RESTRICTED NON-LINEAR APPROXIMATION IN SEQUENCE SPACES AND APPLICATIONS TO WAVELET BASES AND INTERPOLATION

EUGENIO HERNÁNDEZ AND DANIEL VERA

ABSTRACT. Restricted non-linear approximation is a type of N-term approximation where a measure ν on the index set (rather than the counting measure) is used to control the number of terms in the approximation. We show that embeddings for restricted non-linear approximation spaces in terms of weighted Lorentz sequence spaces are equivalent to Jackson and Bernstein type inequalities, and also to the upper and lower Temlyakov property. As applications we obtain results for wavelet bases in Triebel-Lizorkin spaces by showing the Temlyakow property in this setting. Moreover, new interpolation results for Triebel-Lizorkin and Besov spaces are obtained.

1. INTRODUCTION

Thresholding of wavelet coefficients is a technique used in image processing to compress signals or reduce noise. The simplest thresholding algorithm $T_\varepsilon (\varepsilon > 0)$ of a signal f is obtained by eliminating from a representation of f the terms whose coefficients have absolute value smaller than ε.

Although the thresholding approximants $T_\varepsilon(f)$ are sometimes a visually faithful representation of f, they are not exact, and from a theoretical point of view an error is introduced if f is replaced by $T_\varepsilon(f)$. Such errors have initially been measured in the L^2−norm, but it is argued in [19] that procedures having small error in L^p, or as stated in the statistical community, small L^p−risk, may reflect better the visual properties of a signal. Observe that in the usual thresholding the error is measured in the same space as the signal is represented, usually L^2.

A more general situation is considered in [6] where the wavelet coefficients are thresholded from a representation of the signal in the Hardy space H^r, $0 < r < \infty$ (recall that $H^r = L^r$ if $1 < r < \infty$), but the error is measured in the Hardy space H^p, $0 < p < \infty$. They show that this situation is equivalent to a type of nonlinear approximation, called **restricted**, in which a measure ν on the index set of dyadic cubes of $\mathbb{R}^d$ is used to control the number of terms in the approximation. In the classical n−term approximation $\nu(Q) = 1$, $Q \in \mathcal{D}$ (counting measure), and in [6] $\nu(Q) = |Q|^{1-p/r}$.

The article [6] provides a description of the approximation spaces in this setting in terms of certain type of discrete Lorentz spaces, as well as interpolation results for certain pairs of H^p and Besov spaces. One of the novelties of this article is that,

although the error is measured in H^p, the approximation spaces are not necessarily contained in H^p.

The theory of restricted nonlinear approximation was further developed in [20] considering the case of a quasi-Banach space $\mathbb{X}$, an unconditional basis $\mathcal{B} = \{\mathbf{e}_I\}_{I \in \mathcal{D}}$, and a measure ν on the countable set $\mathcal{D}$. They show that, in this abstract setting, restricted thresholding and restricted nonlinear approximation are linked to the $p-$Temlyakov property for ν (see definition in [20]). They also show that this property is equivalent to certain Jackson and Bernstein type inequalities and to have the restricted approximation spaces identified as discrete Lorentz spaces. The approach in [20] is that the approximation spaces are contained in $\mathbb{X}$ and, hence, not all results in [6] can be recovered.

Denote by S the space of all sequences $\mathbf{s} = \{s_I\}_{I \in \mathcal{D}}$ of complex numbers indexed by a countable set $\mathcal{D}$. In the present paper we study restricted nonlinear approximation for quasi-Banach lattices $f \subset S$ (see definition in section 2.1). Given a positive measure ν on $\mathcal{D}$ we define the restricted approximation spaces $\mathcal{A}_\mu^\xi(f, \nu)$, $0 < \xi < \infty$, $0 < \mu \leq \infty$, as subsets of S using ν to control the number of terms in the approximation and f to measure the error (see section 2.2).

Denote by $\mathcal{E} = \{\mathbf{e}_I\}_{I \in \mathcal{D}}$ the canonical basis for S. We use a weight sequence $\mathbf{u} = \{u_I\}_{I \in \mathcal{D}}$, $u_I > 0$, to control the weight of each $\mathbf{e}_I$. Discrete Lorentz spaces $\ell_\eta^\mu(\nu)$ are defined as sequences $\mathbf{s} = \{s_I\}_{I \in \mathcal{D}} \in S$ using the ν distribution function of the sequence $\{u_I s_I\}_{I \in \mathcal{D}}$ (see section 2.5). Here, η is a function in $\mathbb{W}$ (see section 2.4) more general than $\eta(t) = t^{1/p}$, $0 < p < \infty$.

It is shown in subsections 2.6 and 2.7 that the condition

$$C_1 \eta_1(\nu(\Gamma)) \leq \left\| \sum_{I \in \Gamma} \frac{\mathbf{e}_I}{u_I} \right\|_f \leq C_2 \eta_2(\nu(\Gamma)) \tag{1.0.1}$$

for all $\Gamma \subset \mathcal{D}$, with $\nu(\Gamma) < \infty$, $\eta_1 \in \mathbb{W}$ and $\eta_2 \in \mathbb{W}_+$, is equivalent to inclusions between $\mathcal{A}_\mu^\xi(f, \nu)$ and $\ell_{t^\xi \eta(t)}^\mu(\mathbf{u}, \nu)$, and also to some Jackson and Bernstein type inequalities. When $\eta_1(t) = \eta_2(t) = t^{1/p}$, condition (1.0.1) is called in [20] the $p-$Temlyakov property.

Working with sequence spaces is not a restriction. Lebesgue, Sobolev, Hardy and Lipschitz spaces all have a sequence space counterpart when using the $\varphi-$transform ([10], [11]) or wavelets ([23], [26], [7], [16], [24], [1], [22]). More generally, the Triebel-Lizorkin, $f_{p,r}^s$, and Besov, $b_{p,r}^s$, spaces of sequences (see section 2.9) allow faithful representations of Triebel-Lizorkin, $F_{p,r}^s(\mathbb{R}^d)$, and Besov, $B_{p,r}^s(\mathbb{R}^d)$, spaces (these include all the above spaces). When our results are coupled with the abstract transference framework designed in [13] we recover results for distribution or function spaces, as the case may be. One reason to consider such general setting, besides the obvious generalizations, is that measuring the error $\|f - T_\varepsilon(f)\|$ in Sobolev spaces, where the smoothing properties of $f - T_\varepsilon(f)$ are taken into account, may give a visually more faithful representation of f, than when measured in L^p. Observe that two functions may visually be very different although they may be close in the L^p norm.

In subsection 2.10 we show that (1.0.1) holds when $f = f_{p_1,q_1}^{s_1}$ and $u_I = \|\mathbf{e}_I\|_{f_{p_2,q_2}^{s_2}}$, with $\eta_1(t) = \eta_2(t) = t^{1/p_1}$ and $\nu(I) = |I|^\alpha$ if and only if $\alpha = p_1(\frac{s_2 - s_1}{d} - \frac{1}{p_2}) + 1 \neq 1$ or if $\alpha = 1$ then $p_1 = q_1$. When the results of subsection 2.6 and 2.7 are applied to this case, we show that restricted approximation spaces of Triebel -Lizorkin spaces are identified

with discrete Lorentz spaces, which coincide with Besov spaces for some particular values of the parameters (Lemma 2.10.4). The results in [6] and [18] are simple corollaries. We also give a result about interpolation of Triebel-Lizorkin and Besov spaces (section 2.11) with less restrictions on the parameters than those considered in [6].

The organization of the this paper is as follows. Notation, definitions, results and comments are given in section 2 which is divided in subsections 2.x with $1 \leq x \leq 11$. In section 3 we prove the results stated in section 2. If a statement of a result is given in subsection 2.x, its proof can be found in subsection 3.x. Be aware that if a subsection 2.x only contains notation, definitions and/or comments, but no statements of results, the corresponding subsection 3.x does not appear in section 3.

2. Notations, Definitions, Statements of Results and Comments

2.1. **Sequence Spaces.** Let $\mathcal{D}$ be a countable (index) set whose elements will be denoted by I. The set $\mathcal{D}$ could be $\mathbb{N}, \mathbb{Z}, ...$ or, as in the applications we have in mind, the countable set of dyadic cubes on $\mathbb{R}^d$.

Denote by $S = \mathbb{C}^{\mathcal{D}}$ the set of all sequences of complex numbers $\mathbf{s} = \{s_I\}_{I \in \mathcal{D}}$ defined over the countable set $\mathcal{D}$. For each $I \in \mathcal{D}$, we denote by $\mathbf{e}_I$ the element of S with entry 1 at I and 0 otherwise. We write $\mathcal{E} = \{\mathbf{e}_I\}_{I \in \mathcal{D}}$ for the canonical basis of S. We shall use the notation $\sum_{I \in \Gamma} s_I \mathbf{e}_I, \Gamma \subset \mathcal{D}$, to denote the element of S whose entry is s_I when $I \in \Gamma$ and 0 otherwise. Notice that no meaning of convergence is attached to the above notation even when Γ is not finite.

Definition 2.1.1. *A linear space of sequences $f \subset S$ is a **quasi-Banach (sequence) lattice** if there is a quasi-norm $\|\cdot\|_f$ in f with respect to which f is complete and satisfies:*

(a) Monotonicity: if $\mathbf{t} \in f$ and $|s_I| \leq |t_I|$ for all $I \in \mathcal{D}$, then $\mathbf{s} \in f$ and $\|\{s_I\}\|_f \leq \|\{t_I\}\|_f$.

(b) If $\mathbf{s} \in f$, then $\lim_{n \to \infty} \|s_{I_n} \mathbf{e}_{I_n}\|_f = 0$, for some enumeration $\mathfrak{I} = \{I_1, I_2, \dots\}$.

We will say that a quasi-Banach (sequence) lattice f is **embedded** in S, and write $f \hookrightarrow S$ if

$$\lim_{n \to \infty} \|\mathbf{s}^n - \mathbf{s}\|_f = 0 \Rightarrow \lim_{n \to \infty} s_I^{(n)} = s_I \quad \forall I \in \mathcal{D}. \tag{2.1.1}$$

Remark 2.1.2. *When $\mathcal{E} = \{\mathbf{e}_I\}_{I \in \mathcal{D}}$ is a Schauder basis for f, condition (a) in Definition 2.1.1 implies that $\mathcal{E}$ is an unconditional basis for f with constant $C = 1$.*

2.2. **Restricted Non-linear Approximation in Sequence Spaces.** In this paper ν will denote a positive measure on the discrete set $\mathcal{D}$ such that $\nu(I) > 0$ for all $I \in \mathcal{D}$. In the classical N-term approximation ν is the counting measure (i.e. $\nu(I) = 1$ for all $I \in \mathcal{D}$), but more general measures are used in the restricted non-linear approximation case. The measure ν will be used to control the number of terms in the approximation.

Definition 2.2.1. *We say that (f, ν) is a **standard scheme** (for restricted non linear approximation) if*

i) f is a quasi-Banach (sequence) lattice embedded in S.

ii) ν is a measure on $\mathcal{D}$ as explained in the first paragraph in this section.

Let (f, ν) be a standard scheme. For $t > 0$, define

$$\Sigma_{t,\nu} = \{\mathbf{t} = \sum_{I \in \Gamma} t_I \mathbf{e}_I : \nu(\Gamma) \leq t\}.$$

Notice that $\Sigma_{t,\nu}$ is not linear, but $\Sigma_{t,\nu} + \Sigma_{t,\nu} \subset \Sigma_{2t,\nu}$.

Given $\mathbf{s} \in S$, the f-error (or f-risk) of approximation to $\mathbf{s}$ by elements of $\Sigma_{t,\nu}$ is given by

$$\sigma_\nu(t, \mathbf{s}) = \sigma_\nu(t, \mathbf{s})_f := \inf_{\mathbf{t} \in \Sigma_{t,\nu}} \|\mathbf{s} - \mathbf{t}\|_f.$$

Notice that elements $\mathbf{s} \in S$ not in f could have finite f-risk since elements of $\Sigma_{t,\nu}$ could have infinite number of entries.

Definition 2.2.2. *(**Restricted Approximation Spaces**) Let (f, ν) be a standard scheme.*

i) For $0 < \xi < \infty$ and $0 < \mu < \infty$, $\mathcal{A}_\mu^\xi(f, \nu)$ is defined as the set of all $\mathbf{s} \in S$ such that

$$\|\mathbf{s}\|_{\mathcal{A}_\mu^\xi(f,\nu)} := \left(\int_0^\infty [t^\xi \sigma_\nu(t, \mathbf{s})]^\mu \frac{dt}{t} \right)^{1/\mu} < \infty. \tag{2.2.1}$$

ii) For $0 < \xi < \infty$ and $\mu = \infty$, $\mathcal{A}_\infty^\xi(f, \nu)$ is defined as the set of all $\mathbf{s} \in S$ such that

$$\|\mathbf{s}\|_{\mathcal{A}_\infty^\xi(f,\nu)} := \sup_{t > 0} t^\xi \sigma_\nu(t, \mathbf{s}) < \infty. \tag{2.2.2}$$

Notice that the spaces $\mathcal{A}_\mu^\xi(f, \nu)$ depend on the canonical basis $\mathcal{E}$ of S. When f are understood, we will write $\mathcal{A}_\mu^\xi(\nu)$ instead of $\mathcal{A}_\mu^\xi(f, \nu)$.

Remark 2.2.3. *If $\mathbf{s} \in f$, using $\sigma_\nu(t, \mathbf{s}) \leq \|\mathbf{s}\|_f$, it is easy to see that (2.2.1) can be replaced by $\|\mathbf{s}\|_f$ plus the same integral from 1 to ∞. We need to consider the whole range $0 < t < \infty$ since we do not assume $\mathbf{s} \in f$. Similar remark holds for $\mu = \infty$ in (2.2.2). Nevertheless, the properties of the restricted non-linear approximation spaces are the same as the N-term approximation spaces (see [27] or [8]).*

By splitting the integral in dyadic pieces and using the monotonicity of the f-error σ_ν we have an equivalent quasi-norm for the restricted approximation spaces:

$$\|\mathbf{s}\|_{\mathcal{A}_\mu^\xi(\nu)} \approx \left(\sum_{k=-\infty}^\infty [2^{k\xi} \sigma_\nu(2^k, \mathbf{s})]^\mu \right)^{1/\mu}. \tag{2.2.3}$$

2.3. The Jackson and Bernstein type inequalities.
It is well known the fundamental role played by the Jackson and Bernstein type inequalities in non-linear approximation theory. Considering our standard scheme (f, ν) we give the following definitions.

Definition 2.3.1. *Given $r > 0$, a quasi-Banach (sequence) lattice $g \subset S$ satisfies the Jackson's inequality of order r if there exists $C > 0$ such that*

$$\sigma_\nu(t, \mathbf{s}) \leq C t^{-r} \|\mathbf{s}\|_g \quad \text{for all } \mathbf{s} \in g.$$

Definition 2.3.2. *Given $r > 0$, a quasi-Banach (sequence) lattice $g \subset S$ satisfies the Bernstein's inequality of order r if there exists $C > 0$ such that*

$$\|\mathbf{t}\|_g \leq C t^r \|\mathbf{t}\|_f \quad \text{for all } \mathbf{t} \in \Sigma_{t,\nu} \cap f.$$

We do not assume in the above definitions that $g \hookrightarrow f$, but we need to assume $\mathbf{t} \in \Sigma_{t,\nu} \cap f$ for Definition 2.3.2 to make sense.

2.4. Weight functions for discrete Lorentz spaces.

Definition 2.4.1. *We will denote by $\mathbb{W}$ the set of all continuous functions η : $[0, \infty) \mapsto [0, \infty)$ such that*
  i) $\eta(0) = 0$ and $\lim_{t\to\infty} \eta(t) = \infty$
  ii) η is non-decreasing
  iii) η has the doubling property, that is, there exists $C > 0$ such that $\eta(2t) \leq C\eta(t)$ for all $t > 0$.

A typical element of the class $\mathbb{W}$ is $\eta(t) = t^{1/p}$, $0 < p < \infty$. The functions in the class $\mathbb{W}$ will be used to define general discrete Lorentz spaces. Occasionally, we will need to assume a stronger condition on the function $\eta \in \mathbb{W}$. For $\eta \in \mathbb{W}$ we define the dilation function

$$M_\eta(s) = \sup_{t>0} \frac{\eta(st)}{\eta(t)}, \ s > 0.$$

Since η is non-decreasing, $M_\eta(s) \leq 1$ for $0 < s \leq 1$.

Definition 2.4.2. *We say that $\eta \in \mathbb{W}_+$ if $\eta \in \mathbb{W}$ and there exists $s_0 \in (0,1)$ for which $M_\eta(s_0) < 1$.*

Observe that for $\eta \in \mathbb{W}_+$ and $r > 0$, $\eta^r \in \mathbb{W}_+$. Also, if $\eta \in \mathbb{W}$ and $r > 0$, $t^r \eta(t) \in \mathbb{W}_+$.

Lemma 2.4.3. *Let $\eta \in \mathbb{W}_+$ and take s_0 as in the Definition 2.4.2. Then, there exists $C > 0$ such that for all $t > 0$*

$$\sum_{j=0}^{\infty} \eta(s_0^j t) \leq C\eta(t). \tag{2.4.1}$$

Lemma 2.4.4. *Given $\eta \in \mathbb{W}_+$, there exists $g \in C^1, g \in \mathbb{W}_+$ such that $g \approx \eta$ and $g'(t)/g(t) \approx 1/t$, $t > 0$.*

2.5. General Discrete Lorentz Spaces.
We will define the discrete Lorentz spaces we will work with. First, we recall some classical definitions (see e.g. [8] or [3]). For a sequence $\mathbf{s} = \{s_I\}_{I\in\mathcal{D}} \in S$ indexed by the countable set $\mathcal{D}$, the non-increasing rearrangement of $\mathbf{s}$ with respect to a measure ν on $\mathcal{D}$ is

$$\mathbf{s}_\nu^*(t) = \inf\{\lambda > 0 : \nu(\{I \in \mathcal{D} : |s_I| > \lambda\}) \leq t\}.$$

For $\eta \in \mathbb{W}$, ν a measure on $\mathcal{D}$, and $\mu \in (0, \infty]$, the **discrete Lorentz space** $\ell_\eta^\mu(\nu)$ is the set of all $\mathbf{s} = \{s_I\}_{I\in\mathcal{D}} \in S$ such that

$$\|\mathbf{s}\|_{\ell_\eta^\mu(\nu)} := \left(\int_0^\infty [\eta(t)\mathbf{s}_\nu^*(t)]^\mu \frac{dt}{t} \right)^{1/\mu} < \infty, \quad 0 < \mu < \infty \tag{2.5.1}$$

and

$$\|\mathbf{s}\|_{\ell_\eta^\infty(\nu)} := \sup_{t>0} \eta(t)\mathbf{s}_\nu^*(t) < \infty.$$

If $\eta(t) = t^{1/p}, 1 \leq p < \infty$, then $\ell^\mu_\eta(\nu) = \ell^{p,\mu}(\nu)$ are the classical (discrete) Lorentz spaces. For $p = \mu$, $\ell^{p,p}(\nu) = \ell^p(\nu), 0 < p < \infty$, are the spaces of sequences $\mathbf{s} \in S$ such that

$$\|\mathbf{s}\|_{\ell^p(\nu)} = \left(\sum_{I \in \mathcal{D}} |s_I|^p \nu(I)\right)^{1/p}.$$

Notation. For $\xi > 0$ and $\eta \in \mathbb{W}$, $\tilde{\eta}(t) = t^\xi \eta(t) \in \mathbb{W}_+$ and $\ell^\mu_{\tilde{\eta}}(\nu)$ will be denoted by $\ell^\mu_{\xi,\eta}(\nu)$.

Proposition 2.5.1. *Let $\eta \in \mathbb{W}$ and ν a measure on $\mathcal{D}$. For a sequence $\mathbf{s} = \{s_I\}_{I \in \mathcal{D}} \in S$ we have*

$$\|\mathbf{s}\|_{\ell^\infty_\eta(\nu)} \approx \sup_{\lambda > 0} \lambda\eta(\nu(\{I \in \mathcal{D} : |s_I| > \lambda\})).$$

Moreover, if $0 < \mu < \infty$ and $\eta \in \mathbb{W}_+$

$$\|\mathbf{s}\|_{\ell^\mu_\eta(\nu)} \approx \left(\int_0^\infty [\lambda\eta(\nu(\{I \in \mathcal{D} : |s_I| > \lambda\}))]^\mu \frac{d\lambda}{\lambda}\right)^{1/\mu}.$$

A sequence $\mathbf{u} = \{u_I\}_{I \in \mathcal{D}} \in S$ such that $u_I > 0$ for all $I \in \mathcal{D}$ will be called a **weight sequence**.

Definition 2.5.2. *Let $\mathbf{u} = \{u_I\}_{I \in \mathcal{D}}$ be a weight sequence and ν a positive measure as defined in Subsection 2.2. For $0 < \mu \leq \infty$ and $\eta \in \mathbb{W}$ define the space $\ell^\mu_\eta(\mathbf{u}, \nu)$ as the set of all sequences $\mathbf{s} = \sum_{I \in \mathcal{D}} s_I \mathbf{e}_I \in S$ such that*

$$\|\mathbf{s}\|_{\ell^\mu_\eta(\mathbf{u},\nu)} := \|\{u_I s_I\}_{I \in \mathcal{D}}\|_{\ell^\mu_\eta(\nu)} < \infty.$$

These spaces will be used in Subsections 2.6 and 2.7 to characterize Jackson and Bernstein type inequalities in the setting of restricted non-linear approximation. For applications (see Subsections 2.8-2.11) we shall take $u_I = \|\mathbf{e}_I\|_g$, $I \in \mathcal{D}$, where g is a quasi-Banach (sequence) lattice.

Lemma 2.5.3. *Let $\mathbf{u}$ and ν as in Definition 2.5.2 and write $1_{\Gamma,\mathbf{u}} = \sum_{I \in \Gamma} u_I^{-1} \mathbf{e}_I, \Gamma \subset \mathcal{D}$ and $\nu(\Gamma) < \infty$.*
  (a) If $\eta \in \mathbb{W}$, $\|1_{\Gamma,\mathbf{u}}\|_{\ell^\infty_\eta(\mathbf{u},\nu)} = \eta(\nu(\Gamma))$.
  (b) If $0 < \mu < \infty$ and $\eta \in \mathbb{W}$, $\|1_{\Gamma,\mathbf{u}}\|_{\ell^\mu_\eta(\mathbf{u},\nu)} \geq \eta(\nu(\Gamma))$, and if $\eta \in \mathbb{W}_+$, $\|1_{\Gamma,\mathbf{u}}\|_{\ell^\mu_\eta(\mathbf{u},\nu)} \approx \eta(\nu(\Gamma))$.

2.6. **Jackson type inequalities.** We give equivalent conditions for some Jackson type inequalities to hold in the setting of restricted non-linear approximation. Our result generalizes those obtained in [6] and [20] for restricted non-linear approximation, as well as those obtained in [19] and [15] for the case $\nu(I) = 1$ for all $I \in \mathcal{D}$ (the counting measure).

Theorem 2.6.1. *Let (f, ν) be a standard scheme (see Definition 2.2.1)and let $\mathbf{u} = \{u_I\}_{I \in \mathcal{D}}$ be a weight sequence. Fix $\xi > 0$ and $\mu \in (0, \infty]$. Then, for any function $\eta \in \mathbb{W}_+$ the following are equivalent:*
  1) There exists $C > 0$ such that for all $\Gamma \subset \mathcal{D}$ with $\nu(\Gamma) < \infty$

$$\left\|\sum_{I \in \Gamma} \frac{\mathbf{e}_I}{u_I}\right\|_f \leq C\eta(\nu(\Gamma)).$$

2) $\ell^\mu_{\xi,\eta}(\mathbf{u}, \nu) \hookrightarrow \mathcal{A}^\xi_\mu(f, \nu)$.

3) The space $\ell^\mu_{\xi,\eta}(\mathbf{u}, \nu)$ satisfies Jackson's inequality of order ξ, that is, there exists $C > 0$ such that

$$\sigma_\nu(t, \mathbf{s})_f \leq Ct^{-\xi} \|\mathbf{s}\|_{\ell^\mu_{\xi,\eta}(\mathbf{u}, \nu)}, \quad \text{for all} \quad \mathbf{s} \in \ell^\mu_{\xi,\eta}(\mathbf{u}, \nu).$$

Taking $\eta(t) = t^{1/p}, 0 < p < \infty$, and $u_I = \|\mathbf{e}_I\|_f$ in Theorem 2.6.1, condition 1) is called in [20] the (upper) p-Temlyakov property for f. In this case, $\ell^\mu_{\xi,\eta}(\mathbf{u}, \nu) = \ell^{q,\mu}(\mathbf{u}, \nu)$ with $\frac{1}{q} = \xi + \frac{1}{p}$.

Taking ν as the counting measure on $\mathcal{D}$ we recover Theorem 3.6 in [15] from Theorem 2.6.1.

2.7. Bernstein type inequalities.
We give equivalent conditions for some Bernstein type inequalities to hold in the setting of restricted non-linear approximation. This result generalizes those obtained in [6] and [20] for restricted non-linear approximation, as well as those obtained in [19] and [15] for the case $\nu(I) = 1$ for all $I \in \mathcal{D}$ (the counting measure).

We first begin with a representation theorem for the spaces $\mathcal{A}^\xi_\mu(f, \nu)$. The proof follows that in [27] replacing the counting measure by a general positive measure ν.

Proposition 2.7.1. *Let (f, ν) be a standard scheme (see Definition 2.2.1). Fix $\xi > 0$ and $\mu \in (0, \infty]$. The following statements are equivalent*

i) $\mathbf{s} \in \mathcal{A}^\xi_\mu(f, \nu)$.

ii) There exists $\mathbf{s}_k \in \Sigma_{2^k, \nu} \cap f, k \in \mathbb{Z}$, such that $\mathbf{s} = \sum_{k=-\infty}^\infty \mathbf{s}_k$ and $\{2^{k\xi} \|\mathbf{s}_k\|_f\}_{k \in \mathbb{Z}} \in \ell^\mu(\mathbb{Z})$.

Moreover,

$$\|\mathbf{s}\|_{\mathcal{A}^\xi_\mu(f, \nu)} \approx \inf \left\{ \left[\sum_{k=-\infty}^\infty (2^{k\xi} \|\mathbf{s}_k\|_f)^\mu \right]^{1/\mu} \right\}$$

where the infimum is taken over all representations of $\mathbf{s}$ as in ii).

Theorem 2.7.2. *Let (f, ν) be a standard scheme (see Definition 2.2.1) and let $\mathbf{u} = \{u_I\}_{I \in \mathcal{D}}$ be a weight sequence. Fix $\xi > 0$ and $\mu \in (0, \infty]$. Then, for any function $\eta \in \mathbb{W}$ the following are equivalent:*

1) There exists $C > 0$ such that for all $\Gamma \subset \mathcal{D}$ with $\nu(\Gamma) < \infty$,

$$\frac{1}{C}\eta(\nu(\Gamma)) \leq \left\| \sum_{I \in \Gamma} \frac{\mathbf{e}_I}{u_I} \right\|_f.$$

2) The space $\ell^\mu_{\xi,\eta}(\mathbf{u}, \nu)$ satisfies Bernstein's inequality of order ξ, that is, there exists $C > 0$ such that

$$\|\mathbf{s}\|_{\ell^\mu_{\xi,\eta}(\mathbf{u}, \nu)} \leq Ct^\xi \|\mathbf{s}\|_f \quad \text{for all} \quad \mathbf{s} \in \Sigma_{t,\nu} \cap f.$$

3) $\mathcal{A}^\xi_\mu(f, \nu) \hookrightarrow \ell^\mu_{\xi,\eta}(\mathbf{u}, \nu)$.

Taking $\eta(t) = t^{1/p}, 0 < p < \infty$, and $u_I = \|\mathbf{e}_I\|_f$ in Theorem 2.7.2, condition 1) is called in [20] the (lower) p-Temlyakov property for f. In this case, $\ell^\mu_{\xi,\eta}(\mathbf{u}, \nu) = \ell^{q,\mu}(\mathbf{u}, \nu)$ with $\frac{1}{q} = \xi + \frac{1}{p}$.

Theorem 2.7.2 generalizes Theorem 5 in [20] for a standard scheme. The proof of this theorem does not use the theory of real interpolation of quasi-Banach spaces; we will, however, make use of it to shorten our proof.

Taking ν as the counting measure on $\mathcal{D}$ we recover Theorem 4.2 in [15] from Theorem 2.7.2.

2.8. Restricted non-linear approximation and real interpolation.
It is well known that N-term approximation and real interpolation are interconnected. If the Jackson and Bernstein's inequalities hold for $\nu =$ counting measure, N-term approximation spaces are characterized in terms of interpolation spaces (see e.g. Theorem 3.1 in [9] or Section 9, Chapter 7 in [8]).

As pointed out in [6] the above mentioned theory can be developed in a more general setting. In particular, it can be done in the frame of the abstract scheme we have introduced in subsection 2.2. Below we state the results we need in this paper. The proofs are straight-forward modifications of those given in the references cited in the first paragraph of this section.

Theorem 2.8.1. *Let (f, ν) be a standard scheme. Suppose that the quasi-Banach lattice $g \subset S$ satisfies the Jackson and Bernstein's inequalities for some $r > 0$. Then, for $0 < \xi < r$ and $0 < \mu \leq \infty$ we have*

$$\mathcal{A}_\mu^\xi(f, \nu) = (f, g)_{\xi/r, \mu}.$$

It is not difficult to show that the spaces $\mathcal{A}_q^r(f, \nu), 0 < r < \infty, 0 < q \leq \infty$, satisfy the Jackson and Bernstein's inequalities of order r, so that by Theorem 2.8.1,

$$\mathcal{A}_\mu^\xi(f, \nu) = \left(f, \mathcal{A}_q^r(f, \nu)\right)_{\xi/r, \mu}$$

for $0 < \xi < r$ and $0 < \mu \leq \infty$. From here, and using the reiteration theorem for real interpolation we obtain the following result that will be used in the proof of Theorem 2.7.2.

Corollary 2.8.2. *Let $0 < \alpha_0, \alpha_1 < \infty, 0 < q, q_0, q_1 \leq \infty$ and $0 < \theta < 1$. Then,*

$$\left(\mathcal{A}_{q_0}^{\alpha_0}(f, \nu), \mathcal{A}_{q_1}^{\alpha_1}(f, \nu)\right)_{\theta, q} = \mathcal{A}_q^\alpha(f, \nu), \quad \alpha = (1 - \theta)\alpha_0 + \theta\alpha_1$$

for a standard scheme (f, ν).

2.9. Sequence spaces associated with smoothness spaces.
A large number of spaces used in Analysis are particular cases of the Triebel-Lizorkin and Besov spaces.

Given $s \in \mathbb{R}, 0 < p < \infty$, and $0 < r \leq \infty$, the Triebel-Lizorkin spaces on $\mathbb{R}^d$ are denoted by $F_{p,r}^s := F_{p,r}^s(\mathbb{R}^d)$ where s is a smoothness parameter, p measures integrability and r measures a refinement of smoothness. The reader can find the definition of these spaces in [11, 12]. Lebesgue spaces $L^p(\mathbb{R}^d) = F_{p,2}^0, 1 < p < \infty$, Hardy spaces $H^p(\mathbb{R}^d) = F_{p,2}^0, 0 < p \leq 1$, and Sobolev spaces $W_p^s(\mathbb{R}^d) = F_{p,2}^s, s > 0, 1 < p < \infty$, are included in this collection.

Given $s \in \mathbb{R}, 0 < p, r \leq \infty$, the Besov spaces on $\mathbb{R}^d$ are denoted by $B_{p,r}^s := B_{p,r}^s(\mathbb{R}^d)$ with an interpretation of the parameters as in the case of the Triebel-Lizorkin spaces. These spaces include the Lipschitz classes (see [12]).

There are characterizations of $F_{p,r}^s$ and $B_{p,r}^s$ in terms of sequence spaces. Such characterizations were given first in [11] using the φ-transform. Wavelet bases with appropriate regularity and moment conditions also provide such characterizations.

A brief description of wavelet bases in $\mathbb{R}^d$ follows. Let $\mathcal{D}$ be the set of dyadic cubes in $\mathbb{R}^d$ given by

$$Q_{j,k} = 2^{-j}([0,1)^d + k),\ j \in \mathbb{Z},\ k \in \mathbb{Z}^d.$$

A finite collection of functions $\Psi = \{\psi^{(1)}, \ldots, \psi^{(L)}\} \subset L^2(\mathbb{R}^d)$ with $L = 2^d - 1$ is an (orthonormal) wavelet family if the set

$$\mathcal{W} := \{\psi_{Q_{j,k}}^{(\ell)}(x) := 2^{\frac{jd}{2}}\psi^{(\ell)}(2^j x - k):\ Q_{j,k} \in \mathcal{D},\ \ell = 1, 2, \ldots, L\}$$

is an orthonormal basis for $L^2(\mathbb{R}^d)$. This is the definition that appears in [23]. The reader can consult properties of wavelets in [26], [7], [16] and [24].

Definition 2.9.1. *Given $s \in \mathbb{R}, 0 < p < \infty$ and $0 < r \leq \infty$, we let $f_{p,r}^s$ be the space of sequences $\mathbf{s} = \{s_Q\}_{Q \in \mathcal{D}}$ such that*

$$\|\mathbf{s}\|_{f_{p,r}^s} := \left\| \left[\sum_{Q \in \mathcal{D}} (|Q|^{-s/d+1/r-1/2}\, |s_Q| \chi_Q^{(r)}(\cdot))^r \right]^{1/r} \right\|_{L^p(\mathbb{R}^d)} < \infty$$

where $\chi_Q^{(r)}(\cdot) = \chi_Q(\cdot)\, |Q|^{-1/r}$ and $\chi_Q(\cdot)$ denotes the characteristic function of Q.

Definition 2.9.2. *Given $s \in \mathbb{R}, 0 < p, r \leq \infty$, we let $b_{p,r}^s$ be the space of sequences $\mathbf{s} = \{s_Q\}_{Q \in \mathcal{D}}$ such that*

$$\|\mathbf{s}\|_{b_{p,r}^s} := \left[\sum_{j \in \mathbb{Z}} \left(\sum_{|Q|=2^{-jd}} (|Q|^{-s/d+1/p-1/2}\, |s_Q|)^p \right)^{r/p} \right]^{1/r} < \infty$$

with the obvious modifications when $p, r = \infty$.

With appropriate conditions in the elements of a wavelet family $\Psi = \{\psi^{(1)}, \ldots, \psi^{(L)}\}$, $(L = 2^d - 1)$ it can be shown that $\mathcal{W}$ is an unconditional basis of $F_{p,r}^s$ or $B_{p,r}^s$ and if $f = \sum_{\ell=1}^{L} \sum_{Q \in \mathcal{D}} s_Q^\ell \psi_Q^{(\ell)}$, then

$$\|f\|_{F_{p,r}^s} \approx \sum_{\ell=1}^{L} \left\| \{s_Q^\ell\}_{Q \in \mathcal{D}} \right\|_{f_{p,r}^s} \quad \text{and} \quad \|f\|_{B_{p,r}^s} \approx \sum_{\ell=1}^{L} \left\| \{s_Q^\ell\}_{Q \in \mathcal{D}} \right\|_{b_{p,r}^s}. \tag{2.9.1}$$

Conditions on Ψ for these equivalences to hold can be found in [26], [16], [1], [22], [23]. When a wavelet family Ψ provides an unconditional basis for $F_{p,r}^s$ or $B_{p,r}^s$, with equivalences as in (2.9.1), we shall say that Ψ is **admissible** for $F_{p,r}^s$ or $B_{p,r}^s$, respectively.

The equivalences (2.9.1) allow us to work at the sequence level. We shall drop the sum over ℓ since it only changes the constants in the computations below. The results proved for sequence spaces $f_{p,r}^s$ or $b_{p,r}^s$ can be transferred to $F_{p,q}^s$ or $B_{p,r}^s$ by the abstract transference framework developed in [13].

We notice that the Triebel-Lizorkin and Besov spaces characterized as in (2.9.1) are called homogeneous, been often denoted by $\dot{F}_{p,r}^s$ and $\dot{B}_{p,r}^s$. The non-homogeneous case requires small modifications. Also minor modifications will allow for the anisotropic spaces as considered in [13], or the spaces defined by wavelets on bounded domains. We restrict ourselves to the cases characterized by (2.9.1).

2.10. **Restricted approximation for Triebel-Lizorkin sequence spaces.** As consequence of the theorems developed in Sections 2.6 and 2.7 we will obtain results for restricted approximation in Triebel-Lizorkin sequence spaces. When coupled with the abstract transference framework developed in [13], our results generalizes those in [6] and, with minor modifications, those obtained in [18].

Lemma 2.10.1. *Let $\Gamma \subset \mathcal{D}$ (not necessarily finite), $x \in \cup_{Q \in \Gamma} Q$, and $\gamma \neq 0$. Define*

$$S_\Gamma^\gamma(x) = \sum_{Q \in \Gamma} |Q|^\gamma \chi_Q(x).$$

i) If $\gamma > 0$ and there exists Q^x, the biggest cube in Γ that contains x, then $S_\Gamma^\gamma(x) \approx |Q^x|^\gamma \chi_{Q^x}(x) = |Q^x|^\gamma$
ii) If $\gamma < 0$ and there exists Q_x, the smallest cube in Γ that contains x, then $S_\Gamma^\gamma(x) \approx |Q_x|^\gamma \chi_{Q_x}(x) = |Q_x|^\gamma$.

The smallest cube Q_x from Γ that contains $x \in \cup_{Q \in \Gamma} Q$ has been used by other authors in the context of non-linear approximation with wavelet basis (see [17], [6], [13], [14]). As far as we know, the biggest cube Q^x from Γ that contains $x \in \cup_{Q \in \Gamma} Q$ has not been used before.

Theorem 2.10.2. *Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$. For $\alpha \in \mathbb{R}$ and $\Gamma \subset \mathcal{D}$ define $\nu_\alpha(\Gamma) = \sum_{Q \in \Gamma} |Q|^\alpha$. Suppose $\nu_\alpha(\Gamma) < \infty$. Then,*

$$\left\| \sum_{Q \in \Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{f_{p_2,q_2}^{s_2}}} \right\|_{f_{p_1,q_1}^{s_1}} \approx [\nu_\alpha(\Gamma)]^{1/p_1} \tag{2.10.1}$$

if and only if $\alpha \neq 1$ and $\alpha = p_1(\frac{s_2-s_1}{d} - \frac{1}{p_2}) + 1$ or $\alpha = 1, \frac{s_2-s_1}{d} = \frac{1}{p_2}$ and $p_1 = q_1$.

Theorems 2.6.1 and 2.7.2 with $\eta(t) = t^{1/p_1}$ and $u_Q = \|\mathbf{e}_Q\|_{f_{p_2,q_2}^{s_2}}$ together with Theorem 2.10.2 show that non-linear approximation with error measured in $f_{p_1,q_1}^{s_1}$ when the basis is normalized in $f_{p_2,q_2}^{s_2}$ is related to the use of the measure $\nu_\alpha(Q) = |Q|^\alpha$, $Q \in \mathcal{D}$, $\alpha = p_1(\frac{s_2-s_1}{d} - \frac{1}{p_2})$, to control the number of terms in the approximation. Notice that no role is played by the second smoothness parameters q_1, q_2.

Theorems 2.6.1 and 2.7.2 together with Theorem 2.10.2 also allow us to identify the restricted approximation spaces in the Triebel-Lizorkin setting as discrete Lorentz spaces.

Corollary 2.10.3. *Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2 < \infty$, $0 < q_1, q_2 \leq \infty$ and define $\alpha = p_1(\frac{s_2-s_1}{d} - \frac{1}{p_2}) + 1$. For $\Gamma \subset \mathcal{D}$ define $\nu_\alpha(\Gamma) = \sum_{Q \in \Gamma} |Q|^\alpha$. Let $\xi > 0$ and $\mu \in (0, \infty]$. If $\alpha \neq 1$,*

$$\mathcal{A}_\mu^\xi(f_{p_1,q_1}^{s_1}, \nu_\alpha) = \ell^{\tau,\mu}(\mathbf{u}, \nu_\alpha),$$

where $\frac{1}{\tau} = \xi + \frac{1}{p_1}$ and $\mathbf{u} = \{\|\mathbf{e}_Q\|_{f_{p_2,q_2}^{s_2}}\}_{Q \in \mathcal{D}}$. If $\alpha = 1$ the result holds with $p_1 = q_1$.

For particular values of the parameters, the discrete Lorentz spaces that appear in Corollary 2.10.3 can be identified as Besov spaces.

Lemma 2.10.4. *Let $s_1, s_2 \in \mathbb{R}$, $0 < p_1, p_2 < \infty$, $0 < q_2 \leq \infty$ and define $\alpha = p_1(\frac{s_2-s_1}{d} - \frac{1}{p_2}) + 1$. For $\Gamma \subset \mathcal{D}$ define $\nu_\alpha(\Gamma) = \sum_{Q \in \Gamma} |Q|^\alpha$. Given $\tau \in (0, \infty)$ we have*

$$\ell^{\tau,\tau}(\mathbf{u}, \nu_\alpha) = b_{\tau,\tau}^\gamma,$$

(with equal quasi-norms) where $\mathbf{u} = \{\|\mathbf{e}_Q\|_{f_{p_2,q_2}^{s_2}}\}_{Q \in \mathcal{D}}$ and $\gamma = s_1 + d(\frac{1}{\tau} - \frac{1}{p_1})(1 - \alpha)$.

Remark 2.10.5. *If we consider the point of view of* [20], *then we can only prove* $\ell^{\tau,\tau}(\mathbf{u}, \nu_\alpha) = b^\gamma_{\tau,\tau} \cap f^{s_1}_{p1,q1}$ *and the equivalence of quasi-norms holds if we take* $\|\cdot\|_{b^\gamma_{\tau,\tau}}$ *in the right-hand side.*

When $\tau < p_1$ *and* $\frac{\gamma}{d} - \frac{1}{\tau} = \frac{s_1}{d} - \frac{1}{p_1}$ *it is known that* $b^\gamma_{\tau,\tau} \hookrightarrow f^{s_1}_{p_1,q_1}$ *(see* [10] *or* [4]*). This situation occurs when* $\alpha = 0$ *(the counting measure) but it is not true in our more general situation.*

The following result identifies certain non-linear approximation spaces in the restricted setting, when the error is measured in Triebel-Lizorkin spaces, as Besov spaces. It is obtained as an easy corollary to Lemma 2.10.4 and Corollary 2.10.3.

Corollary 2.10.6. *Let* $s_1, s_2 \in \mathbb{R}, 0 < p_1, p_2 < \infty, 0 < q_1 \le \infty$ *and define* $\alpha = p_1(\frac{s_2 - s_1}{d} - \frac{1}{p_2}) + 1$. *For* $\Gamma \subset \mathcal{D}$ *define* $\nu_\alpha(\Gamma) = \sum_{Q \in \Gamma} |Q|^\alpha$. *Given* $\xi > 0$ *define* τ *by* $\frac{1}{\tau} = \xi + \frac{1}{p_1}$. *If* $\alpha \neq 1$,

$$\mathcal{A}^\xi_\tau(f^{s_1}_{p_1,q_1}, \nu_\alpha) = b^\gamma_{\tau,\tau} \quad \text{(equivalent quasi-norms),}$$

where $\gamma = s_1 + d\xi(1 - \alpha)$. *If* $\alpha = 1$ *the result holds with* $\gamma = s_1$ *and* $p_1 = q_1$.

Remark 2.10.7. *If we were to apply Theorem 1 in* [20] *we will obtain* $\mathcal{A}^\xi_\tau(f^{s_1}_{p_1,q_1}, \mathbf{u}, \nu_\alpha) = b^\gamma_{\tau,\tau} \cap f^{s_1}_{p_1,q_1}$ *with equivalence of quasi-norms, as in Remark 2.10.5.*

The results obtained in [6] for restricted non-linear approximation with wavelets in the Hardy space $H^p, 0 < p < \infty$, when the wavelets coefficients are restricted to $H^r, 0 < r < \infty$, are simple consequences of the above results and the abstract transference framework developed in [13]. To see this, notice that the sequence spaces associated to H^p and H^r (r, p as above) are $f^0_{p,2}$ and $f^0_{r,2}$, respectively.

Thus, for a wavelet basis $\mathcal{W} = \{\psi^{(\ell)}_Q : Q \in \mathcal{D}, \ell = 1, \ldots, L\}, (L = 2^d - 1)$ admissible for H^p and $B^\gamma_{\tau,\tau}$,

$$\mathcal{A}^\xi_\tau(H^p, \mathcal{W}, \nu_\alpha) = B^\gamma_{\tau,\tau}, \tag{2.10.2}$$

where $\gamma = \frac{dp}{r}\xi$, τ defined by $\frac{1}{\tau} = \xi + \frac{1}{p}$, and $\alpha = 1 - p/r (\neq 1)$. This is Corollary 6.3 in [6]. Notice that $\mathcal{A}^\xi_\tau(H^p, \mathcal{W}, \nu_\alpha)$ corresponds to an approximation space where the wavelet coefficients are normalized in H^r. In the above notation we have emphasize that the approximation spaces are defined using wavelet basis.

In this situation, The Jackson and Bernstein's inequalities (Theorems 5.1 and 5.2 in [6]) follow from (2.10.2) and the fact that the approximation spaces always satisfy the Jackson and Bernstein's inequalities.

The other situation considered in [6] is $B_p := B^0_{p,p}, 0 < p < \infty$, when the wavelet coefficients are restricted in $H_r, 0 < r < \infty$. In this case, the sequence spaces associated to $B^0_{p,p} = F^0_{p,p}$ and H^r are $f^0_{p,p}$ and $f^0_{r,2}$, respectively. Corollary 2.10.6 then produces

$$\mathcal{A}^\xi_\tau(B_p, \mathcal{W}, \nu_\alpha) = B^\gamma_{\tau,\tau}, \tag{2.10.3}$$

where $\gamma = \frac{dp}{r}\xi$, τ defined by $\frac{1}{\tau} = \xi + \frac{1}{p}$, with ν_α and $\mathcal{W}$ as before. This is more general than Corollary 6.1 in [6] and a comparison with (2.10.2) proves immediately a more general version of Theorem 6.3 in [6]. Of course, the Jackson and Bernstein's inequalities of Theorems 5.4 and 5.5 in [6] also follow from our results.

To show an example not treated in [6] consider the wavelet orthonormal basis $\mathcal{W} = \{\psi^{(\ell)}_Q : Q \in \mathcal{D}, \ell = 1, \ldots, L\}$ ($L = 2^d - 1$) admissible for the Sobolev space $W^s_2, s > 0$.

We want to measure the error in W_2^s but we restrict the wavelet coefficients to L^2. Since the sequence spaces associated to W_2^s and L^2 are $f_{2,2}^s$ and $f_{2,2}^0$, Corollary 2.10.6 and the abstract framework of [13] gives

$$\mathcal{A}_\tau^\xi(W_2^s, \mathcal{W}, \nu_\alpha) = b_{\tau,\tau}^\gamma$$

where $\gamma = s + d\xi(1-\alpha)$, $\alpha = -2s/d$ and τ defined by $\frac{1}{\tau} = \xi + \frac{1}{2}$.

We remark that defining appropriate sequence spaces, a little more work will show the results proved in [18] for the anisotropic case.

2.11. **Application to Real Interpolation.** Once the restricted approximation spaces for Triebel-Lizorkin sequence spaces have been identified (see Corollary 2.10.6) we can use Theorem 2.8.1 to obtain results about real interpolation. This method has been used before (see [9] or [13]). But in the classical case, the parameters of the spaces interpolated are restricted. With the theory of restricted approximation we will prove interpolation results for a much larger set of parameters.

Theorem 2.11.1. *Let $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$. For $0 < \tau < p$, $0 < \theta < 1$ and $\gamma \neq s$ $(\gamma \in \mathbb{R})$ we have*

$$\left(f_{p,q}^s, b_{\tau,\tau}^\gamma\right)_{\theta,\tau_\theta} = b_{\tau_\theta,\tau_\theta}^{(1-\theta)s+\theta\gamma} \quad with \quad \frac{1}{\tau_\theta} = \frac{1-\theta}{p} + \frac{\theta}{\tau}.$$

Remark 2.11.2. *Although the Theorem is presented as a result about interpolation of Triebel-Lizorkin and Besov (sequence) spaces, it is a result about interpolation of Triebel-Lizorkin sequence spaces, since $b_{\tau,\tau}^\gamma = f_{\tau,\tau}^\gamma$. Thus, the result can be stated as*

$$\left(f_{p,q}^s, f_{\tau,\tau}^\gamma\right)_{\theta,\tau_\theta} = f_{\tau_\theta,\tau_\theta}^{(1-\theta)s+\theta\gamma} \quad with \quad \frac{1}{\tau_\theta} = \frac{1-\theta}{p} + \frac{\theta}{\tau}. \qquad (2.11.1)$$

Remark 2.11.3. *Notice that we do not need the restriction $\frac{\gamma}{d} - \frac{1}{\tau} = \frac{s}{d} - \frac{1}{p}$ characteristic of this type of results when classical non-linear approximation is used (see e.g. [9] or [13]).*

Remark 2.11.4. *By the transference framework designed in [13], the result of Theorem 2.11.1 can be translated to a result for (homogeneous) Triebel-Lizorkin spaces. The non-homogeneous case and the case of bounded domains can also be obtained with minor modifications in the proof.*

Remark 2.11.5. *The merit of Theorem 2.11.1 is that proves results for a large set of parameters using the theory of approximation. Nevertheless, many (but not all, as far as we know) have already been proved. One can read from Theorem 3.5 in [4] the following result:*

$$\left(F_{p_0,q_0}^{s_0}, F_{p_1,q_1}^{s_1}\right)_{\theta,p} = F_{p,p}^s = B_{p,p}^s \qquad (2.11.2)$$

when $p_i < q_i$, $i = 0,1$, $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$, $s = (1-\theta)s_0 + \theta s_1$ and $s_0 \neq s_1$. Comparing with (2.11.1) we see that (2.11.2) has a larger set of parameters in the second space, while (2.11.1) does not have the restriction $p < q$ that is required in (2.11.2). Both of these shortcomings are due to the methods of the proofs. On the other hand, Theorem 2.42/1 (page 184) of [28] shows (2.11.2) without $p_i < q_i$ but assuming $1 < p_i, q_i < \infty$.

3. Proofs

3.4. Weight functions for discrete Lorentz spaces (proofs).

Proof of Lemma 2.4.3. Let $\delta := M_\eta(s_0) < 1$. By definition of M_η we have

$$1 > \delta \geq \frac{\eta(s_0^{j+1}t)}{\eta(s_0^j t)} \quad \text{for all} \quad j = 0, 1, 2, \ldots$$

Therefore,

$$\sum_{j=0}^\infty \eta(s_0^j t) \leq \sum_{j=0}^\infty \delta^j \eta(t) = \eta(t)\frac{1}{1-\delta}.$$

∎

Proof of Lemma 2.4.4. Define $g(t) = \int_0^t \frac{\eta(s)}{s}ds$. With s_0 as in Definition 2.4.2

$$g(t) = \sum_{j=0}^\infty \int_{s_0^{j+1}t}^{s_0^j t} \frac{\eta(s)}{s}ds \leq \sum_{j=0}^\infty \eta(s_0^j t) \log(s_0^{-1}) \leq C\eta(t)\log(s_0^{-1})$$

by Lemma 2.4.3 ($C = \frac{1}{1-\delta}$, see the proof of Lemma 2.4.3). On the other hand

$$g(t) \geq \int_{t/2}^t \frac{\eta(s)}{s}ds \geq \eta(t/2)\log 2 \geq D\eta(t)\log 2$$

by the doubling property of η. This shows

$$C_1\eta(t) \leq g(t) \leq C_2\eta(t), \quad t \in (0,\infty) \tag{3.4.1}$$

with $0 < C_1 \leq C_2 < \infty$. As an alternative proof one can see that η satisfies the hypotheses of Lemma 1.4 in [21] (p. 54) to conclude $g \approx \eta$. The function g is clearly non-decreasing and (3.4.1) shows that $g \in \mathbb{W}$. It is clear that $g \in C^1$ with $g'(t) = \eta(t)/t$. Thus

$$\frac{g'(t)}{g(t)} = \frac{\eta(t)/t}{\eta(t)} = \frac{1}{t}.$$

It remains to prove that $g \in \mathbb{W}_+$. To prove this, observe that a function $\eta \in \mathbb{W}$ is an element of $\mathbb{W}_+$ if and only if

$$i_\eta := \lim_{t\to 0^+} \frac{\log M_\eta(t)}{\log t} > 0$$

(i_η is called the lower dilation (Boyd) index of η - see [3]). Using (3.4.1)

$$i_g = \lim_{t\to 0^+} \frac{\log M_g(t)}{\log t} \geq \lim_{t\to 0^+} \frac{\log(\frac{C_1}{C_2}M_\eta(t))}{\log t} = \lim_{t\to 0^+} \frac{\log(M_\eta(t))}{\log t} = i_\eta$$

and, similarly

$$i_g \leq \lim_{t\to 0^+} \frac{\log(\frac{C_2}{C_1}M_\eta(t))}{\log t} = i_\eta.$$

Thus, $i_g = i_\eta > 0$ which proves $g \in \mathbb{W}_+$. ∎

3.5. General discrete Lorentz spaces (proofs).

Proof of Proposition 2.5.1. The case $\mu = \infty$ follows from part iii) of Proposition 2.2.5 in [5]. For $0 < \mu < \infty$, let $w(t) = [\eta(t)]^\mu/t$, $0 < t < \infty$. Writing $\lambda_\nu(t, \mathbf{s}) = \nu(\{I \in \mathcal{D} : |s_I| > t\})$ for the distribution function of $\mathbf{s}$ with respect to the measure ν and $\mathrm{W}(s) = \int_0^s w(t)dt$, $0 < s < \infty$, part ii) of Proposition 2.2.5 in [5] gives

$$\|\mathbf{s}\|_{\ell_\eta^\mu(\nu)} = \left(\int_0^\infty \mu t^\mu \mathrm{W}(\lambda_\nu(t, \mathbf{s})) \frac{dt}{t} \right)^{1/\mu}.$$

Since $\eta \in \mathbb{W}_+$, η^μ satisfies the hypothesis of Lemma 1.4 in [21] (p. 54) so that we conclude

$$\mathrm{W}(s) = \int_0^s \frac{\eta(t)^\mu}{t} dt \approx [\eta(s)]^\mu$$

(see also the proof of Lemma 2.4.4 and the comment that follows Definition 2.4.2). This proves the result. ∎

Proof of Lemma 2.5.3. (a) Writing $1_{\Gamma,\mathbf{u}} = \sum_{I \in \mathcal{D}} s_I \mathbf{e}_I$ we have $s_I = u_I^{-1}$ for all $I \in \Gamma$ and $s_I = 0$ if $I \notin \Gamma$. Thus, $u_I s_I = 1$ for all $I \in \Gamma$ and $u_I s_I = 0$ if $I \notin \Gamma$. This implies

$$\{u_I s_I\}_\nu^*(t) = \left\{ \begin{array}{ll} 1, & 0 < t < \nu(\Gamma) \\ 0, & t \geq \nu(\Gamma) \end{array} \right\}. \tag{3.5.1}$$

By Definition 2.5.2

$$\|1_{\Gamma,\mathbf{u}}\|_{\ell_\eta^\infty(\mathbf{u},\nu)} = \sup_{0 < t < \nu(\Gamma)} \eta(t) = \eta(\nu(\Gamma)).$$

(b) Using 3.5.1 we have

$$\begin{aligned}
\|1_{\Gamma,\mathbf{u}}\|_{\ell_\eta^\mu(\mathbf{u},\nu)} &= \left(\int_0^{\nu(\Gamma)} [\eta(t)]^\mu \frac{dt}{t} \right)^{1/\mu} \geq \left(\int_{\nu(\Gamma)/2}^{\nu(\Gamma)} [\eta(t)]^\mu \frac{dt}{t} \right)^{1/\mu} \\
&\geq \eta(\nu(\Gamma)/2) \log 2 \geq C\eta(\nu(\Gamma))
\end{aligned}$$

since η is doubling. For the reverse inequality, since $\eta \in \mathbb{W}_+$, by Proposition 2.5.1 we obtain

$$\|1_{\Gamma,\mathbf{u}}\|_{\ell_\eta^\mu(\mathbf{u},\nu)} \approx \left(\int_0^1 [\lambda \eta(\nu(\Gamma))]^\mu \frac{d\lambda}{\lambda} \right)^{1/\mu} \approx \eta(\nu(\Gamma)).$$

∎

3.6. Jackson type inequalities (proofs).

2) $\Rightarrow$ 3) This is immediate since $\mathcal{A}_\mu^\xi(\nu) \hookrightarrow \mathcal{A}_\infty^\xi(\nu)$ and 3) is equivalent to $\ell_{\xi,\eta}^\mu(\mathbf{u},\nu) \hookrightarrow \mathcal{A}_\infty^\xi(f,\nu)$.

3) $\Rightarrow$ 1) Let $0 < s_0 < 1$ be such that $M_\eta(s_0) < 1$ as in the definition of $\eta \in \mathbb{W}_+$. Let $\Gamma \subset \mathcal{D}$ with $\nu(\Gamma) < \infty$ and write $1_\Gamma := 1_{\Gamma,\mathbf{u}} = \sum_{I \in \Gamma} \frac{\mathbf{e}_I}{u_I}$. By Lemma 1 in [20] (see also the proof of Theorem 2.1 in [13]), for $\Lambda_0 = \Gamma \subset \mathcal{D}$ one can find a subset $\Lambda_1 \subset \Lambda_0$ with $\nu(\Lambda_1) \leq s_0 \nu(\Lambda_0)$ such that

$$\|1_{\Lambda_0} - 1_{\Lambda_1}\|_f \approx \sigma_\nu(s_0 \nu(\Lambda_0), 1_{\Lambda_0}).$$

We repeat this argument to find nested subsets

$$\Gamma = \Lambda_0 \supset \Lambda_1 \supset \ldots \supset \Lambda_j \supset \Lambda_{j+1} \supset \ldots$$

such that $\nu(\Lambda_{j+1}) \leq s_0 \nu(\Lambda_j)$ and

$$\left\| 1_{\Lambda_j} - 1_{\Lambda_{j+1}} \right\|_f \approx \sigma_\nu(s_0 \nu(\Lambda_j), 1_{\Lambda_j}), \quad j = 0, 1, 2, \ldots$$

By the ρ-power triangle inequality for f we get

$$\left\| 1_\Gamma \right\|_f^\rho \;\leq\; \sum_{j=0}^{\infty} \left\| 1_{\Lambda_j} - 1_{\Lambda_{j+1}} \right\|_f^\rho \approx \sum_{j=0}^{\infty} \sigma_\nu^\rho(s_0 \nu(\Lambda_j), 1_{\Lambda_j}).$$

Using the hypothesis and Lemma 2.5.3 we obtain

$$\sigma_\nu(s_0 \nu(\Lambda_j), 1_{\Lambda_j}) \leq C[s_0 \nu(\Lambda_j)]^{-\xi} \left\| 1_{\Lambda_j} \right\|_{\ell_{\xi,\eta}^\mu(\nu)} \approx \eta(\nu(\Lambda_j)) \,.$$

Concatenating these inequalities we deduce

$$\left\| 1_\Gamma \right\|_f \;\lesssim\; \left[\sum_{j=0}^{\infty} \eta^\rho(\nu(\Lambda_j)) \right]^{1/\rho} \lesssim \left[\sum_{j=0}^{\infty} \eta^\rho(s_0^j \nu(\Lambda_0)) \right]^{1/\rho}$$

$$\lesssim\; \eta(\nu(\Lambda_0)) = \eta(\nu(\Gamma))$$

by Lemma 2.4.3, since η and η^ρ belong to $\mathbb{W}_+$.

   1) $\Rightarrow$ 2) By Lemma 2.4.4 we may assume $\eta \in C^1$ and $\eta'(t)/\eta(t) \approx 1/t$, $t > 0$. We start by bounding $\sigma_\nu(t, \mathbf{s})$ for $\mathbf{s} \in \ell_{\xi,\eta}^\mu(\mathbf{u}, \nu)$. Recall that $\mathbf{d} := \{s_I u_I\}_{I \in \mathcal{D}} \in \ell_{\xi,\eta}^\mu(\nu)$. Since $\nu(\{I \in \mathcal{D} : |u_I s_I| > \mathbf{d}_\nu^*(t)\}) \leq t$ we have

$$\sigma_\nu(t, \mathbf{s}) = \inf_{\mathbf{t} \in \Sigma_{t,\nu}} \| \mathbf{s} - \mathbf{t} \|_f \leq \left\| \sum_{|u_I s_I| \leq \mathbf{d}_\nu^*(t)} s_I \mathbf{e}_I \right\|_f .$$

For $j = 0, 1, 2, \ldots$ let $\Lambda_j = \{ I \in \mathcal{D} : 2^{-j-1} \mathbf{d}_\nu^*(t) < |s_I u_I| \leq 2^{-j} \mathbf{d}_\nu^*(t) \}$. The ρ-power triangle inequality and the monotonicity property of f, together with the hypothesis, imply

$$[\sigma_\nu(t, \mathbf{s})]^\rho \;\leq\; \sum_{j=0}^{\infty} \left\| \sum_{I \in \Lambda_j} s_I \mathbf{e}_I \right\|_f^\rho = \sum_{j=0}^{\infty} \left\| \sum_{I \in \Lambda_j} s_I u_I \frac{\mathbf{e}_I}{u_I} \right\|_f^\rho$$

$$\leq\; C \sum_{j=0}^{\infty} [2^{-j} \mathbf{d}_\nu^*(t)]^\rho \eta^\rho(\nu(\Lambda_j))$$

$$\leq\; C \int_0^{\mathbf{d}_\nu^*(t)} \lambda^\rho [\eta(\nu(\{I \in \mathcal{D} : |s_I u_I| > \lambda\}))]^\rho \frac{d\lambda}{\lambda}.$$

Applying part 2 of Lemma 2 in [20] with $F(\lambda) = \frac{\lambda^\rho}{\rho}$ and $G(\lambda) = [\eta(\lambda)]^\rho$ yields

$$[\sigma_\nu(t, \mathbf{s})]^\rho \;\leq\; \frac{1}{\rho} [\mathbf{d}_\nu^*(t) \eta(t)]^\rho + \frac{1}{\rho} \int_t^\infty [\mathbf{d}_\nu^*(s)]^\rho d\eta^\rho(s)$$

$$\approx\; [\mathbf{d}_\nu^*(t) \eta(t)]^\rho + \int_t^\infty [\mathbf{d}_\nu^*(s)]^\rho \eta^\rho(s) \frac{ds}{s}$$

where we have used $\eta'(s)/\eta(s) \approx 1/s$. Therefore

$$\sigma_\nu(t, \mathbf{s}) \lesssim \mathbf{d}_\nu^*(t) \eta(t) + \left(\int_t^\infty [\mathbf{d}_\nu^*(s) \eta(s)]^\rho \frac{ds}{s} \right)^{1/\rho} .$$

Thus,

$$
\begin{aligned}
\|\mathbf{s}\|_{\mathcal{A}_\mu^\xi(\nu)}^\mu
&= \int_0^\infty [t^\xi \sigma_\nu(t,\mathbf{s})]^\mu \frac{dt}{t} \\
&\lesssim \int_0^\infty [t^\xi \eta(t)\mathbf{d}_\nu^*(t)]^\mu \frac{dt}{t} + \int_0^\infty \left[t^\xi \left(\int_t^\infty [\mathbf{d}_\nu^*(s)\eta(s)]^\rho \frac{ds}{s} \right)^{1/\rho} \right]^\mu \frac{dt}{t} \\
&:= I + II.
\end{aligned}
$$

The first term, I, is precisely $\|\mathbf{s}\|_{\ell_{\xi,\eta}^\mu(\mathbf{u},\nu)}^\mu$. For II use Hardy's inequality (see [3], p.124) with a ρ such that $\mu/\rho > 1$ (notice that this is always possible since if f satisfies the ρ-power triangle inequality it satisfies the ρ'-power triangle inequality for any $0 < \rho' \le \rho$) to obtain

$$
\begin{aligned}
II^{1/\mu}
&= \left[\int_0^\infty t^{\xi\mu} \left(\int_t^\infty [\mathbf{d}_\nu^*(s)\eta(s)]^\rho \frac{ds}{s} \right)^{\mu/\rho} \frac{dt}{t} \right]^{1/\mu} \\
&\le \left[\frac{1}{\rho\xi} \int_0^\infty [\mathbf{d}_\nu^*(s)\eta(s)]^\mu s^{\xi\mu} \frac{ds}{s} \right]^{1/\mu} \\
&= C \|\mathbf{s}\|_{\ell_{\xi,\eta}^\mu(\mathbf{u},\nu)},
\end{aligned}
$$

This proves the result.   ∎

3.7. Bernstein type inequalities (proofs).

Proof of Proposition 2.7.1. i) $\Rightarrow$ ii) Let $\mathbf{s} \in \mathcal{A}_\mu^\xi(f,\nu)$. Choose $\varphi_k \in \Sigma_{2^{k-1},\nu}$ such that $\|\mathbf{s} - \varphi_k\|_f \le 2\sigma_\nu(2^{k-1},\mathbf{s})$. Let $\mathbf{s}_k = \varphi_k - \varphi_{k-1}$, so that $\mathbf{s}_k \in \Sigma_{2^k,\nu}$. Since $\mathbf{s} \in \mathcal{A}_\mu^\xi(f,\nu)$ we have $\sigma_\nu(2^{k-1},\mathbf{s}) \to 0$ as $k \to \infty$; the assumption $f \hookrightarrow S$ implies

$$
\lim_{k\to\infty} \varphi_k = \mathbf{s} \quad \text{in } \mathcal{D} \text{ (term by term).}
$$

On the other hand $\lim_{k\to-\infty} \varphi_k = 0$ since $\nu(\operatorname{supp} \varphi_k) \to 0$ as $k \to -\infty$. Thus,

$$
\mathbf{s} = \lim_{k\to\infty} \varphi_k = \sum_{k=-\infty}^\infty \mathbf{s}_k.
$$

Now,

$$
\|\mathbf{s}_k\|_f^\rho \le \|\mathbf{s} - \varphi_k\|_f^\rho + \|\mathbf{s} - \varphi_{k-1}\|_f^\rho \le 2 \cdot 2^\rho [\sigma_\nu(2^{k-2},\mathbf{s})]^\rho.
$$

Therefore,

$$
\sum_{k\in\mathbb{Z}} [2^{k\xi} \|\mathbf{s}_k\|_f]^\mu \le C \sum_{k\in\mathbb{Z}} [2^{k\xi}\sigma_\nu(2^{k-2},\mathbf{s})]^\mu \approx \|\mathbf{s}\|_{\mathcal{A}_\mu^\xi(\nu)}^\mu,
$$

by the discrete characterization of the restricted approximation spaces given in Subsection 2.2. It is easy to see that the result also holds for $\mu = \infty$.

ii) $\Rightarrow$ i) Observe that $\sum_{k=-\infty}^{\ell-1} \mathbf{s}_k \in \sum_{2^\ell,\nu}$ since each $\mathbf{s}_k \in \Sigma_{2^k,\nu}$. Take ρ such that $0 < \rho < \mu$ and $\|\cdot\|_f$ satisfies the ρ-power triangle inequality. We have

$$
[\sigma_\nu(2^\ell,\mathbf{s})]^\rho \le \left\| \mathbf{s} - \sum_{k=-\infty}^{\ell-1} \mathbf{s}_k \right\|_f^\rho \le \sum_{k=\ell}^\infty \|\mathbf{s}_k\|_f^\rho.
$$

With $p = \mu/\rho > 1$, (here $0 < \mu < \infty$) and $u > 0$ such that $u < \xi\rho$ we have

$$
\begin{aligned}
\|\mathbf{s}\|^{\mu}_{\mathcal{A}^{\xi}_{\mu}(\nu)} &\approx \sum_{\ell=-\infty}^{\infty} [2^{\ell\xi}\sigma_{\nu}(2^{\ell}, \mathbf{s})]^{\mu} \leq \sum_{\ell=-\infty}^{\infty} [2^{\ell\xi}(\sum_{k=\ell}^{\infty} \|\mathbf{s}_k\|^{\rho}_f)^{1/\rho}]^{\mu} \\
&= \sum_{\ell=-\infty}^{\infty} 2^{\ell\xi\mu}(\sum_{k=\ell}^{\infty} 2^{-ku}2^{ku} \|\mathbf{s}_k\|^{\rho}_f)^p \\
&\leq \sum_{\ell=-\infty}^{\infty} 2^{\ell\xi\mu}(\sum_{k=\ell}^{\infty} 2^{-kup'})^{p/p'}(\sum_{k=l}^{\infty} 2^{kup} \|\mathbf{s}_k\|^{\mu}_f) \\
&\approx \sum_{\ell\in\mathbb{Z}} 2^{\ell(\xi\mu-up)} \left(\sum_{k=\ell}^{\infty} 2^{kup} \|\mathbf{s}_k\|^{\mu}_f\right) \\
&= \sum_{k\in\mathbb{Z}} 2^{kup} \|\mathbf{s}_k\|^{\mu}_f (\sum_{\ell=-\infty}^{k} 2^{\ell(\xi\mu-up)}) \approx \sum_{k\in\mathbb{Z}} 2^{k\xi\mu} \|\mathbf{s}_k\|^{\mu}_f.
\end{aligned}
$$

Since the last expression is finite by hypothesis, we have proved $\mathbf{s} \in \mathcal{A}^{\xi}_{\mu}(f, \nu)$ for $0 < \mu < \infty$. For $\mu = \infty$ we have

$$
\begin{aligned}
\|\mathbf{s}\|_{\mathcal{A}^{\xi}_{\infty}(\nu)} &\approx \sup_{\ell\in\mathbb{Z}} 2^{\ell\xi}\sigma_{\nu}(2^{\ell}, \mathbf{s}) \leq \sup_{\ell\in\mathbb{Z}} 2^{\ell\xi}(\sum_{k=\ell}^{\infty} \|\mathbf{s}_k\|^{\rho}_f)^{1/\rho} \\
&= \sup_{\ell\in\mathbb{Z}} 2^{\ell\xi}(\sum_{k=\ell}^{\infty} 2^{-k\xi\rho}2^{k\xi\rho} \|\mathbf{s}_k\|^{\rho}_f)^{1/\rho} \\
&\leq (\sup_{k\in\mathbb{Z}} 2^{k\xi} \|\mathbf{s}_k\|_f) \sup_{\ell\in\mathbb{Z}} 2^{\ell\xi}(\sum_{k=\ell}^{\infty} 2^{-k\xi\rho})^{1/\rho} \approx \sup_{k\in\mathbb{Z}} 2^{k\xi} \|\mathbf{s}_k\|_f.
\end{aligned}
$$

$\blacksquare$

Proof of Theorem 2.7.2. 1) $\Rightarrow$ 2) Let $\mathbf{s} \in \Sigma_{t,\nu} \cap f$ and write $\mathbf{s} = \{s_I\}_{I\in\mathcal{D}}$. Given $0 < \tau \leq \nu(\Gamma)$ choose $\Lambda_{\tau} = \{I \in \Gamma : |u_I s_I| \geq \mathbf{d}^*_{\nu}(\tau)\}$ where $\mathbf{d} = \{u_I s_I\}_{I\in\mathcal{D}}$. We have $\tau \leq \nu(\Lambda_{\tau})$ (see (4) in [20] or [2]). Applying the hypothesis 1) and the monotonicity of f we obtain

$$
\mathbf{d}^*_{\nu}(\tau)\eta(\tau) \leq \mathbf{d}^*_{\nu}(\tau)\eta(\nu(\Lambda_{\tau})) \lesssim \mathbf{d}^*_{\nu}(\tau) \left\|\sum_{I\in\Lambda_{\tau}} \frac{\mathbf{e}_I}{u_I}\right\|_f \leq \left\|\sum_{I\in\Lambda_{\tau}} s_I \mathbf{e}_I\right\|_f \lesssim \|\mathbf{s}\|_f.
$$

Since $\nu(\{I \in \mathcal{D} : |s_I u_I| > 0\}) = \nu(\Gamma) \leq t$ we have $\mathbf{d}^*_{\nu}(\tau) = 0$ for $\tau \geq t$. Thus,

$$
\|\mathbf{s}\|^{\mu}_{\ell^{\mu}_{\xi,\eta}(\mathbf{u},\nu)} = \int_0^t [\tau^{\xi}\eta(\tau)\mathbf{d}^*_{\nu}(\tau)]^{\mu}\frac{d\tau}{\tau} \lesssim \|\mathbf{s}\|^{\mu}_f \int_0^t \tau^{\xi\mu}\frac{d\tau}{\tau} \approx t^{\xi\mu} \|\mathbf{s}\|^{\mu}_f.
$$

The case $\mu = \infty$ is treated similarly.

2) $\Rightarrow$ 1) Let $1_{\Gamma} := 1_{\Gamma,\mathbf{u}} = \sum_{I\in\Gamma} \frac{\mathbf{e}_I}{u_I}$ and $\nu(\Gamma) = t$, so that $1_{\Gamma} \in \Sigma_{t,\nu}$. We may assume $1_{\Gamma} \in f$, since otherwise the right hand side of 1) is ∞, and the result is trivially true. Hypothesis 2) gives

$$
\|1_{\Gamma}\|_f \gtrsim t^{-\xi} \|1_{\Gamma}\|_{\ell^{\mu}_{\xi,\eta}(\mathbf{u},\nu)} \gtrsim \eta(\nu(\Gamma)),
$$

where the last inequality is due to Lemma 2.5.3.

3) $\Rightarrow$ 2) For $\mathbf{s} \in \Sigma_{t,\nu} \cap f$, $\sigma_\nu(\tau, \mathbf{s}) = 0$ if $\tau \geq t$. Thus, by 3)

$$
\begin{aligned}
\|\mathbf{s}\|_{\ell^\mu_{\xi,\eta}(\mathbf{u},\nu)} &\lesssim \|\mathbf{s}\|_{\mathcal{A}^\xi_\mu(\nu)} = \left(\int_0^t [\tau^\xi \sigma_\nu(\tau, \mathbf{s})]^\mu \frac{d\tau}{\tau} \right)^{1/\mu} \\
&\lesssim \|\mathbf{s}\|_f \left(\int_0^t \tau^{\xi\mu} \frac{d\tau}{\tau} \right)^{1/\mu} = t^\xi \|\mathbf{s}\|_f,
\end{aligned}
$$

where we have used $\sigma_\nu(\tau, \mathbf{s}) \leq \|\mathbf{s}\|_f$ for all $\tau > 0$.

2) $\Rightarrow$ 3) We have already proved that 1) $\Leftrightarrow$ 2); since 1) does not depend on ξ, μ, then 2) holds for all $\tilde{\xi} > 0$ and all $\mu \in (0, \infty]$. For any $\tilde{\xi} > 0$ take $\tilde{\rho}$ such that $\ell^\mu_{\tilde{\xi},\eta}(\mathbf{u}, \nu)$ satisfies the $\tilde{\rho}$-power triangular inequality. By Proposition 2.7.1 we can find $\mathbf{s}_k \in \Sigma_{2^k,\nu} \cap f$, $k \in \mathbb{Z}$, such that $\mathbf{s} = \sum_{k \in \mathbb{Z}} \mathbf{s}_k$ (in $\mathcal{D}$) and

$$
\|\mathbf{s}\|_{\mathcal{A}^{\tilde{\xi}}_{\tilde{\rho}}(\nu)} \approx \left(\sum_{k \in \mathbb{Z}} [2^{k\tilde{\xi}} \|\mathbf{s}\|_f]^{\tilde{\rho}} \right)^{1/\tilde{\rho}}.
$$

Applying hypothesis 2) to $\ell^\mu_{\tilde{\xi},\eta}(\mathbf{u}, \nu)$ we obtain

$$
\|\mathbf{s}\|^{\tilde{\rho}}_{\ell^\mu_{\tilde{\xi},\eta}(\mathbf{u},\nu)} \lesssim \sum_{k \in \mathbb{Z}} [\|\mathbf{s}_k\|_{\ell^\mu_{\tilde{\xi},\eta}(\mathbf{u},\nu)}]^{\tilde{\rho}} \lesssim \sum_{k \in \mathbb{Z}} (2^{k\tilde{\xi}} \|\mathbf{s}_k\|_f)^{\tilde{\rho}} \approx \|\mathbf{s}\|^{\tilde{\rho}}_{\mathcal{A}^{\tilde{\xi}}_{\tilde{\rho}}(\nu)}.
$$

This means that for $\mu \in (0, \infty]$ and any $\tilde{\xi} > 0$ we have the continuous inclusion

$$
\mathcal{A}^{\tilde{\xi}}_{\tilde{\rho}}(\nu) \hookrightarrow \ell^\mu_{\tilde{\xi},\eta}(\mathbf{u}, \nu), \tag{3.7.1}
$$

where $\tilde{\rho}$ is the exponent of the $\tilde{\rho}$-power triangle inequality for $\ell^\mu_{\tilde{\xi},\eta}(\mathbf{u}, \nu)$. From Corollary 2.8.2, for $\xi = (\xi_0 + \xi_1)/2$ and any $\rho \in (0, 1]$ we have

$$
(\mathcal{A}^{\xi_0}_\rho(\nu), \mathcal{A}^{\xi_1}_\rho(\nu))_{1/2,\mu} = \mathcal{A}^\xi_\mu(\nu).
$$

Applying (3.7.1) with $\tilde{\xi} = \xi_0$, first, then $\tilde{\xi} = \xi_1$ and $\rho = \min\{\tilde{\rho}_0, \tilde{\rho}_1\}$ we obtain

$$
\mathcal{A}^\xi_\mu(\nu) = (\mathcal{A}^{\xi_0}_\rho(\nu), \mathcal{A}^{\xi_1}_\rho(\nu))_{1/2,\mu} \hookrightarrow (\ell^\mu_{\xi_0,\eta}(\mathbf{u},\nu), \ell^\mu_{\xi_1,\eta}(\mathbf{u},\nu))_{1/2,\mu} = \ell^\mu_{\xi,\eta}(\mathbf{u}, \nu)
$$

where the last equality is a result in real interpolation of discrete Lorentz spaces that can be found in [25] (Theorem 3).   ∎

3.10. Restricted approximation for Triebel-Lizorkin sequence spaces (proofs).

Proof of Lemma 2.10.1. i) It is clear that $|Q^x|^\gamma \chi_{Q^x}(x) \leq S^\gamma_\Gamma(x)$ since the right hand side of this inequality contain at least the cube Q^x (and possibly more). For the reverse inequality we enlarge the sum defining $S^\gamma_\Gamma(x)$ to include all dyadic cubes contained in Q^x. Therefore,

$$
S^\gamma_\Gamma(x) \leq \sum_{Q \subset Q^x : Q \in \mathcal{D}} |Q|^\gamma = \sum_{j=0}^\infty (2^{-jd} |Q^x|)^\gamma = |Q^x|^\gamma \sum_{j=0}^\infty 2^{-jd\gamma} \approx |Q^x|^\gamma
$$

since $\gamma > 0$.

ii) It is clear that $|Q_x|^\gamma \chi_{Q_x}(x) \leq S_\Gamma^\gamma(x)$ since the right hand side of this inequality contains at least the cube Q_x (and possibly more). For the reverse inequality, we enlarge the sum defining $S_\Gamma^\gamma(x)$ to include all dyadic cubes containing Q_x. Therefore,

$$S_\Gamma^\gamma(x) \leq \sum_{Q \supset Q_x : Q \in \mathcal{D}} |Q|^\gamma = \sum_{j=0}^\infty (2^{jd}|Q_x|)^\gamma = |Q_x|^\gamma \sum_{j=0}^\infty 2^{jd\gamma} \approx |Q_x|^\gamma$$

since $\gamma < 0$.   ∎

Proof of Theorem 2.10.2. We start by proving (2.10.1). Write $f_1 := f_{p_1,q_1}^{s_1}$ and $f_2 := f_{p_2,q_2}^{s_2}$ to simplify notation in this proof. By Definition 2.9.1 we have $\|\mathbf{e}_Q\|_{f_2} = |Q|^{-s_2/d+1/p_2-1/2}$ and

$$
\left\| \sum_{Q \in \Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{f_2}} \right\|_{f_1} = \left(\int_{\mathbb{R}^d} \left[\sum_{Q \in \Gamma} |Q|^{\frac{s_2-s_1}{d}q_1} |Q|^{-q_1/p_2} \chi_Q(x) \right]^{p_1/q_1} dx \right)^{1/p_1}
$$

$$
= \left(\int_{\mathbb{R}^d} \left[\sum_{Q \in \Gamma} (|Q|^{\frac{\alpha-1}{p_1}} \chi_Q(x))^{q_1} \right]^{p_1/q_1} dx \right)^{1/p_1}. \qquad (3.10.1)
$$

Consider first the case $\alpha > 1$. In this case, since $\nu_\alpha(\Gamma) < \infty$, the biggest Q^x contained in Γ exists for all $x \in \cup_{Q \in \Gamma} Q$. Applying Lemma 2.10.1, part i), first with $\gamma = \frac{\alpha-1}{p_1}q_1 > 0$ and then with $\gamma = \alpha - 1 > 0$, we obtain

$$
\left[\sum_{Q \in \Gamma} |Q|^{\frac{\alpha-1}{p_1}q_1} \chi_Q(x) \right]^{p_1/q_1} \approx |Q^x|^{\alpha-1} \chi_{Q^x}(x) \approx \sum_{Q \in \Gamma} |Q|^{\alpha-1} \chi_Q(x)
$$

for all $x \in \cup_{Q \in \Gamma} Q$. From (3.10.1) we deduce

$$
\left\| \sum_{Q \in \Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{f_2}} \right\|_{f_1} \approx \left(\int_{\mathbb{R}^d} \sum_{Q \in \Gamma} |Q|^{\alpha-1} \chi_Q(x) dx \right)^{1/p_1}
$$

$$
= \left(\sum_{Q \in \Gamma} |Q|^\alpha \right)^{1/p_1} = [\nu_\alpha(\Gamma)]^{1/p_1}.
$$

Consider now the case $\alpha < 1$. If $\alpha \leq 0$, since $\nu_\alpha(\Gamma) < \infty$, the smallest cube Q_x contained in Γ exists for all $x \in \cup_{Q \in \Gamma} Q$ (notice that $\alpha = 0$ is the classical case of counting measure). If $0 < \alpha < 1$ we can show that the set E_α of all $x \in \cup_{Q \in \Gamma} Q$ for which Q_x does not exists has measure zero. To see this, write $\mathcal{D}_k = \{Q \in \mathcal{D} : |Q| = 2^{-kd}, k \in \mathbb{Z}\}$. Then, for all $m \geq 0$, $E_\alpha \subset \cup_{k \geq m} \cup_{Q \in \Gamma \cap \mathcal{D}_k} Q$; therefore

$$
|E_\alpha| \leq \sum_{k \geq m} \sum_{Q \in \Gamma \cap \mathcal{D}_k} |Q| = \sum_{k \geq m} \sum_{Q \in \Gamma \cap \mathcal{D}_k} |Q|^\alpha |Q|^{1-\alpha}
$$

$$
\leq \nu_\alpha(\Gamma) \sum_{k \geq m} 2^{-kd(1-\alpha)} \approx \nu_\alpha(\Gamma) 2^{-md(1-\alpha)}
$$

since $1 - \alpha > 0$. Letting $m \to \infty$ we deduce $|E_\alpha| = 0$.

Apply Lemma 2.10.1, part ii), first with $\gamma = \frac{\alpha-1}{p_1}q_1 < 0$ and then with $\gamma = \alpha - 1 < 0$ to obtain

$$\left[\sum_{Q\in\Gamma} |Q|^{\frac{\alpha-1}{p_1}q_1} \chi_Q(x)\right]^{p_1/q_1} \approx |Q_x|^{\alpha-1} \chi_{Q_x}(x) \approx \sum_{Q\in\Gamma} |Q|^{\alpha-1} \chi_Q(x)$$

for all $x \in \cup_{Q\in\Gamma}Q$ if $\alpha \le 0$ and all $x \in \cup_{Q\in\Gamma}Q \setminus E_\alpha$ if $0 < \alpha < 1$. In any case, from (3.10.1) we deduce

$$\left\|\sum_{Q\in\Gamma} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{f_2}}\right\|_{f_1} \approx \left(\int_{\mathbb{R}^d} \sum_{Q\in\Gamma} |Q|^{\alpha-1} \chi_Q(x)dx\right)^{1/p_1} = \left(\sum_{Q\in\Gamma} |Q|^\alpha\right)^{1/p_1} = [\nu_\alpha(\Gamma)]^{1/p_1}.$$

For $\alpha = 1$ the set E_1 of all $x \in \cup_{Q\in\Gamma}Q$ for which Q_x exists has also measure zero. Indeed

$$|E_1| \le \sum_{k\ge m} \sum_{Q\in\Gamma\cap\mathcal{D}_k} |Q| = \sum_{k\ge m} \nu_1(\Gamma \cap \mathcal{D}_k)$$

and the last sum tends to zero as $m \to \infty$ since they are the tails of the convergent sum $\sum_{k\ge 1} \nu_1(\Gamma \cap \mathcal{D}_k) \le \nu_1(\Gamma) < \infty$.

Suppose now that (2.10.1) holds. For $N \in \mathbb{N}$ and $L = 2^l$ consider the set $\Gamma_{N,L} = \{[0, L]^d + Lj : j \in \mathbb{N}^d, 0 \le |j| < N\}$ of N^d disjoint dyadic cubes of size length L. For this collection we have

$$\nu_\alpha(\Gamma_{N,L}) = \sum_{\Gamma_{N,L}} |Q|^\alpha = (L^\alpha N)^d. \qquad (3.10.2)$$

Also

$$\left\|\sum_{Q\in\Gamma_{N,L}} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{f_2}}\right\|_{f_1} = \left(\int_{\mathbb{R}^d} \left[S_{\Gamma_{N,L}}^\gamma(x)\right]^{p_1/q_1} dx\right)^{1/p_1}$$

with $\gamma = (\frac{s_2-s_1}{d} - \frac{1}{p_2})q_1$. Since $S_{\Gamma_{N,L}}^\gamma(x) = L^{d\gamma} \sum_{\Gamma_{N,L}} \chi_Q(x) = L^{d\gamma}\chi_{[0,NL]^d}(x)$ we obtain

$$\left\|\sum_{Q\in\Gamma_{N,L}} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{f_2}}\right\|_{f_1} = L^{d\gamma/q_1}(LN)^{d/p_1} = L^{d\left(\frac{\gamma}{q_1}+\frac{1}{p_1}\right)} N^{d/p_1}. \qquad (3.10.3)$$

Choose $N, N' \in \mathbb{N}$, $L = 2^l, L' = 2^{l'}$ such that $L^\alpha N = (L')^\alpha N'$ so that (3.10.2) implies $\nu_\alpha(\Gamma_{N,L}) = \nu_\alpha(\Gamma_{N',L'})$. By (2.10.1) and (3.10.3) we deduce

$$L^{d\left(\frac{\gamma}{q_1}+\frac{1}{p_1}\right)} N^{d/p_1} \approx (L')^{d\left(\frac{\gamma}{q_1}+\frac{1}{p_1}\right)} (N')^{d/p_1} \quad \Leftrightarrow \quad \left(\frac{L}{L'}\right)^{d\left(\frac{\gamma}{q_1}+\frac{1-\alpha}{p_1}\right)} \approx 1.$$

This forces $\frac{\gamma}{q_1} = \frac{\alpha-1}{p_1}$, or equivalently $\frac{s_2-s_1}{d} - \frac{1}{p_2} = \frac{\alpha-1}{p_1}$ as desired.

For $\alpha = 1$ we still have to prove that $p_1 = q_1$. Let $N \in \mathbb{N}$ and $\Gamma_N = \{Q \subset [0,1]^d : 2^{-Nd} < |Q| \le 1\}$. We have $\nu_1(\Gamma_N) = N$ and

$$\left\|\sum_{Q\in\Gamma_N} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{f_2}}\right\|_{f_1} = \left(\int_{\mathbb{R}^d} \left(\sum_{\Gamma_N} \chi_Q(x)\right)^{p_1/q_1} dx\right)^{1/p_1} = N^{1/q_1}. \qquad (3.10.4)$$

For the same $N \in \mathbb{N}$ take $\tilde{\Gamma}_N = \{[0,1]^d + j\vec{\mathbf{e}}_1 : 0 \le j < N\}$ so that $\nu(\tilde{\Gamma}_N) = N$ and

$$\left\| \sum_{Q \in \tilde{\Gamma}_N} \frac{\mathbf{e}_Q}{\|\mathbf{e}_Q\|_{f_2}} \right\|_{f_1} = \left(\int_{\mathbb{R}^d} \chi_{[0,N] \times [0,1]^{d-1}}(x) dx \right)^{1/p_1} = N^{1/p_1} . \tag{3.10.5}$$

By (2.10.1) applied to Γ_N and $\tilde{\Gamma}_N$ together with (3.10.4)and (3.10.5) we obtain $N^{1/q_1} \approx N^{1/p_1}$. This forces $p_1 = q_1$ as we wanted. ∎

Proof of Corollary 2.10.3. Apply Theorems 2.6.1 and 2.7.2 to $f = f_{p_1,q_1}^{s_1}$, $\mathbf{u}$ and ν_α, as given in the statement of the corollary, and with $\eta(t) = t^{1/p_1}$. ∎

Proof of Lemma 2.10.4. Let $f_2 := f_{p_2,q_2}^{s_2}$ to simplify notation. Since $\|\mathbf{e}_Q\|_{f_2} = |Q|^{-s_2/d+1/p_2-1/2} = |Q|^{-\gamma/d+(1-\alpha)/\tau-1/2}$ for $\mathbf{s} = \sum_{Q \in \mathcal{D}} s_Q \mathbf{e}_Q$ we have

$$\begin{aligned}
\|\mathbf{s}\|_{\ell^{\tau,\tau}(\mathbf{u},\nu_\alpha)}^\tau &= \left\| \{\|s_Q \mathbf{e}_Q\|_{f_2}\}_{Q \in \mathcal{D}} \right\|_{\ell^{\tau,\tau}(\nu_\alpha)}^\tau = \sum_{Q \in \mathcal{D}} \|s_Q \mathbf{e}_Q\|_{f_2}^\tau |Q|^\alpha \\
&= \sum_{Q \in \mathcal{D}} (|s_Q| \, |Q|^{-\gamma/d+(1-\alpha)/\tau-1/2} \, |Q|^{\alpha/\tau})^\tau \\
&= \sum_{Q \in \mathcal{D}} (|s_Q| \, |Q|^{-\gamma/d+1/\tau-1/2})^\tau = \|\mathbf{s}\|_{b_{\tau,\tau}^\gamma}^\tau .
\end{aligned}$$

∎

3.11. **Application to real interpolation (proofs).**

Proof of Theorem 2.11.1. Write $\xi = 1/\tau - 1/p > 0$ and choose $\alpha \ne 1$ such that $\gamma = s + (1-\alpha)\xi d$ (*i.e.* $\alpha = 1 - \frac{\gamma-s}{\xi d}$), which is possible since $\gamma \ne s$. Once α is chosen, take $s_2 \in \mathbb{R}$ in such a way that $\alpha = p(\frac{s_2-s}{d})$ (*i.e.* $s_2 = s + \frac{\alpha d}{p}$). Theorem 2.10.2 shows that $f_{p,q}^s$ satisfies 1) of Theorems 2.6.1 and 2.7.2 with $\eta(t) = t^{1/p}$, for the "normalization" space $f_2 := f_{p,p}^{s_2}$ and ν_α (notice that $\alpha = p(\frac{s_2-s}{d})$ is the condition required in Lemma 2.10.4). Thus, the space $\ell^{\tau,\mu}(\mathbf{u}, \nu_\alpha)$ satisfies the Jackson and Bernstein's inequalities of order $\xi = 1/\tau - 1/p > 0$, where $\mathbf{u} = \{\|\mathbf{e}_Q\|_{f_2}\}_{Q \in \mathcal{D}}$. Taking $\mu = \tau$, Lemma 2.10.4 shows that $b_{\tau,\tau}^\gamma$ satisfies the Jackson and Bernstein's inequalities of order $\xi = 1/\tau - 1/p > 0$, since $\gamma = s + (1-\alpha)\xi d$ (the required condition).

By Theorem 2.8.1, for $0 < \theta < 1$,

$$(f_{p,q}^s, b_{\tau,\tau}^\gamma)_{\theta,\tau_\theta} = \mathcal{A}_{\tau_\theta}^{\theta\xi}(f_{p,q}^s, \nu_\alpha).$$

Since $\frac{1}{\tau_\theta} = \frac{(1-\theta)}{p} + \frac{\theta}{\tau} = \theta(\frac{1}{\tau} - \frac{1}{p}) + \frac{1}{p} = \theta\xi + \frac{1}{p}$, Corollary 2.10.6 gives

$$\mathcal{A}_{\tau_\theta}^{\theta\xi}(f_{p,q}^s, \nu_\alpha) = b_{\tau_\theta,\tau_\theta}^{\tilde{\gamma}}$$

with $\tilde{\gamma} = s + d\theta\xi(1-\alpha) = s + d\theta\xi\frac{(\gamma-s)}{\xi d} = s + \theta(\gamma - s) = (1-\theta)s + \theta\gamma$, which proves the result. ∎

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EUGENIO HERNÁNDEZ, DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN
  E-mail address: eugenio.hernandez@uam.es

DANIEL VERA, DEPARTAMENTO DE MATEMÁTICAS, UNIVERSIDAD AUTÓNOMA DE MADRID, 28049 MADRID, SPAIN
  E-mail address: daniel.vera@uam.es