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A Note on Selecting Maximals in Finite Spaces

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Juan Carlos García-Bermejo

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A NOTE ON SELECTING MAXIMALS IN FINITE SPACES

Juan Carlos García-Bermejo
Autonomous University of Madrid
Campus de Cantoblanco
Cra. de Colmenar, km. 15
28049-MADRID

e-mail: juancarlos.gbermejo@uam.es
Telephones: +34-649857896; +34-91 497 43 16/42 92.
Fax: +34- 497 86 16/ 51 61

ABSTRACT

Given a choice problem, the maximization rule may select many alternatives. In such cases, it is common practice to interpret that the final choice will end up being made by some random procedure, assigning to any maximal alternative the same probability of being chosen. However, there may be reasons based on the same original preferences for which it is suitable to select certain maximal alternatives over others. This paper introduces two choice criteria induced by the original preferences such that maximizing with respect to each of them may give a finer selection of alternatives than maximizing with respect to the original preferences. Those criteria are built by means of several preference relations induced by the original preferences, namely, two (weak) dominance relations, two indirect preference relations and the dominance relations defined with the help of those indirect preferences. It is remarkable that as the original preferences approach being complete and transitive, those criteria become both simpler and closer to such preferences. In particular, they coincide with the original preferences when these are complete and transitive, in which case they provide the same solution as those preferences.

KEYWORDS: maximal elements, maximization, acyclical preferences, rational choice, choice function, refinements of maximization rule.

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1.- INTRODUCTION

We know that when preferences are not acyclic the standard maximization choice rule may not solve all the choice problems that can arise within a space of alternatives. We also know that even when preferences are acyclic and the maximization rule selects a non-empty set of alternatives, that set may include more than one option, frequently too many. When this is the case, it is conventional to interpret that the final choice will end up being made by means of some random procedure, assuming that the alternatives selected by the rule are equivalent of sorts. However, Luce (1956) already revealed that in such cases there may be reasons, based on the same original preferences, for which it is suitable to select certain maximal alternatives ahead of others.

Consider the following example:

Example 1.- The alternative x is better than y, y and z are indifferent, and x and z are also indifferent. Maximizing over the set $\{x, y, z\}$ gives $\{x, z\}$. However, there is a certain asymmetry in favour of x because x is strictly better than y. Hence, it makes sense to take $\{x\}$ as a solution instead of the standard solution.

Cases like this do not arise if preferences are complete and transitive. However, the possibility that individual preferences are not complete is familiar to any agent. Likewise, the possibility that individual preferences are not transitive has been extensively discussed in the literature. Fishburn (1991) summarizes this discussion. In social preferences, non-completeness and violations of transitivity are phenomena induced by well known aggregation rules such as the Pareto criterion and the majority method.

Luce's proposal consists of defining a new preference relation which, induced by the original one, serves to reject all maximal alternatives which would be worse than another possible alternative according to the new relation. In fact, the standard maximization rule may be understood as operating in two steps. In step one, all the alternatives that are worse than another alternative in the choice problem are precluded from being chosen. In step two, the non-excluded alternatives are accepted and included in the choice set. Thus, applying additional criteria induced by the original preferences for rejecting alternatives and then accepting the rest may be considered as a coherent way of refining the standard rule of maximizing with respect to the original preferences. This is what this paper aims to do.

The authors of the above example, Begoña Subiza and Josep E. Peris, have followed this kind of strategy in various articles, but they have introduced more general discriminating criteria than the criterion introduced by Luce.[1] It should be noted that Luce's criterion only guarantees to give solutions when the original preferences comply with specific and fairly demanding properties.[2] In contrast, Subiza and Peris use as an additional basic criterion to reject a maximal alternative, the fact of being dominated by another alternative included in the same group of feasible alternatives.

Nevertheless, the additional use of this dominance criterion can result in an excessively large group of maximal alternatives. Because of this, it may be necessary to introduce

[1] Subiza and Peris (2000), (2002), (2005a) and (2005b).
[2] Specifically, Luce assumes that the original preference relation is a semiorder.

some additional rejection criteria. Specifically, Subiza and Peris (2005a) propose excluding an alternative *y* from being chosen if there is any alternative *x* that is better than a number of alternatives higher than *y*.[3]

In many ways, this paper follows the line devised by those authors. In particular, it coincides with them on the relevance of the dominance relation as a rejection criterion. It also agrees with the majority of their mentioned articles by supposing that the space of alternatives is finite, and tries also to make as few demanding assumptions on the original preferences as possible. However, this paper differs in its approach in other ways.

The only assumptions that are made in this paper on the original preferences are that they are reflexive and have no cycles. It is even shown how to apply the approach when such cycles exist. Thus, one relevant difference from Subiza and Peris's approach - and also from the literature on tournaments - is the assumption that the original relation of preference is a weak one which may not be complete. In this manner, there may be indifferent alternatives although the relation is not complete. This assumption leads to distinguish between two dominance relations defined on the basis of the original preferences and, above all, demonstrates that the approach that is proposed in the paper may be more useful when the original preferences are incomplete.

Another important difference has to do with the rejection criteria proposed in addition to the dominance relationships based on the original preferences. As pointed out above, Subiza and Peris (2005a) propose excluding any alternative *y* from being chosen if there is any alternative *x* that is better than a number of alternatives higher than *y*. Instead, this paper proposes to use, as additional rejection criteria, the indirect preference relations that can be defined on the basis of the original preferences, as well as the dominance relations based on those indirect preferences.

The main reason to do so is because considering that a maximal alternative *x* is preferable to another *y* because the number of alternatives that are worse than *x* is higher than the number of alternatives worse than *y* may not be so convincing, as demonstrated in the following example. Alternative x^1 is preferred to x^2, $x^1$is preferred to x^3, x^2 is indifferent to x^3, x^1 is indifferent to y^1, y^1 is preferred to y^2, y^1 is indifferent to x^2, y^1 is indifferent to x^3, y^2 is preferred to x^2 and y^2 is preferred to x^3. Notice that there is an alternative worse than *y* which is preferred to any alternative *x*' that is worse than *x*, namely y^2. If the strict preferences were transitive, then the case would be more convincing because then y^1 would be preferred to x^2 and to x^3, so that the set of the alternatives that are worse than y^1 would be larger than the set of the alternatives that are worse than x^1. Despite this, it is possible to build similar examples in which strict preferences are transitive and where, as in the example above, it is unclear whether we can infer that *x* is preferable to *y*. Suppose, for instance, that the preferences are as described in the former example in this paragraph, except that y^2 is indifferent to x^2 and y^2 is indifferent to x^3. It is then true that the number of the alternatives that are worse

[3] In other papers, Subiza and Peris have proposed other choice criteria. For instance, in Subiza and Peris (2000) propose that *x* should be preferable to *y* if *x* is indirectly preferred to a number of alternatives higher than *y*. Analogously, in Subiza and Peris (2002) define a further alternative rejection criteria for the case in which the original preference relation does not include indifference comparisons between the alternatives.

than *x* is higher than the number of those that are worse than *y*, but every alternative worse than *x* is indifferent to all those worse than *y*.

Given that the space of alternatives is finite, another feature of the paper is that it attempts to exploit as much as possible the information provided by the original preference relation throughout all the space. The information we get from this strategy provides us with stricter rejection criteria than we would get from considering only the information available in each choice problem considered, as shown in the following example:

Example 2.- Let us suppose that in the space {*x*, *y*, *z*}, alternative *x* is better than alternative *y*, which in turn is better than *z*. Alternatives *x* and *z* are indifferent and the problem is which to choose of *x* and *z*. If we use the information available in the set {*x*, *z*} exclusively, neither is dominant which means the dominance relation will give us the same solution, {*x*, *z*}, as the maximization rule. If, on the other hand, we use the information available throughout the entire space {*x*, *y*, *z*}, the solution will be *x*, which is also intuitively the more plausible solution.

Section 2 introduces the notation and the basic notions used in the paper. In particular, the direct dominance relations and the indirect preference and dominance relations are presented there. In section 3, all those relations are combined to give, through maximizing the resulting combined relation, a solution to any decision problem that is at least as selective as the solution achieved by maximizing with respect to the original preferences, and, in cases, it is more selective than this latter solution. However, the fact that the space of alternatives is finite give the additional possibility of defining a more demanding choice criterion such that maximizing in regard to it gives to any decision problem a solution which (1) is at least as selective as that given by the combined preference relation presented in Section 3, (2) may be more selective than it, and (3) in contrast with that relation, satisfies all the standard consistency conditions on choice functions. All this is shown in Section 4. Section 5 is merely a summary.

2.- NOTATION AND BASIC NOTIONS

Let us summarize some basic notions of choice theory. A (weak) preference relation is a reflexive binary relation defined on a space of alternatives *X*. The relation *R* may represent an agent's preferences or it may express a social preference obtained by an aggregation procedure. The expression '*xRy*' can be read 'option *x* is at least as good as option *y*'. The strict preference relation *P* is the asymmetric part of *R*, and the indifference relation *I* is its symmetric part; '*xPy*' means '*x* is preferred to *y*', or '*x* is better than option *y*', and '*xIy*' means '*x* and *y* are indifferent'. The relations *P* and *I* are defined as usual: (a) *xPy* if and only if (hereafter 'iff') *xRy* but not *yRx*; and (b) *xIy* iff *xRy* and *yRx*. By definition, *P* is asymmetric, that is, if *xPy* then not *yPx*; and *I* is symmetric, that is, if *xIy* then *yIx*.

R is reflexive on set *X* iff for all $x \in X$, *xRx*. *R* is said to be complete if and only if for every $x, y \in X$, *xRy* or *yRx*. Let us write '$x \|_R y$' to express that neither *xRy* nor *yRx*.

R is transitive iff for every $x, y, z \in X$, if *xRy* and *yRz*, then *xRz*; it is said to be quasi-transitive when its asymmetric part *P* is transitive; and *R* is acyclic if for every sequence

$x^1, x^2,.., x^n$, such that x^iPx^{i+1} for every i=, 1,2.., n-1, not $x^nP\, x^1$. In addition, let R be PI-transitive when $xPyIz$ implies xPz; and IP-transitive when $xIyPz$ implies xPz. R is said to be a pre-order when it is reflexive and transitive. It is a complete pre-order if in addition, it is also complete.

Any non-empty subset S of X represents a choice problem. A choice rule h assigns to any choice problem S a subset of S. This can be expressed symbolically as $h(S)\subseteq S$. This subset is usually called the choice set.[4] An outstanding property that a choice rule h may have is that it returns a non-empty set as solution for any choice problem S, i.e. for every $S\subseteq X$ such that $S\neq\emptyset$, $h(S)\neq\emptyset$. When rule h satisfies this property it is said to be a choice function.

The standard choice rule is the maximization rule. Given a preference relation R, this rule assigns to any choice problem S the set of the maximal elements in S with respect to P. In symbols,

$$\mu(S|R)=\{x\in S\text{: there is no } y\in S \text{ such that } yPx\}.$$[5]

If R is complete, maximization coincides with optimization. The optimization rule assigns to any choice set S the set of optimal elements in S under relation R. In symbols,

$$\omega(S|R)=\{x\in S\text{: } xRy \text{ for every } y\in S\}.$$

Notice that in Example 2 optimization is a choice function. That means that even if optimization is a choice function, there may be reasons why some optimal alternatives are preferable to others.

Indirect preference relations

Let $(x^q)= (x^1,.., x^n)$, $n\geq 2$ and for any i=1,..,n, $x^i\in X$, be a sequence of alternatives in X. We call it *linear* iff for any i, j=2,..,n-1, (1) $x^1\neq x^i\neq x^n$; and (2) $x^i\neq x^j$.

Let us say that $(x^q)= (x^1,.., x^n)$ is a *downward sequence* if (1) for any j=1,..,n-1, x^jRx^{j+1}, and (2) there exists a h=1,..,n-1, such that x^hPx^{h+1}. We call it *strictly downward* if (1) it is downward, and (2) for any j=1,..,n-1, x^jPx^{j+1}.

Let us say that alternative x is *indirectly preferred* to alternative y, in symbols $x\overleftrightarrow{P}y$, iff (1) there is a linear and downward sequence $(c^q)= (c^1,.., c^r)$ such that $x=c^1$ and $y=c^r$, and there is no linear and downward sequence $(d^q)= (d^1,.., d^s)$ such that $y=d^1$ and $x=d^s$.

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[4] The domain of the choice rule may be restricted to a proper subclass of non-empty subsets of X (*cf.* Richter, 1971; Moulin, 1985: 149). This paper does not follow this restriction.

[5] Notice that in Example 2 optimization is a choice function. This means that even if optimization is a choice function, there may be reasons why some optimal alternatives are preferable to others.

Something similar may be said about the condition proposed in Fuchs-Selinger and Mayer (2003) based on a concept of domination different from that used by Subiza and Peris and used also in this paper. The condition in question is proposed to ensure that there are optimal alternatives in every choice problem. In our framework, it requires that for any choice problem $S\subseteq X$ and any of its subsets $S'\subseteq S$, there is an option $x\in S$ such that xRy for all $y\in S'$. Obviously, the condition holds in Example 2, and in particular it is satisfied by the choice problem $\{x, z\}$. However, it would be preferable to choose $\{x\}$ rather than $\{x, z\}$.

Furthermore, we can say that alternative x is *strongly and indirectly preferred* to alternative y, in symbols $x\vec{P}y$, iff there is a linear and strictly downward sequence $(a^q)=(a^1,.., a^n)$ such that $x=a^1$ and $y=a^n$, and there is no linear and strictly downward sequence $(b^q)=(b^1,.., b^m)$ such that $y=b^1$ and $x=b^m$.[6]

Dominance relations

As pointed out above, Subiza and Peris employ the relation of (weak) dominance as an additional criterion for refining the solution generated by maximizing with respect to the original preference relation.

Given any weak preference relation R (or a strict preference relation P), let us say that option *x [weakly] R* (or *P*)-*quasi-dominates option y* (*in the context X*) - in symbols xR^Dy - iff for every $z\in X$, (1) if yPz then xPz and (2) if zPx then zPy.[7]

We denote by P^D the asymmetric part of R^D, and say that x [*weakly*] R (or P)-*dominates option y* when xP^Dy. It should be noted that, by definition, R^D is reflexive and transitive and P^D is asymmetric and transitive, independently of which properties R may meet. Therefore, if X is finite, then $\mu(.|P^D)$ is a choice function. Furthermore, it is easy to check that if P is transitive, then xPy implies xP^Dy and, therefore, $\mu(.|R^D)\subseteq\subseteq\mu(.|R)$.

When R is not complete it is possible that xRz while $y\|_R\, z$ for some z may occur. Therefore, in the eventuality that preferences may not be complete, the following dominance relation may also be relevant: xR^dy iff (1) xR^Dy , (2)for all z such that $x\neq z\neq y$, if zRx then zRy, and (3) for all v such that $x\neq v\neq y$, if yRv then xRv. Let us denote by P^d the asymmetric part of R^d. As before, those definitions directly imply that (1) R^d is reflexive and transitive, (2) P^d is asymmetric and transitive, and (3) $\mu(.|P^d)$ is a choice function.

Substitution rules and refinements

Let us call *choice environment* any pair (X, R) where X is a non-empty set and R is a preference relation defined on X. Given a class Γ of choice environments, a *substitution rule* is a function f that assigns a choice environment to every choice environment in the class Γ.

Let us say that a substitution rule f *potentially refines the choice environment* (X, R) if for any choice problem $S\subseteq X$, (1) $\mu(S|R')\neq\emptyset$, and (2) $\mu(S|R')\subseteq\mu(S|R)$, where $(X , P')=f(X, R)$. The rule f *refines the choice environment* (X, P) if (1) it potentially refines this environment, and (2) there exists a $S'\subseteq X$ such that $\mu(S'|R')\subset \mu(S'|R)$.

Let us say, in addition, that a substitution rule f (1) is a *potential refinement for the class* Γ if it potentially refines all the choice environments in such a class; and that (2) it is a

[6] Since R is acyclic, $\vec{P}=P^\infty$, where P^∞ is the transitive closure of P.
[7] As Subiza and Peris (2002: 4) point out with regards to the dominance relation: 'This relation is somewhat similar to the 'covering relation' defined by Miller (1977) and Fishburn (1977) for tournaments (complete [and asymmetric] preference relations) and later extended for general preference relations in Schwartz (1986)'.

refinement for the class Γ if it potentially refines all the choice environments in such a class, and it refines at least one of such environments.

Posed in those terms, the question to be addressed is how the preference and dominance relations that can be defined on the basis of R in the way shown in this section - namely, the relations P^D, P^d, $\vec{P}$, $\vec{P}^D$, $\overleftrightarrow{P}$ and $\overleftrightarrow{P}^D$ - can help to refine the class of all the choice environments (X, R) where X is a finite set and R is an acyclic preference relation (let us call such environments here after finite and asymmetric choice environments).

3.- COMBINING THE DIRECT AND INDIRECT DOMINANCE RELATIONS

Let (X, R) any finite and acyclic choice environment. It should be noted that R may not be transitive, nor even quasi-transitive. Notice, moreover, that since R is acyclic then μ is a choice function, *i.e.* for every non-empty $S \subseteq X$, $\mu(S|R) \neq \emptyset$ (Moulin, 1985: 151).

In the case that the original preference relation $R^0 \neq R$ is reflexive but not necessarily acyclic, P, I and R can be defined in the following way:

(1) xPy iff (a) xP^0y and (b) there is not any sequence $(a^1,.., a^n)$ such that $y= a^1$, and $x= a^n$, for any i=1,.., n-1, $a^iP^0a^{i+1}$, and $a^nP^0a^1$. This definition implies that P is acyclic.

(2) xIy iff (a) xI^0y or (b) there is a sequence $(a^1,.., a^n)$ such that $y= a^1$, and $x= a^n$, for any i=1,.., n-1, $a^iP^0a^{i+1}$, and $a^nP^0a^1$.

(3) xRy iff xPy or xIy.

We have remarked above that in the standard framework the strong preference relation P may be understood as a criterion for excluding from choice those alternatives that are worse than another according to it. In an similar way, the induced strict preference and dominance relations that have been introduced in the preceding section - namely, the relations P^D, P^d, $\vec{P}$, $\vec{P}^D$, $\overleftrightarrow{P}$ and $\overleftrightarrow{P}^D$ - may be seen as additional criteria for rejecting those alternatives that, according to any one of them, are worse than another.

Let $(P \cup P^D)$ denote the relation 'xPy or xP^Dy'. Obviously, if $\mu(S|R) \cap \mu(S|R^D) \neq \emptyset$ then that relation potentially refines the environment (X, R) because trivially $\mu(S|R) \cap \mu(S|R^D) \subseteq \mu(S|R)$.

Abusing of notation, let $(P \cup P^D \cup \vec{P} \cup \vec{P}^D)$ denote the relation 'xPy or xP^Dy or $x\vec{P}y$ or $x\vec{P}^Dy$'. It is also trivial that the relation $(P \cup P^D \cup \vec{P} \cup \vec{P}^D)$ potentially refines the environment (X, R), if $\mu(S|R) \cap \mu(S|R^D) \cap \mu(S|\vec{P}) \cap \mu(S|\vec{R}^D) \neq \emptyset$. [8]

It should be noted, however, that since P is acyclic, $\vec{P}$ is transitive. Therefore, $\mu(S|\vec{R}^D) \subseteq \mu(S|\vec{P})$. On the other hand, $\mu(S|\vec{P}) = \mu(S|R)$, because if xPy then $x\vec{P}y$, and for any $z \in X$, if there exists a v such that $v\vec{P}z$ then there exists a w such that wPz. Therefore, the relation $(P^D \cup \vec{P}^D)$ can help in refining the environment (X, R) at least as much as the relation $(P \cup P^D \cup \vec{P} \cup \vec{P}^D)$ might do.

In addition, the following result is straightforward.

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[8] Apologies for this abuse of notation.

Lemma 1.- The relation $(P^D \cup \vec{P}^D)$ is transitive.

HINT.- It follows from the transitivity of P^D and $\vec{P}^D$, and from the following, easily verified fact: if (xP^Dy and $x\vec{P}^Dy$) or ($x\vec{P}^Dy$ and yP^Dz), then $x\vec{P}^Dz$.

Hence, $\mu(S|R^D) \cap \mu(S|\vec{R}^D) \neq \varnothing$, so that the environment $(X|R^D \cup \vec{R}^D)$ potentially refines the environment (X, R).

The following two examples show that the relations P^D and $\vec{P}^D$ are logically independent from the other, in the sense that (1) there are cases where for some alternatives xP^Dy but not $x\vec{P}^Dy$, and (2) there are cases where for some alternatives $x\vec{P}^Dy$ but not xP^Dy.

Example 3.- xPz, xPv, yPv and vPz. In this case, xP^Dy but not $x\vec{P}^Dy$.
Example 4.- xPz, yPv and zPv. In this case, $x\vec{P}^Dy$ but not xP^Dy.

Notice, also, that the relation $\overleftrightarrow{R}^D$ is transitive.[9]

Next, in order to potentially refine the environment (X, R) with the help of the relations P^D, $\vec{P}^D$, $\overleftrightarrow{P}^D$ and P^d, let us now define the relations $\succ$ and $\succcurlyeq$ on their basis:

(1) if $x, y \notin \mu(X|P^D \cup \vec{P}^D)$, then $x \succ y$ iff, xP^Dy or $x\vec{P}^D$;
(2) if $x \in \mu(X|P^D \cup \vec{P}^D)$, and $y \notin \mu(X|P^D \cup \vec{P}^D)$,
then $x \succ y$;
(3) if $x, y \in \mu(X|P^D \cup \vec{P}^D)$, then $x \succ y$ iff $x\overleftrightarrow{P}^Dy$;
(4) if $[x \in \mu(\mu(X|P^D \cup \vec{P}^D)|\overleftrightarrow{P}^D)$ and
$y \notin \mu(\mu(X|P^D \cup \vec{P}^D)|\overleftrightarrow{P}^D)]$, then $x \succ y$;
(5) if $x, y \in \mu(\mu(X|P^D \cup \vec{P}^D)|\overleftrightarrow{P}^D)$, then $x \succ y$ iff xP^dy; and
(6) $x \succcurlyeq y$ iff not $y \succ x$.

With regard to this new relation, consider the following result.

Lemma 2.- The relation $\succ$ is transitive.

PROOF.-
If $x \succ y$ and $y \succ z$, then there are only five possible cases:

(1) $x \in \mu(\mu(X|P^D \cup \vec{P}^D)|\overleftrightarrow{R}^D)$ and $z \notin \mu(\mu(X|P^D \cup \vec{P}^D)|\overleftrightarrow{R}^D)$;
(2) $x \in \mu(X|P^D \cup \vec{P}^D)$, and $z \notin \mu(X|P^D \cup \vec{P}^D)$;
(3) $x, y, z \in \mu(\mu(X|P^D \cup \vec{P}^D)|\overleftrightarrow{R}^D)$;
(4) $x, y, z \in \mu(X|P^D \cup \vec{P}^D)$; and

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[9] In contrast, if R is not complete, then $\overleftrightarrow{P}$ may not be transitive, as the following example shows: $a^1Ia^2Pa^3Ia^4=b^1Pb^2Ib^3Ib^4$, where $a^2=b^3$. It is true that $a^1\overleftrightarrow{P}a^4=b^1$, and $a^4=b^1\overleftrightarrow{P}b^4$, but there is no downward linear sequence $(c^q)=(c^1,.., c^r)$ such that $a^1=c^1$ y $b^4=c^r$.

(5) $x, y, z \notin \mu(X | P^D \cup \vec{P}^D)$.

In cases (1) and (2), $x \succ z$ by definition.
In cases (3), (4) and (5), the transitivity of P^d, $\overleftrightarrow{P}^D$ and $(P^D \cup \vec{P}^D)$ implies $x \succ z$. Q.E.D.

Obviously, Lemma 2 implies that $\mu(. | \succcurlyeq) = \mu(\mu(\mu(S | P^D \cup \vec{P}^D) | \overleftrightarrow{R}^D) | R^d)$ is a choice function.

Let us now turn to the question whether the relation $\succcurlyeq$ can help in refining the class of the finite and acyclic choice environments $(X | R)$.

Theorem 1.- (a) The choice environment $(X | \succcurlyeq)$ potentially refines the choice environment $(X | R)$. (b) Let f be the substitution rule that assigns to any choice environment $(X' | R')$ the environment $(X' | \succcurlyeq')$, where $\succ'$ is induced by R' in the same way that R induces $\succcurlyeq$. The rule f is a potential refinement for the class of all the finite and acyclic choice environments.

PROOF.- In regard with (a), since R is acyclic, if xPy then $x\vec{P}^D y$. Thus, $y \notin \mu(X | P^D \cup \vec{P}^D)$. If $x \in \mu(X | P^D \cup \vec{P}^D)$, then by definition, $x \succ y$. If, on the contrary, $x \notin \mu(X | P^D \cup \vec{P}^D)$, then $x\vec{P}^D y$ implies $x \succ y$. Therefore, if xPy then $x \succ y$. Hence, $\mu(. | \succcurlyeq)$ is a choice function. On the other hand, since $x\vec{P}^D y$ implies $x \succ y$, then for any $S \subseteq X$, $\mu(S | \succcurlyeq) \subseteq \mu(S | P)$. In regard with (b), notice that $(X | P)$ is any finite and acyclic choice environment. Q.E.D.

In order to show that the substitution rule f is a refinement for the class of all the finite and acyclic choice environments $(X' | R')$, notice that in both examples 3 and 4, $\mu(\{x, y, z, v\} | R) = \{x, y\}$ while $\mu(\{x, y, z, v\} | (P^D \cup \vec{P}^D) = \{x\}$.

It should be noted, additionally, that there are cases where
(1) $\mu(\mu(X | P^D \cup \vec{P}^D) | \overleftrightarrow{R}^D) \subset \mu(X | P^D \cup \vec{P}^D)$, and
(2) $\mu(\mu(\mu(X | P^D \cup \vec{P}^D) | \overleftrightarrow{R}^D) | xP^d y) \subset \mu(\mu(X | P^D \cup \vec{P}^D) | \overleftrightarrow{R}^D)$.

With regard to case (1), consider the following example:

Example 5.- xPy, xIz, zPv, vIw, wPr; $\mu(\{x, y, z, v, w, r\} | P^D \cup \vec{P}^D) = \{x, z, w\}$, while $\mu(\mu(\{x, y, z, v, w, r\} | P^D \cup \vec{P}^D) | \overleftrightarrow{R}^D) = \{x, z\}$.

With regard to case (2), consider the example:

Example 6.- xIy, z; $\mu(\mu(\{x, y, z\} | P^D \cup \vec{P}^D) | \overleftrightarrow{R}^D) = \{x, y, z\}$, while $\mu(\mu(\mu(\{x, y, z\} | P^D \cup \vec{P}^D) | \overleftrightarrow{R}^D) | R^d) = \{x, y\}$.

Thus, these examples show that we should not dispense with relation $\overleftrightarrow{R}^D$ nor with relation R^d.

It should also be noted that as R approaches being complete and transitive, the induced combined relation $\succcurlyeq$ becomes both simpler and closer to the original preferences, as the following theorem shows.

Theorem 2.-

(1) If P is transitive, then

(1.b) $\mu(R^D)\subseteq \mu(R)$;

(1.a) $P=\vec{P}$; therefore, $P^D=\vec{P}^D$;

(1.c) hence, the relation $\succ$ can be defined in the following equivalent way:

(1) if $x, y\notin \mu(X|R^D)$, then $x \succ y$ iff, xP^Dy;

(2) if $x\in \mu(X|R^D)$, and $y\notin \mu(X|R^D)$, then $x\succ y$;

(3) if $x, y\in \mu(X|R^D)$, then $x \succ y$ iff $x\overleftrightarrow{P}^Dy$;

(4) if $[x\in \mu(\mu(X|R^D)|\overleftrightarrow{R}^D)$ and $y\notin \mu(\mu(X|P^D)|\overleftrightarrow{R}^D)]$, then $x \succ y$;

(5) if $x, y\in \mu(\mu(X|R^D)|\overleftrightarrow{R}^D)$, then $x \succ y$ iff xP^dy;

and (1.d) for any $S\subseteq X$, $\mu(S|\succcurlyeq)=\mu(\mu(\mu(S|R^D)|\overleftrightarrow{R}^D)|R^d)$.

(2) If P is transitive and R is both PI-transitive and IP-transitive, then

(2.a) $P=\overleftrightarrow{P}$; therefore, $P^D=\overleftrightarrow{P}^D$;

(2.b) $\mu(X|R^D)=\mu(\mu(X|R^D)|\overleftrightarrow{R}^D)$;

(2.c) therefore, the relation $\succ$ can be defined in the following equivalent way:

(1) if $x, y\notin \mu(X|R^D)$, then $x \succ y$ iff, xP^Dy;

(2) if $x\in \mu(X|R^D)$, and $y\notin \mu(X|R^D)$, then $x\succ y$;

(3) if $x, y\in \mu(X|R^D)$, then $x \succ y$ iff xP^dy;

and (2.d) for any $S\subseteq X$, $\mu(S|\succcurlyeq)=\mu(\mu(S|R^D)|R^d)$.

(3) If R is complete, then

(3.a) $P^D=P^d$;

(3.b) $\mu(X|P^D\cup\vec{P}^D)=\mu(\mu(X|P^D\cup\vec{P}^D)|\overleftrightarrow{R}^D)$;

(3.c) hence, the relation $\succ$ can be defined in the following equivalent way:

(1) if $x, y\notin \mu(X|P^D\cup\vec{P}^D)$, then $x \succ y$ iff, xP^Dy or $x\vec{P}^D y$;

(2) if $x\in \mu(X|P^D\cup\vec{P}^D)$, and $y\notin \mu(X|P^D\cup\vec{P}^D)$, then $x\succ y$;

(3.d) therefore, for any $S\subseteq X$, $\mu(S|\succcurlyeq)=\mu(S|P^D\cup\vec{P}^D)$.

(4) If R is complete and P is transitive, then

(4.a) the relation $\succ$ can be defined in the following equivalent way:

(1) if $x, y\notin \mu(X|R^D)$, then $x \succ y$ iff, xP^Dy;

(2) if $x\in \mu(X|R^D)$, and $y\notin \mu(X|R^D)$, then $x\succ y$;

(4.b) therefore, for any $S\subseteq X$, $\mu(S|\succcurlyeq)=\mu(S|R^D)$.

(5) If P is transitive and R is complete, PI-transitive and IP-transitive, then

(5.a) $P=P^D$; thus

(5.b) $x \succ y$ iff xPy, and therefore $x\succcurlyeq y$ iff xRy;

(5.c) hence, for any $S\subseteq X$, $\mu(S|\succcurlyeq)=\mu(S|R)$.

PROOF.-

The only point that may not seem obvious is the point (3.b). To prove it, let (1) R be complete, (2) $x\neq y$, (3) $x, y\in \mu(X|P)$, and (4) $x, y\in \mu(X|R^D)$. Next, let us assume that there is a downward linear sequence $(a^q)=(a^1,..,a^n)$ such that $x=a^1$ and $y=a^n$. Let a^i be the last component of (a^q) such that a^iPa^{i+1}. Since $x, y\in \mu(X|R)$ and R is complete, xRa^i, yRa^i, xRa^{n-1} and $a^{n-1}Iy$. Since $x, y\in \mu(X|R^D)$, if xPa^{n-1} then there is a z such that $yPzRx$ and the the point (3.b) would hold. Thus, $a^{n-1}Ix$. Then, the sequence $(y, a^i, a^{i+1},.., a^{i+1}, a^{i+2},.., a^{n-2},$

a^{n-1}, x) such that yRa^i or $y=a^i$, a^iPa^{i+1}, $a^{i+1}Ia^{i+2}$,.., $a^{n-2}Ia^{n-1}$, $a^{n-1}I\,x$ is linear and downward. Hence, if there is a downward linear sequence (a^1,..,a^n) such that $x=a^1$ and $y=a^n$, then there is another downward linear sequence (b^1,..,b^m) such that $y=b^1$ and $y=b^m$. So, neither $x\overleftrightarrow{P}y$, nor $x\vec{P}^Dy$. Thus, $\mu(X|P^D\cup\vec{P}^D)=\mu(\mu(X|P^D\cup\vec{P}^D)|\overleftrightarrow{R}^D)$. Q.E.D.

It should be noted that when R is complete, neither $\overleftrightarrow{P}^D$ nor P^d helps in selecting a proper subset of $\mu(X|P^D\cup\vec{P}^D)$. We can therefore, in this case, dispense with them in defining the relation $\succ$.

Let us now turn to another question. Given a substitution rule f and a given class Γ^0 of choice environments, let be Γ^i the class $\Gamma^i=\{(X', R')$: there is an environment $(X, R)\in\Gamma^{i-1}$ such that $(X', R')=f(X, R)\}$, for any $i=1,\ldots$. Imagine that for every i, Γ^i is a refinement for the class Γ^{i-1}. At which stage should the search for the finest selection of maximal alternatives stop?[10]

This indeterminacy would be avoided if, for any $i=1$ the rule f is not a refinement for Γ^{i-1}, in the sense that for any environment $(X, R)\in\Gamma^i$, $(X, R)=f(X, R)$. In particular, this implies that that for any environment $(X, R)\in\Gamma^0$, $f(X, R)\subseteq(X, R)$. However, the procedure of inducing the preference relations $\succ$ and $\succcurlyeq$ from a finite and acyclic choice environment does not even satisfy this property for Γ^0, as the following example shows:

Example 7.- xPy, xPw, zPv, zPw, vPw. Notice that R is quasi-transitive, and that neither xP^Dz, nor $x\vec{P}^Dz$, nor zP^Dx, nor $z\vec{P}^Dx$. Therefore, $x\sim z$. But xP^Da for any a such that zP^Da, and there is a b such xP^Db and not zP^Db. Therefore, $x\succ^1 z$, where $\succcurlyeq^1$ is the relation induced by $\succcurlyeq$ in the same way that R induces $\succcurlyeq$.

4.- TOWARDS A MORE DEMANDING CHOICE CRITERION

Whilst $\succ$ is transitive, the induced relation of weak preference $\succcurlyeq$ may be not. As a consequence, maximizing in respect of $\succcurlyeq$ may fail to satisfy the strongest among the standard consistency conditions such as Arrow's Condition [if $S\subseteq T$ and $h(T)\cap S\neq\varnothing$, then $h(T)\cap S=h(S)$] and Sen's β Condition [if $S\subseteq T$ and $h(T)\cap h(S)\neq\varnothing$, then $h(S)\subseteq h(T)$], for instance. (For a summary of such conditions see Moulin, 1985).

Nevertheless, since X is finite and $\mu(.|\succcurlyeq)$ is a choice function, $\mu(.|\succcurlyeq)$ induces a complete pre-order in the way that described below. Therefore, maximizing in respect of that preorder satisfies all of the above mentioned standard consistency conditions on choice functions.

Imagine that S is a proper subset of set T, that x is selected by $\mu(.|\succcurlyeq)$ in T and S, and that y is chosen by $\mu(.|\succcurlyeq)$ in S but not in T. Since T is larger than S, x is chosen in a more demanding environment than y. According to the intuition behind the criterion that we are introducing, this is why x exhibits a certain asymmetry over y. Hence, since x is

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[10] This question was suggested to me by Josep Peris.

also selected in S, it makes sense to advocate that the choice of y in S should be discarded.[11]

In order to apply this suggestion, consider the sequence A^1, A^2, .., A^r of subsets of X such that (a) $A^1 = X$; (b) $A^i = A^{i-1} - \mu(A^{i-1} | \succcurlyeq)$; and (c) $A^r = \emptyset$, $A^{r-1} \neq \emptyset$, where i=2,.., r. Given that X is finite and $\mu(. | \succcurlyeq)$ is a choice function, such a sequence exists and is unique. In addition, it is a partition of X. Thus, for every $x \in X$ there is one and only one A^i such that (1) $x \in \mu(A^i | \succcurlyeq)$ and (2) for every $j > i$, $x \notin \mu(A^j | \succcurlyeq)$. Let A_x denote this set, and let us define the following induced preference relation: $x \tilde{\geqq} y$ iff $A^i = A_x$ and $A^j = A_y$ and $i \leq j$.

It is easy to check that the relation $\tilde{\geqq}$ is reflexive, complete and transitive, in other words, that it is a complete pre-order. In consequence, maximizing with respect to it (1) is a choice function; (2) it coincides with optimizing with respect to $\tilde{\geqq}$; and (c) both choice rules satisfy the most demanding consistency conditions such as those mentioned above.

Like the relation $\succcurlyeq$, the relation $x \tilde{>} y$ helps in potentially refining the class of all the finite and acyclic choice environments $(X | R)$. Notice in this respect that, by definition, if $x \succ y$ then $x \tilde{>} y$. Hence, for all $S \subseteq X$, $\mu(S | \tilde{\geqq}) \subseteq \mu(S | \succcurlyeq)$. Morover, as it is stated in the proof of Theorem 1, xPy implies $x \succ y$. Thus, it is ensued that if xPy then $x \tilde{>} y$, and that $\mu(S' | \tilde{\geqq}) \subseteq \mu(S' | R)$ for any $S' \subseteq X$,. In other words, the choice environment $(X | \tilde{\geqq})$ potentially refines the choice environment $(X | R)$. In consequence, the substitution rule f is a potential refinement for the class of all the finite and acyclic choice environments $(X' | R')$, where f' is the substitution rule that assigns to any choice environment $(X' | R')$ the environment $(X' | \tilde{\geqq}')$ and $\tilde{\geqq}'$ is induced by R' in the same way that R induces $\tilde{\geqq}$.

In addition, each of the examples 3-6 shows the existence of a finite space of alternatives X, a reflexive and acyclic preference relation R defined on X, and a non-empty choice set $S \subseteq X$ such that $\mu(S | \tilde{\geqq}) \subset \mu(S | R)$.

The question, then, is whether introducing relation $\tilde{\geqq}$ adds something to the work done by $\succcurlyeq$ in refining the class of the finite and acyclic choice environments $(X' | R')$, that is, whether are there cases where $\mu(S' | \tilde{\geqq}) \subset \mu(S' | \succcurlyeq)$.

The following example shows the answer to be affirmative:

Example 8.- rPx, rPv, sPt, sPy, sPz, xPv, tPy, tPz, yPz. Notice that $\mu(\{x, y, z, v\} | \succcurlyeq) =$ $= \{x, y\}$ wilst $\mu(\{x, y, z, v\} | \tilde{\geqq}) = \{x\}$, because $x \sim y$, $x \succ v$, $x \succ z$, $y \succ v$, $y \succ z$, but $x \cong t$ and $x \tilde{>} a$ for any $a \in \{y, z, v\}$.

We have shown above that xPy implies $x \tilde{>} y$. Additionally, it is easy to check that if $\succcurlyeq$ satisfies the properties of PI-transitivity and IP-transitivity, then $\succcurlyeq = \tilde{\geqq}$. Thus, as a corollary of Theorem 2, it is also easy to confirm that if R is complete, PI-transitive and IP-transitive, then $R = \succcurlyeq = \tilde{\geqq}$.

[11] These considerations may be also understood as a justification for Sen's Condition β.

It is remarkable, however, that the refinement procedures proposed in this paper may fail to satisfy Arrow's Independence of Irrelevant Alternatives [given any choice rule h, if R and R' have the same restriction on a choice problem $S \subseteq X$, then $h(S|R) = h(S|R')$], in the following sense.

Consider the choice rule $\mu^{\succcurlyeq}(.|R'')$ such that $\mu^{\succcurlyeq}(S|R'') = \mu(S|\succcurlyeq)$, for any $S \subseteq X$, and the choice rule $\mu^{\tilde{\geqq}}(S|R'')$ such that $\mu^{\tilde{\geqq}}(S|R'') = \mu(S|\tilde{\geqq})$. Let (X, R) and (X, R') be two choice environments such that X is finite and both R and R' are reflexive and acyclic. It can be easily checked that there may be an $S \subseteq X$ where R and R' have the same restriction and such that $\mu^{\succcurlyeq}(S|R) \neq \mu^{\succcurlyeq}(S|R')$, or $\mu^{\tilde{\geqq}}(S|R) \neq \mu^{\tilde{\geqq}}(S|R')$, or both.

Notice, for such choice functions, that in order to choose from any S it is necessary to gather information about the available comparisons between any alternatives in X and in S as well as those out of this set. In addition, if S is not a set in the sequence $A^1, A^2, .., A^r$, $\mu^{\tilde{\geqq}}(S|R)$ is the subset of alternatives in S that are chosen because they should be chosen in another larger choice problem.

Indeed, the refinement procedures shown in this paper may fail to satisfy the following weaker condition on reversion of preferences: Let (X, R) and (X', R') be two choice environments such that X' is finite, $X \subseteq X'$ and both R and R' are reflexive and acyclic. If $x \succ y$ ($x \gtrsim y$) then not $y \succ' x$ (not $y \gtrsim' x$).[12]

With this in mind, consider the following example:

Example 9.- $X = \{x, y, z, v, w\}$; $X' = \{x, y, z, v, w\} \cup \{r\}$; xPy, xPz, xPw, vPw; and $xP'y$, $xP'z$, $xP'w$, $vP'w$, $vP'r$, $rP'x$. Therefore, xP^Dy, xP^Dz, xP^Dv, xP^Dw, $x \succ v$, $x \gtrsim v$ and $\{x\} = \mu^{\succcurlyeq}(X|R) = \mu^{\tilde{\geqq}}(X|R)$. However, not $xP^{D\prime}v$; on the contrary, $vP^{D\prime}x$, $v\vec{P}^{D\prime}y$, $v\vec{P}^{D\prime}z$, $v\vec{P}^{D\prime}w$ and $v\vec{P}^{D\prime}r$; hence $v \succ' x$, $v \tilde{\geqq}' x$ and $\{v\} = \mu^{\succcurlyeq}(X'|R') = \mu^{\tilde{\geqq}}(X'|R')$.

Now, let $f^{\succcurlyeq}$ be the function that assigns the environment $(X, \succcurlyeq)$ to the choice environment (X, R) in the way shown in the preceding section, let $f^{\tilde{\geqq}}$ be the function that assigns the environment $(X, \tilde{\geqq})$ to the choice environment $(X, \succcurlyeq)$ in the way shown in this section, and let $f_{\succcurlyeq}^{\tilde{\geqq}} = f^{\tilde{\geqq}} \circ f^{\succcurlyeq}$. Given that $\tilde{\geqq}$ is reflexive, complete and transitive, $(X, \tilde{\geqq}) = f^{\tilde{\geqq}}(X, R) = f^{\tilde{\geqq}}(f^{\tilde{\geqq}}(X, R))$. However, the risk of indeterminacy does not disappear, because its source is $f^{\succcurlyeq}$. For example, it might be the case that $\tilde{\geqq}'' \subset \tilde{\geqq}'$, where $(X, \tilde{\geqq}'') = f^{\tilde{\geqq}}(f^{\succcurlyeq}(f^{\succcurlyeq}(X, R)))$ and $(X, \tilde{\geqq}') = f^{\tilde{\geqq}}(f^{\succcurlyeq}(X, R)))$.

It should be noted, in addition, that for large choice problems S, $\mu(S|\succcurlyeq')$ may be 'too large'.

5.- SUMMARY

When the set of maximal alternatives in a choice problem is not a singleton,

[12] In contrast, this condition is satisfied by each of the six relations used in the paper as additional rejection criteria, namely, P^D, P^d, $\vec{P}$, $\vec{P}^D$, $\overleftrightarrow{P}$ and $\overleftrightarrow{P}^D$.

it is common practice to interpret that the last choice will end up being made by some random procedure, assigning to any maximal alternative the same probability of being chosen. However, there may be reasons based on the same original preferences for which it is suitable to select certain maximal alternatives over others.

In that respect, this paper follows the strategy of defining additional criteria induced by the original preferences and apply them for rejecting alternatives from being chosen and then accepting the rest. Specifically, six preference relations have been introduced to that end, namely: the dominance relation P^D defined on the basis of the original strict preference P, a second dominance relation P^d also defined on the basis of P, the indirect preference relation $\vec{P}$, the dominance relation $\vec{P}^D$ defined on the basis of $\vec{P}$, the indirect preference relation $\overleftrightarrow{P}$ and the corresponding dominance relation $\overleftrightarrow{P}^D$.

First, it has been shown that the relation $(P^D \cup \vec{P}^D)$ is transitive (in Lemma 1) and that maximizing with respect to it is a choice function, and it has been remarked that the relation $(P^D \cup \vec{P}^D)$ can help in refining the finite and acyclic choice environment (X, R) at least as much as the relation $(P \cup P^D \cup \vec{P} \cup \vec{P}^D)$ might do. In addition, by means of examples 3 and 4 it has been also shown that the relations P^D and $\vec{P}^D$ are logically independent one from the other.

Next, given that the relations $(P^D \cup \vec{P}^D)$, $\overleftrightarrow{P}^D$ and P^d are transitive, the combined induced relation $\succcurlyeq$ has been defined using those relations for that task. After showing that $\succ$ is transitive and that maximizing with respect to it is a choice function, Theorem 1 states that the substitution rule $f^{\succcurlyeq}$ that assigns the relation $\succcurlyeq$ to the finite and acyclic choice environment (X, R) potentially refines the class of the finite and acyclic choice environments. Moreover, theorems 3-6 shows that $f^{\succcurlyeq}$ is a refinement for such a class of choice environments, showing also that the relations may $\overleftrightarrow{P}^D$ and P^d may mean that maximizing with respect to $\succcurlyeq$ provides a finer solution that maximizing with respect to $(P^D \cup \vec{P}^D)$.

It has also been shown, in Theorem 2, that as R approaches being complete and transitive, the relation $\succcurlyeq$ becomes both simpler and closer to the original preferences. A salient corollary of this theorem is that the approach proposed in this paper may be more useful when the original preferences are incomplete.

Section 3 finished pointing out that applying $f^{\succcurlyeq}$ for refining the original choice environment may generate indeterminacy as to when should stop the selection of maximal alternatives.

By exploiting the fact that X is finite, the preorder $\gtrsim$ induced by $\succcurlyeq$ has been introduced. Since it is complete and transitive, $\gtrsim$ satisfies the most demanding standard consistency requirements. It has been also stated that the substitution rule $f^{\gtrsim}$ that assigns the relation $\gtrsim$ to the environment (X, R) is a refinement for the class of the finite and acyclic choice environments. And it has been shown that maximizing with respect to $\gtrsim$ not only helps in refining the finite and acyclic choice environment (X, R) at least as much as the maximizing with respect to $\succcurlyeq$, but also that there are cases where maximizing with respect to $\gtrsim$ provides a more selective solution.

Given that ≽ and ≧ use information from any part of the space *X*, it is not surprising that these relations may fail to meet Arrow's Independence of Irrelevant Alternatives. In fact, they may fail likewise to meet the weaker condition on the reversion of preferences introduced at the end of Section 4, a condition satisfied by each of the six relations used as additional rejection criteria in this paper.

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