal{S}^{-1})^{\alpha\beta}F^3_{2,\alpha}\cdot F^3_{2,\beta}\big) + \mathcal{Q}^\alpha{}_\gamma(\mathcal{S}^{-1})^{\gamma\beta}F^3_{2,\alpha}\wedge F^3_{2,\beta}\big]. \end{aligned} \tag{5.50}$$

The gauge symmetries of this effective action are

$$A^1_{1,\alpha} \longrightarrow A^1_{1,\alpha} + d\lambda^1_\alpha, \tag{5.51a}$$

$$A^1_{2,\alpha} \longrightarrow A^2_{1,\alpha} + d\lambda^2_\alpha, \qquad A^3_{1,\alpha} \longrightarrow A^3_{1,\alpha} + \mathcal{A}^{\delta\beta}{}_\alpha\, k^\gamma{}_\delta (A^1_{1,\beta}\lambda^2_\gamma + b^1_\delta d\lambda^2_\beta), \tag{5.51b}$$

$$A^3_{1,\alpha} \longrightarrow A^3_{1,\alpha} + d\lambda^3_\alpha, \tag{5.51c}$$

with the discrete shifts of the scalars being gauged to

$$b^1_\beta \longrightarrow b^1_\beta + k^\alpha{}_\beta \lambda^1_\alpha, \tag{5.51d}$$

$$b^2_\beta \longrightarrow b^2_\beta + k^\alpha{}_\beta \lambda^2_\alpha, \qquad b^3_\alpha \longrightarrow b^3_\alpha + \mathcal{K}^{\beta\delta}{}_\alpha\, k^\alpha{}_\beta \lambda^2_\alpha b^1_\delta, \tag{5.51e}$$

$$b^3_\beta \longrightarrow b^3_\beta + k'^\alpha{}_\beta \lambda^3_\alpha. \tag{5.51f}$$

These correspond to a set of non-commuting discrete $\mathbb{Z}_{r^I_\beta}$ gauge symmetries, where r^I_β is the lower integer for which $(k^{-1})_\alpha{}^\beta r^i_\beta$ (or $(k'^{-1})_\alpha{}^\beta r^3_\beta$ in the case of $I = 3$) is an integer.

We can also work out the transformation of charged fields under these discrete gauge transformations. For that aim, we consider a 4d charged particle $\psi(x)$ with integer charges q^α_I. The 4d covariant derivative is given by

$$D\psi(x) = \left[d + iq^\alpha_I \hat{A}^I_{1,\alpha}\right]\psi(x) \tag{5.52}$$

with $\hat{A}^i_{1,\alpha} = (k^{-1})_\alpha{}^\beta \eta^i_\beta$, $i = 1, 2$ and $\hat{A}^3_{1,\alpha} = (k'^{-1})_\alpha{}^\beta \eta^3_\beta$. Acting on (5.52) with (5.51) we obtain the following transformation properties under the discrete gauge symmetry generators

$$\tilde{T}^\gamma_1 : \qquad \psi(x) \to \exp\left[2\pi i (k^{-1})_\delta{}^\gamma q^\delta_1\right]\psi(x), \tag{5.53a}$$

$$\tilde{T}^\gamma_2 : \qquad \psi(x) \to \exp\left[2\pi i (k^{-1})_\delta{}^\gamma q^\delta_2\right]\mathcal{U}\,\psi(x), \tag{5.53b}$$

$$\tilde{T}^\gamma_3 : \qquad \psi(x) \to \exp\left[2\pi i (k'^{-1})_\delta{}^\gamma q^\delta_3\right]\psi(x), \tag{5.53c}$$

where $\mathcal{U}$ is the charge redefinition

$$\mathcal{U} : \quad \begin{pmatrix} q^\alpha_1 \\ q^\alpha_2 \\ q^\alpha_3 \end{pmatrix} \to \begin{pmatrix} \delta^\alpha_\beta & 0 & \mathcal{A}^{\gamma\alpha}{}_\beta \\ 0 & \delta^\alpha_\beta & 0 \\ 0 & 0 & \delta^\alpha_\beta \end{pmatrix} \begin{pmatrix} q^\beta_1 \\ q^\beta_2 \\ q^\beta_3 \end{pmatrix}. \tag{5.54}$$

6

Discrete gauge symmetries in flux compactifications

String theory compactifications with non-trivial background fluxes have been the subject of extensive explorations (see e.g. [10, 181, 182, 183] for reviews). The presence of non-trivial background fluxes gives rise to interesting physics; for instance, changes in supersymmetric conditions [184, 185, 186, 187, 188, 189], appearance of flux-induced supersymmetry breaking soft terms, [190, 191, 192, 193], closed string moduli stabilization in type IIA [194, 195, 196, 82] and type IIB [197, 198, 199, 200, 201], open string moduli stabilization by open and closed string fluxes [202, 203, 204, 186, 205], moduli stabilization in M-theory [206, 207, 208], and in the heterotic theory [209, 210, 211, 212, 213, 214, 215, 216], etc.

Even if we just consider the theory at topological level, the presence of non-trivial background fluxes modifies the brane wrappings by the Freed-Witen consistency conditions (or their dual versions) [217, 218].

Most of the studies mentioned about flux-induced modifications of brane wrappings have focused on 4d space-time filling branes (used for model building), and brane instantons. In this chapter we will consider wrapped branes with nontrivial topological interplay from fluxes, focusing on general 4d defects (strings, particles, domain walls), whose topological charges turn out to be conserved only modulo an integer p, related to the flux quanta.

Let us consider first 4d strings and particles. As we saw in chapter 3, strings and particles with $\mathbb{Z}_p$-valued charges are related to the existence of $\mathbb{Z}_p$ discrete gauge symmetries. In this chapter they are given by branes wrapped on $\mathbb{Z}$-valued homological cycles; these branes can however decay in sets of p due to the effect of fluxes (dubbed 'flux catalysis'). This is ultimately related to the fact that homology is in general not the right mathematical tool to classify brane charges. For instance, D-brane charges must be classified by K-theory (in the absence of fluxes), or twisted K-theory (in the presence of NSNS 3-form flux) [219].

In general, the mathematical tool classifying D-brane charges in the presence of RR fluxes (or including NS5-branes and fundamental strings) has not been determined. Our analysis can thus be regarded as a physical classification of stable objects and their decay processes in certain 4d flux compactifications (in analogy with the physical derivation of twisted K-theory in [218]), and hence a computation of the groups of brane charges, regardless of their underlying mathematical definition (see e.g. [220, 221, 222, 223, 224, 225, 226] for earlier discussions in this vein). In this respect, one relevant conclusion of our analysis is that these groups can be non-Abelian.

Let us now turn to $\mathbb{Z}_p$ valued domain walls. Domain walls from wrapped branes separate vacua differing in their background fluxes, the fact that they are $\mathbb{Z}_p$ valued means that they can decay in sets of p by nucleation of a string loop. Therefore, our analysis provides a physical derivation of certain $\mathbb{Z}_p$-valued flux quantization conditions. This goes beyond the naive characterization of fluxes as integer cohomology classes, and is related to the fact that integer cohomology is not the right mathematical tool to classify p-form fluxes in string theory (see [227] for a partially more complete definition, in terms of K-theory classes).

In this chapter, we will focus on type II compactifications with non-trivial background fluxes. Some aspects of $\mathbb{Z}_p$ strings in compactifications with fluxes have been considered in [228] for type IIB (with NSNS and RR 3-form flux) and in [229] for the heterotic (with gauge fluxes). Also, we focus on compactifications with p-form fluxes, and do not introduce geometric or non-geometric fluxes

In section 6.1 we explain how discrete symmetries arise in string compactifications with non-trivial background fluxes. In section 6.2 we apply those ideas to 4d type II compactifications, in the case with only one kind of background flux has been turned on, focusing on Abelian groups of the form $\mathbb{Z}_k$. A generalization to non-Abelian discrete symmetries, while keeping only one kind of background flux, is presented in section 6.3. The possibility of having several kinds of fluxes at the same time, together with the possible inconsistencies that naively may arise and how they are avoided, is studied in section 6.4. In section 6.5 we turn to $\mathbb{Z}_p$-valued domain walls, and their relation to string duality symmetries relating vacua with different flux quanta. Finally, in section 6.6 we study gaugings by KK gauge bosons from continuous $U(1)$ isometries in the compactification space in the presence of background fluxes, using the ideas of flux catalysis.

6.1 Flux catalysis

We showed in chapter 3 that the existence of a $\mathbb{Z}_p$ discrete symmetry can be identified by the prince of BF cou[lings in the 4d theory. In type II models with non-trivial background fluxes these 4d BF couplings arise from the Kaluza-Klein reduction of the 10d Chern-Simons couplings, which are of the form

$$\int_{10d} H_3 \wedge F_p \wedge C_{7-p}, \tag{6.1a}$$

$$\int_{10d} B_2 \wedge F_p \wedge F_{8-p}. \tag{6.1b}$$

Here B_2 and H_3 denote the NSNS 2-form potential and its field strength, whereas C_n and F_{n+1} denote the RR n-form potential and its field strength, with n even or odd for type IIA or IIB theories, respectively.

The $\mathbb{Z}_p$ discrete gauge symmetry will prove a useful tool to classify branes in the systems. The $\mathbb{Z}_p$ valued particles and strings charged under the discrete gauge symmetry are given by branes wrapped on homologically non-trivial $\mathbb{Z}$-valued cycles, yet they can decay in sets of p, due to processes allowed by the presence of fluxes (dubbed ‘flux catalysis’). Prototypes of such processes are the decay of D-branes ending on a higher-dimensional brane with non-trivial NSNS flux along its worldvolume [218], due to the Freed-Witten consistency condition[1], or the decay of fundamental strings on a Dp-brane with non-trivial p-form flux along its worldvolume (as in the baryon vertex in [230]).

One may think that the charge carried by the instantonic object sources a flux which is left behind after the day, such that the p charged objects do not decay to the vacuum. We emphasise that in the present 4d case the decay of p charged objects is to the vacuum of the theory, with no flux left behind. This is particularly clear in the language of 4d field theory, where the field theory operators describing the instantonic decays have no long range effect [126]. We do not consider other setups, in other space-time dimensions (actually, other codimensionalities of the $\mathbb{Z}_p$ charged branes) where there is some left-over flux which demands a combined classification of D-branes and fluxes [222]

In the rest of this chapter $\overline{F}_n$ will denote the n-form that has a non-trivial flux and $\hat{F}_n$ will denote an n-form that is obtained from the reduction of some higher form.

6.2 Abelian discrete gauge symmetries from flux catalysis in type II

In this section we will study the possible discrete gauge symmetries that can arise in type II compactifications with only one type of background fluxes.

6.2.1 Flux catalysis in type IIA

6.2.1.1 Massive type IIA

Consider massive type IIA string theory with mass parameter $\overline{F}_0 = p_0$ compactified on a Calabi-Yau $\mathbb{X}_6$. The Chern-Simons couplings (6.1b) contain a term leading to a 4d BF coupling as follows

$$\int_{10d} \overline{F}_0 B_2 \wedge F_8 \quad \longrightarrow \quad p \int_{4d} B_2 \wedge \hat{F}_2 \tag{6.2}$$

where we have defined

$$\hat{F}_2 = \int_{\mathbb{X}_6} F_8. \tag{6.3}$$

The theory automatically has a $\mathbb{Z}_p$ discrete gauge symmetry.

The $\mathbb{Z}_p$-charged particles are D6-branes wrapped on $\mathbb{X}_6$, which can be annihilated in sets of p by an instanton given by an NS5-brane wrapped on $\mathbb{X}_6$.

The $\mathbb{Z}_p$-charged strings correspond to fundamental strings (F1s), and can be annihilated in sets of p by a string junction given by the object charged magnetically under C_7, namely a D0-brane. Indeed, as shown in [231], a D0-brane in the presence of a mass parameter p must be dressed with p fundamental strings.

[1] Actually [217] considered the case of torsion H_3, and the physical picture for general H_3 appeared in [218]. Still, we stick to the widely used term FW anomaly / consistency conditions, even for non-torsion H_3.

6.2.1.2 Type IIA with F_2 flux

Consider type IIA string theory with 2-form flux $\overline{F}_2$ compactified on a Calabi-Yau $\mathbb{X}_6$. It is convenient to introduce a basis of 2-cycles $\{\Pi_k\}$ and dual 4-cycles $\{\gamma_l\}$ with $\Pi_k \cdot \Gamma_l = \delta_{kl}$. Let us define

$$\int_{\Pi_k} \overline{F}_2 = p_k. \tag{6.4}$$

There is a term in the 10d Chern-Simons couplings (6.1b) producing a 4d BF coupling

$$\int_{10d} B_2 \wedge \overline{F}_2 \wedge F_6 \quad \longrightarrow \quad \sum_k p_k \int_{4d} B_2 \wedge \hat{F}_2^k \tag{6.5}$$

where we have defined

$$\hat{F}_2^k = \int_{\Gamma_k} F_6. \tag{6.6}$$

The theory has a $\mathbb{Z}_q$ discrete gauge symmetry with $q = \sum_k \frac{p_k^2}{p}$, where $p = \gcd(p_k)$.

Particles charged under this symmetry with minimal charge correspond to a D4-brane wrapped on $\sum_k n_k \Gamma_k$ with integers n_k satisfying $\sum_k n_k p_k = p$, which always exist by Bezout's lemma; the relevant instanton that annihilates them is given by an NS5-brane wrapped on $\mathbb{X}_6$. Indeed, due to the Freed-Witten anomaly induced by $\overline{F}_2$, the NS5-brane must emit sets of p_k D4-branes wrapped on Γ_k, for all k. Since a brane wrapped on Γ_k has charge +1 under Q_k, the charge violation for a combination $Q_a = \sum_k c_k^a Q_k$ is $\sum_k c_k^a p_k$. This vanishes for massless $U(1)$'s, whereas $Q = \sum_k \frac{p_k}{p} Q_k$ is violated in q units.

The $\mathbb{Z}_q$-charged strings are fundamental strings, and are annihilated in sets of q by a string junction given by a D2-brane wrapped on the 2-cycle $\Pi = \sum_k \frac{p_k}{p} \Pi_k$. Indeed, a D2-brane on Π_k can annihilate fundamental strings in sets of p_k, which would violate the Z_q symmetry, since in general p_k is not a multiple of q. However, such D2-brane is not gauge invariant, since it also carries monopole charge under some unbroken $U(1)$'s (concretely, any linear combination involving Q_k). But the linear combination $\Pi = \sum_k \frac{p_k}{p} \Pi_k$ is gauge invariant, since it has no monopole charge, and emits q fundamental strings, in agreement with the $\mathbb{Z}_q$ symmetry.

6.2.1.3 Type IIA with F_4 flux

Consider type IIA string theory with 4-form flux $\overline{F}_4$ for the field strength of the RR 3-form compactified on a Calabi-Yau $\mathbb{X}_6$. It is convenient to introduce a basis of 2-cycles $\{\Pi_k\}$ and dual 4-cycles $\{\gamma_l\}$ with $\Pi_k \cdot \Gamma_l = \delta_{kl}$. Let us define

$$\int_{\Gamma_k} \overline{F}_4 = p_k. \tag{6.7}$$

There is a term in the 10d Chern-Simons couplings (6.1b) producing a 4d BF coupling

$$\int_{10d} B_2 \wedge \overline{F}_4 \wedge F_4 \quad \longrightarrow \quad \sum_k p_k \int_{4d} B_2 \wedge \hat{F}_2^k \tag{6.8}$$

where we have defined

$$\hat{F}_2^k = \int_{\Pi_k} F_4. \tag{6.9}$$

The theory has a $\mathbb{Z}_q$ discrete gauge symmetry with $q = \sum_k \frac{p_k^2}{p}$, where $p = \gcd(p_k)$.

Particles charged under this symmetry with minimal charge correspond to a D2-brane wrapped on $\sum_k n_k \Pi_k$ with integers n_k satisfying $\sum_k n_k p_k = p$, which always exist by Bezout's lemma; the relevant instanton that annihilates them is given by an NS5-brane wrapped on $\mathbb{X}_6$. Indeed, due to the Freed-Witten anomaly induced by $\overline{F}_4$, the NS5-brane must emit sets of p_k D4-branes wrapped on Π_k, for all k. Since a brane wrapped on Π_k has charge $+1$ under Q_k, the charge violation for a combination $Q_a = \sum_k c_k^a Q_k$ is $\sum_k c_k^a p_k$. This vanishes for massless $U(1)$'s, whereas $Q = \sum_k \frac{p_k}{p} Q_k$ is violated in q units.

The $\mathbb{Z}_q$-charged strings are fundamental strings, and are annihilated in sets of q by a string junction given by a D4-brane wrapped on the 4-cycle $\Gamma = \sum_k \frac{p_k}{p}\Gamma_k$. Indeed, a D4-brane on Γ_k can annihilate fundamental strings in sets of p_k, which would violate the Z_q symmetry, since in general p_k is not a multiple of q. However, such D4-brane is not gauge invariant, since it also carries monopole charge under some unbroken $U(1)$'s (concretely, any linear combination involving Q_k). But the linear combination $\Gamma = \sum_k \frac{p_k}{p}\Gamma_k$ is gauge invariant, since it has no monopole charge, and emits q fundamental strings, in agreement with the $\mathbb{Z}_q$ symmetry.

6.2.1.4 Type IIA with F_6 flux (Freund-Robin)

Consider type IIA string theory compactified on a Calabi-Yau $\mathbb{X}_6$ with p units of $\overline{F}_6$ flux on it (for instance, as in the Freund-Robin compactifications, ubiquitous in the AdS$_4$/CFT$_3$ correspondence initiated by [232]). The Chern-Simons couplings (6.1b) contain a term leading to a 4d BF coupling as follows

$$\int_{10d} B_2 \wedge F_2 \wedge \overline{F}_6 \quad \longrightarrow \quad p\int_{4d} B_2 \wedge F_2. \tag{6.10}$$

The theory automatically has a $\mathbb{Z}_p$ discrete gauge symmetry.

The $\mathbb{Z}_p$-charged particles are D0-branes, which can be annihilated in sets of p by an instanton given by an NS5-brane wrapped on $\mathbb{X}_6$.

The $\mathbb{Z}_p$-charged strings correspond to fundamental strings, which can be annihilated in sets of p by an instanton given by a D6-brane wrapped on $\mathbb{X}_6$ (this is analogous to the AdS$_4$/CFT$_3$ version of the baryon in [230]).

6.2.1.5 Type IIA with NSNS flux

Consider type IIA theory compactified on a Calabi-Yau $\mathbb{X}_6$ with NSNS 3-form flux $\overline{H}_3$. We introduce a symplectic basis of 3-cycles $\{\alpha_k\}$, $\{\beta_l\}$ with $\alpha_k \cdot \beta_l = \delta_{kl}$. Let us define

$$\int_{\alpha_k} \overline{H}_3 = p_k, \qquad \int_{\beta_k} \overline{H}_3 = p'_k. \tag{6.11}$$

There are 4d couplings arising from the reduction of the 10d Chern-Simons term (6.1a)

$$\int_{10d} \overline{H}_3 \wedge F_2 \wedge C_5 \quad \longrightarrow \quad \sum_k \int_{4d} \left(p_k \hat{B}_k - p'_k \hat{B}'_k\right) \wedge F_2 \tag{6.12}$$

where we have defined

$$\int_{\beta_k} C_5 = \hat{B}_k, \qquad \int_{\alpha_k} C_5 = \hat{B}'_k. \tag{6.13}$$

The theory has a $\mathbb{Z}_p$ discrete gauge symmetry with $p = \gcd(P_k, p'_k)$.

The $\mathbb{Z}_p$-charged particles are D0-branes. The Freed-Witten anomaly induced by $\overline{H}_3$ dictates that a D2-brane wrapped on α_k (resp. β_k) has to be dressed with p_k (reps. p'_k) D0-branes; therefore exactly p D0-branes can be annihilated by an instanton given by a D2-brane wrapped on $\sum_k(n_k\alpha_k+m_k\beta_k)$, with integers n_k, m_k, satisfying $\sum_k(n_kp_k+m_kp'_k) = p$, which always exist by Bezout's lemma.

The $\mathbb{Z}_p$-charged strings with minimal charge corresponds to a D4-brane wrapping the linear combination 3-cycle [2]

$$\Pi = \sum_k \left(\frac{p_k}{p}\beta_k - \frac{p'_k}{p}\alpha_k\right) \tag{6.14}$$

which is free from Freed-Witten inconsistencies, $\int_\Pi \overline{H}_3 = 0$. The string junction annihilating p charged strings is a D6-brane wrapped on $\mathbb{X}_6$; due to its Freed-Witten anomaly it emits D4-branes in the total class $\sum_k \left(p_k[\beta_k] - p'_k[\alpha_k]\right)$, namely p minimally charged strings.

6.2.2 Flux catalysis in type IIB

6.2.2.1 Type IIB with NSNS 3-form flux

Consider type IIB compactified in a Calaby-Yau $\mathbb{X}_6$ with NSNS 3-form flux $\overline{H}_3$. Without loss of generality, we introduce a symplectic basis of 3-cycles $\{\alpha_k\}$, $\{\beta_l\}$ such that there is flux only on the α cycles and define

$$\int_{\alpha_k} \overline{H}_3 = p_k. \tag{6.15}$$

There are 4d couplings arising from the reduction of the 10d Chern-Simons term (6.1a)

$$\int_{10d} \overline{H}_3 \wedge C_2 \wedge F_5 \quad \longrightarrow \quad \sum_k p_k \int_{4d} C_2 \wedge \hat{F}_2^k \tag{6.16}$$

where we have defined

$$\int_{\beta_k} F_5 = \hat{F}_2^k. \tag{6.17}$$

The theory has a $\mathbb{Z}_q$ discrete gauge symmetry with $q = \sum_k \frac{p_k^2}{p}$, where $p = \gcd(p_k)$.

[2]D4-branes wrapped on linear combinations of 3-cycles with Freed-Witten inconsistencies have D2-branes attached, and they correspond to the strings bounding domain walls discussed in section 6.5.

Particles charged under this symmetry with minimal charge correspond to a D3-brane wrapped on $\sum_k n_k\beta_k$ with integers n_k satisfying $\sum_k n_k p_k = p$, which always exist by Bezout's lemma; the relevant instanton that annihilates them is given by a D5-brane wrapped on $\mathbb{X}_6$. Indeed, due to the Freed-Witten anomaly induced by $\overline{H}_3$, the D5-brane must emit sets of p_k D3-branes wrapped on β_k, for all k. Since a brane wrapped on β_k has charge +1 under Q_k, the charge violation for a combination $Q_a = \sum_k c_k^a Q_k$ is $\sum_k c_k^a p_k$. This vanishes for massless $U(1)$'s, whereas $Q = \sum_k \frac{p_k}{p} Q_k$ is violated in q units.

The $\mathbb{Z}_q$-charged strings are D1-branes, and are annihilated in sets of q by a string junction given by a D3-brane wrapped on the 3-cycle $\alpha = \sum_k \frac{p_k}{p}\alpha_k$. Indeed, a D3-brane on α_k can annihilate fundamental strings in sets of p_k, which would violate the Z_q symmetry, since in general p_k is not a multiple of q. However, such D3-brane is not gauge invariant, since it also carries monopole charge under some unbroken $U(1)$'s (concretely, any linear combination involving Q_k). But the linear combination $\alpha = \sum_k \frac{p_k}{p}\alpha_k$ is gauge invariant, since it has no monopole charge, and emits q fundamental strings, in agreement with the $\mathbb{Z}_q$ symmetry.

6.2.2.2 Type IIB with RR 3-form flux

Consider type IIB compactified in a Calaby-Yau $\mathbb{X}_6$ with RR 3-form flux $\overline{F}_3$. Without loss of generality, we introduce a symplectic basis of 3-cycles $\{\alpha_k\}$, $\{\beta_l\}$ such that there is flux only on the α cycles and define

$$\int_{\alpha_k} \overline{F}_3 = p_k. \tag{6.18}$$

There are 4d couplings arising from the reduction of the 10d Chern-Simons term (6.1a)

$$\int_{10d} B_2 \wedge \overline{F}_3 \wedge F_5 \quad \longrightarrow \quad \sum_k p_k \int_{4d} B_2 \wedge \hat{F}_2^k \tag{6.19}$$

where we have defined

$$\int_{\beta_k} F_5 = \hat{F}_2^k. \tag{6.20}$$

The theory has a $\mathbb{Z}_q$ discrete gauge symmetry with $q = \sum_k \frac{p_k^2}{p}$, where $p = \gcd(p_k)$.

Particles charged under this symmetry with minimal charge correspond to a D3-brane wrapped on $\sum_k n_k\beta_k$ with integers n_k satisfying $\sum_k n_k p_k = p$, which always exist by Bezout's lemma; the relevant instanton that annihilates them is given by a NS5-brane wrapped on $\mathbb{X}_6$. Indeed, due to the Freed-Witten anomaly induced by $\overline{F}_3$, the NS5-brane must emit sets of p_k D3-branes wrapped on β_k, for all k. Since a brane wrapped on β_k has charge +1 under Q_k, the charge violation for a combination $Q_a = \sum_k c_k^a Q_k$ is $\sum_k c_k^a p_k$. This vanishes for massless $U(1)$'s, whereas $Q = \sum_k \frac{p_k}{p} Q_k$ is violated in q units.

The $\mathbb{Z}_q$-charged strings are fundamental strings, and are annihilated in sets of q by a string junction given by a D3-brane wrapped on the 3-cycle $\alpha = \sum_k \frac{p_k}{p}\alpha_k$. Indeed, a D3-brane on α_k can annihilate fundamental strings in sets of p_k, which would violate the Z_q symmetry, since in general p_k is not a multiple of q. However, such D3-brane is not gauge invariant, since it also carries monopole charge under some unbroken $U(1)$'s (concretely, any linear combination involving Q_k). But the linear combination $\alpha = \sum_k \frac{p_k}{p}\alpha_k$ is gauge invariant, since it has no monopole charge, and emits q fundamental strings, in agreement with the $\mathbb{Z}_q$ symmetry.

6.2.2.3 Comment on type IIB with 3-form fluxes

The two setups discussed above can be obtained from each other by replacing $H_3 \leftrightarrow F_3$ and $C_2 \leftrightarrow B_2$ in the couplings, and $D5 \leftrightarrow NS5$ and $D1 \leftrightarrow F1$ in the wrapped branes, as expected from S-duality.

In fact, acting with transformations in the 10d $SL(2,\mathbb{Z})$ duality group, we can obtain configurations with combined NSNS and RR 3-form fluxes, albeit a restricted class. In particular, since the starting configurations have zero contribution to the D3-brane tadpole $\int_{\mathbb{X}_6} \overline{F}_3 \wedge \overline{H}_3$, and this is $SL(2.\mathbb{Z})$ invariant, this strategy cannot reach configurations with non-zero D3-brane tadpole. General flux configurations and the role of the tadpole will be discussed in section 6.4.

6.3 Non-Abelian discrete gauge symmetries from flux catalysis

When fluxes coexist, there can be several $\mathbb{Z}_p$ factors, which may be non-commuting, resulting in a non-Abelian discrete gauge symmetry[3]. In field theoretical grounds, the non-Abelianity implies that strings associated to non-commuting elements g and h, when crossing each other, produce a new string stretching between them, associated to the commutator $c = ghg^{-1}h^{-1}$.

Microscopically, these strings appear by the Hanany-Witten brane creation effects [180], discussed in appendix C. Although the strings created in the crossing are finite in extent, the theory must also contain stable infinite strings associated to the generators c, which therefore describe genuine elements of the discrete symmetry group of the theory.

Non-Abelian discrete symmetries and brane/string creation effects have been realized in cases with discrete symmetries that arise from p-forms on torsion cohomology classes in [138, 143]. We will now show that the non-Abelianity for discrete symmetries can also arise from flux catalysis.

We will consider models with two $\mathbb{Z}_p$ symmetries, with charged strings given by D4-branes on two 3-cycles of $\mathbb{X}_6$, which upon crossing produce fundamental strings sitting at the 3-cycle intersection points in $\mathbb{X}_6$ and stretching in 4d[4].

Wrapped D4-branes playing the role of $\mathbb{Z}_p$-charged have appeared in section 6.2.1.5. However, the structure of Chern-Simons terms is such that it leads to the gauging of only one $U(1)$, and cannot accommodate two independent generators. To overcome this point, we will consider geometries with non-trivial 1-cycles, which allow a richer set of gauge bosons and Chern-Simons couplings. Such geometries include not only $\mathbb{T}^6$ and $K3 \times \mathbb{T}^2$, but more general cases beyond the Calabi-Yau realm, for instance certain twisted tori, or general $\mathbb{T}^2$ bundles over a base $\mathbb{B}_4$ (such as those which appeared in [233] in the heterotic context). For simplicity, we will have in mind a product $\mathbb{B}_4 \times \mathbb{T}^2$, keeping $\mathbb{B}_4$ arbitrary, since we are not particularly interested in supersymmetry.

[3] See [132, 133, 134, 135, 136, 137] for early field theory literature, and [138, 143] for string realizations in type II and [139, 145] in heterotic orbifolds

[4] Or similarly for other D-branes intersecting at points in $\mathbb{X}_6$, e.g. D3- and D5-branes on 2- and 4-cycles, respectively.

Note that it should be possible to obtain non-Abelian discrete gauge symmetries without resorting to 1-cycles if one admits extra sources of gauging, like torsion homology classes or the presence of additional fluxes (e.g. geometric or non-geometric). However, in this section we will stick to pure flux catalysis and allow for 1-cycles.

Consider type IIA string theory compactified in $\mathbb{X}_6 = \mathbb{B}_4 \times \mathbb{T}^2$. Let us introduce the $\mathbb{T}^2$ two 1-cycles a and b, and two dual basis $\{\Pi_k\}$ and $\{\Pi'_k\}$ of 2-cycles in $\mathbb{B}_4$ such that $\Pi_k \cdot \Pi'_l = \delta_{kl}$. We introduce fluxes for the RR 2-form field strength, with flux quanta given by

$$\int_{\Pi_k} \overline{F}_2 = p_k. \tag{6.21}$$

In the presence of this background, the 10d Chern-Simons term (6.1a) descends to 4d BF couplings as

$$\int_{10d} \overline{F}_2 \wedge H_3 \wedge C_5 \quad \longrightarrow \quad \int_{4d} \left(\sum_k p_k \hat{B}_k \wedge \hat{F}_2^a - \sum_k p_k \hat{B}'_k \wedge \hat{F}_2^b \right) \tag{6.22}$$

where we have defined the 4d forms

$$\int_a H_3 = \hat{F}_2^a, \qquad \int_b H_3 = \hat{F}_2^b, \tag{6.23a}$$

$$\int_{\Pi'_k \times b} C_5 = \hat{B}_k, \qquad \int_{\Pi'_k \times a} C_5 = \hat{B}'_k. \tag{6.23b}$$

If we define $p = \gcd(p_k)$, then we obtain the following $\mathbb{Z}_p$ discrete gauge symmetries:

- The first term in the right-hand side of (6.22) gives rise to a $\mathbb{Z}_p$ discrete gauge symmetry. The $\mathbb{Z}_p$ charged particles are fundamental strings winding around the a 1-cycle, and annihilate on an instantons given by a D2-brane wrapped on the 3-cycles $\Pi'_k \times b$. The $\mathbb{Z}_p$ charged strings are D4-branes wrapped on the 3-cycle $\Delta = \sum_k \frac{p_k}{p} \Pi'_k \times b$, and annihilate on a string junction given by a NS5-brane wrapped on the 5-cycle $b \times \mathbb{B}_4$.
- The second term in the right-hand side of (6.22) gives rise to a $\mathbb{Z}_p$ discrete gauge symmetry. The $\mathbb{Z}_p$ charged particles are fundamental strings winding around the b 1-cycle, and annihilate on an instantons given by a D2-brane wrapped on the 3-cycles $\Pi'_k \times a$. The $\mathbb{Z}_p$ charged strings are D4-branes wrapped on the 3-cycle $\Delta' = \sum_k \frac{p_k}{p} \Pi'_k \times a$, and annihilate on a string junction given by a NS5-brane wrapped on the 5-cycle $a \times \mathbb{B}_4$.

Crossing two 4d strings minimally charged under the two $\mathbb{Z}_p$ factors produces r fundamental strings, with

$$r = \Delta \cdot \Delta' = \sum_{k,l} \frac{p_k p_l}{p^2} \Pi'_k \cdot \Pi'_l. \tag{6.24}$$

The symmetry is a discrete Heisenberg group generated by elements T, T', and a central element C, with relations

$$T^p = T'^p = 1, \qquad TT' = C^r T' T. \tag{6.25}$$

In principle, the element C contains a finite order piece, since fundamental strings carry discrete charges, as follows from a further 4d BF coupling from the 10d Chern-Simons term

$$\int_{10d} B_2 \wedge \overline{F}_2 \wedge F_6 \quad \longrightarrow \quad \sum_k \int_{4d} p_k B_2 \wedge \hat{F}_2^k \tag{6.26}$$

where B_2 is the NSNS 2-form and we have defined

$$\int_{\Pi'_k \times a \times b} F_6 = \hat{F}_2^k. \tag{6.27}$$

This configuration leads to a $\mathbb{Z}_q$ discrete gauge symmetry with $q = \sum_k \frac{p_k^2}{p}$. This suggest the relation $C^q = 1$ in addition to (6.25).

However, C is actually slightly more subtle and involves the continuous part of the group. Indeed, the above group relations imply

$$T' = T^p T' = C^{pr} T' T^p = C^{pr} T' \tag{6.28}$$

which, if C involves just the discrete part of the symmetry, would require $pr = 0 \mod q$, which is not true in general. The point becomes clearer in the physical interpretation of (6.28). Consider crossing one 4d string associated to T' with p 4d strings associated to T, leading to the creation of pr fundamental strings. Since the set of p T-strings is trivial, we would expect the set of pr fundamental strings to be so. Physically, one can indeed annihilate the fundamental strings in sets of p, by using combinations of D2-branes on Π_k (each annihilating p_k strings). However, as noted in section 6.2.1.2, such D2-branes carry non-trivial monopole charge under the unbroken $U(1)$'s. Hence, the central element C contains not only the discrete gauge transformation associated to the fundamental strings, but also a (dual) gauge transformation of the unbroken $U(1)$'s.

6.4 Combining fluxes

The simultaneous presence of several kinds of fluxes in a compactification is often required by the equations of motion, and inconsistent configurations of $\mathbb{Z}_p$ valued wrapped branes may naively arise. However, since string theory is a consistent microscopic theory, such configurations should not be possible. In the following sections we will discuss some of these incompatibilities arising when combining several types of fluxes and the mechanisms by which string theory avoids such configurations in consistent flux compactifications.

6.4.1 NSNS and RR fluxes

Let us consider first combinations of NSNS and RR fluxes. In particular, we will focus in type IIB compactifications with simultaneous NSNS and RR 3-form fluxes, which are a popular setup for moduli stabilization; for other type IIA or IIB with NSNS and RR fluxes, similar lessons will apply.

Type IIB compactifications with NSNS or RR 3-form flux were studied in section 6.2.2. We will consider the same setup but, for simplicity, we will restrict to the case with only one 3-cycle α and its dual β (i.e. $\alpha \cdot \beta = 1$). Let us introduce NSNS and RR 3-form fluxes such that

$$\int_\alpha \overline{F}_3 = p, \qquad \int_\beta \overline{H}_3 = p. \tag{6.29}$$

The 10d Chern-Simons couplings (6.1) contain the terms

$$\int_{10d} \overline{F}_3 \wedge B_2 \wedge F_5, \qquad \int_{10d} \overline{H}_3 \wedge C_2 \wedge F_5, \tag{6.30}$$

which, after performing dimensional reduction, produce the following 4d BF coupling

$$\int_{4d} \left(pB_2 \wedge \hat{F}_2 - p' C_2 \wedge \hat{F}_2' \right) \tag{6.31}$$

where we have defined the 4d field strengths

$$\int_\beta F_2 = \hat{F}_2, \qquad \int_\alpha F_5 = \hat{F}_2'. \tag{6.32}$$

Let us now combine equation (6.31) with the results from section 6.2.2:

- The first BF coupling in equation (6.31) leads to a $\mathbb{Z}_p$ discrete gauge symmetry. The $\mathbb{Z}_p$-charged particles are D3-branes wrapped on β and are annihilated by an instanton given by an NS5-brane wrapped on $\mathbb{X}_6$. The $\mathbb{Z}_p$-charged strings are fundamental strings which are annihilated by a string junction given by a D3-brane wrapped on α.

- The second BF coupling in equation (6.31) leads to a $\mathbb{Z}_{p'}$ discrete gauge symmetry. The $\mathbb{Z}_{p'}$-charged particles are D3-branes wrapped on α and are annihilated by an instanton given by an D5-brane wrapped on $\mathbb{X}_6$. The $\mathbb{Z}_{p'}$-charged strings are D1-branes which are annihilated by a string junction given by a D3-brane wrapped on β.

This naive combination leads to some puzzles, as follows. Charged particles under the $\mathbb{Z}_p$ symmetry play the role of string junctions under the $\mathbb{Z}_{p'}$ and vice versa. Microscopically, the reason is that D3-branes on β have a Freed-Witten anomaly and emit D-strings with $\mathbb{Z}_{p'}$ charge and conversely, D3-branes on α emit fundamental strings with $\mathbb{Z}_p$ charge. This naively allows configurations where pp' $Z_{p'}$-charged strings annihilate on p $\mathbb{Z}_p$-charged particles that annihilate on an instanton; or conversely, pp' $\mathbb{Z}_p$-charged strings annihilate on p' $\mathbb{Z}_{p'}$-charged particles that annihilate on an instanton. However, such configurations must be inconsistent, as the boundary of a string cannot have a boundary itself.

This problem can be rephrased in the following way. Recall from section 3.1 that string junctions are "magnetic monopoles" of the broken $U(1)$ gauge symmetries. The double role of $\mathbb{Z}_p$ electrically charged particles playing as $\mathbb{Z}_{p'}$ string junctions reflects an underlying simultaneous gauging of a gauge boson and its magnetic dual. This follows from the 4d electric-magnetic duality between $\hat{F}_2$ and $\hat{F}_2'$ that arises from the 10d self-duality of the RR 4-form. Such gaugings are inconsistent already on purely 4d grounds:

if we denote ϕ the 4d scalar dual to the 2-form B_2, the BF couplings (6.31) imply the gauge transformations

$$A_1 \to A_1 + d\lambda, \qquad \phi \to \phi + p\lambda; \tag{6.33a}$$
$$C_2 \to C_2 + d\Sigma_1, \qquad A_1 \to A_1 + p'\Sigma_1. \tag{6.33b}$$

However, it is not possible to write a consistent Lagrangian that respects both transformations and the relation $\hat{F}_2 = *_4\hat{F}'_2$. In physical terms, (6.33a) describes the gauge boson A_1 becoming massive by eating up the scalar ϕ, while (6.33b) describes the 2-form C_2 becoming massive by eating the presumed massless gauge boson A_1, which is in fact massive and therefore contains too many degrees of freedom.[5]

The above discussion does not imply that type IIB vacua with NSNS and RR fluxes are inconsistent, but rather that string theory must include ingredients to circumvent these problems.

The combination of NSNS and RR fluxes contributes to the RR tadpole conditions. In the type IIB case we study above, there is a tadpole of D3-brane charge given by

$$N_{flux} = \int_{\mathbb{X}_6} \overline{F}_3 \wedge \overline{H}_3 = pp'. \tag{6.34}$$

It turns out that the extra ingredients required to cancel the tadpole precisely solve the above inconsistencies. We consider two possibilities:

- **Orientifold planes**. The above tadpole can be cancelled by introducing O3-planes, as often done in the context of moduli stabilization. Their effect on the fields relevant to the discrete symmetries is drastic, since they are all projected out and no remnant discrete symmetry is left. The above problems are solved by removing degrees of freedom and rendering the structures trivial.

- **Anti-branes**. Another possibility is to introduce $\bar{\text{D}}3$- (or D3-)branes. They modify the above discussion because their overall worldvolume $U(1)$ couples to the relevant 2-forms through the $\bar{\text{D}}3$-brane Chern-Simons couplings. Denoting respectively by f_2 and f'_2 the field strength of the overall $U(1)$ and its 4d dual on a stack of pp' branes, we have the coupling
$$-\int_{4d} pp' \left(C_2 \wedge f_2 + B_2 \wedge f'_2\right). \tag{6.35}$$
The appearance of new degrees of freedom solves the problems, and leads to non-trivial discrete gauge symmetries as follows. Let us denote by $Q_{[\beta]}$ and $Q_{[\alpha]}$ the electric and magnetic generators that correspond to $\hat{D}_2$ and $\hat{D}'_2$ in equation (6.31), respectively; and by $Q_{U(1)_e}$ and $Q_{U(1)_m}$ the electric and magnetic generators of the $\bar{\text{D}}3$-brane $U(1)$, corresponding to the field strengths f'_2 and f_2 in equation (6.35). And NS5-brane instanton violates these $U(1)$ charges by $\Delta Q_{[\beta]} = p$ and $\Delta Q_{U(1)_e} = -pp'$,

[5] This problem has also been pointed out in the supergravity literature, in the context of embedding tensor formalism (see e.g. [234]). In that case, consistency of the theory requires the embedding tensor to satisfy a quadratic constraint which ensures that the gauging can be turned into a purely electric one in a suitable symplectic frame. In the above discussion we are dealing with genuinely electric/magnetic gaugings, which cannot be rotated into electric ones and which therefore lead to inconsistencies in the 4d theory.

while a D5-brane instanton gives $\Delta Q_{[\alpha]} = -p'$ and $\Delta Q_{U(1)_m} = -pp'$. Now consider the linear combinations

$$Q_1 = pQ_{[\beta]} - Q_{U(1)_e}, \qquad Q_2 = p'Q_{[\alpha]} + Q_{U(1)_m}. \tag{6.36}$$

The magnetic dual of Q_1 is $pQ_{[\alpha]} - Q_{U(1)_m}$, which is preserved by all instantons. Hence, the monopoles of Q_1 do to decay, and can play the role of junctions for the strings. These are associated to the discrete subgroup $\mathbb{Z}_{p^2-pp'} \subset U(1)_{Q_1}$, preserved by the NS5-brane instantons. Similarly $U(1)_{Q_2}$ leads to a discrete $\mathbb{Z}_{p'^2+pp'}$ symmetry.

6.4.2 Purely RR fluxes and symplectic rotations

Not all combinations of fluxes lead to tadpole contributions. Consider a type IIA compactification on a Calabi-Yau $\mathbb{X}_6$ with p' units of $\overline{F}_0$ and p units of $\overline{F}_6$. According to the results of section 6.2.1 one would have the following discrete gauge symmetries:

- $\overline{F}_0$ gives rise to a $\mathbb{Z}_{p'}$ discrete gauge symmetry. The $\mathbb{Z}_{p'}$-charged particles arise from D6-branes wrapped on $\mathbb{X}_6$ and are annihilated by an instanton given by a NS5-brane wrapped on $\mathbb{X}_6$. The $\mathbb{Z}_{p'}$-charged strings are fundamental strings, and are annihilated by a string junction given by a D0-brane.
- $\overline{F}_6$ gives rise to a $\mathbb{Z}_p$ discrete gauge symmetry. The $\mathbb{Z}_p$-charged particles correspond to D0-branes and are annihilated by an instanton given by a NS5-brane wrapped on $\mathbb{X}_6$. The $\mathbb{Z}_{p'}$-charged strings are fundamental strings, and are annihilated by a string junction given by a D6-brane wrapped on $\mathbb{X}_6$.

Naively, the system suffers from the troubles of the previous section, since $\mathbb{Z}_{p'}$-charged particles are also junctions for $\mathbb{Z}_p$-charged strings, and vice versa. However, in this case, the combination of $\overline{F}_0$ and $\overline{F}_6$ fluxes does not contribute to the RR tadpole and no extra ingredients can come to the rescue.

The main difference of this configuration compared to the one in the previous section, is that both $\mathbb{Z}_p$- and $\mathbb{Z}_{p'}$-charged strings are given by fundamental strings, and the instantons are given in both cases by a NS5-brane wrapping the whole $\mathbb{X}_6$. This implies that there is only one scalar that is being gauged by the gauge potential. This can be seen more clearly by looking to the Chern-Simons terms. The 10d Chern-Simons terms (6.1b) contain

$$\int_{10d} \left(\overline{F}_0 B_2 \wedge F_8 + \overline{F}_6 \wedge B_2 \wedge F_2\right) \tag{6.37}$$

which lead to the 4d couplings

$$\int_{4d} \left(p' B_2 \wedge \hat{F}_2' + pB_2 \wedge F_2\right), \tag{6.38}$$

where we have defined

$$\hat{F}_2' = \int_{\mathbb{X}_6} F_8, \tag{6.39}$$

which corresponds to the 4d dual of F_2.

The 4d couplings (6.38) describe a gauging by a combination of the electric and magnetic gauge potentials, which can be turned into a purely electric one by a 4d electric-magnetic symplectic transformation. Extracting $r = \gcd(p, p')$ we have

$$\int_{4d} rB_2 \wedge \left(\frac{p'}{r} \hat{F}_2' + \frac{p}{r} \hat{F}_2 \right) \quad \xrightarrow{sympl.} \quad \int_{4d} rB_2 \wedge f_2 \tag{6.40}$$

where f_2 is the field strength associated to the combination $Q = \frac{p'}{r} Q_e + \frac{p}{r} Q_m$ of the electric and magnetic generators. There is a $\mathbb{Z}_r$ discrete gauge symmetry, matching the annihilation/creation processes as follows. An NS5-brane instanton annihilates p' D6- and p D0-branes, namely r sets of the basic unit of Q-charge ($\frac{p'}{r}$ D6- and $\frac{p}{r}$ D0-branes). Similarly, $\mathbb{Z}_r$ charged strings are fundamental strings that can annihilate in sets of p' on a D0-brane junction of in sets of p on a wrapped D6-brane junction, and are hence conserved modulo r.

Similar conclusions apply to other combinations of type IIA RR fluxes. The above discussion is straightforward in the mirror IIB setup, in which all RR fluxes map into 3-form flux. In particular, 4d electric-magnetic symplectic transformations are simply changes in the symplectic basis of 3-cycles α and β in the IIB picture.

6.5 Strings and unstable domain walls

In the previous section we saw that the orientifold in supersymmetric flux compactifications generically projects out all flux-induced BF couplings and therefore also 4d strings endings on string junctions. However, there is another set of discrete ($\mathbb{Z}_p$ valued) brane wrappings in supersymmetric flux compactifications which usually survive the orientifold projection, namely 4d strings with domain walls attached to them. Nucleation of such 4d strings render some of the domain walls in the flux compactification ustable[6] and vacua separated by the wall, which would naively be different, are actually identical up to discrete identifications. In this section we will describe these 4d objects from a microscopical point of view, starting with a simple illustrative example to make the main ideas manifest, with similar phenomena in more general flux compactifications being discussed subsequently.

6.5.1 F_2 flux quantization in massive IIA

Let us consider massive IIA string theory compactified on a 6d manifold with mass parameter $\overline{F}_0 = p$. In addition to the $\mathbb{Z}_p$-valued fundamental strings discussed in section 6.2.1.1, the system admits also a set of 4d strings arising from NS5-branes wrapped on 4-cycles. The Freed-Witten anomaly on the NS5-brane forces these strings to have domain walls attached to them, given by p D6-branes wrapped on the same 4-cycle and ending on the NS5-brane. Therefore, p such D6-brane domain walls are unstable against nucleation of a string loop, as depicted in figure 6.1.

From standard arguments [235], two vacua separated by a single domain wall differ in one unit of RR flux $\overline{F}_2$ along the 2-cycle dual to the wrapped 4-cycle. The above

[6] In this section, the term “unstable” is used in a merely topological sense, and does not imply any dynamical content.

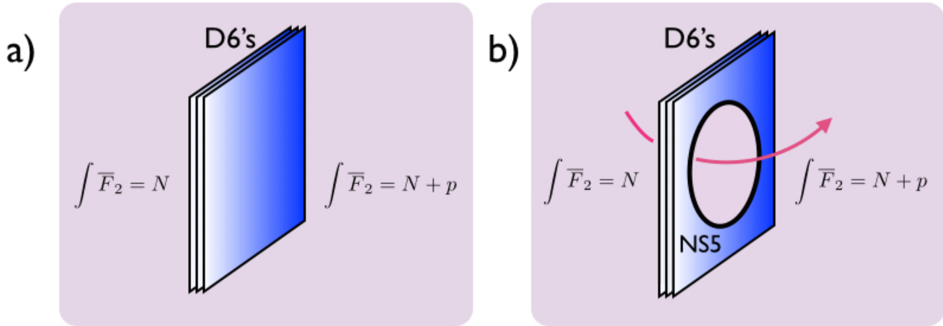

Figure 6.1: a) A 4d domain wall obtained by p D6-branes wrapped on a -cycle in a Calabi-Yau compactification of massive type IIA theory. It separates two regions of 4d space-time which differ by p units of RR F_2 flux on the dual 2-cycle. b)The domain wall is unstable by nucleation of holes bounded by strings, realized as one NS5-brane wrapped on the 4-cycle. The two vacua with differing flux must be equivalent.

instability of a set of p domain walls therefore indicate that vacua differing p units of $\overline{F}_2$ along the 2-cycle are actually equivalent. Using the argument for different 2-cycles, this implies the quantization condition

$$p_k \equiv \int_{\Pi_k} \overline{F}_2 \in p\,\mathbb{Z} \quad \text{for any } \Pi_k \in H_2(\mathbb{X}_6, \mathbb{Z}), \tag{6.41}$$

so the flux is Z_p-valued, rather than $\mathbb{Z}$-valued. This deviation from the cohomological classification of RR fields should be a generalization (suitable for the presence of 0-form flux) of the K-theory classification of RR fields [227].

From the point of view of effective supergravity, these equivalences follow from simple underlying axion-like field identifications. In the above example, the introduction of $\overline{F}_0 = p$ in type IIA theory implies the presence of a Chern-Simons coupling in the definition of the physical RR 2-form (see e.g. [72])

$$\tilde{F}_2 = dC_1 + \overline{F}_2 + pB_2. \tag{6.42}$$

When crossing a domain wall through a hole bounded by a 4d string, as in figure 6.1b, the axion $\phi_k = \int_{\Pi_k} B_2$ experiences a monodromy $\phi_k \to \phi_k + 1$ and a shift in the $\overline{F}_2$ flux, $p_k \to p_k - p$. According to equation (6.42), these two effects cancel each other in the sense that the physical field strength $\tilde{F}_2$ is left invariant, which implies that both vacua are equivalent.

Unstable domain walls typically arise in theories on which a discrete gauge symmetry G is spontaneously broken to a subgroup H (see e.g. [137] for a review). This description encompasses our example by considering the group $G = \mathbb{Z}$ of monodromies generated by the 4d strings, which is broken by $\overline{F}_0$ to a subgroup $H = \mathbb{Z}_p$. Strings with monodromies $a \in G/H$ which lay in the broken generators of G cannot be stable and have domain walls. Those are therefore classified by the cosets aH, so that different strings can bound the same domain wall only if they belong to the same coset. In cases where G contains several factors, these properties lead to an interesting interplay between the different types of 4d strings and domain walls, as we will show in the following section for more general IIA flux vacua.

6.5.2 $\mathbb{Z}_p$-valued domain walls in IIA flux vacua

In a fully consistent flux compactification the equations of motion typically demand different kinds of fluxes to be simultaneously switched on. Unstable domain walls, like those in the previous section, can produce intricate identifications of seemingly flux backgrounds.

The supergravity argument explained above can easily be extended to arbitrary flux backgrounds. The complete set of physical field strengths that appear in the 10d type IIA supergravity action is

$$H_3 = dB_2 + \overline{H}_3, \qquad \tilde{F}_p = dC_{p-1} - H_3 \wedge C_{p-3} + \left(\overline{F} e^{B_2}\right)_p, \tag{6.43}$$

where $\overline{F} = \overline{F}_0 + \overline{F}_2 + \overline{F}_4 + \overline{F}_6$ and $(\cdot)_p$ selects the p-form component of a polyform.

In the absence of 1-cycles, the axion-like fields of the compactification come from dimensionally reducing B_2 along the 2-cycles and C_3 along the 3-cycles of the compact manifold. Strings in 4d are thus given by NS5-branes wrapping 4-cycles and D4-branes wrapping 3-cycles, as those induce the correct monodromy for the corresponding axions. In the presence of general flux backgrounds, these objects develop Freed-Witten anomalies, that are cancelled by attaching domain walls. From equation (6.43) it is straightforward to work out the axion like identifications induced by the flux background and to interpret them in terms of unstable domain walls. To linear order in the shifts, we obtain the structure of unstable domain walls summarised in table 6.1.

Domain wall		String		Rank
type	cycle	type	cycle	
D2	–	NS5	$[\Gamma_4] \in H_4(\mathbb{X}_6, \mathbb{Z})$	$\int_{\Gamma_4} \overline{F}_4$
D4	$[\Pi_2] \in H_2(\mathbb{X}_6, \mathbb{Z})$	NS5	$[\Gamma_4] \in H_4(\mathbb{X}_6, \mathbb{Z})$	$\int_{\Gamma_4} \overline{F}_2 \wedge \pi_2$
D6	$[\Gamma_4'] \in H_4(\mathbb{X}_6, \mathbb{Z})$	NS5	$[\Gamma_4] \in H_4(\mathbb{X}_6, \mathbb{Z})$	$\int_{\Gamma_4} \overline{F}_0 \pi_4$
D2	–	D4	$[\alpha_3] \in H_3(\mathbb{X}_6, \mathbb{Z})$	$\int_{\alpha_3} \overline{H}_3$

Table 6.1: Domain walls in flux compactifications and the type of 4d strings that can nucleate in the presence of fluxes. The last column denotes the number of domain walls that are needed to nucleate a hole bounded by a string.

Having multiple types of domain walls and strings allows for new phenomena that were not present un the example of the previous section.

- **Hole collisions.** If a domain wall can decay via nucleation of different strings, there can be collisions of different type of holes [137]. This is the case of D2-brane domain walls in type IIA compactifications with $\overline{F}_4$ and $\overline{H}_3$ fluxes, as they can decay via nucleation of 4d strings from NS5-branes wrapping a 4-cycle or from D4-branes wrapping a 3-cycle (see table 6.1). The collision of the corresponding holes leads to a configuration with a single hole, crossed by a new 4d string, as shown in figure 6.2. Such string is a NS5/D4 bound state with cancelled Freed-Witten anomalies and therefore has no domain wall attached. In terms of the M-theory upping, the

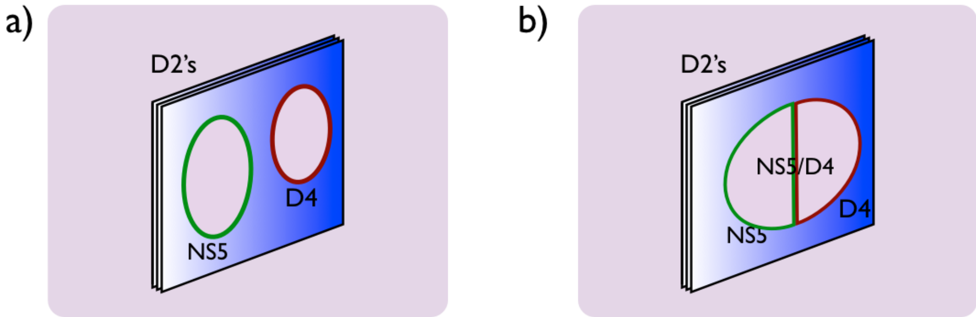

Figure 6.2: Collision of holes in unstable D2-brane domain walls in 4d compactifications with $\overline{F}_4$ and $\overline{H}_3$ fluxes.

new 4d string corresponds to a single M5-brane wrapping a linear combination of 4-cycles such that the total G_4 flux on it cancels.

- **Hanany-Witten effect for strings bounded with walls.** In theories where the unstable domain walls are associated to the spontaneous breaking of a non-Abelian discrete symmetry group G to a subgroup H, there is an interesting interplay between the 4d strings attached to domain walls of this section and the 4d strings that were discussed in section 6.2. Consider two 4d strings with non-commutaive monodromies a and b associated to two broken generators of G, and therefore with attache domain walls. If the commutator $c = a^{-1}b^{-1}ab$ lies in H, then the crossing of the two strings leads to the creation of a new streched 4d string with monodromy c with no domain wall attached. This situation arises in type IIA flux compactification without orientifold planes. To be more precise, consider a compactification with $\overline{H}_3$ flux on two Hodge dual 3-cycles α_3 and β_3. From table 6.1 we see that a D4-brane wrapped on α_3 leads to a 4d string with $\int \alpha_3 \overline{H}_3$ D2-brane domain walls attached, and a similar argument holds for a D4-brane wrapped on β_3. By the Hanany-Witen effect, crossing these 4d strings results in a stretched fundamental string with no domain wall attached, see figure 6.3.

6.5.3 Type IIB $SL(2,\mathbb{Z})$ from unstable domain walls

Consider a generic type IIB compactification on a Calabi-Yau $\mathbb{X}_6$ with NSNS and RR 3-form fluxes. For simplicity we will restrict to a single 3-cycle α_3 and its Hodge dual β_3 with general fluxes

$$\int_{\alpha_3} \overline{H}_3 = N, \qquad \int_{\beta_3} \overline{H}_3 = M, \tag{6.44a}$$

$$\int_{\alpha_3} \overline{F}_3 = N', \qquad \int_{\beta_3} \overline{F}_3 = M'. \tag{6.44b}$$

A D7-brane wrapping $\mathbb{X}_6$ leads to a 4d string with $k = \gcd(N, M)$ domain walls attached. They consist of D5-branes wrapped on the class $\frac{1}{k}\left(M[\alpha_3] - N[\beta_3]\right)$ and are required in order to cancel the $\overline{H}_3$-induced Freed-Witten anomaly on the D7-branes. Hence, a set of of k such domain walls can decay by nucleation of a 4d string.

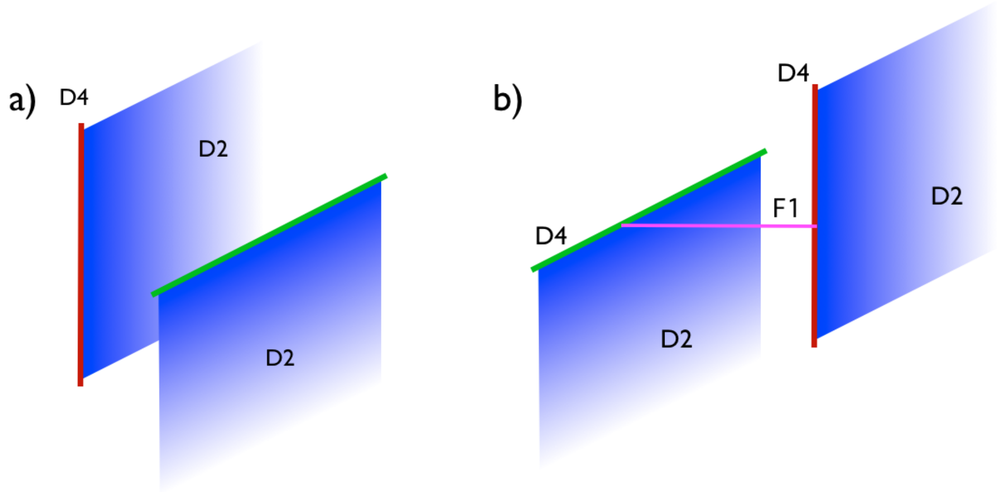

Figure 6.3: In compactifications with $\overline{H}_3$ fluxes, crossing of 4d strings (wrapped D4-branes) with attached domain walls (D2-branes) produces 4d strings with no domain wall (F1s).

As one crosses the domain wall, the complex axion-dilaton and the 3-form fluxes experience a $SL(2,\mathbb{Z})$ transformation

$$\tau \to \tau + 1, \quad N' \to N' + N, \quad M' \to M' + M. \tag{6.45}$$

This can be extended to the entire type IIB $SL(2,\mathbb{Z})$ duality group by considering (sets of) 4d strings arising from (p,q) 7-branes wrapped on $\mathbb{X}_6$ and domain walls arising from general bound states of D5- and NS5-branes wrapped on 3-cycles. Hence, unstable domain walls encode the existence of non-trivial dualities in string theory.

Crossing these unstable domain walls leaves always invariant the contribution of the 3-form fluxes to the RR 4-form tadpole

$$N_{flux} \equiv \int_{\mathbb{X}_6} \overline{H}_3 \wedge \overline{F}_3 = NM' - N'M, \tag{6.46}$$

and therefore the number of O3-planes, D3-branes or other sources required for tadpole cancellation.

However, this is not true in general for the case of stable domain walls, which change the flux background in a way that cannot be undone by a $SL(2,\mathbb{Z})$ transformation[7]. For instance, a domain wall given by one D5-brane on α_3 shifts the $\overline{F}_3$ flux in β_3 by one unit. The change in the 3-form flux contribution to the RR tadpoles is compensated by the appearance of N spacetime filling D3-brane on one side of the domain wall, which are microscopically required by a Freed-Witten anomaly on the D5-brane domain wall.

6.6 Flux catalysis and continuous isometries

In the previous sections we have focused on 4d $U(1)$ gauge bosons arising from higher dimensional p-form gauge potentials. In this section we briefly consider gaugings by KK

[7]See e.g. [236] for an explicit discussion.

gauge bosons from continuous $U(1)$ isometries in the compactification space. These do not arise in CY threefold compactifications, but may be present in other spaces.

6.6.1 A gauging by KK gauge bosons

Consider e.g. type IIB string theory compactified on a space $\mathbb{X}_6$, which for simplicity we take $B_4 \times \mathbb{T}^2$. Let $x^4, \ldots, x^7$ denote (local) coordinates on B_4, and x^8, x^9 coordinates on $\mathbb{T}^2$ (normalized with periodicity 1). We introduce p units NSNS 3-form flux with one leg along x^9 and two legs on a 2-cycle Σ_2 on B_4 (locally along e.g. x^6, x^7),

$$\int_{\Sigma_2 \times (\mathbb{S}^1)_9} \overline{H}_3 = p. \tag{6.47}$$

This flux breaks the KK $U(1)$ associated to x^9 because the 2-form gauge potential for such H_3 is not translationally invariant, as it can be written

$$B_2 = p\, x^9\, dx^6 dx^7. \tag{6.48}$$

Upon dimensional reduction, there is a gauging of the scalar $\phi = \int_{\Sigma_2} B_2$, since a translation in x^9 shifts the value of ϕ. Namely

$$A_1 \to A_1 + d\lambda \quad , \quad \phi \to \phi + p\lambda \tag{6.49}$$

This gauging defines a discrete $\mathbb{Z}_p$ gauge symmetry from a discrete isometry (analogous to the gauging in magnetized D-branes in section 7.3 where we make use of the formalism developed in section 3.3). The gauging should be manifest as a BF coupling upon dimensional reduction, although we will not need this result.

The particles that are charged under this discrete symmetry are states with KK momentum along x^9, and can annihilate in sets of p on a fundamental string wrapped on Σ_2 (and localized in x^9, hence violating momentum conservation). The 4d $\mathbb{Z}_p$-charged strings are NS5-branes on the 4-cycle dual to Σ_2 in $\mathbb{X}_6$, namely along x^4, x^5, x^8, x^9. The junction annihilating p strings is the KK monopole associated to x^9, namely a Taub-NUT (TN) geometry with isometry direction x^9, and base $\mathbb{R}^3$ in the non-compact 4d space, see first three rows in table 6.2. Microscopically, the TN geometry can be shown to emit p NS5-branes as follows. In the absence of the TN, $\overline{H}_3 = p\, dx^6 dx^7 dx^9$, but when the TN is present dx^9 is not well-defined and must be promoted to $\rho_1 \equiv dx^9 + \vec{v} \cdot d\vec{x}$, where $\vec{x} = (x^1, x^2, x^3)$, and $\vec{v}$ is the Dirac monopole potential. This 1-form is not closed, but rather $d\rho_1 = \omega_2$, where ω_2 is a harmonic 2-form supported on the TN center. The Bianchi identity becomes

$$dH_3 \;=\; p\, \omega_2\, dx^6 dx^7. \tag{6.50}$$

This source term induces a FW-like inconsistency, which must be cancelled by extra sources for H_3. These are p NS5-branes along e.g. $x^0, x^1, x^4, x^5, x^8, x^9$, ending on the TN location in x^1 at a four-dimensional boundary along x^0, x^4, x^5, x^8 (notice that the x^9 direction shrinks at the TN center).

The above argument is dual to a standard FW argument, as follows. T-duality along x^8 gives a type IIA configuration consisting of a TN geometry along $x^0, x^4, \ldots, x^8$ (with $\mathbb{S}^1$ fiber along x^9), with $\overline{H}_3 \sim p\, dx^6 dx^7 dx^9$ and NS5-branes along $x^0, x^1, x^4, x^5, x^8, x^9$. We can now perform a lift to M-theory by introducing a new direction denoted by $x^{9'}$, and shrink x^9 to get back to type IIA (a 9-9' flip). After this process, we end up with a D6-brane along $x^0, x^4, \ldots, x^8, x^{9'}$, with NSNS 3-form flux $\overline{H}_3 \sim p\, dx^6 dx^7 dx^{9'}$ inducing a FW anomaly, cancelled by D4-branes along x^0, x^1, x^4, x^5, x^8.

NS5	0	1	$\times$	$\times$	4	5	$\times$	$\times$	8	9
H_3	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	6	7	$\times$	9
TN	0	$\times$	$\times$	$\times$	4	5	6	7	8	$\otimes$
D5	0	$\times$	2	$\times$	4	5	6	7	$\times$	$\times$
F_3	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	7	8	9
TN	0	$\times$	$\times$	$\times$	4	5	6	$\otimes$	8	9
D3	0	$\times$	$\times$	3	4	5	$\times$	$\times$	$\times$	$\times$

Table 6.2: Relative geometry of the objects and fluxes involved in the realization of a non-Abelian discrete gauge symmetry. A number (resp. a cross) denotes that the corresponding brane or flux extends (or does not extend) along the corresponding direction. The circled cross indicates the isometric direction for Taub-NUT geometries producing line operators for string decays. In the top two triplets, the first row is the object producing the $\mathbb{Z}_p$-charged string, while the second and third to the flux and junction catalyzing its decay. The last line describes the D3-branes corresponding to the 4d strings created upon crossing the NS5- and D5-branes.

6.6.2 Engineering a non-Abelian discrete gauge symmetry

The above ingredients allow to engineer a non-Abelian discrete gauge symmetry, in a system with NSNS and RR 3-form fluxes. Consider for simplicity a $\mathbb{T}^6$ compactification (although the construction may generalize e.g. to torus bundles), parametrized by coordinates $x^4, \ldots, x^9$, and introduce 3-form fluxes $\overline{H}_3 \sim p\, dx^6 dx^7 dx^9$ and $\overline{F}_3 \sim p\, dx^7 dx^8 dx^9$, with the same number of flux quanta for simplicity. As in the previous section, $\overline{H}_3$ breaks the KK $U(1)_{x^9}$ down to $\mathbb{Z}_p$ (with charged strings given by NS5-branes wrapped on x^4, x^5, x^8, x^9, and annihilating on a suitable TN, see first three rows in table 6.2). By S-duality, $\overline{F}_3$ breaks the KK $U(1)_{x^8}$ down to $\mathbb{Z}_p$ (with charged strings played by D5-branes wrapped on x^4, x^5, x^6, x^7, and annihilating on a different TN, see middle three rows in table 6.2). As in section 6.3, the two $\mathbb{Z}_p$ factors are non-commuting, because crossing the corresponding 4d strings produces a new string, given by a D3-brane wrapped on x^4, x^5, by the HW effect. The resulting symmetry is a discrete Heisenberg group, with relations $A^p = B^p = 1$, $AB = CBA$, and $C^p = 1$. The non-Abelianity is analogous to that of section 5.2 (see also [138] for an early 5d realization), with the difference that in the present case the two $\mathbb{Z}_p$ factors arise from flux catalysis, rather than from torsion (co)homology. The relation $C^p = 1$ follows because the D3-brane 4d strings are associated to a $\mathbb{Z}_p$ discrete symmetry induced by the 3-form fluxes (in a way different from section 6.2.2, since it requires the presence of non-trivial 1-cycles) through the following Chern-Simons term

$$\int_{10d} C_4 \wedge (F_3 \wedge \overline{H}_3 + \overline{F}_3 \wedge H_3) \;\rightarrow\; \int_{4d} p\, \hat{B}_2 \wedge (\hat{F}_2 + \hat{F}_2'), \tag{6.51}$$

with $\hat{B}_2 = \int_{x^4 x^5} C_4$, and 4d $U(1)$ field strengths $\hat{F}_2 = \int_{x^8} F_3$ and $\hat{F}_2' = \int_{x^9} H_3$.

7

Discrete symmetries from discrete isometries.

An important feature of Kaluza-Klein compactifications is that isometries of the compactification manifold produce gauge symmetries in the lower theory. While this is familiar for continuous isometries, it also holds for discrete isometries, suggesting a natural source for (possibly non-Abelian) discrete gauge symmetries.

In section 3.3 we described discrete gauge symmetries in terms of gaugings of isometries of the moduli space of the scalars of the theory. While this mechanism is seemingly different from the one mentioned above, we will show that the latter nicely fits within the framework of section 3.3.

A general analysis of the mechanism is beyond the scope of this thesis; to illustrate the main ideas, in section 7.1 we will focus on the particular simple case of compactifications on twisted tori. Their realization in terms of gaugings allows to describe the discrete gauge symmetry in the language of gauging of non-Abelian axions of section 3.3. In section 7.2 we discuss the action of discrete symmetries of twisted tori on Kaluza-Klein modes.

As a specific and more realistic example, in section 7.3 we present a model of magnetized D-branes. The continuous isometries of the compactification space are broken to a discrete (non-Abelian) subgroup by the nontrivial flux in the branes. After considering the simpler case of a magnetized $\mathbb{T}^2$ (section 7.3.1) to illustrate the main ideas, we generalize those ideas to the case of a magnetized $\mathbb{T}^6$ in section 7.3.2 and apply them to a specific model in section 7.3.3.

Finally, in section 7.4, we study non-perturbative instanton effects, and how they manage to preserve the non-Abelian discrete gauge symmetry, using the constructions presented in section 7.3 for concreteness.

7.1 Non-Abelian discrete symmetries from discrete isometries of the twisted torus

Prototypical examples of compactification spaces with discrete isometries are twisted tori. For simplicity we focus on the case of a twisted torus $(\mathbb{T}^3)_M$ (where M denotes the first Chern class of the $\mathbb{S}^1$ fibration over the base $\mathbb{T}^2$). This space and its symmetries can be neatly displayed by the following coset construction (see e.g. [237, 238]). Consider the set $\mathcal{H}_3(\mathbb{R})$ of upper triangular matrices

$$g(x,y,z) = \begin{pmatrix} 1 & x & z+\frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \qquad x,y,z \in \mathbb{R} \tag{7.1}$$

which forms a non-compact Heisenberg group under multiplication

$$g(x, y, z)g(x', y', z') = g(x + x', y + y', z + z' + \frac{1}{2}(xy' - x'y)) \tag{7.2}$$

A basis of e.g. right-invariant forms $\eta^x = dx$, $\eta^y = dy$, $\eta^z = dz - \frac{1}{2}(ydx - xdy)$ allows the introduction of a metric $ds^2 = (\eta^x)^2 + (\eta^y)^2 + (\eta^z)^2$ with an isometry group defined by right multiplication, and therefore given by $\mathcal{H}_3(\mathbb{R})$ itself. More precisely, we have that the Killing vectors of this metric are given by the left-invariant vectors of $\mathcal{H}_3(\mathbb{R})$, a simple basis for them being

$$X^x_L = \partial_x - \tfrac{1}{2}y\partial_z, g(x, y, z) \to g(x + \lambda_x, y, z - \tfrac{1}{2}y\lambda_x), \tag{7.3a}$$
$$X^y_L]\partial_y + \tfrac{1}{2}x\partial_z, g(x, y, z) \to g(x, y + \lambda_y, z + \tfrac{1}{2}x\lambda_y), \tag{7.3b}$$
$$X^z_L = \partial_z, g(x, y, z) \to g(x, y, z + \lambda_z), \tag{7.3c}$$

where we have also specified the continuous isometries generated upon exponentiation of such Lie algebra elements.

The twisted torus is obtained as a left coset $(\mathbb{T}^3)_M = \mathcal{H}_3(\mathbb{R})/\mathcal{H}_3(M)$ of the non-compact space $\mathcal{H}_3(\mathbb{R})$ by the infinite discrete subgroup $\hat{\Gamma} = \mathcal{H}_3(M)$ with elements of the form

$$\begin{pmatrix} 1 & Mn_x & Mn_z \\ 0 & 1 & Mn_y \\ 0 & 0 & 1 \end{pmatrix}, \qquad n_x, n_y, n_z \in \mathbb{Z} \tag{7.4}$$

In other words, by imposing the identifications

$$g(x, y, z) \sim g(x + M, y, z - \tfrac{M}{2}y) \sim g(x, y + M, z + \tfrac{M}{2}x) \sim g(x, y, z + M). \tag{7.5}$$

As the metric is made of right-invariant forms, $(\mathbb{T}^3)_M$ has a well-defined quotient metric. On the other hand, some of the isometries of the parent space $\mathcal{H}_3(M)$ are broken in $(\mathbb{T}^3)_M$. The quotient enjoys a continuous U(1) isometry along the $\mathbb{S}^1$ fiber, generated by the invariant Killing vector $X^z_L = X^z_R = \partial_z$. However, the other two vectors X^x_L and X^y_L are not right-invariant, and so the corresponding continuous isometries disappear. Indeed, one can see that the action of X^x_L and X^y_L is in general different for different points of $\mathcal{H}_3(M)$ which are identified under (7.5). For instance,

$$e^{\lambda_x X^x_L} : g(x, y, z) \to g(x + \lambda_x, y, z - \tfrac{1}{2}y\lambda_x), \tag{7.6a}$$
$$\begin{aligned} e^{\lambda_x X^x_L} : g(x, y + M, z + \tfrac{M}{2}x) &\to g(x + \lambda_x, y + M, z + \tfrac{M}{2}x - \tfrac{1}{2}(y + M)\lambda_x) \\ &\sim g(x + \lambda_x, y, z - \tfrac{1}{2}y\lambda_x + M\lambda_x), \end{aligned} \tag{7.6b}$$

and so these two actions are the same only if $\lambda_x \in \mathbb{Z}$. A similar statement holds for the parameter λ_y in (7.3b). Hence, one finds that the identifications (7.5) break two of the continuous isometries of the parent $\mathcal{H}_3(\mathbb{R})$, preserving only the discrete order-M actions generated by

$$\begin{aligned} e^{X^x_L} &: g(x, y, z) \to g(x + 1, y, z - \tfrac{1}{2}y), \\ e^{X^y_L} &: g(x, y, z) \to g(x, y + 1, z + \tfrac{1}{2}x). \end{aligned} \tag{7.7}$$

Just like X_L^x and X_L^y, these generators do not commute, but rather produce an element of the U(1) generated by X_L^z, and realize a discrete Heisenberg group $\mathbf{P} = H_M = \mathcal{H}_3(M = 1)/\mathcal{H}_3(M)$. This discrete non-Abelian isometry group produces a discrete non-Abelian gauge symmetry H_M in the lower-dimensional theory[1].

The above construction is a particular case of a more general setup (see e.g. [239, 240]). Given a non-compact group G, the metric constructed with right-invariant forms has G itself as its isometry group (by right multiplication). In taking the coset G/H by a subgroup H, some of these isometries may survive (in continuous or discrete versions). In general, H is not a normal subgroup of G, so G/H is *not* a group, and cannot be the isometry group. To identify the correct isometry group, note that a point g_1 in G/H is, at the level of G, an equivalence class of points of the form $g_2 = g_1\gamma$, with $\gamma \in H$. An isometry R in G, mapping such g_1 and g_2 to g_1R and g_2R, is an isometry in G/H if the images are in the same equivalence class, namely if $g_2R = g_1R\gamma'$ for some $\gamma' \in H$. This requires R to satisfy $R^{-1}\gamma R = \gamma'$, namely conjugation by R should leave H invariant (although not necessarily pointwise). Those transformations form the so-called normalizer group N_H of H, and define the maximal subgroup of G such that H is normal in N_H. Since H acts trivially on G/H, the actual isometry group of G/H is N_H/H.

It is easy to show that in the twisted torus the group $N_{\mathcal{H}_3(M)}/\mathcal{H}_3(M)$ corresponds to the one identified above, namely $H_M\times$ U(1). The simplicity of the twisted torus allows to explicitly compute interesting restrictions imposed by the discrete symmetry on couplings of the lower-dimensional theory, as analyzed in detail in section 7.2.

It is natural to ask if, besides the above higher-dimensional description, there is a lower-dimensional description of the discrete gauge symmetry in terms of gauging of suitable scalars. Indeed, it is familiar that compactification on a twisted torus can alternatively be viewed as a compactification on $\mathbb{T}^3$ with metric fluxes, which can be described in terms of gauging a Heisenberg algebra [241]. The qualitative structure of the gauging is already manifest in the twisted torus metric, with $g_{xz} \sim y$ and $g_{yz} \sim x$, as follows. A gauge transformation of the KK gauge boson $V_\mu^x \sim g^x{}_\mu$ along the circle parametrized by y (i.e. a translation in y) shifts the vev of the scalar $\phi \sim g_{xz}$, and similarly for the KK gauge boson along x and the scalar g_{yz}. The integer M arises as the ratio of winding numbers of the map between full translations in the geometric circles, and the induced shifts in the scalar manifold. The non-Abelian structure of the isometries of the scalar manifold makes the resulting discrete gauge symmetry non-Abelian.

7.2 KK modes and Yukawas in twisted tori

We have seen in in the previous sections that non-Abelian discrete isometries of the twisted torus $(\mathbb{T}^2)_M = \mathcal{H}_3(\mathbb{R})/\mathcal{H}_3(M)$ lead to non-Abelian discrete gauge symmetries in the compactified effective theory. Thus, we expect the presence of powerful selection rules in this setup for the couplings of KK modes. In this section we work out such selection rules for the three-point couplings, this time exploiting the underlying group structure of the twisted torus.

[1]Note that although the twisted torus geometry has torsion cycles, the discrete gauge symmetry from discrete isometries is associated to components of the metric, and not to p-forms reduced on torsion classes, in contrast with the previous section.

In more general terms, for a compactification on a group manifold G/Γ, where G is a Lie group and $\Gamma \subset G$ a cocompact lattice, we expect 4d KK particles to arrange in irreducible unitary representations of the discrete isometry group $\mathbf{P}$ of G/Γ. Such representations can be explicitly worked out from the irreducible representations of G that are invariant under Γ. In physical terms, the components of these (generically infinite dimensional) representations correspond to wavefunctions of the particles in the 4d theory. The Clebsch-Gordan decomposition of the tensor product of two representations (namely, the operator product expansion, OPE) then allows the computation of superpotential couplings in the 4d effective theory, relating overlaps of n wavefunctions to overlaps of two wavefunctions. Since the Γ-invariant irreducible representations of G are also arranged in finite dimensional irreducible representations of the discrete symmetry group $\mathbf{P}$, the OPE must satisfy the set of selection rules associated to the discrete charge conservation.

In what follows we illustrate this procedure with the twisted torus compactification of section 7.1, for which $G = \mathcal{H}_3(\mathbb{R})$ is the Heisenberg group.

7.2.1 KK wavefunctions in twisted tori

The irreducible unitary representations of the Heisenberg group can be worked out starting from eq. (3.79), for instance by means of Kirillov's orbit method (see e.g. Appendix D of [242] for details). In general, irreducible representations $\pi(g)$ of non-Abelian groups are not simple functions, but rather operators acting on a Hilbert space of functions $u(\vec{s}) \in L^2(\mathbb{R}^{p(\pi)})$ with $p(\pi) \in \mathbb{N}$. For the case of the 3-dimensional Heisenberg group the complete set of irreducible unitary representations is given by

$$\begin{aligned} \pi_k(\vec{\phi})u(s) &= \exp\left[2\pi i k\left(\phi^3 + \frac{M}{2}\phi^1\phi^2 + \phi^2 s\right)\right]u(s+M\phi_1)\,, \qquad u(s) \in L^2(\mathbb{R}) \\ \pi_{k_1,k_2}(\vec{\phi}) &= \exp\left[2\pi i\left(k_1\phi^1 + k_2\phi^2\right)\right]. \end{aligned} \tag{7.8}$$

Γ-invariant irreducible representations can be constructed by taking sums over the lattice Γ

$$B(g) \equiv \sum_{\gamma\in\Gamma} \pi(\gamma g)u(s) \tag{7.9}$$

For the particular case of the Heisenberg group the complete procedure was carried out in [242]. Taking complex coordinates, $z = \phi^1 + U\phi^2$, and imposing $B(g)$ to be eigenstates of the Laplacian (namely, of the quadratic Casimir invariant of $\mathcal{H}_3(\mathbb{R})$), we obtain

$$B^M_{k,n,\delta}(z,\phi^3) = \Psi^{kM}_{n,\delta}(z)\exp\left(2\pi i k\phi^3\right), \tag{7.10a}$$

$$B_{\mathbf{k}}(z) = \exp\left[2\pi i\frac{\mathrm{Im}(\mathbf{k}z)}{\mathrm{Im}(U)}\right], \tag{7.10b}$$

where $\mathbf{k} \equiv -k_2 + \bar{U}k_1$, with $k_{1,2} \in \mathbb{N}$. We have defined

$$\begin{aligned} \Psi^N_{n,\delta}(z) \equiv (2\pi|N|)^{\frac{1}{4}} \sum_{s\in\mathbb{Z}} \psi_n &\left[\sqrt{2\pi|N|}\left(\frac{\delta}{N} + s + \frac{\mathrm{Im}(z)}{\mathrm{Im}(U)}\right)\right] \\ \exp&\left[2\pi i N \mathrm{Re}(z)\left(\frac{\delta}{N} + s + \frac{\mathrm{Im}(z)}{\mathrm{Im}(U)}\right)\right] \end{aligned} \tag{7.11}$$

with $n \in \mathbb{N}$, $\delta \in \mathbb{Z}_N$ and $\psi_n(x)$ the Hermite functions given by

$$\psi_n(x) \equiv \frac{1}{\sqrt{n! 2^n \pi^{1/2}}} H_n(x) e^{-x^2/2} \tag{7.12}$$

where $H_n(x)$ are the standard Hermite polynomials.

7.2.2 Yukawa couplings for KK modes

We are particularly interested in 4d particles with wavefunctions of the type (7.10) as those carry a non-zero KK momentum along the fiber of the twisted torus and therefore see the non-Abelian nature of the gauging. In the language of magnetized D-branes these correspond to particles with non-trivial charge k under the gauge symmetry of the D-brane. In such magnetized brane language, n denotes the Landau level and δ runs over the degeneracy of the corresponding Landau level (namely, it is a flavour index).

For any given set of states (7.10) with fixed k and δ (namely, for any given Γ-invariant representation of G with non-vanishing central charge) the ground state $n = 0$ (i.e., the highest weight of the representation) can be expressed in terms of Jacobi theta functions as

$$B^M_{k,0,\delta}(z, \phi^3) = (2\pi|kM|)^{\frac{1}{4}}\, \vartheta\begin{bmatrix} \frac{\delta}{kM} \\ 0 \end{bmatrix}(kMz;\, kMU)\, \exp\left[i\pi kM \frac{z\,\mathrm{Im}(z)}{\mathrm{Im}(U)} + 2\pi i k \phi^3\right] \tag{7.13}$$

One may easily check that (7.13) transforms under the generators of the gauge lattice $\hat{\Gamma}$ as

$$\phi^1 \to \phi^1 + \frac{1}{M}\,, \qquad \phi^3 \to \phi^3 - \frac{\phi^2}{2}\,, \qquad B^M_{k,0,\delta} \to \omega^\delta B^M_{k,0,\delta}, \tag{7.14a}$$

$$\phi^2 \to \phi^2 + \frac{1}{M}\,, \qquad \phi^3 \to \phi^3 + \frac{\phi^1}{2}\,, \qquad B^M_{k,0,\delta} \to B^M_{k,0,\delta+k}, \tag{7.14b}$$

$$\phi^3 \to \phi^3 + \frac{1}{M}\,, \qquad B^M_{k,0,\delta} \to \omega^k B^M_{k,0,\delta}, \tag{7.14c}$$

with $\omega \equiv \exp(2\pi i/M)$. As we saw in section 7.1, these are the generators of the discrete gauge symmetry $\mathbf{P} = \Gamma/\Gamma'$ for the level k. For instance, for $k = 1$ we have $\mathbf{P} = (\mathbb{Z}_M \times \mathbb{Z}_M) \rtimes \mathbb{Z}_M$, and in the particular case of $M = 3$, $\mathbf{P} = \Delta(27)$.

Let us now focus on the OPE of irreducible representations. For for zero-th Landau levels (7.13) the OPE can be easily worked out from the following relation between theta functions [75]

$$\begin{aligned} &\vartheta\begin{bmatrix} \frac{\delta_1}{N_1} \\ 0 \end{bmatrix}(N_1 z_1;\, N_1 U)\, \vartheta\begin{bmatrix} \frac{\delta_2}{N_2} \\ 0 \end{bmatrix}(N_2 z_2;\, N_2 U) = \\ &= \sum_{m \in \mathbb{Z}_{N_1+N_2}} \vartheta\begin{bmatrix} \frac{\delta_1+\delta_2+N_1 m}{N_1+N_2} \\ 0 \end{bmatrix}(N_1 z_1 + N_2 z_2;\, (N_1+N_2)U) \\ &\cdot \vartheta\begin{bmatrix} \frac{N_2\delta_1 - N_1\delta_2 + N_1 N_2 m}{N_1 N_2 (N_1+N_2)} \\ 0 \end{bmatrix}(N_1 N_2 (z_1 - z_2);\, N_1 N_2 (N_1+N_2) U), \end{aligned} \tag{7.15}$$

which leads to,

$$\Psi^{N_1}_{0,\delta_1}(z_1)\Psi^{N_2}_{0,\delta_2}(z_2) = \sum_{m\in\mathbb{Z}_{N_1+N_2}} \Psi^{N_1+N_2}_{0,\delta_1+\delta_2+N_1 m}\left(\frac{N_1 z_1 + N_2 z_2}{N_1+N_2}\right)\Psi^{N_1N_2(N_1+N_2)}_{0,N_2\delta_1-N_1\delta_2+N_1N_2 m}\left(\frac{z_1-z_2}{N_1+N_2}\right). \tag{7.16}$$

Setting $z_1 = z_2 = z$, $M_1 = M_2 = M$ and multiplying in both sides of this equation by $e^{2\pi i(k+j)\phi^3}$ we obtain

$$B^M_{k,0,\delta_1}(z,\phi^3)\, B^M_{j,0,\delta_2}(z,\phi^3) \;=\; \sum_{m\in\mathbb{Z}_{M(k+j)}} B^M_{k+j,0,\delta_1+\delta_2+kmM}(z,\phi^3)\,\Psi^{kj(k+j)M^3}_{0,M(j\delta_1-k\delta_2+kjmM)}(0) \tag{7.17}$$

This OPE can be used to compute superpotential couplings which only involve 4d KK modes with zero-th Landau level. For instance, we see from the above expansion that, up to an overall normalization factor, 3-particle couplings are given by

$$Y_{(k,0,\delta_1)(j,0,\delta_2)(h,0,\delta_3)} \simeq \Psi^{kjhM^3}_{0,M(j\delta_3-h\delta_2)}(0) \tag{7.18}$$

together with the selection rules,

$$h = k + j, \tag{7.19a}$$

$$\frac{\delta_3-\delta_1-\delta_2}{kM} \in \mathbb{Z}_{hM}\,. \tag{7.19b}$$

Let us now generalize equation (7.17) to KK particles with higher Landau level. The key observation is that higher Landau levels can be obtained by acting with the creation operators on the lowest Landau level (namely, by acting with the lowering operator on the highest weight of the corresponding irreducible representation). The Heisenberg algebra has only one creation operator. This is given by

$$a^\dagger \equiv 2\frac{\partial}{\partial z} - \pi N\bar{z} \tag{7.20}$$

Indeed, from eq. (7.10) one may check that

$$a^\dagger\,\Psi^N_{n,\delta} = i\sqrt{4\pi|N|(n+1)}\,\Psi^N_{n+1,\delta} \tag{7.21}$$

Acting with this operator an arbitrary number of times on both sides of eq. (7.16) and performing some algebra, we obtain

$$\begin{aligned}\Psi^{N_1}_{n,\delta_1}(z_1)\Psi^{N_2}_{p,\delta_2}(z_2) =& \sqrt{\frac{(-1)^{n+p}}{(N_1+N_2)^{n+p+1}}}\sum_{m\in\mathbb{Z}_{N_1+N_2}}\sum_{\ell=0}^{n}\sum_{s=0}^{p}(-1)^s\sqrt{N_1^{n+s-\ell}N_2^{p+\ell-s}}\\ &\cdot\left[\binom{n}{\ell}\binom{p}{s}\binom{n+p-\ell-s}{n-\ell}\binom{\ell+s}{\ell}\right]^{\frac{1}{2}}\\ &\cdot\Psi^{N_1+N_2}_{n+p-\ell-s,\,\delta_1+\delta_2+N_1m}\left(\frac{N_1z_1+N_2z_2}{N_1+N_2}\right)\\ &\cdot\Psi^{N_1N_2(N_1+N_2)}_{\ell+s,\,N_2\delta_1-N_1\delta_2+N_1N_2m}\left(\frac{z_1-z_2}{N_1+N_2}\right).\end{aligned} \tag{7.22}$$

Setting $z_1 = z_2 = z$, $M_1 = M_2 = M$ and multiplying in both sides of the equation by $e^{2\pi i(k+j)\phi^3}$), as we did before, we obtain the OPE for the complete set of KK modes of the 4d theory

$$B^M_{k,n,\delta_1}(z,\phi^3)\,B^M_{j,p,\delta_2}(z,\phi^3) = \sqrt{\frac{(-1)^{n+p}}{(k+j)^{n+p+1}M}} \sum_{m\in\mathbb{Z}_{M(k+j)}} \sum_{\ell=0}^{n}\sum_{s=0}^{p}(-1)^s\sqrt{k^{n+s-\ell}j^{p+\ell-s}} \cdot \left[\binom{n}{\ell}\binom{p}{s}\binom{n+p-\ell-s}{n-\ell}\binom{\ell+s}{\ell}\right]^{\frac{1}{2}} \cdot B^M_{k+j,\,n+p-\ell-s,\,\delta_1+\delta_2+kmM}(z,\phi^3)\,\Psi^{kjM^3(k+j)}_{\ell+s,\,M(j\delta_1-k\delta_2+kjmM)}(0). \tag{7.23}$$

From this expression we easily read the 3-particle couplings for arbitrary KK modes in the 4d theory. Up to combinatorial and overall numeric factors, these are given by

$$Y_{(k,n,\delta_1)(j,p,\delta_2)(h,q,\delta_3)} \sim \Psi^{kjhM^3}_{n+p-q,\,M(j\delta_3-h\delta_2)}(0), \tag{7.24}$$

together with the selection rules (7.19) and

$$q \in \{0,1,\ldots,n+p\}. \tag{7.25}$$

7.3 Magnetized branes and discrete flavor symmetries

In this section we discuss the appearance of non-Abelian discrete symmetries in magnetized toroidal compactifications, focusing on magnetized D-brane systems, although similar conclusions hold for analogous heterotic models and T-dual intersecting brane models (for review of these constructions, see [10] and references therein). These symmetries are analogous to those in the twisted torus in the previous section, since dimensional reduction of the latter on the $\mathbb{S}^1$ fiber produces a $\mathbb{T}^2$ compactification with a constant magnetic field for the KK gauge boson.

We start our analysis with the case of magnetized $\mathbb{T}^2$, to make the main ideas manifest, and also to allow contact with the earlier geometric discussion for twisted tori; subsequently we move on and analyze the more involved system of magnetized $\mathbb{T}^6$ compactifications. For the latter, and via dimensional reduction of the 10d type I supergravity action, we will make direct contact with the formalism of section 3.3.3.

7.3.1 Non-Abelian discrete symmetries and Yukawa couplings in magnetized $\mathbb{T}^2$

As a warm up, let us consider a $\mathbb{T}^2$ compactification with a $U(1)$ gauge field background

$$A_1 = \pi M\,(xdy - ydx), \tag{7.26}$$

so that

$$F_2 = 2\pi M dx \wedge dy \tag{7.27}$$

Before introducing F_2 the translations generated by ∂_x and ∂_y are clearly symmetries of the system. When introducing a non-vanishing F_2, even if constant along $\mathbb{T}^2$, they are no longer so, since A_1 depends explicitly on its coordinates x, y

$$A_1(x+\lambda_x, y) = A_1(x,y) + \lambda_x d\chi_x, \quad \chi_x = \pi M y, \tag{7.28a}$$
$$A_1(x, y+\lambda_y) = A_1(x,y) + \lambda_y d\chi_y, \quad \chi_y = -\pi M x. \tag{7.28b}$$

Hence, if we want to leave our system unchanged, with every translation we need to perform a gauge transformation that compensates the change in A_1. Acting on a wavefunction of charge q, this means that we need to perform the operations

$$\psi(x,y) \longrightarrow e^{-iq\lambda_x\chi_x}\psi(x+\lambda_x, y) = e^{q\lambda_x X_x}\psi(x,y), \tag{7.29a}$$
$$\psi(x,y) \longrightarrow e^{-iq\lambda_y\chi_y}\psi(x, y+\lambda_y) = e^{q\lambda_y X_y}\psi(x,y), \tag{7.29b}$$

instead of plain translations. The above are generated by the operators X_x, X_y, defined as (we also introduce the generator of gauge transformations X_Q)

$$X_x = \partial_x - i\pi M y, \tag{7.30a}$$
$$X_y = \partial_y + i\pi M x, \tag{7.30b}$$
$$X_Q = 2\pi i. \tag{7.30c}$$

These are the analogues of the left-invariant vectors of the twisted torus. Indeed, they satisfy the Heisenberg algebra $[X_x, X_y] = M X_Q$, which exponentiates to the group

$$\begin{aligned} g(\epsilon_x, \epsilon_y, \epsilon_Q) &= \exp\left(\frac{\epsilon_x}{M}X_x + \frac{\epsilon_y}{M}X_y + \frac{\epsilon_Q}{M}X_Q\right) \\ g(\epsilon'_x, \epsilon'_y, \epsilon'_Q)g(\epsilon_x, \epsilon_y, \epsilon_Q) &= g\left(\epsilon_x+\epsilon'_x,\, \epsilon_y+\epsilon'_y,\, \epsilon_Q+\epsilon'_Q + \frac{\epsilon'_x\epsilon_y}{2} - \frac{\epsilon_x\epsilon'_y}{2}\right) \end{aligned} \tag{7.31}$$

Again, the continuous version of this group is not a symmetry of our system. The point is that since the two-torus is compact, we need to impose well-defined boundary conditions on our charged particles, namely

$$\psi(x+1, y) = e^{iq\chi_x}\psi(x,y), \tag{7.32a}$$
$$\psi(x, y+1) = e^{iq\chi_y}\psi(x,y). \tag{7.32b}$$

In order to be actual symmetries of the system, the actions of X_x, X_y and X_Q must be compatible with the above identifications. This is automatic for X_Q, but not for X_x and X_y, since

$$e^{iq\lambda_x X_x}\psi(x, y+1) = e^{iq\chi_y}e^{q\lambda_x X_x}\psi(x,y) \quad \Longleftrightarrow \quad e^{iq\lambda_x M} = 1 \tag{7.33}$$

which is only true if $\lambda_x q M \in \mathbb{Z}$. Similarly, we obtain that $\lambda_y q M \in \mathbb{Z}$ and so, for particles of minimal charge $q = 1$ the symmetry corresponds only to a set of discrete elements together with the gauge transformations generated by X_Q, namely

$$\mathbf{P} = \{g(n_x, n_y, \epsilon_Q) \mid n_x, n_y = 0, \ldots, M-1; \epsilon_Q \in \mathbb{R}\} = H_M \times U(1). \tag{7.34}$$

Notice that in order to arrive to the above conclusion it was not necessary to know the precise form of the wavefunctions in a magnetized torus. This is to be expected because (7.34) is a symmetry group of the background, and not of its fluctuations. Nevertheless such symmetry group should have a well-defined action on the magnetized torus wavefunctions, which should transform as a particular representation under the discrete group H_M. Indeed, by solving for the $q = 1$ wavefunctions of a magnetized $\mathbb{T}^2$ one finds (see, e.g., [75])

$$\psi^{j,M}(z,U) = e^{i\pi M z\, \mathrm{Im}\, z/\mathrm{Im}\, U} \cdot \vartheta \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (Mz, MU) \tag{7.35}$$

where U stands for the complex structure and $z = x + Uy$ the complex coordinate of the $\mathbb{T}^2$, $j \in \mathbb{Z} \mod M$ is a family index and ϑ is the Jacobi theta function

$$\vartheta \begin{bmatrix} r \\ p \end{bmatrix} (\nu, U) = \sum_{l\in\mathbb{Z}} e^{\pi i (r+l)^2 U}\; e^{2\pi i (r+l)(\nu+p)} \tag{7.36}$$

One can now check that the action of the symmetry group (7.34) on this set is given by

$$g(n_x, n_y, \epsilon_Q)\, \psi^{j,M}(z,U) \;=\; e^{\frac{2\pi i}{M}(\epsilon_Q + n_x n_y/2)} e^{2\pi i \frac{n_x j}{M}}\; \psi^{j+n_y,M}(z,U) \tag{7.37}$$

with n_x, n_y and ϵ_Q taken as in (7.34). Notice that acting on the vector of functions

$$\Psi = \begin{pmatrix} \psi^{0,M} \\ \vdots \\ \psi^{M-1,M} \end{pmatrix} \tag{7.38}$$

the action of g preserves the norm $\sum_j |\psi^j|^2$ and corresponds to an element of $U(M)$. In particular, the discrete parameters n_x, n_y that generate the group H_M are mapped to the 't Hooft clock and shift $M \times M$ matrices

$$\mathbf{P}(1,0,0) \longrightarrow \tilde{T}_x \equiv \begin{pmatrix} 1 & & & \\ & \omega & & \\ & & \cdots & \\ & & & \omega^{M-1} \end{pmatrix}, \tag{7.39a}$$

$$\mathbf{P}(0,1,0) \longrightarrow \tilde{T}_y \equiv \begin{pmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ 1 & & & \end{pmatrix}, \tag{7.39b}$$

with ω the M-th root of unity. Hence, via its action on wavefunctions, the discrete gauge group H_M is embedded into a non-Abelian discrete subgroup of $SU(M)$.

The above system can be equivalently described as gaugings of a $\mathbb{T}^2$ compactification (see [241] for a heterotic description, and [243] for a D-brane/F-theory setup). In fact, the gauging structure is already manifest in (7.26), as follows. A gauge transformation of the KK gauge boson $V_\mu^x \sim g^x{}_\mu$ along the circle parametrized by y (i.e. a translation in y) shifts the vev of the Wilson line scalar $\xi_x \sim A_x$, and similarly for the KK gauge boson along x and the Wilson line scalar along y. The integer M arises as the ratio of winding numbers of the map between full translations in the geometric circles and the induced shifts in the Wilson line scalars. The non-Abelian structure is manifest in the above Heisenberg algebra, which corresponds to the gauging algebra (3.80). In this respect, the appearance of the discrete Heisenberg group gauge symmetry in the compactified theory fits within the general perspective in section 3.3.3. Note that such picture implies that performing translations along the coordinates x and y should be equivalent to performing shifts in the corresponding axion scalars, which for the gauging associated to magnetization are the $\mathbb{T}^2$ Wilson lines. Indeed, in the presence of Wilson lines the wavefunction (7.35) generalizes to

$$\psi^{j,M}(z+\xi, U) = e^{i\pi M(z+\xi)\,\mathrm{Im}(z+\xi)/\,\mathrm{Im}\,U} \cdot \vartheta \begin{bmatrix} \frac{j}{M} \\ 0 \end{bmatrix} (M(z+\xi), MU) \tag{7.40}$$

with $\xi = -\xi_y + U\xi_x$, and so a translation in $\mathbb{T}^2$ can be traded for a change in the Wilson line, and viceversa, in agreement with the gauging picture. This qualitative description can be fleshed out by performing the dimensional reduction of the $U(1)$ theory on a magnetized $\mathbb{T}^2$, as we analyze in the next section for the more complete case of magnetized $\mathbb{T}^6$ compactifications.

Before that, we pause to emphasize the effect of these non-Abelian discrete gauge symmetries at the level of the 4d effective action, in particular as selection rules for charged matter Yukawa couplings. For simplicity, we consider the case where all charged matter fields involved have equal range M, and transform under the discrete Heisenberg group with the clock and shift matrices (7.39). Further possibilities, with different field multiplicities and transformations, are illustrated by the example in section 7.3.3. Hence our present case involves couplings

$$\lambda_{ijk}\, \Phi_i^{ab}\Phi_j^{bc}\Phi_k^{ca} \tag{7.41}$$

where $i, j, k = 1, \ldots, M$ are family indices, and a, b, c are Chan-Paton gauge indices. Since the massless 4d fields Φ_i have an internal wavefunction (7.35), they also transform with the matrices (7.39). The constraints imposed by the symmetry are

$$\lambda_{ijk} = 0 \quad \text{if } i+j+k \neq 0 \qquad \text{mod } M, \tag{7.42a}$$

$$\lambda_{ijk} = \lambda_{i+1,j+1,k+1}. \tag{7.42b}$$

These selection rules were obtained by explicit computation in [75, 74] for magnetized and intersecting brane models, respectively; they were suspected to arise from a discrete symmetry in [244] (see also [245]). Our analysis shows that this is not an accidental symmetry but rather a discrete gauge symmetry present in the model.

7.3.2 Dimensional reduction and non-Abelian discrete symmetries

Let us now generalize the above simple picture and consider N magnetized D9-branes on a $\mathbb{T}^6 = (\mathbb{T}^2)_1 \times (\mathbb{T}^2)_2 \times (\mathbb{T}^2)_3$ orientifold compactification with O9 and O5-planes (the conclusions hold for any system leading to the same 4d theory, in particular T-duals with lower-dimensional intersecting/magnetized branes). The 10d effective action for this setup can be suitably described in terms of the type I supergravity action

$$S_{10\mathrm{d}} = \frac{1}{2\kappa^2}\int d^{10}x(-G)^{1/2}\left[e^{-2\phi}(R + 4\partial_\mu\phi\partial^\mu\phi) - \frac{1}{4}|\tilde{F}_3|^2 - \frac{1}{4}|\tilde{F}_7|^2 - e^{-\phi}\,\mathrm{tr}(|F_2|^2)\right] \tag{7.43}$$

where we have doubled the degrees of freedom of $\tilde{F}_3$ by introducing a dual 7-form field-strength $\tilde{F}_7 = -*\tilde{F}_3$, with

$$\tilde{F}_3 = dC_2 - \omega_3, \tag{7.44a}$$

$$\tilde{F}_7 = dC_6 - \frac{1}{12}\omega_7, \tag{7.44b}$$

and ω_3 and ω_7 respectively the 3- and 7-dimensional Chern-Simons forms

$$\omega_3 = \mathrm{tr}_V \left[A \wedge dA - \frac{2i}{3} A \wedge A \wedge A \right], \tag{7.45a}$$

$$\begin{aligned} \omega_7 = \mathrm{tr}_V \Big[& A \wedge dA \wedge dA \wedge dA - \frac{4i}{3} A \wedge A \wedge A \wedge dA \wedge dA - \\ & - \frac{6}{5} A \wedge A \wedge A \wedge A \wedge A \wedge dA + \frac{4i}{7} A \wedge A \wedge A \wedge A \wedge A \wedge A \wedge A \Big]. \end{aligned} \tag{7.45b}$$

In order to achieve a chiral 4d compactification we magnetize the D9-branes by considering a background for the Yang-Mills field strength F_2 of the form

$$F_2 = \sum_{r=1}^{3} \frac{\pi i}{\mathrm{Im}\, U^r} \begin{pmatrix} \frac{m_a^r}{n_a^r} \mathbb{I}_{N_a} & & & \\ & \frac{m_b^r}{n_b^r} \mathbb{I}_{N_b} & & \\ & & \frac{m_c^r}{n_c^r} \mathbb{I}_{N_c} & \\ & & & \ddots \end{pmatrix} dz^r \wedge d\bar{z}^r, \tag{7.46}$$

where $z^r = dx^r + U^r dy^r$ is the complexified coordinate of $(\mathbb{T}^2)_r$, U^r its complex structure and $n_\alpha^r, m_\alpha^r \in \mathbb{Z}$ the D9-brane 'magnetic numbers', with $N_\alpha = n_\alpha^1 n_\alpha^2 n_\alpha^3$, $N = \sum_\alpha N_\alpha$.

Upon dimensional reduction, and focusing on 'diagonal' geometric moduli, the 4d effective theory contains $7+3N$ complex scalars: 3 complex structure moduli U^p, 3 Kähler moduli T^p, 1 axio-dilaton S and $3N$ complex Wilson lines ξ_α^p, that can be defined as [246, 247]

$$T^p = \int_{(\mathbb{T}^2)_p} C_2 + i e^{-\phi} J, \tag{7.47a}$$

$$S = \int_{\mathbb{T}^6} C_6 + i e^{-\phi} \mathrm{Vol}_6, \tag{7.47b}$$

$$\xi_\alpha^p = -\xi_{\alpha,y}^p + U^p \xi_{\alpha,x}^p, \tag{7.47c}$$

with J the Kähler form of $\mathbb{T}^6$, and $\mathrm{Vol}_6 = J^3/3!$ its volume form. The scalars $\xi_{\alpha,x}^p$ and $\xi_{\alpha,y}^p$ are the real Wilson lines along the two 1-cycles of $(\mathbb{T}^2)_p$, with periodicity $[0, 2/n_\alpha^r)$.[2]

There are in addition $6+N$ $U(1)$ gauge bosons in the 4d effective theory: 6 $U(1)$ gauge bosons coming from the isometries of the $\mathbb{T}^6$, that we shall represent by $V_\mu^{x,p}$ and $V_\mu^{y,p}$, and N $U(1)$ gauge bosons from the Cartan generators of the D9-brane $U(N)$ gauge group, denoted by A_μ^α in what follows.

The kinetic terms for the 4d scalars can be obtained by dimensionally reducing the

[2] A different (yet common) convention in the literature for the normalization of the Wilson line scalars is such that $\xi_{\alpha,x}^p, \xi_{\alpha,y}^p$ lay on the interval $[0, 1/n_\alpha^p)$.

10d action (7.43) on the above background (see also [241]), resulting in[3]

$$\begin{aligned}\mathcal{L}_{4\mathrm{d}} =& \frac{1}{(S-\bar{S})^2}\left| DS - \frac{1}{2}\sum_{p=1}^{3}\sum_{\alpha} c_\alpha^p \left(\xi^p_{x,\alpha} D\xi^p_{y,\alpha} - \xi^p_{y,\alpha} D\xi^p_{x,\alpha}\right)\right|^2 \\ &+ \sum_{p=1}^{3}\left[\frac{1}{(U^p-\bar{U}^p)^2}|\partial U^p|^2 + \frac{1}{(T^p-\bar{T}^p)^2}\left| DT^p + \frac{1}{2}\sum_\alpha c_\alpha^0 \left(\xi^p_{x,\alpha} D\xi^p_{y,\alpha} - \xi^p_{y,\alpha} D\xi^p_{x,\alpha}\right)\right|^2 \right. \\ &\left. + \frac{1}{U^p-\bar{U}^p}\sum_\alpha \frac{c_\alpha^0}{T^p-\bar{T}^p}\left|-D\xi^p_{y,\alpha} + U^p D\xi^p_{x,\alpha}\right|^2\right],\end{aligned} \tag{7.48}$$

where we have defined the following covariant derivatives

$$D_\mu S = \partial_\mu S + \sum_\alpha d_\alpha^0 A_\mu^\alpha, \tag{7.49a}$$

$$D_\mu T^p = \partial_\mu T^p - \sum_\alpha d_\alpha^p A_\mu^\alpha, \tag{7.49b}$$

$$D_\mu \xi^p_{x,\alpha} = \partial_\mu \xi^p_{x,\alpha} + \frac{m_\alpha^p}{n_\alpha^p} V_\mu^{y,p}, \tag{7.49c}$$

$$D_\mu \xi^p_{y,\alpha} = \partial_\mu \xi^p_{y,\alpha} - \frac{m_\alpha^p}{n_\alpha^p} V_\mu^{x,p}. \tag{7.49d}$$

Notice that the coefficients of this expression

$$\begin{aligned} c_\alpha^0 &= n_\alpha^1 n_\alpha^2 n_\alpha^3\,, \quad & c_\alpha^1 &= n_\alpha^1 m_\alpha^2 m_\alpha^3\,, \quad & c_\alpha^2 &= m_\alpha^1 n_\alpha^2 m_\alpha^3\,, \quad & c_\alpha^3 &= m_\alpha^1 m_\alpha^2 n_\alpha^3, \\ d_\alpha^0 &= m_\alpha^1 m_\alpha^2 m_\alpha^3\,, \quad & d_\alpha^1 &= m_\alpha^1 n_\alpha^2 n_\alpha^3\,, \quad & d_\alpha^2 &= n_\alpha^1 m_\alpha^2 n_\alpha^3\,, \quad & d_\alpha^3 &= n_\alpha^1 n_\alpha^2 m_\alpha^3, \end{aligned} \tag{7.50}$$

measure the D9-, D5-, D3/$\overline{\text{D3}}$ and D7/$\overline{\text{D7}}$-brane charges of our system, induced on the stack of N D9-branes by the magnetization.

In order to make contact with our general discussion of section 3.3.3, let us analyze the symmetries of the axion-like scalars within (7.48). Due to the shift symmetries of the RR potentials in 10d, the real scalars $\phi^0 \equiv \text{Re } S$ and $\phi^r \equiv \text{Re } T^r$ behave as axions in the 4d effective theory with shift symmetries

$$\phi^P \longrightarrow \phi^P + \epsilon^P \qquad P = 0,1,2,3, \tag{7.51}$$

and discrete identifications

$$\phi^P \simeq \phi^P + 1 \qquad P = 0,1,2,3. \tag{7.52}$$

The same occurs for the Wilson line scalars $\xi^r_{\alpha,x}$ and $\xi^r_{\alpha,y}$, for whom 4d shift symmetries descend from 10d YM gauge invariance. Setting momentarily $d^P_\alpha = 0$, we see that in order to have a symmetry of the action (7.48) a shift in the Wilson lines should be accompanied with a shift in the above RR axions. More precisely we have that

$$\xi^p_{\alpha,x} \to \xi^p_{\alpha,x} + \epsilon^p_{\alpha,x}, \quad \phi^0 \to \phi^0 + \tfrac{1}{2} c^p_\alpha \xi^p_{y,\alpha}\epsilon^p_{\alpha,x}, \quad \phi^p \to \phi^p - \frac{1}{2} c^0_\alpha \xi^p_{y,\alpha}\epsilon^p_{\alpha,x}, \tag{7.53a}$$

$$\xi^p_{\alpha,y} \to \xi^p_{\alpha,y} + \epsilon^p_{\alpha,y}, \quad \phi^0 \to \phi^0 - \tfrac{1}{2} c^p_\alpha \xi^p_{x,\alpha}\epsilon^p_{\alpha,y}, \quad \phi^p \to \phi^p + \frac{1}{2} c^0_\alpha \xi^p_{x,\alpha}\epsilon^p_{\alpha,y}, \tag{7.53b}$$

[3] We have taken the magnetization to be actually along the vector representation of $SO(2N)$, so that sums over α in equation (7.48) and following expressions do not run over the orientifold brane images.

leave (7.48) invariant. We thus have the discrete identifications

$$\xi^p_{\alpha,x} \simeq \xi^p_{\alpha,x} + \frac{2}{n^p_\alpha}, \quad \phi^0 \simeq \phi^0 + c^p_\alpha \frac{\xi^p_{y,\alpha}}{n^p_\alpha}, \quad \phi^p \simeq \phi^p - c^0_\alpha \frac{\xi^p_{y,\alpha}}{n^p_\alpha}, \tag{7.54a}$$

$$\xi^p_{\alpha,y} \simeq \xi^p_{\alpha,y} + \frac{2}{n^p_\alpha}, \quad \phi^0 \simeq \phi^0 - c^p_\alpha \frac{\xi^p_{x,\alpha}}{n^p_\alpha}, \quad \phi^p \simeq \phi^p + c^0_\alpha \frac{\xi^p_{x,\alpha}}{n^p_\alpha}. \tag{7.54b}$$

Switching the coefficients d^P_α back on, the action (7.48) can be rewritten in the form (3.71). In particular, it can be written as a gauged non-Abelian scalar manifold with action (3.76), right-invariant 1-forms (c.f. equations (3.77))

$$\eta^{\phi^p}_\mu = \partial_\mu \phi^p + \frac{1}{2}\sum_\alpha \left(-2d^p_\alpha A^\alpha_\mu + c^0_\alpha \xi^p_{x,\alpha}\eta^{\xi^p_{y,\alpha}}_\mu - c^0_\alpha \xi^p_{y,\alpha}\eta^{\xi^p_{x,\alpha}}_\mu\right), \tag{7.55a}$$

$$\eta^{\phi^0}_\mu = \partial_\mu \phi^0 + \frac{1}{2}\sum_\alpha \left[2d^0_\alpha A^\alpha_\mu - \sum_{p=1}^{3}\left(c^p_\alpha \xi^p_{x,\alpha}\eta^{\xi^p_{y,\alpha}}_\mu - c^p_\alpha \xi^p_{y,\alpha}\eta^{\xi^p_{x,\alpha}}_\mu\right)\right], \tag{7.55b}$$

$$\eta^{\xi^p_{x,\alpha}}_\mu = \partial_\mu \xi^p_{x,\alpha} + \frac{m^p_\alpha}{n^p_\alpha} V^{y,p}_\mu, \tag{7.55c}$$

$$\eta^{\xi^p_{y,\alpha}}_\mu = \partial_\mu \xi^p_{y,\alpha} - \frac{m^p_\alpha}{n^p_\alpha} V^{x,p}_\mu, \tag{7.55d}$$

tangent space metric

$$\begin{aligned}\mathcal{P}_{ab} &= \begin{pmatrix} \mathcal{P}_{\phi^p\phi^p} & 0 & 0 & 0 \\ 0 & \mathcal{P}_{\phi^0\phi^0} & 0 & 0 \\ 0 & 0 & \mathcal{P}_{\xi^p_{x,\alpha}\xi^p_{x,\alpha}} & \mathcal{P}_{\xi^p_{x,\alpha}\xi^p_{y,\alpha}} \\ 0 & 0 & \mathcal{P}_{\xi^p_{y,\alpha}\xi^p_{x,\alpha}} & \mathcal{P}_{\xi^p_{y,\alpha}\xi^p_{y,\alpha}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{(T^p-\bar{T}^p)^2} & 0 & 0 & 0 \\ 0 & \frac{1}{(S-\bar{S})^2} & 0 & 0 \\ 0 & 0 & \frac{c^0_\alpha |U^p|^2}{(U^p-\bar{U}^p)(T^p-\bar{T}^p)} & -\frac{c^0_\alpha (U^p+\bar{U}^p)}{(U^p-\bar{U}^p)(T^p-\bar{T}^p)} \\ 0 & 0 & -\frac{c^0_\alpha (U^p+\bar{U}^p)}{(U^p-\bar{U}^p)(T^p-\bar{T}^p)} & \frac{c^0_\alpha}{(U^p-\bar{U}^p)(T^p-\bar{T}^p)} \end{pmatrix},\end{aligned} \tag{7.56}$$

and algebra of shift symmetries

$$[t_{x^p_\alpha}, t_{y^p_\alpha}] = c^0_\alpha t_{\phi^p} - c^p_\alpha t_{\phi^0}, \tag{7.57}$$

where $t_{x^p_\alpha}$, $t_{y^p_\alpha}$, t_{ϕ^p} and t_{ϕ^0} denote the generators of shifts of the axion-like scalars $\xi^p_{x,\alpha}$, $\xi^p_{y,\alpha}$, ϕ^p and ϕ^0, respectively.

From these expressions we observe that the coefficients c^P_α (i.e., the D9- and D5-brane charges our our model) determine the structure constants of the non-Abelian algebra in the axionic manifold $(\phi^P, \xi^p_{x,\alpha}, \xi^p_{y,\alpha})$. On the other hand the coefficients d^P_α (the D3/$\overline{\text{D3}}$ and D7/$\overline{\text{D7}}$ charges) specify the set of D-brane $U(1)$'s that become massive and the embedding of their gauge lattice into the lattice of scalar shifts. Indeed, as one can check from (7.49), the linear combinations of D9-brane $U(1)$ gauge symmetries

$$Q^P = \sum_\alpha d^P_\alpha Q^\alpha, \qquad P = 0, 1, 2, 3, \tag{7.58}$$

are spontaneously broken to discrete gauge symmetries by eating the RR scalars ϕ^P, as it is familiar from the generalized Green-Schwarz mechanism in magnetized D9-brane compactifications. Similarly, the $U(1)$ Kaluza-Klein isometries $V_\mu^{x,p}$ and $V_\mu^{y,p}$ are spontaneously broken to discrete isometries by eating Wilson line scalars.

From (7.55) one can check that under the $U(1)$ gauge transformations the above axion-like scalars shift according to

Q^α	X^p	Y^p
$A_\mu^\alpha \to A_\mu^\alpha + \partial_\mu \lambda^1$	$V_\mu^{x,p} \to V_\mu^{x,p} + \partial_\mu \lambda^2$	$V_\mu^{y,p} \to V_\mu^{y,p} + \partial_\mu \lambda^3$
$\phi^0 \to \phi^0 - d_\alpha^0 \lambda^1$	$\phi^0 \to \phi^0 - \sum_\alpha d_\alpha^0 \xi_{x,\alpha}^p \lambda^2$	$\phi^0 \to \phi^0 - \sum_\alpha d_\alpha^0 \xi_{y,\alpha}^p \lambda^3$
$\phi^p \to \phi^p + d_\alpha^p \lambda^1$	$\phi^p \to \phi^p + \sum_\alpha d_\alpha^p \xi_{x,\alpha}^p \lambda^2$	$\phi^p \to \phi^p + \sum_\alpha d_\alpha^p \xi_{y,\alpha}^p \lambda^3$
	$\xi_{y,\alpha}^p \to \xi_{y,\alpha}^p + \frac{m_\alpha^p}{n_\alpha^p} \lambda^2$	$\xi_{x,\alpha}^p \to \xi_{x,\alpha}^p - \frac{m_\alpha^p}{n_\alpha^p} \lambda^3$
	$A_\mu^\alpha \to A_\mu^\alpha + \xi_{x,\alpha}^p \partial_\mu \lambda^2$	$A_\mu^\alpha \to A_\mu^\alpha + \xi_{y,\alpha}^p \partial_\mu \lambda^3$

(7.59)

This in turn implies that these gauge generators satisfy the gauge algebra [243]

$$[X^p, Y^p] = -\frac{m_\alpha^p}{n_\alpha^p} Q^\alpha. \tag{7.60}$$

The discrete identifications (7.52) and (7.54) are mapped via the above shifts to the discrete gauge symmetry group of the theory, which can be embedded in the continuous Lie group that arises from (7.60). Rather than describing the most general case, in what follows we illustrate the type of discrete gauge symmetries that one may obtain by analyzing a semi-realistic example.

7.3.3 An example: flavour symmetries in a MSSM-like model

We can illustrate the application of the above general ideas by considering the MSSM-like model of [168, 74, 75] and its global realization in terms of an orientifold of $\mathbb{T}^6/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ [80]. The model consists of two stacks of magnetized D9-branes (stacks a and d), and two stacks of D5-branes (stacks b and c). The wrapping and magnetization numbers are summarized in table 7.1.

If brane b is not on top of the orientifold plane, the gauge group is $SU(3) \times SU(2)_L \times U(1)_Y \times U(1)_{B-L} \times \mathbb{Z}_3$.[4] The two $U(1)$ factors are related to the diagonal $U(1)$ generators of the three stacks a, c and d as

$$Q_Y = \frac{1}{6}(Q_a - 3Q_c + 3Q_d), \tag{7.61a}$$

$$Q_{B-L} = \frac{Q_a}{3} + Q_d, \tag{7.61b}$$

[4] At other particular points of the moduli space, the continuous part of the gauge group can be enhanced to the maximal $SU(4) \times SU(2)_L \times SU(2)_R$ gauge symmetry of this model. See [74] for details.

N_α	(n_α^1, m_α^1)	(n_α^2, m_α^2)	(n_α^3, m_α^3)
$N_a = 3$	$(1,0)$	$(3,1)$	$(3,-1)$
$N_b = 1$	$(0,1)$	$(1,0)$	$(0,-1)$
$N_c = 1$	$(0,1)$	$(0,-1)$	$(1,0)$
$N_d = 1$	$(1,0)$	$(3,-1)$	$(3,1)$

Table 7.1: Wrapping and magnetization numbers of the T-dual model to that of [168, 74] with D5 and magnetized D9-branes.

whereas the remaining orthogonal combination of U(1)'s,

$$Q_{\mathbb{Z}_3} = 3Q_a - Q_d, \tag{7.61c}$$

is anomalous and is spontaneously broken to a discrete $\mathbb{Z}_3$ gauge symmetry [140].[5] Indeed, observe from table 7.1 that the magnetization on the D9-branes induce non-trivial D7/$\overline{\text{D7}}$ charges

$$d_a^2 = d_d^3 = 3, \qquad d_a^3 = d_d^2 = -3 \tag{7.62}$$

so that from equation (7.55) we observe that $3U(1)_a - U(1)_d$ becomes massive by combining with the linear combination of RR axions $\phi^2 - \phi^3$.

Sector	Field	$SU(3) \times SU(2)_L$	Q_Y	Q_{B-L}	$Q_{\mathbb{Z}_3}$
ab	Q_L	$3(\mathbf{3},\mathbf{2})$	1/6	1/3	3
ac	U_R	$3(\bar{\mathbf{3}},\mathbf{1})$	-2/3	-1/3	-3
$ac*$	D_R	$3(\bar{\mathbf{3}},\mathbf{1})$	1/3	-1/3	-3
db	L	$3(\mathbf{1},\mathbf{2})$	-1/2	-1	1
dc	N_R	$3(\mathbf{1},\mathbf{1})$	0	1	-1
$dc*$	E_R	$3(\mathbf{1},\mathbf{1})$	1	1	-1
bc	H_u	$(\mathbf{1},\mathbf{2})$	1/2	0	0
bc	H_d	$(\mathbf{1},\bar{\mathbf{2}})$	-1/2	0	0

Table 7.2: Chiral spectrum, Higgs sector and charges of the model in table 7.1.

The chiral spectrum of the model is summarized in table 7.2, and is exactly that of the MSSM with three generations of quarks and leptons and one vector-like pair of Higgses. As it has been noticed in [140], the $\mathbb{Z}_3$ discrete gauge symmetry of this model is equivalent to baryon triality [130], up to $\text{U}(1)_{B-L}$ and $\text{U}(1)_Y$ transformations. In particular dimension five proton decay operators $Q_LQ_LQ_LL$ and $U_RE_RU_RD_R$ vanish to

[5] More precisely, the anomalous $U(1)$ is broken to a $\mathbb{Z}_9$ discrete gauge symmetry, but a $\mathbb{Z}_3 \subset \mathbb{Z}_9$ subgroup actually corresponds to the center of $SU(3)$. Hence, the only non-trivial discrete symmetry is $\mathbb{Z}_9 / \mathbb{Z}_3 \simeq \mathbb{Z}_3$.

all orders in perturbation theory and, according to the discussion in section 7.4, also at the non-perturbative level.[6]

Besides this $\mathbb{Z}_3$ discrete gauge symmetry, there are additional discrete gauge symmetries in this model that come from the isometries of $\mathbb{T}^2 \times \mathbb{T}^2 \times \mathbb{T}^2$ and act non-trivially on the flavour indices of the MSSM fields. Indeed, following our discussion in the previous section, we observe that the four translational symmetries of the second and third 2-tori are gauged and spontaneously broken down to $\mathbb{Z}_3$ discrete gauge symmetries. Together with the above flavour-universal discrete symmetry, these symmetries form a non-Abelian discrete gauge symmetry algebra

$$[X^2_{\mathbb{Z}_3}, Y^2_{\mathbb{Z}_3}] = -[X^3_{\mathbb{Z}_3}, Y^3_{\mathbb{Z}_3}] = -\frac{Q_{\mathbb{Z}_3}}{3} + \dots \tag{7.63}$$

where the dots in the r.h.s denote possible additional continuous U(1) generators. The four discrete isometry generators act on the MSSM fields as

$$e^{X^2_{\mathbb{Z}_3}} : X^k_R \to e^{-\frac{2\pi i k}{3}} X^k_R, \tag{7.64a}$$
$$e^{X^3_{\mathbb{Z}_3}} : X^k_L \to e^{\frac{2\pi i k}{3}} X^k_L, \tag{7.64b}$$
$$e^{Y^2_{\mathbb{Z}_3}} : (X^1_R, X^2_R, X^3_R) \to (X^2_R, X^3_R, X^1_R), \tag{7.64c}$$
$$e^{Y^3_{\mathbb{Z}_3}} : (X^1_L, X^2_L, X^3_L) \to (X^3_L, X^1_L, X^2_L), \tag{7.64d}$$

where $k = 1, 2, 3$ denotes the three generations of MSSM fields and X_R and X_L denote collectively the right-handed and the left-handed MSSM fields, respectively. The resulting finite discrete symmetry group can be thought as two copies of $\Delta(27)$ acting respectively on the left or the right-handed MSSM fields and sharing a common flavour-universal center that contains $Q_{\mathbb{Z}_3}$.

The most interesting implications of flavour symmetries are the constraints they impose on the flavour structure of the couplings and, more particularly, on Yukawa couplings. In order to describe the structure of Yukawa couplings imposed by the non-Abelian discrete symmetry in this particular model, let us write them schematically as

$$\sum_{i,j=1}^{3} Y_{ij} X^i_L X^j_R H \tag{7.65}$$

where Y_{ij} are holomorphic functions of the complex structure and the complex Wilson line scalars. In general, under a discrete gauge transformation the MSSM fields transform as in (7.64), so that Y_{ij} will also transform accordingly such that the sum (7.65) remains invariant under discrete symmetry transformations. This, together with the fact that Y_{ij} are holomorphic functions, leads to a set of constraints on the structure of the couplings. For the particular case at hand, we find that

$$\frac{Y_{11}}{Y_{21}} = \frac{Y_{12}}{Y_{22}} = \frac{Y_{13}}{Y_{23}}, \qquad \frac{Y_{21}}{Y_{31}} = \frac{Y_{22}}{Y_{32}} = \frac{Y_{23}}{Y_{33}}, \qquad \frac{Y_{31}}{Y_{11}} = \frac{Y_{32}}{Y_{12}} = \frac{Y_{33}}{Y_{13}}, \tag{7.66a}$$
$$\frac{Y_{11}}{Y_{12}} = \frac{Y_{21}}{Y_{22}} = \frac{Y_{31}}{Y_{32}}, \qquad \frac{Y_{12}}{Y_{13}} = \frac{Y_{22}}{Y_{23}} = \frac{Y_{32}}{Y_{33}}, \qquad \frac{Y_{13}}{Y_{11}} = \frac{Y_{23}}{Y_{21}} = \frac{Y_{33}}{Y_{31}}. \tag{7.66b}$$

[6] Baryon or lepton violating operators with dimension less than five are forbidden in this model because of the continuous $\mathrm{U}(1)_{B-L}$ gauge symmetry.

The details on the derivation of these relations can be found in appendix F. These relations imply that Yukawa couplings in this model have the structure

$$(Y_{ij}) = \begin{pmatrix} a_1b_1 & a_1b_2 & a_1b_3 \\ a_2b_1 & a_2b_2 & a_2b_3 \\ a_3b_1 & a_3b_2 & a_3b_3 \end{pmatrix} \tag{7.67}$$

with a_i and b_i, $i = 1, 2, 3$, holomorphic functions of the moduli. Intuitively, the fields X_L^i and X_R^j in (7.65) are triplets under two different non-Abelian factors (albeit with a common center) associated to two different internal $\mathbb{T}^2$'s; their transformations must cancel against those of Y_{ij}, which must be made up of two objects a_i and b_j, transforming as conjugate triplets under the two factors.

The above result is in agreement with what was found in [74, 75] from a direct computation and in particular implies that the Yukawa matrices of this model have rank one. As we have already mentioned, discrete gauge symmetries are exact symmetries of the theory so this rank one structure will be preserved in the complete non-perturbative formulation of the model. In particular, for this model the rank one texture should survive through the instanton effects mentioned in [76]. Indeed, as we discuss in detail later on, and in analogy with the Abelian case, non-perturbative effects will in general induce couplings that violate the underlying continuous symmetries, but are invariant under the discrete gauge symmetry. This results in a very much constrained flavor structure also for those non-perturbative couplings.

7.3.4 Kähler potential and holomorphic variables

In the previous section we have made use of the holomorphic dependence of superpotential Yukawa couplings on the complex structure and complexified Wilson lines in order to obtain the selection rules that the discrete gauge symmetry imposes on them. As we will see in the next section, holomorphicity of the superpotential is also a key ingredient in deriving analogous rules for non-perturbatively induced superpotential couplings. Note however that, while the complex structure and Wilson lines transform holomorphically under the transformations (7.59), the complex axio-dilaton and Kähler scalars defined in eq.(7.47) in general transform non-holomorphically. Thus, the latter are not the right variables in terms of which the superpotential and gauge kinetic functions are holomorphic quantities.

A simple method to obtain the suitable variables consists on expressing the 4d effective action (7.48) in terms of the second derivatives of a Kähler potential

$$\mathcal{L}_{4\text{d}} = -\sum_{i,j} K_{i\bar{j}} \partial M^i \partial \bar{M}^j. \tag{7.68}$$

Indeed, after some algebra we find that the following Kähler potential

$$\begin{aligned} K = &-\sum_{p=1}^{3} \left[\log(U^p - \bar{U}^p) + \log\left(\hat{T}^p - \bar{\hat{T}}^p - \frac{1}{2} \sum_\alpha c_\alpha^0 \frac{(\xi_\alpha^p - \bar{\xi}_\alpha^p)^2}{U^p - \bar{U}^p} \right) \right] \\ &- \log\left(\hat{S} - \bar{\hat{S}} + \frac{1}{2} \sum_\alpha \sum_{p=1}^{3} c_\alpha^p \frac{(\xi_\alpha^p - \bar{\xi}_\alpha^p)^2}{U^p - \bar{U}^p} \right) \end{aligned} \tag{7.69}$$

correctly reproduces equation (7.48),[7] where the redefined fields $\hat{S}$ and $\hat{T}^p$ are given by[8]

$$\hat{S} = S - \frac{1}{2}\sum_{\alpha}\sum_{p=1}^{3} c_\alpha^p \frac{\xi_\alpha^p \operatorname{Im} \xi_\alpha^p}{\operatorname{Im} U^p}, \tag{7.70a}$$

$$\hat{T}^p = T^p + \frac{1}{2}\sum_{\alpha} c_\alpha^0 \frac{\xi_\alpha^p \operatorname{Im} \xi_\alpha^p}{\operatorname{Im} U^p}. \tag{7.70b}$$

In particular, the discrete identifications (7.52) and (7.54) in terms of these variables now correspond to the holomorphic identifications (see also [242])

$$\hat{S} \simeq \hat{S} + 1, \tag{7.71a}$$

$$\hat{T}^p \simeq \hat{T}^p + 1, \tag{7.71b}$$

$$\xi_\alpha^p \simeq \xi_\alpha^p + \frac{2}{n_\alpha^p}, \tag{7.71c}$$

$$\xi_\alpha^p \simeq \xi_\alpha^p + \frac{2U^p}{n_\alpha^p} \qquad \hat{S} \simeq \hat{S} - \frac{c_\alpha^p}{n_\alpha^p}\left(2\xi_\alpha^p + \frac{U^p}{n_\alpha^p}\right) \qquad \hat{T}^p \simeq \hat{T}^p + \frac{c_\alpha^0}{n_\alpha^p}\left(2\xi_\alpha^p + \frac{U^p}{n_\alpha^p}\right). \tag{7.71d}$$

these holomorphic variables will be useful when we discuss instanton effects in next section.

7.4 Instantons and non-Abelian symmetries

As we have seen in the previous section, the presence of non-Abelian discrete gauge symmetries in D-brane and other string theory models directly constrain the structure of Yukawa couplings at the perturbative level. A natural question is then if such discrete symmetries also affect those couplings that are generated at the non-perturbative level, in particular by instanton effects in 4d chiral compactifications. The purpose of this section is to show that this is indeed the case, and that most of the intuition that holds for instanton effects in compactifications with discrete Abelian symmetries generalizes to the non-Abelian case.

Let us recall the structure of instanton induced couplings in 4d chiral D-brane models, which is typically of the form

$$\Phi_1 \Phi_2 \dots \Phi_N \, \mathcal{A} \, e^{-S_{\text{inst.}}} \tag{7.72}$$

where

$$S_{\text{inst.}} = 2\pi(g_s^{-1} V + i\phi) \tag{7.73}$$

[7] In fact, the above Kähler potential leads to an extra term in the kinetic term of the complex Wilson line scalars that is not present in equation (7.48)

$$K_{\xi_\alpha^p \bar{\xi}_\alpha^p} = -\frac{1}{U^p - \bar{U}^p}\sum_{\alpha}\left(\frac{c_\alpha^0}{T^p - \bar{T}^p} - \frac{c_\alpha^p}{S - \bar{S}}\right).$$

This terms perfectly agrees with the CFT result obtained in [171]. From this point of view, this extra term comes from the $\mathrm{tr}(|F_2|^4)$ term that we have neglected in equation (7.43).

[8] Similarly, matter fields are also redefined by the Wilson line scalars. This redefinition can be seen for instance from the perturbative Yukawa couplings [242]. The latter carry an exponential prefactor which depends non-holomorphically on the Wilson line scalars, and that it is absorbed into a redefinition of the bifundamental fields.

is the complexification of the D-instanton volume and Φ_i are 4d chiral open string modes. Finally, the prefactor $\mathcal{A}$ is a non-trivial function of the open and closed string moduli of the compactification, excluding those closed string moduli that enter into D-instanton actions $S_{\text{inst.}}$.

The open string operator $\Phi_1\Phi_2\ldots\Phi_N$ is non-trivially charged under a $U(1)$ gauge symmetry arising from a bulk D-brane, symmetry that becomes massive by eating the axionic closed string modulus ϕ. Both this term and $\exp(-S_{\text{inst.}})$ are not invariant under such $U(1)$ gauge transformations and the corresponding shift in ϕ, but their product is, so that (7.72) is an allowed operator. In case that the massive $U(1)$ symmetry is not totally broken but a $\mathbb{Z}_k$ subgroup remains, then $\exp(-S_{\text{inst.}})$ is invariant under the action of such $\mathbb{Z}_k$ subgroup, and so must be $\Phi_1\ldots\Phi_N$, so that not all operators can be generated in the effective theory [140].

Let us now turn to the non-Abelian case. As we have seen in section 7.3, when considering discrete symmetries in D-brane models we may not only focus on axions ϕ arising from the closed string sector, but also on open string axions ξ_α. Hence, in order to check the transformation properties of each of the factors in (7.72) under non-Abelian transformations we need to consider the prefactor $\mathcal{A}$ and its dependence on those open string axions that enter into the definition of the non-Abelian symmetry.

The prefactor $\mathcal{A}$ is oftentimes difficult to obtain, but it can be explicitly computed in examples like toroidal compactifications with magnetized and/or intersecting D-branes.[9] For instance, let us consider two magnetized D9-branes on an orientifold of $\mathbb{T}^6 = (\mathbb{T}^2)_1 \times (\mathbb{T}^2)_2 \times (\mathbb{T}^2)_3$ with magnetic numbers (n_a^r, m_a^r) and (n_b^r, m_b^r) as in (7.46), and an Euclidean D1-brane wrapping $(\mathbb{T}^2)_p$'s. If this E1-brane has the appropriate zero mode structure and assuming that $d_a^r = -d_b^r = N$, a superpotential coupling like (7.72) will be generated for the open string fields Φ_i^{ab} that transforms in the bifundamental of $U(1)_a \times U(1)_b$. More precisely we will have something of the form

$$e^{-S_{\text{inst.}}}\sum_\alpha \Phi_{\alpha_1}\ldots\Phi_{\alpha_N}\ \mathcal{A}'_{\alpha_1}\ldots\mathcal{A}'_{\alpha_N}, \tag{7.74}$$

where each of the factors $\mathcal{A}_{\alpha_i}$ arises from a three-point function of open string chiral fields, namely two fermionic zero modes of the E1-brane and a 4d chiral multiplet $\Phi_{\alpha_i}^{ab}$. Since $\mathbb{T}^6$ is factorized, such three-point functions are given by the product of three functions of the form

$$\mathcal{A}'_{\delta_{ijk}} = e^{i\pi M\xi\,\mathrm{Im}\,\xi/\mathrm{Im}\,U}\cdot\vartheta\begin{bmatrix}\delta_{ijk}\\ 0\end{bmatrix}(M\xi, MU) \tag{7.75}$$

one for each factor $(\mathbb{T}^2)_r$ $r = 1,2,3$, where for simplicity we have omitted the label r of the $\mathbb{T}^2$ in all these quantities. Here $U = U^r$ is the complex structure modulus of such $\mathbb{T}^2$ and ξ is a linear combination of complex open string moduli in $(\mathbb{T}^2)_r$. Namely,

$$M\xi = (I_{bc}\xi_a + I_{ca}\xi_b)/d \tag{7.76a}$$
$$M = I_{ab}I_{bc}I_{ca}/d^2 \tag{7.76b}$$

with $\xi_\alpha = \xi_\alpha^r$ defined as in (7.47) and $I_{ab} = I_{ab}^r \equiv n_a^r m_b^r - n_b^r m_a^r$ the number of zero modes that arise in the sector ab from $(\mathbb{T}^2)_r$. Similarly, one can define $I_{ca} = I_{ca}^r$ and $I_{bc} = I_{bc}^r$ as

[9] This also applies to elliptically fibered Calabi-Yau compactifications where the interaction between open string chiral fields is localized at the elliptic fiber [169].

the zero modes of the E1-brane charged under $\mathrm{U}(1)_a$ and $\mathrm{U}(1)_b$, respectively, arising from $(\mathbb{T}^2)_r$. Finally, $d = \text{g.c.d.}(I_{ab}, I_{bc}, I_{ca})$ and we have that

$$\delta_{ijk} = \frac{i}{I_{ab}} + \frac{j}{I_{ca}} + \frac{k}{I_{bc}}, \tag{7.77}$$

where i, j, k label the chiral zero modes at each D-brane sector. In particular, the index i labels 4d chiral fields Φ_α in (7.74) and j, k the charged zero modes of the E1-instanton that couple to them.

It is easy to see that (7.75) is not a holomorphic function of the open string moduli ξ_α of the compactification. However, one may absorb the non-holomorphic prefactor $\exp(i\pi M\xi\,\mathrm{Im}\,\xi/\mathrm{Im}\,U)$ into the definition of the instanton classical action $S_{\text{inst.}}$ and the chiral fields Φ_α. Indeed, as first pointed out in [242], the whole expression (7.74) can be rewritten as

$$e^{-\hat{S}_{\text{inst.}}} \sum_\alpha \hat{\Phi}_{\alpha_1} \ldots \hat{\Phi}_{\alpha_N}\, \mathcal{A}_{\alpha_1} \ldots \mathcal{A}_{\alpha_N}, \tag{7.78}$$

where $\hat{S}_{\text{inst.}}$ is a linear function of the holomorphic variables $\hat{S}$, $\hat{T}^r$ defined in (7.70), $\hat{\Phi}_\alpha$ are the redefined 4d chiral fields of [242] and the prefactors $\mathcal{A}_\alpha$ are now holomorphic functions of the moduli. In the example at hand we have that $\hat{S}_{\text{inst.}} = \hat{T}^p$, and that (7.75) gets replaced by

$$\mathcal{A}_{\delta_{ijk}} = \vartheta \begin{bmatrix} \delta_{ijk} \\ 0 \end{bmatrix} (M\xi, MU). \tag{7.79}$$

One can now check how the non-perturbative coupling transforms under the discrete gauge symmetry, and in particular under discrete Wilson line shifts. On the one hand we have

$$\xi \longrightarrow \xi + \frac{1}{M}, \tag{7.80a}$$

$$\mathcal{A}_{\delta_{ijk}} \longrightarrow \mathcal{A}_{\delta_{ijk}}\; e^{2\pi i \delta_{ijk}}. \tag{7.80b}$$

If $I_{ab} = I_{bc} = I_{ca} = d$ then (7.80) corresponds to the third identification in (7.71), under which the other holomorphic variables do not transform. In particular $\hat{S}_{\text{inst.}}$ remains invariant and since the product of $\mathcal{A}$'s saturates all possible values j, k for the charged instanton zero modes we obtain the transformation

$$e^{-\hat{S}_{\text{inst.}}}\, \mathcal{A}_{\alpha_1} \ldots \mathcal{A}_{\alpha_N} \longrightarrow e^{-\hat{S}_{\text{inst.}}}\, \mathcal{A}_{\alpha_1} \ldots \mathcal{A}_{\alpha_N}\, e^{2\pi i \sum_i \frac{\alpha_i}{I_{ab}}}, \tag{7.81}$$

which means this term is invariant only if the flavour indices α_i add up to a multiple of I_{ab}, or in other words if

$$\sum_i \alpha_i = 0 \mod I_{ab}, \tag{7.82}$$

in analogy with the selection rules for perturbative Yukawa couplings.

On the other hand we have the shift

$$\xi \longrightarrow \xi + \frac{U}{M}, \tag{7.83a}$$

$$\mathcal{A}_{\delta_{ijk}} \longrightarrow \mathcal{A}_{\delta_{ijk}+1/M}\; e^{-\pi i U/M} e^{-2\pi i \xi}, \tag{7.83b}$$

that can be partially compensated by a simultaneous shift of the form

$$\hat{S}_{\text{inst.}} \longrightarrow \hat{S}_{\text{inst.}} + 2\xi + U \tag{7.84}$$

as follows from the last identification in (7.71). Hence the product in the right hand side of (7.81) remains invariant under (7.83) except for a shift in the zero mode indices $(i,j,k) \to (i+i_0, j+j_0, k+k_0)$ such that

$$i_0 I_{bc} I_{ca} + j_0 I_{ab} I_{bc} + k_0 I_{ca} I_{ab} = d^2 \tag{7.85}$$

which is always possible. Notice that $\mathcal{A}_{\delta_{ijk}}$ only depends on the value of the l.h.s. of (7.85), so given a prefactor $\mathcal{A}_{\alpha_i}$ in (7.78) there is a unique image $\mathcal{A}_{\alpha'_i}$ under the shift (7.85). We then we have that the second transformation acts as

$$e^{-\hat{S}_{\text{inst.}}} \mathcal{A}_{\alpha_1} \dots \mathcal{A}_{\alpha_N} \longrightarrow e^{-\hat{S}_{\text{inst.}}} \mathcal{A}_{\alpha'_1} \dots \mathcal{A}_{\alpha'_N}, \tag{7.86}$$

and as a permutation of the chiral fields Φ_{α_i} and instanton zero modes. Hence, if the operator (7.78) is not invariant under this shift, the whole instanton amplitude should be a sum of operators of this form invariant under (7.86).

7.4.1 An example

Let us consider an example used in section 5 of [74], namely the case where there is only one $\mathbb{T}^2$ and $I_{ab} = I_{bc} = I_{ca} = 3$. There we have that

$$\mathcal{A}_{\delta_{111}} = \vartheta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (3\xi, 3U) = \mathcal{A}_{\delta_{222}} = \mathcal{A}_{\delta_{333}} \equiv A, \tag{7.87a}$$

$$\mathcal{A}_{\delta_{132}} = \vartheta \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} (3\xi, 3U) = \mathcal{A}_{\delta_{213}} = \mathcal{A}_{\delta_{321}} \equiv B, \tag{7.87b}$$

$$\mathcal{A}_{\delta_{123}} = \vartheta \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} (3\xi, 3U) = \mathcal{A}_{\delta_{231}} = \mathcal{A}_{\delta_{312}} \equiv C, \tag{7.87c}$$

all the other couplings vanishing. This induces a coupling of the form

$$e^{-\hat{S}_{\text{inst.}}} \left[ABC \left(\hat{\Phi}_1^3 + \hat{\Phi}_2^3 + \hat{\Phi}_3^3 \right) + (A^3 + B^3 + C^3) \hat{\Phi}_1 \hat{\Phi}_2 \hat{\Phi}_3 \right], \tag{7.88}$$

which is indeed invariant under the discrete shifts (7.80) and (7.83), acting as

$$\xi \to \xi + \frac{1}{3}, A \to A, \quad B \to e^{2\pi i/3} B, \quad C \to e^{-2\pi i/3} C, \tag{7.89a}$$

$$\xi \to \xi + \frac{U}{3}, A \to B \to C \to A, \tag{7.89b}$$

Notice however that none of the terms of (7.88) is invariant individually. Interestingly, for $\xi = 0$ we have that $A = 0$ and $B = -C$, so (7.88) vanishes identically at that point.

8

Discrete gauge symmetries from higher forms

The structure in chapter 3 is the only one available in four dimensions. However, in higher dimensions there are gauge symmetries carried by higher-rank antisymmetric tensors, and it is reasonable to exploit them to generate discrete $\mathbb{Z}_p$ gauge symmetries. Conversely, higher dimensions allow the existence of $\mathbb{Z}_p$ charged objects with higher worldvolume dimensionality.

The most straightforward possibility is to consider a 1-form gauge field and a $(D-2)$-form gauge field in D dimensions, coupling through a $B_{D-2} \wedge F_2$; this is a trivial addition of dimensions, in which the 4d $\mathbb{Z}_p$ string is extended to real codimension-2 $(D-3)$-brane.

In this chapter we will generalize the analysis done in section 3.2 in order to explore $\mathbb{Z}_p$ discrete gauge symmetries whose underlying continuous symmetry involves genuine higher-rank antisymmetric tensors, in any dual picture.

Subsequently, we present explicit constructions in string theory where we can apply this analysis. In section 8.3 we study higher-rank Z_p symmetries in string theory flux compactifications, and for concreteness we present a similar analysis to that of sections 6.1 and 6.2 but in 6d instead of 4d. In section 8.4 we study the possibility of realising higher-rank Z_p symmetries in string theory compactifications with torsion. Finally, in section 8.5 we consider the non-Abelian case.

8.1 Field theory of higher-rank Abelian $\mathbb{Z}_p$ gauge symmetries

Consider a theory in D dimensions, with a r-form field A_r and a $(r-1)$-form field ϕ_{r-1}, with gauge invariance[1]

$$A_r \longrightarrow d\lambda_{r-1}, \tag{8.1a}$$

$$\phi_{r-1} \longrightarrow \phi_{r-1} + k\lambda_{r-1}. \tag{8.1b}$$

The notation is chosen to recover the familiar one for $r = 1$ (3.3). The gauge invariant action, which generalizes (3.1) is

$$\int_{\mathbb{M}_D} \frac{1}{2} \left(d\phi_{r-1} + kA_r\right) \wedge *_D \left(d\phi_{r-1} + kA_r\right) = \int_{\mathbb{M}_D} \frac{1}{2} \left|d\phi_{r-1} + kA_r\right|^2 . \tag{8.2}$$

[1] These theories have been considered e.g. in [248] (see also [249, 250]). As in there, we consider the gauge symmetries to be compact, namely there is charge stabilization for the extended objects to which they couple. As in the 4d case, stabilization is such that the minimal charge is unity.

Note that in this theory both fields are gauge fields since, in addition to (8.1), (8.2) is invariant under

$$\phi_{r-1} \longrightarrow \phi_{r-1} + d\sigma_{r-2}. \tag{8.3}$$

In the above Lagrangian, the field A_r eats up the field ϕ_{r-1} and becomes massive. This is consistent with the counting of degrees of freedom of antisymmetric tensor gauge fields under the $SO(D-2)$ and $SO(D-1)$ little groups for massless and massive particles:

$$\binom{D-2}{r} + \binom{D-2}{r-1} = \binom{D-1}{r}. \tag{8.4}$$

The gauge symmetry of A_r is broken spontaneously, but a discrete $\mathbb{Z}_k$ symmetry remains. This is a higher-rank analogue of the Higgsing of a $U(1)$ gauge group by eating up the phase of a charge-k scalar.

The action (8.2) can be written in terms of the D dimensional dual fields to ϕ_{r-1} and A_r, given by a $(D-r-1)$-form gauge potential B_{D-r-1} and a $(D-r-2)$-form gauge potential V_{D-r-2}, respectively, as

$$\begin{aligned}\int_{\mathbb{M}_D} \frac{1}{2}\left(dV_{D-r-2} + kB_{D-r-1}\right) \wedge *_D \left(dV_{D-r-2} + kB_{D-r-1}\right) \\ = \int_{\mathbb{M}_D} \frac{1}{2}\left|dV_{D-r-2} + kB_{D-r-1}\right|^2\end{aligned} \tag{8.5}$$

where

$$dB_{D-r-1} = *_D d\phi_{r-1}, \tag{8.6a}$$
$$dV_{D-r-2} = *_D dA_r. \tag{8.6b}$$

This action is invariant under a gauge transformation of the form

$$B_{D-r-1} \longrightarrow B_{D-r-1} + d\Lambda_{D-r-2}, \tag{8.7a}$$
$$V_{D-r-2} \longrightarrow V_{D-r-2} - k\Lambda_{D-r-2}, \tag{8.7b}$$

and it is also invariant under

$$V_{D-r-2} \longrightarrow V_{D-r-2} + d\Xi_{D-r-3}. \tag{8.8}$$

This dual description makes manifest an *emergent* $\mathbb{Z}_k$ discrete gauge symmetry, and as in chapter 3, it may differ from the original one.

In the case of the above $\mathbb{Z}_k$ theory, the analogues of charge n particles with worldline C in 4d are $(r-1)$-branes with worldvolume L_r and charge n under A_r which are described as insertions of the operators

$$\mathcal{O}_{(r-1)\text{-brane(s)}} \sim e^{2\pi i n \int_{L_r} A_r}. \tag{8.9}$$

Their charge is conserved modulo k, since there are gauge invariant generalized junctions which create/annihilate sets of $(r-1)$-branes with total charge k,

$$e^{-2\pi i \int_{P_{r-1}} \phi_{r-1}} e^{2\pi i n \int_{L_r} A_r} = e^{-2\pi i \int_{P_{r-1}} \phi_{r-1}} \mathcal{O}_{(r-1)\text{-brane(s)}}, \tag{8.10}$$

where $P_{r-1} = \partial L_r$.

In addition, the theory contains the analogue of 4d $\mathbb{Z}_k$-charged strings, namely $\mathbb{Z}_k$-charged $(D-r-2)$-branes described as the insertion of operators along a worldvolume Σ_{D-r-1}

$$\mathcal{O}_{(D-r-2)\text{brane(s)}} \sim e^{-2\pi i \int_{\Sigma_{D-r-1}} B_{D-r-1}}, \tag{8.11}$$

where B_{D-r-1} is the $(D-r-1)$-form dual to ϕ_{r-1} and p is defined modulo k. $(D-r-2)$-brane charge is also conserved modulo k, since there are operators describing $(D-r-2)$-branes of total charge k on worldvolumes Σ_{D-r-1} ending along a generalized junction C_{D-r-2} $(\partial\Sigma_{D-r-1} = C_{D-r-2})$

$$e^{-2\pi i \int_{C_{D-r-2}} V_{D-r-2}} e^{2\pi i k \int_{\Sigma_{D-r-1}} B_{D-r-1}}. \tag{8.12}$$

By standard arguments, the quantum amplitude of a process involving a (minimally charged) $(r-1)$-brane with worldvolume Σ_r, and a (minimally charged) $(D-r-2)$-brane with worldvolume Δ_{D-r-1} receives a phase

$$\exp\left[2\pi i \frac{np}{k} L(\Sigma_r, \Delta_{D-r-1})\right], \tag{8.13}$$

where $L(\Sigma_r, \Delta_{D-r-1})$ is the so-called linking number of Σ_r and Δ_{D-r-1} in D dimensions (the number of times Σ_r surrounds Δ_{D-r-1} or vice versa).

As opposed to the 4d case, where the most natural thing is to think of particles as the fundamental objects, we have no reason to consider only the $(r-1)$-branes as fundamental objects and the $(D-r-2)$-branes as the associated topological defects, and not the other way around. Therefore, we will take a democratic approach and treat both cases equally.

8.2 Higher-rank Abelian $\mathbb{Z}_p$ gauge symmetries and BF couplings

8.2.1 The BF coupling

Recall that the D-dimensional action for a $\mathbb{Z}_k$ gauge discrete symmetry coming from a r-form gauge potential is

$$\mathcal{S} = \int_{\mathbb{M}_D} \frac{1}{2} \left(d\phi_{r-1} + kA_r\right) \wedge *_D \left(d\phi_{r-1} + kA_r\right). \tag{8.14}$$

It contains the terms

$$\mathcal{S} \supset \frac{k}{2} \int_{\mathbb{M}_D} \left(d\phi_{r-1} \wedge *_D A_r + A_r \wedge *_D d\phi_{r-1}\right) = k \int_{\mathbb{M}_D} A_r \wedge *_D d\phi_{r-1}, \tag{8.15}$$

where the equality follows from the fact that $\alpha \wedge *\beta = \beta \wedge *\alpha$. If we define $dB_{D-r-1} = *_D d\phi_{r-1}$, then (8.15) can be rewritten as

$$\mathcal{S} \supset k \int_{\mathbb{M}_D} A_r \wedge dB_{D-r-1}. \tag{8.16}$$

Integrating (8.16) by parts, we get

$$\mathcal{S} \supset k \int_{\mathbb{M}_D} dA_r \wedge B_{D-r-1} = k \int_{\mathbb{M}_D} F_{r+1} \wedge B_{D-r-1}, \tag{8.17}$$

where $F_{r+1} = dA_r$.

The same analysis can be done starting with the dual theory. Recall that the action is given by

$$\mathcal{S}' = \int_{\mathbb{M}_D} \frac{1}{2} \left(dV_{D-r-2} + kB_{D-r-1}\right) \wedge *_D \left(dV_{D-r-2} + kB_{D-r-1}\right). \tag{8.18}$$

It contains the terms

$$\begin{aligned} \mathcal{S}' \supset \frac{k}{2} \int_{\mathbb{M}_D} & \left(dV_{D-r-2} \wedge *_D B_{D-r-1} + B_{D-r-1} \wedge *_D dV_{D-r-2}\right) \\ &= k \int_{\mathbb{M}_D} B_{D-r-1} \wedge *_D dV_{D-r-2}, \end{aligned} \tag{8.19}$$

where the equality follows from the fact that $\alpha \wedge *\beta = \beta \wedge *\alpha$. If we define $dA_r = *_D dV_{D-r-2}$, then (8.15) can be rewritten as

$$\mathcal{S}' \supset k \int_{\mathbb{M}_D} B_{D-r-1} \wedge dA_r. \tag{8.20}$$

Integrating (8.16) by parts, we get

$$\mathcal{S}' \supset k \int_{\mathbb{M}_D} dB_{D-r-1} \wedge A_r = k \int_{\mathbb{M}_D} H_{D-r} \wedge A_r, \tag{8.21}$$

where $H_{D-r} = dB_{D-r-1}$.

Looking at the couplings (8.17) and (8.21), we see that both of them involve the forms A_r and B_{D-r-1} under which the fundamental objects and the associated topological defects are charged; they differ in which form the exterior differential operator d is acting on. Therefore, and in order to be consistent with the 4d analysis, we will consider that the fundamental objects correspond to those who are charged under the form which is being acted on by the exterior differential operator d.

8.2.2 Generalization for multiple antisymmetric tensors

The previous argument can easily be generalized to the case where we have multiple fields of each type.

Consider a single $(r-1)$-form field ϕ_{r-1} and several r-form gauge fields A_r^k in D dimensions. The action is given by

$$\int_{\mathbb{M}_D} \sum_{k=1}^{n} \left| d\phi_{r-1} + q_k A_r^k \right|^2. \tag{8.22}$$

This is gauge invariant under

$$A_r^k \longrightarrow A_r^k - d\lambda_{r-1}^k, \tag{8.23a}$$

$$\phi_{r-1} \longrightarrow \phi_{r-1} + \sum_k q_k \lambda_{r-1}^k. \tag{8.23b}$$

and

$$\phi_{r-1} \longrightarrow \phi_{r-1} + d\xi_{r-2}. \tag{8.24}$$

The corresponding BF couplings are

$$\int_{\mathbb{M}_D} \sum_{k=1}^n q_k B_{D-r-1} \wedge F_{r+1}^k \tag{8.25}$$

where $dB_{D-r-1} = *_D d\phi_{r-1}$ and $F_{r+1}^k = dA_r^k$.

The remnant discrete gauge symmetry is not manifest by inspection. Naively, it may seem that each continuous gauge symmetry is broken into a $\mathbb{Z}_{q_k}$ subgroup. This is however not correct, since the different continuous gauge symmetries couple simultaneously to a *single* $r-1$-form field. Indeed, there is only one broken linear combination of them, while the orthogonal $n-1$ linear combinations remain unbroken.

Let $\vec{q} = (q_1, \ldots, q_n)$ be the charge vector of ϕ_{r-1} under the continuous gauge symmetries and let Q_k be the generator of the continuous gauge symmetry corresponding to the r-form gauge potential A_r^k, for $k = 1, \ldots, n$. The unbroken part of the continuous gauge symmetry is generated by Q_a, $a = 1, \ldots, n$, which are given by linear combinations

$$A_1 = \sum_{k=1}^n c_a^k Q_k, \qquad c_a^k \in \mathbb{Z}, \tag{8.26}$$

with $\vec{c}_a \cdot \vec{p} = 0$. The only broken linear combination is the one orthogonal to all the previous ones, namely it is given by

$$Q = \sum_{k=1}^n \frac{q_k}{q} Q_k, \tag{8.27}$$

where the factor $q = \gcd(q_k)$ is included in order to keep the stabilization such that minimal charge is 1. Its BF couplings are

$$\sum_{k=1}^n \frac{(q_k)^2}{q} B_{D-r-1} \wedge F_{r+1}. \tag{8.28}$$

The symmetry is therefore $\mathbb{Z}_r$ with $r = \sum_k \frac{(q_k)^2}{q}$. The $\mathbb{Z}_r$ structure follows from the structure of charged $(r-1)$-brane states, which are created by operators

$$\exp\left(-2\pi i \int_{P_{r-1}} \phi_{r-1}\right) \exp\left(2\pi i \int_{L_r} \sum_k q_k A_r^k\right). \tag{8.29}$$

This violates Q_k charge conservation in q_k units, and hence $Q = \sum_k \frac{q_k}{q} Q_k$ in $r = \sum_k \frac{(q_k)^2}{q}$ units.

Let us now consider the theory dual to (8.22). The action is given by

$$\int_{\mathbb{M}_D} \sum_{k=1}^{m} \left| dV^k_{D-r-2} + q_k B_{D-r-1} \right|^2 , \tag{8.30}$$

where $dB_{D-r-1} = *_D d\phi_{r-1}$ and $dV^k_{D-r-2} = *_D dA^k_r$, and it is gauge invariant under

$$B_{D-r-1} \longrightarrow B_{D-r-1} - d\Lambda_{D-r-2}, \tag{8.31a}$$
$$V^k_{D-r-2} \longrightarrow V^k_{D-r-2} + q_k \Lambda_{D-r-2}, \tag{8.31b}$$

in addition to the gauge transformations

$$V^k_{D-r-2} \longrightarrow V^k_{D-r-2} + d\Xi^k_{D-r-3}. \tag{8.32}$$

The action (8.30) corresponds to a single $(D-r-1)$-form field made massive by coupling to several $(D-r-2)$-form fields.

The corresponding BF couplings are

$$\int_{\mathbb{M}_D} \sum_{k=1}^{m} q_k A^k_r \wedge H_{D-r}, \tag{8.33}$$

where $dA^k_r = *_D dV^k_{D-r-2}$ and $H_{D-r} = dB_{D-r-1}$.

The potential B_{D-r-1} actually eats only one linear combination of the fields V^k_{D-r-2}, while the orthogonal linear combinations remain as massless $(D-r-2)$-form fields. Denoting $q = \gcd(q_j)$, the massive gauge symmetry leaves a remnant $\mathbb{Z}_q$ gauge symmetry. This follows from the structure of $\mathbb{Z}_q$-charged $(D-r-2)$-branes, whose number can be violated by operators

$$\exp\left(-2\pi i \int_{\mathbb{C}_{D-r-2}} V^k_{D-r-2}\right) \exp\left(2\pi i \int_{\Sigma_{D-r-1}} k B_{D-r-1}\right). \tag{8.34}$$

Each such vertex creates q_k $(D-r-2)$-branes, so by Bezout's lemma, there exists a set of vertices which (minimally) violates their number in q units, making the $(D-r-2)$-branes $\mathbb{Z}_q$-valued. In addition, the theory enjoys the continuous gauge invariance associated to the orthogonal combinations of the V^k_{D-r-2}'s.

From this analysis it follows that the emergent gauge symmetry associated to the original $\mathbb{Z}_q$ is a $\mathbb{Z}_r$, and vice versa. Hence, the discrete part of the emergent gauge group in the dual description is different from the original one.

The fact that the original $\mathbb{Z}_q$ (resp. $\mathbb{Z}_r$) and the emergent $\mathbb{Z}_r$ (resp. $\mathbb{Z}_q$) gauge symmetries are different is not in contradiction with charge stabilization of the dual charged objects, because of the presence of additional charges under the additional continuous gauge symmetries in the system.

To finish the analysis, let us consider the case with several fields of each kind. The D-dimensional action is given by

$$\int_{\mathbb{M}_D} \sum_{l=1}^{m} \left| d\phi^l_{r-1} + \sum_{k=1}^{n} p_{lk} A^k_r \right|^2 . \tag{8.35}$$

This is gauge invariant under

$$A_r^k \longrightarrow A_r^k - d\lambda_{r-1}^k, \tag{8.36a}$$

$$\phi_{r-1}^l \longrightarrow \phi_{r-1}^l + \sum_{k=1}^{n} p_{lk}\lambda_{r-1}^k, \tag{8.36b}$$

and

$$\phi_{r-1}^l \longrightarrow \phi_{r-1}^l + d\xi_{r-2}^l. \tag{8.37}$$

The corresponding BF couplings are

$$\int_{\mathbb{M}_D} \sum_{l=1}^{m}\sum_{k=1}^{n} p_{kl} B_{D-r-1}^l \wedge F_{r+1}^k, \tag{8.38}$$

where $dB_{D-r-1}^l = *_D d\phi_{r-1}^l$ and $F_{r+1}^k = dA_r^k$.

Let Q_k be the generator of the symmetry corresponding to the r-form gauge potential A_r^k, $k = 1, \dots, n$, and consider a linear combination $Q = \sum_k c^k Q_k$ such that $\gcd(c_k) = 1$. Then the BF couplings for the field strength F_{r+1} of the gauge symmetry generated by Q are

$$\int_{\mathbb{M}_D} \sum_{l=1}^{m}\left(\sum_{k=1}^{n} p_{lk}c^k\right) B_{D-r-1}^l \wedge F_{r+1} = \int_{4d} \sum_{l=1}^{m} q_l B_{D-r-1}^l \wedge F_{r+1}, \tag{8.39}$$

where $q_l = \sum_k p_{lk}c^k$. Hence, the symmetry generated by Q is broken to a Z_q subgroup where $q = \gcd(q_l)$.

8.3 Flux catalysis in 6d

8.3.1 Generalities

In this section we will study how higher-rank $\mathbb{Z}_p$ discrete gauge symmetries can be realized in compactifications of string theory in the presence of background fluxes. For concreteness we will focus in compactifications from 10d to 6d, and will carry on an analysis analogue to the one in section 6.1, while applying the results of section 8.2.

As in section 6.1, the 10d Chern-Simons couplings we will consider are

$$\int_{10d} B_2 \wedge F_p \wedge F_{8-p}, \tag{8.40a}$$

$$\int_{10d} H_3 \wedge F_p \wedge C_{7-p}. \tag{8.40b}$$

Here B_2 and H_3 denote the NSNS 2-form potential and its field strength, whereas C_n and F_{n+1} denote the RR n-form potential and its field strength, with n even or odd for type IIA or IIB theories, respectively.

In what follows, $\overline{F}_n$ will denote the n-form that has a non-trivial flux, and $\hat{F}_n$ will denote an n-form that is obtained from the reduction of some higher form.

Since we are not particularly interested in supersymmetry, we take the compactification space $\mathbb{X}_4$ to be compact manifold of real dimension 4.

8.3.2 Type IIA compactifications

8.3.2.1 Massive type IIA

Consider massive type IIA theory with mass parameter $\overline{F}_0 = p$ compactified on a real dimension 4 space $\mathbb{X}_4$.There are two 10d Chern-Simons couplings (8.40) that lead to the 6d terms we are interested in.

$$\int_{10d} B_2 \wedge \overline{F}_0 \wedge F_8 \tag{8.41a}$$

$$\int_{10d} H_3 \wedge \overline{F}_0 \wedge C_7 \tag{8.41b}$$

The Chern-Simons coupling (8.41a) produces a 6d BF coupling

$$\int_{10d} B_2 \wedge \overline{F}_0 \wedge F_8 \quad \longrightarrow \quad p\int_{6d} B_2 \wedge \hat{F}_4 \tag{8.42}$$

where

$$\hat{F}_4 = \int_{\mathbb{X}_4} F_8. \tag{8.43}$$

The theory automatically has a $\mathbb{Z}_p$ discrete gauge symmetry. The $\mathbb{Z}_p$-charged fundamental objects are 2-branes given by D6-branes wrapped on $\mathbb{X}_4$, which are annihilated in sets of p by a 2-brane junction given by a NS5-brane wrapped on $\mathbb{X}_4$. The associated topological defects are $\mathbb{Z}_p$-charged 1-branes given by fundamental strings, which are annihilated in sets of p by a 1-brane junction given by a D0-brane.

Let us study now the dual theory. The Chern-Simons coupling (8.41b) gives rise to a 6d BF coupling

$$\int_{10d} H_3 \wedge \overline{F}_0 \wedge C_7 \quad \longrightarrow \quad p\int_{6d} H_3 \wedge \hat{C}_3 \tag{8.44}$$

where

$$\hat{C}_3 = \int_{\mathbb{X}_4} C_7. \tag{8.45}$$

The theory automatically has a $\mathbb{Z}_p$ discrete gauge symmetry. The $\mathbb{Z}_p$-charged fundamental objects are 1-branes given by fundamental strings, which are annihilated in sets of p by a 1-brane junction given by a D0-brane. The associated topological defects are $\mathbb{Z}_p$-charged 2-branes given by D6-branes wrapped on $\mathbb{X}_4$, which are annihilated in sets of p by a 2-brane junction given by a NS5-brane wrapped on $\mathbb{X}_4$.

In this case, both the original theory and its dual give rise to the same discrete gauge symmetry.

8.3.2.2 Type IIA with 2-form flux

Consider type IIA compactifications on a real dimension 4 space $\mathbb{X}_4$ with 2-form flux $\overline{F}_2$. Let us introduce an adapted symplectic basis of 2-cycles $\{\alpha_k\}$, $\{\beta_k\}$, with $\alpha_k \cdot \beta_l = \delta_{kl}$, such that there is flux only in the α cycles, and define

$$\int_{\alpha_k} \overline{F}_2 = p_k \tag{8.46}$$

There are two 10d Chern-Simons couplings (8.40) that lead to the 6d terms we are interested in.

$$\int_{10d} B_2 \wedge \overline{F}_2 \wedge F_6 \tag{8.47a}$$

$$\int_{10d} H_3 \wedge \overline{F}_2 \wedge C_5 \tag{8.47b}$$

The 10d Chern-Simons coupling (8.47a) produces the 6d couplings

$$\int_{10d} B_2 \wedge \overline{F}_2 \wedge F_6 \quad \longrightarrow \quad \sum_k \int_{6d} p_k B_2 \wedge \hat{F}_4^k \tag{8.48}$$

where

$$\hat{F}_4^k = \int_{\beta_k} F_6. \tag{8.49}$$

There is a $\mathbb{Z}_q$ discrete gauge symmetry, where $q = \sum_k \frac{p_k^2}{p}$, with $p = \gcd(p_k)$. The $\mathbb{Z}_q$-charged fundamental objects are 2-branes, the one with minimal charge being a D4-brane wrapped on a 2-cycle $\sum_k n_k \beta_k$, where the n_k are integers satisfying $\sum n_k p_k = p$, which always exist by Bezout's lemma; they are annihilated in sets of q by a 2-brane junction given by a NS5-brane wrapped on $\mathbb{X}_4$. The associated topological defects are $\mathbb{Z}_q$-charged 1-branes given by fundamental strings, which are annihilated in sets of q by a 1-brane junction given by a D2-brane on a 2-cycle $\sum_k \frac{p_k}{p} \alpha_k$.

Let us study now the dual theory. The 10d Chern-Simons coupling (8.47b) gives rise to the 6d couplings

$$\int_{10d} H_3 \wedge \overline{F}_2 \wedge C_5 \quad \longrightarrow \quad \sum_k \int_{6d} p_k H_3 \wedge \hat{C}_3^k. \tag{8.50}$$

where

$$\hat{C}_3^k = \int_{\beta_k} C_5. \tag{8.51}$$

There is a $\mathbb{Z}_p$ discrete gauge symmetry, where $p = \gcd(p_k)$. The $\mathbb{Z}_p$-charged fundamental objects are 1-branes given by fundamental strings, which are annihilated in sets of p by a 1-brane junction given by a D2-brane wrapped on a 2-cycle $\sum_k n_k \alpha_k$, where the n_k are integers satisfying $\sum_k n_k p_k = p$. The associated topological defects are $\mathbb{Z}_p$-charged 2-branes, the one with minimal charge being a D4-brane wrapped on a 2-cycle $\sum_k \frac{p_k}{p} \beta_k$, and they can be annihilated in sets of p by a 2-brane junction given by a NS5-brane wrapped on $\mathbb{X}_4$.

8.3.2.3 Type IIA with 4-form flux

Consider type IIA compactifications on a real dimension 4 space $\mathbb{X}_4$ with p units of $\overline{F}_4$ flux on it. There are two 10d Chern-Simons couplings (8.40) that lead to the 6d terms we are interested in.

$$\int_{10d} B_2 \wedge \overline{F}_4 \wedge F_4 \tag{8.52a}$$

$$\int_{10d} H_3 \wedge \overline{F}_4 \wedge C_3 \tag{8.52b}$$

The Chern-Simons coupling (8.52a) produces a 6d BF coupling

$$\int_{10d} B_2 \wedge \overline{F}_4 \wedge F_4 \quad \longrightarrow \quad p\int_{6d} B_2 \wedge F_4 . \tag{8.53}$$

The theory automatically has a $\mathbb{Z}_p$ discrete gauge symmetry. The $\mathbb{Z}_p$-charged fundamental objects are 2-branes given by D2-branes, which are annihilated in sets of p by a 2-brane junction given by a NS5-brane wrapped on $\mathbb{X}_4$. The associated topological defects are $\mathbb{Z}_p$-charged 1-branes given by fundamental strings, which are annihilated in sets of p by a 1-brane junction given by a D4-brane wrapped on $\mathbb{X}_4$.

Let us study now the dual theory. The Chern-Simons couplings (8.52b) gives rise to a 6d BF coupling

$$\int_{10d} H_3 \wedge \overline{F}_4 \wedge C_3 \quad \longrightarrow \quad p\int_{6d} H_3 \wedge C_3 . \tag{8.54}$$

The theory automatically has a $\mathbb{Z}_p$ discrete gauge symmetry. The $\mathbb{Z}_p$-charged fundamental objects are 1-branes given by fundamental strings, which are annihilated in sets of p by a 1-brane junction given by a D4-brane wrapped on $\mathbb{X}_4$. The associated topological defects are $\mathbb{Z}_p$-charged 2-branes given by D2-branes, which are annihilated in sets of p by a 2-brane junction given by a NS5-brane wrapped on $\mathbb{X}_4$.

In this case, both the original theory and its dual give rise to the same discrete gauge symmetry.

The M-theory version of this system is interesting, and arise naturally in the context of $\mathrm{AdS}_7/\mathrm{CFT}_4$ correspondence. Compactification of M-theory on a 4-manifold down to $D = 7$, with p units of G_4 4-form flux produces a 7d coupling $p\int_{7d} G_4 \wedge C_3$. The corresponding $\mathbb{Z}_p$ discrete gauge symmetry has appeared in [251]. M2-branes correspond to the two kinds if $\mathbb{Z}_p$ topological defects, hence M2-branes pick up $\mathbb{Z}_p$ phases when surrounding each other, in a higher dimensional analogy of anyons in $D = 3$. This interesting behaviour is presumably linked to the elusive system of coincident M5-branes underlying this gauge/gravity duality.

8.3.2.4 Type IIA with NSNS flux

Consider type IIA compactifications on a real dimension 4 space $\mathbb{X}_4$ with NSNS 3-form flux $\overline{H}_3$. Let us introduce a basis of 3-cycles $\{\Gamma_i\}$ and a dual basis of 1-cycles $\{\gamma_i\}$ such that $\gamma_i \cdot \Gamma_j = \delta_{ij}$. Let us define

$$p_i = \int_{\Gamma_i} \overline{H}_3 . \tag{8.55}$$

There are three 10d Chern-Simons couplings (8.40) that lead to the 6d terms we are interested in.

$$\int_{10d} \overline{H}_3 \wedge F_2 \wedge C_5, \tag{8.56a}$$

$$\int_{10d} \overline{H}_3 \wedge C_1 \wedge F_6, \tag{8.56b}$$

$$\int_{10d} \overline{H}_3 \wedge F_4 \wedge C_3. \tag{8.56c}$$

Let us start with (8.56a). The 6d couplings it leads to are

$$\int_{10d} \overline{H}_3 \wedge F_2 \wedge C_5 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i F_2 \wedge \hat{C}_4^i \tag{8.57}$$

where

$$\hat{C}_4^i = \int_{\gamma_i} C_5. \tag{8.58}$$

There is a $\mathbb{Z}_p$ discrete symmetry, where $p = \gcd(p_i)$. The $\mathbb{Z}_p$-charged fundamental objects are particles given by D0-branes, which can be annihilated in sets of p by an instanton given by a D2-brane wrapped on a 3-cycle $\sum_i n_i \Gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$. The associated topological defects are $\mathbb{Z}\,-p$-charged 3-branes, the one with minimal charge being a D4-brane wrapped on a 1-cycle $\sum_i \frac{p_i}{p}\gamma_i$, and they can be annihilated in sets of p by a 3-brane junction given by a D6-brane wrapped on $\mathbb{X}_4$.

Let us study now the dual theory. The 10d Chern-Simons coupling (8.56b) gives rise to the 6d couplings

$$\int_{10d} \overline{H}_3 \wedge C_1 \wedge F_6 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i C_1 \wedge \hat{F}_5^i \tag{8.59}$$

where

$$\hat{F}_5^i = \int_{\gamma_i} F_6. \tag{8.60}$$

There is a $\mathbb{Z}_q$ discrete gauge symmetry, where $q = \sum_i \frac{p_i^2}{p}$, with $p = \gcd(p_i)$. The $\mathbb{Z}_q$-charged fundamental objects are 3-branes, the one with minimal charge being a D4-brane wrapped on 1-cycle $\sum_i n_i \gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$, and they can be annihilated in sets of q by a D6-brane wrapped on $\mathbb{X}_4$. The associated topological defects are $\mathbb{Z}_q$-charged particles given by D0-branes, which can be annihilated in sets of q by an instanton given by a D2-brane wrapped on a 3-cycle $\sum_i \frac{p_i}{p}\Gamma_i$.

In this type of compactifications there is another pair of discrete gauge symmetries that can be obtained.

The 10d Chern-Simons couplings (8.56c) leads to 6d terms

$$\int_{10d} \overline{H}_3 \wedge F_4 \wedge C_3 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i F_4 \wedge \hat{C}_2^i \tag{8.61}$$

where

$$\hat{C}_2^i = \int_{\gamma_i} C_3. \tag{8.62}$$

There is a $\mathbb{Z}_p$ discrete symmetry, where $p = \gcd(p_i)$. The fundamental objects are $\mathbb{Z}_p$-charged 2-branes given by D2-branes, which can be annihilated in sets of p by a 2-brane junction given by a D4-brane wrapped on a 3-cycle $\sum_i n_i \Gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$. The associated topological defects are $\mathbb{Z}_p$-charged 1-branes, the one with minimal charge being a D2-brane wrapped on a 1-cycle $\sum_i \frac{p_i}{p}\gamma_i$, and they can be annihilated in sets of p by a 1-brane junction given by a D4-brane wrapped on $\mathbb{X}_4$.

However, (8.56c) also leads to the dual case, with 6d coupling

$$\int_{10d} \overline{H}_3 \wedge F_4 \wedge C_3 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i \hat{F}_3^i \wedge C_3 \tag{8.63}$$

where

$$\hat{F}_3^i = \int_{\gamma_i} F_4. \tag{8.64}$$

There is a $\mathbb{Z}_q$ discrete gauge symmetry, where $q = \sum_i \frac{p_i^2}{p}$, with $p = \gcd(p_i)$. The $\mathbb{Z}_q$-charged fundamental objects are 1-branes, the one with minimal charge being a D2-brane wrapped on a 1-cycle $\sum_i n_i \gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$, and they can be annihilated in sets of q by a string junction given by a D4-brane wrapped on $\mathbb{X}_4$. The associated topological defects are $\mathbb{Z}_q$-charged 2-branes given by D2-branes, which can be annihilated in sets of q by a 2-brane junction given by a D4-brane wrapped on a 3-cycle $\sum_i \frac{p_i}{p}\Gamma_i$.

8.3.3 Type IIB compactifications

8.3.3.1 Type IIB with 1-form flux

Consider type IIB compactifications on a real dimension 4 space $\mathbb{X}_4$ with NSNS 3-form flux $\overline{H}_3$. Let us introduce a basis of 3-cycles $\{\Gamma_i\}$ and a dual basis of 1-cycles $\{\gamma_i\}$ such that $\gamma_i \cdot \Gamma_j = \delta_{ij}$. Let us define

$$p_i = \int_{\alpha_i} \overline{F}_1. \tag{8.65}$$

There are two 10d Chern-Simons couplings (8.40) that lead to the 6d terms we are interested in.

$$\int_{10d} B_2 \wedge \overline{F}_1 \wedge F_7, \tag{8.66a}$$

$$\int_{10d} H_3 \wedge \overline{F}_1 \wedge C_6. \tag{8.66b}$$

The 10d Chern-Simons coupling (8.66a) leads to 6d terms

$$\int_{10d} B_2 \wedge \overline{F}_1 \wedge F_7 \quad \longrightarrow \quad \sum_i \int_{6d} p_i B_2 \wedge \hat{F}_4^i \tag{8.67}$$

where

$$\hat{F}_4^i = \int_{\Gamma_i} F_7. \tag{8.68}$$

There is a $\mathbb{Z}_q$ discrete gauge symmetry, where $q = \sum_i \frac{p_i^2}{p}$, with $p = \gcd(p_i)$. The $\mathbb{Z}_q$-charged fundamental objects are 2-branes, the one with minimal charge being a D5-brane wrapped on 3-cycle $\sum_i n_i \Gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$, and they can be annihilated in sets of q by a 2-brane junction given by a NS5-brane wrapped on $\mathbb{X}_4$. The associated topological defects are $\mathbb{Z}_q$-charged 1-branes given by fundamental strings, which can be annihilated in sets of q by a 1-brane junction given by a D1-brane wrapped on a 1-cycle $\sum_i \frac{p_i}{p} \gamma_i$.

Let us study the dual theory. The 10d Chern-Simons coupling (8.66b) leads to 6d terms

$$\int_{10d} H_3 \wedge \overline{F}_1 \wedge C_6 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i H_3 \wedge \hat{C}_3^i \tag{8.69}$$

where

$$\hat{C}_3^i = \int_{\Gamma_i} C_6. \tag{8.70}$$

There is a $\mathbb{Z}_p$ discrete symmetry, where $p = \gcd(p_i)$. The $\mathbb{Z}_p$-charged fundamental objects are 1-branes given by fundamental strings, which can be annihilated in sets of p by a 1-brane junction given by a D1-brane wrapped on a 1-cycle $\sum_i n_i \gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$. The associated topological defects are $\mathbb{Z}_p$-charged 2-branes, the one with minimal charge being a D5-brane wrapped on a 3-cycle $\sum_i \frac{p_i}{p} \Gamma_i$, and they can be annihilated in sets of p by a 2-brane junction given by a NS5-brane wrapped on $\mathbb{X}_4$.

8.3.3.2 Type IIB with NSNS flux

Consider type IIB compactifications on a real dimension 4 space $\mathbb{X}_4$ with NSNS 3-form flux $\overline{H}_3$. Let us introduce a basis of 3-cycles $\{\Gamma_i\}$ and a dual basis of 1-cycles $\{\gamma_i\}$ such that $\gamma_i \cdot \Gamma_j = \delta_{ij}$. Let us define

$$p_i = \int_{\Gamma_i} \overline{H}_3. \tag{8.71}$$

There are two 10d Chern-Simons couplings (8.40) that lead to the 6d terms we are interested in.

$$\int_{10d} \overline{H}_3 \wedge C_2 \wedge F_5, \tag{8.72a}$$

$$\int_{10d} \overline{H}_3 \wedge C_4 \wedge F_3. \tag{8.72b}$$

Let us start with (8.72a). The 6d terms it leads to are

$$\int_{10d} \overline{H}_3 \wedge C_2 \wedge F_5 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i C_2 \wedge \hat{F}_4^i \tag{8.73}$$

where

$$\hat{F}_4^i = \int_{\gamma_i} F_5. \tag{8.74}$$

There is a $\mathbb{Z}_q$ discrete gauge symmetry, where $q = \sum_i \frac{p_i^2}{p}$, with $p = \gcd(p_i)$. The $\mathbb{Z}_q$-charged fundamental objects are 2-branes, the one with minimal charge being a D3-brane wrapped on a 1-cycle $\sum_i n_i \gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$, and they can be annihilated in sets of q by a 2-brane junction given by a D5-brane wrapped on $\mathbb{X}_4$. The associated topological defects are $\mathbb{Z}_q$-charged 1-branes given by D1-branes, which can be annihilated in sets of p by a 1-brane junction given by a D3-brane wrapped on a 3-cycle $\sum_i \frac{p_i}{p} \Gamma_i$.

Let us study now the dual theory. The 10d Chern-Simons coupling (8.72b) leads to 6d terms

$$\int_{10d} \overline{H}_3 \wedge C_4 \wedge F_3 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i \hat{C}_3^i \wedge F_3 \tag{8.75}$$

where

$$\hat{C}_3^i = \int_{\gamma_i} C_4. \tag{8.76}$$

There is a $\mathbb{Z}_p$ discrete symmetry, where $p = \gcd(p_i)$. The Z_p-charged fundamental objects are 1-branes given by D1-branes, which can be annihilated in sets of p by a 1-brane junction given by a D3-brane wrapped on a 3-cycle $\sum_i n_i \Gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$. The associated topological defects are $\mathbb{Z}_p$-charged 2-branes, the one with minimal charge being a D3-brane wrapped on a 1-cycle $\sum_i \frac{p_i}{p} \gamma_i$, and they can be annihilated in sets of p by a 2-brane junction given by a D5-brane wrapped on $\mathbb{X}_4$.

In this type of compactifications there is another pair of discrete gauge symmetries that can be obtained.

The 10d Chern-Simons coupling (8.72a) also leads to the 6d terms

$$\int_{10d} \overline{H}_3 \wedge C_2 \wedge F_5 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i \hat{C}_1^i \wedge F_5 \tag{8.77}$$

where

$$\hat{C}_1^i = \int_{\gamma_i} C_2. \tag{8.78}$$

There is a $\mathbb{Z}_p$ discrete symmetry, where $p = \gcd(p_i)$. The $\mathbb{Z}_p$-charged fundamental objects are 3-branes given by D3-branes, which can be annihilated in sets of p by a 3-brane junction given by a D5-brane wrapped on a 3-cycle $\sum_i n_i \Gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$. The associated topological defects are $\mathbb{Z}_p$-charged particles, the one with minimal charge being a D1-brane wrapped on a 1-cycle $\sum_i \frac{p_i}{p} \gamma_i$, and they can be annihilated in sets of p by an instanton given by a D3-brane wrapped on $\mathbb{X}_4$.

Let us study the dual theory. The 10d Chern-Simons couplings (8.72b) also leads to 6d terms

$$\int_{10d} \overline{H}_3 \wedge C_4 \wedge F_3 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i C_4 \wedge \hat{F}_2^i \tag{8.79}$$

where

$$\hat{F}_2^i = \int_{\gamma_i} F_3. \tag{8.80}$$

There is a $\mathbb{Z}_q$ discrete gauge symmetry, where $q = \sum_i \frac{p_i^2}{p}$, with $p = \gcd(p_i)$. The $\mathbb{Z}_q$-charged fundamental objects are particles, the one with minimal charge being a D1-brane wrapped on a 1-cycle $\sum_i n_i \gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$, and they can be annihilated in sets of q by an instanton given by a D3-brane wrapped on $\mathbb{X}_4$. The associated topological defects are $\mathbb{Z}_q$-charged 3-branes given by D3-branes, which can be annihilated in sets of p by a 3-brane junction given by a D5-brane wrapped on a 3-cycle $\sum_i \frac{p_i}{p} \Gamma_i$.

8.3.3.3 Type IIB with RR 3-form flux

Consider type IIB compactifications on a real dimension 4 space $\mathbb{X}_4$ with NSNS 3-form flux $\overline{H}_3$. Let us introduce a basis of 3-cycles $\{\Gamma_i\}$ and a dual basis of 1-cycles $\{\gamma_i\}$ such that $\gamma_i \cdot \Gamma_j = \delta_{ij}$. Let us define

$$p_i = \int_{\Gamma_i} \overline{F}_3. \tag{8.81}$$

There are two 10d Chern-Simons couplings (8.40) that lead to the 6d terms we are interested in.

$$\int_{10d} B_2 \wedge \overline{F}_3 \wedge F_5, \tag{8.82a}$$

$$\int_{10d} H_3 \wedge \overline{F}_3 \wedge C_4. \tag{8.82b}$$

The 10d Chern-Simons coupling (8.82a) leads to the 6d terms

$$\int_{10d} B_2 \wedge \overline{F}_3 \wedge F_5 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i B_2 \wedge \hat{F}_4^i \tag{8.83}$$

where

$$\hat{F}_4^i = \int_{\gamma_i} F_5. \tag{8.84}$$

There is a $\mathbb{Z}_q$ discrete gauge symmetry, where $q = \sum_i \frac{p_i^2}{p}$, with $p = \gcd(p_i)$. The $\mathbb{Z}_q$-charged fundamental objects are 2-branes, the one with minimal charge being a D3-brane wrapped on a 1-cycle $\sum_i n_i \gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$, and they can be annihilated in sets of q by a 2-brane junction given by a NS5-brane wrapped on $\mathbb{X}_4$. The associated topological defects are $\mathbb{Z}_q$-charged 1-branes given by fundamental strings, which can be annihilated in sets of q by a 1-brane junction given by a D3-brane wrapped on a 3-cycle $\sum_i \frac{p_i}{p} \Gamma_i$.

Let us study now the dual theory. The 10d Chern-Simons coupling (8.82b) leads to the 6d terms

$$\int_{10d} H_3 \wedge \overline{F}_3 \wedge C_4 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i H_3 \wedge \hat{C}_3^i \tag{8.85}$$

where

$$\hat{C}_3^i = \int_{\gamma_i} C_4. \tag{8.86}$$

There is a $\mathbb{Z}_p$ discrete symmetry, where $p = \gcd(p_i)$. The $\mathbb{Z}_p$-charged fundamental objects are 1-branes given by fundamental strings, which can be annihilated in sets of p by a string junction given by a D3-brane wrapped on a 3-cycle $\sum_i n_i \Gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$. The associated topological defects are $\mathbb{Z}_p$-charged 2-branes, the one with minimal charge being a D3-brane wrapped on a 1-cycle $\sum_i \frac{p_i}{p}\gamma_i$, and they can be annihilated in sets of p by a 2-brane junction given by a NS5-brane wrapped on $\mathbb{X}_4$.

In this type of compactifications there is another pair of discrete gauge symmetries that can be obtained.

The 10d Chern-Simons coupling (8.82a) also leads to the 6d terms

$$\int_{10d} B_2 \wedge \overline{F}_3 \wedge F_5 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i \hat{B}_1^i \wedge F_5 \tag{8.87}$$

where

$$\hat{B}_1^i = \int_{\gamma_i} B_2. \tag{8.88}$$

There is a $\mathbb{Z}_p$ discrete symmetry, where $p = \gcd(p_i)$. The $\mathbb{Z}_p$-charged fundamental objects are 3-branes given by D3-branes, which can be annihilated in sets of p by a 3-brane junction given by a NS5-brane wrapped on a 3-cycle $\sum_i n_i \Gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$. The associated topological defects are $\mathbb{Z}_p$-charged particles, the one with minimal charge being a fundamental string wrapped on a 1-cycle $\sum_i \frac{p_i}{p}\gamma_i$, and they can be annihilated in sets of p by an instanton given by a D3-brane wrapped on $\mathbb{X}_4$.

Let us study now the dual theory. The 10d Chern-Simons coupling (8.82b) also leads to the 6d terms

$$\int_{10d} H_3 \wedge \overline{F}_3 \wedge C_4 \quad \longrightarrow \quad \sum_{i=1}^{4} \int_{6d} p_i \hat{F}_2^i \wedge C_4 \tag{8.89}$$

where

$$\hat{F}_2^i = \int_{\gamma_i} H_3. \tag{8.90}$$

There is a $\mathbb{Z}_q$ discrete gauge symmetry, where $q = \sum_i \frac{p_i^2}{p}$, with $p = \gcd(p_i)$. The $\mathbb{Z}_q$-charged fundamental objects are particles, the one with minimal charge being a fundamental string wrapped on a 1-cycle $\sum_i n_i \gamma_i$, where the n_i are integers satisfying $\sum_i n_i p_i = p$, and they can be annihilated in sets of q by an instanton given by a D3-brane wrapped on $\mathbb{X}_4$. The associated topological defects are $\mathbb{Z}_q$-charged 3-branes given by D3-branes, which can be annihilated in sets of q by a 3-brane junction given by a NS5-brane wrapped on a 3-cycle $\sum_i \frac{p_i}{p}\Gamma_i$.

8.4 Higher-rank $\mathbb{Z}_p$ symmetries in string compactifications with torsion

Discrete gauge symmetries associated to higher-rank forms are briefly mentioned in [227], although related to torsion in homology or K-theory. This very formal discussion can be made very explicit following [141], at least for torsion homology. To show that compactifications with torsion homology can produce higher-rank discrete symmetries, we consider a simple illustrative example. Consider M-theory on a 4-manifold with torsion 1-cycles (and their dual 2-cycles), $H_1(\mathbb{X}_4, \mathbb{Z}) = H_2(\mathbb{X}_4, \mathbb{Z}) = \mathbb{Z}_p$. We focus on the sector of M2-branes on 1-cycles – 7d strings – and M5-branes on 2-cycles – 7d 3-branes – (there is another sector of M2-branes on 2-cycles and M5-branes on 1-cycles, which can be discussed similarly). Following [141], we introduce the Poincaré dual torsion 2- and 3-forms $\alpha_2^{\rm tor}$, $\tilde{\omega}_3^{\rm tor}$, satisfying the relations

$$d\omega_1^{\rm tor} = p\,\alpha_2^{\rm tor}, \tag{8.91a}$$

$$d\beta_2^{\rm tor} = p\,\tilde{\omega}_3^{\rm tor}, \tag{8.91b}$$

where $\omega_1^{\rm tor}$ and $\beta_2^{\rm tor}$ are globally well-defined 1- and 2-forms. The torsion 2- and 3-forms $\alpha_2^{\rm tor}$ and $\tilde{\omega}_3^{\rm tor}$ are thus trivial in de Rham cohomology, but not in the $\mathbb{Z}$-valued cohomology, i.e. $H_2(\mathbb{X}_4\,\mathbb{R}) = H_3(\mathbb{X}_4, \mathbb{R}) = \varnothing$, $H_2(\mathbb{X}_4\,\mathbb{Z}) = H_3(\mathbb{X}_4, \mathbb{Z}) = \mathbb{Z}_p$. The torsion linking number is encoded in the intersection pairing

$$\int_{\mathbb{X}_4} \alpha_2^{\rm tor} \wedge \beta_2^{\rm tor} = \int_{\mathbb{X}_4} \omega_1^{\rm tor} \wedge \tilde{\omega}_2^{\rm tor} = 1. \tag{8.92}$$

These forms are assumed to be eigenstates of the Laplacian [141], corresponding to massive modes; they can be usefully exploited to describe dimensional reduction of the antisymmetric tensor fields, in particular, the M-theory 3- and 6-forms

$$C_3 = \phi_1 \wedge \alpha_2^{\rm tor} \; + \; A_2 \wedge \omega_1^{\rm tor}, C_6 \qquad = B_4 \wedge \beta_2^{\rm tor} \; + \; V_3 \wedge \tilde{\omega}_3^{\rm tor}. \tag{8.93a}$$

The corresponding field strengths contain the structures

$$dC_3 = (d\phi_1 + pA_2) \wedge \alpha_2^{\rm tor} + \dots, \tag{8.94a}$$

$$dC_6 = (dV3 + pB_4) \wedge \tilde{\omega}_3^{\rm tor} + \dots, \tag{8.94b}$$

which (modulo a trivial sign redefinition) imply the gauge invariances (8.1), (8.7). Accordingly, the 11d kinetic term for $G_4 = dC_3$ (and its dual) lead to 7d actions with the structure (8.2), (8.5). The dimensional reduction we have just sketched thus relates the underlying torsion homology with the $\mathbb{Z}_p$ gauge theory lagrangians of section 8.1.

8.5 The non-abelian case

In 4d, the non-abelian character can be detected by letting two strings (with charges given by non-commuting group elements a, b) cross, and watching the appearance of an stretched string (with charge given by the commutator $c = aba^{-1}b^{-1}$). In string theory realizations, this follows from brane creation processes when the underlying branes are crossed [180].

In general dimension D, we can look for similar effects, the only difference being that the objects have richer dimensionality. Consider the following table, which describes the geometry of two branes (denoted 1 and 2) which cross and lead to the creation of brane 3

	d_1	d_2	d_3	
Brane 1	$-\cdots-$	$-\cdots-$	$\times\cdots\times$	$\times$
Brane 2	$-\cdots-$	$\times\cdots\times$	$-\cdots-$	$\times$
Brane 3	$-\cdots-$	$\times\cdots\times$	$\times\cdots\times$	$-$

The symbols $-$ and $\times$ denote that the brane spans or does not span the corresponding dimension, and obviously $d_1 + d_2 + d_3 + 1 = D$. The last entry corresponds to the single overall transverse dimensions to branes 1 and 2, on which the crossing proceeds, and along which the created brane 3 stretches.

As a concrete example, involving discrete gauge symmetries arising from torsion homology c.f. section 8.4, consider type IIB compactified on a 5-manifold with a $\mathbb{Z}_p$ torsion 3-cycle, self-intersecting over the dual $\mathbb{Z}_p$ torsion 1-cycle (the $\mathrm{AdS}_5 \times \mathbb{S}^5 / \mathbb{Z}_3$ geometry in [138] is a realization for $p = 3$). The theory contains 5d 2-branes arising from NS5-branes on the torsion 3-cycle, a further set of 5d 2-branes from D5-branes on the torsion 3-cycle, and a set of 5d 2-branes from D3-branes on the torsion 1-cycle. The crossing of NS5- and D5-branes produces D3-branes [180], leading to the above 2-brane crossing effect (with $d_1 = 2$, $d_2 = d_3 = 1$ in the above table); the resulting discrete group is non-abelian, and is given by a Δ_{27} (for general $\mathbb{Z}_p$ torsion, a discrete Heisenberg group [143], see also [178]).

One can similarly construct more exotic examples, in which the non-abelian symmetry group elements are associated to objects of *different* dimensionality. For instance, consider type IIA compactified on the same geometry as above, i.e. a 5-manifold with torsion 3- and 1-cycles. The theory contains 5d 2-branes from NS5-branes on the torsion 3-cycle, a set of 5d 1-branes from D4-branes on the torsion 3-cycle, and a further set of 5d 1-branes from D2-branes on the torsion 1-cycle. The crossing of NS5- and D4-branes produces D2-branes; in 5d the process corresponds to crossing a 2-brane with a 1-brane, with the creation of another kind of 1-brane (hence we have $d_1 = 1$, $d_2 = 2$, $d_3 = 1$). The resulting discrete Heisenberg symmetry group is exotic, since its elements are associated to objects of different dimensionality. A similar phenomenon already occurs in the (abelian) context of D-brane charge classification by K-theory, where in certain examples the charges in a K-theory group correspond to branes in cohomology classes of different degree (e.g. [227] quotes the example of $\mathbb{RP}_7$, where the torsion cohomology is $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$, with the torsion K-theory is $\mathbb{Z}_8$).

Similar examples could be worked out involving branes whose charges are $\mathbb{Z}$-valued in (co)homology, but which are actually torsion due to the presence of background fluxes.

9

Discrete gauge symmetries from closed string tachyon condensation

9.1 Introduction

In the previous chapters, it was always possible to embed the discrete gauge symmetries into continuous ones, which were broken to the discrete subgroup by the gauging of (or Higgsing by) scalars of the theory, with the prototypical example being a $\mathbb{Z}_n$ discrete symmetries coming from $U(1)$ groups on D-branes, which are broken by 4d Stückelberg couplings. The fact that discrete gauge symmetries arise from broken continuous ones is very useful for several reasons. For instance, it helps to make contact with the 4d field theory description, and to construct charged particles and other topological defects [126], like in sections 6.1, where the branes giving rise to the charged particles and strings wrap cycles related to those on which the forms that give rise to the gauge potential forms are wrapped. It is also practical in purely field theoretical setups, to study anomaly cancellation conditions [130, 121, 252, 253].

One important point one should take into account is that the mass acquired by the gauge bosons of the broken continuous groups is of the order of the string scale. However, from the 4d effective theory point of view these continuous symmetries can be considered as a useful device to understand better the discrete gauge symmetries; in exchange, the price one has to pay is to 'integrate in' degrees of freedom of the string scale sector which are relevant to the topology of the symmetry breaking and the charged objects.

However, in string theory, both in 10d and in compactifications, there are discrete gauge symmetries that cannot be embedded into continuous groups. The prototypical example would be discrete gauge symmetries arising from large isometries of the compactification space, i.e. isometries that cannot be connected to the identity in a continuous way. The goal of this chapter is to show that the embedding is actually possible, if one integrates in a suitable sector of stringy physics.

The 'suitable sector' one needs to consider will be provided by extra space-time dimensions, where we will be able to promote the discrete gauge symmetry into a continuous one. This will require us to make use of the supercritical string theories described in appendix A (see [254, 255, 256, 257, 258] for more detailed discussions).

While in the previous chapters the continuous gauge symmetry was broken to a discrete subgroup by some gauging/Higgsing mechanism, in the present case this mechanism is replaced by the condensation of the closed string tachyon which is present in the spectrum of all supercritical theories. This process will quench the extra dimensions allowing as to recover the critical theory (as shown in appendix A), while breaking the continuous group to a discrete subgroup.

This closed string tachyon condensation involves string scale physics; however, this also happened in the case of continuous symmetries broken by some gauging/Higgsing mechanism. In fact, as in the latter case, since the effects of the tachyon condensation with respect to the symmetry and its breaking are topological, they can be analyzed reliably.

Associated to a discrete gauge symmetry there are discretely charged topological defects. In the case of discrete gauge symmetries arising from closed string tachyon condensation, they can be constructed as solitons of the closed string tachyon field. This is reminiscent of the construction of D-branes as solitons of open string tachyons, and its connection to K-theory. The nature of these topological defects as well as the underlying mathematical structures classifying them are beyond the scope of this thesis, and will not be considered here.

The rest of this chapter is organised as follows. In section 9.2, as a warmup exercise, we will consider an analogous realization of discrete symmetries, in the more familiar open strain setup, given by a $\mathbb{Z}_2$ symmetry in type I arising from a continuous $U(1)$ broken by open string tachyon condensation. In section 9.3, we present a mechanism to embed $\mathbb{Z}_n$ discrete symmetries into continuous ones that act on the extra dimensions as rotations, which we dub 'quenched rotations', and how to construct $\mathbb{Z}_n$-charged topological defects as closed string tachyon solitions. In section 9.4 we present a more general method of embedding discrete symmetries into continuous groups, which we dub 'quenched translations'; the continuous group acts as a translation in a compact $\mathbb{S}^1$ supercritical dimension, along which the theory picks up a $\mathbb{Z}_n$ holonomy. Section 9.5 contains a generalization to non-Abelian discrete symmetries, and present a explicit example of a discrete Heisenberg group.

9.2 A warmup exercise: a type I $\mathbb{Z}_2$ symmetry from open string tachyon condensation

In this section we present an analogous implementation in the context of *open* string tachyon condensation. We use it in 10d type I theory to derive a discrete $\mathbb{Z}_2$ gauge symmetry from a continuous one (see [259] for a similar phenomenon in type IIB orientifolds).

Recall that in 10d type I theory there exist several $\mathbb{Z}_2$ charged non-BPS branes [219]. In particular we focus on the D7- and D0-branes (denoted by $\widehat{\text{D7}}$- and $\widehat{\text{D0}}$-branes in what follows), which pick up a -1 when moved around each other [219, 260]. Hence, they correspond to a $\mathbb{Z}_2$ charged particle and the dual codimension-2 $\mathbb{Z}_2$ charged defect (the 10d analogue of 4d $\mathbb{Z}_2$ string), associated to a $\mathbb{Z}_2$ discrete gauge symmetry. Since the $\widehat{\text{D0}}$-brane is a spinor under the $SO(32)$ perturbative type I gauge symmetry [261, 262], the $\mathbb{Z}_2$ can be defined as acting as -1 on spinors and leaving tensors invariant.

Naively, this type I $\mathbb{Z}_2$ symmetry cannot be described as a discrete remnant of a broken continuous gauge symmetry. However, this can be achieved by regarding type I theory as the endpoint of open string tachyon condensation, starting from a configuration with additional D9-$\overline{\text{D9}}$ brane pairs [263]; this is natural given the construction of non-BPS branes in terms of brane-antibrane pairs.

Consider e.g. the case of two extra D9-$\overline{\text{D9}}$ brane pairs. The gauge symmetry is enhanced to $SO(34) \times SO(2)$, and there is a complex tachyon in the bifundamental, i.e. a vector of $SO(34)$ with $SO(2) \simeq U(1)$ charge $+1$ (and -1 for the conjugate scalar)[1]. On the other hand, the $\widehat{\text{D0}}$-brane can be shown[2] to transform as a chiral bi-spinor, namely a chiral $SO(34)$ spinor with $U(1)$ charge $+\frac{1}{2}$, and an opposite-chirality $SO(34)$ spinor with $U(1)$ charge $-\frac{1}{2}$. Tachyon condensation imposes the breaking

$$SO(34) \times SO(2) \to SO(32) \times SO(2)' \times SO(2) \xrightarrow{\langle T \rangle} SO(32) \times SO(2)_{\text{diag}} \tag{9.1}$$

The intermediate step just displays the two $SO(2)$ symmetries most relevant in the final breaking.

The phenomenon is very similar to a Higgs mechanism, with the *proviso* that the diagonal subgroup actually disappears from the theory (this is analogous to the disappearance of the diagonal $U(1)$ in the annihilation of spacetime filling brane-antibrane pairs, see e.g. [264] for discussions). The anti-diagonal combination $U(1)_{\text{anti}}$ of $SO(2)' \times SO(2)$, generated by $Q_{\text{anti}} = Q_{SO(2)} - Q_{SO(2)'}$, is Higgsed down by the tachyon, which carries charge $+2$, thereby leaving a remnant $\mathbb{Z}_2$ discrete gauge symmetry. The $\mathbb{Z}_2$ charged particles are the $\widehat{\text{D0}}$-brane states, which transform as a chiral $SO(32)$ spinor with $U(1)_{\text{anti}}$ charge $+1$.

The main lesson is that this $\mathbb{Z}_2$ symmetry of 10d type I theory[3] can be derived as an unbroken discrete symmetry of a continuous gauge symmetry, by regarding the theory as the endpoint of (open string) tachyon condensation. The 'parent' theory includes extra degrees of freedom on which the continuous symmetry acts, and which disappear upon tachyon condensation.

9.3 Quenched rotations

In general, discrete symmetries cannot be regarded as a discrete subgroup of a continuous group action on the theory. This happens for instance for discrete large isometries in compactifications, namely discrete isometries associated to *large* diffeomorphisms of the geometry. In compactifications, discrete isometries of the internal space become discrete gauge symmetries of the lower-dimensional theory. Hence, discrete gauge symmetries from large isometries cannot in principle be regarded as unbroken remnants of some continuous gauge symmetry.

In this section we actually show that even such discrete symmetries can be embedded into continuous groups, which however act on extra dimensions in a supercritical extension of the theory. The continuous orbits come out of the critical spacetime slice, and

[1]For future convenience, we mention that in models with n extra D9-$\overline{\text{D9}}$ pairs, there are massless fermions transforming under $SO(32+n) \times SO(n)$ as follows [263]: one set of chiral spinors in the representation $(\begin{smallmatrix}\square\\\square\end{smallmatrix}, 1) + (1, \square\square)$, and one opposite-chirality spinor in the $(\square, \square)$.

[2]This follows from the quantization of the fermion zero modes in $\widehat{\text{D0}}$-D9 and $\widehat{\text{D0}}$-$\overline{\text{D9}}$ sectors, and restricting onto states invariant under the worldvolume $O(1)$ gauge symmetry.

[3]Incidentally, other $\mathbb{Z}_2$ charged branes can be associated to discrete $\mathbb{Z}_2$ subgroups of continuous symmetries, albeit associated not to gauge bosons but to higher RR p-forms, when described as K-theory valued objects [227].

are quenched by a closed string tachyon condensation process, which thus breaks the continuous symmetry, yet preserves the discrete one. The appearance of extra dimensions is very intuitive in extending discrete isometries to a continuous action, since these symmetries are related to properties of the metric; still our constructions apply to fairly general discrete symmetries.

9.3.1 Spacetime parity

The prototypical example of large diffeormorphisms is that of orientation reversing actions, for instance, spacetime parity. Consider the action

$$(x^0, x^1, \ldots, x^{2n-1}) \longrightarrow (x^0, -x^1, \ldots, -x^{2n-1}) \tag{9.2}$$

in $2n$-dimensional Minkowski space-time. Since it has determinant -1, it lies in a component of the Lorentz group disconnected from the identity. For simplicity, we will focus on the action

$$(x^0, x^1, \ldots, , x^{2n-2}, x^{2n-1}) \longrightarrow (x^0, x^1, \ldots, x^{2n-2}, -x^{2n-1}), \tag{9.3}$$

which only reverses the orientation of x^{2n-1}, that lies in the same component disconnected from the identity.

This $\mathbb{Z}_2$ symmetry is easily embedded in a continuous symmetry by adding one extra dimension x^{2n} and considering the $SO(2)$ rotations in the plane (x^{2n-1}, x^{2n}). This is easily implemented in string theory, for instance, using the supercritical bosonic string.[4]

Consider the supercritical bosonic string theory from section A.1 with one extra dimension denoted by x^{26}, with the appropriate timeline linear dilation. The theory is invariant under $SO(2)$ rotations in the (x^{25}, x^{26}) 2-plane plane. We can connect this theory with the critical 26d bosonic theory by a closed string tachyon profile

$$T(X^+, X^{26}) = \frac{\mu^2}{2\alpha'} \exp(\beta X^+)(X^{26})^2. \tag{9.4}$$

At $X^+ \to -\infty$ the tachyon vanishes and we have a 27d theory with a continuous $SO(2)$ rotational invariance in the 2-plane (x^{25}, x^{26}). At $X^+ \to \infty$, the onset of the tachyon truncates the dynamics to the slice $X^{26} = 0$, breaking the $SO(2)$ symmetry to the $\mathbb{Z}_2$ subgroup $X^{25} \to -X^{25}$. Hence the $\mathbb{Z}_2$ parity symmetry can be regarded as a discrete subgroup of a continuous higher dimensional rotation group, broken by the tachyon condensation removing the extra dimension.

The analogy of this breaking with a Higgs mechanism can be emphasized by using polar coordinates, $W = X^{25} + iX^{26} = |W|e^{i\theta}$. Then $X^{26} \sim W - \overline{W}$ and

$$T \sim (X^{26})^2 \sim W^2 - 2W\overline{W} + \overline{W}^2 \sim e^{2i\theta} - 2 + e^{-2i\theta}. \tag{9.5}$$

The tachyon background implies vevs only for modes of even charge under the $U(1)$ symmetry, hence a $\mathbb{Z}_2$ symmetry remains. This description will be useful for the construction of $\mathbb{Z}_2$ charged defects in section 9.3.3.

Finally, note that although this construction embeds the discrete group into a continuous one, there is no actual 26d $SO(2)$ gauge boson.

[4] Supercritical superstrings cannot be used since they are not parity invariant, due to chiral fermions or Chern-Simons couplings; however, parity can be combined with other actions to give symmetries of those theories, see section 9.4.4.5

9.3.2 A heterotic $\mathbb{Z}_2$ from closed tachyon condensation

In the 10d $SO(32)$ heterotic string, the gauge group is actually $Spin(32)/\mathbb{Z}_2$, and there is a $\mathbb{Z}_2$ symmetry under which the (massive) spinor states are odd, while fields in the adjoint are even[5]. We now propose a realization of this $\mathbb{Z}_2$ symmetry as a discrete remnant of a continuous $U(1)$ symmetry, exploiting the supercritical heterotic strings introduced in section A.2.

For that purpose, it suffices to focus on the case of $D = 12$, i.e. two extra dimensions, denoted x^{10}, x^{11}. We complexify the extra worldsheet fields into a complex scalar $Z \equiv X^{10} + iX^{11}$, and a complex fermion $\Lambda = \lambda^{33} + i\lambda^{34}$. We consider the $U(1)$ action

$$Z \to e^{i\alpha} Z \quad , \quad \Lambda \to e^{-i\alpha}\Lambda \tag{9.6}$$

namely, the anti-diagonal

$$SO(2)_{\rm anti} \subset SO(2)_{\rm rot} \times SO(2)_{\rm gauge} \subset SO(2)_{\rm rot} \times SO(2+32). \tag{9.7}$$

Consider now the tachyon background (A.18) producing the $SO(32)$ heterotic

$$T^{33}(X) \sim e^{\beta X^+} X^{10}, \qquad T^{34}(X) \sim e^{\beta X^+} X^{11} \to T(X) \sim e^{\beta X^+} Z, \tag{9.8}$$

which we have recast in terms of a holomorphic complex tachyon $T \equiv T^{33} + iT^{34}$. The order parameter $\partial_Z T$ transforms in the bifundamental of $SO(2)_{\rm rot} \times SO(2)_{\rm gauge}$, and breaks the $SO(2)_{\rm rot} \times SO(34)$ symmetry down to $SO(32)$ (times a diagonal factor which 'disappears'). The anti-diagonal $SO(2)_{\rm anti}$, generated by $Q_{\rm anti} = Q_{SO(2)_{\rm gauge}} - Q_{SO(2)_{\rm rot}}$, is broken by a charge $+2$ the tachyon background. To show that there is an unbroken $\mathbb{Z}_2$ acting as -1 on the $SO(32)$ spinors, it suffices to show that they descend from states with $SO(2)_{\rm anti}$ charge ± 1. Indeed, they descend from the massive groundstates in the g_2 twisted (and g_1-untwisted) sector, which has λ^a, ψ^m fermion zero modes; the states transform as a $SO(34) \times SO(2)_{\rm rot}$ bi-spinor, hence have $SO(2)_{\rm anti}$ charge ± 1, and descend to $\mathbb{Z}_2$ odd $SO(32)$ spinors. Note that the discussion parallels that of the type I $\mathbb{Z}_2$ in section 9.2 (as suggested by the duality proposed in [254]).

A slightly unsatisfactory aspect of the construction is that there are actually no gauge bosons associated to the $SO(n)_{\rm rot}$ group. This could be achieved by curving the geometry of the extra dimensions in the radial direction, with the angular coordinates asymptoting to a finite size $\mathbb{S}^{n-1}$. Its $SO(n)$ isometry group would then produce an $SO(n)$ gauge symmetry (in 11d). Although the geometric curvature will render the worldsheet theory non-solvable, we expect basic intuitions of the flat space case to extend to the curved situation, in what concerns the relevant topology of symmetry breaking. In any event, we will eventually turn to a more general construction, with physical gauge bosons, in section 9.4.

[5] Recall that the $\mathbb{Z}_2$ by which $Spin(32)$ is quotiented prevents the presence of states in the vector representation; also notice that this is *not* the $\mathbb{Z}_2$ discrete gauge symmetry we are interested in.

9.3.3 Topological defects from closed tachyon condensation

In the embedding of a discrete symmetry into a continuous one acting on extra dimensions, all relevant degrees of freedom are eventually removed by the closed string tachyon condensation. We may therefore ask what we gain by such construction. The answer is that, in analogy with open string tachyon condensation, certain branes of the final theory can be constructed as solitons of the tachyon field. Since our focus is on aspects related to $\mathbb{Z}_n$ discrete gauge symmetries, in this section we describe the tachyon profiles corresponding to the codimension-2 $\mathbb{Z}_n$ charged defects (real codimension-2 objects around which the theory is transformed by a discrete $\mathbb{Z}_n$ holonomy, e.g. the 4d $\mathbb{Z}_n$ strings).

However, we cannot refrain from pointing out that in the setup from section 9.3.2, the basic symmetries and their breaking are precisely as in type I theory with extra brane-antibrane pairs. In particular, this suggests a classification of topological brane charges in 10d heterotic in terms of a KO-theory (dovetailing heterotic/type I duality), in this case associated a pair of bundles $SO(32+n)\times SO(n)$ (namely, the gauge bundle and the normal bundle of the 10d slice in the supercritical $(10+n)$-dimensional spacetime), which annihilate via *closed* string tachyon condensation.

Focusing back on the construction of $\mathbb{Z}_n$ defects, let us consider again the analysis of parity in the bosonic theory in section 9.3.1. This $\mathbb{Z}_2$ symmetry is embedded into a $U(1)$ rotating the 2-plane (x^{25}, x^{26}). The critical vacuum is recovered by a tachyon background $T \sim (\mathrm{Im}\, w)^2$ c.f. (9.5), with $w \equiv x^{25}+ix^{26} = |w|e^{i\theta}$, whose zero cuts out the slice $\sin\theta = 0$. In order to describe a $\mathbb{Z}_2$ defect transverse to another 2-plane, e.g. (x^{23}, x^{24}), we write $z \equiv x^{23} + ix^{24} = |z|e^{i\varphi}$ and consider a closed string tachyon background vanishing at the locus $\sin\theta' = 0$, with $\theta' = \theta - \frac{1}{2}\varphi$. The remaining spacetime is still critical, with a new angular coordinate in (x^{23}, x^{24}) given by $\varphi' \sim \varphi + \frac{1}{2}\theta$. A rotation α in φ' (keeping θ' fixed at the tachyon minimum) is secretly a rotation $\delta\phi = \alpha$, $\delta\theta = \alpha/2$; hence, a full 2π rotation results in a π rotation in θ, i.e. a $\mathbb{Z}_2$ parity operation.

9.4 Discrete gauge symmetries as quenched translations

The set of discrete symmetries amenable to the quenched rotation construction in the previous section is limited. In this section we present a far more universal embedding of discrete symmetries into continuous ones, which are realized as continuous translations in an extra (supercritical) $\mathbb{S}^1$ dimension. The extra $\mathbb{S}^1$ is subsequently eaten up by closed string tachyon condensation, which breaks the KK $U(1)$ symmetry down to the discrete subgroup. The basic strategy is to use periodic tachyon profiles, c.f. (A.5) to yield condensation processes which are well-defined on $\mathbb{S}^1$. As mentioned at the end of section A.1, we do not mind giving up exact solvability of the worldsheet CFT, and rely on the main lesson that condensation truncates dynamics to the vanishing locus of the 2d potential energy[6].

[6]This is analogous to the applications of open string tachyon condensation in annihilation processes, even if they are not exactly solvable BCFTs.

9.4.1 The mapping torus

The basic ingredient in the construction is the mapping torus, whose construction we illustrate in fairly general terms. Although we apply it in the string theory setup, most of the construction can be carried out in the quantum field theory framework[7]; the ultimate removal of the extra dimensions by tachyon condensation is however more genuinely stringy.

Consider an N-dimensional theory on a spacetime $\mathbb{X}_N$, and let Θ be the generator of a discrete gauge symmetry $\mathbb{Z}_n$. We consider extending the theory to $\mathbb{X}_N \times I$, where I is a one-dimensional interval[8] parametrized by a coordinate $0 \leqslant y \leqslant 2\pi R$. We subsequently glue the theories at $y=0$ and $y=2\pi R$, but up to the action of Θ. The final configuration is the theory on $\mathbb{X}_N$ non-trivially fibered over $\mathbb{S}^1$. The fibration is locally $\mathbb{X}_N \times \mathbb{R}$, but there is a non-trivial discrete holonomy implementing the action of Θ. For example, if Θ is a discrete large isometry, the (purely geometric) glueing is

$$(x, y=0) \simeq (\Theta(x), y=2\pi R). \tag{9.9}$$

Returning to the general situation, the extra dimension produces a KK $U(1)$ gauge boson from the N-dimensional viewpoint. The orbit of the associated translational vector field ∂_y clearly contains the discrete $\mathbb{Z}_n$ transformations, which are thus embedded as a discrete subgroup $\mathbb{Z}_N \subset U(1)$. States of the theory $\mathbb{X}_N$ transforming with phase $e^{2\pi i \frac{p}{n}}$ under the discrete symmetry generator Θ extend as states with fractional momentum p/n (mod $\mathbb{Z}$) along $\mathbb{S}^1$. We note that the minimal $U(1)$ charge unit is $1/n$.

In the supercritical string theory construction, the extra dimension is removed by a tachyon profile with periodicity $2\pi R$, which truncates the theory to the slice $Y=0$ (mod $2\pi R$). For instance, in the bosonic string theory, we sketchily write[9]

$$T \sim \mu^2\big[1-\cos\left(\tfrac{Y}{R}\right)\big] = 2\mu^2 \sin^2\left(\tfrac{Y}{2R}\right), \tag{9.10}$$

and similarly in other supercritical string theories. Concretely, we use the heterotic $HO^{+(n)}$ theory (since the $HO^{+(n)/}$ breaks translational invariance in the extra dimensions) and take $T \sim \sin(\frac{Y}{2R})$, as in the LHS of (A.14); for type II extended as supercritical type 0 orbifold, we take $T \sim \sin(\frac{Y}{2R})X'$, obtained from (A.22) by renaming $X \to Y$ and taking a suitable $k' \to 0$ limit.

Since the periodic tachyon profile only excites components of integer KK momentum, it mimics a breaking of the $U(1)$ symmetry by fields of integer charge. Normalizing the minimal charge to $+1$, the breaking is implemented by fields of charge n. Hence, the continuous symmetry is broken to a discrete $\mathbb{Z}_n$ symmetry in the slice $y=0$, i.e. to the $\mathbb{Z}_n$ of the original theory at $\mathbb{X}_N$.

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[7]Incidentally, the mapping torus (a.k.a. cylinder) is widely used in the study of global anomalies [265, 266, 267]. It would be interesting to explore possible connections with our physical realization.

[8]This is easily implemented in the supercritical bosonic and heterotic $HO^{+(n)}$ theories; for the supercritical type 0 orbifolds decaying to type II we must add another extra dimension, which can be kept non-compact; for the $HO^{+(n)/}$ theory, the orbifolding breaks the translational invariance and there is no actual continuous KK gauge symmetry, so we do not consider it here.

[9]Note that the radius R can be kept arbitrary; for instance, the operator can be made marginal by turning on a lightlike dependence and adjusting the β coefficient appropriately.

9.4.2 Sum over disconnected theories

There is an alternative orbifold description of the above construction (hence, particularly well-suited for string theory), as follows. We start with the theory extended to a trivial product $\mathbb{X}_N \times (\mathbb{S}^1)'$, with $(\mathbb{S}^1)'$ a circle of length $2\pi nR$, parametrized by a coordinate y'. Subsequently, we mod out by the discrete action Θ on $\mathbb{X}_N$, accompanied by a shift $y' \to y' + 2\pi R$. The unit cell under this action is an $\mathbb{S}^1$ of length $2\pi R$, along which the theory is twisted by the action of Θ, as in the previous section. Similarly, we consider a tachyon profile with periodicity $2\pi R$.

We may regard the quotient theory on $\mathbb{S}^1$ as the set of $\mathbb{Z}_n$ invariant configurations[10] in the parent theory on $\mathbb{X}_N \times (\mathbb{S}^1)'$. In this description, states with charge p under Θ have integer KK momentum along $(\mathbb{S}^1)'$, while the tachyon profile has KK momentum multiple of n.

This viewpoint leads to an interesting interplay with the description of discrete gauge symmetries as a 'sum over disconnected theories' [176], (see appendix D for a review). This 'sum over theories' prescription is reproduced by the tachyon condensation on the orbifold of the theory on $\mathbb{X}_N \times (\mathbb{S}^1)'$, as follows. There is a tachyon profile with n zeroes, located at $y' = 0, 2\pi R, \ldots, 2\pi R(n-1)$ in $(\mathbb{S}^1)'$. Tachyon condensation produces n disconnected copies of the theory on $\mathbb{X}_N$ differing by the action of Θ^k, $k = 0, \ldots, n-1$. These copies correspond to the different 'theories' which coexist in the superposition (which in this language, is nothing but restricting to orbifold invariant amplitudes).

9.4.3 Topological $\mathbb{Z}_n$ defects and quenched fluxbranes

A basic property of theories with discrete $\mathbb{Z}_n$ gauge symmetries is the existence of $\mathbb{Z}_n$ charged defects, real codimension-2 objects around which the theory is transformed by a discrete $\mathbb{Z}_n$ holonomy. This non-trivial behaviour of the theory around the $\mathbb{S}^1$ surrounding the $\mathbb{Z}_n$ defect, is identical to the fibration over $\mathbb{S}^1$ in the mapping torus in the previous sections. This may be regarded as an underlying reason for the universality of the mapping torus construction, which applies to fairly general discrete symmetries (as opposed to those in section 9.3). In this section we use this relation to construct tachyon condensation profiles which produce the $\mathbb{Z}_n$ charged defects of the theory, generalizing section 9.3.3 to the more universal mapping torus setup.

The construction is very reminiscent of the fluxbranes[11] in [269, 270, 271, 272] explained in section B.

The $\mathbb{Z}_n$ defects can be constructed by using the same strategy in dimensional quenching, rather than in dimensional reduction. We start with the mapping torus of $\mathbb{X}_N$ fibered over a $\mathbb{S}^1$ parametrized by y, c.f. section 9.4.1. We choose a 2-plane (x^8, x^9), or $z \equiv x^8 + ix^9 \equiv re^{i\varphi}$. Finally, we turn on a closed string tachyon profile (9.10), but now depending on $y' = y + \varphi R$, to remove one extra $\mathbb{S}^1$ dimension. The resulting configuration

[10] In orbifold language, this corresponds to restricting to the untwisted sector. Twisted states stretching between different zero loci of the tachyon will disappear in the process of tachyon condensation, so they can be ignored in the discussion.

[11] We use the term 'fluxbrane' in the original sense of extended solutions with non-trivial (compactly supported) magnetic fields in their transverse dimensions, rather than in the recent use as branes carrying worldvolume magnetic flux [268].

contains a $\mathbb{Z}_n$ defect at the origin of the 2-plane, since a rotation in φ' results in a $\mathbb{Z}_n$ holonomy. Note that the disappearance the KK gauge bosons in dimensional quenching (as compared with dimensional reduction, recall section A.4), implies that there is no actual magnetic flux on the 2-plane, yet there is a non-trivial holonomy, as required to describe a $\mathbb{Z}_n$ charged object. We refer to these $\mathbb{Z}_n$ defects as 'quenched fluxbranes'.

Note that in contrast with actual fluxbranes, we do not require quenched fluxbranes to solve the equations of motion of the spacetime effective theory. Instead, we use the construction to characterize the relevant topology describing $\mathbb{Z}_n$ defects.

The construction makes manifest that $\mathbb{Z}_n$ defects are conserved modulo n. Indeed, n $\mathbb{Z}_n$ defects correspond to a fluxbrane with trivial monodromy: going around it once implies moving n times around $\mathbb{S}^1$ in the mapping torus. The configuration can be trivialized by a coordinate reparametrization.

9.4.4 Examples

9.4.4.1 Spacetime parity revisited

As an example, consider the realization of a spacetime $\mathbb{Z}_2$ parity in e.g. the bosonic theory. Differently from section 9.3.1, the 27d supercritical geometry has the dimension x^{25} fibered non-trivially along an extra $\mathbb{S}^1$ parametrized by y, forming a Möbius strip. In this non-orientable geometry, spacetime parity is a $\mathbb{Z}_2$ subgroup of a continuous KK $U(1)$. The symmetry breaking is triggered by closed string tachyon condensation. The $\mathbb{Z}_2$ defects of the 26d theory, regions around which spacetime parity flips, can be constructed as quenched fluxbranes with a tachyon condensate (9.10), with the replacement $Y \to Y' = Y - R\varphi$, where φ is the angle in the 2-plane transverse to the defect.

9.4.4.2 $\mathbb{Z}_2$ symmetries of heterotic theories

We can easily implement the mapping torus construction to realize continuous versions of certain discrete symmetries of 10d heterotic theories. In order to have an extra (translational invariant) $\mathbb{S}^1$, rather than an orbifold, we exploit the $HO^{+(1)}$ theory c.f. section A.2. For instance, we can propose a different $U(1)$ embedding[12] of the $\mathbb{Z}_2$ symmetry of the 10d $SO(32)$ theory of section 9.3.2, as follows. To reproduce the $\mathbb{Z}_2$ holonomy along the $\mathbb{S}^1$, the supercritical $HO^{+(1)}$ theory should have an $\mathbb{Z}_2$ Wilson line introducing a -1 phase on $SO(32)$ spinors, e.g.

$$A = \tfrac{1}{R}\,\mathrm{diag}(\,i\sigma_2, 0, \ldots, 0) \quad \to \quad \exp\left(\tfrac{1}{2}\int_{\mathbb{S}^1} i\sigma_2\, dy\right) = -\mathbf{1} \qquad (9.11)$$

where the factor $\frac{1}{2}$ corresponds to the charge of spinors.

Consider a second example, given by the $\mathbb{Z}_2$ symmetry exchanging the two E_8's in the 10d $E_8 \times E_8$ heterotic, i.e. an outer automorphism. The gauge nature of this symmetry,

[12] Incidentally, the same discrete symmetry may have different supercritical embeddings into continuous symmetries. This is similar to embedding the same $\mathbb{Z}_n$ symmetry into different continuous $U(1)$ groups, in fixed dimension.

argued in [273], can be made manifest using the mapping torus construction in the supercritical $E_8 \times E_8$ theory mentioned in section A.2. In this case, we must introduce a permutation Wilson line along the $\mathbb{S}^1$, similar to those in CHL strings in lower dimensional compactifications [274, 275] (see also [276, 85, 84] for permutation Wilson lines in toroidal orbifolds).

9.4.4.3 Discrete isometries of $\mathbb{T}^2$

We now give new examples, based on the discrete isometries of $\mathbb{T}^2$ in section E. To prevent a notational clash, we use $\beta \simeq \beta + 2\pi$ to parametrize the supercritical $\mathbb{S}^1$.

The construction of the mapping torus for $\mathbb{T}^2$ is basically an orbifold of $\mathbb{T}^3 = \mathbb{T}^2 \times \mathbb{S}^1$, by a rotation in $\mathbb{T}^2$ and a simultaneous shift in $\mathbb{S}^1$. They are described by free worldsheet CFTs and are familiar in orbifold constructions (in critical strings), see e.g. [277]. Instead, we recast the construction in a language which will admit an easy generalization to CYs in projective spaces. In order to exploit the power of complex geometry, let us extend $\mathbb{S}^1$ to $\mathbb{C}^* \equiv \mathbb{C} - \{0\}$, by introducing a variable $w = |w|e^{i\beta}$ (which can eventually be fixed to $|w| = 1$ to retract onto $\mathbb{S}^1$). The mapping torus is associated to an elliptic fibration over $\mathbb{C}^*$, with constant τ parameter on the fiber, and suitable $SL(2, \mathbb{Z})$ monodromies around the origin. It turns out that holomorphic fibrations suffice for our purposes. Indeed, the constant τ holomorphic fibrations discussed in the context of F-theory [278, 279], can be readily adapt to the present (non-compact) setup.

Consider a Weierstrass fibration over a complex plane w

$$y^2 = x^3 + f(w)x + g(w) \tag{9.12}$$

Our base space is non-compact, so we do not fix the degrees of the polynomials f, g. From (E.2), a constant τ fibration is achieved by [278]

$$f(w) = \alpha\phi(w)^2, \tag{9.13a}$$
$$g(w) = \phi(w)^3, \tag{9.13b}$$

with $\phi(z)$ some polynomial. The value of τ is encoded in α.

In order to describe the $\mathbb{Z}_2$ in table E, which exists for generic values of τ, we simply choose $\phi(w) = w$, and have

$$y^2 = x^3 + \alpha w^2 x + w^3. \tag{9.14}$$

Moving along $\mathbb{S}^1$ (namely, $w \to e^{i\delta\beta}w$), the coordinates transform as $x \to e^{i\delta\beta}x$, $y \to e^{3i\delta\beta/2}y$. The holonomy along $\mathbb{S}^1$ is $x \to x$, $y \to -y$, precisely the desired $\mathbb{Z}_2$ action.

The construction of $\mathbb{Z}_2$ defects is now straightforward. We simply introduce a complex coordinate z for the two real transverse dimensions, and consider the configuration obtained from (9.14) by the replacement $w \to w + z$. Note that two $\mathbb{Z}_2$ strings are described by a fibration with $\phi = w^2$, which can be made trivial by a reparametrization,

$$y^2 = x^3 + \alpha\, w^4\, x \,+\, w^6 \quad \longrightarrow \quad y'^2 = x'^3 + \alpha x' + 1, \tag{9.15}$$

with $y = w^3 y$, $x = w^2 x$.

A similar discussion can be carried out for the remaining holomorphic $\mathbb{Z}_n$ actions in section E. Skipping further details, we simply quote the relevant fibrations:

$$\mathbb{Z}_4 : y^2 = x^3 + wx, \tag{9.16a}$$

$$\mathbb{Z}_6 : y^2 = x^3 + w, \tag{9.16b}$$

$$\mathbb{Z}_3 : y^2 = x^3 + w^2. \tag{9.16c}$$

These fibrations differ slightly from those in [279], because the latter describe crystallographic actions on global geometries. We note that the local behavior of their fibrations around fixed points on the base is equivalent to ours, modulo reparametrizations describing creation/annihilation of n $\mathbb{Z}_n$ defects.

9.4.4.4 Discrete isometries in CYs: the quintic

The above strategy generalizes easily to more general CYs, as we illustrate for the quintic $\mathbb{X}_6 = \mathbb{P}_5[5]$. Recall its expression (E.3) at the Fermat point,

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0. \tag{9.17}$$

We simply focus on the $\mathbb{Z}_5$ generated by $z_1 \to e^{2\pi i/5} z_1$, with $z_2, \ldots, z_5$ invariant. The fibration associated to the mapping torus can be written

$$w\, z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0. \tag{9.18}$$

Setting $w = e^{i\delta y}$, moving along the $\mathbb{S}^1$ gives $w \to e^{i\delta y} w$ and $z_1 \to e^{-i\delta y/5} z_1$, so that completing the circle implements the desired $\mathbb{Z}_5$ monodromy. The construction of $\mathbb{Z}_5$ strings in the 4d theory amounts to a reinterpretation of w in terms of the transverse coordinates, as in the previous section.

Note the important point that motion in w does not correspond to changing the moduli of $\mathbb{X}_6$. Recall that complex structure moduli are described by deformations of the defining equation corresponding to monomials $\prod_i (z_i)^{n_i}$'s with $n_i < 4$. This means that as one moves around the string there is no physical scalar which is shifting. This is fine because the monodromy is not part of a continuous gauge symmetry acting on any scalar of the 4d theory (as it disappears from the theory in the tachyon condensation).

These discrete isometries are relevant, since they often correspond to discrete R-symmetries of the 4d effective theory. The discussion of possible applications of our tools to phenomenologically interesting discrete R-symmetries is beyond the scope of this thesis.

9.4.4.5 Antiholomorphic $\mathbb{Z}_2$ and CP as a gauge symmetry

A final class of discrete isometries of CY compactifications are given by antiholomorphic $\mathbb{Z}_2$ actions, e.g. $z_i \to \bar{z}_i$, which are large isometries of the CY spaces with defining equations with real coefficients. These are orientation-reversing, and hence are not symmetries of the superstrings, but can be actual symmetries if combined with an extra action. For instance, their combination with 4d parity gives a discrete symmetry, which in heterotic compactifications corresponds to a CP transformation [280]. Applying the mapping torus construction to this $\mathbb{Z}_2$ symmetry (i.e. combining ingredients of the previous section and section 9.4.4.1) results in a description of CP as a discrete gauge symmetry explicitly embedded in a (supercritical) $U(1)$ symmetry. This is a new twist in the history of realizing CP as a gauge symmetry, see e.g. [273, 281].

9.4.4.6 $\mathbb{Z}_n$ symmetries already in $U(1)$ groups

Although we have focused on discrete symmetries from (large) isometries, the constructions can be applied to general $\mathbb{Z}_n$ discrete symmetries, even those embedded in continuous $U(1)$ factors already in the critical string theory (see [140, 142, 143, 144, 282, 149, 148] for such symmetries in string setups). Focusing on the 4d setup for concreteness, recall from section 3.1 that the key ingredient is a $U(1)$ group with potential A_1, acting on a real periodic scalar $\phi \simeq \phi + 1$ as

$$A_1 \longrightarrow A_1 + d\lambda, \tag{9.19a}$$

$$\phi \longrightarrow \phi + n\lambda. \tag{9.19b}$$

The $U(1)$ is broken, with ϕ turning into the longitudinal component of the massive gauge boson. But there is an unbroken $\mathbb{Z}_n$, preserved even by non-perturbative effects. For instance, gauge invariance forces the amplitude of an instanton at a point P to be dressed as

$$e^{-2\pi i\phi} \exp(2\pi in \int_L A_1) \tag{9.20}$$

which describes the emission of electrically charged particles of total charge n (i.e. preserving the $\mathbb{Z}_n$) along semi-infinite worldlines L starting at P.

Let us now consider embedding this $\mathbb{Z}_n$ symmetry as a mapping torus construction in a supercritical extension of the theory. Along the extra $\mathbb{S}^1$ there is a non-trivial $U(1)$ transformation (integrating to the $\mathbb{Z}_n$ generator) and a corresponding shift $\phi \to \phi + 1$. In other words there is one unit of flux for the field strength 1-form $F_1 = d\phi$

$$\int_{\mathbb{S}^1} F_1 = 1. \tag{9.21}$$

Notice that the mapping torus of length $2\pi R$ and the n-cover circle of length $2\pi nR$ (c.f. section 9.4.2) provide a physical realization of the two $\mathbb{S}^1$'s in section 3.3.1 associated to the periodicity of ϕ and of the $U(1)$.

Fields with charge q under the $\mathbb{Z}_n$ have $\mathbb{S}^1$ boundary conditions twisted by $e^{2\pi i\, q/n}$, hence have KK momenta $k + q/n$, with $k \in \mathbb{Z}$, and so carry charge under the KK $U(1)$. This piece allows to recover (9.20) in this picture, as follows. The operator $e^{-2\pi i\phi}$ at a point in the $\mathbb{S}^1$ (and at point P in the critical spacetime) picks up phase rotations under translation, i.e. under a KK $U(1)$ transformation, which must be cancelled by those insertions of KK modes, with total KK momentum 1, e.g. n states of minimal $\mathbb{Z}_n$ charge.

The discussion in the previous paragraph has a nice string theory realization in the context of $\mathbb{Z}_n$ symmetries arising from the $U(1)$ gauge groups on D-branes. In particular we focus on D6-brane models c.f. [140], where A_1 is the gauge field on D6-branes, ϕ is the integral of the RR 3-form C_3 over some 3-cycle Σ_3, and the instanton is an euclidean D2-brane on Σ_3. The non-trivial shift of ϕ, namely the flux (9.21), corresponds to a 4-form field strength flux

$$\int_{\Sigma_3 \times \mathbb{S}^1} F_4 = 1 \tag{9.22}$$

The D2-brane instanton on Σ_3 is not consistent by itself, but must emit particles with one unit of total KK momentum. This is more clear in the T-dual picture, which contains a D3-brane on $\Sigma_3 \times \mathbb{S}^1$ with one unit of F_3 flux over Σ_3, which must emit a fundamental string with one unit of winding charge [219].

9.5 The non-Abelian case

In this section we will generalize the above ideas to the non-Abelian case. First we will show an intuitive generalization of the mapping torus construction in section 9.4.1, and present an explicit example of how it works for a discrete Heisenberg group. Afterwards, we will present a more formal description of the method in terms of cosets; subsequently, we will apply this general framework description, which has a broad applicability, to the discrete Heisenberg group.

9.5.1 Mapping torus generalization

Consider an N-dimensional theory $\mathbb{X}_N$ with a discrete gauge symmetry group Γ (in general, not realized as a subgroup of a broken continuous symmetry). We would like to add extra dimensions in a supercritical extension of the theory, such that Γ is embedded in a continuous non-abelian group G.

A possibility is that G is an isometry group acting on the extra dimensional space $\mathbb{Y}$, as $g : y \to g(y)$. We may consider the trivial product $\mathbb{X}_N \times \mathbb{Y}$ and quotient by the simultaneous action of the subgroup Γ on the theory $\mathbb{X}_N$ and the space $\mathbb{Y}$, namely $\gamma : (x, y) \to (\gamma(x), \gamma(y))$, for $\gamma \in \Gamma$. If the action of G on $\mathbb{Y}$ leaves no fixed points, the quotient generalizes the mapping torus construction in section 9.4.2, in the sense that non-trivial loops in the quotient define circles along which the theory $\mathbb{X}_N$ is twisted by the action of an element of Γ. Subsequently turning on a non-trivial tachyon background invariant under Γ (and hence with zeroes of the worldsheet potential related by Γ) would truncate the theory back to the critical theory $\mathbb{X}_N$ with the desired symmetry breaking.

A simple example of this construction is obtained by considering $\mathbb{Y}$ to be the group manifold G itself, with action given by e.g. right multiplication; but any other $\mathbb{Y}$ on which G acts transitively suffices. A more important and subtle point is that the action of G on $\mathbb{Y}$ defined above does not descend in general to a globally well-defined action in the quotient. However, if Γ is a normal subgroup[13] of G, then a globally well defined action in the quotient can be constructed (see section 9.5.2). Unfortunately, a normal discrete subgroup of a path-connected Lie group G necessarily belongs to the center of G, and is therefore abelian. Hence, we cannot embed the non-abelian discrete symmetry in a continuous isometry/symmetry acting on the extra dimensions, broken by tachyon condensation when truncating to the critical theory.

Configurations with local group actions, which are however not symmetries of the system have appeared in the context of discrete gauge symmetries in [143]. They describe a non-abelian continuous symmetry which is broken and has become massive by gauging a set of scalars. This broken continuous symmetry may have an unbroken discrete subgroup, which manifests as a discrete gauge symmetry of the theory. This perspective is useful to deal with the non-abelian discrete symmetry Γ of the theory $\mathbb{X}_N$, and its embedding into a continuous group G acting on the quotient $(\mathbb{X}_N \times \mathbb{Y})/\Gamma$. This construction provides an embedding of Γ into a continuous symmetry which is broken and made massive by a process of gauging. The tachyon condensation would then truncate the gauged theory to the critical spacetime slice, triggering no additional symmetry breaking. Note that the

[13] Recall that a group N is a normal subgroup of G, written $N \lhd G$, if $g^{-1}ng \in N$ for all $n \in N$, $g \in G$.

construction in section 9.4.4.6 can be regarded as a particular realization of this idea in the abelian case.

9.5.1.1 Discrete Heisenberg group from supercritical magnetized tori

It is easy to provide concrete examples of this realization; in this section we present an embedding of a discrete Heisenberg group in terms of a supercritical extension with magnetized $\mathbb{T}^2$, in a setup close to [143].

Consider the theory $\mathbb{X}_N$ to have a $U(1)$ gauge symmetry and a discrete gauge symmetry, generated by two order-n elements A, B (with $A^n = B^n = 1$) which commute to an element of $U(1)$, that is $AB = CBA$ with $C \in U(1)$ (note that C is required to be of order n as well). This is a discrete Heisenberg group, which we denote by H_n. We assume that H_n is not embedded into any continuous (massive or not) symmetry of $\mathbb{X}_N$.

Consider now a supercritical extension of the theory with two extra dimensions (two plus two for type 0 extensions of type II models), which parametrize a $\mathbb{T}^2$ (for simplicity taken square with unit length coordinates x, y). In analogy with the mapping torus construction, we specify that the theory $\mathbb{X}_N$ picks up the action of the generators A, B, as one moves along the two fundamental cycles of $\mathbb{T}^2$. This embeds the two discrete generators into continuous translational $U(1)$ actions, the would-be KK gauge symmetries of the theory. These symmetries are however broken, even before the process of tachyon condensation. To see this, note that moving around the whole $\mathbb{T}^2$ results in an action $ABA^{-1}B^{-1} = C$, namely there is a circulation of the $U(1)$ gauge potential. This implies that there is a non-trivial $U(1)$ magnetic field (with n units of flux [143]) on the $\mathbb{T}^2$. Although the field strength can be taken constant and the configuration seems translational invariant, the gauge potential can be written

$$\mathcal{A} = \pi n(xdy - ydx), \tag{9.23}$$

so that translations in x imply a change in the Wilson line of $\mathcal{A}$ along y, and viceversa. This can be regarded as an action of the Kaluza-Klein $U(1)$'s on the Wilson line scalars, which define a gauging that breaks the continuous symmetry and makes the Kaluza-Klein gauge bosons massive [143, 148]. There is however an unbroken $H_n \rtimes U(1)$ symmetry, corresponding to the original group of $\mathbb{X}_N$.

We may subsequently turn on a non-trivial periodic tachyon profile to remove the extra dimensions, and truncate the dynamics to the origin. For instance, we may take the supercritical heterotic theory $HO^{+(2)}$ with $T^{33} \sim \sin(\pi x)$, $T^{34} \sim \sin(\pi y)$. We emphasize that the tachyon background does not lead to any additional breaking of the symmetry. It should be straightforward to find other examples of non-Abelian discrete symmetries embedded in continuous massive symmetries.

9.5.2 General framework

Consider a theory $\mathcal{M}$, with a (right-acting) discrete symmetry group Γ, which we want to embed as part of a supercritical theory. We also have a group G with a normal subgroup N. As a first step in the construction, we build the trivial fiber bundle $E \equiv G \times \mathcal{M}$, with canonical projection map onto the first factor $\pi(g, m) = g$. There is a canonical action A of G, so that for each $g' \in G$, $A(g') : (g, m) \to (g'g, m)$.

We now pick a homomorphism $\phi : N \to \Gamma$ and for each $n \in N$ consider the action $F_n : (g, m) \to (ng, m \cdot \phi(g^{-1}ng))$. We quotient $E = G \times \mathcal{M}$ by the equivalence relation $e_1 \sim e_2$ if $F_n(e_1) = e_2$ for some $n \in N$. The quotient, denoted by E/F, has a natural projection π' onto G/N with preimage $\mathcal{M}$, so the configuration is a non-trivial fiber bundle over G/N. Furthermore, thanks to the normality of N, the action $A(g)$ descends to a well-defined action in the quotient (namely, the images under $A(g)$ of F-equivalent points are F-equivalent).

The topology of the bundle is specified by the holonomies around non-trivial loops in G/N. Consider a one-parameter curve $g(t)$ in G, with $t \in [0, 1]$, going from the identity to some $n \in N$, namely $g(0) = 1$, $g(1) = n$. This descends to a closed loop in G/N, with a non-trivial action on the fiber $\mathcal{M}$. In particular, a point of E/F with representative $(g, m) \in E$ comes back as the point $(ng, m) \in E$, which is in the class of $(g, m \cdot \phi(g^{-1}n^{-1}g))$ in E/F. Namely, the fiber suffers a monodromy given by an element in $\phi(N) = \Gamma$, the discrete symmetry group. The construction succeeds in embedding this symmetry as part of the continuous group G acting on the coset G/N. Notice however, that the continuous group may act as a non-trivial shift of scalars, resulting in a gauging which makes the symmetry massive.

The last step would be to introduce a tachyon restricting the dynamics to the identity class in G/N. This is difficult to describe in general, but can be worked out in detail in examples.

9.5.2.1 The Heisenberg group

We now describe a particular example, which eventually corresponds to the Heisenberg group in section 9.5.1.1. This illustrates some of the ingredients ultimately leading to massive continuous symmetries.

We take G to be the Heisenberg group $H_3(\mathbb{R})$ and introduce the normal subgroup N, which has a normal subgroup N_N, i.e. $N_N \lhd N \lhd G$. They are defined by the sets of matrices of the form

$$G = \begin{pmatrix} 1 & x & z + \frac{xy}{2} \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}, \tag{9.24a}$$

$$N = \begin{pmatrix} 1 & n_x & h_z + \frac{n_x n_y}{2} \\ 0 & 1 & n_y \\ 0 & 0 & 1 \end{pmatrix}, \tag{9.24b}$$

$$N_N = \begin{pmatrix} 1 & n_x & n_z + \frac{n_x n_y}{2} \\ 0 & 1 & n_y \\ 0 & 0 & 1 \end{pmatrix}, \tag{9.24c}$$

with $x, y, z, h_z \in \mathbb{R}$, $n_x, n_y, n_z \in \mathbb{Z}$. The quotients are $G/N \simeq \mathbb{T}^2$, $N/N_N \simeq U(1)$.

We consider a theory $\mathcal{M}$ with a symmetry $H_n \rtimes U(1)$, with H_n a discrete Heisenberg group. The action of N on this theory (the homomorphism ϕ above) includes a $U(1)$ gauge transformation with parameter $e^{2\pi i h_z}$. Incidentally, note that the homomorphism ϕ has a non-trivial kernel, given by N_N. We now take the product $G \times \mathcal{M}$, and quotient by the equivalence relation $(g, e^{2\pi i\theta}) \sim (hg, m \cdot \phi(g^{-1}hg), e^{2\pi i(h_z+\theta)})$, where the last entry describes the gauge $U(1)$ fiber, and m represents other sectors of the $\mathcal{M}$ on which ϕ acts. The resulting quotient space is a non-trivial fiber bundle over $\mathbb{T}^2$. The non-trivial glueing is manifest in the identifications $(x+1, y, z+\frac{y}{2})$, $(x, y+1, z-\frac{x}{2})$, so the $U(1)$ fiber transforms as $e^{2\pi iz} \to e^{2\pi iz}e^{i\pi y}$ under $x \to x+1$, and as $e^{2\pi iz} \to e^{2\pi iz}e^{-i\pi x}$ under $y \to y+1$. A connection on this bundle must satisfy $\mathcal{A}(x+1, y) = A(x, y) - \pi dy$, $\mathcal{A}(x, y+1) = A(x, y) + \pi dx$, which imply that the solutions carry a nonzero magnetic flux.

The action of G does not correspond to a true symmetry of the background, as follows. On $G \times \mathcal{M}$, it takes $(g, e^{2\pi i\theta})$ to $(g'g, e^{2\pi i\theta})$. To find the action on the quotient, we take without loss of generality $g = (x, y, 0)$ and $g' = (a, b, c)$, and have

$$(g'g, e^{2\pi i\theta}) \;=\; \big(x+a, y+b, c+\frac{1}{2}(ay-bx), e^{2\pi i\theta}e^{\pi i(-2c+bx-ay+(a+x)y-(b+y)x)}\big). \tag{9.25}$$

It maps fibers at different points, and also acts by moving along the fiber, i.e. translations plus gauge transformations, as usual in the magnetized torus. The true symmetries are given by the transformations $A(n)$ for $n \in N$, which are precisely $H_n \rtimes U(1)$, recovering the results in [143, 148].

10
Conclusions

In this work we have explored different ways Abelian and non-Abelian discrete gauge symmetries can be realized in the context of string theory. We have also discussed how the discrete gauge symmetries can be embedded into continuous groups.

In chapter 3 we have reviewed discrete gauge symmetries in 4d field theory. We have analyzed two different but equivalent ways to study discrete gauge symmetries, the BF formulation and discrete symmetries as gaugings of isometries of the moduli space of the scalars in the theory.

In chapter 4 we have studied how to realize discrete gauge symmetries in intersecting D-brane models. We have shown that in semi-realistic (MS)SM type II orientifold constructions (sections 4.3 and 4.4) discrete gauge symmetries arise in a natural way, in the form of Z_n subgroups of continuous $U(1)$'s of the models. The list of the possible discrete gauge symmetries is limited, and it agrees with the anomaly-free classification of discrete gauge symmetries in [121]. In particular, we can easily get:

- The discrete groups R_N, which arise as subgroups of $U(1)_{B-L}$, with R_2 being R-parity.
- The baryon triality Z_3 generated by $B_3 = R_3 L_3$.
- The lepton triality L_3.
- The $R_3 L_3^2$ symmetry.

All these symmetries forbid proton decay through dimension four operators, but only R-parity and baryon triality allow for neutrino Majorana masses, which makes them phenomenologically preferred.

We have also explored the realization of the Z_2 R-parity from different sources, like the existence of instanton sectors with minimal instanton number 2 (section 4.5), and constraints from cancellation of K-theory charge (section 4.6).

In chapter 5 we extend the realization in [141, 138] of discrete gauge symmetries from NSNS and RR p-form fields in compactifications with torsion homology. In particular, we have shown that dimensional reduction of type IIB supergravity on a manifold with torsion produces the 4d Lagrangian for non-Abelian discrete gauge symmetries constructed in section 3.3.

In chapter 6 we have studied wrapped branes in the presence of non-trivial background fluxes. While they are wrapped on homologically non-trivial $\mathbb{Z}$-valued cycles, there are flux-induced topological effects associated to the Freed-Witten anomaly that render their physical charges $\mathbb{Z}_p$ valued rather than $\mathbb{Z}$-valued, indicating that homology is not the right mathematical tool to classify brane charges. It was already known that D-brane charges must be classified by K-theory (in the absence of fluxes), or twisted K-theory (in the presence of NSNS 3-form flux); since we also consider the presence of RR p-form fluxes, the groups that are obtained can be considered a generalization of twisted K-theory, and their physical construction provides an interesting way to explore the mathematical formulation of such groups.

We have shown in sections 6.1 and 6.2 that in the cases with only one kind of background fluxes, for $\mathbb{Z}_p$-charged particles and strings there is an underlying $\mathbb{Z}_p$ discrete gauge symmetry that arises from the 10d Chern-Simons couplings. He have also constructed systems where the discrete group classifying brane charges is non-Abelian in section 6.3.

We have also discussed $\mathbb{Z}_p$-valued domain walls (section 6.5) and their relation to string duality symmetries relating vacua with different flux quanta.

In chapter 7 we have considered discrete symmetries arising from discrete isometries of the compactification manifold, exemplified by twisted tori compactifications (sections 7.1 and 7.2).

As an application of these ideas to semi-realistic models, we have focused on magnetized D-brane models (section 7.3), where the continuous isometries of the compactification manifold are broken into a non-Abelian discrete subgroup by the presence of the magnetic fluxes. These discrete groups typically have a Heisenberg-like structure, with generators associated to discrete isometries of the torus geometry, commuting to discrete symmetries generated by the D-brane $U(1)$'s. The symmetry has been derived microscopically by analyzing charged matter wave functions, and from dimensional reduction.

We have also shown that these symmetries imply powerful selection rule on the Yukawa couplings of charged matter fields, including those observed in [74, 75] (and their interpretation in [244, 245]), which are thus exact even at the non-perturbative level (section 7.4).

In chapter 8 we have considered discrete gauge symmetries remaining from broken continuous gauge symmetries carried by general antisymmetric tensor fields in arbitrary dimensions. We have described the field theory for these general Z_p gauge theories, generalising the analysis in chapter 3. In section 3.3 we saw that the language of gaugings in supergravity is an elegant way to describe 1-form gauge symmetries broken by scalars; it would be interesting to develop such a description of the higher-rank case.

We have also studied several Abelian and non-Abelian realizations in string theory, for instance, in compactifications with torsion cycles, or in compactifications with non-trivial background fluxes using the mechanism of flux catalysis explained in chapter 6. In particular, we have generalized the 4d analysis of section 6.2 to the 6d case.

In chapter 9 we have shown how genuinely discrete gauge symmetries in string theory can be embedded into continuous symmetries, by extending the theories beyond the critical dimension and using the closed string tachyon condensation. We have discussed several different examples of such symmetries and embeddings.

We have presented two different ways to realize the embedding, dubbed 'quenched rotations' (section 9.3) and 'quenched translations' (section 9.4). We have also shown that the stabilization of discrete gauge symmetries as quenched translations allows us to make contact with the alternative description of discrete gauge symmetries as a sum over disconnected theories [176].

We have also discussed several aspects of how the realization of discrete gauge symmetries as quenched translations can be generalized to non-Abelian discrete symmetries, and worked out the case of a discrete Heisenberg group; we have seen that we usually require an additional gauging mechanism to break the continuous group into the discrete subgroup.

All these embedding of discrete symmetries into continuous ones would be pointless if we would not gain anything from them. In fact, one of the main results of this chapter is that we can construct charged topological defects as closed string tachyon solitons. This is reminiscent of the stabilization of D-branes as open string tachyon solitons, and K-theory. However in our case the underlying mathematical structures of closed string tachyon condensation and its topological solitons are not well known; therefore, one line of investigation that should be pursued is the clarification of such structures. One of the first steps would be to construct objects that are already known in string theory as solitons of closed string tachyons, in analogy with D-branes as open string tachyon solitons. There has been recent work in this direction showing that heterotic NS5-branes can be obtained from closed string tachyon condensation [283].

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Conclusiones

En esta tesis hemos explorado distintas maneras en las que simetrÃas gauge discretas abelianas y no abelianas pueden realizarse en el contexto de teoría de cuerdas. También hemos discutido cómo las simetrías gauge discretas pueden encajarse en grupos continuos.

En el capítulo 3 hemos repasado las simetrías gauge discretas en teoría de campos en cuatro dimensiones. Hemos analizado dos maneras diferentes pero equivalente de estudiar las simetr'ias gauge discretas, la formulación BF y simetrías discretas provenientes de gaugings de las isometrías del espacio de moduli de los escalares de la teoría.

En el capítulo 4 hemos estudiado cómo obtener simetrías gauge discretas en modelos de D-branas intersecantes. Hemos mostrado que en construcciones semi-realistas de orientifolds de supercuerdas de tipo II que dan lugar al Modelo Estándar o a su versión mínimamente supersimétrica (secciones 4.3 y 4.4), las simetrías gauge discretas surgen de manera natural, en la forma de subgrupos $\mathbb{Z}_n$ de los $U(1)$ continuos de los modelos. La lista de posibles simetrías gauge discretas es limitada, y está de acuerdo con la clasificación de simetrías gauge discretas libres de anomalías en [121]. Más concretamente, podemos obtener fácilmente simetrás como estas:

- Los grupos discretos R_N, que surgen como subgrupos de $U(1)_{B-L}$, con R_2 siendo paridad R.
- La trialidad bariónica Z_3 generada por $B_3 = R_3 L_3$.
- La trialidad leptónica L_3.
- La simetría $R_3 L_3^2$.

Todas estas simetrías prohíben la desintegración del protón a través de operadores de dimensión cuatro, pero solo la paridad R y la trialidad bariónica permiten masas de neutrino de tipo Majorana, por lo que son las opciones preferidas desde el punto de vista fenomenológico.

También hemos explorado la posibilidad de obtener el $\mathbb{Z}_2$ de paridad R a partir de distintas fuentes, como la existencia de sectors de instantones con un número de instantón mínimo igual a dos (sección 4.5), y restricciones que provienen de la cancelación de cargas de teoría K (sección 4.6).

En el capítulo 5 hemos extendido la obtención de simetrías gauge discretas a partir de campos de p-forma NSNS y RR en compactifications con homología de torsión en [141, 138]. En particular, hemos mostrado que reducción dimensional de supergravedad tipo IIB en una variedad con torsión produce el lagrangiano en cuatro dimensiones para simetrías discretas no abelianas construído en la sección 3.3.

En el capítulo 6 hemos estudiado branas enrolladas en la presencia de flujos de fondo no triviales. Aunque están enrolladas en ciclos $\mathbb{Z}$ valuados homologicamente no triviales, existen efectos topológicos inducidos por los flujos asociados a la anomalía de Freed-Witten que vuelven sus cargas físicas $\mathbb{Z}_p$ valuadas en vez de $\mathbb{Z}$ valuadas, indicando que homología no es la herramienta matemática adecuada para clasificar las cargas de las branas. Ya se sabía de antes que las cargas de D-brana han de clasificarse usando teoría K (en ausencia de flujos), o de teoría K retorcida (en la presencia de flujo de 3-forma NSNS); puesto que nosotros también consideramos la presencia de flujos de p-forma RR, los grupos obtenidos pueden considerarse como una generalización de teoría K retorcida, y su cosntrucción física proporciona una manera interesante de explorar la formulación matemática de tales grupos.

En las secciones 6.1 y 6.2 hemos mostrado que en casos con un solo tipo de flujos de fondo, para partículas y cuerdas con cargas $\mathbb{Z}_p$ valuadas existe una simetría $\mathbb{Z}_p$ subyacente que emerge de los acoplos de Chern-Simons en diez dimensiones. También hemos construído sistemas donde el grupo discreto que clasifica las cargas de las branas es no abeliano en la sección 6.3.

También hemos discutido sobre paredes de dominio $\mathbb{Z}_p$ valuadas (sección 6.5) y su relación con simetrías de dualidad en teoría de cuerdas relaciionando vacíos con distintos cuantos de flujo.

En el cap'itulo 7 hemos considerado las simetrías discertas que surgen de isometrías discretas de la variedad de compactificación, ejemplificándolo con el caso de compactificationes en toros retorcidos (secciones 7.1 y 7.2).

Como una aplicación de estas ideas a modelos semi-realistas, nos hemos centrado en modelos de D-branas magnetizadas (sección 7.3), donde las isometrías continuas del espacio de compactificación están rotas a un subgrupo discreto no abeliano por la presencia de flujos magnéticos. Estos grupos discretos tienen típicamente una estructura de tipo Heisenberg, con generadores asociados a isometrías discretas de la geometría del toro, conmutando a isometrías discretas generadas por los $U(1)$ de las D-branas. La simetría ha sido dervidada microscópicamente analizando las funciones de onda de la materoa cargada, y a partir de reducción dimensional.

También hemos mostrado que estas simetrías implican poderosas reglas de selección en los acoplos de Yukawa de los campos de materia cargados, incluyendo los observados en [74, 75] (y su interpretación en [244, 245]), que son por lo tanto exactos incluso a nivel no perturbativo (sección 7.4).

En el capítulo 8 hemos considerado simetrías gauge discretas que quedan como remanentes de simetr'ías gauge continuas rotas asociadas a campos tensoriales antisimétricos generales en un número arbitrario de dimensiones. Hemos descrito la teoría de campos para estas teorías gauge $\mathbb{Z}_p$ generales, generalizando el análisis del capítulo 3. Sería interesante el poder desarrollar una descripción de estas simetrías discretas asocaidas a tensores de alto rango análoga a la elegante descripción de simetrías gauge asociadas a 1-formas rotas por escalares en el lenguaje de gaugings en supergraviedad de la sección 3.3.

También hemos estudiado diversas maneras de obtener estas simetrías, tanto abelianas como no abelianas, en teoría de cuerdas, por ejemplo, en compactificaciones con ciclos de torsión, o en compactifications con flujos de fondo no triviales usando el mecanismo de catálisis de flujo (flux catalysis) explicado en el capítulo 6. Concretamente, hemos generalizado el análisis en cuatro dimensiones de la sección 6.2 al caso de seis dimensiones.

En el capítulo 9 hemos mostrado cómo simetrías gauge discretas auténticas pueden ser encajadas en simetrías continuas, extendiendo las teorías más allá de la dimensión crítica y usando condensación de taquiones de cuerda serrada. Hemos discutido diversos ejemplos de tales simetrías y encajes.

Hemos presentado dos maneras diferentes de llevar a cabo el encaje, denominadas 'quenched rotations' (section 9.3) y 'quenched translations' (section 9.4). También hemos mostrado que la obtención de simetrías gauge discretas a través de 'quenched translations' nos permite conectar con la descripción alternativa de simetrías gauge discretas como una suma sobre teorías inconexas [176].

También hemos discutido diversos aspectos sobre cómo la obtenci
on de simetrías gauge discretas a través de 'quenched translations' se piede generar a simetrías discretas no abelianas, y hemos estudiado el caso particular de un grupo de Heisenberg discreto; hemos visto que generalmente se requiere un mecanismo de gauging extra para romper el grupo continuo al subgrupo discreto.

Todos estos encajes de simetrías discretas en continuas no tendrían sentido si no fuéramos capaces de beneficiarnos de algún modo. Uno de los principales resultados de este capítulo es que podemos construir defectos topológicos cargados como solitones de taquiones de cuerda cerrada. Esto evoca la realización de D-branas como solitiones de taquiones de cuerda abierta, y teoría K. Sin embargo, en nuestro caso las estructuras matemáticas subyacentes de la condensación de taquiones de cuerda cerrada y sus solitiones topológicos no son conocidas plenamente; por lo tanto, una línea de investigación a seguir sería la clarificación de dichas estructuras. Uno de los posibles primeros pasos sería construir objetos que ya se conocen en teoría de cuerdas a través de condensación de taquiones de cuerda cerrada, en analog'ía con las D-branas y la condensación de taquiones de cuerda abierta. Trabajos recientes en este sentido has conseguido obtener las NS5-branas heteróticas a partir de condensaciones de cuerda cerrada [283].

A

Supercritical string theories

The 26d bosonic string theory, 10d type II, type I, type 0 and heterotic string theories, are the so-called *critical* string theories; in the conformal field theory language, it means that the number of dimensions of those theories is such that the total central charge is zero, with the matter central charge being given by

$$c_{matter} = D = 26, \tag{A.1a}$$

$$c_{matter} = \frac{3}{2}D = 15, \tag{A.1b}$$

for the bosonic and superstring theories, respectively.

If one introduces a linear background for the dilation,

$$\phi = \phi_0 + V_\mu X^\mu, \tag{A.2}$$

the expression for the matter central charge is given by

$$c_{matter} = D + 6\alpha' V_\mu V^\mu, \tag{A.3a}$$

$$c_{matter} = \frac{3}{2}D + 6\alpha' V_\mu V^\mu, \tag{A.3b}$$

for the bosonic and superstring theories, respectively. Therefore, with an appropriate dilaton background we can get theories with a number of dimensions different from the critical one (26 for bosonic and 10 for superstring theory). These theories are called *non-critical* and were first considered in [284]. They are separated into *subcritial* string theories if the number of dimensions is lower than the critical one, and *supercritical* string theories if it is higher.

In chapter 9 we will use supercritical string theories to be able to embed discrete symmetries into continuous ones, so let us review them here[1].

Supercritical string theories are defined by generalising the worldsheet content of the 26d bosonic string, or 10d superstring, to D dimensions (with extra changes for supercritical heterotics), and introducing an appropriate linear dilaton background, which we take to be timelike (i.e. choose coordinates such that $V_\mu = 0$ for $\mu \neq 0$) to produce supercritical theories rather than subcritical ones.

[1] For a more detailed analysis of supercritical string theories see [254, 255, 256, 257, 258].

As usual, consistent theories must fulfil the requirement of modular invariance, with leads to different supercritical theories, discussed below. The pattern of GSO-like projections (when required) determines the space-time spectrum, in particular massless[2] and tachyonic fields. An important aspect of all these theories is that closed string tachyons are ubiquitous.

Unlike open string tachyons, closed string instabilities are less understood (see e.g. [285, 286, 287, 288, 289, 290] for some discussions). Fortunately, there is a quite precise description of the different effects of closed string tachyon condensation in supercritical strings, which is even quantitative for light like tachyon profiles [255, 256, 257, 258][3]. In particular, they have been shown to trigger a reduction in the number of dimensions [256], dubbed 'dimension quenching' [258].

The standard 26d bosonic string theory, the 10d $SO(32)$ heterotic, and the 10d type 0 and type II superstrings can be regarded as the endpoint of closed string tachyon condensation of suitable supercritical string theories.

A.1 Supercritical bosonic strings

The supercritical bosonic string in D-dimensional space-time is defined by D worldsheet bosons X^M, $M = 0, \ldots, D-11$, and an appropriate timeline linear dilaton. As in the familiar 26d theory, the light spectrum contains a (real) closed string tachyon $T(X)$, abad massless graviton, 2-form and dilaton fields, G_{MN}, B_{MN}, ϕ.

The modular invariant partition function for the supercritical bosonic string in D dimensions is

$$Z(\tau) = \frac{(4\pi^2\alpha'\tau_2)^{-\frac{D-2}{2}}}{|\eta|^{2(D-2)}}. \tag{A.4}$$

Since the tachyon vertex operator is the identity, a tachyon background couples as a world sheet potential. Tachyon condensation can be followed quantitatively for specific 'light-like' profiles, in which the tachyon $T(X^)$ depends on the direction X^+. It describes the dynamics close to the boundary of an expanding bubble interpolating between two vacua: the 'parent' with no tachyon and the endpoint of tachyon condensation.

We will focus on tachyon profiles describing the disappearance of space-time dimensions, see [256] for additional details. To remove e.g. one dimension $Y \equiv X^{D-1}$, we deform the worldsheet action by a superposition of conformal operators

$$T(X^+, Y) = \mu_0^2 \exp\left(\beta X^+\right) - \mu_k^2 \cos\left(kY\right) \exp\left(\beta_k X^+\right). \tag{A.5}$$

The parameters μ_0, μ_k are tuned to achieve a stationary (yet not stable, in this bosonic case) endpoint, while β and β_k are fixed to make the operators marginal, i.e. to satisfy the tachyon space-time equations of motion

$$\partial^\mu\partial_\mu T - 2V^\mu\partial_\mu T + \frac{4}{\alpha'}T = 0. \tag{A.6}$$

[2]Since the dilaton background breaks Poincaré invariance, one should be careful in talking about the mass. We follow the convention in [254] of meaning the mass term arising in the equations of motion of the space-time field.

[3]For other works on light-like tachyon condensation, see [291, 292, 293].

The theory simplifies in the limit $k \to 0$ (i.e. the wavelength of the tachyon k^{-1} is long compared to the string scale) while keeping fixed $\alpha' k^2 \mu_k^2 \equiv \mu^2$ and $\mu'^2 \equiv \mu_0^2 - \mu_k^2$. The tachyon profile becomes

$$T(X^+, Y) = \frac{\mu^2}{2\alpha'} \exp\left(\beta X^+\right) Y^2 + T_0(X^+), \tag{A.7}$$

where we have defined

$$T_0(X^+) = \frac{\mu^2 X^+}{\alpha' q \sqrt{2}} \exp\left(\beta X^+\right) + \mu'^2 \exp\left(\beta X^+\right), \tag{A.8}$$

with

$$q\beta = \frac{2\sqrt{2}}{\alpha'}. \tag{A.9}$$

The operator $T_0(X^+)$ should be thought as of representing tachyon condensation along a lightlike direction in $D-1$ space-time dimensions. Then, the second term in (A.8) represents a mode $\exp(\beta X^+)$ of the tachyon with amplitude μ'^2, which can be fine-tuned to zero by setting μ'^2 to vanish. The meaning of the first term in (A.8) is less transparent; however, it can be shown [256] that this term is precisely cancelled by the quantum effective potential generated upon integrating out the Y field. Therefore, the tachyon profile essentially becomes

$$T(X^+, Y) = \frac{\mu^2}{2\alpha'} \exp\left(\beta X^+\right) Y^2. \tag{A.10}$$

At $X^+ \to -\infty$ we have the parent bosonic theory in D dimensions and vanishing tachyon profile, while at $X^+ \to \infty$ the strings are pinned at $Y = 0$, so space-time effectively loses one dimension. The quantum correction produced when we integrate out the world sheet field Y at late X^+ readjusts the metric and dilaton background, correctly accounting for the change in central charge [256].

The bottom line is that the dimension Y disappears via closed string tachyon condensation. It is straightforward to generalize to the disappearance of several dimensions, and in particular to decay down to the familiar 26d bosonic string theory.

The general lesson about closed string tachyon condensation is the reduction of space-time dimensions onto the locus of the vanishing tachyon. This lesson can be applied to more general tachyon profiles even if the corresponding worldsheet theory is not exactly solvable; in other words, the quadratic profile (A.10) is a local approximation for any tachyon profile sufficiently near a simple zero, around which the background is otherwise trivial (except for the linear dilaton). From this perspective, the trigonometric profile (A.5) would lead to a periodic array of zeroes, and the purpose of the $k \to 0$ limits to decouple them and extract an isolated zero.

A.2 Supercritical heterotic strings

In this section we will review the supercritical heterotics strings in $D = 10 + n$ dimensions, dubbed $HO^{+(n)}$ and $HO^{+(n)/}$ in [254]. In both, the world sheet theory generalizes the

10d $SO(32)$ heterotic in its fermionic formulation. There are D (left- and right-moving) bosons X^M, $M = 0, \ldots, D-1$, and D right-moving fermions $\tilde{\psi}^M$, all in the vector representation of the $SO(1, D-1)$ space-time Lorentz group. In addition, there are $(32+n)$ left-moving fermions, which we split as λ^1, $a = 1, \ldots, 32$ and χ^m, $m = 1, \ldots, n$ (the latter are required to cancel the 2d gravitational anomaly). Finally, a timelike linear dilaton is introduced to achieve the correct matter central charges. The two theories differ in their GSO projections, and their properties are discussed separately in the following.

A.2.1 $HO^{+(n)}$ theory

In the $HO^{+(n)}$ theory there are two GSO-like projections, which can be described as orbifolds by the $\mathbb{Z}_2$ actions g_1, g_2 in table A.1. The GSO g_1 acts on the 32 left-moving fermions as in the 10d $SO(32)$ heterotic string, while g_2 is an extension of the standard GSO projection in the right-moving sector (as recovered by 'forgetting' the $\tilde{\psi}^m$, χ^m).

Field	g_1	g_2
X^μ	+	+
X^m	+	+
$\tilde{\psi}^\mu$	+	+
$\tilde{\psi}^m$	+	-
λ^a	-	+
χ^m	+	-

Table A.1: Charge assignments for the GSO projections in the $HO^{+(n)}$ theory. For convenience we use indices $\mu = 0, \ldots, 9$, and $m = 10, \ldots, D-1$. Note that the index m for the χs labels their multiplicity, but it is not a Lorentz index.

The symmetry is $SO(1, 9+n)_{X,\tilde{\psi}} \times SO(n)_\chi \times SO(32)_\lambda$. Tachyonic and massless states arise only in the g_1-untwisted sector, and are as follows

Sector	State		D-dim. field	Comment
Unt.	$\chi^m_{-\frac{1}{2}}\,\|0\rangle$	$\rightarrow$	T^m	Tachyons
	$\alpha^M_{-1}\,\tilde{\psi}^N_{-\frac{1}{2}}\,\|0\rangle$	$\rightarrow$	G_{MN}, B_{MN}, ϕ	graviton, 2-form, dilaton
	$\lambda^a_{-\frac{1}{2}}\lambda^b_{-\frac{1}{2}}\,\tilde{\psi}^M_{-\frac{1}{2}}\,\|0\rangle$	$\rightarrow$	A^{ab}_M	$SO(32)$ gauge bosons
g_2-twist.	$\alpha^M_{-1}\,\|\text{spinor}\rangle$	$\rightarrow$	Ψ^M_α	Rarita-Schwinger + Dirac
	$\lambda^a_{-\frac{1}{2}}\lambda^b_{-\frac{1}{2}}\|\text{spinor}\rangle$	$\rightarrow$	λ^{ab}_α	Spinor in adj. of $SO(32)$

The tachyons T^m are singlets under the $SO(32)$ gauge group. The spinor groundstates arise from fermion zero modes for $\tilde{\psi}^M$, χ^m, and have fixed overall chirality under $SO(1, 9+n) \times SO(n)$.

Since the actions of g_1 and g_2 commute with each other, the partition function factorizes in these two sectors. A bosonic piece which does not see the world sheet symmetries also factors out. In other words, the partition functions has a structure $Z = Z_{CM} \times Z_X \times Z_{\tilde{\psi},\chi} \times Z_\lambda$, and is given by

$$\begin{aligned} Z_{HO^{+(n)}}(\tau) =& \frac{1}{4}(4\pi^2\alpha'\tau_2)^{-4-\frac{n}{2}}|\eta|^{-16-2n} \\ &\times \frac{1}{\bar{\eta}^4|\eta|^n}\left(\bar{\vartheta}\begin{bmatrix}0\\0\end{bmatrix}^4\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^n - \bar{\vartheta}\begin{bmatrix}0\\1/2\end{bmatrix}^4\left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^n\right. \\ &\left. - \bar{\vartheta}\begin{bmatrix}1/2\\0\end{bmatrix}^4\left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^n \pm \bar{\vartheta}\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^n\right) \\ &\times \frac{1}{\eta^{16}}\left(\vartheta\begin{bmatrix}0\\0\end{bmatrix}^{16} + \vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^{16} + \vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^{16} + \vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^{16}\right) \end{aligned} \tag{A.11}$$

This theory is related to the 10d $SO(32)$ heterotic theory by (light-like) closed string tachyon condensation quenching the n extra dimensions. In this case, the tachyon background couples as a worldsheet superpotential

$$\Delta\mathcal{L}_{2d} = -\frac{1}{2\pi}\int d\theta^+ \sum_m \Lambda^m T^m(X) \tag{A.12}$$

where T^m are functions of the $(0,1)$ superfields $X^M + i\theta^+\tilde{\psi}^M$, and we have Fermi superfields $\Lambda^m = \chi^m + \theta^+ F^m$, with F^m being auxiliary fields (see e.g. [294]). Using the kinetic term to integrate out the latter, the extra terms in components are a worldsheet potential for the worldsheet bosons and Yukawa couplings for worldsheet fermions

$$\mathcal{L}_{\rm pot} \sim \sum_m (T^m(X))^2, \tag{A.13a}$$

$$\mathcal{L}_{\rm Yuk} \sim \partial_M T^m(X)\,\chi^m\,\tilde{\psi}^M \tag{A.13b}$$

To describe the disappearance of all dimensions X^m, we consider a profile for the tachyons, sketchily given by

$$T^m(X^+, X) = \mu_k^2 \sin(kX^m)\exp(\beta_k X^+) \quad \rightarrow \quad T^m(X^+, X) = \mu^2 X^m \exp(\beta_k X^+)$$

The LHS describes the deformation by exponential operators (taken marginal by tuning β_k), while the RHS describes the configuration after a $k \to 0$ limit, similar to the earlier one in section A.1. At $X^+ \to -\infty$ we have the theory in D dimensions and vanishing tachyon profile, while at $X^+ \to \infty$ the dynamics truncates to the slice $X^m = 0$, since all the extra worldsheet fields are made massive by (A.13). The endpoint of tachyon condensation is the 10d supersymmetric $SO(32)$ heterotic string theory[4]. It is worth noting that from the spacetime perspective, the 10d fermions arise as zero modes of the $(10+n)$d fermions coupled to the tachyon kink.

A.2.2 $HO^{+(n)/}$ theory

In the $HO^{+(n)/}$ theory, the GSO projection is given by the action g_1 in table A.2. All left-moving fermions λ, χ are on equal footing, so they are collectively denoted λ^a, $a = 1, \ldots, 32+n$.

[4] Actually, a BPS state in this 10d theory, c.f. [256]

Field	g_1	g_2
X^μ	+	+
X^m	+	-
$\tilde{\psi}^\mu$	-	+
$\tilde{\psi}^m$	-	-
λ^a	-	-

Table A.2: Charge assignments for the GSO projections in the $HO^{+(n)/}$ theory. For convenience we use indices $\mu = 0, \ldots, 9$, and $m = 10, \ldots, D-1$.

The symmetry is $SO(1, 9+n)_{X,\tilde{\psi}} \times SO(32+n)_\lambda$. Light states arise only in the g_1-untwisted sector, and are as follows

Sector	State		D-dim. field	Comment
g_1-untwisted	$\lambda^a_{-\frac{1}{2}}\,|0\rangle$	$\rightarrow$	T^a	Tachyons
	$\alpha^M_{-1}\,\tilde{\psi}^N_{-\frac{1}{2}}\,|0\rangle$	$\rightarrow$	G_{MN}, B_{MN}, ϕ	graviton, 2-form, dilaton
	$\lambda^a_{-\frac{1}{2}}\lambda^b_{-1/2}\,\tilde{\psi}^N_{-\frac{1}{2}}\,|0\rangle$	$\rightarrow$	A^{ab}_M	$SO(32+n)$ gauge bosons

The tachyons T^a transform in the vector representation of the $SO(32+n)$ gauge group.

The modular invariant partition function factorizes as $Z = Z_{CM} \times Z_X \times Z_{\tilde{\psi},\lambda}$, and is given by

$$\begin{aligned} Z(\tau) =& \frac{1}{2}(4\pi^2\alpha'\tau_2)^{-4-\frac{n}{2}}|\eta|^{-16-2n}\frac{1}{\bar{\eta}^4\eta^{16}|\eta|^n} \\ &\times \left(\bar{\vartheta}\begin{bmatrix}0\\0\end{bmatrix}^4\vartheta\begin{bmatrix}0\\0\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^n - \bar{\vartheta}\begin{bmatrix}0\\1/2\end{bmatrix}^4\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^n\right. \\ &\left. - \bar{\vartheta}\begin{bmatrix}1/2\\0\end{bmatrix}^4\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^n \pm \bar{\vartheta}\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^n\right) \end{aligned} \tag{A.14}$$

The above theory has no space-time fermions, so it is not a supercritical extension of the supersymmetric $SO(32)$ heterotic. However, the latter is related to a $\mathbb{Z}_2$ orbifold of the $HO^{+(n)/}$ theory, defined by the element g_2 in table A.2.

The original $HO^{+(n)/}$ spectrum is the g_2-untwisted sector, so it propagates in D dimensions, and must be projected onto g_2-invariant states. In particular, the tachyons T^a, as well as the 'mixed tensors' $G_{m\mu}, B_{m\mu}$, are forced to vanish at the fixed locus $X^m = 0$. The only additional massless states arise from the $g_1 g_2$-twisted sector, and correspond to the following 10d massless fields localized at the fixed locus $X^m = 0$

State		10d field	Comment
$\alpha^{\mu}_{-1}\lvert\text{spinor}\rangle$	$\rightarrow$	ψ^{μ}_{α}	Gravitino+dilatino
$\lambda^{a}_{-\frac{1}{2}}\lambda^{b}_{-\frac{1}{2}}\lvert\text{spinor}\rangle$	$\rightarrow$	χ^{ab}_{α}	chiral fermion in $(\begin{smallmatrix}\Box\\\Box\end{smallmatrix},1)$
$\lambda^{a}_{-\frac{1}{2}}\alpha^{m}_{-\frac{1}{2}}\lvert\text{spinor}'\rangle$	$\rightarrow$	$\chi_{\dot\alpha}$	opp. ch. fermion in $(\Box,\Box)$
$\alpha^{m}_{-\frac{1}{2}}\alpha^{n}_{-\frac{1}{2}}\lvert\text{spinor}\rangle$	$\rightarrow$	χ^{mn}_{α}	chiral fermion in $(1,\Box\Box)$

The spinor groundstate arises from the fermion zero modes of $\tilde\psi^{\mu}$, and the 10d chirality is fixed by the GSO projection. The representations of the fermions is under the $SO(32+n)$ gauge group and the $SO(n)_{\rm rot}$ rotational group in the coordinates X^m. Regarding the latter as some kind of gauge group, the fermion content motivated ref. [254] to propose a duality with type I with n D9-$\bar{\rm D}$9 brane pairs.

The modular invariant partition function is given by

$$
\begin{aligned}
Z(\tau) =&\frac{1}{4}(4\pi^2\alpha'\tau_2)^{-4}|\eta|^{-16}\\
&\times\Bigg\{(4\pi^2\alpha'\tau_2)^{-\frac{n}{2}}|\eta|^{-2n}\frac{1}{\bar\eta^4\eta^{16}|\eta|^n}\cdot\Bigg(\bar\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4\vartheta\begin{bmatrix}0\\0\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^n\\
&-\bar\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^4\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^n-\bar\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^n\pm\bar\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^n\Bigg)\\
&+\left|\frac{\eta}{\vartheta\left[{1/2\atop 0}\right]}\right|^n\frac{1}{\bar\eta^4\eta^{16}|\eta|^n}\cdot\Bigg(\bar\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^n\\
&-\bar\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^4\vartheta\begin{bmatrix}0\\0\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^n-\bar\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^n\pm\bar\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^n\Bigg)\\
&+\left|\frac{\eta}{\vartheta\left[{0\atop 1/2}\right]}\right|^n\frac{1}{\bar\eta^4\eta^{16}|\eta|^n}\cdot\Bigg(\bar\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^n\\
&-\bar\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4\vartheta\begin{bmatrix}0\\0\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^n-\bar\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^4\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^n\pm\bar\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^n\Bigg)\\
&+\left|\frac{\eta}{\vartheta\left[{0\atop 0}\right]}\right|^n\frac{1}{\bar\eta^4\eta^{16}|\eta|^n}\cdot\Bigg(-\bar\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^4\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^n\\
&-\bar\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}^4\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^n+\bar\vartheta\begin{bmatrix}0\\0\end{bmatrix}^4\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^n\pm\bar\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}^4\vartheta\begin{bmatrix}0\\0\end{bmatrix}^{16}\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^n\Bigg)\Bigg\}.
\end{aligned}
\tag{A.15}
$$

This orbifold configuration relates to the 10d $SO(32)$ heterotic theory by (light-like) tachyon condensation. The tachyon background couples as a worldsheet superpotential

$$
\Delta\mathcal{L}_{2d} \;=\; -\frac{1}{2\pi}\int d\theta^+\,\sum_a \Lambda^a\,T^a(X) \tag{A.16}
$$

(with Fermi superfields $\Lambda^a=\lambda^a+\theta^+F^a$), and the 2d scalar potential and fermion couplings are given by

$$\mathcal{L}_{\rm pot} \sim \sum_a (\,T^a(X)\,)^2, \tag{A.17a}$$

$$\mathcal{L}_{\rm Yuk} \sim \partial_M T^a(X)\,\lambda^a\,\tilde\psi^M \tag{A.17b}$$

The removal of the dimensions X^m, $m = 10, \ldots, D-1$, requires a profile for the tachyons $T^{a=23+m}$, i.e. $a = 33, \ldots, 32+n$, sketchily

$$T^{23+m}(X^+, X) = \mu_k^2 \sin(kX^m)\exp(\beta_k X^+) \quad \to \quad T^{23+m}(X^+, X) = \mu^2 X^m \exp(\beta_k X^+)$$

before and after the familiar $k \to 0$ limit. Note that we must restrict to tachyon profiles invariant under the orbifold $\mathbb{Z}_2$ symmetry $X^m \to -X^m$, $T^a \to -T^a$.

At $X^+ \to -\infty$ we have the parent orbifold configuration with D dimensions and vanishing tachyon, while at $X^+ \to \infty$ the dynamics truncates to the slice $X^m = 0$. The endpoint of tachyon condensation is the 10d supersymmetric $SO(32)$ heterotic theory[5].

A.3 Supercritical type 0 strings and decay to type II

In this section we describe 10d type II theories as the endpoint of closed string tachyon condensation in some supercritical theory. Supercritical type II theories exist, but only in dimensions $D = 8p + 2$ (because of the specific structure of their modular invariant partition function). Fortunately, a more generic extension can be obtained by considering supercritical type 0 theories in $D = 10 + 2p$ dimensions, and performing a suitable orbifold to relate them to the 10d type II theories [256], as we now review.

The supercritical type 0 theories in D dimensions are described by D worldsheet bosons X^M, and left and right fermions ψ^M, $\tilde\psi^M$, with $M = 0, \ldots, D-1$, and the usual timelike linear dilaton background. There is a GSO projection, associated to $(-1)^{F_w}$, where F_w is total worldsheet fermion number, with two choices that correspond to the type 0A or 0B theories, as in the critical type 0 theories.

The spacetime spectrum contains a real tachyon $T(X)$ given by the groundstate in the NSNS sector. The massless NSNS sector contains the D-dimensional graviton, 2-form and dilaton. In the RR sector, the states are the tensor products of left and right (non-chiral) spinor groundstates, decomposing as a bunch of p-forms (different in the 0A and 0B theories).

The modular invariant partition function for the supercritical type 0A/B in $D = 10+n$ dimensions is given by

$$\begin{aligned} Z_0(\tau) =& \frac{1}{2}(4\pi^2\alpha'\tau_2)^{-4-\frac{n}{2}}\,|\eta|^{-16-2n}\frac{1}{|\eta|^{8+n}} \\ &\left(\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^{8+n} + \left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^{8+n} + \left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^{8+n} \mp \left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^{8+n} \right). \end{aligned} \tag{A.18}$$

The tachyon couples as a worldsheet superpotential,

$$\Delta\,\mathcal{L} = \frac{i}{2\pi}\int d\theta^+ d\theta^- T(X), \tag{A.19}$$

[5] Actually, a BPS state in this 10d theory, c.f. [256]

where $T(X)$ depends on the $(1,1)$ superfields $X^M + i\theta^-\psi^M + i\theta^+\tilde{\psi}^M + i\theta^-\theta^+F^M$ (see e.g. [294]). Upon integrating out the auxiliary fields F^M, the terms in components describe a worldsheet potential and Yukawa couplings

$$\mathcal{L}_{\rm pot} \sim \partial^M T(X)\partial_M T(X), \tag{A.20a}$$

$$\mathcal{L}_{\rm Yuk} \sim \partial_M\partial_N \mathcal{T}(X)\,\tilde{\psi}^M\psi^N \tag{A.20b}$$

Condensation of this tachyon can produce dimension quenching in type 0 theories, but cannot connect down to 10d type II theories. In order to achieve the latter, we must instead consider a $\mathbb{Z}_2$ quotient of the above supercritical configuration.

In particular, we focus on even dimensions $D = 10 + 2p$, and split the extra $2p$ coordinates in two sets, denoted by X^m, X'^m. For such even D, there is a global symmetry on the worldsheet, corresponding to left-moving worldsheet fermion number $(-1)^{F_{L_w}}$ (i.e. under which ψ^M are odd and $\tilde{\psi}^M$ are even). In the critical $D = 10$ case, orbifolding by $(-1)^{F_{L_w}}$ produces the 10d type II theories (since this projection combines with the type 0 GSO to produce independent left and right GSO projections). In the supercritical case, modular invariance requires to mod out by $g \equiv (-1)^{F_{L_w}} \cdot \mathcal{R}$, where $\mathcal{R}$ is the spacetime $\mathbb{Z}_2$ action $X'^m \to -X'^m$, $X^m \to X^m$. The g-untwisted sector corresponds to the ($\mathbb{Z}_2$ projected) old NSNS and RR sectors, and describe fields in $D = 10+2p$ dimensions, while the twisted sectors are NS-R and R-NS sectors localized at $X'^m = 0$, i.e. in $10+p$ dimensions. The NSNS and RR sectors contain bosonic fields, whereas the NS-R and R-NS sectors contain fermion fields. The choices of 0A or 0B as starting point determine the IIA or IIB - like projections in this twisted sector.

Focusing on the supercritical type 0B, the massless fields correspond to $(10+p)$-dimensional vector-spinor fields ψ^M_α, where M runs through $D = 10+2p$ coordinates, and α denotes a bi-spinor of $SO(1,9+p) \times SO(p)$, with an overall chirality projection (i.e. the decomposition of an $SO(1,9+2p)$ chiral spinor). The vector-spinor field splits as a gravitino and a Weyl spinor, as usual.

The modular invariant partition function for the supercritical type 0A/B in $D = 10+n$

dimensions orbifolded by the $\mathbb{Z}_2$ defined above is given by

$$
\begin{aligned}
Z_0^{orb}(\tau) =& \frac{1}{4}(4\pi^2\alpha'\tau_2)^{-4}|\eta|^{-16} \\
&\times \Bigg\{ (4\pi^2\alpha'\tau_2)^{-\frac{n}{2}}|\eta|^{-2n}\frac{1}{|\eta|^{8+n}}\left(\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^{8+n} + \left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^{8+n} + \left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^{8+n} \mp \left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^{8+n}\right) \\
&+ \left|\frac{\eta}{\vartheta\left[\begin{smallmatrix}1/2\\0\end{smallmatrix}\right]}\right|^n \frac{1}{|\eta|^{8+n}} \cdot \Bigg(\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^n \\
&+ \left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^n + \left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^n \mp \left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^n \Bigg) \\
&+ \left|\frac{\eta}{\vartheta\left[\begin{smallmatrix}0\\1/2\end{smallmatrix}\right]}\right|^n \frac{1}{|\eta|^{8+n}} \cdot \Bigg(\left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^n \\
&+ \left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^n + \left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^n \mp \left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^n \Bigg) \\
&+ \left|\frac{\eta}{\vartheta\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]}\right|^n \frac{1}{|\eta|^{8+n}} \cdot \Bigg(\left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^n \\
&+ \left|\vartheta\begin{bmatrix}1/2\\0\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}0\\1/2\end{bmatrix}\right|^n + \left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^n \mp \left|\vartheta\begin{bmatrix}1/2\\1/2\end{bmatrix}\right|^8 \left|\vartheta\begin{bmatrix}0\\0\end{bmatrix}\right|^n \Bigg) \Bigg\}.
\end{aligned}
\tag{A.21}
$$

These supercritical non-compact orbifolds can be connected with 10d type II theories by closed string tachyon condensation. The local worldsheet coupling to the tachyon background is still given by (A.19) and (A.20). The superpotential must be $\mathbb{Z}_2$-odd, so the tachyon is a superposition of marginal operators with sine dependence on X'^m's. For instance, choosing a sine dependence also on X^m, and taking two extra dimensions for simplicity, we have

$$T = \mu \exp(\beta X^+)\, \mu_{k,k'}\, \sin(kX)\, \sin(k'X'). \tag{A.22}$$

In the familiar $k \to 0$ limit, we have the tachyon profile $T \sim XX'$, or in general (setting some constants equal for simplicity)

$$T = \mu \exp(\beta X^+) \sum_{m=1}^{p} X^m X'^m. \tag{A.23}$$

This removes the coordinates X^m, X'^m and produces a 10d superstring theory at $X^m = X'^m = 0$. It contains NSNS, NSR, RNS and RR sectors with a type II GSO projection, i.e. we recover the 10d type II theories. In particular, notice that the tachyon vanishes at $X = X' = 0$ and does not give any dynamical mode after condensation.

A.4 Dimensional reduction vs dimension quenching

We conclude the discussion several conceptual remarks: Dimension quenching is drastically different from Kaluza-Klein dimensional reduction. From the spacetime viewpoint, dimension quenching causes the extra dimensions to completely disappear from the theory.

In particular, there remain no towers of extra-dimensional momentum modes. Even at the level of massless modes, its effect on spacetime bosons differs from a truncation to the zero mode sector; for instance, mixed components $G_{\mu m}$ disappear completely (whereas they can survive in KK dimensional reduction). However, it is important to emphasize that, dimension quenching *does* behave like dimensional reduction for massless spacetime fermions; indeed, the 10d spinors arise from zero modes of the higher-dimensional Dirac operator coupled to the tachyon background [254].

Hence, from the spacetime perspective, the natural order parameter measuring the tachyon background is the derived quantity coupling to the spacetime fermions, namely $\partial_m T^a$ in heterotic, c.f. (A.13), and $\partial_m \partial_n T$ in type 0/II, c.f. (A.20). The symmetry breaking pattern is mostly encoded in the quantum numbers of this quantity. For instance, in section A.2 the background $\partial_m T^a$ breaks the $SO(32+n) \times SO(n)$ gauge and rotational symmetries down to $SO(32)$ (times a diagonal $SO(n)_{\rm diag}$). Similarly, in section A.3 the background $\partial_m \partial_n T$ breaks the $SO(n) \times SO(n)$ rotational symmetries (seemingly with a left-over diagonal $SO(n)_{\rm diag}$). Although the diagonal symmetries $SO(n)_{\rm diag}$ would seem unbroken from a Higgsing perspective in spacetime, the microscopic worldsheet computation shows that they actually disappear.

B

Fluxbranes

In this section we will review some basic concepts on fluxbranes. See [271, 272] for more detailed discussions.

B.1 The IIA F7-brane

Consider M-theory on an space $\mathbb{X}_{11} = \mathbb{M}_{10} \times \mathbb{R} = \mathbb{M}_8 \times \mathbb{R}_2 \times \mathbb{R}$ (flat space) with metric

$$ds_{11}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dr^2 + r^2 d\phi^2 + dx_{10}^2 \tag{B.1}$$

where the x^μ, $\mu = 0, \ldots, 7$ parametrize $\mathbb{M}_8$, r, ϕ parametrize $\mathbb{R}^2$ and x_{10} parametrizes $\mathbb{R}$.

We will perform a Kaluza-Klein reduction from 11 to 10 dimensions involving the following shift identification:

$$x_{10} \sim x_{10} + 2\pi n_1 R \tag{B.2a}$$

$$\phi \sim \phi + 2\pi n_2 + 2\pi n_1 B R^2. \tag{B.2b}$$

In other words, the dimensional reduction is performed along orbits of the Killing vector $q = \partial_{x_{10}} + BR\partial_\phi$. As usual $R = \frac{g_s}{M_s}$ is the ratio of the string coupling to the string mass, and B will be identified with the strength of the magnetic field at the origin ($r = 0$).

In order to perform the Kaluza-Klein reduction it is convenient to introduce the coordinate $\tilde{\phi} = \phi - BRx_{10}$ which is canonically identified and constant along orbits of q,

$$x_{10} \sim x_{10} + 2\pi n_1 R \tag{B.3a}$$

$$\tilde{\phi} \sim \tilde{\phi} + 2\pi n_2. \tag{B.3b}$$

In terms of $\tilde{\phi}$ the eleven dimensional metric is

$$ds_{11}^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dr^2 + r^2 (d\tilde{\phi} + BRdx_{10})^2 + dx_{10}^2. \tag{B.4}$$

The Kaluza-Klein ansatz metric is

$$ds_{11}^2 = e^{\frac{4\Phi}{3}} (dx_{11} + RA_\rho dx^\rho)^2 + e^{-\frac{2\Phi}{3}} ds_{10}^2 \tag{B.5}$$

where Φ and A are the ten dimensional dilaton and RR 1-form, respectively, and $x^\rho = \{x^\mu, r, \tilde{\phi}\}$.

Comparing (B.4) and (B.5) one gets

$$
\begin{aligned}
ds_{10}^2 &= \sqrt{\Lambda}(\eta_{\mu\nu}dx^\mu dx^\nu + dr^2) + \frac{r^2 d\tilde{\phi}^2}{\sqrt{\Lambda}} && \text{(B.6a)}\\
A_{\tilde{\phi}} &= \frac{Br^2}{\Lambda} && \text{(B.6b)}\\
e^{\frac{4\Phi}{3}} &= \Lambda && \text{(B.6c)}\\
\Lambda &\equiv 1 + B^2R^2r^2. && \text{(B.6d)}
\end{aligned}
$$

The solution describes a 8-dimensional Poincaré invariant configuration of non-vanishing RR 1-form field strength flux turned on in the $(r, \tilde{\phi})$ plane. The total magnetic flux is

$$\int_{\mathbb{R}^2} F = \frac{2\pi}{R^2 B} \tag{B.7}$$

where

$$F = dA = \frac{2Br}{(1 + B^2R^2r^2)^2} dr \wedge d\tilde{\phi} \tag{B.8}$$

Since the flux is given by $\int B(r)dS$, and $dS = rdr \wedge d\tilde{\phi}$, $B(r) = 2B/(1 + B^2R^2r^2)^2$ and $B(0) = 2B$.[1]

Near the origin the metric is smooth and has small curvature, and the RR field strength is approximately constant. At large r, the fluxbrane induces strong coupling, since the radius of compactification (coefficient of dx_{10}^2) grows like r.

B.2 Lower dimensional F-branes

We will now construct fluxbranes with non-vanishing flux for the $2k$-form $F \wedge \ldots \wedge F$, the F$(9 - 2k)$-branes.

We first write the 11-dimensional metric as

$$ds_{11}^2 = dx_{10}^2 + \sum_{i=1}^{k}(dr_i^2 + r_i^2 d\phi_i^2) + \eta_{\mu\nu}dx^\mu dx^\nu \tag{B.9}$$

where r_i, ϕ_i are coordinates in the i^{th} $\mathbb{R}^2$ and the x^μ parametrize $\mathbb{M}^{10-2k}$.

We will perform a Kaluza-Klein reduction from 11 to 10 dimensions involving the following shift identification:

$$
\begin{aligned}
x_{10} &\sim x_{10} + 2\pi n_0 R && \text{(B.10a)}\\
\phi_i &\sim \phi_i + 2\pi n_i + 2\pi n_0 B_i R^2. && \text{(B.10b)}
\end{aligned}
$$

Defining the adapted coordinates

$$\tilde{\phi}_i = \phi_i - B_i R x_{10} \tag{B.11}$$

[1] Is this right?

the metric becomes

$$ds_{11}^2 = \eta_{\mu\nu}dx^\mu dx^\nu + \sum_{i=1}^{k}\left(dr_i^2 + r_i^2(d\tilde{\phi}_i + B_i R dx_{10})^2\right) + dx_{10}^2. \quad \text{(B.12)}$$

Comparing this expression with (B.5) we obtain

$$e^{\frac{4\Phi}{3}} = \lambda \quad \text{(B.13a)}$$

$$A_{\tilde{\phi}_i} = \frac{B_i r_i^2}{1 + R62\sum_{j=1}^{k} B_j^2 r_j^2} \quad \text{(B.13b)}$$

$$ds_{10}^2 = \sqrt{\Lambda}\left(\eta_{\mu\nu}dx^\mu dx^\nu = \sum_{i=1}^{k}(dr_i^2 + r_i^2 d\tilde{\phi}_i^2)\right) - \frac{R^2}{\sqrt{\Lambda}}\left(\sum_{i=1}^{k} B_i r_i^2 d\tilde{\phi}_i\right)^2 \quad \text{(B.13c)}$$

$$\Lambda \equiv 1 + R^2\sum_{i=1}^{k} B_i^2 r_i^2 \quad \text{(B.13d)}$$

The solution describes a $(10-2k)$-dimensional Poincaré invariant object with nonzero F^k flux in the transverse $\mathbb{R}^{2k}$, a F$(9-2k)$-brane.

As in the previous section, the dilaton blows up away from the core of the fluxbrane.

B.3 IIA↔0A duality

For $B = \frac{2}{R^2}$ one has a 4π rotation in (B.2), which is equivalent to no rotation at all. Therefore IIA with this critical magnetic field is dual to the IIA vacuum.

For $B = \frac{1}{R^2} = \frac{M_s^2}{g_s^2}$ the rotation in (B.2) has no effect on bosons but gives a minus sign for fermions. According to [295] it is conjetured that M-theory compactification on S^1 with twisted fermion boundary conditions is the 0A string theory at half string coupling; and it has been conjectured in [269] that

$$IIA(B, g_s) \leftrightarrow 0A(B - \frac{M_s^2}{g_s^2}, \frac{g_s}{2}) \quad \text{(B.14)}$$

Therefore we obtained a duality between IIA theory with $B = \frac{1}{R^2}$ and 0A string theory with no flux.

B.4 F7↔IIA cone duality

The circle used in section B.1 to reduce from 11d to 10d is not unique. The above IIA↔0A duality can be viewed as different choices of this circle. In both cases the circle lies in the torus parametrised by x^{11} and the angle ϕ, but they differ by a modular transformation. Therefore we can obtain more general alternate IIA descriptions on the theory transforming the torus (B.3) by a $SL(2,\mathbb{Z})$ transformation

$$\hat{x}_{10} = a x_{10} + bR\tilde{\phi}, \quad \text{(B.15a)}$$

$$\hat{\phi} = \frac{c}{R}x_{10} + d\tilde{\phi}, \quad \text{(B.15b)}$$

identified as

$$\hat{x}_{10} \sim \hat{x}_{10} + 2\pi n_1 R, \tag{B.16a}$$
$$\hat{\phi} \sim \hat{\phi} + 2\pi n_2, \tag{B.16b}$$

where $a, b, c, d \in \mathbb{Z}$ obey $ad - bc = 1$.

The flat 11-dimensional metric (B.4) becomes

$$ds_{11}^2 = \eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + r^2((a - bBR^2)d\hat{\phi} + (dBR-]\frac{c}{R}d\hat{x}_{11}^2 + (d\, d\hat{x}_{10} - bRd\hat{\phi})^2. \tag{B.17}$$

Consider the case

$$BR^2 = \frac{1}{N} \tag{B.18}$$

with $N \in \mathbb{Z}$. Then choosing

$$a = 1, \quad b = N - 1, \quad c = 1, \quad d = N, \tag{B.19}$$

reduces (B.17) to

$$ds_{11}^2 = \eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + \frac{r^2}{N^2}d\hat{\phi}^2 + (Nd\hat{x}_{10} - (N-1)Rd\hat{\phi})^2. \tag{B.20}$$

This corresponds to an orbifold flat 10-dimensional space, with orbifold action generated by

$$z \to e^{2\pi i BR^2} z \tag{B.21}$$

where $z = re^{i\phi}$. The string coupling is $g_s = (NRM_p)^{\frac{3}{2}}$. There is also a flat $U(1)$ connection $A_\phi = \frac{N-1}{N}$.

One can consider a more general case defined by

$$BR^2 = \frac{m}{N} \tag{B.22}$$

with $m \in \mathbb{Z}$. Then we can take

$$c = m, \quad d = N, \quad Na - mb = 1. \tag{B.23}$$

Since there are infinite many a, b satisfying this relation, we will choose the smallest b that reduces (B.17) to

$$ds_{11}^2 = \eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + \frac{r^2}{N^2}d\hat{\phi}^2 + (Nd\hat{x}_{10} - bRd\hat{\phi})^2. \tag{B.24}$$

This differs from (B.20) only in the flat connection.

Therefore a F7-brane is dual to IIA on a flat cone with RR flux at the origin.

In a completely analogous way, it can be shown that the F$(9-2k)$-branes are dual, for suitable B_i, to $\mathbb{C}^k/\mathbb{Z}_N$ orbifold singularities.

C
Freed-Witten and Hanany-Witten effects

In this appendix we collect the main brane topological effects used in the text.

C.1 Freed-Witten effects

Many string theory compactifications include D-branes wrapped on cycles in the internal space. One of the possible questions that one may ask is what the possible cycles which can be wrapped by D-branes are, for the theory to be consistent.

One condition which must be satisfied is that the field theory on the D-brane must be consistent. For instance, it must be anomaly free. This puts restrictions on the possible cycles on which "free D-branes" (i.e. D-branes with no other branes ending on them) can wrap

In [217] D. Freed and E. Witten showed that a D-brane cannot wrap a sub manifold which in turn supports some units of NSNS three-form flux, since that configuration is anomalous, and therefore inconsistent.

Nevertheless, let us assume that we do have such a D-brane. Let $\mathcal{W}'$ be the cycle wrapped by it, and let $[H]|_{\mathcal{W}'}$ be the class of the NSNS three-form flux on it. It was shown in [218] by Maldacena, Moore and Seiberg that the anomaly can be cancelled if one also considers D-branes ending on the first on a cycle $\mathcal{W} \subset \mathcal{W}'$ such that

$$PD(\mathcal{W} \subset \mathcal{W}') = W_3(\mathcal{W}') + [H]|_{\mathcal{W}'}, \tag{C.1}$$

where $PD(\mathcal{W} \subset \mathcal{W}')$ denotes the Poincaré dual of $\mathcal{W}$ in $\mathcal{W}$ and $W_3(\mathcal{W}')$ is the integral Stiefel-Whitney class[1] of $T\mathcal{W}$.

This argument can be generalized to D-branes wrapping cycles which support RR p-form fluxes.

The different versions of this effect that will be useful throughout the text are the following:

- **FW1.** A Dp-brane with worldvolume S_{p+1} with homologically non-trivial $\overline{H}_3|_S$ must emit D$(p-2)$-branes on the Poincaré dual class [218] (morally, along directions of S_{p+1} transverse to $\overline{H}_3|_S$).

[1] This class is a torsion class, and it is included for completeness. In this thesis, when we only consider flux compactifications with no torsion; therefore, this term can be ignored.

- **FW2.** An NS5-brane with worldvolume S_6 with homologically non-trivial $\overline{F}_p|_S$ emits D$(6-p)$-branes spanning the Poincaré dual class. This follows by considering FW1 for D5-branes with $\overline{H}_3$ emitting D3-branes, S-dualizing to NS5-branes with $\overline{F}_3$ emitting D3-branes, and T-dualizing to general $\overline{F}_p$.

- **FW3.** A Dp-brane with worldvolume S with homologically non-trivial $\overline{F}_p$ emits F1s along the Poincaré dual class [219]. This follows by considering FW1 for D3-branes with $\overline{H}_3$ emitting D1-branes, S-dualizing to D3-branes with $\overline{F}_3$ emitting F1s, and T-dualizing to general Dp-branes with $\overline{F}_p$.

C.2 Hanany-Witten or brane creation effects

The Hanany-Witten effect is a process in superstring theory in which a p-brane and a q-brane crossing each other along one dimension produce (or annihilate) an r-brane extending along the dimension where the crossing took place and the dimensions shared by both the p-brane and the q-brane. It was first proposed in [180] by A. Hanany and E. Witten, where they considered the crossing of a D5-brane and a NS5-brane producing a D3-brane, and give several arguments to support their claim.

Using T-duality and S-duality the result of [180] can be generalized to other types of p-branes. The versions of this effect that will be used throughout the main text are:

- **HW1.** An NS5-brane along directions $x^0, x^1, x^2, x^3, x^4, x^5$ and a D$(p+3)$-brane along directions $x^0, x^1, \ldots, x^p, x^6, x^7, x^8$, with $p \leqslant 5$ can be crossed in the direction x^9 leading to the creation of a D$(p+1)$-brane along $x^0, x^1, \ldots, x^p, x^9$ (see figure C.1 for the $p = 5$ case). This effect follows from [180] by T-duality (and coordinate relabeling).

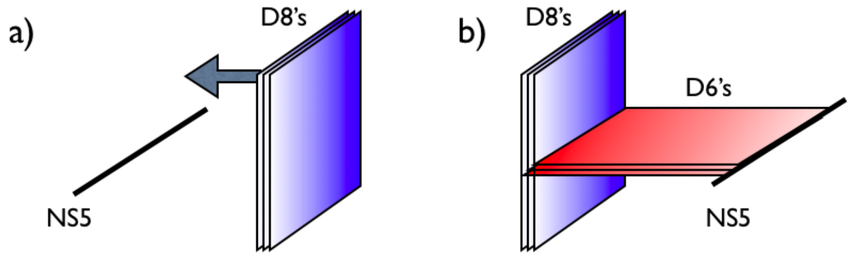

Figure C.1: Brane creation effect by crossing D8-branes and NS5-branes.

- **HW2.** A Dp-brane along $x^0, \ldots, x^p$ and a D$(8-p)$-brane along $x^0, x^{p+1}, \ldots, x^8$ can be crossed in x^9 leading to the creation of F1s in the directions x^0, x^9 (see figure C.2 for the $p = 0$ and $p = 8$ cases). This follows by considering HW1 for creation of D1-branes from NS5- and D3-brane crossing, S-dualizing to the creation of F1s from D5- and D3-brane crossing, and T-duality to general p.

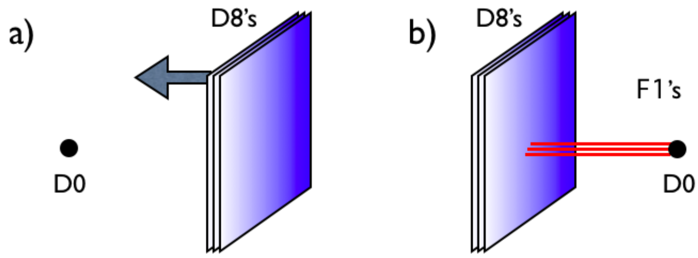

Figure C.2: Brane creation effect by crossing D8- and D0-branes.

C.3 Remarks

We conclude by mentioning that the FW and HW effects are related, as follows (see figures C.1, C.2). In the HW effect, we regard one of the branes as a source of flux, which the second brane (compactified by closing it at infinity) picks up in the form of FW anomaly, or not, depending on whether we are before or after the HW crossing. The HW brane creation is required to cancel this FW anomaly, conversely the FW consistency condition is required to explain the HW brane creation.

D

Sum over disconnected theories

Discrete gauge symmetries can be described as a 'sum over disconnected theories' [176]. In this appendix we will describe them, in the 4d avatar described in appendix A of that reference.

Consider a 4d gauge theory, with a non-perturbative sector restricted to instantons numbers multiple of n. The theory can be described as a sum over n disconnected theories (with unconstrained instanton sector) with rotating θ angle, $\theta_k = \theta + 2\pi\frac{k}{n}$. Schematically, an amplitude mediated by instanton number p configurations reads

$$
\begin{aligned}
&\sum_{p\in\mathbb{Z}} \frac{1}{n} \sum_{k=1}^{n-1} \langle \mathrm{out}|(\ldots)\exp[\mathrm{i}(\theta + 2\pi\frac{\mathrm{k}}{\mathrm{n}}\mathrm{p}]|\mathrm{in}\rangle = \\
&= \sum_{p\in\mathbb{Z}} \left[\frac{1}{n}\sum_{k=1}^{n-1} \exp\left(2\pi i \frac{kp}{n}\right)\right) \langle \mathrm{out}|(\ldots)\exp(\mathrm{i}\theta\mathrm{p})|\mathrm{in}\rangle \\
&= \sum_{p\in n\,\mathbb{Z}} \langle \mathrm{out}|(\ldots)e^{\mathrm{i}\theta\mathrm{p}}|\mathrm{in}\rangle .
\end{aligned}
\tag{D.1}
$$

The projection operator in the second expression reflects the existence of a $\mathbb{Z}_n$ discrete symmetry, rotating the θ angle; pictorially, mapping the k^{th} disconnected theory to the $(k+1)^{th}$ one. The construction admits a straightforward generalization to other field theories or string models.

E

Some examples of discrete isometries

In chapter 9 we will study different methods to embed discrete gauge symmetries coming from discrete isometries corresponding to large diffeomorphisms. In this appendix we gather a few standard yet illustrative examples of such isometries in compact manifolds. We first review the case $\mathbb{T}^2$, and move on to the quintic. Extension to other CY hypersurfaces is straightforward.

E.1 Discrete isometries of $\mathbb{T}^2$

The 2-torus can be described as a quotient $\mathbb{R}^2/\Gamma$ of the 2-plane by a lattice Γ of translations. Beyond the $U(1)^2$ continuous isometry group, there are possible discrete isometries from crystallographic symmetries of Γ, which correspond to large diffeomorphisms. Introducing a complex coordinate z with periodicities $z \simeq z+1$, $z \simeq z+\tau$, these isometries are subgroups of the $SL(2,\mathbb{Z})$ modular group, leaving the lattice invariant (possibly for some specific choice of τ). An alternative description of $\mathbb{T}^2$ is via the Weierstrass equation

$$y^2 = x^3 + fx + g. \tag{E.1}$$

The coordinates x, y relate to z by the so-called 'uniformization mapping'. Here f, g are complex constants, in terms of which the j-function of the complex structure parameter τ is

$$j(\tau) = \frac{2(24\,f)^3}{27g^2 + 4f^3}. \tag{E.2}$$

The discrete symmetries, and the values of the complex structure modulus at which they hold, are familiar from the construction of toroidal orbifolds. They are

Symmetry	τ	Generator	f, g	Weierstrass	Generators	$SL(2,\mathbb{Z})$
$\mathbb{Z}_2$	arbitr.	$z \to -z$	arbitr.	$y^2 = x^3 + fx + g$	$x \to x$, $y \to -y$	$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathbb{Z}_4$	$\tau = i$	$z \to e^{2\pi i/4} z$	$g = 0$	$y^2 = x^3 - x$	$x \to -x$, $y \to iy$	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$
$\mathbb{Z}_6$	$\tau = e^{\pi i/3}$	$z \to e^{2\pi i/6} z$	$f = 0$	$y^2 = x^3 + 1$	$x \to e^{2\pi i/3} x$, $y \to -y$	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$
$\mathbb{Z}_2$	$\tau_1 = 1$	$z \to \overline{z}$	real	$y^2 = x^3 + fx + g$	$x \to \overline{x}$ $y \to \overline{y}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$\mathbb{Z}_2$	$\tau_1 = \frac{1}{2}$	$z \to \overline{z}$	real	$y^2 = x^3 + fx + g$	$x \to \overline{x}$ $y \to \overline{y}$	$\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}$

Here we have applied certain rescalings to simplify the Weierstrass equation for $\mathbb{Z}_4$ and $\mathbb{Z}_6$. Note that, by squaring the $\mathbb{Z}_6$ generator, there is a $\mathbb{Z}_3$ symmetry for $\tau = e^{\pi i/3}$. Finally, the last two entries correspond to orientation-reversing actions, and they are actually not in $SL(2,\mathbb{Z})$.

E.2 Discrete isometries of the quintic

Consider the quintic CY $\mathbb{X}_6 = \mathbb{P}_5[5]$ at the Fermat point, i.e. the hypersurface with defining equation

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0 \tag{E.3}$$

This has a discrete symmetry group $(\mathbb{Z}_5)^4 \times S_5$. The S_5 is the group of permutations of 5 elements, the homogeneous coordinates z_i. The $\mathbb{Z}_5$'s are generated by independent phase rotations $z_i \to e^{2\pi i/5} z_i$, with the removal of an overall phase rotation which is part of the projective action to define the ambient $\mathbb{P}_5$.

In addition, there is an antiholomorphic action

$$z_i \to \overline{z}_i \tag{E.4}$$

These actions, and their products, have been extensively exploited in the literature.

F
Details on the derivation of (7.66)

In order to obtain the set of constraints that the discrete symmetry imposes on the holomorphic Yukawa couplings Y_{ij} it is important to recall that if $f(\xi)$ is a holomorphic function of ξ with domain on $\mathbb{C}$ and is invariant under some discrete lattice Γ then $f(\xi)$ is actually independent of ξ. Knowing how Y_{ij} transform under the isometry generators, we can then build holomorphic invariants of $\{X^2_{\mathbb{Z}_3}, Y^2_{\mathbb{Z}_3}\}$ and/or $\{X^3_{\mathbb{Z}_3}, Y^3_{\mathbb{Z}_3}\}$ that are independent of the corresponding complex Wilson line scalars and that satisfy particular relations.

Let us first illustrate the procedure on a similar model with only two generations of fields transforming as

$$X^2_{\mathbb{Z}_3} : X^k_R \to e^{-i\pi k} X^k_R, \qquad Y^2_{\mathbb{Z}_3} : (X^1_R, X^2_R) \to (X^2_R, X^1_R), \tag{F.1a}$$

$$X^3_{\mathbb{Z}_3} : X^k_L \to e^{i\pi k} X^k_L, \qquad Y^3_{\mathbb{Z}_3} : (X^1_L, X^2_L) \to (X^2_L, X^1_L), \tag{F.1b}$$

with $k = 1, 2$. Yukawa couplings are of the form

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix} \tag{F.2}$$

Taking into account the above transformations of the fields, we observe that the following function

$$A \equiv \frac{Y_{11}}{Y_{21}} - \frac{Y_{12}}{Y_{22}} \tag{F.3}$$

is invariant under $X^3_{\mathbb{Z}_3}$ and $(Y^3_{\mathbb{Z}_3})^2$, so that a is independent of the complex Wilson line scalar ξ^3. Moreover, under $Y^3_{\mathbb{Z}_3}$ it transforms as

$$A \to -A \tag{F.4}$$

but since acting with $Y^3_{\mathbb{Z}_3}$ is equivalent to performing a shift in ξ^3, this means that a has to be identically zero. We have therefore shown that

$$\frac{Y_{11}}{Y_{21}} = \frac{Y_{12}}{Y_{22}} \tag{F.5}$$

For the three generation model of section 7.3.3 the proof follows the same logic. For instance, let us consider the following functions

$$A \equiv -\frac{Y_{11}}{Y_{21}} + \frac{Y_{12}}{Y_{22}} + \frac{Y_{13}}{Y_{23}}, \tag{F.6a}$$

$$B \equiv \frac{Y_{11}}{Y_{21}} - \frac{Y_{12}}{Y_{22}} + \frac{Y_{13}}{Y_{23}}, \tag{F.6b}$$

$$C \equiv \frac{Y_{11}}{Y_{21}} + \frac{Y_{12}}{Y_{22}} - \frac{Y_{13}}{Y_{23}}, \tag{F.6c}$$

invariant under $X^3_{\mathbb{Z}_3}$ and $(Y^3_{\mathbb{Z}_3})^3$ and therefore independent of the complex Wilson line scalar ξ^3. Under $Y^3_{\mathbb{Z}_3}$ they transform as

$$A \to B \to C \to A \tag{F.7}$$

and therefore we must have $A = B = C$, from which the first relation in (7.66) follows. The other relations in (7.66) are proven similarly.

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