Extension of the adiabatic regularization method to spin-1/2 fields

Aitor Landete

Instituto de Física Teórica, UAM-CSIC. Universidad Autonoma de Madrid, Cantoblanco, 28049 Madrid, Spain

E-mail: `aitor.landete@csic.es`

Abstract. The adiabatic regularization method was designed by L. Parker [1] for scalar fields in order to to subtract the potentially UV divergences that appear in the particle number operator. After that the method was generalized [2] to remove, in a consistent way, the UV divergences that appear in the expectation value of the stress-energy tensor $\langle T_{\mu\nu} \rangle$ in homogeneous cosmological backgrounds. We are going to provide here the extension of the adiabatic regularization method to spin-1/2 fields first given in [3]. In order to achieve this extension we will show the generalization of the adiabatic expansion for fermionic fields which differs significantly from the WKB-type expansion that works for the scalar modes. We will also show the consistency of the extended method computing well-known results, computed by other renormalization methods for a Dirac field in a FLRW spacetime, like the conformal and axial anomalies. Finally we will compute the expectation value of the stress-energy tensor for a Dirac field in a de Sitter spacetime.

1. Introduction

It is well-know that in General Relativity matter exerts its gravitational influence by curving the spacetime. Thus, It is natural to study the propagation of quantum fields in curved spacetieme in order to search for new effects of gravitation in this approach, where the gravitational field is not quantized. So quantum field theory in curved spacetime could be seen as a first step to coalesce Einstein's theory of GR and QFT in Minkowski space within a self-consistent and successful framework [4, 5]. It plays a crucial role in the understanding of the low-energy regime in quantum cosmology [6]. and scrutinize the understanding of the power spectrum in inflationary cosmology [7]. Moreover it constitutes the driving mechanism to explain the quantum radiance of black holes [8]

Like in the well-known Minkowskian QFT, physical quantities of physical interest, such as the expectation value of the stress-energy tensor diverges. So it is necessary to employ regularization and renormalization methods to handle this divergences. But renormalization in Minkowski spacetime is different than the renormalization in curved spacetime because the inherent gravitatory interaction introduces additional divergences that cannot be renormalized in the same way as in Minkowski. As a toy example if we try to renormalize the expectation value of the component 00 of the stress energy tensor of a free scalar field, we see that normal ordering renormalizes the divergence that appear. Meanwhile we can see that in curved spacetime appear more divergent terms in the expectation value that cannot be renormalized by the usual techniques in Minkowski. The gravitationally created particles generate an energy

density with new ultraviolet (UV) divergences, as compared with the UV divergences present in Minkowski space. This requires more sophisticated methods of renormalization, adapted to the time-dependent or curved background.

Adiabatic regularization was first introduced in Parker's pioneer work on particle creation in the expanding universe [1] as a way to overcome the rapid oscillation of the particle number operator and UV divergences during the expansion. The method was later systematized and generalized to consistently deal with the UV divergences of the stress-energy tensor of scalar fields. The adiabatic method identifies the UV subtraction terms by first considering a slowly varying expansion factor $a(t)$. This naturally leads to a Liouville or WKB-type asymptotic expansion for the modes characterized by the comoving momentum $\vec{k}$. The subtraction terms identified in this way are valid for arbitrary smooth expansions [9]. This method has the distinguishing features of renormalize the particle number operator and allow the use of numerical methods. It has been shown that when the method is applied to renormalize local expectation values, as the stress-energy tensor, it turns out to be equivalent to the DeWitt-Schwinger point-splitting method for scalar fields [10, 11].

2. Adiabatic expansion for spin-$1/2$ fields.

As we have seen, the adiabatic method is only applicable to scalar fields. Our main goal is to generalize the method in order to study the Dirac field in curved spacetime. First of all, consider the Dirac equation in a spatially flat FLRW spacetime $ds^2 = dt^2 - a(t)d\vec{x}^2$ (for simplicity we will develop all the computations in this background)

$$(i\gamma^0\partial_0 + \frac{3i}{2}\frac{\dot{a}}{a}\gamma^0 + \frac{i}{a}\vec{\gamma}\vec{\nabla} - m)\psi = 0 \, , \tag{1}$$

where γ^μ are the Dirac matrices in Minkowski spacetime. For our purposes it is convenient work with the standard Dirac-Pauli representation. After momentum expansion $\psi = \sum_{\vec{k}} \psi_{\vec{k}}(t)e^{i\vec{k}\vec{x}}$ it is convenient to write the Dirac field in terms of two two-components spinors

$$\psi_{\vec{k}}(t) = \begin{pmatrix} \frac{1}{\sqrt{L^3 a^3}} h_k^I(t)\xi_\lambda(\vec{k}) \\ \frac{1}{\sqrt{L^3 a^3}} h_k^{II}(t)\frac{\vec{\sigma}\vec{k}}{k}\xi_\lambda(\vec{k}) \end{pmatrix} \tag{2}$$

where $\vec{\sigma}$ are the usual Pauli matrices. $\xi_\lambda(\vec{k})$ is a constant normalized bispinor $\xi_\lambda^\dagger \xi_\lambda = 1$ such that $\frac{\vec{\sigma}\vec{k}}{2k}\xi_\lambda = \lambda\xi_\lambda$. $\lambda = \pm 1/2$ represents the eigenvalue for the helicity, or spin component along the $\vec{k}$ direction. Thus, with the expansion (2) we can express the Dirac equation (1) as first order coupled equation system of the scalar functions h_k^I and h_k^{II}

$$h_k^{II} = \frac{ia}{k}(\partial_t + im)h_k^I \tag{3}$$

$$h_k^I = \frac{ia}{k}(\partial_t - im)h_k^{II} \, , \tag{4}$$

we are able to uncouple the former system and obtain two Klein-Gordon-type equations: $(\partial_t^2 + \frac{\dot{a}}{a}\partial_t + im\frac{\dot{a}}{a} + m^2 + \frac{k^2}{a^2})h_k^I = 0$ and a similar one for h_k^{II}, obtained by changing $m \to -m$ in the equation for h_k^I. The normalization condition for the four-spinor is

$$|h_k^I(t)|^2 + |h_k^{II}(t)|^2 = 1 \, . \tag{5}$$

One could be tempted to use the above second order equations to generate a WKB-type expansion for h_k^I and h_k^{II}. However a WKB-type ansatz, like in the scalar case, is specifically

designed to preserve the Klein-Gordon product, and hence the associated Wronskian condition, but not to preserve the Dirac product and the normalization condition (5). Therefore one should follow a different way.

The zeroth adiabatic order should generalize in a natural way the standard solution in Minkowski space. Thus, it has to be of the form

$$
\begin{aligned}
g_k^{I(0)}(t) &= \sqrt{\frac{\omega(t)+m}{2\omega(t)}} e^{-i\int^t \omega(t')dt'} \\
g_k^{II(0)}(t) &= \sqrt{\frac{\omega(t)-m}{2\omega(t)}} e^{-i\int^t \omega(t')dt'} \ .
\end{aligned}
\tag{6}
$$

It is easy to see that the zeroth order obeys the normalization condition $|g_k^{I(0)}(t)|^2 + |g_k^{II(0)}(t)|^2 = 1$. The form of the zeroth order and the field equations (3) and (4) suggests the following alternative ansatz for the adiabatic expansion (at adiabatic order n)

$$
\begin{aligned}
g_k^{I(n)}(t) &= \sqrt{\frac{\omega+m}{2\omega}} e^{-i\int^t (\omega(t')+\omega^{(1)}+...+\omega^{(n)})dt'} \\
&\times \ (1+F^{(1)}+...+F^{(n)}) \\
g_k^{II(n)}(t) &= \sqrt{\frac{\omega-m}{2\omega}} e^{-i\int^t (\omega(t')+\omega^{(1)}+...+\omega^{(n)})dt'} \\
&\times \ (1+G^{(1)}+...+G^{(n)}) \ ,
\end{aligned}
$$

where $\omega^{(n)}$, $F^{(n)}$ and $G^{(n)}$ are local functions of adiabatic order n. Imposing equations (3) and (4) and keeping terms of fixed adiabatic order one gets a system of equations at each order that one has to solve. Moreover, the solution should also respect the normalization condition $|g_k^{I(n)}(t)|^2 + |g_k^{II(n)}(t)|^2 = 1$ (at the given adiabatic order n), which we impose as a new equation in our system. In order to show how the method works we are going to compute explicitly the first and the second adiabatic orders of the expansion.

For the first adiabatic order we obtain that $\omega^{(1)} = 0$. Moreover, the functions $F^{(1)}$, $G^{(1)}$ should have a vanishing real part and verify the relation $G^{(1)} = F^{(1)} + i\frac{m\dot{a}}{2\omega^2 a}$. This solution can be parametrized as $F^{(1)} = -Ai\frac{m\dot{a}}{\omega^2 a}$, $G^{(1)} = Bi\frac{m\dot{a}}{\omega^2 a}$, where A, B are arbitrary real constants obeying $A + B = 1/2$. We note that, although the solution at first order is not univocally determined, local observables are actually independent of the ambiguity in $A-B$. We find useful for simplifying expressions and for computational purposes to fix the parameters as $A = B$. This implies $F^{(1)}(-m) = G^{(1)}(m)$, we are able to apply this kind of simplification at every adiabatic order for simplicity.

If we go to second adiabatic order and we impose $F^{(2)}(-m) = G^{(2)}(m)$ for simplicity. The solutions are then

$$
\omega^{(2)} = \frac{5m^4\dot{a}^2 - 3\omega^2 m^2\dot{a}^2 - 2\omega^2 m^2\ddot{a}a}{8\omega^5 a^2}
\tag{7}
$$

$$
F^{(2)} = \frac{m^2 R}{48\omega^4} - \frac{5m^4\dot{a}^2}{16\omega^6 a^2} - \frac{m^2\dot{a}^2}{32\omega^4 a^2} - \frac{mR}{48\omega^3} + \frac{5m^3\dot{a}^2}{16\omega^5 a^2} \ .
\tag{8}
$$

As in the scalar case, we obtain that $\omega^{(odd)} = 0$. The explicit solutions to third and fourth adiabatic orders are given in [12].

The nth adiabatic order for the fermionic modes defined by $g_k^{I(n)}$ and $g_k^{II(n)}$ allows us to define the subtraction terms to cancel the UV divergences. The covariance notion of adiabatic invariance ensures the underlying covariance of the subtraction procedure. In the next section we will show as a hint of the consistency of the extended method the agreement of the conformal and chiral anomalies computed with the adiabatic scheme for spin-1/2 fields with the well-known results computed by other regularization methods.

3. Consistency tests. Computation of the conformal al chiral anomalies

Renormalization methods are crucial in the computation of quantum anomalies in QFT in curved spacetime. The study of this methods is likely to obtain a deeper understanding of the known anomalies and, moreover, look for new ones [13]. It is well-known that the chiral and conformally anomalies have been computed by other renormalization methods. As a consistency test of the extension method shown before we are going to compute via our extended method this well-known results. Concerning local observables in a generic FLRW spacetime, the adiabatic regularization axioms tell us that the second adiabatic order is required to renormalize the two-point function and the fourth order is required when the observable is related to the stress-energy tensor.

Note that, due to the way that the adiabatic regularization method is constructed, the regularized expressions also allow for an efficient numerical estimation when the modes for the quantum state are difficult to manage analytically. [14]

3.1. Conformal anomaly

At the classical level we know that the trace of the stress-energy tensor for a Dirac field takes the simple expression

$$T_\mu^\mu = m\bar{\psi}\psi . \tag{9}$$

When the field is massless the trace classically vanishes, pointing the emergence of the conformal invariance. However, in the quantum theory the expectation value $\langle T_\mu^\mu \rangle = m\langle \bar{\psi}\psi \rangle$ takes a nonzero value even in the massless limit due to the conservation of the expectation value of the stress-energy tensor. Our purpose is to perform the calculation of this anomalous trace using the extension of the adiabatic regularization method for spin-1/2 introduced above. Since the expectation value $\langle \bar{\psi}\psi \rangle$ is now regarded as a piece of the average value of the stress-energy tensor $\langle T_{\mu\nu} \rangle$, the renormalization should be performed up to the fourth adiabatic order according to the adiabatic regularization axioms. So one have to evaluate the trace anomaly by taking the massless limit in the above expression

$$
\begin{aligned}
\langle T_\mu^\mu \rangle_r &= \lim_{m\to 0} \frac{-2m}{(2\pi)^3 a^3} \int d^3k (|h_k^I|^2 - |h_k^{II}|^2) \\
&\quad - |g_k^{I(4)}|^2 + |g_k^{II(4)}|^2) .
\end{aligned}
$$

Using the adiabatic expansion of the fermionic field obtained in the former section and integrating we obtain

$$
\begin{aligned}
\langle T_\mu^\mu \rangle_r &= \frac{1}{2880\pi^2} \left[-11\left(R_{\alpha\beta}R^{\alpha\beta} - \frac{1}{3}R^2\right) + 6\Box R \right] \\
&= \frac{1}{2880\pi^2} \left[\frac{11}{2}G + 6\Box R \right] ,
\end{aligned} \tag{10}
$$

where in the second line we have rewritten the former expression in terms of the Gauss-Bonnet invariant G, which for a FLRW spacetime is given by $G = -2(R_{\mu\nu}R^{\mu\nu} - R^2/3)$. The conformal

anomaly is generically given for a conformal field of spin $0, 1/2$ or 1 in terms of three parameters

$$\langle T_\mu^\mu \rangle_r = \frac{1}{2880\pi^2}\left(AC_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + BG + C\Box R\right) \ . \tag{11}$$

The result obtained for a Dirac spin-1/2 field by other renormalization procedures is $A = -9, B = 11/2, C = 6$ [5]. Our above result (10) agrees exactly with the results obtained from other methods. We note that in a FLRW spacetime the Weyl tensor $C_{\mu\nu\rho\sigma}$ vanishes identically. This can be regarded as a nontrivial test of the robustness of our proposal.

3.2. Chiral anomaly
In curved spacetime the axial vector current $J_A^\mu \equiv \bar{\psi}\gamma^\mu\gamma^5\psi$ obeys the covariant equation $\nabla_\mu J_A^\mu = 2im\bar{\psi}\gamma^5\psi$. For a massless Dirac field the classical axial current is conserved, due to the chiral symmetry. At the quantum level the expectation value $\langle\nabla_\mu J_A^\mu\rangle$ may acquire a non-zero value in the massless limit. We want to evaluate this quantity using the adiabatic regularization for fermions. The strategy is similar to the evaluation of the conformal or trace anomaly. Since the divergences of $\langle\nabla_\mu J_A^\mu\rangle$ are of fourth adiabatic order we have to compute $\langle 2im\bar{\psi}\gamma^5\psi\rangle$ also at fourth adiabatic order. In this case we have to compute

$$\langle\bar{\psi}\gamma^5\psi\rangle_r = \frac{-2}{(2\pi)^3 a^3}\int d^3k(h_k^{I*}h_k^{II} - h_k^{II*}h_k^I - g_k^{I(4)*}g_k^{II(4)} + g_k^{II(4)*}g_k^{I(4)}) \tag{12}$$

Concerning the axial current anomaly, the subtraction terms of fourth adiabatic order cancel out while the third order terms are, after integration in momenta, still proportional to the mass. Therefore, in the massless limit $\langle\nabla_\mu J_A^\mu\rangle_r = 0$, in agreement with the fact that the axial current anomaly obtained from other renormalization prescriptions $\epsilon^{\mu\nu\alpha\beta}R_{\mu\nu}^{\ \ \lambda\xi}R_{\alpha\beta\lambda\xi}$ vanishes for a FLRW spacetime.

4. Expectation value of the stress-energy tensor of a free Dirac field in a de Sitter spacetime in a Bunch-Davies vacuum

For de Sitter spacetime with scale factor $a(t) = e^{Ht}$ and H constant, the coupled differential equations (3) and (4) take the form

$$h_k^{II} = \frac{ie^{Ht}}{k}(\partial_t + im)h_k^I \ , \ \ h_k^I = \frac{ie^{Ht}}{k}(\partial_t - im)h_k^{II}, \tag{13}$$

We define the following dimensionless variables

$$z \equiv kH^{-1}e^{-Ht} \qquad \mu \equiv \frac{m}{H} \ . \tag{14}$$

In terms of these variables, the exact modes are expressed in terms of the Hankel functions $H_\alpha^{(1)}$ in the following way

$$h_k^I = i\frac{\sqrt{\pi z}}{2}e^{\frac{\pi\mu}{2}}H_{\frac{1}{2}-i\mu}^{(1)}(z) \tag{15}$$

and

$$h_k^{II} = \frac{\sqrt{\pi z}}{2}e^{\frac{\pi\mu}{2}}H_{-\frac{1}{2}-i\mu}^{(1)}(z) \ . \tag{16}$$

Equations (15) and (16) determine a vacuum for spin-1/2 fields analogous to the Bunch-Davies vacuum for scalars, because it is the solution that reproduces the adiabatic modes for initial times. Note that $h_k^{II}(m) = h_k^I(-m)$. In the following computations we are going to simplify

the ambiguity that appears in the adiabatic expansion assuming for simplicity the condition $F^{(n)}(m) = G^{(n)}(-m)$.

Due to the symmetries of de Sitter spacetime, the renormalized expectation value of the stress-energy tensor is related to the the renormalization of its quantum trace in the following way

$$\langle T^{\mu\nu} \rangle_r = \frac{1}{4} g^{\mu\nu} \langle T^\rho_\rho \rangle_r \; . \tag{17}$$

As analyzed in the Section 3.1, the unrenormalized formal expression for $\langle T^\rho_\rho \rangle$ contains UV divergences. So the unrenormalized stress-energy tensor is given by

$$\langle T^{\mu\nu} \rangle = \frac{1}{4} g^{\mu\nu} \frac{-mH^3}{\pi^2} \int_0^\infty dz \left[\left(|h_k^I|^2 - |h_k^{II}|^2 \right) z^2 \right] \tag{18}$$

If we substitute the exact modes in the former expression we see that contains quadratic and logarithmic UV divergences for $z \to \infty$ due to

$$\left(|h_k^I|^2 - |h_k^{II}|^2 \right) z^2 \sim z - \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^3} \right) \tag{19}$$

Therefore, in order to obtain the renormalized trace we have to substract the corresponding adiabatic expansion up to fourth order

$$\langle T^\rho_\rho \rangle_r = \frac{-mH^3}{\pi^2} \int_0^\infty dz z^2 \left(|h_k^I|^2 - |h_k^{II}|^2 - \left| g_k^{I\,(4)} \right|^2 + \left| g_k^{II\,(4)} \right|^2 \right) \; . \tag{20}$$

This integral is convergent and can be solved numerically. However, it can also be evaluated analytically by introducing an auxiliary regulator σ (20). The integration of (18) gives

$$\langle T^\rho_\rho \rangle = \lim_{\sigma \to 0} \left(\frac{H^2 m^2}{\pi^2 \sigma^2} - \frac{m^2 \left(H^2 + m^2 \right)}{4\pi^2} \cdot \frac{1 + 2\gamma + 2\log\left(\frac{\sigma}{2}\right) + 2\Re\mathfrak{e}\left[\psi\left(-2 - i\frac{m}{H} \right) \right]}{4\pi^2} \right) \tag{21}$$

which is divergent in the $\sigma \to 0$ limit as we already know. If we integrate the adiabatic counterterms that we have added in (20), we obtain

$$\langle T^\rho_\rho \rangle_{Ad} = \lim_{\sigma \to 0} \left(\frac{H^2 m^2}{\pi^2 \sigma^2} - \frac{11H^4 + 190H^2m^2 + 60m^4}{240\pi^2} - \frac{m^2 \left(H^2 + m^2 \right) \left(2\gamma + 2\log\left(\frac{\sigma}{2}\right) + 2\log\left(\frac{m}{H}\right) \right)}{4\pi^2} \right) . \tag{22}$$

Equation (22) is also divergent when $\sigma \to 0$. However, if (22) is subtracted from (21) the result is finite in the $\sigma \to 0$ limit and gives the quantum trace. From it we can immediately obtain an analytic expression for the renormalized stress-energy tensor

$$\langle T^{\mu\nu} \rangle_r = \frac{1}{960\pi^2} g^{\mu\nu} \left(11H^4 + 130H^2m^2 + 120m^2(H^2 + m^2) \left(\log\left(\frac{m}{H} \right) - \Re\mathfrak{e}\left[\psi\left(-1 + i\frac{m}{H} \right) \right] \right) \right). \tag{23}$$

where $\psi(z)$ is the digamma function.

References

[1] Parker L., *The creation of particles in an expanding universe*, Ph.D. thesis, Harvard University (1966).

[2] Parker L. and Fulling S.A., *Phys. Rev. D* **9**, 341 (1974). Fulling S. A. and Parker L., *Ann. Phys. (N. Y.)* **87**, 176 (1974). Fulling S. A., Parker L. and Hu B. L. *Phys. Rev. D* **10**, 3905 (1974).

[3] A. Landete, J. Navarro-Salas and F. Torrenti *Phys. Rev. D***88**, 061501 (2013).

[4] Parker L. and Toms D.J., *Quantum field theory in curved spacetime: quantized fields and gravity*, Cambridge University Press, (2009).

[5] Birrell N.D. and Davies P.C.W., *Quantum fields in curved space*, Cambridge University Press, (1982).

[6] Agulló I., Ashtekar A. and Nelson W., *Phys. Rev. Lett.* **109**, 251301 (2012).

[7] Parker L., Amplitude of perturbations from inflation, hep-th/0702216. Agulló I., Navarro-Salas J., Olmo G.J. and Parker L., *Phys. Rev. Lett.* **103**, 061301 (2009); *Phys. Rev. D* **81**, 043514, (2010).

[8] Hawking S. W. *Commun. Math. Phys.* **43**, 199 (1975).

[9] Parker L., *Phys.Rev.Lett.* **21** 562 (1968); *Phys. Rev.* **183**, 1057(1969); *Phys. Rev. D* **3**, 346 (1971)

[10] Birrell N.D. *Proc. R. Soc. Lond B* **361**, 513 (1978).

[11] Anderson P. R. and Parker L., *Phys. Rev. D* **36**, 2963 (1987).

[12] A. Landete, J. Navarro-Salas and F. Torrenti *Phys. Rev. D***89**, 044030 (2014).

[13] I. Agullo, A. Landete and J. Navarro-Salas *Preprint* gr-qc/1409.6406.

[14] Anderson. P.R. and Eaker W., *Phys. Rev. D* **61**, 024003 (1999); Habib S., Molina-Paris C. and Mottola E., *Phys. Rev. D* **61**, 024010 (1999); Anderson P.R., Molina-Paris C., Evanich D., Cook, G.B., *Phys. Rev. D* **78**, 083514 (2008); Bates, J.D. and Anderson, P.R., *Phys. Rev. D* **82**, 024018 (2010); and referenced cited in these papers.