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Dyadic harmonic analysis

—non-doubling and noncommutative aspects—

Memoria de Tesis Doctoral presentada por

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Resumen

La presente tesis contiene resultados sobre análisis armónico diádico en distintos contextos; proporcionando estimaciones *a priori* para modelos diádicos de integrales singulares. La exposición de los resultados se divide en tres partes. En la primera se caracterizan las medidas de Borel en $\mathbb{R}$ para las cuales la transformada de Hilbert diádica asociada es de tipo débil $(1,1)$. Sorprendentemente, la clase de medidas obtenida contiene estrictamente a las medidas diádicamente doblantes y está contenida estrictamente en la clase de Borel. Se demuestra además que la clase dual caracteriza el tipo débil $(1,1)$ del adjunto de la transformada de Hilbert diádica. La herramienta principal es una nueva descomposición de Calderón-Zygmund válida para medidas de Borel generales y de interés independiente. Caracterizaciones análogas del tipo débil $(1,1)$ para operadores Haar shift multidimensionales son obtenidas en términos de dos sistemas de Haar generalizados y no necesariamente cancelativos. Los paraproductos diádicos y sus adjuntos figuran como casos particulares importantes. Por otro lado, es bien sabido que operadores de Calderón-Zygmund con núcleos matriciales — incluso aquellos con buenas propiedades de tamaño y suavidad o cancelación — carecen de estimaciones en L_p semiconmutativas para $p \neq 2$. En la segunda parte de la tesis se obtienen estimaciones de tipo débil $(1,1)$ de operadores perfectamente diádicos y, en general para operadores Haar shift, en términos de una descomposición fila/columna de la función de partida. Se muestra también que operadores de Calderón-Zygmund generales satisfacen estimaciones de tipo $\mathrm{H}_1 \to L_1$, que junto con estimaciones de tipo $L_\infty \to \mathrm{BMO}$, implican estimaciones fila/columna en espacios L_p semiconmutativos. El enfoque presentado es aplicable a transformadas de martingala y paraproductos con símbolos no conmutativos, para los que obtenemos estimaciones análogas. La tercera parte está dedicada a la generalización semiconmutativa de los resultados obtenidos en la primera parte. Esto es, a la caracterización del tipo débil $(1,1)$ de operadores Haar shift definidos en términos de dos sistemas de Haar generalizados adaptados a una medida de Borel y con símbolos conmutativos. Así como en el caso conmutativo, el principal recurso técnico es una versión no conmutativa de la descomposición de Calderón-Zygmund introducida en la primera parte.

Abstract

This thesis is divided into three parts, each presenting results on dyadic harmonic analysis in different settings. More specifically, it provides *a priori* estimates of dyadic and singular integral operators in the non-doubling and semicommutative frameworks. In Part I we characterize the locally finite Borel measures μ on $\mathbb{R}$ for which the associated dyadic Hilbert transform satisfy $L^1(\mu) \to L^{1,\infty}(\mu)$ estimates. Surprisingly, the class of such measures is strictly bigger than the standard class of dyadically doubling measures and strictly smaller than the whole Borel class. We further show that a dual class characterizes the weak-type $(1,1)$ of the adjoint of the dyadic Hilbert transform. In higher dimensions, we provide a complete characterization of the weak-type $(1,1)$ of arbitrary Haar shift operators — cancellative or not — written in terms of two generalized Haar systems, including dyadic paraproducts. The main tool used in Part I is a new Calderón-Zygmund decomposition valid for arbitrary Borel measures which is of independent interest. On the other hand, it is well known that Calderón-Zygmund operators with noncommuting kernels may fail to be L_p bounded in semicommutative L_p spaces for $p \neq 2$, even for kernels with good size and smoothness properties or having dyadic cancellation properties. In Part II we obtain weak-type $(1,1)$ estimates for perfect dyadic Calderón-Zygmund operators associated to noncommuting kernels in terms of a row/column decomposition of the input function. Analogous estimates are also proved for arbitrary Haar shift operators. General Calderón-Zygmund operators satisfy $\mathrm{H}_1 \to L_1$ type estimates. In conjunction with $L_\infty \to \mathrm{BMO}$ type estimates, we get similar row/column L_p estimates. The approach here presented also applies to martingale transforms and paraproducts with noncommuting symbols for which we obtain analogous estimates. In Part III we obtain a complete characterization of the weak-type $(1,1)$ of commuting Haar shift operators in terms of generalized Haar systems adapted to a Borel measure μ in the semicommutative setting. The main technical tool in our method is a noncommutative Calderón-Zygmund decomposition that generalizes the Calderón-Zygmund decomposition used in the first part.

Contents

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Prefacio

Las técnicas diádicas juegan actualmente un papel fundamental en análisis armónico. El origen del *análisis armónico diádico* se remonta a los trabajos de Hardy, Littlewood, Paley y Walsh entre otros. Algunos resultados en el área pueden ser contextualizados en la teoría de desigualdades de martingalas. Por ejemplo, el maximal diádico y la función cuadrado diádica son casos particulares del maximal de Doob y de la función cuadrado de Burkholder relativos a filtraciones diádicas; siendo modelos relativamente simples del maximal de Hardy-Littlewood y de funciones cuadrado. De manera análoga, las integrales singulares — con la transformada de Hilbert como arquetipo — son modelizadas mediante transformadas de martingala y paraproductos de martingala. Tales operadores son representados en términos de operadores de diferencia de martingalas y esperanzas condicionadas, por lo que potentes métodos probabilísticos pueden ser aplicados al análisis de sus propiedades de acotación. En el marco euclídeo — y con mayor generalidad en el contexto de martingalas relativas a filtraciones atómicas — los operadores de diferencia de martingalas descomponen como una suma de proyecciones de rango uno que son perfectamente localizadas, i.e., en proyecciones de Haar. De esta manera, en el contexto euclídeo, las transformadas de martingala corresponden a operadores diagonales relativos al sistema de Haar. Este enfoque puede ser extendido al considerar operadores compactos cuya representación matricial con respecto al sistema de Haar sea dispersa. Dichos operadores, llamados *operadores Haar shift*, conforman una fuente rica de modelos de integrales singulares.

La presente disertación trata principalmente modelos diádicos y de Haar para operadores y objetos clásicos de análisis armónico. En particular, se estudiarán las propiedades de acotación de operadores Haar shift en distintos contextos. En la siguiente sección haremos un breve repaso de los conceptos básicos de análisis armónico diádico. En las secciones subsiguientes presentamos y discutimos los resultados obtenidos durante el desarrollo de esta tesis.

Análisis armónico diádico clásico

En los últimos años, modelos diádicos han recibido una atención considerable por la comunidad matemática, debido principalmente a su utilidad en la resolución de la llamada conjetura A_2, que afirma que ciertos operadores satisfacen una estimación en $L^2(w)$ para todo peso $w \in A_2$ con constante que depende linealmente de la característica A_2 de w. Técnicas de extrapolación pueden luego ser empleadas para obtener la dependencia óptima en la característica A_p para la estimación en $L^p(w)$ correspondiente. Este problema ha recibido una atención creciente desde su planteamiento por Buckley en [7] debido principalmente al trabajo de Astala, Iwaniec y Saksman [1], en el cual demostraron que si el operador de Beurling-Ahlfors satisface estimaciones óptimas en $L^p(w)$, se podrían entonces obtener resultados de regularidad para soluciones de la ecuación de Beltrami.

La solución de la conjetura A_2 para la función maximal fue obtenida por Buckley en [7]. Wittwer probó la conjetura A_2 para multiplicadores de Haar en una dimensión en [78]. La conjetura A_2 para el operador de Beurling-Ahlfors, la transformada de Hilbert y las transformadas de Riesz fue demostrada por Petermichl y Volberg en [65, 63, 64] (véase también [22]) mediante una representación de dichos operadores en términos de operadores Haar shift; obteniendo así una respuesta positiva al problema propuesto en [1]. La conjetura para paraproductos fue probada por Beznosova en [5] y por Cruz-Uribe, Martell y Pérez en [17] usando un enfoque distinto. La solución final de la conjetura A_2 para operadores de Calderón-Zygmund generales fue obtenida por Hytönen en [29]. Un ingrediente clave para la demostración final de la conjetura es que operadores de Calderón-Zygmund pueden ser representados como una serie rápidamente convergente de operadores Haar shift y paraproductos diádicos, resultado conocido como el *teorema de representación de Hytönen*. Este resultado está estrechamente relacionado con el tratamiento de Figiel [24] del teorema $T(1)$ y también con el trabajo de Beylkin, Coifman y Rokhlin [4] y se basa en una descomposición de operadores de Calderón-Zygmund obtenida por Nazarov, Treil y Volberg en [57] para probar el teorema $T(1)$ en espacios no homogéneos. Sin embargo, el teorema de representación de Hytönen difiere de las descomposiciones obtenidas en [24, 4] en cuanto a que las series asociadas convergen rápidamente tanto para operadores suaves como para operadores no suaves. Dicha propiedad hace que el teorema de representación de Hytönen sea un resultado importante en sí mismo, independientemente de su utilidad en la resolución de la conjetura A_2.

Antes del novedoso trabajo de Petermichl en [62] los únicos modelos diádicos disponibles para integrales singulares eran los *multiplicadores de Haar* y los *paraproductos diádicos*. En una dimensión estos operadores son de la siguiente forma

$$T_\alpha f(x) = \sum_{I\in\mathscr{D}} \alpha_I \langle f, h_I\rangle h_I(x) \quad \text{y} \quad \Pi_\rho f(x) = \sum_{I\in\mathscr{D}} \langle f\rangle_I \langle \rho, h_I\rangle h_I(x).$$

Aquí, $\mathscr{D}$ denota un cierto retículo diádica en $\mathbb{R}$, los símbolos α_I son escalares uniformemente acotados, $\rho \in \mathrm{BMO}_{\mathscr{D}}$, $\langle f, h_I\rangle$ denota la forma $\int_{\mathbb{R}} f(x)h_I(x)\,dx$, $\langle f\rangle_I$ denota el promedio de f en I y h_I es la función de Haar asociada a $I \in \mathscr{D}$:

$$h_I = \frac{1}{|I|^{1/2}}\big(1_{I_-} - 1_{I_+}\big).$$

Aquí, I_- y I_+ denotan los hijos diádicos izquierdo y derecho de I. Por supuesto, el sistema de Haar $\{h_I\}_{I\in\mathscr{D}}$ es un sistema ortonormal en $L^2(\mathbb{R})$. Petermichl introdujo en [62] la *transformada de Hilbert diádica*, dada por la expresión

$$H_{\mathscr{D}} f(x) = \sum_{I \in \mathscr{D}} \langle f, h_I \rangle \big(h_{I_-}(x) - h_{I_+}(x) \big).$$

El hecho notable es que es posible recuperar la transformada de Hilbert promediando $H_{\mathscr{D}}$ sobre retículos diádicos aleatorios. Es gracias a este resultado que las técnicas diádicas juegan actualmente un papel central en la teoría de estimaciones con pesos, puesto que establece la pauta de "transferir" pruebas relativamente sencillas del contexto diádico al continuo. Lacey, Petermichl y Reguera introdujeron en [44] una clase de operadores a la cual pertenecen los multiplicadores de Haar y la transformada de Hilbert diádica: los *operadores Haar shift*. Un operador de Haar shift de complejidad $(j, k) \in \mathbb{Z}_+^2$ es de la forma

$$Ш_{j,k} f(x) = \sum_{I \in \mathscr{D}} A_I f = \sum_{I \in \mathscr{D}} \sum_{\substack{J \in \mathscr{D}_j(I) \\ K \in \mathscr{D}_k(I)}} \alpha_{J,K}^I \langle f, h_J \rangle h_K(x),$$

donde $\mathscr{D}_j(I)$ denota la familia de los j-ésimos descendientes diádicos de I, es decir, los elementos de la partición de I en subintervalos $J \in \mathscr{D}$ de longitud $\ell(J) = 2^{-j}\ell(J)$. De esta manera, los multiplicadores de Haar y la transformada de Hilbert diádica son operadores Haar shift de complejidad $(0, 0)$ y $(0, 1)$. Usualmente se restringe la atención a operadores cuyos símbolos $\alpha_{J,K}^I$ satisfacen la normalización

$$\big|\alpha_{J,K}^I\big| \leq \frac{\sqrt{|J||K|}}{|I|},$$

lo que garantiza que $Ш_{j,k}$ sea un operador contractivo en L^2 y que sus componentes localizadas A_I sean contractivas en L^p para $1 \leq p \leq \infty$. Si en la definición de operador Haar shift se permite también el uso de funciones características normalizadas en L^2 — i.e. $|I|^{-1/2} 1_I$ para $I \in \mathscr{D}$ — como bloques constituyentes de operadores diádicos, se obtiene la clase de *operadores Haar shift no cancelativos*, una clase de operadores diádicos que incluye los paraproductos diádicos y los paraproductos adjuntos

$$\Pi_\rho^* f(x) = \sum_{I \in \mathscr{D}} \langle f, h_I \rangle \langle \rho, h_I \rangle \frac{1_I(x)}{|I|},$$

como operadores de complejidad $(0, 0)$. Sin embargo, al considerar funciones características se pierde ortogonalidad por lo que la acotación en L^2 es no trivial, requiriendo de teoremas parecidos al teorema de encaje de Carleson. Por esta razón la acotación en L^2 de operadores Haar shift no cancelativos es por lo general asumida.

Los operadores Haar shift no cancelativos también incluyen a la clase de *operadores dispersos positivos* introducida por Lerner en [45], donde proporciona una prueba alternativa y más elemental de la conjetura A_2. Su demostración se basa en el notable hecho de que la norma de un operador de Calderón-Zygmund en un retículo de Banach es dominada por la norma de una combinación de operadores dispersos positivos. Más todavía, dicho control por operadores dispersos positivos es puntual como se demuestra en [15, 43]. Los operadores dispersos positivos vienen definidos por

$$Sf(x) = \sum_{I \in \mathcal{S}} \langle f \rangle_{I^{(j)}} 1_I(x)$$

donde $I^{(j)}$ es el ancestro diádico j-ésimo de I y $\mathcal{S} \subset \mathscr{D}$ denota una familia dispersa de intervalos diádicos en el sentido que para todo $I \in \mathcal{S}$ se tiene que

$$\sum_{J \in \mathscr{D}_1(I) \cap \mathcal{S}} |J| \leq \frac{1}{2}|I|.$$

Es fácil ver que los operadores dispersos positivos son operadores Haar shift de complejidad $(j, 0)$ con coeficientes dados por

$$\alpha_J^I = \begin{cases} \dfrac{|J|^{1/2}}{|I|^{1/2}} & \text{si} \quad I \in \mathcal{S} \\ 0 & \text{de otro modo.} \end{cases}$$

El tipo débil $(1, 1)$ sin pesos de operadores Haar shift juega un papel esencial en ambos métodos de prueba de la conjetura A_2 — ya sea por aproximación o dominación. Explícitamente se requiere que

$$\lambda |\{x \in \mathbb{R} : |Ш_{j,k} f(x)| > \lambda\}| \leq C \|f\|_{L^1(\mathbb{R})},$$

donde la constate C sólo depende de la complejidad (j, k) del operador de manera lineal o incluso polinomial. Por supuesto, la acotación en L^p para $1 < p < \infty$ se obtiene como corolario usando los argumentos de interpolación y dualidad. El tipo débil $(1, 1)$ de operadores diádicos se obtiene utilizando la descomposición de Calderón-Zygmund estándar como en [17, 29, 44]. Repasemos brevemente esta técnica. Dada $f \in L^1(\mathbb{R})$ y $\lambda > 0$, consideremos el conjunto de nivel

$$\Omega_\lambda = \Big\{ x \in \mathbb{R} : M_{\mathscr{D}} f(x) > \lambda \Big\} = \bigcup_i Q_i,$$

donde $M_{\mathscr{D}}$ denota el maximal diádico $M_{\mathscr{D}} f = \sup_{I \in \mathscr{D}} \langle |f| \rangle_I$ y $\{Q_i\}_i$ es la familia de cubos diádicos maximales asociada a Ω_λ. Entonces f descompone como $f = g + b$, donde g es conocida como la “parte buena” de f y b la “parte mala” y vienen dadas por

$$g = f 1_{\mathbb{R} \setminus \Omega_\lambda} + \sum_i \langle f \rangle_{Q_i} 1_{Q_i} \quad \text{y} \quad b = \sum_i (f - \langle f \rangle_{Q_i}) 1_{Q_i}.$$

Si escribimos $b_i = (f - \langle f \rangle_{Q_i}) 1_{Q_i}$, entonces tenemos que

- $\|g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}$ y $\|g\|_{L^\infty(\mathbb{R})} \leq 2\lambda$.
- $\operatorname{supp}(b_i) \subset Q_i$, $\int_{Q_i} b_i(x)\, dx = 0$ y $\sum_i \|b_i\|_{L^1(\mathbb{R})} \leq 2\|f\|_{L^1(\mathbb{R})}$.

En el análisis del comportamiento de un operador Haar shift $Ш_{j,k}$ cerca de la escala L^1, las estimaciones satisfechas por la parte buena permiten obtener constantes del orden de la norma de $Ш_{j,k}$ en L^2. Las propiedades de localización y de media cero de los términos b_i y la estructura diádica de $Ш_{j,k}$ y de sus componentes A_I permiten obtener estimaciones con constantes que dependen linealmente de la complejidad (j,k), y de la norma de operador de A_I en $L^1(\mathbb{R})$. De hecho, como se demuestra en [30], la dependencia lineal en j es en realidad óptima.

El motivo principal de esta tesis es el contribuir a esta línea de investigación al proporcionar estimaciones de tipo débil $(1,1)$ para operadores Haar shift en los contextos de análisis armónico no doblante y análisis armónico semiconmutativo. Si bien las estimaciones obtenidas parecen no ser óptimas en su dependencia de la complejidad del operador, nuestros resultados indican — e incluso caracterizan — la estructura básica que los espacios ambiente deben satisfacer para que existan estimaciones *a priori* de operadores diádicos.

Análisis armónico diádico para medidas no doblantes

Consideremos una medida de Borel positiva μ en $\mathbb{R}$, es posible entonces definir un sistema de Haar mediante las funciones

$$h_I^\mu = \sqrt{m(I)}\left(\frac{1_{I_-}}{\mu(I_-)} - \frac{1_{I_+}}{\mu(I_+)}\right), \qquad \text{con} \quad m(I) = \frac{\mu(I_-)\mu(I_+)}{\mu(I)};$$

formando un sistema ortonormal en $L^2(\mu)$. Podemos así considerar una transformada de Hilbert diádica asociada al sistema $\{h_I^\mu\}_{I\in\mathscr{D}}$:

$$H_{\mathscr{D}}^\mu f(x) = \sum_{I\in\mathscr{D}} \langle f, h_I^\mu\rangle \big(h_{I_-}^\mu(x) - h_{I_+}^\mu(x)\big)$$

y examinar sus propiedades de acotación, de las cuales la acotación en $L^2(\mu)$ es inmediata por ortogonalidad. La teoría estándar de Calderón-Zygmund puede ser fácilmente extendida a contextos en los que la medida subyacente es doblante. Puesto que el operador en cuestión es diádico, la condición sobre μ puede ser relajada a ser diádicamente doblante. En tal caso uno puede transcribir literalmente la prueba clásica y obtener el tipo débil $(1,1)$ de la transformada de Hilbert diádica con respecto de μ. Una pregunta natural es determinar si existen medidas μ que no sean necesariamente diádicamente doblantes para las cuales $H_{\mathscr{D}}^\mu$ mapea $L^1(\mu)$ continuamente en $L^{1,\infty}(\mu)$.

La caracterización de las medidas para las cuales un determinado operador es acotado es generalmente un problema difícil. Tal es el caso, por ejemplo, de la acotación en L^2 de la transformada de Cauchy y la clase de medidas de crecimiento lineal obtenida por Tolsa en [75]. Este descubrimiento permitió la formulación de teorías de Calderón-Zygmund no estándares — en las cuales la medida subyacente μ obedece una propiedad de crecimiento polinomial — desarrolladas por Nazarov, Treil, Volberg y Tolsa y que podrían ser aplicadas a la presente situación. Sin embargo, la aplicación de dichas teorías requeriría añadir suposiciones, que serían probablemente innecesarias *a posteriori*, puesto que estamos tratando con un operador diádico. Algunos operadores diádicos tienen buenas propiedades de acotación incluso en contextos no doblantes. Por ejemplo, el maximal diádico y la función cuadrado diádica son de tipo débil $(1,1)$ para cualquier medida de Borel μ, como se demuestra en el marco de desigualdades de martingalas en [20] y [10] respectivamente. Con esto en mente, uno podría suponer que $H^{\mu}_{\mathscr{D}}$ es de tipo débil $(1,1)$ para cualquier medida de Borel μ, sea doblante, no doblante o de crecimiento polinomial. Es natural plantearse el mismo problema para otros operadores diádicos tales como el adjunto de la transformada de Hilbert diádica, operadores Haar shift cancelativos, paraproductos diádicos, adjuntos de paraproductos diádicos, o en general para operadores Haar shift no cancelativos. Esto motiva el problema que tratamos en la Parte I de esta tesis:

> *Determinar la familia de medidas μ para las cuales un operador diádico es de tipo débil* $(1,1)$.

Como hemos mencionado, si la medida μ es diádicamente doblante, uno puede aplicar la teoría estándar de Calderón-Zygmund para probar que estos operadores satisfacen estimaciones de tipo débil $(1,1)$. De esta manera, es natural preguntarse si la condición doblante es en realidad necesaria o si es sólo conveniente. Como mostraremos en la Parte I no existe una respuesta universal a tal cuestión: la clase de medidas asociada a las propiedades de acotación de cierto operador depende fuertemente del operador considerado. Ilustremos esto con algunos ejemplos:

- **Paraproductos diádicos y multiplicadores de Haar en dimensión 1.** Veremos en los Teoremas 1.5, 1.11 y 4.8 que estos operadores son de tipo débil $(1,1)$ para cualquier medida de Borel localmente finita.

- **La transformada de Hilbert diádica y su adjunto.** En el Teorema 1.5 demostraremos que cada uno de estos operadores tiene asociada una clase de medidas que dicta el tipo débil $(1,1)$ del operador. En el Capítulo 3 construiremos medidas que pertenecen a cada una de estas clases, y mostraremos que las clase asociada a la transformada de Hilbert diádica y aquella asociada a su adjunto son distintas y que ninguna contiene a la otra. Mostraremos también que la clase de medidas diádicamente doblantes está contenida estrictamente en la intersección de estas dos clases.

- **Adjuntos de paraproductos diádicos.** En el Teorema 4.8 mostraremos que el tipo débil $(1,1)$ de adjuntos de paraproductos diádicos implica que la medida subyacente es doblante.

- **Operadores Haar shift.** Demostraremos caracterizaciones análogas para operadores de Haar shift cancelativos en el Teorema 1.11 y para operadores Haar shift no cancelativos en el Teorema 4.3.

Así, nuestros resultados principales de la Parte I presentan la caracterización de las medidas para las cuales cualquiera de esos operadores es de tipo débil $(1,1)$. Cabe mencionar que las pruebas de estos resultados son relativamente simples al tener una descomposición de Calderón-Zygmund para medidas generales.

En el Teorema 1.1 proponemos una nueva descomposición de Calderón-Zygmund válida en contextos no doblantes y de interés independiente, con una parte buena modificada que continua siendo p-integrable para $p \gg 1$. Esta modificación es necesaria puesto que, en esta situación, la parte buena de la descomposición de Calderón-Zygmund usual pierde esta propiedad debido a que el promedio de f en un cubo diádico maximal no puede ser uniformemente controlado a menos que la medida ambiente sea doblante o diádicamente doblante. Esta parte buena modificada debe ser "balanceada" por una parte mala adicional que a su vez debe ser controlada. Seamos más precisos. Dada $f \in L^1(\mu)$ y $\lambda > 0$, consideremos la familia $\{Q_j\}_j$ de cubos diádicos maximales — con respecto a la propiedad de que el promedio de $|f|$ en Q con respecto a μ sea $\langle |f| \rangle_Q > \lambda$ — asociados al conjunto de nivel Ω_λ. Entonces podemos descomponer f como $f = g + b + \beta$ donde

$$
\begin{aligned}
g(x) &= f(x)\, 1_{\mathbb{R}\setminus\Omega_\lambda}(x) + \sum_j \langle f \rangle_{\widehat{Q}_j}\, 1_{Q_j}(x) \\
&\qquad\qquad + \sum_j \big(\langle f \rangle_{Q_j} - \langle f \rangle_{\widehat{Q}_j}\big)\, \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)}\, 1_{\widehat{Q}_j}(x), \\
b(x) &= \sum_j b_j(x) \; = \; \sum_j \big(f(x) - \langle f \rangle_{Q_j}\big)\, 1_{Q_j}(x), \\
\beta(x) &= \sum_j \beta_j(x) \; = \; \sum_j \big(\langle f \rangle_{Q_j} - \langle f \rangle_{\widehat{Q}_j}\big) \Big(1_{Q_j}(x) - \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)}\, 1_{\widehat{Q}_j}(x)\Big).
\end{aligned}
$$

Esta descomposición es tal que

- $g \in L^p(\mu)$ para todo $1 \le p < \infty$ con
$$\|g\|_{L^p(\mu)} \le C_p\, \lambda^{p-1}\, \|f\|_{L^1(\mu)};$$
- $b = \sum_j b_j$, con
$$\operatorname{supp}(b_j) \subset Q_j, \quad \int_{\mathbb{R}} b_j(x)\, d\mu(x) = 0, \quad \sum_j \|b_j\|_{L^1(\mu)} \le 2\, \|f\|_{L^1(\mu)};$$
- $\beta = \sum_j \beta_j$, con
$$\operatorname{supp}(\beta_j) \subset \widehat{Q}_j, \quad \int_{\mathbb{R}} \beta_j(x)\, d\mu(x) = 0, \quad \sum_j \|\beta_j\|_{L^1(\mu)} \le 4\, \|f\|_{L^1(\mu)},$$

  donde para cada j, denotamos por $\widehat{Q}_j$ al padre diádico de Q_j.

Comparemos esta descomposición con la descomposición de Calderón-Zygmund clásica. Primeramente, perdemos la cota en L^∞ para la parte buena. Sin embargo, esto no supone problema alguno, puesto que en la práctica típicamente se utiliza la estimación en L^2 de g. Respecto a los términos malos, el término b tiene la misma forma y propiedades que la parte mala clásica. Los términos constituyentes de la parte mala adicional β están soportados en los cubos diádicos $\{\widehat{Q}_j\}_j$, que no son disjuntos, pero sin embargo poseen cierta cancelación. Esta descomposición de Calderón-Zygmund es la clave para obtener las estimaciones débiles $(1,1)$ que consideramos en la Parte I.

Confiamos en que los resultados presentados en la primera parte de esta tesis son también válidos para otros retículos diádicos, en particular para filtraciones diádicas en espacios geométricamente doblantes construidas a partir de los cubos diádicos de Christ [11], o por medio de otras construcciones diádicas, como por ejemplo la construida por David en [19]. Una pregunta interesante es si estos resultados también son válidos en los contextos teóricos de medida recientemente estudiados por Treil [76]; Thiele, Treil y Volberg [74] y por Lacey [43].

Operadores de Calderón-Zygmund asociados a núcleos matriciales

Entendido en un sentido amplio, el análisis armónico semiconmutativo trata el estudio de integrales singulares que actúan sobre funciones que toman valores matriciales o en álgebras de operadores. Históricamente la teoría matricial ha formado parte de la teoría vectorial, que resulta ser un enfoque inadecuado para proporcionar estimaciones de tipo débil $(1,1)$ y en general para estimaciones extremales — de tipo Hardy/BMO. Esto se debe principalmente a que la teoría vectorial apenas considera la estructura algebraica de las funciones con valores matriciales. La perspectiva adecuada para tratar estos problemas es ofrecida por el análisis no conmutativo, un área motivada por von Neumann al unificar las formulaciones de Heisenberg y de Schrödinger de la mecánica cuántica. El quid de esta teoría consiste en sustituir funciones por operadores en un espacio de Hilbert; lo que en física se conoce como *cuantización*. El considerar operadores en lugar de funciones conlleva un producto no conmutativo dado por la composición de operadores. Para nuestro objetivo particular, la cuantización de la teoría de integración y de la teoría L^p conduce a reemplazar espacios L^∞ por álgebras de von Neumann, que son C^*-álgebras de operadores en un espacio de Hilbert que contienen la identidad y son cerradas en la topología débil-$*$. Así, trazas juegan el papel de integrales y proyecciones ortogonales el de funciones características. Asociados a un álgebra de von Neumann $\mathcal{M}$ con traza τ, los espacios $L_p(\mathcal{M})$ no conmutativos — en la teoría L_p no conmutativa el parámetro de escala se suele indicar como subíndice — son los espacios de operadores para los cuales la norma

$$\|x\|_{L_p(\mathcal{M})} = \tau(|x|^p)^{1/p}$$

es finita. Aquí $|x| = (x^*x)^{1/2}$ es el módulo de x y $|x|^p$ es definido por cálculo funcional para operadores positivos. Remitimos a [67, 53] y a algunas de sus referencias para una exposición más detallada y precisa de la teoría de integración no conmutativa.

Por simplicidad, consideremos el álgebra de funciones

$$\mathcal{A}_B = \Big\{ f : \mathbb{R} \to \mathcal{B}(\ell^2) : f \text{ es fuertemente medible y } \operatorname*{ess\,sup}_{x\in\mathbb{R}} \|f(x)\|_{\mathcal{B}(\ell^2)} < \infty \Big\},$$

donde $\mathcal{B}(\ell^2)$ es el álgebra de operadores acotados en el espacio de Hilbert ℓ^2. El cierre débil-$*$ $\mathcal{A}_B$ de $\mathcal{A}$ es un álgebra de von Neumann isomorfa a $L_\infty(\mathbb{R})\overline{\otimes}\mathcal{B}(\ell^2)$, que por lo tanto es equipada con la traza $\tau(f) = \int_{\mathbb{R}} \mathrm{Tr}(f(x))\,dx$, donde Tr denota la traza estándar en $\mathcal{B}(\ell^2)$ y el espacio L_p no conmutativo asociado $L_p(\mathcal{B}(\ell^2))$ es precisamente la p-clase de Schatten. Denotemos $\mathcal{B}(\ell^2)$ por $\mathcal{M}$. Por lo mencionado en la discusión inicial, el espacio $L_p(\mathcal{A})$ es el cierre del espacio generado por funciones simples apropiadamente definidas. Para tales funciones tenemos que

$$\tau(|f|^p) = \int_{\mathbb{R}} \mathrm{Tr}(|f(x)|^p)dx = \int_{\mathbb{R}} \|f(x)\|_{L_p(\mathcal{M})}^p\,dx.$$

De esto deducimos que para $1 \leq p < \infty$, el espacio $L_p(\mathcal{A})$ es isométricamente isomorfo al espacio de Bochner clásico $L^p(\mathbb{R}; L_p(\mathcal{M}))$. Por supuesto, los resultados que obtenemos son también ciertos para funciones que toman valores en álgebras de von Neumann arbitrarias, siempre que tengan una traza *normal, semifinita y fiel*.

Cabe preguntarse si operadores de Haar shift que actúen en funciones con valores matriciales admiten estimaciones *a priori* en $L_p(\mathcal{A})$. En otras palabras si

$$Ш_{j,k}f(x) = \sum_{I\in\mathscr{D}} A_I f = \sum_{I\in\mathscr{D}} \sum_{\substack{J\in\mathscr{D}_j(I)\\ K\in\mathscr{D}_k(I)}} \alpha^I_{J,K}\langle f, h_J\rangle h_K(x)$$

es un operador acotado en $L_p(\mathcal{A})$. En la presente situación $\langle f, h_J\rangle$ denota la forma $\int_{\mathbb{R}} f(x)h_I(x)\,dx$, que tiene valores matriciales y los símbolos $\alpha^I_{J,K}$ son escalares uniformemente acotados. Puesto que las p-clases de Schatten son espacios de Banach con la propiedad UMD para $1 < p < \infty$, la acotación en L_p de operadores Haar shift, como la de operadores de Calderón-Zygmund , es resuelta por la teoría vectorial clásica desarrollada por Burkholder en [8, 9], Bourgain [6] y Figiel [24]. Puesto que la clase traza (i.e., la clase 1 de Schatten) no es UMD, la teoría vectorial es insuficiente para proporcionar estimaciones de tipo débil $(1, 1)$ adecuadas. Para atajar dicho problema la estructura no conmutativa es esencial. El espacio $L_1(\mathcal{A})$ débil es definido por medio de la cuasi-norma

$$\|f\|_{L_{1,\infty}(\mathcal{A})} = \sup_{\lambda>0} \lambda\tau(\{|f| > \lambda\}),$$

donde $\tau(\{|f| > \lambda\})$ denota la traza de la proyección espectral de $|f|$ asociada al intervalo (λ, ∞). Con esto se define una función de distribución no conmutativa que comparte las propiedades de su contrapunto conmutativo, siendo esta la razón por la que hemos elegido esta notación. El espacio $L_{1,\infty}(\mathcal{A})$ así construido satisface las propiedades de interpolación esperadas. Es necesario enfatizar que el espacio de Boncher débil $L^{1,\infty}(\mathbb{R}; L_1(\mathcal{M}))$ no es de utilidad para nuestros propósitos, pues $L_1(\mathcal{M})$ no es UMD, razón por la cual incluso los multiplicadores de Haar pueden ser no acotados. El mismo razonamiento descarta utilizar el espacio $L^{1,\infty}(\mathbb{R}; \mathcal{M})$.

En [58] Parcet demostró el tipo débil $(1, 1)$ apropiado para operadores de Calderón-Zygmund que actúan en funciones con valores matriciales y de núcleos escalares, es decir, que para tal operador T se tiene que

$$\lambda\tau(\{|Tf| > \lambda\}) \lesssim \|f\|_{L_1(\mathcal{A})}$$

uniformemente en $\lambda > 0$. Su demostración se basa en la aplicación de una versión no conmutativa de la descomposición de Calderón-Zygmund clásica, que introdujo él mismo adaptando la ingeniosa construcción que Cuculescu [18] utilizó para probar la extensión no conmutativo del tipo débil $(1,1)$ del maximal de Doob; un resultado de gran importancia en la teoría de martingalas no conmutativas. La construcción de Cuculescu permite obtener una proyección q y una familia de proyecciones $(p_k)_{k\in\mathbb{Z}}$ disjuntas a pares que corresponden al conjunto de nivel clásico Ω_λ y a su descomposición en cubos maximales:

$$q \sim \mathbb{R} \setminus \Omega_\lambda \quad \text{and} \quad p_k \sim \big\{ Q_j \, \text{es un cubo maximal de } \, \Omega_\lambda \, y \, Q_j \in \mathscr{D}_k \big\},$$

donde $\mathscr{D}_k$ denota la familia de cubos diádicos de longitud $\ell(Q) = 2^{-k}$. Además, estas proyecciones cumplen que $\sum_k p_k = 1_\mathcal{A} - q$, donde $1_\mathcal{A}$ es la unidad de $\mathcal{A}$. La descomposición de Calderón-Zygmund no conmutativa es obtenida emulando la descomposición clásica acorde a la interpretación de estas proyecciones. Esto es que $f \in L_1(\mathcal{A})$ descompone como $f = g + b$, donde las partes buena y mala son dadas por:

$$g = \sum_{i,j\in\mathbb{Z}} p_i f_{i\vee j} p_j \quad \text{y} \quad b = \sum_{i,j\in\mathbb{Z}} p_i (f - f_{i\vee j}) p_j,$$

donde $i \vee j = \max(i,j)$ y f_k denota la esperanza condicionada

$$f_k = \sum_{Q\in\mathscr{D}_k} \langle f \rangle_Q 1_Q.$$

Aquí $\langle f \rangle_Q$ denota la media de f en Q y es por tanto un operador. La no conmutatividad en este contexto es explícita en esta descomposición dada la presencia de términos fuera de la diagonal, i.e., aquellos tales que $i \neq j$. En efecto, en el caso conmutativo los términos fuera de la diagonal desaparecen, puesto que las proyecciones p_k son disjuntas a pares. Los términos diagonales satisfacen las mismas propiedades que las partes buena y mala de la descomposición clásica:

- $\|g_\Delta\|_{L_1(\mathcal{A})} \leq \|f\|_{L_1(\mathcal{A})}$ y $\|g_\Delta\|_{L_\infty(\mathcal{A})} \leq 2\lambda$.
- $b_\Delta = \sum_i b_{\Delta,i}$, donde los términos $b_{\Delta,i}$ tienen media cero y satisfacen la estimación $\sum_i \|b_{\Delta,i}\|_{L_1(\mathcal{A})} \leq 2\|f\|_{L_1(\mathcal{A})}$.

Estas propiedades son utilizadas para obtener estimaciones de tipo débil para los términos diagonales, procediendo tal como en el caso conmutativo. Por otro lado, existen indicios de que los términos fuera de la diagonal no satisfacen las estimaciones clásicas, contando únicamente con estimaciones truncadas. Esto sin embargo supone una dificultad sorteable en la práctica, puesto que estimaciones de tipo débil para los términos fuera de la diagonal son obtenidas mediante principios de sesudo-localización en el caso de operadores de Calderón-Zygmund y por las buenas propiedades de localización de los operadores diádicos. Antes de los resultados obtenidos en [58], las únicas estimaciones de tipo débil $(1,1)$ conocidas en contextos no conmutativos eran las asociadas a transformadas de martingala y funciones cuadrado de martingala obtenidas por Parcet y Randrianantoanina en [59] y por Randrianantoanina en [70, 71, 72], además de las obtenidas en [69] para la transformada de Hilbert no conmutativa asociada a álgebras sub-diagonales maximales y desigualdades de tipo débil para el maximal ergódico, demostradas por Junge y Xu [39].

Los mismos métodos utilizados en [58] proporcionan estimaciones para operadores Haar shift $Ш_{j,k}$, esta vez con constantes que dependen polinomialmente de la complejidad (j,k). Cabría preguntarse si operadores Haar shift con símbolos matriciales $\alpha^I_{J,K} \in \mathcal{M}$ satisfacen estimaciones análogas. Entre estos objetos, los paraproductos con valores matriciales han atraído particular atención. Para empezar, la no conmutatividad nos fuerza a considerar distintos operadores dependiendo si, por ejemplo, los símbolos multiplican por la izquierda o por la derecha a las formas $\langle f, h_I\rangle \in \mathcal{M}$. Consideremos símbolos $\alpha_I \in \mathcal{M}$ uniformemente acotados. Asociados a estos, un par fila/columna de multiplicadores de Haar pueden ser definidos mediante

$$T^{\mathrm{r}}_\alpha(f) = \sum_{Q\in\mathscr{D}} \langle f, h_I\rangle \alpha_I h_I, \qquad T^{\mathrm{c}}_\alpha(f) = \sum_{I\in\mathscr{D}} \alpha_I \langle f, h_I\rangle h_I.$$

Como veremos inmediatamente, es posible construir multiplicadores Haar fila/columna que no sean de tipo débil $(1,1)$ ni de tipo fuerte (p,p) para $p \neq 2$, demostrando que en este tipo de cuestiones la naturaleza no conmutativa es predominante. En el caso de paraproductos con símbolos matriciales, tal y como demostraron Katz en [41]; Nazarov, Pisier, Treil y Volberg en [56] y Mei [50], la acotación en L_2 es violada incluso por paraproductos con símbolos razonablemente elegidos. El contraejemplo que hemos mencionado para multiplicadores de Haar es bien conocido en la teoría de martingalas no conmutativas y es el siguiente. Sea $\mathcal{A} = L^\infty([0,1))\overline{\otimes}\mathcal{M}$, y consideremos el multiplicador columna asociado a los símbolos $\alpha_I = e_{k,1}$ para $I \in \mathscr{D}_{k-1}$ — recordemos que $e_{i,j}$ denota la matriz cuya única entrada no nula es la (i,j). Entonces, si

$$f_n = \sum_{k=1}^n \Bigg(\sum_{I\in\mathscr{D}_{k-1}} |I|^{1/2} h_I \Bigg) e_{1,k},$$

encontramos que

$$\|f_n\|_{L_1(\mathcal{A})} = \sqrt{n} \quad \text{y} \quad \|T_{\mathrm{c}} f_n\|_{L_{1,\infty}(\mathcal{A})} = n \gg \|f_n\|_{L_1(\mathcal{A})}$$

para n lo suficientemente grande. Este problema motiva la cuestión principal que abordamos en la Parte II:

> *¿Existen subespacios o subconjuntos $A_{\mathrm{r}}/A_{\mathrm{c}}$ de $L_1(\mathcal{A})$ tales que $f = f_{\mathrm{r}} + f_{\mathrm{c}}$, donde $f_{\mathrm{r}} \in A_{\mathrm{r}}$ y $f_{\mathrm{c}} \in A_{\mathrm{c}}$ y que*
>
> $$T_{\mathrm{r}} : A_{\mathrm{r}} \to L_{1,\infty}(\mathcal{A}) \quad y \quad T_{\mathrm{c}} : A_{\mathrm{c}} \to L_{1,\infty}(\mathcal{A})?$$

A pesar de que no hemos sido capaces de responder esta pregunta, hemos encontrado resultados interesantes en esta dirección.

- **Operadores Haar shift con símbolos matriciales/no conmutativos.** En el Teorema 5.1 (i) veremos que para $f \in L_1(\mathcal{A})$ existe una descomposición explícita $f = f_{\mathrm{r}} + f_{\mathrm{c}}$ tal que

$$\|\text{Ш}^{\mathrm{r}}_{j,k} f_{\mathrm{r}}\|_{L_{1,\infty}(\mathcal{A})} + \|\text{Ш}^{\mathrm{c}}_{j,k} f_{\mathrm{c}}\|_{L_{1,\infty}(\mathcal{A})} \leq C_{j,k} \|f\|_{L_1(\mathcal{A})}.$$

  para operadores Haar shift fila/columna, incluyendo multiplicadores de Haar, paraproductos y adjuntos de paraproductos.

- **Transformadas de martingala y paraproductos de martingala.** Adaptamos las técnicas utilizadas en la resolución del punto anterior al contexto de martingalas no conmutativas, obteniendo resultados análogos para transformadas de martingala y paraproductos de martingala en el Teorema 5.3 (i), siempre que sean definidos con respecto a una filtración regular.

La descomposición *fila/columna* $f = f_{\mathrm{r}} + f_{\mathrm{c}}$ obtenida en el Teorema 5.1 (i) viene dada por truncaciones triangulares complementarias en términos de proyecciones apropiadamente elegidas. Estas proyecciones son obtenidas modificando la construcción de Cuculescu con el fin de que las proyecciones asociadas a distintos conjuntos de nivel sean comparables, propiedad trivialmente satisfecha por los conjuntos de nivel clásicos. Las constantes que obtenemos en las estimaciones del Teorema 5.1 (i) son de orden exponencial, $C_{j,k} \sim 2^j$. Los argumentos clásicos utilizados para encontrar constantes de dependencia óptima (sea lineal o polinomial) es obstruido por la presencia de truncaciones triangulares, que no son acotadas en L_1 como; un resultado clásico de Kwapień y Pełczyński [42]. Esta misma razón previene extender los argumentos utilizados a operadores de Calderón-Zygmund genéricos, dejándolo como problema abierto.

Por otro lado, técnicas complementarias nos permiten obtener estimaciones para operadores de Calderón-Zygmund genéricos con núcleos no conmutativos, es decir, que para un par de operadores fila/columna formalmente dados por

$$T_{\mathrm{r}} f(x) \sim \int_{\mathbb{R}} f(y) k(x,y)\, dy \quad \text{and} \quad T_{\mathrm{c}} f(x) \sim \int_{\mathbb{R}} k(x,y) f(y)\, dy,$$

asociadas a un núcleo $k(x,y) \in \mathcal{M}$, para $x \neq y$, que satisface las las condiciones clásicas de tamaño y suavidad. Estas estimaciones son obtenidas utilizando la teoría de espacios de Hardy fila/columna desarrollada por Mei [51], su versión en el marco de la teoría de martingalas desarrollada por Pisier y Xu en [66] y la teoría de interpolación y dualidad asociada [32, 38, 55].

- **Estimaciones $H_1 \to L_1$.** En el Teorema 5.1 (ii) obtenemos que un par fila/columna $(T_{\mathrm{r}}, T_{\mathrm{c}})$ mapea continuamente el espacio de Hardy fila/columna en $L_1(\mathcal{A})$. Argumentos de interpolación y dualidad proporcionan estimaciones en L_p obtenidas en el Teorema 5.2.

- **Transformadas de martingala y paraproductos de martingala.** Estimaciones en H_1 y L_p para transformadas de martingala y paraproductos con símbolos no conmutativos asociados a filtraciones arbitrarias son obtenidas en el Teorema 5.3 (ii).

Análisis armónico diádico semiconmutativo no doblante

Es natural preguntarse si los resultados expuestos en la Parte I son también válidos en el contexto semiconmutativo de la Parte II. Es decir:

> *¿Es posible determinar la clase de medidas para las cuales un operador diádico que actúa sobre funciones con valores matriciales es de tipo débil* $(1,1)$*?*

Tratamos esta cuestión en la Parte III de esta tesis. Para abordarla introducimos una descomposición de Calderón-Zygmund no conmutativa que es versión cuantizada de la descomposición de Calderón-Zygmund del Teorema 1.1. Tal y como en el caso de la descomposición de Calderón-Zygmund obtenida en [58] — que es válida para la medida de Lebesgue y para medidas doblantes — la descomposición que presentamos se fundamenta en una adaptación de la construcción de Cuculescu a esta situación. En el Teorema 9.2 obtenemos la descomposición $f = g + b + \beta$, donde cada parte tiene un término diagonal y un término fuera de la diagonal dados por

- $g = g_\Delta + g_{\mathrm{off}}$, donde

$$g_\Delta = qfq + \sum_{k\in\mathbb{Z}} \mathsf{E}_{k-1}\left(p_k f_k p_k\right),$$
$$g_{\mathrm{off}} = (1_{\mathcal{A}} - q)fq + qf(1_{\mathcal{A}} - q) + \sum_{i\neq j} \mathsf{E}_{i\vee j-1}\left(p_i f_{i\vee j} p_j\right);$$

- $b = b_\Delta + b_{\mathrm{off}}$, donde

$$b_\Delta = \sum_{k\in\mathbb{Z}} p_k(f - f_k)p_k\,, \qquad b_{\mathrm{off}} = \sum_{i\neq j} p_i(f - f_{i\vee j})p_j;$$

- $\beta = \beta_\Delta + \beta_{\mathrm{off}}$, donde

$$\beta_\Delta = \sum_{k\in\mathbb{Z}} \mathsf{D}_k(p_k f_k p_k), \qquad \beta_{\mathrm{off}} = \sum_{i\neq j} \mathsf{D}_{i\vee j}\left(p_i f_{i\vee j} p_j\right).$$

Como en la descomposición [58], los términos diagonales satisfacen las siguientes propiedades clásicas:

- $\|g_\Delta\|_{L_1(\mathcal{A})} \leq \|f\|_{L_1(\mathcal{A})}$ y $\|g_\Delta\|_{L_2(\mathcal{A})} \leq C\lambda\|f\|_{L_1(\mathcal{A})}$.
- $b_\Delta = \sum_i b_{\Delta,i}$, donde $b_{\Delta,i}$ es de media cero y $\sum_i \|b_{\Delta,i}\|_{L_1(\mathcal{A})} \leq 2\|f\|_{L_1(\mathcal{A})}$.
- $\beta_\Delta = \sum_i \beta_{\Delta,i}$, con $\beta_{\Delta,i}$ es de media cero y $\sum_i \|\beta_{\Delta,i}\|_{L_1(\mathcal{A})} \leq 2\|f\|_{L_1(\mathcal{A})}$.

En la práctica, los términos fuera de la diagonal son controlados por principios de localización. Esta descomposición de Calderón-Zygmund es nuestra herramienta principal para resolver la cuestión inicial de la tercera parte de esta tesis:

- **Operadores Haar shift con símbolos conmutativos.** El tipo débil $(1,1)$ de estos operadores es caracterizado de la misma manera que el caso conmutativo, tal y como demostramos en el Teorema 9.4.

Dificultades considerables son añadidas al problema de obtener estimaciones débiles *a priori* cuando se consideran operadores con símbolos no conmutativos. Los métodos utilizados en el Teorema 5.1 (i) no son aplicables en esta situación ni siquiera para multiplicadores de Haar. Discutiremos estas dificultades al concluir la Parte III, dejándolo como problema abierto.

Declaración

Los resultados presentados en esta tesis están basados en investigación original llevada a cabo con otros investigadores durante mis estudios de doctorado, realizado entre marzo del 2011 y febrero del 2015. Dichos artículos de investigación corresponden a las referencias [14, 27, 46], que alistamos a continuación.

[14] J.M. Conde-Alonso and L.D. López-Sánchez. Operator-valued dyadic harmonic analysis beyond doubling measures. Preprint. arXiv:1412.4937, 2014. Enviado para su publicación.

[27] G. Hong, L.D. López-Sánchez, J.M. Martell, and J. Parcet. Calderón-Zygmund operators associated to matrix-valued kernels. *Int. Math. Res. Not. IMRN*, 2014(5):1221–1252, 2014.

[46] L.D. López-Sánchez, J.M. Martell, and J. Parcet. Dyadic harmonic analysis beyond doubling measures. *Adv. Math.*, 267(0):44 – 93, 2014.

La organización de esta tesis es como sigue: el contenido de Parte I está basado en los resultados obtenidos en [46], la Parte II se basa en la referencia [27] y la Parte III en [14].

Preface

Dyadic techniques are nowadays fundamental in harmonic analysis. Their origin dates back to Hardy, Littlewood, Paley and Walsh among others. In the context of martingale inequalities, the dyadic maximal and square functions arise as particular cases of Doob's maximal function and Burkholder's square function for martingales associated to a dyadic filtration; furnishing relatively simple models of the Hardy-Littlewood maximal function and of square functions. Similarly, singular integral operators — with the Hilbert transform standing as a prominent example — have been traditionally modeled by martingale transforms and martingale paraproducts. These last operators can be written in terms of martingale differences and conditional expectations, so that the full strength of probability methods applies in the analysis of their boundedness properties. In the Euclidean setting — and more generally in the atomic martingale setting — dyadic martingale differences decompose as a sum of rank one perfectly localized projections, to wit, Haar projections. Therefore, in the Euclidean setting martingale transforms are in fact diagonal operators relative to the classical Haar system. In this spirit and somewhat roughly, one may consider compact operators having a structured sparse matrix representation relative to the Haar system. These operators are known as *Haar shift operators* and provide a slightly more complex and yet fruitful model of singular integral operators.

In this thesis we will be chiefly interested in dyadic and Haar analogues of the classical objects in harmonic analysis and study their boundedness properties in several settings. In the following section we will recall some basic background from the classical theory. Right afterwards we will discuss the results obtained in this thesis.

Classical dyadic harmonic analysis

In recent years dyadic operators have attracted a lot of attention related to the so-called A_2-conjecture. This seeks to establish that some operators obey an $L^2(w)$ estimate for every $w \in A_2$ with a constant that grows linearly in the A_2-characteristic of w. Extrapolation techniques can then be used to obtain the optimal dependence on the A_p-characteristic for the corresponding $L^p(w)$ estimate. This problem attracted increased attention after its introduction by Buckley in [7] due to the work of Astala, Iwaniec and Saksman [1]. There they showed that if sharp weighted estimates were satisfied by the Beurling-Ahlfors operator, then one could get regularity results for solutions of the Beltrami equation.

For the maximal function the A_2 conjecture was proved by Buckley [7]. In [78] Wittwer proved the A_2-conjecture for Haar multipliers in one dimension. The Beurling-Ahlfors transform, the Hilbert transform and the Riesz transforms were then considered by Petermichl and Volberg in [65, 63, 64] (see also [22]) thus giving a positive answer to the question posed in [1]. The A_2-conjecture for these operators was shown by representing them as averages of certain dyadic operators called Haar shifts. Paraproducts were treated in [5], and with a different approach in [17]. The final solution to the A_2-conjecture for general Calderón-Zygmund operators was obtained by Hytönen in his celebrated paper [29]. A key ingredient in the proof of those results is that Calderón-Zygmund operators can be expanded as a rapidly convergent series of Haar shift operators and dyadic paraproducts. This result in its full generality is known as *Hytönen's representation theorem.* It is related to the approach to the $T(1)$ theorem as treated by Figiel [24] and by Beylkin, Coifman and Rokhlin in [4], and is based on a decomposition provided by Nazarov, Treil and Volberg in [57] to tackle the $T(1)$ theorem in non-homogeneous spaces. It differs however from the approaches in [24, 4] in that the associated expansions are rapidly convergent for smooth and non-smooth operators, yielding another proof of the $T(1)$ theorem. This property makes this representation an outstanding result by itself.

Prior to the groundbreaking work of Petermichl in [62], *martingale transforms/Haar multipliers* and *martingale/dyadic paraproducts* were the only available dyadic models for singular integral operators. In the one dimensional setting these operators are of the form

$$T_\alpha f(x) = \sum_{I\in\mathscr{D}} \alpha_I \langle f, h_I\rangle h_I(x) \quad \text{and} \quad \Pi_\rho f(x) = \sum_{I\in\mathscr{D}} \langle f\rangle_I \langle \rho, h_I\rangle h_I(x).$$

Here $\mathscr{D}$ denotes some dyadic grid in $\mathbb{R}$, α_I are uniformly bounded scalars, $\rho \in \mathrm{BMO}_{\mathscr{D}}$, $\langle f, h_I\rangle$ denotes the pairing $\int_{\mathbb{R}} f(x)h_I(x)\,dx$, $\langle f\rangle_I$ is the average of f over I and h_I is the Haar function associated with $I \in \mathscr{D}$:

$$h_I = \frac{1}{|I|^{1/2}}\big(1_{I_-} - 1_{I_+}\big),$$

where I_- and I_+ are the left and right dyadic children of I. Obviously, the Haar system is an orthonormal system on $L^2(\mathbb{R})$. In [62] Petermichl introduced the *dyadic Hilbert transform* given by

$$H_{\mathscr{D}} f(x) = \sum_{I\in\mathscr{D}} \langle f, h_I\rangle \big(h_{I_-}(x) - h_{I_+}(x)\big).$$

The importance — and the name — of this operator comes from the fact that the classical Hilbert transform can be obtained via averaging $H_{\mathscr{D}}$ over randomized dyadic grids. This result settled the central rôle dyadic models play in providing sharp estimates for singular integral operators, by allowing to transfer the rather simple proofs in the dyadic setting to the continuous setting. A larger class of operators of which Haar multipliers and the dyadic Hilbert transform are particular instances was introduced by Lacey, Petermichl and Reguera in [44]. A *Haar shift operator* of complexity $(j,k) \in \mathbb{Z}_+^2$ has the form

$$Ш_{j,k} f(x) = \sum_{I \in \mathscr{D}} A_I f = \sum_{I \in \mathscr{D}} \sum_{\substack{J \in \mathscr{D}_j(I) \\ K \in \mathscr{D}_k(I)}} \alpha^I_{J,K} \langle f, h_J \rangle h_K(x),$$

where $\mathscr{D}_j(I)$ denotes the family of j-dyadic descendants of I, i.e., the partition of I into subintervals $J \in \mathscr{D}$ of length $\ell(J) = 2^{-j}\ell(J)$. Haar multipliers and the dyadic Hilbert transform arise as Haar shift operators of complexity $(0,0)$ and $(0,1)$. The symbols $\alpha^I_{J,K}$ are usually subject to the normalization

$$|\alpha^I_{J,K}| \leq \frac{\sqrt{|J||K|}}{|I|},$$

which ensures that the Haar shift operator $Ш_{j,k}$ is contractive in L^2 and that the components A_I are contractive in L^p for $1 \leq p \leq \infty$. If in the definition of Haar shift operators one allows L^2-normalized indicator functions $|I|^{-1/2} 1_I$ to stand alongside Haar functions as building blocks, one then obtains *non-cancellative Haar shift operators*; an even larger class that includes dyadic paraproducts and their adjoints,

$$\Pi^*_\rho f(x) = \sum_{I \in \mathscr{D}} \langle f, h_I \rangle \langle \rho, h_I \rangle \frac{1_I(x)}{|I|},$$

as instances of complexity $(0,0)$. However, orthogonality is lost and thus L^2 boundedness becomes non-trivial, relying on Carleson embedding-type theorems. Hence, for this extended class of dyadic operators L^2 boundedness is generally assumed.

On the other hand, non-cancellative Haar shift operators also include the class of the so-called *positive sparse operators* introduced by Lerner in [45] to provide an alternative and more elementary proof of the A_2 conjecture. His proof rests on the remarkable fact that the operator norm of Calderón-Zygmund operators in a Banach lattice is dominated by the norm of a positive sparse operator. Furthermore, this control by positive sparse operators can be proved to hold pointwise [15, 43]. Positive sparse operators are defined by

$$Sf(x) = \sum_{I \in \mathcal{S}} \langle f \rangle_{I^{(j)}} 1_I(x)$$

where $I^{(j)}$ is the j-dyadic ancestor of I and $\mathcal{S} \subset \mathscr{D}$ is a sparse family of dyadic cubes in the sense that for all $I \in \mathcal{S}$

$$\sum_{J \in \mathscr{D}_1(I) \cap \mathcal{S}} |J| \leq \frac{1}{2}|I|.$$

It is then easy to see that positive sparse operators are non-cancellative Haar shift operators of complexity $(j,0)$ with coefficients

$$\alpha^I_J = \begin{cases} \dfrac{|J|^{1/2}}{|I|^{1/2}} & \text{if} \quad I \in \mathcal{S} \\ 0 & \text{otherwise.} \end{cases}$$

Essential to either, the approximation or domination approach, is that Haar shift operators are of weak-type $(1,1)$. Namely, that

$$\lambda|\{x \in \mathbb{R} : |Ш_{j,k}f(x)| > \lambda\}| \leq C\|f\|_{L^1(\mathbb{R})}$$

with C depending only on the complexity (j,k), ideally in a linear or even polynomial way. Of course, as a by-product one obtains L^p boundedness for $1 < p < \infty$ by standard interpolation and duality arguments. Weak-type $(1,1)$ estimates can be obtained by using the standard Calderón-Zygmund decomposition (see for instance [17, 29, 44]). Let us overview this procedure. Given $f \in L^1(\mathbb{R})$ and $\lambda > 0$, consider the level set

$$\Omega_\lambda = \Big\{x \in \mathbb{R} : M_{\mathscr{D}}f(x) > \lambda\Big\} = \bigcup_i Q_i.$$

Here $M_{\mathscr{D}}$ is the dyadic Hardy-Littlewood maximal function $M_{\mathscr{D}}f = \sup_{I\in\mathscr{D}}\langle|f|\rangle_I$ and $\{Q_i\}_i$ is the associated disjoint collection of maximal dyadic intervals. Then f decomposes as $f = g + b$, where the good and bad parts are given by

$$g = f1_{\mathbb{R}\setminus\Omega_\lambda} + \sum_i \langle f\rangle_{Q_i} 1_{Q_i} \quad \text{and} \quad b = \sum_i (f - \langle f\rangle_{Q_i})1_{Q_i}.$$

Letting $b_i = (f - \langle f\rangle_{Q_i})1_{Q_i}$, we have

- $\|g\|_{L^1(\mathbb{R})} \leq \|f\|_{L^1(\mathbb{R})}$ and $\|g\|_{L^\infty(\mathbb{R})} \leq 2\lambda$.
- $\operatorname{supp}(b_i) \subset Q_i$, $\int_{Q_i} b_i(x)\,dx = 0$ and $\sum_i \|b_i\|_{L^1(\mathbb{R})} \leq 2\|f\|_{L^1(\mathbb{R})}$.

The properties of this (non-linear) decomposition are crucial for the analysis of classical operators, as it is the case for Haar shift operators. Indeed, the estimates satisfied by the good part deliver constants of the same order of the L^2 operator norm of $Ш_{j,k}$. One then exploits (as done in [29]) the localization and mean zero properties of the bad terms b_i. The dyadic structure of $Ш_{j,k}$ and of its localized components A_I permits to get constants depending linearly on the complexity (j,k), and on the operator norm of A_I on $L^1(\mathbb{R})$. In fact, as shown in [30], linear dependence on j is actually sharp.

It is the *leitmotif* of this thesis to contribute to this line of research by yielding analogous weak-type $(1,1)$ estimates for Haar shift operators in the generalized settings of semicommutative harmonic analysis and non-doubling harmonic analysis. If not of optimal dependence on the complexity of the operator, the weak-type $(1,1)$ estimates we obtain point out — and even characterize — the basic structure the ambient spaces should have in order for *a priori* weak-type $(1,1)$ estimates to hold.

Dyadic harmonic analysis beyond doubling measures

Let us consider a Borel measure μ in $\mathbb{R}$. One can define a Haar system in a similar manner by

$$h_I^\mu = \sqrt{m(I)}\left(\frac{1_{I_-}}{\mu(I_-)} - \frac{1_{I_+}}{\mu(I_+)}\right), \qquad \text{with} \quad m(I) = \frac{\mu(I_-)\mu(I_+)}{\mu(I)};$$

which is now orthonormal in $L^2(\mu)$. Hence, we may consider a dyadic Hilbert transform relative to the Haar system $\{h_I^\mu\}_{I\in\mathscr{D}}$, viz.

$$H_{\mathscr{D}}^\mu f(x) = \sum_{I\in\mathscr{D}} \langle f, h_I^\mu \rangle \big(h_{I_-}^\mu(x) - h_{I_+}^\mu(x) \big)$$

and ask about its boundedness properties. The boundedness on $L^2(\mu)$ is again automatic by orthogonality. The standard Calderón-Zygmund theory can be easily extended to settings where the underlying measure is doubling. In the present situation, since the operator is dyadic, one could even relax that condition and assume that μ is dyadically doubling. In such a case, we can almost copy verbatim the standard proof and conclude the weak-type $(1,1)$ (with respect to μ) and therefore obtain the same bounds as before. Suppose next that the measure μ is not dyadically doubling, and we would like to find the class of measures μ for which $H_{\mathscr{D}}^\mu$ maps continuously $L^1(\mu)$ into $L^{1,\infty}(\mu)$.

Characterizing the class of measures for which a given operator is bounded is in general a hard problem. For instance, that is the case for the L^2 boundedness of the Cauchy integral operator in the plane and the class of linear growth measures obtained by Tolsa [75]. This led to non-standard Calderón-Zygmund theories (where μ has some polynomial growth *á la* Nazarov-Treil-Volberg and Tolsa) that one could try to apply in the present situation. This would probably require some extra (and *a posteriori* unnecessary) assumptions on μ. On the other hand, let us recall that $H_{\mathscr{D}}^\mu$ is a dyadic operator. Sometimes dyadic operators behave well even without assuming doubling: the dyadic Hardy-Littlewood maximal function and the dyadic square function are of weak-type $(1,1)$ for general Borel measures μ, see respectively [20] and [10]. In view of that, one could be tempted to conjecture that $H_{\mathscr{D}}^\mu$ is of weak-type $(1,1)$ for general measures μ without assuming any further doubling property (or polynomial growth). One could also ask the same questions for some other dyadic operators: the adjoint of the dyadic Hilbert transform, (cancellative) Haar shift operators, dyadic paraproducts or their adjoints or, more in general, non-cancellative Haar shift operators. This motivates one of the main questions we address in Part I:

> *Determine the family of measures μ for which a given dyadic operator (e.g., the dyadic Hilbert transform or its adjoint, a dyadic paraproduct or its adjoint, a cancellative or non-cancellative Haar shift operator) maps continuously $L^1(\mu)$ into $L^{1,\infty}(\mu)$.*

We know already that if μ is dyadically doubling these operators satisfy weak-type estimates by a straightforward use of the standard Calderón-Zygmund theory. Therefore, it is natural to wonder whether the doubling condition is necessary or it is just convenient. As we will see in Part I there is no universal answer to that question for all the previous operators: the class of measures depends heavily on the operator in question. Let us illustrate this phenomenon with some examples:

- **Dyadic paraproducts and 1-dimensional Haar multipliers.** We shall see in Theorems 1.5, 1.11 and 4.8 that these operators are of weak-type $(1,1)$ for every locally finite Borel measure.

- **The dyadic Hilbert transform and its adjoint.** We shall prove in Theorem 1.5 that each operator gives rise to a family of measures governing the corresponding weak-type $(1,1)$. In Chapter 3 we shall provide some examples of measures, showing that the two classes (the one for the dyadic Hilbert transform and the one for its adjoint) are different and none of them is contained in the other. Further, the class of dyadically doubling measures is strictly contained in the intersection of the two classes.

- **Adjoints of dyadic paraproducts.** We shall obtain in Theorem 4.8 that the weak-type $(1,1)$ of these operators leads naturally to the dyadically doubling condition for μ.

- **Haar shift operators.** Analogous characterizations for cancellative Haar shift operators are obtained in Theorem 1.11 and in Theorem 4.3 for non-cancellative Haar shift operators.

Beside these examples, our main results will answer the question above providing a characterization of the measures for which any of the previous operators is of weak-type $(1,1)$. It should be pointed out that the proofs of such results are relatively simple, once we have obtained the appropriate Calderón-Zygmund decomposition valid for general measures. In Theorem 1.1 we propose a new Calderón-Zygmund decomposition, interesting on its own right, with a new good part which will be still higher integrable. We need to do this, since the usual "good part" in the classical Calderón-Zygmund decomposition is no longer good in a general situation: the L^∞ bound (or even any higher integrability) is ruined by the fact that the average of f on a given maximal cube cannot be bounded unless the measure is assumed to be doubling or dyadically doubling. This new good part leads to an additional bad term that needs to be controlled. More precisely, fixed $\lambda > 0$, let $\{Q_j\}_j$ be the corresponding family of maximal dyadic cubes of the level set Ω_λ (maximal with respect to the property that the μ average of $|f|$ is $\langle |f| \rangle_Q > \lambda$). Then we write $f = g + b + \beta$ with

$$
\begin{aligned}
g(x) &= f(x)\, 1_{\mathbb{R}\setminus\Omega_\lambda}(x) + \sum_j \langle f\rangle_{\widehat{Q}_j}\, 1_{Q_j}(x) \\
&\qquad\qquad + \sum_j \big(\langle f\rangle_{Q_j} - \langle f\rangle_{\widehat{Q}_j}\big)\, \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)}\, 1_{\widehat{Q}_j}(x), \\
b(x) &= \sum_j b_j(x) \;=\; \sum_j \big(f(x) - \langle f\rangle_{Q_j}\big)\, 1_{Q_j}(x), \\
\beta(x) &= \sum_j \beta_j(x) \;=\; \sum_j \big(\langle f\rangle_{Q_j} - \langle f\rangle_{\widehat{Q}_j}\big) \Big(1_{Q_j}(x) - \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)}\, 1_{\widehat{Q}_j}(x)\Big).
\end{aligned}
$$

The decomposition is such that

- $g \in L^p(\mu)$ for every $1 \le p < \infty$ with
$$\|g\|_{L^p(\mu)} \le C_p\, \lambda^{p-1}\, \|f\|_{L^1(\mu)};$$

- $b = \sum_j b_j$, with
$$\operatorname{supp}(b_j) \subset Q_j, \qquad \int_{\mathbb{R}} b_j(x)\, d\mu(x) = 0, \qquad \sum_j \|b_j\|_{L^1(\mu)} \le 2\, \|f\|_{L^1(\mu)};$$

- $\beta = \sum_j \beta_j$, with
$$\operatorname{supp}(\beta_j) \subset \widehat{Q}_j, \qquad \int_{\mathbb{R}} \beta_j(x)\, d\mu(x) = 0, \qquad \sum_j \|\beta_j\|_{L^1(\mu)} \le 4\, \|f\|_{L^1(\mu)},$$

  where, for each j, we write $\widehat{Q}_j$ to denote the dyadic parent of Q_j.

Let us compare this with the classical Calderón-Zygmund decomposition. First, we lose the L^∞ bound for the good part, however, for practical purposes this is not a problem since in most of the cases one typically uses the L^2 estimate for g. We now have two bad terms: the typical one b; and the new one β, whose building blocks are supported in the dyadic cubes $\{\widehat{Q}_j\}_j$, which are not pairwise disjoint, but still possess some cancelation. This new Calderón-Zygmund decomposition is key to obtaining the weak-type estimates for the Haar shift operators we consider.

We are confident that these results should also hold for other dyadic lattices and, more in general, in the context of geometrically doubling metric spaces in terms of Christ's dyadic cubes [11], or some other dyadic constructions like that of David in [19]. It is an interesting question whether these results also hold in the general measure-theoretic setting of Lacey in [43] and of Thiele, Treil and Volberg in [74] and Treil in [76].

Calderón-Zygmund operators associated to matrix-valued kernels

In a general sense, semicommutative harmonic analysis study of singular integrals acting on matrix or operator-valued functions. Historically, the matrix-valued theory has been treated part of the vector-valued theory. However, the vector-valued setting offers a limited approach to prove adequate weak-type $(1,1)$ estimates. This is mostly due to the fact that vector-valued theory is oblivious of the intrinsic algebraic structure of matrix-valued functions. A better suited perspective is supplied by noncommutative analysis, a field motivated by von Neumann after unifying Heisenberg and Schrödinger formulations of quantum mechanics. The gist of this theory is to replace functions with operators on a Hilbert space; this replacement entails a noncommutative multiplication given by composition of operators. More specifically, the *quantization* of L^p theory translates the rôle of L^∞ spaces to von Neumann algebras, i.e., weak*-closed unital C^*-algebras of operators on a Hilbert space. In this setting, traces hold the place of integrals and orthogonal projections mirror characteristic functions. Thus, associated to a von Neumann algebra $\mathcal{M}$ with trace τ, the noncommutative $L_p(\mathcal{M})$ spaces — in noncommutative L_p theory the scale parameter is traditionally displayed as a subscript — are the spaces of operators for which the norm

$$\|x\|_{L_p(\mathcal{M})} = \tau(|x|^p)^{1/p}$$

is finite. Here $|x| = (x^*x)^{1/2}$ is the modulus of x and $|x|^p$ is defined by functional calculus of positive operators. A much more detailed and precise discussion is given in [67, 53] and in references therein.

The connection to our setting is provided by the tensor product theory of von Neumann algebras. For simplicity, let us consider the algebra of functions

$$\mathcal{A}_B = \left\{ f : \mathbb{R} \to \mathcal{B}(\ell^2) : f \text{ strongly measurable s.t. } \operatorname*{ess\,sup}_{x\in\mathbb{R}} \|f(x)\|_{\mathcal{B}(\ell^2)} < \infty \right\},$$

where $\mathcal{B}(\ell^2)$ is the space of bounded operators on the sequence Hilbert space ℓ^2. The weak*-operator closure $\mathcal{A}$ of $\mathcal{A}_B$ is a von Neumann algebra isomorphic to $L_\infty(\mathbb{R})\overline{\otimes}\mathcal{B}(\ell^2)$ and it is thus equipped with the trace $\tau(f) = \int_{\mathbb{R}} \operatorname{Tr}(f(x))\, dx$. Here Tr is the standard trace on $\mathcal{B}(\ell^2)$ and the associated noncommutative L_p space $L_p(\mathcal{B}(\ell^2))$ corresponds to the Schatten p-class. Denote $\mathcal{B}(\ell^2)$ by $\mathcal{M}$. By the above discussion, the corresponding noncommutative $L_p(\mathcal{A})$ space is then the closure of appropriately chosen simple functions. For such functions we have that

$$\tau(|f|^p) = \int_{\mathbb{R}} \operatorname{Tr}(|f(x)|^p) dx = \int_{\mathbb{R}} \|f(x)\|^p_{L_p(\mathcal{M})}\, dx.$$

It thus deduced that for $1 \le p < \infty$ the space $L_p(\mathcal{A})$ is isometrically isomorphic to the classical Bochner space $L^p(\mathbb{R}; L_p(\mathcal{M}))$. Of course, the results here discussed are also valid for functions taking values on an arbitrary von Neumann algebra with a *n.s.f.* trace.

One might then wonder if Haar shift operators acting on matrix-valued functions admit *a priori* estimates in $L_p(\mathcal{A})$. In other words, whether

$$Ш_{j,k} f(x) = \sum_{I\in\mathscr{D}} A_I f = \sum_{I\in\mathscr{D}} \sum_{\substack{J\in\mathscr{D}_j(I)\\ K\in\mathscr{D}_k(I)}} \alpha^I_{J,K} \langle f, h_J\rangle h_K(x),$$

acts boundedly on $L_p(\mathcal{A})$. Here by $\langle f, h_J\rangle$ we denote the pairing $\int_{\mathbb{R}} f(x) h_I(x)\, dx$, which is matrix-valued and $\alpha^I_{J,K}$ are uniformly bounded scalars. Since Schatten p-classes are UMD Banach spaces for $1 < p < \infty$, the question of L_p boundedness of Haar shift operators and Calderón-Zygmund operators is settled by the classical vector-valued theory as developed by Burkholder in [8, 9], Bourgain [6] and Figiel [24]. It is at the point of seeking a suitable weak-type estimate for $p = 1$ where the vector-valued theory fails, since the Schatten 1-class is not UMD. To deal with such questions the noncommutative structure becomes essential. Following the construction of noncommutative symmetric spaces (see [53]), a noncommutative weak $L_1(\mathcal{A})$ is defined via the quasi-norm

$$\|f\|_{L_{1,\infty}(\mathcal{A})} = \sup_{\lambda>0} \lambda\tau(\{|f| > \lambda\}).$$

Here $\tau(\{|f| > \lambda\})$ denotes the trace of the spectral projection of $|f|$ associated to the interval (λ, ∞). This defines a noncommutative distribution function that shares the same properties of its classical counterpart, hence the notation. The resulting space $L_{1,\infty}(\mathcal{A})$ has the expected interpolation properties. We emphasize that the weak Bochner space $L^{1,\infty}(\mathbb{R}; L_1(\mathcal{M}))$ is of no use for our purposes since $L_1(\mathcal{M})$ is not a UMD space and thus even Haar multipliers may not be bounded. The same reasoning rules out working with $L^{1,\infty}(\mathbb{R}; \mathcal{M})$.

In [58] Parcet provided the adequate weak-type $(1,1)$ estimates for Calderón-Zygmund operators with scalar or commuting kernels acting on matrix-valued functions. Namely, he got that for such a Calderón-Zygmund operator T

$$\lambda\tau(\{|Tf| > \lambda\}) \lesssim \|f\|_{L_1(\mathcal{A})}$$

uniformly over $\lambda > 0$. This is shown by constructing a noncommutative extension of the Calderón-Zygmund decomposition. Essential to this result is the ingenious construction of Cuculescu in [18] with which he rendered the analogue of the weak-type $(1,1)$ Doob's maximal inequality in the intimately related field of noncommutative martingale theory. Cuculescu's construction enables to obtain a projections q and a family of pairwise disjoint projections $(p_k)_{k\in\mathbb{Z}}$ related to the decomposition of classical level set Ω_λ in the following way

$$q \sim \mathbb{R} \setminus \Omega_\lambda \quad \text{and} \quad p_k \sim \big\{Q_j \text{ maximal cube in } \Omega_\lambda : Q_j \in \mathscr{D}_k\big\},$$

and such that $\sum_k p_k = 1_\mathcal{A} - q$, with $1_\mathcal{A}$ being the unit in $\mathcal{A}$. Here, $\mathscr{D}_k$ denotes the family of dyadic cubes of sidelength $\ell(Q) = 2^{-k}$. The noncommutative Calderón-Zygmund decomposition is given in terms of these projections by $f = g + b$, where the good and the bad parts are

$$g = \sum_{i,j\in\mathbb{Z}} p_i f_{i\vee j} p_j \quad \text{and} \quad b = \sum_{i,j\in\mathbb{Z}} p_i (f - f_{i\vee j}) p_j,$$

where $i \vee j = \max(i,j)$ and by f_k we denote the conditional expectation

$$f_k = \sum_{Q\in\mathscr{D}_k} \langle f\rangle_Q 1_Q.$$

Here $\langle f\rangle_Q$ is the mean of f over Q, hence an operator. In the form of this decomposition the noncommutativity of this setting is explicit. Indeed, in a commutative situation the disjointness of the projections p_k reduces to the diagonal case, namely that in which $i = j$. The diagonal terms satisfy the same estimates of the classical decomposition. Namely,

- $\|g_\Delta\|_{L_1(\mathcal{A})} \leq \|f\|_{L_1(\mathcal{A})}$ and $\|g_\Delta\|_{L_\infty(\mathcal{A})} \leq 2\lambda$.

- $b_\Delta = \sum_i b_{\Delta,i}$, with $b_{\Delta,i}$ of mean zero and $\sum_i \|b_{\Delta,i}\|_{L_1(\mathcal{A})} \leq 2\|f\|_{L_1(\mathcal{A})}$.

These properties render weak-type estimates of the diagonal terms by proceeding as with the classical Calderón-Zygmund decomposition. On the other hand, the off-diagonal terms — those for which $i \neq j$ — seem to lack the above classical estimates. In practice, the off-diagonal terms are dealt by using a pseudo-localization principle and certain truncated estimates satisfied by the off-diagonal terms. Prior to the remarkable developments in [58], beside of course [18], weak-type $(1,1)$ estimates in the noncommutative setting were only known for noncommutative martingale transforms and square functions, as developed by Parcet and Randrianantoanina [59] and by Randrianantoanina in [70, 71, 72]; in [69] for the noncommutative Hilbert transform associated to maximal sub-diagonal algebras and by Junge and Xu [39] in the context of maximal ergodic theorems.

The same methods Parcet used in [58] give the analogous estimate for a Haar shift operator $\text{Ш}_{j,k}$ with polynomial dependence on the complexity. One might then ask about the boundedness properties of Haar shift operators with matrix-valued symbols $\alpha^I_{J,K} \in \mathcal{M}$. Matrix-valued paraproducts are prominent examples that have attracted some attention. Different operators arise depending on whether the symbols act by right or left multiplication on each coefficient $\langle f, h_I \rangle \in \mathcal{M}$. Consider for example Haar multipliers with uniformly bounded symbols $\alpha_I \in \mathcal{M}$. A pair of column/row operators are introduced by

$$T^{\mathrm{c}}_\alpha(f) = \sum_{I\in\mathscr{D}} \alpha_I \langle f, h_I\rangle h_I, \qquad T^{\mathrm{r}}_\alpha(f) = \sum_{Q\in\mathscr{D}} \langle f, h_I\rangle \alpha_I h_I.$$

Even in the Lebesgue setting, Haar multipliers with noncommuting symbols may lack weak-type $(1,1)$ and strong (p,p) estimates for $p \neq 2$, highlighting the fact that the noncommutative nature of the context predominates. In the case of dyadic paraproducts, L_2 boundedness fails even for reasonably chosen symbols as proved by Katz in [41], by Nazarov, Pisier, Treil and Volberg [56] and Mei [50]. Let us illustrate this by giving a classical counterexample coming from noncommutative martingale theory. Let $\mathcal{A} = L^\infty([0,1))\overline{\otimes}\mathcal{M}$, and consider the column multiplier with symbol $\alpha_I = e_{k,1}$ for $I \in \mathscr{D}_{k-1}$. If

$$f_n = \sum_{k=1}^n \Bigg(\sum_{I\in\mathscr{D}_{k-1}} |I|^{1/2} h_I \Bigg) e_{1,k},$$

it is easily seen that

$$\|f_n\|_{L_1(\mathcal{A})} = \sqrt{n} \quad \text{and} \quad \|T_{\mathrm{c}} f_n\|_{L_{1,\infty}(\mathcal{A})} = n \gg \|f_n\|_{L_1(\mathcal{A})}$$

for sufficiently large n. This motivates the problem that we intend address in Part II, namely:

> *Does there exist subspaces or subsets $A_{\mathrm{r}}/A_{\mathrm{c}}$ of $L_1(\mathcal{A})$ such that $f = f_{\mathrm{r}} + f_{\mathrm{c}}$ for $f_{\mathrm{r}} \in A_{\mathrm{r}}$ and $f_{\mathrm{c}} \in A_{\mathrm{c}}$ and that*
>
> $$T_{\mathrm{r}} : A_{\mathrm{r}} \to L_{1,\infty}(\mathcal{A}) \quad \text{and} \quad T_{\mathrm{c}} : A_{\mathrm{c}} \to L_{1,\infty}(\mathcal{A})?$$

Despite falling short to solve this question, we get interesting results in this direction.

- **Haar shift operators.** In Theorem 5.1 (i) we shall see that for $f \in L_1(\mathcal{A})$ there exist an explicit decomposition $f = f_{\mathrm{r}} + f_{\mathrm{c}}$ such that

$$\|\text{Ш}^{\mathrm{r}}_{j,k} f_{\mathrm{r}}\|_{L_{1,\infty}(\mathcal{A})} + \|\text{Ш}^{\mathrm{c}}_{j,k} f_{\mathrm{c}}\|_{L_{1,\infty}(\mathcal{A})} \leq C_{j,k} \|f\|_{L_1(\mathcal{A})}.$$

  for row/column Haar shift operators including Haar multipliers, paraproducts and their adjoints.

- **Martingale multipliers and paraproducts.** By extending the same techniques to the noncommutative martingale setting in Theorem 5.3 (i) we shall obtain analogous results for martingale difference operators and martingale paraproducts with non commuting symbols for regular filtrations.

The decomposition $f = f_{\mathrm{r}} + f_{\mathrm{c}}$ obtained above is given in terms of triangular truncations relative to suitably chosen projections depending on f. More precisely, the projections used are obtained by adapting Cuculescu's construction so that projections associated to different heights are comparable. In Theorem 5.1 (i) we get constants of order $C_{j,k} \sim 2^j$, which seem far from being optimal. The classical argument giving constants of linear or even polynomial order encounters a major obstacle due to the presence of triangular truncations, which are not bounded in L_1 by the classical result of Kwapień and Pełczyński [42]. This is also the reason why we did not succeed in extended the argument above to generic Calderón-Zygmund operators, leaving it as an open problem.

On the other hand, by complementary techniques we are able to proof estimates for generic noncommuting Calderón-Zygmund operators. That is, for a pair of row/column operators given formally by

$$T_{\mathrm{r}} f(x) \sim \int_{\mathbb{R}} f(y) k(x,y)\, dy \quad \text{and} \quad T_{\mathrm{c}} f(x) \sim \int_{\mathbb{R}} k(x,y) f(y)\, dy,$$

with kernels such that $k(x,y) \in \mathcal{M}$ for $x \neq y$ satisfying standard size and smoothness estimates. This is done by using the theory of row/column Hardy spaces of Mei [51], its martingale analogues, developed earlier in [66] and the associated interpolation and duality properties [32, 38, 55].

- **Hardy space estimates.** In Theorem 5.1 (ii) we will show that $(T_{\mathrm{r}}, T_{\mathrm{c}})$ maps continuously row/column Hardy spaces into $L_1(\mathcal{A})$. Interpolation and duality arguments provide L_p estimates in Theorem 5.2.

- **Martingale multipliers and paraproducts.** H_1 and L_p estimates are obtained for noncommuting martingale transforms and paraproducts for arbitrary filtrations in Theorem 5.3 (ii).

Non-doubling semicommutative dyadic harmonic analysis

It is then natural to ask to what degree the results of Part I can be carried to the semicommutative context of Part II. Namely,

> *Can one determine the class of measures for which a dyadic operator acting on operator-valued functions is of weak-type* $(1,1)$*?*

This question motivates Part III of this thesis. To answer it we introduce a noncommutative Calderón-Zygmund decomposition that generalizes the Calderón-Zygmund decomposition used in Part I. As the noncommutative Calderón-Zygmund decomposition introduced in [58] – which is valid for the Lebesgue measure — this decomposition relies in adapting Cuculescu's construction to this setting. We obtain in Theorem 9.2 that $f = g + b + \beta$ with each term having a diagonal and an off-diagonal part given by

- $g = g_\Delta + g_{\mathrm{off}}$, where

$$g_\Delta = qfq + \sum_{k\in\mathbb{Z}} \mathsf{E}_{k-1}\left(p_k f_k p_k\right),$$
$$g_{\mathrm{off}} = (1_{\mathcal{A}} - q)fq + qf(1_{\mathcal{A}} - q) + \sum_{i\neq j} \mathsf{E}_{i\vee j-1}\left(p_i f_{i\vee j} p_j\right);$$

- $b = b_\Delta + b_{\mathrm{off}}$, where

$$b_\Delta = \sum_{k\in\mathbb{Z}} p_k(f - f_k)p_k\,, \qquad b_{\mathrm{off}} = \sum_{i\neq j} p_i(f - f_{i\vee j})p_j;$$

- $\beta = \beta_\Delta + \beta_{\mathrm{off}}$, where

$$\beta_\Delta = \sum_{k\in\mathbb{Z}} \mathsf{D}_k(p_k f_k p_k), \qquad \beta_{\mathrm{off}} = \sum_{i\neq j} \mathsf{D}_{i\vee j}\left(p_i f_{i\vee j} p_j\right).$$

As in the decomposition obtained in [58], the diagonal terms satisfy the classical properties

- $\|g_\Delta\|_{L_1(\mathcal{A})} \leq \|f\|_{L_1(\mathcal{A})}$ and $\|g_\Delta\|_{L_2(\mathcal{A})} \leq C\lambda\|f\|_{L_1(\mathcal{A})}$.
- $b_\Delta = \sum_i b_{\Delta,i}$, with $b_{\Delta,i}$ of mean zero and $\sum_i \|b_{\Delta,i}\|_{L_1(\mathcal{A})} \leq 2\|f\|_{L_1(\mathcal{A})}$.
- $\beta_\Delta = \sum_i \beta_{\Delta,i}$, with $\beta_{\Delta,i}$ of mean zero and $\sum_i \|\beta_{\Delta,i}\|_{L_1(\mathcal{A})} \leq 2\|f\|_{L_1(\mathcal{A})}$.

In practice, the off-diagonal terms are controlled by localization principles. We use this decomposition to answer the motivating question.

- **Commuting Haar shift operators.** The weak-type $(1,1)$ of Haar shift operators with commuting symbols is characterized as in the commutative non-doubling setting as we shall see in Theorem 9.4.

The consideration of noncommuting symbols introduces considerable additional difficulties when trying to provide *a priori* weak-type estimates. The methods of Theorem 5.1 (i) are not applicable to this setting even when considering Haar multipliers. We discuss this at the end of Part III and leave it as an open problem.

Statement

The results presented in this thesis are based on original work done in collaboration with other researchers during the course of my PhD, from March 2011 to February 2015. These results based on the released publications [14, 27, 46]. These references are listed below.

The organization of thesis is as follows; the content of Part I is based on [46], Part II is based on reference [27] and Part III based on [14].

[14] J.M. Conde-Alonso and L.D. López-Sánchez. Operator-valued dyadic harmonic analysis beyond doubling measures. Preprint. arXiv:1412.4937, 2014. Submitted.

[27] G. Hong, L.D. López-Sánchez, J.M. Martell, and J. Parcet. Calderón-Zygmund operators associated to matrix-valued kernels. *Int. Math. Res. Not. IMRN*, 2014(5):1221–1252, 2014.

[46] L.D. López-Sánchez, J.M. Martell, and J. Parcet. Dyadic harmonic analysis beyond doubling measures. *Adv. Math.*, 267(0):44 – 93, 2014.

Part I

Dyadic harmonic analysis beyond doubling measures

Chapter 1

Introduction and main results

In this Part of the dissertation we study the boundedness behavior of dyadic operators with respect to Borel measures that are not necessarily doubling. For simplicity we will restrict ourselves to the Euclidean setting with the standard dyadic grid $\mathscr{D}$ in $\mathbb{R}^d$. Of course, our results should also hold for other dyadic lattices and, more in general, in the context of geometrically doubling metric spaces in terms of Christ's dyadic cubes [11], or some other dyadic constructions [19, 31]. We will use the following notation, for every $Q \in \mathscr{D}$, we let $\mathscr{D}_k(Q)$, $k \geq 1$, be the family of dyadic subcubes of side-length $2^{-k}\,\ell(Q)$. We shall work with Borel measures μ such that $\mu(Q) < \infty$ for every dyadic cube Q (equivalently, the μ-measure of every compact set is finite). To go beyond the well-known framework of the Calderón-Zygmund theory for doubling measures, the first thing we do is to develop a Calderón-Zygmund decomposition adapted to μ and to the associated dyadic maximal function

$$M_{\mathscr{D}}f(x) = \sup_{x\in Q\in\mathscr{D}} \langle |f| \rangle_Q = \sup_{x\in Q\in\mathscr{D}} \frac{1}{\mu(Q)} \int_Q |f(x)|\,d\mu(x).$$

Here we have used the notation $\langle g \rangle_Q$ for the μ-average of g on Q and we set $\langle g \rangle_Q = 0$ if $\mu(Q) = 0$. As usual, if $f \in L^1(\mu)$ and $\lambda > 0$, we cover $\{M_{\mathscr{D}}f > \lambda\}$ by the maximal dyadic cubes $\{Q_j\}_j$. In the general setting that we are considering, such maximal cubes exist (for every $\lambda > 0$) if the μ-measure of every d-dimensional quadrant is infinity. Otherwise, maximal cubes exist for λ large enough. For the sake of clarity in exposition, in the following result we assume that each d-dimensional quadrant has infinite μ-measure. The general case will be addressed in Section 2.4 below.

One could try to use the standard Calderón-Zygmund decomposition, $f = g + b$ where g and b are respectively the "good" and "bad" parts. As usual, in each Q_j the "good" part would agree with $\langle f \rangle_{Q_j}$. However, this good part would not be bounded (or even higher integrable) and therefore this decomposition would be of no use. Our new Calderón-Zygmund decomposition solves the problem with the "good" part and adds a new "bad" part whose building blocks have vanishing integrals and each of them is supported in $\widehat{Q}_j$, the dyadic parent of Q_j.

Theorem 1.1. *Let μ be a Borel measure on $\mathbb{R}^d$ satisfying that $\mu(Q) < \infty$ for all $Q \in \mathscr{D}$ and that each d-dimensional quadrant has infinite μ-measure. Given* *an integrable function* $f \in L^1(\mu)$ *and* $\lambda > 0$*, consider the standard covering of* $\Omega_\lambda = \{M_{\mathscr{D}} f > \lambda\}$ *by maximal dyadic cubes* $\{Q_j\}_j$*. Then we can write* $f = g + b + \beta$ *with*

$$\begin{aligned} g(x) &= f(x)\, 1_{\mathbb{R}^d \setminus \Omega_\lambda}(x) + \sum_j \langle f \rangle_{\widehat{Q}_j} 1_{Q_j}(x) \\ &\qquad\qquad + \sum_j \big(\langle f \rangle_{Q_j} - \langle f \rangle_{\widehat{Q}_j}\big) \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} 1_{\widehat{Q}_j}(x), \\ b(x) &= \sum_j b_j(x) \;=\; \sum_j \big(f(x) - \langle f \rangle_{Q_j}\big) 1_{Q_j}(x), \\ \beta(x) &= \sum_j \beta_j(x) \;=\; \sum_j \big(\langle f \rangle_{Q_j} - \langle f \rangle_{\widehat{Q}_j}\big) \Big(1_{Q_j}(x) - \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} 1_{\widehat{Q}_j}(x)\Big). \end{aligned}$$

Then, we have the following properties:

(a) *The function* g *satisfies*

$$\|g\|_{L^p(\mu)}^p \le C_p \lambda^{p-1} \|f\|_{L^1(\mu)} \quad \textit{for every} \quad 1 \le p < \infty.$$

(b) *The function* b *decomposes as* $b = \sum_j b_j$*, where*

$$\operatorname{supp}(b_j) \subset Q_j, \qquad \int_{\mathbb{R}^d} b_j(x)\, d\mu(x) = 0, \qquad \sum_j \|b_j\|_{L^1(\mu)} \le 2 \|f\|_{L^1(\mu)}.$$

(c) *The function* β *decomposes as* $\beta = \sum_j \beta_j$*, where*

$$\operatorname{supp}(\beta_j) \subset \widehat{Q}_j, \qquad \int_{\mathbb{R}^d} \beta_j(x)\, d\mu(x) = 0, \qquad \sum_j \|\beta_j\|_{L^1(\mu)} \le 4 \|f\|_{L^1(\mu)}.$$

Theorem 1.1 is closely related to Gundy's martingale decomposition [26] and was obtained in the unpublished manuscript [49] (see also [16]). It is however more flexible because the building blocks are the maximal cubes in place of the martingale differences. This feature is crucial when considering Haar shift operators allowing us to characterize their weak-type $(1,1)$ for general Borel measures.

A baby model of the mentioned characterization —which will be illustrative for the general statement— is given by the dyadic Hilbert transform in $\mathbb{R}$ and its adjoint. To define this operator we first need to introduce some notation. First, to simplify the exposition, let us assume that $\mu(I) > 0$ for every $I \in \mathscr{D}$, below we will consider the general case. Given $I \in \mathscr{D}$ we write I_-, I_+ for the (left and right) dyadic children of I, and, as before, $\widehat{I}$ is the dyadic parent of I. We set

$$h_I \;=\; \sqrt{m(I)} \left(\frac{1_{I_-}}{\mu(I_-)} - \frac{1_{I_+}}{\mu(I_+)} \right), \qquad \text{with} \quad m(I) = \frac{\mu(I_-)\mu(I_+)}{\mu(I)}. \tag{1.2}$$

Let us first observe that the system $\mathcal{H} = \{h_I\}_{I \in \mathscr{D}}$ is orthonormal. Additionally, for every $I \in \mathscr{D}$ we have

$$\|h_I\|_{L^1(\mu)} = 2\sqrt{m(I)}, \qquad\qquad \|h_I\|_{L^\infty(\mu)} \approx \frac{1}{\sqrt{m(I)}}. \tag{1.3}$$

Therefore we obtain the following condition which will become meaningful later

$$\sup_{I\in\mathscr{D}} \|h_I\|_{L^\infty(\mu)}\|h_I\|_{L^1(\mu)} < \infty. \tag{1.4}$$

We define the dyadic Hilbert transform by

$$H_{\mathscr{D}} f(x) = \sum_{I\in\mathscr{D}} \langle f, h_I\rangle \big(h_{I_-}(x) - h_{I_+}(x)\big) = \sum_{I\in\mathscr{D}} \sigma(I)\langle f, h_{\widehat{I}}\rangle h_I(x),$$

where $\sigma(I) = 1$ if $I = (\widehat{I}\,)_-$ and $\sigma(I) = -1$ if $I = (\widehat{I}\,)_+$. Another toy model in the 1-dimensional setting is the adjoint of $H_{\mathscr{D}}$ which can be written as

$$H_{\mathscr{D}}^* f(x) \;=\; \sum_{I\in\mathscr{D}} \sigma(I)\langle f, h_I\rangle h_{\widehat{I}}(x).$$

We are going to show that the increasing or decreasing properties of m characterize the boundedness of $H_{\mathscr{D}}$ and $H_{\mathscr{D}}^*$. This motivates the following definition. We say that μ is *m-increasing* if there exists $0 < C < \infty$ such that

$$m(I) \le C\, m(\widehat{I}\,), \qquad I \in \mathscr{D}.$$

We say that μ is *m-decreasing* if there exists $0 < C < \infty$ such that

$$m(\widehat{I}\,) \le C\, m(I), \qquad I \in \mathscr{D}.$$

Finally, we say that μ is *m-equilibrated* if μ is both m-increasing and m-decreasing.

Let us note that if μ is the Lebesgue measure, or in general any dyadically doubling measure, we have that $m(I) \approx \mu(I)$ and therefore μ is m-equilibrated. As we will show below, the converse is not true. In general, we observe that $m(I)$ is half the harmonic mean of the measures of the children of I and therefore,

$$\begin{aligned} m(I) = \left(\frac{1}{\mu(I_-)} + \frac{1}{\mu(I_+)}\right)^{-1} &\approx \left(\max\left\{\frac{1}{\mu(I_-)}, \frac{1}{\mu(I_+)}\right\}\right)^{-1} \\ &= \min\big\{\mu(I_-), \mu(I_+)\big\} < \mu(I). \end{aligned}$$

Thus, m gives quantitative information about the degeneracy of μ over I: $m(I)/\mu(I) \ll 1$ implies that μ mostly concentrates on only one child of I, and $m(I)/\mu(I) \gtrsim 1$ gives that $\mu(I_-) \approx \mu(I_+) \approx \mu(I)$.

We are ready to state our next result which characterizes the measures for which $H_{\mathscr{D}}$ and $H_{\mathscr{D}}^*$ are bounded for $p \neq 2$.

Theorem 1.5. *Let μ be a Borel measure on $\mathbb{R}$ satisfying that $0 < \mu(I) < \infty$ for every $I \in \mathscr{D}$.*

(i) $H_{\mathscr{D}} : L^1(\mu) \to L^{1,\infty}(\mu)$ if and only if μ is m-increasing.

(ii) $H_{\mathscr{D}}^* : L^1(\mu) \to L^{1,\infty}(\mu)$ if and only if μ is m-decreasing.

Moreover, if $1 < p < 2$ we have:

(iii) $H_{\mathscr{D}} : L^p(\mu) \to L^p(\mu)$ if and only if μ is m-increasing.

(iv) $H_{\mathscr{D}}^* : L^p(\mu) \to L^p(\mu)$ *if and only if* μ *is* m*-decreasing.*

If $2 < p < \infty$*, by duality, the previous equivalences remain true upon switching the conditions on* μ*.*

Furthermore, given two non-negative integers r*,* s*, let* $\text{Ш}_{r,s}$ *be a Haar shift of complexity* (r, s)*, that is,*

$$\text{Ш}_{r,s} f(x) = \sum_{I \in \mathscr{D}} \sum_{\substack{J \in \mathscr{D}_r(I) \\ K \in \mathscr{D}_s(I)}} \alpha_{J,K}^I \langle f, h_J \rangle h_K(x) \quad \text{with} \quad \sup_{I,J,K} |\alpha_{J,K}^I| < \infty. \tag{1.6}$$

If μ *is* m*-equilibrated then* $\text{Ш}_{r,s}$ *is bounded from* $L^1(\mu)$ *to* $L^{1,\infty}(\mu)$ *and from* $L^p(\mu)$ *to* $L^p(\mu)$ *for every* $1 < p < \infty$*.*

Let us observe that our assumption on the coefficients of the Haar shift operator is not standard, below we shall explain why this is natural (see Theorem 1.11 and the comment following it).

Let us observe that using the notation in the previous result $H_{\mathscr{D}}$ is a Haar shift of complexity $(0, 1)$ whereas $H_{\mathscr{D}}^*$ is a Haar shift of complexity $(1, 0)$. As noted above, dyadically doubling measures are m-equilibrated. Therefore, in this case, $H_{\mathscr{D}}$, $H_{\mathscr{D}}^*$, and all 1-dimensional Haar shifts $\text{Ш}_{r,s}$ with arbitrary complexity are of weak-type $(1, 1)$ and bounded on $L^p(\mu)$ for every $1 < p < \infty$. In Section 3.1 we shall present examples of measures in $\mathbb{R}$ as follows:

- μ is m-equilibrated, but μ is neither dyadically doubling nor of polynomial growth. Thus, we have an example of a measure that is out of the classical theory for which the dyadic Hilbert transform, its adjoint and any Haar shift is of weak-type $(1, 1)$ and bounded on $L^p(\mu)$ for every $1 < p < \infty$.

- μ is m-increasing, but μ is not m-decreasing, not dyadically doubling, not of polynomial growth. Thus, $H_{\mathscr{D}}$ is of weak-type $(1, 1)$, bounded on $L^p(\mu)$ for every $1 < p \le 2$ and unbounded on $L^p(\mu)$ for $2 < p < \infty$; $H_{\mathscr{D}}^*$ is bounded on $L^p(\mu)$ for $2 \le p < \infty$, not of weak-type $(1, 1)$ and unbounded on $L^p(\mu)$ for every $1 < p < 2$.

- μ is m-decreasing, but μ is not m-increasing, not dyadically doubling, not of polynomial growth. Thus, $H_{\mathscr{D}}$ is bounded on $L^p(\mu)$ for $2 \le p < \infty$, not of weak-type $(1, 1)$ and unbounded on $L^p(\mu)$ for every $1 < p < 2$; $H_{\mathscr{D}}^*$ is of weak-type $(1, 1)$, bounded on $L^p(\mu)$ for every $1 < p \le 2$ and unbounded on $L^p(\mu)$ for $2 < p < \infty$.

- μ is not m-decreasing, not m-increasing, not dyadically doubling, but μ has polynomial growth. Thus, this is an example of a measure *á la* Nazarov-Treil-Volberg and Tolsa for which $H_{\mathscr{D}}$ and $H_{\mathscr{D}}^*$ are bounded on $L^2(\mu)$, unbounded on $L^p(\mu)$ for $1 < p < \infty$, $p \ne 2$, and not of weak-type $(1, 1)$.

Our next goal is to extend the previous result to higher dimensions. In this case we do not necessarily assume that the measures have full support. The building blocks, that is, the Haar functions are not in one-to-one correspondence to the dyadic cubes: associated to every cube Q we expect to have at most $2^d - 1$ linearly independent Haar functions. Moreover, there are different ways to construct a Haar system (see Section 3.2 below). We next define the Haar systems that we are going to use:

Definition 1.7. Let μ be a Borel measure on $\mathbb{R}^d$, $d \geq 1$, satisfying that $\mu(Q) < \infty$ for every $Q \in \mathscr{D}$. We say that $\Phi = \{\phi_Q\}_{Q\in\mathscr{D}}$ is a *generalized Haar system* in $\mathbb{R}^d$ if the following conditions hold:

(a) For every $Q \in \mathscr{D}$, $\operatorname{supp}(\phi_Q) \subset Q$.

(b) If Q', $Q \in \mathscr{D}$ and $Q' \subsetneq Q$, then ϕ_Q is constant on Q'.

(c) For every $Q \in \mathscr{D}$, $\displaystyle\int_{\mathbb{R}^d} \phi_Q(x)\, d\mu(x) = 0$.

(d) For every $Q \in \mathscr{D}$, either $\|\phi_Q\|_{L^2(\mu)} = 1$ or $\phi_Q \equiv 0$.

Remark 1.8. The following comments pertain to the previous definition.

- Note that (b) implies that ϕ_Q is constant on the dyadic children of Q. In particular, ϕ_Q is a simple function which takes at most 2^d different values.

- Given a generalized Haar system $\Phi = \{\phi_Q\}_{Q\in\mathscr{D}}$, we write $\mathscr{D}_\Phi$ for the set of dyadic cubes Q for which $\phi_Q \not\equiv 0$. By assumption, we allow $\mathscr{D}_\Phi$ to be a proper subcollection of $\mathscr{D}$. Note that $\{\phi_Q\}_{Q\in\mathscr{D}}$ is an orthogonal system whereas $\{\phi_Q\}_{Q\in\mathscr{D}_\Phi}$ is orthonormal.

  Let us point out that we allow the measure μ to vanish in some dyadic cubes. If $\mu(Q) = 0$, we must have $\phi_Q \equiv 0$ and therefore $Q \in \mathscr{D} \setminus \mathscr{D}_\Phi$. If $\mu(Q) = \mu(Q')$ for some child Q' of Q (i.e., every brother of Q' has null μ-measure) then $\phi_Q \equiv 0$ and thus $Q \in \mathscr{D} \setminus \mathscr{D}_\Phi$. Suppose now that $Q \in \mathscr{D}_\Phi$ (therefore $\mu(Q) > 0$), by convention, we set $\phi_Q \equiv 0$ in every dyadic child of Q with vanishing measure.

- Let us suppose that for every $Q \in \mathscr{D}_\Phi$, ϕ_Q takes exactly 2 different non-zero values (call Φ a 2-*value* generalized Haar system). In view of the previous remark, ϕ_Q is "uniquely" determined modulo a multiplicative ± 1. That is, we can find E_Q^+, $E_Q^- \subset Q$, such that $E_Q^+ \cap E_Q^- = \emptyset$, $E_Q^\pm$ is comprised of dyadic children of Q, $\mu(E_Q^\pm) > 0$ and

$$\phi_Q = \sqrt{m_\Phi(Q)}\left(\frac{1_{E_Q^-}}{\mu(E_Q^-)} - \frac{1_{E_Q^+}}{\mu(E_Q^+)}\right), \quad \text{with } m_\Phi(Q) = \frac{\mu(E_Q^-)\mu(E_Q^+)}{\mu(E_Q^- \cup E_Q^+)}. \tag{1.9}$$

  Then, for every $Q \in \mathscr{D}_\Phi$ we have

$$\|\phi_Q\|_{L^1(\mu)} = 2\sqrt{m_\Phi(Q)}, \qquad \|\phi_Q\|_{L^\infty(\mu)} \approx \frac{1}{\sqrt{m_\Phi(Q)}}. \tag{1.10}$$

- In dimension 1, if we assume as before that $\mu(I) > 0$ for every $I \in \mathscr{D}$, we then have that $\mathcal{H}$ defined above is a generalized Haar system in $\mathbb{R}$ with $\mathscr{D}_{\mathcal{H}} = \mathscr{D}$. The previous remark and the fact every dyadic interval has two children say that $\mathcal{H}$ is "unique" in the following sense: let Φ be a generalized Haar system in $\mathbb{R}$, then $\phi_I = \pm h_I$ for every $I \in \mathscr{D}_\Phi$. Note that we can now allow the measure to vanish on some dyadic intervals. In such a case we will have that $\phi_I \equiv 0$ for every $I \in \mathscr{D}$ for which $\mu(I_-) \cdot \mu(I_+) = 0$. Also, $\phi_I = \pm h_I$ and $m_\Phi(I) = m(I)$ for every $I \in \mathscr{D}_\Phi$.

Our main result concerning general Haar shift operators characterizes the weak-type $(1,1)$ in terms of the measure μ and the generalized Haar systems that define the operator. In Section 4.1 we shall also consider non-cancellative Haar shift operators where condition (c) in Definition 1.7 is dropped for the Haar systems Φ and Ψ. This will allow us to obtain similar results for dyadic paraproducts.

Theorem 1.11. *Let μ be a Borel measure on $\mathbb{R}^d$, $d \geq 1$, such that $\mu(Q) < \infty$ for every $Q \in \mathscr{D}$. Let $\Phi = \{\phi_Q\}_{Q \in \mathscr{D}}$ and $\Psi = \{\psi_Q\}_{Q \in \mathscr{D}}$ be two generalized Haar systems in $\mathbb{R}^d$. Given two non-negative integers r, s we set*

$$\Xi(\Phi, \Psi; r, s) = \sup_{Q \in \mathscr{D}} \left\{ \|\phi_R\|_{L^\infty(\mu)} \|\psi_S\|_{L^1(\mu)} : R \in \mathscr{D}_r(Q), S \in \mathscr{D}_s(Q) \right\}.$$

Let $\text{Ш}_{r,s}$ be a Haar shift of complexity (r,s), that is,

$$\text{Ш}_{r,s} f(x) = \sum_{Q \in \mathscr{D}} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha^Q_{R,S} \langle f, \phi_R \rangle \psi_S(x) \qquad \textit{with} \quad \sup_{Q,R,S} |\alpha^Q_{R,S}| < \infty.$$

If $\Xi(\Phi, \Psi; r, s) < \infty$, then $\text{Ш}_{r,s}$ maps continuously $L^1(\mu)$ into $L^{1,\infty}(\mu)$, and by interpolation $\text{Ш}_{r,s}$ is bounded on $L^p(\mu)$, $1 < p \leq 2$.

Conversely, let $\text{Ш}_{r,s}$ be a Haar shift of complexity (r,s) satisfying the non-degeneracy condition $\inf_{Q,R,S} |\alpha^Q_{R,S}| > 0$. If $\text{Ш}_{r,s}$ maps continuously $L^1(\mu)$ into $L^{1,\infty}(\mu)$ then $\Xi(\Phi, \Psi; r, s) < \infty$.

Let us point out that in the Euclidean setting with the Lebesgue measure one typically assumes that $|\alpha^Q_{R,S}| \lesssim (|R|\,|S|)^{1/2}/|Q|$. Our condition, with a general measure, is less restrictive and more natural: having assumed the corresponding condition with respect to μ, $H_{\mathscr{D}}$ and $H^*_{\mathscr{D}}$ would not be 1-dimensional Haar shift operators unless μ is dyadically doubling.

To illustrate the generality and the applicability of Theorem 1.11 we consider some examples. Before doing that we need to introduce some notation. Let Φ be a generalized Haar system in $\mathbb{R}^d$, we say that Φ is *standard* if

$$\sup_{Q \in \mathscr{D}} \|\phi_Q\|_{L^1(\mu)} \, \|\phi_Q\|_{L^\infty(\mu)} < \infty. \tag{1.12}$$

Note that we can restrict the supremum to $Q \in \mathscr{D}_\Phi$. Also, if $Q \in \mathscr{D}_\Phi$, Hölder's inequality and (d) imply that each term in the supremum is bounded from below by 1. Thus, Φ being standard says that the previous quantity is bounded from below and from above uniformly for every $Q \in \mathscr{D}_\Phi$. Notice that in the language of Theorem 1.11, Φ being standard is equivalent to $\Xi(\Phi, \Phi; 0, 0) < \infty$.

Remark 1.13. If Φ is a 2-value generalized Haar system, (1.10) implies that Φ is standard. Note that in $\mathbb{R}$ (since every dyadic interval has two children) every generalized Haar system, including $\mathcal{H}$ introduced above, is of 2-value type and therefore standard.

Example 1.14 (Haar multipliers)**.** Let $\Phi = \{\phi_Q\}_Q$ be a generalized Haar system in $\mathbb{R}^d$. We take the Haar shift operator of complexity $(r, s) = (0, 0)$, usually referred to as a Haar multiplier,

$$Ш_{0,0} f(x) = \sum_{Q \in \mathscr{D}} \alpha_Q \langle f, \phi_Q \rangle \phi_Q(x), \qquad \text{with} \quad \sup_Q |\alpha_Q| < \infty.$$

Then $\Xi(\Phi, \Phi; 0, 0) < \infty$ is equivalent to the fact that Φ is standard. Therefore Theorem 1.11 says that $Ш_{0,0}$ is of weak-type $(1, 1)$ provided Φ is standard. We also have the converse for non-degenerate Haar shifts of complexity $(0, 0)$. As a consequence of these we have the following characterization: "Φ is standard if and only if all Haar multipliers are of weak-type $(1, 1)$". As observed above this can be applied to any 2-value generalized Haar system in $\mathbb{R}^d$. In particular, for an arbitrary measure in $\mathbb{R}$ such that $\mu(I) > 0$ for every $I \in \mathscr{D}$, all Haar multipliers of the form

$$Ш_{0,0} f(x) = \sum_{I \in \mathscr{D}} \alpha_I \langle f, h_I \rangle h_I(x), \qquad \text{with} \quad \sup_I |\alpha_I| < \infty,$$

are of weak-type $(1, 1)$. In higher dimensions, taking an arbitrary measure such that $\mu(Q) > 0$ for every $Q \in \mathscr{D}$, any Haar multiplier as above defined in terms of a 2-value generalized Haar system in $\mathbb{R}^d$ is of weak-type $(1, 1)$. We note that we cannot remove the assumption that the system is 2-value: in Section 3.2 we shall give an example of a generalized Haar system that is not standard and a Haar multiplier that is not of weak-type $(1, 1)$. All these comments can be generalized to measures without full support.

Example 1.15 (The dyadic Hilbert transform I)**.** For simplicity, we first suppose that $\mu(I) > 0$ for every $I \in \mathscr{D}$. The dyadic Hilbert transform in $\mathbb{R}$ can be seen as the non-degenerate Haar shift $H_{\mathscr{D}} = Ш_{0,1}$ with $\alpha^I_{I,I_\pm} = \mp 1$. Theorem 1.11 says that $H_{\mathscr{D}}$ is of weak-type $(1, 1)$ if and only if $\Xi(\mathcal{H}, \mathcal{H}; 0, 1) < \infty$, which in view of (1.3) is equivalent to the fact that μ is m-increasing. For the adjoint of the dyadic Hilbert transform $H^*_{\mathscr{D}} = Ш_{1,0}$ with $\alpha^I_{I_\pm,I} = \mp 1$ and this is a non-degenerate Haar shift. Again, Theorem 1.11 characterizes the weak-type $(1, 1)$ of $H^*_{\mathscr{D}}$ in terms of $\Xi(\mathcal{H}, \mathcal{H}; 1, 0) < \infty$, which this time rewrites into the property that μ is m-decreasing.

Example 1.16 (The dyadic Hilbert transform II)**.** We now consider the dyadic Hilbert transform but with respect to measures that may vanish. Let Φ be a generalized Haar system in $\mathbb{R}$ and let $\mathscr{D}_\Phi$ be as before. By the discussion above we may suppose that $\phi_I = h_I$ for every $I \in \mathscr{D}_\Phi$. Then, the corresponding dyadic Hilbert transform can be written as

$$H_{\mathscr{D},\Phi} f = \sum_{I \in \mathscr{D}} \langle f, \phi_I \rangle \big(\phi_{I_-} - \phi_{I_+}\big) = \sum_{I \in \mathscr{D}_\Phi : \widehat{I} \in \mathscr{D}_\Phi} \sigma(I) \langle f, h_{\widehat{I}} \rangle h_I,$$

where $\sigma(I) = 1$ if $I = (\widehat{I}\,)_-$ and $\sigma(I) = -1$ if $I = (\widehat{I}\,)_+$. As before we have that $H_{\mathscr{D},\Phi} = Ш_{0,1}$ is non-degenerate. Therefore its weak-type $(1,1)$ is characterized in terms of the finiteness of $\Xi(\Phi,\Phi;0,1)$. Thus, we obtain that

$$H_{\mathscr{D},\Phi} : L^1(\mu) \longrightarrow L^{1,\infty}(\mu) \quad \Longleftrightarrow \quad m(I) \le C\, m(\widehat{I}\,),\ I, \widehat{I} \in \mathscr{D}_\Phi.$$

Note that the latter condition says that μ is m-increasing on the family $\mathscr{D}_\Phi$ (so in particular the intervals with zero μ-measure or those with one child of zero μ-measure do not count).

For the adjoint of $H_{\mathscr{D},\Phi}$ we have

$$H^*_{\mathscr{D},\Phi} f(x) = \sum_{I\in\mathscr{D}} \sigma(I)\langle f, \phi_I\rangle \phi_{\widehat{I}} = \sum_{I\in\mathscr{D}_\Phi : \widehat{I}\in\mathscr{D}_\Phi} \sigma(I)\langle f, h_I\rangle h_{\widehat{I}}$$

and we can analogously obtain

$$H^*_{\mathscr{D},\Phi} : L^1(\mu) \longrightarrow L^{1,\infty}(\mu) \quad \Longleftrightarrow \quad m(\widehat{I}\,) \le C\, m(I),\ I, \widehat{I} \in \mathscr{D}_\Phi.$$

Example 1.17 (Haar Shifts in $\mathbb{R}$)**.** We start with the case $\mu(I) > 0$ for every $I \in \mathscr{D}$. Let us consider $Ш = Ш_{r,s}$ as in (1.6), that is, a Haar shift operator of complexity (r,s) defined in terms of the system $\mathcal{H}$. By Theorem 1.11 we know that $\Xi(\mathcal{H},\mathcal{H};r,s) < \infty$ is sufficient (and necessary if we knew that $Ш$ is non-degenerate) for the weak-type $(1,1)$. We can rewrite this condition as follows: $m(K) \lesssim m(J)$ for every $I \in \mathscr{D}$, $J \in \mathscr{D}_r(I)$, $K \in \mathscr{D}_s(I)$. If μ is m-equilibrated then $m(J) \approx m(I)$ and $m(K) \approx m(I)$ for every $I \in \mathscr{D}$, $J \in \mathscr{D}_r(I)$, $K \in \mathscr{D}_s(I)$. All these and (1.4) give at once $\Xi(\mathcal{H},\mathcal{H};r,s) < \infty$ for every r, $s \ge 0$. Thus, in dimension 1, the fact μ is m-equilibrated implies that every Haar shift operator is of weak-type $(1,1)$. We would like to recall that in Chapter 3 we shall construct measures that are m-equilibrated but are neither dyadically doubling nor of polynomial growth. Thus, Haar shift operators are a large family of (dyadic) Calderón-Zygmund operators obeying a weak-type $(1,1)$ bound with underlaying measures that do not satisfy those classical conditions.

For measures vanishing in some cubes, Theorem 1.11 gives us a sufficient (and often necessary) condition. However, it is not clear whether in such a case one can write that condition in terms of μ being m-equilibrated. We would need to be able to compare $m(K)$ and $m(J)$ for K and J as before with the additional condition that J, $K \in \mathscr{D}_\Phi$. Note that the fact that μ is m-equilibrated gives information about jumps of order 1 in the generations and it could happen that we cannot "connect" J and K with "1-jumps" within $\mathscr{D}_\Phi$. Take for instance $I = [0,1)$, $J = [0,4)$, $d\mu(x) = 1_{[0,1)\cup[2,4)}(x)\,dx$, $\Phi = \{h_I, h_J\}$ and $Ш_{2,0} = \langle f, h_I\rangle h_J$. Then Theorem 1.11 says that $Ш_{2,0}$ is of weak-type $(1,1)$ since $\Xi(\Phi,\Phi;2,0) = 4\,(m[0,4)\cdot m[0,1))^{1/2} = 4/\sqrt{6} < \infty$. However, $\mathscr{D}_\Phi = \{I, J\}$ and these two dyadic intervals are 2-generation separated.

Example 1.18 (Haar Shifts in $\mathbb{R}^d$ for 2-value generalized Haar systems)**.** Let us suppose that Φ and Ψ are 2-value generalized Haar systems. Write $E^{\pm}_Q$ (resp. $F^{\pm}_Q$)

for the sets associated with $\phi_Q \in \mathscr{D}_\Phi$ (resp. $\psi_Q \in \mathscr{D}_\Psi$), see (1.9). By (1.10) we have that $\Xi(\Phi, \Psi; r, s) < \infty$ if an only if μ satisfies

$$m_\Psi(S) = \frac{\mu(F_S^-)\mu(F_S^+)}{\mu(F_S^- \cup F_S^+)} \lesssim \frac{\mu(E_R^-)\mu(E_R^+)}{\mu(E_Q^- \cup E_R^+)} = m_\Phi(R) \tag{1.19}$$

for every $Q \in \mathscr{D}$, $R \in \mathscr{D}_r(Q)$, $S \in \mathscr{D}_s(Q)$, $R \in \mathscr{D}_\Phi$ and $S \in \mathscr{D}_\Psi$. Therefore Theorem 1.11 says that $Ш_{r,s}$ is of weak-type $(1, 1)$ provided μ satisfies the condition (1.19). The converse holds provided $Ш_{r,s}$ is non-degenerated.

The organization of this Part of the dissertation is as follows. Chapter 2 contains the proof of our main results as listed in this Introduction. In Chapter 3 we shall present some examples of measures in $\mathbb{R}$ that are not dyadically doubling (neither have polynomial growth) for which either the dyadic Hilbert transform, its adjoint or both are of weak-type $(1, 1)$. In the higher dimensional case we will review some constructions of Haar systems. We shall see that the obtained characterization depends also on the Haar system that we work with. That is, if we take a Haar shift operator (i.e., we fix the family of coefficients) and write it with different Haar systems, the conditions on the measure for the weak-type $(1, 1)$ depend on the chosen Haar system. Finally, in Chapter 4 we present some further results including non-cancellative Haar shift operators and therefore dyadic paraproducts, and some comments about the relationship between Haar shifts and martingale transforms.

Chapter 2

Proofs of the main results

Before proving our main results and for later use, we observe that for any measurable set $E \subset \mathbb{R}^d$ we have $\|1_E\|_{L^{1,\infty}(\mu)} = \|1_E\|_{L^1(\mu)} = \mu(E)$. This easily implies that if f is a simple function, then

$$\|f\|_{L^{1,\infty}(\mu)} \leq \|f\|_{L^1(\mu)} \leq \#\{f(x) : x \in \mathbb{R}^d\} \, \|f\|_{L^{1,\infty}(\mu)}. \tag{2.1}$$

2.1 A new Calderón-Zygmund decomposition

As pointed out before, we shall work with the standard dyadic filtration $\mathscr{D} = \bigcup_{k\in\mathbb{Z}} \mathscr{D}_k$ in $\mathbb{R}^d$, but all our results hold for any other dyadic lattice. If $k \geq 0$ is a nonnegative integer, we write $\mathscr{D}_k(Q)$ for the partition of Q into dyadic subcubes of side-length $2^{-k}\ell(Q)$ and $Q^{(k)}$ for its k-th dyadic ancestor, i.e., the only cube of side-length $2^k\ell(Q)$ that contains Q. The cubes in $\mathscr{D}_1(Q)$ are called dyadic children of Q and $\widehat{Q} = Q^{(1)}$ is the dyadic parent of Q.

By μ we will denote any positive Borel measure on $\mathbb{R}^d$ such that $\mu(Q) < \infty$ for all $Q \in \mathscr{D}$. Write $\mathscr{B}$ for the class of such measures. Once μ is fixed, we set for $Q \in \mathscr{D}$

$$\langle f \rangle_Q = \frac{1}{\mu(Q)} \int_Q f(x)\, d\mu(x) \qquad \text{with} \quad \langle f \rangle_Q = 0 \ \text{when} \ \mu(Q) = 0.$$

The dyadic maximal operator for $\mu \in \mathscr{B}$ is then $M_{\mathscr{D}}f(x) = \sup_{x\in Q\in\mathscr{D}} \langle |f| \rangle_Q$.

Let us write $\mathbb{R}^d_j$, $1 \leq j \leq 2^d$, for the d-dimensional quadrants in $\mathbb{R}^d$. It will be convenient to consider temporarily the subclass $\mathscr{B}_\infty$ of measures $\mu \in \mathscr{B}$ such that $\mu(\mathbb{R}^d_j) = \infty$ for all $1 \leq j \leq 2^d$. We will prove our main results under the assumption that $\mu \in \mathscr{B}_\infty$ and sketch in Section 2.4 the modifications needed to adapt our arguments for any $\mu \in \mathscr{B}$.

Assuming now that $\mu \in \mathscr{B}_\infty$, we know that $\langle |f| \rangle_Q \to 0$ as $\ell(Q) \to \infty$ whenever $f \in L^1(\mu)$. In particular, given any $\lambda > 0$, there exists a collection of disjoint maximal dyadic cubes $\{Q_j\}_j$ such that

$$\Omega_\lambda = \left\{ x \in \mathbb{R}^d : \ M_{\mathscr{D}}f(x) > \lambda \right\} = \bigcup_j Q_j,$$

where the cubes $\{Q_j\}_j$ are maximal in the sense that for all dyadic cubes $Q \supsetneq Q_j$ we have

$$\langle |f| \rangle_Q \leq \lambda < \langle |f| \rangle_{Q_j}, \tag{2.2}$$

Using this covering of the level set Ω_λ, we can reproduce the classical estimate to show the weak-type $(1,1)$ boundedness of the dyadic Hardy-Littlewood maximal operator. Note that maximal cubes have positive measure by construction.

Proof of Theorem 1.1. We are currently assuming that $\mu \in \mathscr{B}_\infty$, see Section 2.4 for the modifications needed in the general case. By construction, $f = g+b+\beta$. Moreover, the support and mean-zero conditions for b_j and β_j can be easily checked. On the other hand, since the cubes Q_j are pairwise disjoint

$$\sum_j \|b_j\|_{L^1(\mu)} \leq 2 \sum_j \int_{Q_j} |f(x)| \, d\mu(x) \leq 2 \, \|f\|_{L^1(\mu)}.$$

Similarly, by the maximality of the Calderón-Zygmund cubes, see (2.2), we obtain

$$\sum_j \|\beta_j\|_{L^1(\mu)} \leq \sum_j 2\big(\langle |f| \rangle_{Q_j} + \langle |f| \rangle_{\widehat{Q}_j}\big)\mu(Q_j) \leq 4 \sum_j \int_{Q_j} |f| \, d\mu \leq 4\|f\|_{L^1(\mu)}.$$

It remains to prove the norm inequalities for g. Write g_1, g_2 and g_3 for each of the terms defining g and let us estimate these in turn. It is immediate that $\|g_1\|_{L^1(\mu)} \leq \|f\|_{L^1(\mu)}$. Since $M_{\mathscr{D}}$ is of weak-type $(1,1)$, Lebesgue's differentiation theorem yields $\|g_1\|_{L^\infty(\mu)} \leq \|M_{\mathscr{D}} f \cdot 1_{\mathbb{R}^d \setminus \Omega_\lambda}\|_{L^\infty(\mu)} \leq \lambda$. The estimates for g_2 are similar. Since $\langle |f| \rangle_{\widehat{Q}_j} \leq \lambda$, we obtain

$$\|g_2\|_{L^1(\mu)} \leq \lambda \, \mu(\Omega_\lambda) \leq \|f\|_{L^1(\mu)} \qquad \text{and} \qquad \|g_2\|_{L^\infty(\mu)} \leq \lambda.$$

These estimates immediately yield the corresponding $L^p(\mu)$-estimates for g_1 and g_2.

The estimate for g_3 is not straightforward: each term in the sum is supported in $\widehat{Q}_j$, and these sets are not pairwise disjoint in general. In particular, an L^∞ estimate is not to be expected. However, we do have that

$$\begin{aligned} |g_3(x)| &\leq \sum_j \big(\langle |f| \rangle_{Q_j} + \langle |f| \rangle_{\widehat{Q}_j}\big) \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} 1_{\widehat{Q}_j}(x) \\ &\leq 2 \sum_j \Big(\int_{Q_j} |f(y)| \, d\mu(y) \Big) \frac{1}{\mu(\widehat{Q}_j)} 1_{\widehat{Q}_j}(x) =: 2\, Tf(x). \end{aligned}$$

The following lemma contains the relevant estimates for T:

Lemma 2.3. *Let $\{Q_j\}_j$ be a family of pairwise disjoint dyadic cubes and set*

$$Tf(x) = \sum_j \Big(\int_{Q_j} |f(y)| \, d\mu(y) \Big) \frac{1}{\mu(\widehat{Q}_j)} 1_{\widehat{Q}_j}(x).$$

For every $m \in \mathbb{N}$, T satisfies the estimate

$$\|Tf\|_{L^m(\mu)}^m \leq m! \Big(\sup_j \frac{1}{\mu(\widehat{Q}_j)} \int_{\widehat{Q}_j} |f(y)| \, d\mu(y) \Big)^{m-1} \int_{\bigcup_j Q_j} |f(x)| \, d\mu(x)$$

Assume this result momentarily. The case $m = 1$ implies that $\|g_3\|_{L^1(\mu)} \leq 2\,\|f\|_{L^1(\mu)}$. On the other hand, applying it for a general integer m, we get by (2.2)

$$\|g_3\|_{L^m(\mu)}^m \leq 2^m\, m!\, \lambda^{m-1}\, \|f\|_{L^1(\mu)}.$$

Now, if $1 < p < \infty$ is not an integer, we take $m = [p] + 1$ and let $0 < \theta < 1$ be such that $p = \theta + (1-\theta)m$. Then, by Hölder's inequality with indices $\frac{1}{\theta}$ and $\frac{1}{1-\theta}$, we obtain as desired

$$\|g_3\|_{L^p(\mu)}^p \leq \|g_3\|_{L^1(\mu)}^\theta \|g_3\|_{L^m(\mu)}^{(1-\theta)m} \leq 2^p (m!)^{\frac{p-1}{m-1}} \lambda^{p-1} \|f\|_{L^1(\mu)}.$$

□

Proof of Lemma 2.3. The case $m = 1$ is trivial. Let us proceed by induction and assume that the estimate for m holds. Write $\varphi_j = \frac{1}{\mu(\widehat{Q}_j)} \int_{Q_j} |f|\, d\mu$ and define the sets

$$\Lambda_k = \Big\{ (j_1, j_2, \ldots, j_{m+1}) \in \mathbb{N}^{m+1} : \widehat{Q}_{j_k} = \widehat{Q}_{j_1} \cap \widehat{Q}_{j_2} \cap \cdots \cap \widehat{Q}_{j_{m+1}} \Big\}$$
$$= \Big\{ (j_1, j_2, \ldots, j_{m+1}) \in \mathbb{N}^{m+1} : \widehat{Q}_{j_k} \subset \widehat{Q}_{j_1}, \ldots, \widehat{Q}_{j_{m+1}} \Big\}.$$

By symmetry we obtain

$$\|Tf\|_{L^{m+1}(\mu)}^{m+1} \leq \sum_{k=1}^{m+1} \sum_{\Lambda_k} \varphi_{j_1} \cdots \varphi_{j_{m+1}}\, \mu(\widehat{Q}_{j_1} \cap \cdots \cap \widehat{Q}_{j_{m+1}})$$
$$= (m+1) \sum_{\Lambda_{m+1}} \varphi_{j_1} \cdots \varphi_{j_m} \int_{Q_{j_{m+1}}} |f(x)|\, d\mu(x)$$
$$= (m+1) \sum_{j_1, \ldots, j_m} \varphi_{j_1} \cdots \varphi_{j_m} \sum_{j_{m+1} : (j_1, \ldots, j_{m+1}) \in \Lambda_{m+1}} \int_{Q_{j_{m+1}}} |f(x)|\, d\mu(x).$$

Notice that for a fixed m-tuple $(j_1, \ldots, j_m)$, it follows that

$$\bigcup_{j_{m+1} : (j_1, \ldots, j_{m+1}) \in \Lambda_{m+1}} Q_{j_{m+1}} \subset \bigcup_{j_{m+1} : (j_1, \ldots, j_{m+1}) \in \Lambda_{m+1}} \widehat{Q}_{j_{m+1}} \subset \widehat{Q}_{j_1} \cap \cdots \cap \widehat{Q}_{j_m},$$

and, moreover, the cubes in the first union are pairwise disjoint. Thus, the fact that $\widehat{Q}_{j_1} \cap \cdots \cap \widehat{Q}_{j_m} = \widehat{Q}_{j_i}$, for some $1 \leq i \leq m$, gives

$$\|Tf\|_{L^{m+1}(\mu)}^{m+1} \leq (m+1) \sum_{j_1, \ldots, j_m} \varphi_{j_1} \cdots \varphi_{j_m} \int_{\widehat{Q}_{j_1} \cap \cdots \cap \widehat{Q}_{j_m}} |f(x)|\, d\mu(x)$$
$$\leq (m+1) \Big(\sup_j \frac{1}{\mu(\widehat{Q}_j)} \int_{\widehat{Q}_j} |f|\, d\mu \Big) \sum_{j_1, \ldots, j_m} \varphi_{j_1} \cdots \varphi_{j_m}\, \mu(\widehat{Q}_{j_1} \cap \cdots \cap \widehat{Q}_{j_m})$$
$$= (m+1) \Big(\sup_j \frac{1}{\mu(\widehat{Q}_j)} \int_{\widehat{Q}_j} |f|\, d\mu \Big) \|Tf\|_{L^m(\mu)}^m.$$

This and the induction hypothesis yield at once the desired estimate and the proof is complete. □

The new Calderón-Zygmund decomposition in Theorem 1.1 can be used to obtain that some classical operators are of weak-type $(1,1)$ for general Borel measures: the ℓ^q-valued dyadic Hardy-Littlewood maximal function with $1 < q < \infty$, the dyadic square function, and 1-dimensional Haar multipliers. For the first operator, one needs a straightforward sequence-valued extension of the new Calderón-Zygmund decomposition and the reader is referred to [16]. Let us then look at the *dyadic square function*

$$\mathcal{S}_{\mathscr{D}} f(x) = \left(\sum_{Q \in \mathscr{D}} \left| \langle f \rangle_Q - \langle f \rangle_{\widehat{Q}} \right|^2 1_Q(x) \right)^{1/2}.$$

It is well-known that $\mathcal{S}_{\mathscr{D}}$ is bounded from $L^1(\mu)$ to $L^{1,\infty}(\mu)$ with a proof adopting a probabilistic point of view. However, using our Calderón-Zygmund decomposition one can reprove this result using harmonic analysis techniques as follows. We decompose $f = g + b + \beta$ as in Theorem 1.1. The estimate for the good part is standard using that $\mathcal{S}_{\mathscr{D}}$ is bounded on $L^2(\mu)$ and (a) in Theorem 1.1. For the bad terms, using the weak-type $(1,1)$ of $M_{\mathscr{D}}$, it suffices to restrict the level set to $\mathbb{R}^d \setminus \Omega_\lambda$. Theorem 1.1 parts (b) and (c) yield respectively that $(\mathcal{S}_{\mathscr{D}} b_j)\, 1_{\mathbb{R}^d \setminus Q_j} \equiv 0$ and $(\mathcal{S}_{\mathscr{D}} \beta_j)\, 1_{\mathbb{R}^d \setminus \widehat{Q}_j} \equiv 0$. Thus everything is reduced to the following

$$\begin{aligned} \mu\{x \in \mathbb{R}^d \setminus \Omega_\lambda : \mathcal{S}_{\mathscr{D}}\beta(x) > \lambda/2\} &\leq \frac{2}{\lambda} \sum_j \int_{\widehat{Q}_j \setminus Q_j} |\mathcal{S}_{\mathscr{D}} \beta_j| \, d\mu \\ = \frac{2}{\lambda} \sum_j |\langle f \rangle_{Q_j} - \langle f \rangle_{\widehat{Q}_j}| \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} \mu(\widehat{Q}_j \setminus Q_j) &\leq \frac{4}{\lambda} \sum_j \int_{Q_j} |f| \, d\mu \leq \frac{4}{\lambda} \|f\|_{L^1(\mu)}. \end{aligned}$$

All these ingredients allow one to conclude that $\mathcal{S}_{\mathscr{D}}$ is of weak-type $(1,1)$. Details are left to the reader

Finally, under the assumption that $0 < \mu(I) < \infty$ for all $I \in \mathscr{D}$, we consider the 1-dimensional Haar multipliers defined as

$$T_\alpha f(x) = \sum_{I \in \mathscr{D}} \alpha_I \langle f, h_I \rangle h_I(x), \qquad \sup_I |\alpha_I| < \infty.$$

This operator is bounded on $L^2(\mu)$ by orthonormality. A probabilistic point of view, see Section 4.3, yields that T_α is a dyadic martingale transform and therefore of weak-type $(1,1)$. Again, our new decomposition gives a proof with a "harmonic analysis" flavor. We first observe that $T_\alpha b_j(x) = 0$ for every $x \in \mathbb{R} \setminus Q_j$. Therefore, using Theorem 1.1 and proceeding as above everything reduces to the following estimate

$$\begin{aligned} \mu\{x \in \mathbb{R} : |T_\alpha \beta(x)| > \lambda/2\} &\leq \frac{2}{\lambda} \sum_j |\alpha_{\widehat{I}_j}| \, |\langle f \rangle_{I_j} - \langle f \rangle_{\widehat{I}_j}| \sqrt{m(\widehat{I}_j)} \|h_{\widehat{I}_j}\|_{L^1(\mu)} \\ &\leq \sup_I |\alpha_I| \frac{8}{\lambda} \sum_j \langle |f| \rangle_{I_j} m(\widehat{I}_j) \leq \sup_I |\alpha_I| \frac{8}{\lambda} \|f\|_{L^1(\mu)}, \end{aligned}$$

where we have used (2.5) below, (2.2), (1.3) and that $m(\widehat{I}_j) < \mu(I_j)$.

2.2 The dyadic Hilbert transform

In this section we prove Theorem 1.5. Although the estimates for $H_{\mathscr{D}}$ and $H_{\mathscr{D}}^*$ follow from Theorem 1.11 as explained above, we believe that it is worth giving the argument: the proofs for our toy models $H_{\mathscr{D}}$ and $H_{\mathscr{D}}^*$ are much simpler and have motivated our general result. We will skip, however, the last statement in the result since it follows from Theorem 1.11, as explained in Example 1.17, and interpolation.

Before starting the proof we observe that by the orthonormality of the system $\mathcal{H}$ we have

$$\|H_{\mathscr{D}} f\|_{L^2(\mu)}^2 = \sum_{I\in\mathscr{D}} |\langle f, h_{\widehat{I}}\rangle|^2 \leq 2\|f\|_{L^2(\mu)}^2. \tag{2.4}$$

Thus, $H_{\mathscr{D}}$ and $H_{\mathscr{D}}^*$ are bounded on $L^2(\mu)$.

Proof of Theorem 1.5 *(i)*. We first prove the necessity of μ being m-increasing. Take $f = h_I$ so that $H_{\mathscr{D}} f = h_{I_-} - h_{I_+}$. Using that h_I is constant on dyadic subintervals of I, (2.1) and that $H_{\mathscr{D}}$ is of weak-type $(1,1)$ we obtain that μ is m-increasing:

$$\begin{aligned}\left(\sqrt{m(I_-)} + \sqrt{m(I_+)}\right) &\approx \|h_{I_-}\|_{L^1(\mu)} + \|h_{I_+}\|_{L^1(\mu)} \\ &\approx \|H_{\mathscr{D}} h_I\|_{L^{1,\infty}(\mu)} \lesssim \|h_I\|_{L^1(\mu)} \approx \sqrt{m(I)}.\end{aligned}$$

Next we obtain that if μ is m-increasing then $H_{\mathscr{D}}$ is of weak-type $(1,1)$. In order to use Theorem 1.1, we shall assume that $\mu \in \mathscr{B}_\infty$, that is, $\mu[0,\infty) = \mu(-\infty,0) = \infty$. The general case will be considered in Section 2.4 below. Fix $\lambda > 0$ and decompose f by means of the Calderón-Zygmund decomposition in Theorem 1.1. Hence,

$$\begin{aligned}\mu\{x\in\mathbb{R} : |H_{\mathscr{D}} f(x)| > \lambda\} &\leq \mu\{x\in\mathbb{R} : |H_{\mathscr{D}} g(x)| > \lambda/3\} + \mu(\Omega_\lambda) \\ &\quad + \mu\{x\in\mathbb{R}\setminus\Omega_\lambda : |H_{\mathscr{D}} b(x)| > \lambda/3\} + \mu\{x\in\mathbb{R} : |H_{\mathscr{D}}\beta(x)| > \lambda/3\} \\ &= S_1 + S_2 + S_3 + S_4.\end{aligned}$$

Using the weak-type $(1,1)$ for $M_{\mathscr{D}}$, Theorem 1.1 part (a) and (2.4) it is standard to check that $S_1 + S_2 \leq (C/\lambda)\|f\|_{L^1(\mu)}$. Using that each b_j has vanishing integral and that h_I is constant on each $I_\pm$ it is easy to see that $H_{\mathscr{D}} b_j(x) = 0$ whenever $x \in \mathbb{R}\setminus I_j$ and thus $S_3 = 0$. To estimate S_4 we first observe that

$$\langle \beta_j, h_I\rangle = \sigma(I_j)(\langle f\rangle_{I_j} - \langle f\rangle_{\widehat{I}_j})\sqrt{m(\widehat{I}_j)}\,\delta_{\widehat{I}_j, I}. \tag{2.5}$$

This can be easily obtained using that β_j and h_I have vanishing integrals; that β_j is supported on $\widehat{I}_j$ and constant on each dyadic children of $\widehat{I}_j$; and that h_I is supported on I. Thus,

$$H_{\mathscr{D}}\beta_j = \sum_{I\in\mathscr{D}} \sigma(I)\langle \beta_j, h_{\widehat{I}}\rangle h_I = (\langle f\rangle_{I_j} - \langle f\rangle_{\widehat{I}_j})\sqrt{m(\widehat{I}_j)}\;(h_{I_j} - h_{I_j^b}),$$

where $I^b = \widehat{I} \setminus I \in \mathscr{D}$ is the dyadic brother of $I \in \mathscr{D}$. Using (2.2), (1.3), the assumption that μ is m-increasing and the fact that $m(\widehat{I}) \leq \mu(I)$ for every $I \in \mathscr{D}$ we conclude as desired

$$S_4 \leq \frac{3}{\lambda}\sum_j \|H_{\mathscr{D}}\beta_j\|_{L^1(\mu)} \lesssim \frac{1}{\lambda}\sum_j \langle |f|\rangle_{I_j} m(\widehat{I}_j) \lesssim \frac{1}{\lambda}\sum_j \int_{I_j} |f|\, d\mu \leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}.$$

This completes the proof of (i). □

Proof of Theorem 1.5 (ii). Take $f = h_I$ so that $H^*_{\mathscr{D}} f = \sigma(I)\, h_{\widehat{I}}$. Assuming that $H^*_{\mathscr{D}}$ is of weak-type $(1,1)$, we obtain by (2.1) that μ is m-decreasing:

$$2\sqrt{m(\widehat{I}\,)} = \|h_{\widehat{I}}\|_{L^1(\mu)} \approx \|h_{\widehat{I}}\|_{L^{1,\infty}(\mu)} = \|H^*_{\mathscr{D}} f\|_{L^{1,\infty}(\mu)} \lesssim \|h_I\|_{L^1(\mu)} \approx \sqrt{m(I)}.$$

To prove the converse we proceed as above. We shall assume that $\mu \in \mathscr{B}_\infty$, the general case will be considered in Section 2.4 below. The estimates for S_1 and S_2 are standard (since $H^*_{\mathscr{D}}$ is bounded on $L^2(\mu)$). For S_3 we first observe that if $x \in \mathbb{R} \setminus I_j$

$$H^*_{\mathscr{D}} b_j(x) = \sum_{I \in \mathscr{D}} \sigma(I)\langle b_j, h_I\rangle h_{\widehat{I}}(x) = \sigma(I_j)\langle b_j, h_{I_j}\rangle h_{\widehat{I}_j}(x) = \sigma(I_j)\langle f, h_{I_j}\rangle h_{\widehat{I}_j}(x).$$

We use this expression, (1.3) and that μ is m-decreasing:

$$\begin{aligned} S_3 &\leq \frac{3}{\lambda}\sum_j \int_{\mathbb{R}\setminus I_j} |H^*_{\mathscr{D}} b_j(x)|\, d\mu(x) \leq \frac{3}{\lambda}\sum_j \|h_{I_j}\|_{L^\infty(\mu)} \|h_{\widehat{I}_j}\|_{L^1(\mu)} \int_{I_j} |f|\, d\mu \\ &\approx \frac{1}{\lambda}\sum_j \sqrt{\frac{m(\widehat{I}_j)}{m(I_j)}} \int_{I_j} |f|\, d\mu \lesssim \frac{1}{\lambda}\sum_j \int_{I_j} |f|\, d\mu \leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}. \end{aligned}$$

To estimate S_4 we use (2.5),

$$H^*_{\mathscr{D}}\beta_j = \sum_{I \in \mathscr{D}} \sigma(I)\langle \beta_j, h_I\rangle h_{\widehat{I}} = \sigma(\widehat{I}_j)\sigma(I_j)\big(\langle f\rangle_{I_j} - \langle f\rangle_{\widehat{I}_j}\big)\sqrt{m(\widehat{I}_j)}\, h_{I_j^{(2)}},$$

where we recall that $I_j^{(2)}$ is the 2nd-dyadic ancestor of I_j. We use that μ is m-decreasing and $m(\widehat{I}\,) \leq \mu(I)$ to conclude that

$$\begin{aligned} S_4 &\leq \frac{3}{\lambda}\sum_j \|H^*_{\mathscr{D}}\beta_j\|_{L^1(\mu)} \leq \frac{12}{\lambda}\sum_j \langle |f|\rangle_{I_j} \sqrt{m(\widehat{I}_j) m\big(I_j^{(2)}\big)} \\ &\lesssim \frac{1}{\lambda}\sum_j \langle |f|\rangle_{I_j} m(\widehat{I}_j) \lesssim \frac{1}{\lambda}\sum_j \int_{I_j} |f|\, d\mu \leq \frac{1}{\lambda}\|f\|_{L^1(\mu)}. \end{aligned}$$

This completes the proof of (ii). □

Proof of Theorem 1.5 (iii). If μ is m-increasing we can use (i) to interpolate with the $L^2(\mu)$ bound to conclude estimates on $L^p(\mu)$ for every $1 < p < 2$. Conversely, we note that

$$\|h_I\|_{L^p(\mu)} = \sqrt{m(I)}\left(\frac{1}{\mu(I_-)^{p-1}} + \frac{1}{\mu(I_+)^{p-1}}\right)^{\frac{1}{p}} \approx m(I)^{\frac{1}{2}-\frac{1}{p'}}. \tag{2.6}$$

On the other hand, if we then assume that $H_{\mathscr{D}}$ is bounded on $L^p(\mu)$ we conclude that

$$m(I_-)^{\frac{1}{2}-\frac{1}{p'}} + m(I_+)^{\frac{1}{2}-\frac{1}{p'}} \approx \|h_{I_-} - h_{I_+}\|_{L^p(\mu)} = \|H_{\mathscr{D}} h_I\|_{L^p(\mu)} \lesssim \|h_I\|_{L^p(\mu)} \approx m(I)^{\frac{1}{2}-\frac{1}{p'}}.$$

This and the fact that $1 < p < 2$ imply that μ is m-increasing. □

Proof of Theorem 1.5 *(iv).* For $H^*_{\mathscr{D}}$ we can proceed in the same way. By interpolation and *(ii)*, μ being m-decreasing gives boundedness on $L^p(\mu)$ for $1 < p < 2$. Conversely, if $H^*_{\mathscr{D}}$ is bounded on $L^p(\mu)$ for some $1 < p < 2$, then

$$m(\widehat{I})^{\frac{1}{2}-\frac{1}{p'}} \approx \|h_{\widehat{I}}\|_{L^p(\mu)} = \|H^*_{\mathscr{D}} h_I\|_{L^p(\mu)} \lesssim \|h_I\|_{L^p(\mu)} \approx m(I)^{\frac{1}{2}-\frac{1}{p'}},$$

and therefore μ is m-decreasing. □

2.3 Haar shift operators in higher dimensions

We first see that $\text{Ш}_{r,s}$ is a bounded operator on $L^2(\mu)$. Following [29], we write

$$\text{Ш}_{r,s} f(x) = \sum_{Q\in\mathscr{D}} \Bigg(\sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S} \langle f, \phi_R\rangle \psi_S(x) \Bigg) =: \sum_{Q\in\mathscr{D}} A_Q f(x)$$

As observed before, Φ and Ψ are orthogonal systems. This implies

$$\begin{aligned} \|A_Q f\|^2_{L^2(\mu)} &= \sum_{S\in\mathscr{D}_s(Q)} \Bigg| \Bigg\langle f, \sum_{R\in\mathscr{D}_r(Q)} \alpha^Q_{R,S} \phi_R \Bigg\rangle \Bigg|^2 \|\psi_S\|^2_{L^2(\mu)} \\ &\le \|f\|^2_{L^2(\mu)} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \big|\alpha^Q_{R,S}\big|^2 \|\phi_R\|^2_{L^2(\mu)} \\ &\le 2^{(r+s)\,d} \Big(\sup_{Q,R,S} \big|\alpha^Q_{R,S}\big|^2 \Big) \|f\|^2_{L^2(\mu)}. \end{aligned} \tag{2.7}$$

For $Q \in \mathscr{D}$ and non-negative integer r, s, we write $P^r_{\Phi,Q}$ and $P^s_{\Psi,Q}$ for the projections

$$P^r_{\Phi,Q} f = \sum_{R\in\mathscr{D}_r(Q)} \langle f, \phi_R\rangle \phi_R, \qquad P^s_{\Psi,Q} f = \sum_{S\in\mathscr{D}_s(Q)} \langle f, \psi_S\rangle \psi_S.$$

We then have

$$\begin{aligned} P^s_{Q,\Psi} A_Q P^r_{Q,\Phi} f &= \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \langle A_Q(\phi_R), \psi_S\rangle \, \langle f, \phi_R\rangle \psi_S \\ &= \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \|\phi_R\|^2_{L^2(\mu)} \|\psi_S\|^2_{L^2(\mu)} \alpha^Q_{R,S} \langle f, \phi_R\rangle \psi_S = A_Q f. \end{aligned}$$

Fixed r and s, we notice that the projections $P^r_{\Phi,Q}$ are orthogonal on the index Q and the same occurs with $P^s_{\Psi,Q}$. Hence, by (2.7) and orthogonality

$$\begin{aligned}\|\text{Ш}_{r,s}f\|^2_{L^2(\mu)} &= \sum_{Q\in\mathscr{D}} \left\|P^s_{Q,\Psi}A_Q P^r_{Q,\Phi}f\right\|^2_{L^2(\mu)} \leq C \sum_{Q\in\mathscr{D}} \left\|P^r_{Q,\Phi}f\right\|^2_{L^2(\mu)} \\ &\leq C \sum_{Q\in\mathscr{D}} \sum_{R\in\mathscr{D}_r(Q)} \left|\langle f,\phi_R\rangle\right|^2 \leq C\,\|f\|^2_{L^2(\mu)},\end{aligned}$$

and this shows that $\text{Ш}_{r,s}$ is bounded on $L^2(\mu)$.

Proof of Theorem 1.11. We first show that $\Xi(\Phi,\Psi;r,s)<\infty$ implies that $\text{Ш}_{r,s}$ is of weak-type $(1,1)$. We shall assume that $\mu\in\mathscr{B}_\infty$ and the general case will be considered in Section 2.4 below. Let $\lambda>0$ be fixed and perform the Calderón-Zygmund decomposition in Theorem 1.1. Then,

$$\begin{aligned}\mu\{x\in\mathbb{R}^d : |\text{Ш}_{r,s}f(x)|>\lambda\} &\leq \mu\{x\in\mathbb{R}^d : |\text{Ш}_{r,s}g(x)|>\lambda/3\} + \mu(\Omega_\lambda) \\ &\quad + \mu\{x\in\mathbb{R}^d\setminus\Omega_\lambda : |\text{Ш}_{r,s}b(x)|>\lambda/3\} \\ &\quad + \mu\{x\in\mathbb{R}^d : |\text{Ш}_{r,s}\beta(x)|>\lambda/3\} \\ &= S_1+S_2+S_3+S_4.\end{aligned}$$

Using the weak-type $(1,1)$ for $M_{\mathscr{D}}$, Theorem 1.1 part (a) and that $\text{Ш}_{r,s}$ is bounded on $L^2(\mu)$ it is standard to check that

$$S_1+S_2 \leq \frac{C_{r,s}}{\lambda}\|f\|_{L^1(\mu)}.$$

We next consider S_3. Let $x\in\mathbb{R}^d\setminus Q_j$ and observe that

$$\begin{aligned}|\text{Ш}_{r,s}b_j(x)| &\leq \sup_{Q,R,S}|\alpha^Q_{R,S}| \sum_{Q\in\mathscr{D}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \left|\langle b_j,\phi_R\rangle\right| |\psi_S(x)| \\ &\lesssim \sum_{Q_j\subsetneq Q\subset Q_j^{(r)}} \sum_{\substack{R\in\mathscr{D}_r(Q), R\subset Q_j\\ S\in\mathscr{D}_s(Q)}} \left|\langle b_j,\phi_R\rangle\right| |\psi_S(x)|.\end{aligned} \tag{2.8}$$

In the last inequality we have used that each non-vanishing term leads to $Q_j\subsetneq Q\subset Q_j^{(r)}$ and $R\subset Q_j$ since ϕ_R is supported in R and constant on the children of R, b_j is supported in Q_j and has vanishing integral, and ψ_S is supported in S. This, Chebyshev's inequality and Theorem 1.1 imply

$$\begin{aligned}S_3 &\leq \frac{3}{\lambda}\sum_j \int_{\mathbb{R}^d\setminus Q_j} |\text{Ш}_{r,s}b_j|\,d\mu \\ &\lesssim \frac{1}{\lambda}\sum_j \sum_{Q_j\subsetneq Q\subset Q_j^{(r)}} \sum_{\substack{R\in\mathscr{D}_r(Q), R\subset Q_j\\ S\in\mathscr{D}_s(Q)}} \|b_j\|_{L^1(\mu)}\,\|\phi_R\|_{L^\infty(\mu)}\|\psi_S\|_{L^1(\mu)} \\ &\leq \frac{2^{(r+s)\,d}\,r}{\lambda}\,\Xi(\Phi,\Psi;r,s)\sum_j \|b_j\|_{L^1(\mu)} \leq \frac{C_{r,s}}{\lambda}\,\|f\|_{L^1(\mu)}.\end{aligned}$$

We finally estimate S_4. Let us observe that β_j and ϕ_R have vanishing integral. Besides, β_j is supported in $\widehat{Q}_j$ and constant on each dyadic child of $\widehat{Q}_j$, and ϕ_R is supported in R and constant on each dyadic child of R. All these imply that $\langle \beta_j, \phi_R \rangle = 0$ unless $R = \widehat{Q}_j$. Then,

$$(2.9) \quad |Ш_{r,s}\beta_j(x)| \le \sup_{Q,R,S} |\alpha^Q_{R,S}| \sum_{S\in\mathscr{D}_s(Q_j^{(r+1)})} \big|\langle \beta_j, \phi_{\widehat{Q}_j}\rangle\big| \, |\psi_S(x)| \lesssim \|\beta_j\|_{L^1(\mu)} \sum_{S\in\mathscr{D}_s(Q_j^{(r+1)})} \|\phi_{\widehat{Q}_j}\|_{L^\infty(\mu)} \, |\psi_S(x)|.$$

Therefore, Chebyshev's inequality and Theorem 1.1 imply

$$S_4 \le \frac{3}{\lambda}\sum_j \|Ш_{r,s}\beta_j\|_{L^1(\mu)} \lesssim \frac{1}{\lambda}\sum_j \|\beta_j\|_{L^1(\mu)} \sum_{S\in\mathscr{D}_s(Q_j^{(r+1)})} \|\phi_{\widehat{Q}_j}\|_{L^\infty(\mu)} \|\psi_S\|_{L^1(\mu)} \le \frac{2^{sd}}{\lambda}\Xi(\Phi,\Psi;r,s)\sum_j \|\beta_j\|_{L^1(\mu)} \le \frac{C_{r,s}}{\lambda}\,\|f\|_{L^1(\mu)}.$$

Gathering the obtained estimates this part of the proof is complete.

We now turn to the converse, that is, we show that if a non-degenerate Haar shift $Ш_{r,s}$ is of weak-type $(1,1)$ then $\Xi(\Phi,\Psi;r,s) < \infty$. For every $Q \in \mathscr{D}_\Phi$, we pick $Q_\infty \in \mathscr{D}_1(Q)$ such that ϕ_Q (which we recall that is constant on the dyadic children of Q) attains its maximum in Q_∞. Define

$$\widetilde{\varphi}_Q(x) = \big(\varphi_Q(x) - \langle \varphi_Q\rangle_Q\big)\, 1_Q(x), \quad \varphi_Q(x) = \operatorname{sgn}\big(\phi_Q(x)\big)\frac{1_{Q_\infty}(x)}{\mu(Q_\infty)},$$

where $\operatorname{sgn}(t) = t/|t|$ if $t \ne 0$ and $\operatorname{sgn}(0) = 0$. We note that by construction $\widetilde{\varphi}_Q$ is supported on Q, constant on dyadic children of Q and has vanishing integral. These imply that $\langle \widetilde{\varphi}_Q, \phi_R\rangle = 0$ if $Q \ne R$. Also,

$$\langle \widetilde{\varphi}_Q, \phi_Q\rangle = \langle \varphi_Q, \phi_Q\rangle = \frac{1}{\mu(Q_\infty)}\int_{Q_\infty} |\phi_Q(x)|\, d\mu(x) = \|\phi_Q\|_{L^\infty(\mu)},$$

where we have used that ϕ_Q has vanishing integral and is constant on the dyadic children of Q. On the other hand,

$$\|\widetilde{\varphi}_Q\|_{L^1(\mu)} \le 2\int_Q |\varphi_Q(x)|\, d\mu(x) = 2.$$

Let us now obtain that $\Xi(\Phi,\Psi;r,s) < \infty$. In the definition of $\Xi(\Phi,\Psi;r,s)$ we may clearly assume that $R \in \mathscr{D}_\Phi$ and $S \in \mathscr{D}_\Psi$. Thus, we fix $Q_0 \in \mathscr{D}$, $R_0 \in \mathscr{D}_r(Q_0)$ and $S_0 \in \mathscr{D}_s(Q_0)$ with $\|\phi_{R_0}\|_{L^2(\mu)} = 1$ and $\|\psi_{S_0}\|_{L^2(\mu)} = 1$. We use the properties of the function $\widetilde{\varphi}_{R_0}$ just defined and the non-degeneracy of $Ш_{r,s}$ to obtain that for every $x \in \mathbb{R}^d$

$$\begin{aligned}|\text{Ш}_{r,s}\widetilde{\varphi}_{R_0}(x)| &= \Big|\sum_{Q\in\mathscr{D}}\sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\,\langle\widetilde{\varphi}_{R_0},\phi_R\rangle\psi_S(x)\Big|\\ &= \|\phi_{R_0}\|_{L^\infty(\mu)}\Big|\sum_{S\in\mathscr{D}_s(Q_0)}\alpha^{Q_0}_{R_0,S}\psi_S(x)\Big| \geq \inf_{Q,R,S}|\alpha^Q_{R,S}|\,\|\phi_{R_0}\|_{L^\infty(\mu)}\,|\psi_{S_0}(x)|,\end{aligned}$$

where we have used that $\mathscr{D}_s(Q_0)$ is comprised of pairwise disjoint cubes. Using that $\text{Ш}_{r,s}$ is of weak-type $(1,1)$ and that ψ_{S_0} is constant on dyadic children of S_0, and (2.1) we obtain

$$\begin{aligned}\|\phi_{R_0}\|_{L^\infty(\mu)}\,\|\psi_{S_0}\|_{L^1(\mu)} &\approx \big\|\|\phi_{R_0}\|_{L^\infty(\mu)}\,\psi_{S_0}\big\|_{L^{1,\infty}(\mu)}\\ &\lesssim \|\text{Ш}_{r,s}\widetilde{\varphi}_{R_0}\|_{L^{1,\infty}(\mu)} \lesssim \|\widetilde{\varphi}_{R_0}\|_{L^1(\mu)} \leq 2.\end{aligned}$$

This immediately implies that $\Xi(\Phi,\Psi;r,s) < \infty$. ☐

Remark 2.10. From the previous proof and a standard homogeneity argument on the parameter $\|\text{Ш}_{r,s}\|_{\mathcal{B}(L_2(\mu))}$; the operator norm of $\text{Ш}_{r,s}$ on $L^2(\mu)$. We obtain that, under the conditions of Theorem 1.11,

$$\begin{aligned}\|\text{Ш}_{r,s}\|_{\mathcal{B}(L^1(\mu),L^{1,\infty}(\mu))} \leq C_0\,\big(&\|\text{Ш}_{r,s}\|_{\mathcal{B}(L_2(\mu))}\\ &+ 2^{s\,d}\,(r\,2^{r\,d}+1)\,\Xi(\Phi,\Psi;r,s)\sup_{Q,R,S}|\alpha^Q_{R,S}|\big),\end{aligned}$$

where C_0 is a universal constant (independent of the dimension, for instance, in the previous argument one can safely take $C_0 \leq 217$.)

Remark 2.11. One can obtain an analog of Theorem 1.5 parts (*iii*), (*iv*) for non-degenerate Haar shift operators defined in terms of 2-value Haar systems Φ and Ψ. To be more precise, let $\text{Ш}_{r,s}$ be a non-degenerate Haar shift of complexity (r,s) associated to two 2-value generalized Haar systems. If $\text{Ш}_{r,s}$ is of weak-type (p,p) for some $1 < p < 2$ then $\Xi(\Phi,\Psi;r,s) < \infty$. The proof is very similar to what we did for the dyadic Hilbert transform. Fix $Q_0 \in \mathscr{D}$, $R_0 \in \mathscr{D}_r(Q_0)$, $S_0 \in \mathscr{D}_s(Q_0)$. Then, using that the cubes in $\mathscr{D}_s(Q_0)$ are pairwise disjoint,

$$\begin{aligned}|\text{Ш}_{r,s}\phi_{R_0}(x)| &= \Big|\sum_{Q\in\mathscr{D}}\sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\,\delta_{R_0,R}\,\psi_S(x)\Big|\\ &= \Big|\sum_{S\in\mathscr{D}_s(Q_0)}\alpha^{Q_0}_{R_0,S}\psi_S(x)\Big| \geq \inf_{Q,R,S}|\alpha^Q_{R,S}|\,|\psi_{S_0}(x)|.\end{aligned}$$

Using that $\text{Ш}_{r,s}$ is of weak-type (p,p) and that ψ_{S_0} is constant on dyadic children of S_0 we obtain

$$\|\psi_{S_0}\|_{L^p(\mu)} \lesssim \|\text{Ш}_{r,s}\phi_{R_0}\|_{L^{p,\infty}(\mu)} \lesssim \|\phi_{R_0}\|_{L^p(\mu)}.$$

Also, by (1.9), (1.10) and proceeding as in (2.6) we obtain

$$\|\psi_{S_0}\|_{L^1(\mu)}^{1-\frac{2}{p'}} \approx m_\Psi(S_0)^{\frac{1}{2}-\frac{1}{p'}} \approx \|\psi_{S_0}\|_{L^p(\mu)} \lesssim \|\phi_{R_0}\|_{L^p(\mu)} \approx m_\Phi(R_0)^{\frac{1}{2}-\frac{1}{p'}} \approx \|\phi_{R_0}\|_{L^\infty(\mu)}^{-(1-\frac{2}{p'})}.$$

This easily implies that $\Xi(\Phi, \Psi; r, s) < \infty$.

2.4 The case $\mu \in \mathscr{B} \setminus \mathscr{B}_\infty$

The Calderón-Zygmund decomposition in Theorem 1.1 has been obtained under the assumption that every d-dimensional quadrant has infinite μ-measure, $\mu \in \mathscr{B}_\infty$ in the language of Section 2.1. Also, Theorems 1.5 and 1.11 have been proved under this assumption. Here we discuss how to remove this constraint and work with arbitrary measures in $\mathscr{B}$.

Due to the nature of the standard dyadic grid, $\mathbb{R}^d$ splits naturally in 2^d components each of them being a d-dimensional quadrant. Let $\mathbb{R}^d_k$, $1 \le k \le 2^d$, denote the d-dimensional quadrants in $\mathbb{R}^d$: that is, the sets $\mathbb{R}^\pm \times \cdots \times \mathbb{R}^\pm$ where $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{R}^- = (-\infty, 0)$. Let $\mathscr{D}^k$ be the collection of dyadic cubes contained in $\mathbb{R}^d_k$. We set

$$M_{\mathscr{D}^k} f(x) = \sup_{x \in Q \in \mathscr{D}^k} \frac{1}{|Q|} \int_Q |f(y)|\, d\mu(y) = M_{\mathscr{D}}\big(f\, 1_{\mathbb{R}^d_k}\big)(x)\, 1_{\mathbb{R}^d_k}(x).$$

Hence, given a function f we have that

$$f(x) = \sum_{k=1}^{2^d} f(x)\, 1_{\mathbb{R}^d_k}(x), \qquad M_{\mathscr{D}} f(x) = \sum_{k=1}^{2^d} M_{\mathscr{D}^k} f(x)\, 1_{\mathbb{R}^d_k}(x),$$

and in each sum there is at most only one non-zero term. Because of this decomposition, to extend our results it will suffice to assume that f is supported in some $\mathbb{R}^d_k$ and obtain the corresponding decompositions and estimates in $\mathbb{R}^d_k$.

Notice that if f is supported in $\mathbb{R}^d_k$, $M_{\mathscr{D}} f = M_{\mathscr{D}^k} f$ and this function is supported in $\mathbb{R}^d_k$. In particular, for any $\lambda > 0$,

$$\Omega_\lambda = \{x \in \mathbb{R}^d : M_{\mathscr{D}} f(x) > \lambda\} = \{x \in \mathbb{R}^d_k : M_{\mathscr{D}^k} f(x) > \lambda\},$$

and so any decomposition of this set will consist of cubes in $\mathscr{D}^k$. We modify our notation and define $\langle f\rangle_{\mathbb{R}^d_k} = \frac{1}{\mu(\mathbb{R}^d_k)} \int_{\mathbb{R}^d_k} f\, d\mu$ if $\mu(\mathbb{R}^d_k) < \infty$ and $\langle f\rangle_{\mathbb{R}^d_k} = 0$ if $\mu(\mathbb{R}^d_k) = \infty$.

The following result is the analog of Theorem 1.1.

Theorem 2.12. *Given $1 \le k \le 2^d$, $\mu \in \mathscr{B}$ and $f \in L^1(\mu)$ with $\operatorname{supp} f \subset \mathbb{R}^d_k$, so that for every $\lambda > \langle |f|\rangle_{\mathbb{R}^d_k}$ there exists a covering of $\Omega_\lambda = \{M_{\mathscr{D}} f > \lambda\}$ by maximal dyadic cubes $\{Q_j\}_j \subset \mathscr{D}^k$. Then, we may find a decomposition $f = g + b + \beta$ with g, b and β as defined in Theorem 1.1 and satisfying the very same properties.*

Proof. If $\mu(\mathbb{R}^d_k) = \infty$, then the proof given above goes through without change. If $\mu(\mathbb{R}^d_k) < \infty$, then in the notation used above, $\langle |f| \rangle_Q \to \langle |f| \rangle_{\mathbb{R}^d_k} < \lambda$ as $\ell(Q) \to \infty$ for $Q \in \mathscr{D}^k$. Hence, if $Q \in \mathscr{D}^k$ is such that $\langle |f| \rangle_Q > \lambda$, then Q must be contained in a maximal cube with the same property. Hence, we can easily form the collection of maximal cubes $\{Q_j\}_j \subset \mathscr{D}^k$. We observe that this covering gives the right estimate for the level sets of $M_{\mathscr{D}} f = M_{\mathscr{D}^k} f$ if $\lambda > \langle |f| \rangle_{\mathbb{R}^d_k}$. For $0 < \lambda \leq \langle |f| \rangle_{\mathbb{R}^d_k}$ we immediately have

$$\mu(\Omega_\lambda) \leq \mu(\mathbb{R}^d_k) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d_k} |f(x)| \, d\mu(x).$$

These in turn imply that $M_{\mathscr{D}^j}$ is of weak-type $(1,1)$. From here we repeat the arguments in the proof Theorem 1.1 to complete the proof without change. □

Proof of Theorems 1.5 and 1.11 for $\mu \in \mathscr{B}$. We obtain the weak-type $(1,1)$ estimate for $\text{Ш}_{r,s}$, the arguments for $H_{\mathscr{D}}$ and $H^*_{\mathscr{D}}$ are identical.

Suppose first that $\operatorname{supp} f \subset \mathbb{R}^d_k$ with $1 \leq k \leq 2^d$. If $\mu(\mathbb{R}^d_k) = \infty$, then the arguments above go through without change. Assume otherwise that $\mu(\mathbb{R}^d_k) < \infty$. If $\lambda > \langle |f| \rangle_{\mathbb{R}^d_k}$ then we repeat the same proof using Theorem 2.12 in place of Theorem 1.1. If $0 < \lambda \leq \langle |f| \rangle_{\mathbb{R}^d_k}$ we cannot form the Calderón-Zygmund decomposition. Nevertheless, the estimate is immediate after observing that by construction $\text{Ш}_{r,s} f$ is supported in $\mathbb{R}^d_k$ since so is f. Then,

$$\mu(\{x \in \mathbb{R}^d : |\text{Ш}_{r,s} f(x)| > \lambda\}) \leq \mu(\mathbb{R}^d_k) \leq \frac{1}{\lambda} \int_{\mathbb{R}^d_k} |f(x)| \, d\mu(x).$$

To prove the weak-type estimate in the general case, fix f and write $f = \sum_{k=1}^{2^d} f \, 1_{\mathbb{R}^d_k}$. By construction we then have

$$\text{Ш}_{r,s} f(x) = \sum_{k=1}^{2^d} \text{Ш}_{r,s}(f \, 1_{\mathbb{R}^d_k})(x) \, 1_{\mathbb{R}^d_k}(x).$$

Therefore, by the above argument applied to each $\mathbb{R}^d_k$

$$\begin{aligned} \mu(\{x \in \mathbb{R}^d : |\text{Ш}_{r,s} f(x)| > \lambda\}) &= \sum_{k=1}^{2^d} \mu(\{x \in \mathbb{R}^d_k : |\text{Ш}_{r,s}(f \, 1_{\mathbb{R}^d_k})(x)| > \lambda\}) \\ &\lesssim \frac{1}{\lambda} \sum_{k=1}^{2^d} \int_{\mathbb{R}^d_k} |f(x)| \, d\mu(x) = \frac{1}{\lambda} \int_{\mathbb{R}^d} |f(x)| \, d\mu(x), \end{aligned}$$

and conclude as desired □

Remark 2.13. As explained above, the standard dyadic grid splits $\mathbb{R}^d$ in 2^d components, each of them being a d-dimensional quadrant. These components are defined with respect to the property that if a given cube is in a fixed component, all of its relatives (ascendants and descendants) remain in the same component. This connectivity property depends on the dyadic grid chosen, and one can find other dyadic grids with other number of components. Let us work for simplicity in

$\mathbb{R}$ and suppose that we want to find dyadic grids "generated" by $I_0 = [0,1)$. We need to give the ascendants of I_0, say I_k, $k \le -1$. Once we have them, we translate each I_k by $j\,2^{-k}$ with $j \in \mathbb{Z}$ and these define the cubes of the fixed generation 2^{-k} for $k \le 0$. The small cubes are obtained by subdivision. Hence, in the present scenario, we only need to define the I_k's. Let us start by finding the parent of I_0: we just have two choices $[0,2)$ or $[-1,1)$, and once we choose one, which we call I_{-1}, we need to pass to the next level and decide which is the parent of I_{-1}, for which again we have two choices. Continuing this we have a sequence of cubes I_k, $k \le 0$, which determines the dyadic grid. In the classical dyadic grid one always choose the parent of I_k "to the right", that is, so that I_k is the left half of I_{k-1}. This eventually gives two components. One way to obtain a dyadic grid with one component is to alternatively take parents "to the left" and "to the right". That is, if we take $I_0 = [0,1)$, $I_{-1} = [-1,1)$, $I_{-2} = [-1,3)$, $I_{-3} = [-5,3)$, ... we obtain one component. More precisely, take the family of intervals $I_k = [0, 2^{-k})$ for $k \ge 0$ and for $k \le -1$ let $I_k = [a_k - 2^{-k}, a_k)$ with $a_k = (2^{-k}+1)/3$ if $-k$ is odd and $a_k = (2^{-k+1}+1)/3$ if $-k$ is even. Notice that $\{I_k\}_{k\in\mathbb{Z}}$ is a decreasing family of intervals of dyadic side-length. Notice that each I_k is one of the halves of I_{k-1}. Using I_k we generate the dyadic cubes of generation 2^{-k} by taking the intervals $I_{j,k} = j\,2^{-k} + I_k$ with $j \in \mathbb{Z}$. Finally we set $\widetilde{\mathscr{D}} = \{I_{j,k} : j,k \in \mathbb{Z}\}$. This is clearly a dyadic grid in $\mathbb{R}$. Let us observe that $a_k \to \infty$ and $a_k - 2^{-k} \to -\infty$ as $k \to -\infty$ and therefore $I_k \nearrow \mathbb{R}$ as $k \to -\infty$. This means that this dyadic grid induces just one component (in the sense described above) since for any I_1, $I_2 \in \widetilde{\mathscr{D}}$ we can find a large k such that both I_1 and I_2 are contained in $I_{-k} \in \widetilde{\mathscr{D}}$. We finally observe that the dyadic grids with one component occur more often than those with two, as the classical dyadic grid. Indeed, if at each generation we select randomly the parent (among the possibilities "to the left" and "to the right"), the probability of ending with a system with one component is 1.

Chapter 3

Examples of measures and Haar systems

3.1 The one dimensional case

As we have seen above the 1-dimensional case is somehow special since the Haar system is “uniquely” determined. Let us work with the measures in Theorem 1.5, that is, μ is a Borel measure in $\mathbb{R}$ with $0 < \mu(I) < \infty$ for every $I \in \mathscr{D}$. As we have seen in that result, m-increasing, m-decreasing and m-equilibrated measures are the ones governing the boundedness of $H_{\mathscr{D}}$, $H_{\mathscr{D}}^*$ and Haar shift operators. We are going to describe some examples of non-standard measures satisfying those conditions.

We can easily obtain examples of m-equilibrated measures. Let μ be a dyadically doubling measure, i.e., $\mu(\widehat{I}) \lesssim \mu(I)$ for all $I \in \mathscr{D}$ where $\widehat{I}$ is the dyadic parent of I. Then, $m(I) \approx \mu(I)$ and clearly μ is m-equilibrated. This applies straightforwardly to the Lebesgue measure.

We next construct some measures that are m-increasing, m-decreasing or m-equilibrated without being dyadically doubling or of polynomial growth. Set $d\nu = dx 1_{\mathbb{R}\setminus[0,1)} + d\mu$, where μ is a measure supported on the interval $[0,1)$ defined as follows. Let $\{I_k\}_{k\geq 0}$ be the decreasing sequence of dyadic intervals $I_k = [0, 2^{-k})$ and let $\{a_k\}_{k\geq 1}$ be such that $0 < a_k < 1$ and $a_1 = 1/2$. Set $b_k = 1 - a_k$. Define μ recursively by setting $\mu(I_0) = 1$ and

$$\mu(I_k) = a_k \mu(\widehat{I}_k) = a_k \mu(I_{k-1}) \quad \text{and} \quad \mu(I_k^b) = b_k \mu(\widehat{I}_k) = b_k \mu(I_{k-1}), \tag{3.1}$$

for $k \geq 1$, where we recall that $I_k^b = [2^{-k}, 2^{-k+1})$ is the dyadic brother of I_k. On I_k^b, μ is taken to be uniform, i.e., $\mu(J) = \mu(I_k^b)\,|J|/|I_k^b|$ for any $J \in \mathscr{D}$, $J \subset I_k^b$. We illustrate this procedure in Figure 3.1.

By construction, if $I \cap I_0 = \emptyset$ or $I_0 \subset I$ we have

$$\frac{m(I)}{m(\widehat{I})} = \frac{|I|/4}{|\widehat{I}|/4} = \frac{1}{2}$$

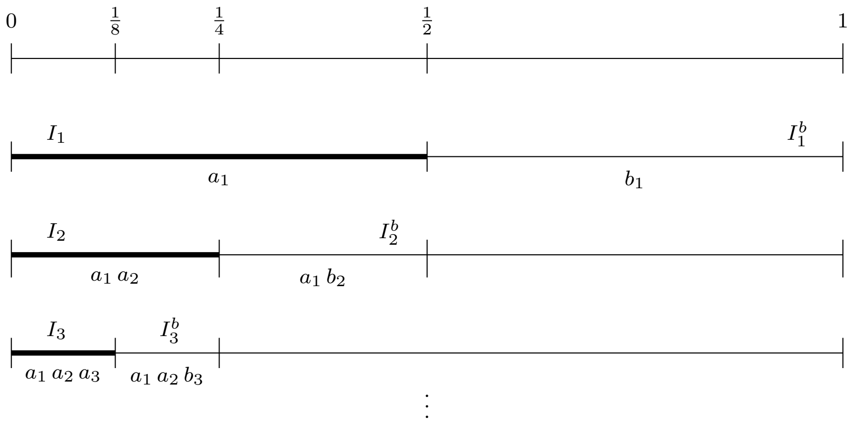

Figure 3.1: Construction of μ

Also, if $I \in \mathscr{D}$ and $\widehat{I} \subset I_k^b$ for some $k \geq 1$ then

$$\frac{m(I)}{m(\widehat{I}\,)} = \frac{\frac{\mu(I_k^b)\,|I|}{4\,|I_k^b|}}{\frac{\mu(I_k^b)\,|\widehat{I}|}{4\,|I_k^b|}} = \frac{1}{2}$$

In the remainder cases we always have that $\widehat{I} = \widehat{I}_k$ for some $k \geq 1$ and I is either I_k or I_k^b. Note that by (3.1) we get

$$m(I_k^b) = \frac{\mu\big((I_k^b)_-\big)\mu\big((I_k^b)_+\big)}{\mu(I_k^b)} = \frac{\mu(I_k^b)}{4} = \frac{1}{4}\,b_k\,\mu(\widehat{I}_k),$$

$$m(I_k) = \frac{\mu\big((I_k)_-\big)\mu\big((I_k)_+\big)}{\mu(I_k)} = \frac{\mu(I_{k+1})\mu(I_{k+1}^b)}{\mu(I_k)} = a_{k+1}\,b_{k+1}\,a_k\,\mu(\widehat{I}_k),$$

$$m(\widehat{I}_k) = \frac{\mu(I_k)\mu(I_k^b)}{\mu(\widehat{I}_k)} = a_k\,b_k\,\mu(\widehat{I}_k).$$

Hence,

$$\frac{m(I_k)}{m(\widehat{I}_k)} = \frac{a_{k+1}b_{k+1}}{b_k} \qquad \text{and} \qquad \frac{m(I_k^b)}{m(\widehat{I}_k)} = \frac{1}{4a_k}. \tag{3.2}$$

We now proceed to study the previous ratios associated to measures given by particular choices of the defining sequences $\{a_k\}_k$ and $\{b_k\}_k$. We shall construct three non-dyadically doubling and of non-polynomial growth measures. In the first example μ is m-equilibrated, in the second μ is m-increasing and is not m-decreasing, in the third μ is m-decreasing and is not m-increasing. Finally, in the last example we give a measure μ which is of polynomial growth but is neither dyadically doubling, nor m-increasing, nor m-decreasing.

(a) Let $b_k = \frac{1}{k}$ for $k \geq 2$. The measure μ is **non-dyadically doubling** since by (3.1), if $k \geq 2$

$$\frac{\mu(\widehat{I}_k)}{\mu(I_k^b)} = \frac{1}{b_k} = k \underset{k\to\infty}{\longrightarrow} \infty.$$

From substituting a_k and b_k in (3.2) we get that,

$$\frac{m(I_k)}{m(\widehat{I}_k)} = \Big(1 - \frac{1}{k+1}\Big)\frac{k}{k+1}, \qquad \frac{m(I_k^b)}{m(\widehat{I}_k)} = \frac{1}{4\Big(1-\frac{1}{k}\Big)}.$$

Both sequences are bounded from above and from below, which implies that μ is **m-equilibrated**. Besides, for $0 < t < \infty$

$$\frac{\mu(I_k)}{|I_k|^t} = \frac{a_1 \dots a_k}{2^{-kt}} = \frac{1}{2}\frac{2^{kt}}{k} \underset{k\to\infty}{\longrightarrow} \infty.$$

Thus, μ does **not have polynomial growth**.

(b) Set $b_k = 2^{-k^2}$. In this case μ is **non-dyadically doubling**, since by (3.1)

$$\frac{\mu(\widehat{I}_k)}{\mu(I_k^b)} = 2^{k^2} \underset{k\to\infty}{\longrightarrow} \infty.$$

Since $\frac{1}{2} \leq a_k < 1$, by (3.2) we get that $m(\widehat{I}_k) \approx m(I_k^b)$. However,

$$4 < \frac{m(\widehat{I}_k)}{m(I_k)} = \frac{2^{-k^2}}{\big(1-2^{-(k+1)^2}\big)2^{-(k+1)^2}} \underset{k\to\infty}{\longrightarrow} \infty.$$

Thus, μ is **m-increasing** but is **not m-decreasing**. Notice that for $t > 1$,

$$\frac{\mu(I_k)}{|I_k|^t} = \frac{a_1 \dots a_k}{2^{-kt}} = 2^{kt}\prod_{j=1}^{k}(1-2^{-j^2}) \geq 2^{kt}\Big(1-\frac{1}{2}\Big)^k = 2^{k(t-1)} \underset{k\to\infty}{\longrightarrow} \infty.$$

For $0 < t \leq 1$, let n and m be positive integers such that $\frac{1}{n+1} < t \leq \frac{1}{n}$ and $k = 2(n+1)m$. Then, $2^{kt} > 2^{2m}$ and

$$\begin{aligned}\frac{\mu(I_k)}{|I_k|^t} &\geq \Big(2^m\prod_{j=1}^{m}(1-2^{-j^2})\Big)\cdot\Big(2^m\prod_{j=m+1}^{k}(1-2^{-j^2})\Big) \geq 2^m\prod_{j=m+1}^{k}(1-2^{-j^2}) \\ &\geq 2^m\big(1-2^{-m^2}\big)^{k-m} = \Big(2\big(1-2^{-m^2}\big)^{(2(n+1)-1)}\Big)^m \underset{m\to\infty}{\longrightarrow} \infty\end{aligned}$$

Thus, μ does **not have polynomial growth**.

(c) Let $n \in \mathbb{N}$ and set $f(n) = \frac{n(n+1)}{2}$. For $k \geq 2$ define

$$b_k = \frac{1}{2}\frac{1}{k - f(n-1)},$$

where $n \geq 2$ is such that $f(n-1) < k \leq f(n)$. Fix $n \geq 2$ and $f(n-1) < k \leq f(n)$. Then $k = f(n-1)+r$, with $1 \leq r \leq f(n)-f(n-1) = n$ and $b_k = 1/(2r)$. Hence,

$$\frac{1}{2n} \leq b_k \leq \frac{1}{2}.$$

and $\liminf_{k\to\infty} b_k = 0$. By (3.1) this choice of b_k defines a **non-doubling** measure. Since $\frac{1}{2} \le a_k < 1$, by (3.2) we get that $m(\widehat{I}_k) \approx m(I_k^b)$ for every k. On the other hand,

$$\frac{b_{k+1}}{b_k} = \begin{cases} \dfrac{k - f(n-1)}{k+1-f(n-1)} = \dfrac{r}{r+1} \approx 1, & \text{if } k < f(n); \\ \dfrac{k - f(n-1)}{k+1-f(n)} = n \to \infty, & \text{if } k = f(n). \end{cases}$$

Hence, by (3.2) μ is **not** m**-increasing**. However, μ is m**-decreasing** since $b_k/b_{k+1} \le 2$.

We finally see that μ has no polynomial growth. We start with the case $t > 1$. For $s, j \ge 2$ such that $f(s-1) < j = f(s-1) + r \le f(s)$ with $1 \le r \le s$, we have that $a_j = \frac{2r-1}{2r}$. Then, if $k = f(n)$

$$\begin{aligned} \frac{\mu(I_k)}{|I_k|^t} = \frac{a_1 \dots a_k}{2^{-kt}} &= 2^{kt} \prod_{s=1}^{n} \prod_{r=1}^{s} \frac{2r-1}{2r} \\ &= 2^{k(t-1)} \prod_{s=1}^{n} \prod_{r=1}^{s} \frac{2r-1}{r} \\ &\ge 2^{k(t-1)} = 2^{f(n)\,(t-1)} \underset{n\to\infty}{\longrightarrow} \infty. \end{aligned}$$

Consider now $0 < t \le 1$ and let $m \ge 2$ be the unique integer such that $\frac{2}{f(m)} < t \le \frac{2}{f(m-1)}$. Let $k = f(n)$ with n large enough so that $k \ge f(m)^2$. Then $2^{kt} \ge 2^{2f(m)}$ and

$$\begin{aligned} \frac{\mu(I_k)}{|I_k|^t} &\ge \Big(2^{f(m)} \prod_{s=1}^{m} \prod_{r=1}^{s} \frac{2r-1}{2r}\Big) \cdot \Big(2^{f(m)} \prod_{s=m+1}^{n} \prod_{r=1}^{s} \frac{2r-1}{2r}\Big) \\ &\ge 2^{f(m)} \prod_{s=m+1}^{n} \prod_{r=1}^{s} \frac{2r-1}{2r} \\ &= 2^{f(m)} \prod_{s=m+1}^{n} \frac{(2s)!}{2^{2s}(s!)^2} \\ &= 2^{f(m)} 2^{-2(f(n)-f(m))} \prod_{s=m+1}^{n} \frac{(2s)!}{(s!)^2} \\ &\ge 2^{f(m)} 2^{-2(f(n)-f(m))} 2^{3(f(n)-f(m))} = 2^{f(n)} \underset{n\to\infty}{\longrightarrow} \infty, \end{aligned}$$

where in the last inequality we have used that $(2s)!/(s!)^2$ is increasing and therefore bounded from below by 8. Thus, μ does **not have polynomial growth**.

(*d*) Let $b_2 = b_3 = 1/2$, and for every $k \ge 2$, $b_{2k} = 1/k$, $b_{2k+1} = 1 - 1/k$. The measure μ is **non-dyadically doubling** since by (3.1), if $k \ge 2$, then

$$\frac{\mu(\widehat{I}_{2k})}{\mu(I_{2k}^b)} = \frac{1}{b_{2k}} = k \underset{k\to\infty}{\longrightarrow} \infty.$$

From substituting a_k and b_k in (3.2) we get that,

$$\frac{m(I^b_{2k+1})}{m(\widehat{I}_{2k+1})} = \frac{1}{4a_{2k+1}} = \frac{k}{4} \xrightarrow[k\to\infty]{} \infty,$$

which implies that μ is **not** m**-increasing**. Also,

$$\frac{m(\widehat{I}_{2k+1})}{m(I_{2k+1})} = \frac{b_{2k+1}}{a_{2(k+1)}\, b_{2(k+1)}} = \frac{(k+1)^2\,(k-1)}{k^2} \xrightarrow[k\to\infty]{} \infty,$$

which implies that μ is **not** m**-decreasing**.

We finally see that μ **has linear growth**, that is, $\mu(I)/|I| \le C$ for every I. We first notice that it suffices to consider $I \in \mathscr{D}$ since any arbitrary interval J can be covered by a bounded number of $I \in \mathscr{D}$ with $|I| \approx |J|$. Let us now fix $I \in \mathscr{D}$. The cases $I \cap [0,1) = \emptyset$ or $[0,1) \subset I$ are trivial since $\mu(I) = |I|$. Suppose next that $I \subsetneq [0,1)$. Then, either $I = I_k$ or $I \subset I^b_k$ for some $k \ge 1$. In the latter scenario we have that by construction $\mu(I)/|I| = \mu(I^b_k)/|I^b_k|$, therefore we only have to consider $I = I_k$ or $I = I^b_k$ for k large. Let us fix $k \ge 6$. Notice that

$$\frac{\mu(I^b_k)}{|I^b_k|} = \frac{\mu(I_k)}{|I_k|}\,\frac{b_k}{a_k} = \frac{\mu(I_{k-1})}{|I_{k-1}|}\,2\,b_k.$$

Thus,

$$\frac{\mu(I^b_{2k})}{|I^b_{2k}|} = \frac{\mu(I_{2k})}{|I_{2k}|}\,\frac{b_{2k}}{a_{2k}} \le \frac{\mu(I_{2k})}{|I_{2k}|}, \qquad \frac{\mu(I^b_{2k+1})}{|I^b_{2k+1}|} = \frac{\mu(I_{2k})}{|I_{2k}|}\,2\,b_{2k+1} \le 2\frac{\mu(I_{2k})}{|I_{2k}|}.$$

Additionally,

$$\frac{\mu(I_{2k+1})}{|I_{2k+1}|} = \frac{\mu(I_{2k})}{|I_{2k}|}\,2\,a_{2k+1} \le 2\,\frac{\mu(I_{2k})}{|I_{2k}|}.$$

All these together show that it suffices to bound $\mu(I_{2k})/|I_{2k}|$ for $k \ge 3$. Let $k \ge 3$, then we obtain as desired

$$\mu(I_{2k}) = \prod_{j=1}^{2k} a_j = 2^{-3}\Big(\prod_{j=2}^{k} a_{2j}\Big)\Big(\prod_{j=2}^{k-1} a_{2j+1}\Big) = 2^{-3}\frac{1}{k!} \le \frac{4}{3}\,2^{-2k} = \frac{4}{3}\,|I_{2k}|.$$

3.2 The higher dimensional case: specific Haar system constructions

As we have shown in Theorem 1.11, the weak-type $(1,1)$ estimate for Haar shifts is governed by the finiteness of the quantities $\Xi(\Phi,\Psi;r,s)$. In the 1-dimensional case, these can be written only in terms of the measure μ since the Haar system $\mathcal{H}$ is "unique" (see Remark 1.8). However in higher dimensions we have different choices of the Haar system and each of them may lead to a different condition. Therefore, before getting into that let us construct some specific Haar systems.

Among the μ-Haar systems in higher dimensions, two of them are relatively easy to construct: Wilson's Haar system and Mitrea's Haar system [77, 21, 12, 54, 29].

Following [25], we present a simplified way of obtaining this two μ-Haar systems for measures $\mu \in \mathscr{B}$.

To construct Wilson's Haar system, start with some enumeration $(Q_j)_{j=1}^{2^d}$ of the dyadic children of Q and build a dyadic (or logarithmic) partition tree on it. The partition is given as follows: set $\mathscr{W}_0(Q) = \{\{1, 2, \ldots, 2^d\}\}$ and let $\mathscr{W}_1(Q) = \{\{1, \ldots, 2^{d-1}\}, \{2^{d-1}+1, \ldots, 2^d\}\}$. Proceed recursively to get the partition $\mathscr{W}_k(Q)$, obtained upon halving the elements of $\mathscr{W}_{k-1}(Q)$ and ending up with $\mathscr{W}_d(Q) = \{\{1\}, \{2\}, \ldots, \{2^d\}\}$. Set

$$E_Q^\omega = \bigcup_{j\in\omega} Q_j \qquad \text{with} \qquad \omega \in \mathscr{W}(Q) = \bigcup_{k=0}^{d-1} \mathscr{W}_k(Q).$$

We are going to see that the family of sets $\{E_Q^\omega\}_{\omega\in\mathscr{W}(Q)}$ behaves like a one-dimensional dyadic grid. Form construction, any $\omega \in \mathscr{W}_{k-1}(Q)$, $1 \le k \le d$, has two disjoint children $\omega_-, \omega_+ \in \mathscr{W}_k(Q)$ such that $\omega = \omega_- \cup \omega_+$. Thus, following the notation of the 1-dimensional case, we write $(E_Q^\omega)_- = E_Q^{\omega_-}$ and $(E_Q^\omega)_+ = E_Q^{\omega_+}$. Note that these two sets are disjoint and $E_Q^\omega = (E_Q^\omega)_- \cup (E_Q^\omega)_+$. We call $(E_Q^\omega)_-$ and $(E_Q^\omega)_+$ the dyadic children of E_Q^ω. Besides, for every $\omega \in \mathscr{W}_k(Q)$, $1 \le k \le d$, there exists a unique $\widehat{\omega} \in \mathscr{W}_{k-1}(Q)$ such that $\widehat{\omega} \supset \omega$ and thus $E_Q^\omega \subset \widehat{E_Q^\omega} = E_Q^{\widehat{\omega}}$. We call $\widehat{E_Q^\omega}$ the dyadic parent of E_Q^ω. Moreover, E_Q^ω and $E_Q^{\omega'}$ are either disjoint or one is contained in the other.

We define the Haar functions adapted to the family of sets $\{E_Q^\omega\}_{\omega\in\mathscr{W}(Q)}$: for every $\omega \in \mathscr{W}(Q)$ we set

$$h_Q^\omega = \sqrt{m(E_Q^w)} \left(\frac{1_{(E_Q^\omega)_-}}{\mu((E_Q^\omega)_-)} - \frac{1_{(E_Q^\omega)_+}}{\mu((E_Q^\omega)_+)} \right),$$

where

$$m(E_Q^\omega) = \frac{\mu((E_Q^\omega)_-)\,\mu((E_Q^\omega)_+)}{\mu(E_Q^\omega)} = \left(\frac{1}{\mu((E_Q^\omega)_-)} + \frac{1}{\mu((E_Q^\omega)_+)} \right)^{-1} \approx \min\big\{\mu((E_Q^\omega)_-), \mu((E_Q^\omega)_+)\big\}.$$

Note that this makes sense provided $\mu((E_Q^\omega)_-)\,\mu((E_Q^\omega)_+) > 0$. For otherwise, we set $h_Q^\omega \equiv 0$.

Note that for a fixed $Q \in \mathscr{D}$ and $\omega \in \mathscr{W}(Q)$, one can easily verify that h_Q^ω satisfies the properties (a)–(d) in Definition 1.7. Let us further observe that h_Q^ω is orthogonal to $h_Q^{\omega'}$ for $\omega \ne \omega'$. We would like to emphasize that here we have $2^d - 1$ generalized Haar functions associated to each Q (one for each $\omega \in \mathscr{W}(Q)$). In this way, if for every Q we pick $\omega_Q \in \mathscr{W}(Q)$, we have that $\{h_Q^{\omega_Q}\}_{Q\in\mathscr{D}}$ is a 2-value generalized Haar system in $\mathbb{R}^d$ (see Definition 1.7 and Remark 1.8) and therefore standard (see (1.12)).

Mitrea's Haar system is constructed in the following way. Let us fix an enumeration $(Q_j)_{j=1}^{2^d}$ of the dyadic children of Q. For every $2 \le j \le 2^d$ we set

$\widetilde{Q}_j = \cup_{k=j}^{2^d} Q_k$. We define Mitrea's Haar system as follows: for every $1 \le j \le 2^d - 1$ we set

$$H_Q^j = \sqrt{m(Q_j)} \left(\frac{1_{Q_j}}{\mu(Q_j)} - \frac{1_{\widetilde{Q}_{j+1}}}{\mu(\widetilde{Q}_{j+1})} \right),$$

where

$$m(Q_j) = \frac{\mu(Q_j)\,\mu(\widetilde{Q}_{j+1})}{\mu(\widetilde{Q}_j)} = \left(\frac{1}{\mu(Q_j)} + \frac{1}{\mu(\widetilde{Q}_{j+1})} \right)^{-1} \approx \min\{\mu(Q_j), \mu(\widetilde{Q}_{j+1})\}.$$

This definition makes sense provided $\mu(Q_j)\,\mu(\widetilde{Q}_{j+1}) > 0$. For otherwise, we set $H_Q^j \equiv 0$.

Again, for a fixed $1 \le j \le 2^d - 1$ and $Q \in \mathscr{D}$, one can easily verify that H_Q^j satisfies the properties (a)–(d) in Definition 1.7 and also that H_Q^j is orthogonal to $H_Q^{j'}$ for $j \neq j'$. As before, we have $2^d - 1$ generalized Haar functions associated to each Q (one for each j). Hence, if for every Q we pick j_Q, $1 \le j_Q \le 2^d - 1$, we have that $\{H_Q^{j_Q}\}_{Q \in \mathscr{D}}$ is a 2-value generalized Haar system in $\mathbb{R}^d$ (see Definition 1.7 and Remark 1.8) and therefore standard (see (1.12)).

We finally present another way to construct Haar systems in the spirit of the wavelet construction. For this example, we assume that μ is a product measure, that is, $\mu = \mu_1 \times \cdots \times \mu_d$ where $\mu_1, \dots, \mu_d$ are Borel measures in $\mathbb{R}$ satisfying $\mu_j(I) < \infty$ for every $I \in \mathscr{D}$. We will use the following notation, given $Q \in \mathscr{D}(\mathbb{R}^d)$ we have that $Q = I_1^Q \times \cdots \times I_d^Q$ with $I_j^Q \in \mathscr{D}(\mathbb{R})$. Hence, $\mu(Q) = \prod_{j=1}^d \mu_j(I_j^Q)$. Associated to each μ_j we consider a μ_j-generalized Haar system $\Phi_j = \{\phi_{j,I}^1\}_{I \in \mathscr{D}(\mathbb{R})}$. For every $I \in \mathscr{D}(\mathbb{R})$ with $\mu_j(I) > 0$ we set $\phi_{j,I}^0 = 1_I / \mu_j(I)^{\frac{1}{2}}$ and $\phi_{j,I}^0 \equiv 0$ otherwise. For every $\epsilon = (\epsilon_1, \dots, \epsilon_d) \in \{0,1\}^d \setminus \{0\}^d$ and $Q \in \mathscr{D}(\mathbb{R}^d)$ we define

$$\phi_Q^\epsilon(x) = \prod_{j=1}^d \phi_{j,I_j^Q}^{\epsilon_j}(x_j).$$

We have that each ϕ_Q^ϵ satisfies the properties (a)–(d) in Definition 1.7 and also that ϕ_Q^ϵ is orthogonal to $\phi_Q^{\epsilon'}$ for $\epsilon \neq \epsilon'$. Hence, if for every Q we pick ϵ_Q, as above, we have that $\{\phi_Q^{\epsilon_Q}\}_{Q \in \mathscr{D}}$ is a generalized Haar system in $\mathbb{R}^d$, see Definition 1.7. Note that Remark 1.8 says each Φ_j is a 2-value generalized Haar system in $\mathbb{R}$. However, unless some further condition is imposed in each measure μ_j, one has that ϕ_Q^ϵ may take more than 2 non-vanishing values (this is quite easy if we take $\epsilon = \{1\}^d$). Nevertheless, if $Q \in \mathscr{D}_\Phi$ then

$$\|\phi_Q^\epsilon\|_{L^1(\mu)} = \prod_{j=1}^d \|\phi_{j,I_j^Q}^{\epsilon_j}\|_{L^1(\mu_j)}, \qquad \|\phi_Q^\epsilon\|_{L^\infty(\mu)} = \prod_{j=1}^d \|\phi_{j,I_j^Q}^{\epsilon_j}\|_{L^\infty(\mu_j)}.$$

Let $m_j(I) = \mu_j(I_-)\mu_j(I_+)/\mu_j(I)$ for $I \in \mathscr{D}_{\Phi_j}$. Then we have that, for every $I \in \mathscr{D}_{\Phi_j}$,

$$\|\phi_{j,I}^0\|_{L^1(\mu_j)} = \sqrt{\mu_j(I)}, \qquad \|\phi_{j,I}^0\|_{L^\infty(\mu_j)} = \frac{1}{\sqrt{\mu_j(I)}},$$

and, as in 1.3,

$$\|\phi^1_{j,I}\|_{L^1(\mu_j)} = 2\sqrt{m_j(I)}, \qquad \|\phi^1_{j,I}\|_{L^\infty(\mu_j)} \approx \frac{1}{\sqrt{m_j(I)}}.$$

Thus, despite the fact that Φ is not a 2-value generalized Haar system in general, we obtain that Φ is standard.

To conclude this section we observe that although the generalized Haar systems we have constructed above are all standard, this is not the case in general. We work in $\mathbb{R}^2$ and for $k \geq 2$ we let $Q_k = [k, k+1) \times [k, k+1)$. Fix an enumeration $Q_k^1, Q_k^2, Q_k^3, Q_k^4$ of the dyadic children of Q_k. Define $F(x) \equiv 1$ if $x \notin \cup_{k\geq 2} Q_k$ and elsewhere

$$F(x) = \sum_{k=2}^{\infty} \Big(\frac{4}{k^2} \big(1_{Q_k^1}(x) + 1_{Q_k^2}(x) \big) + \frac{2(k^2-2)}{k^2} \big(1_{Q_k^3}(x) + 1_{Q_k^4}(x) \big) \Big).$$

We consider $d\mu(x) = F(x)\, dx$ which is a Borel measure such that $0 < \mu(Q) < \infty$ for every $Q \in \mathscr{D}$. By construction we have

$$\mu(Q_k^1) = \mu(Q_k^2) = \frac{1}{k^2}, \qquad \mu(Q_k^3) = \mu(Q_k^4) = \frac{k^2-2}{2k^2}, \qquad \mu(Q_k) = 1.$$

Next we consider the system $\Phi = \{\phi_{Q_k}\}_{k\geq 2}$ with

$$\begin{aligned} \phi_{Q_k} = \frac{1}{2k} \Big(\frac{1_{Q_k^1}}{\mu(Q_k^1)} - \frac{1_{Q_k^2}}{\mu(Q_k^2)} \Big) + \sqrt{\frac{k^2-2}{8\,k^2}} \Big(\frac{1_{Q_k^3}}{\mu(Q_k^3)} - \frac{1_{Q_k^4}}{\mu(Q_k^4)} \Big) \\ = \frac{k}{2} \big(1_{Q_k^1} - 1_{Q_k^2} \big) + \sqrt{\frac{k^2}{2\,(k^2-2)}} \big(1_{Q_k^3} - 1_{Q_k^4} \big). \end{aligned}$$

By construction each ϕ_{Q_k} satisfies (a)–(d) in Definition 1.7 where we observe that in (d) we have $\|\phi_{Q_k}\|_{L^2(\mu)} = 1$. Thus, Φ is a generalized Haar system in $\mathbb{R}^2$. On the other hand,

$$\begin{aligned} \|\phi_{Q_k}\|_{L^1(\mu)} \|\phi_{Q_k}\|_{L^\infty(\mu)} = \left(\frac{1}{k} + \sqrt{\frac{k^2-2}{2\,k^2}} \right) \max \left\{ \frac{k}{2}, \sqrt{\frac{k^2}{2\,(k^2-2)}} \right\} \\ \geq \sqrt{\frac{k^2-2}{2\,k^2}}\, \frac{k}{2} = \frac{\sqrt{k^2-2}}{2\,\sqrt{2}} \underset{k\to\infty}{\longrightarrow} \infty. \end{aligned}$$

Therefore, Φ is not standard. We note that in view of Example 1.14 we have that the Haar multiplier

$$T_\epsilon f = \sum_{Q\in\mathscr{D}} \epsilon_Q \,\langle f, \phi_Q\rangle\, \phi_Q, \qquad \epsilon_Q = \pm 1 \tag{3.3}$$

is not of weak-type $(1,1)$. We can obtain this from Theorem 1.11. However, here the situation is very simple: we just take $\varphi_{Q_k} = 1_{Q_k^1}/\mu(Q_k^1)$ and obtain that

$$T_\epsilon \varphi_{Q_k} = \epsilon_{Q_k} \,\langle \varphi_{Q_k}, \phi_{Q_k}\rangle\, \phi_{Q_k} = \epsilon_{Q_k}\, \frac{k}{2}\, \phi_{Q_k}.$$

Thus, by (2.1),

$$\frac{\|T_\epsilon \varphi_{Q_k}\|_{L^{1,\infty}(\mu)}}{\|\varphi_{Q_k}\|_{L^1(\mu)}} \approx \frac{k\,\|\phi_{Q_k}\|_{L^1(\mu)}}{2} = \frac{k}{2}\left(\frac{1}{k} + \sqrt{\frac{k^2-2}{2\,k^2}}\right) \underset{k\to\infty}{\longrightarrow} \infty,$$

and therefore T_ϵ is not of weak-type $(1,1)$.

Let us finally point out that in the classical situation (i.e., when μ is the Lebesgue measure and we take a standard Haar system) these operators are usually referred to as a martingale transforms. As it is well known, martingale transforms are of weak-type $(1,1)$ for any measure μ by the use of probability methods. Surprisingly, T_ϵ is not of weak-type $(1,1)$ and therefore T_ϵ cannot be written as a "martingale transform" operator in terms of martingale differences (see (4.16) below for further details).

3.3 Examples of measures in higher dimensions

Taking into account the previous constructions, we are going to give some examples of non trivial measures so that the conditions in Theorem 1.11 hold. We first notice that if μ is dyadically doubling then $\mu(Q) \approx \mu(Q')$ for every dyadic children Q' of Q. In particular, for any generalized Haar system Φ, one can show that $\|\Phi_Q\|_{L^1(\mu)} \approx \mu(Q)^{1/2}$ and $\|\Phi_Q\|_{L^\infty(\mu)} \approx \mu(Q)^{-1/2}$ for every $Q \in \mathscr{D}_\Phi$. This clearly implies that we always have that $\Xi(\Phi, \Psi; r, s) \leq C_{r,s}$ for any choices of generalized Haar systems. Thus, the problem becomes interesting when μ is not dyadically doubling. The general case admits too many choices, and we just want to give an illustration of the kind of issues that one can find. Therefore we are going to restrict ourselves to dimension $d = 2$ with $0 < \mu(Q) < \infty$ for every $Q \in \mathscr{D}(\mathbb{R}^2)$ and $\Phi = \Psi$ with $\mathscr{D}_\Phi = \mathscr{D}$. We are going to consider the complexities $(1,0)$ and $(0,1)$ (since these are related to the model operators $H_\mathscr{D}$ and $H_\mathscr{D}^*$ in 1-dimension).

We consider Wilson's construction. We halve each Q horizontally and write Q_N for the northern "hemisphere" and Q_S the southern "hemisphere". If for every cube Q we take the anti-clockwise enumeration starting with the west-south corner then $Q_S = E_Q^{\{1,2\}}$ and $Q_N = E_Q^{\{3,4\}}$. We now take Wilson's system $\Phi = \{h_Q^{\{1,2,3,4\}}\}_{Q\in\mathscr{D}}$, that is,

$$h_Q^{\{1,2,3,4\}} = \sqrt{m_{N,S}(Q)}\left(\frac{1_{Q_S}}{\mu(Q_S)} - \frac{1_{Q_N}}{\mu(Q_N)}\right), \qquad m_{N,S}(Q) = \frac{\mu(Q_S)\,\mu(Q_N)}{\mu(Q)}.$$

Suppose that $d\mu(x,y) = dx\,d\nu(y)$ then μ is dyadically doubling iff ν is dyadically doubling. If $Q = I \times J$ then

$$m_{N,S}(Q) = |I|\,m_\nu(J) = |I|\,\frac{\nu(J_-)\,\nu(J_+)}{\nu(J)}.$$

Then $\Xi(\Phi, \Phi; 0, 1) < \infty$ if and only if ν is m_ν-increasing and $\Xi(\Phi, \Phi; 1, 0) < \infty$ if and only if ν is m_ν-decreasing. Using the examples we constructed above we find measures μ in $\mathbb{R}^2$ which are non-dyadically doubling but they satisfy one (or both) conditions.

However if we use another Haar system we get a different behavior. Suppose now that our enumeration is clockwise and starts with the west-south corner then $Q_W = E_Q^{\{1,2\}}$ and $Q_E = E_Q^{\{3,4\}}$ are respectively the western and eastern "hemispheres". If now take Wilson's system $\Phi = \{h_Q^{\{1,2,3,4\}}\}$ then we get the same definitions as before replacing Q_S by Q_W and Q_N by Q_E. In particular,

$$m_{E,W}(Q) = \frac{|I|}{4}\,\nu(J)$$

Then we always have $\Xi(\Phi, \Phi; 0, 1) \leq 1/\sqrt{2} < \infty$, whereas $\Xi(\Phi, \Phi; 1, 0) < \infty$ if and only if ν is dyadically doubling.

Similar examples can be constructed using Mitrea's Haar shifts.

We finally look at the Haar system using the wavelet construction. If our system is comprised of $\phi_Q^{i,1}(x,y) = \phi_{1,I}^i(x)\phi_{2,J}^1(y)$ with $i = 0$ or 1 we obtain

$$\|\phi_Q^{i,1}\|_{L^1(dx\times d\nu)} = 2\sqrt{|I|\,m_\nu(J)}, \qquad \|\phi_Q^{i,1}\|_{L^\infty(dx\times d\nu)} \approx \frac{1}{\sqrt{|I|\,m_\nu(J)}},$$

and then we have the same behavior as before: $\Xi(\Phi, \Phi; 0, 1) < \infty$ if and only if ν is m_ν-increasing and $\Xi(\Phi, \Phi; 1, 0) < \infty$ if and only if ν is m_ν-decreasing. On the other hand, if we take $\phi_Q^{1,0}(x,y) = \phi_{1,I}^1(x)\phi_{2,J}^0(y)$ and obtain

$$\|\phi_Q^{1,0}\|_{L^1(dx\times d\nu)} = \sqrt{|I|\,\nu(J)}, \qquad \|\phi_Q^{1,0}\|_{L^\infty(dx\times d\nu)} = \frac{1}{\sqrt{|I|\,\nu(J)}}.$$

Then we always have $\Xi(\Phi, \Phi; 0, 1) \leq 1/\sqrt{2} < \infty$, whereas $\Xi(\Phi, \Phi; 1, 0) < \infty$ if and only if ν is dyadically doubling.

Chapter 4

Further Results

4.1 Non-cancellative Haar shift operators

One can consider Haar shift operators defined in terms of generalized Haar systems that are not required to satisfy the vanishing integral condition. To elaborate on this, let us first consider the case of the dyadic paraproducts and their adjoints. The space $\mathrm{BMO}_{\mathscr{D}}(\mu)$ is the space of locally integrable functions ρ such that

$$\|\rho\|_{\mathrm{BMO}_{\mathscr{D}}(\mu)} = \sup_{Q\in\mathscr{D}} \Big(\frac{1}{\mu(Q)}\int_Q \left|\rho(x) - \langle\rho\rangle_Q\right|^2 d\mu(x)\Big)^{\frac{1}{2}} < \infty,$$

where as usual the terms where $\mu(Q) = 0$ are assumed to be 0. Given $\rho \in \mathrm{BMO}_{\mathscr{D}}(\mu)$, and $\Theta = \{\theta_Q\}_{Q\in\mathscr{D}}$, $\Psi = \{\psi_Q\}_{Q\in\mathscr{D}}$, two (cancellative) generalized Haar systems, we define the *dyadic paraproduct* Π_ρ:

$$\Pi_\rho f(x) = \sum_{Q\in\mathscr{D}} \langle\rho,\theta_Q\rangle\langle f\rangle_Q \psi_Q(x).$$

Note that for each cube Q, θ_Q and ψ_Q are cancellative generalized Haar functions. However, the term $\langle f\rangle_Q$ can be viewed, after renormalization, as f paired with the non-cancellative generalized Haar function $1_Q/\mu(Q)^{1/2}$. That is the reason why we call this operator a non-cancellative Haar shift, see below for further details.

Alternatively, one can consider dyadic paraproducts by incorporating μ-Carleson sequences. Given a sequence $\gamma = \{\gamma_Q\}_{Q\in\mathscr{D}}$, we say that γ is a μ-Carleson sequence, which is denoted by $\gamma \in \mathscr{C}(\mu)$, if for every $Q \in \mathscr{D}$ we have that $\gamma_Q = 0$ if $\mu(Q) = 0$ and

$$\|\gamma\|_{\mathscr{C}(\mu)} = \sup_{Q\in\mathscr{D},\, \mu(Q)>0} \left(\frac{\sum_{Q'\in\mathscr{D}(Q)} |\gamma_{Q'}|^2}{\mu(Q)}\right)^{\frac{1}{2}} < \infty.$$

Typical examples of μ-Carleson sequences are given by $\mathrm{BMO}_{\mathscr{D}}(\mu)$ functions. Indeed if $\rho \in \mathrm{BMO}_{\mathscr{D}}(\mu)$, $\Theta = \{\theta_Q\}_{Q\in\mathscr{D}}$ is a generalized Haar system and we set $\gamma_Q = \langle\rho,\theta_Q\rangle$ we have that γ is μ-Carleson measure: if $Q_0 \in \mathscr{D}$ such that $\mu(Q_0) > 0$, we have by orthogonality

$$\sum_{Q\in\mathscr{D}(Q_0)} |\gamma_Q|^2 = \sum_{Q\in\mathscr{D}(Q_0)} \big|\big\langle(\rho - \langle\rho\rangle_{Q_0})1_{Q_0}, \theta_Q\big\rangle\big|^2$$
$$\leq \|(\rho - \langle\rho\rangle_{Q_0})1_{Q_0}\|^2_{L^2(\mu)} \leq \|\rho\|^2_{\mathrm{BMO}(\mu)}\, \mu(Q_0)$$

and therefore $\|\gamma\|_{\mathscr{C}(\mu)} \leq \|\rho\|_{\mathrm{BMO}_{\mathscr{D}}(\mu)}$. One can also reverse this procedure. Indeed, given $\gamma \in \mathscr{C}(\mu)$ and a generalized Haar system $\Theta = \{\theta_Q\}_{Q\in\mathscr{D}}$ we can define a function ρ which is a Haar expansion using Θ with the coefficients given by the sequence γ as follows. It suffices to consider the function ρ in any d-dimensional quadrant, say for simplicity that we are in $\mathbb{R}^d_1 = [0,\infty)^d$. Let $Q_k = [0, 2^{-k})^d$ and set

$$\rho(x) = \sum_{k\in\mathbb{Z}} \Big(\sum_{Q\in\mathscr{D}(Q_k)\setminus\mathscr{D}(Q_{k+1})} \gamma_Q\, \theta_Q(x)\Big) 1_{Q_k\setminus Q_{k+1}}(x).$$

Note that for every $x \in \mathbb{R}^d_1$, the sum in k contains only one non-vanishing term. From orthogonality and the Carleson condition it follows that for every $k_0 \in \mathbb{Z}$,

$$\|\rho\|^2_{L^2(Q_{k_0})} \leq \sum_{k\geq k_0} \sum_{Q\in\mathscr{D}(Q_k)\setminus\mathscr{D}(Q_{k+1})} |\gamma_Q|^2 \tag{4.1}$$
$$= \sum_{Q\in\mathscr{D}(Q_{k_0})} |\gamma_Q|^2 \leq \|\gamma\|^2_{\mathscr{C}(\mu)}\, \mu(Q_{k_0}).$$

In particular ρ is locally integrable. We next take an arbitrary $R \in \mathscr{D}$, $R \subset \mathbb{R}^d_1$. Assume first that $R = Q_{k_0}$ for some $k_0 \in \mathbb{Z}$. Then easy calculations and (4.1) lead to

$$\frac{1}{\mu(R)} \int_R \big|\rho(x) - \langle\rho\rangle_R\big|^2\, d\mu(x) = \langle|\rho|^2\rangle_{Q_{k_0}} - \big|\langle\rho\rangle_{Q_{k_0}}\big|^2$$
$$\leq \langle|\rho|^2\rangle_{Q_{k_0}} = \mu(Q_{k_0})^{-1}\, \|\rho\|^2_{L^2(Q_{k_0})} \leq \|\gamma\|^2_{\mathscr{C}(\mu)}.$$

On the other hand if $R \notin \{Q_k\}_k$, then there exists a unique k such that $R \subset Q_k \setminus Q_{k+1}$. Then for every $x \in R$ we have

$$\rho(x) = \sum_{Q\in\mathscr{D}(Q_k)\setminus\mathscr{D}(Q_{k+1})} \gamma_Q\, \theta_Q(x)$$
$$= \sum_{Q\in\mathscr{D}(R)} \gamma_Q\, \theta_Q(x) + \sum_{\substack{Q\in\mathscr{D}(Q_k)\setminus\mathscr{D}(Q_{k+1})\\ R\subsetneq Q}} \gamma_Q\, \theta_Q(x) = I(x) + II.$$

Note that II is constant and that $\int_R I(x)\, d\mu(x) = 0$ then

$$\frac{1}{\mu(R)} \int_R |\rho(x) - \langle\rho\rangle_R|^2\, d\mu(x) = \frac{1}{\mu(R)} \int_R \Big| \sum_{Q\in\mathscr{D}(R)} \gamma_Q\, \theta_Q(x)\Big|^2\, d\mu(x)$$
$$\leq \frac{1}{\mu(R)} \sum_{Q\in\mathscr{D}(R)} |\gamma_Q|^2 \leq \|\gamma\|^2_{\mathscr{C}(\mu)}.$$

Gathering the two cases it follows that $\rho \in \mathrm{BMO}_{\mathscr{D}}(\mu)$ with $\|\rho\|_{\mathrm{BMO}_{\mathscr{D}}(\mu)} \leq \|\gamma\|_{\mathscr{C}(\mu)}$. Further details are left to the reader.

Given γ a μ-Carleson sequence and $\Psi = \{\psi_Q\}_{Q\in\mathscr{D}}$ a generalized Haar system we define the *dyadic paraproduct* Π_γ as follows

$$\Pi_\gamma f(x) = \sum_{Q\in\mathscr{D}} \gamma_Q \langle f\rangle_Q \psi_Q(x).$$

If we set $\widetilde{\phi}_Q = 1_Q/\mu(Q)^{1/2}$ if $\mu(Q) > 0$ and $\widetilde{\phi}_Q \equiv 0$ otherwise we have that $\widetilde{\Phi} = \{\widetilde{\phi}_Q\}_{Q\in\mathscr{D}}$ satisfies (a), (b) and (d) in Definition 1.7. Since (c) does not hold we call $\widetilde{\Phi}$ a non-cancellative generalized Haar system. In such a way we can write

$$\Pi_\gamma f(x) = \sum_{Q\in\mathscr{D}} \alpha_Q \langle f, \widetilde{\phi}_Q\rangle \psi_Q(x), \qquad \alpha_Q = \frac{\gamma_Q}{\mu(Q)^{\frac{1}{2}}}.$$

Note that

$$|\alpha_Q| \leq \left(\frac{\sum_{Q'\in\mathscr{D}(Q)} |\gamma_{Q'}|^2}{\mu(Q)}\right)^{\frac{1}{2}} \leq \|\gamma\|_{\mathscr{C}(\mu)}.$$

Thus, we can see Π_γ as a Haar shift of complexity $(0,0)$ with respect to the non-cancellative generalized Haar system $\widetilde{\Phi}$ and the (cancellative) generalized Haar system Ψ. Notice that the adjoint of the paraproduct can be written as

$$\Pi^*_\gamma f(x) = \sum_{Q\in\mathscr{D}} \gamma_Q \langle f, \psi_Q\rangle \frac{1_Q(x)}{\mu(Q)} = \sum_{Q\in\mathscr{D}} \alpha_Q \langle f, \psi_Q\rangle \widetilde{\phi}_Q(x).$$

Again Π^*_γ is a Haar shift of complexity $(0,0)$ with respect to a (cancellative) generalized Haar system Ψ and the non-cancellative generalized Haar system $\widetilde{\Phi}$. This motivates the definition of a non-cancellative Haar shift operator:

$$\widetilde{Ш}_{r,s} f(x) = \sum_{Q\in\mathscr{D}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S} \langle f, \widetilde{\phi}_R\rangle \widetilde{\psi}_S(x), \qquad \sup_{Q,R,S} |\alpha^Q_{R,S}| < \infty, \tag{4.2}$$

with $\widetilde{\Phi} = \{\widetilde{\phi}_Q\}_{Q\in\mathscr{D}}$ and $\widetilde{\Psi} = \{\widetilde{\psi}_Q\}_{Q\in\mathscr{D}}$ being two non-cancellative generalized Haar systems, i.e., both of them satisfies (a), (b) and (d) in Definition 1.7. We would like to stress that $\widetilde{\Phi}$ and $\widetilde{\Psi}$ do not necessarily satisfy (c), therefore the $L^2(\mu)$ boundedness does not automatically follow from the assumed conditions. Thus, is natural to impose that $\widetilde{Ш}_{r,s}$ is bounded on $L^2(\mu)$ along with some local boundedness property and these condition will be checked in any specific situation.

Theorem 4.3. *Let μ be a Borel measure on $\mathbb{R}^d$, $d \geq 1$, satisfying that $\mu(Q) < \infty$ for every $Q \in \mathscr{D}$. Let $\widetilde{\Phi} = \{\widetilde{\phi}_Q\}_{Q\in\mathscr{D}}$ and $\widetilde{\Psi} = \{\widetilde{\psi}_Q\}_{Q\in\mathscr{D}}$ be two non-cancellative generalized Haar systems in $\mathbb{R}^d$. Let r, s be two non-negative integers and consider $\widetilde{Ш}_{r,s}$ as in (4.2). Assume that $\widetilde{Ш}_{r,s}$ is bounded on $L^2(\mu)$ and also that $\widetilde{Ш}_{r,s}$ satisfies the following restricted local $L^2(\mu)$ boundedness: for every $Q_0 \in \mathscr{D}$ we have that*

$$\|\widetilde{Ш}^{Q_0}_{r,s}(1_{Q_0})\|_{L^2(\mu)} \lesssim \mu(Q_0)^{\frac{1}{2}}, \tag{4.4}$$

where the constant is uniform on Q_0 and

$$\widetilde{\text{Ш}}_{r,s}^{Q_0} f(x) = \sum_{Q \in \mathscr{D}(Q_0)} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha_{R,S}^Q \langle f, \widetilde{\phi}_R \rangle \widetilde{\psi}_S(x).$$

If $\Xi(\widetilde{\Phi}, \widetilde{\Psi}; r, s) < \infty$, then $\widetilde{\text{Ш}}_{r,s}$ maps continuously $L^1(\mu)$ into $L^{1,\infty}(\mu)$.

Remark 4.5. Let us observe that $\widetilde{\text{Ш}}_{r,s}^{Q_0}$ is the non-cancellative Haar shift operator associated with the sequence $\gamma_{R,S}^Q = \alpha_{R,S}^Q$ for $Q \in \mathscr{D}(Q_0)$, $R \in \mathscr{D}_r(Q)$, $S \in \mathscr{D}_s(Q)$; and $\gamma_{R,S}^Q = 0$ otherwise. Also, the $L^2(\mu)$ boundedness of $\widetilde{\text{Ш}}_{r,s}^{Q_0}$ clearly implies (4.4).

Remark 4.6. Notice that if we further assume that both Haar systems $\widetilde{\Phi}$ and $\widetilde{\Psi}$ are cancellative, then we automatically obtain (4.4) and the $L^2(\mu)$ boundedness of $\widetilde{\text{Ш}}_{r,s}^{Q_0}$ (see Section 2.3). In such a case Theorem 4.3 becomes Theorem 1.11

Proof. The proof is similar to that of Theorem 1.11, therefore we only give the parts of the argument that are different. Again we may assume that $\mu \in \mathscr{B}_\infty$, the general case follows as before. Follow the proof of Theorem 1.11. For S_1 we use our assumption that $\widetilde{\text{Ш}}_{r,s}$ is bounded on $L^2(\mu)$. The estimate for S_2 is the same. Let us observe that the estimate for S_3 is entirely analogous since in (2.8) we have not used the vanishing integral of ϕ_Q. We are then left with estimating S_4, for which we first observe that

$$\begin{aligned} S_4 &\le \mu(\Omega_\lambda) + \mu\{x \in \mathbb{R}^d \setminus \Omega_\lambda : |\widetilde{\text{Ш}}_{r,s}\beta(x)| > \lambda/3\} \\ &\le \frac{1}{\lambda}\|f\|_{L^1(\mu)} + \frac{3}{\lambda}\sum_j \Big(\int_{\mathbb{R}^d \setminus \widehat{Q}_j} |\widetilde{\text{Ш}}_{r,s}\beta_j| d\mu + \int_{\widehat{Q}_j \setminus Q_j} |\widetilde{\text{Ш}}_{r,s}\beta_j| d\mu \Big) \end{aligned}$$

and we estimate each term in the interior sum. Proceeding as in (2.8) and using Theorem 1.1 we can analogously obtain

$$\begin{aligned} &\sum_j \int_{\mathbb{R}^d \setminus \widehat{Q}_j} |\widetilde{\text{Ш}}_{r,s}\beta_j| \, d\mu \\ &\qquad \lesssim \sum_j \sum_{\widehat{Q}_j \subsetneq Q \subset Q_j^{(r+1)}} \sum_{\substack{R \in \mathscr{D}_r(Q), R \subset \widehat{Q}_j \\ S \in \mathscr{D}_s(Q)}} \|\beta_j\|_{L^1(\mu)} \|\widetilde{\phi}_R\|_{L^\infty(\mu)} \|\widetilde{\psi}_S\|_{L^1(\mu)} \\ &\qquad \le 2^{2+(r+s)\,d}\, r\, \Xi(\widetilde{\Phi}, \widetilde{\Psi}; r, s) \|f\|_{L^1(\mu)}. \end{aligned}$$

On the other hand, for every $x \in \widehat{Q}_j \setminus Q_j$ we have

$$|\widetilde{\text{Ш}}_{r,s}\beta_j(x)| \le \Bigg| \sum_{Q \subsetneq \widehat{Q}_j} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \dots \Bigg| + \Bigg| \sum_{\widehat{Q}_j \subset Q} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \dots \Bigg| = F_j(x) + G_j(x)$$

and we estimate each function in turn. For $F_j(x)$ we note that the terms $Q \subset Q_j$ vanish and therefore $R \subset Q \subset \widehat{Q}_j \setminus Q_j$. Thus β_j is constant on R and then

$$F_j(x) = \Bigg| \sum_{Q \subsetneq \widehat{Q}_j \setminus Q_j} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha_{R,S}^Q \langle \beta_j, \widetilde{\phi}_R \rangle \, \widetilde{\psi}_S(x) \Bigg|$$

$$
\begin{aligned}
&= \big|\langle f\rangle_{Q_j} - \langle f\rangle_{\widehat{Q}_j}\big| \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} \Big| \sum_{Q\subsetneq \widehat{Q}_j\setminus Q_j} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\langle 1_{\widehat{Q}_j\setminus Q_j}, \widetilde{\phi}_R\rangle\, \widetilde{\psi}_S(x)\Big| \\
&\le 2\,\langle |f|\rangle_{Q_j} \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} \sum_{\substack{Q'\in\mathscr{D}_1(\widehat{Q}_j)\\ Q'\neq Q_j}} \Big| \sum_{Q\in\mathscr{D}(Q')} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\langle 1_{Q'}, \widetilde{\phi}_R\rangle\, \widetilde{\psi}_S(x)\Big| \\
&= 2\,\langle |f|\rangle_{Q_j} \frac{\mu(Q_j)}{\mu(\widehat{Q}_j)} \sum_{\substack{Q'\in\mathscr{D}_1(\widehat{Q}_j)\\ Q'\neq Q_j}} \big|\widetilde{\text{III}}^{Q'}_{r,s}(1_{Q'})(x)\big|.
\end{aligned}
$$

This, the fact that supp $\widetilde{\text{III}}^{Q'}_{r,s}(1_{Q'}) \subset Q'$ and that these cubes and pairwise disjoint, and (4.4) yield

$$
\begin{aligned}
\int_{\widehat{Q}_j\setminus Q_j} F_j\,d\mu &\le 2\,\langle |f|\rangle_{Q_j} \left(\frac{\mu(Q_j)}{\mu(\widehat{Q}_j)}\right) \sum_{\substack{Q'\in\mathscr{D}_1(\widehat{Q}_j)\\ Q'\neq Q_j}} \int_{Q'} \big|\widetilde{\text{III}}^{Q'}_{r,s}(1_{Q'})\big|\,d\mu \\
&\le 2\,\langle |f|\rangle_{Q_j} \left(\frac{\mu(Q_j)}{\mu(\widehat{Q}_j)}\right) \sum_{\substack{Q'\in\mathscr{D}_1(\widehat{Q}_j)\\ Q'\neq Q_j}} \big\|\widetilde{\text{III}}^{Q'}_{r,s}(1_{Q'})\big\|_{L^2(\mu)}\, \mu(Q')^{\frac{1}{2}} \\
&\lesssim \int_{Q_j} |f|\,d\mu.
\end{aligned}
$$

For G_j we proceed as before

$$
\begin{aligned}
\int_{\widehat{Q}_j\setminus Q_j} G_j\,d\mu &\lesssim \sum_{\widehat{Q}_j\subset Q\subset Q_j^{(r+1)}} \sum_{\substack{R\in\mathscr{D}_r(Q), R\subset\widehat{Q}_j\\ S\in\mathscr{D}_s(Q)}} \|\beta_j\|_{L^1(\mu)}\, \|\widetilde{\phi}_R\|_{L^\infty(\mu)} \|\widetilde{\psi}_S\|_{L^1(\mu)} \\
&\le 2^{(r+s)\,d}\, r\,\Xi(\widetilde{\Phi},\widetilde{\Psi}; r,s)\|\beta_j\|_{L^1(\mu)}.
\end{aligned}
$$

Gathering the previous estimates we conclude that

$$
\begin{aligned}
\sum_j \int_{\widehat{Q}_j\setminus Q_j} |\widetilde{\text{III}}_{r,s}\beta_j|\,d\mu &\le \sum_j \int_{\widehat{Q}_j\setminus Q_j} (F_j + G_j)\,d\mu \\
&\lesssim \sum_j \int_{Q_j} |f|\,d\mu + \sum_j \|\beta_j\|_{L^1(\mu)} \lesssim \|f\|_{L^1(\mu)}. \qquad \square
\end{aligned}
$$

Remark 4.7. As above, if we keep track of the constants and use a standard homogeneity argument we obtain that, under the conditions of Theorem 1.11,

$$
\begin{aligned}
\|\widetilde{\text{III}}_{r,s}\|_{\mathcal{B}(L^1(\mu),L^{1,\infty}(\mu))} &\le C_0 \Big(\|\widetilde{\text{III}}_{r,s}\|_{\mathcal{B}(L_2(\mu))} \\
&\quad + \sup_{Q\in\mathscr{D},\mu(Q)\neq 0} \frac{\|\widetilde{\text{III}}^{Q}_{r,s}(1_Q)\|_{L^2(\mu)}}{\sqrt{\mu(Q)}} + 2^{(s+r)\,d}\, r\ \Xi(\Phi,\Psi; r,s) \sup_{Q,R,S} |\alpha^Q_{R,S}| \Big),
\end{aligned}
$$

where C_0 is a universal constant (independent of the dimension, for instance, in the previous argument one can safely take $C_0 \le 220$).

4.2 Dyadic paraproducts

As a consequence of Theorem 4.3 we can obtain the following result for dyadic paraproducts.

Theorem 4.8. *Let μ be a Borel measure on $\mathbb{R}^d$, $d \geq 1$, satisfying that $\mu(Q) < \infty$ for every $Q \in \mathscr{D}$. Let $\Psi = \{\psi_Q\}_{Q\in\mathscr{D}}$ be a generalized Haar system. Given a sequence $\gamma = \{\gamma_Q\}_{Q\in\mathscr{D}}$ we consider the* dyadic paraproduct Π_γ *and its adjoint* Π_γ^*:

$$\Pi_\gamma f(x) = \sum_{Q\in\mathscr{D}} \gamma_Q \langle f\rangle_Q \psi_Q(x), \qquad \Pi_\gamma^* f(x) = \sum_{Q\in\mathscr{D}} \gamma_Q \langle f, \psi_Q\rangle \frac{1_Q(x)}{\mu(Q)}.$$

Then we have the following:

(*i*) *For every $\gamma \in \mathscr{C}(\mu)$, Π_γ is of weak-type $(1,1)$ and there exists a universal constant C_0 (one can take for instance $C_0 \leq 288$) such that*

$$\|\Pi_\gamma f\|_{L^{1,\infty}(\mu)} \leq C_0 \|\gamma\|_{\mathscr{C}(\mu)} \|f\|_{L^1(\mu)}.$$

Consequently, Π_γ is bounded on $L^p(\mu)$, $1 < p \leq 2$ (the constant is dimension free and depends linearly on $\|\gamma\|_{\mathscr{C}(\mu)}$).

(*ii*) *If*

$$\sup_{Q\in\mathscr{D}} \|\psi_Q\|_{L^\infty(\mu)} \mu(Q)^{\frac{1}{2}} < \infty, \tag{4.9}$$

then Π_γ^* is of weak-type $(1,1)$ for every $\gamma \in \mathscr{C}(\mu)$ with boundedness constant depending linearly on $\|\gamma\|_{\mathscr{C}(\mu)}$. Conversely, if Π_γ^* is of weak-type $(1,1)$ with operator norm $\|\Pi_\gamma^*\|_{\mathcal{B}(L^1(\mu),L^{1,\infty}(\mu))} \leq C \|\gamma\|_{\mathscr{C}(\mu)}$ for every $\gamma \in \mathscr{C}(\mu)$, then (4.9) holds. Additionally, if (4.9) holds then Π_γ^* is bounded on $L^p(\mu)$ for $1 < p < 2$ (the case $p \geq 2$ follows from (i) without assuming (4.9)).

(*iii*) *Suppose in particular that $d = 1$, $\mu(I) > 0$ for every $I \in \mathscr{D}$ and that $\Psi = \mathcal{H}$. Then, Π_γ is of weak-type $(1,1)$ and bounded on $L^p(\mu)$, $1 < p \leq 2$, for every $\gamma \in \mathscr{C}(\mu)$. However, if for every $\gamma \in \mathscr{C}(\mu)$ we have that Π_γ^* is of weak-type $(1,1)$ or weak-type (p,p) for some $1 < p < 2$, then μ is dyadically doubling. Conversely, if μ is dyadically doubling then Π_γ^* is of weak-type $(1,1)$ and bounded on $L^p(\mu)$, $1 < p < 2$, for every $\gamma \in \mathscr{C}(\mu)$.*

(*iv*) *In (i), (ii), (iii) we can replace the condition "$\gamma \in \mathscr{C}(\mu)$" by "$\gamma_Q = \langle \rho, \theta_Q\rangle$ with $\rho \in \mathrm{BMO}_{\mathscr{D}}(\mu)$ and $\Theta = \{\theta_Q\}_{Q\in\mathscr{D}}$ a generalized Haar system"; and in the boundedness constants $\|\gamma\|_{\mathscr{C}(\mu)}$ by $\|\rho\|_{\mathrm{BMO}_{\mathscr{D}}(\mu)}$.*

Before starting the proof, let us state the $L^2(\mu)$ boundedness of the paraproduct (and its adjoint) along with the corresponding restricted local boundedness as a lemma:

Lemma 4.10. *Under the assumptions of Theorem 4.8, for every $\gamma \in \mathscr{C}(\mu)$ we have*

$$\|\Pi_\gamma f\|_{L^2(\mu)} \leq 2 \|\gamma\|_{\mathscr{C}(\mu)} \|f\|_{L^2(\mu)}. \tag{4.11}$$

Moreover, for every $Q_0 \in \mathscr{D}$ we obtain

$$\|\Pi_\gamma^{Q_0} f\|_{L^2(\mu)} \leq 2\,\|\gamma\|_{\mathscr{C}(\mu)}\,\|f\|_{L^2(\mu)}, \quad \Pi_\gamma^{Q_0} f = \sum_{Q\in\mathscr{D}(Q_0)} \gamma_Q\,\langle f\rangle_Q \psi_Q. \tag{4.12}$$

Proof. We claim that it suffices to obtain (4.11). Indeed, we consider a new sequence $\widetilde{\gamma} = \{\widetilde{\gamma}_Q\}_{Q\in\mathscr{D}}$ with $\widetilde{\gamma}_Q = \gamma_Q$ if $Q \in \mathscr{D}(Q_0)$ and $\widetilde{\gamma}_Q = 0$ otherwise. We clearly have that $\widetilde{\gamma} \in \mathscr{C}(\mu)$ with $\|\widetilde{\gamma}\|_{\mathscr{C}(\mu)} \leq \|\gamma\|_{\mathscr{C}(\mu)}$ and also $\Pi_\gamma^{Q_0} = \Pi_{\widetilde{\gamma}}$. Thus, (4.11) applied to $\widetilde{\gamma}$ implies (4.12).

We obtain (4.11) using ideas from [60]. Let us first suppose that $\mu \in \mathscr{B}_\infty$. The argument is somehow standard, but, since our setting is very general, we give the argument for completeness. Given $f \in L^2(\mu)$ and $\lambda > 0$, as in Theorem 1.1, we can find a maximal collection of dyadic cubes $\{Q_j^\lambda\}_j$ such that $\Omega_\lambda = \cup_j Q_j^\lambda$. We notice that the existence of such maximal cubes follows from the fact that $\langle |f|\rangle_Q \leq \langle |f|^2\rangle_Q^{1/2} \to 0$ as $\ell(Q) \to \infty$, given our current assumption $\mu \in \mathscr{B}_\infty$. Next we use that Ψ is cancellative, therefore orthogonal,

$$\begin{aligned}
\|\Pi_\gamma f\|_{L^2(\mu)}^2 &= \sum_{Q\in\mathscr{D}} |\gamma_Q|^2 \left|\langle f\rangle_Q\right|^2 \|\psi_Q\|_{L^2(\mu)}^2 \\
&\leq \int_0^\infty \sum_{Q\in\mathscr{D}} 1_{\{\langle|f|\rangle_Q>\lambda\}}(\lambda)\,|\gamma_Q|^2\,2\,\lambda\,d\lambda \\
&\leq \int_0^\infty \sum_j \sum_{Q\in\mathscr{D}(Q_j^\lambda)} |\gamma_Q|^2\,2\lambda\,d\lambda \\
&\leq \|\gamma\|_{\mathscr{C}(\mu)}^2 \int_0^\infty \sum_j \mu(Q_j^\lambda)\,2\lambda\,d\lambda \\
&\leq \|\gamma\|_{\mathscr{C}(\mu)}^2 \int_0^\infty \mu(\Omega_\lambda)\,2\,\lambda\,d\lambda \\
&= \|\gamma\|_{\mathscr{C}(\mu)}^2\,\|M_{\mathscr{D}} f\|_{L^2(\mu)}^2 \leq 4\,\|\gamma\|_{\mathscr{C}(\mu)}^2\,\|f\|_{L^2(\mu)}^2,
\end{aligned} \tag{4.13}$$

and this completes the proof of the fact that Π_γ is bounded on $L^2(\mu)$ provided $\mu \in \mathscr{B}_\infty$. To consider the general case, as before we may suppose that $\operatorname{supp} f \subset \mathbb{R}_k^d$, $1 \leq k \leq 2^d$ with $\mu(\mathbb{R}_k^d) < \infty$. In (4.13) we split the integral in two: $0 < \lambda \leq \langle|f|\rangle_{\mathbb{R}_k^d}$ and $\lambda > \langle|f|\rangle_{\mathbb{R}_k^d}$. In the second case we can find the maximal cubes $\{Q_j^\lambda\}$ and the previous argument goes through. Let us next consider the integral in the range $0 < \lambda \leq \langle|f|\rangle_{\mathbb{R}_k^d}$. Let $\{Q_n\}_{n\geq 1} \subset \mathscr{D}(\mathbb{R}_k^d)$ be an increasing sequence such that $\cup_n Q_n = \mathbb{R}_k^d$. Then, we proceed as above

$$\begin{aligned}
\int_0^{\langle|f|\rangle_{\mathbb{R}_k^d}} \sum_{Q\in\mathscr{D}(\mathbb{R}_k^d)} 1_{\{\langle|f|\rangle_Q>\lambda\}}(\lambda)\,|\gamma_Q|^2\,2\,\lambda\,d\lambda &\leq \langle|f|\rangle_{\mathbb{R}_k^d}^2 \sup_n \sum_{Q\in\mathscr{D}(Q_n)} |\gamma_Q|^2 \\
&\leq \|\gamma\|_{\mathscr{C}(\mu)}^2\,\langle|f|^2\rangle_{\mathbb{R}_k^d} \sup_n \mu(Q_n) = \|\gamma\|_{\mathscr{C}(\mu)}^2\,\|f\|_{L^2(\mu)}^2.
\end{aligned}$$

This completes the proof of (4.11). □

Proof of Theorem 4.8. We start with Π_γ. Set $\widetilde{\phi}_Q = 1_Q/\mu(Q)^{1/2}$ if $\mu(Q) > 0$ and $\widetilde{\phi}_Q = 0$ otherwise and consider the non-cancellative generalized Haar system

$\widetilde{\Phi} = \{\widetilde{\phi}_Q\}_{Q\in\mathscr{D}}$. As explained above, in the notation of Theorem 4.3, Π_γ is a non-cancellative Haar shift operator of complexity $(0,0)$ with respect to the systems $\widetilde{\Phi}$ and $\widetilde{\Psi} = \Psi$. By Lemma 4.10 we have the required $L^2(\mu)$ bounds in Theorem 4.3. Thus the weak-type $(1,1)$ (and by interpolation the boundedness on $L^p(\mu)$, $1 < p < 2$) of Π_γ follows from the property $\Xi(\widetilde{\Phi}, \Psi; 0, 0) < \infty$. But this is in turn trivial: by Hölder's inequality we have for every $Q \in \mathscr{D}_\Psi$

$$\|\widetilde{\phi}_Q\|_{L^\infty(\mu)}\|\psi_Q\|_{L^1(\mu)} = \mu(Q)^{-\frac{1}{2}}\|\psi_Q\|_{L^1(\mu)} \le \|\psi_Q\|_{L^2(\mu)} = 1.$$

This completes the proof of (i). For the boundedness constant we can use Remark 4.7 along with Lemma 4.10 to obtain the linear dependence on $\|\gamma\|_{\mathscr{C}(\mu)}$.

We now turn to (ii). We have shown that Π_γ is bounded on $L^2(\mu)$ and so is its adjoint Π_γ^*. Notice that $(\Pi_\gamma^*)^{Q_0} = (\Pi_\gamma^{Q_0})^*$ and therefore $(\Pi_\gamma^*)^{Q_0}$ satisfies (4.12). Then, we apply again Theorem 4.3 to Π_γ^* which is a non-cancellative Haar shift operator of complexity $(0,0)$ with respect to the non-cancellative generalized Haar systems Ψ, $\widetilde{\Phi}$. Thus, $\Xi(\Psi, \widetilde{\Phi}; 0, 0) < \infty$, which coincides with (4.9), implies that Π_γ^* is of weak-type $(1,1)$. The linear dependence on $\|\gamma\|_{\mathscr{C}(\mu)}$ uses the same argument as above. Let us now obtain the converse. Notice that in (4.9) we can restrict the supremum to $Q \in \mathscr{D}_\Psi$ and in particular $\mu(Q) > 0$. Fix one of these cubes Q_0 and let $\gamma_Q = \delta_{Q,Q_0}\sqrt{\mu(Q_0)}$. Then, $\gamma \in \mathscr{C}(\mu)$ with $\|\gamma\|_{\mathscr{C}(\mu)} = 1$. Take

$$f = \operatorname{sgn}\left(\psi_{Q_0}(x)\right)\frac{1_{Q_{0,\infty}}(x)}{\mu(Q_{0,\infty})},$$

where $Q_{0,\infty} \in \mathscr{D}_1(Q_0)$ is a cube where ψ_{Q_0} attains its maximum. Then, as in the proof of Theorem 1.11 and using that Π_γ^* is of weak-type $(1,1)$ with uniform constant (since $\|\gamma\|_{\mathscr{C}(\mu)} = 1$) we obtain

$$\begin{aligned}\|\psi_{Q_0}\|_{L^\infty(\mu)}\sqrt{\mu(Q_0)} &= \left\|\langle f, \psi_{Q_0}\rangle \frac{1_{Q_0}}{\sqrt{\mu(Q_0)}}\right\|_{L^{1,\infty}(\mu)} \\ &= \|\Pi_\gamma^* f\|_{L^{1,\infty}(\mu)} \le C\,\|f\|_{L^1(\mu)} = C.\end{aligned}$$

Repeating this for every $Q_0 \in \mathscr{D}_\Psi$ we obtain (4.9) as desired.

To complete the proof of (ii) we first observe that for $p \ge 2$, duality and (i) give the $L^p(\mu)$ boundedness of Π_γ^* with no further assumption on μ. For $1 < p < 2$, assuming (4.9), we already know that Π_γ^* is of weak-type $(1,1)$. The desired estimates now follow by interpolation with the $L^2(\mu)$ bound from Lemma 4.10.

To obtain (iii) we apply (i) and (ii) and observe that (4.9) can be written as

$$\sup_{I\in\mathscr{D}}\frac{\sqrt{\mu(I)}}{\sqrt{\min\{\mu(I_-), \mu(I_+)\}}} \approx \sup_{I\in\mathscr{D}}\frac{\sqrt{\mu(I)}}{\sqrt{m(I)}} < \infty,$$

which in turn is equivalent to the fact that μ is dyadically doubling. To complete the proof of (iii) it remains to show that if Π_γ^* is of weak type (p,p) for some $1 < p < 2$ then μ is dyadically doubling. Fix then $1 < p < 2$ and $I_0 \in \mathscr{D}$. Let

$\gamma_I = \delta_{I,\widehat{I}_0}\sqrt{\mu(\widehat{I}_0)}$ and observe that $\gamma \in \mathscr{C}(\mu)$ with $\|\gamma\|_{\mathscr{C}(\mu)} = 1$. Taking $f = h_{\widehat{I}_0}$, by (2.6) we have

$$\mu(\widehat{I}_0)^{\frac{1}{p}-\frac{1}{2}} = \Big\|\gamma_{\widehat{I}_0}\frac{1_{\widehat{I}_0}}{\mu(\widehat{I}_0)}\Big\|_{L^{p,\infty}(\mu)} = \|\Pi^*_\gamma f\|_{L^{p,\infty}(\mu)}$$
$$\leq C\,\|f\|_{L^p(\mu)} \approx m(\widehat{I}_0)^{\frac{1}{2}-\frac{1}{p'}} = m(\widehat{I}_0)^{\frac{1}{p}-\frac{1}{2}} \leq \mu(I_0)^{\frac{1}{p}-\frac{1}{2}}.$$

This estimate holds for every $I_0 \in \mathscr{D}$ and therefore μ is dyadically doubling as desired.

We finally show (iv). As observed before if we set $\gamma_Q = \langle \rho, \theta_Q\rangle$ with $\rho \in \mathrm{BMO}_{\mathscr{D}}(\mu)$ and $\Theta = \{\theta_Q\}_{Q\in\mathscr{D}}$ being a generalized Haar system we have that $\gamma \in \mathscr{C}(\mu)$ with $\|\gamma\|_{\mathscr{C}(\mu)} \leq \|\rho\|_{\mathrm{BMO}_{\mathscr{D}}(\mu)}$. Therefore the only assertion that is not contained in the previous items is the converse implication in (ii). As before, in (4.9), we can restrict the supremum to $Q \in \mathscr{D}_\Psi$ and in particular $\mu(Q) > 0$. Fix one of these cubes Q_0, take $\Theta = \Psi$ and let $\rho = \psi_{Q_0}\sqrt{\mu(Q_0)}$. Then,

$$\|\rho\|^2_{\mathrm{BMO}_{\mathscr{D}}(\mu)} = \sup_{Q_0\subset Q\in\mathscr{D}} \frac{1}{\mu(Q)}\int_Q \big|\rho - \langle\rho\rangle_Q\big|^2\,d\mu = \sup_{Q_0\subset Q\in\mathscr{D}}\frac{\mu(Q_0)}{\mu(Q)} = 1.$$

We take the same function f as in (ii), use that Π^*_ρ is of weak-type $(1,1)$ with uniform constant (since $\|\rho\|_{\mathrm{BMO}_{\mathscr{D}}(\mu)} = 1$) and obtain

$$\|\psi_{Q_0}\|_{L^\infty(\mu)}\,\mu(Q_0)^{\frac{1}{2}} = \Big\|\langle f, \psi_{Q_0}\rangle\frac{1_{Q_0}}{\sqrt{\mu(Q_0)}}\Big\|_{L^{1,\infty}(\mu)}$$
$$= \|\Pi^*_\rho f\|_{L^{1,\infty}(\mu)} \leq C\,\|f\|_{L^1(\mu)} = C.$$

Repeating this for every $Q_0 \in \mathscr{D}_\Psi$ we obtain as desired (4.9). This completes the proof of (iv). □

4.3 On the probabilistic approach

We shall work with a fixed Borel measure μ on $\mathbb{R}^d$ such that $\mu(Q) < \infty$ for every dyadic cube Q. The dyadic system $\mathscr{D} = (\mathscr{D}_k)_{k\in\mathbb{Z}}$ is a filtration on $\mathbb{R}^d$. The conditional expectation operator E_k associated to $\mathscr{D}_k$ is defined by

$$\mathsf{E}_k f(x) = \sum_{Q\in\mathscr{D}_k}\mathsf{E}_Q f(x) = \sum_{Q\in\mathscr{D}_k}\langle f\rangle_Q 1_Q(x),$$

where $\langle f\rangle_Q = 0$ if $\mu(Q) = 0$. The martingale difference operators D_k are given by $\mathsf{D}_k = \mathsf{E}_k - \mathsf{E}_{k-1}$. It is clear from the definitions that the operators E_k form an increasing family projections that preserve integrals and that D_k are orthogonal projections. Thus, if $f \in L^p(\mu)$, $1 \leq p < \infty$, the sequence $(\mathsf{E}_k f)_{k\in\mathbb{Z}}$ is an L^p-martingale and

$$f(x) = \sum_{k\in\mathbb{Z}}\mathsf{D}_k f + \mathsf{E}_{-\infty} f = \sum_{n>k}\mathsf{D}_n f + \mathsf{E}_k f, \tag{4.14}$$

where the convergence is in $L^p(\mu)$ and μ-almost everywhere, and where $\mathsf{E}_{-\infty}f = \sum_{j=1}^{2^d}\langle f\rangle_{\mathbb{R}^d_j}1_{\mathbb{R}^d_j}$. Let $Q \in \mathscr{D}_{k-1}$ and denote by D_Q the projection

$$D_Q f(x) = \mathsf{D}_k f(x)\, 1_Q(x) = \Big(\sum_{Q'\in\mathscr{D}_1(Q)} \mathsf{E}_{Q'}f(x)\Big) - \mathsf{E}_Q f(x).$$

Hence $\mathsf{D}_k = \sum_{Q\in\mathscr{D}_{k-1}}\mathsf{D}_Q$. Observe that we may set $\mathsf{D}_Q f \equiv 0$ if $\mu(Q) = 0$. We easily obtain that $\phi \in \mathsf{D}_Q(L^2(\mu))$ (by this we mean the image of $L^2(\mu)$ by the operator D_Q) if and only if ϕ is supported on Q, constant on dyadic subcubes of Q, and has vanishing μ-integral. In such a case we may write

$$\phi(x) = \sum_{Q'\in\mathscr{D}_1(Q)} a_{Q'}\frac{1_{Q'}(x)}{\mu(Q')}, \tag{4.15}$$

with $\sum_{Q'\in\mathscr{D}_1(Q)} a_{Q'} = 0$, and where it is understood that $a'_Q = 0$ if $\mu(Q') = 0$ and we use the standard convention that $0\cdot\infty = 0$. Hence, $\mathsf{D}_Q(L^2(\mu))$ is a vector space of dimension at most $2^d - 1$.

If we are in dimension $d = 1$ and $I \in \mathscr{D}$ satisfies $\mu(I) > 0$, then $h_I \in \mathsf{D}_I(L^2(\mu))$ (since $\mathsf{D}_I h_I = h_I$). Note that in such a case $\mathsf{D}_I(L^2(\mu))$ is 1-dimensional and therefore $\mathsf{D}_I f = \langle f, h_I\rangle h_I$, for every $f \in L^2(\mu)$.

In the higher dimensional case, assume for simplicity that $\mu(Q) > 0$ for every $Q \in \mathscr{D}$. Let us consider the Wilson's Haar system $\{h_Q^\omega : \omega \in \mathscr{W}(Q), Q \in \mathscr{D}\}$. By othonormality of the Wilson's Haar system and the fact that the cardinality of $\mathscr{W}(Q)$ is $2^d - 1$ we immediately obtain that $\{h_Q^\omega : \omega \in \mathscr{W}(Q)\}$ is an orthonormal basis of $\mathsf{D}_Q(L^2(\mu))$. Thus,

$$\mathsf{D}_Q f = \sum_{\omega\in\mathscr{W}(Q)} \langle f, h_Q^\omega\rangle h_Q^\omega, \qquad f \in L^2(\mu).$$

The same can be done with Mitrea's Haar system (see above), in which case we obtain

$$\mathsf{D}_Q f = \sum_{j=1}^{2^d-1} \langle f, H_Q^j\rangle H_Q^j, \qquad f \in L^2(\mu).$$

Finally, if $\mu = \mu_1 \times \cdots \times \mu_d$ with μ_j Borel measures in $\mathbb{R}$ such that $0 < \mu_j(I) < \infty$ for every $I \in \mathscr{D}(\mathbb{R})$ and we consider the Haar system in the spirit of the wavelet construction $\{\phi_Q^\epsilon : \epsilon \in \{0,1\}^d \setminus \{0\}^d, Q \in \mathscr{D}\}$ we analogously have

$$\mathsf{D}_Q f = \sum_{\epsilon\in\{0,1\}^d\setminus\{0\}^d} \langle f, \phi_Q^\epsilon\rangle \phi_Q^\epsilon, \qquad f \in L^2(\mu).$$

We next see that martingale transforms can be written as Haar multipliers (i.e., Haar shifts of complexity $(0,0)$). A martingale transform is defined as

$$Tf(x) = \sum_{k\in\mathbb{Z}} \xi_k(x)\,\mathsf{D}_k f(x)$$

where the sequence $\{\xi_k\}_{k\in\mathbb{Z}}$ is predictable with respect to the dyadic filtration $(\mathscr{D}_k)_{k\in\mathbb{Z}}$, that is, ξ_k is $\sigma(\mathscr{D}_{k-1})$-measurable. Then ξ_k is constant on the cubes

$Q \in \mathscr{D}_{k-1}$. Namely, $\xi_k(x) = \sum_{Q\in\mathscr{D}_{k-1}} \alpha_Q 1_Q(x)$. Thus, by definition of the projections D_Q we get then that the martingale transform defined by $\{\xi_k\}_{k\in\mathbb{Z}}$ can be equivalently written as

$$(4.16) \qquad Tf(x) = \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathscr{D}_{k-1}} \alpha_Q\, \mathsf{D}_Q f(x) = \sum_{Q\in\mathscr{D}} \alpha_Q \left(\sum_{j=1}^{2^d-1} \langle f, \psi_Q^j\rangle \psi_Q^j(x) \right),$$

with $\{\psi_Q^j\}_{1\le j\le 2^d-1}$ being any orthonormal basis of $\mathsf{D}_Q(L^2(\mu))$. Thus, every martingale transform can be represented as a sum of 2^d-1 Haar multipliers, i.e., a Haar shift operators of complexity $(0,0)$ (see Example 1.14). Note that each Haar shift operator in the sum is written in terms of the system $\{\psi_Q^{j_Q}\}_{Q\in\mathscr{D}}$ where for each $Q\in\mathscr{D}$ we chose j_Q with $1\le j_Q\le 2^d-1$.

It is easy to see that any orthonormal basis $\{\psi_Q^j\}_{1\le j\le 2^d-1}$ of $\mathsf{D}_Q(L^2(\mu))$ is also a basis of $\mathsf{D}_Q(L^p(\mu))$ for $1\le p\le\infty$. Assuming further that $\mu\in\mathscr{B}_\infty$, (4.14) says that $\{\psi_Q^j\}_{1\le j\le 2^d-1, Q\in\mathscr{D}}$ is a basis of $L^p(\mu)$, $1\le p<\infty$. However, in view of (4.16), Burkholder's theorem of L^p boundedness of martingale transforms, $1<p<\infty$, does not suffice to show that a given Haar basis is unconditional in $L^p(\mu)$. In fact, unconditionality of a Haar basis is not true in general. We take the last example in Section 3.2 of a non-standard generalized Haar system and the Haar multiplier in (3.3). We can easily see that for every $1<p<2$,

$$\frac{\|T_\epsilon \varphi_{Q_k}\|_{L^p(\mu)}}{\|\varphi_{Q_k}\|_{L^p(\mu)}} = \frac{k\,\|\phi_{Q_k}\|_{L^p(\mu)}}{2\,\mu(Q_k^1)^{\frac{1}{p}-1}} = \frac{k}{2\,k^{2\,\frac{p-1}{p}}} \left(\frac{k^{p-2}}{2^{p-1}} + \left(\frac{k^2}{k^2-2}\right)^{\frac{p}{2}-1} 2^{-\frac{p}{2}} \right)^{\frac{1}{p}}$$
$$\gtrsim k^{\frac{2-p}{p}} \underset{k\to\infty}{\longrightarrow} \infty.$$

Also, if we now take $\widetilde{\varphi}_{Q_k} = 1_{Q_k^3}/\mu(1_{Q_k^3})$ then, for $2<p<\infty$,

$$\frac{\|T_\epsilon \widetilde{\varphi}_{Q_k}\|_{L^p(\mu)}}{\|\widetilde{\varphi}_{Q_k}\|_{L^p(\mu)}} = \left(\frac{k^2}{2\,(k^2-2)}\right)^{\frac{1}{2}} \frac{\|\phi_{Q_k}\|_{L^p(\mu)}}{\mu(Q_k^3)^{\frac{1}{p}-1}}$$
$$= 2^{\frac{1}{p}-\frac{3}{2}} \left(\frac{k^2-2}{k^2}\right)^{\frac{1}{2}-\frac{1}{p}} \left(\frac{k^{p-2}}{2^{p-1}} + \left(\frac{k^2}{k^2-2}\right)^{\frac{p}{2}-1} 2^{-\frac{p}{2}} \right)^{\frac{1}{p}} \gtrsim k^{\frac{p-2}{p}} \underset{k\to\infty}{\longrightarrow} \infty.$$

These imply that $\Phi = \{\phi_{Q_k}\}_{k\ge 2}$ is not an unconditional basis (on its span) on $L^p(\mu)$ for $1<p<\infty$ with $p\ne 2$.

Nevertheless, the standardness property

$$\sup_{1\le j\le 2^d-1} \sup_{Q\in\mathscr{D}} \|\psi_Q^j\|_{L^1(\mu)} \|\psi_Q^j\|_{L^\infty(\mu)} < \infty,$$

implies, by Theorem 1.11, that every Haar multiplier is of weak type $(1,1)$ and, by interpolation and duality, $L^p(\mu)$ bounded for every $1<p<\infty$. This, in turn, gives that $\{\psi_Q^j\}_{1\le j\le 2^d-1, Q\in\mathscr{D}}$ is an unconditional basis for $L^p(\mu)$, $1<p<\infty$.

Let us now look at the case of the dyadic Hilbert transform an its adjoint in dimension $d=1$. Assume that $\mu(I)>0$ for every $I\in\mathscr{D}$. One can easily see that

$$h_{I_\pm}(x) = \mp \frac{\mu(I_\pm)}{\sqrt{m(I)}} h_{I_\pm}(x) h_I(x).$$

Hence,

$$
\begin{aligned}
H_{\mathscr{D}} f(x) &= \sum_{I\in\mathscr{D}} \langle f, h_I\rangle \big(h_{I_-}(x) - h_{I_+}(x)\big)\\
&= \sum_{I\in\mathscr{D}} \frac{1}{\sqrt{m(I)}} \big(\mu(I_-)h_{I_-}(x) + \mu(I_+)h_{I_+}(x)\big)\langle f, h_I\rangle h_I(x)\\
&= \sum_{k\in\mathbb{Z}} \Bigg(\sum_{I\in\mathscr{D}_{k-1}} \frac{\mu(I_-)h_{I_-}(x) + \mu(I_+)h_{I_+}(x)}{\sqrt{m(I)}} \Bigg) \Bigg(\sum_{J\in\mathscr{D}_{k-1}} \langle f, h_J\rangle h_J(x) \Bigg)\\
&= \sum_{k\in\mathbb{Z}} \xi_k(x) \mathsf{D}_k f(x),
\end{aligned}
$$

where we have used that $\mathsf{D}_k = \sum_{I\in\mathscr{D}_{k-1}} \mathsf{D}_I$ and that $\mathsf{D}_I f = \langle f, h_I\rangle h_I$. The coefficient ξ_k is $\sigma(\mathscr{D}_{k+1})$-measurable, defining a non predictable sequence. One may thus regard the dyadic Hilbert transform as a “generalized martingale transform”. Let us finally observe that for the adjoint of the Hilbert transform, since D_k is a projection, we have

$$
H^*_{\mathscr{D}} f(x) = \sum_{k\in\mathbb{Z}} \mathsf{D}_k\big(\xi_k f\big)(x).
$$

Similar expressions can be obtained for other Haar shift operators in every dimension provided the coefficients can be split as $\alpha^Q_{R,S} = \gamma^Q_R \, \beta^Q_S$. This procedure shows that Haar shift operators of arbitrary complexity “fill” the space of “martingale transforms” with arbitrary measurable coefficients, further details are left to the interested reader. In particular, we see why classical tools coming from martingale L^p-theory do not apply in the present contexts, and our Calderón-Zygmund decomposition establishes the right substitute of Gundy’s martingale decomposition in such a general setting.

Part II

Calderón-Zygmund operators associated to matrix-valued kernels

Chapter 5

Introduction and main results

A *semicommutative* Calderón-Zygmund operator has the formal expression

$$Tf(x) \sim \int_{\mathbb{R}^d} k(x,y)(f(y))\,dy,$$

where the kernel acts linearly on the matrix-valued function $f = (f_{ij})$ and satisfies standard size/smoothness Calderón-Zygmund type conditions. This is the operator model for quite a number of problems which have attracted some attention in recent years, including matrix-valued paraproducts, operator-valued Calderón-Zygmund theory or Fourier multipliers on group von Neumann algebras, see [33, 36, 50, 56, 58] and the references therein. To be more precise, let $\mathcal{B}(\ell_2)$ stand for the matrix algebra of bounded linear operators on ℓ_2. Consider the algebra formed by essentially bounded functions $f : \mathbb{R}^d \to \mathcal{B}(\ell_2)$. Its weak operator closure is a von Neumann algebra $\mathcal{A}$ and as such we may construct noncommutative L_p spaces over it. Let us highlight a few significant examples:

- **Scalar kernels:** $k(x,y) \in \mathbb{C}$ and

$$k(x,y)(f(y)) = \Big(k(x,y)f_{ij}(y)\Big).$$

- **Schur product actions:** $k(x,y) \in \mathcal{B}(\ell_2)$ and

$$k(x,y)(f(y)) = \Big(k_{ij}(x,y)f_{ij}(y)\Big).$$

- **Fully noncommutative model:** $k(x,y) \in \mathcal{B}(\ell_2)\bar{\otimes}\mathcal{B}(\ell_2)$ and

$$k(x,y)(f(y)) = \bigg(\sum_m \mathrm{Tr}\big(k''_m(y)f(y)\big)k'_m(x)_{ij}\bigg).$$

- **Partial traces, noncommuting kernels:** $k(x,y) \in \mathcal{B}(\ell_2)$ and

$$k(x,y)(f(y)) = \begin{cases} \bigg(\displaystyle\sum_s k_{is}(x,y)f_{sj}(y)\bigg), \\ \bigg(\displaystyle\sum_s f_{is}(y)k_{sj}(x,y)\bigg) \end{cases}.$$

Scalar kernels require a matrix-valued Calderón-Zygmund decomposition in terms of noncommutative martingales and a pseudo-localization principle to control the tails of Tf in the L_2-metric [58]. Hilbert space-valued kernels were later considered in [53], see also [51, 66, 70] for previous related results. The second case refers to the Schur matrix product $k(x,y) \bullet f(y)$, considered in [36] to analyze cross product extensions of classical Calderón-Zygmund operators. It is instrumental for Hörmander-Mihlin theorems on Fourier multipliers associated to discrete groups and for Schur multipliers with a Calderón-Zygmund behavior [36, 35]. In the fully noncommutative model, we approximate $k(x,y)$ by a sum of elementary tensors $\sum_m k'_m(x) \otimes k''_m(y)$ and the action is given by

$$Tf(x) \sim \int_{\mathbb{R}^d} (id \otimes \mathrm{Tr}) \Big(k(x,y) \big(1_{\mathcal{B}(\ell_2)} \otimes f(y) \big) \Big) \, dy.$$

In this case, we regard the space $L_p(\mathcal{A}) = L_p(\mathbb{R}^d; L_p(\mathcal{B}(\ell_2)))$ as a whole. In other words, the noncommutative nature of $L_p(\mathcal{A})$ predominates and the presence of a Euclidean subspace is ignored. That is what happens for purely noncommutative Calderón-Zygmund operators [34] and justifies the presence of $id \otimes \mathrm{Tr}$, to integrate over the full algebra $\mathcal{A}$ and not just over the Euclidean part. The last case refers to matrix-valued kernels acting on f by left/right multiplication, $k(x,y)f(y)$ and $f(y)k(x,y)$. Matrix-valued paraproducts are prominent examples [41, 50, 52, 56, 68]. This is the only case in which the kernel does not commute with f, since the Schur product is abelian and we find that $(id \otimes \mathrm{Tr})\big(k(x,y)(1_{\mathcal{B}(\ell_2)} \otimes f(y))\big) = (id \otimes \mathrm{Tr})\big((1_{\mathcal{B}(\ell_2)} \otimes f(y))k(x,y)\big)$ as a consequence of the tracial property.

Our main goal is to obtain endpoint estimates for Calderón-Zygmund operators with noncommuting kernels, motivated by a recent estimate from [36] for semicommutative Calderón-Zygmund operators. If $k(x,y)$ acts linearly on $\mathcal{B}(\ell_2)$ and satisfies the Hörmander smoothness condition in the norm of bounded linear maps on $\mathcal{B}(\ell_2)$, the following results were recently proved in [36]

- If T is $L_\infty(\mathcal{B}(\ell_2); L_2^{\mathrm{r}}(\mathbb{R}^d))$-bounded, then $T : L_\infty(\mathcal{A}) \to \mathrm{BMO_r}(\mathcal{A})$,
- If T is $L_\infty(\mathcal{B}(\ell_2); L_2^{\mathrm{c}}(\mathbb{R}^d))$-bounded, then $T : L_\infty(\mathcal{A}) \to \mathrm{BMO_c}(\mathcal{A})$.

Here, the $L_\infty(L_2^{\mathrm{c}})$-boundedness assumption refers to

$$\Big\| \Big(\int_{\mathbb{R}^d} Tf(x)^* Tf(x) \, dx \Big)^{\frac12} \Big\|_{\mathcal{B}(\ell_2)} \lesssim \Big\| \Big(\int_{\mathbb{R}^d} f(x)^* f(x) \, dx \Big)^{\frac12} \Big\|_{\mathcal{B}(\ell_2)},$$

while the column-BMO norm of a matrix-valued function g is given by

$$\sup_{Q \,\mathrm{cube}} \Big\| \Big(\frac{1}{|Q|} \int_Q \big(g(x) - \langle g \rangle_Q \big)^* \big(g(x) - \langle g \rangle_Q \big) \, dx \Big)^{\frac12} \Big\|_{\mathcal{B}(\ell_2)}.$$

Taking adjoints we find $L_\infty(L_2^{\mathrm{r}})$-boundedness and row-BMO norm. The noncommutative BMO space $\mathrm{BMO}(\mathcal{A}) = \mathrm{BMO_r}(\mathcal{A}) \cap \mathrm{BMO_c}(\mathcal{A})$ was introduced in [66].

According to [55] such a BMO space satisfies the expected interpolation behavior with the corresponding L_p scale. Therefore, standard interpolation and duality arguments show that $T : L_p(\mathcal{A}) \to L_p(\mathcal{A})$ for $1 < p < \infty$ provided the kernel is smooth enough in both variables and T is a normal self-adjoint map satisfying the $L_\infty(L_2^{\mathrm{r}})$ and $L_\infty(L_2^{\mathrm{c}})$ boundedness assumptions. In other words, the row/column boundedness conditions essentially play the role of the L_2-boundedness assumption in classical Calderón-Zygmund theory.

Although this certainly works for non-scalar kernels — Schur product actions were used e.g. in [36] — the boundedness assumptions impose nearly commuting conditions on the kernel. Namely, given $k : \mathbb{R}^{2d} \setminus \Delta \to \mathcal{B}(\ell_2)$ smooth and given $x \notin \mathrm{supp}_{\mathbb{R}^d} f$, let us set formally the row/column Calderón-Zygmund operators

$$T_{\mathrm{r}} f(x) = \int_{\mathbb{R}^d} f(y) k(x,y)\, dy \quad \text{and} \quad T_{\mathrm{c}} f(x) = \int_{\mathbb{R}^d} k(x,y) f(y)\, dy.$$

It is not difficult to construct noncommuting kernels such that

- T_{r} and T_{c} are $L_2(\mathcal{A})$-bounded,
- T_{r} and T_{c} are not $L_p(\mathcal{A})$-bounded for $1 < p \neq 2 < \infty$,

see e.g. [58, Section 6.1] for specific examples. Therefore, the $L_\infty(L_2^{\mathrm{r}})$ and $L_\infty(L_2^{\mathrm{c}})$ boundedness assumption is in general too restrictive when kernel and function do not commute. Assume in what follows that T_{r} and T_{c} are $L_2(\mathcal{A})$-bounded. We are interested in weakened forms of L_p boundedness and endpoint estimates for these Calderón-Zygmund operators.

Let $\mathscr{D}$ denote some dyadic grid in $\mathbb{R}^d$. A *dyadic noncommuting* Calderón-Zygmund operator will be a $L_2(\mathcal{A})$-bounded pair $(T_{\mathrm{r}}, T_{\mathrm{c}})$ associated to a noncommuting kernel satisfying one of the following conditions:

- **Perfect dyadic kernels** are such that

$$\big\| k(x,y) - k(z,y) \big\|_{\mathcal{B}(\ell_2)} + \big\| k(y,x) - k(y,z) \big\|_{\mathcal{B}(\ell_2)} = 0$$

  whenever $x, z \in Q$ and $y \in R$ for some disjoint dyadic cubes $Q, R \in \mathscr{D}$.

- **Haar shift kernels** are given in terms of two *generalized Haar systems* $\Phi = \{\phi_Q\}_{Q \in \mathscr{D}}$ and $\Psi = \{\psi_Q\}_{Q \in \mathscr{D}}$ as defined in Section 6.2. For some fixed $r, s \in \mathbb{Z}_+$ let

$$k(x,y) = \sum_{Q \in \mathscr{D}} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha_{R,S}^Q \phi_R(y) \psi_S(x),$$

  with uniformly bounded matrix-valued symbols $\alpha_{R,S}^Q \in \mathcal{B}(\ell_2)$. Here $\mathscr{D}_k(Q)$ denotes the family of k-dyadic descendants of Q, i.e., the partition of Q into subcubes $R \in \mathscr{D}$ of side-length $\ell(R) = 2^{-k}\ell(Q)$.

Perfect dyadic kernels were introduced in [2] and include Haar multipliers, as well as paraproducts and their adjoints. If I_- and I_+ denote the left/right halves of a dyadic interval $I \subset \mathbb{R}$, the standard model for Haar shifts is the dyadic Hilbert transform with kernel $\sum_I (h_{I_-}(y) - h_{I_+}(y))h_I(x)$. It appeared after Petermichl's crucial result [62], showing the classical Hilbert transform as a certain average of dyadic Hilbert transforms. Hytönen's representation theorem [29] extends this result to arbitrary Calderón-Zygmund operators.

By a *generic noncommuting* Calderón-Zygmund operator wi will refer to $L_2(\mathcal{A})$-bounded pairs $(T_\mathrm{r}, T_\mathrm{c})$ with a noncommuting kernel satisfying the standard size and smoothness conditions:

- if $x, y \in \mathbb{R}^d$, we have
$$\|k(x,y)\|_{\mathcal{B}(\ell_2)} \lesssim \frac{1}{|x-y|^d}.$$
- There exists $0 < \gamma \leq 1$ such that
$$\|k(x,y) - k(x',y)\|_{\mathcal{B}(\ell_2)} \lesssim \frac{|x-x'|^\gamma}{|x-y|^{d+\gamma}} \quad \text{if} \quad |x-x'| \leq \frac{1}{2}|x-y|,$$
$$\|k(x,y) - k(x,y')\|_{\mathcal{B}(\ell_2)} \lesssim \frac{|y-y'|^\gamma}{|x-y|^{d+\gamma}} \quad \text{if} \quad |y-y'| \leq \frac{1}{2}|x-y|.$$

We will refer to γ as the Lipschitz smoothness parameter of the kernel.

Theorem 5.1. *The following inequalities hold:*

(i) *Given $f \in L_1(\mathcal{A})$, there exists an explicit decomposition $f = f_\mathrm{r} + f_\mathrm{c}$ so that the following inequality holds for any row/column pair $T_\mathrm{r}/T_\mathrm{c}$ of dyadic noncommuting Calderón-Zygmund operators*
$$\|T_\mathrm{r} f_\mathrm{r}\|_{L_{1,\infty}(\mathcal{A})} + \|T_\mathrm{c} f_\mathrm{c}\|_{L_{1,\infty}(\mathcal{A})} \lesssim \|f\|_{L_1(\mathcal{A})}.$$

(ii) *Given any row/column pair $T_\mathrm{r}/T_\mathrm{c}$ of generic noncommuting Calderón-Zygmund operators, we have $T_\mathrm{r} : \mathrm{H}_1^\mathrm{r}(\mathcal{A}) \to L_1(\mathcal{A})$ and $T_\mathrm{c} : \mathrm{H}_1^\mathrm{c}(\mathcal{A}) \to L_1(\mathcal{A})$. In particular, if $\|f\|_{\mathrm{H}_1(\mathcal{A})} \sim \|f_\mathrm{r}\|_{\mathrm{H}_1^\mathrm{r}(\mathcal{A})} + \|f_\mathrm{c}\|_{\mathrm{H}_1^\mathrm{c}(\mathcal{A})}$ we get*
$$\|T_\mathrm{r} f_\mathrm{r}\|_{L_1(\mathcal{A})} + \|T_\mathrm{c} f_\mathrm{c}\|_{L_1(\mathcal{A})} \lesssim \|f\|_{\mathrm{H}_1(\mathcal{A})}.$$

The noncommutative forms of $L_{1,\infty}$ and the Hardy space H_1 are well-known in the subject, but we will remind the definitions later on. Our main result is the inequality in Theorem 5.1 (i) and its noncommutative martingale generalization in Theorem 5.3 below. The argument we use simplifies that of [58] for dyadic Calderón-Zygmund operators with commuting kernels. The following result easily follows from Theorem 5.1 by interpolation and duality arguments. Nevertheless, it is worth mentioning the L_p estimates derived by our main results.

Theorem 5.2. *The following inequalities hold for generic noncommuting Calderón-Zygmund operators:*

(i) *If $1 < p < 2$ and $f \in L_p(\mathcal{A})$*
$$\inf_{f = f_\mathrm{r} + f_\mathrm{c}} \big\|T_\mathrm{r} f_\mathrm{r}\big\|_{L_p(\mathcal{A})} + \big\|T_\mathrm{c} f_\mathrm{c}\big\|_{L_p(\mathcal{A})} \lesssim \|f\|_{L_p(\mathcal{A})}.$$
In fact, we also have that $T_\mathrm{r} : \mathrm{H}_p^\mathrm{r}(\mathcal{A}) \to L_p(\mathcal{A})$ and $T_\mathrm{c} : \mathrm{H}_p^\mathrm{c}(\mathcal{A}) \to L_p(\mathcal{A})$.

(*ii*) *If* $2 < p < \infty$ *and* $f \in L_p(\mathcal{A})$

$$\big\|T_{\mathrm{r}} f\big\|_{\mathrm{H}_p^{\mathrm{r}}(\mathcal{A})} + \big\|T_{\mathrm{c}} f\big\|_{\mathrm{H}_p^{\mathrm{c}}(\mathcal{A})} \lesssim \|f\|_{L_p(\mathcal{A})}.$$

(*iii*) *Given* $f \in L_\infty(\mathcal{A})$, *we also have* $\|T_{\mathrm{r}} f\|_{\mathrm{BMO}_{\mathrm{r}}(\mathcal{A})} + \|T_{\mathrm{c}} f\|_{\mathrm{BMO}_{\mathrm{c}}(\mathcal{A})} \lesssim \|f\|_{\mathcal{A}}$.

Theorems 5.1 and 5.2 also hold for other operator-valued functions, replacing $\mathcal{B}(\ell_2)$ by any semifinite von Neumann algebra $\mathcal{M}$. Our proof will be written in this framework. Let us now consider a weak-$*$ dense filtration $\Sigma_{\mathcal{A}} = (\mathcal{A}_n)_{n\geq 1}$ of von Neumann subalgebras of an arbitrary semifinite von Neumann algebra $\mathcal{A}$. In the following result, we will consider two kinds of operators in $L_p(\mathcal{A})$:

- **Noncommuting martingale transforms**

$$M_\xi^{\mathrm{r}} f = \sum_{k\geq 1} \mathsf{D}_k(f)\xi_{k-1} \quad \text{and} \quad M_\xi^{\mathrm{c}} f = \sum_{k\geq 1} \xi_{k-1}\mathsf{D}_k(f).$$

- **Paraproducts with noncommuting symbol**

$$\Pi_\rho^{\mathrm{r}}(f) = \sum_{k\geq 1} \mathsf{E}_{k-1}(f)\mathsf{D}_k(\rho) \quad \text{and} \quad \Pi_\rho^{\mathrm{c}}(f) = \sum_{k\geq 1} \mathsf{D}_k(\rho)\mathsf{E}_{k-1}(f).$$

Here D_k denotes the martingale difference operator $\mathsf{E}_k - \mathsf{E}_{k-1}$ and $\xi_k \in \mathcal{A}_k$ is an adapted sequence. Of course, the symbols ξ and ρ do not necessarily commute with the function. Randrianantoanina considered in [70] noncommutative martingale transforms with commuting coefficients. As for paraproducts with noncommuting symbols, Mei studied the L_p-boundedness for $p > 2$ and regular filtrations in [50] and analyzed in [52] the case $p < 2$ in the dyadic matrix-valued case under a strong BMO condition on the symbol. Our theorem below goes beyond these results, see also [53] for related results.

Theorem 5.3. *Consider the pairs:*

(*i*) *martingale transforms* $(M_\xi^{\mathrm{r}}, M_\xi^{\mathrm{c}})$, *with* $\sup_k \|\xi_k\|_{\mathcal{M}} < \infty$;

(*ii*) *martingale paraproducts* $(\Pi_\rho^{\mathrm{r}}, \Pi_\rho^{\mathrm{c}})$, *with* $\Pi_\rho^{\mathrm{r}/\mathrm{c}}$ $L_2(\mathcal{A})$*-bounded.*

If $\Sigma_{\mathcal{A}}$ *is regular, we obtain weak type* $(1,1)$ *inequalities like in Theorem 5.1* (*i*) *for martingale transforms and paraproducts. The estimates in Theorems 5.1* (*ii*) *and 5.2 also hold for both families and for arbitrary filtrations* $\Sigma_{\mathcal{A}}$. *Moreover, the martingale paraproducts* Π_ρ^{r} *and* Π_ρ^{c} *are* L_p*-bounded for* $2 < p < \infty$ *and* $L_\infty \to \mathrm{BMO}$.

For martingale transforms, there are also examples of noncommuting kernels lacking L_p-boundedness for $p \neq 2$. In the case of regular filtrations, our weak type estimates extend those in [70] with appropriate substitutes for noncommuting coefficients. Our strong type estimates — including the analog of Theorem 5.1 (*ii*) — may be derived from the results in [66]. We use nevertheless a different argument using atomic decompositions, which is also valid for paraproducts. Our result for paraproducts goes beyond [50, Theorem 1.2] in two aspects. First, our estimates for $p > 2$ hold for arbitrary martingales, not just for regular ones. Second, we partially answer Mei's question in [50] after the proof of Theorem 1.2 for the case $p < 2$ and also for weak type $(1, 1)$ estimates.

This part of the dissertation is organized following the order presented in this Introduction, describing the basic setting in Chapter 6. We shall assume some familiarity with basic notions from noncommutative integration. The content of [58, Section 1] is enough for our purposes, more can be found in [40, 67, 73].

Chapter 6

Noncommuting dyadic operators

Let $\mathcal{M}$ be a semifinite von Neumann algebra equipped with a normal semifinite faithful trace ν. Consider the algebra of essentially bounded functions $\mathbb{R}^d \to \mathcal{M}$ equipped with the normal semifinite faithful (*n.s.f.*) trace

$$\tau(f) = \int_{\mathbb{R}^d} \nu(f(x))\, dx.$$

Its weak-operator closure is a von Neumann algebra $\mathcal{A}$. If $1 \leq p \leq \infty$, we write $L_p(\mathcal{M})$ and $L_p(\mathcal{A})$ for the noncommutative L_p spaces associated to the pairs $(\mathcal{M}, \nu)$ and $(\mathcal{A}, \tau)$. The lattices of projections are written $\mathcal{P}(\mathcal{M})$ and $\mathcal{P}(\mathcal{A})$, while $1_{\mathcal{M}}$ and $1_{\mathcal{A}}$ stand for the unit elements.

The set of dyadic cubes in $\mathbb{R}^d$ is denoted by $\mathscr{D}$ and we use $\mathscr{D}_k$ for the k-th generation, formed by cubes Q with side-length $\ell(Q) = 2^{-k}$. If $f : \mathbb{R}^d \to \mathcal{M}$ is integrable on $Q \in \mathscr{D}$, we set the average

$$\langle f \rangle_Q = \frac{1}{|Q|} \int_Q f(y)\, dy = \fint_Q f(y)\, dy.$$

Let us write $(\mathsf{E}_k)_{k\in\mathbb{Z}}$ for the family of conditional expectations associated to the classical dyadic filtration on $\mathbb{R}^d$. E_k will also stand for the tensor product $\mathsf{E}_k \otimes id_{\mathcal{M}}$ acting on $\mathcal{A}$. If $1 \leq p \leq \infty$ and $f \in L_p(\mathcal{A})$

$$\mathsf{E}_k(f) =: f_k = \sum_{Q\in\mathscr{D}_k} \langle f \rangle_Q 1_Q,$$

$$\mathsf{D}_k(f) =: df_k = \sum_{Q\in\mathscr{D}_k} \big(\langle f \rangle_Q - \langle f \rangle_{\widehat{Q}}\big) 1_Q,$$

where $\widehat{Q}$ denotes the dyadic parent of Q. We will write $(\mathcal{A}_k)_{k\in\mathbb{Z}}$ for the filtration $\mathcal{A}_k = \mathsf{E}_k(\mathcal{A})$. The noncommutative weak L_1-space, denoted by $L_{1,\infty}(\mathcal{A})$, is the set of all τ-measurable operators f for which $\|f\|_{L_{1,\infty}(\mathcal{A})} = \sup_{\lambda>0} \lambda\tau(\{|f| > \lambda\}) < \infty$, see [23] for a more in depth discussion. In this case, we write $\tau(\{|f| > \lambda\})$ to denote the trace of the spectral projection of $|f|$ associated to the interval (λ, ∞). We find this terminology more intuitive, since it is reminiscent of the classical one. The space $L_{1,\infty}(\mathcal{A})$ is a quasi-Banach space and satisfies the quasi-triangle inequality below which will be used with no further reference

$$\lambda\,\tau(\{|f_1+f_2|>\lambda\})\le\lambda\tau(\{|f_1|>\lambda/2\})+\lambda\tau(\{|f_2|>\lambda/2\}).$$

Let us consider the dense subspace

$$\mathcal{A}_{+,K}=L_1(\mathcal{A})\cap\Big\{f:\mathbb{R}^d\to\mathcal{M}:\ f\in\mathcal{A}_+,\,\mathrm{supp}_{\mathbb{R}^d}f\text{ is compact}\Big\}\subset L_1^+(\mathcal{A}).$$

Here $\mathrm{supp}_{\mathbb{R}^d}$ means the support of f as a vector-valued function in $\mathbb{R}^d$. In other words, we have $\mathrm{supp}_{\mathbb{R}^d}f=\mathrm{supp}\,\|f\|_{\mathcal{M}}$. We employ this terminology to distinguish from $\mathrm{supp}\,f$, the support of f as an operator in $\mathcal{A}$.

Any function $f\in\mathcal{A}_{+,K}$ gives rise to a martingale $(f_k)_{k\in\mathbb{Z}}$ with respect to the dyadic filtration. Moreover, it is clear that given $f\in\mathcal{A}_{+,K}$ and $\lambda>0$, there must exist $m_\lambda(f)\in\mathbb{Z}$ so that $0\le f_k\le\lambda$ for all $k\le m_\lambda(f)$. The noncommutative analogue of the weak type $(1,1)$ boundedness of Doob's maximal function is due to Cuculescu. Here we state it in the context of operator-valued functions from $\mathcal{A}$.

Cuculescu's construction [18]. *Let $f\in\mathcal{A}_{+,K}$ and consider the corresponding martingale $(f_k)_{k\in\mathbb{Z}}$ relative to the filtration $(\mathcal{A}_k)_{k\in\mathbb{Z}}$. Given $\lambda\in\mathbb{R}_+$, there exists a decreasing sequence of projections $(q_k(\lambda))_{k\in\mathbb{Z}}$ in $\mathcal{A}$ satisfying*

(a) q_k is a projection in $\mathcal{A}_k$,

(b) q_k commutes with $q_{k-1}f_kq_{k-1}$,

(c) $q_kf_kq_k\le\lambda q_k$.

(d) $q=\bigwedge_k q_k$ satisfies

$$\|qf_kq\|_{\mathcal{A}}\le\lambda\text{ for all }k\ge1\quad\text{and}\quad\tau(1_{\mathcal{A}}-q)\le\frac{1}{\lambda}\|f\|_{L_1(\mathcal{A})}.$$

Explicitly,

$$q_k(\lambda)=1_{(0,\lambda]}(q_{k-1}(\lambda)f_kq_{k-1}(\lambda))q_{k-1}(\lambda)$$

with $q_k(\lambda)=1_{\mathcal{A}}$ for $k\le m_\lambda(f)$.

Given $f\in\mathcal{A}_{+,K}$, consider the Cuculescu's sequence $(q_k(\lambda))_{k\in\mathbb{Z}}$ associated to (f,λ) for a given $\lambda>0$. Since λ will be fixed most of the time, we will shorten the notation by q_k and only write $q_k(\lambda)$ when needed. Define the sequence $(p_k)_{k\in\mathbb{Z}}$ of disjoint projections $p_k=q_{k-1}-q_k$, so that

$$\sum_{k\in\mathbb{Z}}p_k=\mathbf{1}_{\mathcal{A}}-q\quad\text{with}\quad q=\bigwedge_{k\in\mathbb{Z}}q_k.$$

Calderón-Zygmund decomposition [58]. *Given $f\in\mathcal{A}_{+,K}$ and $\lambda>0$, we may decompose $f=g+b$ as the sum of operators defined in terms of Cuculescu's construction, where each term has a diagonal and an off-diagonal part given by*

- $g = g_\Delta + g_{\mathrm{off}}$, *where*

$$g_\Delta = qfq + \sum_{k\in\mathbb{Z}} p_k f_k p_k,$$
$$g_{\mathrm{off}} = \sum_{i\neq j} p_i f_{i\vee j} p_j \ + \ qf(\mathbf{1}_{\mathcal{A}} - q) + (\mathbf{1}_{\mathcal{A}} - q)fq.;$$

- $b = b_\Delta + b_{\mathrm{off}}$, *where*

$$b_\Delta = \sum_{k\in\mathbb{Z}} p_k(f - f_k)p_k\,, \qquad b_{\mathrm{off}} = \sum_{i\neq j} p_i(f - f_{i\vee j})p_j.$$

Moreover, we have the diagonal estimates

$$\Big\| qfq + \sum_{k\in\mathbb{Z}} p_k f_k p_k \Big\|_{L_2(\mathcal{A})}^2 \leq 2^d \lambda \|f\|_{L_1(\mathcal{A})},$$
$$\sum_{k\in\mathbb{Z}} \big\| p_k(f - f_k)p_k \big\|_{L_1(\mathcal{A})} \leq 2\|f\|_{L_1(\mathcal{A})}.$$

The expression below for g_{off} will be also instrumental

$$g_{\mathrm{off}} = \sum_{s=1}^{\infty} \sum_{k=m_\lambda+1}^{\infty} p_k df_{k+s} q_{k+s-1} + q_{k+s-1} df_{k+s} p_k.$$

The key result of this Part is Theorem 5.1, since the remaining theorems follow from it or by using analog ideas. We begin with the proof of the weak type estimates for perfect dyadic Calderón-Zygmund operators and then make the necessary adjustments to make it work for Haar shift operators. The proof of Theorem 5.1 (*ii*) will require to recall some recent results on square function and atomic Hardy spaces.

6.1 Perfect dyadic Calderón-Zygmund operators

To the best of our knowledge, the notion of perfect dyadic Calderón-Zygmund operator was rigorously defined for the first time in [2] by Auscher, Hofmann, Muscalu, Tao and Thiele. Accordingly, we define a *perfect dyadic* Calderón-Zygmund operator *with noncommuting kernel* as a pair $(T_{\mathrm{r}}, T_{\mathrm{c}})$ formally given by

$$T_{\mathrm{r}} f(x) \sim \int_{\mathbb{R}^d} f(y)k(x,y)\,dy, \qquad T_{\mathrm{c}} f(x) \sim \int_{\mathbb{R}^d} k(x,y)f(y)\,dy;$$

with an $\mathcal{M}$-valued kernel satisfying the perfect dyadic conditions

$$\big\| k(x,y) - k(z,y) \big\|_{\mathcal{M}} + \big\| k(y,x) - k(y,z) \big\|_{\mathcal{M}} = 0$$

whenever $x, z \in Q$ and $y \in R$ for some disjoint dyadic cubes Q, R. Alternatively, we may think of perfect dyadic kernels $k : \mathbb{R}^{2d} \backslash \Delta \to \mathcal{M}$ as those which are constant on $2d$-cubes of the form $Q \times R$, where Q, R are distinct dyadic cubes in $\mathbb{R}^d$ with the same side-length and sharing the same dyadic parent. Classical perfect dyadic

Calderón-Zygmund operators include Haar multipliers and dyadic paraproducts. In fact these operators and adjoints of paraproducts essentially build up the class of perfect dyadic Calderón-Zygmund operators as proved in [2]. In this respect, as we shall taler see, Haar shift operators generalize perfect dyadic operators. To further emphasize the perfect cancellation property of the kernel, we can express the associated scalar operators in the following form

$$T_\alpha f(x) = \int_{\mathbb{R}^d} \left(\sum_{Q \in \mathscr{D}} \frac{\alpha_{\widehat{Q}}}{|Q|} 1_Q(x)(1_Q - 2^{-d} 1_{\widehat{Q}})(y) \right) f(y)\, dy,$$

$$\Pi_\rho f(x) = \int_{\mathbb{R}^d} \left(\sum_{Q \in \mathscr{D}} \frac{1}{|Q|} (\langle \rho \rangle_Q - \langle \rho \rangle_{\widehat{Q}}) 1_Q(x) 2^{-d} 1_{\widehat{Q}}(y) \right) f(y)\, dy,$$

with $\sup_Q |\alpha_Q| < \infty$ and $\rho : \mathbb{R}^d \to \mathbb{C}$ in dyadic BMO. In the noncommuting setting, the coefficients α_Q and the symbol ρ become operators in $\mathcal{M}$ and a $\mathcal{M}$-valued function respectively which a priori do not commute with $f \in L_p(\mathcal{A})$. Nevertheless, the perfect dyadic condition for the kernel is still satisfied in these cases.

Proof of Theorem 5.1 (i) ***- Perfect dyadic operators***. Since f can be split as a sum of four positive operators and by density of the span of $\mathcal{A}_{+,K}$ in $L_1(\mathcal{A})$, we may clearly assume that $f \in \mathcal{A}_{+,K}$. A well-known lack of Cuculescu's construction is that we do not necessarily have $q_k(\lambda_1) \leq q_k(\lambda_2)$ for $\lambda_1 \leq \lambda_2$. This is typically solved restricting our attention to lacunary values for λ. Define

$$\pi_{j,k} = \bigwedge_{s \geq j} q_k(2^s) - \bigwedge_{s \geq j-1} q_k(2^s) \quad \text{for} \quad j,k \in \mathbb{Z}.$$

We have $\sum_j \pi_{j,k} = 1_\mathcal{A} - \psi_k$ in the SOT sense, where

$$\psi_k = \bigwedge_{s \in \mathbb{Z}} q_k(2^s).$$

Observe that $\psi_k df_k = df_k \psi_k = 0$ for $k \in \mathbb{Z}$. Indeed, we have

$$\begin{aligned} \|\psi_k df_k\|_\mathcal{A} &\leq \|\psi_k f_k^{\frac12}\|_\mathcal{A} \|f_k\|_\mathcal{A}^{\frac12} + \|\psi_k f_{k-1}^{\frac12}\|_\mathcal{A} \|f_{k-1}\|_\mathcal{A}^{\frac12} \\ &= \|\psi_k f_k \psi_k\|_\mathcal{A}^{\frac12} \|f_k\|_\mathcal{A}^{\frac12} + \|\psi_k f_{k-1} \psi_k\|_\mathcal{A}^{\frac12} \|f_{k-1}\|_\mathcal{A}^{\frac12} \leq \lim_{s \to -\infty} 2^{1+\frac{s}{2}} \|f\|_\mathcal{A}^{\frac12}. \end{aligned}$$

In particular, we find $f = \sum_k (1_\mathcal{A} - \psi_{k-1}) df_k (1_\mathcal{A} - \psi_{k-1})$ and set $f = f_\mathrm{r} + f_\mathrm{c}$ with

$$f_\mathrm{r} = \sum_{k \in \mathbb{Z}} \mathsf{LT}_{k-1}(df_k) = \sum_{k \in \mathbb{Z}} \left(\sum_{i > j} \pi_{i,k-1} df_k \pi_{j,k-1} \right),$$

$$f_\mathrm{c} = \sum_{k \in \mathbb{Z}} \mathsf{UT}_{k-1}(df_k) = \sum_{k \in \mathbb{Z}} \left(\sum_{i \leq j} \pi_{i,k-1} df_k \pi_{j,k-1} \right).$$

This is the decomposition we will use for any perfect dyadic Calderón-Zygmund operator. Given such an operator $T = (T_\mathrm{r}, T_\mathrm{c})$ and $\lambda > 0$, the goal is to show that there exists an absolute constant c_0 so that

$$\lambda \tau(\{|T_\mathrm{r} f_\mathrm{r}| > \lambda\}) + \lambda \tau(\{|T_\mathrm{c} f_\mathrm{c}| > \lambda\}) \leq c_0 \|f\|_{L_1(\mathcal{A})}$$

for any $f \in \mathcal{A}_{+,K}$ and any $\lambda > 0$. By symmetry in the argument, we will just prove the inequality for $T_{\mathrm{c}} f_{\mathrm{c}}$. Moreover, replacing c_0 by $2c_0$ we may also assume that $\lambda = 2^\ell$ for some $\ell \in \mathbb{Z}$. Having fixed the value of $\lambda = 2^\ell$, we may consider the Calderón-Zygmund decomposition $f = g_\Delta + g_{\mathrm{off}} + b_\Delta + b_{\mathrm{off}}$ and set

$$\begin{aligned} g_\Delta^{\mathrm{c}} &= \sum_{k\in\mathbb{Z}} \mathsf{UT}_{k-1}\big(\mathsf{D}_k(g_\Delta)\big), \quad & g_{\mathrm{off}}^{\mathrm{c}} &= \sum_{k\in\mathbb{Z}} \mathsf{UT}_{k-1}\big(\mathsf{D}_k(g_{\mathrm{off}})\big), \\ b_\Delta^{\mathrm{c}} &= \sum_{k\in\mathbb{Z}} \mathsf{UT}_{k-1}\big(\mathsf{D}_k(b_\Delta)\big), \quad & b_{\mathrm{off}}^{\mathrm{c}} &= \sum_{k\in\mathbb{Z}} \mathsf{UT}_{k-1}\big(\mathsf{D}_k(b_{\mathrm{off}})\big). \end{aligned}$$

By the quasi-triangle inequality it suffices to show

$$\lambda\Big(\tau\Big\{|T_{\mathrm{c}} g_\Delta^{\mathrm{c}}| > \lambda\Big\} + \tau\big\{|T_{\mathrm{c}} b_\Delta^{\mathrm{c}}| > \lambda\big\} + \tau\big\{|T_{\mathrm{c}} g_{\mathrm{off}}^{\mathrm{c}}| > \lambda\big\} + \tau\big\{|T_{\mathrm{c}} b_{\mathrm{off}}^{\mathrm{c}}| > \lambda\big\}\Big) \lesssim \|f\|_{L_1(\mathcal{A})}.$$

The first term is first estimated by Chebychev's inequality in $\mathcal{A}$

$$\lambda\tau\big\{|T_{\mathrm{c}} g_\Delta^{\mathrm{c}}| > \lambda\big\} \le \frac{1}{\lambda}\big\|T_{\mathrm{c}} g_\Delta^{\mathrm{c}}\big\|_{L_2(\mathcal{A})}^2 \lesssim \frac{1}{\lambda}\|g_\Delta^{\mathrm{c}}\|_{L_2(\mathcal{A})}^2.$$

We use that $\mathsf{UT}_{k-1}\big(\mathsf{D}_k(g_\Delta)\big)$ are in fact martingale differences, so that

$$\begin{aligned} \frac{1}{\lambda}\|g_\Delta^{\mathrm{c}}\|_{L_2(\mathcal{A})}^2 &= \frac{1}{\lambda}\sum_{k\in\mathbb{Z}} \big\|\mathsf{UT}_{k-1}\big(\mathsf{D}_k(g_\Delta)\big)\big\|_{L_2(\mathcal{A})}^2 \\ &\le \frac{1}{\lambda}\sum_{k\in\mathbb{Z}} \|\mathsf{D}_k(g_\Delta)\|_{L_2(\mathcal{A})}^2 = \frac{1}{\lambda}\Big\|\sum_{k\in\mathbb{Z}} \mathsf{D}_k(g_\Delta)\Big\|_{L_2(\mathcal{A})}^2 \\ &= \frac{1}{\lambda}\Big\|qfq + \sum_{k\in\mathbb{Z}} p_k f_k p_k\Big\|_{L_2(\mathcal{A})}^2 \le 2^d\|f\|_{L_1(\mathcal{A})}. \end{aligned}$$

Indeed, the first inequality above follows from the fact that triangular truncations are contractive in $L_2(\mathcal{A})$, while the last inequality arise from the diagonal estimates in the noncommutative Calderón-Zygmund decomposition stated above. To handle the remaining terms, we introduce the projection

$$\widehat{q} = \bigwedge_{s\ge\ell} q(2^s) = \bigwedge_{s\ge\ell}\bigwedge_{k\in\mathbb{Z}} q_k(2^s).$$

According to Cuculescu's construction, we find

$$\tau\big(1_{\mathcal{A}} - \widehat{q}\big) \le \sum_{s\ge\ell} \tau\big(1_{\mathcal{A}} - q(2^s)\big) \le \sum_{s\ge\ell} \frac{1}{2^s}\|f\|_{L_1(\mathcal{A})} = \frac{2}{\lambda}\|f\|_{L_1(\mathcal{A})}.$$

This reduces our problem to show that

$$\lambda\Big(\tau\big\{\big|T_{\mathrm{c}}(b_\Delta^{\mathrm{c}})\widehat{q}\big| > \lambda\big\} + \tau\big\{\big|T_{\mathrm{c}}(g_{\mathrm{off}}^{\mathrm{c}})\widehat{q}\big| > \lambda\big\} + \tau\big\{\big|T_{\mathrm{c}}(b_{\mathrm{off}}^{\mathrm{c}})\widehat{q}\big| > \lambda\big\}\Big) \lesssim \|f\|_{L_1(\mathcal{A})}.$$

The perfect dyadic nature of T_{c} comes now into scene. Indeed, we claim that the three terms $T_{\mathrm{c}}(b_\Delta^{\mathrm{c}})\widehat{q}$, $T_{\mathrm{c}}(g_{\mathrm{off}}^{\mathrm{c}})\widehat{q}$ and $T_{\mathrm{c}}(b_{\mathrm{off}}^{\mathrm{c}})\widehat{q}$ vanish whenever T_{c} is perfect dyadic. This will be enough to conclude the proof. If $Q_k(x)$ is the only cube in $\mathscr{D}_k$ containing x, we find a.e. x

$$\begin{aligned} T_{\mathrm{c}}(b_\Delta^{\mathrm{c}})(x)\widehat{q}(x) &= \sum_{k\in\mathbb{Z}} T_{\mathrm{c}}\big(\mathsf{UT}_{k-1}(\mathsf{D}_k(b_\Delta))\big)(x)\,\widehat{q}(x) \\ &= \sum_{k\in\mathbb{Z}} T_{\mathrm{c}}\Big(\mathsf{UT}_{k-1}(\mathsf{D}_k(b_\Delta))1_{Q_{k-1}(x)}\Big)(x)\,\widehat{q}(x) \\ &\qquad + \sum_{k\in\mathbb{Z}} \sum_{\substack{Q\in\mathscr{D}_{k-1} \\ x\notin Q}} \left(\int_Q k(x,y)\mathsf{UT}_{k-1}(\mathsf{D}_k(b_\Delta))(y)\,dy\right)\widehat{q}(x). \end{aligned}$$

The last term on the right vanishes since the term $\mathsf{UT}_{k-1}(\mathsf{D}_k(b_\Delta))$ has mean 0 in any $Q \in \mathscr{D}_{k-1}$, so that we may replace $k(x,y)$ by $k(x,y) - k(x,c_Q)$, which is 0 when $x \notin Q$ by the perfect dyadic cancellation of the kernel. On the other hand, if we define the projection

$$\widehat{q}_{k-1} = \bigwedge_{s\geq \ell} q_{k-1}(2^s),$$

we see that $\widehat{q}(x) = \widehat{q}_{k-1}(x)\widehat{q}(x) = \widehat{q}_{k-1}(y)\widehat{q}(x)$ for any $y \in Q_{k-1}(x)$. This gives

$$T_{\mathrm{c}}(b_\Delta^{\mathrm{c}})(x)\widehat{q}(x) = \sum\nolimits_k T_{\mathrm{c}}\Big(\mathsf{UT}_{k-1}(\mathsf{D}_k(b_\Delta))\widehat{q}_{k-1}1_{Q_{k-1}(x)}\Big)(x)\,\widehat{q}(x).$$

The exact same argument applies for $g_{\mathrm{off}}^{\mathrm{c}}$ and $b_{\mathrm{off}}^{\mathrm{c}}$, so that it suffices to prove

$$\begin{aligned} \mathsf{UT}_{k-1}(\mathsf{D}_k(b_\Delta))\,\widehat{q}_{k-1} &= 0, \\ \mathsf{UT}_{k-1}(\mathsf{D}_k(g_{\mathrm{off}}))\,\widehat{q}_{k-1} &= 0, \\ \mathsf{UT}_{k-1}(\mathsf{D}_k(b_{\mathrm{off}}))\,\widehat{q}_{k-1} &= 0, \end{aligned}$$

for all $k \in \mathbb{Z}$. In all these cases we will be using the following two key identities

- $\widehat{q}_{k-1}\pi_{i,k-1} = \pi_{j,k-1}\widehat{q}_{k-1} = 0$ for $i,j > \ell$ and $k \in \mathbb{Z}$,
- $\pi_{i,k-1}p_{k-s} = p_{k-s}\pi_{j,k-1} = 0$ for $s \geq 1$, $i,j \leq \ell$ and $k \in \mathbb{Z}$.

The proof is straightforward and left to the reader. It only requires to apply the monotonicity properties of $\bigwedge_{s\geq j} q_k(2^s)$, which increases in j and decreases in k. If we apply the first identity to $\mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\,\widehat{q}_{k-1}$ for any γ, we get

$$\mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\,\widehat{q}_{k-1} = \sum_{i\leq j\leq \ell} \pi_{i,k-1}d\gamma_k\pi_{j,k-1}\widehat{q}_{k-1}.$$

Therefore, if we know that $d\gamma_k = A_k + B_k$ where the left support of A_k and the right support of B_k are dominated by $\sum_{s\geq 1} p_{k-s} = 1_{\mathcal{A}} - q_{k-1}$, then we deduce that $\mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\,\widehat{q}_{k-1} = 0$. In other words, it suffices to prove that

$$q_{k-1}\mathsf{D}_k(\gamma)q_{k-1} = 0 \quad \text{for} \quad \gamma =\in \{b_\Delta, g_{\mathrm{off}}, b_{\mathrm{off}}\}.$$

We have

$$\mathsf{D}_k(b_\Delta) = \sum_j \mathsf{D}_k\big(p_j(f - f_j)p_j\big)$$

$$
\begin{aligned}
&= \sum_{j<k} p_j(f_k - f_j)p_j - \sum_{j<k-1} p_j(f_{k-1} - f_j)p_j \\
&= \sum_{j\le k-1} p_j df_k p_j = (1_{\mathcal{A}} - q_{k-1})\mathsf{D}_k(b_\Delta)(1_{\mathcal{A}} - q_{k-1}).
\end{aligned}
$$

To calculate the martingale differences for g_{off}, we invoke the formula

$$
g_{\text{off}} = \sum_{s=1}^{\infty}\sum_{j\in\mathbb{Z}} p_j df_{j+s} q_{j+s-1} + q_{j+s-1} df_{j+s} p_j
$$

given in the statement of the Calderón-Zygmund decomposition. Then we find

$$
\begin{aligned}
\mathsf{D}_k(g_{\text{off}}) &= \sum_{s=1}^{\infty} p_{k-s} df_k q_{k-1} + q_{k-1} df_k p_{k-s} \\
&= (1_{\mathcal{A}} - q_{k-1}) df_k q_{k-1} + q_{k-1} df_k (1_{\mathcal{A}} - q_{k-1}).
\end{aligned}
$$

Finally, it remains to consider the martingale differences of b_{off}

$$
\begin{aligned}
\mathsf{D}_k(b_{\text{off}}) &= \sum_{s=1}^{\infty}\sum_{j\in\mathbb{Z}} \mathsf{D}_k\big(p_j(f - f_{j+s})p_{j+s} + p_{j+s}(f - f_{j+s})p_j\big) \\
&= \sum_{s=1}^{\infty}\sum_{j<k-s} p_j(f_k - f_{j+s})p_{j+s} + p_{j+s}(f_k - f_{j+s})p_j \\
&\qquad - \sum_{s=1}^{\infty}\sum_{j<k-s-1} p_j(f_{k-1} - f_{j+s})p_{j+s} + p_{j+s}(f_{k-1} - f_{j+s})p_j \\
&= \sum_{s=1}^{\infty}\sum_{j<k-s} p_j df_k p_{j+s} + \sum_{s=1}^{\infty}\sum_{j<k-s} p_{j+s} df_k p_j = A_k + A_k^*.
\end{aligned}
$$

So $q_{k-1}A_k = A_k^* q_{k-1} = 0$ and $q_{k-1}\mathsf{D}_k(\gamma)q_{k-1} = 0$ for $\gamma = b_\Delta, g_{\text{off}}, b_{\text{off}}$ as desired. □

6.2 Haar shift operators

We say that $\Phi = \{\phi_Q\}_{Q\in\mathscr{D}}$ is a *generalized Haar system* in $\mathbb{R}^d$ adapted to $\mathscr{D}$ if the following conditions hold:

(a) For every $Q \in \mathscr{D}$, $\operatorname{supp}(\phi_Q) \subset Q$.

(b) If Q', $Q \in \mathscr{D}$ and $Q' \subsetneq Q$, then ϕ_Q is constant on Q'.

(c) For every $Q \in \mathscr{D}$, $\displaystyle\int_{\mathbb{R}^d} \phi_Q(x)\,dx = 0$.

(d) For every $Q \in \mathscr{D}$, we have $\|\phi_Q\|_{L^2(\mu)} = 1$.

Such Haar systems yield orthonormal systems in $L_2(\mathbb{R}^d)$. If the vanishing integral condition (c) is not imposed, the Haar system is said to be *non-cancellative*. Particular constructions of Haar systems are considerer in Part I.

Let $\Phi = \{\phi_Q\}_{Q\in\mathscr{D}}$ and $\Psi = \{\psi_Q\}_{Q\in\mathscr{D}}$ be two non-necessarily cancellative generalized Haar systems in $\mathbb{R}^d$. A column *noncommuting Haar shift with complexity* (r,s) has the form

$$\text{Ш}^{\mathrm{c}}_{r,s}f(x) = \sum_{Q\in\mathscr{D}} A_Q f = \sum_{Q\in\mathscr{D}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\langle f,\phi_R\rangle\psi_S(x), \quad \sup \big\|\alpha^Q_{R,S}\big\|_{\mathcal{M}} < \infty.$$

where $\langle f,\phi_R\rangle = \int f\phi_R\,dx \in \mathcal{M}$. A row operator is likewise defined. If the underlying Haar systems are cancellative, the Har shift operator is said to be *cancellative*. Several objects in commutative dyadic harmonic analysis admit this general form, including Haar multipliers, dyadic paraproducts, the dyadic Hilbert transform and their adjoints. As in the classical case, orthogonality arguments are enough to show L_2 boundedness. Further, if the symbols $\alpha^Q_{R,S}\in\mathcal{M}$ satisfy the estimate

$$\big\|\alpha^Q_{R,S}\big\|_{\mathcal{M}} \leq \frac{\sqrt{|R||S|}}{|Q|} = 2^{-\frac{1}{2}(r+s)\,d}$$

the associated Haar shift operator is contractive in $L_2(\mathcal{A})$. We proceed to show this.

Lemma 6.1. *A cancellative row/column Haar shift operator* $\text{Ш}_{r,s}$ *satisfies the* L_2 *estimate*

$$\|\text{Ш}_{r,s}f\|_{L_2(\mathcal{A})} \leq 2^{\frac{1}{2}(r+s)\,d}\sup\big\|\alpha^Q_{R,S}\big\|_{\mathcal{M}}\|f\|_{L_2(\mathcal{A})}.$$

Proof. The argument is standard. Observe that for a row/column operator we have

$$\text{Ш}_{r,s}f = \sum_{Q\in\mathscr{D}} P^s_{\Psi,Q}A_QP^r_{\Phi,Q}f,$$

where $P^r_{\Phi,Q}$ and $P^r_{\Psi,Q}$ denote the projections

$$P^r_{\Phi,Q}f = \sum_{R\in\mathscr{D}_r(Q)}\langle f,\phi_R\rangle\phi_R \quad\text{and}\quad P^s_{\Psi,Q}f = \sum_{S\in\mathscr{D}_s(Q)}\langle f,\psi_S\rangle\psi_S;$$

thus obtaining families of projections orthogonal on the index Q. Therefore

$$\|\text{Ш}_{r,s}f\|^2_{L_2(\mathcal{A})} = \sum_{Q\in\mathscr{D}}\|A_Qf\|^2_{L_2(\mathcal{A})} = \sum_{Q\in\mathscr{D}}\big\|A_QP^r_{\Phi,Q}f\big\|^2_{L_2(\mathcal{A})}.$$

It is easily seen that A_Q is a bounded operator on $L_2(\mathcal{A})$. Indeed, by Hölder's and triangular inequalities we have

$$\begin{aligned}
\|A_Qg\|^2_{L_2(\mathcal{A})} &= \sum_{S\in\mathscr{D}_s(Q)}\Bigg\|\sum_{R\in\mathscr{D}_r(Q)}\alpha^Q_{R,S}\langle g,\phi_R\rangle\Bigg\|^2_{L_2(\mathcal{M})}\\
&\leq \sum_{S\in\mathscr{D}_s(Q)}\Bigg(\sum_{R\in\mathscr{D}_r(Q)}\big\|\alpha^Q_{R,S}\big\|_{\mathcal{M}}\int_R\|g(x)\|_{L_2(\mathcal{M})}\,|\phi_R(x)|dx\Bigg)^2\\
&\leq \sum_{S\in\mathscr{D}_s(Q)}\sum_{R\in\mathscr{D}_r(Q)}\big\|\alpha^Q_{R,S}\big\|^2_{\mathcal{M}}\int_Q\|g(x)\|^2_{L_2(\mathcal{M})}dx\\
&\leq 2^{(r+s)\,d}\sup\big\|\alpha^Q_{R,S}\big\|^2_{\mathcal{M}}\|g\|^2_{L_2(\mathcal{A})}.
\end{aligned}$$

This yields

$$\begin{aligned}\|\text{Ш}_{r,s}f\|^2_{L_2(\mathcal{A})} &\leq 2^{(r+s)\,d}\sup\big\|\alpha^Q_{R,S}\big\|_{\mathcal{M}}\sum_{Q\in\mathscr{D}}\big\|P^r_{\Phi,Q}f\big\|^2_{L_2(\mathcal{A})}\\ &\leq 2^{(r+s)\,d}\sup\big\|\alpha^Q_{R,S}\big\|_{\mathcal{M}}\|f\|^2_{L_2(\mathcal{A})}.\end{aligned}$$ □

As in the case of paraproducts, in what follows we will assume that noncancellative Haar shift operators are bounded on $L_2(\mathcal{A})$.

The next lemma is crucial to analyze Haar shifts and general Calderón-Zygmund operators with noncommuting kernels. We take here the opportunity to slightly modify the argument in [58, Lemma 4.2], which was not entirely correct.

Lemma 6.2. *Given $r\in\mathbb{Z}_+$, there exists $\zeta\in\mathcal{P}(\mathcal{A})$ such that:*

(i) $\lambda\tau(1_{\mathcal{A}}-\zeta)\leq 2^{rd}\|f\|_{L_1(\mathcal{A})}$,

(ii) If $Q_0\in\mathscr{D}_{k_0}$ and $x\in Q_0^{(r)}$, then $\zeta(x)\leq\widehat{q}_{k_0}(y)$ for all $y\in Q_0$.

In the second property, we write $Q_0^{(r)}$ for the unique r-th dyadic ancestor of Q_0.

Proof. We have

$$1_{\mathcal{A}}-\widehat{q}_k=\sum_{j\leq k}\big(\widehat{q}_{j-1}-\widehat{q}_j\big)=\sum_{j\leq k}\sum_{Q\in\mathscr{D}_j}\rho_Q\otimes 1_Q=\sum_{Q\in\mathscr{D}_k}\Big[\sum_{R\supset Q}\rho_R\Big]\otimes 1_Q$$

for some family of projections $\rho_Q\in\mathcal{P}(\mathcal{M})$. Define

$$\zeta=\bigwedge_{k\in\mathbb{Z}}\zeta_k\quad\text{with}\quad\zeta_k=1_{\mathcal{A}}-\bigvee_{j\leq k}\bigvee_{Q\in\mathscr{D}_j}\rho_Q 1_{Q^{(s)}}.$$

It is clear that the ζ_k's are decreasing in k and we find

$$\begin{aligned}\lambda\tau(1_{\mathcal{A}}-\zeta) &= \lambda\lim_{k\to\infty}\tau(1_{\mathcal{A}}-\zeta_k)\\ &\leq \lambda\lim_{k\to\infty}\sum_{j\leq k}\sum_{Q\in\mathscr{D}_j}\nu(\rho_Q)\big|Q^{(r)}\big|\\ &= 2^{rd}\lim_{k\to\infty}\lambda\sum_{j\leq k}\sum_{Q\in\mathscr{D}_j}\tau(\rho_Q\otimes 1_Q)\\ &= 2^{rd}\lambda\,\tau\big(1_{\mathcal{A}}-\widehat{q}\big)=2^{rd}\lambda\sum_{m\geq\ell}\tau\big(1_{\mathcal{A}}-q(2^m)\big)\lesssim 2^{rd}\|f\|_{L_1(\mathcal{A})}.\end{aligned}$$

To prove the second property, it will be useful to observe that $Q_1\subsetneq Q_2$ implies that ρ_{Q_1} and ρ_{Q_2} are orthogonal projections. Indeed, according to the definition of ρ_Q above, we have $\rho_{Q_1}\rho_{Q_2}1_{Q_1}=(\widehat{q}_{j_1-1}-\widehat{q}_{j_1})(\widehat{q}_{j_2-1}-\widehat{q}_{j_2})1_{Q_1}=0$ for $\ell(Q_1)=2^{-j_1}$ and $\ell(Q_2)=2^{-j_2}$. Then, we find

$$\begin{aligned}\zeta(x)\leq\zeta_{k_0}(x) &= 1_{\mathcal{M}}-\bigvee_{j\leq k_0}\bigvee_{Q\in\mathscr{D}_j}\rho_Q 1_{Q^{(r)}}(x)\\ &\leq 1_{\mathcal{M}}-\bigvee_{R\supset Q_0}\rho_R=1_{\mathcal{M}}-\sum_{R\supset Q_0}\rho_R\\ &= \Bigg(1_{\mathcal{A}}-\sum_{Q\in\mathscr{D}_{k_0}}\Bigg(\sum_{R\supset Q}\rho_R\Bigg)\otimes 1_Q\Bigg)(y)=\widehat{q}_{k_0}(y).\end{aligned}$$ □

Proof of Theorem 5.1 (i) ***- Haar shift operators.*** As in the perfect dyadic case, we assume $f \in \mathcal{A}_{+,K}$ and decompose $f = f_{\mathrm{r}} + f_{\mathrm{c}}$ in the same way. Once more the argument is row/column symmetric, and we just consider the column part. After fixing $\lambda = 2^\ell$ for some $\ell \in \mathbb{Z}$, we construct the corresponding Calderón-Zygmund decomposition for $f_{\mathrm{c}} = g_\Delta^{\mathrm{c}} + g_{\mathrm{off}}^{\mathrm{c}} + b_\Delta^{\mathrm{c}} + b_{\mathrm{off}}^{\mathrm{c}}$. According to Lemma 6.1, we may control the term $\text{Ш}_{r,s}^{\mathrm{c}}(g_\Delta^{\mathrm{c}})$ in as in Theorem 5.1 (i). Given $\gamma \in \{b_\Delta, g_{\mathrm{off}}, b_{\mathrm{off}}\}$, the other terms can be decomposed as follows

$$\begin{aligned}
\text{Ш}_{r,s}^{\mathrm{c}}(\gamma^{\mathrm{c}}) &= \sum_{k\in\mathbb{Z}} \text{Ш}_{r,s}^{\mathrm{c}}\big(\mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\big) \\
&= \sum_{k\in\mathbb{Z}} \sum_{Q\in\mathscr{D}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha_{R,S}^Q \Big(\int_{\mathbb{R}^d} \mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\phi_R\, dy \Big) \psi_S(x) \\
&= \sum_{k\in\mathbb{Z}} \Bigg(\sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leq 2^{-k+1}}} + \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)>2^{-k+1}\\ \ell(Q)\leq 2^{r-k+1}}} + \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)>2^{r-k+1}}} \Bigg) = A_\gamma + B_\gamma + C_\gamma.
\end{aligned}$$

We claim that $C_\gamma = 0$. Namely, we have $\ell(R) = 2^{-s}\ell(Q) > 2^{-k+1}$. This means that $\mathsf{E}_{k-1}(\phi_R) = \phi_R$ since the Haar functions ϕ_R are constant in the dyadic children of S, whose length sides are greater or equal than $2^{-(k-1)}$. This yields

$$\begin{aligned}
\int_{\mathbb{R}^d} \mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\phi_R\, dy &= \int_{\mathbb{R}^d} \mathsf{E}_{k-1}\big(\mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\phi_R\big)\, dy \\
&= \int_{\mathbb{R}^d} \mathsf{E}_{k-1}\big(\mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\big)\phi_R\, dy \\
&= \int_{\mathbb{R}^d} \big(\mathsf{UT}_{k-1}(\mathsf{E}_{k-1}\mathsf{D}_k(\gamma))\phi_R\, dy = 0.
\end{aligned}$$

In order to deal with the remaining terms A_γ and B_γ, we invoke the identity

$$q_{k-1}\mathsf{D}_k(\gamma)q_{k-1} = 0$$

which was already justified in the perfect dyadic case whenever $\gamma = b_\Delta, g_{\mathrm{off}}, b_{\mathrm{off}}$. Namely, since $\pi_{i,k-1}(1_{\mathcal{A}} - q_{k-1}) = (1_{\mathcal{A}} - q_{k-1})\pi_{j,k-1} = 0$ for $i, j \leq \ell$, we find

$$\mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma)) = \sum_{i\leq j} \pi_{i,k-1}\mathsf{D}_k(\gamma)\pi_{j,k-1} = \sum_{\substack{i\leq j\\ j>\ell}} \pi_{i,k-1}\mathsf{D}_k(\gamma)\pi_{j,k-1}.$$

Let us now consider the term A_γ, we have

$$\lambda\tau\big\{|A_\gamma| > \lambda\big\} \leq \lambda\,\tau\big(1_{\mathcal{A}} - \widehat{q}\big) + \lambda\tau\Big\{\big|A_\gamma\widehat{q}\big| > \frac{\lambda}{2}\Big\}.$$

We already know that the first term on the right is dominated by $\|f\|_{L_1(\mathcal{A})}$ and

$$A_\gamma\widehat{q} = \sum_{k\in\mathbb{Z}} \sum_{\substack{Q\in\mathscr{D}\\ \ell(Q)\leq 2^{-k+1}}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha_{R,S}^Q \bigg(\int_{\mathbb{R}^d} \mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\phi_R\, dy \bigg) \psi_S(x)\, \widehat{q}(x).$$

Given $Q \in \mathscr{D}$ with $\ell(Q) \leq 2^{-k+1}$ let

$$k_Q \geq k-1 \quad \text{determined by} \quad \ell(Q) = 2^{-k_Q}.$$

It is clear that $\widehat{q}(x) = \widehat{q}_{k_Q}(x)\widehat{q}(x) = \widehat{q}_{k_Q}(y)\widehat{q}(x) = \widehat{q}_{k-1}(y)\widehat{q}(x)$ whenever x, y belong to Q. However, the presence of $\psi_S(x), \phi_R(y)$ implies (unless the corresponding term is 0) that the pair $(x,y) \in S \times R \subset Q \times Q$ so that we may write

$$A_\gamma \widehat{q} = \sum_{k \in \mathbb{Z}} \sum_{\substack{Q \in \mathscr{D} \\ \ell(Q) \leq 2^{-k+1}}} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha_{R,S}^Q \Big(\int_{\mathbb{R}^d} \mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\widehat{q}_{k-1}\phi_R \, dy \Big) \psi_S(x) \, \widehat{q}(x).$$

Therefore, we conclude

$$\mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\widehat{q}_{k-1} = \sum_{\substack{i \leq j \\ j > \ell}} \pi_{i,k-1}\mathsf{D}_k(\gamma)\pi_{j,k-1}\widehat{q}_{k-1} = 0$$

since $\pi_{j,k-1}\widehat{q}_{k-1} = 0$ when $j > \ell$. This shows that $A_\gamma \widehat{q} = 0$. Let us finally consider the term B_γ. We will follow a similar argument with the projection ζ from Lemma 6.2 instead. Namely, we have

$$\lambda\tau\big\{|B_\gamma| > \lambda\big\} \leq \lambda\tau\big(1_{\mathcal{A}} - \zeta\big) + \lambda\tau\Big\{\big|B_\gamma\zeta\big| > \frac{\lambda}{2}\Big\}.$$

According to property i) of Lemma 6.2, it suffices to show that $B_\gamma \zeta = 0$. Now we know that $\ell(Q) \leq 2^{r-k+1}$, so that $k_Q \geq k-r-1$. Let us now consider the 2^{rd} dyadic cubes T_j having Q as their r-th dyadic ancestor. This gives rise to the identities

$$\zeta(x) = \zeta_{k_Q+r}(x)\zeta(x) = \zeta_{k_Q+r}(y)\zeta(x) = \widehat{q}_{k_Q+r}(z)\zeta(x) = \widehat{q}_{k-1}(z)\zeta(x)$$

for $(x,y,z) \in Q \times Q \times T_j$. Indeed, the second identity follows from the fact that $\mathsf{E}_{k_Q}(\zeta_{k_Q+r}) = \zeta_{k_Q+r}$, the third one from the second property in Lemma 6.2 and the last one from the inequality $k_Q \geq k-r-1$. Hence, given $y \in S \subset Q$ we pick the unique j for which $R = T_j$ and deduce that $\zeta(x) = \widehat{q}_{k-1}(y)\zeta(x)$. Then it yields the identity

$$B_\gamma \zeta = \sum_{k \in \mathbb{Z}} \sum_{\substack{Q \in \mathscr{D} \\ \ell(Q) > 2^{-k+1} \\ \ell(Q) \leq 2^{r-k+1}}} \sum_{\substack{R \in \mathscr{D}_r(Q) \\ S \in \mathscr{D}_s(Q)}} \alpha_{R,S}^Q \Big(\int_{\mathbb{R}^d} \mathsf{E}_{k-1}(\mathsf{D}_k(\gamma))\widehat{q}_{k-1}\phi_R \, dy \Big) \psi_S(x) \, \zeta(x).$$

The integrand $\mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\widehat{q}_{k-1}$ vanishes for the same reason as it did above. □

Remark 6.3. Our constants $\sim 2^{rd}$ seem far from being sharp. The classical argument giving constants $\sim r$ unfortunately encounters a major obstacle due to the presence of triangular truncations, which are not bounded in L_1. This is also the reason why we did not succeed in extended the argument above to generic Calderón-Zygmund operators. In fact, we leave this as an open problem for any interested reader.

Remark 6.4. Note that our decomposition $f = f_\mathrm{r} + f_\mathrm{c}$ is completely determined by the projections $\pi_{j,k}$, which in turn depend on f. According to the statement of Theorem 5.1 (*ii*), it would be desirable to identify subspaces or even subsets $A_\mathrm{r}/A_\mathrm{c}$ of $L_1(\mathcal{A})$ for which we have

$$T_\mathrm{r} : A_\mathrm{r} \to L_{1,\infty}(\mathcal{A}) \quad \text{and} \quad T_\mathrm{c} : A_\mathrm{c} \to L_{1,\infty}(\mathcal{A}).$$

Note however that our use of Calderón-Zygmund decomposition provides estimates of the form $\|T_\mathrm{r} f_\mathrm{r}\|_{L_{1,\infty}(\mathcal{A})} \lesssim \|f\|_{L_1(\mathcal{A})}$. Morally, f can not be replaced by f_r on the right hand side since triangular truncations are not bounded in L_1. On the other hand, the sets A_r and A_c are not empty since both contain

$$A = \Big\{ f \in L_1^+(\mathcal{A}) \, \big| \, f = \sum_{j,k \in \mathbb{Z}} \pi_{j,k-1} df_k \pi_{j,k-1} \Big\},$$

which in turn contains all $f \in L_1^+(\mathcal{A})$ such that $f(x)$ belongs to the center of $\mathcal{M}$ for all $x \in \mathbb{R}^d$. Note that A is not a linear subspace since the $\pi_{j,k}$'s depend on f. It is an interesting problem to determine larger sets $A_\mathrm{r}/A_\mathrm{c}$ in $L_1(\mathcal{A})$.

Chapter 7

Noncommuting Calderón-Zygmund operators

7.1 Operator-valued Hardy spaces

The proofs of Theorems 5.1 (*ii*), 5.2 and 5.3 arise from a careful combination of recent results in the theory of noncommutative Hardy spaces. Let us begin introducing Mei's notion [51] of row and column Hardy spaces for our algebra of operator-valued functions $\mathcal{A}$. In order to distinguish from order Hardy spaces to be introduced below, let us follows Mei's notation and define

$$\mathrm{H}_1(\mathbb{R}^d;\mathcal{M}) = \mathrm{H}_1^{\mathrm{r}}(\mathbb{R}^d;\mathcal{M}) + \mathrm{H}_1^{\mathrm{c}}(\mathbb{R}^d;\mathcal{M})$$

as the space of functions $f \in L_1(\mathcal{A})$ for which we have

$$\|f\|_{\mathrm{H}_1(\mathbb{R}^d;\mathcal{M})} = \inf_{f=g+h} \|g\|_{\mathrm{H}_1^{\mathrm{r}}(\mathbb{R}^d;\mathcal{M})} + \|h\|_{\mathrm{H}_1^{\mathrm{c}}(\mathbb{R}^d;\mathcal{M})} < \infty,$$

where the row/column norms are given by

$$\|g\|_{\mathrm{H}_1^{\mathrm{r}}(\mathbb{R}^d;\mathcal{M})} = \Big\| \Big(\int_\Gamma \Big[\frac{\partial \widehat{g}}{\partial t} \frac{\partial \widehat{g}^*}{\partial t} + \sum_j \frac{\partial \widehat{g}}{\partial x_j} \frac{\partial \widehat{g}^*}{\partial x_j} \Big] (x+\cdot,t) \, \frac{dxdt}{t^{d-1}} \Big)^{\frac12} \Big\|_{L_1(\mathcal{A})},$$

$$\|h\|_{\mathrm{H}_1^{\mathrm{c}}(\mathbb{R}^d;\mathcal{M})} = \Big\| \Big(\int_\Gamma \Big[\frac{\partial \widehat{h}^*}{\partial t} \frac{\partial \widehat{h}}{\partial t} + \sum_j \frac{\partial \widehat{h}^*}{\partial x_j} \frac{\partial \widehat{h}}{\partial x_j} \Big] (x+\cdot,t) \, \frac{dxdt}{t^{d-1}} \Big)^{\frac12} \Big\|_{L_1(\mathcal{A})},$$

with $\Gamma = \{(x,t) \in \mathbb{R}_+^{d+1} \mid |x| < y\}$ and $\widehat{f}(x,t) = P_t f(x)$ for the Poisson semigroup $(P_t)_{t \ge 0}$. In other words, operator-valued forms of Lusin's square function. We say that $a \in L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$ is a *column atom* if there exists a cube Q so that

- $\mathrm{supp}_{\mathbb{R}^d}\, a = Q$,
- $\displaystyle\int_Q a(y)\, dy = 0$,
- $\displaystyle\|a\|_{L_1(\mathcal{M};L_2^c(\mathbb{R}^d))} = \nu\bigg(\Big(\int_Q |a(y)|^2 \, dy \Big)^{\frac12} \bigg) \le \frac{1}{\sqrt{|Q|}}$.

According to [51, Theorem 2.8], we have

$$\|f\|_{\mathrm{H}_1^c(\mathbb{R}^d;\mathcal{M})} \sim \inf \Big\{ \sum\nolimits_k |\lambda_k| \, \big| \, f = \sum\nolimits_k \lambda_k a_k \text{ with } a_k \text{ column atoms} \Big\}.$$

On the other hand, we have already settled a dyadic filtration $(\mathcal{A}_k)_{k\in\mathbb{Z}}$ for our algebra of operator-valued functions $\mathcal{A}$. Then, we may follow [66] to define the corresponding noncommutative Hardy space $\mathrm{H}_1(\mathcal{A})$ as the completion of the space of finite martingales in $L_1(\mathcal{A})$ with respect to the norm

$$\|f\|_{\mathrm{H}_1(\mathcal{A})} = \inf_{\substack{f=g+h \\ g,h \text{ martingales}}} \Big\| \Big(\sum_{k\in\mathbb{Z}} dg_k dg_k^* \Big)^{\frac12} \Big\|_{L_1(\mathcal{A})} + \Big\| \Big(\sum_{k\in\mathbb{Z}} dh_k^* dh_k \Big)^{\frac12} \Big\|_{L_1(\mathcal{A})}.$$

In other words, $\mathrm{H}_1(\mathcal{A}) = \mathrm{H}_1^r(\mathcal{A}) + \mathrm{H}_1^c(\mathcal{A})$, where the spaces on the right are the completions of the spaces of finite L_1-martingales with respect to the norms in L_1 of the corresponding row/column square functions given above. By the use of a dyadic covering [13, 51], it can be shown that there exist $d+1$ dyadic filtrations $\Sigma_{\mathcal{A}}^j$ $(0 \le j \le n)$ in $\mathbb{R}^d$ so that

$$\mathrm{H}_1(\mathbb{R}^d;\mathcal{M}) \simeq \sum_{j=0}^d \mathrm{H}_1(\mathcal{A}, \Sigma_{\mathcal{A}}^j),$$

where the latter spaces are defined as $\mathrm{H}_1(\mathcal{A})$ after replacing the standard filtration $\Sigma_{\mathcal{A}}^0$ by any other dyadic filtration in our family. Moreover, this isomorphism also holds independently for row/column Hardy spaces.

Proof of Theorem 5.1 *(ii)*. We will show that

$$T_r : \mathrm{H}_1^r(\mathcal{A}) \to L_1(\mathcal{A}) \quad \text{and} \quad T_c : \mathrm{H}_1^c(\mathcal{A}) \to L_1(\mathcal{A}),$$

for any generic noncommuting Calderón-Zygmund operator (T_r, T_c). Indeed, in that case we decompose $f = f_r + f_c \in \mathrm{H}_1(\mathcal{A})$, so that

$$\|f\|_{\mathrm{H}_1(\mathcal{A})} \sim \|f_r\|_{\mathrm{H}_1^r(\mathcal{A})} + \|f_c\|_{\mathrm{H}_1^c(\mathcal{A})}$$

and we deduce that

$$\|T_r f_r\|_{L_1(\mathcal{A})} + \|T_c f_c\|_{L_1(\mathcal{A})} \lesssim \|f_r\|_{\mathrm{H}_1^r(\mathcal{A})} + \|f_c\|_{\mathrm{H}_1^c(\mathcal{A})} \sim \|f\|_{\mathrm{H}_1(\mathcal{A})}.$$

According to our observation above, $\mathrm{H}_1(\mathcal{A})$ embeds isomorphically into $\mathrm{H}_1(\mathbb{R}^d;\mathcal{M})$ by means of a suitable choice of dyadic coverings of $\mathbb{R}^d$, and the same holds for row and column spaces isolatedly. Thus, it suffices to show that

$$T_r : \mathrm{H}_1^r(\mathbb{R}^d;\mathcal{M}) \to L_1(\mathcal{A}) \quad \text{and} \quad T_c : \mathrm{H}_1^c(\mathbb{R}^d;\mathcal{M}) \to L_1(\mathcal{A})$$

boundedly. Both estimates are identical, let us prove the column case. According to the atomic decomposition of $\mathrm{H}_1^c(\mathbb{R}^d;\mathcal{M})$ we just find a uniform upper estimate for the L_1 norm of $T_c(a)$ valid for an arbitrary column atom a

$$\|T_c(a)\|_{L_1(\mathcal{A})} \le \big\| T_c(a) 1_{2Q} \big\|_{L_1(\mathcal{A})} + \big\| T_c(a) 1_{\mathbb{R}^d \setminus 2Q} \big\|_{L_1(\mathcal{A})}.$$

The second term is dominated by

$$\begin{aligned}
\big\|T_{\mathrm{c}}(a)1_{\mathbb{R}^d\setminus 2Q}\big\|_{L_1(\mathcal{A})} &= \nu \int_{\mathbb{R}^d\setminus 2Q} \left| \int_Q k(x,y)a(y)\,dy \right| dx \\
&\le \int_Q \left(\int_{\mathbb{R}^d\setminus 2Q} \big\|k(x,y)-k(x,c_Q)\big\|_{\mathcal{M}}\,dx \right) \nu|a(y)|\,dy \\
&\lesssim \nu\left(\int_Q |a(y)|\,dy \right) \le \sqrt{|Q|}\nu\left(\left(\int_Q |a(y)|^2\,dy \right)^{\frac12} \right) \le 1,
\end{aligned}$$

where the next to last estimate follows from Hansen's inequality or as a consequence of the operator-convexity of the function $a \mapsto |a|^2$. As for the first term, it suffices to show that $T_{\mathrm{c}} : L_1(\mathcal{M}; L_2^c(\mathbb{R}^d)) \to L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$, since then we find again

$$\begin{aligned}
\big\|T_{\mathrm{c}}(a)1_{2Q}\big\|_{L_1(\mathcal{A})} &= \nu\left(\int_{2Q} |T_{\mathrm{c}}(a)(x)|\,dx \right) \\
&\le \sqrt{|2Q|}\,\nu\left(\left(\int_{2Q} |T_{\mathrm{c}}(a)(x)|^2\,dx \right)^{\frac12} \right) \\
&\lesssim \sqrt{|2Q|}\,\nu\left(\left(\int_Q |a(x)|^2\,dx \right)^{\frac12} \right) \lesssim 1.
\end{aligned}$$

The $L_1(\mathcal{M}; L_2^c(\mathbb{R}^d))$-boundedness of T_{c} follows from anti-linear duality

$$\big\|T_{\mathrm{c}}(f)\big\|_{L_1(\mathcal{M};L_2^c(\mathbb{R}^d))} \le \left(\sup_{\|g\|_{L_\infty(L_2^c)}\le 1} \big\|T_{\mathrm{c}}^*(g)\big\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}^d))} \right) \|f\|_{L_1(\mathcal{M};L_2^c(\mathbb{R}^d))}.$$

It is easily checked that the adjoint $T_{\mathrm{c}}^*(g)$ has the form

$$T_{\mathrm{c}}^* g(x) \sim \int_{\mathbb{R}^d} k(y,x)^* g(y)\,dy$$

when we construct it with respect to the anti-linear bracket $[f,g] = \tau(f^*g)$. This means in particular that T_{c}^* is still an L_2-bounded column Calderón-Zygmund operator associated to a kernel satisfying Hörmander smoothness. This gives rise to

$$\begin{aligned}
\big\|T_{\mathrm{c}}^*(g)\big\|_{L_\infty(\mathcal{M};L_2^c(\mathbb{R}^d))} &= \left\| \left(\int_{\mathbb{R}^d} |T_{\mathrm{c}}^*(g)(x)|^2\,dx \right)^{\frac12} \right\|_{\mathcal{M}} \\
&= \sup_{\|u\|_{L_2(\mathcal{M})}\le 1} \left(\int_{\mathbb{R}^d} \Big[|T_{\mathrm{c}}^*(g)(x)|^2 u, u \Big]_{L_2(\mathcal{M})} dx \right)^{\frac12} \\
&= \sup_{\|u\|_{L_2(\mathcal{M})}\le 1} \left(\int_{\mathbb{R}^d} \big\|T_{\mathrm{c}}^*(gu)(x)\big\|_{L_2(\mathcal{M})}^2\,dx \right)^{\frac12} \\
&\lesssim \sup_{\|u\|_{L_2(\mathcal{M})}\le 1} \left(\int_{\mathbb{R}^d} \big\|g(x)u\big\|_{L_2(\mathcal{M})}^2\,dx \right)^{\frac12}
\end{aligned}$$

$$= \left\| \left(\int_{\mathbb{R}^d} |g(x)|^2 \, dx \right)^{\frac{1}{2}} \right\|_{\mathcal{M}}.$$

The third identity above uses the right $\mathcal{M}$-module nature of column Calderón-Zygmund operators. □

Remark 7.1. The proof above also shows that $L_1(L_2^\dagger)$ and $L_\infty(L_2^\dagger)$ boundedness of $T_\dagger$ for $\dagger \in \{\mathrm{r}, \mathrm{c}\}$ follow from the corresponding L_2 boundedness of the same operator. As noticed in [36], this is very specific of Calderón-Zygmund operators with noncommuting kernels since other semicommutative Calderón-Zygmund operators fail to satisfy this implication. The key property here is left/right $\mathcal{M}$-modularity, so that

$$uT_{\mathrm{r}}(f) = T_{\mathrm{r}}(uf) \quad \text{and} \quad T_{\mathrm{c}}(f)u = T_{\mathrm{c}}(fu).$$

7.2 Row/column L_p estimates

Theorem 5.2 follows as an easy consequence of Theorem 5.1 after applying suitable interpolation/duality results. Thus, we will only outline the definition of the involved spaces and the necessary results to deduce Theorem 5.2 from Theorem 5.1. Given $1 < p < \infty$, the noncommutative Hardy space $\mathrm{H}_p(\mathcal{A})$ is defined as

$$\mathrm{H}_p(\mathcal{A}) = \begin{cases} \mathrm{H}_p^{\mathrm{r}}(\mathcal{A}) + \mathrm{H}_p^{\mathrm{c}}(\mathcal{A}) & \text{if } 1 < p \leq 2, \\ \mathrm{H}_p^{\mathrm{r}}(\mathcal{A}) \cap \mathrm{H}_p^{\mathrm{c}}(\mathcal{A}) & \text{if } 2 \leq p < \infty, \end{cases}$$

where the corresponding row/column Hardy spaces arise as the completion of the subspace of finite martingales in $L_p(\mathcal{A})$ with respect to the norms given by the row and column square functions

$$\|f\|_{\mathrm{H}_p^{\mathrm{r}}(\mathcal{A})} = \left\| \left(\sum_{k \in \mathbb{Z}} df_k df_k^* \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{A})},$$

$$\|f\|_{\mathrm{H}_p^{\mathrm{c}}(\mathcal{A})} = \left\| \left(\sum_{k \in \mathbb{Z}} df_k^* df_k \right)^{\frac{1}{2}} \right\|_{L_p(\mathcal{A})}.$$

Pisier/Xu obtained in [66] the noncommutative Burkholder-Gundy inequalities which can be formulated as $L_p(\mathcal{A}) \simeq \mathrm{H}_p(\mathcal{A})$ for $1 < p < \infty$. On the other hand, we know from [32, 38] that $\mathrm{H}_p^\dagger(\mathcal{A})^* \simeq \mathrm{H}_{p'}^\dagger(\mathcal{A})$ for $\dagger \in \{\mathrm{r}, \mathrm{c}\}$ and $1 < p < \infty$. Regarding interpolation, we know from Musat [55] that

$$\mathrm{H}_p^\dagger(\mathcal{A}) \simeq \big[\mathrm{H}_{p_0}^\dagger(\mathcal{A}), \mathrm{H}_{p_1}^\dagger(\mathcal{A}) \big]_\theta,$$

where $\dagger \in \{\mathrm{r}, \mathrm{c}\}$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. The proof of Theorem 5.2 is now straightforward.

Proof of Theorem 5.2. We know that

$$T_{\mathrm{r}} : \mathrm{H}_1^{\mathrm{r}}(\mathcal{A}) \to L_1(\mathcal{A}) \quad \text{and} \quad T_{\mathrm{c}} : \mathrm{H}_1^{\mathrm{c}}(\mathcal{A}) \to L_1(\mathcal{A}).$$

If $1 < p < 2$, we find $T_{\mathrm{r}} : \mathrm{H}_p^{\mathrm{r}}(\mathcal{A}) \to L_p(\mathcal{A})$ and $T_{\mathrm{c}} : \mathrm{H}_p^{\mathrm{c}}(\mathcal{A}) \to L_p(\mathcal{A})$ by interpolation with $L_2(\mathcal{A}) = \mathrm{H}_2^{\mathrm{c}}(\mathcal{A}) = \mathrm{H}_2^{\mathrm{c}}(\mathcal{A})$. Hence, taking a decomposition $f = f_{\mathrm{r}} + f_{\mathrm{c}}$ satisfying

$$\|f\|_{L_p(\mathcal{A})} \sim \|f\|_{\mathrm{H}_p(\mathcal{A})} \sim \|f_{\mathrm{r}}\|_{\mathrm{H}_p^{\mathrm{r}}(\mathcal{A})} + \|f_{\mathrm{c}}\|_{\mathrm{H}_p^{\mathrm{c}}(\mathcal{A})}$$

we get $\|T_{\mathrm{r}} f_{\mathrm{r}}\|_{L_p(\mathcal{A})} + \|T_{\mathrm{c}} f_{\mathrm{c}}\|_{L_p(\mathcal{A})} \lesssim \|f\|_{L_p(\mathcal{A})}$. Now if $2 < p < \infty$, recalling that $T_{\mathrm{r}}^*, T_{\mathrm{c}}^*$ are again row/column Calderón-Zygmund operators with the same properties, duality gives $T_{\mathrm{r}} : L_p(\mathcal{A}) \to \mathrm{H}_p^{\mathrm{r}}(\mathcal{A})$ and $T_{\mathrm{c}} : L_p(\mathcal{A}) \to \mathrm{H}_p^{\mathrm{c}}(\mathcal{A})$. This immediately yields the inequality in Theorem 5.2 (*ii*). The $L_\infty \to \mathrm{BMO}$ type estimates were originally proved in [36], these also follow by duality from Theorem 5.1. □

Remark 7.2. We may also find L_p boundedness for $T_{\mathrm{r}}/T_{\mathrm{c}}$ after composing with suitable smooth Fourier multipliers approximating the identity. Let us illustrate this assertion for T_{c} and $2 < p < \infty$. Indeed, if A is the infinitesimal generator of a Markov semigroup $\mathcal{S} = (S_t)_{t \geq 0}$ acting on $\mathcal{A}$, it will be proved in [34] — refining the argument in [36, Theorem A] — that the operator

$$\frac{A^{\varepsilon}}{(1+A)^{2\varepsilon}}$$

takes $\mathrm{H}_p^{\mathrm{c}}(\mathcal{S})$ to $L_p(\mathcal{A})$, with constants depending on $\varepsilon > 0$. We refer e.g. to [36] for the definition of the semigroup Hardy space $\mathrm{H}_p^{\mathrm{c}}(\mathcal{S})$. When $\mathcal{A} = L_\infty(\mathbb{R}^d) \bar{\otimes} \mathcal{M}$ and the generator $-A$ is the Laplacian, $\mathrm{H}_p^{\mathrm{c}}(\mathcal{S})$ is isomorphic to $\mathrm{H}_p^{\mathrm{c}}(\mathcal{A})$ and the operator above is the Fourier multiplier with symbol $|\xi|^{2\varepsilon}/(1+|\xi|^2)^{2\varepsilon}$.

Chapter 8

Noncommuting martingale transforms and paraproducts

In this chapter we turn our attention to noncommutative martingale transforms and paraproducts. In particular, the former pair $(\mathcal{A}, \tau)$ will refer in what follows to an arbitrary semifinite von Neumann algebra equipped with a normal faithful semifinite trace. Our filtration $\Sigma_{\mathcal{A}} = (\mathcal{A}_k)_{k \geq 1}$ will be any increasing family of von Neumann subalgebras, whose union is weak-$*$ dense in $\mathcal{A}$. The operators E_k and D_k still denote the corresponding conditional expectations and martingale difference operators. As mentioned in the Introduction, we will deal with

- **Noncommuting martingale transforms**

$$M_\xi^{\mathrm{r}} f = \sum_{k \geq 1} \mathsf{D}_k(f) \xi_{k-1} \quad \text{and} \quad M_\xi^{\mathrm{c}} f = \sum_{k \geq 1} \xi_{k-1} \mathsf{D}_k(f).$$

- **Paraproducts with noncommuting symbol**

$$\Pi_\rho^{\mathrm{r}}(f) = \sum_{k \geq 1} \mathsf{E}_{k-1}(f) \mathsf{D}_k(\rho) \quad \text{and} \quad \Pi_\rho^{\mathrm{c}}(f) = \sum_{k \geq 1} \mathsf{D}_k(\rho) \mathsf{E}_{k-1}(f).$$

The martingale coefficients $\xi_k \in \mathcal{A}_k$ form an adapted sequence and it is easy to show that L_2-boundedness of M_ξ^{r} and M_ξ^{c} hold iff the ξ_k's are uniformly bounded in the norm of $\mathcal{A}$. On the other hand, the classical characterization $\Pi_\rho : L_2 \to L_2$ iff $\rho \in \mathrm{BMO}$ was disproved by Nazarov, Pisier, Treil and Volberg [56], see also Mei's paper [50]. Hence, the L_2-boundedness of Π_ρ^{r} and Π_ρ^{c} will be simply assumed in what follows.

8.1 Weak-type $(1,1)$ estimates

Regarding Cuculescu's construction and the Calderón-Zygmund decomposition, no essential changes are needed. Namely, given $f \in L_1^+(\mathcal{A})$ (the former space $\mathcal{A}_{+,K}$ is unnecessary since our filtration starts now at $k = 1$) and $\lambda \in \mathbb{R}_+$, Cuculescu's construction is verbatim the same. The only difference is on the diagonal estimate

$$\Big\| qfq + \sum_{k=1}^{\infty} p_k f_k p_k \Big\|_{L_2(\mathcal{A})}^2 \lesssim \lambda \|f\|_{L_1(\mathcal{A})}.$$

This inequality requires to work with regular filtrations, which are defined through the additional condition $\mathsf{E}_k(f) \le c\mathsf{E}_{k-1}(f)$ for some absolute constant $c > 0$ and every pair $(f,k) \in \mathcal{A}_+ \times \mathbb{Z}_+$. Of course, the reader might think that it is more appropriate to use in this case the noncommutative form of Gundy's decomposition [59], which does not require any regularity assumption on the martingale. This leads unfortunately to new difficulties related to our use of triangular truncations.

Proof of Theorem 5.3 (i). The argument is essentially the same as in the perfect dyadic case. Given $f \in L_1^+(\mathcal{A})$, we construct the same decomposition $f = f_{\mathrm{r}} + f_{\mathrm{c}}$ via the projections $\pi_{j,k}$ and fix $\lambda = 2^\ell$ for some $\ell \in \mathbb{Z}$. A further Calderón-Zygmund decomposition gives $f_{\mathrm{c}} = g_\Delta^{\mathrm{c}} + g_{\mathrm{off}}^{\mathrm{c}} + b_\Delta^{\mathrm{c}} + b_{\mathrm{off}}^{\mathrm{c}}$ as usual. According to our regularity assumption, we still have

$$\begin{aligned} \max \Big\{ \|g_\Delta^{\mathrm{r}}\|_{L_2(\mathcal{A})}^2, \|g_\Delta^{\mathrm{c}}\|_{L_2(\mathcal{A})}^2 \Big\} &\le \|g_\Delta\|_{L_2(\mathcal{A})}^2 \\ &= \Big\| qfq + \sum_{k\ge1} p_k f_k p_k \Big\|_{L_2(\mathcal{A})}^2 \lesssim \lambda \|f\|_{L_1(\mathcal{A})}. \end{aligned}$$

Thus, arguing as in the proof of Theorem 5.1 it suffices to show that

$$\widehat{q} M_\xi^{\mathrm{r}}(\gamma^{\mathrm{r}}) = M_\xi^{\mathrm{c}}(\gamma^{\mathrm{c}})\widehat{q} = \widehat{q}\Pi_\rho^{\mathrm{r}}(\gamma^{\mathrm{r}}) = \Pi_\rho^{\mathrm{c}}(\gamma^{\mathrm{c}})\widehat{q} = 0$$

for any $\gamma \in \{g_{\mathrm{off}}, b_\Delta, b_{\mathrm{off}}\}$. As usual, we just consider the column case by symmetry. Let us begin with martingale transforms. Since $\gamma^{\mathrm{c}} = \sum_j \mathsf{UT}_{j-1}(\mathsf{D}_j(\gamma))$ and the triangular truncation UT_{j-1} is built with j-predictable projections, we see that $\mathsf{UT}_{j-1}(\mathsf{D}_j(\gamma))$ is a j martingale difference, so that

$$\mathsf{D}_k(\gamma^{\mathrm{c}}) = \mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma)).$$

By the proof of Theorem 5.1, we know $\mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\widehat{q}_{k-1} = 0$ and

$$M_\xi^{\mathrm{c}}(\gamma^{\mathrm{c}})\widehat{q} = \sum_{k=1}^{\infty} \xi_{k-1}\mathsf{D}_k(\gamma^{\mathrm{c}})\widehat{q} = \sum_{k=1}^{\infty} \xi_{k-1}\mathsf{UT}_{k-1}(\mathsf{D}_k(\gamma))\widehat{q}_{k-1}\widehat{q} = 0.$$

For martingale paraproducts, we observe that $\mathsf{E}_{k-1}(\gamma^{\mathrm{c}}) = \sum_{j<k} \mathsf{UT}_{j-1}(\mathsf{D}_j(\gamma))$ and

$$\Pi_\rho^{\mathrm{c}}(\gamma^{\mathrm{c}})\widehat{q} = \sum_{k=1}^{\infty} \mathsf{D}_k(\rho) \sum_{j<k} \mathsf{UT}_{j-1}(\mathsf{D}_j(\gamma))\widehat{q}_{j-1}\widehat{q} = 0. \qquad \square$$

Remark 8.1. Is really the regular filtration assumption in Theorem 5.3 necessary?

Remark 8.2. Adjoints of martingale paraproducts have the form

$$[\Pi_\rho^{\mathrm{c}}]^* f = \sum_{k\ge1} \mathsf{E}_{k-1}\big(\mathsf{D}_k(\rho^*)\mathsf{D}_k(f)\big) \quad \text{and} \quad [\Pi_\rho^{\mathrm{r}}]^* f = \sum_{k\ge1} \mathsf{E}_{k-1}\big(\mathsf{D}_k(f)\mathsf{D}_k(\rho^*)\big)$$

when using the anti-linear duality bracket. It is easy to adapt the argument above for these maps, to obtain weak type inequalities for adjoints of noncommutative paraproducts associated to regular filtrations

$$\inf_{f=f_{\mathrm{r}}+f_{\mathrm{c}}} \big\| [\Pi_\rho^{\mathrm{r}}]^* f_{\mathrm{r}} \big\|_{L_{1,\infty}(\mathcal{A})} + \big\| [\Pi_\rho^{\mathrm{c}}]^* f_{\mathrm{c}} \big\|_{L_{1,\infty}(\mathcal{A})} \leq \|f\|_{L_1(\mathcal{A})}.$$

8.2 Atoms and John-Niremberg inequality

We defined above the noncommutative Hardy spaces $\mathrm{H}_1(\mathcal{A})$. Alternatively, we may also consider the noncommutative form $\mathrm{h}_1(\mathcal{A}) = \mathrm{h}_1^{\mathrm{r}}(\mathcal{A}) + \mathrm{h}_1^{\mathrm{c}}(\mathcal{A}) + \mathrm{h}_1^{\Delta}(\mathcal{A})$ of the conditional Hardy space h_1, where the norms are given by

$$\|f\|_{\mathrm{h}_1^{\mathrm{r}}(\mathcal{A})} = \Big\| \Big(\sum_{k\geq 1} \mathsf{E}_{k-1}\big(df_k df_k^*\big) \Big)^{\frac{1}{2}} \Big\|_{L_1(\mathcal{A})},$$

$$\|f\|_{\mathrm{h}_1^{\mathrm{c}}(\mathcal{A})} = \Big\| \Big(\sum_{k\geq 1} \mathsf{E}_{k-1}\big(df_k^* df_k\big) \Big)^{\frac{1}{2}} \Big\|_{L_1(\mathcal{A})},$$

$$\|f\|_{\mathrm{h}_1^{\Delta}(\mathcal{A})} = \Big\| \sum_{k\geq 1} |df_k| \Big\|_{L_1(\mathcal{A})} = \sum_{k\geq 1} \|df_k\|_{L_1(\mathcal{A})}.$$

The space $\mathrm{h}_1(\mathcal{A})$ was studied in [33, 61], it was independently proved that

$$\mathrm{H}_1^{\mathrm{r}}(\mathcal{A}) \simeq \mathrm{h}_1^{\mathrm{r}}(\mathcal{A}) + \mathrm{h}_1^{\Delta}(\mathcal{A}),$$
$$\mathrm{H}_1^{\mathrm{c}}(\mathcal{A}) \simeq \mathrm{h}_1^{\mathrm{c}}(\mathcal{A}) + \mathrm{h}_1^{\Delta}(\mathcal{A}).$$

In conjunction, these isomorphisms could be regarded as a noncommutative form of Davis' decomposition for martingales. Shortly after, it was found in [3] an atomic decomposition for the spaces $\mathrm{h}_1^{\mathrm{r}}(\mathcal{A})$ and $\mathrm{h}_1^{\mathrm{c}}(\mathcal{A})$. More precisely, an element a in $L_1(\mathcal{A}) \cap L_2(\mathcal{A})$ is called a *column atom* with respect to the filtration $(\mathcal{A}_k)_{k\geq 1}$ if there exists $k_0 \in \mathbb{Z}_+$ and a finite projection $e \in \mathcal{A}_{k_0}$ such that

- $a = ae$,
- $\mathsf{E}_{k_0}(a) = 0$,
- $\|a\|_{L_2(\mathcal{A})} \leq \tau(e)^{-\frac{1}{2}}$.

An element $a \in L_1(\mathcal{A})$ is called a c-atom if it is a column atom or $a \in \mathcal{A}_1$ with $\|a\|_{L_1(\mathcal{A})} \leq 1$. Row atoms are defined to satisfy $a = ea$ instead and r-atoms are defined similarly. We also refer to [28] for q-analogs of these notions. In the following result, we collect some norm equivalences coming from atomic decompositions and John-Nirenberg type inequalities. Recall that

$$\|f\|_{\mathrm{BMO_c}(\mathcal{A})} = \sup_{k\geq 1} \Big\| \mathsf{E}_k\big[(f-f_{k-1})^*(f-f_{k-1})\big] \Big\|_{\mathcal{A}}^{\frac{1}{2}},$$

$$\|f\|_{\mathrm{bmo_c}(\mathcal{A})} = \max\Big\{ \big\|\mathsf{E}_1(f)\big\|_{L_1(\mathcal{A})}, \ \sup_{k\geq 1} \Big\| \mathsf{E}_k\big[(f-f_k)^*(f-f_k)\big] \Big\|_{\mathcal{A}}^{\frac{1}{2}} \Big\}.$$

As usual, the corresponding row norms of f arise as the column norms of f^*. If we also define $\|f\|_{\mathrm{bmo}_\Delta(\mathcal{A})} = \sup_k \|df_k\|_{\mathcal{A}}$, then we can define the spaces $\mathrm{BMO}(\mathcal{A})$ and $\mathrm{bmo}(\mathcal{A})$ as follows

$$\|f\|_{\mathrm{BMO}(\mathcal{A})} = \max\Big\{\|f\|_{\mathrm{BMO_r}(\mathcal{A})}, \|f\|_{\mathrm{BMO_c}(\mathcal{A})}\Big\},$$
$$\|f\|_{\mathrm{bmo}(\mathcal{A})} = \max\Big\{\|f\|_{\mathrm{bmo_r}(\mathcal{A})}, \|f\|_{\mathrm{bmo_c}(\mathcal{A})}, \|f\|_{\mathrm{bmo}_\Delta(\mathcal{A})}\Big\}.$$

The isomorphism $\mathrm{BMO}(\mathcal{A}) \simeq \mathrm{bmo}(\mathcal{A})$ was independently proved in [33, 61].

Atoms and John-Nirenberg inequality [3, 28]. *We have*

$$\|f\|_{\mathrm{h}_1^{\mathrm{r}}} \sim \inf\Big\{\sum_k |\lambda_k| \, \Big| \, f = \sum\nolimits_k \lambda_k a_k \text{ and } a_k \text{ r-}atom\Big\},$$
$$\|f\|_{\mathrm{h}_1^{\mathrm{c}}} \sim \inf\Big\{\sum_k |\lambda_k| \, \Big| \, f = \sum\nolimits_k \lambda_k a_k \text{ and } a_k \text{ c-}atom\Big\},$$
$$\|f\|_{\mathrm{bmo}(\mathcal{A})} \sim \sup_{k\geq 1}\Big\{\|df_k\|_\infty \vee \sup_{\substack{\beta\in\mathcal{A}_k \\ \|\beta\|_1\leq 1}} \big\|\beta(f-f_k)\big\|_{L_1(\mathcal{A})} \vee \sup_{\substack{\beta\in\mathcal{A}_k \\ \|\beta\|_1\leq 1}} \big\|(f-f_k)\beta\big\|_{L_1(\mathcal{A})}\Big\}.$$

The last equivalence is a John-Nirenberg type inequality, which differs from [37].

Proof of Theorem 5.3 *(ii)*. Let us begin with $\mathrm{H}_1 \to L_1$ type inequalities. We will show that $T_\dagger : \mathrm{H}_1^\dagger(\mathcal{A}) \to L_1(\mathcal{A})$ with $\dagger \in \{\mathrm{r}, \mathrm{c}\}$ for both martingale transforms and paraproducts. Since we have

$$\mathrm{H}_1^\dagger(\mathcal{A}) \simeq \mathrm{h}_1^\dagger(\mathcal{A}) + \mathrm{h}_1^\Delta(\mathcal{A}),$$

it suffices to show that $T_\dagger : \mathrm{X} \to L_1(\mathcal{A})$ with X any of the two spaces appearing on the right. Once more, the argument is row/column symmetric and we just consider columns. To see that $T_{\mathrm{c}} : \mathrm{h}_1^{\mathrm{c}}(\mathcal{A}) \to L_1(\mathcal{A})$ we may use the atomic decomposition above, so that it suffices to find a uniform upper bound for $\|T_{\mathrm{c}}(a)\|_{L_1(\mathcal{A})}$ with a being a c-atom. If $a \in \mathcal{A}_1$ with $\|a\|_{L_1(\mathcal{A})} \leq 1$, then we see that

$$M_\xi^{\mathrm{c}}(a) = \xi_0 a_1 \quad \text{and} \quad \Pi_\rho^{\mathrm{c}}(a) = \rho a = \Pi_\rho^{\mathrm{c}}(u|a|^{\frac12})|a|^{\frac12} \quad \text{for} \quad a = u|a|.$$

In particular, $\|M_\xi^{\mathrm{c}}(a)\|_{L_1(\mathcal{A})} + \|\Pi_\rho^{\mathrm{c}}(a)\|_{L_1(\mathcal{A})} \lesssim \|a\|_{L_1(\mathcal{A})} \leq 1$. If a is a column atom, we find

$$M_\xi^{\mathrm{c}}(a) = \sum_{k>k_0} \xi_{k-1}\mathsf{D}_k(a) = \sum_{k>k_0} \xi_{k-1}\mathsf{D}_k(a)e = M_\xi^{\mathrm{c}}(a)e,$$
$$\Pi_\rho^{\mathrm{c}}(a) = \sum_{k>k_0+1} \mathsf{D}_k(\rho)\mathsf{E}_{k-1}(a) = \sum_{k>k_0+1} \mathsf{D}_k(\rho)\mathsf{E}_{k-1}(a)e = \Pi_\rho^{\mathrm{c}}(a)e.$$

This gives rise to

$$\|T_{\mathrm{c}}(a)\|_{L_1(\mathcal{A})} = \|T_{\mathrm{c}}(a)e\|_{L_1(\mathcal{A})} \leq \|T_{\mathrm{c}}(a)\|_{L_2(\mathcal{A})}\|e\|_{L_2(\mathcal{A})} \lesssim \|a\|_{L_2(\mathcal{A})}\|e\|_{L_2(\mathcal{A})} \leq 1$$

for both martingale transforms and paraproducts. We have already justified the $\mathrm{h}_1^{\mathrm{c}} \to L_1$ boundedness. Let us now look at h_1^{Δ}

$$\|M_\xi^{\mathrm{c}}(f)\|_{L_1(\mathcal{A})} \leq \sum_{k\geq 1} \|\xi_k\|_{\mathcal{A}} \|\mathsf{D}_k(f)\|_{L_1(\mathcal{A})} \leq \left(\sup_{k\geq 1} \|\xi_k\|_{\mathcal{A}}\right) \|f\|_{\mathrm{h}_1^{\Delta}(\mathcal{A})}$$

As for the paraproduct, we use the John-Nirenberg inequality above

$$\begin{aligned}\|\Pi_\rho^{\mathrm{c}}(f)\|_{L_1(\mathcal{A})} &= \Big\| \sum_{k\geq 1} \mathsf{D}_k(\rho) \sum_{j<k} \mathsf{D}_j(f) \Big\|_{L_1(\mathcal{A})} \\ &= \Big\| \sum_{k\geq 1} (\rho - \rho_k) \mathsf{D}_k(f) \Big\|_{L_1(\mathcal{A})} \lesssim \|\rho\|_{\mathrm{bmo}(\mathcal{A})} \|f\|_{\mathrm{h}_1^{\Delta}(\mathcal{A})}.\end{aligned}$$

According to [33, 61] and [50, 56], we have

$$\|\rho\|_{\mathrm{bmo}(\mathcal{A})} \sim \|\rho\|_{\mathrm{BMO}(\mathcal{A})} \lesssim \max\Big\{ \|\Pi_\rho^{\mathrm{r}}\|_{\mathcal{B}(L_2(\mathcal{A}))}, \|\Pi_\rho^{\mathrm{c}}\|_{\mathcal{B}(L_2(\mathcal{A}))} \Big\}.$$

All this together gives that M_ξ^{c} and Π_ρ^{c} take $\mathrm{H}_1^{\mathrm{c}}(\mathcal{A})$ into $L_1(\mathcal{A})$ as we claimed. In fact slight modifications of the given argument yield the same result for $[\Pi_\rho^{\mathrm{c}}]^*$, details are left to he reader. This is all what is needed to produce analog inequalities in this setting to those in Theorems 5.1 and 5.2, we just need to follow the arguments verbatim. It remains to show that $\Pi_\rho^{\mathrm{c}} : L_p(\mathcal{A}) \to L_p(\mathcal{A})$ for $p > 2$, for which it will be enough to prove $L_\infty \to \mathrm{BMO}$ boundedness and use interpolation. The $L_\infty \to \mathrm{BMO}_{\mathrm{c}}$ boundedness follows by duality from the $\mathrm{H}_1^{\mathrm{c}} \to L_1$ boundedness of $[\Pi_\rho^{\mathrm{c}}]^*$. On the other hand, the $L_\infty \to \mathrm{BMO}_{\mathrm{r}}$ boundedness is very simple

$$\begin{aligned}\|\Pi_\rho^{\mathrm{c}} f\|_{\mathrm{BMO}_{\mathrm{r}}(\mathcal{A})} &= \sup_{k\geq 1} \Big\| \mathsf{E}_k \Big(\sum_{j\geq k} \mathsf{D}_j(\Pi_\rho^{\mathrm{c}}(f)) \mathsf{D}_j(\Pi_\rho^{\mathrm{c}}(f))^* \Big) \Big\|_{\mathcal{A}}^{\frac{1}{2}} \\ &= \sup_{k\geq 1} \Big\| \mathsf{E}_k \Big(\sum_{j\geq k} \mathsf{D}_j(\rho) \mathsf{E}_{j-1}(f) \mathsf{E}_{j-1}(f)^* \mathsf{D}_j(\rho)^* \Big) \Big\|_{\mathcal{A}}^{\frac{1}{2}} \\ &\leq \sup_{k\geq 1} \Big\| \mathsf{E}_k \Big(\sum_{j\geq k} \mathsf{D}_j(\rho) \mathsf{D}_j(\rho)^* \Big) \Big\|_{\mathcal{A}}^{\frac{1}{2}} \|f\|_\infty \leq \|\rho\|_{\mathrm{BMO}_{\mathrm{r}}(\mathcal{A})} \|f\|_\infty.\end{aligned}$$

Now we majorize $\|\rho\|_{\mathrm{BMO}_{\mathrm{r}}(\mathcal{A})}$ by the $L_2 \to L_2$ norm of Π_ρ as we did above. □

Observe that we have not needed to assume regularity of our martingale filtration and we find that $[\Pi_\rho^{\mathrm{r}}]^*, [\Pi_\rho^{\mathrm{c}}]^*$ take $\mathrm{H}_1 \to L_1$ and $L_p \to L_p$ for $1 < p < 2$ by duality. In some sense, row/column noncommutative paraproducts present a similar behavior as row/column square functions in the noncommutative Burkholder-Gundy and Khintchine inequalities [47, 48, 66]. On the other hand, [71, Theorem 5.7] yields $L \log L \to L_1$ type estimates for a finite von Neumann algebra $\mathcal{A}$ with $(T_{\mathrm{r}}, T_{\mathrm{c}})$ a martingale transform/paraproduct with noncommuting coefficients/symbol

$$\inf_{f = f_{\mathrm{r}} + f_{\mathrm{c}}} \big\| T_{\mathrm{r}} f_{\mathrm{r}} \big\|_{L_1(\mathcal{A})} + \big\| T_{\mathrm{c}} f_{\mathrm{c}} \big\|_{L_1(\mathcal{A})} \lesssim \|f\|_{L \log L(\mathcal{A})}.$$

Part III

Non-doubling semicommutative dyadic harmonic analysis

Chapter 9

Introduction and results

Recall that $\Phi = \{\phi_Q\}_{Q\in\mathscr{D}}$ is a generalized Haar system in $\mathbb{R}^d$ adapted to a locally finite Borel measure $\mu \in \mathscr{B}$ and a dyadic lattice $\mathscr{D}$ if the following conditions hold:

(a) For every $Q \in \mathscr{D}$, $\operatorname{supp}(\phi_Q) \subset Q$.

(b) If Q', $Q \in \mathscr{D}$ and $Q' \subsetneq Q$, then ϕ_Q is constant on Q'.

(c) For every $Q \in \mathscr{D}$, $\displaystyle\int_{\mathbb{R}^d} \phi_Q \, d\mu = 0$.

(d) For every $Q \in \mathscr{D}$, either $\|\phi_Q\|_{L^2(\mu)} = 1$ or $\phi_Q \equiv 0$ and $\mu(Q) = 0$.

If the vanishing integral condition (c) is not imposed, the Haar system is said to be *non-cancellative*. Let $\Phi = \{\phi_Q\}_{Q\in\mathscr{D}}$ and $\Psi = \{\psi_Q\}_{Q\in\mathscr{D}}$ be two non-necessarily cancellative generalized Haar systems in $\mathbb{R}^d$. A Haar shift operator of complexity $(r, s) \in \mathbb{N} \times \mathbb{N}$ is an operator of the form

$$(9.1) \qquad Ш_{r,s} f(x) = \sum_{Q\in\mathscr{D}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S} \langle f, \phi_R\rangle \psi_S(x), \quad \text{with} \quad \sup_{Q,R,S} |\alpha^Q_{R,S}| < \infty;$$

where $\langle f, g\rangle = \int_{\mathbb{R}^d} fg \, d\mu$ and $\mathscr{D}_k(Q)$, $k \in \mathbb{N}$, denotes the family of k-dyadic descendants of Q: the partition of Q into subcubes $R \in \mathscr{D}$ of side-length $\ell(R) = 2^{-k}\ell(Q)$. Several objects in dyadic harmonic analysis have the general form (9.1), including Haar multipliers, dyadic paraproducts, the dyadic model of the Hilbert transform and their adjoints. Haar shift operators have served as important tools in the study of many different problems in harmonic analysis since the form (9.1) is a fruitful source of models of Calderón-Zygmund operators. In particular, in the case where μ is the Lebesgue measure, Calderón-Zygmund operators can be expressed as weak limits of certain averages of cancellative Haar shift operators and paraproducts [29] and are pointwise dominated by positive dyadic operators, which are Haar shift operators relative to non-cancellative Haar systems [15].

The boundedness behavior of Haar shift operators with respect to arbitrary locally finite Borel measures in the commutative setting was studied in [46] as presented here in Part I, where the weak-type $(1, 1)$ of such operators is characterized. In this Part of this thesis we extend the scope of this result to the setting of semicommutative L_p spaces. The main technique that we will use in our approach is a generalization of the Calderón-Zygmund decomposition stated in Theorem 1.1 which is valid for operator-valued functions, in the spirit of the Calderón-Zygmund decomposition constructed in [58].

Let us briefly recall the semicommutative framework and adapt it to the non-doubling setting. Consider a pair $(\mathcal{M}, \nu)$ where $\mathcal{M}$ is a von Neumann algebra and ν is a normal semifinite faithful trace on $\mathcal{M}$ and let μ be a locally finite Borel measure on $\mathbb{R}^d$. Let $\mathcal{A}_B$ be the algebra of essentially bounded $\mathcal{M}$-valued functions

$$\mathcal{A}_B = \left\{ f : \mathbb{R}^d \to \mathcal{M} : f \text{ strongly measurable s.t. } \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} \|f(x)\|_{\mathcal{M}} < \infty \right\}$$

equipped with the n.s.f. trace $\tau(f) = \int_{\mathbb{R}^d} \nu(f)\, d\mu$. The weak-operator closure $\mathcal{A}$ of $\mathcal{A}_B$ is a von Neumann algebra isomorphic to $L_\infty(\mathbb{R}^d, \mu)\overline{\otimes}\mathcal{M}$. Given a rearrangement invariant quasi-Banach function space X, let us write $X(\mathcal{M})$ and $X(\mathcal{A})$ for their associated noncommutative symmetric spaces. In particular $L_p(\mathcal{M})$ and $L_p(\mathcal{A})$ denote the noncommutative L_p spaces associated to the pairs $(\mathcal{M}, \nu)$ and $(\mathcal{A}, \tau)$. It can be readily seen that for $1 \leq p < \infty$ the noncommutative L_p space $L_p(\mathcal{A})$ is isometric to the Bochner L_p space $L_p(\mathbb{R}^d, \mu; L_p(\mathcal{M}))$. The lattices of projections are denoted by $\mathcal{P}(\mathcal{M})$ and $\mathcal{P}(\mathcal{A})$, while $1_{\mathcal{M}}$ and $1_{\mathcal{A}}$ stand for the unit elements and $\mathcal{M}'$ and $\mathcal{A}'$ stand for their respective commutants. For a more detailed discussion on noncommutative L_p spaces we refer to [53] and references therein. The reader unfamiliar with the theory of noncommutative L_p spaces may think of $\mathcal{M}$ as the algebra $\mathcal{B}(\ell_2^n)$ of $n \times n$ matrices equipped with the standard trace Tr, thereby recovering the classical Schatten p-classes. The reader should take into account that, with this setting in mind, we provide estimates uniform on n.

Before stating our results let us reintroduce some notation. By $(\mathsf{E}_k)_{k\in\mathbb{Z}}$ we will denote the family of conditional expectations associated to $\mathscr{D}_k$ — the dyadic cubes Q of side-length $\ell(Q) = 2^{-k}$ — relative to μ and write D_k for the corresponding martingale difference operators. The tensor product $\mathsf{E}_k \otimes id_{\mathcal{M}}$ acting on $\mathcal{A}$ will also be denoted by E_k, which yields a filtration $(\mathcal{A}_k)_{k\in\mathbb{Z}}$ on $\mathcal{A}$. We thus have that

$$\mathsf{E}_k(f) =: f_k = \sum_{Q \in \mathscr{D}_k} \langle f \rangle_Q 1_Q,$$
$$\mathsf{D}_k(f) =: df_k = \sum_{Q \in \mathscr{D}_k} \big(\langle f \rangle_Q - \langle f \rangle_{\widehat{Q}}\big) 1_Q,$$

which correspond to projections to the class of operators constant at scale $\mathscr{D}_k$. Here 1_Q denotes the characteristic function of Q, $\langle f \rangle_Q = \mu(Q)^{-1} \int_Q f \, d\mu$ and $\widehat{Q}$ is the dyadic parent of Q: the only dyadic cube that contains Q with twice its side-length.

We will construct the Calderón-Zygmund decomposition for functions in the class

$$\mathcal{A}_{+,K} = \{ f : \mathbb{R}^d \to \mathcal{M} \,|\, f \geq 0, \ \operatorname{supp}_{\mathbb{R}^d}(f) \text{ is compact} \},$$

whose span is dense in $L_1(\mathcal{A})$. Here $\operatorname{supp}_{\mathbb{R}^d}(f)$ stands for the support of f as an operator-valued function, as opposed to its support projection as an element of a von Neumann algebra. As the Calderón-Zygmund decomposition introduced in [58] — which is suitable for the Lebesgue measure and doubling measures — the Calderón-Zygmund decomposition here presented is comprised of diagonal and off-diagonal terms, reflecting the lack of commutativity in the operator-valued framework. Taking $i \vee j = \max\{i, j\}$ and $i \wedge j = \min\{i, j\}$ for $i, j \in \mathbb{Z}$ we have:

Theorem 9.2. *Let $f \in \mathcal{A}_{+,K}$ and let $\lambda > 0$. Then there exist a family of pairwise disjoint projections $(p_k)_{k\in\mathbb{Z}}$ adapted to $(\mathcal{A}_k)_{k\in\mathbb{Z}}$ and a projection $q := 1_{\mathcal{A}} - \sum_k p_k \in \mathcal{P}(\mathcal{A})$ such that f can be decomposed as $f = g+b+\beta$, where each term has a diagonal and an off-diagonal part given by*

- $g = g_\Delta + g_{\mathrm{off}}$, *where*

$$g_\Delta = qfq + \sum_{k\in\mathbb{Z}} \mathsf{E}_{k-1}\left(p_k f_k p_k\right),$$
$$g_{\mathrm{off}} = (1_{\mathcal{A}} - q)fq + qf(1_{\mathcal{A}} - q) + \sum_{i\neq j} \mathsf{E}_{i\vee j-1}\left(p_i f_{i\vee j} p_j\right);$$

- $b = b_\Delta + b_{\mathrm{off}}$, *where*

$$b_\Delta = \sum_{k\in\mathbb{Z}} p_k(f - f_k)p_k\,, \qquad b_{\mathrm{off}} = \sum_{i\neq j} p_i(f - f_{i\vee j})p_j;$$

- $\beta = \beta_\Delta + \beta_{\mathrm{off}}$, *where*

$$\beta_\Delta = \sum_{k\in\mathbb{Z}} \mathsf{D}_k(p_k f_k p_k), \qquad \beta_{\mathrm{off}} = \sum_{i\neq j} \mathsf{D}_{i\vee j}\left(p_i f_{i\vee j} p_j\right).$$

The diagonal terms satisfy the classical properties

(*a*) $g_\Delta \in L_1(\mathcal{A}) \cap L_2(\mathcal{A})$ *with*

$$\|g_\Delta\|_{L_1(\mathcal{A})} = \|f\|_{L_1(\mathcal{A})}, \quad \|g_\Delta\|^2_{L_2(\mathcal{A})} \leq 39\lambda\|f\|_{L_1(\mathcal{A})};$$

(*b*) $b_\Delta = \sum_{k\in\mathbb{Z}} b_k$, *with* $\int_{\mathbb{R}^d} b_k \, d\mu = 0$ *and satisfies the estimate*

$$\|b_\Delta\|_{L_1(\mathcal{A})} = \sum_{k\in\mathbb{Z}} \|b_k\|_{L_1(\mathcal{A})} \leq 2\|f\|_{L_1(\mathcal{A})};$$

(*c*) $\beta_\Delta = \sum_{k\in\mathbb{Z}} \beta_k$, *with each β_k a k martingale difference, and is such that*

$$\|\beta_\Delta\|_{L_1(\mathcal{A})} \leq \sum_{k\in\mathbb{Z}} \|\beta_k\|_{L_1(\mathcal{A})} \leq 2\|f\|_{L_1(\mathcal{A})}.$$

The off-diagonal terms are such that

(*d*) g_{off} *decomposes as* $g_{\mathrm{off}} = \sum_{k\in\mathbb{Z},h\geq 1} g_{k,h}$, *where $g_{k,h}$ is the $(k+h)$ martingale difference* $g_{k,h} = \mathsf{D}_{k+h}(p_k f_{k+h} q_{k+h} + q_{k+h} f_{k+h} p_k)$, *and satisfies the estimate*

$$\sup_{h\geq 1} \sum_{k\in\mathbb{Z}} \|g_{k,h}\|^2_{L_2(\mathcal{A})} \leq 16\lambda\|f\|_{L_1(\mathcal{A})};$$

(*e*) $b_{\mathrm{off}} = \sum_{k\in\mathbb{Z},h\geq 1} b_{k,h}$, *where* $b_{k,h} = p_k(f - f_{k+h})p_{k+h} + p_{k+h}(f - f_{k+h})p_k$, $\int_{\mathbb{R}^d} b_{k,h}\, d\mu = 0$ *and*

$$\sum_{k\in\mathbb{Z}} \|b_{k,h}\|_{L_1(\mathcal{A})} \leq 8(h+1)\|f\|_{L_1(\mathcal{A})};$$

(*f*) $\beta_{\mathrm{off}} = \sum_{k\in\mathbb{Z},h\geq 1} \beta_{k,h}$, *where* $\beta_{k,h} = \mathsf{D}_{k+h}(p_k f_{k+h} p_{k+h} + p_{k+h} f_{k+h} p_k)$ *and*

$$\sum_{k\in\mathbb{Z}} \|\beta_{k,h}\|_{L_1(\mathcal{A})} \leq 8(h+1)\|f\|_{L_1(\mathcal{A})}.$$

Observe that the diagonal terms satisfy estimates similar to those of their commutative counterparts found in [46]. However, in contrast to the classical setting, there are additional difficulties in proving the estimates even for diagonal terms due to the noncommutativity of $\mathcal{A}$. In particular, the estimates of g_Δ are proved in a different way and only hold for $p \leq 2$. In addition, the fact that μ is allowed to be nondoubling brings other difficulties not present in [58]. On the other hand, at first glance the off-diagonal estimates in (*d*), (*e*) and (*f*) seem to be insufficient, since they are weaker than the expected ones: $\|g_{\mathrm{off}}\|_{L_2(\mathcal{A})} \lesssim \lambda\|f\|_{L_1(\mathcal{A})}$, $\sum_{k,h}\|b_{k,h}\|_{L_1(\mathcal{A})} \lesssim \|f\|_{L_1(\mathcal{A})}$ and $\sum_{k,h}\|\beta_{k,h}\|_{L_1(\mathcal{A})} \lesssim \|f\|_{L_1(\mathcal{A})}$. Moreover, estimates of this nature seem to fail as hinted in [58]. However, the estimates at hand will prove to be sufficient for our purposes as the operators under consideration are localized in a sense stronger than in [53, 58]. In that respect, one can think of our result as a partial answer to the question posed in [53] about the existence of a Littlewood-Paley theory for nondoubling measures in the semicommutative context.

Let $\Phi = \{\phi_Q\}_{Q\in\mathscr{D}}$ and $\Psi = \{\psi_Q\}_{Q\in\mathscr{D}}$ be two non-necessarily cancellative generalized Haar systems. A *commuting Haar shift operator* is an $L_2(\mathcal{A})$ bounded operator of the form

$$\text{(9.3)}\qquad Ш_{r,s}f(x) = \sum_{Q\in\mathscr{D}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\langle f,\phi_R\rangle\psi_S(x), \qquad \sup_{Q,R,S}\|\alpha^Q_{R,S}\|_{\mathcal{M}} < \infty,$$

where the symbols $\alpha^Q_{R,S}$ lie in $\mathcal{M}\cap\mathcal{M}'$, the center of $\mathcal{M}$. Notice that in this definition the pairing $\langle f,g\rangle = \int_{\mathbb{R}^d} fg\, d\mu$ is in fact a partial trace and whence operator-valued. Our second result determines conditions for which the weak-type $(1,1)$ for these operators hold.

Theorem 9.4. *Let* $Ш_{r,s}$ *be given as in* (9.3). *Assume that* $Ш_{r,s}$ *satisfies the restricted local vector-valued* L_2 *estimate*

$$\text{(9.5)}\qquad \int_{\mathbb{R}^d} \|Ш^{Q_0}_{r,s}(1_{Q_0})(x)\|^2_{\mathcal{M}}\, d\mu(x) \leq C\mu(Q_0),$$

uniformly over $Q_0\in\mathscr{D}$. *Here*

$$Ш^{Q_0}_{r,s}f(x) := \sum_{Q\in\mathscr{D}(Q_0)} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\langle f,\phi_R\rangle\psi_S(x),$$

where $\mathscr{D}(Q)$ *denotes the family of all dyadic subcubes of* Q *including* Q *itself. If*

$$\Xi(\Phi,\Psi,r,s) := \sup_{Q\in\mathscr{D}} \{\|\phi_R\|_{L_\infty(\mu)}\|\psi_S\|_{L_1(\mu)} : R\in\mathscr{D}_r(Q), S\in\mathscr{D}_s(Q)\} < \infty. \tag{9.6}$$

then $\text{Ш}_{r,s}$ *maps* $L_1(\mathcal{A})$ *continuously into* $L_{1,\infty}(\mathcal{A})$.

Remark 9.7. A testing argument with simple functions is used in [46] to show that the condition (9.6) is also necessary when the symbols are all nonzero. One can show that this is also the case in the present setting by following similar ideas. Indeed, the validity of the testing arguments relied on the fact that (2.1) holds for simple functions. For a simple tensor it is clear that

$$\begin{aligned}\|1_E\otimes p\|_{L_{1,\infty}(\mathcal{A})} &= \sup_{\lambda>0}\lambda\tau(1_{(\lambda,\infty)}(1_E\otimes p))\\ &= \tau(1_E\otimes p) = \mu(E)\nu(p) = \|1_E\|_{L^1(\mu)}\|p\|_{L_1(\mathcal{M})}.\end{aligned}$$

And thus

$$\|f\|_{L^{1,\infty}(\mathcal{A})} \le \|f\|_{L^1(\mathcal{A})} \le \#\{f(x) : x\in\mathbb{R}^d\}\,\|f\|_{L^{1,\infty}(\mathcal{A})}$$

holds for operator-valued functions of the form

$$f = \sum_{i=1}^n a_i 1_{E_i}\otimes p_i,$$

where $a_i\in\mathbb{C}$, $E_i\subset\mathbb{R}^d$ are pairwise disjoint μ-measurable sets and $p_i\in\mathcal{P}(\mathcal{M})$ are ν-finite pairwise disjoint projections.

Remark 9.8. As in the commutative case, if the Haar systems $\Phi=\{\phi_Q\}_{Q\in\mathscr{D}}$ and $\Psi=\{\psi_Q\}_{Q\in\mathscr{D}}$ are cancellative, orthogonality arguments may be used to verify that the condition (9.5) and the L^2 boundedness of $\text{Ш}_{r,s}$ are satisfied.

As discussed in Part I, the condition (9.6) may be interpreted as certain restriction on the measure μ in terms of its degeneracy over generations of dyadic cubes. The resulting class of measures depends strongly on the Haar shift operator in question. For some operators the associated class of measures is shown to be strictly bigger than the doubling class, but nevertheless disjoint from the class of measures of polynomial growth, for which non-standard Calderón-Zygmund theories are available.

Chapter 10

The Calderón-Zygmund decomposition

This section is devoted to the proof of Theorem 9.2. First, some reductions are in order. For simplicity we will assume that $\mu(\mathbb{R}^d) = \infty$ and that the dyadic lattice $\mathscr{D}$ has no quadrants. Namely, that $\mathscr{D}$ is such that for every compact K there exists $Q \in \mathscr{D}$ with $K \subset Q$. These assumptions can be removed arguing as in [46]. However, we find the second assumption very natural since — in a probabilistic sense — almost all dyadic lattices satisfy it. Also, as argued in [46], we are confident that our results also hold in the context of geometrically doubling metric spaces. From the previous assumptions, it can be seen that for a fixed $f \in \mathcal{A}_{+,K}$ and $\lambda > 0$ there exists $m_\lambda(f) \in \mathbb{Z}$ such that $f_k \leq \lambda 1_{\mathcal{A}}$ for all $k \leq m_\lambda(f)$ (see [58]). Without loss of generality, we may also assume that f has only finite non-vanishing martingale differences.

Remark 10.1. To ease notation, we will use the normalization $m_\lambda(f) = 0$. It is safe to assume so since in the proofs of Theorems 9.2 and 9.4 both $f \in \mathcal{A}_{+,K}$ and $\lambda > 0$ will remain fixed, but otherwise arbitrary.

We start with the construction of the projections $(p_k)_{k\in\mathbb{Z}}$ and q of Theorem 9.2. To that end we will use the so-called Cuculescu's construction. Here we state it in the precise form that we will use, although the construction can be done in any semifinite von Neumann algebra.

Cuculescu's construction [18]. *Let $f \in \mathcal{A}_{+,K}$ and consider the associated positive martingale $(f_k)_{k\in\mathbb{Z}}$ relative to the dyadic filtration $(\mathcal{A}_k)_{k\in\mathbb{Z}}$. Given $\lambda > 0$, the decreasing sequence of projections $(q_k)_{k\in\mathbb{Z}}$ defined recursively by $q_k = 1_{\mathcal{A}}$ for $k \leq 0$ and*

$$q_k = q_k(f, \lambda) := 1_{(0,\lambda]}\big(q_{k-1} f_k q_{k-1}\big)$$

is such that

(*a*) *q_k is a projection in $\mathcal{A}_k$,*

(*b*) *q_k commutes with $q_{k-1} f_k q_{k-1}$,*

(c) $q_k f_k q_k \leq \lambda q_k$,

(d) $q = \bigwedge_k q_k$ *satisfies*

$$\|q f_k q\|_{\mathcal{A}} \leq \lambda \text{ for all } k \geq 1 \quad \text{and} \quad \tau(1_{\mathcal{A}} - q) \leq \frac{1}{\lambda}\|f\|_{L_1(\mathcal{A})}.$$

Define the sequence $(p_k)_{k\geq 1}$ *of pairwise disjoint projections by*

$$p_k = q_{k-1} - q_k.$$

In particular

$$\sum_{k\geq 1} p_k = 1_{\mathcal{A}} - q$$

and also $p_k f_k p_k \geq \lambda p_k$.

Remark 10.2. Since the projection q_k is $\sigma(\mathscr{D}_k)$-measurable, we have the following useful expression

$$q_k = \sum_{Q\in\mathscr{D}_k} q_Q \otimes 1_Q,$$

where $q_Q = q_Q(f, Q)$ are projections in $\mathcal{M}$ defined by

$$q_Q = \begin{cases} 1_{\mathcal{M}} & \text{if } k < 0 \\ 1_{(0,\lambda]}(q_{\widehat{Q}} \langle f\rangle_Q \, q_{\widehat{Q}}) & \text{if } k \geq 0. \end{cases}$$

As in Cuculescu's construction, these projections satisfy

(a) $q_Q \leq q_{\widehat{Q}}$.

(b) q_Q commutes with $q_{\widehat{Q}} \langle f\rangle_Q \, q_{\widehat{Q}}$.

(c) $q_Q \langle f\rangle_Q \, q_Q \leq \lambda q_Q$.

One then can express the projections p_k as

$$p_k = \sum_{Q\in\mathscr{D}_k} (q_{\widehat{Q}} - q_Q) 1_Q =: \sum_{Q\in\mathscr{D}_k} p_Q \otimes 1_Q, \tag{10.3}$$

and we analogously have that $p_Q \in \mathcal{P}(\mathcal{M})$ is such that $p_Q\langle f\rangle_Q \, p_Q \geq \lambda p_Q$. As detailed in [58], one could interpret the projections p_k as the union dyadic cubes of side-length 2^{-k} into which the classical level set $\Omega_\lambda = \{\sup_k f_k > \lambda\}$ is decomposed. One can thus view q as the complementary set of Ω_λ.

Proof of Theorem 9.2. By construction $f = g + b + \beta$. We now turn to the estimates of the diagonal part. For the L_1 estimate of g_Δ observe that by the tracial property

$$\begin{aligned} \|g_\Delta\|_{L_1(\mathcal{A})} &= \tau(fq) + \sum_{k\geq 1} \tau(\mathsf{E}_{k-1}(p_k f_k p_k)) \\ &= \tau(fq) + \tau(f(1_{\mathcal{A}} - q)) = \|f\|_{L_1(\mathcal{A})}, \end{aligned}$$

since E_k preserves the trace. The proof of the L_2 estimate of g_Δ is a bit more involved since μ is not necessarily doubling. Also, the lack of commutativity of $\mathcal{M}$ prevents us from following the argument that appeared in [46]. However, standard arguments in noncommutative martingale theory apply. First notice that since q_k commutes with $q_{k-1}f_kq_{k-1}$,

$$\mathsf{E}_{k-1}(p_kf_kp_k) = q_{k-1}f_{k-1}q_{k-1} - \mathsf{E}_{k-1}(q_kf_kq_k).$$

Thus,

$$\begin{aligned}\Big\| \sum_{k\geq 1} \mathsf{E}_{k-1}(p_kf_kp_k)\Big\|_{L_2(\mathcal{A})}^2 &\leq 2\bigg(\Big\| \sum_{k\geq 1} q_kf_kq_k - \mathsf{E}_{k-1}(q_kf_kq_k)\Big\|_{L_2(\mathcal{A})}^2 \\ &\qquad + \Big\| \sum_{k\geq 1} q_kf_kq_k - q_{k-1}f_{k-1}q_{k-1}\Big\|_{L_2(\mathcal{A})}^2 \bigg) \\ &= 2(I + II).\end{aligned}$$

As it is proved in [70, Lemma 3.4], we have that

$$\begin{aligned}\|q_kf_kq_k - \mathsf{E}_{k-1}(q_kf_kq_k)\|_{L_2(\mathcal{A})}^2 \leq 2\big(\|q_kf_kq_k\|_{L_2(\mathcal{A})}^2 - \|q_{k-1}f_{k-1}q_{k-1}\|_{L_2(\mathcal{A})}^2\big) \\ + 6\lambda\tau(q_{k-1}f_{k-1}q_{k-1} - q_kf_kq_k).\end{aligned}$$

Therefore, by orthogonality of martingale differences and the previous estimate, summation over k gives

$$\begin{aligned} I &= \sum_{k\geq 1} \|q_kf_kq_k - \mathsf{E}_{k-1}(p_kf_kp_k)\|_{L_2(\mathcal{A})}^2 \\ &\leq \lim_{k\to\infty} \Big(2\Big(\|q_kf_kq_k\|_{L_2(\mathcal{A})}^2 - \|q_0f_0q_0\|_{L_2(\mathcal{A})}^2 \Big) + 6\lambda\tau(q_0f_0q_0 - q_kf_kq_k)\Big) \\ &\leq \lim_{k\to\infty} \Big(2\|q_kf_kq_k\|_{L_2(\mathcal{A})}^2 + 6\lambda\tau(q_0f_0)\Big) \leq 8\lambda\|f\|_{L_1(\mathcal{A})},\end{aligned}$$

where Hölder's inequality and (c) of Cuculescu's construction were used. To estimate II we perform the telescopic sum in order to get

$$II \leq 2\|qfq\|_{L_2(\mathcal{A})}^2 + 2\|q_0f_0q_0\|_{L_2(\mathcal{A})}^2 \leq 4\lambda\|f\|_{L_1(\mathcal{A})},$$

which follows from the estimate $qfq \leq \lambda q$, which in turn can be deduced from Cuculescu's construction (see [58, Section 4.1]). By this last estimate and using that $(a+b+c)^2 \leq 3a^2 + 3b^2 + 3c^2$ for a, b, c positive numbers, we finally obtain

$$\|g_\Delta\|_{L_2(\mathcal{A})}^2 \leq 39\lambda\|f\|_{L_1(\mathcal{A})}.$$

The bad terms are easier to handle. Clearly the bad term b_Δ is comprised of the self-adjoint terms $b_k = p_k(f - f_k)p_k$ with the mean zero property $\mathsf{E}_k(b_k) = 0$, so that $\int_{\mathbb{R}^d} b\, d\mu = 0$. Moreover, by the orthogonality of the projections p_k, the tracial property of τ and since conditional expectations are bimodular and trace preserving, we have that

$$\|b_\Delta\|_{L_1(\mathcal{A})} = \sum_{k\geq 1} \|b_k\|_{L_1(\mathcal{A})} \leq \sum_{k\geq 1} \tau\big(p_k(f+f_k)p_k\big)$$
$$= 2\tau(f(1_{\mathcal{A}}-q)) \leq 2\|f\|_{L_1(\mathcal{A})}.$$

Similarly, $\beta_\Delta = \sum_k \beta_k$, where $\beta_k = \mathsf{D}_k(p_k f_k p_k) = \mathsf{D}_k \beta_\Delta$ is a k martingale difference — and hence of mean zero. Moreover, as conditional expectations are contractive on $L_1(\mathcal{A})$

$$\|\beta_\Delta\|_{L_1(\mathcal{A})} \leq \sum_{k\geq 1} \|\beta_k\|_{L_1(\mathcal{A})} \leq 2\sum_{k\geq 1} \tau(p_k f_k p_k) = 2\tau(f(1_{\mathcal{A}}-q)) \leq 2\|f\|_{L_1(\mathcal{A})}.$$

We now turn to the off-diagonal terms, which require some more work. To get the appropriate estimate for g_{off}, first we need to obtain a manageable expression for its k martingale difference. Rewrite g_{off} as

$$g_{\mathrm{off}} = (1_{\mathcal{A}}-q)fq + qf(1_{\mathcal{A}}-q) + \sum_{k\geq 1}\sum_{h\geq 1} \mathsf{E}_{k+h-1}(p_k f_{k+h} p_{k+h} + p_{k+h} f_{k+h} p_k).$$

Since $p_{i\wedge j}, p_{i\vee j} \leq q_{i\wedge j-1}$ and by the commutation property (b) of Cuculescu's construction we have that

$$p_i f_{i\wedge j} p_j = p_i q_{i\wedge j-1} f_{i\wedge j} q_{i\wedge j-1} p_j = 0, \quad i\neq j, \quad i,j\in\mathbb{N}\cup\{\infty\}. \tag{10.4}$$

Thus,

$$\sum_{k\geq 1}\sum_{h\geq 1} \mathsf{E}_{k+h-1}(p_k f_{k+h} p_{k+h} + p_{k+h} f_{k+h} p_k)$$
$$= \sum_{k\geq 1}\sum_{h\geq 1} \mathsf{E}_{k+h-1}\big(p_k(f_{k+h}-f_k)p_{k+h} + p_{k+h}(f_{k+h}-f_k)p_k\big)$$
$$= \sum_{k\geq 1}\sum_{h\geq 1}\sum_{i=1}^{h} \mathsf{E}_{k+h-1}(p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k).$$

We may now proceed to calculate $\mathsf{D}_j(g_{\mathrm{off}})$ for $j\geq 1$. Taking into account that, for $h\geq 1$, $\mathsf{D}_j\mathsf{E}_{k+h-1} = \mathsf{D}_j$ if $j<k+h$ and zero otherwise, we get that

$$\mathsf{D}_j(g_{\mathrm{off}}) = \mathsf{D}_j((1_{\mathcal{A}}-q)fq + qf(1_{\mathcal{A}}-q))$$
$$+ \sum_{k<j}\sum_{h>j-k}\sum_{i=1}^{h} \mathsf{D}_j(p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k)$$
$$+ \sum_{k\geq j}\sum_{h\geq 1}\sum_{i=1}^{h} \mathsf{D}_j(p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k) = I + II + III.$$

We deal first with II. By Fubini's theorem we obtain that

$$II = \sum_{k<j}\left(\sum_{i=1}^{j-k}\sum_{h>j-k} \mathsf{D}_j(p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k)\right.$$

$$+ \sum_{i>j-k} \sum_{h \geq i} \mathsf{D}_j (p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k) \Bigg)$$

$$= \sum_{k<j} \Bigg(\sum_{i=1}^{j-k} \mathsf{D}_j (p_k df_{k+i} q_j + q_j df_{k+i} p_k)$$

$$+ \sum_{i>j-k} \mathsf{D}_j (p_k df_{k+i} q_{k+i-1} + q_{k+i-1} df_{k+i} p_k)$$

$$- \sum_{i \geq 1} \mathsf{D}_j (p_k df_{k+i} q + q df_{k+i} p_k) \Bigg) = II_1 + II_2 + II_3.$$

After summing over i in II_1 and noticing that by (b) of Cuculescu's construction (recall that $k < j$)

$$p_k f_k q_j = p_k q_{k-1} f_k q_{k-1} q_j = 0 = q_j f_k p_k,$$

we find that

$$II_1 = \sum_{k<j} \mathsf{D}_j (p_k f_j q_j + q_j f_j p_k) = \mathsf{D}_j \big((1_\mathcal{A} - q_{j-1}) f_j q_j + q_j f_j (1_\mathcal{A} - q_{j-1}) \big).$$

The term II_2 vanishes since

$$p_k df_{k+i} q_{k+i-1} + q_{k+i-1} df_{k+i} p_k = \mathsf{D}_{k+i} (p_k f q_{k+i-1} + q_{k+i-1} f p_k) \tag{10.5}$$

and $\mathsf{D}_j \mathsf{D}_{k+i} = 0$, as $k + i > j$. Performing the summation over i in II_3 and using (10.4) with $i \wedge j = k$ and $i \vee j = \infty$, we get that

$$II = \mathsf{D}_j \big((1_\mathcal{A} - q_{j-1}) f_j q_j + q_j f_j (1_\mathcal{A} - q_{j-1}) \big)$$
$$- \mathsf{D}_j \big((1_\mathcal{A} - q_{j-1}) f q + q f (1_\mathcal{A} - q_{j-1}) \big).$$

Changing the order of summation

$$III = \sum_{k \geq j} \sum_{i \geq 1} \sum_{h \geq i} \mathsf{D}_j (p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k)$$

$$= \sum_{k \geq j} \Bigg(\sum_{i \geq 1} \mathsf{D}_j (p_k df_{k+i} q_{k+i-1} + q_{k+i-1} df_{k+i} p_k)$$

$$- \sum_{i \geq 1} \mathsf{D}_j (p_k df_{k+i} q + q df_{k+i} p_k) \Bigg)$$

$$= -\mathsf{D}_j \big((q_{j-1} - q) f q + q f (q_{j-1} - q) \big).$$

Here, we have also used (10.5), as $k + i > j$, and (10.4) with $i \vee j = \infty$. Finally, summing everything we get that for $j \geq 1$

$$\mathsf{D}_j (g_{\text{off}}) = \mathsf{D}_j \big((1_\mathcal{A} - q_{j-1}) f_j q_j \big) + \mathsf{D}_j \big(q_j f_j (1_\mathcal{A} - q_{j-1}) \big).$$

On the other hand, $\mathsf{D}_j(g_{\mathrm{off}}) = 0$ for $j \le 0$. Indeed,

$$\mathsf{D}_j(g_{\mathrm{off}}) = \mathsf{D}_j((1_{\mathcal{A}} - q)fq + qf(1_{\mathcal{A}} - q)) \\ + \sum_{k\ge 1}\sum_{h\ge 1}\sum_{i=1}^{h} \mathsf{D}_j(p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k)$$

and, arguing as with III above and since $q_0 = 1_{\mathcal{A}}$, we have that

$$\sum_{k\ge 1}\sum_{h\ge 1}\sum_{i=1}^{h} \mathsf{D}_j(p_k df_{k+i} p_{k+h} + p_{k+h} df_{k+i} p_k) = -\mathsf{D}_j\big((1_{\mathcal{A}} - q)fq + qf(1_{\mathcal{A}} - q)\big).$$

Thus, in L_2 sense

$$\begin{aligned} g_{\mathrm{off}} = \sum_{j\ge 1} \mathsf{D}_j(g_{\mathrm{off}}) &= \sum_{j\ge 1}\sum_{k<j} \mathsf{D}_j(p_k f_j q_j + q_j f_j p_k) \\ &= \sum_{k\ge 1}\sum_{h\ge 1} \mathsf{D}_{k+h}(p_k f_{k+h} q_{k+h} + q_{k+h} f_{k+h} p_k) =: \sum_{k\ge 1}\sum_{h\ge 1} g_{k,h}. \end{aligned}$$

We are now in the position to prove the estimate in (d) of Theorem 9.2. Notice first that by Hölder's inequality, the C^*-algebra property and (c) of Cuculescu's construction

$$\begin{aligned} \|g_{k,h}\|_{L_2(\mathcal{A})}^2 &\le 16\|q_{k+h} f_{k+h} p_k\|_{L_2(\mathcal{A})}^2 \\ &= 16\tau(p_k f_{k+h} q_{k+h} f_{k+h} p_k) \\ &\le 16\big\|f_{k+h}^{1/2} q_{k+h} f_{k+h}^{1/2}\big\|_{\mathcal{A}}\, \tau\big(f_{k+h}^{1/2} p_k f_{k+h}^{1/2}\big) \\ &= 16\|q_{k+h} f_{k+h} q_{k+h}\|_{\mathcal{A}}\, \tau(p_k f_{k+h} p_k) \le 16\lambda\tau(f p_k). \end{aligned}$$

This proves that for all $h \ge 1$

$$\sum_{k\ge 1} \|g_{k,h}\|_{L_2(\mathcal{A})}^2 \le 16\lambda\tau(f(1_{\mathcal{A}} - q)) \le 16\lambda\|f\|_{L_1(\mathcal{A})}.$$

For the bad terms we follow [58]. First, rewrite b_{off} as

$$b_{\mathrm{off}} = \sum_{h\ge 1}\sum_{k\ge 1} p_k(f - f_{k+h})p_{k+h} + p_{k+h}(f - f_{k+h})p_k =: \sum_{h\ge 1}\sum_{k\ge 1} b_{k,h}.$$

Clearly, the terms $b_{k,h}$ have mean zero and satisfy the estimate

$$\|b_{k,h}\|_{L_1(\mathcal{A})} \le 2\|p_k f p_{k+h} + p_{k+h} f p_k\|_{L_1(\mathcal{A})}.$$

Next, observe that we can decompose the off-diagonal terms $p_k f p_{k+h} + p_{k+h} f p_k$ into a sum of four positive overlapping box-diagonal terms

$$\begin{aligned} p_k f p_{k+h} + p_{k+h} f p_k = &\Big(\sum_{j=0}^{h} p_{k+j}\Big) f \Big(\sum_{j=0}^{h} p_{k+j}\Big) - \Big(\sum_{j=0}^{h-1} p_{k+j}\Big) f \Big(\sum_{j=0}^{h-1} p_{k+j}\Big) \\ &- \Big(\sum_{j=1}^{h} p_{k+j}\Big) f \Big(\sum_{j=1}^{h} p_{k+j}\Big) + \Big(\sum_{j=1}^{h-1} p_{k+j}\Big) f \Big(\sum_{j=1}^{h-1} p_{k+j}\Big). \end{aligned}$$

The previous expression implies that

$$\begin{aligned}\sum_{k\geq 1}\|p_k f p_{k+h} + p_{k+h} f p_k\|_{L_1(\mathcal{A})} &\leq 4\sum_{k\geq 1}\sum_{j=0}^{h}\tau(f p_{k+j})\\ &= 4\sum_{j=0}^{h}\tau(f(q_j - q)) \leq 4(h+1)\|f\|_{L_1(\mathcal{A})},\end{aligned}$$

and hence the estimate in (e) holds. On the other hand, we have

$$\beta_{\mathrm{off}} = \sum_{k\geq 1}\sum_{h\geq 1}\mathsf{D}_{k+h}(p_k f_{k+h} p_{k+h} + p_{k+h} f_{k+h} p_k) =: \sum_{k\geq 1}\sum_{h\geq 1}\beta_{k,h}.$$

Each term in the previous sum satisfies the same estimate

$$\|\beta_{k,h}\|_{L_1(\mathcal{A})} \leq 2\|p_k f p_{k+h} + p_{k+h} f p_k\|_{L_1(\mathcal{A})},$$

which yields the corresponding estimate for β_{off}. □

Chapter 11

Commuting Haar shift operators

We now turn to the proof of Theorem 9.4. Namely that

$$\lambda\tau(\{|\text{Ш}_{r,s}f| > \lambda\}) \lesssim \|f\|_{L_1(\mathcal{A})}$$

for all $\lambda > 0$. Here $\tau(\{|f| > \lambda\})$ denotes the trace of the spectral projection of $|f|$ associated to the interval (λ, ∞), which defines a noncommutative distribution function. We find this terminology more intuitive, since it is reminiscent of the classical one. Following the construction of noncommutative symmetric spaces (see [53] and references therein), the resulting $L_{1,\infty}(\mathcal{A})$ space is a quasi-Banach space with quasi-norm $\|f\|_{L_{1,\infty}(\mathcal{A})} = \sup_{\lambda>0} \lambda\tau(\{|f| > \lambda\})$ which interpolates with $L_2(\mathcal{A})$. It should be mentioned that the weak Bochner space $L_{1,\infty}(\mathbb{R}^d, \mu; L_1(\mathcal{M}))$ is of no use for our purposes since $L_1(\mathcal{M})$ is not a UMD space and thus even Haar multipliers may not be bounded, which rules out the use of this space as an appropriate setting for providing weak-type $(1, 1)$ estimates for the operators in question. The same applies if one considers $\mathcal{M}$ instead of $L_1(\mathcal{M})$ as target space.

Proof of Theorem 9.4. Let $f \in \mathcal{A}_{+,K}$. The general case follows by the density of the span of $\mathcal{A}_{+,K}$ in $L_1(\mathcal{A})$. Consider the Calderón-Zygmund decomposition $f = g_\Delta + b_\Delta + \beta_\Delta + g_{\text{off}} + b_{\text{off}} + \beta_{\text{off}}$ associated to (f, λ) for a given $\lambda > 0$. By the quasi-triangle inequality in $L_{1,\infty}(\mathcal{A})$ it suffices to show that

$$\lambda\tau(\{|\text{Ш}_{r,s}(\gamma)| > \lambda\}) \lesssim \|f\|_{L_1(\mathcal{A})}$$

for all $\gamma \in \{g_\Delta, b_\Delta, \beta_\Delta, g_{\text{off}}, b_{\text{off}}, \beta_{\text{off}}\}$. We start with the diagonal terms, for which the estimates are very similar to the classical ones. For g_Δ we use Chebyshev's inequality, the L_2 boundedness of $\text{Ш}_{r,s}$ and the L_2 estimate in (a) of Theorem 9.2 to get

$$\lambda\tau(\{|\text{Ш}_{r,s}(g_\Delta)| > \lambda\}) \leq 39\|\text{Ш}_{r,s}\|^2_{\mathcal{B}(L_2(\mathcal{A}))}\|f\|_{L_1(\mathcal{A})},$$

where $\|\text{Ш}_{r,s}\|_{\mathcal{B}(L_2(\mathcal{A}))}$ denotes the operator norm of $\text{Ш}_{r,s}$ on $L_2(\mathcal{A})$. For the remaining γ, we decompose $\text{Ш}_{r,s}(\gamma)$ as

$$\begin{aligned}\text{Ш}_{r,s}(\gamma) = (1_\mathcal{A} - q)\text{Ш}_{r,s}(\gamma)(1_\mathcal{A} - q) + q\text{Ш}_{r,s}(\gamma)q \\ + q\text{Ш}_{r,s}(\gamma)(1_\mathcal{A} - q) + (1_\mathcal{A} - q)\text{Ш}_{r,s}(\gamma)q.\end{aligned}$$

Since the distribution function is adjoint-invariant and by the second estimate in (d) of Cuculescu's construction, we get that

$$\lambda\tau(\{|\text{Ш}_{r,s}(\gamma)| > \lambda\}) \le 12\|f\|_{L_1(\mathcal{A})} + \lambda\tau(\{|q\text{Ш}_{r,s}(\gamma)q| > \lambda/4\}).$$

To prove the estimate for $\gamma = b_\Delta$, observe that we may further decompose each term b_k in (b) of Theorem 9.2 as

$$b_k = \sum_{L\in\mathscr{D}_k} p_L(f - \langle f\rangle_L)p_L 1_L =: \sum_{L\in\mathscr{D}_k} b_L,$$

where the projections p_L are defined as in (10.3). Since the Haar function ϕ_R is constant on dyadic subcubes of R and b_L has zero integral, $\langle b_L, \phi_R\rangle$ is nonzero only for $R \subset L$, i.e., $R^{(r)} \subset L^{(r)}$ for their respective r-dyadic ancestors. On the other hand, if $x \in L$ we have that $q(x) \le q_k(x) = q_L$ in the order of the lattice $\mathcal{P}(\mathcal{M})$. This together with (10.3) gives that for $x \in L$

$$q(x)\langle b_L, \phi_R\rangle q(x) = q(x)q_L\, p_L\langle b_L, \phi_R\rangle p_L q_L q(x) = 0. \tag{11.1}$$

Using that $\alpha^Q_{R,S} \in \mathcal{M}\cap\mathcal{M}'$ we find the estimate

$$\begin{aligned}
&\|q\text{Ш}_{r,s}(b_L)q\|_{L_1(\mathcal{A})} \\
&\quad\le \sum_{\substack{Q\in\mathscr{D}\\ L\subsetneq Q\subset L^{(r)}}} \sum_{\substack{R\in\mathscr{D}_r(Q),\, R\subset L\\ S\in\mathscr{D}_s(Q)}} \|\alpha^Q_{R,S}\|_{\mathcal{M}} \|\langle b_L, \phi_R\rangle\|_{L_1(\mathcal{M})}\|\psi_S\|_{L_1(\mu)} \\
&\quad\le \sup_{Q,R,S}\|\alpha^Q_{R,S}\|_{\mathcal{M}} \sum_{\substack{Q\in\mathscr{D}\\ L\subsetneq Q\subset L^{(r)}}} \sum_{\substack{R\in\mathscr{D}_r(Q),\, R\subset L\\ S\in\mathscr{D}_s(Q)}} \|\phi_R\|_{L_\infty(\mu)}\|\psi_S\|_{L_1(\mu)}\|b_L\|_{L_1(\mathcal{A})} \\
&\quad\le r2^{(r+s)d}\sup_{Q,R,S}\|\alpha^Q_{R,S}\|_{\mathcal{M}}\ \Xi(\Phi,\Psi; r,s)\|b_L\|_{L_1(\mathcal{A})}.
\end{aligned} \tag{11.2}$$

This, Chebyshev's inequality, the fact that dyadic cubes in $\mathscr{D}_k$ are disjoint and (b) of Theorem 9.2 give the estimate

$$\lambda\tau(\{|q\text{Ш}_{r,s}b_\Delta q| > \lambda\}) \le r2^{1+(r+s)d}\sup_{Q,R,S}\|\alpha^Q_{R,S}\|_{\mathcal{M}}\ \Xi(\Phi,\Psi; r,s)\|f\|_{L_1(\mathcal{A})}.$$

For $\gamma = \beta_\Delta$ we proceed likewise by writing

$$\begin{aligned}
\beta_k = \mathsf{D}_k(\beta_\Delta) &= \sum_{L\in\mathscr{D}_{k-1}} \sum_{J\in\mathscr{D}_1(L)} p_J\langle f\rangle_J\, p_J\left(1_J - \frac{\mu(J)}{\mu(L)}1_L\right) \\
&=: \sum_{L\in\mathscr{D}_{k-1}} \sum_{J\in\mathscr{D}_1(L)} \beta_{L,J} =: \sum_{L\in\mathscr{D}_k} \beta_L,
\end{aligned}$$

where each term β_L is supported (as an operator-valued function) on L, is constant on the dyadic descendants of L and has mean zero. By Chebyshev's inequality we have

$$\begin{aligned}
\lambda\tau(\{|q\text{Ш}_{r,s}(\beta_\Delta)q| > \lambda\}) \le \sum_{k\ge1}\sum_{L\in\mathscr{D}_{k-1}} \Big(&\int_{\mathbb{R}^d\setminus L} \nu\big(|q(x)\text{Ш}_{r,s}\beta_L(x)q(x)|\big)d\mu(x) \\
&+ \int_L \nu\big(|q(x)\text{Ш}_{r,s}\beta_L(x)q(x)|\big)d\mu(x)\Big).
\end{aligned}$$

Since $\langle \beta_L, \phi_R \rangle$ is nonzero only for dyadic cubes $R \subset L$, proceeding as in (11.2) we obtain

$$\int_{\mathbb{R}^d \setminus L} \nu\big(|q(x)\text{Ш}_{r,s}\beta_L(x)q(x)|\big)\,d\mu(x) \leq r2^{(r+s)d} \sup_{Q,R,S} \|\alpha^Q_{R,S}\|_{\mathcal{M}}\ \Xi(\Phi,\Psi;r,s)\|\beta_L\|_{L_1(\mathcal{A})}.$$

Arguing as above and recalling that $\alpha^Q_{R,S} \in \mathcal{M} \cap \mathcal{M}'$, for $x \in L$ we obtain

$$\begin{aligned} q(x)\text{Ш}_{r,s}(\beta_L)(x)q(x) &= \sum_{\substack{Q\in\mathscr{D}\\ L\subset Q\subset L^{(r)}}} \sum_{\substack{R\in\mathscr{D}_r(Q),\,R\subset L\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\, q(x)\langle \beta_L, \phi_R\rangle q(x)\psi_S(x) \\ &\quad + \sum_{\substack{Q\in\mathscr{D}\\ Q\subsetneq L}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\, q(x)\langle \beta_L, \phi_R\rangle q(x)\psi_S(x) \\ &= F_L(x) + G_L(x). \end{aligned}$$

As in (11.2) we get the estimate

$$\int_L \nu\big(|F_L(x)|\big)\,d\mu(x) \leq (r+1)2^{(r+s)d} \sup_{Q,R,S} \|\alpha^Q_{R,S}\|_{\mathcal{M}}\ \Xi(\Phi,\Psi;r,s)\|\beta_L\|_{L_1(\mathcal{A})}.$$

To estimate $G_L(x)$ we further decompose β_L and get

$$G_L(x) = \sum_{J\in\mathscr{D}_1(L)} \sum_{\substack{Q\in\mathscr{D}\\ Q\subsetneq L}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\, q(x)\langle \beta_{L,J}, \phi_R\rangle q(x)\psi_S(x) = \sum_{J\in\mathscr{D}_1(L)} G_{L,J}(x).$$

Given $J \in \mathscr{D}_1(L)$ and a dyadic cube $Q \subsetneq L$ we either have $Q \subset J$ or $Q \subset L \setminus J$. Yet the former case leads to zero terms since, as in (11.1), for $x \in Q \subset J$ we have $q(x) \leq q_J$ and thus $q(x)\langle \beta_{L,J}, \phi_R\rangle q(x) = 0$. Hence,

$$\begin{aligned} G_{L,J}(x) &= -p_J\langle f\rangle_J\, p_J\,\frac{\mu(J)}{\mu(L)} \sum_{\substack{Q\in\mathscr{D}\\ Q\subset L\setminus J}} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\, q(x)\langle 1_{L\setminus J}, \phi_R\rangle q(x)\psi_S(x) \\ &= -p_J\langle f\rangle_J\, p_J\,\frac{\mu(J)}{\mu(L)} \sum_{\substack{Q'\in\mathscr{D}_1(L)\\ Q'\neq J}} \sum_{Q\in\mathscr{D}(Q')} \sum_{\substack{R\in\mathscr{D}_r(Q)\\ S\in\mathscr{D}_s(Q)}} \alpha^Q_{R,S}\, q(x)\langle 1_{Q'}, \phi_R\rangle q(x)\psi_S(x) \\ &= -p_J\langle f\rangle_J\, p_J\,\frac{\mu(J)}{\mu(L)} \sum_{\substack{Q'\in\mathscr{D}_1(L)\\ Q'\neq J}} q(x)\text{Ш}^{Q'}_{r,s}(1_{Q'})(x)q(x). \end{aligned}$$

Then, by Hölder's inequality and the fact that $\operatorname{supp}_{\mathbb{R}^d}\big(\text{Ш}^{Q'}_{r,s}(1_{Q'})\big) \subset Q'$

$$\begin{aligned} &\int_L \nu\big(|G_L(x)|\big)\,d\mu(x) \\ &\quad = \int_L \nu\left(\left|\sum_{J\in\mathscr{D}_1(L)} p_J\langle f\rangle_J\, p_J\,\frac{\mu(J)}{\mu(L)} \sum_{\substack{Q'\in\mathscr{D}_1(L)\\ Q'\neq J}} q(x)\text{Ш}^{Q'}_{r,s}(1_{Q'})(x)q(x)\right|\right)d\mu(x) \end{aligned}$$

$$\leq \sum_{J\in\mathscr{D}_1(L)} \|p_J\langle f\rangle_J\, p_J\|_{L_1(\mathcal{M})}\,\frac{\mu(J)}{\mu(L)} \sum_{\substack{Q'\in\mathscr{D}_1(L)\\ Q'\neq J}} \int_L \|\text{Ш}^{Q'}_{r,s}(1_{Q'})(x)\|_{\mathcal{M}}\, d\mu(x)$$

$$\leq \sum_{J\in\mathscr{D}_1(L)} \|p_J\langle f\rangle_J\, p_J\|_{L_1(\mathcal{M})}\,\frac{\mu(J)}{\mu(L)}$$

$$\times \left(\sum_{\substack{Q'\in\mathscr{D}_1(L)\\ Q'\neq J}} \left(\int_{\mathbb{R}^d} \|\text{Ш}^{Q'}_{r,s}(1_{Q'})(x)\|^2_{\mathcal{M}}\, d\mu(x)\right)^{\frac12} \mu(Q')^{\frac12}\right)$$

$$\leq \sup_{\substack{Q\in\mathscr{D},\\ \mu(Q)\neq 0}} \frac{1}{\mu(Q)^{\frac12}} \left(\int_{\mathbb{R}^d} \|\text{Ш}^{Q}_{r,s}(1_{Q})(x)\|^2_{\mathcal{M}}\, d\mu(x)\right)^{\frac12} \left\|\sum_{J\in\mathscr{D}_1(L)} p_J\langle f\rangle_J\, p_J 1_J\right\|_{L_1(\mathcal{A})},$$

which is finite by the local vector-valued L_2 estimate (9.5). By the estimate in (c) of the Calderón-Zygmund decomposition

$$\sum_{k\geq 1}\sum_{L\in\mathscr{D}_k}\left(\|\beta_L\|_{L_1(\mathcal{A})} + \left\|\sum_{J\in\mathscr{D}_1(L)} p_J\langle f\rangle_J\, p_J 1_J\right\|_{L_1(\mathcal{A})}\right)$$

$$\leq \sum_k \big(\|\beta_k\|_1 + \|p_k f_k p_k\|_1\big) \leq 3\|f\|_1.$$

Thus, gathering the previous estimates

$$\lambda\tau(\{|q\text{Ш}_{r,s}(\beta_\Delta)q| > \lambda\})$$

$$\leq \Bigg((r+2)2^{1+(r+s)d} \sup_{Q,R,S} \|\alpha^Q_{R,S}\|_{\mathcal{M}}\ \Xi(\Phi,\Psi;r,s)$$

$$+ \sup_{\substack{Q,\in\mathscr{D}\\ \mu(Q)\neq 0}} \frac{1}{\mu(Q)^{\frac12}} \left(\int_{\mathbb{R}^d} \|\text{Ш}^{Q}_{r,s}(1_{Q})(x)\|^2_{\mathcal{M}}\, d\mu(x)\right)^{\frac12}\Bigg)\|f\|_{L_1(\mathcal{A})}.$$

We now turn to the weak-type estimates for the off-diagonal terms, starting with g_{off}. By Chebyshev's inequality

$$\lambda\tau(\{|q\text{Ш}_{r,s}(g_{\text{off}})q| > \lambda\}) \leq \frac{1}{\lambda}\left(\sum_{h\geq 1}\left\|\sum_{k\geq 1} q\text{Ш}_{r,s}(g_{k,h})q\right\|_{L_2(\mathcal{A})}\right)^2.$$

We further decompose the terms $g_{k,h}$ as

$$g_{k,h} = \sum_{L\in\mathscr{D}_k}\sum_{J\in\mathscr{D}_h(L)} \big(p_L\langle f\rangle_J\, q_J + q_J\langle f\rangle_J\, p_L\big)\left(1_J - \frac{\mu(J)}{\mu(\widehat{J})}1_{\widehat{J}}\right) =: \sum_{L\in\mathscr{D}_k} g_{L,h}.$$

Clearly, each term $g_{L,h}$ is such that $\operatorname{supp}_{\mathbb{R}^d}(g_{L,h}) \subset L$ and has mean zero on the $(h-1)$-descendants of L. Thus, $\langle g_{L,h}, \phi_R\rangle$ is nonzero only for $R \subset \widehat{J}$ for some $J \in \mathscr{D}_h(L)$, which amounts to say that $R \in \mathscr{D}_{h+j-1}(L)$ for some $j \geq 0$. Furthermore $g_{L,h} = p_L A_{L,h} + A^*_{L,h} p_L$, where

$$A_{L,h} = p_L\langle f\rangle_J\, q_J\left(1_J - \frac{\mu(J)}{\mu(\widehat{J})}1_{\widehat{J}}\right).$$

Proceeding as in (11.1) we get that $q(x)\langle g_{L,h}, \phi_R\rangle q(x) = 0$ if $x \in L$. In other words, only the cubes R such that $R^{(r)} \supsetneq L$ lead to nonzero terms. These two observations in terms of side-lengths provide that h must be such that $\ell(L) = 2^{-k} < \ell(R^{(r)}) = 2^{-(k+h+j-1-r)}$, namely $h \leq r$. This and the assumption $\alpha_{R,S}^Q \in \mathcal{M} \cap \mathcal{M}'$ allow us to deduce that $q(x)\text{Ш}_{r,s} g_{k,h}(x)q(x) = 0$ whenever $h > r$. This localization property and the orthogonality of martingale differences in $L_2(\mathcal{A})$, enable us to obtain that

$$\sum_{h\geq 1}\Big\| \sum_{k\geq 1} q\text{Ш}_{r,s}(g_{k,h})q\Big\|_{L_2(\mathcal{A})} \leq \|\text{Ш}_{r,s}\|_{\mathcal{B}(L_2(\mathcal{A}))} \sum_{h=1}^{r} \Big\|\sum_{k\geq 1} g_{k,h}\Big\|_{L_2(\mathcal{A})}$$
$$= \|\text{Ш}_{r,s}\|_{\mathcal{B}(L_2(\mathcal{A}))} \sum_{h=1}^{r}\bigg(\sum_{k\geq 1}\|g_{k,h}\|^2_{L_2(\mathcal{A})}\bigg)^{1/2}.$$

Therefore, by the estimate in (d) of Theorem 9.2 we arrive at

$$\lambda\tau(\{|q\text{Ш}_{r,s}(g_{\text{off}})q| > \lambda\}) \leq 16r^2\|\text{Ш}_{r,s}\|^2_{\mathcal{B}(L_2(\mathcal{A}))}\|f\|_{L_1(\mathcal{A})}.$$

To get the estimate for b_{off} we proceed in an entirely similar way by decomposing the terms $b_{k,h}$ in (e) of Theorem 9.2 as

$$b_{k,h} = \sum_{L\in\mathscr{D}_k}\sum_{J\in\mathscr{D}_h(L)} \big(p_L(f - \langle f\rangle_J)p_J + p_J(f-\langle f\rangle_J)p_L\big)1_J =: \sum_{L\in\mathscr{D}_k} b_{L,h}.$$

It is clear that $\text{supp}_{\mathbb{R}^d}(b_{L,h}) \subset L$, that $b_{L,h}$ has mean zero over the h-dyadic descendants of L and that $b_{L,h} = p_L B_{L,h} + B_{L,h}^* p_L$, with $B_{L,h} = p_L(f-\langle f\rangle_J)p_J 1_J$. Arguing as above, $q(x)\langle b_{L,h}, \phi_R\rangle q(x)$ is nonzero only for $R \subsetneq L \subsetneq R^{(r)} \subsetneq L^{(r)}$ and hence $q(x)\text{Ш}_{r,s}(b_{L,h})(x)q(x)$ vanishes if $h > r$. Thus, for $h \leq r$ we follow the steps in (11.2) to get the estimate

$$\sum_{L\in\mathscr{D}_k}\|q\text{Ш}_{r,s}(b_{L,h})q\|_{L_1(\mathcal{A})}$$
$$\leq (r-1)2^{(r+s)d}\sup_{Q,R,S}\|\alpha_{R,S}^Q\|_{\mathcal{M}}\ \Xi(\Phi,\Psi;r,s)\|b_{k,h}\|_{L_1(\mathcal{A})}.$$

By Chebyshev's inequality and the estimate in (e) of the Calderón-Zygmund decomposition we obtain

$$\lambda\tau(\{|q\text{Ш}_{r,s}(b_{\text{off}})q| > \lambda\})$$
$$\leq (r-1)2^{3+(r+s)d}\sup_{Q,R,S}\|\alpha_{R,S}^Q\|_{\mathcal{M}}\ \Xi(\Phi,\Psi;r,s)\sum_{h=1}^{r}(h+1)\|f\|_{L_1(\mathcal{A})}$$
$$= r(r-1)(r+3)2^{2+(r+s)d}\sup_{Q,R,S}\|\alpha_{R,S}^Q\|_{\mathcal{M}}\ \Xi(\Phi,\Psi;r,s)\|f\|_{L_1(\mathcal{A})}.$$

Finally, for $\gamma = \beta_{\text{off}}$ observe that

$$\beta_{k,h} = \sum_{L\in\mathscr{D}_k}\sum_{J\in\mathscr{D}_h(L)}\big(p_L\langle f\rangle_J\, p_J + p_J\langle f\rangle_J\, p_L\big)\bigg(1_J - \frac{\mu(J)}{\mu(\widehat{J})}1_{\widehat{J}}\bigg)$$
$$=: \sum_{L\in\mathscr{D}_k}\beta_{L,h} = \sum_{L\in\mathscr{D}_k}\big(p_L C_{L,h} + C_{L,h}^* p_L\big).$$

Here we may repeat the analysis made for $b_{L,h}$, as each $\beta_{L,h}$ is a $(k+h)$-martingale difference operator with $\mathrm{supp}_{\mathbb{R}^d}(\beta_{L,h}) \subset L$. This and (f) of Theorem 9.2 render the desired estimate

$$\begin{aligned}&\lambda\tau(\{|q\text{Ш}_{r,s}(\beta_{\mathrm{off}})q| > \lambda\}) \\ &\qquad \leq r(r-1)(r+3)2^{2+(r+s)d} \sup_{Q,R,S} \|\alpha^Q_{R,S}\|_{\mathcal{M}}\, \Xi(\Phi,\Psi;r,s)\|f\|_{L_1(\mathcal{A})},\end{aligned}$$

with which we complete the proof of Theorem 9.4. □

Remark 11.3. It is worth mentioning that we have not truly needed the assumption that the symbols are commuting to obtain the estimates for the diagonal terms. Indeed, all the calculations for the diagonal terms in the proof of Theorem 9.4 can be done without this assumption simply by rearranging multiplications. Unlike in (11.1), in the case when $\gamma \in \{g_{\mathrm{off}}, b_{\mathrm{off}}, \beta_{\mathrm{off}}\}$ and $x \in L$, $q(x)$ is required to be multiplied on **both** sides of $\langle \gamma_{L,h}, \phi_R\rangle$ in order to annihilate it.

Remark 11.4. The consideration of noncommuting symbols in (9.3) introduces considerable additional difficulties when trying to provide *a priori* estimates. First, different operators arise depending on whether the symbols act by right or left multiplication on each coefficient $\langle f, \phi_R\rangle$. More specifically, in the case of Haar multipliers, a pair of column/row operators are introduced by

$$T^{\mathrm{c}}_\alpha(f) = \sum_{Q\in\mathscr{D}} \alpha_Q\langle f,\phi_Q\rangle\phi_Q, \qquad T^{\mathrm{r}}_\alpha(f) = \sum_{Q\in\mathscr{D}} \langle f,\phi_Q\rangle\alpha_Q\phi_Q,$$

with uniformly bounded $\alpha_Q \in \mathcal{M}$. Even in the Lebesgue setting, Haar multipliers with noncommuting symbols may lack weak-type $(1,1)$ and strong (p,p) estimates for $p \neq 2$. This problem was solved in [27] as presented in Part II. There, weak-type $(1,1)$ estimates for Haar shift operators relative to the Lebesgue measure were obtained in terms of a column/row decomposition of the input function. Let us recall that decomposition, given $f \in \mathcal{A}_{+,K}$ and $a,k \in \mathbb{Z}$ consider the Cuculescu's projections $q_k(2^c) = q_k(f,2^c)$ and

$$\pi_{a,k} = \bigwedge_{c\geq a} q_k(2^c) - \bigwedge_{c\geq a-1} q_k(2^c).$$

For fixed k the projections $\pi_{a,k}$ are pairwise disjoint. Thus, f decomposes in column/row components as $f = f_{\mathrm{c}} + f_{\mathrm{r}}$ in terms of the multiscale triangle truncations

$$f_{\mathrm{c}} = \sum_{k\geq 1}\sum_{a\leq b} \pi_{a,k-1} df_k \pi_{b,k-1}\,, \quad f_{\mathrm{r}} = \sum_{k\geq 1}\sum_{a>b} \pi_{a,k-1} df_k \pi_{b,k-1}.$$

This decomposition is used in conjunction with the Calderón-Zygmund decomposition found in [58] to obtain that $\|M_{\mathrm{r}} f_{\mathrm{r}}\|_{1,\infty} + \|M_{\mathrm{c}} f_{\mathrm{c}}\|_{1,\infty} \lesssim \|f\|_1$, among analogous estimates for other Haar shift operators. Key to this argument is that the terms γ in the Calderón-Zygmund decomposition not having a proper L_2 estimate are such that $\mathsf{D}_k(\gamma) = (1_{\mathcal{A}} - q_{k-1})A_k + A_k^*(1_{\mathcal{A}} - q_{k-1})$, which leads to vanishing triangular truncations. A major setback for extending this argument to the nondoubling setting is that $\mathsf{D}_k(\beta_\Delta) = \beta_k = q_{k-1}\beta_k q_{k-1}$, reflecting that its classical counterpart decomposes into terms supported in the dyadic parents of the maximal dyadic cubes of Ω_λ. This forces to estimate L_1 norms of triangular truncations of β_k, which in the $\mathcal{B}(\ell_2^n)$-valued setting brings constants at best of order $\log(n+1)$. Furthermore, higher integrability of β_k — such as $L \log L$ (see [69])— might be hindered since μ is permitted to be nondoubling.

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