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Research Article

Concerning Asymptotic Behavior for Extremal Polynomials Associated to Nondiagonal Sobolev Norms

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Let $\mathbb{P}$ be the space of polynomials with complex coefficients endowed with a nondiagonal Sobolev norm $\|\cdot\|_{W^{1,p}(V\mu)}$, where the matrix V and the measure μ constitute a p-admissible pair for $1 \le p \le \infty$. In this paper we establish the zero location and asymptotic behavior of extremal polynomials associated to $\|\cdot\|_{W^{1,p}(V\mu)}$, stating hypothesis on the matrix V rather than on the diagonal matrix appearing in its unitary factorization.

1. Introduction

In the last decades the asymptotic behavior of Sobolev orthogonal polynomials has been one of the main topics of interest to investigators in the field. In [1] the authors obtain the nth root asymptotic of Sobolev orthogonal polynomials when the zeros of these polynomials are contained in a compact set of the complex plane; however, the boundedness of the zeros of Sobolev orthogonal polynomials is an open problem, but as was stated in [2], it could be obtained as a consequence of the boundedness of the multiplication operator $Mf(z) = zf(z)$. Thus, finding conditions to ensure the boundedness of M would provide important information about the crucial issue of determining the asymptotic behavior of Sobolev orthogonal polynomials (see, e.g., [3–13]). The more general result on this topic is [3, Theorem 8.1] which characterizes in terms of equivalent norms in Sobolev spaces the boundedness of M for the classical diagonal norm

$$\|q\|_{W^{N,p}(\mu_0,\mu_1,\dots,\mu_N)} := \left(\sum_{k=0}^{N} \left\|q^{(k)}\right\|_{L^p(\mu_k)}\right)^{1/p} \tag{1}$$

(see Theorem 3 below, which is [3, Theorem 8.1] in the case $N = 1$). The rest of the above mentioned papers provides conditions that ensure the equivalence of norms in Sobolev spaces, and consequently, the boundedness of M.

Results related to nondiagonal Sobolev norms may be found in [5, 6, 14–19]. Particularly, in [5, 6, 15, 18, 19] the authors establish the asymptotic behavior of orthogonal polynomials with respect to nondiagonal Sobolev inner products and the authors in [5] deal with the asymptotic behavior of extremal polynomials with respect to the following nondiagonal Sobolev norms.

Let $\mathbb{P}$ be the space of polynomials with complex coefficients and let μ be a finite Borel positive measure with compact support $S(\mu)$ consisting of infinitely many points in the complex plane; let us consider the diagonal matrix $\Lambda := \operatorname{diag}(\lambda_j)$, $j = 0, \dots, N$, with λ_j being positive μ-almost everywhere measurable functions, and $U := (u_{jk})$, $0 \le j, k \le N$, a matrix of measurable functions such that the matrix $U(x) = (u_{jk}(x))$, $0 \le j, k \le N$ is unitary μ-almost everywhere. If $V := U\Lambda U^*$, where U^* denotes the transpose conjugate of U (note that then V is a positive definite matrix

μ-almost everywhere), and $1 \le p < \infty$ we define the Sobolev norm on the space of polynomials $\mathbb{P}$

$$\begin{aligned}\|q\|_{W^{N,p}(V\mu)} &:= \left(\int \left[\left(q, q', \ldots, q^{(N)}\right) V^{2/p} \times \left(q, q', \ldots, q^{(N)}\right)^* \right]^{p/2} d\mu \right)^{1/p} \\ &:= \left(\int \left[\left(q, q', \ldots, q^{(N)}\right) U \Lambda^{2/p} U^* \times \left(q, q', \ldots, q^{(N)}\right)^* \right]^{p/2} d\mu \right)^{1/p}.\end{aligned} \quad (2)$$

In [20, Chapter XIII] certain general conditions imposed on the matrix V are requested in order to guarantee the existence of an unitary representation with measurable entries.

If U is not the identity matrix μ-almost everywhere, then (2) defines a nondiagonal Sobolev norm in which the product of derivatives of different order appears. We say that $q_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is an nth monic extremal polynomial with respect to the norm (2) if

$$\|q_n\|_{W^{N,p}(V\mu)} = \inf \left\{ \|q\|_{W^{N,p}(V\mu)} : q(z) = z^n + b_{n-1}z^{n-1} + \cdots + b_1 z + b_0,\ b_j \in \mathbb{C} \right\}. \quad (3)$$

It is clear that there exists at least an nth monic extremal polynomial. Furthermore, it is unique if $1 < p < \infty$. If $p = 2$, then the nth monic extremal polynomial is precisely the nth monic Sobolev orthogonal polynomial with respect to the inner product corresponding to (2).

In [5, Theorem 1] the authors showed that the zeros of the polynomials in $\{q_n\}_{n\ge0}$ are uniformly bounded in the complex plane, whenever there exists a constant C such that $\lambda_j \le C\lambda_k$, μ-almost everywhere for $0 \le j, k \le N$. This property made possible to obtain the nth root asymptotic behavior of extremal polynomials (see [5, Theorems 2 and 6]). Although it is required compact support for μ, this is, certainly, a natural hypothesis: if $S(\mu)$ is not bounded, then we cannot expect to have zeros uniformly bounded, not even in the classical case (orthogonal polynomials in L^2); see [21].

Taking $N = 1$, $1 \le p \le 2$ and setting up hypothesis on the matrix V (see (4)) rather than on the diagonal matrix λ, the authors of [22] the following equivalent result to [5, Theorem 1].

Theorem 1 (see [22, Theorem 4.3]). *Let γ be a finite union of rectifiable compact curves in the complex plane, μ a finite Borel measure with compact support $S(\mu) = \gamma$, V a positive definite matrix μ-almost everywhere and*

$$V^{2/p} = \begin{pmatrix} a_p & b_p \\ \overline{b_p} & c_p \end{pmatrix}. \quad (4)$$

Assume that $1 \le p \le 2$, $(c_p^{p/2} d\mu/ds)^{-1} \in L^{1/(p-1)}(\gamma)$, and the norms in $W^{1,p}((a_p^{p/2} + c_p^{p/2})\mu, c_p^{p/2}\mu)$ and $W^{1,p}(a_p^{p/2}\mu, c_p^{p/2}\mu)$ are equivalent on $\mathbb{P}$. Let $\{q_n\}_{n\ge0}$ be a sequence of extremal polynomials with respect to (2). Then the multiplication operator is bounded with the norm $W^{1,p}(V\mu)$ and the zeros of $\{q_n\}_{n\ge0}$ lie in the bounded disk $\{z : |z| \le 2 \| M \|\}$.

In this paper we improve Theorem 1 in two directions: on the one hand, we enlarge the class of measures μ considered and, on the other hand, we prove our result for $1 \le p < \infty$ (see Theorem 19). In order to describe the measures we will deal with, we introduce the definition of p-admissible pairs as follows: given $1 \le p < \infty$, we say that the pair (V, μ) is p-admissible if μ is a finite Borel measure which can be written as $\mu = \mu_1 + \mu_2$, its support $S(\mu)$ is a compact subset of the complex plane which contains infinitely many points, and V is a positive definite matrix μ-almost everywhere with $|b_p|^2 \le (1-\varepsilon_0)a_p c_p$, μ_1-almost everywhere for some fixed $0 < \varepsilon_0 \le 1$; the support $S(\mu_2)$ is contained in a finite union of rectifiable compact curves γ with $(c_p^{p/2} d\mu_2/ds)^{-1} \in L^{1/(p-1)}(\gamma)$ if $\gamma \ne \emptyset$, $V^{2/p} := \begin{pmatrix} a_p & b_p \\ \overline{b_p} & c_p \end{pmatrix}$ and $d\mu_2/ds$ is the Radon-Nykodim derivative of μ_2 with respect to the Euclidean length in γ.

We want to make three remarks about this definition. First of all, since $V = \begin{pmatrix} a_2 & b_2 \\ \overline{b_2} & c_2 \end{pmatrix}$ is a positive definite matrix μ-almost everywhere, $V^{2/p}$ also has this property and hence $|b_p|^2 < a_p c_p$, μ-almost everywhere.

In order to obtain $(c_p^{p/2} d\mu_2/ds)^{-1} \in L^{1/(p-1)}(\gamma)$ the best choice for μ_2 is the restriction of μ to γ.

Note that the support of μ is an arbitrary compact set: we just require that $S(\mu_2)$ (the part of $S(\mu)$ in which $V^{2/p}$ is about to be a degenerated quadratic form, when $|b_p|^2$ is very close to $a_p c_p$) is a union of curves.

Therefore, with the results on p-admissible pairs we complement and improve the study started in [22], where the case $\mu = \mu_2$ with $1 \le p \le 2$ was considered.

Another interesting property which could be studied is the asymptotic estimate for the behavior of extremal polynomials because, in this setting, there does not exist the usual three-term recurrence relation for orthogonal polynomials in L^2 and this makes it really difficult to find an explicit expression for the extremal polynomial of degree n. In this regard, Theorems 22 and 23 deduce the asymptotic behavior of extremal polynomials as an application of Theorems 18 and 19. More precisely, we obtain the nth root and the zero counting measure asymptotic both of those polynomials and their derivatives to any order. The study of the nth root asymptotic is a classical problem in the theory of orthogonal polynomials; see for instance, [1, 2, 5, 23, 24].

Furthermore, in Theorem 23 we find the following asymptotic relation:

$$\lim_{n\to\infty} \frac{q_n^{(j+1)}(z)}{n q_n^{(j)}(z)} = \int \frac{d\omega_{S(\mu)}(x)}{z - x} \quad (5)$$

for any $j \ge 0$.

The main idea of [5, 6, 22] and this paper is to compare nondiagonal and diagonal norms.

When it comes to compare nondiagonal and diagonal norms, [25] is remarkable, since the authors show that symmetric Sobolev bilinear forms, like symmetric matrices, can be rewritten with a diagonal representation; unfortunately, the entries of these diagonal matrices are real measures, and we cannot use this representation since we need positive measures for the Sobolev norms.

Finally, we would like to note that the central obstacle in order to generalize the results given in this paper and [22] to the case of more derivatives is that there are too many entries in the matrix V and just a few relations to control them (see Lemma 8 and notice that some limits appearing in that Lemma do not provide any new information). In that case we have just three entries (a_p, b_p, c_p), but in the simple case of two derivatives ($N = 2$) we have

$$V := \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ \overline{a_{12}} & a_{22} & a_{23} \\ \overline{a_{13}} & \overline{a_{23}} & a_{33} \end{pmatrix}, \qquad (6)$$

and we would need to control six functions $(a_{11}, a_{12}, a_{13}, a_{22}, a_{23}, a_{33})$; in the general case with N derivatives, we would need to control $(N+1)(N+2)/2$ functions.

The outline of the paper is as follows. In Section 2 we provide some background and previous results on the multiplication operator and the location of zeros of extremal polynomials. We have devoted Section 3 to some technical lemmas in order to simplify the proof of Theorem 17 about the equivalence of norms; in fact, in these lemmas the hardest part of this proof is collected. In Section 4 we give the proof of that Theorem and in Section 5 we deduce some results on asymptotic of extremal polynomials.

2. Background and Previous Results

In what follows, given $1 \le p < \infty$ we define

$$\begin{aligned} &\|f\|_{W^{1,p}(a_p^{p/2}\mu, c_p^{p/2}\mu)} \\ &\quad := \left(\int \left(a_p^{p/2} |f|^p + c_p^{p/2} |f'|^p \right) d\mu \right)^{1/p}, \\ &\|f\|_{W^{1,p}(D\mu)} := \left(\int \left(a_p |f|^2 + c_p |f'|^2 \right)^{p/2} d\mu \right)^{1/p}, \qquad (7) \\ &\|f\|_{W^{1,p}(V\mu)} \\ &\quad := \left(\int \left(a_p |f|^2 + c_p |f'|^2 2\Re \left(b_p f \overline{f'} \right) \right)^{p/2} d\mu \right)^{1/p}, \end{aligned}$$

for every polynomial f.

It is obviously much easier to deal with the norms $\|\cdot\|_{W^{1,p}(a_p^{p/2}\mu, c_p^{p/2}\mu)}$ and $\|\cdot\|_{W^{1,p}(D\mu)}$ than with the one $\|\cdot\|_{W^{1,p}(V\mu)}$. Therefore, one of our main goals is to provide weak hypotheses to guarantee the equivalence of these norms on the linear space of polynomials $\mathbb{P}$ (see Section 4).

In order to bound the zeros of polynomials, one of the most successful strategies has certainly been to bound the multiplication operator by the independent variable $Mf(z) = zf(z)$, where

$$\|M\| := \sup \left\{ \|zq(z)\|_{W^{1,p}(V\mu)} : \|q\|_{W^{1,p}(V\mu)} = 1 \right\}. \qquad (8)$$

Regarding this issue, the following result is known.

Theorem 2 (see [5, Theorem 3]). *Let μ be a finite Borel measure in $\mathbb{C}$ with compact support let and $1 \le p < \infty$. Let $\{q_n\}_{n\ge0}$ be a sequence of extremal polynomials with respect to (2). Then the zeros of $\{q_n\}_{n\ge0}$ lie in the disk $\{z : |z| \le 2 \| M \|\}$.*

It is also known the following simple characterization of the boundedness of M.

Theorem 3 (see [3, Theorem 8.1]). *Let μ be a finite Borel measure in $\mathbb{C}$ with compact support; α, β nonnegative measurable functions; and $1 \le p < \infty$. Then the multiplication operator is bounded in $W^{1,p}(\alpha\mu, \beta\mu)$ if and only if the following condition holds:*

$$\begin{aligned} &\textit{the norms in } W^{1,p}\left((\alpha+\beta)\mu, \beta\mu\right) \\ &\textit{and } W^{1,p}\left(\alpha\mu, \beta\mu\right) \textit{ are equivalent on } \mathbb{P}. \end{aligned} \qquad (9)$$

It is clear that if there exists a constant C such that $\beta \le C\alpha$ μ-almost everywhere, then (9) holds. In [8, 13] some other very simple conditions implying (9) are shown.

In what follows, we will fix a p-admissible pair (V, μ) with $1 \le p < \infty$; then $S(\mu_2)$ is contained in a finite union of rectifiable compact curves γ in the complex plane; each of these connected components of γ is not required to be either simple or closed.

3. Technical Lemmas

For the sake of clarity and readability, we have opted for proving all the technical lemmas in this section. This makes the proof of Theorem 17 much more understandable.

The following result is well known.

Lemma 4. *Let us consider $1 \le \alpha < \infty$. Then*

$$(x+y)^\alpha \le 2^{\alpha-1}\left(x^\alpha + y^\alpha\right) \quad \textit{for every } x, y \ge 0. \qquad (10)$$

Lemma 5 (see [22, Lemma 3.1]). *Let us consider $0 < \alpha \le 1$. Then*

(1) *$|y|^\alpha - |x|^\alpha \le |y - x|^\alpha$ for every $x, y \in \mathbb{R}$;*

(2) *$2^{\alpha-1}(y^\alpha + x^\alpha) \le (y+x)^\alpha \le y^\alpha + x^\alpha$ for every $x, y \ge 0$.*

Lemma 6 (see [22, Lemma 3.2]). *Let $\{s_n\}_n$ and $\{t_n\}_n$ be two sequences of positive numbers. Then*

$$\lim_{n\to\infty} \frac{2 s_n t_n}{s_n^2 + t_n^2} = 1 \quad \textit{iif} \lim_{n\to\infty} \frac{s_n}{t_n} = 1. \qquad (11)$$

In what follows a_p, b_p, and c_p refer to the coefficients of the fixed matrix $V^{2/p}$.

Definition 7. We say that $\{f_n\}_n \subset \mathbb{P}$ is an extremal sequence for p if, for every n, $\| f_n\|_{L^\infty(\mu_2)} = 1$ and

$$\lim_{n\to\infty} \frac{\int \left|2b_p f_n f_n'\right|^{p/2} d\mu_2}{\int \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} = 1. \tag{12}$$

Lemma 8. *If* $1 \le p < \infty$ *and* $\{f_n\}_n$ *is an extremal sequence for* p*, then*

$$\lim_{n\to\infty} \frac{\int \left|b_p f_n f_n'\right|^{p/2} d\mu_2}{\int \left(\sqrt{a_p c_p}\,|f_n f_n'|\right)^{p/2} d\mu_2} = 1,$$

$$\lim_{n\to\infty} \frac{\int \left(\sqrt{a_p c_p}\,|f_n f_n'|\right)^{p/2} d\mu_2}{\left(\int \left(a_p|f_n|^2\right)^{p/2} d\mu_2\right)^{1/2}\left(\int \left(c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{1/2}} = 1,$$

$$\lim_{n\to\infty} \frac{2\left(\int \left(a_p|f_n|^2\right)^{p/2} d\mu_2\right)^{1/2}\left(\int \left(c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{1/2}}{\int \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2} = 1,$$

$$\lim_{n\to\infty} \frac{2^{p/2-1}\int \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2}{\int \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} = 1,$$

$$\lim_{n\to\infty} \frac{\int \left(a_p|f_n|^2\right)^{p/2} d\mu_2}{\int \left(c_p|f_n'|^2\right)^{p/2} d\mu_2} = 1. \tag{13}$$

Proof. The case $1 \le p \le 2$ is a consequence of [22, Lemmas 3.5 and 3.6]. We deal now with the case $p > 2$. First note that we can rewrite limit (12) in Definition 7 as the limit of the following product:

$$\lim_{n\to\infty} \frac{\int \left|b_p f_n f_n'\right|^{p/2} d\mu_2}{\int \left(\sqrt{a_p c_p}\,|f_n f_n'|\right)^{p/2} d\mu_2} \cdot \frac{\int \left(\sqrt{a_p c_p}\,|f_n f_n'|\right)^{p/2} d\mu_2}{\left(\int \left(a_p|f_n|^2\right)^{p/2} d\mu_2\right)^{1/2}\left(\int \left(c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{1/2}} \cdot \frac{2\left(\int \left(a_p|f_n|^2\right)^{p/2} d\mu_2\right)^{1/2}\left(\int \left(c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{1/2}}{\int \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2} \cdot \frac{2^{p/2-1}\int \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2}{\int \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} = 1. \tag{14}$$

Since the limit of the product is 1, if we prove that the first, third, and fourth factors tend to 1 as n tends to infinity, then the limit of the second factor must also be 1.

So, our problem is reduced to show

$$\lim_{n\to\infty} \frac{\int \left|b_p f_n f_n'\right|^{p/2} d\mu_2}{\int \left(\sqrt{a_p c_p}\,|f_n f_n'|\right)^{p/2} d\mu_2} = 1, \tag{15}$$

$$\lim_{n\to\infty} \frac{2\left(\int \left(a_p|f_n|^2\right)^{p/2} d\mu_2\right)^{1/2}\left(\int \left(c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{1/2}}{\int \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2} = 1, \tag{16}$$

$$\lim_{n\to\infty} \frac{2^{p/2-1}\int \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2}{\int \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} = 1, \tag{17}$$

$$\lim_{n\to\infty} \frac{\int \left(a_p|f_n|^2\right)^{p/2} d\mu_2}{\int \left(c_p|f_n'|^2\right)^{p/2} d\mu_2} = 1. \tag{18}$$

Again, we can rewrite the limit in the definition of extremal sequence as the limit of the following product:

$$\lim_{n\to\infty} \frac{\int \left|b_p f_n f_n'\right|^{p/2} d\mu_2}{\int \left(\sqrt{a_p c_p}\,|f_n f_n'|\right)^{p/2} d\mu_2} \cdot \frac{\int \left(2\sqrt{a_p c_p}\,|f_n f_n'|\right)^{p/2} d\mu_2}{\int \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} = 1. \tag{19}$$

The two factors above are nonnegative and less than or equal to 1 using, respectively, that $|b_p|^2 < a_p c_p$ μ_2-almost everywhere and $2xy \le x^2 + y^2$. Thus,

$$\lim_{n\to\infty} \frac{\int \left|b_p f_n f_n'\right|^{p/2} d\mu_2}{\int \left(\sqrt{a_p c_p}\,|f_n f_n'|\right)^{p/2} d\mu_2} = 1 = \lim_{n\to\infty} \frac{\int \left(2\sqrt{a_p c_p}\,|f_n f_n'|\right)^{p/2} d\mu_2}{\int \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2}, \tag{20}$$

and (15) holds.

Given $\varepsilon > 0$, for each n let us define the following sets:

$$F_{n,\varepsilon} := \left\{ z \in S(\mu_2) : \frac{1}{1+\varepsilon} \le \frac{\sqrt{a_p}\,|f_n|}{\sqrt{c_p}\,|f_n'|} \le 1+\varepsilon \right\}, \qquad F_{n,\varepsilon}^c := S(\mu_2) \setminus F_{n,\varepsilon}. \tag{21}$$

Let us consider the strictly decreasing function $A(t) := 2t/(t^2+1)$ on $[1,\infty)$. If $t \ge 1+\varepsilon$, then $A(t) \le A(1+\varepsilon) =: C_\varepsilon < A(1) = 1$. Consequently, if $x/y \ge 1+\varepsilon$, then

$2xy/(x^2+y^2) = A(x/y) \le C_\varepsilon$, and if $x/y \le 1/(1+\varepsilon)$, then $y/x \ge 1+\varepsilon$ and $2xy/(x^2+y^2) = A(y/x) \le C_\varepsilon$. Therefore,

$$\frac{\int_{F_{n,\varepsilon}^c} \left(2\sqrt{a_p c_p}\left|f_n f_n'\right|\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} \le C_\varepsilon^{p/2} < 1. \quad (22)$$

Using this fact and (20), we have

$$\begin{aligned}
1 = {} & \lim_{n\to\infty}\Bigg(\Bigg(\Bigg(\Bigg(\int_{F_{n,\varepsilon}} \left(2\sqrt{a_p c_p}\left|f_n f_n'\right|\right)^{p/2} d\mu_2\Bigg) \\
& \times\Bigg(\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\Bigg)^{-1}\Bigg) \\
& + \Bigg(\Bigg(\int_{F_{n,\varepsilon}^c} \left(2\sqrt{a_p c_p}\left|f_n f_n'\right|\right)^{p/2} d\mu_2\Bigg) \\
& \times\Bigg(\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\Bigg)^{-1}\Bigg)\Bigg) \\
& \times\Bigg(1 + \Bigg(\Bigg(\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\Bigg) \\
& \times\Bigg(\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu\Bigg)^{-1}\Bigg)\Bigg)^{-1}\Bigg) \\
\le {} & \Bigg(C_\varepsilon^{p/2} + \liminf_{n\to\infty}\Bigg(\Bigg(\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\Bigg) \\
& \times\Bigg(\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\Bigg)^{-1}\Bigg)\Bigg) \\
& \times\Bigg(1 + \liminf_{n\to\infty}\Bigg(\Bigg(\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\Bigg) \\
& \times\Bigg(\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\Bigg)^{-1}\Bigg)\Bigg)^{-1}.
\end{aligned} \quad (23)$$

If we assume that $\liminf_{n\to\infty}((\int_{F_{n,\varepsilon}} (a_p|f_n|^2 + c_p|f_n'|^2)^{p/2} d\mu_2)(\int_{F_{n,\varepsilon}^c} (a_p|f_n|^2 + c_p|f_n'|^2)^{p/2} d\mu_2)^{-1}) = l < \infty$, then from the previous inequality we have $1 \le ((C_\varepsilon^{p/2} + l)/(1+l)) < 1$, and this is a contradiction. Hence, $\lim_{n\to\infty}((\int_{F_{n,\varepsilon}} (a_p|f_n|^2 + c_p|f_n'|^2)^{p/2} d\mu_2)(\int_{F_{n,\varepsilon}^c} (a_p|f_n|^2 + c_p|f_n'|^2)^{p/2} d\mu_2)^{-1}) = \infty$ and consequently,

$$\lim_{n\to\infty}\frac{\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} = 0. \quad (24)$$

Since for each n, we have

$$\begin{aligned}
& \frac{\int_{F_{n,\varepsilon}^c} \left(2\sqrt{a_p c_p}\left|f_n f_n'\right|\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} \\
& \quad \le \frac{\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2},
\end{aligned} \quad (25)$$

then (24) implies that

$$\lim_{n\to\infty}\frac{\int_{F_{n,\varepsilon}^c} \left(2\sqrt{a_p c_p}\left|f_n f_n'\right|\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} = 0. \quad (26)$$

On the other hand, using (20) it is easy to deduce that

$$\begin{aligned}
& \lim_{n\to\infty}\Bigg(\Bigg(\Bigg(\Bigg(\int_{F_{n,\varepsilon}} \left(2\sqrt{a_p c_p}\left|f_n f_n'\right|\right)^{p/2} d\mu_2\Bigg) \\
& \quad \times\Bigg(\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\Bigg)^{-1}\Bigg) \\
& \quad + \Bigg(\Bigg(\int_{F_{n,\varepsilon}^c} \left(2\sqrt{a_p c_p}\left|f_n f_n'\right|\right)^{p/2} d\mu_2\Bigg) \\
& \quad \times\Bigg(\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\Bigg)^{-1}\Bigg) \\
& \quad \times\Bigg(1 + \Bigg(\Bigg(\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\Bigg) \\
& \quad \times\Bigg(\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\Bigg)^{-1}\Bigg)\Bigg)^{-1}\Bigg) \\
& = 1.
\end{aligned} \quad (27)$$

Consequently, (24), (26), and (27) give

$$\lim_{n\to\infty}\frac{\int_{F_{n,\varepsilon}} \left(2\sqrt{a_p c_p}\left|f_n f_n'\right|\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} = 1. \quad (28)$$

Furthermore, since

$$\begin{aligned}
0 & \le \frac{1}{(1+\varepsilon)^{p/2}\int_{F_{n,\varepsilon}} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2} \\
& \le \frac{1}{\int_{F_{n,\varepsilon}} \left(\sqrt{a_p c_p}\left|f_n f_n'\right|\right)^{p/2} d\mu_2},
\end{aligned} \quad (29)$$

we obtain

$$0 \le \frac{\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2}{(1+\varepsilon)^{p/2} \int_{F_{n,\varepsilon}} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2} \le \frac{\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(\sqrt{a_p c_p}\,|f_n f_n'|\right)^{p/2} d\mu_2}. \tag{30}$$

Therefore, (24), (28), and (30) give

$$\lim_{n\to\infty} \frac{\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2} = 0. \tag{31}$$

Similar arguments allow us to show

$$\lim_{n\to\infty} \frac{\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2\right)^{p/2} d\mu_2} = 0. \tag{32}$$

From (31) and (32) we obtain

$$\lim_{n\to\infty} \frac{\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2} = 0 = \lim_{n\to\infty} \frac{\int_{F_{n,\varepsilon}^c} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2}, \tag{33}$$

$$\lim_{n\to\infty} \frac{\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2\right)^{p/2} d\mu_2} = 0 = \lim_{n\to\infty} \frac{\int_{F_{n,\varepsilon}^c} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2\right)^{p/2} d\mu_2}. \tag{34}$$

As a consequence of (33) we have

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{\int \left(a_p|f_n|^2\right)^{p/2} d\mu_2}{\int \left(c_p|f_n'|^2\right)^{p/2} d\mu_2} \\
&= \limsup_{n\to\infty} \Bigg(\Bigg(\Bigg(\left(\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2\right)^{p/2} d\mu_2\right) \times \left(\int_{F_{n,\varepsilon}} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{-1}\Bigg) \\
&\quad + \Bigg(\left(\int_{F_{n,\varepsilon}^c} \left(a_p|f_n|^2\right)^{p/2} d\mu_2\right) \times \left(\int_{F_{n,\varepsilon}} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{-1}\Bigg)\Bigg) \\
&\quad \times \Bigg(1 + \Bigg(\left(\int_{F_{n,\varepsilon}^c} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2\right) \times \left(\int_{F_{n,\varepsilon}} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{-1}\Bigg)\Bigg)^{-1}\Bigg) \\
&= \limsup_{n\to\infty} \frac{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2} \le (1+\varepsilon)^p.
\end{aligned} \tag{35}$$

In a similar way we obtain

$$\frac{1}{(1+\varepsilon)^p} \le \liminf_{n\to\infty} \frac{\int \left(a_p|f_n|^2\right)^{p/2} d\mu_2}{\int \left(c_p|f_n'|^2\right)^{p/2} d\mu_2}. \tag{36}$$

Since these inequalities hold for every $\varepsilon > 0$, we conclude that (18) holds. Applying now Lemma 6 we obtain (16).

Using Lemma 4, (18), and (34) we obtain that for every $\varepsilon, \eta > 0$ there exists N such that for every $n \ge N$ the following holds:

$$\begin{aligned}
1 &\le \frac{2^{p/2-1} \int \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2}{\int \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} \\
&\le \frac{2^{p/2-1} \int \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2}{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} \\
&\le \frac{2^{p/2-1} \int \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2}{\left(1 + \left(1/(1+\varepsilon)^2\right)\right)^{p/2} \int_{F_{n,\varepsilon}} \left(a_p|f_n|^2\right)^{p/2} d\mu_2} \\
&\le \frac{2^{p/2} (1+\eta) \int \left(a_p|f_n|^2\right)^{p/2} d\mu_2}{\left(1 + \left(1/(1+\varepsilon)^2\right)\right)^{p/2} \int_{F_{n,\varepsilon}} \left(a_p|f_n|^2\right)^{p/2} d\mu_2}
\end{aligned}$$

$$\leq \frac{2^{p/2}(1+\eta)^2}{\left(1+\left(1/(1+\varepsilon)^2\right)\right)^{p/2}} \cdot \frac{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2\right)^{p/2} d\mu_2}{\int_{F_{n,\varepsilon}} \left(a_p|f_n|^2\right)^{p/2} d\mu_2}$$
$$= \frac{2^{p/2}(1+\eta)^2}{\left(1+\left(1/(1+\varepsilon)^2\right)\right)^{p/2}}. \tag{37}$$

Then (17) follows from the previous inequalities, since $\varepsilon, \eta > 0$ are arbitrary.

This completes the proof. □

Definition 9. For each $0 < \varepsilon < 1$, we define the sets A_ε and A_ε^c as

$$A_\varepsilon := \left\{z \in S(\mu_2) : \left|b_p\right| > (1-\varepsilon)\sqrt{a_p c_p}\right\},$$
$$A_\varepsilon^c := S(\mu_2) \setminus A_\varepsilon. \tag{38}$$

Lemma 10. *If $1 \leq p < \infty$ and $\{f_n\}_n$ is an extremal sequence for p and ε is small enough, then*

$$\lim_{n\to\infty} \frac{\int_{A_\varepsilon} \left|b_p f_n f_n'\right|^{p/2} d\mu_2}{\int \left|b_p f_n f_n'\right|^{p/2} d\mu_2} = 1. \tag{39}$$

Remark 11. The statement of the lemma might seem strange, because we could have a priori $\mu_2(A_\varepsilon) = 0$; however, the existence of the fundamental sequence implies $\mu_2(A_\varepsilon) > 0$.

Proof. If $1 \leq p \leq 2$, then the result follows from [22, Lemma 3.8]. For the case $p > 2$ it suffices to follow the proof of [22, Lemma 3.8] applying Lemma 8 to conclude the result. □

Lemma 12. *If $1 \leq p < \infty$, $\{f_n\}_n$ is an extremal sequence for p and ε is small enough, then*

$$\lim_{n\to\infty} \frac{\int_{A_\varepsilon} \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2}{\int \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2} = 1,$$
$$\lim_{n\to\infty} \frac{\int_{A_\varepsilon^c} \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2}{\int_{A_\varepsilon} \left(\left(a_p|f_n|^2\right)^{p/2} + \left(c_p|f_n'|^2\right)^{p/2}\right) d\mu_2} = 0. \tag{40}$$

Proof. If $1 \leq p \leq 2$, then the result follows from [22, Lemma 3.10]. For the case $p > 2$ it suffices to follow the proof of [22, Lemma 3.10] applying Lemmas 8 and 10 to conclude the result. □

Lemma 13. *If $1 \leq p < \infty$, $\{f_n\}_n$ is an extremal sequence for p and ε is small enough, then*

$$\lim_{n\to\infty} \frac{\int_{A_\varepsilon} \left(a_p|f_n|^2\right)^{p/2} d\mu_2}{\int \left(a_p|f_n|^2\right)^{p/2} d\mu_2} = 1,$$
$$\lim_{n\to\infty} \frac{\int_{A_\varepsilon} \left(c_p|f_n'|^2\right)^{p/2} d\mu_2}{\int \left(c_p|f_n'|^2\right)^{p/2} d\mu_2} = 1. \tag{41}$$

Proof. If $1 \leq p \leq 2$, then the result follows from [22, Lemma 3.11]. For the case $p > 2$ it suffices to follow the proof of [22, Lemma 3.11] applying Lemmas 8, 10, and 12 to conclude the result. □

Lemma 14. *If $1 \leq p < \infty$ and $\{f_n\}_n$ is an extremal sequence for p, then for every $\varepsilon > 0$ small enough with $\mu_2(A_\varepsilon^c) > 0$ and for every $t \in (0,1)$ there exists N such that $\inf_{z\in A_\varepsilon^c}|f_n(z)| < t$ for every $n \geq N$.*

Proof. If $1 \leq p \leq 2$, then the result follows from [22, Lemma 3.12]. For the case $p > 2$ it is sufficient to follow the proof of [22, Lemma 3.12] applying Lemma 13 to conclude the result. □

Definition 15. If f is a continuous function on γ, we define the oscillation of f on γ, and we denote it by $\operatorname{osc}(f)$, as

$$\operatorname{osc}(f) := \sup_{z,w\in\gamma} \left|f(z) - f(w)\right|. \tag{42}$$

Lemma 16 (see [22, Lemma 3.14]). *For $1 \leq p < \infty$, let us assume that γ is connected and $(c_p^{p/2}\, d\mu_2/ds)^{-1} \in L^{1/(p-1)}(\gamma)$, where $d\mu_2/ds$ is the Radon-Nykodim derivative of μ_2 with respect to the Euclidean length in γ. (According to one's notation, if $p = 1$ then $1/(p-1) = \infty$.) Then*

$$\int_\gamma \left|f'\right|^p c_p^{p/2} d\mu_2 \geq k \cdot osc^p(f),$$
$$\text{with } \frac{1}{k} = \left\|\frac{1}{\left(c_p^{p/2}(d\mu_2/ds)\right)}\right\|_{L^{1/(p-1)}(\gamma)}, \tag{43}$$

for every polynomial f.

4. Equivalent Norms

Now we prove the announced result about the equivalence of norms for $1 \leq p < \infty$.

Theorem 17. *Let one consider $1 \leq p < \infty$ and (V, μ) a p-admissible pair. Then the norms $W^{1,p}(a_p^{p/2}\mu, c_p^{p/2}\mu)$, $W^{1,p}(D\mu)$, and $W^{1,p}(V\mu)$ defined as in (3) are equivalent on the space of polynomials $\mathbb{P}$.*

Proof. The equivalence of the two first norms is straightforward, by Lemmas 4 and 5. We prove now the equivalence of the two last norms.

Let us prove that there exists a positive constant $C := C(V, \mu, p)$ such that

$$C\,\|f\|_{W^{1,p}(D\mu)} \le \|f\|_{W^{1,p}(V\mu)} \le \sqrt{2}\,\|f\|_{W^{1,p}(D\mu)}, \quad \text{for every } f \in \mathbb{P}. \tag{44}$$

Let us prove first the second inequality $\|f\|_{W^{1,p}(V\mu)} \le \sqrt{2}\,\|f\|_{W^{1,p}(D\mu)}$.

Note that $|2\Re(b_p f\overline{f'})| \le |2b_p ff'| \le 2\sqrt{a_p c_p}|ff'| \le a_p|f|^2 + c_p|f'|^2$; therefore, for every polynomial f it holds that

$$\begin{aligned}\|f\|^p_{W^{1,p}(V\mu)} &= \int \left(a_p|f|^2 + c_p|f'|^2 + 2\Re\left(b_p f\overline{f'}\right)\right)^{p/2} d\mu \\ &\le 2^{p/2}\int\left(a_p|f|^2 + c_p|f'|^2\right)^{p/2} d\mu \\ &= 2^{p/2}\|f\|^p_{W^{1,p}(D\mu)}.\end{aligned} \tag{45}$$

In order to prove the first inequality, $C\,\|f\|_{W^{1,p}(D\mu)} \le \|f\|_{W^{1,p}(V\mu)}$, note that

$$\begin{aligned}\|f\|^p_{W^{1,p}(V\mu_1)} &= \int \left(a_p|f|^2 + c_p|f'|^2 + 2\Re\left(b_p f\overline{f'}\right)\right)^{p/2} d\mu_1 \\ &\ge \int\left(a_p|f|^2 + c_p|f'|^2 - 2\,\left|b_p ff'\right|\right)^{p/2} d\mu_1 \\ &\ge \int\left(a_p|f|^2 + c_p|f'|^2 - 2\sqrt{1-\varepsilon_0}\sqrt{a_p c_p}\,\left|ff'\right|\right)^{p/2} d\mu_1 \\ &\ge \left(1-\sqrt{1-\varepsilon_0}\right)^{p/2}\int\left(a_p|f|^2 + c_p|f'|^2\right)^{p/2} d\mu_1 \\ &= \left(1-\sqrt{1-\varepsilon_0}\right)^{p/2}\|f\|^p_{W^{1,p}(D\mu_1)}.\end{aligned} \tag{46}$$

If $\gamma = \emptyset$ (i.e., $\mu = \mu_1$), then we have finished the proof. Assume that $\gamma \neq \emptyset$ then we prove $C\ \|f\|_{W^{1,p}(D\mu_2)} \le \|f\|_{W^{1,p}(V\mu_2)}$, seeking for a contradiction. It is clear that it suffices to prove it when γ is connected, that is, when γ is a rectifiable compact curve. Let us assume that there exists a sequence $\{f_n\}_n \subset \mathbb{P}$ such that

$$\lim_{n\to\infty}\frac{\int\left(a_p|f_n|^2 + c_p|f_n'|^2 + 2\Re\left(b_p f_n\overline{f_n'}\right)\right)^{p/2} d\mu_2}{\int\left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2}\, d\mu_2} = 0. \tag{47}$$

If $1 \le p \le 2$, then [22, Lemma 3.1] (with $\alpha = p/2$) gives

$$\begin{aligned}&\int\left(a_p|f_n|^2 + c_p|f_n'|^2 + 2\Re\left(b_p f_n\overline{f_n'}\right)\right)^{p/2} d\mu_2 \\ &\quad\ge \int\left(a_p|f_n|^2 + c_p|f_n'|^2 - \left|2b_p f_n f_n'\right|\right)^{p/2} d\mu_2 \\ &\quad\ge \int\left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2 - \int\left|2b_p f_n f_n'\right|^{p/2} d\mu_2.\end{aligned} \tag{48}$$

This right-hand side of the inequality is positive, because $|2b_p ff'| \le a_p|f|^2 + c_p|f'|^2 \mu_2$-almost everywhere. This implies

$$\lim_{n\to\infty}\left(\left(\int\left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2 - \int\left|2b_p f_n f_n'\right|^{p/2} d\mu_2\right) \times\left(\int\left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{-1}\right) = 0, \tag{49}$$

and hence

$$\lim_{n\to\infty}\frac{\int\left|2b_p f_n f_n'\right|^{p/2} d\mu_2}{\int\left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2} = 1. \tag{50}$$

If $p > 2$, then

$$\begin{aligned}&\left(\int\left(a_p|f_n|^2 + c_p|f_n'|^2 + 2\Re\left(b_p f_n\overline{f_n'}\right)\right)^{p/2} d\mu_2\right)^{2/p} \\ &\quad\ge \left(\int\left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{2/p} - \left(\int\left|2b_p f_n f_n'\right|^{p/2} d\mu_2\right)^{2/p} \\ &\quad\ge 0,\end{aligned} \tag{51}$$

since $|2b_p ff'| \le a_p|f|^2 + c_p|f'|^2 \mu_2$-almost everywhere. Therefore,

$$\lim_{n\to\infty}\left(\left(\left(\int\left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{2/p} - \left(\int\left|2b_p f_n f_n'\right|^{p/2} d\mu_2\right)^{2/p}\right) \times\left(\left(\int\left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{2/p}\right)^{-1}\right) = 0, \tag{52}$$

or, equivalently,

$$\lim_{n\to\infty}\frac{\left(\int\left|2b_p f_n f_n'\right|^{p/2} d\mu_2\right)^{2/p}}{\left(\int\left(a_p|f_n|^2 + c_p|f_n'|^2\right)^{p/2} d\mu_2\right)^{2/p}} = 1, \tag{53}$$

and (50) also holds for $p > 2$.

If f_n is constant for some n, then $\int |2b_p f_n f_n'|^{p/2} d\mu_2 = 0$; therefore, taking a subsequence if it is necessary, without loss of generality we can assume that f_n is nonconstant and $\|f_n\|_{L^\infty(\mu_2)} = 1$ for every n. Then $\{f_n\}_n$ is an extremal sequence for p. Applying Lemma 8,

$$\lim_{n\to\infty} \frac{\int \left(a_p |f_n|^2\right)^{p/2} d\mu_2}{\int \left(c_p |f_n'|^2\right)^{p/2} d\mu_2} = 1. \tag{54}$$

By Lemma 14, there exists $\{z_n\}_n \subset S(\mu_2)$ such that $|f_n(z_n)| \le 1/2$ for every $n \ge N_1$. Now, taking into account that $\|f_n\|_{L^\infty(\mu_2)} = 1$ and that γ is connected, we can apply Lemma 16, and then

$$\begin{aligned} \int \left|f_n'\right|^p c_p^{p/2} d\mu_2 &\ge k \cdot \operatorname{osc}^p(f_n) \\ &\ge k\left(\|f_n\|_{L^\infty(\mu_2)} - |f_n(z_n)|\right)^p \\ &\ge k\left(1 - \frac{1}{2}\right)^p = \frac{k}{2^p} > 0 \end{aligned} \tag{55}$$

for every $n \ge N_1$, with $1/k = \|1/(c_p^{p/2} d\mu_2/ds)\|_{L^{1/(p-1)}(\gamma)}$.

Let us fix ε small enough. On the one hand, by Lemma 13 it holds that

$$\begin{aligned} \int \left(a_p |f_n|^2\right)^{p/2} d\mu_2 &\le 2\int_{A_\varepsilon} \left(a_p |f_n|^2\right)^{p/2} d\mu_2 \\ &\le 2\,\|f_n\|_{L^\infty(\mu_2)}^p \int_{A_\varepsilon} a_p^{p/2} d\mu_2 \\ &= 2\int_{A_\varepsilon} a_p^{p/2} d\mu_2 \end{aligned} \tag{56}$$

for every $n \ge N_2 = N_2(\varepsilon)$.

On the other hand, we have

$$\begin{aligned} \lim_{\varepsilon\to 0^+} \mu_2(A_\varepsilon) &= \lim_{\varepsilon\to 0^+} \mu_2\left(\left\{|b_p| > (1-\varepsilon)\sqrt{a_p c_p}\right\}\right) \\ &= \mu_2\left(\left\{|b_p| \ge \sqrt{a_p c_p}\right\}\right) = 0. \end{aligned} \tag{57}$$

This implies

$$\lim_{\varepsilon\to 0^+} \int_{A_\varepsilon} a_p^{p/2} d\mu_2 = 0. \tag{58}$$

Given any $\delta > 0$ there exists ε_1 with $\int_{A_{\varepsilon_1}} a_p^{p/2}\, d\mu_2 < \delta$. Hence, $\int (a_p|f_n|^2)^{p/2} d\mu_2 < 2\delta$ for every $n \ge N_2(\varepsilon_1)$. Therefore, $\lim_{n\to\infty} \int (a_p|f_n|^2)^{p/2} d\mu_2 = 0$, which is a contradiction with (54) and (55). □

The following result is a direct consequence of Theorems 3 and 17.

Theorem 18. *Let one consider $1 \le p < \infty$ and (V, μ) a p-admissible pair. Then the multiplication operator is bounded in $W^{1,p}(V\mu)$ if and only if the following condition holds:*

$$\begin{aligned} &\text{the norms in } W^{1,p}\left(\left(a_p^{p/2} + c_p^{p/2}\right)\mu, c_p^{p/2}\mu\right) \\ &\text{and } W^{1,p}\left(a_p^{p/2}\mu, c_p^{p/2}\mu\right) \text{ are equivalent on } \mathbb{P}. \end{aligned} \tag{59}$$

This latter theorem and Theorem 2 give the following result.

Theorem 19. *Let one consider $1 \le p < \infty$ and (V, μ) a p-admissible pair such that (59) takes place. Let $\{q_n\}_{n\ge 0}$ be a sequence of extremal polynomials with respect to (2). Then the multiplication operator is bounded and the zeros of $\{q_n\}_{n\ge 0}$ lie in the bounded disk $\{z : |z| \le 2\,\|M\|\}$.*

In general, it is not difficult to check wether or not (59) holds. It is clear that if there exists a constant C such that $c_p \le C a_p$ μ-almost everywhere, then (59) holds. In [8, 13] some other very simple conditions implying (59) are shown.

The following is a direct consequence of Theorem 19.

Corollary 20. *Let one consider $1 \le p < \infty$ and (V, μ) a p-admissible pair. Assume that $c_p \le C a_p$, μ-almost everywhere for some constant C. Let $\{q_n\}_{n\ge 0}$ be a sequence of extremal polynomials with respect to (2). Then the zeros of $\{q_n\}_{n\ge 0}$ are uniformly bounded in the complex plane.*

Finally, we have the following particular consequence for Sobolev orthogonal polynomials.

Corollary 21. *Let (V, μ) be a 2-admissible pair. Assume that there exists a constant C such that $c_2 \le C\, a_2$,μ-almost everywhere. Let $\{q_n\}_{n\ge 0}$ be the sequence of Sobolev orthogonal polynomials with respect to $V\mu$. Then the zeros of the polynomials in $\{q_n\}_{n\ge 0}$ are uniformly bounded in the complex plane.*

5. Asymptotic of Extremal Polynomials

We start this section by setting some notation. Let Q_n, $\|\cdot\|_{L^2(\mu)}$, $\operatorname{cap}(S(\mu))$, and $\omega_{S(\mu)}$ denote, respectively, the nth monic orthogonal polynomial with respect to $L^2(\mu)$, the usual norm in the space $L^2(\mu)$, the logarithmic capacity of $S(\mu)$, and the equilibrium measure of $S(\mu)$. Furthermore, in order to analyze the asymptotic behavior for extremal polynomials we will use a special class of measures, "regular measures," denoted by **Reg** and defined in [24]. In that work, the authors proved (see Theorem 3.1.1) that, for measures supported on a compact set of the complex plane, $\mu \in$ **Reg** if and only if

$$\lim_{n\to\infty} \|Q_n\|_{L^2(\mu)}^{1/n} = \operatorname{cap}(S(\mu)). \tag{60}$$

Finally, if $z_1, z_2, \ldots, z_n$ denote the zeros, repeated according to their multiplicity, of a polynomial q whose degree is exactly n, and δ_{z_j} is the Dirac measure with mass one at the point z_j, the expression

$$\nu(q) := \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j} \tag{61}$$

defines the normalized zero counting measure of q.

We can already state the first result in this section.

Theorem 22. *Let one consider* $1 \leq p < \infty$, (V, μ) *a* p*-admissible pair and* $\{q_n\}_{n\geq 0}$ *the sequence of extremal polynomials with respect to* $\|\cdot\|_{W^{1,p}(V\mu)}$. *Assume that the following conditions hold:*

(i) $a_p^{p/2}\mu \in \mathbf{Reg}$;

(ii) $S(\mu)$ *is regular with respect to the Dirichlet problem;*

(iii) *condition* (59) *takes place.*

Then,

$$\lim_{n\to\infty} \left\| q_n^{(j)} \right\|_{S(\mu)}^{1/n} = cap\,(S(\mu)), \quad j \geq 0. \tag{62}$$

Furthermore, if the complement of $S(\mu)$ *is connected, then*

$$\lim_{n\to\infty} \nu\left(q_n^{(j)}\right) = \omega_{S(\mu)}, \quad j \geq 0. \tag{63}$$

in the weak star topology of measures.

Proof. Note that, in our context, the hypothesis removed with respect to [5, Theorem 2] is equivalent to the following two facts: on the one hand, the multiplication operator is bounded (see Theorem 3), and on the other hand, the norms of $W^{1,p}(a_p^{p/2}\mu, c_p^{p/2}\mu)$ and $W^{1,p}(V\mu)$ defined as in (3) are equivalent (see Theorem 18). With this in mind, we just need to follow the proof of [5, Theorem 2] to conclude the result. □

In the following theorem, we use $g_\Omega(z;\infty)$ to denote the Green's function for Ω with logarithmic singularity at ∞, where Ω is the unbounded component of the complement of $S(\mu)$. Notice that, if $S(\mu)$ is regular with respect to the Dirichlet problem, then $g_\Omega(z;\infty)$ is continuous up to the boundary and it can be extended continuously to all $\mathbb{C}$, with value zero on $\mathbb{C} \setminus \Omega$.

Theorem 23. *Let one consider* $1 \leq p < \infty$, (V, μ) *a* p*-admissible pair and* $\{q_n\}_{n\geq 0}$ *the sequence of extremal polynomials with respect to* $\|\cdot\|_{W^{1,p}(V\mu)}$. *Assume that the following conditions hold:*

(i) $a_p^{p/2}\mu \in \mathbf{Reg}$;

(ii) $S(\mu)$ *is regular with respect to the Dirichlet problem;*

(iii) *condition* (59) *takes place.*

Then, for each $j \geq 0$,

$$\limsup_{n\to\infty} \left| q_n^{(j)}(z) \right|^{1/n} \leq cap\,(S(\mu))\, e^{g_\Omega(z;\infty)}, \tag{64}$$

uniformly on compact subsets of $\mathbb{C}$. *Furthermore, for each* $j \geq 0$,

$$\lim_{n\to\infty} \left| q_n^{(j)}(z) \right|^{1/n} = cap\,(S(\mu))\, e^{g_\Omega(z;\infty)}, \tag{65}$$

uniformly on each compact subset of $\{z : |z| > 2\|M\|\} \cap \Omega$. *Finally, if the complement of* $S(\mu)$ *is connected, one has equality in* (64) *for all* $z \in \mathbb{C}$, *except for a set of capacity zero,* $S(\omega_{S(\mu)}) \subset \{z : |z| \leq 2\|M\|\}$ *and*

$$\lim_{n\to\infty} \frac{q_n^{(j+1)}(z)}{n q_n^{(j)}(z)} = \int \frac{d\omega_{S(\mu)}(x)}{z - x} \tag{66}$$

uniformly on each compact subset of $\{z : |z| > 2\|M\|\}$.

Proof. Note that, in our context, the multiplication operator is bounded (see Theorem 3) and the norms of $W^{1,p}(a_p^{p/2}\mu, c_p^{p/2}\mu)$ and $W^{1,p}(V\mu)$ defined as in (3) are equivalent (see Theorem 18). This is the crucial fact in the proof of this theorem; once we know this, we just need to follow the proof given in [5, Theorem 6] point by point to conclude the result. □

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