SELF-SIMILAR SOLUTIONS TO COAGULATION AND FRAGMENTATION EQUATIONS

A thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Mathematics

Author
Giancarlo Breschi

Advisor
Dr. Marco Antonio Fontelos López

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Figure o.o.1: Zirconia agglomerate chains in a transmission electron microscopy image (the image is taken from Eggersdorfer [106]). Notice the complex ramificated patterns: these structures are self-similar or scale invariant over a limited size range. Self-similarity in coagulating-fragmentating systems is the object of this theses.
"I once wanted to give a few words in the foreword which now actually are not in it, which, however, I'll write to you now because they might be a key for you: I wanted to write that my work consists of two parts: of the one which is here, and of everything which I have not written. And precisely this second part is the important one."

Ludwig Wittgenstein on his Tractatus Logico-Philosophicus, in a letter to Ludwig von Ficker (1919).
Premisa y Objeto del Trabajo

La teoría de la coagulación permite el estudio de un gran número de fenómenos difusos en la naturaleza a varias escalas y en ámbitos lejanos entre sí; se basa en la descripción de procesos de agregación de unidades pequeñas (partículas) a través de sucesivas fusiones en unidades mayores. En los modelos más sencillos, los eventos de coagulación obedecen a un único mecanismo binario caracterizado por las propiedades físicas del sistema. A nivel de descripción cinética, el fenómeno de coagulación se puede resumir así: la interacción en un sistema de partículas se verifica cuando dos partículas chocan y, si se verifican las condiciones adecuadas, en el lugar de encuentro se juntan y dan lugar a una nueva partícula más pesada (la suma de las anteriores) que las sustituye. La nueva partícula puede seguir agregándose según el mismo procedimiento cinético si entra en contacto con otras partículas del sistema.

A partir del mecanismo básico, es posible enriquecer la coagulación binaria con otros mecanismos, como la fragmentación, la difusión o el transporte, la creación-destrucción de partículas (fuentes), crecimiento o erosión de las mismas, etcétera, y derivar aproximaciones de campo medio de las concentraciones (medias estadísticas en sistemas con un número elevado de partículas) para describir su evolución. Esta tesis se ocupa de la ecuación de coagulación de Smoluchowski, célebre modelo de campo medio que estudia la distribución de las partículas de un sistema en función de su masa, y de la ecuación de fragmentación, su modelo dual. En particular, el interés se centra en el comportamiento universal de las soluciones de estas ecuaciones que se aproximan, gracias a oportunos rescalamientos, a regímenes auto-similares.

La teoría que se desarrolla alrededor de estos modelos es sorprendentemente rica, y, a pesar de la aparente simplicidad de los mecanismos básicos, los comportamientos de las soluciones son variados y su comprensión precisa de herramientas que barren un amplio espectro de disciplinas aplicadas y teóricas. Después de casi un siglo desde el célebre trabajo del fundador de la teoría, el físico polaco Marian Smoluchowski [276, 277], esta ha alcanzado un notable grado de madurez, extensión y autonomía con respeto a otros campos; aún quedan abiertos desafíos y cuestiones matemáticas interesantes, por lo que se puede afirmar que la teoría ha conseguido consistencia y una importante atención en el sector. Para un matemático interesado en ecuaciones diferenciales resulta particularmente estimulante trabajar con un modelo que mezcla operadores diferenciales y no locales (y además no lineales), pudiendo valerse de la intuición física por un lado y de la
generalidad del análisis por el otro.

Este trabajo empieza con una presentación general de la materia tratada: una introducción matemática a la coagulación-fragmentación y los mecanismos adicionales, las principales propiedades de los modelos deterministas así como un estado del arte de los resultados de existencia de soluciones. La gran cuestión abierta para estas ecuaciones es el comportamiento para largos tiempos y en presencia de singularidades. Más allá de estimaciones asintóticas y las soluciones explícitas para pocos casos aislados, el comportamiento universal de las soluciones no está comprendido satisfactoriamente. En la Sección 1.3 de la introducción se describen los dos problemas autosimilares tratados y el enfoque que se ha propuesto en la literatura, así como sus limitaciones. En particular, proponemos la hipótesis de soluciones autosimilares de segundo tipo (en el sentido de Barenblatt) para la ecuación de Smoluchowski y detallamos los resultados alcanzados; estos se distribuyen en los capítulos 2 y 3, relativos al caso con y sin singularidad en tiempo finito. En el caso de gelación en tiempo finito, el resultado obtenido es un teorema de punto fijo para la existencia de soluciones autosimilares de segundo tipo con infinitos momentos acotados (pero con masa infinita, como explicaremos en la Sección 1.3) que corresponden a una perturbación del único caso explícito conocido: el núcleo multiplicativo. En el Capítulo 3 proponemos el problema de soluciones autosimilares de segundo tipo para núcleos que no producen gelación en tiempo finito. Los resultados consisten en la construcción de soluciones globales que combina los desarrollos asintóticos y los comportamientos intermedios. En ambos casos (con y sin gelación), los exponentes de autosimilaridad se computan solucionando un adecuado problema de autovalores no lineal que resolvemos con método perturbativo cerca de soluciones explícitas y numéricamente allí donde no se puede extraer información del análisis perturbativo.

Finalmente, el Capítulo 4 trata el problema de soluciones autosimilares para la ecuación lineal de fragmentación irreversible. Este modelo presenta un amplio desarrollo desde el punto de vista de la ecuación de evolución, o, por lo menos, su estudio se encuentra en un estadio más maduro que el modelo no-lineal de coagulación. Además, el comportamiento de universalidad está más o menos satisfactoriamente establecido y se tienen resultados de existencia para la ecuación autosimilar. Nos hemos por tanto centrado en una investigación más específica: determinamos fórmulas explícitas para las soluciones en forma de productos infinitos a través del método de Wiener-Hopf con el cálculo de residuos en el plano complejo tanto para el caso de fragmentación con fragmentos infinitésimos como para el de fragmentación con fracciones mínimas acotadas (también llamado el caso de soporte compacto) que presenta propiedades de analiticidad distintas. Este estudio nos lleva a determinar con precisión los comportamientos asintóticos de las soluciones en los dos casos.
Premise and Object of the Work

Coagulation theory allows the study of a large number of diverse phenomena in nature at various scales and in areas distant to each other; it is based on the description of aggregation processes of smaller units (particles) through successive fusions into larger units. In the simplest models, coagulation events follow a fixed binary mechanism whose characteristics are determined by the physical properties of the system. From a kinetic point of view, the phenomenon of coagulation can be summarized like this: the interaction in a system of particles takes place when two particles collide and, when proper conditions are verified, they stick together in an active site of the surface and give rise to a heavier cluster (with the sum of the previous masses) that replaces them. The new particle can continue aggregating by the same kinetic process if it comes in contact with other particles of the system.

Starting from the basic mechanism, it is possible to enrich the binary coagulation with other mechanisms such as, among others, fragmentation, diffusion or transport, particle creation-destruction (sources), growth or erosion, and derive mean field approximations for the concentrations (statistical averages in systems with a large number of particles) to describe their evolution. This thesis deals with the Smoluchowski coagulation equation, famous mean field model that studies the distribution of the particles of a system in terms of their mass, and the equation of fragmentation, its dual model. In particular, the focus is set on the universal behavior of the solutions of these equations that approach, via proper rescalings, specific self-similar regimes.

The theory developing around these models is surprisingly rich, and, despite the apparent simplicity of the basic mechanisms, the behaviors of the solutions are varied and their understanding requires tools that cover a broad spectrum of theoretical and applied disciplines. After nearly a century since the famous work of the founder of the theory, the Polish physicist Marian Smoluchowski [276, 277], this has reached a remarkable degree of maturity, extension and autonomy with respect to other fields; there remain open challenges and interesting mathematical questions, so we can say that the theory has achieved consistent and significant attention in the sector. For a mathematician interested in differential equations, it is particularly stimulating to consider a model that combines differential and non-local non-linear operators, taking advantage both of the physical intuition and the generality of the analysis.

This work begins with an overview of the subject: a mathematical introduction to coagulation-
fragmentation and their additional mechanisms, the main properties of the deterministic models and a state of the art about the results of existence of solutions. The main open question for these equations, however, is the behavior for long times and in the presence of singularities. Beyond asymptotic estimates and explicit solutions for a few isolated cases, the universal behavior of the solutions is not satisfactorily understood. The two self-similar problems addressed and the approach that has been proposed in the literature (and its limitations) are described in Section 1.3. In particular, we propose there the hypothesis of self-similar solutions of the second kind (in the sense defined by Barenblatt) for the Smoluchowski equation and we give more details on the results achieved; these are therefore distributed in Chapters 2 and 3, respectively to the cases of singularity in finite time or global solutions. As for the gelation in finite time, the result is a fixed point theorem for the existence of self-similar solutions of the second kind with infinite bounded moments (but with infinite mass, as explained in Section 1.3) corresponding to a perturbation of the only known explicit case: the multiplicative kernel. In Chapter 3, we propose the problem of self-similar solutions of the second kind for kernels that do not produce gelation in finite time. The results consist in the construction of global solutions matching asymptotic expansions and intermediate behaviors. In both cases (with and without gelation), the exponents of self-similarity are computed by solving a suitable nonlinear eigenvalue problem. We obtain the latter with perturbation methods close to explicit solutions and rely on numerics where it is not possible to extract information from the perturbative analysis.

Finally, Chapter 4 deals with the problem of self-similar solutions to the linear equation of irreversible fragmentation. This model has been studied thoroughly from the point of view of the evolution equation, or at least its study is far more mature then the non-linear coagulation model. Also, the universality behavior of the solutions is more or less well understood and the self-similar equation has extensive well-posedness results. We focused at carrying over more specific investigations: we rigorously determine an explicit formula in the form of infinite product for the self-similar solutions through a Wiener-Hopf method and through the calculation of residues in the complex plane; this is done for both the case of fragmentation which allows infinitesimal fragments and for fragmentation which bounds from below the minimum fragments (this is called the compact support case and presents different analyticity properties). This study leads us to accurately determine the asymptotic behaviors of the solutions in both cases.
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CHAPTER 1
GENERAL INTRODUCTION

Many interesting mathematical problems arise from experimental sciences. They develop thanks to the cooperation of mathematicians and theoretical scientists that share the interest in formalizing nature’s phenomena and creating new abstract and general frameworks for the issues that concern transversally both disciplines. The characteristics of each discipline provide specific approaches and the advances in an area have influence on the other. When the groundwork establishes questions, models and methods of analysis, these basic elements and the new understandings of a theory can sprout into a new sub-discipline. Processes of this kind are particularly effective in mathematical physics, where the mathematical and physical paradigms, among others, exchange ideas and solutions for the study of groups of differential equations.

With regard to the issues that concern us here, in the development of Coagulation Theory occurred something analogous. The coagulation equation arose in the early twentieth century from the pioneering work of Marian Smoluchowski \[276, 277\] on the physical chemistry of aerosols, and later was employed in the modeling of a constantly growing variety of phenomena (to which we will dedicate a through description in Section \[1.2\]); afterwards, the model has been enriched with different fundamental mechanisms (such as fragmentation) and it inspired the formalization of other kinetic processes that also belong to the field of coagulation and fragmentation. As this trend got stronger, it was also attempted a theoretical merging of different processes with the purpose of a general understanding of all the operators involved. Thanks to the work of pure mathematicians, many new complex models can now be studied. In this sense we can say that analytical methods have been derived from mathematical physics and now their new developments have influence back on it. To name which mathematical fields are most relevantly interested in Coagulation Theory, important contributions to it come from Functional Analysis and PDE, but also from Probability Theory and Stochastic Processes, Dynamical Systems and Numeric Analysis.

The goal of the General Introduction is to present the processes that are subject of this study, reflect the main lines of research and summarize some aspects in the state-of-the-art. In the development of this first chapter, we discuss the basic concepts of self-similar solution and focus on their importance for pure coagulation and pure fragmentation equations. On the one hand, we describe the models and the areas of application as well as list some fundamental bibliographical references; on the other hand, we devote the rest of this introduction to known results on existence and uniqueness problems and on the phenomenon of gelation (Section \[1.2\]); the most important topic is self-similarity and it will be introduced in Section \[1.3\].
Figure 1.0.1: Heterogeneous aerosol coagulation where droplets contain smaller particles (enclosures). The image is taken from Efendiev [101], and represents the growth of nano-composites \( \text{SiO}_2/\text{Fe}_2\text{O}_3 \) in aerosol medium \( \text{SiO}_2 \). This process presents a multi-scale nature and the proposed model uses two different coagulation kernels.

Table 1.1: Simplified classification of colloids dispersed in gasses, liquids and solids with relevant examples.
Finally, let us point out that we have not treated many areas of Coagulation Theory because the scopes of this work are far more restricted than those of a monograph. Probabilistic questions represent a quite independent branch of study as well as applied modelization. Also the numeric analysis topics are left out of this presentation due to the fact that self-similarity issues (that are our main interest) are poorly or unsatisfactorily covered in the literature.

1.1. Coagulation and fragmentation processes

1.1.1. Description of the coagulation model

The coagulation-fragmentation models study the time evolution of a system of aggregates based on their mass or, when considering inhomogeneous systems, their evolution based on the couple of variables mass-position. From the statistical point of view, the particles can be identified by their mass (or mass-position) and, therefore, the function that is important to determine is the average concentration of aggregates in terms of mass (mass-position) related to time. Mass can be substituted by other appropriate physical magnitudes characterizing the units involved in the coagulation system (e.g., there are models that employ a two-dimensional mass-moment couple and others using energy or the radius length); the mathematical literature usually denotes the independent variable as mass for two reasons: the preponderance of applications to real physical systems characterized by mass and the original work of Smoluchowski equation on the physics of aerosols.

Consider first the case of pure or irreversible coagulation: let an homogeneous system of particles distributed according to an initial density $c_0(x)$, where $x$ denotes the mass of the particles. When two units (clusters) of mass $x$ and $y$ get close and interact, they can merge and form a new aggregate with mass $x+y$ which replaces the two preceding particles. This binary mechanism is repeated with nonnegative frequency $K(x,y)$ and depends only on the mass-value of the two involved clusters; therefore, coagulation can be schematized as follows:

$$ \left(1_{x}, 1_{y}, 0_{(x+y)}\right) \xrightarrow{K(x,y)} \left(0_{x}, 0_{y}, 1_{(x+y)}\right). $$

Richer information on the single events requires incorporating spatial magnitudes (position, speed and rate of diffusion) as well as the geometric characteristics or the chemical structure of the clusters, the dependence from macroscopic magnitudes which influence the kinetics of the particles (temperature, pressure, etc.), and the inclusion of additional phenomena such as fragmentation. It must be remarked, here, that the validity and usefulness of this simplified pure coagulation mechanism should be checked empirically for each system; the reason to consider this basic model is that introducing more details into the analysis leads to derive almost intractable equations (at least in a first stage), and it is preferable to undertake the study of this theory with only its essential ingredients.

It is immediate to observe that each event of coagulation reduces the number of clusters in the system, it keeps the total mass invariant while it increases the average size of the aggregates: it is therefore necessary that any description of the evolution of the system derived from law (1.1.1)
still verifies these properties. A further observation is that the frequency $K$ must be a symmetric function $K(x, y) = K(y, x)$ and the mass may range in a continuous or discrete set of values, such as, for example, when the system under consideration presents polymers built out of monomers. The theory developed for the discrete case is analogous to the one for the continuous case (except for some technical details that may vary) and can be treated within the latter. This work is restricted exclusively to the continuous case; therefore, throughout this introduction we will avoid discerning specifically the theory of discrete coagulation from the continuous one. Consider, however, that many techniques and demonstrations are not the same in the two settings, and even some technical assumptions may change: therefore, the interested reader should refer to the specific literature on discrete cases.

Following the useful classification of Laurençot and Mischler [204], the modelization of coagulation processes can be divided into three levels of description (see Figure 1.1.1):

- $\mu$, microscopic level: the system consists of $N$ particles interacting stochastically and the interest lies primarily on the behavior at large numbers $N$. This theory is referred to as finite-volume with mean-field description and differs from the infinite-volume description in that this latter arises at mesoscopic level after deriving a deterministic approximation. There are some principal stochastic models that describe such systems (the Smoluchowski and Markus-Lushnikov processes, for example), and it turned out that it is possible to establish some results of convergence and connect some (few) explicit kernels to suitable processes present in branches of Probability Theory (e.g., continuous and discrete random trees, random graphs). We refer to Aldous [8] and Bertoin [59]. From the applications point of view, one has significant freedom in proposing specific models of this kind, but deriving significant results is a difficult task and even direct simulations suffer from heavy limitations due to computation capabilities (usually the number of real systems particles is many orders of magnitude larger than that of possible simulations).

- $m$, mesoscopic level: the details of the events of coagulation are averaged in the context of a statistical description: the quantity of interest is now the density $c(x, t)\, dx$, defined as the average number of clusters with mass belonging to the continuous range $[x, x + dx]$, and a kinetic mass balance equation is derived for its evolution. The function $K(x, y)$, or coagulation kernel, allows to calculate the rate of occurrence of events of type $\{x, y\}$: the quantity $\left(\frac{1}{2}c(x, t)\, dx\, c(y, t)\, dy\, K(x, y)\right)$ is the average quantity of coagulating mass that, interacting between the intervals $[x, x + dx]$ and $[y, y + dy]$, forms aggregates with mass in the range $\left[ (x + y), (x + y + d(x + y)) \right]$. In this sense, the interest is not focused on the behavior of single clusters, but on the interaction of mass with itself. The exact balance of the variation of $c(x, t)$ leads to the continuous version of the Smoluchowski equation, also known as population balance equation (PBE in some references):

$$
\frac{d}{dt}c(x, t) = \frac{1}{2} \int_0^x K(x - y, y) c(x - y, t) c(y, t)\, dy - c(x, t) \int_0^\infty K(x, y) c(y, t)\, dy. \quad (1.1.2)
$$

Equation (1.1.2) shows two contributions of non-linear and non-local kind: the balance is determined by a gain operator (processes between pairs $(y)\cdot(y - x)$ whose mass sums new $(x)$ clusters) minus the loss term: the interactions between $(x)$-massed clusters and all the others to create heavier aggregates. There are also some alternative mean-field equations.
mainly interested in the microscopic and mesoscopic scales, as well.

Figure 1.1: Early and later stages of coagulation strongly depend on the specific details of the system, such as microscopic description.

- Strong interaction between nanoparticles, leading to larger clusters.
- Well-defined system with a high number of clusters indicated by their mass.
- Specific processes, geometries, and physical contexts.

Interests in observable properties that strongly depend on the specific physical chemistry of the system.

- Polymer chains of gels (in complex of modes).
- Catalyzed growth of polymer chains of gels (in complex of modes).
- Nuclear and monomers at early stages produce almost spherical clusters.

- Well-defined systems with a high number of clusters indicated by their mass.
from Smoluchowski’s one, like the Becker-Döring clusters equation, but we will not enter into the details and refer to Laurençot and Mischler’s review [204].

- *M*, macroscopic level: the interest concerns physically observable quantities, such as the moments of the system, but also the chemical and mechanical properties distinctive of each system: some examples are the aerosol pressure and temperature (if the coagulating medium is a gas), the shape and dispersion of the aggregates, the change of phase (macrostructures emerging or a sudden change of other properties of the solution), the gel phase (porosity, elasticity, density of the gel), etcetera. Since it is not clear how to derive consistently the macroscopic evolution from the microscopic details of coagulation, there are two alternative strategies that can be followed: the first is the physical or chemical phenomenological description (properties are experimentally observed and classified by groups of reagents or types of systems); the second is the physical-mathematical one, in which the equations provide the time evolution of some interesting parameters, such as the moments or other quantities related to the solutions. This last approach makes use of the mean field approximation and thus requires the provision of as many details of the model as possible, which in general yields untractable equations, whenever their derivation can even be achieved. In general, it has not been possible to amalgamate the two approaches further than the following attainment (nonetheless fundamental): the mean field equations predict singularities (or changes of phase) that actually correspond to physically observed transitions, known as *gelation* and *shattering*. In the mathematical literature, consequently, such singularities are named after these phenomena. Even if they have been extensively studied, there is still much work to fit the mathematical behaviors within the real complexity of experimental observations, a complexity that, at later stages, is very system-depending. See Figure 1.1.1 for a scheme of possible situations.

We mention here that an interesting question affects the physical consistency of mesoscopic and microscopic models. Their relationship is studied through the problem of convergence of the Marcus-Lushnikov process to the Smoluchowski equation. It can be seen that for the kernels for which explicit solution and rigorous results are known the hydrodynamic limits of microscopic models can be settled (in a suitable sense, see for example Norris [250] or the recent works of Cepeda and Fournier [5] with references therein). Besides, probabilistic models are also known for which the ergodic means match the explicit solutions (1.1.2) and a general central limit for these stochastic processes can be found in the recent work of Kolokoltsov [185]. A more detailed explanation can be found in the work of Aldous and Norris [6, 250].

### 1.1.2. Fields of application

Progresses in the coagulation-fragmentation theory entail also fruitful consequences for the development of both theoretical disciplines like pure analysis, mathematical physics and probability, and more practical fields among which we can include integral equations, numerical analysis and stochastic processes (random graphs and SPDEs). However, as mentioned above, the origins of this theory lie in aerosol physics and a variety of applications have risen even before the first rigorous formal results could be established. Here we provide a tentative list, incomplete due to the extensive development of applications, of representative fields of application integrating...
references from [8, 204, 215] with some more recent study; further references can also be found in [94, 305, 37].

**Pure coagulation:** Early studies were focused on physical chemistry, but from the eighties and nineties new applications to astrophysics multiplied and, more recently, biology and other sciences concerned with the distribution and evolution of populations employ coagulation models.

- **Aerosols:** they are liquid or solid particles suspended in a gas (cf.: books [171, 250], articles [94, 224, 251]). This field includes applications such as dust smoke and haze, clouds, drops and snowflakes formation, atmospheric precipitations [46, 59, 172, 224, 217].

- **Polymerization:** formation of chains and macromolecules from simple monomers. Growth can be: by chain addition processes (for example, polyethylene, polypropylene, polystyrene) or by condensation processes at specific chemical groups (for example, polyesters, polyamides, polyurethanes, polycarbonates). Basic formulations are available in chemistry text books and specific applications can be found in [315, 112].

- Drops coalescence in sprays [263].

- Astrophysics: formation of large scale structures in the universe [273]; stars and protoplanets formation from clouds of dusts and gasses [7, 32, 272]. A different model from Smoluchowski coagulation equation was proposed by the Oort-van Hulst in 1946 [252] in the framework of protoplanetary formation; its connections to Smoluchowski equation were recently explained in [193].

- Economy and graph theory: size distribution of firms, networks, connected components in graphs [37].

- Biology, molecular biology: algae [8], evolution of phytoplankton populations [4, 263], abundance of species or behaviors of animal groups. Also, at the molecular scale: interaction of proteins in lipid membranes [159] and aggregation of lipids forming membranes and other structures [254, 255].

- Population dynamics and population genetics: [36, 57].

**Fragmentation and coagulation-fragmentation:** Studies have focused on issues that correspond to the inverse of the above listed phenomena or on the complex interaction that include both fragmentation and coagulation:

- Depolymerization: [318, 316].

- Erosion: [186].

- Sprays and drop dispersion: [69]; breakage of immiscible fluids in turbulent flows [100, 193]; fragmentation–coagulation–scattering model for the dynamics of stirred liquid–liquid dispersions [132].

- Economy and social systems: [4, 8, 313].
Figure 1.1.2: Scanning electron micrographs of a population of cells *Chlorella vulgaris*, image from [177]. A, cell aggregation. B, *C. vulgaris* cells under different magnification. C, cell division. Cell cultures can show diverse phenomena as aggregation/fragmentation, mitosis (births by cell division) and death events. See the recent work of Perhame [206, 256] for information on the growth-fragmentation/cell-division equation.

- Cell cultures, multicellular growing systems and biological tissues [34, 35, 36].

See also the Kinetic theory of inertial particles [309] and the modelization of different phenomena in the framework of the mathematical kinetic theory for active particles [56, 57, 50]. This theory (called KTAP in short) deals with the modelling of large systems of active units characterized by microscopic states of activity in addition to the other typical variables (position and mass). This additional description enables specific interactions and individualistic behaviors for each “living” entity, who can develop, together with other active particles, collective identity and systemic functions. The KTAP theory is developed with the aim of envisaging the complexities of structured systems in system biology.

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### 1.1.3. MEAN FIELD THEORY FOR PURE COAGULATION

It may be appreciated the extension of the mathematical theory of coagulation-fragmentation from the articulation of different sub-sectors that constitute it and from the connections that have been established with other areas (Section 1.1.2 makes it evident): a first distinction one finds is between stochastic and deterministic models and between microscopic and mesoscopic levels of description to which various authors consider generalizations and introduce variations. Today, it would be difficult to systematically collect all those approaches and, in any manner, is not the objective of this thesis to exhaustively compile them. We consider here only the Smoluchowski equation for pure coagulation (mean field approximation with infinite volume and homogeneous
injected into a high Re water jet. The four pictures were taken from different cameras. For more information on statistical models of

Figure 1.13: Image taken from Eastwood, Armit Lakeshers [100] on the breakup of an immiscible dispersed fluid (silicone oil) injected

(p)
distribution in space) and, separately, we have also studied its dual process (in a sense explained in Section 1.1.5): the pure linear fragmentation. Mixed models of coagulation-fragmentation receive much interest and the literature analyzes specific questions, as, for example, the problem of establishing equilibria and the dynamics in presence of dominating fragmentation, dominating coagulation, with mass transport, etc. The reason to restrict to two basic models is that, despite the progressive enrichment of results for increasingly complex models, key issues are still open for simpler cases of pure coagulation and fragmentation: for the Smoluchowski equation there is no general knowledge of the evolution of systems (in particular: explicit formulas or regime behaviors with respect to classes of initial data), but only a few explicit solutions for specific models have been found out. And with respect to the linear equation of fragmentation, for which more results are available, the study of self-similar solutions has yet to be completed (conditions for convergence, rigorous properties). In this thesis the focus is on the study of the universal behavior of these two equations and, before addressing the issue in detail, we put forth a quick review of other fundamental theoretical aspects: the development of singularities in finite time, the existence of solutions and their regularity.

Examining the Smoluchowski equation, one remarks that the coagulation kernel $K(x, y)$ is not fixed, so that this function is the only parameter that determines the dynamics of (1.1.2) and the properties of the two nonlocal operators. The modeling work, hence, consists of setting the functional form of the kernel in order to summarize the relevant physical information of the system; on the other hand, the mathematical aim is to classify possible kernels by the properties of the solutions and the operators. In Table 1.2 we provide examples of relevant kernels (previously collected in [274]) proposed for physical applications.

For almost all physical kernels of Table 1.2 no explicit solutions are available. Numerical methods have steadily improved, but have to deal with the important issue of approximating operators defined on infinite intervals of mass. We can cite some recent works in [108, 189, 190, 191, 101, 14, 133, 135, 227].

Standard kernels in mathematical references are the constant $K_1(x, y) = 1$, the additive $K_+(x, y) = x + y$ and the product kernel $K_*(x, y) = x \cdot y$; all these can be included in the general bilinear kernel $K(x, y) = a + b(x + y) + c \cdot xy$ which has been extensively investigated and for which explicit solutions, at least for specific data, are available (in the discrete setting: [278, 279]). We note that the kernels $K_1, K_+$ and $K_*$ fall into the class of homogeneous kernels defined in the following:

$$K(ax, ay) = a^\lambda K(x, y).$$  \hfill (1.1.3)

Also, we will often refer to the following sub-class of homogeneous kernels:

$$K_{\mu+\nu}(x, y) := \frac{1}{2} (x^\mu y^\nu + x^\nu y^\mu).$$  \hfill (1.1.4)

From the mathematical point of view, homogeneous kernels (1.1.3) denote a particularly important class since they provide a prototype to be studied and grant a tool to catalog the fundamental dynamic properties depending on a single parameter: the degree of homogeneity $\lambda := \mu + \nu$. If $0 \leq \lambda \leq 1$, in fact, the Smoluchowski equation admits global-in-time existence and uniqueness of mass conserving solutions; however, if $1 < \lambda \leq 2$, existence is valid only for bounded times and blow-up in finite time may occur. This is interpreted as a change of phase of some particles forming a gel (this interpretation comes from systems in physical chemistry where this phenomenon is observed).
<table>
<thead>
<tr>
<th>Kernel $K(x, y)$:</th>
<th>Application:</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x^{\frac{1}{3}} + y^{\frac{1}{3}}) \left( x^{-\frac{1}{3}} + y^{-\frac{1}{3}} \right)$</td>
<td>Brownian motion in continuous regime;</td>
</tr>
<tr>
<td>$(x^{\frac{1}{3}} + y^{\frac{1}{3}})^2 (x^{-1} + y^{-1})^{\frac{1}{2}}$</td>
<td>Brownian motion in free molecular regime;</td>
</tr>
<tr>
<td>$(x^{\frac{1}{3}} + y^{\frac{1}{3}})^3$</td>
<td>Shear (linear velocity profile);</td>
</tr>
<tr>
<td>$(x^{\frac{1}{3}} + y^{\frac{1}{3}})^{\frac{7}{4}}$</td>
<td>Shear (nonlinear velocity profile);</td>
</tr>
<tr>
<td>$(x^{\frac{1}{3}} + y^{\frac{1}{3}})^2 \left</td>
<td>x^{\frac{1}{3}} + y^{\frac{1}{3}} \right</td>
</tr>
<tr>
<td>$(x^{\frac{1}{3}} + y^{\frac{1}{3}})^2 \left</td>
<td>x^{\frac{2}{3}} + y^{\frac{2}{3}} \right</td>
</tr>
<tr>
<td>$(x - y)^2 (x + y)^{-1}$</td>
<td>Analytic approximation of Berry’s kernel;</td>
</tr>
<tr>
<td>$(x + c) (y + c)$</td>
<td>Condensation and/or branched-chain polymerization;</td>
</tr>
<tr>
<td>$(x^{\frac{1}{3}} + y^{\frac{1}{3}}) (xy)^{\frac{1}{3}} (x + y)^{-\frac{3}{2}}$</td>
<td>Used in Kinetic theory.</td>
</tr>
</tbody>
</table>

More kernels have been studied to deal with Van der Waal or viscous forces and in presence of diffusiophoresis and thermophoresis (flow of aerosol particles down concentration of gas or from warm to cool gas).

Table 1.2: Kernels in physical literature. Remark that variables $x$ and $y$ often appear with the fixed exponent $\frac{1}{3}$; this is due to the fact that the corresponding physical models characterize the interaction between moving particles (spheres) depending their radio $r$, and obviously $r \propto x^{\frac{1}{3}}$.

Table 1.3 presents the explicit solutions for the kernels $K_1$, $K_+$ and $K_\times$ in the discrete and continuous settings; it also shows the time intervals of existence; we can see that the first two (degree of homogeneity less than 1) have $t \in \mathbb{R}^+$, that the solution of the additive kernel has the moment of order $\frac{1}{2}$ unbounded and that the multiplicative kernel (continuous setting) does not have bounded mass at any time $t > 0$ (it is a non-physical solution). The mathematical mechanism of
singularity development therefore distinguishes between the two fundamental dynamic possibilities for coagulation equations. Following also the physical intuition, its occurrence depends on the growth rate of the kernel. In the following we describe some basic facts about gelation; later, in Section 1.2, we discuss in greater details the results of existence and uniqueness of solutions.

<table>
<thead>
<tr>
<th>EXPLICIT KERNEL:</th>
<th>$K_1 = 1$</th>
<th>$K_+ = x + y$</th>
<th>$K_\times = x \cdot y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>DISCRETE FORM:</td>
<td>$(1 + \frac{t}{2})^{-2} \left( \frac{t}{2\pi} \right)^{-1}$</td>
<td>$e^{-t} B(1 - e^{-t}, x)$</td>
<td>$x^{-1} B(1, x)$</td>
</tr>
<tr>
<td>CONTINUOUS FORM:</td>
<td>$4t^{-2}e^{-2t}$</td>
<td>$\frac{1}{\sqrt{2\pi}} e^{-t} x^{-\frac{3}{2}} e^{-e^{-2t}}$</td>
<td>$\frac{1}{\sqrt{2\pi}} e^{-t^2/2}$</td>
</tr>
<tr>
<td>TIME RANGE:</td>
<td>$t \in [0, \infty)$</td>
<td>$t \in [0, \infty)$</td>
<td>$t \in [0, 1)$</td>
</tr>
</tbody>
</table>

Table 1.3: Explicit solutions to (1.1.2) with monodisperse initial conditions (this means that $c_0(x) = \delta_1(x)$). The function $B(\lambda, x)$ is the Borel distribution, defined as: $B(\lambda, x) = (\lambda x)^{-1} e^{-\lambda x/\pi} x^\frac{3}{2}$.

Gelation phenomenon and its relationship with the blow-up of solutions:

The fact that the blow-up in Smoluchowski equation receives the name gelation is due to the phenomenon of gelation in colloids. From the physical point of view, aggregation processes can lead to the formation of a diffused gel phase. A gel is a solid continuous structure immersed in a percolating liquid phase; the solid phase usually represents no more than 10\% of the volume of the gel. Gels can be classified in terms of the cross-linked structure they present (see Figure 1.1.4 for a quick look) and the fluid that permeates the free space in the network (hydrogels for water, aerogels for gasses, etcetera); their colloidal size scale (the volume of the open spaces between the gel solid filaments) is in the majority of cases smaller than a micrometer. Depending on the kind of network and the chemical properties of the solid phase, the diffusion of salts in the liquid gel can result almost the same as they diffuse in the corresponding free liquid. Another feature is that the solid-fluid interface between the phases has an area on the order of 1000 $m^2/gr$ of solid. This means that the dominating forces are of interfacial and short-range nature, such as hydrogen-bonding, van der Waals and electrostatic forces; the liquid medium is ensnared through surface tension effects.

Accordingly to the IUPAC definition [175], a gel can contain:

1. a covalent polymer network, e.g., a network formed by cross-linking polymer chains or by nonlinear polymerization;
2. a polymer network formed through the physical aggregation of polymer chains, caused by hydrogen bonds, crystallization, helix formation, complexation, etc., that results in regions of local order acting as the network junction points. The resulting swollen network may be termed a “thermoreversible gel” if the regions of local order are thermally reversible;

3. a polymer network formed through glassy junction points, e.g., one based on block copolymers. If the junction points are thermally reversible glassy domains, the resulting swollen network may also be termed a thermoreversible gel;

4. lamellar structures including mesophases, e.g., soap gels, phospholipids, and clays;

5. particulate disordered structures, e.g., a flocculent precipitate usually consisting of particles with large geometrical anisotropy, such as in V₂O₅ gels and globular or fibrillar protein gels.

Being a mesoscopic approximation, Smoluchowski equation does not allow describing neither the special features of the gel phase nor its local structures. Such limitations of the model do not preclude the formation of singularities being interpreted as a mathematical expression of the phenomenon of change of phase and, until the blow-up, that the predictions stand with experimentally testable validity.

The heuristic explanation of gelling solutions is this: during regular evolution, heavy aggregates with increasing density merge in the system; at a certain time the effect, in the system, of the formation of a different phase manifests through an irregular dynamics: the gel reduces the mass of the system, because smaller clusters start to aggregate to it and, aggregating, they also become part of the gel phase. It is usually said that in the system appears a particle of infinite density which absorbs finite mass. Envisaging this singularity development, one can explicitly state the desired features of the mathematical investigation: it is interesting to determine solutions that
conservethe total mass of the system, to predict the moment at which the mass begins to lessen and characterize the escape to infinity of mass in terms of algebraic tails of the solution.

Foremost, the main way to address the issue of how mass redistributes over time is the analysis of the moments of the solutions \( c(x,t) \). Hence we introduce:

\[
M_\alpha(t) := \int_{\mathbb{R}^+} y^\alpha c(y,t) \, dy, \quad \alpha \in \mathbb{R},
\]

and, in particular, we emphasize that \( M_0(t) \) denotes the equivalent of the amount of particles per unit volume of the coagulating system; \( M_1(t) \) its total mass and \( M_2(t) \) is the second moment (a measure of dispersion of mass used to detect blow-up). The central question is whether and when solutions verify \( \frac{d}{dt} M_1 = 0 \), congruously to the microscopic mechanism of coagulation defined in (1.1.1) which prescribes the conservation of mass in each event of aggregation. To see how the matter is delicate, we construct now two clarifying examples.

**Example 1.1 (Toy model).** Let \( K(x,y) = (xy)^{\frac{1}{2}} \) and consider the initial monodisperse datum \( c_0(x) = \delta_1(x) \), so that \( M_\alpha(0) = 1 \) for any \( \alpha \). Multiply equation (1.1.2) by \( x \) and integrate; formally, we obtain the variation of the first moment:

\[
\frac{d}{dt} M_1(t) = \frac{1}{2} \int_0^\infty (x-y+y) \int_0^x y^\frac{1}{2} (x-y)^{\frac{1}{2}} c(y,t) c(x-y,t) \, dy - \left( \int_0^x \int_0^{\frac{x}{2}+1} c(x,t) \, dx \right) \left( \int_0^\infty y^\frac{1}{2} c(y,t) \, dy \right)
\]

Under the hypothesis needed to apply Fubini’s Theorem in the first double integral, and also, that both \( M_\frac{1}{2}+1(t) < \infty \) and \( M_\frac{1}{2}(t) < \infty \), it turns out that \( \frac{d}{dt} M_1(t) = 0 \) and, hence, \( M_1(t) = 1 \) as long as the regular solution exists.

A possible mechanism for gelation to take place (if \( \lambda > 1 \)) is that, in finite time, \( M_\frac{1}{2}+1(t) \rightarrow \infty \), whence this implies that the solution develops in finite time an algebraic non-integrable tail at infinity even if the initial datum has compact support.

It is significant to mention that even for the simplified case described above, it has not been possible yet to understand how gelation happens and what yields it, nor at what critical time gelation does occur with general initial distribution and homogeneity degree \( \lambda \), whereas it is clear that the sufficient condition for it to take place is that the kernel presents super-linear growth at infinity (\( \lambda > 1 \)). Furthermore, notice that Smoluchowski equation for post-gelation times has also been considered (for example by imposing a law for the interaction between sol and gel); a branch of literature deals with this topic [16, 301, 204, 315]. However, from the physical and statistical viewpoint, there is neither an accepted description of the microscopic interactions between particles of finite and infinite mass [15], nor a different mean field model that is able to take infinite masses into account. The post-gelation solutions reflect therefore a line of research of essentially mathematical nature.

Let us define the critical gelation time \( T^* \) precisely in terms of the existence of solutions to equation (1.1.2) with the property of preserving the total mass:

\[
T^* := \inf \{ t \geq 0, \, M_1(t) < M_1(0) \}.
\]
As mentioned, in general it is not possible to compute explicitly the value of $T^*$ and it must be sought for an estimation. As for explicit kernels plenty of detailed information is available and, in particular, it is simple to determine the critical time for the product kernel: we show it in the following example.

**Example 1.2** (Product kernel). Consider the product kernel $K(x, y) = x \cdot y$ and an initial datum $c_0(x)$ with the first $n$ moments finite, with $n = 1, 2, 3$. Multiply Smoluchowski equation (1.1.2) by $x^2 = ((x - y)^2 + 2y(x - y) + y^2)$ and, integrating, one gets, repeating the considerations of Example 1.1:

\[
\frac{d}{dt} M_2(t) = \frac{1}{2} \int_0^\infty \int_0^z \left((x - y)^2 + 2y(x - y) + y^2\right) y(x - y) c(y, t) c(x - y, t) \, dy - M_3(t) \cdot M_1(t),
\]

so that, supposing that the solution $c(x, t)$ exists and the first $n$ moments stay finite with $n = 1, 2, 3$, we can conclude that

\[
\frac{d}{dt} M_2(t) = M_2^2(t).
\]

The solution for $M_2(t)$ is $M_2(t) = \left(\frac{1}{M_2^{(0)}} - t\right)^{-1}$ so evidently the blow-up time for the second moment is $\tau := M_2^{-1}(0)$. It can be shown rigorously [112, 248] that critical gelation time $T^* = M_2^{-1}(0)$ coincides with the second moment blow-up.

In the applied literature it is commonly assumed that, as in the case of the product kernel, the gelation time coincides with the time of blow-up of some positive moment; although it may possibly result true, it has not been established precisely what moment should manifest the singularity (cf. [165]).

As for the exact result on gelation, Escobedo, Mischler and Perthame prove in [117] that for homogeneous kernels as in formula (1.1.4) with $\lambda = \mu + \nu \in (1, 2]$ it holds:

\[
M_\alpha(t) \in L^2 \text{ if } \alpha \in \left(\frac{\lambda}{2}, \frac{\lambda + 1}{2}\right),
\]

and hence for that range of $\lambda$ values it follows in particular that $M_1(t) \in L^2$, so that $M_1(t)$ cannot be constant for all time. In other words, there are simple a priori estimates which, together with rigorous technical proves, permit us to obtain a very relevant piece of information on the system based only on the level of homogeneity of the kernel (or in the case of more general kernels that can be estimated asymptotically with homogeneous functions).

Another relevant feature of Smoluchowski equation is that, when $\lambda > 2$, an instantaneous gelation phenomenon occurs with $T^* = 0$. Spouge [280, 282, 283] had been the first to conjecture the instantaneous gelation, but the exact result was proven by van Dongen, da Costa and Carr (see [292, 60]). However, as van Dongen points out, instantaneous gelling kernels are not physically acceptable in real world applications to clusters coagulation, since these kernels require the active surface sites for aggregation to be bigger than the total surface of the clusters.

We conclude, therefore, that it is known how to determine the occurrence of gelation depending on the growth of the kernel at infinity, that is, depending on the strength of the interaction between heavy masses. Singular kernels at $x \approx 0$ can also develop shock concentrations for the solutions $c(x, t)$ depending on the initial conditions and some work of refinement has to be developed yet to characterize the exact conditions for each case.
1.1.4. Additional mechanisms

Numerous additional or alternative processes to irreversible Smoluchowski coagulation, as it has been described in Section 1.1.3, have also been considered. Since in real applications they often result interlaced, these new mechanisms do not only constitute separate models, but respond to the necessity to estimate and understand the relevance that each of them holds in the interplay of complex phenomena; whether it is possible to drop them out of the model or not follows from the analysis of mean field equations like Smoluchowski equation with increasingly complex mathematics. We report here a list of some of the most relevant phenomena, and others can also be found in the work of Laurençot and Mischler [204].

Ternary or $n$–ary reactions:

Apart from coagulation events involving only two clusters at the same time, one can also consider the interaction of multiple clusters per time. Such events may be infrequent, but nonetheless they occur in some systems in a relevant way. Smoluchowski equation can be rewritten accordingly, with generalized kernels (of more than two variables) and double (triple, etc.) integrals.

An interesting related kinetic equation comes from the study of the wave turbulence for the interaction of wave fields. The evolution of the number $n_k$ of average spectral wave-action in a set of dispersive waves is given by this equation (in the case with dominant 3–interactions):

$$
\frac{\partial n_k}{\partial t} = 4\pi \int |T_{k_1 k_2 k_3}|^2 n_{k_1} n_{k_2} n_{k_3} \mathcal{F}_3 [n] \delta \left( k - k_1 - k_2 - k_3 \right) dk_1 dk_2,
$$

$$
\mathcal{F}_3 [n] = \left( \frac{1}{n_k} - \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} \right) \delta \left( \omega_k - \omega_{k_1} - \omega_{k_2} \right) + \left( \frac{1}{n_k} - \frac{1}{n_{k_1}} + \frac{1}{n_{k_2}} \right) \delta \left( \omega_{k_1} - \omega_k - \omega_{k_2} \right) + \left( \frac{1}{n_k} + \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} \right) \delta \left( \omega_{k_2} - \omega_{k_1} - \omega_k \right),
$$

where $\omega_k$ represents the dispersion law and, in a similar fashion, it is possible to deduce the equation for the dominant $N$–interactions. There are also many variations to this model and their implications are relevant to the study of turbulence in fluids. Examples of 3-wave systems are sound and magnetic sound in three dimensions, anisotropic quasi-two-dimensional Alfvén waves in plasmas or capillary waves on deep water. An example of 4-wave system comes from Langmuir waves in isotropic plasmas and an example of 5-wave system are one-dimensional gravity waves [72]. For more information refer to the work of Balk, Zakharov and Connaughton among others [18, 24, 74, 92, 103, 111]. Smoluchowski aggregation (in the stationary cases with a source and a sink) has also been studied by Connaughton from the point of view of Kolmogorov cascade [76, 77].

The shape of the aggregates:

The morphology of the aggregates can be extremely diverse, depending on the physical system under consideration, so that the shape of the aggregates can be characterized, in some cases, by a fractal distribution. This is not the case of the fluid drop coagulation or of other species which can rearrange after coagulating: when two drops enter in contact, they form a new object subject to the effect of the surface tension; this force provides a simple shape (a sphere if the drop is
suspended in the atmosphere, for example). However, coagulation of polymers can determine the growth of more or less twisted chains, and, when there are many reactant sites (in the formation of gels, for example), the appearance of multiple branches; on the top of that, coagulation of solid particles can be even more complex: depending on the stiffness of the bounds and the number of active sites on the surface (points where aggregation between clusters is permitted), the morphology can be described only at a statistical level (see Figure 1.1.5). The quantity that one can attempt to determine is the fractal dimension of the objects that characterizes the mass–radius relation of aggregated objects:

\[
\frac{r^{D_f}}{m} = \text{const},
\]

In this kind of models, \( D_f \) depends, apart from the geometry of the coagulation mechanism, also on other physical effects present in the system, like the details of transport or diffusion of particles and the affinity to aggregate of the active sites on the surfaces. As a general rule of thumb, lower affinities and good stirring transport yield slightly more compact structures (and higher fractal dimensions), while strong affinities and brownian diffusion produce the opposite. In the literature one can find numerous studies of these phenomena \[176, 187\].

**Inhomogeneous systems:**

It is possible that during the evolution of coagulation an important spatial correlation between clusters can appear. In this case, the transport mechanism stirring the system can reduce that correlation, but there are many systems in which spatially localized distributions infringe the mean
field hypothesis. A complete model should take into account these contributions, and introduce diffusion/transport effects. A class of models proposes that the aggregates travel in space according to a Brownian motion with diffusivity $a(x)$ that depends on the value of mass. A model of Brownian coagulation appears for example in the work of Norris [249], together with results of convergence, existence and uniqueness. At mesoscopic level, Norris proposes a process of measures $(\mu_t)_{t\geq0}$ over $\mathbb{R}^D \times \mathbb{R}^+$ (with $p$ being the spatial variable and $x$ the mass) that satisfies the following equation:

$$\partial_t (\mu_t) = \frac{1}{2} a \Delta_p \mu_t + C (\mu_t). \quad (1.1.11)$$

Here $a = a(x) \propto x^{-\frac{1}{2}}$ is the diffusivity and the coagulation operator $C (\mu)$ in the sense of measures is defined as the immediate generalization of the two coagulation integrals in equation (1.1.2). After rearranging terms, one can write:

$$\langle \varphi, C (\mu) \rangle := \frac{1}{2} \int \int \{ \varphi (x + y) - \varphi (x) - \varphi (y) \} K (x, y) \mu (dx) \mu (dy). \quad (1.1.12)$$

A different possibility is the introduction of terms responsible of transport mechanisms; consider a velocity field $v \in \mathbb{R}^D$ depending on position $p$, time $t$ and mass $x$. In order for this field to "carry" the measure-valued solution $\mu(x, p, t)$, the term $v(x, p, t) \cdot \nabla_p \mu (x, p, t)$ is introduced into the equation. A physical example is raindrops falling in presence of wind.

**Multicomponent coagulation:**

If in the system there are $l$ different coagulating species that also interact with each other, the $x$ variable can be generalized to have $l$–vector values. A particular case is that of the kinetic coagulation, which considered that the particles that coagulate should be characterized by the pair of values mass-moment $\{x, p\}$. Aggregation is therefore determined by the creation of a new particle with $\{x, p\} + \{y, q\} \rightarrow \{(x + y), (p + q)\}$. To this respect, see [262, 129, 127, 129].

**Cluster production, mass-transport and growth/erosion:**

Sources and sinks can be introduced when the effect of new particles entering or leaving the system is relevant; these sources may be homogeneously distributed throughout the system or localized in certain regions. Consider two examples: in the formation of clouds, a model can account for the soil moisture evaporating and reaching the cloud system homogeneously, providing uniformly water molecules; or, in rainfall formation, the cloud height can be considered a localized layer source of drops, wind and gravity determine the velocity field in the air during the coagulation of raindrops and the ground can be represented with a bottom layer that completely absorbs the aggregates dropping to zero height. A few references: [23, 96, 274, 275].

Another possible mechanism is the growth of aggregates from the exchange of matter with the environment. This has been applied, for example, in populations of cells and bubbles and drops with environmental humidity (cf. [254, 51, 199, 206, 275]). Let $I (u, x)$ represent the rate of mass interchange between the environment and clusters of mass $x$; let also $u$ be the density of the medium so that concentration $c(x, t)$ satisfies the mass transport equation $\partial_t c + \partial_x (I c) = 0$. Different physical laws provide the function $I$ determining the evolution law for the quantity $u$. 

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The paper of Banasiak [22] brings together previous results and presents in an unified setting the coagulation-fragmentation equations with mass transport (in cases of strong fragmentation).

### 1.1.5. Fragmentation and Coagulation-Fragmentation

A special interest deserve the deterministic models of fragmentation for which, as for coagulation, appears a rich variety of different cases to identify. We first limit ourselves to the description of the spontaneous (or linear) binary fragmentation and, later, we derive the equation for the multiple fragmentation. Useful recent references for further details can be consulted in [64, 156, 192]. Different possibilities for physical models include collision induced fragmentation which are intrinsically nonlinear models (see the references [100, 195, 68]). The reaction scheme that represents the single event of fragmentation is:

\[
\begin{pmatrix}
1_{\{x\}}, 0_{\{x-y\}}, 0_{\{y\}} \\
0_{\{x\}}, 1_{\{x-y\}}, 1_{\{y\}}
\end{pmatrix}
\]

with \( y < x \), and, analogously to the considerations developed before for the Smoluchowski equation, one can write down a coagulation-fragmentation equation of balance of mass:

\[
\frac{d}{dt}c(x, t) = C(c) - BF(c),
\]

where we have adopted the compact notation with operators denoting with \( C(c) \) the right hand side of Smoluchowski equation (1.1.2) and with \( BF(c) \) the binary fragmentation operator:

\[
BF(c)(x) := \left( \frac{1}{2} \int_{0}^{x} F(x - y, y) \, dy \right) \times c(x, t) - \int_{0}^{\infty} F(x, y) c(x + y, t) \, dy.
\]

As before, the kernel \( F(x - y, y) \) is a symmetric function and the amount \( \int_{0}^{\infty} F(x - y, y) c(x, t) \, dy \, dt \) is equal to the average number of fragmentation events of type (1.1.13) within the time interval \((t, t + dt)\) that, given particles of mass in \((x, x + dx)\) produce pairs of fragments with mass in \((x - y, x - y + dx)\) and in \((y, y + dy)\). Like equation (1.1.2), also the coagulation-fragmentation equation is a population balance (in this case constituted by four contributions). It is assumed: firstly, that the fragments are distributed homogeneously throughout the system at all times; secondly, that the geometric and spatial features can be neglected as any other effect depending on other phenomena; in this way, the only magnitude distinguishing the dynamic behavior of the fragments is mass.

In absence of coagulation, that is \( K \equiv 0 \), the equation

\[
\frac{d}{dt}c(x, t) = -BF(c)
\]

is called pure or irreversible fragmentation equation.

It is possible to observe that reaction (1.1.13) represents the opposite of (1.1.1) and, correspondingly, one can establish a formal duality between Smoluchowski’s coagulation and pure fragmentation
mesoscopic level. To see it, consider, as in [6], fragmentation and coagulation kernels that can be explicitly time dependent: $K(x, y; t)$ and $F(x - y, y; t)$; then, whenever $c(x, t)$ is a solution of (1.1.2) with kernel $K(x, y; t)$, if one reverses the direction of time and defines $F(x - y, y; t)$ through the following duality formula:

$$c(x + y, t) F(x, y; t) = c(x - y) c(y) K(x - y, y; t),$$

one can consequently show that $c(x, t)$ is also a solution to (1.1.16) and vice-versa. Since fragmentation equation is linear and easier to study, some authors attempted to employ this relation to detect new solutions and properties for the coagulation equation. Nevertheless, in general, starting with kernels $F$ that are not time dependent, one obtains new kernels $K$ in which the time dependence is explicit and vice-versa, which yields an extra difficulty.

The multiple fragmentation:

The multiple fragmentation of a $x$–massed particle can also be considered and, from now on, when referring to fragmentation, we will always simply consider the general process. Let $S_x$ be the space of non-increasingly ordered, finite sequences $y_x = \{y_1, \ldots, y_k\}$ such that $x = \sum y_x = y_1 + \ldots + y_k$, and let also $\sigma(y)$ represent a positive measure of the breakage of a total mass $\sum y$ into the sequence of fragments $y$. At the same time, we can also consider the measure $\sigma_i(x, y)$, accounting for the breakage of some mass $y = \sum y$ into a sequence $y$ such that $y_i = x$ (this measure plays the role of a marginal distribution, see Cañizo [6] for more details). Then, we can consider the spontaneous rate of fragmentation of all the clusters of size $x$:

$$F^-(c) = c(x, t) \int_{S_x} \sigma(y),$$

and, correspondingly, the contribution to $x$–clusters from heavier ones is

$$F^+(c) = \int_x^\infty c(y, t) \sum_{i=1}^\infty \sigma_i(x, y).$$

We remark that, for any $y$ such that $x = \sum y$, we can extend $y$ to an infinite sequence $\tilde{y} = \{y_1, \ldots, y_k, 0, \ldots\}$ so that:

$$x \int_{S_x} \sigma(y) = \int_x^\infty \left( \sum_{i=1}^\infty y_i \right) \sigma(\tilde{y}) = \sum_{i=1}^\infty \int_{S_x} y_i \sigma_i(\tilde{y}) = \sum_{i=1}^\infty \int_0^x y_i \sigma_i(y_i, x)$$

$$= \int_0^x y \sum_{i=1}^\infty \sigma_i(y, x)$$

and this motivates the introduction of density $b(y, x)$ such that, at least formally:

$$\sum_{i=1}^\infty \sigma_i(y, x) = b(y, x) \, dy.$$
Terms (1.1.18)-(1.1.19) can now be rewritten depending on \( b(y,x) \) yielding the coagulation-multiple fragmentation equation:

\[
\frac{d}{dt} c(x,t) = C(c) - F(c),
\]

\[
F(c) = \beta(x)c(x,t) - \int_{y}^{\infty} c(y,t) b(x,y) \, dy,
\]

where \( \beta(x), x > 0 \), is the spontaneous fragmentation rate of \( x \)-masses given by:

\[
\beta(x) = \int_{0}^{x} \frac{y}{x} b(y,x) \, dy.
\]

We remark that also the binary coagulation-fragmentation equation (1.1.14) can be straightforwardly written in a similar fashion.

After mean-field equations (1.1.22-1.1.23) are settled, we can also introduce the following probability distribution, for \( x \geq y > 0 \):

\[
P(y, x) = \frac{b(y, x)}{\beta(x)} = \frac{b(y, x)}{\int_{0}^{y} b(y', x) \, dy'}
\]

which defines the probability of obtaining a fragment of size \( [y, y+dy] \) from a particle of size \( x \). A mathematically interesting case is when this probability depends only on the fraction \( y/x \), that is, when \( b(y,x) \) is of the form:

\[
b(y,x) = \beta(x) \frac{1}{x} B\left(\frac{y}{x}\right),
\]

with \( B : [0, 1] \rightarrow \mathbb{R}^+ \). This kernel is called the relative fragmentation rate and verifies the normalization property:

\[
\int_{0}^{1} u B(u) \, du = 1.
\]

Using the newly introduced function \( B(u) \), we can consider the probability that the fragmentation event of a \( x \)-massed particle produces a fragment of size between 0 and \( \alpha x \), with \( \alpha \in (0, 1) \). This probability is computed as:

\[
\mathbb{P}(y \leq \alpha x \mid x) = \frac{1}{x} \int_{0}^{\alpha x} B\left(\frac{y}{x}\right) \, dy = \int_{0}^{\alpha} B(u) \, du.
\]

Formula (1.1.28) shows that the probability introduced in (1.1.25) does not depend on the particle size \( x \), but only on the fraction \( \alpha \). For this reason, a fragmentation kernel \( b(y,x) \) satisfying (1.1.26) is called self-similar. In Chapter 4 we will assume that the fragmentation equation is in self-similar form and restrict \( \beta(x) \) to a suitable algebraic function of \( x \).
1.1.6. Shattering Phase Transition in Fragmentation Equations

Equation (1.1.22–1.1.23) and, notably, even the simpler linear fragmentation equation:

$$\frac{d}{dt} c(x, t) = -\beta(x) c(x, t) + \int_x^\infty c(y, t) \frac{\beta(y)}{y} B\left(\frac{x}{y}\right) \, dy$$  \hspace{1cm} (1.1.29)

may predict a phase transition phenomenon analogous to gelation singularity relative to the Smoluchowski coagulation equation (cf. [228, 68]). This finite time singularity is called \textit{shattering} and corresponds to the formation of an infinite quantity of small dust-like fragments (zero massed) which, ultimately, has the effect of interrupting mass conservation. The mass of the system starts diminishing due to the growth of a \textit{dust phase}; mathematically, one sees that small fragments dominate the asymptotic distribution with an algebraic expansion at the origin. Analogously to the gelling case, the blow-up of some suitable inverse moment $M_\alpha$, $\alpha < 0$, at some critical time $T^*$, corresponds to the mentioned loss of mass for the first moment $M_1(t) < M_1(0)$ when $t > T^*$.

The relevant case of homogeneous fragmentation rates $\beta(x)$ permits characterizing the conditions of such a blow-up occurrence and analyzing the asymptotic behaviors. To shed light on the shattering mechanism, we therefore consider $\beta(x) = x^\lambda$, multiply both sides of equation (1.1.29) times $x^\alpha$ and integrate. We suppose \textit{a priori} that all the integrals converge and Fubini’s theorem applies so that the following formulas must be considered in formal sense. Since:

$$\int_0^\infty x^\alpha \left( \int_x^\infty c(y, t) y^{\lambda-1} B\left(\frac{x}{y}\right) \, dy \right) \, dx = \int_0^\infty y^{\lambda+\alpha} c(y, t) \, dy \int_0^y \left(\frac{x}{y}\right)^\alpha B\left(\frac{x}{y}\right) \, dx,$$

we get the moments formula for this class of fragmentation equation:

$$\frac{d}{dt} M_\alpha(t) = (\Theta(\alpha) - 1) M_{\alpha+\lambda}(t),$$  \hspace{1cm} (1.1.30)

with $\Theta(\alpha)$ the moment transform (Mellin transform) of $B(u)$:

$$\Theta(\alpha) := \int_0^1 u^\alpha B(u) \, du.$$  \hspace{1cm} (1.1.31)

Then we can enunciate the following proposition:

\textbf{Proposition 1.3.} \textit{There exists a finite shattering time $T^*$ for equation (1.1.29) and $\beta(x) = x^\lambda$, with $T^*$ defined as in equation (1.1.22), if and only if $\lambda < 0$.}

This result has been known, even for non homogeneous kernels, since the work of Filippov [136]. Here we follow the proof of Cheng and Redner [68] for its simplicity. We avoid, for now, a precise statement of the technical conditions for the existence of $\Theta(\alpha)$ and refer to the results in Chapter 4 and to [68, 118], among others, for a correct formalization of the details. We also point out that the condition $\lambda < 0$, independently of the properties of the relative distribution of fragmented particles, physically means that small clusters fragment faster and faster up to an infinite rate.
Proof. Regarding the necessary condition, we wish to locate the shattering transition. Fix $\alpha = 1 + \varepsilon$ with $\varepsilon > 0$ and suppose $T^* < \infty$. Therefore, under mild assumptions on the regularity of $B\left( u \right)$,

$$\Theta (1 + \varepsilon) \rightarrow \Theta (1) = 1.$$ 

Then $M_{1+\lambda+\varepsilon} (t)$ cannot be bounded on $(0, T^*)$ as $\varepsilon \rightarrow 0^+$: in fact, mass is conserved up to $T^*$, but equation $\frac{d}{dt}M_{1+\varepsilon} (t) = (\Theta (1 + \varepsilon) - 1) M_{1+\lambda+\varepsilon} (t)$ cannot be valid for any fixed $t > T^*$ when $\varepsilon$ go to zero since $\frac{d}{dt}M_1$ is necessarily $\frac{d}{dt}M_1 = 0$ on $(0, T^*)$.

We have to show now that $\lambda$ is negative. Consider an initial datum such that $\text{supp} \left( c_o \right) \subseteq [0, 1]$. This hypothesis do not suppose a loss of generality thanks to linearity of the equation and the possibility of performing a rescaling. Thus, for any positive time $t > 0$, the fragmentation solution must still verify $\text{supp} \left( c\left( x, t \right) \right) \subseteq [0, 1]$, and, for any fixed $t$ and for any $\alpha_1 < \alpha_2$, we have:

$$M_{\alpha_1} (t) \geq M_{\alpha_2} (t).$$

However, $M_1 < \infty$ at any time, so that $M_{1+\lambda} (t) \rightarrow \infty$ can possibly be unbounded only if $\lambda < 0$.

As for the sufficiency, assume now $\lambda < 0$ and suppose by contradiction that no shattering occurs. Then $T^* = \infty$ and the mass $M_1 (t)$ is equal to a fixed quantity $M$ constantly; then we can consider:

$$\frac{d}{dt}M_{1-\lambda} (t) = (\Theta (1 - \lambda) - 1) M = \text{const.}$$

On the other hand, due to the monotonicity in the interval $(0, 1)$ of $u^{1-\lambda} < u$, we see that

$$\Theta (1 - \lambda) = \frac{\int_0^1 u^{1-\lambda} B\left( u \right) \, du}{\int_0^1 u B\left( u \right) \, du} < 1,$$

and thus the derivative $\frac{d}{dt}M_{1-\lambda} (t)$ is equal to a negative constant: $\frac{d}{dt}M_{1-\lambda} (t) < 0$. Also $M_{1-\lambda} (t)$ must be zero at time $T^0$ as determined by the following formula:

$$T^0 = \frac{M_{1-\lambda} (0)}{\left( 1 - \Theta (1 - \lambda) \right) M}. \quad (1.1.32)$$

However, we remark that for any non-zero (and non-negative) initial distribution $c_o$, an obvious bound to the solution of equation (1.1.20) is to neglect the gain term (always non-negative). Doing so, one can see that any solution $c\left( x, t \right)$ verifies an exponential decay estimate:

$$c\left( x, t \right) \geq c_o\left( x \right) e^{-x^{\lambda} t}, \quad (1.1.33)$$

which makes it impossible for any moment to be zero, and in particular $M_{1-\lambda} (T^0) \neq 0$. Therefore we have a contradiction and shattering occurs with $T^* = T^0$.

Remark 1.4. Inequality (1.1.33) can obviously be re-obtained dropping the assumption on $\beta$; its consequence is that the solution is non-negative whenever the initial datum is. As for the coagulation equation, non-negativity of solutions is not easily obtained with a priori bounds (it is obtained, for example, from cut-off and convergence techniques).

In the next section we review the main classic and recent results in the literature on the coagulation and fragmentation equations.
1.2. COAGULATION AND FRAGMENTATION: STATE OF ART

Several different approaches appear in the literature studying the well-posedness of pure coagulation, pure fragmentation and mixed coagulation-fragmentation equations. Regarding the existence problem, the first rigorous results came with the work of White [507, 508], Spouge [280, 282] and Ernst, Hendriks and Ziff [116, 114]. It is interesting to observe that most of the known explicit solutions were found earlier [223, 278, 279, 11] and the first solution (constant kernel in the discrete setting) had already been deduced by Smoluchowski in his original work. In this sense, they represent the cornerstones which oriented later researches. Another interesting related question is the non-existence of solutions or, from another viewpoint, the existence of solutions with infinite mass (which are not physical). There are also cases in pure coagulating systems in which instantaneous gelation takes place due to the fact that the kernel grows too quickly (for example $K(x, y) \geq x^{2+\alpha} + y^{2+\alpha}$, with $\alpha > 0$, or $K(x, y) \geq x^a y^b$ with $a > 1$ or $b > 1$); see [293, 60]. The question whether, under sufficiently strong fragmentation, the instantaneous gelation can be avoided, is to our knowledge an open problem. Instead it is known that diffusion (even degenerate diffusion) can prevent gelation (see Cañizo, Desvillettes and Fellner [55, 54]).

1.2.1. Qualitative properties

Before dealing with theorems of existence and uniqueness with and without gelation, there is some qualitative analysis that can be done almost effortlessly and that will come to use later. We begin with a formal identity for Smoluchowski equation extensively employed by Escobedo, Fournier, Laurençot and Mischler as in [204]: multiply both sides of equation (1.1.2) with a test function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}$, integrate and apply (formally) Fubini’s theorem to obtain:

$$
\frac{d}{dt} \int_0^\infty \phi(x) c(x, t) \, dx = \frac{1}{2} \int_0^\infty \int_0^\infty K(x, y) \left( \phi(x+y) - \phi(x) - \phi(y) \right) c(x, t) c(y, t) \, dx \, dy,
$$

(1.2.1)

for test functions with suitable regularity.

Similar identities can be also derived for the other mean field equations of Section 1.1. If we impose $\phi(x) = x^{\alpha}$ in (1.2.1), we get the moments formula for the Smoluchowski equation. Given that:

$$
\text{sign} \left\{ (x+y)^\alpha - x^{\alpha} - y^{\alpha} \right\} = \text{sign} \{ \alpha - 1 \},
$$

we can observe that, as long as solutions exist and are sufficiently regular, the moments $M_{\alpha}(t)$ are non-increasing with $\alpha \leq 1$, non-decreasing with $\alpha \geq 1$ and constant with $\alpha = 1$; this fact, of course, goes accordingly to the physical intuition of the coagulation phenomenon.

Moreover, under an additional growth condition on $K$, that is:

$$
K(x, y) \leq K(x, x+y) + K(y, x+y),
$$

(1.2.2)

then, setting $\phi(x, t) = p e^{p-1}(x, t)$, $p \geq 1$, we can observe that the $L^p$-norm of the solution $\| c(\cdot, t) \|_p$ is non-increasing in time, whenever it is finite. This prevents the solution from concentrating at
some finite point, even if it can still develop an algebraic tail at infinity. An example of Kernel which does not verify condition (1.2.2) is \( K(x, y) = x^{-\gamma}y^{-\gamma} \) with \( \gamma > 1 \) (strong coagulation for small clusters but slow interaction with heavy masses).

An interesting qualitative property that we should address in this moment concerns the non-negativity results for non-negative initial data. Given the non-linearity of the coagulation operator, this property can not be obtained a priori from equation (1.1.2), but, instead, has been shown with the existence theorems in suitable classes of non-negative functions. Differently, Da Costa \[8\] for the discrete case showed that solutions to the Cauchy problem with all bounded moments existing for some time must be non-negative during that time. Another useful strategy corresponds to the estimation of some norms of the negative part of the solution, \( c^{-}(x, t) := -\min(c(x, t), 0) \), and the application of Gronwall’s Lemma to show that, if such a norm is zero for \( t = 0 \), then it must remain zero for all time. See for example Lemma 5.3.1 in Cañizo \[56\] for more details.

1.2.2. WELL-POSEDNESS

The interplay between coagulation and fragmentation processes leads to a variegated classification of behaviors, from singularities in finite time (gelation, shattering) to convergence to equilibriums (for example, under the so-called detailed balance condition). A complete list is not entirely possible yet, since there are still open problems and some answers can only be conjectured, particularly in the non-homogeneous setting. However, we can point out some relevant cases and try to describe the Big Picture of the mathematical theory of coagulation. The literature on the well-posedness problem is extensively developed and occupies a prominent position; due to the quantity of branches and different techniques, we consider here only the part which is more useful to our study.

The classic approach for the cases without gelation consists, as in Dubowski and Stewart \[98\], in studying the existence and non-negativity of equation (1.1.14) when \( K(x, y) \leq a(1 + x + y) \), for some \( a > 0 \), and when the fragmentation kernel is bounded by some function (in this case, \( F(x - y', y'') \leq b_1(1 + x^{f_1}) \), for some exponent \( f_1 \)). They implement a cut-off technique with respect to the kernel. They also point out an optimal condition on the binary fragmentation term for the solution to be mass-conserving during the interval \([0, T]\) in the class of functions with bounded moments or of exponentially decaying functions:

\[ \int_{0}^{x} y F(x - y, y) \, dy \leq b_2(1 + x^{f_2}) , \quad f_2 \geq 2 \]  

(1.2.3)

and

\[ M_2(t) < \infty, \quad t \in [0, T]. \]  

(1.2.4)

Unicity, instead, rely on a much more restrictive condition: \( c_o(x) \) must decay at least exponentially and \( \int_{0}^{\infty} F(x - y, y) \, dy \leq b(1 + x^{\alpha}) \) with \( \alpha < 1 \). This seemingly excessive limitation corresponds to the fact that mixing both coagulation and fragmentation make the equation lose its good properties for coagulation or fragmentation mechanism alone.
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<td>$\exists !, T^* = \infty$ $\exists !, T^* &lt; \infty$ (bounded from below)</td>
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<td>HOMOGENEOUS NON-GELLING COAGULATION:</td>
<td>$K(x, y) \leq \kappa_0 (x^\lambda + y^\lambda)$, $\lambda \in (-\infty, 1]$</td>
<td>$\exists !$ in the weak sense, $T^* = \infty$</td>
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<td></td>
<td>STRONG FRAGMENTATION AND WEAK COAGULATION:</td>
<td>$\beta(x) = x^\alpha$, $B(u) = (\nu + 2) u^\nu$, $\nu \in (-1, 0], 0 \leq \sigma \leq 1, \sigma &lt; \alpha$, and $K(x, y) \leq \kappa_2 ((1 + x)^\sigma + (1 + y)^\sigma)$</td>
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<td>STRONG FRAGMENTATION AND STRONG COAGULATION:</td>
<td>the same as above and $0 \leq \sigma \leq \tau &lt; \alpha$ and $K(x, y) \leq \kappa_3 (x^\sigma y^\tau + x^\tau y^\sigma)$</td>
<td>local solution and uniqueness as above: [27]</td>
</tr>
<tr>
<td></td>
<td>DETAILED BALANCE CONDITION:</td>
<td>let $M \in L^1$ be a non-zero function such that: $K(x, y) M(x) M(y) = b(x, y) M(x + y)$</td>
<td>any solution converges to a single equilibrium $C<a href="x">M_{t(0)}</a>$ depending only on the initial mass</td>
</tr>
<tr>
<td></td>
<td>OTHER EQUILIBRIA:</td>
<td>strong fragmentation, small initial mass (the latter is probably a technical condition)</td>
<td>any solution converges to a single equilibrium $C<a href="x">M_{t(0)}</a>$ with exponential trend in $L^2$</td>
</tr>
</tbody>
</table>

Table 1.4: The Big Draft of coagulation-fragmentation theory most relevant existence, uniqueness and convergence results.
For coagulation alone, Norris ([250] Theorem 2.1) shows the following classic general result:

**Theorem 1.5** (Unicity for Smoluchowski equation, Norris 1999). Assume that \( K (x, y) \leq \kappa (x) \kappa (y) \) with \( \kappa (\cdot) \) a sub-additive function. Let \( c_0 (x) \) be a non-negative Radon measure satisfying \( \langle c_0, \kappa \rangle < \infty \). Then for every \( T < \infty \) there is at most one solution to Smoluchowski equation with initial condition \( c_0 (x) \) satisfying:

\[
\begin{align*}
t &\rightarrow \langle c (t, \cdot) , \kappa \rangle \text{ is in } C ([0, T]), \\
t &\rightarrow \langle c (t, \cdot) , \kappa^2 \rangle \text{ is in } L^1 ([0, T]).
\end{align*}
\]

The unicity class requires the finiteness of two moments which provides, basically, information on small and large \( x \). Also, Norris complement Theorem 1.5 with the existence counterpart:

**Theorem** (Existence and unicity for Smoluchowski equation, Norris 1999). Under the assumptions of Theorem 1.5, if \( \kappa (x) > \varepsilon x \) for some \( \varepsilon \), then any strong solution is conservative.

Let also \( \langle c_0, \kappa^2 \rangle < \infty \). Then:

1. There exists a unique strong maximal solution for any \( t \leq T [c_0] \), with \( T [c_0] \) a lower estimate of \( T^* \) satisfying \( T [c_0] \geq \langle c_0, \kappa^2 \rangle^{-1} \);

2. if \( \kappa^2 \) is also sub-linear or if \( K (x, y) \leq \kappa (x) + \kappa (y) \), then \( T [c_0] \equiv T^* = \infty \).

The notion of weak solution seems therefore a good tool to establish results for the integral equations we are interested in.

### 1.2.2.1. Existence, uniqueness and gelation occurrence in the weak sense for the coagulation-fragmentation equation

Recent developments rely on weak and strong compactness methods by Escobedo, Laurençot, Mischler and Perthame [194, 174, 204, 115] (adapting the DiPerna-Lions techniques for the Boltzmann equation to coagulation setting). Such methods have proven useful for the study of the conditions leading to the behaviors of Table 1.4 for solutions of (1.1.22-1.1.23) also in the inhomogeneous dispersive case. In the following, we briefly summarize their works which are optimal in the sense that they require only the weakest known technical hypothesis. Existence of coagulation-fragmentation equations (global-in-time or until blow-up) is classified depending on the interplay of the strengths the two kernels. A proper notion of weak solutions is needed in order to attain the greatest generality and to include the previous results.

Compactness methods in Lebesgue spaces are particularly suited for the integral operators in equation (1.1.22-1.1.23). The functional spaces involved here are the following:

\[
\begin{align*}
L^1_{\phi} &:= \left\{ f \geq 0 \text{ a.e.} : f \cdot \phi \in L^1 (\mathbb{R}^+) \right\}, \\
L^1_k &:= L^1_{\phi_k}, \quad \phi_k (x) = (1 + x^k).
\end{align*}
\]

We remark that the use of these classes implicitly requires the non-negativity of the solutions. It is now necessary to state exactly the definition of a weak solution:


**Definition.** A solution \( c (x, t) \) to the coagulation-fragmentation equation with initial profile \( c_0 \in L^1 \) and such that \( c \in L^\infty ([0, T], L^1) \) for every \( T > 0 \), is a function verifying \( Q (c) := C (c) − F (c) \) is in \( L^1 ([0, T], [0, L]) \) for every \( L > 0 \) and moreover \( c (x, t) \) satisfies, for every \( t \geq 0 \),

\[
M_1 (t) \leq M_1 (0),
\]

\[
\int_{\mathbb{R}^+} c \, \hat{c}_t \psi \, dx \, dt + \int_{\mathbb{R}^+} c_0 \, \psi (\cdot , 0) \, dx + \int_{\mathbb{R}^+} Q (c) \, \psi \, dx \, dt = 0,
\]

for each test function \( \psi (x, t) \in C_0^\infty (\mathbb{R}^+ \times \mathbb{R}^+) \).

We remark that in the previous definition the requirement is explicitly \( M_1 (t) \leq M_1 (0) \) and not strictly \( M_1 (t) = M_1 (0) \). This means that weak solutions do not pretend to distinguish between gelling and non-gelling cases. If \( K (x, y) \leq \kappa (x) \kappa (y) \) where \( \kappa \) is a sub-additive function, that is \( \kappa (x + y) \leq \kappa (x) + \kappa (y) \), then the initial profile can actually include more initial distributions: \( c_0 \in L^1_w \) with \( w (x) = \min (1 + x, \kappa (x)) \) (see Norris [259] to this respect).

A technique to obtain solutions to (1.2.9), called the **Stability Principle**, is enunciated in Laurencot-Mischler [204] and can be summarized as follows. One considers a sequence \( \{c_n (x, t)\} \) of solutions of an approximating problem of (1.1.22-1.1.23) and proves that \( \{c_n\} \) and \( \{Q (c_n)\} \) belong to a weakly compact subset of \( L^1_{\text{loc}} \) and that, for every fixed \( n \), the time depending functions \( \left( \int_{\mathbb{R}^+} c_n (x, t) \psi (x, t) \, dx \right) \) do belong to a strongly compact subset of \( L^1 \) for any test function \( \psi \). Then there exist a subsequence \( c_n \) and a function \( c (x, t) \) such that the subsequence converges weakly \( c_n \rightharpoonup c \) in \( L^1_{\text{loc}} \), the subsequence of operators converges weakly \( Q (c_n) \rightharpoonup Q (c) \) in \( L^1_{\text{loc}} \) and, finally, \( c \) is a solution of (1.2.9).

This technique has been employed under a variety of technical conditions and, for a comprehensive list of references, see the already mentioned work by Laurencot and Mischler. The main result in this sense is the following:

**Theorem 1.6.** Let initial data be of the class \( c_0 \in L^1 \). Then, the coagulation-fragmentation problem (1.1.22-1.1.23) admits at least one global-in-time, mass-conserving, weak solution in the sense of Definition 1.2.2.1 under the assumptions of weak coagulation or strong fragmentation.

Instead, under the assumption of strong coagulation and under additional assumptions on the initial mass \( M_1 (0) \), any weak solution is mass-conserving only for a finite time \( T^* < \infty \).

As for the mass-conservation, it is easy to see the reason why weak coagulation and strong fragmentation yield a strong control on the moments and on the behavior of the tails of the solutions. The proof of this fact is similar to computations developed in Examples 1.1.1.2 and we can summarize it quickly thanks to identity (1.2.1) extended for the coagulation-fragmentation equation (1.1.22-1.1.23).

As a general notation rule throughout this Introduction, let the values \( C_1 , C_2 , \ldots \) represent suitably positive finite constants that we do not specify.

**Proof sketch.** For the weak coagulation, consider the case \( \phi (x) = x^2 \). The second moment equation can be bounded above thanks to the fact that the fragmentation term is non-positive. This yields:

\[
\frac{d}{dt} M_2 (t) \leq C_1 M_1 (t) (M_1 (t) + M_2 (t)).
\]
In this way, if $c_0 \in L^1_+$, one proves that $M_2 (t) \leq C_2 M_2 (0) e^t$ and a strong control on the decay of the solution permits arguing that $M_1 (t) = M_1 (0)$ is constant.

To conclude our explanation about the weak formulation, we have to point out that the availability of uniqueness results is only limited to global mass-conserving cases or, for coagulation, to the homogeneous kernels. When gelation is expected to occur, there are unicity results up to the critical time. Post-gelling solutions are not widely studied, although there are some studies for the explicit case of the product kernel $K_x (x, y) = xy$.

A typical uniqueness result is the following theorem by Laurençot and Mischler (in [204]).

**Theorem 1.7.** Let the coagulation kernel $K (x, y)$ satisfy $K (x, y) \leq \kappa (x) \kappa (y)$ where $\kappa$ is a subadditive function, that is: $\kappa (x + y) \leq \kappa (x) + \kappa (y)$. If the fragmentation kernel $b (x, y)$ satisfies

$$
\int_0^x b (y, x - y) (\kappa (y) - \kappa (x) + \kappa (x - y)) \, dy \leq C \kappa (x), \tag{1.2.11}
$$

then there exists a unique solution $c (x, t)$ to the coagulation-fragmentation problem (1.1.22-1.1.23) of the class:

$$
c \in C ([0, T] , L^1_\kappa) \cap L^1 ([0, T] , L^1_{\kappa^2}), \tag{1.2.12}
$$

for each $T > 0$.

The proof follows from a suitable extension of the fundamental identity for the Smoluchowski equation (1.2.8) to the coagulation-fragmentation equation; then one can apply the Gronwall Lemma. For the details, see [204]. We remark also that the solutions of the coagulation-fragmentation equation can fail to belong to the class (1.2.12) for all time, so the applicability of Theorem 1.7 depends on estimating the behavior of the solution.

**Remark 1.8.** In fragmentation and coagulation problems, the two most important spaces (for their physical meaning) are the Lebesgue spaces of functions with finite 0th and finite first moments. It turns out that the technical difficulties arising for the mixed equations derive from the fact that the two operators show different behaviors on the two spaces: the coagulation operator behaves well on the classic $L^1$ space (since the norm corresponds to the total amount of clusters in the system and we intuitively know that it decreases) and, as we have seen, on $L^1_{(1+x)dx}$; instead the fragmentation operator is ill-posed on $L^1$ spaces (since it increases the number of fragments and so the norm too), but shows good properties on $L^1_{edx}$. This leads to the need of different proofs for the cases with strong or weak fragmentation and extra restrictions to the so called fragmentation with finite number of fragments. Fragmentation with infinite number of fragments is still well-posed for the pure fragmentation and for fragmentation with bounded coagulation. See the work of Banasiak and Lamb [14, 25, 20, 21] for this discussion and in particular [27] and refer to Cepeda [64] for the infinite fragmentation case.

### 1.2.2.2. Existence, uniqueness and gelation occurrence for pure coagulation with homogeneous kernel

Homogeneous kernels in pure coagulation are particularly important for the analysis of our work and, in general, represent the most relevant class considered both in applications and in the-
We recall that \( L^1_\phi \) is the space of non-negative Radon measures \( \| f \|_\phi = \| f \cdot \phi \|_{L^1} < \infty \); we change slightly the definition of the second functional space accordingly to our new needs: let \( L_\lambda \) be defined as

\[
L_\lambda := L^1_{\phi_\lambda}, \quad \text{where} \; \phi_\lambda (x) = x^\lambda. \tag{1.2.13}
\]

Remember that \( L_\lambda \) includes all functions with finite \( \lambda \)–moment and in fact \( M_\lambda (f) < \infty \) is the norm of non-negative functions verifying \( f \in L_\lambda \). Let \( K \) be a kernel of homogeneity degree in the interval \( \lambda \in (-\infty, 2] \) and consider the space of test functions \( \psi \in C(\lambda) \), defined as:

\[
\begin{align*}
\lambda &\leq 0 \quad \left\{ \psi \in C : \sup_x \frac{|\psi(x)|}{x^\lambda} < \infty \right\}, \\
0 < \lambda &\leq 1 \quad \left\{ \psi \in C : \psi(0) = 0 \text{ and} \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x - y|^\lambda} < \infty \right\}, \\
1 < \lambda &\leq 2 \quad \left\{ \psi \in C : \psi(0) = 0 \text{ and} \sup_{x \neq y} \frac{|\psi(x) - \psi(y)|}{|x^\lambda - y^\lambda|} < \infty \right\}.
\end{align*}
\]

(1.2.14)

(1.2.15)

(1.2.16)

We need two technical estimations on the growth and regularity of the kernel:

**Assumption 1.9.** The \( \lambda \)–homogeneous kernel \( K(x, y) \) satisfies \( K(x, y) \leq \tilde{K}(x, y) \), with

\[
\tilde{K}(x, y) := \kappa_0 (x + y)^\lambda, \quad \lambda \leq 1, \quad \text{or} \quad \tilde{K}(x, y) := \kappa_0 (xy^{\lambda-1} + x^{\lambda-1}y), \quad 1 < \lambda \leq 2,
\]

for some constant \( \kappa_0 > 0 \).

Moreover:

**Assumption 1.10.** The \( \lambda \)–homogeneous kernel \( K(x, y) \) belongs to \( W^{1, \infty} ((\varepsilon, \frac{1}{2}) \times (\varepsilon, \frac{1}{2})) \) for every \( \varepsilon \in (0, 1) \); moreover, for some constant \( \kappa_1 > 0 \), it verifies:

\[
|\delta (x, y) \partial_x K(x, y)| \leq \kappa_1 x^{\lambda-1} y^\lambda
\]

and \( \delta (x, y) \) is defined as:

\[
\delta (x, y) := \begin{cases} x^\lambda + y^\lambda, & 0 < \lambda \leq 1, \\ \min (x^\lambda, y^\lambda), & 1 < \lambda \leq 2. \end{cases}
\]

(1.2.18)

(1.2.19)

We redefine the notion of weak solution to Smoluchowski equation with respect to Definition 1.2.8, consider an initial profile \( c_0 \in L_\lambda \) and a measure \( c(x, t) \) with \( t \in [0, T) \). Then \( c(x, t) \) is a weak solution if:

1. \( c(\cdot, 0) = c_0 \);
2. the mapping \( t \mapsto \| c(\cdot, t) \|_\psi \) is differentiable on \( t \in [0, T) \) for every test-function \( \psi \in C(\lambda) \);
3. for every \( t \in [0, T) \) one has \( \sup_{s \leq t} M_\lambda (c(\cdot, t)) < \infty \);
4. For every test-function $\psi \in C_c(\lambda)$, the weak Smoluchowski equation is verified:

$$\frac{d}{dt} \int_{\mathbb{R}^+} c(x, t) \psi(x) \, dx - \frac{1}{2} \int_{\mathbb{R}^+} \int_{\mathbb{R}^+} K(x, y) (\psi(x + y) - \psi(x) - \psi(y)) \, c(x, t) \, c(y, t) \, dx \, dy = 0.$$  

(1.2.20)

Such a weak solution $c(x, t)$ is referred to as the $(\lambda, c_0, T)$ -weak solution of (1.2.20).

We observe that under Assumption 1.10, all the integrals in the weak Smoluchowski equation (1.2.20) are absolutely convergent and bounded in time intervals $[0, \tau], \tau < T$.

We conclude with the result by Fournier and Laurençot [143]: the general theorem of well-posedness (uniqueness under moments estimation) for homogeneous coagulation kernels:

**Theorem 1.11.** Let $K$ verify Assumptions 1.9-1.10 and the initial profile be such that $c_0 \in L_\lambda$.

- If $\lambda \in (-\infty, 1]$, then there exists a unique $(\lambda, c_0, \infty)$ -weak solution to (1.2.20).

- If $\lambda \in (1, 2]$, then there exists a critical time $T^* \geq (\lambda \kappa_0 M_\lambda(c_0))^{-1}$, which could still be $T^* = \infty$, such that:

$$\lim_{t \to T^*} M_\lambda(c(\cdot, t)) = \infty$$

(1.2.21)

and such that there exists a unique $(\lambda, c_0, T^*)$ -weak solution to (1.2.20).

Notice that if we impose a lower bound on the growth of $K$ (e.g. $K \geq \kappa_0 x^{\lambda/2} y^{\lambda/2}$), then, in addition to $T^* \geq (\lambda \kappa_0 M_\lambda(c_0))^{-1}$, it also follows that $T^* < \infty$ (gelation occurs) from moment estimations like those of Examples 1.1-1.2. Theorem 1.11, even if similar to the results of Section 1.2.2, requires less restrictions on the functional classes. In particular, it characterizes the weak solutions as being $\lambda$-weak solutions: one needs the control over the $\lambda$ moment, guaranteeing that the approach to gelation occurs with the blow-up of the moment of order $\lambda$. It shows therefore this intuitive feature that reflects the homogeneity property of the kernel and its growth condition.

The proof of Theorem 1.11 is based on a suitable definition of a Weierstrass distance for the uniqueness part and, for the global in time existence, on a tightness argument. The case $\lambda \in (1, 2]$ follows a different strategy: the truncation of the coagulation kernel permits to prove a local existence result for initial data with fast decay at infinity by a strong compactness method; one finds a lower bound for the existence time which depends on the moment $M_\lambda(c_0)$ of the initial data, and then extends the solution to a maximal existence time. We summarize now the main steps to prove the global in time existence and sketch the uniqueness part. We skip the gelling case and refer the reader to [143].

**Uniqueness:**

Let’s introduce some notation:

$$F[f](x) := \int_x^\infty f(y) \, dy, \quad G[f](x) := \int_x^\infty \int_0^x f(y) \, dy \, dx,$$

(1.2.22)
and, given two functions $f (c, t)$ and $g (x, t)$, we also define for $\lambda < 0$:

\[
E_{f,g} (x, t) := G [f (\cdot, t)] (x) - G [g (\cdot, x)] (x),
\]

\[
R (x, t) := \int_{x}^{\infty} y^{\lambda - 1} \text{sign} (E_{f,g} (y, t)) \, dy
\]

and, similarly, if $0 < \lambda \leqslant 2$,

\[
E_{f,g} (x, t) := F [f (\cdot, t)] (x) - F [g (\cdot, x)] (x),
\]

\[
R (x, t) := \int_{0}^{x} y^{\lambda - 1} \text{sign} (E_{f,g} (y, t)) \, dy.
\]

Then, if $\lambda < 0$ and $f \in L_{\lambda}$,

\[
\int_{0}^{\infty} x^{\lambda - 1} G [f (\cdot, t)] (x) \, dx = \frac{M_{\lambda} (f)}{|\lambda|}, \quad \lim_{x \to 0} x^{\lambda} G = \lim_{x \to \infty} x^{\lambda} G = 0,
\]

and the same formulas are true, when $0 < \lambda \leqslant 2$, replacing $F$ by $G$. Moreover, for each $\varepsilon > 0$ we have that $F \in L^{\infty} (\varepsilon, \infty)$ and $G \in L^{\infty} (0, \frac{1}{\varepsilon})$. As for the properties of $E$ and $R$, we have:

**Lemma 1.12.** Suppose that the conditions of Theorem 1.11 are verified and that $c_{0}$ and $\gamma_{0}$ are two initial conditions in $L_{\lambda}$. If $c (x, t)$ and $\gamma (x, t)$ are the corresponding $(\lambda, c_{0}, T)$ and $(\lambda, \gamma_{0}, T)$ weak solutions, then for each $t < T$ the function $R (\cdot, t)$ belongs to the test-class $C_{(\lambda)}$ and the following distance inequality holds:

\[
\frac{d}{dt} \int_{0}^{\infty} x^{\lambda - 1} |E_{c,\gamma} (x, t)| \, dx
\]

\[
\leqslant \frac{1}{2} \int_{0}^{\infty} \int_{0}^{\infty} K (x, y) \left( (x+y)^{\lambda - 1} - x^{\lambda - 1} \right) (c (x, t) - \gamma (x, t)) |E_{c,\gamma} (x, t)| \, dx \, dy
\]

\[
+ \frac{1}{2} \int_{0}^{\infty} \int_{0}^{\infty} (\partial_{x} K (x, t)) (R (x+y, t) - R (x, t) - R (y, t)) (c (x, t) - \gamma (x, t)) |E_{c,\gamma} (x, t)| \, dx \, dy.
\]

Under the Assumptions 1.9 and 1.10, one can fix a constant $C (\lambda, \kappa_{0}, \kappa_{1})$ depending only on the parameters, and derive, with standard estimations:

\[
K (x, y) |(R (x+y, t) - R (x, t) - R (y, t))| \leqslant C x^{\lambda} y^{\lambda};
\]

\[
\lambda < 0 \quad K (x, y) \left( (x+y)^{\lambda - 1} + x^{\lambda - 1} \right) \leqslant C x^{\lambda - 1} y^{\lambda};
\]

\[
0 \leqslant \lambda \leqslant 2 \quad K (x, y) \left( (x+y)^{\lambda - 1} - x^{\lambda - 1} \right) \leqslant C x^{\lambda - 1} y^{\lambda}.
\]
These estimations can be applied to Lemma \ref{lem:estimation}. In that case, the distance inequality between two solutions \( c(x, t) \) and \( \gamma(x, t) \) reads:

\[
\frac{d}{dt} \int_0^\infty x^{\lambda-1} |E_{c, \gamma}(x, t)| \, dx \leq C M_\lambda (c(\cdot, t) + \gamma(\cdot, t)) \int_0^\infty x^{\lambda-1} |E_{c, \gamma}(x, t)| \, dx, \tag{1.2.32}
\]

so, under the hypothesis of Theorem \ref{thm:existence} for two weak solutions in \( L_\lambda \) with the same initial profile Gronwall’s Lemma readily implies that \( |E_{c, \gamma}(x, t)| = 0 \).

**Existence:**

**Proof.** We remind that we prove here existence only for \( \lambda \leq 1 \). For any \( n \geq 1 \) we consider a sequence of initial data

\[
c_{0, \{n\}}(x) := \mathbf{1}_{\left[\frac{1}{n}, n\right]} c_0,
\]

where \( \mathbf{1}_{\left[\frac{1}{n}, n\right]} \) is the indicator function of the interval. All \( \{c_{0, \{n\}}(x)\} \) belong to \( L_\lambda \) for all \( \lambda \in \mathbb{R} \).

We use the notation \( G_{t, \{n\}} = G[c_{\{n\}}(\cdot, t)] \) and \( F_{t, \{n\}} = F[c_{\{n\}}(\cdot, t)] \); for \( \lambda < 0 \), we see that \( (G_{0, \{n\}}) \to G_0 \) as \( n \to \infty \) in \( L_\lambda^{-1} \) and for \( 0 \leq \lambda \leq 1 \), analogously, \( (F_{0, \{n\}}) \to F_0 \) as \( n \to \infty \) in \( L_\lambda^{-1} \). Then, we also see that under Assumption \ref{ass:1} we can introduce \( \varphi = \sqrt{u_0 x^{\lambda/2}} \), if \( \lambda < 0 \), and \( \varphi = \sqrt{u_0 (1+x)^\lambda} \) otherwise: in both cases, \( \varphi \) is sub-additive functions and \( K(x, y) \leq \varphi(x) \varphi(y) \), so that we can recall Norris existence Theorem \ref{thm:norris} to conclude that, for each \( n \), there exists a \( (\lambda, c_{0, \{n\}}, \infty) \)–weak solution \( c_{\{n\}}(x, t) \) to Smoluchowski equation.

We can now consider inequality \ref{ineq:distance} for two solutions \( c_{\{n\}} \) and \( c_{\{m\}} \). Suppose that \( \lambda < 0 \) to fix ideas. Then, imposing \( \phi = x^\lambda \) in equation \ref{ineq:distance} and since \((x + y)^\lambda - x^\lambda - y^\lambda \leq 0 \), the moments show monotonicity:

\[
M_\lambda (c_{\{n\}}) \leq M_\lambda (c_{\{n\}, 0}) \leq M_\lambda (c_0) \tag{1.2.33}
\]

and this yields:

\[
\int_0^\infty x^{\lambda-1} \left| G_{t, \{n\}}(x, t) - G_{t, \{m\}}(x, t) \right| \, dx \leq e^{Ct} \int_0^\infty x^{\lambda-1} \left| G_{0, \{n\}}(x) - G_{0, \{m\}}(x) \right| \, dx. \tag{1.2.34}
\]

The next step (we skip the details) is to show that, from estimates \ref{ineq:distance} and \ref{ineq:monotonicity} and the fact that the mapping \( t \mapsto G_{t, \{n\}} \) belongs to \( C \left( \mathbb{R}^+; L_\lambda^{-1} \right) \), the sequence of mappings \( \{t \mapsto G_{t, \{n\}}\} \) is a Cauchy sequence in \( C \left( \mathbb{R}^+; L_\lambda^{-1} \right) \). Therefore, there exists a measure \( g \in C \left( \mathbb{R}^+; L_\lambda^{-1} \right) \) such that, for each \( t \), \( \sup_{s \leq t} \int_0^\infty x^{\lambda-1} \left| G_{t, \{n\}}(x, s) - g(x) \right| \, dx \to 0 \), and

\[
\lim_{\varepsilon \to 0} \sup_{s \leq t} \left[ \int_0^\varepsilon x^{\lambda-1} g(x, s) \, dx + \int_{\frac{\varepsilon}{2}}^\infty x^{\lambda-1} g(x, s) \, dx \right] = 0. \tag{1.2.35}
\]

We want to show that \( c_{\{n\}} \) is tight, on finite time intervals, in \( L_\lambda^1 \). For this we consider the extreme intervals of the \( L_\lambda^1 \) norm \( \left( \int_{0}^{\varepsilon} x^\lambda c_{\{n\}}(x, s) \, dx + \int_{\frac{\varepsilon}{2}}^{\infty} x^\lambda c_{\{n\}}(x, s) \, dx \right) \) and we want this contribution to go uniformly to zero.
It can be shown, thanks to identity (1.2.27), that:

\[
\int_0^\varepsilon x^\lambda c_{\{n\}} (x, s) \, dx + \int_\varepsilon^\infty x^\lambda c_{\{n\}} (x, s) \, dx \\
\leq C \left( \int_0^{2\varepsilon} x^{\lambda-1} G_{s,\{n\}} \, dx + \int_\varepsilon^\infty x^{\lambda-1} G_{s,\{n\}} \, dx \right),
\]

and, since \(G_{s,\{n\}}\) is Cauchy sequence and \(g(x,s)\) satisfy (1.2.35), one can apply Lebesgue dominated convergence theorem to conclude that \(c_{\{n\}}\) is tight. Its limit in the weak sense is \(c(x,t) := \partial_x g(x,t)\) and \(c \in L^1_t\) with \(M_\lambda (c) \leq M_\lambda (c_0)\).

This part of the proof can be repeated almost identically for the case \(0 \leq \lambda \leq 1\) substituting appropriately \(F_{s,\{n\}}\) in place of \(G_{s,\{n\}}\). It is still necessary to show that the limit measure is a solution to the weakly formulated Smoluchowski equation (1.2.20). In order to conclude, we need to show that, for every test-function \(\phi\):

\[
\limsup_{n \to \infty} \sup_{s \leq t} \int_0^\infty \phi (x) \left( c_{\{n\}} (x, s) - c(x, s) \right) \, dx = 0 \tag{1.2.37}
\]

and

\[
\limsup_{n \to \infty} \sup_{s \leq t} \int_0^\infty \int_0^\infty K (x,y) \left( \phi (x+y) - \phi (x) - \phi (y) \right) (c_{\{n\}} - c \cdot c) \, dx \, dy = 0, \tag{1.2.38}
\]

so that, integrating the weak Smoluchowski equation (1.2.20) with respect of time, each term can be individually satisfied. The two limits above can be finally be checked thanks to the tightness \(c_{\{n\}}\) and manipulating the integrals as follows:

\[
\limsup_{n \to \infty} \sup_{s \leq t} \int_0^\infty \phi (x) \left( c_{\{n\}} (x, s) - c(x, s) \right) \, dx \\
= \limsup_{n \to \infty} \sup_{s \leq t} \int_0^\infty \phi ' (x) \left( G_{s,\{n\}} (x, s) - G_s (x, s) \right) \, dx \\
\leq \limsup_{n \to \infty} C \int_0^\infty x^{\lambda-1} \left( G_{s,\{n\}} (x, s) - g(x,s) \right) \, dx = 0. \tag{1.2.39}
\]

This concludes the proof.
1.3. **Self-similar solutions**

The experimental physics has highlighted the occurrence of universal behaviors in many of the models appearing in coagulation theory. We briefly provide a mathematical introduction to self-similarity and then pass to the self-similar fragmentation equation and to the role of self-similarity for the Smoluchowski equation.

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### 1.3.1. Self-similarity: a brief overview

In this Section we deal with a general recipe for the analysis of self-similarity. It is not possible to develop here all the fundamental ideas behind self-similarity and related issues, such as dimensional analysis, perturbation methods and renormalization groups. We must instead limit ourselves to a simplified introduction to the role of self-similarity in differential equations pointing out only the issues that represent a substantial element in this thesis. For a broader account, it is recommended to consult the excellent monographs by Barenblatt [312, 253, 250], but also Sachdev [266, 267] and Sedov [269]; refer also to the papers by Angenent and Velázquez [10] and, more recently, by Eggers and Fontelos [103] that have extensively surveyed rigorous, formal and numerical results on the formation of singularities for a wide range of evolution equations. A specific selection of papers involving self-similarity and singularities can also be found in the bibliography (for example, [31, 119, 120]).

Self-similarity paradigm, intermediate asymptotics framework, universality behaviors and, more generally, the mathematics of scaling laws grant a decisive understanding of many linear and non-linear phenomena. These theoretical approaches introduced (or at least drafted) by the work of great physicists and mathematicians - and among the greatest we can recall Kolmogorov, Zeldovich and Barenblatt - open methodological ways of research and promote a systematical insight in some remarkable principles of nature. We can attempt to state them in few words:

1. Many phenomena exhibit their *specific footprint*: it is the particular way in which they reproduce themselves in time and space. This geometric feature - i.e., *shape* - is made evident thanks to systematic rescalings of the observation (and of the theoretical solutions). One can assert that the recognition of such properties is a specific manifestation of the *covariance principle* in physics: different observers in different frames of reference can unambiguously correlate their measurements.

2. Scaling laws show two faces of dynamics: the geometric and the analytic one. This interplay yields a simplified, *essential reduction* to low dimensional equations exhibiting typical qualitative properties that classify and characterize both the long-term regimes and the singularity dynamics. From the viewpoint of self-similarity, they can both respond to a single mechanism that belongs to the theory of dynamical systems: the presence of *attractors*.

3. Perturbations around explicit solutions generalize to non-linear fields the concept of *superposition principle* characteristic of linear problems. They are useful for the understanding of regular and singular behaviours with respect to variations of conditions (boundary, initial, ...) and of parametrized operators.
4. **Universality** occurs when broad classes of initial data converge to specific shapes or profiles, usually depending on some macroscopic or conserved quantity of the initial data (for example the initial mass). In this way systems forget the details via a coarsening simplification process. We remark here that in no way the self-similar reduction of the problem is compelled to respect the same laws as the original physical problem: a very important feature of self-similarity is that it sometimes permits to understand the violation of conservation laws at the occurrence of singularities. This point is particularly important in our viewpoint on coagulation gelling dynamics.

Self-similarity is not a theory in the proper sense of the word but, differently from other fields like “perturbation methods”, it does not consist only on isolated techniques to carry on specific analysis: the study of intermediate asymptotics of phenomena is based on a transversal framework and, as such, it may or may not become successful. Moreover, many different variations of the self-similarity “recipe” (or canon) have to be forged and thus a physical intuition can provide the fundamental orientation. In this thesis, the study of Smoluchowski equation follows the physical intuition expressed in Principle 4 that self-similarity can explain the violation of conservation of mass for Smoluchowski equation and we will explain this viewpoint in Section 1.3.3. If the approach of second kind similarity turns out to be valid for gelling kernels (as in Chapter 2), it is natural to apply it to all range of homogeneity degrees (as in the results of Chapter 3) even without violation of conservation of mass.

### 1.3.1.1. Self-similarity: a first steps guide

To begin with, following the steps recompiled by Eggers and Fontelos, a time dependent abstract evolution problem as

$$
\phi_t = \mathbf{O}[\phi]
$$

is called **self-similar** if the spatial distribution of the solutions at different moments can be obtained from one another by similarity transformation. We are interested in similarity regimes asymptotically in time, whether we look at large times or close to some singularity, but the following can easily be adapted for spatially depending behaviors and singularities occurring at some point $x_*$. We introduce the new time variable $t' = T^* - t$ if there is a singularity at time $T^*$, or, for globally existing solutions, simply $t' = t$; in this way, one investigates the following ansatz for the structure of the solutions that is suitable both for blow-ups and for asymptotic regimes:

$$
\phi(x, t) = (t')^{\alpha} \Phi \left( \frac{x}{(t')^\beta} \right),
$$

where $\alpha$ and $\beta$ are called the similarity exponents. We remark that it is also possible to consider more general functions of $t'$ as for example in the rescaling $\sigma(t') \Phi \left( \frac{x}{\sigma(t')} \right)$ (with similarity exponent $\gamma$). In this abstract setting, one then imposes the ansatz given by (1.3.2) in the evolution problem and try to fix the similarity exponents via dimensional analysis or symmetry.

In this sense, a very important question arises: which conditions on $\alpha$ and $\beta$ lead to the existence of self-similar solutions $\Phi$ for the problem (1.3.1)? If there is only a set of values for $\alpha$ and $\beta$ that actually grant solutions, they usually take rational values and the problem is said to present a self-similarity of the first kind. Otherwise, it is possible
that solutions exist locally for a continuous set of exponents; in this case, some extra requisites can be needed in order to provide consistency with respect to the conditions of the problem. Imposing these restrictions corresponds to solving a nonlinear eigenvalue problem (self-similarity of the second kind), and this class of problems, in general, is characterized by irrational exponents. Analytic difficulties dramatically increase with respect to first kind similarity, and only few models are today completely solved even if, the more self-similarity is studied and understood, the greater their number is.

More insight on the problem can be obtained considering a slightly different formulation of the self-similar function $\Phi$. We also introduce the self-similar time $\tau = -\ln \left( t' \right)$ and call $\xi = x / \left( t' \right)^{\beta}$ and rewrite:

$$
\phi (x, t) = (t')^\alpha \Phi (\xi, \tau);
$$

therefore, we get a new self-similar evolution equation:

$$
\dot{\alpha} \Phi + \beta \xi \dot{\xi} \Phi - \alpha \Phi = \mathbf{O} \left[ \Phi \right]
$$

(1.3.3)

where, in the language of dynamical systems, solutions $\Phi$ like (1.3.2) are fixed points of the orbits of (1.3.4). We can now follow the steps for the stability analysis as in Eggers-Fontelos [103]: one is interested to the attractors of (1.3.4) and, if $\Phi$ is a solution, then:

$$
\alpha \Phi - \beta \xi \dot{\xi} \Phi + \mathbf{O} \left[ \Phi \right] = 0
$$

(1.3.5)

independently of $\tau$. The basic linear $\varepsilon-$expansion of $\Phi$ consists in linearizing around $\Phi$:

$$
\Phi = \Phi + \varepsilon \mathbf{P} \left( \xi, \tau \right),
$$

(1.3.6)

giving the linear evolution problem

$$
\hat{\mathcal{L}} \Phi = \mathcal{L}_\Phi \left[ \mathbf{P} \right].
$$

(1.3.7)

If $\mathcal{N}_\Phi = \{ \nu \left( \alpha, \beta \right) \}_{\mathbb{N}}$ represents the discrete set of eigenvalues of $\mathcal{L}_\Phi \left[ \mathbf{P} \right]$, then the solution of the linearized problem can be written in terms of the eigenfunctions: $e^{\nu \tau} P_i \left( \xi; \alpha, \beta \right)$. In the known cases the spectrum turns out to be discrete and, for the self-similar solution $\Phi$ to be stable (exponentially or algebraically fast), the analysis of the eigenvalues must prove them to be negative. Evidently, such analysis can be rather complex and lead to different situations (for a full account, we refer again to [103]).

1.3.2. SELF-SIMILARITY AND THE SELF-SIMILAR FRAGMENTATION EQUATION

As one can deduce from the literature, the mathematical theory of fragmentation is far more prosperously developed than its coagulation counterpart, evidently due to the fact that the theory of linear operators gives general and stronger results. One can recover in just a few relevant works the most relevant results concerning cases with or without singularity. In Table 1.4, one can see the principal results on the fragmentation equation (well-posedness and regularity). Regarding the study of the self-similar fragmentation equation, we list now some references: Cheng and Redner [68] give a first general discussion of the kinetics of continuous, irreversible fragmentation
processes; Treat in [200] and mainly in [283] settles some fundamental facts about the similarity solutions; Bertoin [89] develops the probabilistic counterpart (fragmentation stochastic processes); Escobedo, Mischler and Rodriguez-Ricard [118], notably, completely solve the existence of self-similar solutions and the convergence problem; later, Michel [242, 244] deals with the cell division eigenproblem; Laurençot and Perthame [206] develop bounds for the exponential decay for the growth-fragmentation or cell-division equation; these problems (in particular the growth-fragmentation equations) and the rate of convergence are later considered by Cáceres, Cañizo and Mischler [54] and Balagué, Cañizo and Gabriel [17].

A brief account of some results will be summarized in this Section after the derivation of the self-similar equation that is the object of our study.

### 1.3.2.1. The self-similar equation

We consider here the fragmentation equation with self-similar break-up rate (1.1.29). That is:

\[
\frac{d}{dt} c(x,t) = Q_f[c](x,t),
\]

\[
Q_f[c](x,t) = -x^\alpha c(x,t) + \int_x^\infty c(y,t) y^{\alpha-1} B\left(\frac{x}{y}\right) \, dy.
\]

As Lushnikov had shown, one can use an *invariance argument* to derive the self-similar equation. Consider, in fact, the scale transformations with new variables \( \dot{x} \) and \( \dot{t} \) given by \( x = A \dot{x} \) and \( t = A^{-\alpha} (\dot{t} + t_0) \). Then, relabeling the variables,

\[
\frac{d}{d\dot{t}} c_A(x,t) = -x^\alpha c_A(x,t) + \int_x^\infty c_A(y,t) y^{\alpha-1} B\left(\frac{x}{y}\right) \, dy,
\]

where we also consider the family of functions \( c_A(x,t) \) given by formula:

\[
c_A(x,t) = A^2 c(Ax, A^{-\alpha}(t + t_0)).
\]

The pre-factor \( A^2 \) preserves total mass of the one-parameter family of solutions, as one can directly check, so that the renormalization group (1.3.11) provides a formula to generate solutions with different values for the moments \( M_\beta, \beta \in \mathbb{R} \). The key point now is that we seek a particular function like (1.3.2) such that not only the total mass moment \( M_1 \) is constant under renormalization (and time evolution), but also all the other moments are conserved. The role of this self-similar function is that of a special physical solution to the fragmentation equation.

As we show now, the above requirement is a symmetry property that permits determining function (1.3.2) in an unique way: in this sense, the similarity problem does actually belong to the first kind self-similarity. To do so, we seek a family \( c_A \) independent of \( A \); calling \( q = Ax \) and \( p = A^{-\alpha}(t + t_0) \), we formally apply the derivative with respect to \( A \) and get:

\[
0 = \frac{d}{dA} c_A(x,t) = \frac{1}{A} \frac{d}{dA} c_A(x,t) = 2c(q,p) + q\dot{c}_q c - \alpha p\dot{c}_p c.
\]
This standard first order PDE has general solution which depends on $\alpha$:

$$c_*(x, t) = C_1 \left( \frac{t + t_0}{A^2} \right) \Phi \left( \frac{C_2 x (t + t_0)^{\frac{1}{2}}}{A^2} \right),$$  \hspace{1cm} (1.3.13)

where $C_1$ and $C_2$ are free constants and $A$ can still be arbitrarily chosen. A possible way to fix $C_1$ and $C_2$, however, is deciding the values of $M_1$ and $M_0 (0)$. Let $M_\beta$ be formally the moments of the self-similar function $\Phi$ (notice the difference in notation between $M_\alpha$ and $M_\beta$):

$$M_\beta = \int_0^\infty \xi^\beta \Phi (\xi) \, d\xi,$$  \hspace{1cm} (1.3.14)

and, fixing $t_0 = 0$ (absence of singularities), straightforward computations lead to:

$$M_0 (0) = \frac{C_1 M_0}{C_2 A^2}, \quad M_0 (t) = M_0 (0) \left( \frac{t}{A^2} \right)^{\frac{1}{2}},$$  \hspace{1cm} (1.3.15)

$$M_1 = \frac{C_1 M_1}{(C_2 A^2)^2}.$$  \hspace{1cm} (1.3.16)

Without assuming anything yet on $M_0$ and $M_1$, we can fix $A^2 = C_1$, introduce the self-similar variable $\xi = xt^{\frac{1}{2}}$ and relabel $\Phi (\xi)$ in order to get:

$$c_*(x, t) = t^{\frac{1}{2}} \Phi (\xi),$$  \hspace{1cm} (1.3.17)

that, imposed in (1.3.16), yields the desired self-similar fragmentation equation:

$$2\Phi (\xi) + \xi \frac{d}{d\xi} \Phi (\xi) = - \alpha \xi^\alpha \Phi (\xi) + \int_\xi^\infty \alpha \eta^{a-1} B \left( \frac{\xi}{\eta} \right) \Phi (\eta) \, d\eta.$$  \hspace{1cm} (1.3.18)

Analogous conclusions on the self-similar form of this equation can also be carried over in the framework of non-constant mass (see Treat [290]).

### 1.3.2.2. Semigroup approach to the existence of solutions

We now consider the results relative to equation (1.3.18). Both existence, uniqueness and convergence to the self-similar profiles can be obtained under general conditions and we will focus on the work by Escobedo, Mischler and Rodriguez Ricard [118]; it is necessary to remark that this work also extends to more cases (self-similar solutions for coagulation equation and convergence to equilibrium in coagulation-fragmentation), but we are now interested specifically on fragmentation. To prepare our tool box, we first introduce the functional spaces we need: on one hand $L^1_{\text{loc}}$ is the set of functions from $\mathbb{R}^+$ to $\mathbb{R}$ that are integrable on any interval $[\delta, \frac{1}{2}]$, for any $\delta \in (0, 1)$; one can also associate to $L^1_{\text{loc}}$ the measure space $ML_{\text{loc}}$. Thus, for any continuous function $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$, we can consider $ML_\phi$ as the set of $ML_{\text{loc}}$-measures such that the generalized $\phi$-moment is finite:

$$M_\phi (|\mu|) := \int_0^\infty \phi (y) \, d|\mu| (y) < \infty.$$
Correspondingly, \( L_{\phi} := ML_{\phi} \cap L_{\text{loc}}^1 \), and we will use the notation \( ML_k \) if \( \phi = x^k \) as we have previously done; also, we will use the standard space of bounded total variation
\[
BV^1 := \{ f \in L_{\text{loc}}^1 \text{ such that } f' \in ML_1 \}.
\]
All these Banach spaces can be defined analogously over the interval \((0, 1)\) (as we will do when working with the daughter distribution function \(B\)), in which case we will write explicitly the interval after the space (we will write \( BV^1 (0, 1) \) for instance). It is therefore necessary to develop suitable estimates for some of these spaces regarding the behavior of the fragmentation and coagulation operators; from this viewpoint, we also consider the linear transport operator that appear at the left hand side of equation (1.3.18):
\[
T (f) (x) := 2 f (x) + x \partial_x f (x).
\] (1.3.19)
Not surprisingly, the estimates that we need are similar to what we have already obtained from the moments equations and from fundamental identities like (1.2.1). They show the “good behavior” of these operators and of their respective dual operators. For a full account of \textit{a priori} properties, see also Cañizo \cite{Ca9}. Thanks to the definition of operator \( T (f) \), we can rewrite the self-similar fragmentation equation more concisely:
\[
\alpha Q_F [\Phi] (\xi) - T [\Phi] (\xi) = 0
\] (1.3.20)
and introduce, analogously to the (linear evolution) fragmentation equation (1.3.8–1.3.9), the self-similar evolution fragmentation equation:
\[
\partial_\tau \Phi (\xi, \tau) = \alpha Q_F [\Phi] (\xi, \tau) - T [\Phi] (\xi, \tau)
\] (1.3.21)
whose stationary solutions are exactly the solutions to (1.3.20).
Consider now the following assumptions:
\[
\alpha \geq 0, \quad \text{non—shattering;}
\] (1.3.22)
\[
B \geq 0, \quad \text{supp} (B) \subset [0, 1], \quad \int_0^1 u \, dB (u) = 1, \quad \text{regularity I};
\] (1.3.23)
\[
B \in ML_m (0, 1), \quad m \leq 1, \quad \text{regularity II};
\] (1.3.24)
\[
B \in BV^1 (0, 1) \cap ML_m, \quad m \leq \alpha, \quad \text{extra regularity.}
\] (1.3.25)
Therefore, we can state the following resuming \textit{Universality Theorem} on the pure fragmentation model:

**Theorem 1.13.** Assume the non—shattering condition and, moreover, the regularity \( I – II \) on the daughter-distribution function \( B \). Hence:
1. [EXISTENCE AND UNIQUENESS] For each mass-value $M > 0$ there exists a unique $\Phi_M \in ML_1$ solution to (1.3.20) with $M_1 (\Phi_M) = M$. Moreover $\Phi_M$ belongs to the space of measures:

$$\Phi_M \in M_{\infty; m} := \bigcap_{\gamma \geq m} ML_{\gamma}. \quad (1.3.26)$$

If Assumption (1.3.25) holds, then the self-similar solution gains additional regularity: $\Phi_M \in Z_{\infty; (\alpha+1)}$ with

$$Z_{\infty; (\alpha+1)} := BV^1 \cap M_{\infty; (\alpha+1)}. \quad (1.3.27)$$

2. [FRAGMENTATION EVOLUTION EQUATION] For each $c_0 \in ML_m \cap ML_1$ and mass $M > 0$, there exists a unique global in time, mass conserving solution $c \in C ((0, T); L_1^1) \cap L_1^1 (0, T; L_1^{\alpha+2})$. It can be, moreover, rewritten in selfsimilar variables:

$$g (\xi, \tau) = e^{-\frac{\tau}{\alpha} \gamma} c \left( e^\tau, \xi e^{-\frac{1}{\alpha} \tau} \right), \quad \text{and } g \text{ is uniformly bounded in } ML_{\gamma}, \gamma \geq m. \quad (1.3.28)$$

3. [EXTRA REGULARITY] Assume also (1.3.25). Then $\Phi_M \in BV^1$ and, for any $c_0 \in ML_m \cap ML_1 \cap BV^1$, then $g (\xi, \tau)$ is also uniformly bounded in $BV^1$.

4. [CONVERGENCE] Finally, assuming (1.3.25), any initial data is such that the $M$–massed solution $c (x, t)$ and the corresponding $\Phi$ satisfy:

$$\lim_{t \to \infty} \left( \int_0^\infty x |c (x, t) - (1 + t)^{\frac{1}{2}} \Phi \left( x (1 + t)^{\frac{1}{2}} \right) | \, dx \right) = 0. \quad (1.3.29)$$

It is not beneficial to reproduce here the complete proof of this Theorem, since it would lead quite far from the purposes of this Introduction. The proof of (2.), for example, goes in direction of the well-posedness results of Section 1.2 and oversets the interest to self-similarity of this section. As for (3.) and (4.), we can just point out that the technique employed in [118] is the following: one defines (and analyses) the strict Lyapunov functional on $L_1$

$$H [g] (\tau) := \int_0^\infty \xi |g (\cdot, \tau) - \Phi| \, d\xi$$

and uses a Lasalle’s invariance principle to show that $H [g] (\tau) \to 0$. After deducing $c (x, t)$ via $g (\xi, \tau)$ thanks to formula (1.3.28), one can show that $c (x, t)$ is actually the solution to the (evolution) fragmentation equation and then substitute in $H [c] (t)$, concluding the convergence stated in Theorem 1.13. We skip the details and refer to [118] for the complete proof. We can instead consider the existence and uniqueness problem for $\Phi_M$ under the simplifying Assumption (1.3.25).

**Existence:**

We address now the issue of existence of solutions to (1.3.20). The strategy that is followed in [118] is that of considering the evolution problem (1.3.21), introducing the associated semigroup $S_t$ and applying the following very useful abstract Theorem (see also Gamba, Panferov and Villani [152] in the Boltzmann equation setting):
Theorem 1.14. Let $Y$ be a Banach space and $(S_t)_{t \geq 0}$ be a continuous semigroup in $Y$. Assume that $S_t$ is weakly (sequentially) continuous and that there exists $Z$ a non-empty, convex and weakly (sequentially) compact subset of $Y$ such that it is invariant under the action of $S_t$. Then, there exists a stationary element $\zeta \in Z$ under the action of $S_t$, that is, for any $t \geq 0$, $S_t \zeta = \zeta$.

As for the self-similar problem (1.3.21), the key observation is that $S_t$ maintains invariant the measure spaces $BV^1 \cap ML_1$, with $\gamma \geq 1 + \alpha$. To see this, we recall that we can multiply equation (1.3.21) by any function $\phi \in C^1$ and integrate formally to get the (usual) general identity:

$$
\frac{d}{dt} \int_{0}^{\infty} \phi \Phi \, d\xi = \int_{0}^{\infty} \Phi ( \partial_t \zeta (\xi \phi) - 2\phi ) \, d\xi + \alpha \int_{0}^{\infty} \int_{0}^{\xi} \xi^\alpha \Phi B \left( \frac{\eta}{\xi} \right) \left( \phi (\eta) - \frac{\eta}{\xi} \phi (\xi) \right) \, d\eta d\xi.
$$

(1.3.30)

One can therefore consider the $\gamma$th moment taking $\phi = \xi^\gamma$:

$$
\frac{d}{dt} M_\gamma = (\gamma - 1) M_\gamma + C (1 - \gamma) M_{\alpha + \gamma},
$$

(1.3.31)

where $C$ is a constant depending on $\gamma$, $\alpha$ and on the value of an integral of B: $\int_{0}^{1} B (u) (u^\gamma - u) \, du$ (and hence the regularity conditions on B are imposed here). We observe that:

$$
\frac{d}{dt} M_1 = 0,
$$

(1.3.32)

so also equation (1.3.21) conserves the initial mass. As for moments with $\gamma > 1$, one applies Hölder’s inequality and get:

$$
\frac{d}{dt} M_\gamma \leq (\gamma - 1) M_\gamma - C (1 - \gamma) M_\gamma^{\frac{1}{\theta}},
$$

(1.3.33)

with $\theta \in (0, 1)$. Whenever the initial moments are bounded, an application of Gronwall’s Lemma shows the uniform estimate

$$
\sup M_\gamma (t) \leq \mu (\gamma, M_\gamma (0)).
$$

(1.3.34)

It is also possible to extend uniform estimates to all moments with $\gamma \geq m$, where $m$ is given by (1.3.24). Then one differentiates equation (1.3.21) with respect to $\xi$ and gets

$$
\frac{d}{dt} \int_{0}^{\infty} \xi |\partial_\xi \zeta \Phi| \, d\xi \leq - \int_{0}^{\infty} \left( \xi + \left( \int_{0}^{1} u B (u) \, du \right) \xi^{1 + \alpha} \right) |\partial_\xi \zeta \Phi| \, d\xi + C \int_{0}^{\infty} \xi^\alpha \Phi \, d\xi
$$

(1.3.35)

and a new application of Gronwall’s Lemma leads to the uniform estimate for $BV^1$.

To conclude the proof of Theorem 1.14, one still needs to check that the mapping $S_t$ is weakly sequentially continuous in $ML_m \cap ML_1$. In order to demonstrate this semigroup property, one can follow these schematic steps:

1. Define the concepts of dual, mild and distributional solution to equation (1.3.21); show that they coincide in $L^\infty \left( [0, T], X_{dual} \cap Y_{mild} \right)$ for suitable choices $X_{dual}$ and $Y_{mild}$. The mild solution is constructed thanks to the semigroup.
2. Show that the truncated fragmentation rate leads to a unique global solution in a space $\tilde{X}$.

3. Prove a stability principle: if $\tilde{X}$ is weakly compact in $X$ and $f_n$ are solutions to \textcolor{red}{(1.3.21)} which are bounded in $L^\infty ([0,T],X)$, then there is a solution $f$ to \textcolor{red}{(1.3.21)} which is bounded in $L^\infty ([0,T],X)$ and corresponds to the weak limit of a subsequence $f_{n_k} \rightharpoonup f$ in the distribution sense $D' ((0,T) \times (0,\infty))$. Therefore, conclude that the semigroup is weakly sequentially continuous.

Point 1.: definitions and equivalences:

Consider some initial profile $\Phi_M (\xi, 0)$ belonging to $Y^{\text{mild}} := ML_\alpha \cap ML_1$. A mild solution to equation \textcolor{red}{(1.3.21)} is a function $\Phi_M$ which conserves the first moment and belongs to $C ([0, T]; L^1_1) \cap L^\infty (0, T; Y^{\text{mild}})$ and verifies:

$$\Phi_M (\xi, t) = \mathcal{T}_0 (t) (\Phi_M (\xi, 0)) + \int_0^t \mathcal{T}_0 (t-s) (Q_F (\xi, s)) \, ds, \quad (1.3.36)$$

where $\mathcal{T}_0 (t)$ is the semigroup associated to the linear transport operator $T [f]$. The standard definition of the dual solution $\mu$ is, with $X^{\text{dual}} \equiv Y^{\text{mild}}$ for the self-similar fragmentation case,

$$\int_0^\infty \int_0^\infty \mu \bar{c} \psi \, d\xi \, dt + \int_0^\infty \mu (\xi, 0) \psi \, d\xi + \int_0^\infty \left( \langle \alpha Q_F [\mu], \psi \rangle - \langle T [\mu], \psi \rangle \right) \, dt = 0, \quad (1.3.37)$$

with $\mu$ belonging to $L^\infty (0, T; Y^{\text{mild}})$ and $\psi \in C_0^\infty$.

Point 2.: truncated problem:

Suppose here that $B \in L^\infty$ and supp $(B) \subset (0, 1]$. Then, for sufficiently small values of $T > 0$, one can see that

$$F (f) (\xi, t) = \mathcal{T}_0 (t) (f (\xi, 0)) + \int_0^t \mathcal{T}_0 (t-s) (Q_F (f) (\xi, s)) \, ds$$

is a contraction, which is the operator at the right hand side of formula \textcolor{red}{(1.3.36)}. By the Banach fixed point Theorem, there is therefore a unique solution $\Phi_M (\xi, t)$ such that $F (\Phi_M) = \Phi_M$, $t \leq T$, with $\Phi_M \in C ([0, T]; L^1_1)$. This is a mild solution to \textcolor{red}{(1.3.21)} and one can iteratively extend $T$ to be arbitrarily large with the additional restriction that sup $\Phi_M \leq C \Phi_M (\cdot, 0) \leq Y^{\text{mild}}$.

Point 3.: stability principle:

We finally address how to pass to the limit. One can consider a sequence of solutions $\Phi_M^{(\infty)}$ to problems with truncated $B^{(\infty)}$ approximating $B$ under the more general Assumptions \textcolor{red}{(1.3.23-1.3.24, 1.3.25)}. Therefore, consider $X = ML_\alpha \cap ML_1$ and we wish to pass to the limit to a solution $\Phi_M \in C ((0,T), X)$ for any $T > 0$. Thanks to the equivalence between mild and dual solutions,
one can consider solutions $\mu(n)$ via formula (1.3.37) and use the strongly compact sequence in $C([0, T])$ defined as:

$$\left\{ \int_0^\infty \psi (\xi) \mu(n) (\xi, t) \ dx \right\},$$

(1.3.38)

for each $\psi \in D$. Now it is possible to pass to the limit (the limit $\mu$ is still a dual solution). Moreover, one can use again the equivalency with mild solutions to define the mild limit solution and conclude that $S_t$ is weakly sequentially continuous.

**Uniqueness:**

Uniqueness holds thanks to a contraction property in $L^1_1$ that we can easily summarize:

**Proof.** Consider the operator at the right hand side of (1.3.21), that is:

$$S (f) = \alpha \Phi (f (\xi)) - T (f (\xi)),$$

(1.3.39)

and, if $f$ belongs to $C_c$, one can consider the following:

$$\sigma (f) := \int_0^\infty S (f (\xi)) \xi \text{sign} (f (\xi)) \ d\xi.$$  

(1.3.40)

Substituting, with $\phi (\xi) := \xi \text{sign} (f (\xi))$, one gets:

$$\sigma (f) = \int_0^\infty f (\xi) \left( \int_0^\xi b (\xi, \eta) \left( \phi (\eta) - \frac{\eta}{\xi} \phi (\xi) \right) \ d\eta \right) \ d\xi,$$

(1.3.41)

since one can pass the derivatives to $\phi$ with standard calculations and also notice that

$$(\partial_\xi (\xi \phi) - 2 \phi) = 0.$$  

Remark now that, for each $\xi, \eta$, one has $b (\xi, \eta) \geq 0$ and:

$$f (\xi) \left( \phi (\eta) - \frac{\eta}{\xi} \phi (\xi) \right) = \eta (f (\xi) \text{sign} (f (\eta)) - |f (\xi)|) \leq 0,$$

so that:

$$\sigma (f) \leq 0.$$  

(1.3.42)

By density, one can extend this property to each function in $L^1_1 \cap L^1_{1+\alpha}$ and conclude that any solution given by Theorem 1.13 must also satisfy the contraction property (1.3.42). Moreover, since $S (f)$ is linear, consider two different solutions $\Phi_{M,1}$ and $\Phi_{M,2}$ to (1.3.20) with the same mass so that their difference verifies a stronger property than (1.3.42): $\sigma (\Phi_{M,1} - \Phi_{M,2}) = 0$ since $S (\Phi_{M,1} - \Phi_{M,2}) = 0$.  

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We can therefore introduce $\hat{\Phi}_M = \Phi_{M,1} - \Phi_{M,2}$ and then:

$$
\begin{cases}
\sigma \left( \frac{\partial}{\partial \xi} \hat{\Phi}_M \right) = 0, \\
\xi \hat{\Phi}_M (\xi) \, d\xi = M - M = 0.
\end{cases}
$$

(1.3.43)

Hence, by (1.3.41), one has $\hat{\Phi}_M (\xi) \left( \phi (\eta) - \frac{\partial}{\partial \xi} \phi (\xi) \right) = 0$, so that:

$$
\hat{\Phi}_M (\xi) \text{sign} \left( \frac{\partial}{\partial \xi} \phi (\eta) \right) - \left| \hat{\Phi}_M (\xi) \right| = 0,
$$

which permits to conclude that, for every $\xi$ and $\eta$, sign $\left( \hat{\Phi}_M (\eta) \right) = \text{sign} \left( \hat{\Phi}_M (\xi) \right)$ is constant.

The only possibility for the first moment of $\hat{\Phi}_M$ to be zero, therefore, is that $\hat{\Phi}_M = 0$. \hfill \Box

**Remark 1.15.** Uniqueness follows a standard procedure once contraction formula (1.3.42) is established. We can also observe that in all cases when mass preserving coagulation-fragmentation solutions exist globally in time, they seem to converge to a unique equilibrium or self-similar profile (in the case of pure fragmentation) determined by the constant mass. Such property (which is very natural) is not easily deduced in the pure coagulation setting and we will come back to it in the next Section.

## 1.3.3. Self-similarity and Smoluchowski Equation

We can finally consider issues of self-similarity for the Smoluchowski equation. We are concerned here with the issues stemming from the so called scaling hypothesis, a very crucial and problematic framework that should provide a wide understanding of the dynamical behavior of mean field coagulation. The principal aim of this theses is to propose a second kind self-similarity framework for Smoluchowski equation and to work out some first results supporting its validity. We believe that, on the long term, a proper scaling hypothesis is the required tool for the comprehension of both regular coalescence in mass-transportation regimes and of the gelation phenomenon.

It is widely believed in physical literature (we can stress out the work by van Dongen and Ernst [300], and cite also [46, 77, 133, 251]) and numeric experiments (Filbet, Laurençot and Lee [135, 214]) that most relevant systems approach self-similar regimes under not too dispersed initial conditions. We will come back later on the exact notion of dispersion that is needed, remarking now only that it requires an initial profile to decay faster than some algebraic function. Let $s (t)$ be some non-negative growing-in-time function and consider the self-similar function $\psi (\xi) : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$
c (x, t) \approx \frac{1}{s^2 (t)} \psi \left( \frac{x}{s (t)} \right),
$$

(1.3.44)

where the approx. sign “$\approx$” must be formalized in some sense. The function $s (t)$ is also called the typical size, identifying the isoline along which the rescaled $c$ solution is constant.

It is very important to observe, here, that imposing a strict equality in (1.3.44) and assuming that $c (x, t)$ is an actual mass-conserving solution of Smoluchowski equation given by some initial
constraint is the typical hypothesis leading to a self-similar problem of the first kind, in the sense we introduced in Section 1.3.1.1. In this setting, \( \psi \) must be a non-negative function satisfying:

\[
\mathcal{M}_1 := \int_0^\infty \xi \psi (\xi) \, d\xi < \infty, \tag{1.3.45}
\]

which is the same mass conservation law required for solutions \( c(x,t) \); doing so fixes the self-similarity exponent \( \theta = 2 \) (since the \( L^1 \)-norm of \( c(x,t) \) has to be constant) and permits determining the function \( s(t) \).

Nevertheless, this first-kind scaling framework presents some major flaws. While the fact that it has not been possible to prove convergence to unique self-similar solutions could still be regarded as a mathematic lack of knowledge, we can instead point out that gelling kernels do not produce mass-conserving solutions: it is for example evident that the explicit solution for the multiplicative kernel has infinite mass for any \( t > 0 \) (see Table 1.3). Moreover, mass-conservation is not a required condition for a function to be the solution of a non-gelling kernel, but rather an extra property that one gains \textit{a posteriori}. All this discussion can induce some doubts on the validity of the first kind scaling, and we propose a more general scaling theory, of the second kind, that follows from a weaker interpretation of the “\( \approx \)” sign in equation (1.3.44) and that obviously owes its name to the second type self-similar problems we introduced in Section 1.3.1.1. In order to state precisely our departing point, it is first useful to review the available work from previous literature. Much of this work bears useful insight and can guide us in future studies.

1.3.3.1. First kind scaling

Assuming that the coagulation kernel \( K \) is exactly homogeneous of homogeneity degree \( \lambda \), we can observe that Smoluchowski equation presents a two-parameters renormalization group that we can exploit, analogously to the way Lushnikov’s invariance statement for the fragmentation equation lead to the self-similar derivation of the self-similar equation. We can in fact check that whenever \( c(x,t) \) is a solution, then

\[
\tilde{c}_{a,b} (x,t) := \frac{a^{\lambda+1}}{b} c(ax, bt) \tag{1.3.46}
\]

is also a solution, for arbitrary positive values of \( a \) and \( b \). We require, however, that only the members of this family with the same mass represent the same solution, and this condition \( (b = a^{\lambda-1}) \) restricts the symmetry group to a one parameter family. On the other hand, let’s suppose that \( c \) verifies formula (1.3.44). Due to the invariance of \( a \), one gets:

\[
\frac{1}{s^2(t)} \psi \left( \frac{x}{s(t)} \right) = c(x,t) = a^2 c(ax, a^{\lambda-1} t) = \frac{1}{s^2(a^{\lambda-1} t)} \psi \left( \frac{ax}{s(a^{\lambda-1} t)} \right),
\]

with \(-\infty < \lambda < 1\),

\[
\psi \left( \frac{t}{t^{1-\lambda}} \right), \tag{1.3.47}
\]

with constant pre-factor \( w \). Also, when \( \lambda = 1 \) (the additive case), \( s(t) \) is proportional to \( e^{wt} \). In this sense the additive case is a limit case. Instead, the gelling kernels are considered irregular cases and a different approach is adopted to treat them. We can now proceed with the main claim, rephrasing Leyvraz [215]:

\[
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\]
Claim 1.16. The dynamical scaling hypothesis states that any rescaled solution \( c(x, t) \) starting from a cluster distribution \( c_0(x) \) with a sufficiently high moment \( M_n(0) < \infty, n \geq n_0(\lambda) \), converges weakly to the self-similar profile given by equation (1.3.47).

The exact measure that should be considered for the weak convergence has still to be determined, but it is commonly suspected that a proper choice could be the moment convergence, which is expressed by

\[
\frac{1}{M_k(t)} \int_0^\infty x^k c(x, t) f \left( \frac{x}{t^{1-\alpha}} \right) \, dx \rightarrow w \int_0^\infty \xi^k \psi(\xi) \, d\xi
\]

for every bounded continuous function \( f(x) \). If the above is true for some \( k > k_0 \), then it is known that it holds also for every \( k \geq k \). The reasons to use a weak convergence could be only a technical limitation but, in the explicit cases, one can work with suitably defined probability distributions (cf. Menon and Pego [233]). Then, in a subsequent work, Menon and Pego also showed that the rescaled solutions converge strongly to self-similar profiles, see [238], Theorem 1.1. We will come back to the explicit cases after we address the existence of first-kind self-similar solutions and its consequences.

Imposing (1.3.47) in Smoluchowski equation, one can get through routine manipulation and obtain a first self-similar Smoluchowski equation:

\[
Q_C[\psi](\xi) - wT[\psi](\xi) = 0,
\]

with transport operator \( T[\psi] = 2\psi + \xi\psi' \). While developing the steps, one has to use the fact that \( s(t) \propto t^{1/(1-\alpha)} \), if \( \alpha < 1 \), or \( s(t) \propto e^{\lambda t} \), \( \lambda = 1 \), in order to simplify at both sides of Smoluchowski equation the factors \( s'(t) \) and \( w(s(t)) \lambda \) that appear.

We wish to inspect the existence of solutions to equation (1.3.49) as it has been done in the key articles of Escobedo, Mischler and Rodriguez-Ricard [18] and Fournier and Laurençot [14]. In the former, the authors consider the self-similar evolution equation:

\[
\partial_\psi \psi = T[\psi] + (1 - \lambda) Q_C[\psi]
\]

under homogeneous kernels of the class \( K(x, y) = x^\alpha y^\beta + y^\alpha x^\beta, \alpha \in [-1, 0], \beta \in [0, 1] \) and \( \alpha + \beta = \lambda \in [0, 1] \) and establish the existence of steady state solutions to it. They also proceed from unique solutions \( c(x, t) \) of the Smoluchowski equation and set conditions for the rescaled profiles given by formula (1.3.47) to be solutions (not necessarily steady solutions) of (1.3.50). Consider the space \( X_\infty = \bigcap_{k>2-\beta} X_k \) and \( X_k \) defined as:

- if \( \alpha = 0 \) then \( X_k = ML_\lambda \cap ML_k \);
- if \( \alpha < 0 \) then \( X_k = ML_{-\lambda} \cap ML_k \).

Then one has the following (see [18] for the the proof):

**Theorem 1.17.** For any \( M > 0 \), there exists at least one self-similar profile \( F_M \in X_\infty \) verifying \( M_1(F_M) = M \) and equation (1.3.49) with \( w = -(1 - \lambda) \). The solution is in the weak-dual sense of Section 1.3.2.2.
Moreover, for any \( c_0 (x) \in Y \) where \( Y \) is the “not too dispersed functions space” given by \( Y := L_{2a}^1 \cap \bigcap_{\max (2,3,2-\beta)}^1\ ), then there exists a unique solution to the Smoluchowski equation \( c (x,t) \in C ([0,T] ; L_1^1) \cap L^{\infty} (0,T;Y) \). This solution is such that the associated \( \psi \) defined as:

\[
c (x,t) = \frac{1}{\ln (1 + t)} \psi \left( \frac{x}{t^{\frac{1}{1-\alpha}}}, \frac{1}{1-\lambda} \ln (1 + t) \right)
\]

is a solution to (1.3.51) and \( \psi (\cdot, t) \) is uniformly bounded in \( X_k \) for every \( k \geq 2 - \beta \).

This result cannot guarantee neither uniqueness nor stability of self-similar solutions, which instead could be obtained for the self-similar fragmentation equation as we have seen in Section 1.3.2. The existence result proposed by Fournier and Laurençot is slightly more general: they consider three classes of homogeneous kernels and include also the case \( \lambda = 1 \), but it has not been possible yet to investigate the well-posedness any further. Moreover, the scaling hypothesis for gelling cases still remains untractable in this framework. It must also be remarked that the exactly solved models show the existence of both uniparametric families of algebraically decaying self-similar solutions and special exponentially decaying ones, so, in general, we do not expect the self-similar solutions to be unique. Quite on the contrary, finding and understanding the dynamical role of special decaying self-similar solutions is the complex and interesting question that is still unresolved for the Smoluchowski equation.

### 1.3.3.2. Scaling theory for explicitly solvable kernels

We turn now our attention to the three explicitly solvable kernels \( K_1 (x,y) := 2, K_+ (x,y) := x+y \) and \( K_\times (x,y) := xy \) and in this section we restrict the homogeneity degrees only to the three discrete corresponding values \( \lambda = 0,1,2 \). Menon and Pego dedicate three comprehensive works [230, 237, 240] to them, fully characterizing their self-similarity features. We present some issues and techniques that also play an important role in our work: in fact, a remarkable characteristic of Smoluchowski equation is that both integrals in the coagulation operator can be easily transformed thanks to the Laplace transform. In the explicit cases, the problem turns out to be completely solvable in an elegant fashion and the weak convergence of generic data follows from the convergence of the Laplace solution.

The Laplace transform of a positive Radon measure \( \nu \) on the interval \((0, \infty)\) is given by the well-known integral:

\[
\sigma (\eta) = \int_{0}^{\infty} e^{-\eta \xi} \nu (d\xi) = \int_{0}^{\infty} e^{-\eta \xi} n (\xi) \, d\xi
\]

where the last equality is true if the measure \( \nu \) admits density function \( n \). We are not interested here in the general setting of measure-valued solutions to Smoluchowski equation, so for simplicity we restrict ourselves to the classic Laplace transform of integrable functions. In order to include also functions \( n (\xi) \) which present a stronger singularity at the origin, we also introduce the “desingularized” Laplace transforms

\[
\phi (\eta) := \int_{0}^{\infty} (1 - e^{-\eta \xi}) \, n (\xi) \, d\xi
\]
and
\[ D^\rho \Phi (\eta) := \int_0^\infty (1 - e^{-\eta \xi}) \xi^{1+\rho n (\xi)} \, d\xi \]  
(1.3.54)
for even stronger singularities at the origin. We also simply denote with \( \Phi (\eta) \) the non-fractional case with \( \rho = 0 \), that is: \( \Phi (\eta) = D^0 \Phi (\eta) \). It should be remarked (without entering in the details) that when \( \rho \neq 0 \) the Laplace transform introduced by formula \( (1.3.54) \) acts like a fractional derivative operator of order \( \rho \) of \( \Phi (\eta) \). Controlling its behavior represents the main difficulty when trying to extend the results that Menon and Pego have shown or collected about the explicit cases.

We can formally apply formula \( (1.3.53) \) to Smoluchowski equation when \( \lambda = 0, 1 \) and formula \( (1.3.54) \) with \( \rho = 0 \) and \( \lambda = 2 \) to obtain the Smoluchowski equation with Laplace variable, then solve it, check whether the transform is well-defined and invert the transforms to obtain the regular solutions. The equations in terms of the new variable are appealingly simpler then the original ones:

\[
\begin{align*}
\overline{\partial}_t \phi (\eta, t) & = - (\phi (\eta, t))^2, & \lambda = 0; \\
\overline{\partial}_t \phi (\eta, t) - \phi (\eta, t) \overline{\partial}_\eta \phi (\eta, t) & = - \phi (\eta, t), & \lambda = 1; \\
\overline{\partial}_t \Phi (\eta, t) - \Phi (\eta, t) \overline{\partial}_\eta \Phi (\eta, t) & = 0, & \lambda = 2.
\end{align*}
\]
(1.3.55-1.3.57)

We will not enter into details on the resolution steps for such equations and refer to [238], Sections 2.3-2.5, for the reader to consult them. For each case, a specific existence and uniqueness theorem is proved for solutions starting with bounded \( \lambda \)-th moment, and each solution corresponds to the inverse transform of the solutions of \( (1.3.55-1.3.57) \) with initial data \( \phi_0 (\eta) := \int_0^\infty (1 - e^{-\eta x}) c_0 (x) \, dx \) or \( \Phi_0 (\eta) := \int_0^\infty (1 - e^{-\eta x}) x c_0 (x) \, dx \) respectively. In the last case, we can recognize Burger’s equation and, notably, the time taken for characteristics to intersect gives the gelation time \( T_{\text{gel}} = \overline{\partial}_t \Phi_0 (0) = \int_0^\infty x^2 c_0 (x) \, dx = M_2 (0) \); in this sense, we recover the previously known result.

We are instead interested in studying the convergence of the solutions \( \phi \) and \( \Phi \) for large times, since the point-wise convergence of such Laplace transforms is equivalent to the convergence of time dependent probabilities depending on the rescaled solutions to a suitably defined non-trivial probability distribution function. More specifically, thanks to the uniqueness of the solution \( c (x, t) \) to each Smoluchowski initial value problem, we can unambiguously define the following time-parametrized family of probability distribution:

\[
S_\lambda (x, t) := \frac{1}{M_\lambda (t)} \int_0^x y^\gamma c (y, t) \, dy.
\]
(1.3.58)

Consider the \( \lambda \)-th family of time-rescalings for each case: given \( 0 < \rho \leq 1 \), the function \( \gamma_{\lambda, \rho} (t) \) is now defined as:

\[
\begin{align*}
\gamma_{0, \rho} (t) & = t^{\frac{1}{2}}, & \lambda = 0; \\
\gamma_{1, \rho} (t) & = e^{\frac{\rho t}{\rho - 1}}, & \lambda = 1; \\
\gamma_{2, \rho} (t) & = (1 - t)^{-\frac{2}{\rho-1}}, & \lambda = 2.
\end{align*}
\]
(1.3.59-1.3.61)
and correspondingly we can introduce the self-similar variable \( \xi := x (\gamma_{\lambda, \rho} (t))^{-1} \).

In their work, Menon and Pego do not assume the first kind scaling hypothesis, since they do not fix the scaling exponents a priori as in formula (1.3.47); quite on the contrary, their scaling ansatz explicitly depends on the free parameter \( \rho \), a definitely important detail since it identifies the long time limit of the rescaled solution, its asymptotic behaviors and the domains of attraction of the self-similar solutions. The three theorems they show can be summarized in a single theorem as follows:

**Theorem 1.18** (Menon-Pego’s theorem). For any solution to Smoluchowski initial value problem there is a \( \rho \)–dependent rescaling \( \gamma_{\lambda, \rho} (t) \) and a non-trivial probability distribution function \( S^\xi_{\lambda, \rho} (x) \) such that:

\[
\lim_{t \to T^*} S_\lambda ((\gamma_{\lambda, \rho} (t) x), t) = S^\xi_{\lambda, \rho} (x), \text{ at all continuity points of } S^\xi_{\lambda, \rho}
\]

if and only if the initial datum \( c_0 \) is such that:

\[
\int_0^x y^{\lambda + 1} c_0 (y) \, dy \sim x^{1-\rho} L (x)
\]

for the same \( \rho \in (0, 1] \) and some function \( L \), also depending on \( \rho \), such that \( \lim_{x \to \infty} (L (\alpha x) / L (x)) = 1 \) for each \( \alpha > 0 \).

In the converse implication of Theorem 1.18, moreover, the rescaled limit distribution \( S^\xi_{\lambda, \rho} (x) \) is known in the explicit cases and thus, remarkably, the initial decaying behavior of the solution (the divergence rate of the \( (\lambda + 1)^{\text{th}} \) moment) characterizes the dynamical behavior in a fixed way.

**The constant kernel:**

It is known that the original explicit solution for the Smoluchowski equation with constant kernel given by:

\[
c (x, t) = t^{-2} \exp \left( -\frac{x}{t} \right)
\]

is just a particular member of the one-parameter family:

\[
\mathcal{C}_\rho (x, t) = t^{-\frac{\alpha + 1}{\rho}} f_\rho \left( \frac{x t^{-\frac{1}{\rho}}}{\bar{\alpha}} \right),
\]

where \( f_\rho = \frac{d}{dx} F_\rho (x) \) and \( F_\rho \) is the Mittag-Leffler distribution defined, for \( 0 < \rho \leq 1 \), as

\[
F_\rho (x) := \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^{\rho k}}{\Gamma (1 + \rho k)}.
\]

The functions belonging to \( \{ \mathcal{C}_\rho \} \) present infinite mass for \( \rho < 1 \) and only for the special case \( \rho = 1 \) the solution has all bounded moments, since \( \mathcal{C}_1 (x, t) \) is exactly the solution originally found by Smoluchowski (formula (1.3.64)). We can also compute the divergence of the first moment for the family \( \{ \mathcal{C}_\rho \} \):

\[
\int_0^x y \mathcal{C}_\rho (y, 0) \, dy \sim \frac{\rho x^{1-\rho}}{\Gamma (2 - \rho)}
\]
which is exactly equal to unity when $\rho = 1$. Theorem 1.18 applies, and the rescaled limit distribution in the converse implication is actually given by $S_{0,\rho}(x) = F_\rho(x)$, the Mittag-Leffler distribution. Thanks to this theorem, the condition for a solution to be attracted to the exponentially decaying one is that

$$
\int_0^x y \ c_o(y) \ dy \sim_{x \to \infty} L(x)
$$

where $L(x)$, verifying $\lim_{x \to x} (L(\alpha x)/L(x)) = 1$ for each $\alpha > 0$, is called a slowly varying function and can be a logarithmic function, a power of a logarithm or an iterated logarithm, for example. Therefore, it is not necessary for the initial mass to be finite, but it can also diverge slowly.

The additive kernel:

The explicit solution proposed by Golovin contains a $x^{-\frac{3}{2}}$ factor:

$$
c(x, t) = \frac{1}{\sqrt{2\pi}} x^{-\frac{3}{2}} e^{-t} \exp \left(-e^{-2t} \frac{x}{2}\right), \quad (1.3.68)
$$

so the Laplace transform to be used is clearly the desingularized one, as defined in formula (1.3.53). Again, this explicit solution is just only one member of a uniparametric family of solutions, with non-integrable singularity at the origin, given by:

$$
\mathcal{C}_\rho(x, t) = (e^{-2t})^{\frac{\rho+1}{\rho}} f_\rho \left(x (e^{-t})^{\frac{\rho+1}{\rho}}\right),
$$

where

$$
f_\rho(x) = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} x^{\rho k-2} \frac{\pi e^{\frac{\rho k}{\rho+1}}}{k!} \Gamma \left(1 + k - \frac{\rho}{\rho+1} k\right) \sin \left(\pi k \cdot \frac{\rho}{\rho+1}\right),
$$

with first and second moment behaving as:

$$
M_1 |_{0}^{x} = \int_{0}^{x} y f_\rho(y) \ dy = \sum_{k=1}^{\infty} (-1)^{k+1} x^{\frac{\rho k}{\rho+1}} \frac{\pi e^{\frac{\rho k}{\rho+1}}}{k!} \Gamma \left(1 + k - \frac{\rho}{\rho+1} k\right) \sin \left(\pi k \cdot \frac{\rho}{\rho+1}\right),
$$

and

$$
M_2 |_{0}^{x} = \int_{0}^{x} y^2 f_\rho(y) \ dy \sim_{x \to \infty} \frac{\rho (\rho+1)}{2 - \rho} x^{1-\rho}.
$$

In equation (1.3.71) we denote with $M_1 |_{0}^{x}$ the truncated mass, so that $M_1 |_{0}^{\infty} = M_1$, the conserved mass of the solutions; we also recognize that for the family of explicit solutions $M_1 |_{0}^{x} = S_1 ((\gamma_{1,\rho}(t) x), t)$ with time scaling $\gamma_{1,\rho}(t) = (e^{-t})^{\frac{\rho+1}{\rho}}$. In this context, we have already characterized all the elements necessary for Theorem 1.18 to be repeated, but it is first useful to analyze the scaling solutions in Laplace variable for equation (1.3.56).

Consider some rescaling $\gamma(t)$, the self-similar variable $\xi = \eta \gamma(t)$ and the self-similar ansatz $\phi(\eta, t) = \gamma^{-1} \phi(\xi)$. Imposing in (1.3.56), we get:

$$
\frac{\gamma'(t)}{\gamma(t)} \left(\phi' - \xi \phi''\right) + \phi \xi \phi' - \phi = 0,
$$

In equation (1.3.71) we denote with $M_1 |_{0}^{x}$ the truncated mass, so that $M_1 |_{0}^{\infty} = M_1$, the conserved mass of the solutions; we also recognize that for the family of explicit solutions $M_1 |_{0}^{x} = S_1 ((\gamma_{1,\rho}(t) x), t)$ with time scaling $\gamma_{1,\rho}(t) = (e^{-t})^{\frac{\rho+1}{\rho}}$. In this context, we have already characterized all the elements necessary for Theorem 1.18 to be repeated, but it is first useful to analyze the scaling solutions in Laplace variable for equation (1.3.56).
and, separating variables with $\gamma'/\gamma = w$ so that $\gamma(t) = k_1 e^{wt}$, we can write:

$$
\left(w \xi - \hat{\phi}\right) \ddot{\xi} \hat{\phi} + (1 - w) \dot{\phi} = 0.
$$

(1.3.74)

Equation (1.3.74), though elementary, is not solvable explicitly for $\hat{\phi}(\xi)$; we instead find an implicit solution for $\xi(\phi)$ since we can formally consider the linear equation

$$
\xi' (\phi) - \frac{w}{w-1} \xi (\phi) + \frac{1}{w-1} = 0,
$$

(1.3.75)

and write its solution:

$$
\xi = \phi_\rho + k_2 \phi_\rho^{1+\rho}.
$$

(1.3.76)

Here $\rho = (w - 1)^{-1}$ and $\phi_\rho(\xi)$ is the implicitly defined self-similar solution depending on the parameter $\rho$. By scaling, we can set $k_1 = k_2 = 1$, while the range of values of $\rho$ is limited to $\rho > 0$ (due to mass conservation); an extra requirement is that $\rho \leq 1$: if $\rho > 1$, the Laplace transform cease to satisfy complete monotonicity, a necessary and sufficient condition for its inverse transform, that is $c(x, t)$, to be a non-negative solution. For more details, see Section 6.2 in [236].

**Remark 1.19.** We therefore remark a very interesting fact that will come to use later: the parameter range is such that $\rho = 1$ is a special value, since it represents the unique point of frontier between solutions $c(x, t)$ with negative values (when $\rho > 1$) and solutions decaying slowly with power tails (when $\rho \in (0, 1)$). The values $\rho > 1$ are unphysical (since the solution has negative values); in the limit $\rho \rightarrow 1^+$, we can heuristically say that the first zero $\underline{x}$ of the solution defined as $x := \inf \{x \in \text{Supp}(c) \text{ such that } c(x, t) < 0\}$ is pushed to infinity, in a such way that $\rho = 1$ is the maximal value for $\rho$ so that $\underline{x}$ is still unbounded. On the other side, $\rho < 1$, in the limit $\rho \rightarrow 1^-$, we find that the self-similar solutions present a faster and faster algebraic decaying tail so that, for the critical value $\rho = 1$, only the exponential decay remains.

As an extra remark, we also see that for $\rho = 1$ equation (1.3.76) is quadratic and can be explicitly solved as $\phi(\xi) = \frac{1}{2} \left(-1 \pm \sqrt{1 + 4\xi}\right)$. We keep only the solution with the plus sign and notice that it is analytic, for complex values of $\xi$, in a neighborhood of zero. All these issues will be explained extensively in the analysis of self-similar solutions in Chapter 2.

Consider now the truncated first moment of a solution $c(x, t)$ to Smoluchowski equation, which we denote with

$$
M(x, t) = \int_0^x y c(y, t) \, dy.
$$

(1.3.77)

Theorem 1.18 can be reformulated as follows:

**Theorem 1.20.** Suppose that there exists a time-rescaling $\gamma(t)$ such that $M(\gamma(t) x, t) = S_1((\gamma(t) x), t)$ converges point-wise as $t \rightarrow \infty$ to some non-trivial probability distribution $S_{1,\rho}(x)$. Then there is some value $\rho$ and a slowly varying function $L$ such that

$$
M_{2_{10}} := \int_0^x y^2 c_0(y) \, dy \sim L(x) x^{1-\rho}.
$$

(1.3.78)
Conversely, suppose that the second moment of the initial datum diverges, for some \( \rho \) and some slowly varying function \( L \), as in formula (1.3.78). Then there is a strictly increasing rescaling \( \gamma_{1,\rho} (t) \approx \exp \left( \frac{\rho+1}{\rho} t \right) \) (possibly modified by a slowly varying function) such that:

\[
\lim_{t \to \infty} M \left( \gamma_{1,\rho} (t) x, t \right) = M_{1,0} = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{\rho^k}{k!} \Gamma \left( 1 + k - \frac{\rho}{\rho+1} k \right) \frac{\sin \left( \pi k \frac{\rho}{\rho+1} \right)}{\pi k \frac{\rho}{\rho+1}}. \tag{1.3.79}
\]

The same remarks about the constant kernel Theorem can be repeated here. In particular, we point out that the classical self-similar solution associated to Golovin’s solution attracts all solutions with initial second moment diverging as a slowly varying function.

**Multiplicative kernel:**

We conclude this Section with the unique case of successfully analyzed self-similar problem for a gelling kernel. This result is possible thanks to the easily deducible self-similar solution that is known. Consider in fact McLeod’s solution to the discrete Smoluchowski equation with multiplicative kernel and monodisperse initial datum \( c_0 (x) = \delta (x - 1) \):

\[
c (x, t) = \sum_{k=1}^{\infty} \tilde{c}_k (t) \delta (x - k), \quad \tilde{c}_k (t) = \frac{1}{k!} t^{k-1} k^{k-2} e^{-tk}. \tag{1.3.80}
\]

We want a rescaling for \( t \to 1^- \) and large \( k \) since we look close to the gelling time. Using Stirling’s approximation \( k! \approx \sqrt{2\pi k} e^{-k} k^k \) we find

\[
\tilde{c}_k (t) \approx \frac{1}{\sqrt{2\pi}} k^{-\frac{k}{2}} \exp \left( k (1 - t + \log t) \right), \tag{1.3.81}
\]

so that if we pass to the limit \( t \to 1, k \to \infty \) holding fixed the self-similar variable \( \xi = k (1 - t)^2 \), multiplying each \( \tilde{c}_k (t) \) times \( (1 - t)^{-5} \), we get:

\[
\lim_{t \to 1, k \to \infty} (1 - t)^{-5} \tilde{c}_k (t) \sim \psi (\xi) := \frac{1}{\sqrt{2\pi}} \xi^{-\frac{5}{2}} e^{-\frac{1}{2} \xi}. \tag{1.3.82}
\]

The function \( \psi (\xi) \) is the self-similar profile to which converges the following explicit solution for the continuous Smoluchowski equation:

\[
c (x, t) = \frac{1}{\sqrt{2\pi}} x^{-\frac{5}{2}} \exp \left( -\frac{1}{2} (1 - t)^2 x \right). \tag{1.3.83}
\]

Analogously, this solution is just a special member of a uniparametric family of solutions. However, one can use a classic relation between the additive and multiplicative cases to skip the intermediate steps and get directly to the conclusions: in fact, one can always relate solutions \( c \) and \( \tilde{c} \) (respectively for the multiplicative and the additive case) thanks to the formula:

\[
x^2 c (x, t) = \frac{x}{(1 - t)} \tilde{c} (x, -\log (1 - t)). \tag{1.3.84}
\]
Therefore, we can easily see that the uniparametric family to be considered is

\[ \mathbf{c}_\rho (x, t) = (1 - t)^{-1 + \frac{2\rho}{\rho + 1}} f_\rho \left( x (1 - t)^{\frac{\rho + 1}{\rho}} \right), \]

\[ f_\rho (x) = \frac{1}{\pi} \sum_{k=1}^{\infty} \left( -1 \right)^{k+1} \frac{x^{\rho+k-3}}{k!} \Gamma \left( 1 + k - \frac{\rho}{\rho + 1} k \right) \sin \left( \pi k - \frac{\rho}{\rho + 1} k \right), \]

and that the probability distribution associated to the second moment for the multiplicative case verifies:

\[ S_2 (x, t) = S_1 (x, - \log (1 - t)). \]

Theorem 1.18 is straightforwardly adapted, so we skip its statement. Instead we remark that the gelling solutions that conserve finite mass up to the critie time are attracted by self-similar profiles with infinite mass whenever their initial third moment diverges suitably.

In conclusion, the analysis of the explicit cases shows the non-triviality of the problem of defining a correct scaling hypothesis that should not be simply deduced by imposing a single desired decay \textit{a priori}.

1.3.3. Towards a second kind scaling hypothesis

The study of the explicit cases can support the need of higher flexibility in deriving the self-similar equations. In particular, the existence of families of solutions with rich behavior (algebraic decay except for a single member of the family with exponential decay) suggests that Claim 1.16 may result too restrictive.

As mentioned above, the idea is not to impose any condition on the family (1.3.44) and try to solve a suitable formal equation under general rescaling

\[ c_\theta (x, t) = s^\theta (t) \psi \left( s \left( t \right) x \right). \]

The only technical requirement we impose to Smoluchowski equation is that we limit the homogeneous class of kernels to the multiplicative kernel \( K_\lambda (x, y) = (xy)^\lambda \) and \( \lambda \leq 1, \lambda \neq \frac{1}{2} \). Even if there are other families of general homogeneous kernels that have been successfully treated in the literature and which also include \( K_\lambda \) as a subfamily (as, for example, \( \frac{1}{2} \left( x^\alpha y^\beta + x^\beta y^\alpha \right) \)), we point out that multiplicative kernels retain the principal dynamic characteristic of all homogeneous kernels (the degree of homogeneity \( \lambda \) and the interaction strength of growing masses), while avoiding the arguably unnecessary technical difficulty of a more sophisticated family of functions. On the other hand, it should also be pointed out that our choice excludes kernels with a singularity at the origin like the original Smoluchowski’s kernel (see Table 1.2 for a list of possibilities). We believe that the results of this thesis can be considered sufficiently general as to roughly represent the main properties of all homogenous kernels, at least in an asymptotic sense.

In order to cover also the gelling cases, we can consider explicitly \( \tilde{s} (T^* - t) = s (t) \). The first necessary step is to consider the formal conditions for \( s^\theta (t) \psi \left( s \left( t \right) x \right) \) to satisfy Smoluchowski equation, and, at a later stage, we may ask in which sense the objects we find actually represent valid solutions. Let \( \xi \) be the self-similar variable \( \xi := s \left( t \right) x \) so that we easily obtain:

\[ s' (t) T_\theta [\psi] (\xi) = s^{-2\lambda + \theta} (t) C [\psi] (\xi), \quad \lambda < \frac{1}{2} \]

\[ -s' (T^* - t) T_\theta [\psi] (\xi) = s^{-2\lambda + \theta} (T^* - t) C [\psi] (\xi), \quad \frac{1}{2} < \lambda \leq 1 \]
where the transport operator depend on $\theta$ and the coagulation operator has now the $\lambda-$multiplicative kernel:

\[
T_\theta [\psi] (\xi) = \theta \psi (\xi) + \xi \psi' (\xi), \quad (1.3.91)
\]

\[
C [\psi] (\xi) = \frac{1}{2} \int_0^\xi ( (\xi - \eta) \eta^\lambda \psi (\xi - \eta) \psi (\eta) \, d\eta - \xi^\lambda \psi (\xi) \int_0^\infty \eta^\lambda \psi (\eta) \, d\eta. \quad (1.3.92)
\]

By separating variables, in both cases we have algebraic solutions for the typical size function $s(t)$, unless $\theta = 1 + 2\lambda$ which implies a special rescaling $s(t) \propto e^{wt}$. Thus:

\[
s(t) = ((1 + 2\lambda - \theta) wt)^{\frac{1}{1+2\lambda - \theta}}, \quad (1.3.93)
\]

\[
\pm wT_\theta [\psi] (\xi) = C [\psi] (\xi), \quad (1.3.94)
\]

where the plus or minus sign depends on $\lambda < \frac{1}{2}$ or $\frac{1}{2} < \lambda \leq 1$.

The algebraic exponent of the typical size function is called the self-similarity exponent and, in order to develop a simple notation, we can call it $\beta$. In that way, setting $w = \beta = \frac{1}{1+2\lambda-\theta}$, we finally get to the self-similar problem we want to study:

\[
s(t) = t^\beta, \quad (1.3.95)
\]

\[
c\left(-\frac{1}{2}+1+2\lambda\right) (x, t) = t^{\beta(1+2\lambda)-1} \psi \left(t^\beta x\right), \quad \lambda < \frac{1}{2}; \quad (1.3.96)
\]

\[
c\left(-\frac{1}{2}+1+2\lambda\right) (x, t) = (T^x - t)^{\beta(1+2\lambda)-1} \psi \left((T^x - t)^\beta x\right), \quad \frac{1}{2} < \lambda \leq 1, \quad (1.3.97)
\]

\[
(-1)^{1\left\{\frac{1}{2} < \lambda \leq 1\right\}} ((\beta (1 + 2\lambda) - 1) \psi (\xi) + \beta \xi \psi' (\xi)) = C [\psi] (\xi), \quad (1.3.98)
\]

where we use $(-1)^{1\left\{\frac{1}{2} < \lambda \leq 1\right\}}$ in order to distinguish the cases with plus or minus sign. Equation (1.3.98) can be written in a much nicer way if we remark that, when $\lambda > \frac{1}{2}$ then we are interested only on values of $\beta > 0$ while, when $\lambda < \frac{1}{2}$, we only consider $\beta < 0$. Remembering that we have assumed $(-1)^{1\left\{\frac{1}{2} < \lambda \leq 1\right\}} \beta < 0$, we can write compactly:

\[
-|\beta| \left( 1 + 2\lambda - \frac{1}{\beta} \right) \psi (\xi) + \xi \psi' (\xi) = C [\psi] (\xi). \quad (1.3.99)
\]

Notice that whenever $\psi (\xi)$ is a solution of (1.3.99) with $M_\alpha < \infty$ for some $\alpha$, then for any $\mu > 0$ the expression

\[
\psi_\mu (\xi) = \mu^{1+2\lambda} \psi (\mu \xi) \quad (1.3.100)
\]

also defines a solution of (1.3.99) with the same bounded moments. The solutions for a given $\lambda$ can be grouped as one-parameter families and, for simplicity, we will prove results for a particular member of the family uniquely defined by a specific value in its asymptotics near the origin, giving for granted that any result of existence can be extended automatically to the complete family.

Our main tool of investigation is, similarly to Menon and Pego, the regularized Laplace transform and we focus at studying the families of solutions to equation (1.3.99) depending on the values of $\beta$; in this sense, it can be stimulating to fix a long term program that aims at showing the role of second-kind self-similarity and the role of each member of the family of solutions in the dynamics of Smoluchowski coagulation.
We try therefore to conjecture some desirable results (in a similar fashion to the still open problems proposed by Aldous [8] and Leyvraz [215] in their works) and then we briefly summarize what we have shown that can support these ideas.

Claim 1.21 (Similarity exponent in the non-gelling case). Consider equation (1.3.99) with multiplicative kernel of homogeneity degree $\lambda < \frac{1}{2}$. There exists a continuous, decreasing function $\beta^* : (-\infty, \frac{1}{2}) \to (-\infty, 0]$ such that for each $\lambda$ there is a unique non-negative function $\psi$ satisfying (1.3.99) with $\beta = \beta^* (\lambda)$ and

$$\int_0^\infty \xi^{2\lambda+\delta} \psi (\xi) \ d\xi < \infty, \quad (1.3.101)$$

for each $\delta > 0$. Moreover, $\beta^* (0) = -1$ and $\beta^* (\lambda) = -\frac{1}{1-2\lambda}$ when $\lambda < 0$. The similarity exponent $\beta^* (\lambda)$ is critical in the sense that, for each $\lambda$, defining $\xi_0 (\beta, \lambda) = \inf \{\xi \in \text{supp} (\psi) \ s.t. : \ \psi (\xi) < 0\}$,

$$\beta^* = \sup \{\beta \leq 0 \ s.t. : \xi_0 (\beta, \lambda) = \infty\} = \inf \{\beta \leq 0 \ s.t. : \xi_0 (\beta, \lambda) < \infty\}. \quad (1.3.102)$$

For each $\beta \neq \beta^*$ the self-similar solutions to (1.3.99) have algebraic decay.

The fact that the finiteness of moments require a specific value for $\beta$ makes evident the nature of this self-similar problem as of the second kind. In the gelling case we can conjecture almost the same:

Claim 1.22 (Similarity exponent in the gelling case). When $\frac{1}{2} < \lambda \leq 1$, there exists a continuous, decreasing function $\beta^* : (\frac{1}{2}, 1] \to (0, \infty)$ such that for each $\lambda$ there is a unique non-negative function $\psi$ satisfying (1.3.99) with $\beta = \beta^* (\lambda)$ and

$$\int_0^\infty \xi^{2\lambda-\frac{1}{2}+\delta} \psi (\xi) \ d\xi < \infty, \quad (1.3.103)$$

for each $\delta > 0$. Moreover, $\beta^* (1) = 2$. The similarity exponent $\beta^* (\lambda)$ is critical in the sense that, for each $\lambda$,

$$\beta^* = \inf \{\beta > 0 \ s.t. : \xi_0 (\beta, \lambda) = \infty\} = \sup \{\beta > 0 \ s.t. : \xi_0 (\beta, \lambda) < \infty\}. \quad (1.3.104)$$

For each $\beta \neq \beta^*$ the self-similar solutions to (1.3.99) have algebraic decay.

As for the case with global in time existence, in that part of the work (Chapter 3) we relied on an appropriate matching of different asymptotics for the selfsimilar solution that yields to a global description of the solutions. We could find explicitly the function $\beta^* (\lambda) = -\frac{1}{1-2\lambda}$ for $\lambda \leq 0$, while, for $0 < \lambda < \frac{1}{2}$, the similarity exponent has to be computed by solving a nonlinear eigenvalue problem. This study can be performed via perturbative methods around the particular case $\lambda = 0$ and $\beta^* = -1$ and numerically when this first technique cannot be applied any more. The function $\beta^*$ seems to be at least continuous, constantly decreasing, negative and unbounded below when $\lambda \to \frac{1}{2}^-$. Also, $-1 - 2\lambda > \beta^* > -\frac{1}{1-2\lambda}$ in this range of values.

As for the solutions, when $0 < \lambda < \frac{1}{2}$ they present a singularity at the origin and decay at infinity as:

$$\psi (\xi) \approx A\xi^{-1-2\lambda}, \ \psi (\xi) \approx A\xi^{-1-2\lambda+\frac{1}{2}} \quad (1.3.105)$$

unless $\beta = \beta^*$, when they have exponential decay at infinity and the same expansion at zero. Therefore, the solution has all bounded moments starting from the $2\lambda$th.
Remark 1.23. The interesting characterization of $\beta^*$ as critical boundary between positive and negative solutions has been observed numerically and should be supported by rigorous proofs.

Figure 1.3.1: Resuming scheme of known and guessed properties of $\beta^*(\lambda)$. The dotted line represent values of the critical exponent obtained from numerical experiments, while the continuous line cover the cases where we have obtained rigorous results via analytical methods.

In the gelling case, $\frac{1}{2} < \lambda \leq 1$ (Chapter 2), the singularity at the origin is stronger and also depending on $\beta$:

$$
\psi (\xi) \xrightarrow{\xi \to 0} \xi^{-1-2\lambda + \frac{1}{\beta}} + K\xi^{-1-2\lambda + \frac{2}{\beta}} + l.o.t.
$$

(1.3.106)

This made the asymptotic analysis much harder and the solutions show a rich set of behaviors. Again, from numerical tests, $\beta^*$ seemed to be the critical value since for $\beta > \beta^*$ but $\beta \to \beta^*$ the corresponding numerical solutions appear to decay faster and faster. The function $\beta^*(\lambda)$ seems to be regular, decreasing, positive and such that $\beta^*(\frac{1}{2}^+) = +\infty$.

We studied the self-similar problem of the perturbative case ($\lambda \approx 1$) via a fixed point technique giving existence and uniqueness of both the critical value $\beta^*$ and the corresponding self-similar solution $\psi$ with all the moments bounded starting from a sufficiently high one. Given the fact that the expansion at zero is given by formula (1.3.106), we can motivate the small distinction between property (1.3.101) and (1.3.102) for non-gelling and gelling cases.
We would like to conclude this introduction saying that a weak convergence result for solutions of the Smoluchowski equation and the characterization of the domains of attraction is absolutely required in order to confirm the validity of the second kind scaling hypothesis but, at present time, it is still lacking. We can only make guesses of how the generalization of Theorem 1.18 should be like, and it remains a stimulating problem to study in future investigations.
CHAPTER 2

SELF-SIMILAR SOLUTIONS OF THE SECOND KIND REPRESENTING GELATION IN FINITE TIME FOR THE SMOLUCHOWSKI EQUATION

We study the selfsimilar solutions of the Smoluchowski’s equation with kernel $K(x, y) = x^{1-\varepsilon}y^{1-\varepsilon}$ for $\varepsilon > 0$ and $\varepsilon \ll 1$. We prove that by choosing the similarity exponents as a suitable function of $\varepsilon$, the selfsimilar solutions present correct behaviours at the origin and at infinity, which amounts to solving a nonlinear eigenvalue problem. This characterizes the selfsimilar solution found as being of the second kind in the notation introduced by Barenblatt.

2.1. DERIVATION OF THE SELF-SIMILAR EQUATION AND MAIN THEOREM

It is known that the degree of homogeneity $\gamma$ is very important to determine the properties of the dynamics. In this article we restrict ourselves to kernels of the form $K(x, y) = (xy)^\gamma$ with $\gamma = 1 - \varepsilon$ and $\varepsilon \in (0, \frac{1}{2})$; it is known that, under this assumption, gelation in finite time occurs (see [17] and references therein). On the other hand, if $\varepsilon > \frac{1}{2}$ solutions exist globally in time [16]. However, in this case the selfsimilar solutions have not been determined explicitly and, more remarkably, from numerical experiments, they seem to belong to the class of the so called selfsimilarity of the second kind (cf. [236], [231]) where the similarity exponents cannot be determined from simple dimensional considerations. It is also interesting to notice that, if $\varepsilon > 0$, a phenomenon of instantaneous gelation takes place (cf. [293]).

We will consider the case $\gamma = 1 - \varepsilon$ with $\varepsilon > 0$ and $\varepsilon \ll 1$. Henceforth, the Smoluchowski equation takes the form:
\[
c_t(x, t) = \frac{1}{2} \int_0^x (x - y)^{1-\varepsilon} y^{1-\varepsilon} c(x - y, t) c(y, t) dy - x^{1-\varepsilon} c(x, t) \int_0^\infty y^{1-\varepsilon} c(y, t) dy, \tag{2.1.1}
\]
and the selfsimilar solutions are of the form:

\[
c(x, t) = (t_0 - t)^\alpha \psi \left((t_0 - t)^\beta x\right), \tag{2.1.2}
\]
with selfsimilar variable:

\[
\xi = (t_0 - t)^\beta x.
\]

Notice that we can write (2.1.1) in the form

\[
c_t(x, t) = \int_0^{\xi/2} \left((x - y)^{1-\varepsilon} y^{1-\varepsilon} c(x - y, t) c(y, t) - (xy)^{1-\varepsilon} c(x, t) y^{1-\varepsilon} c(y, t)\right) dy
\]

\[
- x^{1-\varepsilon} c(x, t) \int_{\xi/2}^\infty y^{1-\varepsilon} c(y, t) dy.
\]

Conservation of the total mass, \(\frac{d}{dt} \int_0^\infty xc(x, t) dx = 0\), requires that

\[
\psi(\xi) = C \xi^{-\frac{\beta}{\alpha}} + o(\xi^{-\frac{\beta}{\alpha}}), \text{ as } \xi \to 0. \tag{2.1.3}
\]

From dimensional analysis we deduce that

\[
\alpha = \beta (-2\varepsilon + 3) - 1,
\]

and then \(\psi(\xi)\) must obey the following integrodifferential equation:

\[
- (\beta (-2\varepsilon + 3) - 1) \psi(\xi) - \beta \xi \psi'(\xi) = \int_0^{\xi/2} \left((\xi - y)^{1-\varepsilon} y^{1-\varepsilon} \psi(\xi - y) \psi(y) - (xy)^{1-\varepsilon} \psi(\xi) \psi(y)\right) dy
\]

\[
- \xi^{1-\varepsilon} \psi(\xi) \int_{\xi/2}^\infty y^{1-\varepsilon} \psi(y) \ dy. \tag{2.1.4}
\]

In this article, we will prove that, for \(\varepsilon\) sufficiently small, there exist solutions to equation (2.1.1) satisfying (2.1.3), such that

\[
\beta = 2 + 2\varepsilon + O(\varepsilon^2) \tag{2.1.5}
\]

and

\[
\int_0^\infty \xi^{7-\delta} \psi(\xi) \, d\xi < \infty \tag{2.1.6}
\]

for any \(0 < \delta < 1\). The solutions satisfying (2.1.6) are unique among an infinite family of solutions to (2.1.1) satisfying (2.1.3) which, in general present algebraic decay at infinity (as we will show) such that (2.1.6) is not satisfied. Moreover, we will prove that the solutions satisfying (2.1.3), (2.1.6) are such that all moments (from the second moment) of \(\psi(\xi)\) are bounded, that is:

\[
\int_0^\infty \xi^N \psi(\xi) \, d\xi < \infty, \text{ for any } N \geq 2. \tag{2.1.7}
\]
Notice that selfsimilar solutions satisfying (2.1.7) are the proper ones in order to describe gelation in finite time: the corresponding $c(x,t)$ will be such that all moments $M_i (i \geq 2)$ are bounded up to time $t_0$, when all moments $M_i (i \geq 2)$ become unbounded.

Notice that given a solution $\psi(\xi)$ of (2.1.4) satisfying (2.1.7), the expression $\mu^{2-\epsilon}\psi(\mu\xi)$ (with $\mu > 0$) is also a solution of (2.1.4) satisfying (2.1.7). Hence, the solutions for a given $\epsilon$ form a one-parameter family. In this paper, for the sake of simplicity, we will prove results for a particular member of the family uniquely defined by a certain value of $C$ ($C = 1$, specifically) in the asymptotics (2.1.3) near the origin and understand that existence results extend automatically to the complete family.

This article can be divided in two parts. The first part, including sections 2 to 5, presents asymptotic arguments to compute the behaviour of selfsimilar solutions at the origin and at infinity and the construction of selfsimilar solutions for a suitable choice of $\beta$. For a generic $\beta$, solutions decay at infinity in an algebraic manner, while for an appropriate choice of $\beta$ (formula (2.1.3)) the solution will decay exponentially fast. The arguments, based on the use of Laplace transform, allow a matched asymptotics construction of globally defined solutions. We remark that previous studies in the literature (cf. [300], [109], [293], [215]) focused on the asymptotics at the origin or at infinity and did not attempt to construct a global solution and do not provide a criterion for the choice of the similarity exponent $\beta$. The second part of the paper, including sections 6-10, is devoted to a fully rigorous proof of existence of selfsimilar solutions with all moments bounded. Previous studies focused on the exactly integrable case $\epsilon = 0$.

The strategy to prove the existence of selfsimilar solutions satisfying (2.1.6) is perturbative. We take as an order zero (in $\epsilon$) approximation the well known selfsimilar solution corresponding to $\epsilon = 0$. The equation (2.1.1), when $\epsilon = 0$, transforms into inviscid Burgers’ equation for the Laplace transform of $c$: $u_t + (u^2/2)_x = 0$. This fact allows to perform an explicit integration and analysis of its solution by performing an inverse Laplace transform. In this way, it was shown (cf. [236], [238], [240]) that the selfsimilar solution is a dynamical attractor for the coagulation equation (2.1.1). When $\epsilon > 0$, the use of Laplace transform leads to a nonlocal Burgers’ equation where the nonlinearity is of the form $((D^{-\epsilon}u)^2/2)_x$, i.e. involves a nonlocal operator $D^{-\epsilon}u$ which is a formal equivalent to a fractional derivative of order $-\epsilon$. When looking for selfsimilar solutions, the exponent $\beta$ must be found as a function of $\epsilon$. If we expand $\beta$ in powers of $\epsilon$ and expand the operator $D^{-\epsilon}u$ in powers of $\epsilon$ we arrive, at first (linear) order, to a non-homogeneous equation which must be solved under the condition that solutions are such that sufficiently high moments are bounded. This problem can be viewed as a sort of Fredholm alternative condition from which $\beta$ is found at linear order as a function of $\epsilon$. A fixed point argument allows us to find the solution to the nonlinear problem as a small correction to the linear solution. This requires the study of nonlocal and nonlinear operators in suitable weighed Sobolev spaces. Finally, we provide an argument to show that the solutions found must be positive and use (2.1.4) to find inequalities involving moments of different orders which serve to prove that all moments of the solution found are bounded. More precisely, positive solutions satisfy the following Theorem:

**Theorem 2.1.** There exists an $\epsilon_0 > 0$ and a function

$$\lambda(\epsilon) = 2 + O(\epsilon)$$

such that for any $0 < \epsilon < \epsilon_0$ and with $\beta(\epsilon) = 2 + \epsilon \lambda(\epsilon)$ there exists a solution to (2.2.6) (up to rescaling) with all its moments $M_\alpha (\alpha \geq 2)$ bounded and positive.
The rest of this article is organized as follows: in Section 2.2 we reformulate the problem in terms of Laplace transform; this approach is essential for our analysis, as it permits to write Smoluchowski’s equation as a Burgers-type differential equation. Section 2.3 is devoted to the study of the selfsimilar solution in the particular case \( \varepsilon = 0 \), with special emphasis on the asymptotic behaviour close to the origin and at infinity. A similar study is done in Section 2.4 for the case \( \varepsilon > 0 \). In section 2.5 we present the argument of the existence of selfsimilar solutions decaying fast at infinity by an appropriate choice of the similarity exponent \( \beta \). Sections 2.6–2.8 are then devoted to construct the rigorous proof of the existence of such selfsimilar solution. In Section 2.6, we formulate the nonlinear problem in an appropriate setting consisting of a fixed point argument based on the estimates for an auxiliary linear problem. Such linear problem is studied in Section 2.7, while Section 2.8 develops the fixed point argument and completes the proof of the nonlinear problem, based on some results in functional analysis deduced in Section 2.10. Finally, in Section 2.9 we discuss the possibility of a faster decay of the selfsimilar solution at infinity.

### 2.2. Laplace Transform Representation

A particularly useful way to study equation (2.1.1) is by means of the Laplace transform. Applying it to Smoluchowski equation yields

\[
\int_{\mathbb{R}^+} e^{-\lambda x} c_t(x, t) \, dx = \frac{1}{2} \int_{\mathbb{R}^+} e^{-\lambda x} \int_{[0,x]} y^{1-\varepsilon} (x-y)^{1-\varepsilon} c(y, t) c(x-y, t) \, dy \, dx
\]

\[
- \int_{\mathbb{R}^+} e^{-\lambda x} x^{1-\varepsilon} c(x, t) \int_{\mathbb{R}^+} y^{1-\varepsilon} c(y, t) \, dy \, dx,
\]

and defining formally

\[
C(\lambda, t) = \int_{\mathbb{R}^+} e^{-\lambda x} c(x, t) \, dx,
\]

\[
D^{\varepsilon}_{\lambda} C(\lambda, t) = \int_{\mathbb{R}^+} e^{-\lambda x} x^{-\varepsilon} c(x, t) \, dx,
\]

(2.2.1)

(2.2.2)

to which we can formally associate the corresponding derivatives with respect to \( \lambda \):

\[
C_{\lambda}(\lambda, t) = \frac{dC(\lambda, t)}{d\lambda} = -\int_{\mathbb{R}^+} e^{-\lambda x} x c(x, t) \, dx,
\]

\[
D^{\varepsilon}_{\lambda} C_{\lambda}(\lambda, t) = \frac{d}{d\lambda} D^{\varepsilon}_{\lambda} C(\lambda, t) = \frac{d}{d\lambda} \int_{\mathbb{R}^+} e^{-\lambda x} x^{-\varepsilon} c(x, t) \, dx = -\int_{\mathbb{R}^+} e^{-\lambda x} x^{1-\varepsilon} c(x, t) \, dx.
\]

We arrive then at a new PDE

\[
C_t(\lambda, t) = \frac{1}{2} \left[ D^{\varepsilon}_{\lambda} C_{\lambda}(\lambda, t) \right]^2 - D^{\varepsilon}_{\lambda} C(\lambda, t) \, D^{\varepsilon}_{\lambda} C_{\lambda}(\lambda, t),
\]

(2.2.3)

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where we have manipulated the integrals in such a way that:

\[
\int_{\mathbb{R}^+} e^{-\lambda x} \frac{1}{2} \int_{[0,x]} y^{1-\epsilon} (x - y)^{1-\epsilon} c(y, t) c(x - y, t) \, dy \, dx \\
= \frac{1}{2} \int_{\mathbb{R}^+} e^{-\lambda(z+y)} z^{1-\epsilon} c(y, t) \, dz = \frac{1}{2} \left[ \int_{\mathbb{R}^+} e^{-\lambda x} x^{1-\epsilon} c(x, t) \, dx \right]^2 \\
= \frac{1}{2} \left[ D_{-\epsilon}^\lambda C_\lambda(\lambda, t) \right]^2.
\]

Notice now that we can rewrite equation (2.2.3) in the form:

\[
C_t(\lambda, t) = \frac{1}{2} \left[ D_{-\epsilon}^\lambda C_\lambda(\lambda, t) - D_{-\epsilon}^\lambda C_\lambda(0, t) \right]^2 - \frac{1}{2} \left[ D_{-\epsilon}^\lambda C_\lambda(0, t) \right]^2 \tag{2.2.4}
\]

and, by taking the \(\lambda\)-derivative of (2.2.4), we eliminate the last term at the right hand side, we obtain:

\[
C_{\lambda\lambda}(\lambda, t) = \left[ D_{-\epsilon}^\lambda C_\lambda(\lambda, t) - D_{-\epsilon}^\lambda C_\lambda(0, t) \right] \left[ D_{-\epsilon}^\lambda C_\lambda(\lambda, t) - D_{-\epsilon}^\lambda C_\lambda(0, t) \right]_{\lambda}.
\]

Again, at least formally, one can evaluate the resulting equation at \(\lambda = 0\) and obtain

\[
C_{\lambda\lambda}(0, t) = 0,
\]

so that, by defining

\[
\omega(\lambda, t) \equiv C_\lambda(\lambda, t) - C_\lambda(0, t) = -\int_{\mathbb{R}^+} (e^{-\lambda x} - 1) x c(x, t) \, dx \\
D_{-\epsilon}^\lambda \omega(\lambda, t) \equiv -\int_{\mathbb{R}^+} (e^{-\lambda x} - 1) x^{1-\epsilon} c(x, t) \, dx
\]

we conclude with the equation:

\[
\omega_t(\lambda, t) = \frac{1}{2} c_\lambda \left( D_{-\epsilon}^\lambda \omega(\lambda, t) \right)^2 \tag{2.2.5}
\]

which, in the particular case \(\epsilon = 0\), is the Burgers equation for \(\omega(\lambda, t)\).

The selfsimilar solutions of (2.2.5) can be sought for in the form:

\[
\omega(\lambda, t) = (t_0 - t)^{\alpha'} \Phi \left( \eta \equiv \frac{\lambda}{(t_0 - t)^{\beta'}} \right)
\]

that, substituted into equation (2.2.5) with \(\alpha' = (1 - 2\epsilon)\beta' - 1\), yields the following ordinary differential equation for \(\eta\):

\[
-(1 - 2\epsilon)\beta' - 1) \Phi(\eta) + \beta' \eta \Phi'(\eta) = \frac{1}{2} \frac{d}{d\eta} \left[ D_{-\epsilon}^{-\epsilon} \Phi(\eta) \right]^2. \tag{2.2.6}
\]

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From dimensional analysis, we can also easily find the relation between \( \beta' \) in (2.2.6) and \( \beta \) in (2.1.4):

\[
\omega (\lambda, t) = - \int_{\mathbb{R}^+} (e^{-\lambda x} - 1) xc(x, t) \, dx = - \int_{\mathbb{R}^+} (e^{-\lambda x} - 1) x(t_0 + t)\beta(-2\varepsilon + 1) - 1 \psi \left( \left( t_0 + t \right)^\beta x \right) \, dx
\]

\[
= \left( t_0 + t \right)^{1-2\varepsilon} \beta^{-1} \Phi \left( \eta = \frac{\lambda}{(t_0 + t)\beta} \right)
\]

and conclude that it must be \( \beta = \beta' \). It results moreover that the selfsimilar solution \( \psi \) of equation (2.1.4) in the ordinary space and the corresponding selfsimilar solution \( \Phi \) in equation (2.2.6) are related by the following formula:

\[
\Phi (\eta) = - \int_{\mathbb{R}^+} (e^{-\eta \xi} - 1) \xi \psi (\xi) \, d\xi.
\]  

(2.2.7)

Our task will then be to construct a solution \( \Phi \) to (2.2.6) such that its inverse transform has the desired properties. In order to obtain \( \psi (\xi) \) from \( \Phi (\eta) \), first take the \( \eta \)-derivative of formula (2.2.7)

\[
\Phi' (\eta) = \int_{\mathbb{R}^+} e^{-\eta \xi} \xi^2 \psi (\xi) \, d\xi;
\]

i.e. \( \Phi' \) is the Laplace transform of \( \xi^2 \psi (\xi) \). Henceforth:

\[
\psi (\xi) = \frac{1}{2\pi i} \frac{1}{\xi^2} \int_{-i\infty}^{i\infty} e^{\eta \xi} \Phi' (\eta) \, d\eta,
\]

(2.2.8)

where the integration takes place along the imaginary axis (cf. formula 29.2.2 of [1]).

### 2.3. Seltsimilar Solutions for \( \varepsilon = 0 \) and Their Asymptotics

In this section we study the selfsimilar solutions of Schmoluohowski equation in the particular case \( \varepsilon = 0 \). We will derive the same results that have been previously obtained in [236] in a somewhat different fashion; we include here our derivation since it will be easily adapted in the case \( \varepsilon \neq 0 \).

The equation for the selfsimilar solutions is:

\[
-(3\beta - 1) \psi (\xi) - \beta \xi \psi' (\xi) = \int_{\xi}^{\xi/2} ((\xi - y) y \psi (\xi - y) - \xi y \psi (\xi) \psi (y)) \psi (y) \, dy - \xi \psi (\xi) \int_{\xi/2}^{\xi} y \psi (y) \, dy.
\]

(2.3.1)
Correspondingly, in Laplace transform representation (2.2.6), one has the following equation:

\[ - (\beta - 1) \Phi (\eta) + \beta \eta \Phi' (\eta) = \Phi (\eta) \Phi' (\eta) \, , \]  

(2.3.2)

As we will show, there exist solutions to (2.3.4) such that, as \( \xi \to 0 \),

\[ \psi (\xi) = C\xi^{-\frac{3\beta-1}{\beta}} + o(\xi^{-\frac{3\beta-1}{\beta}}) \, , \]  

(2.3.3)

according to condition (2.1.3). Nevertheless, not all solutions will decay sufficiently fast at infinity as to guarantee that all moments of the corresponding \( c(x, t) \), except for the first (that is always singular by the asymptotics given by (2.3.3)), are bounded.

We write the self-similar solution in the more convenient form:

\[ \psi (\xi) = \frac{G (\xi)}{\xi^{\frac{2}{\beta}}} \]  

(2.3.4)

for a suitable \( G (\xi) \). If \( G \) is such that all the integrals in equation (2.3.4) converge, then it must obey

\[ G' (\xi) = -\frac{1}{\beta} \frac{1}{\xi^{1-\frac{1}{\beta}}} \left( \int_{[0,\frac{1}{2}]} \left( \frac{G (\xi y) G (\xi (1 - y))}{y^{\frac{2}{\beta} - \frac{1}{\beta}} (1 - y)^{\frac{2}{\beta} - \frac{1}{\beta}}} - \frac{G (\xi) G (\xi y)}{y^{\frac{2}{\beta} - \frac{1}{\beta}}} \right) dy - G (\xi) \int_{[\frac{1}{2}, \infty]} \frac{G (\xi y)}{y^{\frac{2}{\beta} - \frac{1}{\beta}}} dy \right) \]  

(2.3.5)

2.3.1. Asymptotics close to the origin

Analyzing equation (2.3.5) for \( \xi \ll 1 \) we will be able to find an asymptotic expansion for \( G (\xi) \). Indeed, it is possible, from the definition of \( G (\xi) \) in (2.3.4) and the asymptotics in (2.3.3), to seek a function \( G (\xi) \) such that

\[ G(\xi) = G(0) + o(1) \, , \text{ as } \xi \to 0. \]  

(2.3.6)

Assuming \( 2 - \frac{1}{\beta} > 1 \), let \( C (\beta) \) be a constant depending only on \( \beta \) defined as follows:

\[ C (\beta) = \int_{[0,\frac{1}{2}]} \left( \frac{1}{y^{\frac{2}{\beta} - \frac{1}{\beta}}} (1 - y)^{\frac{2}{\beta} - \frac{1}{\beta}} - \frac{1}{y^{\frac{2}{\beta} - \frac{1}{\beta}}} \right) dy - \int_{[\frac{1}{2}, \infty]} \frac{1}{y^{\frac{2}{\beta} - \frac{1}{\beta}}} dy \]  

(2.3.7)

We will show now that for a sufficiently small constant \( \chi (\beta) \), the function \( G(\xi) \) verifies the following expansion, in the range \( \xi \ll \chi (\beta) = O (1) \) :  

\[ G(\xi) = G(0) - G^2 (0) C (\beta) \xi^{\frac{4}{\beta}} + o \left( \xi^{\frac{4}{\beta}} \right) \]  

(2.3.8)
Looking at equation (2.3.5), we can define

\[ A(\xi) := \frac{G(\xi y) G(\xi (1 - y))}{y^2 - \frac{1}{3}} \left( \frac{y}{1 - y} \right) dy, \]  
\[ B(\xi) := \int_{\left[ \frac{1}{2}, \xi^\frac{1}{2} \right]} \frac{G(\xi y)}{y^2 - \frac{1}{3}} dy. \]  
\[ (2.3.9) \]
\[ (2.3.10) \]

Therefore, for \( \xi \ll 1 \) (and hence \( \xi y \ll 1 \) for \( y \in \left[ 0, \frac{1}{2} \right] \)), we can approximate the value of \( A(\xi) \) in (2.3.9) at leading order in \( \xi \) by replacing \( G(\xi (1 - y)) \), \( G(\xi y) \) as well as \( G(\xi y) \) by their asymptotic values as \( \xi \to 0 \), that is by \( G(0) \). Concerning \( B(\xi) \), we can split the integration domain \( \left[ \frac{1}{2}, \infty \right) \) as \( \left[ \frac{1}{2}, \varrho(\xi) \right) \cup [\varrho(\xi), +\infty) \) where \( \varrho(\xi) \) will be chosen appropriately so that \( \varrho(\xi) \to \infty \) for \( \xi \to 0 \) but \( \varrho(\xi) \ll 1 \). Let’s take, for instance, \( \varrho(\xi) = \frac{1}{\sqrt{\xi}} \). Then, for \( \xi \ll 1 \), equation (2.3.5) yields

\[ G'(\xi) \sim -\frac{1}{\beta \xi^{\frac{1}{3}}} G^2(0) \left( \int_{\left[ 0, \frac{1}{2} \right]} \left( \frac{1}{y^{2 - \frac{1}{3}}} - \frac{1}{y^{2 - \frac{1}{3}}} \right) dy \right. \]
\[ \left. - \int_{\left[ \frac{1}{2}, \xi^{\frac{1}{2}} \right]} \frac{1}{y^{2 - \frac{1}{3}}} dy \right) + \frac{G(0)}{\beta \xi^{\frac{1}{3}}} \int_{\left[ \xi^{\frac{1}{2}}, \infty \right)} G(\xi y) \frac{dy}{y^{2 - \frac{1}{3}}}. \]  
\[ (2.3.11) \]

On one hand, \( G(\xi) \) has to be bounded according to the conditions imposed on the asymptotic behaviour of \( \psi(\xi) \), namely assuming \( \psi(\xi) \sim O(\xi^{-\alpha/\beta}) \) as \( \xi \to \infty \); therefore we have:

\[ \int_{\left[ \xi^{\frac{1}{2}}, \infty \right)} \frac{G(\xi y)}{y^{2 - \frac{1}{3}}} dy < C \int_{\left[ \xi^{\frac{1}{2}}, \infty \right)} \frac{1}{y^{2 - \frac{1}{3}}} dy = O(\xi^{\frac{1}{2} - \frac{1}{3}}), \]

which we use to conclude that

\[ \int_{\left[ 0, \frac{1}{2} \right]} \left( \frac{1}{y^{2 - \frac{1}{3}}} - \frac{1}{y^{2 - \frac{1}{3}}} \right) dy - \int_{\left[ \frac{1}{2}, \xi^{\frac{1}{2}} \right]} \frac{1}{y^{2 - \frac{1}{3}}} dy = C(\beta) + O(\xi^{\frac{1}{2} - \frac{1}{3}}). \]

Thus, expression (2.3.11) is at leading order

\[ G'(\xi) \sim -\frac{G^2(0) C(\beta)}{\beta \xi^{\frac{1}{3}}}, \]

so that, provided \( C(\beta) \neq 0 \), we find

\[ G(\xi) \sim G(0) - G^2(0) C(\beta) \xi^{\frac{1}{2}}. \]

In the particular case of \( \beta = 2 \), a straightforward calculation allows us to obtain

\[ C(2) = \int_{\left[ 0, \frac{1}{2} \right]} \left( \frac{1}{y^2} - \frac{1}{y^2} \right) dy - \int_{\left[ \xi^{\frac{1}{2}}, \infty \right)} \frac{1}{y^2} dy = 2\sqrt{2} - 2\sqrt{2} = 0. \]
2.3.2. ASYMPTOTICS AT INFINITY

We will turn now to consider the asymptotic behaviour of the selfsimilar solution at infinity. The study of this asymptotics from equation (2.3.1) is much simplified by using the Laplace transform representation (2.3.2). Since equation (2.3.2) is a first order ordinary differential equation, we can easily solve it to obtain the implicit expression

\[ \eta (\Phi) = \Phi \left( k \Phi^{\frac{1}{\beta}} + 1 \right) . \]  

(2.3.12)

where \( k \) is an arbitrary real constant. In order to compute \( \psi(\xi) \) we can use (2.2.8) with \( \Phi (\eta) \) given implicitly by (2.3.12).

In the particular case \( \beta = 2 \), equation (2.3.12) reduces to a second order polynomial; we can find its zeros and obtain two possible solutions:

\[ \Phi (\eta) = \Phi_\pm (\eta) = -1 \pm \frac{\sqrt{1 + 4k\eta}}{2k} . \]

Then, using (2.2.8),

\[ \psi(\xi) = \pm \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{\eta\xi} \frac{1}{\sqrt{1 + 4k\eta}} d\eta \]

and standard deformation of the integration contour in the complex plane towards \( \Re(\eta) < -\frac{1}{4k} \), \( \Im(\eta) = 0 \) yields then

\[ \psi(\xi) = \frac{e^{-\xi/(4k)}}{\sqrt{4\pi k\xi^2}} , \]

(2.3.13)

where we have chosen the function \( \psi(\xi) \) which is positive.

When \( \beta \neq 2 \), a direct inversion formula to obtain \( \Phi (\eta) \) from (2.3.12) is not available, but still we can argue using analyticity to compute \( \psi(\xi) \). First, note that \( \eta (\Phi) \) given by (2.3.12) is a complex analytic function in any domain that excludes the origin \( \Phi = 0 \) and therefore \( \Phi (\eta) \) is analytic except in \( \eta = 0 \). Indeed, from the implicit solution (2.3.12) it follows

\[ \Phi(\eta) = \eta - k\eta^{\frac{1}{\beta}} + o(\eta^{\frac{1}{\beta}}) , \]

(2.3.14)

so that a branch cut is present in the line \( (\Re(\eta) < 0, \Im(\eta) = 0) \) and \( \eta^{\frac{1}{\beta}} \) jumps form \( e^{-\frac{1}{\beta} \pi i} |\eta|^{\frac{1}{\beta}} \) to \( e^{\frac{1}{\beta} \pi i} |\eta|^{\frac{1}{\beta}} \) across that cut. We can deform the integration contour as in Figure 2.3.4 so that the main contribution to the inverse Laplace transform of \( \Phi(\eta) \), as \( \xi \to \infty \), is produced by \( \Gamma_{3,4} \) and hence

\[ \psi(\xi) \sim -\frac{k}{\pi} \frac{\beta}{\beta - 1} \sin \left( \frac{\beta \pi}{\beta - 1} \right) \frac{\Gamma \left( \frac{\beta}{\beta - 1} \right)}{\xi^{2 + \frac{1}{\beta}}} . \]

Notice the existence of two free parameters, \( k \) and \( \beta \), in (2.3.14) and two parameters, \( G(0) \) and \( C(\beta) \), in the asymptotics for \( G(\xi) \) (and hence for \( \psi(\xi) \)) near \( \xi = 0 \) in (2.3.8). Therefore, if we impose \( G(0) = 1 \), in order to select a particular member of the family of solutions equivalent under rescaling, then, for a given value of \( \beta \), it turns out that \( k \) is the only free parameter that serves to match the local solutions at the origin and at infinity. In order to get a function \( \psi(\xi) \)
that decays faster than any power of \( \xi \) at infinity and preserves all its moments bounded, we must choose \( \beta = 2 \). We stress here the fact that the similarity exponent \( \beta \) which provides a suitable solution to (2.1.4) does not follow from dimensional arguments, but has to be determined so that a suitable condition for the decay of the self-similar solution at infinity is satisfied. This criterion characterizes the self-similar solution as one of the second kind in the notation of Barenblatt (see [29], [103]). In the next sections we will show that the situation is exactly the same, and the matching procedure similar, for \( \varepsilon > 0 \) (at least if \( \varepsilon \) is sufficiently small).

### 2.4. Asymptotics for Selfsimilar Solutions with \( \varepsilon > 0 \)

In this section we study the asymptotic behaviour, both as \( \xi \to 0 \) and \( \xi \to \infty \), of the solutions to (2.1.4). The knowledge of these behaviours will serve, in the next sections, to construct global selfsimilar solutions.
First we introduce the ansatz
\[ \psi (\xi) = \frac{G (\xi)}{\xi^{-2\varepsilon+3-\frac{1}{\beta}}} \] (2.4.1)
so that equation (2.1.4) can be rewritten as:
\[
G' (\xi) = -\frac{1}{\beta} \frac{1}{\xi^{1-\frac{1}{\beta}}} \left( \int_{[0, \frac{1}{2}]} \left( \frac{G (\xi y)}{y^{-\varepsilon+2-\frac{1}{\beta}}} \frac{G (\xi (1-y))}{(1-y)^{-\varepsilon+2-\frac{1}{\beta}}} - \frac{G (\xi) G (\xi y)}{y^{-\varepsilon+2-\frac{1}{\beta}}} \right) dy - G (\xi) \int_{[\frac{1}{2}, \infty)} \frac{G (\xi y)}{y^{-\varepsilon+2-\frac{1}{\beta}}} dy \right) \]
which is a more appropriate form to study the asymptotics of the solutions.

### 2.4.1. Formal Expansion for $\xi \to 0$.

Assuming $2 - \frac{1}{\beta} - \varepsilon > 1$, we can expand $G(\xi)$, for $\xi \to 0$, as in (2.3.6). Let
\[
C (\beta, \varepsilon) = \int_{[0, \frac{1}{2}]} \left( \frac{1}{y^{-\varepsilon+2-\frac{1}{\beta}}} \frac{1}{(1-y)^{-\varepsilon+2-\frac{1}{\beta}}} - \frac{1}{y^{-\varepsilon+2-\frac{1}{\beta}}} \right) dy - \int_{[\frac{1}{2}, \infty)} \frac{1}{y^{-\varepsilon+2-\frac{1}{\beta}}} dy. \tag{2.4.3}
\]
We can introduce, analogously to (2.3.9), (2.3.16),
\[
A (\xi) : = \int_{[0, \frac{1}{2}]} \left( \frac{G (\xi y)}{y^{-\varepsilon+2-\frac{1}{\beta}}} \frac{G (\xi (1-y))}{(1-y)^{-\varepsilon+2-\frac{1}{\beta}}} - \frac{G (\xi) G (\xi y)}{y^{-\varepsilon+2-\frac{1}{\beta}}} \right) dy, \tag{2.4.4}
\]
\[
B (\xi) : = \int_{[\frac{1}{2}, \infty)} \frac{G (\xi y)}{y^{-\varepsilon+2-\frac{1}{\beta}}} dy. \tag{2.4.5}
\]
and, repeating the same procedure as in the case $\varepsilon = 0$, it follows
\[
G' (\xi) \sim -\frac{1}{\beta} \frac{1}{\xi^{1-\frac{1}{\beta}}} G^2 (0) \left( \int_{[0, \frac{1}{2}]} \left( \frac{1}{y^{-\varepsilon+2-\frac{1}{\beta}}} \frac{1}{(1-y)^{-\varepsilon+2-\frac{1}{\beta}}} - \frac{1}{y^{-\varepsilon+2-\frac{1}{\beta}}} \right) dy \right.
\]
\[
- \int_{[\frac{1}{2}, \xi^{\frac{1}{2}}]} \frac{1}{y^{-\varepsilon+2-\frac{1}{\beta}}} dy \right) + \frac{G (0)}{\beta \xi^{-\varepsilon+2-\frac{1}{\beta}}} \int_{[\frac{1}{2}, \infty)} \frac{G (\xi y)}{y^{-\varepsilon+2-\frac{1}{\beta}}} dy. \tag{2.4.6}
\]
Since
\[
\int_{[0, \frac{1}{2}]} \left( \frac{1}{y^{-\varepsilon+2-\frac{1}{\beta}}} \frac{1}{(1-y)^{-\varepsilon+2-\frac{1}{\beta}}} - \frac{1}{y^{-\varepsilon+2-\frac{1}{\beta}}} \right) dy - \int_{[\frac{1}{2}, \xi^{\frac{1}{2}}]} \frac{1}{y^{-\varepsilon+2-\frac{1}{\beta}}} dy
\]
\[
= C (\beta, \varepsilon) + O(\xi^{\frac{1}{2}-\frac{1}{2\beta}+\varepsilon}).
\]

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and
\[
\int_{|\xi^{-\frac{1}{2}}|}^{C} \frac{G'(\xi y)}{y^{-\frac{1}{2} + 2\epsilon}} dy < C \int_{|\xi^{-\frac{1}{2}}|}^{C} \frac{1}{y^{-\frac{1}{2} + 2\epsilon}} dy = O(\xi^{\frac{1}{2} + \frac{1}{\epsilon} + \epsilon}).
\]
we conclude from (2.4.6) the differential equation
\[
G'(\xi) = -\frac{G^2(0)}{\beta \xi^{1-\frac{1}{\epsilon}}} C(\beta, \epsilon) + O(\xi^{-\frac{1}{2} + \frac{1}{\epsilon} + \epsilon})
\]
provided \( C(\beta, \epsilon) \neq 0 \). Hence,
\[
G(\xi) = G(0) - G^2(0) C(\beta, \epsilon) \xi^{\frac{1}{\epsilon}} + o(\xi^{\frac{1}{\epsilon}}).
\]
(2.4.7)

As discussed in the Introduction, the scaling properties of equation (2.1.4) permit to fix \( G(0) = 1 \) so that a particular of the family of solutions identical under rescaling can be considered. The remaining free parameter \( C(\beta, \epsilon) \) or, equivalently, \( \beta \) can be chosen so that \( \psi(\xi) \) decays faster enough at infinity. We remark that it is not necessarily the case that \( C(\beta(\epsilon), \epsilon) = 0 \) as shown numerically in [21], [215].

---

**2.4.2. Algebraic decay asymptotics as \( \xi \to \infty \).**

Concerning the behaviour of \( \psi(\xi) \) at infinity, the crucial observation is the existence of an explicit solution in the form
\[
\psi(\xi) = A \xi^\gamma
\]
with \( A \) and \( \gamma \) determined below. By plugging (2.4.8) into (2.1.4) we obtain:
\[
\gamma = 2\epsilon - 3
\]
(2.4.9)
and
\[
A = \frac{1}{\Gamma(2 + 2\epsilon)} \int_0^1 \frac{(1 - y)^{-2 + 2\epsilon}}{y^{2 + 2\epsilon}} y^{-2 + 2\epsilon} dy = \frac{2\Gamma(-2 + 2\epsilon)}{\Gamma^2(1 + \epsilon)}.
\]
(2.4.10)
Notice that \( A(\epsilon) = \frac{1}{2\epsilon} + O(\epsilon^2) \) as \( \epsilon \to 0^+ \), so that the solution (2.4.8) is not present when \( \epsilon = 0 \).

Of course, one can also find this solution in the Laplace representation from equation (2.2.6). First, we compute the Laplace transform of the algebraic solution in (2.4.8)
\[
\Phi(\eta) = -A \int_{R^+} (e^{-\eta \xi} - 1) \xi^{1+\gamma} d\xi = -A \eta^{-\gamma-2}\Gamma(2 + \gamma),
\]
(2.4.11)
and we apply to them the fractional derivative operator \( D_{\eta}^{-\epsilon} \)
\[
D_{\eta}^{-\epsilon}(\eta) = -A \int_{R^+} (e^{-\eta \xi} - 1) \xi^{1-\epsilon+\gamma} d\xi = -A \eta^{-\gamma+\epsilon-2}\Gamma(2 + \gamma - \epsilon),
\]
(2.4.12)
so that a substitution in to \((2.2.6)\) yields the value of \(A\) given by \((2.4.10)\).

Besides this explicit solution, one can also find more solutions that approach it as \(\eta \to 0\). Let \(\Phi(\eta)\) be of the form

\[
\Phi(\eta) = -A\eta^{1-2\varepsilon}\Gamma(-1+2\varepsilon) + B\eta^\delta.
\]

(2.4.13)

Then using \((2.4.12)\) we have:

\[
D_\eta^{-\varepsilon}\Phi(\eta) = -A\eta^{1-\varepsilon}\Gamma(-1+\varepsilon) + B\frac{\delta}{\delta-\varepsilon} \left( \frac{\Gamma(1-\varepsilon-\delta)}{\Gamma(1-\delta)} \right) \eta^{\delta+\varepsilon}.
\]

(2.4.14)

Plugging it into \((2.2.6)\) yields at leading order the relation:

\[
-((1+2\varepsilon)\beta - 1) + \beta\delta = \frac{\Gamma(1-2\varepsilon)}{(1+\varepsilon)\Gamma(-1+\varepsilon)} \left( \frac{(1+\delta)\delta \Gamma(1+\varepsilon-\delta)}{(1-\varepsilon-\delta)} \right) = \frac{\nu_n}{\nu_n + 1} - \frac{1}{\nu_n + 1} \left( \gamma - (n-1)!\Psi(n) + O(\varepsilon^2) \right),
\]

(2.4.15)

from which one can determine \(\delta(\beta,\varepsilon)\) as a function of \(\beta\) and \(\varepsilon\). We may express that solution as a power series in \(\varepsilon\): there are countably many of them, i.e. \(\delta_n = n + \nu_n\varepsilon + O(\varepsilon^2)\). Thus, an expansion like that yields:

\[
\frac{\Gamma(1-\varepsilon-\delta)}{\Gamma(1-\delta)} = \frac{\Gamma(1-\varepsilon-n-\nu_n\varepsilon)}{\Gamma(1-n-\nu_n\varepsilon)} = \frac{\nu_n}{\nu_n + 1} - \frac{1}{\nu_n + 1} \left( \gamma - (n-1)!\Psi(n) + O(\varepsilon^2) \right),
\]

with \(\gamma\) the Euler constant and \(\Psi(n)\) the digamma function evaluated at the natural numbers. Keeping only the first order, we find that the solutions \(\delta_n\) are:

\[
\delta_n = n - \varepsilon \frac{(\beta - 1)(n-1) + n}{(\beta - \frac{1}{2})(n-1)} + O(\varepsilon^2), \quad n \geq 2.
\]

(2.4.16)

There is also a particular solution with \(\delta_1 = 1 + \nu_1\varepsilon\). This solution is such that, for \(\varepsilon\) small and at leading order,

\[
\frac{\nu_1}{\nu_1 + 1} + \frac{1}{\nu_1 + 1} \left( \nu_1 + 2 \right) \varepsilon = 1 + \varepsilon \left( 2\nu_1 + 4 \right),
\]

which implies:

\[
\delta_1 = \delta_1^+ = 1 + \varepsilon \nu_1^+ := 1 + \varepsilon \left( \frac{5}{3} \pm \sqrt{\frac{1}{9} - \frac{2}{3\varepsilon}} \right) \approx 1 + \varepsilon \left( \frac{5}{3} \pm \frac{2}{3\varepsilon} i \right).
\]

Hence we can write:

\[
\Phi(\eta) = -A\eta^{1-2\varepsilon}\Gamma(-1+2\varepsilon) + \left( C_1^+ \eta^{\delta_1^+} + C_1^- \eta^{\delta_1^-} \right) + C_2\eta^{\delta_2} + ....
\]

(2.4.17)

The constants \(C_1^+\) and \(C_2\) will be chosen later on so that the matching with the local solutions discussed in section \(2.4.2\) is achieved at \(O(\varepsilon)\). For this purpose, notice that expanding \(\Phi(\eta)\) in \(\varepsilon\) and keeping leading order terms, we get

\[
\Phi(\eta) = \left( \frac{1}{4} + C_1^+ + C_1^- \right) \eta + \left( -\frac{1}{2} + C_1^+\nu_1^+ + C_1^-\nu_1^- \right) \varepsilon\eta \log \eta + C_2\eta^{\delta_2} + ....
\]

(2.4.18)

The expansion \((2.4.18)\) will only be valid for \(\varepsilon |\log \eta| \ll O(1)\). It is this region where we will match.
2.4.3. **Exponential decay asymptotics as $\xi \to \infty$.**

All the behaviours as $\xi \to \infty$ that we found so far are of power-like decay and hence $\psi(\xi)$ is not such that all its moments (from the second one) are bounded. We wish to find also a possible solution with exponential-type decay at infinity. In order to show that such exponential decay is possible, we rewrite (2.4.2) in the form

$$
G''(\xi) = -\frac{1}{\beta \xi^{1-\frac{1}{\beta}}} \left[ \int_0^1 \left( \frac{G(\xi y) G(\xi (1-y))}{y^{2-\frac{1}{\beta}} (1-y)^{2-\frac{1}{\beta}}} - \frac{G(\xi)}{y^{2-\frac{1}{\beta}}} \right) dy - \int_{\frac{1}{2}}^{\infty} \frac{G(\xi)}{y^{2-\frac{1}{\beta}}} dy \right] + \frac{G(\xi)}{\beta \xi^{1-\frac{1}{\beta}}} \int_0^{\infty} \frac{G(\xi y) - 1}{y^{2-\frac{1}{\beta}}} dy, \quad (2.4.19)
$$

where $G(\xi)$ is assumed to be such that $G(0) = 1$; we have already remarked that any other solution can be obtained from this by rescaling. The last term at the right hand side of (2.4.19) can be rewritten as

$$
\frac{G(\xi)}{\beta \xi^{1-\frac{1}{\beta}}} \int_0^{\infty} \frac{G(\xi y) - 1}{y^{2-\frac{1}{\beta}}} dy = \left( \frac{1}{\beta} \int_0^{\infty} \frac{G(y) - 1}{y^{2-\frac{1}{\beta}}} dy \right) \xi^{-\varepsilon} G(\xi).
$$

A direct comparison of this term with the one at the left hand side of (2.4.19) suggests to seek for a solution of exponential kind:

$$
G(\xi) \sim e^{-\mu \xi^{1-\varepsilon}} \text{ as } \xi \to \infty. \quad (2.4.20)
$$

The first term at the right hand side of (2.4.19) can be estimated in this case, as $\xi \to \infty$, as

$$
\text{R.H.S.} = -\frac{e^{-\mu \xi^{1-\varepsilon}}}{\beta \xi^{1-\frac{1}{\beta}}} \left[ \int_0^{\frac{1}{2}} \left( \frac{e^{-\mu (-1+y^{1-\varepsilon}+(1-y)^{1-\varepsilon}) \xi^{1-\varepsilon}}}{y^{2-\frac{1}{\beta}} (1-y)^{2-\frac{1}{\beta}}} - \frac{1}{y^{2-\frac{1}{\beta}}} \right) dy - \int_{\frac{1}{2}}^{\varepsilon} \frac{1}{y^{2-\frac{1}{\beta}}} dy \right].
$$

Since the function $g(y) = -1+y^{1-\varepsilon}+(1-y)^{1-\varepsilon}$ is positive and increasing, the asymptotic behaviour of the integral as $\xi \to \infty$ is dominated by the contribution to it of the interval $[0, \delta]$, $\delta \ll 1$. One can then estimate

$$
\int_0^{\frac{1}{2}} \left( \frac{e^{-\mu (-1+y^{1-\varepsilon}+(1-y)^{1-\varepsilon}) \xi^{1-\varepsilon}}}{y^{2-\frac{1}{\beta}} (1-y)^{2-\frac{1}{\beta}}} - \frac{1}{y^{2-\frac{1}{\beta}}} \right) dy \sim \int_0^{\delta} \left( \frac{e^{-\mu (y^{1-\varepsilon}+(1-y)\xi^{1-\varepsilon})}{y^{2-\frac{1}{\beta}} (1-y)^{2-\frac{1}{\beta}}} - \frac{1}{y^{2-\frac{1}{\beta}}} \right) dy \\
\sim \xi^{1-\varepsilon-\frac{1}{\beta}} \int_0^{\delta} e^{-\mu s^{1-\varepsilon}+\mu (1-\varepsilon)\xi^{-\varepsilon} \delta} \frac{1}{s^{2-\frac{1}{\beta}}} ds = \frac{\varepsilon^{1-\varepsilon-\frac{1}{\beta}}}{s^{2-\frac{1}{\beta}}} ds\left[ e^{-\mu s^{1-\varepsilon}+\mu (1-\varepsilon)\xi^{-\varepsilon} \delta} \frac{1}{s^{2-\frac{1}{\beta}}} ds \right].
$$

$$
\sim \xi^{1-\varepsilon-\frac{1}{\beta}} \int_0^{\infty} e^{-\mu s^{1-\varepsilon} - 1} \frac{1}{s^{2-\frac{1}{\beta}}} ds = \frac{1}{\xi^{\varepsilon+\frac{1}{\beta}-1}} \int_0^{\infty} e^{-\mu s^{1-\varepsilon} - 1} ds. \quad (2.4.21)
$$
as $\xi \to \infty$. Using (2.4.20) and (2.4.21) in (2.4.10) we get the equation

$$-\mu(1-\varepsilon) = -\frac{1}{\beta} \mu^{-\frac{1-\varepsilon}{1-\varepsilon}} \left(\int_0^\infty \frac{e^{-s^{1-\varepsilon}}-1}{s^{2-\varepsilon-\frac{1}{\pi}}} dy\right) + \frac{1}{\beta} \int_0^\infty \frac{G(y)-1}{y^{2-\varepsilon-\frac{1}{\pi}}} dy. \quad (2.4.22)$$

The function $G(\xi)$ is unknown, but we expect it to be close to the selfsimilar solution in the case $\varepsilon = 0$, $\beta = 2$ so that equation (2.4.22) is, in the limit $\varepsilon = 0$, $\beta = 2$:

$$\mu + \sqrt{\pi} \mu^{\frac{1}{2}} - \pi = 0,$$

where we have used:

$$\int_0^\infty \frac{e^{-y}-1}{y^{\frac{3}{2}}} dy = -2\sqrt{\pi}.$$

Therefore

$$\mu = \left(\frac{\sqrt{\pi}-1}{2}\right)^2 \pi. \quad (2.4.23)$$

In the general case $\varepsilon > 0$, $\beta = 2+O(\varepsilon)$, the value of $\mu$ is a $O(\varepsilon)$ correction to (2.4.23) and therefore

$$G(\xi) \sim e^{-\left((\frac{\sqrt{\pi}-1}{2})^{\varepsilon} + O(\varepsilon)\right)^{\xi^{1-\varepsilon} + o(\xi^{1-\varepsilon})}} \quad (2.4.24)$$

for a selfsimilar solution that decays exponentially as $\xi \to \infty$.

The precise value of the $\beta$ can be obtained by matching (2.4.24) with the solution for finite values of $\xi$. More precisely, we will impose the condition that $G(\xi)$ does not present an algebraic decay at infinity in order to find $\beta$ as a function of $\varepsilon$. This will be done in a formal way in the next section and rigorously in the final sections of the paper.

## 2.5. Formal Construction of the Selfsimilar Solutions for $\varepsilon > 0$

In order to find solutions that are valid globally for all $\xi > 0$, we will use a perturbative argument by constructing them close to the solution for $\varepsilon = 0$:

$$\psi_0(\xi) = \frac{e^{-\pi \xi}}{\xi^\pi}, \quad \left(\Phi_0(\eta) = 2\pi \left(-1 + \sqrt{1 + \frac{\eta}{\pi}}\right)\right).$$

We will work with the Laplace transform version of the equation, namely equation (2.2.6), and seek the solutions in the form of perturbations of $\Phi_0(\eta)$. More precisely, we write

$$\Phi(\eta) = \Phi_0(\eta) + \varepsilon \Phi_1(\eta) + O(\varepsilon^2), \quad (2.5.1)$$

where $\Phi_1$ is independent of $\varepsilon$. Observe that, at leading order in $\eta$, we have $\Phi_0(\eta) = \eta + O(\eta^2)$. The solution expanded as a power series (2.5.1), thus, is more precise than the one-parameter family.
in a region far from zero. We will refer to (2.4.18) as the inner expansion and to (2.5.1) as the outer expansion.

The inner expansion (2.4.18) depending on $B_2$ can potentially be matched with the outer expansion. This will be done by comparing the $O(\eta^2 \log \eta)$ in the inner expansion with the equivalent term that will appear when expanding (2.5.1).

### 2.5.1. Analysis of $\Phi_1(\eta)$: Formal Expansions

We consider an expansion as in (2.5.1) where we want to determine the first order term $\Phi_1(\eta)$ and look for a $\beta$ of the form:

$$
\beta(\varepsilon) = 2 + \lambda \varepsilon + O(\varepsilon^2).
$$

At leading order in $\varepsilon$ we obtain the equation:

$$
\Phi_1(\eta) - 2\eta \frac{d\Phi_1(\eta)}{d\eta} + \frac{d}{d\eta} (\Phi_0(\eta) \Phi_1(\eta))
= \lambda \eta \Phi_0(\eta) - (4 + \lambda) \Phi_0(\eta) - \frac{d}{d\eta} (\Phi_0(\eta) \Phi_{0,\log}(\eta)),
$$

(2.5.2)

where we have used

$$
D_\eta^{-\varepsilon} \Phi_0(\eta) \equiv -\int_{\mathbb{R}^+} \xi \left[ \sum_{n=0}^{\infty} (-\varepsilon)^n \frac{(\log \xi)^n}{n!} \right] (e^{-\eta \xi} - 1) \psi_0(\xi) \, d\xi
= \Phi_0(\eta) - \varepsilon \Phi_{0,\log}(\eta) + O(\varepsilon^2)
$$

(2.5.3)

with

$$
\Phi_{0,\log}(\eta) = -\int_{\mathbb{R}^+} \xi (\log \xi) (e^{-\eta \xi} - 1) \psi_0(\xi) \, d\xi.
$$

(2.5.4)

We remark that the expansion (2.5.3) is, at the moment, only formal. The $O(\varepsilon^2)$ might eventually become dominant for certain values of $\eta$. We will see below that this is indeed the case for $\eta$ sufficiently small. Nevertheless, one can expect that the expansion is valid in broad regions of the complex $\eta$-plane.

By direct calculation one can find:

$$
\Phi_{0,\log}(\eta) = 2\pi \left( \gamma_2 - \ln \left( 1 + \frac{\eta}{\pi} \right) \right) \left( -1 + \left( 1 + \frac{\eta}{\pi} \right)^{\frac{1}{2}} \right) - 2\pi \ln \left( 1 + \frac{\eta}{\pi} \right),
$$

(2.5.5)

where

$$
\gamma_2 = -2 - \ln \pi + \gamma + 2 \ln 2
$$

(2.5.6)

and $\gamma$ is Euler’s constant.
Moreover an expression for $D^{-\varepsilon}_\eta \Phi_0$ can be directly computed from the definition (2.2.2):

$$D^{-\varepsilon}_\eta \Phi_0 (\eta) = \Gamma \left( -\frac{1}{2} + \varepsilon \right) \eta^\frac{1}{2} \left( 1 - \left( 1 + \frac{\eta}{\pi} \right)^\frac{1}{2} \varepsilon \right). \quad (2.5.7)$$

Equation (2.5.2) is an inhomogeneous first order ordinary differential equation that can be solved explicitly. We are specially interested in the asymptotic of the solutions as $\eta \to 0$, because a particular behaviour in this region of the Laplace space corresponds to the type of decay of the solution at infinity in the ordinary space. In order to describe it, we must first expand each term at the right hand side of (2.5.2) as a power series near $\eta = 0$. Thanks to the fact that both $\Phi_0 (\eta)$ and $D^{-\varepsilon}_\eta \Phi_0 (\eta)$ are analytic in a neighborhood of the origin, we can expand them:

$$\Phi_0 (\eta) = \eta - \frac{1}{4\pi} \eta^2 + ... \quad (2.5.8)$$

$$D^{-\varepsilon}_\eta \Phi_0 (\eta) = \frac{\Gamma \left( -\frac{1}{2} + \varepsilon \right) \left( -\varepsilon - \frac{1}{2} \right)}{\pi^{\frac{1}{2} - \varepsilon}} \eta + \frac{1 - 2\varepsilon \Gamma \left( -\frac{1}{2} + \varepsilon \right) \left( \varepsilon + \frac{1}{2} \right)}{4\pi} \eta^2 + ... \quad (2.5.9)$$

and also compute, using (2.5.5),

$$\Phi_{0, \log} (\eta) = (-4 - \ln \pi + 2 \ln 2 + \gamma) \eta + (-2 - \ln \pi + 2 \ln 2 + \gamma) \eta^2 + ... \quad (2.5.10)$$

We seek the solution $\Phi_1 (\eta)$ to (2.5.2) also as a power series with unknown coefficients:

$$\Phi_1 (\eta) = \sum_{i=1}^{\infty} A_i \eta^i + \sum_{j=2}^{\infty} B_j \eta^j \log \eta, \quad (2.5.11)$$

with $B_1 = 0$. Thus, by expanding the left and right hand sides of (2.5.2) (using (2.5.8)-(2.5.11)) and identifying the same orders $\eta^i$ and $\eta^j \log \eta$ we straightforwardly find:

$$A_1 = 4 - 2\gamma - 4 \ln 2 + 2 \ln \pi \quad (2.5.12)$$

$$B_2 = \frac{1}{4\pi} (\lambda - 2). \quad (2.5.13)$$

### 2.5.2. Matching and positivity of selfsimilar solutions

By comparing now the outer expansion (2.5.1) with $\Phi_0 (\eta)$ given by (2.5.8) and $\Phi_1 (\eta)$ given by (2.5.11) ($A_1$, $B_2$ given by (2.5.12), (2.5.13) respectively) with the inner expansion (2.4.19) we obtain linear equations that allow us to match both expressions at $O(\varepsilon)$:

$$\frac{1}{4} + C_1^+ + C_1^- = 1 + A_1 \varepsilon,$$

$$-\frac{1}{2} + C_1^+ \nu_1^+ + C_1^- \nu_1^- = 0,$$

$$\nu_2 C_2 = \frac{1}{4\pi} (\lambda - 2),$$

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which form a linear regular system for $C^\pm_1$ and $C$. Hence, global solutions to (2.2.6) can be obtained by the matched asymptotics procedure described above.

The fact that the expansion of $\Phi_1 (\eta)$ near $\eta = 0$ contains terms proportional to $\eta^2 \log \eta$ implies that $\Phi_1 (\eta)$ cannot be analytic close to the origin unless the coefficient $B_2$ defined by (2.5.13) is identically zero. In order to show this, consider a term in $\Phi (\eta)$ of the form

$$\tilde{\Phi} (\eta) = (D_1 + D_2 \varepsilon + O(\varepsilon^2)) \eta^{2 + \alpha \varepsilon + O(\varepsilon^2)}$$

(2.5.14)

which can be expanded in $\varepsilon$ as

$$\tilde{\Phi} (\eta) = D_1 \eta^{2} + D_2 \varepsilon \eta^2 + D_1 \varepsilon \alpha \varepsilon \log \eta + O(\varepsilon^2)$$

(2.5.15)

so that $D_1 = -\frac{1}{4\pi}$, $\alpha = \lambda - 2$ in our case. The inverse Laplace transform of (2.5.14) yields

$$\tilde{\psi} (\xi) \sim \frac{(1/2 + O(\varepsilon)) \sin((\lambda - 2) \varepsilon \pi)}{\xi^{\lambda(2 - \lambda)} + O(\varepsilon^2)} + \ldots, \text{ as } \xi \to \infty.$$  

(2.5.16)

If one wishes to eliminate the possibility of having an algebraic decay of the form (2.5.16) it is then necessary to impose $\alpha = 0$ which implies:

$$\lambda = 2$$

(2.5.17)

and $B_2 = 0$.

Finally, we will argue that the selfsimilar solution with exponential decay must be positive. The starting point of the argument is equation (2.4.10), that can be rewritten as

$$G' (\xi) = -\frac{1}{\beta \xi^{1 - \beta}} \left[ \int_0^{\frac{1}{2}} \frac{G (\xi y) G (\xi (1 - y))}{y^{2 - \varepsilon - \frac{1}{2}} (1 - y)^{2 - \varepsilon - \frac{1}{2}}} dy - \int_{\frac{1}{2}}^{\infty} \frac{G (\xi)}{y^{2 - \varepsilon - \frac{1}{2}}} dy \right] + \frac{G (\xi)}{\beta \xi^{\varepsilon}} \int_0^{\frac{1}{2}} \frac{G (y) - 1}{y^{2 - \varepsilon - \frac{1}{2}}} dy.$$  

(2.5.18)

The first term at the right hand side of (2.5.18) can be written as

$$I = \int_0^{\frac{1}{2}} \frac{G (\xi y) G (\xi (1 - y))}{y^{2 - \varepsilon - \frac{1}{2}} (1 - y)^{2 - \varepsilon - \frac{1}{2}}} dy + \int_{\frac{1}{2}}^{1} dy = I_{1, \delta} + I_{2, \delta}.$$

Consider next the possibility that at some point $\xi_0$ we have $G (\xi_0) = G' (\xi_0) = 0$ with $G (\xi) > 0$ for $\xi < \xi_0$. In this situation, all terms in (2.5.18) except for the first integral at the right hand side cancel. Notice that,

$$I_{2, \delta} = \int_{\delta}^{\frac{1}{2}} \frac{G (\xi_0 y) G (\xi_0 (1 - y))}{y^{2 - \varepsilon - \frac{1}{2}} (1 - y)^{2 - \varepsilon - \frac{1}{2}}} dy > 0,$$

$$I_{1, \delta} = \lim_{\xi \to \xi_0} \int_{0}^{\delta} \frac{G (\xi y) G (\xi (1 - y))}{y^{2 - \varepsilon - \frac{1}{2}} (1 - y)^{2 - \varepsilon - \frac{1}{2}}} dy = \lim_{\xi \to \xi_0} \int_{0}^{\delta} \frac{G (\xi (1 - y)) - G (\xi)}{y^{2 - \varepsilon - \frac{1}{2}}} dy + O(\delta)$$

$$= G' (\xi_0) O(\delta^{\frac{1}{2} + \varepsilon}) + O(\delta).$$

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Hence, by taking \( \delta \) sufficiently small, \( I > 0 \). This is a contradiction with the fact that all other terms in (2.5.18) are zero and therefore it is not possible that \( G(\xi_0) = G'(\xi_0) = 0 \) with \( G(\xi) > 0 \) for \( \xi < \xi_0 \). The conclusion is then that, if we change \( \beta \) continuously starting from some \( \beta_0 \) so that \( G(\xi) \) is strictly positive, it is not possible that \( G \) becomes zero at \( \xi_0 < \infty \) for some \( \beta_1 \). Therefore, a zero of \( G(\xi) \) can only form at infinity and then propagate to finite values of \( \xi \) as \( \beta \) changes. But solutions with algebraic decay are positive at infinity (since the leading order term at infinity (2.4.8) is positive, due to the fact that \( A > 0 \)). This implies that a zero can only appear if \( \beta \) goes through a value for which \( \psi(\xi) \) has not an algebraic decay but the exponential decay (2.4.24). Since the solution with exponential decay is positive as \( \xi \to \infty \), we conclude that it must be globally positive.

The argument above is, at the moment formal. Nevertheless, this argument has been used to implement a numerical shooting method for the calculation of the selfsimilar solutions and compare with the result of full numerical simulations of (2.1.4). The results of these numerical studies will be published somewhere else.

Of course, the arguments based on matched asymptotic expansions serve to construct (at least formally) selfsimilar solutions with algebraic decay at infinity and yield the condition (2.5.17) so that the construction procedure breaks down. The rest of the paper will be devoted to prove that condition (2.5.17) is necessary for a solution to present a decay at infinity which is faster than algebraic so that all its moments are bounded.

## 2.6. Setting of the nonlinear problem

We now get back to equation (2.2.6):

\[
-((1 - 2\varepsilon)\beta - 1) \Phi(\eta) + \beta \eta \Phi'(\eta) = \frac{1}{2} \frac{d}{d\eta} \left[D_\eta^{-\varepsilon} \Phi(\eta)\right]^2. \tag{2.6.1}
\]

We will look for a fixed point argument to show that if \( \Phi(\eta) = \Phi_0(\eta) + \varepsilon \Phi_1(\eta; \varepsilon) \) then the \( \Phi_1 \) part of the solution (now consisting of the full perturbation of \( \Phi_0 \)) may be controlled in a suitable norm. We consider \( \beta \) as a function of \( \varepsilon \):

\[
\beta(\varepsilon) = 2 + \varepsilon \lambda(\varepsilon). \tag{2.6.2}
\]

Then equation (2.2.6) may be written as:

\[
\Phi_1 - 2\eta \frac{d}{d\eta} \Phi_1 + \frac{d}{d\eta} [\Phi_0 \Phi_1] = \frac{1}{2\varepsilon} \frac{d}{d\eta} \left[\Phi_0^2 - \left[D_\eta^{-\varepsilon} \Phi_0\right]^2\right] + \lambda(\varepsilon) \eta \frac{d}{d\eta} \Phi_0 - \left(4 + \lambda(\varepsilon) + 2\varepsilon^2 \lambda(\varepsilon)\right) \Phi_0 \\
+ \varepsilon \lambda(\varepsilon) \eta \frac{d}{d\eta} \Phi_1 - \left(4\varepsilon + \varepsilon \lambda(\varepsilon) + 2\varepsilon^2 \lambda(\varepsilon)\right) \Phi_1 + \frac{d}{d\eta} \left[\Phi_0 \Phi_1 - D_\eta^{-\varepsilon} \Phi_0 D_\eta^{-\varepsilon} \Phi_1\right] \\
- \frac{\varepsilon}{2} \frac{d}{d\eta} \left[D_\eta^{-\varepsilon} \Phi_1\right]^2. \tag{2.6.3}
\]
Note that the right hand side of equation \((2.6.3)\) consists of sources (first line at the right hand side), linear terms in \(\Phi_1\) (second line) and a quadratic term in \(\Phi_1\) (last line). We will denote them by \(F_0(\eta), L\Phi_1\) and \(Q(\Phi_1, \Phi_1)\) respectively, so that equation \((2.6.3)\) may be rewritten as:

\[
\Phi_1 - 2\eta \frac{d}{d\eta} \Phi_1 + \frac{d}{d\eta} [\Phi_0 \Phi_1] = F_0(\eta) + L\Phi_1 + Q(\Phi_1, \Phi_1). \tag{2.6.4}
\]

In the previous sections, we have studied equation \((2.6.4)\) with a right hand side consisting only of \(F_0(\eta)\). We relied on the assumption that all other terms \(L\Phi_1\) and \(Q(\Phi_1, \Phi_1)\) were negligibly small in comparison with \(F_0(\eta)\). Of course, this fact needs to be proved and the rest of the article will be devoted to show it. Our strategy will be a fixed point argument involving a suitable choice of \(\lambda(\varepsilon)\). Such an argument involves a mapping that assigns to a given \(\Psi_1\) the solution \(\Phi_1\) to the problem:

\[
\Phi_1 - 2\eta \frac{d}{d\eta} \Phi_1 + \frac{d}{d\eta} [\Phi_0 \Phi_1] = F_0(\eta) + L\Phi_1 + Q(\Phi_1, \Phi_1) \equiv N\Phi_1 \tag{2.6.5}
\]

in a suitable functional space. The criterion for the choice of \(\lambda(\varepsilon)\) will be such that the solution to \((2.6.5)\) does not contain terms of the form \(\eta^2 \log \eta\). As we have shown in the previous section, a term of this sort would imply a slow power like decay for \(\psi\) that we want to avoid.

Equation \((2.6.5)\) can be readily integrated between two complex points \(\eta_0\) and \(\eta\) to yield:

\[
\frac{\sqrt{1 + \frac{\pi}{\eta}}}{(\sqrt{1 + \frac{\pi}{\eta}} - 1)}^2 \Phi_1(\eta) - \frac{\sqrt{1 + \frac{\pi}{\eta_0}}}{(\sqrt{1 + \frac{\pi}{\eta_0}} - 1)}^2 \Phi_1(\eta_0) = \int_{\eta_0}^{\eta} \frac{\sqrt{1 + \frac{\pi}{w}}}{(\sqrt{1 + \frac{\pi}{w}} - 1)}^2 (N\Phi_1)(w) dw, \tag{2.6.6}
\]

where \(\eta_0\) is taken as some point in the strip \(|\Re(\eta)| < 1\) and the integral is performed along a path in the complex plane connecting \(\eta_0\) to \(\eta\) (in fact, we will take both \(\eta_0\) and \(\eta\) as purely imaginary and integrals along the imaginary axis in the next sections). Notice that the integrand in \((2.6.6)\) becomes singular at \(w = 0\) unless the modulus of the right hand side of \((2.6.5)\) is bounded by \(|w|^{2+\delta}\) for some \(\delta > 0\). Since we will be interested in making \(\eta_0 = 0\), we can rewrite \((N\Phi_1)(w)\) as

\[
(N\Phi_1)(w) = (N\Phi_1)(w) - \rho(w) + \rho(w)
\]

where \(\rho(w)\) is a suitably chosen function so that, as \(\eta \to 0:\)

\[
\rho(\eta) = \overline{b_1} \eta + \overline{b_2} \eta^2 + O(|\eta|^3) \tag{2.6.7}
\]

and \(\rho(\eta)\) is bounded and analytic in the strip \(|\Re(\eta)| < 1\). We can take, for instance,

\[
\rho(\eta) = \frac{\overline{b_1} \eta + \overline{b_2} \eta^2}{1 - \frac{\eta^2}{\pi}}. \tag{2.6.8}
\]

Therefore, equation \((2.6.6)\) may be written as:

\[
\frac{\sqrt{1 + \frac{\pi}{\eta}}}{(\sqrt{1 + \frac{\pi}{\eta}} - 1)}^2 (\Phi_1(\eta) - h(\eta)) - \frac{\sqrt{1 + \frac{\pi}{\eta_0}}}{(\sqrt{1 + \frac{\pi}{\eta_0}} - 1)}^2 \Phi_1(\eta_0) = \int_{\eta_0}^{\eta} \frac{\sqrt{1 + \frac{\pi}{w}}}{(\sqrt{1 + \frac{\pi}{w}} - 1)}^2 \frac{(N\Phi_1)(w) - \rho(w)}{\Phi_0(w) - 2w} dw \tag{2.6.9}
\]
where
\[ h(\eta) = \frac{\left(\sqrt{1 + \frac{\eta}{\pi}} - 1\right)^2}{\sqrt{1 + \frac{\eta}{\pi}}} \int_0^\eta \frac{\sqrt{1 + \frac{\eta}{\pi}}}{(\sqrt{1 + \frac{\eta}{\pi}} - 1)^2} \frac{\rho(w)}{\Phi_0(w) - 2w} dw. \] (2.6.10)

If \( \overline{b}_1, \overline{b}_2 \) are such that
\[ (N \Phi_1)(\eta) = \overline{b}_1 \eta + \overline{b}_2 \eta^2 + O(|\eta|^3) \text{ as } \eta \to 0, \] (2.6.11)
then the integral at the right hand side of (2.6.9) can be rewritten as
\[
\text{R.H.S.} = \int_0^\eta \frac{\sqrt{1 + \frac{\eta}{\pi}}}{(\sqrt{1 + \frac{\eta}{\pi}} - 1)^2} \frac{(N \Phi_1)(w) - \rho(w)}{\Phi_0(w) - 2w} dw
- \int_0^n \frac{\sqrt{1 + \frac{\eta}{\pi}}}{(\sqrt{1 + \frac{\eta}{\pi}} - 1)^2} \frac{(N \Phi_1)(w) - \rho(w)}{\Phi_0(w) - 2w} dw.
\] (2.6.12)

We will choose the free parameter \( \Phi_1(\eta_0) \) in (2.6.9) in such a way that the second term at the left hand side of (2.6.9) cancels out with the second integral in (2.6.12). We arrive then at the following expression:
\[
\Phi_1(\eta) - h(\eta) = \frac{\left(\sqrt{1 + \frac{\eta}{\pi}} - 1\right)^2}{\sqrt{1 + \frac{\eta}{\pi}}} \int_0^\eta \frac{\sqrt{1 + \frac{\eta}{\pi}}}{(\sqrt{1 + \frac{\eta}{\pi}} - 1)^2} \frac{(N \Phi_1)(w) - \rho(w)}{\Phi_0(w) - 2w} dw,
\] (2.6.13)
providing a particular solution to (2.6.5) on which we will base our fixed point argument. If \( \rho \) approximates \( N \Phi_1 \) with an error of order \( O(|\eta|^3) \), then we can easily see, sending \( \eta \) to zero in (2.6.13), that \( (\Phi_1 - h) \) is a \( o(|\eta|^2) \) term.

The definition of \( h(\eta) \) given by (2.6.10) and of \( \rho(\eta) \) given by (2.6.7) imply
\[
h(\eta) = \overline{b}_1 \eta - \frac{1}{\pi} \overline{b}_1 \eta^2 - \left( \overline{b}_2 + \frac{3}{4\pi} \overline{b}_1 \right) \eta^2 \ln \eta + O(|\eta|^3 \log |\eta|). \] (2.6.14)

The condition that terms of the form \( \eta^2 \ln \eta \) are not present in the expansion of \( \Phi_1(\eta) \) (and hence of \( h(\eta) \)) near the origin implies the equation:
\[
\overline{b}_2 + \frac{3}{4\pi} \overline{b}_1 = 0,
\] (2.6.15)
from which the condition on \( \lambda (\varepsilon) \) follows. Hence:
\[
\Phi_1(\eta) = \overline{b}_1 \eta - \frac{1}{\pi} \overline{b}_1 \eta^2 + o(|\eta|^2).
\]

Remembering that we want to apply a fixed point argument, we shall impose that the function \( \Phi_1(\eta) \) presents a similar behaviour near the origin:
\[
\Phi_1(\eta) = \overline{a}_1 \eta - \frac{1}{\pi} \overline{a}_1 \eta^2 + o(|\eta|^2).
\]
Next we will seek a formula relating the coefficient \( \overline{a}_1 \) with \( \overline{b}_1 \) and \( \overline{b}_2 \). By the definition of \( (N \Phi_1)(\eta) \) in (2.6.5), performing Taylor expansion of \( F_0(\eta) + L \Phi_1 + Q(\Phi_1, \overline{\Phi}_1) \) and comparing it with (2.6.11)
we can obtain equations for \( \overline{b}_1, \overline{b}_2 \) as a function of \( \lambda (\varepsilon) \) and \( \overline{a}_1 \). In order to achieve this, we straightforwardly compute

\[
F_0(\eta) = \left( -4 - \frac{1}{\varepsilon} \left( 1 - \frac{\Gamma^2(-\frac{1}{2}+\varepsilon)(\varepsilon+\frac{1}{2})^2}{\pi^{1-2\varepsilon}} \right) - 2\varepsilon \lambda (\varepsilon) \right) \eta \\
+ \frac{1}{4\pi} \left( 4 - \lambda (\varepsilon) + 2\varepsilon \lambda (\varepsilon) + \frac{3}{\varepsilon} \left( 1 - \frac{\Gamma^2(-\frac{1}{2}+\varepsilon)(\varepsilon+\frac{1}{2})^2}{\pi^{1-2\varepsilon}} (1 - 2\varepsilon) \right) \right) \eta^2 + o(|\eta|^2). \tag{2.6.16}
\]

Observe that

\[
\frac{1}{\varepsilon} \left( \frac{\Gamma^2\left(-\frac{1}{2}+\varepsilon\right)\left(\varepsilon+\frac{1}{2}\right)^2}{\pi^{1-2\varepsilon}} \right) = 8 + 2 \ln \pi - 4 \ln 2 - 2\gamma + O(\varepsilon), \tag{2.6.17}
\]

\[
\frac{1}{\varepsilon} \left( \frac{\Gamma^2\left(-\frac{1}{2}+\varepsilon\right)\left(\varepsilon+\frac{1}{2}\right)^2}{\pi^{1-2\varepsilon}} (1 - 2\varepsilon) \right) = 6 + 2 \ln \pi - 4 \ln 2 - 2\gamma + O(\varepsilon), \tag{2.6.18}
\]

and both are \( O(1) \) terms. As for the linear functional:

\[
L \overline{\Phi}_1 = \left( \varepsilon (4 + 2\varepsilon \lambda (\varepsilon)) \overline{a}_1 + 2 \left( \overline{a}_1 + \overline{a}_1^2 \Gamma (-\frac{1}{2} + \varepsilon) \left( \varepsilon + \frac{1}{2} \right) \pi^{-\varepsilon} \right) \right) \eta \\
- \left( \frac{1}{\pi} \left( 4 + \lambda (\varepsilon) + 2\varepsilon \lambda (\varepsilon) \right) \overline{a}_1 + 3 \left( \frac{5}{4\pi} \overline{a}_1 + \Gamma (-\frac{1}{2} + \varepsilon) \left( \varepsilon + \frac{1}{2} \right) \pi^{-\varepsilon} \left( \overline{a}_1 - \overline{a}_2^2 \right) \right) \right) \eta^2 \\
+ o(|\eta|^2), \tag{2.6.19}
\]

where \( \overline{a}_1, \overline{a}_2^2 \) are the first two Taylor series coefficients of \( D^{-\varepsilon}_\eta \overline{\Phi}_1 \), i.e.

\[
D^{-\varepsilon}_\eta \overline{\Phi}_1(\eta) = \overline{a}_1 \eta + \overline{a}_2^2 \eta^2 + o(|\eta|^2). \tag{2.6.20}
\]

We will see below that \( |\overline{a}_1 - \overline{a}_1| \) and \( |\overline{a}_2^2 + \frac{1}{\pi}| \) may be both controlled by a constant which is, under suitable conditions, of order \( O(\varepsilon) \); this makes the coefficients of \( \eta \) and \( \eta^2 \) in (2.6.19) to be of order \( O(\varepsilon) \), too.

Finally the quadratic term is:

\[
Q(D^{-\varepsilon}_\eta \overline{\Phi}_1, D^{-\varepsilon}_\eta \overline{\Phi}_1) = \varepsilon \left( \overline{a}_1 \right)^2 \eta + 3\varepsilon \overline{a}_1 \overline{a}_2 \eta^2 + o(|\eta|^2).
\]

Therefore we can reorganize all the terms and conclude:

\[
\overline{b}_1 = -4 - \frac{1}{\varepsilon} \left( \frac{\Gamma^2\left(-\frac{1}{2}+\varepsilon\right)\left(\varepsilon+\frac{1}{2}\right)^2}{\pi^{1-2\varepsilon}} \right) - 2\varepsilon \lambda (\varepsilon) + \varepsilon (4 + 2\varepsilon \lambda (\varepsilon)) \overline{a}_1 \\
+ 2 \left( \overline{a}_1 + \overline{a}_1^2 \Gamma \left( -\frac{1}{2} + \varepsilon \right) \left( \varepsilon + \frac{1}{2} \right) \pi^{-\varepsilon} \right) + \varepsilon \left( \overline{a}_1 \right)^2, \tag{2.6.21}
\]

\[
\overline{b}_2 = \frac{1}{4\pi} \left( 4 + \lambda (\varepsilon) + 2\varepsilon \lambda (\varepsilon) + \frac{3}{\varepsilon} \left( 1 - \frac{\Gamma^2\left(-\frac{1}{2}+\varepsilon\right)\left(\varepsilon+\frac{1}{2}\right)^2}{\pi^{1-2\varepsilon}} \right) (1 - 2\varepsilon) \right) \\
- \left( \frac{1}{\pi} \left( 4 + \lambda (\varepsilon) + 2\varepsilon \lambda (\varepsilon) \right) \overline{a}_1 + 3 \left( \frac{5}{4\pi} \overline{a}_1 + \Gamma \left( -\frac{1}{2} + \varepsilon \right) \left( \varepsilon + \frac{1}{2} \right) \pi^{-\varepsilon} \left( \overline{a}_1 - \overline{a}_2^2 \right) \right) \right) \\
+ 3\varepsilon \overline{a}_1 \overline{a}_2. \tag{2.6.22}
\]
Notice that, as $\varepsilon \to 0^+$, equation (2.6.21) yields the following leading order value for $\overline{b}_1$:

$$\overline{b}_1 = 4 + 2 \ln \pi - 4 \ln 2 - 2\gamma + O(\varepsilon)$$

recovering the result in (2.5.12).

Now condition (2.6.15) implies:

$$0 = \frac{4}{3} (-2 - \varepsilon \overline{a}_1) - \frac{1}{3} (-1 + 4\varepsilon (1 + \overline{a}_1) + 2\varepsilon^2 \overline{a}_1) \lambda(\varepsilon) + 2 \Gamma^2 \left( -\frac{1}{2} + \varepsilon \right) \left( \varepsilon + \frac{1}{2} \right)^2 \left( \overline{a}_1 + \overline{a}_1 \Gamma \left( -\frac{1}{2} + \varepsilon \right) \left( \varepsilon + \frac{1}{2} \right)^{-\frac{1}{2} + \varepsilon} \right) + 3\pi \varepsilon \overline{a}_1 \overline{a}_2$$

$$+ \left( 5\overline{a}_1 + ((1 - 2\varepsilon) \overline{a}_1^2 - 4\pi \overline{a}_2) \Gamma \left( -\frac{1}{2} + \varepsilon \right) \left( \varepsilon + \frac{1}{2} \right)^{-\frac{1}{2} + \varepsilon} \right) + \left( \varepsilon (\overline{a}_1^2 + \pi \varepsilon \overline{a}_1 \overline{a}_2) \right)$$

and, therefore,

$$\lambda(\varepsilon) = \frac{N}{D} \quad (2.6.23)$$

with

$$N = 8 - 6 \Gamma^2 \left( -\frac{1}{2} + \varepsilon \right) \left( \varepsilon + \frac{1}{2} \right)^2 \left( \overline{a}_1 + \overline{a}_1 \Gamma \left( -\frac{1}{2} + \varepsilon \right) \left( \varepsilon + \frac{1}{2} \right)^{-\frac{1}{2} + \varepsilon} \right)$$

$$+ 3 \left( 5\overline{a}_1 + ((1 - 2\varepsilon) \overline{a}_1^2 - 4\pi \overline{a}_2) \Gamma \left( -\frac{1}{2} + \varepsilon \right) \left( \varepsilon + \frac{1}{2} \right)^{-\frac{1}{2} + \varepsilon} \right)$$

$$- 3 \left( \varepsilon (\overline{a}_1^2 + \pi \varepsilon \overline{a}_1 \overline{a}_2) \right), \quad (2.6.24)$$

$$D = 1 - 4\varepsilon (1 + \overline{a}_1) - 2\varepsilon^2 \overline{a}_1. \quad (2.6.25)$$

Letting $\varepsilon$ go to zero, we easily see that $\lambda(\varepsilon) \to 2$ as we obtained in (2.5.17).

We are now in position to introduce the fixed point problem. Let $B$ represent a closed set in an adequate Banach space to be defined below and $I$ an interval in the real line, then we define the mapping $G$ that maps the pair $(\overline{\Phi}_1(\eta), \overline{a}_1) \in B \times I$ to the pair $(\overline{\Phi}_1(\eta), \overline{b}_1) \in B \times I$ defined by (2.6.13), (2.6.21) with $\lambda(\varepsilon)$ defined by (2.6.23) and $\overline{a}_1$, $\overline{a}_2$ defined by (2.6.20). Then, by showing that for $\varepsilon$ sufficiently small $G$ is indeed well defined from $B \times I$ into itself and is a contraction in an appropriate norm, we will find a unique solution to the nonlinear problem (2.2.6). In the next sections we shall provide the precise definition of the functional spaces involved in this argument and all the necessary estimates to prove the following theorem:

**Theorem 2.2.** There exists an $\varepsilon_0 > 0$ and a function

$$\lambda(\varepsilon) = 2 + O(\varepsilon)$$

such that for any $0 < \varepsilon < \varepsilon_0$ and with $\beta(\varepsilon) = 2 + \varepsilon \lambda(\varepsilon)$ there exists a solution to (2.2.6) (up to rescaling) satisfying

$$\int_0^\infty \xi^{\beta - \frac{1}{2}} \psi(\xi) d\xi < \infty \quad (2.6.26)$$

for any $0 < \delta \ll 1$. 

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Remark 2.3. Notice that boundedness of the integral in (2.6.26) excludes the possibility to the solution of Theorem 2.2 to present the asymptotic behaviour (2.5.16). Henceforth, one is led to conjecture that the solution found in Theorem 2.2 presents in fact the exponential decay given by equations (2.4.4) and (2.4.24). In this direction, we will prove in the last section that higher moments of \( \psi (\xi) \) than the one in (2.6.26) are also bounded.

The solution provided by Theorem 2.2 is unique (up to rescaling) in a class of functions to be defined in the next sections.

2.7. THE AUXILIARY LINEAR PROBLEM

In order to solve the nonlinear problem, it is necessary to obtain estimates for \( (\Phi_1 (\eta) - h(\eta)) \) in terms of \( J(\eta) = (N^2) (\eta) - \rho(\eta) \) from equation (2.6.13). Therefore we consider in this section the auxiliary linear problem:

\[
\Phi_1 (\eta) - h(\eta) = \left( \frac{\sqrt{1 + \frac{\eta^2}{\pi}} - 1}{\sqrt{1 + \frac{\eta^2}{\pi}}} \right)^2 \int_0^\eta \frac{\sqrt{1 + \frac{\eta^2}{\pi}}}{(\sqrt{1 + \frac{\eta^2}{\pi}} - 1)^2} J(w)dw
\]  

(2.7.1)

where \( \eta = ik \) and the integration runs along the purely imaginary axis.

Definition 2.4. Let \( X \) be the space of functions \( f \) such that:

\[
\| f(k) \|_X = \int_{-\infty}^{\infty} \left( \frac{1}{|k|^3} + \frac{1}{|k|^2} \right)^2 |f(k)|^2 dk < \infty .
\]

Let \( Y \) be the subspace of \( X \) whose functions are such that:

\[
\| f(k) \|_Y \equiv \| f(k) \|_X + \left\| k \frac{d}{dk} f(k) \right\|_X < \infty .
\]

We next prove the following Lemma:

Lemma 2.5. Let \( J(ik) \) have bounded \( X \)-norm. Then, the function \( (\Phi_1 (ik) - h(ik)) \) defined by (2.7.1) satisfies:

\[
\| \Phi_1 (ik) - h(ik) \|_Y \leq C \| J(ik) \|_X
\]

for some finite constant \( C > 0 \).

Proof. First we note that for \( \eta = ik \), \( k \in \mathbb{R} \) we have

\[
C_1 W(k) \leq \left| \frac{\sqrt{1 + \frac{ik}{\pi}}}{(\sqrt{1 + \frac{ik}{\pi}} - 1)^2} \right| \leq C_2 W(k)
\]  

(2.7.2)

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where \( W(k) = \left( \frac{1}{|k|^2} + \frac{1}{|k|^2} \right) \), and \( C_1, C_2 > 0 \) are suitable constants.

Hence, from formula (2.7.1),

\[
|\Phi_1(ik) - h(ik)|^2 W^2(k) \leq \left| \int_0^{ik} \frac{\sqrt{1 + \frac{ik}{\pi}}}{(\sqrt{1 + \frac{ik}{\pi}} - 1)^2} \frac{J(w)}{\Phi_0(w) - 2w} dw \right|^2. \tag{2.7.3}
\]

By Hardy’s inequality \( \int_{\mathbb{R}} \frac{f^2}{x^2} dx \leq C \int_{\mathbb{R}} (f')^2 dx \) applied to the right hand side of (2.7.3) we obtain

\[
\int_{\mathbb{R}} \left| \int_0^{ik} \frac{\sqrt{1 + \frac{ik}{\pi}}}{(\sqrt{1 + \frac{ik}{\pi}} - 1)^2} \frac{J(w)}{\Phi_0(w) - 2w} dw \right|^2 \leq \int_{\mathbb{R}} \left| \frac{\sqrt{1 + \frac{ik}{\pi}}}{(\sqrt{1 + \frac{ik}{\pi}} - 1)^2} \frac{J(ik)}{\Phi_0(ik) - 2ik} \right|^2 dk. \tag{2.7.4}
\]

Since

\[
C_1 \frac{1}{|k|} \leq \left| \frac{1}{\Phi_0(ik) - 2ik} \right| \leq C_2 \frac{1}{|k|}
\]

for suitable constants \( C_1, C_2 > 0 \), then, using (2.7.2) we conclude:

\[
C_1 \left| \frac{W(k)}{k} \right|^2 |J(ik)|^2 \leq \left| \frac{\sqrt{1 + \frac{ik}{\pi}}}{(\sqrt{1 + \frac{ik}{\pi}} - 1)^2} \frac{J(ik)}{\Phi_0(ik) - 2ik} \right|^2 \leq C_2 \left| \frac{W(k)}{k} \right|^2 |J(ik)|^2 \tag{2.7.5}
\]

for suitable constants \( C_1, C_2 > 0 \). Moreover, by (2.7.3), (2.7.4) and (2.7.5), we obtain:

\[
\int_{\mathbb{R}} |\Phi_1(ik) - h(ik)|^2 \frac{W^2(k)}{k^2} dk \leq C \int_{\mathbb{R}} |J(ik)|^2 \frac{W^2(k)}{k^2} dk. \tag{2.7.6}
\]

Finally, since

\[
1 + \frac{d}{d\eta} \Phi_0(\eta) (\Phi_1(\eta) - h(\eta)) + \eta \frac{d}{d\eta} (\Phi_1(\eta) - h(\eta)) = \frac{J(\eta)}{\Phi_0(\eta) - 2} \tag{2.7.7}
\]

we can estimate \( \| h \|_{k^{-\delta}} (\Phi_1(ik) - h(ik)) \|_X \) in terms of the \( X \)-norm of the right hand side of (2.7.7) and the \( X \)-norm of the first term at the left hand side of (2.7.7), both of which are bounded by the \( X \)-norm of \( J(\eta) \).

In the previous Lemma, we made use of the classical Hardy’s inequality. In fact, one can also use a generalized form of Hardy’s inequality (see Lemma 2.16 below and its Remark 2.14) to obtain the estimates:

\[
\int_{\mathbb{R}^+} |\Phi_1(ik) - h(ik)|^2 \frac{W^2(k)}{k^{2+\delta}} dk \leq C \int_{\mathbb{R}^+} |J(ik)|^2 \frac{W^2(k)}{k^{2+\delta}} dk,
\]

\[
\int_{\mathbb{R}^-} |\Phi_1(ik) - h(ik)|^2 \frac{W^2(k)}{k^{2+\delta}} dk \leq C \int_{\mathbb{R}^-} |J(ik)|^2 \frac{W^2(k)}{k^{2+\delta}} dk
\]

for \( \delta > -1 \), and then prove the following general version of Lemma 2.5.
Lemma 2.6. Let \( J(ik) = f(k) \) be such that
\[
\| f(k) \|_X \equiv \int_{-\infty}^{\infty} \left( \frac{1}{|k|^{3+\delta_1}} + \frac{1}{|k|^{3+\delta_2}} \right)^2 |f(k)|^2 \, dk < \infty
\]
with \( \delta_1, \delta_2 > -1 \). Then, the function \( \left( \Phi_1(ik) - h(ik) \right) \) defined by (2.7) satisfies
\[
\| \Phi_1(ik) - h(ik) \|_Y \leq C \| f(k) \|_X
\]
for some constant \( C > 0 \).

## 2.8. THE FIXED POINT ARGUMENT

In this section we complete the fixed point argument and finish the proof of Theorem 2.2. The first step is to use all the functional analysis results that are collected in the Appendix on functional analysis in order to estimate \( \| (N \Phi_1)(ik) - \rho(ik) \|_X \) in terms of \( \| \Phi_1(ik) - h(ik) \|_Y \). This is done in the following Lemma:

Lemma 2.7. Let \( \Phi_1(ik) \) be such that
\[
\| \Phi_1(ik) - h(ik) \|_Y < \infty.
\]
Then, for \( \varepsilon \) sufficiently small,
\[
\| (N \Phi_1)(ik) - \rho(ik) \|_X \leq C \left( 1 + \varepsilon \| \Phi_1(ik) - h(ik) \|_Y + \varepsilon \| \Phi_1(ik) - h(ik) \|_Y^2 \right). \tag{2.8.1}
\]

Proof. By formula (2.6.3), \( (N \Phi_1)(ik) - \rho(ik) \) can be written as the sum of three terms
\[
(N \Phi_1)(ik) - \rho(ik) = F_0(ik) + L \Phi_1 + Q(\Phi_1, \Phi_1) - \rho(ik) \tag{2.8.2}
\]
that we will estimate separately.

The terms \( L \Phi_1 \) and \( Q(\Phi_1, \Phi_1) \) can be written as
\[
L \Phi_1 = L(\Phi_1 - h(ik)) + L(h(ik))
\]
\[
Q(\Phi_1, \Phi_1) = Q(h(ik), h(ik)) + 2Q(\Phi_1 - h(ik), h(ik)) + Q(\Phi_1 - h(ik), \Phi_1 - h(ik))
\]
so that (2.8.2) can be rewritten in the following fashion:
\[
(N \Phi_1)(ik) - \rho(ik) = \widetilde{F}_0(ik) + \tilde{L}(\Phi_1 - h(ik)) + Q(\Phi_1 - h(ik), \Phi_1 - h(ik))
\]
with
\[
\begin{align*}
\widetilde{F}_0(ik) &= F_0(ik) + L(h(ik)) + Q(h(ik), h(ik)) - \rho(ik), \tag{2.8.3} \\
\tilde{L}(\Phi_1 - h(ik)) &= L(\Phi_1 - h(ik)) + 2Q(\Phi_1 - h(ik), h(ik)). \tag{2.8.4}
\end{align*}
\]
By the definition of $F_0$ in formula (2.6.3), it easily follows that $|F_0(i k)| \sim C |k|^\frac{1}{2}$ as $|k| \to \infty$. The term $L(h(i k))$, by the definition of $L$ in formula (2.6.3), the definition of $\rho(\eta)$ in formula (2.6.8) and the definition of $h(\eta)$ in formula (2.6.10), is such that $|L(h(i k))| \sim C |k|^\frac{1}{2}$ as $|k| \to \infty$. The term $Q(h(i k), h(i k))$, by the definition of $Q$ in formula (2.6.3), is such that $|Q(h(i k), h(i k))| \sim C |k|^{-2\epsilon}$ as $|k| \to \infty$. On the other hand, the function $\rho(i k)$ was chosen in (2.6.7) so that $|\widehat{F_0}(i k)| \sim C |k|^3$ as $|k| \to 0$. From all these considerations, we conclude that

$$\|\widehat{F_0}(i k)\|_X \leq C$$  \hspace{1cm} (2.8.5)

for some constant $C > 0$ independent of $\epsilon$.

The estimate of $\tilde{L}(\Phi_1 - h(i k))$ follows from the separate estimates of $L(\Phi_1 - h(i k))$ and $2Q(\Phi_1 - h(i k), h(i k))$. By the definition of the linear operator $L$ in (2.6.3), we deduce that

$$\|L(\Phi_1 - h(i k))\|_X \leq \epsilon \|\frac{d}{d\eta}(\Phi_1 - h(i k))\|_X + \epsilon \|\Phi_1 - h(i k)\|_X + \frac{d}{d\eta}[\Phi_0(\Phi_1 - h(i k)) - D_{\eta}^{-\epsilon} \Phi_0 D_{\eta}^{-\epsilon}(\Phi_1 - h(i k))]\|_X.$$  \hspace{1cm} (2.8.6)

The first two terms at the right hand side of (2.8.6) are bounded by $C\epsilon \|\Phi_1 - h(i k)\|_Y$, and the third term can be bounded as follows:

$$\|\frac{d}{d\eta}[\Phi_0(i k)(\Phi_1 - h(i k)) - D_{\eta}^{-\epsilon} \Phi_0(i k) D_{\eta}^{-\epsilon}(\Phi_1 - h(i k))]\|_X,$$

$$\leq \|\frac{d}{d\eta}[\Phi_0(i k) - D_{\eta}^{-\epsilon} \Phi_0(i k) D_{\eta}^{-\epsilon}(\Phi_1 - h(i k))]\|_X + \|\frac{d}{d\eta}[\Phi_0(i k)(\Phi_1 - h(i k)) - D_{\eta}^{-\epsilon} (\Phi_1 - h(i k)))]\|_X$$

$$= J_1 + J_2.$$  

In order to estimate $J_1$, we take into account the following estimates for $(\Phi_0 - D_{\eta}^{-\epsilon} \Phi_0)$:

$$|\Phi_0 - D_{\eta}^{-\epsilon} \Phi_0| \leq C\epsilon \frac{|\eta|}{1 + |\eta|^{\frac{1}{2} + \epsilon}},$$

$$|\frac{d}{d\eta}(\Phi_0 - D_{\eta}^{-\epsilon} \Phi_0)| \leq C\epsilon \frac{1}{1 + |\eta|^{\frac{1}{2} + \epsilon}}.$$

so that,

$$J_1 \leq C\epsilon \frac{D_{\eta}^{-\epsilon} (\Phi_1 - h(i k))}{1 + |\eta|^{\frac{1}{2} + \epsilon}} + C\epsilon \frac{|\eta| \frac{d}{d\eta} D_{\eta}^{-\epsilon} (\Phi_1 - h(i k))}{1 + |\eta|^{\frac{1}{2} + \epsilon}} = J_{1,1} + J_{1,2}.$$  

By Lemma 2.14,

$$J_{1,1} = C\epsilon \|D_{\eta}^{-\epsilon} (\Phi_1 - h(i k))\|_X \leq C\epsilon \|\eta|^{-\epsilon} D_{\eta}^{-\epsilon} (\Phi_1 - h(i k))\|_X \leq C\epsilon \|\Phi_1 - h(i k)\|_X,$$  \hspace{1cm} (2.8.7)
\[ J_{1,2} = C\varepsilon \left\| \frac{d}{dk} D_{\eta}^{-\varepsilon} (\Phi_1 - h) \right\|_X \leq C\varepsilon \left\| k \right\|^{1-\varepsilon} \frac{d}{dk} D_{\eta}^{-\varepsilon} (\Phi_1 - h) \right\|_X \]
\[ \leq C\varepsilon \left\| k \right\| \frac{d}{dk} (\Phi_1 - h) \right\|_X \, . \] (2.8.8)

Now
\[ J_2 = \left\| \frac{d}{dk} \left[ \Phi_0 \left( (\Phi_1 - h) - D_{\eta}^{-\varepsilon} (\Phi_1 - h) \right) \right] \right\|_X \]
\[ \leq \left\| \frac{d}{dk} \Phi_0 \left( (\Phi_1 - h) \right) \right\|_X + \left\| \Phi_0 \frac{d}{dk} (\Phi_1 - h) \right\|_X \]
\[ + \left\| \frac{d}{dk} \Phi_0 \left( D_{\eta}^{-\varepsilon} (\Phi_1 - h) \right) \right\|_X + \left\| \Phi_0 \frac{d}{dk} D_{\eta}^{-\varepsilon} (\Phi_1 - h) \right\|_X \]
\[ \leq \left\| \Phi_1 - h \right\|_X + \left\| k \right\| \frac{d}{dk} (\Phi_1 - h) \right\|_X + \left\| k \right\|^{-\varepsilon} \left( D_{\eta}^{-\varepsilon} (\Phi_1 - h) \right) \right\|_X + \left\| k \right\|^{1-\varepsilon} \frac{d}{dk} D_{\eta}^{-\varepsilon} (\Phi_1 - h) \right\|_X \]
\[ \leq C \left( \left\| \Phi_1 - h \right\|_X + \left\| k \right\| \frac{d}{dk} (\Phi_1 - h) \right\|_X \right) = C \left\| \Phi_1 - h \right\|_Y \] (2.8.9)

where we have used the elementary estimates
\[ \left| \Phi_0 \right| \leq |\eta|, \quad \left| \Phi_0 \right| \leq |\eta|^{1-\varepsilon}, \]
\[ \left| \Phi_0^\prime \right| \leq 1, \quad \left| \Phi_0^\prime \right| \leq |\eta|^{-\varepsilon}, \]
and followed by application of Lemma 2.14.

The estimate of \( Q(\Phi_1 - h(ik), h(ik)) \) follows from the following calculation:
\[ \left\| Q(\Phi_1 - h(ik), h(ik)) \right\|_X = \varepsilon \left\| \frac{d}{dk} \left[ D_{\eta}^{-\varepsilon} (\Phi_1 - h(ik)) D_{\eta}^{-\varepsilon} h(ik) \right] \right\|_X \]
\[ \leq \varepsilon \left( \left\| \frac{d}{dk} D_{\eta}^{-\varepsilon} (\Phi_1 - h(ik)) \right\|_X + \left\| D_{\eta}^{-\varepsilon} (\Phi_1 - h(ik)) \frac{d}{dk} D_{\eta}^{-\varepsilon} h(ik) \right\|_X \right) \]
\[ \leq C\varepsilon \left( \left\| \frac{d}{dk} D_{\eta}^{-\varepsilon} (\Phi_1 - h(ik)) \right\|_X + \left\| \frac{1}{1 + |k|^{\frac{1}{2}+\varepsilon}} D_{\eta}^{-\varepsilon} (\Phi_1 - h(ik)) \right\|_X \right) \]
\[ \leq C\varepsilon \left( \left\| k \right\|^{1-\varepsilon} \left| D_{\eta}^{-\varepsilon} h(ik) \right| \right\|_X + \left\| \Phi_1 - h(ik) \right\|_X \right) \]
\[ \leq C\varepsilon \left( \left\| k \right\| \frac{d}{dk} (\Phi_1 - h(ik)) \right\|_X + \left\| \Phi_1 - h(ik) \right\|_X \right) \] (2.8.10)

where we have used the following estimates for \( h(k) \):
\[ \left| D_{\eta}^{-\varepsilon} h(ik) \right| \leq C \frac{|k|}{1 + |k|^{\frac{1}{2}+\varepsilon}} \]
\[ \left| \frac{d}{dk} D_{\eta}^{-\varepsilon} h(ik) \right| \leq C \frac{1}{1 + |k|^{\frac{1}{2}+\varepsilon}} \]

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and Lemma 2.14. We conclude then from (2.8.7), (2.8.8), (2.8.9) and (2.8.10) that
\[
\| \mathcal{L} (\overline{\Phi} - h (ik))\|_X \leq C \varepsilon \left( \| \overline{\Phi} - h (ik)\|_X + \| \varepsilon \frac{d}{dk} (\overline{\Phi} - h (ik))\|_X \right). \quad (2.8.11)
\]

The estimate of \( Q (\overline{\Phi} - h (ik) , \overline{\Phi} - h (ik)) \) is the result of the following calculation:
\[
\begin{align*}
\| Q (\overline{\Phi} - h (ik) , \overline{\Phi} - h (ik))\|_X &= \varepsilon \left( \| \varepsilon \frac{d}{dk} (\overline{\Phi} - h (ik))\|_X \right) \\
&\leq \varepsilon \sup_{k \in \mathbb{R}} \| k \|^{-1+\varepsilon} \| D^{-\varepsilon} (\overline{\Phi} - h (ik))\|_X \\
&\leq \varepsilon \| k \|^{-1+\varepsilon} \| \frac{d}{dk} (D^{-\varepsilon} (\overline{\Phi} - h (ik)))\|_X \\
&\leq \varepsilon \| \frac{d}{dk} \left( \overline{\Phi} - h (ik) \right)\|_X^2, \quad (2.8.12)
\end{align*}
\]

where we have used Lemma 2.12 with \( \alpha = -1 + \frac{\varepsilon}{2} \), \( \mu = -\varepsilon \) and \( \delta = 0 \) to estimate
\[
\sup_{k \in \mathbb{R}} \| k \|^{-1+\varepsilon} \| D^{-\varepsilon} (\overline{\Phi} - h (ik))\|_X \leq C \| k \|^{-1+\varepsilon} \| \frac{d}{dk} (D^{-\varepsilon} (\overline{\Phi} - h (ik)))\|_X
\]
and Lemma 2.14 to estimate
\[
C \| k \|^{-1+\varepsilon} \| \frac{d}{dk} (D^{-\varepsilon} (\overline{\Phi} - h (ik)))\|_X \leq C \| \frac{d}{dk} \left( \overline{\Phi} - h (ik) \right)\|_X.
\]
From inequalities (2.8.5), (2.8.11) and (2.8.12) we conclude (2.8.1).

Therefore, by Lemmas 2.5 and 2.7, we conclude
\[
\| \Phi (ik) - h(ik)\|_Y \leq C \left( 1 + \varepsilon \| \overline{\Phi} (ik) - h(ik)\|_Y + \varepsilon \| \overline{\Phi} (ik) - h(ik)\|_Y^2 \right) \quad (2.8.13)
\]
and thus, if \( \| \overline{\Phi} (ik) - h(ik)\|_Y^2 \leq 2C \) then
\[
\| \Phi (ik) - h(ik)\|_Y \leq C(1 + 2\varepsilon C + 4\varepsilon C^2) \leq 2C \quad (2.8.14)
\]
for \( \varepsilon \) sufficiently small.

In equations (2.6.21) and (2.6.22) there appear the coefficients \( \overline{a}_1^2, \overline{a}_2^2 \) and we need to estimate them in terms of \( \overline{a}_1, \overline{a}_2 \) and the \( Y \)-norm of \( \Phi (\eta) \). The following Lemma deals with this estimate:

**Lemma 2.8.** Let \( \phi_1 (x) \in C_0^\infty (\mathbb{R}^+) \) and
\[
\begin{align*}
\overline{a}_1^2 &= \int_{\mathbb{R}^+} x^{2-\varepsilon} \phi_1 (x) \, dx, \quad \overline{a}_2^2 = \int_{\mathbb{R}^+} x^{3-\varepsilon} \phi_1 (x) \, dx, \\
\overline{a}_1 &= \int_{\mathbb{R}^+} x^2 \phi_1 (x) \, dx, \quad \overline{a}_2 = \int_{\mathbb{R}^+} x^3 \phi_1 (x) \, dx.
\end{align*}
\]

Then
\[
| \overline{a}_1^2 - \overline{a}_1 | \leq C \varepsilon (1 + | \overline{a}_1 | + \| \Phi (ik) - h(ik)\|_Y) \quad (2.8.15)
\]
and
\[
| \overline{a}_2^2 - \overline{a}_2 | \leq C \varepsilon (1 + | \overline{a}_2 | + \| \Phi (ik) - h(ik)\|_Y)
\]
where \( \Phi (\eta) = -\int_{\mathbb{R}^+} (e^{\eta x} - 1) x \phi_1 (x) \, dx \) and \( h (\eta) \) is given by (2.6.10).
Proof. Notice that, since \( \phi_1(x) \in C_0^\infty(\mathbb{R}^+) \), we can write \( \bar{a}_1 \) in terms of the second primitive of \( \phi_1(x) \):

\[
\bar{a}_1 = \int_{\mathbb{R}^+} x^{2-\varepsilon} \phi_1(x) \, dx = (2-\varepsilon)(1-\varepsilon) \int_{\mathbb{R}^+} x^{-\varepsilon} \phi_1^{**}(x) \, dx = (2-\varepsilon)(1-\varepsilon) \left( \frac{\pi}{2} + \int_{\mathbb{R}^+} (x^{-\varepsilon} - 1) \phi_1^{**}(x) \, dx \right)
\]

\[
= (2 - \varepsilon) (1 - \varepsilon) \left( \bar{a}_1 + \sum_{n=1}^{\infty} \frac{(-\varepsilon)^n}{n!} \int_{\mathbb{R}^+} \ln^n x \cdot \phi_1^{**}(x) \, dx \right)
\]

where

\[
\phi_1^{**}(x) = \int_{x}^{\infty} \left( \int_{x'} \phi_1(x'') \, dx'' \right) \, dx'.
\]

Hence, for a given \( \nu > 0, \nu \ll 1 \),

\[
|\bar{a}_1 - a_1| \leq \left| \frac{3}{2} \varepsilon^2 - \varepsilon^2 \right| \bar{a}_1 + \varepsilon \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \int_{\mathbb{R}^+} |\ln x|^n |\phi_1^{**}(x)| \, dx
\]

with

\[
\sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \int_{\mathbb{R}^+} |\ln x|^n |\phi_1^{**}(x)| \, dx
\]

\[
\leq \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \left( \int_{\mathbb{R}^+} \frac{|\ln x|^{2n}}{x^{1+\nu} + x^{1-\nu}} \, dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^+} \left( x^{1+\nu} + x^{1-\nu} \right) |\phi_1^{**}(x)|^2 \, dx \right)^{\frac{1}{2}}
\]

\[
\leq C\varepsilon \left( \int_{\mathbb{R}^+} \left( x^{1+\nu} + x^{1-\nu} \right) |\phi_1^{**}(x)|^2 \, dx \right)^{\frac{1}{2}}
\]

where we have used

\[
\sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \left( \int_{\mathbb{R}^+} \frac{|\ln x|^{2n}}{x^{1+\nu} + x^{1-\nu}} \, dx \right)^{\frac{1}{2}} \leq \frac{C}{2} \sum_{n=1}^{\infty} \frac{\varepsilon^n}{n!} \left( n^2 \right)^{\frac{1}{2}} = \frac{C}{2} \varepsilon \varepsilon^2 \leq C\varepsilon
\]

for \( \varepsilon \ll 1 \). Next we can estimate, using Lemma 2.10,

\[
\int_{\mathbb{R}^+} x^{1-\nu} |\phi_1^{**}(x)|^2 \, dx \leq C \int_{\mathbb{R}^+} x^{5-\nu} |\phi_1(x)|^2 \, dx,
\]

\[
\int_{\mathbb{R}^+} x^{1+\nu} |\phi_1^{**}(x)|^2 \, dx \leq C \int_{\mathbb{R}^+} x^{3+\nu} |\phi_1^{*}(x)|^2 \, dx,
\]

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and using (2.10.6) in Lemma 2.13,
\[ \int_{\mathbb{R}^+} x^{5-\nu} |\phi_1 (x)|^2 \, dx \leq C \int_{\mathbb{R}} |k|^{-1+\nu} \left| \frac{d\Phi_1 (k)}{dk} \right|^2 \, dk \]
\[ \leq 2C \left( \int_{\mathbb{R}} |k|^{-1+\nu} \left| \frac{d(\Phi_1 (k) - h (ik))}{dk} \right|^2 \, dk + \int_{\mathbb{R}} |k|^{-1+\nu} \left| \frac{dh (ik)}{dk} \right|^2 \, dk \right) \]
\[ \leq 2C \left( 1 + \int_{\mathbb{R}} W^2 (k) \left| \frac{d(\Phi_1 (k) - h (ik))}{dk} \right|^2 \, dk \right) \]
\[ \leq C \left( 1 + \|\Phi_1 (k) - h (ik)\|_Y \right), \]
and using (2.10.7) in Lemma 2.13,
\[ \int_{\mathbb{R}^+} x^{3+\nu} |\phi_1^* (x)|^2 \, dx \leq C \int_{\mathbb{R}} |k|^{-1-\nu} \left| \frac{d\Phi_1 (k)}{dk} \right|^2 \, dk \]
\[ \leq 2C \left( \int_{\mathbb{R}} |k|^{-1-\nu} \left| \frac{d(\Phi_1 (k) - h (ik))}{dk} \right|^2 \, dk \right) \]
\[ \leq 2C \left( 1 + \int_{\mathbb{R}} W^2 (k) \left| \frac{d(\Phi_1 (k) - h (ik))}{dk} \right|^2 \, dk \right) \]
\[ \leq C \left( 1 + \|\Phi_1 (k) - h (ik)\|_Y \right). \]

Therefore, for \( \varepsilon \) sufficiently small, we obtain
\[ |\overline{\alpha}_1^2 - \overline{\alpha}_1^1| \leq C \varepsilon \left( 1 + |\overline{\alpha}_1^1| + \|\Phi_1 (k) - h (ik)\|_Y \right). \quad (2.8.16) \]

The very same argument may be repeated to obtain:
\[ |\overline{\alpha}_2^2 - \overline{\alpha}_2^1| \leq C \varepsilon \left( 1 + |\overline{\alpha}_2^1| + \|\Phi_1 (k) - h (ik)\|_Y \right). \quad (2.8.17) \]

By defining
\[ a_1^0 = 4 + 2 \ln \pi - 4 \ln 2 - 2\gamma, \]
and making \( \overline{\alpha}_2 = -\frac{1}{4\pi} \overline{\alpha}_1 \) and \( \overline{b}_2 = -\frac{1}{4\pi} \overline{b}_1 \) (which is implied by the choice of \( \lambda (\varepsilon) \) in equation (2.6.23)–(2.6.25)), it is clear, from equation \( (2.6.21) \) together with Lemma 4.18 and the inequality
\[ |\overline{\alpha}_1| \leq |a_1^0| + |\overline{\alpha}_1 - a_1^0|, \]
that
\[ |\overline{b}_1 - a_1^0| \leq C \varepsilon \left( 1 + |\overline{\alpha}_1 - a_1^0| + \|\Phi_1 (k) - h (ik)\|_Y + \left( |\overline{\alpha}_1 - a_1^0| + \|\Phi_1 (k) - h (ik)\|_Y \right)^2 \right). \quad (2.8.18) \]
for sufficiently large $C$.

By inequalities (2.8.14) and (2.8.18), it follows that $G$ maps a ball $B$ of radius $3C$ around $(\Phi_1(\eta), \Phi_1) = (h(ik), a_1^0)$ into itself, provided that $\varepsilon$ is sufficiently small.

The mapping $G$ is also a contraction: given two functions $\Phi_1^{(1)}$ and $\Phi_1^{(2)}$, the difference of the corresponding images $\Phi_1^{(1)}$ and $\Phi_1^{(2)}$ is, by formula (2.6.12),

$$
\left( \Phi_1^{(1)}(\eta) - \Phi_1^{(2)}(\eta) \right) - \left( h^{(1)}(\eta) - h^{(2)}(\eta) \right)
$$

$$
= \left( \frac{\sqrt{1 + \frac{\pi}{\eta} - 1}}{\sqrt{1 + \frac{\pi}{\eta} + 1}} \right) \int_{0}^{\eta} \left( \frac{N_{\Phi_1^{(1)}}(w) - (N_{\Phi_1^{(2)}}(w))}{\Phi_0(w) - 2w} \right) dw
$$

and hence,

$$
\left\| \Phi_1^{(1)}(ik) - \Phi_1^{(2)}(ik) \right\|_Y
\leq C \left( \left\| h^{(1)}(ik) - h^{(2)}(ik) \right\|_X + \left\| N_{\Phi_1^{(1)}}(ik) - (N_{\Phi_1^{(2)}}(ik)) \right\|_X + \left\| \rho^{(1)}(ik) - \rho^{(2)}(ik) \right\|_X \right)
$$

$$
\leq C \left( \varepsilon \left\| \bar{a}_1^{(1)} - \bar{a}_1^{(2)} \right\| + \varepsilon \left\| \Phi_1^{(1)} - \Phi_1^{(2)} \right\|_Y + \varepsilon \left\| \Phi_1^{(1)} - \Phi_1^{(2)} \right\|^2_Y + \varepsilon \left| \bar{b}_1^{(1)} - \bar{b}_1^{(2)} \right| \right)
$$

$$
\leq C \varepsilon \left( \left\| \bar{a}_1^{(1)} - \bar{a}_1^{(2)} \right\| + \left\| \Phi_1^{(1)} - \Phi_1^{(2)} \right\|_Y \right)
$$

On the other hand it is clear that:

$$
\left\| \bar{a}_1^{(1)} - \bar{a}_1^{(2)} \right\| \leq C \varepsilon \left\| \bar{a}_1^{(1)} - \bar{a}_1^{(2)} \right\|_Y
$$

and we conclude then, by Banach’s fixed point theorem, the existence of a unique fixed point for the mapping $G$ if $\varepsilon$ is sufficiently small. Therefore, there exists a unique solution to the nonlinear problem such that $\left\| \Phi_1(ik) - h(ik) \right\|_Y < \infty$ and hence,

$$
\int_{-\infty}^{\infty} \left( \frac{1}{|k|^2} + \frac{1}{|k|^2} \right)^2 |\Phi'_1(ik) - h'(ik)|^2 dk < \infty.
$$

(2.8.19)

A direct application of Lemma 2.14 with $\gamma = \frac{3}{2} - \delta$, $0 < \delta \ll 1$, yields then (2.6.26). This concludes the proof of Theorem 2.2.

Remark 2.9. We can use (2.8.16) and bootstrap in equation (2.6.3) (using Lemma 2.14) to estimate higher derivatives of $\Phi_1$ in weighted Lebesgue spaces and then use

$$
\xi^n \frac{d^n}{d\xi^n} (\xi^2 \psi(\xi)) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \xi^n e^{ik\xi}(ik)^n ' (ik) dk = \frac{(ik)^n}{2\pi i} \int_{-\infty}^{\infty} e^{ik\xi} \frac{d^n}{d\xi^n} (k^n ' (ik)) dk.
$$

and 2.13 to translate into weighted estimates for derivatives of $\psi(\xi)$ and hence conclude that any derivative of $\psi(\xi)$ is bounded at any arbitrary $\xi > 0$.

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Finally, we present a shooting argument showing that the solution found must be positive. The Theorem 2.2 yields, for any given $\varepsilon$ sufficiently small, a unique $\beta^*$ (depending on $\varepsilon$) in an interval $I = (\beta_0, \beta_1)$ such that solutions with $\beta = \beta^*$ satisfy the estimate (2.6.26). Any other solutions for $\beta \in I \setminus \beta^*$ can be constructed by the formal matched asymptotics procedure in Section 2.5 and are such that the asymptotics at infinity is given by (2.4.8), (2.4.9), (2.4.10) implying that they are positive for $\xi > \xi^*$ (depending on $\beta$). The intermediate asymptotics given by (2.5.16), which yields a positive function provided $\lambda > 2 + O(\varepsilon)$ implies that matched asymptotic solutions are globally positive for $\beta = \beta_1$ taken sufficiently large and $\varepsilon$ sufficiently small (an equivalent argument based on total monotonicity of the matched asymptotic solutions in term of Laplace transforms, formulas (2.4.17), (2.5.14), yields the same conclusion). Hence, since zeros cannot develop at any $\xi < \infty$ as $\beta$ changes continuously from $\beta_1$ to $\beta^*$ (see the argument at the end of Section 2.5) we conclude that the solutions must remain positive as $\beta \to \beta^{*+}$.

### 2.9. FASTER DECAY FOR THE SELFSIMILAR SOLUTION

Theorem 2.2 provides a selfsimilar solution to Smoluchowski’s equation such that its second and third moments are bounded up to the gelation time. Here we show how we can obtain a bound for all the moments given the $(\frac{7-\delta}{2})$th moment provided by Theorem 2.2.

We multiply equation (2.1.4) times $\xi^\alpha$ with $\alpha$ to be fixed, $\alpha \neq 2 + 2\varepsilon - \frac{1}{\beta}$ and $\alpha > \frac{5}{2}$ and integrate to obtain:

\[
-\left(2\varepsilon + 3\right) - \frac{1}{\beta} \int_{\mathbb{R}^+} \xi^\alpha \psi(\xi) d\xi - \int_{\mathbb{R}^+} \xi^{\alpha+1} \psi'(\xi) d\xi = \frac{1}{2\beta} \int_{\mathbb{R}^+} \xi^\alpha \int_{0}^{\xi} (\xi - y)^{-\varepsilon} y^{1-\varepsilon} \psi(\xi - y) \psi(y) dy d\xi
\]

\[
- \frac{1}{\beta} \left( \int_{\mathbb{R}^+} \xi^{\alpha+1-\varepsilon} \psi(\xi) d\xi \right) \left( \int_{\mathbb{R}^+} y^{1-\varepsilon} \psi(y) dy \right).
\]

We can then manipulate this formula to get:

\[
\left( \alpha - (2\varepsilon + 2) + \frac{1}{\beta} \right) \mathcal{M}_\alpha
= \frac{1}{2\beta} \int_{\mathbb{R}^+} \int_{[0,\xi]} ((\xi - y) + y)^\alpha (\xi - y)^{1-\varepsilon} y^{1-\varepsilon} \psi(\xi - y) \psi(y) dy d\xi - \frac{1}{\beta} \mathcal{M}_{\alpha+1-\varepsilon} \mathcal{M}_{1-\varepsilon}
\]

Now we need to find two finite positive constants $C_1$, $C_2$ depending on $\alpha$, such that

\[
(\xi - y)^\alpha + y^\alpha + C_1 ((\xi - y)^{\alpha-1} y + y^{\alpha-1} (\xi - y))
\leq ((\xi - y) + y)^\alpha
\leq (\xi - y)^\alpha + y^\alpha + C_2 ((\xi - y)^{\alpha-1} y + y^{\alpha-1} (\xi - y)),
\]

Therefore, one has to study the function:

\[
f(x) = \frac{(1 + x)^\gamma - 1 - x^\gamma}{x + x^{\gamma-1}}, \quad x > 0.
\]

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We can observe that, if $\gamma > 2$, the limits of $f$ are both:

$$\lim_{x \to 0^+} f(x) = \lim_{x \to \infty} f(x) = \gamma.$$ 

By direct computation we obtain

$$f'(x) = \left( \frac{(1 + x)^{\gamma-1} - x^{\gamma-1}}{(1 + x)^\gamma - 1 - x\gamma} - \frac{1 + (\gamma - 1) x^{\gamma-2}}{(x + x^{\gamma-1})} \right) f(x)$$

and setting $f'(x) = 0$ we arrive at the equation

$$\gamma \left( (1 + x)^{\gamma-1} - x^{\gamma-1} \right) \left( x + x^{\gamma-1} \right) = \left( (1 + x)^\gamma - 1 - x\gamma \right) \left( 1 + (\gamma - 1) x^{\gamma-2} \right)$$

with a unique real zero at $x_0 = 1$ so that $f(x_0) = 2^{\gamma-1} - 1 > 0$ for $\gamma > 2$. If $\gamma \in (3, \infty)$, $f$ is initially increasing, reaching a maximum in $x_0 = 1$ and then decreasing as $x$ goes to infinity to $\gamma$. In both cases we can obtain the explicit upper and lower bounds:

- $\gamma \in (2, 3)$: $C_1 = 2^{\gamma-1} - 1 > 1$, $C_2 = \gamma < 3$;
- $\gamma \in (3, \infty)$: $C_1 = \gamma > 3$, $C_2 = 2^{\gamma-1} - 1 < \infty$.

Using the preceding inequalities, we can bound:

$$\frac{C_1}{\beta} M_{\alpha-\varepsilon} M_{2-\varepsilon} \leq \left( \alpha - (-2\varepsilon + 2) + \frac{1}{\beta} \right) M_{\alpha} \leq \frac{C_2}{\beta} M_{\alpha-\varepsilon} M_{2-\varepsilon}.$$ 

Let’s choose, for instance, $\alpha = \frac{7}{2} + \varepsilon - \delta$, with $\delta < 1$; then, as we know that $\frac{C_{1,2}}{\beta} M_{2-\varepsilon} < \infty$ and $0 < M_{\frac{7}{2} - \delta} \leq \infty$, we find that:

$$C_1 M_{\frac{7}{2} - \delta} \leq M_{\frac{7}{2} + \varepsilon - \delta} \leq C_2 M_{\frac{7}{2} - \delta},$$

which means that we can control a moment of $(\nu + \varepsilon)^{th}$ order if we already control the moment of $\nu^{th}$ order, gaining a little but positive $\varepsilon$. This process can be repeated as many times as needed to control any finite moment (which is therefore positive); moreover, no further restriction is placed upon $\varepsilon$ or $\beta$, meaning that this operation may be carried over any time that a sufficient amount of bounded moments is available. In particular, we can request that $M_{2-\varepsilon} < \infty$ and that

$$- \int_{\mathbb{R}^+} \xi^{\alpha+1} \psi'(\xi) d\xi = (\alpha + 1) M_{\alpha}. $$ 

We therefore conclude Theorem \ref{thm:main}.

### 2.10. Appendix: Results on Functional Analysis

In Lemma \ref{lem:hardy} we proved an estimate involving weighted norms of $\Phi_1(ik)$ which is the Fourier transform of a certain function defined in $\mathbb{R}^+$. Since the function $N \Phi_1$ defined by \ref{eq:fourier} involves terms which are linear and quadratic in $\Phi_1$, we will need to show that the $X$-norm allows to estimate the nonlinearities in terms of powers of the $Y$-norm of $\Phi_1$. This will require a series of properties of the $X$ and $Y$ norms concerning their relation and embeddings with other more familiar functional spaces such as weighed Sobolev and Lebesgue spaces. The purpose of this section is to provide all the needed functional analytic results.

We begin with a generalized Hardy’s inequality valid for a broad range of weights:
Lemma 2.10. Let \( f \in C_0^\infty (\mathbb{R}^+) \). Then for any \( \delta \neq -1 \),
\[
\int_{\mathbb{R}^+} \frac{f^2}{x^{2+\delta}} dx \leq C \int_{\mathbb{R}^+} \frac{(f')^2}{x^\delta} dx. \tag{2.10.1}
\]

Proof. We begin remarking that the l.h.s. integral in equation (2.10.1) is the \( L^2 \)-norm of the function \( \frac{f}{x^{1+\frac{\delta}{2}}} \) so that, applying the Mellin transform in the strip \( \lambda = i\lambda - \frac{1}{2} \), the Plancherel theorem holds true. This means that:
\[
\int_{\mathbb{R}^+} \frac{f^2}{x^{2+\delta}} dx = \frac{1}{2\pi} \int_{\mathbb{R}} \left| \int_{\mathbb{R}^+} x^{i\lambda - \frac{1}{2} - \frac{\delta}{2}} f(x) dx \right|^2 d\lambda.
\]

In the inner integral we may now apply the integration-by-parts formula, and then:
\[
\left| \int_{\mathbb{R}} \frac{1}{i\lambda - \frac{1+\delta}{2}} \left| \int_{\mathbb{R}^+} x^{i\lambda - \frac{1}{2} - \frac{\delta}{2}} f(x) dx \right|^2 d\lambda = \frac{1}{(1+\delta)^2} \int_{\mathbb{R}^+} x^{-\delta} (f')^2 dx.
\]

Since \( |i\lambda - \frac{1+\delta}{2}|^2 = \lambda^2 + (\frac{1+\delta}{2})^2 \geq (\frac{1+\delta}{2})^2 \) we have:
\[
\int_{\mathbb{R}} \frac{1}{(1+\delta)^2} \left| \int_{\mathbb{R}^+} x^{i\lambda - \frac{1}{2} - \frac{\delta}{2}} f(x) dx \right|^2 d\lambda \leq \frac{1}{(1+\delta)^2} \int_{\mathbb{R}^+} x^{-\delta} (f')^2 dx.
\]

Remark 2.11. Inequality (2.10.1) extends to the closure of \( C_0^\infty (\mathbb{R}^+) \) under the norm
\[
\|f\|_Z \equiv \int_{\mathbb{R}^+} \frac{(f')^2}{x^\delta} dx.
\]

Lemma 2.12. Let \( \alpha \in (-2 - \delta + \mu, -\frac{1}{2} + \frac{\delta}{2} + \mu) \) with \( \delta > -1 \), and \( \mu < -\frac{\delta}{2} \); also consider the weight function \( W(x) = \left( \frac{1}{|x|^{2+\delta}} + \frac{1}{|x|^2(1-\delta)} \right) \). Then, for any \( f \in C_0^\infty (\mathbb{R}^+) \),
\[
\sup_{\mathbb{R}^+} (x^{2\alpha + 1} |f(x)|^2) \leq C \int_{\mathbb{R}^+} x^{2+2\mu} \left| \frac{d}{dx} f(x) \right|^2 W^2(x) dx. \tag{2.10.2}
\]

Proof. First, by the Fundamental Theorem of Calculus:
\[
x^{2\alpha + 1} |f(x)|^2 \leq \int_0^x \left( (2\alpha + 1) y^{2\alpha} f^2(y) + 2y^{2\alpha + 1} f(y) \frac{d}{dy} f(y) \right) dy.
\]
Now, by Young’s inequality, 

\[ 2y^{2\alpha+1} f(y) \frac{d}{dy} f(y) \leq (y^\alpha f(y))^2 + \left( y^{\alpha+1} \frac{d}{dy} f(y) \right)^2. \]

Thus we deduce:

\[
\int_0^x \left( (2\alpha + 1) f^2(y) y^{2\alpha} + 2y^{2\alpha+1} f(y) \frac{d}{dy} f(y) \right) dy \leq \int_0^x \left( (2\alpha + 2) f^2(y) y^{2\alpha} + y^{2\alpha+2} \left( \frac{d}{dy} f(y) \right)^2 \right) dy. \tag{2.10.3}
\]

Under the considered hypothesis it is possible to apply Lemma 2.10 to equation (2.10.3). Hence:

\[ x^{2\alpha+1} f^2(x) \leq C \int_{\mathbb{R}^+} y^{2\alpha+2} \left( \frac{d}{dy} f(y) \right)^2 dy. \]

As \( \alpha \) lies in the interval \( \alpha \in (-2 - \delta + \mu, -\frac{1}{2} + \frac{\delta}{2} + \mu) \), the power \( y^{2\alpha+2} \) is bounded by \( y^{2+2\mu} |W(y)|^2 \); then we take the sup on the l.h.s. and we conclude (2.10.2).

We are going to prove now some lemmas connecting weighted Sobolev norms for functions \( \psi(x) \) defined in \( \mathbb{R}^+ \) and the weighted norms the corresponding transforms

\[ \phi(k) \equiv (ik) = - \int_{\mathbb{R}^+} (e^{-ikx} - 1)x\psi(x) \, dx \tag{2.10.4} \]

and

\[ \tilde{\phi}(k) = - \int_{\mathbb{R}^+} (e^{-ikx} - 1 + ikx) x\psi(x) \, dx. \tag{2.10.5} \]

Notice that \( \frac{d}{dk} \phi(k) = i \int_{\mathbb{R}^+} e^{-ikx} x^2 \psi(x) \, dx \) which is proportional to the Fourier transform of \( x^2 \psi(x) \). Hence the lemmas proved below can be translated into results for Fourier transforms and vice-versa.

**Lemma 2.13.** If \( \phi(k) \) is the transform of \( \psi(x) \in C_0^\infty (\mathbb{R}^+) \) as defined in (2.10.4), \( \tilde{\phi}(k) \) is the transform of \( \psi(x) \) as defined in (2.10.5) and \( \alpha \in (0, 1) \) then:

\[
\int_{-\infty}^\infty |k|^{2\alpha-1} \left| \frac{d\phi(k)}{dk} \right|^2 \, dk \geq C \int_{0}^\infty x^{5-2\alpha} |\psi(x)|^2 \, dx \tag{2.10.6}
\]

and

\[
\int_{-\infty}^\infty |k|^{-2\alpha-1} \left| \frac{d\tilde{\phi}(k)}{dk} \right|^2 \, dk \geq C \int_{0}^\infty x^{3+2\alpha} |\psi^*(x)|^2 \, dx, \tag{2.10.7}
\]

where \( \psi^*(x) = -\int_x^\infty \psi(x') \, dx' \).

**Proof.** In order to prove (2.10.6), we write

\[
\int_{-\infty}^\infty |k|^{2\alpha-1} \left| \frac{d\phi(k)}{dk} \right|^2 \, dk = \int_{0}^\infty |k|^{2\alpha-1} |\phi'(k)|^2 \, dk + \int_{-\infty}^0 |k|^{2\alpha-1} |\phi'(k)|^2 \, dk \equiv I_1 + I_2.
\]

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One can compute:

\[ I_1 = \int_{-\infty}^{\infty} |k|^{2\alpha - 1} |\phi'(k)|^2 dk = \int_{-\infty}^{\infty} |\tilde{\phi}'(\lambda)|^2 d\lambda \]

with

\[ \tilde{\phi}'(\lambda) = \int_{0}^{\infty} |k|^{i\lambda + \alpha - 1} \phi'(k) dk, \]

where

\[ \phi'(k) = i \int_{0}^{\infty} e^{-ikx} x^2 \psi(x) dx \]

and then

\[ \tilde{\phi}'(\lambda) = i \int_{0}^{\infty} \int_{0}^{\infty} |k|^{i\lambda + \alpha - 1} e^{-ikx} x^2 \psi(x) dk dx. \]

We can compute next

\[ \int_{0}^{\infty} e^{-ikx} |k|^{i\lambda + \alpha - 1} dk = x^{-i\lambda - \alpha} e^{i \frac{\pi}{2} (i\lambda + \alpha - 1)} \Gamma(i\lambda + \alpha). \]

Analogously we can compute for \( I_2 \):

\[ \int_{0}^{\infty} e^{ikx} |k|^{i\lambda + \alpha - 1} dk = x^{-i\lambda - \alpha} e^{-i \frac{\pi}{2} (i\lambda + \alpha - 1)} \Gamma(i\lambda + \alpha) \]

and conclude

\[ I_1 + I_2 = \int_{-\infty}^{\infty} 4 \cosh^2 \left( \frac{\pi}{2} \lambda \right) |\Gamma(i\lambda + \alpha)|^2 |\tilde{\psi}(\lambda)|^2 d\lambda, \]

where

\[ \tilde{\psi}(\lambda) = \int_{0}^{\infty} x^{-i\lambda - \alpha + 2} \psi(x) dx. \]

Since

\[ \cosh^2 \left( \frac{\pi}{2} \lambda \right) |\Gamma(i\lambda + \alpha)|^2 \simeq (1 + |\lambda|^\alpha)^2 \geq 1 \text{ for } \alpha > 0, \]

then inequality (2.10.6) follows.

Next we proceed to prove (2.10.7). As above, we can write:

\[ \int_{-\infty}^{\infty} |k|^{-2\alpha - 1} \left| \frac{d\tilde{\phi}(k)}{dk} \right|^2 dk = \int_{0}^{\infty} |k|^{-2\alpha - 1} \left| \tilde{\phi}'(k) \right|^2 dk + \int_{0}^{0} |k|^{-2\alpha - 1} \left| \tilde{\phi}'(k) \right|^2 dk = I_1 + I_2 \]

and then compute:

\[ I_1 = \int_{0}^{\infty} |k|^{-2\alpha - 1} \left| \tilde{\phi}'(k) \right|^2 dk = \int_{-\infty}^{\infty} |\tilde{\phi}(\lambda)|^2 d\lambda \]

with

\[ \tilde{\phi}(\lambda) = \int_{0}^{\infty} |k|^{i\lambda - \alpha - 1} \tilde{\phi}'(k) dk \]

where

\[ \tilde{\phi}'(k) = i \int_{0}^{\infty} (e^{-ikx} - 1) x^2 \psi(x) dx \]

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and then
\[ \tilde{\phi}(\lambda) = -\int_0^\infty \int_0^\infty |k|^\lambda - \alpha - 1 (e^{-ikx} - 1) x^2 \psi(x) dk dx. \]

We can compute next
\[ -\int_0^\infty (e^{-ikx} - 1) |k|^\lambda - \alpha - 1 dk = \frac{ix}{i\lambda - \alpha} \int_0^\infty e^{-ikx} |k|^\lambda - \alpha dk = \frac{i}{i\lambda - \alpha} x^{-i\lambda + \alpha} e^{\frac{i\pi}{2}(\lambda - \alpha)} \Gamma(i\lambda - \alpha + 1). \]

Analogously we can compute for \( I_2 \)
\[ -\int_0^\infty (e^{ikx} - 1) |k|^\lambda - \alpha - 1 dk = \frac{i}{i\lambda - \alpha} x^{-i\lambda + \alpha} e^{-\frac{i\pi}{2}(\lambda - \alpha)} \Gamma(i\lambda - \alpha + 1) \]
and conclude
\[ I_1 + I_2 = \int_{-\infty}^\infty \frac{\cosh^2 \left( \frac{\pi}{2} \lambda \right)}{\lambda^2 + \alpha^2} |\Gamma(i\lambda - \alpha + 1)|^2 \left| \tilde{\psi}(\lambda) \right|^2 d\lambda, \]
where
\[ \tilde{\psi}(\lambda) = \int_0^\infty x^{-i\lambda + \alpha + 2} \psi(x) dx = -(-i\lambda + \alpha + 2) \int_0^\infty x^{-i\lambda + \alpha + 1} \psi^*(x) dx. \]

Since
\[ \frac{\lambda^2 + (\alpha + 2)^2}{\lambda^2 + \alpha^2} \cosh^2 \left( \frac{\pi}{2} \lambda \right) |\Gamma(i\lambda - \alpha + 1)|^2 \simeq (1 + |\lambda|^{1-\alpha})^2 \geq 1 \text{ for } \alpha > 0, \]
then inequality (2.10.7) follows.

We will also need to estimate the norm of the operator \( D_k^{-\varepsilon} \phi(k) \) defined as
\[ D_k^{-\varepsilon} \phi(k) \equiv -\int_{\mathbb{R}^+} (e^{-ikx} - 1) x^{1-\varepsilon} \psi(x) dx \quad (2.10.8) \]
in terms of the norm of \( \phi(k) \).

**Lemma 2.14.** Let \( \varepsilon > 0 \) and \( \alpha \neq -n, n \in \mathbb{N}, \) and \( \phi(k), D_k^{-\varepsilon} \phi(k) \) defined by (2.10.4) and (2.10.8), with \( \psi(x) \in C_0^\infty (\mathbb{R}^+) \). Then
\[ \int_{-\infty}^\infty |k|^{2\alpha - 1} \left| \frac{dD_k^{-\varepsilon} \phi(k)}{dk} \right|^2 dk \leq C \int_{-\infty}^\infty |k|^{2\alpha - 1 + 2\varepsilon} \left| \frac{d\phi(k)}{dk} \right|^2 dk \quad (2.10.9) \]
and
\[ \int_{-\infty}^\infty |k|^{2\alpha - 3} \left| D_k^{-\varepsilon} \phi(k) \right|^2 dk \leq C \int_{-\infty}^\infty |k|^{2\alpha - 3 + 2\varepsilon} \left| \phi(k) \right|^2 dk. \quad (2.10.10) \]
Proof. We prove first inequality (2.10.6). By definitions (2.10.4) and (2.10.8),

\[
\int_{-\infty}^{\infty} |k|^{2\alpha-1} \left| \frac{dD_{k}^{\alpha} \phi(k)}{dk} \right|^{2} dk = \int_{\mathbb{R}} |k|^{2\alpha-1} \left| \frac{d}{dk} \left( \int_{\mathbb{R}^+} \left( e^{ikx} - 1 \right) x^{1-\epsilon} \psi_1(x) dx \right) \right|^{2} dk \\
= \int_{\mathbb{R}} |k|^{2\alpha-1} \left( \int_{\mathbb{R}^+} e^{ikx} x^{-\epsilon} \psi_1(x) dx \right)^2 dk \equiv I_1, \quad (2.10.11)
\]

\[
\int_{-\infty}^{\infty} |k|^{2\alpha-1+2\epsilon} \left| \frac{d\phi(k)}{dk} \right|^{2} dk = \int_{\mathbb{R}} |k|^{2\alpha-1+2\epsilon} \left| \frac{d}{dk} \left( \int_{\mathbb{R}^+} \left( e^{ikx} - 1 \right) x \psi_1(x) dx \right) \right|^{2} dk \\
= \int_{\mathbb{R}} |k|^{2\alpha-1+2\epsilon} \left( \int_{\mathbb{R}^+} e^{ikx} x^{2-\epsilon} \psi_1(x) dx \right)^2 dk \equiv I_2. \quad (2.10.12)
\]

We estimate next $I_1$ and $I_2$ defined by formulae (2.10.11) and (2.10.12) respectively. First, by splitting the $k$ integral in two integrals over $\mathbb{R}^+$ and $\mathbb{R}^-$, and, respectively, applying the Mellin Transform for each of these two integrals, we obtain:

\[
I_1 = \int_{\mathbb{R}} \int_{\mathbb{R}^+} |k|^{\lambda-1+\alpha} \left( \int_{\mathbb{R}^+} e^{ikx} x^{2-\epsilon} \psi_1(x) dx \right) dk \left| \int_{\mathbb{R}^+} k^{\lambda-1+\alpha} \left( \int_{\mathbb{R}^+} e^{-ikx} x^{2-\epsilon} \psi_1(x) dx \right) dk \right|^2 d\lambda \\
\equiv I_{1,1} + I_{1,2}, \quad (2.10.13)
\]

\[
I_2 = \int_{\mathbb{R}} \int_{\mathbb{R}^+} |k|^{\lambda-1+\alpha+\epsilon} \left( \int_{\mathbb{R}^+} e^{ikx} x^{2} \psi_1(x) dx \right) dk \left| \int_{\mathbb{R}^+} k^{\lambda-1+\alpha+\epsilon} \left( \int_{\mathbb{R}^+} e^{-ikx} x^{2} \psi_1(x) dx \right) dk \right|^2 d\lambda \\
\equiv I_{2,1} + I_{2,2}. \quad (2.10.14)
\]

By hypothesis, $\psi_1$ is approximated by $C_0^\infty$ functions, and then $e^{i\mu x} x^{2-\epsilon} \psi_1$ is approximated by them too, with $\mu > 0$, $\mu << 1$. Therefore we can write an approximation for $I_1$ and $I_2$ in the form:

\[
I_{1,\mu} = \int_{\mathbb{R}} \int_{\mathbb{R}^+} |k|^{\lambda+\alpha-1} \left( \int_{\mathbb{R}^+} e^{ikx} x^{2+\epsilon} \psi_1(x) dx \right) dk \left| \int_{\mathbb{R}^+} k^{\lambda+\alpha-1} e^{ikx} \psi_1(x) dx \right|^2 d\lambda,
\]

\[
I_{2,\mu} = \int_{\mathbb{R}} \int_{\mathbb{R}^+} |k|^{\lambda+\alpha-\epsilon} \left( \int_{\mathbb{R}^+} e^{ikx} x^{2} \psi_1(x) dx \right) dk \left| \int_{\mathbb{R}^+} k^{\lambda+\alpha-\epsilon} e^{ikx} \psi_1(x) dx \right|^2 d\lambda
\]

respectively. Next we change the order of integration, which is allowed by the regularity assumptions on $\psi_1$, and obtain:

\[
I_{1,\mu} = \int_{\mathbb{R}} \int_{\mathbb{R}^+} \left( e^{i\mu x} x^{2-\epsilon} \psi_1(x) \right) \left( \int_{\mathbb{R}} |k|^{\lambda+\alpha-1} e^{ikx} dk \right) \left| \int_{\mathbb{R}} |k|^{\lambda+\alpha-1} e^{ikx} \psi_1(x) dx \right|^2 d\lambda, \quad (2.10.15)
\]

\[
I_{2,\mu} = \int_{\mathbb{R}} \int_{\mathbb{R}^+} \left( e^{i\mu x} x^{2} \psi_1(x) \right) \left( \int_{\mathbb{R}} |k|^{\lambda+\alpha-\epsilon} e^{ikx} dk \right) \left| \int_{\mathbb{R}} |k|^{\lambda+\alpha-\epsilon} e^{ikx} \psi_1(x) dx \right|^2 d\lambda. \quad (2.10.16)
\]
We define next \( A ( x, \lambda; \alpha, \mu ) = \int_{\mathbb{R}} | k |^{i \lambda + \alpha - 1} e^{i (k + i \mu x) x} dk \) and \( B ( x, \lambda; \alpha, \mu, \varepsilon ) = \int_{\mathbb{R}} | k |^{i \lambda + \alpha - 1 + \varepsilon} e^{i (k + i \mu x) x} dk \).

In both integrals we change the integration variable, \( u = ( k + i \mu ) x \), to obtain:

\[
A ( x, \lambda; \alpha, \mu ) = x^{-i \lambda - \alpha} e^{-\mu x} \int_{\mathbb{R}} | u - i \mu x |^{i \lambda + \alpha - 1} e^{i (u - i \mu x) x} du,
\]
\[
B ( x, \lambda; \alpha, \mu, \varepsilon ) = x^{-i \lambda - \alpha - \varepsilon} e^{-\mu x} \int_{\mathbb{R}} | u - i \mu x |^{i \lambda + \alpha - 1 + \varepsilon} e^{i (u - i \mu x) x} du.
\]

In order to compute the integrals, we split the domain so that \( \int_{\mathbb{R}} = \int_{0}^{\infty} + \int_{-\infty}^{0} \) and then deform the integration path \([0, \infty)\) to make it lay on the imaginary axis and turn it into \([0, i \infty)\). More precisely:

\[
\int_{\mathbb{R}^+} | u - i \mu x |^{i \lambda + \alpha - 1} e^{i (u - i \mu x) x} du = e^{i x} e^{i \pi (i \lambda + \alpha - 1)} \int_{\mathbb{R}^+} t^{i \lambda + \alpha - 1} e^{-t} dt, \quad (2.10.17)
\]
\[
\int_{\mathbb{R}^-} | u - i \mu x |^{i \lambda + \alpha - 1 + \varepsilon} e^{i (u - i \mu x) x} du = e^{i x} e^{i \pi (i \lambda + \alpha - 1 - \varepsilon)} \int_{\mathbb{R}^+} t^{i \lambda + \alpha - 1 - \varepsilon} e^{-t} dt. \quad (2.10.18)
\]

Similarly, for the integrals along the paths \((-\infty, 0)\), we deform towards \((-i \infty, 0)\) and obtain:

\[
\int_{\mathbb{R}^-} | u - i \mu x |^{i \lambda + \alpha - 1} e^{i (u - i \mu x) x} du = e^{-i x} e^{-i \pi (i \lambda + \alpha - 1)} \int_{\mathbb{R}^+} t^{i \lambda + \alpha - 1} e^{-t} dt, \quad (2.10.19)
\]
\[
\int_{\mathbb{R}^-} | u - i \mu x |^{i \lambda + \alpha - 1 + \varepsilon} e^{i (u - i \mu x) x} du = e^{-i x} e^{-i \pi (i \lambda + \alpha - 1 + \varepsilon)} \int_{\mathbb{R}^+} t^{i \lambda + \alpha - 1 + \varepsilon} e^{-t} dt. \quad (2.10.20)
\]

Adding (2.10.17) with (2.10.19) and (2.10.18) with (2.10.20), we conclude:

\[
A ( x, \lambda; \alpha, \mu ) = x^{-i \lambda - \alpha} e^{-\mu x} \Gamma ( i \lambda + \alpha ) \cosh ( i \lambda + \alpha ) ,
\]
\[
B ( x, \lambda; \alpha, \mu, \varepsilon ) = x^{-i \lambda - \alpha - \varepsilon} e^{-\mu x} \Gamma ( i \lambda + \alpha + \varepsilon ) \cosh ( i \lambda + \alpha + \varepsilon ).
\]

Thus, the integrals in (2.10.15) and (2.10.16) are

\[
I_{1, \mu} = \left| \int_{\mathbb{R}} | \Gamma ( i \lambda + \alpha ) \cosh ( i \lambda + \alpha ) |^2 \left| \int_{\mathbb{R}^+} e^{\mu x} \psi_1 ( x ) x^{-i \lambda + 2 - \alpha} e^{-\mu x} dx \right|^2 \right| \quad (2.10.21)
\]
\[
I_{2, \mu} = \left| \int_{\mathbb{R}} | \Gamma ( i \lambda + \alpha + \varepsilon ) \cosh ( i \lambda + \alpha + \varepsilon ) |^2 \left| \int_{\mathbb{R}^+} e^{\mu x} \psi_1 ( x ) x^{-i \lambda + 2 - \alpha} e^{-\mu x} dx \right|^2 \right| \quad (2.10.22)
\]

respectively. Finally, since

\[
| \Gamma ( i \lambda + \alpha + \varepsilon ) \cosh ( i \lambda + \alpha + \varepsilon ) |^2 \geq | \Gamma ( i \lambda + \alpha ) \cosh ( i \lambda + \alpha ) |^2 \quad (2.10.23)
\]

for \( \varepsilon > 0 \) and \( \alpha \neq -n, \ n \in \mathbb{N} \) (see pg. 1220, eq. 2.20 of [138] for instance) we finally conclude \( I_{1, \mu} \leq I_{2, \mu} \) and, passing to the limit \( \mu \to 0 \), inequality (2.10.9) holds.
The proof of inequality \((2.10.10)\) is analogous, except for the fact that it involves integrals of the form
\[
\int_{\mathbb{R}^+} t^{\lambda+\alpha-1} (e^{-t} - 1) \, dt, \quad \int_{\mathbb{R}^+} t^{\lambda+\alpha-1-\varepsilon} (e^{-t} - 1) \, dt
\]
(2.10.24)
in the expressions \((2.10.17)-(2.10.20)\). After one integration by parts, the integrals \((2.10.24)\) yield:
\[
A(x, \lambda; \alpha, \mu) = x^{-i\lambda-\alpha} e^{-\mu x} \frac{\Gamma(i\lambda + \alpha)}{i\lambda + \alpha} \cosh(i\lambda + \alpha),
\]
\[
B(x, \lambda; \alpha, \mu, \varepsilon) = x^{-i\lambda-\alpha-\varepsilon} e^{-\mu x} \frac{\Gamma(i\lambda + \alpha + \varepsilon)}{i\lambda + \alpha + \varepsilon} \cosh(i\lambda + \alpha + \varepsilon).
\]
Since the following inequality
\[
\left| \frac{\Gamma(i\lambda + \alpha + \varepsilon)}{i\lambda + \alpha + \varepsilon} \cosh(i\lambda + \alpha + \varepsilon) \right|^2 \geq \left| \frac{\Gamma(i\lambda + \alpha)}{i\lambda + \alpha} \cosh(i\lambda + \alpha) \right|^2
\]
holds true for \(\varepsilon > 0\) and \(\alpha \neq -n, n \in \mathbb{N}\), inequality \((2.10.10)\) follows and it ends the proof of the Lemma. \(\Box\)

Let \(\psi(\xi)\) and \(\Phi(\eta)\) be related by \((2.2.8)\), so that
\[
\xi^2\psi(\xi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{ik\xi} \Phi'(ik) \, dk.
\]
(2.10.26)

Then, the following lemma holds:

**Lemma 2.15.** If
\[
\int_{\mathbb{R}} \left| \Phi'(ik) \right|^2 \left( |k|^{-1-2\gamma-\delta} + |k|^{-1-2\gamma+\delta} \right) \, dk < \infty
\]
for some \(\delta > 0\) and some \(\gamma > -1\), then the following inequality holds:
\[
\int_{\mathbb{R}^+} \xi^{(\gamma+2)} \psi(\xi) \, d\xi < C \int_{\mathbb{R}} \left| \Phi'(k) \right|^2 \left( |k|^{-1-2\gamma-\delta} + |k|^{-1-2\gamma+\delta} \right) \, dk
\]
(2.10.27)
for some constant \(C\).

**Proof.** Let \(\varepsilon > 0\) be a small constant, then by \((2.2.8)\)
\[
\int_{\mathbb{R}^+} e^{-\varepsilon \xi}(\gamma+2) \psi(\xi) \, d\xi = \int_{\mathbb{R}^+} e^{-\varepsilon \xi} \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{ik\xi} \Phi'(ik) \, dk \, d\xi = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Phi'(ik) \int_{\mathbb{R}^+} \xi^\gamma e^{(ik-\varepsilon)\xi} \, d\xi \, dk
\]
\[
= \frac{1}{2\pi i} \int_{-\infty}^{0} \Phi'(ik) I_1 \, dk + \frac{1}{2\pi i} \int_{0}^{\infty} \Phi'(ik) I_2 \, dk
\]
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breaking the integration dominion of $k$ in $(-\infty, 0)$ and $[0, +\infty)$. In both integrals, observe that

$$-ik + \varepsilon = (k^2 + \varepsilon^2)^{\frac{1}{2}} e^{-i \arccot \frac{k}{\varepsilon}},$$

so that, if we change the integration path to make $\xi = s e^{i \cot \frac{k}{\varepsilon}}$, we obtain:

$$I_{1,2} = e^{(\gamma+1)i} \arccot \frac{k}{\varepsilon} \int_{\mathbb{R}^+} s^\gamma e^{-(k^2+\varepsilon^2)^{\frac{1}{2}}} ds = \frac{e^{(\gamma+1)i} \arccot \frac{k}{\varepsilon}}{(k^2 + \varepsilon^2)^{\frac{\gamma+1}{2}}} \Gamma' (\gamma + 1).$$

Thus, as $\left| e^{(\gamma+1)i} \arccot \frac{k}{\varepsilon} \right| = 1$,

$$\int_{\mathbb{R}^+} e^{-\varepsilon \xi (\gamma+2)} \psi (\xi) d\xi = \frac{\Gamma (\gamma + 1)}{2\pi i} \int_{-\infty}^{\infty} \Phi' (ik) \frac{e^{(\gamma+1)i} \arccot \frac{k}{\varepsilon}}{(k^2 + \varepsilon^2)^{\frac{\gamma+1}{2}}} dk \leq \frac{\Gamma (\gamma + 1)}{2\pi i} \int_{-\infty}^{\infty} \left| \Phi' (ik) \frac{1}{(k^2 + \varepsilon^2)^{\frac{\gamma+1}{2}}} \right| dk,$$

by Cauchy-Schwartz inequality,

$$\leq C \left( \int_{-\infty}^{\infty} \| k \|^{1-\delta} \frac{1}{|k|^{1+\delta}} dk \right)^{\frac{1}{2}} \left( \int_{-\infty}^{\infty} \| \Phi' (ik) \|^2 \frac{1}{|k^2 + \varepsilon^2|^{\gamma+1}} \left( |k|^{1-\delta} + |k|^{1+\delta} \right) dk \right)^{\frac{1}{2}}$$

and taking $\lim_{\varepsilon \to 0}$, we conclude the proof of the Lemma. \qed

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CHAPTER 3
ON global in time self-similar solutions
of Smoluchowski equation with mutli-
pllicative kernel

Here we study the similarity solutions (SS) of Smoluchowski coagulation equation with multi-
pllicative kernel \( K(x, y) = (xy)^s \) for \( s < \frac{1}{2} \). When \( s < 0 \), the SS consists of three regions with dis-
tinct asymptotic behaviours. The appropriate matching yields a global description of the solution
consisting of a Gamma distribution tail, an intermediate region described by a log-normal distribu-
tion and a region of very fast decay of the solutions to zero near the origin. When \( s \in (0, \frac{1}{2}) \),
the SS is unbounded at the origin. It also presents three regions: a Gamma distribution tail, an
intermediate region of power-like (or Pareto distribution) decay and the region close to the ori-
gen where a singularity occurs. The similarity exponents have to be computed solving a nonlinear
eigenvalue problem. We can solve it explicitly when \( s < \varepsilon, |\varepsilon| \ll 1 \), and numerically otherwise. Fi-
nally, full numerical simulations of Smoluchowski equation serve to verify our theoretical results
and show the convergence of solutions to the selfsimilar regime.

3.1. INTRODUCTION

In this chapter we consider Smoluchowski equation with a multiplicative kernel:

\[
c_t(x, t) = \frac{1}{2} \int_0^x (x - y)^s y^s c(x - y)c(y)dy - x^s c(x) \int_0^\infty y^s c(y)dy,
\]

in the case \( s < \frac{1}{2} \). In this range of parameters, solutions with all their moments bounded are
expected to exist for all time (that is \( T^* = +\infty \)) and behave asymptotically as \( t \to \infty \) in a selfsimilar
manner that can be expressed as

\[
c(x, t) \sim t^\alpha f(t^\beta x),
\]

(3.1.2)
in a sense to be precisely defined and for suitable exponents $\alpha, \beta$. The scaling of equation (3.1.1) leads automatically to the relation $\alpha = (2s+1)\beta - 1$, but $\beta$ remains as a free parameter that needs to be determined as part of the solution. From the physical point of view, a result as (3.1.2) contains the essential information on the behaviour of the system under consideration and measurable quantities such as exponents and similarity profiles $f(\xi)$ that can be measured experimentally and lead to direct physical consequences. It is therefore essential to elucidate whether such solutions exist and, if so, what are their shape and essential properties.

We compute, by means of matched asymptotic expansions, the similarity solutions $f(\xi)$ together with the similarity exponent $\beta$ to equation (3.1.1) for $s \leq 0$. For $s \in (0, \frac{1}{2})$ we compute the similarity solutions and develop asymptotic expansions for $\beta$ as a function of $s$ with $s$ sufficiently small. In the range where $\beta$ cannot be computed analytically, we compute it numerically. Finally, full numerical simulation of (3.1.1) is carried out in order to further support our matched asymptotic expansions and to show convergence of the solution $c(x, t)$ of (3.1.1) towards the selfsimilar regime. Our results coincide with results obtained by Cañizo and Mischler [38] (see also [16]) in the range $s \in (-\frac{1}{2}, \frac{1}{2})$ concerning asymptotic behaviour of selfsimilar solutions at the origin and generalize them to other regions in the parameter space as well as provide further information on the asymptotics away from the origin. In particular, the case $s \in (0, \frac{1}{2})$ requires a novel procedure (previously developed and justified with full mathematical rigor in the context of gelation in finite time in [15]) for the computation of $\beta$ and this translates into special asymptotics for the solution.

A summary of our results is provided in Figures 3.1.1 and 3.1.2. For $s < 0$, $\beta(s) = -1/(1-2s)$ and the similarity solutions consist of three regions: I) a region of very fast decay to zero near the origin, II) an intermediate region where the solution approximates a Lognormal distribution function and III) a region extending to infinity where the solution approaches a Gamma distribution function. For $s > 0$ and sufficiently small, $\beta(s) = -1 - 2s + O(s^2)$ and the solutions also consist of three regions: I) a singularity developing at the origin, II) a power-like decay or Pareto distribution function, III) a Gamma distribution extending up to infinity.

![Figure 3.1.1: Structure of the selfsimilar solution for $s < 0$. There exist three regions whose respective behaviours can be described as (I) very vast decay at the origin, (II) Lognormal distribution function, (III) Gamma distribution function.](image-url)
Figure 3.1.2: Structure of the selfsimilar solution for $s \in \left(0, \frac{1}{2}\right)$. There exist three regions whose respective behaviours can be described as (I) singularity at the origin, (II) Pareto distribution function, (III) Gamma distribution function.

### 3.2. The Integro-differential Equation for Selfsimilar Solutions

By plugging the selfsimilar expression

$$c(x, t) = t^\gamma f(t^\beta x),$$

into (3.1.1), choosing

$$\gamma = (2s + 1) \beta - 1,$$

and defining

$$\xi := t^\beta x,$$

we obtain the integrodifferential ordinary differential equation

$$(2s + 1) \beta - 1) f(\xi) + \beta \xi f_\xi(\xi) = \frac{1}{2} \int_0^\xi (\xi - \eta)^s \eta^s f(\xi - \eta) f(\eta) d\eta - \xi^s f(\xi) \int_0^\infty \eta^s f(\eta) d\eta. \quad (3.2.1)$$

$\beta$ is a free parameter that has to be chosen, for a given $s$, from the condition that all the moments

$$M_n = \int_0^\infty x^n c(x, t) dx, \quad n = 1, 2, \ldots,$$

remain bounded for $0 < t < \infty$.

Notice that one can rearrange terms in the more convenient (for the purpose of analysis) form

$$((2s + 1) \beta - 1) f(\xi) + \beta \xi f_\xi(\xi)$$

$$= \frac{1}{2} \int_0^\xi \eta^s \left[(\xi - \eta)^s f(\xi - \eta) f(\eta) - 2\chi_\xi^s(\eta) \xi^s f(\xi) f(\eta)\right] d\eta - \xi^s f(\xi) \int_\frac{\xi}{2}^\infty \eta^s f(\eta) d\eta. \quad (3.2.2)$$

where $\chi_\xi^s(\eta)$ is the characteristic function so that $\chi_\xi^s(\eta) = 1$ for $\eta \leq \frac{\xi}{2}$ and zero elsewhere.
A different approach to the problem is through the use of Laplace transform:

\[ C(\mu, t) = \int_0^\infty (e^{-\mu x} - 1)c(x, t)\,dx. \]

By multiplying equation \([3.1]\) \((e^{-\mu x} - 1)\), integrating in \(x\) and using

\[
\begin{align*}
&\int_0^\infty e^{-\mu x} \left( \int_0^x (x - y)^s y^s c(x - y, t)c(y, t)\,dy\,dx \right) \\
&= \int_0^\infty \int_0^x e^{-\mu y}e^{-\mu(x-y)}(x - y)^s y^s c(x - y, t)c(y, t)\,dy\,dx \\
&= \left( \int_0^\infty e^{-\mu y}y^s c(y, t)\,dy \right)^2,
\end{align*}
\]

we arrive at the equation

\[ C_t(\mu, t) = \frac{1}{2} \left( D^{\mu s}_C(\mu, t) \right)^2, \]

where

\[ D^{\mu s}_C(\mu, t) = \int_0^\infty (e^{-\mu x} - 1)x^s c(x, t)\,dx, \]

formally represents a \((-s)\)-derivative operator. Selfsimilar solutions would be of the form

\[ C(\mu, t) = t^{2s\beta - 1}g(t^{-\beta}\mu), \]

and satisfy the equation

\[ (2s\beta - 1) g - \beta \lambda D_\lambda g = \frac{1}{2} \left( D^{\lambda s}_g \right)^2, \tag{3.2.3} \]

where

\[ \lambda := t^{-\beta}\mu. \]

If \(f(\xi)\) is a solution of \([3.2.1]\), then \(\ell^{1+2s} f (\ell \xi)\) is also a solution for any \(\ell > 0\). Analogously, if \(g(\lambda)\) is a solution of \([3.2.3]\), then \(\ell^{2s} g (\ell^{-1}\lambda)\) is also a solution for any \(\ell > 0\). For the rest of this article, when we refer to the selfsimilar solution, we will be referring to this 1-parameter family (with parameter \(\ell\)). For the purpose of analysis, we will consider a specific unique member identified by its first moment \(M_1\).

### 3.3. Asymptotic behaviour of selfsimilar solutions

The particular case \(s = 0\) with \(\beta = -1\) allows direct integration of both equation \([3.2.1]\) and equation \([3.2.3]\) so that

\[ f(\xi) = 2e^{-\xi}, \tag{3.3.1} \]

and

\[ g(\lambda) = \frac{-2\lambda}{1 + \lambda}, \tag{3.3.2} \]
are their solutions (with first moment given and equal to 2) respectively. Of course, (3.3.2) is the Laplace transform of (3.3.1) as can be easily verified. If \( \beta \neq 1 \) then the solution of (3.2.3) is given by

\[
g(\lambda) = \frac{-2\lambda^{-1/\beta}}{1 + \lambda^{-1/\beta}},
\]

and hence

\[
-\int_0^\infty e^{-\lambda\xi} \xi f(\xi)d\xi = g'(\lambda) = \frac{2\beta\lambda^{-1/\beta-1}}{(1 + \lambda^{-1/\beta})^2},
\]

so that, inverting the Laplace transform (see [4]), and performing contour deformation in the complex plane,

\[
\xi f(\xi) = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} e^{-\lambda\xi} \frac{2\beta^{-1} \lambda^{-1/\beta-1}}{(1 + \lambda^{-1/\beta})^2} d\lambda
\]

\[
= \frac{1}{2\pi i} \int_0^\infty e^{-\lambda\xi} \left( \frac{2\beta^{-1} e^{-i\pi/\beta} \lambda^{-1/\beta-1}}{(1 + e^{-i\pi/\beta} \lambda^{-1/\beta})^2} - \frac{2\beta^{-1} e^{-i\pi/\beta} \lambda^{-1/\beta-1}}{(1 + e^{-i\pi/\beta} \lambda^{-1/\beta})^2} \right) d\lambda.
\]

We find then

\[
\xi f(\xi) \sim -\frac{2\beta^{-1} \sin(\pi/\beta) \Gamma(-1/\beta)}{\pi} \xi^{1/\beta},
\]

as \( \xi \to \infty \), and

\[
\xi f(\xi) \sim \frac{1}{2\pi i} \int_0^\infty e^{-\lambda\xi} \left( 2\beta^{-1} e^{-i\pi/\beta} \lambda^{-1/\beta-1} - 2\beta e^{-i\pi/\beta} \lambda^{-1/\beta-1} \right) d\lambda
\]

\[
= \frac{2\beta^{-1} \sin(\pi/\beta) \Gamma(1/\beta)}{\pi} \xi^{-1/\beta},
\]

as \( \xi \to 0 \). The power-like decay given by (3.3.3) implies that sufficiently high moments will diverge and therefore solutions with \( \beta \neq -1 \) cannot be allowed. The fact that boundedness of all moments requires \( \beta = -1 \) serves to characterize (3.3.1) as a similarity solution of the second kind in the notation introduced by Barenblatt [29].

For \( 0 < s < \frac{1}{2} \) and \( s < 0 \), explicit integration is not possible and one has to rely upon perturbation and asymptotic methods in order to study the solutions.

**3.3.1. Case** \( 0 < s < \frac{1}{2} \)

By introducing \( f(\xi) = A\xi^\delta \) into (3.2.2) and letting \( \xi \to 0 \) we find that the left hand side of (3.2.2) behaves as

\[
((2s + 1) \beta - 1) f(\xi) + \beta \xi f(\xi) \sim ((2s + 1) \beta - 1 + \delta \beta) A\xi^\delta,
\]

(3.3.5)
while the right hand side behaves as

\[
\frac{1}{2} \int_0^\xi (\xi - \eta)^s \eta^s \left[ f(\xi - \eta)f(\eta) - 2\chi_2 (\eta)f(\xi)f(\eta) \right] d\eta - \xi^s f(\xi) \int_\xi^\infty \eta^s f(\eta) d\eta
\]

\[
\sim A^2 \int_0^\xi \left[ (\xi - \eta)^{s+\delta}\eta^{\delta+s} - \xi^{\delta+s}\eta^{\delta+s} \right] d\eta + A^2 \frac{2^{-\delta+s+1}}{\delta + s + 1} \xi^{2\delta+2s+1}
\]

\[
= \left( \int_0^2 \left[ (1 - \eta)^{-1-s}\eta^{-1-s} - \eta^{-1-s} \right] d\eta + \frac{2^{-\delta+s+1}}{\delta + s + 1} \right) A^2 \xi^{2\delta+2s+1}. \tag{3.3.6}
\]

By comparing (3.3.5) and (3.3.6) we conclude

\[
f(\xi) \sim A \xi^{-1-2s} \text{ as } \xi \to 0, \tag{3.3.7}
\]

with

\[
A = \frac{-1}{\int_0^2 [(1 - \eta)^{-1-s}\eta^{-1-s} - \eta^{-1-s}] d\eta - \frac{2^s}{s}} = \frac{-2\Gamma(-2s)}{\Gamma^2(-s)}.	ag{3.3.8}
\]

Notice that \( A = s + o(s^3) \) for \( s \ll 1 \).

On the other hand, for \( \xi \gg 1 \), by introducing the ansatz \( f(\xi) \sim B \xi^\delta e^{-\xi} \) into (3.2.1) we find that the leading order contributions from the right and left hand sides are such that

\[-\beta BA \xi^{\delta+1} e^{-B\xi} \sim \frac{B^2}{2} \xi^{2\delta+1+2s} e^{-B\xi} \int_0^1 (1 - \eta)^{s+\delta}\eta^{s+\delta} d\eta\]

so that

\[\delta = -2s, \quad B = -2\beta A \frac{\Gamma(2+2s)}{\Gamma^2(1+s)},\]

and hence

\[f(\xi) \sim -2\beta A \frac{\Gamma(2+2s)}{\Gamma^2(1+s)} \xi^{-2s} e^{-A\xi} \text{ as } \xi \to \infty. \tag{3.3.9}\]

As in the case \( s = 0 \), there are also solutions that do not decay exponentially fast but instead decay algebraically fast. For them, the left hand side of (3.2.2) vanishes at leading order, i.e.

\[f(\xi) \sim A \xi^{-1-2s+1/\beta} \text{ as } \xi \to \infty. \tag{3.3.10}\]

The next order can be computed by plugging (3.3.10) at the right hand side of (3.2.2) and solving the resulting equation for the correction \( \tilde{f}(\xi) \) to (3.3.10):

\[((2s + 1) \beta - 1) \tilde{f}(\xi) + \beta \xi \tilde{f}(\xi) \sim A^2 c_{s,\beta} \xi^{-1-2s+2/\beta},\]

where \( c_{s,\beta} \) is a numerical constant that can be easily computed. Hence,

\[f(\xi) \sim A \xi^{-1-2s+1/\beta} + O(\xi^{-1-2s+2/\beta}) \text{ as } \xi \to \infty. \]

Notice that the asymptotics (3.3.10) agrees with (3.3.3) in the limit \( s \to 0 \). As in (3.3.3), \( A \) will be a function of \( \beta \) and it will be the condition that \( A \) vanishes (so that (3.3.9) holds) what serves to select the value of \( \beta \).
3.3.2. Case $s < 0$

For $\xi \ll 1$, the last term at the right hand side of (3.2.1) is more singular than the first term at the left hand side. Hence, by comparing $\beta \xi f(\xi)$ with $-\xi^s f(\xi)$ where $G$ stands for $\int_0^\xi \eta^s f(\eta) d\eta$ we find a solution with the leading order behaviour $e^{-\frac{G\xi^s}{s\xi}}$. Since $s > 0$ and $s < 0$, one expects a very fast decay to zero as $\xi \to 0$ and hence we should neglect the first term at the right side of (3.2.1). By doing so, we obtain an ordinary differential equation with solution:

$$f(\xi) \sim \frac{A}{\xi^{2s+1-\frac{1}{s}}} e^{-\frac{G\xi^s}{s\xi}} \text{ as } \xi \to 0,$$

(3.3.11)

(see also [215] and [28] where the same behaviour is shown) and, indeed the integral term is such that

$$\int_0^\xi (\xi - \eta)^s \eta^s f(\xi - \eta) f(\eta) d\eta \sim \int_0^\xi (\xi - \eta)^{\frac{1}{2} - 1 - s} \eta^{\frac{1}{2} - 1 - s} e^{-\frac{G\xi^s}{s\xi}} - \frac{G(\xi - \eta)^s}{s\xi} d\eta$$

$$\leq \xi^{\frac{1}{2} - 1 - 2s} e^{-\frac{G\xi^s}{s\xi}} \ll \xi^{2s - 1 + \frac{1}{s}} e^{-\frac{G\xi^s}{s\xi}}.$$

Concerning the behaviour as $\xi \to \infty$, the same argument that applied for the case $0 < s < 1$ also applies to the present case and hence the asymptotics is given by (3.3.9).

Notice that the asymptotics given by (3.3.7), (3.3.9) and (3.3.11) contain two free parameters: $A$ and $\beta$. The first parameter can be fixed from the condition that the first moment of $f(\xi)$ (that is, the total mass) is given and, say, equal to 2:

$$\int_0^\infty \xi f(\xi) d\xi = 2.$$

The second parameter, the similarity exponent $\beta$, has to be chosen so that all moments of $f(\xi)$ are bounded. Unfortunately this can only be done once a global solution to equation (3.2.1) is found. This will be done, in the next section, explicitly for $s < 0$ and by means of a perturbative approach for $|s| \ll 1$.

3.4. The Selection of the Similarity Exponent $\beta$

The similarity exponent $\beta$, which so far is free, can be found in the case $s < 0$ by imposing the condition that all moments of the solution to (3.2.1) are bounded. This yields a nonlinear eigenvalue problem that can, nevertheless, be easily solved based on the asymptotics developed in the previous section. If we multiply equation (3.2.1) by $\xi$, integrate by parts the term $\xi^2 f_{\xi}$ using the cancellation of boundary terms (due to the fast decay of $f$ at the origin and infinity) as well
as the relation
\[
\frac{1}{2} \int_0^\infty \int_0^\xi (\xi - \eta)^s \eta^s f(\xi - \eta) f(\eta) \, d\eta d\xi - \int_0^\infty \xi^s f(\xi) \, d\xi \int_0^\infty \eta^s f(\eta) \, d\eta
\]
\[
= \frac{1}{2} \int_0^\infty \int_0^\xi (\xi - \eta)^{s+1} \eta^s f(\xi - \eta) f(\eta) \, d\eta d\xi + \frac{1}{2} \int_0^\infty \int_0^\xi (\xi - \eta)^s \eta^{s+1} f(\xi - \eta) f(\eta) \, d\eta d\xi - \int_0^\infty \xi^{s+1} f(\xi) \, d\xi \int_0^\infty \eta^s f(\eta) \, d\eta
\]
\[
= \frac{1}{2} M_{1+s} M_s + \frac{1}{2} M_s M_{1+s} - \frac{1}{2} M_{1+s} M_s = 0;
\]
we conclude
\[
((2s - 1) \beta - 1) M_1 = 0.
\]
Since \( M_1 > 0 \), the relation
\[
\beta = -\frac{1}{1 - 2s}, \quad (3.4.1)
\]
follows.

The argument above cannot be extended to the case \( s \in (0, \frac{1}{2}) \) where equation \((3.2.2)\) holds. Neither the solution is bounded at the origin nor cancellations of moments at the right hand side of \((3.2.2)\) takes place. We present next the analysis for \( s \ll 1 \) and later we will find \( \beta \) numerically for other positive values of \( s \). Note that equation \((3.4.1)\) for \( \beta \) does not need to hold for \( s > 0 \). For the purpose of analysis, it will be more convenient to consider the equation for selfsimilar solutions in the Laplace transform, that is (equation \((3.2.3)\)):
\[
(2s \beta - 1) g - \beta \lambda D_\lambda g = \frac{1}{2} \left( D_\lambda^{-s} g \right)^2. \quad (3.4.2)
\]

We introduce
\[
\beta = -1 + B \varepsilon + O(\varepsilon^2), \quad g = g_0 + \varepsilon g_1 + O(\varepsilon^2),
\]
with
\[
g_0(\lambda) = \frac{-2\lambda}{1 + \lambda},
\]
in \((3.4.2)\) and obtain
\[
\left( -2\varepsilon - 1 + O(\varepsilon^2) \right) \left( g_0 + \varepsilon g_1 + O(\varepsilon^2) \right) \left( 1 - B \varepsilon + O(\varepsilon^2) \right) \lambda \left( g_0 + \varepsilon g_1 + O(\varepsilon^2) \right)
\]
\[
= \frac{1}{2} \left( g_0 + \varepsilon g_1 + \varepsilon g_{0, \log} + O(\varepsilon^2) \right)^2,
\]
where
\[
g_{0, \log} = 2 \int_0^\infty (e^{-\lambda x} - 1) \log x e^{-x} \, dx. \quad (3.4.4)
\]
Since \( g_0 \) satisfies \((3.4.2)\) with \( \varepsilon = 0, \beta = -1 \), we obtain by retaining the \( O(\varepsilon) \) terms in \((3.4.3)\) the equation
\[
-2g_0 - g_1 - B \lambda g_0' + \lambda g_1' = g_0 g_1 + g_0 g_{0, \log}.
\]
which can be rewritten as
\[ Lg_1 = g_0g_0, \log + 2g_0 + B\lambda g_0' \quad (3.4.5) \]
with
\[ Lg_1 := -g_1 + \lambda g_1' - g_0 g_1. \]
The question is then: what is the value of $B$ in equation (3.4.5) so that $g_1(\lambda)$ is the Laplace transform of a function with all its moments bounded? In this way, $B$ appears as a compatibility condition for (3.4.5).

Notice that, for $|\lambda|$ sufficiently small, we can expand $g_0(\lambda)$ in the form:
\[ g_0 = -2\lambda + 2\lambda^2 - 2\lambda^3 + \ldots. \]

Likewise, $g_{0, \log}(\lambda)$ can be expanded as
\[
\begin{align*}
g_{0, \log} &= 2\int_0^\infty (e^{-\lambda x} - 1) \log(xe^{-x})dx = -2\lambda \int_0^\infty x \log xe^{-x}dx \\[
+ \lambda^2 \int_0^\infty x^2 \log xe^{-x}dx + \ldots = 2(\gamma - 1)\lambda + (3 - 2\gamma)\lambda^2 + \ldots,
\end{align*}
\]
where $\gamma \approx 0.5772\ldots$ is the Euler’s constant (cf. [1]). Hence, the right hand side of (3.4.5) can be expanded as
\[
\begin{align*}
g_0g_{0, \log} + 2g_0 + B\lambda g_0' &= (-2\lambda + 2\lambda^2 - 2\lambda^3 + \ldots) \{ 2 + (2(\gamma - 1)\lambda + (3 - 2\gamma)\lambda^2 + \ldots \} + B(-2\lambda + 4\lambda^2 + \ldots) \\
&= -(2B + 4)\lambda + (8 - 4\gamma + 4B)\lambda^2 + \ldots. \quad (3.4.6)
\end{align*}
\]

If we look for a solution $g_1(\lambda)$ to (3.4.5) that is analytic in a neighborhood of $\lambda = 0$, we write
\[ g_1 = a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + \ldots, \quad (3.4.7) \]
and by straightforward calculation one finds
\[ Lg_1 = -g_1 + \lambda g_1' - g_0 g_1 = (a_2 + a_1)\lambda^2 + \ldots, \quad (3.4.8) \]
so that it is not possible to match the $O(\lambda)$ term in (3.4.6) with an equivalent term in (3.4.8) unless $B = -2$. Therefore, the similarity exponent $\beta$ has to be chosen, as a function of $\varepsilon$, as
\[ \beta(\varepsilon) = -1 - 2\varepsilon + O(\varepsilon^2). \]

By comparing the coefficients of $\lambda^2$ in (3.4.6) and (3.4.8) we obtain
\[ a_2 + 2a_1 = -4\gamma, \]
and provided $a_1 = 0$ (which implies $\int_0^\infty \xi f_1(\xi) d\xi = 0$), one has
\[ a_2 = -4\gamma. \]
By using:

\[
\int_0^\infty (e^{-\lambda \xi} - 1) \log \xi e^{-\xi} d\xi = -\frac{\gamma + \log(\lambda + 1)}{\lambda + 1} + \gamma,
\]

\[
\int_0^\infty (e^{-\lambda \xi} - 1) e^{-\xi} d\xi = -\frac{\lambda}{\lambda + 1},
\]

we can get an explicit expression for \( g_1(\lambda) \):

\[
g_1(\lambda) = -\frac{4\lambda}{(1 + \lambda)^2} \int_0^\lambda \left( \frac{(\gamma + 1)z - \log(1 + z)}{z} \right) dz
\]

\[
= -\frac{4(\gamma + 1)\lambda^2}{(1 + \lambda)^2} - \frac{4\lambda}{(1 + \lambda)^2} Li_2(-\lambda),
\]

where \( Li_2(z) \) is is the dilogarithmic function defined as (see \[\text{[1]}\]):

\[
Li_2(z) = -\int_0^z \frac{\log(1 - u)}{u} du,
\]

with the integration contour in the complex plane avoiding the branch-cut singularity at \( \Re(u) < -1, \Im(u) = 0 \).

In the case that \( B \neq -2 \), the expansion \([3.4.7]\) has to be replaced by

\[
g_1 = a_1 \lambda + a_2 \lambda \log \lambda + a_3 \lambda^2 + ..., \]

and by computing the left and right hand sides of \([3.4.5]\) we obtain

\[
a_2 = -(2B + 4). \]

Therefore,

\[
g(\lambda) = -2\lambda - \varepsilon(2B + 4)\lambda \log \lambda + O(\lambda^2),
\]

which is the first order of the expansion in \( \varepsilon \) of

\[
g(\lambda) = -2\lambda^{1+\varepsilon(B+2)} + O(\lambda^2),
\]

and whose inverse Laplace transform is proportional to \( \xi^{-2-\varepsilon(B+2)} = \xi^{-1-2\varepsilon+\varepsilon^2+O(\varepsilon^2)} \), in agreement with \([3.3.10]\).

### 3.5. Matching at Infinity and Refined Asymptotics at the Origin

In this section we will determine, from the expression for \( g_1(\lambda) \) given by \([3.4.9]\), the free coefficients \( A \) in the asymptotic behaviours given by \([3.3.9]\). This will be done for \( s = \varepsilon, |\varepsilon| < 1 \). First, note that by writing \( \lambda = -1 + r \) we can expand \([3.4.9]\), for \(|r| < 1\), in the form

\[
g_1(-1 + r) = -\frac{4}{r^2} \left( -Li_2(1) + \gamma + 1 \right) + 4 \frac{\log r}{r} + \frac{Li_2(1)}{r} + o(r^{-1}).
\]
Observe next that

\[ g_1'(\lambda) = - \int_0^\infty e^{-\lambda \xi} \xi f_1(\xi) d\xi. \]

Hence, inverting the Laplace transform, we get

\[ \xi f_1(\xi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} e^{\lambda \xi} g_1'(\lambda) d\lambda = \frac{e^{-\xi}}{2\pi i} \int_{-\infty}^{\infty} e^{\xi} \left( \frac{8(-Li_2(1) + \gamma + 1)}{r^3} - 4 \log \frac{r}{2} + 4 \frac{1 + Li_2(1)}{r^2} + o(r^{-2}) \right) dr, \tag{3.5.1} \]

and by using integration contour deformation, the residue theorem and letting \( \xi \to \infty \), one can easily estimate

\[ f_1(\xi) \sim -4(-Li_2(1) + \gamma + 1) \xi e^{-\xi} - 4 \log \xi e^{-\xi} - 4(1 - \gamma - Li_2(1)) e^{-\xi}. \]

On the other hand, writing \( A = 1 + a\varepsilon \) and expanding (3.3.9) in \( \varepsilon \) we get

\[ f(\xi) = 2e^{-\xi} - 2\varepsilon a\xi e^{-\xi} - 4\varepsilon \ln \xi e^{-\xi} + O(\varepsilon e^{-\xi}). \]

Therefore

\[ a = 2(-Li_2(1) + \gamma + 1) = 2 \left( -\frac{\pi^2}{6} + \gamma + 1 \right), \]

and then

\[ f(\xi) \sim 2 \left( 1 + \left( \frac{\pi^2}{3} + 2\gamma - 2 \right) \varepsilon + O(\varepsilon^2) \right) \xi^{-2\varepsilon + O(\varepsilon^2)} e^{-1 + \left( -\frac{\pi^2}{3} + 2\gamma + 2 \right) \varepsilon + O(\varepsilon^2)}, \text{ as } \xi \to \infty. \tag{3.5.2} \]

Next we discuss how to match (3.5.2) with the behaviours (3.3.7) (for \( \varepsilon > 0 \)) and (3.3.11) (for \( \varepsilon < 0 \)) near the origin. The procedure will yield intermediate regions with distinct features that we analyze separately.

### 3.5.1. Case \( \varepsilon > 0 \)

An explicit solution to (3.2.2) is given by

\[ f(\xi) = A \xi^{-1-2\varepsilon}, \]

with \( A \) defined in (3.3.8). If we introduce a small perturbation \( W \) in the form

\[ f(\xi) = A \frac{1 + W}{\xi^{1+2\varepsilon}}, \tag{3.5.3} \]

and linearize equation (3.2.2) for \( W \) we deduce

\[ -W + \beta \xi W_{\xi}(\xi) = A \int_0^{\frac{1}{2}} \left( \frac{W(\xi(1 - \eta)) + W(\xi\eta)}{(1 - \eta)^{1+\varepsilon} \eta^{1+\varepsilon}} - \frac{W(\xi\eta) + W(\xi)}{\eta^{1+\varepsilon}} \right) d\eta \]

\[ -AW(\xi) \xi^\varepsilon \int_0^{\infty} \frac{1}{\eta^{1+\varepsilon}} d\eta - A\xi^\varepsilon \int_0^{\xi} \frac{W(\eta)}{\eta^{1+\varepsilon}} d\eta. \tag{3.5.4} \]
We look for solutions to (3.5.4) in the form

\[ W = \xi^\alpha, \]

yielding the following equation for \( \alpha \):

\[ \frac{1}{|\beta|} + \alpha = A \frac{1}{|\beta|} \frac{\Gamma(\alpha - \varepsilon)\Gamma(-\varepsilon)}{\Gamma(\alpha - 2\varepsilon)}, \tag{3.5.5} \]

with \( A \) given by (3.3.8). The relation (3.5.5) and this analysis of small perturbations of (3.3.7) near the origin are not limited to small values of \( \varepsilon \) and is valid if we replace \( \varepsilon \) by an arbitrary \( s \in (0, \frac{1}{2}) \). Equation (3.5.5) cannot be solved for \( \alpha \) in closed form. Nevertheless, if \( \varepsilon \) is small, one can find the solution

\[ \alpha = \frac{1}{2|\beta|} \left( \varepsilon^2 |\beta|^2 + 8\varepsilon |\beta| + 4 + \varepsilon |\beta| - 2 \right) = \varepsilon + O(\varepsilon^2). \]

Notice then that the general form of \( W \) is

\[ W(\xi) = C\xi^{\varepsilon + O(\varepsilon^2)}. \tag{3.5.6} \]

The constants \( C \) and \( \beta \) in (3.5.6) are free and should be chosen so that the first moment is given (which chooses \( C \)) and all other moments \( M_n \) (\( n > 1 \)) are bounded (that is, the solution decays exponentially fast at infinity, formula (3.5.2)). The exact computation of \( C \) can only be done numerically, but we can nevertheless provide a rough sketch the matching procedure. From (3.4.9) it is possible to approximate

\[ g_1(\lambda) \sim \frac{2(\log \lambda)^2}{(\lambda - 1)} \text{ as } |\lambda| \to \infty, \tag{3.5.7} \]

and by contour deformation we conclude

\[ f_1(\xi) \sim \frac{1}{2\pi i} \int_0^\infty e^{-\xi r} \left( \frac{2(\log r + \pi i)^2}{1 + r} - \frac{2(\log r - \pi i)^2}{1 + r} \right) dr \]

\[ = 4 \int_0^\infty e^{-\xi r} \frac{\log r}{1 + r} dr \sim -4 \log \xi, \text{ as } \xi \to 0. \]

Hence,

\[ f_0(\xi) + \varepsilon f_1(\xi) \sim 2 - 4\varepsilon \log \xi + O(\varepsilon^2), \tag{3.5.8} \]

for \( \xi \gg e^{-\varepsilon^{-1}} \). Since \( 2\xi^{-2\varepsilon} \sim 2 - 4\varepsilon \log \xi + O(\varepsilon^2) \) for \( \xi \gg e^{-\varepsilon^{-1}} \), we conclude that (3.3.7) and (3.5.8) are of the same order of magnitude for \( \xi = O(\varepsilon) \), and this sets the size of the inner boundary layer where (3.3.7) represents the asymptotic behaviour for the solution. Between this inner layer and the external region where (3.3.2) holds, there is an intermediate region where the perturbation \( W \) in (3.5.3) becomes dominant and therefore \( f(\xi) \sim C'(\xi^{-1+\alpha-2\varepsilon} \right). \]

### 3.5.1. Case \( \varepsilon < 0 \)

In the case \( \varepsilon < 0 \) we obtained the asymptotic term near the origin:

\[ f(\xi) \sim \frac{A}{\xi^{2\varepsilon+1-\frac{4}{\beta}}} e^{-\frac{\xi\varepsilon}{\beta^2} \varepsilon^2} \text{ as } \xi \to 0. \]
By expanding
\[
e^{-\frac{G\xi}{\pi}} = e^{-\frac{2G}{\pi}} e^{-\frac{G}{\pi} \log \xi} e^{-\frac{G}{\pi} (\log \xi)^2 + \ldots} = e^{-\frac{2G}{\pi}} \xi^{-\frac{G}{\pi}} e^{-\frac{G}{\pi} (\log \xi)^2 + \ldots},
\]
which is convergent if \(\xi \leq e^{-|\varepsilon|^{-\frac{1}{2}}}\), and defining
\[
A = e^{\frac{G}{\pi}} a(\varepsilon),
\]
we conclude \(f(\xi) = aO(1)\) and hence
\[
Q(\xi) = \int_0^\xi (\xi - \eta)^{1-\varepsilon} f(\xi - \eta) f(\eta) d\eta \leq A^2 \xi^{2-1-2\varepsilon} e^{-\frac{2G}{\pi} \xi^{2-1-\varepsilon}} \approx \frac{a^2}{24} \xi^{1+O(\varepsilon)},
\]
for \(\xi \leq e^{-|\varepsilon|^{-\frac{1}{2}}}\). By integrating the equation for self-similar solution we arrive at the formula
\[
f(\xi) = e^{\frac{G}{\pi}} \xi^{2-1-\varepsilon} e^{-\frac{G}{\pi} \xi^{2-1-\varepsilon}} \left[ a(\varepsilon) + \int_0^\xi \frac{\eta^{2\varepsilon-\frac{1}{2}}} {2\beta} e^{-\frac{G}{\pi} (\eta^{\varepsilon-1})} Q(\eta) d\eta \right]. \tag{3.5.9}
\]
Given the asymptotics for \(Q(\xi)\) we find that the integral at the right hand side of (3.5.9) is \(O(e^{-|\varepsilon|^{-\frac{1}{2}}} - \varepsilon)\) for \(\xi \leq e^{-|\varepsilon|^{-\frac{1}{2}}}\). Hence, we can neglect the contribution to the integral from the region \(\xi \leq e^{-|\varepsilon|^{-\frac{1}{2}}}\) and integrate outside this region using (3.5.2) so that
\[
Q(\xi) \approx 4(1 + 2 \left(\frac{\pi^2}{3} + 2\gamma - 2\right) \varepsilon + O(\varepsilon^2)) e^{-\left(1 + (-\frac{2\pi^2}{3} + 2\gamma + 2)\varepsilon + O(\varepsilon^2)\right)} \xi^{1-2\varepsilon} d\eta.
\]
Since
\[
\int_0^\xi (\xi - \eta)^{-\varepsilon} \eta^{-\varepsilon} d\eta = (1 + 2\varepsilon + O(\varepsilon^2)) \xi^{1-2\varepsilon},
\]
we will have a solution to (3.5.9) provided
\[
a(\varepsilon) = -\int_0^\infty \frac{\eta^{2\varepsilon-\frac{1}{2}}}{2\beta} e^{-\frac{G}{\pi} \eta^{\varepsilon}} Q(\eta) d\eta \approx -\int_0^\infty \frac{\eta^{2\varepsilon-\frac{1}{2}} + G}{2\beta} Q(\eta) d\eta
\]
\[
= -\frac{2}{\beta} \left(1 + 2 \left(\frac{\pi^2}{3} + 2\gamma - 1\right) \varepsilon + O(\varepsilon^2)\right) \int_0^\infty \eta^{1-\frac{1}{2} + \frac{G}{\beta}} e^{-\left(1 + (-\frac{2\pi^2}{3} + 2\gamma + 2)\varepsilon + O(\varepsilon^2)\right)} \eta^{\varepsilon} d\eta
\]
\[
= -\frac{2}{\beta} \left(1 + 2 \left(\frac{\pi^2}{3} + 2\gamma - 1\right) \varepsilon + O(\varepsilon^2)\right) \Gamma(2 - \frac{1}{\beta} + \frac{G}{\beta}),
\]
and using
\[
\beta = -1 - 2\varepsilon + O(\varepsilon^2), \tag{3.5.10}
\]
we find
\[
G = 2 \left(1 + \left(\frac{\pi^2}{3} + 2\gamma - 2\right) \varepsilon + O(\varepsilon^2)\right) \Gamma(1 - \varepsilon) \tag{3.5.11}
\]
\[
= 2 + \left(2\gamma + \frac{4}{\pi^2} - 8\right) \varepsilon + O(\varepsilon^2),
\]
and
\[
\]
providing the value of the free parameter $G$ for the asymptotic value of the solution at the origin. Hence, the matching is now complete and all parameters determined for the selfsimilar solution $f(\xi)$. We can also find

$$a(\varepsilon) = 2 + \left(4\gamma^2 - 16\gamma + \frac{8}{3}\gamma\pi^2 + 2\pi^2 - 12\right) \varepsilon + O(\varepsilon^2). \quad (3.5.12)$$

Finally, by expanding the first factor at the right hand side of (3.5.9) and using (3.5.10), (3.5.11) and (3.5.12) we conclude

$$\frac{e^{\frac{2\pi^2}{3\xi^2}}}{\xi^{1+\frac{1}{2}\varepsilon}} e^{\frac{\varepsilon}{\pi}} \sim \frac{1}{\xi^{(2\gamma + \frac{1}{2}\pi^2 - 4)\pi}} e^{e(\log \xi)^2}, \quad (3.5.13)$$

for any $e^{-|\varepsilon|^{-1}} \ll \xi \ll e^{-|\varepsilon|^{-1/2}}$. Notice that we can rewrite the right hand side of (3.5.13) as $e^{e(\log \xi)^2+e(2\gamma + \frac{1}{2}\pi^2 - 4)\pi} \log \xi$, which is a function of $\log \xi$ that decays at $\pm \infty$ and whose maximum value is $e^{e(2\gamma + \frac{1}{2}\pi^2 - 2)^2} = 1 + O(\varepsilon)$ (as one can easily verify). For $\xi \lesssim e^{-|\varepsilon|^{-1/2}}$, the integral at the right hand side of (3.5.9) is still negligible, while (3.5.13) is $1 + O(\varepsilon)$. At some $\xi > e^{-|\varepsilon|^{-1/2}}$, $f(\xi)$ reaches its maximum and starts to decrease due to the increase of the integral at the right hand side of (3.5.9) and eventually decays exponentially fast as given by (3.5.12). Hence, we can distinguish three regions: a) the region $\xi \ll e^{-|\varepsilon|^{-1}}$ where

$$f(\xi) \sim a(\varepsilon) e^{\frac{2+O(\varepsilon)}{\varepsilon}} e^{\frac{2+O(\varepsilon)}{\varepsilon}} e^{(2+O(\varepsilon))(\log \xi)^2},$$

which decays extremely fast to zero as $\xi \to 0$ (faster than any power), b) the region $e^{-|\varepsilon|^{-1}} \ll \xi \ll e^{-|\varepsilon|^{-1/2}}$ where

$$f(\xi) \sim a(\varepsilon) e^{e(\log \xi)^2+e(2\gamma + \frac{1}{2}\pi^2 - 4)\pi} \log \xi, \quad (3.5.14)$$

and where a transition between the first region and the maximum value of $f(\xi)$ takes place, and c) outer region $\xi \gg e^{-|\varepsilon|^{-1/2}}$ where $f(\xi)$ is a small perturbation of $2e^{-\xi}$ for $\xi \lesssim e^{e^{-|\varepsilon|^{-1}}}$ and the asymptotic behaviour is given by (3.5.2).

It is worth noting that the behaviour implied by (3.5.14) is similar to that of a lognormal distribution, while the asymptotics (3.5.2) corresponds to a gamma distribution.

### 3.6. Selfsimilar Solutions for $\varepsilon = -n$

In the particular case when the kernel is of the form $K(x, y) = (xy)^{-n}, (n = 1, 2, ...)$, equation (3.2.2), written in terms of $F(\xi) = f(\xi)/\xi^n$, takes the form

$$((-n + 1) - 1) \xi^n F + \beta \xi^{n+1} F(\xi) = \frac{1}{2} \int_0^{\xi} F(\xi - \eta)F(\eta)d\eta - F(\xi) \int_0^{\xi} F(\eta)d\eta. \quad (3.6.1)$$

By defining the Laplace transform

$$G(\lambda) = \int_0^{\infty} e^{-\lambda \xi} F(\xi)d\xi,$$
equation (3.6.1) takes the form
\[ (-1)^n ((-n + 1) \beta - 1) \frac{d^n G(\lambda)}{d\lambda^n} + (-1)^{n+1} \beta \frac{d^{n+1}}{d\lambda^{n+1}} (\lambda G(\lambda)) = \frac{1}{2} G^2(\lambda) - G(0)G(\lambda). \] \tag{3.6.2}

By suitably rescaling variable and function, we can assume \( G(0) = 1. \) If we seek for a solution that is analytic near the origin \( \lambda = 0, \)
\[ G(\lambda) = 1 + \sum_{m=1}^{\infty} a_m \lambda^m, \]
we find that the right hand side of (3.6.2) is
\[ RHS = -\frac{1}{2} + \frac{a^2}{2} \lambda^2 + O(\lambda^3). \]
That is, there is no \( O(\lambda) \) term. Hence, the linear right hand side of (3.6.2) cannot contain \( O(\lambda) \) term. This implies \( a_{n+1} = 0 \) or
\[ \beta = \beta_n = -\frac{1}{2n+1}. \] \tag{3.6.3}
The first possibility \( (a_{n+1} = 0) \) would imply that the \( M_{n+1} \) moment vanishes, which is not possible for a positive solution. Therefore, the similarity exponent will generically be given by (3.6.3) as we know from (3.4.1) will verify numerically in the next section.

Finally, notice the possibility of a pole of \( G(\lambda) \) at \( \lambda = -a \) (a real and positive) which is a local solution to (3.6.2) where the dominant contributions balance:
\[ (-1)^{n+1} \beta_n \frac{d^{n+1}}{d\lambda^{n+1}} (\lambda G(\lambda)) \approx \frac{1}{2} G^2(\lambda). \] \tag{3.6.4}
By inserting \( G(\lambda) = c/(\lambda + a)^\alpha \) into (3.6.4) we find, at leading order, \( \alpha = -(n+1), c = 2a\beta_n \frac{(2n+1)!}{n!} \) and therefore
\[ G(\lambda) \sim \frac{2(2n)!}{n!} \frac{a}{(\lambda + a)^{n+1}}, \text{ as } \lambda \to -a. \] \tag{3.6.5}
This implies a generic behaviour of \( f(\xi) \) (inverting Laplace transform) of the form
\[ f(\xi) \sim C_n a \xi^n e^{-a\xi}, \text{ as } \xi \to \infty, \]
where \( C_n \) can be computed straightforwardly by evaluation of the residue given by the pole of \( G(\lambda) \) at \( \lambda = -a \) when inverting the Laplace transform. The parameter \( a \) is free, but should be estimated from the condition \( G(0) = 1 \) once \( G(\lambda) \) is evaluated in \( \mathbb{R} \lambda < 0. \) This selection of the free parameter \( a \) can be done analytically for \( n \gg 1. \) In this case, by evaluating the right hand side of (3.6.5) at \( \lambda = 0 \) we find the identity
\[ \frac{(2n)!}{n!} a^{-n} = 1, \]
which yields, using Stirling’s formula,
\[ a = \left( \frac{(2n)!}{n!} \right)^{\frac{1}{n}} \approx \left( \frac{e^{-2n}(2n)^{2n}\sqrt{4\pi n}}{e^{-n}n^n\sqrt{2\pi n}} \right)^{\frac{1}{n}} \approx 4e^{-1}n. \]
Since
\[ C_n \sim \frac{2(2n)!}{(n!)^2} \sim \frac{2e^{-2n}(2n)^{2n}\sqrt{4\pi n}}{e^{-2n}n^{2n}(2\pi n)} = 2^{2n+1} \frac{1}{\sqrt{n}}, \]
we can conclude
\[ f(\xi) \sim \frac{8e^{-1}}{\sqrt{\pi}} \sqrt{n} \xi^ne^{-4e^{-1}n\xi} \text{ as } \xi \to \infty, \]
(3.6.6)
If we take the right hand side of (3.6.6) as valid for any \( \xi > 0 \), then we find a local behaviour near the \( n \)-dependent maximum of \( f(\xi) \) described by
\[ f(\xi) \sim \frac{8e^{-1}}{\sqrt{\pi}} \sqrt{n} \xi^ne^{-4e^{-1}n\xi} \Phi \left( \sqrt{n}(\xi - \xi_0) \right), \]
(3.6.7)
with \( \xi_0 = e/4 \), and \( \Phi \) a gaussian function. Hence, \( f(\xi) \) would approach a Dirac delta as \( n \to \infty \). Of course, the assumption that (3.6.6) is valid for any \( \xi > 0 \) is not correct, but the conclusion that \( f(\xi) \) converges to a certain rescaled (with \( n \)) function \( \Phi \) as \( n \to \infty \) will be verified numerically in the next section.

### 3.7. Computation of numerical solutions to the self-similar problem (3.5.9)

Equation (3.5.9), which is valid for \( s < 0 \), provides a simple way to numerically compute the selfsimilar solutions. Notice first that the term \( Q(\eta) \) involves an integral over the interval \([0, \xi/2]\). Hence, all information at the right hand side of (3.5.9) concerning values of \( f(\eta) \) for \( \eta > \xi \) is limited to the real parameter \( G = \int_{\xi/2}^{\infty} \eta^s f(\eta)d\eta \). We will take an arbitrary value of \( G \) (remember that the selfsimilar solutions, for a given \( s \), are indeed a 1-parameter family \( \ell^{1+2s}f(\xi) \) so that the arbitrariness of \( \ell \) can be translated into the arbitrariness of \( G \)), an arbitrary value of \( \beta \) and an arbitrary value of \( a(s) \), compute the solution \( f(\xi_i), \xi_i = hi \) for \( i = 1, ..., N \) and \( h = L/N \) with \( L \) and \( N \) sufficiently large (where \( L \) represents the length of the domain and will be taken large) by computing the integral at the right hand side of (3.5.9), and check whether \( f(L) \) is positive or negative. By shooting with the parameter \( a(s) \) we obtain a solution \( f(\xi) \) which is positive and such that \( f(L) \) gets as close as desired to zero. If \( L \) is sufficiently large, such solution is very close to our selfsimilar solution. After such solution is computed, we numerically evaluate
\[ G_{out} = \int_0^{\infty} \eta^s f(\eta)d\eta. \]

In general \( G_{out} \neq G \) so that the solution constructed is not consistent with the value of \( G \) taken a priori, but by choosing \( \beta \) appropriately we can make \( G_{out} = G \) therefore finding the similarity exponent \( \beta \). To summarize, our method is a shooting procedure with two parameters, \( a(s) \) and \( \beta \), and the two conditions to find these parameters (or nonlinear eigenvalues) are: 1) the resulting solution is positive and \( f(L) = 0, 2) G_{out} = G \).

As it was expected, the numerical values of \( \beta \) as a function of \( s \) approach the curve
\[ \beta = \frac{1}{1 - 2s}, \]
(3.7.1)
within less than 1% of relative error. In Figures 3.7.1, 3.7.2 we represent the similarity solutions for various values of $s$. Notice the existence of a change in the shape of the similarity solutions as $|s|$ increases. For small values of $|s|$ the maximum decreases, but eventually, as $|s|$ increases, the maximum starts to grow and the shape of the similarity solutions can be very well represented by

$$f(\xi) \approx |s|^2 \Phi \left( |s|^{1/2} (\xi - 1) \right),$$

for large values of $|s|$, as anticipated by (3.6.7). In Figure 3.7.3 we show the collapse of the rescaled (with $\sqrt{|s|}$) profiles towards a certain function $\Phi$.

![Graph 3.7.1](image)

**Figure 3.7.1:** Similarity solutions for $s = -0.1, -0.2, ..., -1$. The arrow indicates increasing values of $-s$.

![Graph 3.7.2](image)

**Figure 3.7.2:** Similarity solutions for $s = -1, -2, ..., -5$. The arrow indicates increasing values of $-s$. 

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Figure 3.7.3: Rescaled similarity solutions for $-s = 4, 5, \ldots, 10$. Inset: value of $f^2_{\text{max}}$ vs. $|s|$ and comparison with a linear law.

### 3.8. Computation of Numerical Solutions of Smoluchowski Equation

In this section we follow the time evolution of an arbitrary initial distribution $c_0$, numerically treating Smoluchowski’s equation as a differential equation of the form $\partial_t c(x,t) = F(c(x,t), x,t)$ with $F$ given as the right hand side of (3.1.1). Our approach has been to adopt a standard predictor-corrector, fourth order and variable time step integrator. In order to produce the numerical results, we have used almost the same scheme as that originally designed by Lee in [21]. Other authors have worked out more stable and sophisticated versions of this algorithm: we point out the recent contribution of Filbet and Laurençot [135] among them.

Let $x$ be a spatially uniform grid ranging from $x_1 = \delta_x$ to $x_N = N\delta_x$; we will call mass sites or mass bins such $x_k$ following the way they are commonly referred to in the literature. When the possible mass numbers are multiples of a minimum $\delta_x$, Smoluchowski’s equation reduces to a discrete form:

$$\partial_t c(x_j, t) = \frac{1}{2} \delta_x \sum_{l+k=j} K(x_l, x_k) c(x_l, t) c(x_k, t) - c(x_j, t) \delta_x \sum_{k=1}^N K(x_j, x_k) c(x_k, t),$$

whose right hand side can be easily computed numerically. It is also evident that this choice cuts off an infinite quantity of mass sites that, sooner or later, will become dynamically relevant in the system. To avoid this restriction, a change of variable $x \to 1/(1 + x)$ was used to map the positive $x$-axis on the bounded interval $(0, 1)$, but, as it has been clearly pointed out in [135], it is not clear how to control the distribution of the new mesh points or the mass distribution among them. See also [108] and references therein for this kind of approach.
In order to determine the cut mass $x_N$, our empirical criteria has been the following: given $T_f$ the desired ending time, if the solution has to reach a selfsimilar regime $c(x, t) \sim t^{\gamma} \psi \left( (T_f)^{\beta} x \right)$, one can find a proper value for $x$ such that $(T_f)^{\gamma} \psi \left( (T_f)^{\beta} x \right) \leq tol$, where $tol$ is a numerical parameter indicating the maximum permitted density of $x_N$-massed clusters at the final time; the value of $\beta$ can be taken coarsely as $\beta \sim -1 - 2\varepsilon$, giving $\gamma \sim -2 - 4\varepsilon - 4\varepsilon^2$ and a low accuracy $\psi$ can be computed via a previous low order simulation.

A great advantage of an uniformly distributed bin model is that the integrodifferential problem is reduced to a $N$-dimensional vector valued ordinary differential equation. Therefore, standard integration algorithms can be applied with good performances. A predictor-corrector method quickly brings an approximation of an implicit scheme, avoiding the heavy workload that computing $F(c(t, x), t, x)$ at each step would impose; it is, moreover, almost possible to guarantee the conservation of the first moment until the initial mass spreads over the $x$-line, augmenting significantly the lost mass that have reached the tail. As for the variable time step method, such an implementation is highly desirable since the peaks of variation in the distribution of $c$ tend to reduce quickly as the time passes. It is thus possible to gradually augment $\delta_f$ and still maintain a relative $c$-variation small enough. We refer to the huge numeric receipts literature for the reader to find further informations on those classical methods.

To compute the $N$-dimensional vector $F$ we consider all possible binary interactions $\{i, k\}$ between active bins of mass: given a small numerical threshold $\mu$, we define at each time $t$ the set $v = \{ i : c_i(t) x_i \geq \mu \}$. Therefore, in a cycle for $i$ ranging on $v$, we consider $v_i = \{ k \in v : k > i \}$ and for each pair $\{i, k\}_{k \in v_i}$,

$$
F_i = F_i - \delta_x K(x_i, x_k) c_i(t) c_k(t), 
F_k = F_k - \delta_x K(x_i, x_k) c_i(t) c_k(t),
$$

and, if $i + k \leq N$,

$$
F_{i+k} = F_{i+k} + \delta_x K(x_i, x_k) c_i(t) c_k(t).
$$

Notice that we have not included the $\{i, i\}$ pair. It is also necessary to consider it, but it provides only half of the coagulating mass:

$$
F_i = F_i - \delta_x K(x_i, x_i) c_i^2(t), 
F_{2i} = F_{2i} + \frac{1}{2} \delta_x K(x_i, x_i) c_i^2(t), \quad \text{if } 2i \leq N.
$$

A new time step $\Delta t$ is established if the absolute variation between $c(x, t)$ and $c(x, t + \Delta t)$ is less or equal than a given tolerance. It is useful to keep track of the evolution of some relevant moments $M_\alpha(t + \Delta t)$. Since it is impossible to do it exactly with this finite scheme, we define some approximated values $m_\alpha(t + \Delta t)$ which resembles $M_\alpha(t + \Delta t)$, and, after each new step, we compute:

$$
m_\alpha(t + \Delta t) = \sum_{i=1}^{N} x_i^\alpha c_i(t + \Delta t) + \lambda_\alpha(t + \Delta t),
$$

where we consider an associated quantity $\lambda_\alpha(t + \Delta t)$ as the cumulative lost contribution to $m_\alpha$. It is computed in the following way: we consider again all possible binary interactions $\{i, k\}$ between active bins of mass at previous time $t$ and run a cycle for $i$ ranging on $v$, but this time we look only for $v_i^k = \{ k \in v : k > i, k + i > N \}$. This set takes into account only the active pairs that form clusters which exceed the cut mass $x_N$. Since $\delta_x K(x_i, x_k) c_i(t) c_k(t)$ represents the velocity at which clusters of mass $x_{i+k}$ are being produced and $\Delta t$ is the interval of time that has passed,
we can approximately consider that the pair \( \{i, k\} \) has produced \( n_{i,k} = \Delta t \delta_x K (x_i, x_k) c_i (t) c_k (t) \) new clusters of mass \( x_{i + k} \). This rough estimate will only be used to compute the lost contribution to \( m_\alpha \):

\[
\lambda_\alpha (t + \Delta t) = \lambda_\alpha (t) + \sum_{i \in \mathbb{V}} \sum_{k \in \mathbb{V}} ((x_i + x_k)^\alpha - x_i^\alpha - x_k^\alpha) n_{i,k}.
\]

We remark now that, from the instant when a sufficient mass escapes the finite coagulating system (infinite mass region), there are three interactions that are dynamically relevant: finite-finite, infinite-finite and infinite-infinite mass region coagulation. The former can be numerically simulated with our scheme while our knowledge of the tail distribution can only be driven forward via an ansatz (an arbitrary fast decay or a self-similar regime). We preferred nevertheless not to introduce such a tail into play and make the mass leaving the finite coagulating system completely stop coagulating. In that resides the need of a \( x_N \) big enough to harbour the relevant distribution of \( c \) for the solution to go as far as the self-similar regime. In Figure 3.8.4 we present the result of the evolution of an initial data concentrated close to the origin and for \( s = -0.2 \), together with the rescaled profiles. As we can see, the convergence towards the self-similar solution computed by the procedure described in the previous section is remarkable.

We finally describe how to analyze the long term solutions and extract information about \( \beta \). If a self-similar regime has been reached, it can be remarked that:

\[
M_\alpha (t) = \int_0^\infty x^\alpha c (x, t) \, dx = t^{(\alpha+1)\beta} \int_0^\infty \xi^\alpha f (\xi) \, d\xi = t^{(2s-\alpha)\beta -1} N_\alpha,
\]

where \( N_\alpha \) is the corresponding moment of the self-similar solution \( f \) of equation (3.2.1). It is therefore possible to fix a time interval \([t_1, t_F]\) and look for a value \( \iota \) such that \( (m_\alpha (t))^\iota \) is approximately linear. We numerically look for an optimal \( \iota \) such that, in the mean square sense, the error of \( at + b \) fitting \( (m_\alpha (t))^\iota \) for \( t \in [t_1, t_F] \) is minimal. Unfortunately, different values of \( \alpha \) can produce varying approximations for \( \beta \). This is due to the fact that \( m_\alpha \) is a coarse approximation of \( M_\alpha \) for big values of \( \alpha \). Our results have been obtained studying mainly the second moment \( \alpha = 2 \). We found, for \( \beta \) as a function of \( s \).
Figure 3.8.1: Solution of the evolution problem with $\varepsilon = -0.2$ for 7 different times (top) and the rescaled profiles (bottom) together with the similarity solution (dotted line, bottom).
Figure 3.8.2: Solution of the evolution problem in a log-log scale and comparison with the theoretical prediction for the intermediate power-like regime (top) and solution in a log scale showing exponential decay at infinity.
CHAPTER 4
ON THE SELF-SIMILAR SOLUTIONS TO THE SPONTANEOUS FRAGMENTATION EQUATION

In this final chapter, we consider the self-similar equation of the linear fragmentation equation. This model has been studied throughly from the point of view of the evolution equation, or at least its study is far more mature than the non-linear coagulation model. Also, the universality behavior of the solutions is more or less well understood and the self-similar equation has extensive well-posedness results. We focus on carrying over specific investigations: we rigorously determine an explicit formula in the form of infinite product for the self-similar solutions $\varphi (\xi)$ thanks to a Wiener-Hopf method and through the calculation of residues in the complex plane; this is done for both the case of fragmentation which allows infinitesimal fragments and for fragmentation which bounds from below the minimum fragments (this is called the compact support case and presents different analyticity properties). This study leads us to accurately determine the asymptotic behaviors of the solutions in both cases.

4.1. MELLIN TRANSFORM AND THE DERIVED EQUATION

Let $c (t, x)$ be the function representing mass distribution at time $t$. We wish to study its asymptotic behavior when it obeys the fragmentation equation, the linear model that we already introduced in (1.1.29):

$$\frac{d}{dt} c (x, t) = - \beta (x) c (x, t) + \int_x^\infty c (y, t) \frac{\beta (y)}{y} B \left( \frac{x}{y} \right) \, dy. \quad (4.1.1)$$

We recall that $\beta (x)$ defines the spontaneous fragmentation rate of $x$—sized clusters and $B (u)$ the relative distribution density of the split products. Moreover, $B$ must satisfy the consistency condition:

$$B (s) \geq 0, \quad \text{supp} (B) \subset [0, 1] \quad (4.1.2)$$
\[ \int_{0}^{1} s B (s) \, ds = 1. \quad (4.1.3) \]

Equation (4.1.3) represents mass conservation after breakage events and, together with conditions (4.1.2), it corresponds to regularity I condition as it has been used in Section 1.3.2.2. In this Chapter, we will develop our knowledge of the self-similar solution \( \varphi (\xi) \) to the self-similar fragmentation equation:

\[ \partial_{\xi} \varphi (\xi) = -\frac{\beta (\xi) + 2}{\xi} \varphi (\xi) + \frac{1}{\xi} \int_{\xi}^{\infty} \frac{\beta (s)}{s} \varphi (s) \, B \left( \frac{\xi}{s} \right) \, ds, \quad (4.1.4) \]

where \( \beta (x) \) is supposed not to produce singularity in finite time. Let us remark here that it is also possible to study the asymptotic behavior of the solutions of the fragmentation equation when shattering does occur: it has been investigated recently, for example by Haas in [106], in a probabilistic framework, via self-similar Markov processes.

Here instead we consider global-in-time problems of the general class considered by Escobedo, Mischler and Rodriguez Ricard [118] characterized, also, by the extra condition of Section 1.3.2.2. That permitted the authors to obtain the \textit{Universality Theorem for fragmentation without shattering}, Theorem 1.13. We recall that, if we assume \( \beta (\xi) = c \xi^\gamma, \gamma > 0 \), and, for some \( m \leq \gamma \),

\[ B \in BV^1 (0, 1) \cap ML_m, \quad (4.1.5) \]

then, for each mass-value \( M > 0 \) there exists a unique solution \( \varphi_M \) solution to (4.1.4) with \( M_1 = M \). Moreover the regularity space for the solution is \( \varphi_M \in Z_{\infty; m} \) with:

\[ Z_{\infty; m} := BV^1 \cap M_{\infty; m}, \quad (4.1.6) \]

where the space of decaying functions \( M_{\infty; m} \) is defined as:

\[ M_{\infty; m} := \bigcap_{y \geq x} ML_y. \quad (4.1.7) \]

Under different working regularity hypothesis for \( B (s) \)-that are: continuity and asymptotic behaviors near \( s = 1 \) and near \( s = 0 \), but \( B \) needs not to decay as in (4.1.5), we will obtain estimations, explicit formulas and stronger \( C^\infty \)-regularity for \( \varphi \). Our strategy is to analyze the properties of the Mellin transform \( \Phi (\eta) \) of \( \varphi (\xi) \), and then deduce from it the relevant information.

We first recall how Mellin transform is defined:

\[ \Phi (z) = \int_{0}^{\infty} x^{z-1} \varphi (x) \, dx, \quad (4.1.8) \]

for all those \( z \in \mathbb{C} \) such that the integral at the right hand side converges. If we apply Mellin transform to the self-similar fragmentation equation we can write:

\[
[\xi^z \varphi (\xi)]_{0}^{\infty} - z \Phi (z) = - \int_{0}^{\infty} \xi^{z-1} \beta (\xi) \, d\xi + \int_{0}^{\infty} s^{z-1} \beta (s) \varphi (s) \left( \int_{0}^{1} \frac{d\xi}{s^{\xi-1}} B \left( \frac{\xi}{s} \right) \right) \, ds
\]

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and after a change of variable, with \( \hat{\beta}\varphi \) the Mellin transform of the product \( \beta (\xi) \cdot \varphi (\xi) \),

\[
-z\Phi (z) = -2\Phi (z) - \left( \hat{\beta}\varphi \right) (z) + \left( \hat{\beta}\varphi \right) (z) \int_0^1 u^{z-1} B (u) \, du. \tag{4.1.9}
\]

Here it is useful to define the moment transform of \( B (s) \), which is

\[
\Theta (z) = \int_0^1 u^{z-1} B (u) \, du. \tag{4.1.10}
\]

Therefore

\[
(2 - z) \Phi (z) = (\Theta (z) - 1) \left( \hat{\beta}\varphi \right) (z), \tag{4.1.11}
\]

and, in particular, if \( \beta (\xi) = c\xi^\gamma \), we deduce the following functional problem in the complex space:

\[
(2 - z) \Phi (z) = (\Theta (z) - 1) c\Phi (z + \gamma). \tag{4.1.12}
\]

We can notice here that we have supposed \( \beta (\xi) = c\xi^\gamma \) with unspecified constant \( c \) rather that \( \beta (\xi) = \gamma \xi^\gamma \), as in Section 1.3.2.2, where \( c = \gamma \) was fixed. Here we work with this little extra freedom and, in such way, we will see the role of the constant \( c \) in terms of its effect on the solutions \( \Phi (z) \) to problem 4.1.13.

Notice that we are interested in evaluating \( \Phi (z) \) at \( z = \delta + is \) in order to apply an inverse Mellin transform. This is defined as:

\[
\varphi (x) = \frac{1}{2\pi i} \int_{-i\infty+\delta}^{i\infty+\delta} x^{-z} \Phi (z) \, dz, \tag{4.1.13}
\]

where the integration is done along the vertical line \( \mathcal{R} (z) = \delta \) in the complex plane. The reason for this is that \( \delta \) is greater than the real part of all singularities of \( \Phi (z) \). This is due to the fact that we request at least that:

\[
\int_0^\infty x^N \varphi (x) \, dx < \infty, \quad N \geq m.
\]

Much information on the Mellin transform and the so-called Mellin-Barnes integrals can be found in the monograph of Paris and Kaminski [253] to which we refer for details on asymptotic expansions and theoretical results.

— 4.1.1. A more general Mellin transform —

In order to proceed, we need to define a slightly more general Mellin transform and deduce the equation corresponding to 4.1.12.

\[
\Phi_{[\varepsilon]} (z) = \Phi_{[0]} (z - \varepsilon) = \int_0^\infty x^{z-1} \frac{\varphi (x)}{x^{\varepsilon}} \, dx. \tag{4.1.14}
\]
Applying it to (4.1.4) gives:

\[-(z - \varepsilon) \Phi_\varepsilon (z) = -2 \Phi_\varepsilon (z) - c \Phi_\varepsilon (z + \gamma) + c \Phi_\varepsilon (z + \gamma) \int_0^1 u^{z-1} B(u) \, du.\]

Correspondingly, we define the new moment transform:

\[\Theta_\varepsilon (z) = \Theta_0 (z - \varepsilon) = \int_0^1 u^{z-1} \frac{B(u)}{u^\varepsilon} \, du. \quad (4.1.15)\]

Finally we obtain the equation for \( \Phi_\varepsilon \):

\[(2 + \varepsilon - z) \Phi_\varepsilon (z) = c \left( \Theta_\varepsilon (z) - 1 \right) \Phi_\varepsilon (z + \gamma). \quad (4.1.16)\]

We imposed that \( \xi^{z-\varepsilon} \varphi (\xi) \) decay to zero both at infinity and at the origin in the convergence region of the complex plane. We will determine later such region. The reason to introduce this \( \varepsilon \)-translation is that we wish to avoid a singularity on the vertical line of the integration path; when the integral is computed, we are able to let \( \varepsilon \) go to \( \varepsilon \rightarrow 0 \) thanks to the continuity of the result with respect to \( \varepsilon \).

### 4.1.2. Carleman problem

Non-local equation (4.1.16) that we consider belongs to a class of problems called Carleman’s problem. They are widely spread and we recall a particular way of thinking that bring forth useful techniques to solve them: in this work we apply a variation to the Wiener-Hopf methods and manage to find explicit solutions. More information on this class of problems can be found in Escobedo and Velázquez [26] and in the references therein.

We now proceed introducing the new variable \( \zeta (z) = e^{z+2\pi i} \) and we correspondingly denote \( f (\zeta) \) with \( f (\zeta) = \log (\Phi_\varepsilon (z)) \), and rename \( T (\zeta) = \Theta_\varepsilon (z) \). We remark that, thanks to this new variable, we can identify \( \zeta (z + \gamma) \) since \( \zeta (z + \gamma) \) performs a complete rotation from \( \zeta (z) \) around the origin. We can formally rewrite Carleman problem (4.1.16) as follows:

\[f (\zeta + 0i) - f (\zeta - 0i) = \log \left( c \frac{T (\zeta) - 1}{2 + \varepsilon - \frac{\gamma}{2\pi i} \log \zeta} \right) = \log c + \log (T (\zeta) - 1) - \log \left( 2 + \varepsilon - \frac{\gamma}{2\pi i} \log \zeta \right), \quad (4.1.17)\]

where \( f \) is analytic in \( \mathbb{C} \setminus \mathbb{R}^+ \). The problem (4.1.17) belongs to the general class of problems of the form:

\[f (\zeta + 0i) - f (\zeta - 0i) = G(\zeta)\]

with solution

\[f' (\zeta) = \frac{1}{2\pi i} \int_0^{\infty} G(s) \frac{ds}{s - \zeta},\]

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provided \( G(s) \) decays sufficiently fast at infinity so that the integral is well defined. Unfortunately, this is not the case in general and one must write \( G(s) \) as the sum of a part whose Cauchy integral can be defined and a part that one has to deal with separately. An alternative approach is to write the equation for \( f' \) so that

\[
f'(\zeta + 0 \cdot i) - f'(\zeta - 0 \cdot i) = \frac{T'(\zeta)}{T(\zeta)} - 1 + \frac{\gamma}{2\pi i} \frac{1}{(2 + \varepsilon - \frac{z}{2\pi i}) \text{Log}(\zeta)}.
\]

It is now possible, assuming suitable boundedness conditions, to apply Cauchy’s formula:

\[
f'(\zeta) = \frac{1}{2\pi i} \int_0^\infty \left[ \frac{T'(s)}{T(s) - 1} + \frac{\gamma}{2\pi i} \frac{1}{2 + \varepsilon - \frac{z}{2\pi i} \text{Log}(s)} \right] \frac{ds}{s - \zeta}.
\]

Then we make a change of variable and get:

\[
(\log(\Phi_\varepsilon(z)))' = -\frac{1}{\gamma} \int_{i\infty}^{i\infty} \left[ \frac{\Theta'(z')}{\Theta(z') - 1} + \frac{1}{2 + \varepsilon - z'} \right] \frac{dz'}{1 - e^{\frac{2\pi i}{\varepsilon}(z-z')}} \tag{4.1.19}
\]

thus resulting in an integral solution formula. It is possible to treat the integral in equation (4.1.19) as a standard path-integral in the complex plane. We can determine the residues at the poles inside a proper closed curve \( \Gamma_N \), integrate the asymptotic contribution along \( \Gamma_N \setminus (-iR_N, iR_N) \) and send \( N \) to infinity. However, we remark here that we are interested in evaluating \( \Phi_\varepsilon \) along the imaginary axis, so that, if \( z = is \), then the pole \( z' = is \) appears in the integration path. This is the reason for the definition of the shifted Mellin transform. We evaluate equation (4.1.19) at \( z + \varepsilon \) and, thanks to \( \Phi_\varepsilon(z) = \Phi_0(z - \varepsilon) \), we get:

\[
(\log(\Phi(z)))' = (\log(\Phi_\varepsilon(z + \varepsilon)))' = -\frac{1}{\gamma} \int_{i\infty}^{i\infty} \left[ \frac{\Theta'(z' - \varepsilon)}{\Theta(z' - \varepsilon) - 1} + \frac{1}{2 + \varepsilon - z'} \right] \frac{dz'}{1 - e^{\frac{2\pi i}{\varepsilon}(z+\varepsilon-z')}} \tag{4.1.20}
\]

with \( \Phi(z) = \Phi_0(z) \) and \( \Theta(z) = \Theta_0(z) \).

**Remark 4.1.** Let’s point out that we are apparently forgetting the pole introduced by the contribution \( r(z') := (2 + \varepsilon - z')^{-1} \) in (4.1.20). Instead, the consistency condition expressed by equation (4.1.13) implies that \( \Theta_0(2) = 1 \) and we can easily see that a formal cancellation prevents the term \( \frac{\Theta'(z' - \varepsilon)}{\Theta(z' - \varepsilon) - 1} + \frac{1}{2 + \varepsilon - z'} \) from being unbounded in a neighborhood of \( z' = 2 + \varepsilon \). Given \( \Theta(2 + \delta) = 1 + \delta \Theta'(2) + o(\delta) \) and \( \Theta'(\omega) \sim \Theta'(2) + \delta \Theta''(2 + \omega) \), under sufficient regularity conditions,

\[
\frac{\Theta'(\omega)}{\Theta(\omega) - 1} + \frac{1}{2 - \omega} = \frac{\Theta'(2) + \delta \Theta''(2)}{\delta \Theta'(2)} - \frac{1}{\delta} \sim \frac{\Theta''(2)}{\Theta'(2)} \tag{4.1.21}
\]

Therefore, \( \omega = 2 \) is not a pole if we admit \( \Theta(\omega) \) being sufficiently regular.

If we derive formally under the integral sign, we can easily obtain:

\[
\frac{d^n}{dz^n} \Theta(z) = \int_0^1 u^{z-1} (\log u)^n \, B(u) \, du, \tag{4.1.22}
\]
and this expression has the same analyticity region of $\Theta(z)$. In the previous remark, the quantity $\frac{\Theta''(z)}{\Theta'(z)}$ is required to be finite and, since both $\Theta'(2)$ and $\Theta''(2)$ are finite and non-zero, the fraction $\frac{\Theta''(z)}{\Theta'(z)}$ is defined for $z$ close to $z = 2$.

Before passing to the next Section, two more remarks are necessary.

Remark 4.2. Until now we have not introduced strong extra hypothesis with respect to the regularity hypothesis of fragmentation Universality Theorem. However, we will often use the fact that the relative distribution $B(u)$ is a continuous function. Such regularity restriction is particularly important when considering asymptotic properties with respect to $z$ since, in those cases, only the behaviors of $B(u)$ close to the extremes of the support of $B(u)$ influence the transform $\Theta(z)$. When $B(u)$ has discontinuities at some points $u_n$, in the contribution of $\Theta(z)$ appear extra terms that could be like $(1 - u_n^\gamma) / z$ (which is the transform of a Heaviside function with jump at $u = u_n$).

The second remark is:

Remark 4.3. Notice that the Carleman problem (4.1.10) cannot fix the multiplicative constant of the solution $\Phi(z)$. Therefore if we find a solution $\varphi(\xi)$ to the self-similar equation as the inverse transform of $\Phi$, also $k_\varphi(\xi)$ will be solution as inverse transform. This is in accordance to fragmentation Universality Theorem that explicitly state that unicity is up to multiplicative constant and that, for any value of the mass of the self-similar solution, that particular self-similar solution attracts solutions to the evolution problem with fixed initial value of their mass.

### 4.2. AN EXPLICIT CASE

We develop here all the details on an explicit case so that the main ideas of this method can be seen at work without cumbersome estimations and technical difficulties. Thus we investigate $B(u) = (\lambda + 2) u^\lambda$, with $\lambda > 0$, so that $\lambda$ is the only parameter governing the fragmentation distribution. Obviously the multiplicative constant $(\lambda + 2)$ is due to the compatibility condition (4.1.3).

Let us say that this case is not original in the literature (see Filippov [136] for example), but we repeat the derivation of the solution in this new framework and, as we will see in Section 4.3 and 4.4, many general cases behave asymptotically as this specific explicit case, a feature that remarks its importance. Instead, a second class of general cases, when $\text{supp}(B)$ is strictly contained in $[0, 1]$, presents new and different behaviors. In his works (cf. [316, 317]), Ziff considers some slightly more general explicit solution, such as the case $B(u) = [\alpha (\lambda + 2) u^\lambda + (1 - \alpha) (\delta + 2) u^\delta]$, but the results are somewhat fragmented into special sub-cases for the parameters.

After the following inspection of (4.1.12), we will obtain that supposing $B(u) = (\lambda + 2) u^\lambda$ yields (up to arbitrary multiplicative constant),

$$
\Phi(z) = \Gamma\left(\frac{z + \lambda}{\gamma}\right),
$$

(4.2.1)
whose inverse Mellin transform is
\[ \varphi (\xi) = \xi^\lambda e^{-\xi}. \quad (4.2.2) \]

In order to show it, we can first see that
\[ \frac{\Theta (z) - 1}{2 - z} = \frac{\lambda + 1}{2 - z} \cdot \frac{1}{z + \lambda} \quad (4.2.3) \]
and equation (4.1.20) is now:
\[ \text{Log} (\Phi (z)) \left( \int_{-iR}^{iR} \frac{1}{\gamma} \text{Log} \left( \frac{z' - \varepsilon + \lambda}{c} \right) \frac{dz'}{1 - e^{\frac{2\pi}{\gamma} i(z + \varepsilon - z')}}, \quad (4.2.4) \]

with poles at \( z' = z + \varepsilon \pm n\gamma \) for \( n = 0 \ldots \infty \) and \( z' = \varepsilon - \lambda \). We remark here that all the integrals have to be considered in the Cauchy Principal Value sense. Moreover, we need \( 0 < \varepsilon < \lambda \) in order to find the known solution (otherwise we would encounter an extra pole in the integration path that we will construct).

Let’s define a sequence of integrals \( J_N (z) \) over a finite interval as:
\[ J_N (z) = \frac{1}{\gamma} \int_{-iR_N}^{iR_N} \text{Log} \left( \frac{z' - \varepsilon + \lambda}{c} \right) \frac{dz'}{1 - e^{\frac{2\pi}{\gamma} i(z + \varepsilon - z')}}, \quad (4.2.5) \]

with \( R_N = \gamma \left( N + \frac{1}{2} \right) \). A simple change of variable permits to write
\[ J_N (z) = \frac{1}{\gamma} \int_{z - iR_N}^{z + iR_N} \text{Log} \left( \frac{z - \chi - \varepsilon + \lambda}{c} \right) \frac{d\chi}{1 - e^{\frac{2\pi}{\gamma} i(z + \varepsilon)}}, \]

so that, deriving \( J_N \) yields:
\[ \frac{d}{dz} J_N (z) = \frac{1}{\gamma} \left( \frac{\text{Log} \left( \frac{-iR_N - \varepsilon + \lambda}{c} \right)}{1 - e^{\frac{2\pi}{\gamma} R_N e^{\frac{2\pi}{\gamma} i(z + \varepsilon)}}} - \frac{\text{Log} \left( \frac{iR_N - \varepsilon + \lambda}{c} \right)}{1 - e^{\frac{2\pi}{\gamma} R_N e^{\frac{2\pi}{\gamma} i(z + \varepsilon)}}} + \int_{-iR_N + z}^{iR_N + z} \frac{1}{z - \chi - \varepsilon + \lambda} \frac{d\chi}{1 - e^{\frac{2\pi}{\gamma} i(z + \varepsilon)}}, \quad (4.2.6) \]

When \( N \) goes to infinity, the two logarithms provide two contributions of different magnitudes:
\[ \frac{\text{Log} \left( \frac{-iR_N - \varepsilon + \lambda}{c} \right)}{1 - e^{\frac{2\pi}{\gamma} R_N e^{\frac{2\pi}{\gamma} i(z + \varepsilon)}}} \sim \log \frac{R_N}{c} - \frac{i\pi}{2} + o(1), \]
\[ \frac{\text{Log} \left( \frac{iR_N - \varepsilon + \lambda}{c} \right)}{1 - e^{\frac{2\pi}{\gamma} R_N e^{\frac{2\pi}{\gamma} i(z + \varepsilon)}}} \sim O \left( e^{-\frac{2\pi}{\gamma} R_N} \log (R_N) \right). \quad (4.2.7) \]

The first has an imaginary part that will be cancelled and the second one is exponentially fast decaying; we shall, therefore, evaluate the remaining integral. First, change back the integration variable to \( z' \) and let:
\[ E_N := \int_{-iR_N}^{iR_N} \frac{1}{z' - \varepsilon + \lambda} \frac{dz'}{1 - e^{\frac{2\pi}{\gamma} i(z + \varepsilon - z')}}, \quad (4.2.8) \]
We are interested in evaluating $z$ at some pure imaginary value $z = is$ and we proceed now analyzing the integrand function $f(z') = \frac{1}{z' - \varepsilon + \lambda - e^{-\frac{i\pi}{2}}e^{-\frac{i\pi}{4}i(z' + \gamma)}}$. It presents simple poles at $z' = \varepsilon - \lambda \in \mathbb{R}$ and $z' = is + \varepsilon + k\gamma$ for each $k$ integer. We can integrate $f$ along the path shown in Figure 4.2.1 that include exactly the poles $z' = is + \varepsilon + k\gamma$ for $k = 0, \ldots, N$ when $N$ is sufficiently large.

Figure 4.2.1: Integration circuit in the complex plane after a translation shift by $v = -\varepsilon$. This figure shows the poles of the integrand function $f(z' + \varepsilon)$ (which are the “natural” poles).

### 4.2.1. The integral $\mathcal{E}_N$: poles and computation of the contribution along the path

The pole at $z = \varepsilon - \lambda < 0$ does not appear inside the integration path $\Gamma_N$, which corresponds to: the imaginary interval $[-i\gamma (N + \frac{1}{2}), i\gamma (N + \frac{1}{2})]$ and the semicircle of radius $(N + \frac{1}{2})\gamma$ centered at 0. When $N$ is large enough, $\Gamma_N$ contains exactly the poles at $z' = is + \varepsilon + k\gamma$ with $k = 0, \ldots, N$. Cauchy residue theorem, when the integration is in the clockwise sense, states that:

$$\oint_{\Gamma_N} f(z') \, dz' = -2\pi i \sum_{k=0}^{N} \text{Res} (\varphi_N; z + k\gamma)$$

with

$$2\pi i \sum_{k=0}^{N} \text{Res} (\varphi_N; z + k\gamma) = \sum_{k=0}^{N} \frac{\gamma}{(z - \varepsilon + \lambda + k\gamma)}$$
thus:
\[
\int_{\Gamma_N} f(z') \, dz' = -\sum_{k=0}^{N} \frac{\gamma}{(z + k\gamma - \varepsilon + \lambda)}.
\] (4.2.9)

### 4.2.1.1. Semicircular contribution of the path

Now it is necessary to compute the contribution to the integral given by the half circle. Let \( \tau = \arcsin \left( \left( N + \frac{1}{2} \right)^{-\frac{1}{2}} \right) \) and \( S_{R_N} \) represents the half circle of \( \Gamma_N \). The minus sign in the following formula is given by the reversed order of the integration limits:

\[
\int_{S_{R_N}} f(z') \, dz' = -i \left( N + \frac{1}{2} \right) \gamma \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \varphi_N \left( \left( N + \frac{1}{2} \right) e^{i\theta} \right) e^{i\theta} \, d\theta
\]

\[
= -i \left( N + \frac{1}{2} \right) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{1}{(N + \frac{1}{2}) e^{i\theta} - \varepsilon + \lambda} \frac{1}{1 - e^{-\frac{2\pi s}{\gamma}} e^{-2\pi i(N + \frac{1}{2}) \cos \theta} e^{\frac{2\pi \gamma}{\tau} (N + \frac{1}{2}) \sin \theta} e^{i\theta} \right) \, d\theta.
\] (4.2.10)

We split now the integral into three parts:

\[
\left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right\} = \left\{ \int_{-\tau}^{0} + \int_{0}^{\tau} + \int_{-\tau}^{0} \right\}.
\]

First, let \( \theta \in (-\tau, \tau) \), \( z = i\sigma \) and \( R_N \gg |\sigma|^2 \). Then \( z' = R_N e^{i\theta} + z \sim R_N + iR_N \sin \theta + i\sigma \) with \( R_N \sin \theta = v \in [-\sqrt{R_N}, \sqrt{R_N}] \), that is, \( z' \) is almost a vertical segment. Analyzing the terms, we can estimate:

\[
\sim \left| \int_{-\tau}^{\tau} \frac{1}{1 - e^{-\frac{2\pi s}{\gamma}} e^{-2\pi i(N + \frac{1}{2}) \cos \theta} e^{2\pi (N + \frac{1}{2}) \sin \theta} e^{i\theta} \, d\theta} \right| \\
\leq 2\tau \min_{\theta \in [\pm \tau]} \left| 1 - e^{-\frac{2\pi s}{\gamma}} e^{-2\pi i(N + \frac{1}{2}) \cos \theta} e^{2\pi (N + \frac{1}{2}) \sin \theta} \right| = 2C\tau
\]

and this term goes to zero when \( N \) goes to infinity because \( \tau \) does. Then we compute the \( (-\frac{\pi}{2}, -\tau) \) contribution. We omit \( \varepsilon \) since it is small compared to the other terms.

\[
\int_{-\frac{\pi}{2}}^{-\tau} f \left( \left( N + \frac{1}{2} \right) e^{i\theta} \right) e^{i\theta} \, d\theta
\]

\[
= -i \left( N + \frac{1}{2} \right) \int_{-\frac{\pi}{2}}^{-\tau} \frac{1}{(N + \frac{1}{2}) e^{i\theta} + 1 - e^{-\frac{2\pi s}{\gamma}} e^{-2\pi i(N + \frac{1}{2}) \cos \theta} e^{\frac{2\pi \gamma}{\tau} (N + \frac{1}{2}) \sin \theta} e^{i\theta} \, d\theta
\]

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\[
\sim -i \left( N + \frac{1}{2} \right) \int_{-\frac{\pi}{2}}^{\tau} \frac{1}{(N + \frac{1}{2}) + e^{-i\theta}} d\theta \\
\sim -i \left[ \frac{\pi}{2} + i\text{Log} \left( \frac{1}{1 + \frac{1}{N}} + \frac{1}{1 + N} i \right) \right] \\
= -i \frac{\pi}{2} + \frac{1}{2} \log \left( 1 - \frac{2N}{(1+N)^2} \right) + i\text{Arctan} \left( \frac{1}{N} \right) \quad (4.2.11)
\]

since \( \theta < -\tau < 0 \) and \( -\tau \to 0 \). We also remark that both the logarithm term and the Arctan function tend to zero when \( N \to \infty \). Therefore this contribution asymptotically only reduces to the imaginary value \(-i\frac{\pi}{2}\).

Finally, the integral over the last interval, \((\tau, \frac{\pi}{2})\), can be controlled in modulus as:

\[
\left| \int_{\tau}^{\frac{\pi}{2}} f \left( \left( N + \frac{1}{2} \right) \gamma e^{i\theta} \right) e^{i\theta} d\theta \right| \sim \int_{\tau}^{\frac{\pi}{2}} \frac{1}{1 - e^{-2\pi s} e^{-2\pi(N+\frac{1}{2}) \cos \theta} e^{2\pi(N+\frac{1}{2}) \sin \theta}} d\theta \\
\sim \int_{\tau}^{\frac{\pi}{2}} \frac{1}{e^{2\pi(N+\frac{1}{2}) \sin \theta}} \left| e^{-2\pi(N+\frac{1}{2}) \sin \theta} - e^{-2\pi s} e^{-2\pi(N+\frac{1}{2}) \cos \theta} \right| d\theta \\
\leq \int_{\tau}^{\frac{\pi}{2}} e^{-2\pi s} e^{2\pi(N+\frac{1}{2}) \frac{1}{2}} d\theta \leq C e^{-2\pi \sqrt{N}}, \quad (4.2.12)
\]

because \((N + \frac{1}{2}) \sin \theta \geq \sqrt{N + \frac{1}{2}}\), with \( \theta \in (\tau, \frac{\pi}{2}) \). This contribution goes to zero exponentially fast and we discard it.

So, finally, the computation lead to the asymptotic contribution of the semicircular part

\[
\lim_{N \to \infty} \int_{S_N} f \left( z' \right) dz' = -i \frac{\pi}{2}, \quad (4.2.13)
\]

that, notably, cancel with the imaginary contribution in \((4.2.7)\). We go back to function \( J_N (z) \) and apply the results obtained so far even if many of them (like formula \((4.2.13)\)) are valid only in the limit and we have not rigorously controlled the error terms.

\subsection*{4.2.2. The infinite product and the inversion formula}

Putting everything together, equation \((4.2.6)\) is, with large \( N \),

\[
\frac{d}{dz} J_N (is) \sim \frac{1}{\gamma} \left( \log \frac{R_N}{c} - \frac{1}{(\frac{z+\lambda}{\gamma})} - \sum_{k=1}^{N} \frac{1}{k} \left( 1 + \frac{1}{k} \left( \frac{z-\lambda}{\gamma} \right) \right) \right). \quad (4.2.14)
\]
Then, calling $g_{Eu}$ the Euler gamma constant and remembering that $R_N = \gamma (N + \frac{1}{2})$:

$$J_N (z) = \left( \frac{z}{\gamma} \log \frac{\gamma}{c} \left( N + \frac{1}{2} \right) - \log \left( \frac{z - \varepsilon + \lambda}{\gamma} \right) - \sum_{k=1}^{N} \log k \left( 1 + \frac{1}{k} \left( \frac{z - \varepsilon + \lambda}{\gamma} \right) \right) \right) + \text{const}$$

$$\sim -g_{Eu} \frac{z}{\gamma} + \frac{z}{\gamma} \sum_{k=1}^{N} \frac{1}{k} - \frac{z}{\gamma} \log \frac{c}{\gamma} + \log \left( \frac{z - \varepsilon + \lambda}{\gamma} \right)^{-1}$$

$$+ \sum_{k=1}^{N} \log \left( 1 + \frac{1}{k} \left( \frac{z - \varepsilon + \lambda}{\gamma} \right) \right)^{-1} + \sum_{k=1}^{N} \log k + \text{const}$$

and we can set the arbitrary constant $\text{const}$, for each $N$, to be $\text{const} (N) = -\sum_{k=1}^{N} \log k + C$. We want also to specify the value of $C$ in such a way that it includes the constant terms that we manipulate in order to obtain:

$$J_N (z) = -g_{Eu} \frac{z + \lambda}{\gamma} + \frac{z + \lambda}{\gamma} \sum_{k=1}^{N} \frac{1}{k} - \frac{z + \lambda}{\gamma} \log \frac{c}{\gamma} + \log \left( \frac{z + \lambda - \varepsilon + \lambda}{\gamma} \right)^{-1}$$

$$+ \sum_{k=1}^{N} \log \left( 1 + \frac{1}{k} \left( \frac{z + \lambda - \varepsilon + \lambda}{\gamma} \right) \right)^{-1}.$$

(4.2.15)

We are now in position to set $c = \gamma$ or $c \neq \gamma$. The election of the former, $c = \gamma$, fixes the homogeneous fragmentation kernel $\beta (\xi) = \gamma \xi^{\gamma}$ and leads to the truncated solution formula:

$$\Phi_N (z) = e^{-g_{Eu} \frac{z + \lambda}{\gamma}} \prod_{k=1}^{N} \frac{1}{\frac{z - \varepsilon + \lambda}{\gamma} + 1} e^{\frac{z + \lambda}{k\gamma}}.$$ 

(4.2.16)

Thus, using a known formula for the Gamma function (cf. Abramovitz-Stegun [2]), we obtain the convergence of (4.2.16) to

$$\Phi_N (z) \to \Gamma \left( \frac{z - \varepsilon + \lambda}{\gamma} \right) \text{ as } N \to \infty.$$ 

(4.2.17)

Thanks to the continuity of the gamma function, we can let $\varepsilon \to 0$ and conclude that:

$$\Phi (z) = \Gamma \left( \frac{z + \lambda}{\gamma} \right),$$ 

(4.2.18)

the explicit solution to the Carleman problem. We remark that the Gamma function is, by definition, the Mellin transform of the negative exponential:

$$\Gamma (z) = \int_{0}^{\infty} e^{-t} t^{z-1} dt,$$ 

(4.2.19)

so that, in our specific case, trivial calculations lead us to:

$$\varphi_{\text{explicit}} (x) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \Gamma \left( \frac{z + \lambda}{\gamma} \right) x^{-z-\lambda+\lambda} \, dz = \frac{1}{2\pi i} \int_{-i\infty+\lambda}^{i\infty+\lambda} \Gamma \left( \frac{\zeta}{\gamma} \right) x^{-\zeta} \, d\zeta.$$
\[
\frac{\gamma}{2\pi} x^\lambda \int_{-i\infty + \frac{\lambda}{2}}^{i\infty + \lambda} \Gamma \left( \frac{\xi}{\gamma} \right) x^{\gamma(-\xi)} \, d\xi
= \gamma x^\lambda \frac{1}{2\pi i} \int_{-i\infty + \frac{\lambda}{2}}^{i\infty + \lambda} \Gamma (w) (x^{\gamma})^{-w} \, dw = \gamma x^\lambda e^{-x^\gamma}, \quad (4.2.20)
\]

where we have lastly applied the Cahen-Mellin formula. This solution has mass \( \mathcal{M}_1 (\varphi_{\text{explicit}}) = \Gamma \left( \frac{2+\lambda}{\gamma} \right) \) and different solutions with different masses can be obtained setting a different value of \( C \) in the expression \( \text{const} (N) = -\sum_{k=1}^{N} \log k + C. \)

If we do not impose the condition \( c = \gamma \), we get an extra exponential factor:

\[
\Phi_c (z) = \left( \frac{c}{\gamma} e^{-\frac{z+\lambda}{\gamma}} \right) \Gamma \left( \frac{z+\lambda}{\gamma} \right).
\]

Since \( \Phi \) is of the form:

\[
\Phi_c (z) = \frac{c}{\gamma} \rho \left( \frac{z+\lambda}{\gamma} \right), \quad \rho (\zeta) = e^{-\zeta} \Gamma (\zeta)
\]

then:

\[
\varphi_c (x) = \frac{c}{\gamma} \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \rho \left( \frac{z+\lambda}{\gamma} \right) x^{-z-\lambda+\lambda} \, dz = c x^\lambda \frac{1}{2\pi i} \int_{-i\infty + \frac{\lambda}{2}}^{i\infty + \lambda} \rho (w) (x^{\gamma})^{-w} \, dw.
\]

\[
\frac{1}{2\pi i} \int_{-i\infty + \frac{\lambda}{2}}^{i\infty + \frac{\lambda}{2}} \rho (w) (x^{\gamma})^{-w} \, dw = \lim_{N \to \infty} \sum_{k=0}^{N} \text{Res} \left( \rho (w) (x^{\gamma})^{-w} \, w = -k \right) = \lim_{N \to \infty} \sum_{k=0}^{N} \frac{(-1)^k}{k!} (e x^{\gamma})^k = e^{-e x^{\gamma}}
\]

\[
\varphi_c (\xi) = \xi e^{-\xi}.
\]

### 4.3. A REPRESENTATION FORMULA FOR THE SOLUTION WHEN \( B(u) = O(u^\mu), u \ll 1 \)

In this section we seek to extend product formula \((4.2.16)\) in order to solve the Carleman problem with greater generality. Let therefore \( B(u) \) be a continuous function and \( B(u) = O(u^\mu) \) as \( u \to 0 \) for some \( \mu > 0 \), \( B(1-u) = O(u^\nu) \) as \( u \to 1^- \), for some \( \nu > -1 \).

We proceed defining \( J_N \) as the integral over a finite interval. Let \( R_N = \gamma \left( N + \frac{1}{2} \right) \) again; this time, instead, we fix from the beginning \( \beta (\xi) = \gamma \xi^\gamma \) and write:

\[
J_N (z) = -\frac{1}{\gamma} \int_{-iR_N+z}^{iR_N+z} \text{Log} \left( \frac{\Theta (z - w - \varepsilon) - 1}{2 + \varepsilon + w - z} \right) \frac{dw}{1 - e^{\frac{2\pi i}{\gamma} (w+\varepsilon)}}, \quad (4.3.1)
\]

\[141\]
From now on, we omit $\varepsilon$ when it is clear that its contribution is small with respect to other terms. In fact, as we have seen in Section 4.2, the translation by $-\varepsilon$ is only needed in order to include inside the integration path the first pole at $z' = is$, but later, thanks to the continuity of the solution $\Phi_{\varepsilon}(z)$, one can let $\varepsilon$ go to zero.

As before, we derive $J_N(z)$ with respect to $z$. It can be seen that one of the two logarithmic contributions goes to zero exponentially fast, as in (4.2.7), so we obtain that:

$$
\frac{d}{dz} J_N(z) \sim -\frac{1}{\gamma} \left( \log \left( \frac{\Theta ( -i R_N ) - 1}{2 + i R_N} \right) - 0 + \int_{-i R_N}^{i R_N} \left( \frac{d}{d\omega} \log \left( \frac{\Theta ( \omega - \varepsilon ) - 1}{2 + \varepsilon - \omega} \right) \right) \frac{d\omega}{1 - e^{\frac{2\pi i (z + \varepsilon - \omega)}}} \right).
$$

(4.3.2)

We could avoid computing the derivative of $\log \left( \frac{\Theta (\omega - \varepsilon ) - 1}{2 + \varepsilon - \omega} \right)$ for the reason that later we will have to compute its integral. However, to reduce confusion, we consider

$$
E_N(z) := \int_{-i R_N}^{i R_N} \left( \frac{\Theta'(\omega - \varepsilon)}{\Theta(\omega - \varepsilon) - 1} + \frac{1}{2 + \varepsilon - \omega} \right) \frac{d\omega}{1 - e^{\frac{2\pi i (z + \varepsilon - \omega)}}}
$$

and compute its approximate value.

### 4.3.1. Computing the Integral $E_N$

First we have to inspect whether $\frac{\Theta'(z)}{\Theta(z)}$ may produce different poles than before. Assuming $B(u)$ has the two aforementioned behaviors near zero and near one, one readily sees that $\Theta(z)$ is analytic for $\Re(z) > -\mu$. Since $\Theta'(z)$ has the same analyticity region (see Remark 4.1), we only need to check whether $\Theta(z) - 1 = 0$ for $z \neq 2$. However, it is clear from the monotonicity of $\Theta(\Re(z))$ with respect to $x = \Re(z)$, that this case cannot occur.

The only existing poles are produced by the factor $\left(1 - e^{\frac{2\pi i (z + \varepsilon - \omega)}}\right)^{-1}$ and they are located at $\omega = z + \varepsilon + \gamma k$ for each $k$ in $\mathbb{Z}$. Now, the corresponding residues can be computed giving:

$$
\oint_{\Gamma_N} \left( \frac{\Theta'(\omega - \varepsilon)}{\Theta(\omega - \varepsilon) - 1} + \frac{1}{2 + \varepsilon - \omega} \right) \frac{d\omega}{1 - e^{\frac{2\pi i (z + \varepsilon - \omega)}}}
$$

$$
= -2\pi i \sum_{k=0}^{N} \text{Res} \left( f(\omega) ; z + \varepsilon + k\gamma \right)
$$

$$
= -\gamma \sum_{k=0}^{N} \left( \frac{\Theta'(z + k\gamma)}{\Theta(z + k\gamma) - 1} + \frac{1}{2 - z - k\gamma} \right).
$$

(4.3.4)

Due to continuity of $\Theta(z)$ (that is the integral of the continuous function $B(u)$), we let $\varepsilon$ go to zero so that on the right hand side of formula (4.3.4) there is not explicit dependence on $\varepsilon$.
4.3.1.1. Semicircular contribution of the path

We consider again the closed integration curve $\Gamma_N$ introduced before, as shown in Figure 4.2.1. If $\omega = R_N e^{i\theta}$, with $\theta$ ranging from $\frac{\pi}{2}$ to $-\frac{\pi}{2}$ with the clockwise orientation, the contour integration along $S_{RN}$ gives:

$$\int_{S_{RN}} f(z') \, dz' = iR_N \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\Theta'(R_N e^{i\theta})}{\Theta(R_N e^{i\theta}) - 1} + \frac{1}{2 - R_N e^{i\theta}} - \frac{1}{2} \right) \frac{e^{i\theta}}{1 - e^{i(\omega - R_N \cos \theta - iR_N \sin \theta)}} \, d\theta$$

(4.3.5)

and, splitting as before,

$$\left\{ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \right\} = \left\{ \int_{-\frac{\pi}{2}}^{-\tau} + \int_{-\tau}^{0} + \int_{0}^{\tau} \right\}.$$

As it can be seen, only the first contribution $\int_{-\frac{\pi}{2}}^{-\tau}$ yields asymptotically a relevant contribution that must be compared with $\log \left( \frac{\Theta(-iR_N) - 1}{2 + iR_N} \right)$. We see that:

$$\mathcal{L}_N := \log \left( \frac{\Theta(-iR_N) - 1}{2 + iR_N} \right) - \int_{S_{RN}} f(z') \, dz'$$

(4.3.6)

$$\mathcal{L}_N \sim \log \left( \frac{\Theta(-iR_N) - 1}{2 + iR_N} \right) + i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\Theta'(R_N e^{i\theta})}{\Theta(R_N e^{i\theta}) - 1} - \frac{R_N e^{i\theta}}{2 - R_N e^{i\theta}} \right) \, d\theta$$

$$\sim \log (\Theta(-iR_N) - 1) - \log R_N - i\frac{\pi}{2} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\Theta'(R_N e^{i\theta})}{\Theta(R_N e^{i\theta}) - 1} iR_N e^{i\theta} \, d\theta - i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \, d\theta$$

$$\sim \log (\Theta(-iR_N) - 1) - \log R_N - i\frac{\pi}{2} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{\omega} \, d\omega - i \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \, d\theta$$

$$\sim \log (\Theta(-iR_N) - 1) + \log (\Theta(R_N) - 1) - \log (\Theta(-iR_N) - 1) - \log R_N - i\pi$$

and finally:

$$\mathcal{L}_N \sim \log (\Theta(R_N) - 1) - \log R_N - i\pi$$

(4.3.7)

We remark here that, since $\Theta(R_N) \to 0$ as $R_N \to \infty$, then $\log (\Theta(R_N) - 1) \to \log (|1 + \Theta(R_N)|) + i\pi \sim i\pi$. This gives exactly the constant contribution that we need: $\mathcal{L}_N$ defined in (4.3.6) is such that $\lim_{N \to \infty} \mathcal{L}_N = 0$.

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### 4.3.2. Infinite Product Asymptotic Formula

We can sum all the contributions in (4.3.2) and obtain:
\[
\frac{d}{dz} J_N(z) \sim -\frac{1}{\gamma} \left( -\log \left( N + \frac{1}{2} \right) - \gamma \left( \frac{\Theta'(z)}{\Theta(z)} - 1 + \frac{1}{2 - z} \right) - \gamma \sum_{k=1}^N \left( \frac{\Theta'(z + k\gamma)}{\Theta(z + k\gamma)} - 1 + \frac{1}{2 - z - k\gamma} \right) \right)
\]
and
\[
J_N(z) = \frac{z}{\gamma} \sum_{k=1}^N \frac{1}{k} + \frac{z}{\gamma} \left( \log N - \sum_{k=1}^N \frac{1}{k} \right)
\]
\[
+ \log \left( \frac{\Theta(z) - 1}{2 - z} \right) + \sum_{k=1}^N \log \left( \frac{\Theta(z + k\gamma) - 1}{2 - k\gamma - z} \right) + \text{const.}
\] (4.3.8)

We have now two possible ways to proceed. On one hand, we can seek solutions close, in some sense, to previously known solutions; this is done in Appendix A, defining a function \( \delta(z) \) such that \( \Theta(z) = \delta(z)(\Theta_0(z) - 1) + 1 \), with \( \Theta_0(z) \) the moment transform of \( B_0(u) \) and associated solution \( \Phi_0(z) \). For example, this can be done close to \( B_0(u) = (\lambda + 2) u^\lambda \). This way of constructing solutions is unsatisfactory and now we obtain a more general result first constructing asymptotically valid solutions and then refining them.

Equation (4.3.8) leads also to:
\[
J_N(z) = \frac{z}{\gamma} \sum_{k=1}^N \frac{1}{k} - gE u \frac{z}{\gamma} - \Theta(z) - \sum_{k=1}^N \Theta(z + k\gamma) + \log \left( \frac{1}{z - 2} \right) + \sum_{k=1}^N \log \left( \frac{1}{z + k\gamma - 2} \right) + c_N,
\] (4.3.9)

since \( \log(1 - \Theta(z)) \sim -\Theta(z) \) when \( \Re(z) \gg 1 \). The problem now is that we have obtained a formula with two kinds of terms: some of them can be used as before to obtain the \( \Gamma \) function, but \( \left( \Theta(z) + \sum_{k=1}^N \Theta(z + k\gamma) \right) \) gives no guarantees that it converges. Therefore, we follow the strategy of summing and subtracting for each \( N \) the constant value \( \left( \Omega(0) + \sum_{k=1}^N \Omega(k\gamma) \right) \), where we have introduced:
\[
\Omega(z) = \int_0^1 u^{z-1} u^\mu \left( B(1) + c(1-u)^\lambda \right) \, du = \frac{B(1)}{z + \mu} + c \frac{\Gamma(1 + \lambda)}{\Gamma(1 + z + \mu + \lambda)},
\] (4.3.10)

where \( \mu > 0 \) is such that it represents the order of decay at zero of \( B \), that is \( B(u) = O(u^\mu) \); moreover, \( \lambda \) and \( c \) are constants such that there exist a \( \varepsilon > 0 \):
\[
\lim_{u \to 1^-} \frac{B(u) - B(1) - c(1-u)^\lambda}{(1-u)^{\lambda+\varepsilon}} = 0.
\] (4.3.11)

In this sense, the two exponents \( \mu \) and \( \lambda \) retract the behavior of \( B(u) \). Choosing appropriately \( c_N \), one can now easily obtain:
\[
\Phi(z) \approx \Gamma \left( \frac{z}{\gamma} \right) e^{-\sum_{k=0}^{\infty} \left( \Theta(z + k\gamma) - \Omega(k\gamma) \right)}
\] (4.3.12)

with large \( z \). We will refine in the next Section this asymptotic formula and now show that the series \( \sum_{k=0}^{\infty} \left[ \Theta(z + k\gamma) - \Omega(k\gamma) \right] \) actually converges for \( \Re(z) > -\mu \).
Proposition 4.4. Under the conditions $B(u) = O(u^\mu)$, with $\mu > 0$, and \(\ref{eq:lambda>0}\) with $\lambda > 0$, the series $\sum |\Theta (z + k\gamma) - \Omega (k\gamma)|$ that appear in formula \(\ref{eq:definition} \) converges for $\Re z > -\mu$.

Proof. By definition of $\Theta (z)$,

$$|\Theta (z + k\gamma) - \Omega (k\gamma)| \leq \int_0^1 u^{\text{Re}(z)+k\gamma-1} \left| B(u) - B(1) u^\mu - cu^\mu (1-u)^\lambda \right| \, du. \quad \tag{4.3.13}$$

we must show that for each $z$ the integral must go to zero sufficiently fast for large $k$. Let:

$$I := \sum \int_0^1 u^{\text{Re}(z)+k\gamma-1} \left| B(u) - B(1) - c (1-u)^\lambda \right| \, du$$

$$= \frac{1}{\gamma} \sum \int_0^{\int \frac{u^{\text{Re}(z)}}{1-v} |B \left( \frac{1}{v^{\gamma}} \right) - B(1) v^{\lambda_1} - cv^{\lambda_1} (1-v^{\frac{1}{\gamma}}) \lambda_2 \right| \, dv$$

then we split the integration domain in two parts: $I = I_1 + I_2$ covering $(0, 1-\delta)$ and $(1-\delta, 1)$ respectively. On $(0, 1-\delta)$, for some small $\delta > 0$, we can place the summation sign under the integral and obtain:

$$I_1 = \frac{1}{\gamma} \int_0^{1-\delta} \left( 1-v^{\text{Re}(z)} \right)^{-1} \left| B \left( \frac{1}{v^{\gamma}} \right) - B(1) v^{\lambda_1} - cv^{\lambda_1} (1-v^{\frac{1}{\gamma}}) \lambda_2 \right| \, dv$$

and, if $\delta$ is sufficiently small, we can approximate the term in the modulus of the second integral $I_2$ as

$$\left| B \left( \frac{1}{v^{\gamma}} \right) - B(1) v^{\lambda_1} - cv^{\lambda_1} (1-v^{\frac{1}{\gamma}}) \right| \ll \left| (1-v^{\frac{1}{\gamma}})^{\lambda+\varepsilon} \right|. \quad \tag{4.3.14}$$

In this way, the singularity at $\nu = 1$ is now integrable. Therefore, we can conclude that the series converges:

$$I \leq \frac{1}{\gamma} \int_0^{1-\delta} \left( 1-v^{\text{Re}(z)} \right)^{-1} \left| B \left( \frac{1}{v^{\gamma}} \right) - B(1) v^{\lambda_1} - cv^{\lambda_1} (1-v^{\frac{1}{\gamma}}) \lambda_2 \right| \, dv + \frac{1}{\gamma} \left( 1-\delta \right)^{-\frac{\varepsilon}{\gamma}} \int_1^{1-\delta} \left( 1-v^{\frac{1}{\gamma}} \right)^{\lambda+\varepsilon-1} \, dv \quad \tag{4.3.15}$$

provided $\Re z > -\mu$.

### 4.3.3. Infinite Product Formulas

Defining $\Phi (z)$ in this way does not give the exact solution since it should verify Carleman problem \(\ref{eq:Carleman}\). Substituting, we see that, instead:

$$\Gamma \left( \frac{z}{\gamma} \right) e^{-\sum_{k=0}^{\infty} [\Theta (z+k\gamma) - \Omega (k\gamma)]} = \gamma \frac{1-\Theta (z)}{z-2} \Phi (z+\gamma)$$

$$= \gamma \frac{1-\Theta (z)}{z-2} \Gamma \left( \frac{z}{\gamma} + 1 \right) e^{-\sum_{k=1}^{\infty} [\Theta (z+k\gamma) - \Omega (k\gamma)]} e^{\Omega (0)},$$

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which is almost exact when \( z \) is large:

\[
e^{-\Theta(z)} \Gamma \left( \frac{z}{\gamma} \right) \approx \gamma \left( 1 - \Theta(z) \right) \frac{\Gamma \left( \frac{z+1}{\gamma} \right)}{z-2},
\]

\[
e^{-\Theta(z)} \approx \frac{1 - \Theta(z)}{z-2}.
\]

From Proposition 4.4, we know that \( \sum_{k=0}^{\infty} [\Theta(z+k\gamma) - \Omega(k\gamma)] \) converges absolutely. If we want an exact solution, we look for \( h(z) \) such that

\[
\Phi(z) = \Gamma \left( \frac{z}{\gamma} \right) e^{-\sum_{k=0}^{\infty} [\Theta(z+k\gamma) - \Omega(k\gamma)]} h(z),
\]

which leads to a new Carleman problem:

\[
e^{-\Theta(z)} h(z) = \frac{z}{z-2} \left( 1 - \Theta(z) \right) h(z + \gamma).
\]

Repeating as before, one can arrive at

\[
h_N(z) = \exp \left( -\log \left( \frac{R_N}{\frac{1-\Theta(R_N)}{R_N-2} e^{-\Theta(R_N)}} \right) \frac{z}{\gamma} + \log \left( \frac{z+1}{z-2} \right) \log \left( \frac{z+k\gamma}{z+k\gamma-2} \right) + \sum_{k=1}^{N} \log \left( \frac{z+k\gamma}{z+k\gamma-2} \right) + c_N \right)
\]

and letting \( N \) go to infinity, we can discard the term \( \log \left( \frac{R_N}{\frac{1-\Theta(R_N)}{R_N-2} e^{-\Theta(R_N)}} \right) \) since \( \log \left( \frac{R_N}{\frac{1-\Theta(R_N)}{R_N-2} e^{-\Theta(R_N)}} \right) \to \log(1) = 0 \). Then \( c_N \) can be fixed such that, at each step,

\[
h_N(z) = k \left( \frac{z}{z-2} \frac{1 - \Theta(z)}{1 - \Omega(z)} e^{\Theta(z) - \Omega(0)} \right) \times
\]

\[
\times \prod_{k=1}^{N} \left( \frac{z+k\gamma}{z+k\gamma-2} \frac{1 - \Theta(z+k\gamma)}{1 - \Omega(z+k\gamma)} e^{\Theta(z+k\gamma) - \Omega(k\gamma)} \right) \frac{z}{z-2} \gamma
\]

We must check the convergence of the infinite product in (4.3.18). We can see that:

\[
\sum_{k=1}^{\infty} \left| \log \left( \frac{z+k\gamma}{z+k\gamma-2} \frac{1 - \Theta(z+k\gamma)}{1 - \Omega(z+k\gamma)} e^{\Theta(z+k\gamma) - \Omega(k\gamma)} \right) \right|
\]

\[
= \sum_{k=1}^{\infty} \left| \Theta(z+k\gamma) - \Omega(k\gamma) - \frac{2}{k\gamma} + \log \left( \frac{1 - \Theta(z+k\gamma)}{1 - \Omega(z+k\gamma)} \right) + \log \left( 1 + \frac{2}{z+k\gamma} \right) \right|
\]

\[
\approx \sum_{k=1}^{\infty} \left| \frac{2}{z+k\gamma - 2} - \frac{2}{k\gamma} \right| < \infty
\]

Therefore, we can reconstruct \( \Phi(z) \) and from the convergence of both infinite products we can write:

\[
\Phi(z) = \Gamma \left( \frac{z}{\gamma} \right) \frac{1 - \Theta(z)}{1 - \Omega(0)} \prod_{k=1}^{\infty} \frac{1 - \Theta(z+k\gamma)}{1 - \Omega(z+k\gamma)}.
\]

(4.3.19)
Also, we can find two other useful representations for $\Phi(z)$:

$$
\Phi(z) = \lim_{N \to \infty} k_N \frac{e^{-8\mu} \frac{z}{\gamma} 1 - \Theta(z)}{1 - \Omega(0)} \prod_{k=1}^{N} \frac{1 - \Theta(z + k\gamma)}{1 - \Omega(k\gamma)} \frac{e^{z^2/2}}{1 + \frac{z^2}{2\gamma}}.
$$

(4.3.20)

where $k_N$ has to be chosen according to the desired multiplicative constant of the solution, and

$$
\Phi(z) = \Gamma\left(\frac{z + \mu}{\gamma}\right) \frac{z + \mu 1 - \Theta(z)}{z - 2} \prod_{k=1}^{\infty} \frac{z + \mu + k\gamma 1 - \Theta(z + k\gamma)}{z - 2 + k\gamma} \frac{1}{1 - \Omega(k\gamma)} e^{-z^2/2\gamma}.
$$

(4.3.21)

Thus, (4.3.19) satisfies exactly equation (4.1.12).

4.4. PROPERTIES OF THE SELFSIMILAR SOLUTION: $C^\infty$ REGULARITY AND EXPONENTIAL DECAY AT INFINITY

4.4.1. Properties of $\Phi(\eta)$

From the inversion formula,

$$
\xi^\varphi(\xi) = \frac{1}{2\pi i} \int_{-i\varepsilon + \delta}^{i\varepsilon + \delta} \xi^{-z} \Phi(z + x) \, dz,
$$

(4.4.1)

then, with $\delta = 0$, we can estimate:

$$
\xi^\varphi |\varphi(\xi)| \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} |\Phi(is + x)| \, ds.
$$

(4.4.2)

It is therefore useful to determine the interval for $x$ in which the integral at the right hand side converges. This is done in a similar way to Proposition 4.4. It is useful to introduce a function $\omega(u)$,

$$
\omega(u) = B(1) + c_1 (1 - u)^{\lambda}
$$

(4.4.3)

to compare it to $B(1 - u)$. The following Lemma shows the analyticity region of the solution.

**Lemma 4.5.** Suppose that $B(1 - u) \sim \omega(u) + o\left((1 - u)^\lambda\right)$ for some $\lambda > 0$ and $B(u) \sim c_2 u^\mu$, $\mu > 0$. Then formula (4.3.21) defines an analytical solution to (4.1.16) for $\Re z > -\mu$. 

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Proof. From (4.3.24), we can remark that \( \Gamma \left( \frac{z + \mu}{\gamma} \right) \) is analytical in the desired region. Then we need to check that
\[
\pi(z) := \frac{1 - \Theta(z)}{1 - \Omega(0)} \prod_{k=1}^{\infty} \frac{1 - \Theta(z + k\gamma)}{1 - \Omega(k\gamma)} \sim \exp \left( - \sum_{k=0}^{\infty} (\Theta(z + k\gamma) - \Omega(k\gamma)) \right)
\]
(4.4.4)
converges when \( \Re z = x > -\mu \). Therefore, splitting the integration domain,
\[
\pi(z) = \exp \left( - \sum_{k=0}^{\infty} \left( \int_{0}^{1-\varepsilon} \left( u^{x+k\gamma} B(u) - u^{k\gamma} \omega(u) \right) \, du + \int_{1-\varepsilon}^{1} o\left((1-u)^{\nu}\right) u^{x+k\gamma} \, du \right) \right),
\]
(4.4.5)
and change variable and pass to the limit:
\[
\pi(z) = \exp \left( - \frac{1}{\gamma} \sum_{k=0}^{\infty} \left( \int_{0}^{(1-\varepsilon)^{\gamma}} \left( v^{rac{1}{\gamma}+1+k} B(v^{1/\gamma}) - v^{rac{1}{\gamma}} \omega(v^{1/\gamma}) \right) \, dv + \int_{(1-\varepsilon)^{\gamma}}^{1} o\left((1-v^{1/\gamma})^{\nu}\right) v^{rac{1}{\gamma}+1+k} \, dv \right) \),
\]
(4.4.6)
\[
\pi(z) = \exp \left( - \frac{1}{\gamma} \left( \int_{0}^{(1-\varepsilon)^{\gamma}} \frac{v^{rac{1}{\gamma}+1}}{1-v} \left( B\left(v^{1/\gamma}\right) - v^{rac{1}{\gamma}} \omega\left(v^{1/\gamma}\right) \right) \, dv + \int_{0}^{(1-\varepsilon)^{\gamma}} o\left((1-x)^{\nu}\right) \frac{(1-x)^{\frac{1}{\gamma}+1}}{x} \, dx \right) \),
\]
(4.4.7)
and, since \( (1 - (1 - x)^{\frac{1}{\gamma}})^{\nu} \sim \left(\frac{1}{\gamma}x\right)^{\nu} \), the second integral is such that:
\[
\int_{0}^{(1-\varepsilon)^{\gamma}} o\left((1-x)^{\nu}\right) \frac{(1-x)^{\frac{1}{\gamma}+1}}{x^{1-\nu}} \, dx \sim \left(\frac{1}{\gamma}\right)^{\nu} \int_{0}^{(1-\varepsilon)^{\gamma}} (1-x)^{\frac{1}{\gamma}+1} \, dx,
\]
(4.4.8)
which is integrable when \( \nu > 0 \). Also, the first integral is convergent when \( x > 1 - \frac{1}{\gamma} - \mu \).

Now we can derive formally and obtain:
\[
\left( - \sum_{k=0}^{\infty} \Theta'(z + k\gamma) \right) \exp \left( - \sum_{k=0}^{\infty} (\Theta(z + k\gamma) - \Omega(k\gamma)) \right).
\]
We should check also that the series \( \sum_{k=0}^{\infty} \Theta'(z + k\gamma) \) is convergent. Thus, consider:
\[
\sum_{k=0}^{\infty} \Theta'(z + k\gamma) = \sum_{k=0}^{\infty} \int_{0}^{1} u^{x+k\gamma} \log u B(u) \, du;
\]
(4.4.9)
the logarithmic contribution, close to \( u = 1 \), can be treated as before, approximating this time \( \log(1-x) \sim (-x) \). Then:
\[
\sum_{k=0}^{\infty} \Theta'(z + k\gamma) = \frac{1}{\gamma^{2}} \sum_{k=0}^{\infty} \left( \int_{0}^{(1-\varepsilon)^{\gamma}} v^{\frac{1}{\gamma}+1+k-1} \log v B\left(v^{1/\gamma}\right) \, dv + \int_{(1-\varepsilon)^{\gamma}}^{1} v^{\frac{1}{\gamma}+1+k-1} \log v B\left(v^{1/\gamma}\right) \, dv \right),
\]
\[ \cong \frac{1}{\gamma^2} \left( \int_{0}^{1} v^{\frac{x+1}{\gamma}-1} \log v \, B \left( \frac{v}{\gamma} \right) \, dv + \int_{0}^{1} (1 - x)^{\frac{x+1}{\gamma}-1} \, B \left( \frac{(1 - x)}{\gamma} \right) \, dx \right), \]

which has still the same convergence region as \( \pi(z) \).

In order to study whether (4.4.2) is valid in the analyticity region, we will use:

\[ \frac{1 + \left| \frac{z}{\gamma} \right|}{\cosh \left( \frac{\pi z}{2 \gamma} \right)} \leq \left| \Gamma \left( \frac{i s + x}{\gamma} \right) \right| \leq \Gamma \left( \frac{x}{\gamma} \right) \frac{1 + \left| \frac{z}{\gamma} \right|}{\cosh \left( \frac{\pi z}{2 \gamma} \right)} \]  (4.4.10)

with equality on the right when \( s = 0 \) and on the one on the left when \( s \to \infty \). Both bounds are integrable with respect to \( s \) when \( x > -\gamma \), but we also need \( x > 0 \) for \( \Gamma \left( \frac{x}{\gamma} \right) < \infty \).

**Lemma 4.6.** Under the same hypothesis of Lemma 4.5, the solution \( \Phi \) given by formula (4.3.21) verifies for \( \Re z = x > -\mu \):

\[ \int_{-\infty}^{\infty} \left| \Phi (is + x) \right| \, ds < \infty. \]  (4.4.11)

**Proof.** This is done using bound (4.4.10) and the well-known formula for the Euler-Mascheroni constant:

\[ -g_{\text{Eu}} = \sum_{n=1}^{\infty} \left( \log \frac{n+1}{n} - \frac{1}{n} \right). \]

We can consider separately the function \( \prod_{k=1}^{\infty} \frac{z^{\mu+k\gamma}}{z-2^{k+2}\gamma} e^{-\frac{\mu+2}{k}} \); this motivates the introduction of the constant

\[ -g_{\text{Eu}} (z) := \sum_{k=1}^{\infty} \left( \log \left( 1 + \frac{\mu+2}{k} \right) - \left( \frac{\mu+2}{\gamma} \right) \frac{1}{k} \right) \]  (4.4.12)

so that (4.3.21) is now:

\[ \Phi (z) = \Gamma \left( \frac{z + \mu}{\gamma} \right) \exp (-g_{\text{Eu}} (z)) \frac{z + \mu}{z - 2} \frac{1 - \Theta (z)}{1 - \Omega (0)} \prod_{k=1}^{\infty} \frac{1 - \Theta (z + k\gamma)}{1 - \Omega (k\gamma)}. \]  (4.4.13)

Now:

\[ \int_{-\infty}^{\infty} \left| \Phi (is + x) \right| \, ds \leq \Gamma \left( \frac{x + \mu}{\gamma} \right) \exp (-g_{\text{Eu}} (z)) \times \]

\[ \int_{-\infty}^{\infty} \left( 1 + \left| \frac{z}{\gamma} \right| \right) \left| \frac{x + is + \mu}{x + is - 2} \prod_{k=0}^{\infty} \frac{1 - \Theta (z + k\gamma)}{1 - \Omega (k\gamma)} \right| \, ds. \]  (4.4.14)
We have also to consider the infinite product of terms \( \frac{1-\Theta(z+k\gamma)}{1-\Omega(k\gamma)} \). This can be done as before:

\[
\left| \prod_{k=0}^{\infty} \frac{1-\Theta(z+k\gamma)}{1-\Omega(k\gamma)} \right| \sim |\epsilon \sum_{k=0}^{\infty} (-\Theta(is+x+k\gamma)+\Omega(k\gamma))| \\
= \exp \left( - \sum_{0}^{1} \left( u^{x+k\gamma} b(u) - u^{k\gamma} \omega(u) \right) du \right),
\] (4.4.15)

which is constant in \( s \). Therefore, in (4.4.14), the factor \( (\cosh \left( \frac{x}{2} \right) ) \) ensures the convergence of the integral.

### 4.4.2. \( L^1 \)-WEIGHTED NORM OF THE SELFSIMILAR SOLUTION \( \varphi (\xi) \) AND ASYMPTOTICS

Consider the explicit case, \( \Phi (z) = \Gamma \left( \frac{z+\mu}{\gamma} \right) \); let \( \tau < \min \{ \mu, \gamma \} \) and \( \alpha = \frac{\tau}{\gamma} < 1 \) and this leads to:

\[
\sum_{n=1}^{\infty} \frac{\Phi (\gamma n - \mu - \tau)}{n!} = \sum_{n=1}^{\infty} \frac{\Gamma (n - \alpha)}{n!} \approx \sum_{n=1}^{\infty} \frac{n^{-\alpha}}{n} < \infty.
\] (4.4.16)

In fact, for all \( \alpha \in \mathbb{R} \),

\[
\lim_{n \to \infty} \frac{\Gamma (n + \alpha)}{\Gamma (n) n^\alpha} = 1.
\] (4.4.17)

In equation (4.4.16), the symbol \( \approx \) between the two series means that their terms are asymptotically the same. In the next Lemma we show that, thanks to the convergence of this series, the selfsimilar solution belongs to a suitable \( L^1 \)-weighted space. Thus, the exponential decay at infinity will be a direct consequence. In the particular case of the explicit self-similar solution \( \varphi \text{explicit} (\xi) = \gamma^\mu e^{-\xi^\tau} \), we can easily check that it verifies

\[
\int_{0}^{\infty} e^{\xi^\gamma} \frac{\varphi \text{explicit} (\xi)}{\xi^{1+\mu+\tau} + \xi^{1+\mu-\tau}} d\xi < \infty.
\]

Analogously, we want to show that it is a generic property of these solutions.

**Lemma.** Under the hypothesis of Lemma [4.5], the self-similar solution \( \varphi (\xi) \) given by (4.1.13) from the infinite product formula (4.3.21) verifies:

\[
\int_{0}^{\infty} e^{\xi^\gamma} \frac{\varphi (\xi)}{\xi^{1+\mu-\tau} + \xi^{1+\mu+\tau}} d\xi < \infty
\] (4.4.18)

for any \( 0 < \tau < \min \{ \mu, \gamma \} \).
Proof. If we show that, for all \(0 < x < \mu\),
\[
\sum_{n=0}^{\infty} \frac{\Phi (\gamma n - x)}{n!} < \infty \tag{4.4.19}
\]
then, excluding the first term of the previous series, we can, for any \(0 < \tau < \min \{\mu, \gamma\}\), truncate:
\[
\sum_{n=1}^{\infty} \frac{\Phi (\gamma n - \mu - \tau)}{n!} < \infty. \tag{4.4.20}
\]
We can see that, formally:
\[
\sum_{n=1}^{\infty} \frac{\Phi (\gamma n - \mu - \tau)}{n!} = \int_{0}^{\infty} (e^{\xi \gamma} - 1) \frac{\varphi (\xi)}{\xi^{1+\mu+\tau}} d\xi < \infty. \tag{4.4.21}
\]
Suppose now that (4.4.21) is valid. Then:
\[
\inf_{\xi} \left( \frac{\xi^{1+\mu+\tau} + \xi^{1+\mu+\tau} (e^{\xi \gamma} - 1)}{\xi^{1+\mu+\tau}} \right) \times \int_{0}^{\infty} e^{\xi \gamma} \frac{\varphi (\xi)}{\xi^{1+\mu+\tau}} d\xi \leqslant \int_{0}^{\infty} \left( \frac{\xi^{1+\mu+\tau} + \xi^{1+\mu+\tau} (e^{\xi \gamma} - 1)}{e^{\xi \gamma}} \right) \left( \frac{\varphi (\xi)}{\xi^{1+\mu+\tau}} \right) d\xi = \int_{0}^{\infty} (e^{\xi \gamma} - 1) \frac{\varphi (\xi)}{\xi^{1+\mu+\tau}} d\xi < \infty. \tag{4.4.22}
\]
We still need to verify that (4.4.19) holds. We can in fact substitute and estimate
\[
\sum_{n=0}^{\infty} \frac{|\Phi (\gamma n - x)|}{n!} = \sum_{n=0}^{\infty} e^{-g_{EU}(\gamma n - x)} \frac{\gamma n - x + \mu}{\gamma n - x - 2} \frac{1 - \Theta (\gamma n - x)}{1 - \Theta (k \gamma)} \prod_{k=1}^{n} \frac{1 - \Theta (\gamma n - x)}{1 - \Theta (k \gamma)} \leqslant \sum_{n=0}^{\infty} \frac{1}{n! \left( 1 + \frac{\mu - 2}{\gamma} \right)} < \infty, \tag{4.4.23}
\]
since \(\frac{1 - \Theta (\gamma n - x)}{1 - \Theta (k \gamma)}\) is decreasing with \(n\) and \(C = \sup_{\mu} e^{-g_{EU}(\gamma n - x)} \frac{2 n + \mu}{\gamma n - x - 2}\). From \(4.4.12\), when \(n\) is increasing, we can check the properties of \(g_{EU} (z)\). A single term is approximately
\[
\log \left( 1 + \frac{\mu + 2}{k + z - 2} \right) - \frac{\mu + 2}{k} \approx \left( \mu + 2 \right) \left( 1 - \frac{1}{k + z - 2} \right) = - z - 2 \frac{\mu + 2}{\gamma} \left( 1 - \frac{1}{k + z - 2} \right),
\]
so that the series at \(4.4.12\) is convergent and \(- g_{EU} (z) > 0\).

This guarantees a decay at zero for \(\varphi (\xi)\) faster than \(\xi^\mu\) and a decay at infinity faster than \(\xi^\mu e^{-\xi \gamma}\), yielding two other consequences: first, from (4.4.18), we can ensure that:
\[
\int_{x}^{\infty} e^{\xi \gamma} \frac{\varphi (\xi)}{\xi^{1+\mu+\tau}} d\xi < \infty \tag{4.4.24}
\]
for some $\delta > \mu + \tau + \gamma$.

Moreover, we can consider the moments of the selfsimilar solution:

$$M_{\alpha} = \int_{0}^{\infty} \xi^{\alpha} \varphi (\xi) \, d\xi$$

(4.4.25)

and see that:

**Corollary 4.7.** For all $\alpha > -\mu - 1$, $M_{\alpha} < \infty$.

**Proof.** Let $\alpha = -\mu - 1 + \delta$ for some $\delta > 0$. From Lemmas 4.5 and 4.6, since $\delta/2 > 0$,

$$\int_{-\infty}^{\infty} \left| \Phi \left( is - \mu + \frac{\delta}{2} \right) \right| \, ds < \infty$$

so that formula (4.4.2) yields

$$\varphi (\xi) \leq C_{0} \xi^{\frac{-\delta}{2}}.$$  

(4.4.26)

Therefore, splitting the integration domain in formula (4.4.25),

$$M_{\alpha} \leq C_{0} \int_{0}^{1} \xi^{\alpha+\frac{\mu}{2}} \, d\xi + \int_{1}^{\infty} \xi^{\alpha} \varphi (\xi) \, d\xi$$

$$\leq C_{0} \int_{0}^{1} \frac{1}{\xi^{1-\frac{\delta}{2}}} \, d\xi + \sup_{\xi \geq 1} \left( \frac{\xi^{1-\mu-\tau} + \xi^{1+\mu+\tau}}{\xi^{1+\mu-\tau} + \xi^{1+\mu+\tau}} \right) \int_{1}^{\infty} e^{\xi^{-\gamma}} \varphi (\xi) \, d\xi,$$

(4.4.27)

and, thanks to (4.4.18), $M_{\alpha} < \infty$. \hfill \Box

Another property of the selfsimilar solution is its regularity belonging to the $C^{\infty}$ class. Consider, in fact, inversion formula for $\varphi$ and derive $n$ times:

$$\varphi[n] (\xi) = \frac{d^{n}}{d\xi^{n}} (\xi^{\alpha} \varphi (\xi)) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \left( \frac{d^{n}}{d\xi^{n}} \xi^{-z+n} \right) \Phi (z + x - n) \, dz$$

$$= \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} (n - z) (n - 1 - z) \cdots (1 - z) \xi^{-z} \Phi (z + x - n) \, dz.$$  

(4.4.28)

One can now repeat Lemmas 4.5 and 4.6 in this context since estimate (4.4.18) can be straightforwardly improved to show that $\left| \frac{d}{dz} \right| \left( \xi^{\frac{\mu}{2}} \right)$ can be bounded by exponentially decaying functions. Hence we conclude that, for $x > -\mu + n$,

$$\frac{d^{n}}{d\xi^{n}} (\xi^{\frac{\mu}{2}} \varphi (\xi)) \leq \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| (n - is) (n - 1 - is) \cdots (1 - is) \right| \left| \Phi(is + x - n) \right| ds < \infty,$$

(4.4.29)

or also

$$\left| \frac{d^{n}}{d\xi^{n}} \varphi (\xi) \right| \leq C_{n,\delta} \xi^{\mu-n-\delta}, \quad \text{with } \delta > 0, \delta \ll 1.$$  

(4.4.30)
Asymptotics at infinity

We proceed now showing that the solution decays with an exponential tail at infinity. From equation (4.1.4), we can write:

\[
\xi^2 e^{\xi^\gamma} \partial_\xi \varphi (\xi) + \varphi (\xi) \left( 2\xi e^{\xi^\gamma} + \xi^2 e^{\xi^\gamma} \right) = \xi e^{\xi^\gamma} \int_\xi^\infty \gamma s \gamma - 1 \varphi (s) B \left( \frac{\xi}{s} \right) \, ds
\]  

(4.4.31)

and

\[
\frac{d}{d\xi} \left( \xi^2 e^{\xi^\gamma} \varphi (\xi) \right) = \gamma \xi e^{\xi^\gamma} \int_\xi^\infty u \gamma - 1 \varphi (u) B \left( \frac{\xi}{u} \right) \, du.
\]  

(4.4.32)

Under the extra assumption that \( B (u) \leq \kappa u^\delta \), and, multiplying and dividing times \( e^{\xi^\gamma} \) inside the integral on the right hand side,

\[
\frac{d}{d\xi} \left( \xi^2 e^{\xi^\gamma} \varphi (\xi) \right) \leq \gamma \kappa \xi^{\delta + 1} \int_\xi^\infty e^{\xi^\gamma} u \gamma - 1 - \delta \varphi (u) \, du
\]

it is now possible to use (4.4.24),

\[
\frac{d}{d\xi} \left( \xi^2 e^{\xi^\gamma} \varphi (\xi) \right) \leq \gamma \kappa C \xi^{\delta + 1},
\]

and then:

\[
\varphi (\xi) \leq \frac{\gamma \kappa C}{2 + \delta} \xi^\delta e^{-\xi^\gamma}.
\]  

(4.4.33)

Asymptotics at zero

To conclude this section, we also refine the asymptotics at zero. From equation (4.4.32), we can remark that:

\[
\int_\xi^\infty u \gamma - 1 \varphi (u) B \left( \frac{\xi}{u} \right) \, du \simeq c_2 \xi^\mu \int_0^\infty u \gamma - 1 - \mu \varphi (u) \, du
\]

Therefore, we see that formally:

\[
\frac{d}{d\xi} \left( \xi^2 e^{\xi^\gamma} \varphi (\xi) \right) \simeq \gamma c_1 \mathcal{M}_{\gamma - \mu - 1} \xi^{1 + \mu} e^{\xi^\gamma},
\]

and this yields the behaviour \( \varphi (\xi) \simeq C \xi^\mu \) when \( \xi \ll 1 \).
4.5. **COMPACT SUPPORT CASE**

In the last Section we consider a rather different class of problems. Here $B(u)$ is a function that does not admit the production of very small fragments, because its support is included in the interval $[u_0, 1]$. Intuitively, this fact should influence the regularity of the problem in two senses: on one hand the decay at zero of the self-similar function should be faster, as we have seen in the previous section that the behavior of $B(u) = O(u^\mu)$, for $u$ close to zero, determined the decay as a power $\varphi(\xi) \approx C\xi^\mu$; on the other hand, also the analyticity properties of the Mellin transform $\Phi(z)$ should be improved since, again, the behavior of $B(u)$ determines the semi-plane $\Re(z) > -\mu$.

To state the problem more precisely, let $B(u)$ be such that:

$$B(u) = \begin{cases} 
0, & u \in [0, u_0] \\
\frac{c_1}{u} (u - u_0) + o\left((u - u_0)^\mu\right), & u \in (u_0, u_0 + \delta), \\
B(1) + \frac{c_2}{u} (1 - u) + o\left((1 - u)^\mu\right), & u \in (1 - \delta, 1) 
\end{cases} \quad (4.5.1)$$

for some $\delta > 0$ sufficiently small. It may be useful to define, accordingly,

$$\begin{cases} 
\omega_{u_0}(u) = \frac{c_1}{u} (u - u_0) \\
\omega(u) = B(1) + \frac{c_2}{u} (1 - u)
\end{cases} \quad (4.5.2)$$

By definition of $\Theta(z)$:

$$\Theta(z) = \int_{u_0}^{1} u^{z-1} B(u) \, du, \quad \Theta'(z) = \int_{u_0}^{1} u^{z-1} \log u B(u) \, du.$$ 

Observe that now the region of analyticity of $\Theta(z)$ is all the complex plane since, when $\Re(z) < 0$, no singularity arises in the integrals. Also, when $\Re(z) < 0$, the dominating part of the integral is close to $u_0$.

We are interested in the asymptotic behavior of $\Theta(z)$ that will allow us to analyze Carleman problem (4.1.17). We show now that:

**Lemma 4.8.** Let $B(u)$ verify (4.5.1), Then, for $\Re(z) < 0$ and $|\Re(z)| \gg 1$, we have

$$\Theta(z) \approx c_1 \Gamma(1 + \mu) \frac{u_0^{z+\mu}}{(-z - \mu)^{1+\mu}}. \quad (4.5.3)$$

**Proof.** To see how the main order term is obtained, we write:

$$\Theta(z) = \left( \int_{u_0}^{1} u^{z-1} (B(u) - \omega_{u_0}(u)) \, du \right) + \left( c_1 \int_{u_0}^{1} u^{z-1} (u - u_0)^\mu \, du \right). \quad (4.5.4)$$
In this way, one can see, following the same kind of estimation as in Sections 4.3 and 4.4, that

\[
\left| \int_{u_0}^{1} u^{z-1} (B(u) - \omega_{u_0}(u)) \, du \right| \leq \frac{u_0^{z+\mu}}{(-z-\mu)^{1+\mu}} \left\{ \int_{u_0}^{u_0+\delta} o\left((u - u_0)^{\mu}\right) \right\} + \left\{ \int_{u_0+\delta}^{1} u^{z-1} (B(u) - \omega_{u_0}(u)) \, du \right\} \leq C \left( \frac{u_0 + \delta}{u_0} \right)^z < 1 \quad (4.5.5)
\]

when \(|z| \gg 1\). Computing from (4.5.4) the asymptotics (4.5.3) for \(\Theta(z)\), we can thus drop the first contribution \(\left( \int_{u_0}^{1} u^{z-1} (B(u) - \omega_{u_0}(u)) \, du \right)\) which is a lower order term. We now briefly check that the second term in (4.5.4), \(c_1 \int_{u_0}^{1} u^{z-1} (u - u_0)^{\mu} \, du\), gives (4.5.3):

\[
\int_{u_0}^{1} u^{z-1} (u - u_0)^{\mu} \, du = u_0^{z+\mu} \int_{0}^{1} \frac{s^{(1+\mu)-1}}{(1+s)^{(1+\mu)-z}} ds = u_0^{z+\mu} \left( \text{Beta} \left( 1 + \mu, -z - \mu \right) + \int_{\frac{1}{u_0}-1}^{\infty} \frac{s^{(1+\mu)-1}}{(1+s)^{(1+\mu)-z}} ds \right). \quad (4.5.6)
\]

Now observe that when \(\Re(z) \to -\infty\) the second term goes to zero. In fact, we can see that:

\[
\left| u_0^{z} \int_{\frac{1}{u_0}-1}^{\infty} \frac{s^{(1+\mu)-1}}{(1+s)^{(1+\mu)-z}} ds \right| \leq u_0^{z} \int_{\frac{1}{u_0}-1}^{\infty} \frac{(1+s)^{(1+\mu)-1}}{(1+s)^{(1+\mu)-z}} ds \leq u_0^{z} s^{z+\mu-1} ds = u_0^{z} \left| \frac{1}{z + \mu} \left( \frac{1}{u_0} \right)^{z+\mu} \right| \approx \frac{1}{|z|}. \quad (4.5.7)
\]

Therefore, using the known representation of Beta function as product of Gamma functions,

\[
\text{Beta} \left( 1 + \mu, -z - \mu \right) = \frac{\Gamma \left( 1 + \mu \right) \Gamma \left( -z - \mu \right)}{\Gamma \left( 1 - z \right)} \sim \Gamma \left( 1 + \mu \right) \left( \frac{1}{-z - \mu} \right)^{1+\mu} + \text{l.o.t.}, \quad (4.5.8)
\]

which leads to the asymptotic behavior for \(\Theta(z)\) written in equation (4.5.3). \(\square\)
4.5.1. Asymptotic solution to the Carleman problem (4.1.17)

Our strategy to analyze Carleman problem (4.1.17) is to look for a function $f$ such that:

$$f (\zeta + 0 \cdot i) - f (\zeta - 0 \cdot i) = \log \left( \frac{\Theta \left( \frac{\pi}{2} \log \zeta \right) - 1}{2 - \frac{\gamma}{2\pi i} \log \zeta} \right) \approx \log \left( \frac{c_1 \Gamma (1 + \mu)}{2 - \frac{\gamma}{2\pi i} \log \zeta} \right). \quad (4.5.9)$$

As a first remark, we see that (4.5.9) can be approximated in the following way:

$$f (\zeta + 0 \cdot i) - f (\zeta - 0 \cdot i) = \log (\gamma c_1 \Gamma (1 + \mu)) + \frac{\gamma}{2\pi i} \log u_0 - (2 + \mu) \log \left( -\frac{\gamma}{2\pi i} \log \zeta \right). \quad (4.5.10)$$

Now $f$ can be sought for as

$$\hat{f} (\zeta) = A (\log \zeta)^2 + B \log \zeta \log \left( -\frac{\gamma}{2\pi i} \log \zeta \right) + C \log \zeta, \quad (4.5.11)$$

for some constant values $A$, $B$ and $C$ to be determined. Substituting in (4.5.10):

$$\hat{f} (\zeta + 0 \cdot i) - \hat{f} (\zeta - 0 \cdot i) =$$

$$= -2\pi i B \left( \log \left( -\frac{\gamma}{2\pi i} \log \zeta \right) + \log \left( 1 + \frac{2\pi i}{\log \zeta} \right) \right) - B \log \zeta \log \left( 1 + \frac{2\pi i}{\log \zeta} \right)$$

$$- 4\pi i A \log \zeta + A 4\pi^2 - 2\pi i C. \quad (4.5.12)$$

Some logarithmic functions can be approximated using the first order Taylor expansion: since

$$\log \left( 1 + \frac{2\pi i}{\log \zeta} \right) = o \left( (\log \zeta)^{-1} \right)$$

and

$$(\log \zeta) \times \left( \log \left( 1 + \frac{2\pi i}{\log \zeta} \right) \right) \approx 2\pi i,$$

we get:

$$\hat{f} (\zeta + 0 \cdot i) - \hat{f} (\zeta - 0 \cdot i) \approx -2\pi i B - 4\pi i A \log \zeta + A 4\pi^2 - 2\pi i B \log \left( -\frac{\gamma}{2\pi i} \log \zeta \right) - 2\pi i C. \quad (4.5.13)$$

Therefore, the approximation (4.5.13) can be compared to (4.5.10) yielding:

$$f (\zeta) = \frac{\gamma}{2\pi^2} \log u_0 (\log \zeta)^2 + \left( \frac{2 + \mu}{2\pi i} \right) \log \zeta \log \left( -\frac{\gamma}{2\pi i} \log \zeta \right) - C \frac{\gamma}{\pi i} \log \zeta, \quad (4.5.14)$$

where the constant $C$ is now fixed as:

$$C = \frac{1}{\gamma} \left( 2 + \mu - \frac{\gamma}{2} \log u_0 + \log (\gamma c_1 \Gamma (1 + \mu)) \right). \quad (4.5.15)$$

Finally this ansatz leads to the approximated solution

$$\tilde{\Phi} (z) = e^{-\frac{\gamma}{2} \log u_0 z^2 + (2 + \mu) z \log (-z) - C z}, \quad (4.5.16)$$

which should be considered as a formal computation for large values of $|\Re (z)|.$
Asymptotic evaluation of the self-similar solution \( \varphi (\xi) \)

We wish now to determine how (4.5.16) influence the self-similar solution \( \varphi (\xi) \). In order to compute it, one applies the inverse transform to (4.5.16), giving:

\[
\varphi (\xi) = \int_{-i\infty}^{i\infty} e^{-z\log\xi} e^{-\frac{z}{2} \log u_0} z^2 + (2 + \mu) z \log(-z) - C z \, dz,
\]

and, to evaluate the asymptotic value of this integral, we will employ the Steepest Descent Method (cf. Bender and Orszag \[28\] or Paris and Kaminsky \[253\]).

When \( \xi \ll 1 \), we seek \( z_\ast \) such that \( f' (z_\ast) = 0 \) with \( f(z) = -z \log \xi - \frac{z}{2} \log u_0 \, z^2 + (2 + \mu) \, z \log \left(-z\right) - C \, z \) and \( f(z) \simeq f(z_\ast) + \frac{f''(z_\ast)}{2} \left(z-z_\ast\right)^2 + o \left( \left(z-z_\ast\right)^2 \right) \). Clearly, we have to solve:

\[
z_\ast = \frac{-\log \xi + C - (2 + \mu) \left( \log \left(-z_\ast\right) - 1 \right)}{\gamma \log u_0},
\]

which, since \( \xi \ll 1 \), can be reduced to:

\[
z_\ast \simeq \frac{-\log \xi + C - (2 + \mu) \left( \log \left( \frac{\log \xi}{\gamma \log u_0} \right) - 1 \right)}{\gamma \log u_0} \ll 0.
\]

In order to simplify the notation, we introduce two values \( f_\ast \) and \( f''_\ast \):

\[
f_\ast = f(z_\ast) \simeq \frac{(\log \xi)^2}{2 \gamma \log u_0} - (2 + \mu) \frac{\log \xi}{\gamma \log u_0} \log \left( \frac{\log \xi}{\gamma \log u_0} \right) + C \frac{\log \xi}{\gamma \log u_0},
\]

\[
f''_\ast = f''(z_\ast) = -\gamma \log u_0 - \frac{2 + \mu}{z_\ast} \simeq \gamma \log u_0 \left( -1 + \frac{2 + \mu}{\log \xi} \right) > 0.
\]

Therefore, we can formally modify the contour of integration having it cross \( z_\ast \): applying Cauchy formula, and observing that the term \( e^{-\frac{z}{2} \log u_0 \, z^2} \) is sufficient to ensure that the arch contributions tend to zero, we have:

\[
\int_{-i\infty}^{0} e^{-z\log\xi} e^{-\frac{z}{2} \log u_0} z^2 + (2 + \mu) z \log(-z) - C z \, dz =
\]

\[
= 0 + \int_{C_N} e^{-z\log\xi} e^{-\frac{z}{2} \log u_0} z^2 \, dz + \int_{-N}^{0} e^{-z\log\xi} e^{-\frac{z}{2} \log u_0} z^2 + (2 + \mu) z \log(-z) - C z \, dz.
\]

This procedure can be repeated with the second contribution. Therefore,

\[
\varphi (\xi) = \int_{-\infty}^{\infty} e^{-z\log\xi} e^{-\frac{z}{2} \log u_0} z^2 + (2 + \mu) z \log(-z) - C z \, dz,
\]
and we can now apply Laplace method and write the leading term of \( \varphi (\xi) \): 

\[
\varphi (\xi) \simeq \int_{z_0 - \varepsilon}^{z_0 + \varepsilon} e^{-z \log \xi} e^{-\frac{d}{2} \log u_0 z^2 + (2 + \mu) z \log (z) - C z} \, dz
\]

\[
= e^{f_0} \int_{-\infty}^{\infty} e^{i \pi (z - z_0)^2} \, dz = \frac{\sqrt{2\pi}}{\sqrt{f_0}} e^{f_0}
\]  

(4.5.23)

then, we see that \( \varphi (\xi) \) decays at zero with log-normal main term, with a non-trivial \( \log \xi \times \log (\log \xi) \) correction:

\[
\varphi (\xi) \simeq \frac{\sqrt{2\pi}}{\left( \frac{2 + \mu}{\log \xi} - 1 \right) \gamma \log u_0} e^{-\frac{2 + \mu}{\log \xi} \log \xi \log \left( \frac{\log \xi}{\gamma \log u_0} \right) \xi \gamma u_0}.
\]  

(4.5.24)

### 4.5.2. Conclusions: The New Carleman Problem and Formal Results for Future Work

In Section 4.5 we considered the compact support case for fragmentation and worked on the asymptotics for the solution. It seems possible to improve this analysis with the techniques that produced infinite product formulas in (4.3.2i). We dedicate some last formal computations to track out how this result could be established.

We can, in general, look for a correction to (4.5.16):

\[
\Phi (z) = e^{-\frac{d}{2} \log u_0 z^2 + (2 + \mu) z \log (z) - C z} e^g(z)
\]  

(4.5.25)

such that

\[(2 - z) \Phi (z) = \gamma (\Theta (z) - 1) \Phi (z + \gamma),\]

which yields a new functional problem in the complex plane for the function \( g(z) \):

\[
e^g(z) = e^{-C\gamma \gamma} \frac{\Theta (z) - 1}{2 - z} e^{-\frac{d}{2} \log u_0 z^2 \log u_0 + (2 + \mu) z \log (1 - \frac{z}{2}) + (2 + \mu) \gamma \log (-\gamma) \gamma e^{g(z + \gamma)}.}
\]  

(4.5.26)

To simplify the notation, we remark that it is possible to write:

\[
\Theta (z) - 1 = \left( \int_{u_0}^{1} u^{z-2} (uB(u)) \, du \right) - 1 = \left( \hat{B}(1) - u_0^{z-2}\hat{B}(u_0) \right) - (z - 2) \int_{u_0}^{1} u^{z-3} \hat{B}(u) \, du - 1
\]

with \( \hat{B}(u) = \int_{u_0}^{u} sB(s) \, ds \), so that \( \hat{B}(u_0) = 0 \) and \( \hat{B}(1) = 1 \). This gives an easy representation of \( \Theta(z) - 1 \):

\[
W(z) = \frac{\Theta (z) - 1}{2 - z} = \int_{u_0}^{1} u^{z-1} \frac{\hat{B}(u)}{u^2} \, du,
\]  

(4.5.27)
so that \( W(z) \) is the moments transform of \( \hat{B}(u) / u^2 \) just like \( \Theta(z) \) is the transform of \( B(u) \).

It is clear that \( W(z) \) is analytic over all the complex plane. Hence, we can rewrite equation \( 4.5.26 \) in this fashion:

\[
\begin{aligned}
\left\{ \begin{array}{l}
ge^{g(z)} = \gamma W(z) e^{f(z)} e^{g(z+\gamma)}, \\
     f(z) = -C\gamma - \frac{\mu}{\gamma} \log u_0 - \gamma^2 z \log u_0 + (2 + \mu) z \log (-1 - \frac{z}{2}) + (2 + \mu) \gamma \log (-z - \gamma)
\end{array} \right.
\end{aligned}
\]
i.e.

\[
\begin{aligned}
    f'(z) = -\gamma^2 \log u_0 + (2 + \mu) \left( \log \left(1 - \frac{\gamma}{z} \right) + \frac{z}{1 - \frac{\gamma}{2}} (\gamma z^{-2}) \right) + \frac{(2 + \mu) \gamma}{z + \gamma} \\
    = -\gamma^2 \log u_0 + (2 + \mu) \log \left(1 - \frac{\gamma}{z} \right).
\end{aligned}
\]

Remark 4.9. Observe that, in agreement to Lemma 4.8, when \(|\Re(z)| \gg 1\),

\[
W(z) \approx c_1 (1 + \mu) \frac{u_0^{z+\mu}}{|z|^{2+\mu}}
\]

and we also introduce \( \tilde{C} \) such that:

\[
\log \left( \gamma W(z) e^{f(z)} \right) \approx \tilde{C} + (z - \gamma^2 z) \log u_0 + (2 + \mu) z \left( \log (-z - \gamma) - \log z \right).
\]

The solution to the bounded support case: a new infinite product formula

Repeating the Wiener-Hopf methods developed in Section 4.3, Problem \( 4.5.28 \) yields:

\[
\frac{d}{dz} g_N(z) \sim -\frac{1}{\gamma} \left( \log (\gamma W(-iR_N) e^{f(-iR_N)}) + \int_{-iR_N}^{iR_N} \frac{d\omega}{1 - e^{\frac{2\pi}{\gamma}(z-\omega)}} \right).
\]

In order to obtain a different result with respect to formula \( 4.3.21 \), we now include the poles at \( \omega = z - k\gamma \), with \( k \gg 0 \), instead of the poles that we considered in Figure 4.2.1. In this sense we exploit the extension of the analyticity region for the compact support case. This yields:

\[
\int_{\Gamma_N} \frac{d\omega}{1 - e^{\frac{2\pi}{\gamma}(z-\omega)}} = 2\pi i \sum_{k=0}^{N} \text{Res} (\varphi_N; z - k\gamma) = \sum_{k=0}^{N} \frac{d}{dz} \left( \log (\gamma W(z - k\gamma) e^{f(z-k\gamma)}) \right).
\]

We remark that the sign has changed with respect to Section 4.3 due to the fact that the orientation in the integration path is the opposite. Finally, we can get:

\[
g_N(z) = -\tilde{C} \frac{z}{\gamma} - \frac{1}{\gamma} \sum_{k=0}^{N} \log (\gamma W(z - k\gamma) e^{f(z-k\gamma)}) + \text{const.}
\]

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We can choose, at each step, the constant such that:

\[
g_N(z) \sim \tilde{C} \left( \frac{z}{\gamma} \right) - \frac{1}{\gamma} \sum_{k=0}^{N} \left( \log \left( W(z - k\gamma) \right) - \log \left( W(-k\gamma) \right) + f(z - k\gamma) - f(-k\gamma) \right) \tag{4.5.34}
\]

Where \( W(-k\gamma) \) is a function chosen analogously to \( \Omega(k\gamma) \) from definition (4.3.16) in order for the series to converge.

Hence, the solutions to (4.5.28) and to (4.1.12) are:

\[
g(z) = -\tilde{C} \left( \frac{z}{\gamma} \right) + \frac{1}{\gamma} \sum_{k=0}^{\infty} \log \left( \frac{W(-k\gamma)}{W(z - k\gamma)} e^{f(-k\gamma) - f(z - k\gamma)} \right), \tag{4.5.35}
\]

\[
\Phi(z) = e^{-\left( \frac{z}{\gamma} \log \nu_0 \left( \frac{1}{\nu} \right) \right)^2 + \left( 2+\mu \right) \frac{z}{\gamma} \log \left( \frac{|z|}{\gamma} \right) - C \frac{z}{\gamma}} \left( \prod_{k=0}^{\infty} \frac{W(-k\gamma)}{W(z - k\gamma)} e^{f(-k\gamma) - f(z - k\gamma)} \right)^{\frac{1}{\gamma}}. \tag{4.5.36}
\]

Concluding this chapter, we remark that the analyticity region of \( \Phi(z) \) could be shown to cover the complex plane, so that all the moments of the self-similar solution \( \varphi(\xi) \) are bounded. Also, all properties of Section 4.4.2 can be adapted and the asymptotics at zero expressed by formula (4.5.24) can be rigorously proved.

### APPENDIX A: SOLUTIONS CLOSE TO EXISTING SOLUTIONS

A particular formula can be obtained when we consider distributions \( B(u) \) that are close to some other \( B_0(u) \) for which the solution is known. First denote as \( \delta(z) \) the function verifying \( \Theta(z) = \delta(z) (\Theta_0(z) - 1) + 1 \), with \( \Theta_0(z) \) the moment transform of \( B_0(u) \). In that case we have:

**Proposition 4.10.** Suppose that

\[
\lim_{s \to 0} \frac{|B_0(1 - s) - B(1 - s)|}{s^\nu} = c, \tag{4.5.37}
\]

with \( \nu > 0 \) and \( B_0(u) \) such that \( \Phi_0(z) \) is solution to the Carleman problem for \( \Theta_0(z) \). Therefore \( \Phi(z) \) defined as

\[
\Phi(z) = \Phi_0(z) \left[ \delta(z) \prod_{k=1}^{\infty} \delta(z + k\gamma) \right] \tag{4.5.38}
\]

is the solution to the Carleman problem for \( \Theta(z) \).

**Proof.** As we know,

\[
J_N(z) \sim \left( \frac{z}{\gamma} \right) \sum_{k=1}^{N} \frac{1}{k} - g_{Eu} \left( \frac{z}{\gamma} \right) + \log \left( \frac{\Theta(z) - 1}{2 - z} \right) + \sum_{k=1}^{N} \left[ \log \left( \frac{\Theta(z + k\gamma) - 1}{2 - k\gamma - z} \right) - \log k\gamma \right] + \text{const}, \tag{4.5.39}
\]
for any $N$, and the integration constant can be fixed as before. Therefore:
\[
J_N(z) \sim z^N \frac{1}{\gamma} \sum_{k=1}^{N} \frac{1}{k} - g \mathcal{E} u \frac{z}{\gamma} + \log \left( \frac{\Theta_0(z) - 1}{2 - z} \right) + \sum_{k=1}^{N} \left( \log \left( \frac{\Delta(z + k\gamma) \Theta_0(z + k\gamma) - 1}{2 - k\gamma - z} \right) \right)
\]
and
\[
\Phi_N(z) = \Phi_N(z) \sim k e^{-g \mathcal{E} u \frac{z}{\gamma}} \left( \frac{\Theta_0(z) - 1}{2 - z} \right) \prod_{k=1}^{N} \left( \frac{\Theta_0(z + k\gamma) - 1}{2 - k\gamma - z} \right) e^{k \frac{z}{\gamma}} \left[ \Delta(z) \prod_{k=1}^{N} \Delta(z + k\gamma) \right]
\]
which leads to:
\[
\Phi(z) = \Phi_N(z) \left[ \delta(z) \prod_{k=1}^{\infty} \delta(z + k\gamma) \right].
\]
We still need to verify the convergence of $\Delta(z) := \delta(z) \prod_{k=1}^{\infty} \delta(z + k\gamma)$. In order for the product $\prod_{k=1}^{\infty} \Delta(z + k\gamma)$ to converge, the series $\sum \log \Delta(z + k\gamma)$ must converge. Thus:
\[
\sum \log \Delta(z + k\gamma) = \sum \log (1 - \Theta(z + k\gamma)) - \sum \log (1 - \Theta_0(z + k\gamma))
\]
and, for $k$ large enough,
\[
\log (1 - \Theta(z + k\gamma)) - \log (1 - \Theta_0(z + k\gamma)) \sim \Theta_0(z + k\gamma) - \Theta(z + k\gamma).
\]
We will impose the condition that the series converges absolutely, that is, for a sufficiently large $k$,
\[
\lim_{k \to \infty} k^{1+\varepsilon} |\Theta_0(z + k\gamma) - \Theta(z + k\gamma)| = 0,
\]
for some nonnegative $\varepsilon$, that is:
\[
k^{1+\varepsilon} \int_{0}^{1} u^{-1+k\gamma} |b_0(u) - b(u)| du \leq k^{1+\varepsilon} \int_{0}^{1} u^{-\nu(z)-1+k\gamma} |b_0(u) - b(u)| du
\]
\[
\sim k^{1+\varepsilon} \int_{0}^{1} u^{k\gamma-1} |b_0(u) - b(u)| du.
\]
The relevant contribution to this integral is, for $k \gg 1$, close to $u \approx 1$. By hypothesis there is a $c < \infty$ such that:
\[
\lim_{s \to 0} \frac{|b_0(1-s) - b(1-s)|}{s^\nu} = c.
\]
Therefore:
\[
k^{1+\varepsilon} \int_{0}^{1} u^{-1+k\gamma} |b_0(u) - b(u)| du \sim c k^{1+\varepsilon} \int_{0}^{1} (1-s)^{k\gamma-1} s^\nu ds
\]
\[
= c k^{1+\varepsilon} \text{Beta} \left( k\gamma, \nu + 1 \right) \sim c k^{1+\varepsilon} \Gamma(\nu + 1) (k\gamma)^{-\nu-1}
\]
so it is sufficient that $\varepsilon = \frac{\nu}{\gamma}$, since $\nu > 0$. \qed
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