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A test for convex dominance with respect to the exponential class based on an L^1 distance

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Abstract

We consider the problem of testing if a non-negative random variable is dominated, in the convex order, by the exponential class. Under the null hypothesis, the variable is *harmonic new better than used in expectation* (HNBUE), a well-known class of ageing distributions in reliability theory. As a test statistic, we propose the L^1 norm of a suitable distance between the empirical and the exponential distributions and we completely determine its asymptotic properties. The practical performance of our proposal is illustrated with simulation studies, which show that the asymptotic test has a good behavior and power even for small sample sizes. Finally, three real data sets are analyzed.

Keywords: Ageing classes of distributions; convex order; exponential distribution; harmonic new better than used in expectation; reliability theory.

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1 Introduction

The exponential distribution plays a key role in reliability theory. Its lack of memory makes it the appropriate benchmark to analyze and compare the ageing properties of other probability distributions. The non-ageing property of an exponential variable essentially means that a used exponential component is as good as a new one. For this reason, almost all the classes of distributions used in reliability theory are constructed by means of a suitable comparison with exponential distributions. For instance, the classes NBU (new better than used), NBUE (new better than used in expectation), IFR (increasing failure rate), IFRA (increasing failure rate average), among others, are generated by comparing certain characteristics of the variable of interest with the corresponding ones of an exponential distribution. The classic book by Barlow and Proschan (1975) [2] provides a detailed study of these lifetime distributions.

A non-negative random variable X such that $0 < \mu := \mathrm{E}X < \infty$ is said to be *harmonic new better than used in expectation* (HNBUE) if the harmonic mean of its mean residual life function is smaller than μ, that is,

$$\frac{1}{\frac{1}{t}\int_0^t \mu(s)^{-1}\,\mathrm{d}s} \leq \mu, \qquad t \geq 0,$$

where $\mu(t) := \mathrm{E}(X|X > t)$. Observe that μ is actually the harmonic mean of $\mu(t)$ for an exponential variable with expectation μ. The class of random variables with the HNBUE property, denoted in the following by $\mathcal{H}$, is fairly large in reliability theory because it contains the class of NBUE distributions and, in consequence, it also includes all IFR, IFRA and NBU distributions.

Taking into account that ageing classes are usually constructed by stochastic comparisons, the theory of stochastic orders provides the perfect framework to deal with specific families of lifetime distributions. Stochastic orders are partial order relations in the set of probability distribution functions. They compare random variables in terms of their global size, different notions of dispersion or variability, uniformity, etc. We refer the reader to the books by Müller and Stoyan (2002) [16] and Shaked and Shanthikumar (2006) [21]. It is common to include a variable in an ageing class if it is below (or above) an exponential variable with respect to a specific stochastic ordering. Therefore, the exponential distribution is always a boundary member of the corresponding class. The choice of the order depends on the interests of the researcher and the problem at hand.

One of the most important variability orders is the convex order. Given two integrable random variables X and Y, it is said that X is less than or equal to Y in the *convex order* if $\mathrm{E}(\phi(X)) \leq \mathrm{E}(\phi(Y))$, for every convex function ϕ for which the previous expectations are well defined. In particular, the definition implies that $\mathrm{E}X = \mathrm{E}Y$ and $\mathrm{Var}(X) \leq \mathrm{Var}(Y)$ (whenever the variables have finite second moment). Hence, the convex order arranges distributions in terms of their variability.

It can be easily proved (see Müller and Stoyan (2002) [16, Theorem 1.8.7]) that $X \in \mathcal{H}$ if and only if X is dominated in the convex order by an exponential random variable. In particular, since $\phi(x) = x^\alpha$ ($x \geq 0$ and $\alpha \geq 1$) is convex, given $X \in \mathcal{H}$ with mean μ, we obtain $\mathrm{E}X^\alpha \leq \mu^\alpha \Gamma(\alpha + 1)$, where $\Gamma(\cdot)$ is Euler's gamma function. Therefore, the members of $\mathcal{H}$ have finite moments of all orders. Further, this characterization of the variables in $\mathcal{H}$ shows that if X has expectation $\mu > 0$ and distribution function F, then $X \in \mathcal{H}$ if and only if

$$\int_t^\infty (G_\mu(x) - F(x))\,\mathrm{d}x \leq 0, \qquad \text{for all } t \geq 0, \tag{1}$$

where $G_\mu(x) := 1 - e^{-x/\mu}$ ($x > 0$) is the distribution function of an exponential random variable with mean μ.

In this paper, we focus on the test with null hypothesis

$$H_0 : X \in \mathcal{H}. \tag{2}$$

Given $\mathcal{C}$, a class of lifetime distributions for which the exponential distribution is a boundary member, there is an extensive literature on the test

$$\begin{aligned} &H_0 : X \text{ is exponentially distributed.} \\ &H_1 : X \in \mathcal{C}, \text{ but } X \text{ is } \textit{not} \text{ exponentially distributed.} \end{aligned} \tag{3}$$

For an thorough review of statistical tests for univariate ageing classes we refer the reader to Lai and Xie (2006) [12, Chapter 7]. The specific case of $\mathcal{C} = \mathcal{H}$ has been widely considered (see Berrendero and Cárcamo (2009) [4] and the references therein). However, note that the test proposed in (2) is a natural first step before proceeding to (3) with $\mathcal{C} = \mathcal{H}$, since in (3) it is implicitly assumed that X has the HNBUE property, which is always unknown in practice. Therefore, accepting the null in the problem (2) would allow the practitioner to proceed to (3), a more specific statistical test to assess if the variable has the exponential distribution.

As far as the authors know, there are no results yet on the test (2). Denuit *et al.* (2007) [6] considered an analogous problem for the NBUE assumption. These authors proposed a Kolmogorov-Smirnov-type test statistic, defining the discrepancy in terms of the supremum metric. Although there is no clear way of ranking tests (the achieved power obviously depends on the alternative hypothesis, see Nikitin (1995) [17]), usually Kolmogorov-Smirnov-type statistics do not achieve a high power. In this paper we construct a test statistic for the HNBUE hypothesis in (2) based on an L^1 distance.

Note that testing H_0 in (2) amounts to verifying that the function in (1) is not positive. Hence, as a discrepancy measure we consider the L^1 norm of the positive part of this function, that is,

$$\theta(X) := \int_0^\infty \left(\int_t^\infty (G_\mu(x) - F(x))\,\mathrm{d}x \right)_+ \mathrm{d}t, \tag{4}$$

where $a_+ := \max\{0, a\}$. Obviously, $\theta(X) = 0$ if and only if $X \in \mathcal{H}$, that is, if H_0 holds. Let us also remark that $\theta(X)$ is finite if and only if $\mathrm{E}X^2 < \infty$ (see Lemma 5). In the context of testing for positive quadrant dependence, Scaillet (2005) [19] suggested the possibility of using a discrepancy measure resembling (4), in the spirit of Cramér-von Mises-type statistics.

Throughout this paper, X is a positive random variable with finite mean $\mu > 0$ and distribution function F, and $X_1, \ldots, X_n$ $(n \geq 1)$ is a random sample from X. In practice, the unknown terms in (4), μ and F, are substituted by sample estimates. Consequently, the (normalized) empirical counterpart of (4) is

$$T_n(X) := \sqrt{n} \int_0^\infty \left(\int_t^\infty (G_{\hat{\mu}}(x) - F_n(x))\,\mathrm{d}x \right)_+ \mathrm{d}t, \tag{5}$$

where

$$\hat{\mu} := \frac{1}{n} \sum_{i=1}^n X_i \qquad \text{and} \qquad F_n(x) := \frac{1}{n} \sum_{i=1}^n I_{\{X_i \leq x\}}, \quad x \geq 0,$$

are the sample mean and the empirical distribution function, respectively. Here I_A stands for the indicator function of the set A. We denote $\theta(X)$ by θ (and $T_n(X)$ by T_n) when the dependency on X is clear from the context.

Although, for a fixed n, it is almost imposible to derive the exact distribution of the statistic T_n in (5), in Subsection 2.1 we determine its asymptotic distribution. We believe the main ideas in this subsection could be important to obtain other results for similar problems (such as tests for other stochastic orderings). A positive feature of the methodology is that its implementation is extremely simple due to a homogeneity property satisfied by the asymptotic distribution of the statistic. In Subsection 2.2, we prove the strong universal consistency of the test

T_n. Actually, the statistic satisfies a *double consistency* property since, for the HNBUE variables strictly dominated by the exponential distribution, it holds that $T_n \to 0$ in probability, so in this case, asymptotically, the null is never rejected. Further, we show that, for a fixed $X \notin \mathcal{H}$, the probability of accepting H_0 usually tends to zero at an exponential speed. Therefore, the proposed testing procedure is expected to have a good power. Indeed, this is confirmed by the simulation studies of Section 3, where we check the practical finite-sample performance of our proposal. Finally, some real data examples are analyzed in Section 4.

Let us finish this introduction by remarking that all the ideas in this work admit at least two direct extensions. On the one hand, the results can be easily adapted to distributions with the HNWUE (*harmonic new worse than used in expectation*) property, that is, the distributions that dominate an exponential variable in the convex order. Further, observe that the exponential distribution is the only one belonging to both the HNBUE and HNWUE classes. Hence, if for a certain data set we accept the two hypotheses (the HNBUE and HNWUE property), we conclude that the data follow an exponential distribution. In other words, applying the HNBUE test and the HNWUE test we obtain a test for exponentiality. This is illustrated in Section 4. On the other hand, the test statistic in (5) can be modified to deal with censored or truncated data, a common problem in practice. It suffices to use the Kaplan-Meier estimator (see e.g. Lee and Wang (2013) [14]) for the distribution function F and the Gill (1983) [10] estimator for the mean μ, as in Denuit *et al.* (2007) [6, Section 4].

2 Asymptotic behavior of the test statistic

2.1 The case where the hypothesis $\mathbf{H}_0$ is true

In this subsection, we determine the asymptotic distribution of the test statistic T_n defined in (5) under the null hypothesis in (2). Observe that it can be expressed as

$$T_n = \|(\mathbb{H}_n + h_n)_+\|_1, \tag{6}$$

where $\|\cdot\|_1$ stands for the usual norm in $L^1 := L^1([0,\infty))$, $\mathbb{H}_n$ is the stochastic process

$$\mathbb{H}_n(t) := \int_t^\infty \sqrt{n}(G_{\hat{\mu}}(x) - G_\mu(x))\,\mathrm{d}x - \int_t^\infty \sqrt{n}(F_n(x) - F(x))\,\mathrm{d}x, \quad t \geq 0, \tag{7}$$

and h_n is the function

$$h_n(t) := \sqrt{n}\int_t^\infty (G_\mu(x) - F(x))\,\mathrm{d}x. \tag{8}$$

The process $\mathbb{H}_n$ represents the stochastic part of T_n while h_n is deterministic. We note that $\mathbb{H}_n(0) = \mathbb{H}_n(\infty) = 0$, and $\mathbb{H}_n$ has differentiable paths a.e. Hence, $\mathbb{H}_n$ looks like a smooth bridge on $[0,\infty)$ (see Figures 1 (a), 2 (a) and 3 (a)). It is also easy to see that the trajectories of $\mathbb{H}_n$ belong to L^1 (a.s.) if and only if $\mathrm{E}X^2 < \infty$. Consequently, L^1 is the natural space to analyze the asymptotic behavior of $\mathbb{H}_n$. The key to obtain the asymptotic distribution of T_n is to discuss first the asymptotic behavior of $\mathbb{H}_n$ in L^1. This is done in Theorem 4 in the Appendix, which provides the necessary and sufficient integrability condition on X for $\mathbb{H}_n$ to converge in distribution in L^1. Based on this asymptotic result and (6), we obtain the limiting distribution of T_n, which allows us to derive the asymptotic rejection region for the test (2).

The role played by the function h_n in T_n allows detecting whether the HNBUE condition is fulfilled or not, since $h_n(t)$ goes to infinity for all t such that the condition (1) does not hold. Further, if X is strictly under the exponential distribution in the convex order, then $h_n(t) \to -\infty$ for all $t > 0$ and T_n converges to 0, as it is stated in Corollary 2. These two possibilities can be clearly observed in Figures 2 (b) and 3 (b).

We use the notation "$\to_{\mathrm{d}}$" for the usual convergence in distribution of random variables. Here $\mathbb{B}_F := \mathbb{B} \circ F$ stands for the *F-Brownian bridge*, where $\mathbb{B}$ is a standard Brownian bridge on $[0, 1]$.

Theorem 1. *For $X \in \mathcal{H}$, we have*

$$T_n \to_d \tau_F := \int_{I_0} \left(\mathbb{H}_F(t)\right)_+ \, \mathrm{d}t,$$

where $\mathbb{H}_F$ is the centered Gaussian process given by

$$\mathbb{H}_F(t) := \mathbb{I}_F(0)\,(1 + t/\mu)\, e^{-t/\mu} - \mathbb{I}_F(t), \qquad t \geq 0, \tag{9}$$

$\mathbb{I}_F$ is the reverse integrated F-Brownian bridge defined by

$$\mathbb{I}_F(t) := \int_t^\infty \mathbb{B}_F(x)\, \mathrm{d}x, \qquad t \geq 0, \tag{10}$$

and

$$I_0 := \left\{ t \in [0, \infty) : \int_t^\infty (G_\mu(x) - F(x))\, \mathrm{d}x = 0 \right\}. \tag{11}$$

The following two corollaries are a direct consequence of Theorem 1 and play a key role in the implementation of the asymptotic test in practice (see Sections 3 and 4).

Corollary 1. *If X is exponentially distributed with mean μ, then*

$$T_n \to_d \tau(\mu) := \int_0^\infty \left(\mathbb{H}_{G_\mu}(t)\right)_+ \, \mathrm{d}t. \tag{12}$$

Moreover, τ is a homogeneous function of degree 2, i.e., $\tau(\mu) = \mu^2 \tau(1)$.

In Figure 1 (b) we have displayed 30 trajectories of the process $\mathbb{H}_{G_1}$. The similarity between the trajectories of $\mathbb{H}_n$, when X is exponential, and its limit $\mathbb{H}_{G_1}$ is remarkable.

Let us consider the class

$$\mathcal{H}^* := \left\{ X : \int_t^\infty (G_\mu(x) - F(x))\,\mathrm{d}x < 0 \quad \text{a.e. in } [0,\infty) \right\}$$

composed of those random variables with the HNBUE property and strictly dominated by the exponential distribution. All the non-exponential HNBUE distributions used in practice belong to $\mathcal{H}^*$.

Corollary 2. *If $X \in \mathcal{H}^*$, then $T_n \to 0$ in probability.*

2.2 The case where the hypothesis $\mathbf{H}_0$ is false

In this subsection we concentrate on the consistency and power of T_n, when X is *not* HNBUE. In the next result, we show the strong universal consistency of the test statistic T_n.

Theorem 2. *If $X \notin \mathcal{H}$ has finite mean $\mu > 0$, then $T_n \to \infty$ a.s. More precisely:*

(a) There exists a constant $c > 0$ such that $\liminf_{n\to\infty} T_n/\sqrt{n} \geq c$ a.s.

(b) If $\mathrm{E}X^2 > 2\mu^2$, then $\liminf_{n\to\infty} T_n/\sqrt{n} \geq \mathrm{E}X^2/2 - \mu^2$ a.s. In particular, if $\mathrm{E}X^2 = \infty$, we have that $T_n/\sqrt{n} \to \infty$ a.s.

The next result concerns the power of the test. We see that the probability of accepting the null hypothesis for a distribution not belonging to $\mathcal{H}$ is controlled by a parametric part, which is similar to that corresponding to a test of equality of the mean, plus a nonparametric part which is exponentially bounded.

Theorem 3. *Let $X \notin \mathcal{H}$ be fixed. There exist $c_1, c_2, T > 0$ (dependent only on F) such that for all $0 \le t < T$ and $n \ge 1$, it holds*

$$\mathrm{P}\left(T_n \le t\sqrt{n}\right) \le \mathrm{P}\left(|\mu - \hat{\mu}| \ge c_1(T-t)\right) + 2\,\exp\left(-c_2\,(T-t)^2 n\right).$$

Actually, with the help of large deviation techniques, we can show this overall probability usually tends to zero at an exponential speed. We consider the *logarithmic moment generating function* of X, $\Lambda(\lambda) := \log \mathrm{E}e^{\lambda X}$, $\lambda \in \mathbb{R}$, and the *Fenchel-Legendre transform of* $\Lambda(\lambda)$, that is, $\Lambda^*(x) := \sup_{\lambda\in\mathbb{R}}\{\lambda x - \Lambda(\lambda)\}$, $x \in \mathbb{R}$.

The next corollary, which is a large deviation-type result, follows from Theorem 3 and Cramér's theorem (see Dembo and Zeitouni (1998) [5, Theorem 2.2.3 and Remark (c), p.27]).

Corollary 3. *Let $X \notin \mathcal{H}$ be fixed. There exist $c_1, c_2, T > 0$ (dependent only on F) such that, for all $0 \le t < T$ and $n \ge 1$,*

$$\mathrm{P}\left(T_n \le t\sqrt{n}\right) \le 2\,\exp\left(-n \inf_{x\in C} \Lambda^*(x)\right) + 2\,\exp\left(-c_2\,(T-t)^2 n\right),$$

where $C := \{x \in \mathbb{R} : |x-\mu| \ge c_1(T-t)\}$. In particular, we have

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{P}\left(T_n \le t\sqrt{n}\right) \le -\min\left\{\inf_{x\in C} \Lambda^*(x), c(T-t)^2\right\}.$$

3 A Monte Carlo study

The purpose of this section is to illustrate the practical behavior of the HNBUE test based on the statistic T_n defined in (5). The construction of the rejection region for the null hypothesis given in (2) relies on the results stated in Corollaries 1 and 2 in Subsection 2.1. The key idea is that, since the discrepancy measure $\theta(X)$ in (4) is 0 if and only if $X \in \mathcal{H}$, then we should reject this hypothesis if the

α	0.01	0.025	0.05	0.1
$c_{1,\alpha}$	2.307952	1.928978	1.628628	1.292343

Table 1: Approximate $(1-\alpha)$-quantiles of $\tau(1)$

normalized empirical counterpart of $\theta(X)$, T_n, is too large. Specifically, for a fixed significance level $0 < \alpha < 1$, we reject $\mathrm{H}_0 : X \in \mathcal{H}$ if $T_n > c_{\mu,\alpha}$, where $\mu = \mathrm{E}X$ and $c_{\mu,\alpha}$ denotes the $(1-\alpha)$-quantile of the distribution $\tau(\mu)$ defined in (12) (that is, $\mathrm{P}(\tau(\mu) > c_{\mu,\alpha}) = \alpha$).

Since $\tau(\mu) = \mu^2\,\tau(1)$, the rejection region of H_0 in (2) becomes $\{T_n > \mu^2\,c_{1,\alpha}\}$. Thus, carrying out this HNBUE test, which amounts to checking the condition $T_n > \mu^2\,c_{1,\alpha}$, is extremely fast from a computational viewpoint. In practice, the unknown mean μ is approximated by $\hat{\mu}$, the sample mean of X. Due to the involved expression of $\tau(1)$ (see Corollary 1), we have approximated its $(1-\alpha)$-quantile, $c_{1,\alpha}$, by sampling 50000 times from $\tau(1)$ and computing the corresponding sample quantile. In Table 1 we show the approximate values of $c_{1,\alpha}$ for the usual significance levels α.

We have computed the proportion of rejections of the null hypothesis in (2) for different probability distributions. In Tables 2 and 3 we display the results for some HNBUE and non-HNBUE distributions, respectively. The number of Monte Carlo samples is 5000 in all cases. The significance level is fixed as $\alpha = 0.05$. We considered three different values of n: 50, 100 and 200. The distributions considered and their probability densities are:

- the exponential distribution with mean $\mu > 0$:

$$f(x) = \frac{1}{\mu} e^{-x/\mu}, \qquad x > 0;$$

- the Weibull distribution with shape parameter $a > 0$ and scale parameter 1:

$$f(x) = a\, x^{a-1}\, e^{-x^a}, \qquad x > 0;$$

- the gamma distribution with shape parameter $a > 0$ and scale parameter 1:

$$f(x) = \frac{1}{\Gamma(a)}\, x^{a-1}\, e^{-x}, \qquad x > 0;$$

- a mixture of two exponential distributions with means μ_1 and μ_2 and mixing parameter $\pi \in (0,1)$:

$$f(x) = \pi\, \frac{1}{\mu_1} e^{-x/\mu_1} + (1-\pi)\, \frac{1}{\mu_2} e^{-x/\mu_2}, \qquad x > 0;$$

- the linear failure rate (LFR) distribution with shape parameter $a > 0$ and scale parameter 1:

$$f(x) = (1+ax)e^{-(x+ax^2/2)}, \qquad x > 0;$$

- the generalized Pareto distribution with shape parameter $a > 0$ and scale parameter 1:

$$f(x) = (1+ax)^{-(1/a+1)}, \qquad x > 0.$$

In Table 2 we observe that for the exponential distribution, the "boundary" between HNBUE and non-HNBUE distributions, we achieve a proportion of H_0 rejections close to the nominal value $\alpha = 0.05$, even when the sample size is moderately low. For the rest of the three HNBUE distributions, which are strictly dominated by the exponential one, the power is clearly below the significance level and diminishes as n increases, as expected taking into account Corollary 2.

Table 3 shows that any departure from the HNBUE hypothesis automatically translates into an increase of the power with respect to $\alpha = 0.05$. As we could

Distribution	n	Power
Exponential(1)	50	0.0478
	100	0.0510
	200	0.0534
Weibull($a = 1.1\,, 1$)	50	0.0064
	100	0.0046
	200	0.0018
Gamma($a = 1.1\,, 1$)	50	0.0278
	100	0.0204
	200	0.0148
LFR($a = 0.1$)	50	0.0106
	100	0.0072
	200	0.0014

Table 2: Proportion of $H_0 : X \in \mathcal{H}$ rejections for some HNBUE distributions.

expect, the power also increases with the sample size and when the discrepancy of the distribution with respect to the exponential is larger. The proposed test shows a good performance when the sample size is relatively low, although the construction of the rejection region is based on the asymptotic distribution of the test statistic.

4 Analysis of real data sets

In this section, we illustrate the practical implementation of the proposed methodology by discussing the convex domination (with respect to the exponential class) of three sets of real data from very different contexts.

Distribution	n	Power
Weibull($a = 0.9\,, 1$)	50	0.1728
	100	0.2728
	200	0.4492
Weibull($a = 0.7\,, 1$)	50	0.7552
	100	0.9608
	200	0.9986
Gamma($a = 0.9\,, 1$)	50	0.0982
	100	0.1376
	200	0.1930
Gamma($a = 0.7\,, 1$)	50	0.3336
	100	0.5628
	200	0.8104
Mixture of exponentials	50	0.0616
$\mu_1 = 1$, $\mu_2 = 1.5$	100	0.0862
$\pi = 0.9$	200	0.1064
Mixture of exponentials	50	0.1680
$\mu_1 = 1$, $\mu_2 = 2$	100	0.2626
$\pi = 0.8$	200	0.3898
Mixture of exponentials	50	0.4636
$\mu_1 = 1$, $\mu_2 = 3$	100	0.7044
$\pi = 0.7$	200	0.9284
Mixture of exponentials	50	0.0692
$\mu_1 = 1$, $\mu_2 = 0.5$	100	0.0762
$\pi = 0.9$	200	0.1118
Pareto($a = 0.2$)	50	0.3702
	100	0.5918
	200	0.8198
Pareto($a = 0.4$)	50	0.7344
	100	0.9288
	200	0.9964

Table 3: Proportion of $H_0 : X \in \mathcal{H}$ rejections for non-HNBUE distributions.

4.1 *Chandra* Orion Ultradeep Project (COUP)

At X-ray wavelengths of light, young stars may be violently variable on rapid timescales, due to explosive releases of energy at the stellar surface. These stellar flares occur when magnetic fields from the interior of the star erupt on the surface, and plasma trapped in magnetic loops is heated to X-ray emitting temperatures. The Chandra X-ray Observatory (`http://chandra.harvard.edu`), a telescope designed to detect X-ray emission from very hot regions of the universe, provides an excellent record of these flares. In the Chandra Orion Ultradeep Project, the Orion Nebula region of young stars was observed for nearly two weeks continuously in January 2003. The COUP study revealed 1616 individual X-ray sources (Getman *et al.* (2005a) [7]). The analysis of these emissions is important, for instance, to discriminate the nature of the X-ray source (see Getman *et al.* (2005b) [8]).

We first consider the 208 inter-arrival times (in seconds) of the photons in the COUP series 263 (available, for instance, at `http://astrostatistics.psu.edu/datasets/Chandra_flares.html`). This corresponds to a source not exhibiting flares and was classified as extragalactic, since this type of sources are isotropically distributed (Getman *et al.* (2005b) [8]). The sample mean for these data is $\hat{\mu} = 4071.48$ seconds and the value of the test statistic is $T_n = 555527.3$. The resulting p-value of the HNBUE test is 0.781, so we accept that the data follow a HNBUE distribution (which includes the exponential as a boundary case). We have also carried out the analogous testing procedure to determine whether the distribution of photon inter-arrivals dominates the exponential distribution in the convex order. For the test $\mathrm{H}_0 : X$ is HNWUE, we have obtained a p-value of 0.615, so we cannot reject this null hypothesis. Since the exponential is the unique distribution which is HNBUE and HNWUE simultaneously, we conclude that the exponential hypothesis cannot be rejected. Indeed, we have also performed the exponentiality test (3) with $\mathcal{C} = \mathcal{H}$ using the test statistic $\hat{\Delta}_2$ and the rejection region proposed in Berrendero and Cárcamo (2009) [4]. We obtained a p-value of 0.575, which indicates that the inter-arrival times of photons are exponentially distributed (see Figure 4). This agrees with the classification of this COUP data series as an apparently constant X-ray source.

The results are different for the COUP series 551, which corresponds to a faint flaring Orion star. There is a variability in the photon arrival rate: periods with few photons are suddenly interrupted by stellar flares. Here the sample mean is $\hat{\mu} = 1284.25$ seconds. The resulting p-values of the HNBUE and HNWUE tests are less than 0.00002 and 0.984, respectively, so the data are HNWUE, but not HNBUE (see Figure 5). Therefore, they are not exponential.

4.2 Compressor failures

Rausand and Høyland (2004) [18, p. 235], report failure time data for a compressor at a Norwegian process plant. The data were the 90 critical failure times (in operating days) of the compressor from 1968 until 1989 (see Figure 6). The sample mean is $\hat{\mu} = 70.83$ days, the HNBUE test statistic is $T_n = 3091.39$ (p-value < 0.00002) and the value of the HNWUE test statistic is 0.108 (p-value 0.92). Consequently, the data from this second set are strictly HNWUE. This is due to the fact that there is a group of failures that have occurred within short intervals but, for the rest of the data, the time between failures apparently increases with the time in operation.

4.3 Wind speed

In the context of wind energy production, it is necessary to characterize the wind in locations surrounding a wind turbine. This is used to calculate the optimal cut-in and cut-out speed of the turbine and its likely power output. The Weibull distribution is commonly used to fit the probability distribution of the wind speed in a specific location (see, e.g., Seguro and Lambert (2000) [20]). Then the Weibull scale and shape parameters can be interpreted in terms of certain weather or geographical characteristics of the location. For instance, for typical wind speed distributions over a homogeneous terrain the shape parameter usually has a value between 2 and 3. Thus, we would expect the distribution of wind speed to be strictly dominated by the exponential in the convex order.

The Green Grid Report studies the financial viability of installing a wind farm on the Eyre Peninsula (Australia). As part of the study, wind speed data were recorded at several stations of the Australian Bureau of Meteorology (BoM) on the peninsula. We have considered the 2009 hourly wind speeds (in m/s) of the Whyalla Aero BoM station (available at `http://www.oz-energy-analysis.org`). The sample mean wind speed is $\hat{\mu} = 5.094$ m/s and the maximum likelihood estimators (m.l.e.) of the Weibull shape and scale parameters fitted to these data are 1.996 and 5.735 respectively. The HNBUE test statistic is $T_n = 0.01$ (p-value 0.9395) and the HNWUE test statistic is 835.5 (p-value < 0.00002). Consequently, the distribution of these wind speed data is strictly dominated in the convex order by the exponential distribution, as Figure 7 clearly reflects.

5 Appendix: Proofs and technical results

Let us introduce some notation. In the sequel, $\mathcal{L}^p := \{X : \mathrm{E}X^p < \infty\}$ $(p > 0)$ is the usual $\mathcal{L}^p$ space. Also, we consider the *Lorenz space*

$$\mathcal{L}^{4,2} := \{X : \Lambda_{4,2}(X) < \infty\},$$

where

$$\Lambda_{4,2}(X) := \int_0^\infty t\sqrt{\mathrm{P}(X > t)}\,\mathrm{d}t$$

(see Ledoux and Talagrand (2002) [13, p. 279]). It can be shown that, for all $\epsilon > 0$, $\mathcal{L}^{4+\epsilon} \subset \mathcal{L}^{4,2} \subset \mathcal{L}^4$ (see for instance Grafakos (2008) [9, Section 1.4]). Therefore, the condition $\Lambda_{4,2}(X) < \infty$ is slightly stronger than $\mathrm{E}X^4 < \infty$.

To obtain the limiting distribution of $\mathbb{H}_n$ defined in (7), note that $\mathbb{H}_n = \mathbb{G}_n - \mathbb{I}_n$, where

$$\begin{aligned} \mathbb{G}_n(t) &:= \int_t^\infty \sqrt{n}(G_{\hat\mu}(x) - G_\mu(x))\,\mathrm{d}x, \\ \mathbb{I}_n(t) &:= \int_t^\infty \sqrt{n}(F_n(x) - F(x))\,\mathrm{d}x. \end{aligned} \tag{13}$$

The asymptotic behavior in L^1 of $\mathbb{G}_n$ defined in (13) is clarified in Lemma 1.

Lemma 1. *Assume that $X \in \mathcal{L}^{4/3}$ and let us consider the process*

$$\widetilde{\mathbb{G}}_n(t) := \sqrt{n}(\mu - \hat\mu)\,(1 + t/\mu)\,e^{-t/\mu}, \qquad t \geq 0. \tag{14}$$

The processes $\mathbb{G}_n$ and $\widetilde{\mathbb{G}}_n$ are a.s. asymptotically equivalent in L^1, that is, $\|\mathbb{G}_n - \widetilde{\mathbb{G}}_n\|_1 \to 0$, a.s., as $n \to \infty$.

Proof. By the mean value theorem, we can write

$$\mathbb{G}_n(t) = \sqrt{n}(\mu - \hat\mu)\,(1 + t/\mu_t)\,e^{-t/\mu_t}, \tag{15}$$

where μ_t is a point between μ and $\hat{\mu}$. From (15) and using again the mean value theorem, we get a.s.

$$\begin{aligned}\|\mathbb{G}_n - \widetilde{\mathbb{G}}_n\|_1 &= \sqrt{n}|\mu - \hat{\mu}| \int_0^\infty \left|(1 + t/\mu)\, e^{-t/\mu} - (1 + t/\mu_t)\, e^{-t/\mu_t}\right| \mathrm{d}t \\ &\le \sqrt{n}(\mu - \hat{\mu})^2 \int_0^\infty \frac{t^2}{\xi_t^3} e^{-t/\xi_t}\, \mathrm{d}t,\end{aligned} \tag{16}$$

where ξ_t is another point between μ and $\hat{\mu}$. Now, we have

$$\begin{aligned}\int_0^\infty \frac{t^2}{\xi_t^3} e^{-t/\xi_t}\, \mathrm{d}t &\le \frac{1}{\min\{\mu, \hat{\mu}\}^3} \int_0^\infty t^2 e^{-t/\max\{\mu,\hat{\mu}\}}\, \mathrm{d}t \\ &= 2\left(\frac{\max\{\mu, \hat{\mu}\}}{\min\{\mu, \hat{\mu}\}}\right)^3.\end{aligned} \tag{17}$$

Since $\hat{\mu} \to \mu$ a.s., we conclude that the bound of the integral in (17) converges to 2 a.s. Finally, the conclusion follows by (16) and the Kolmogorov, Marcinkiewicz and Zygmund strong law of large numbers (see Kallemberg (1997) [11, Theorem 3.23]), as

$$\sqrt{n}(\mu - \hat{\mu})^2 = \left(\frac{1}{n^{3/4}} \sum_{i=1}^n (X_i - \mu)\right)^2 \to 0 \qquad \text{a.s.}$$

whenever $\mathrm{E}|X|^{4/3} < \infty$. □

Next we consider the *empirical process* associated to X, that is

$$\mathbb{E}_n(t) := \sqrt{n}(F_n(t) - F(t)), \qquad t \ge 0, \quad n \ge 1.$$

Lemma 2. *It holds that $\mathbb{E}_n \to_w \mathbb{B}_F$ in W^1 if and only if $X \in \mathcal{L}^{4,2}$, where*

$$W^1 := \left\{ f \in L^1 : \|f\|_{W^1} := \int_0^\infty (1 + t)|f(t)|\, \mathrm{d}t < \infty \right\}. \tag{18}$$

Proof. First note that $\mathbb{E}_n = \sum_{i=1}^n \mathbb{X}_i/\sqrt{n}$, where $\mathbb{X}_1, \ldots, \mathbb{X}_n$ are independent copies of the process $\mathbb{X}(t) = \mathrm{P}(X > t) - I_{\{X>t\}}$, $t \ge 0$. We observe that $W^1 = L^1([0, \infty), \mathcal{A}, \mu)$, with $\mathrm{d}\mu(t) = (1 + t)\mathrm{d}t$, and $\mathcal{A}$ the Lebesgue σ-algebra on $[0, \infty)$, and that $([0, \infty), \mathcal{A}, \mu)$ is σ-finite. Then, we conclude that $\mathbb{X}$ satisfies the CLT in

W^1 if and only if $\int_0^\infty \sqrt{\mathrm{E}\mathbb{X}(t)^2}\,\mathrm{d}\mu(t) < \infty$ (see Araujo and Giné (1980) [1, Exercise 14, p. 205]). It is easy to check that this integrability condition amounts to $\Lambda_{4,2}(X) < \infty$. Finally, the limiting Gaussian process of $\mathbb{E}_n$ is $\mathbb{B}_F$ because they have the same covariance function. $\square$

We are ready to state and prove the key asymptotic result of this work in which "$\to_{\mathrm{w}}$ in L^1" stands for the weak convergence in L^1.

Theorem 4. *The following assertions are equivalent:*

(a) $X \in \mathcal{L}^{4,2}$.

(b) $\mathbb{H}_n \to_w \mathbb{H}_F$ *in* L^1*, where the process* $\mathbb{H}_F$ *is defined in (9).*

Proof. Assume (a) is satisfied. By Lemma 1, and taking into account van der Vaart (1998) [22, Theorem 18.10], the asymptotic distribution in L^1 of $\mathbb{H}_n$ is the same of that of the process $\widetilde{\mathbb{H}}_n = \widetilde{\mathbb{G}}_n - \mathbb{I}_n$, where $\widetilde{\mathbb{G}}_n$ and $\mathbb{I}_n$ are defined in (14) and (13), respectively. Moreover, observe that $\widetilde{\mathbb{H}}_n = \rho(\mathbb{E}_n)$, where $\mathbb{E}_n$ is the empirical process and ρ is the functional defined by

$$\rho(f)(t) = (1 + t/\mu)\, e^{-t/\mu} \int_0^\infty f(x)\,\mathrm{d}x - \int_t^\infty f(x)\,\mathrm{d}x, \qquad t \geq 0. \tag{19}$$

It is readily checked that ρ is a continuous mapping from W^1 onto L^1. Therefore, Lemma 2 joint with the continuous mapping theorem (see for instance Van der Vaart (1998) [22, Theorem 18.11]) imply that $\widetilde{\mathbb{H}}_n = \rho(\mathbb{E}_n) \to_{\mathrm{w}} \rho(\mathbb{B}_F) = \mathbb{H}_F$ in L^1, and, consequently $\mathbb{H}_n \to_{\mathrm{w}} \mathbb{H}_F$ in L^1, and (b) follows.

Conversely, let us assume that (b) holds. This implies that $X \in \mathcal{L}^2$ since this is the necessary and sufficient condition for $\mathbb{H}_n$ to have its trajectories in L^1 a.s. Further, $\widetilde{\mathbb{G}}_n$ can be expressed as a normalized sum in the following way

$$\widetilde{\mathbb{G}}_n(t) = 1/\sqrt{n} \sum_{i=1}^n (1 + t/\mu)\, e^{-t/\mu}(\mu - X_i). \tag{20}$$

Using the equality $\mathrm{E}_{F_Y}(Y-t)_+ = \int_t^\infty (1-F_Y(x))\,\mathrm{d}x$, which is satisfied for any integrable variable Y with distribution function F_Y, we have

$$\begin{aligned}\mathbb{I}_n(t) &= \sqrt{n}\,[\mathrm{E}_F(X-t)_+ - \mathrm{E}_{F_n}(X-t)_+] \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^n [\mathrm{E}(X_i-t)_+ - (X_i-t)_+]\,.\end{aligned} \tag{21}$$

From (20) and (21), we obtain $\widetilde{\mathbb{H}}_n = \sum_{i=1}^n \mathbb{Y}_i/\sqrt{n}$, where the $\mathbb{Y}_i$ $(i=1,\ldots,n)$ are n independent copies of the centered process

$$\mathbb{Y}(t) := (1+t/\mu)\,e^{-t/\mu}(\mu - X) + (X-t)_+ - \mathrm{E}(X-t)_+, \qquad t \ge 0. \tag{22}$$

By hypothesis (and Lemma 1), we have that $\widetilde{\mathbb{H}}_n \to_{\mathrm{w}} \mathbb{H}_F$ in L^1. Hence, we have

$$\int_0^\infty \sqrt{\mathrm{E}\mathbb{Y}(t)^2}\,\mathrm{d}t < \infty \tag{23}$$

and $\mathbb{H}_F$ is a centered Gaussian process (see Araujo and Giné (1980) [1, Exercise 14, p. 205]). Finally, from (22), Cauchy's inequality, and taking the square root, for $t \ge 0$ we obtain

$$\sqrt{\mathrm{E}(X-t)_+^2} \le \sqrt{\mathrm{E}\mathbb{Y}(t)^2} + \mathrm{E}(X-t)_+ + \sqrt{2\mathrm{E}X^2(1+t/\mu)}\,e^{-t/(2\mu)}.$$

Taking into account (23) and the fact that $\int_0^\infty \mathrm{E}(X-t)_+\,\mathrm{d}t = \mathrm{E}X^2/2 < \infty$, we conclude that the function $\sqrt{\mathrm{E}(X-t)_+^2}$, for $t \ge 0$, belongs to L^1. Finally,

$$t^2\,\mathrm{P}(X > 2t) \le \int_{\{X-t>t\}} (X-t)^2\,\mathrm{dP} \le \mathrm{E}(X-t)_+^2,$$

and therefore $\Lambda_{4,2}(X) < \infty$. Thus, $X \in \mathcal{L}^{4,2}$ and the proof is complete. □

Remark 1. By del Barrio *et al.* (1999) [3, Theorem 2.1], $\mathbb{E}_n \to_{\mathrm{w}} \mathbb{B}_F$ in L^1 if and only if $\Lambda_{2,1}(X) := \int_0^\infty \sqrt{\mathrm{P}(X>t)}\,\mathrm{d}t < \infty$. However, this result cannot be directly applied to derive the limiting distribution of $\mathbb{H}_n$ since the mapping ρ defined in

(19) is *not* continuous from L^1 to L^1. Note also that the constant 1 in the weight function $1+t$ that defines W^1 in (18) is necessary to ensure that the convergence in W^1 implies the convergence in L^1.

Lemma 3. *Let $X \in \mathcal{H}$. For $n \geq 1$, consider the sequence of functionals $\xi_n : L^1 \longrightarrow \mathbb{R}$ given by*

$$\xi_n(f) := \int_0^\infty (f + h_n)_+, \tag{24}$$

where the function h_n (dependent on X) is defined in (8). If $f_n \to f$ in L^1, then $\xi_n(f_n) \to \xi_0(f)$, where $\xi_0(f) := \int_{I_0} f_+$ and the set I_0 is defined in (11).

Proof. Since $h_n \equiv 0$ on I_0, we have

$$\begin{aligned} |\xi_n(f_n) - \xi_0(f)| &\leq \int_{I_0} |f_n - f| + \int_{I_0^c} (f_n + h_n)_+ \\ &\leq \int_{I_0} |f_n - f| + \int_{I_0^c} (f_n - f)_+ + \int_{I_0^c} (f + h_n)_+ \\ &\leq \int_0^\infty |f_n - f| + \int_{I_0^c} (f + h_n)_+. \end{aligned} \tag{25}$$

The first integral in (25) goes to 0 because $f_n \to f$ in L^1. Also, taking into account that $X \in \mathcal{H}$, we have that $(f + h_n)_+$ decreases to 0 on the set I_0^c (as n increases to infinity). Therefore, by the monotone convergence theorem, the second integral in (25) converges to 0, and we conclude that $|\xi_n(f_n) - \xi_0(f)| \to 0$ (as $n \to \infty$). □

Proof of Theorem 1. Note that if $X \in \mathcal{H}$, then $X \in \mathcal{L}^{4,2}$. Observe also that $T_n = \xi_n(\mathbb{H}_n)$, with $\mathbb{H}_n$ and ξ_n defined in (7) and (24), respectively. Therefore, using Theorem 4, Lemma 3 and an extended version of the continuous mapping theorem (see Van der Vaart (1998) [22, Theorem 18.11]), we obtain $T_n \to_{\mathrm{d}} \xi_0(\mathbb{H}_F) = \tau_F$. □

Remark 2. If $X \in \mathcal{H}$, it can be checked that $I_0 \subseteq \{t \in [0, \infty) : F(t) = G_\mu(t)\}$, the set of *crossing points* of the two distribution functions.

In the following lemma $\|\cdot\|_\infty$ stands for the supremum norm on $[0,\infty)$.

Lemma 4. *Let us consider $X \notin \mathcal{H}$ with finite mean $\mu > 0$. There exist positive constants $a, b, c > 0$ such that*

$$T_n \geq c\sqrt{n} - (A_n + B_n),$$

where $A_n := a\sqrt{n}\,|\mu - \hat{\mu}|$ and $B_n := b\sqrt{n}\|F_n - F\|_\infty$.

Proof. We rewrite T_n in terms of integrals on $[0,t]$ in the following way:

$$T_n = \int_0^\infty (\mathbb{G}_n^r + \mathbb{I}_n^r + h_n^r)_+, \tag{26}$$

where, for $t \geq 0$,

$$\begin{aligned}
\mathbb{G}_n^r(t) &:= \int_0^t \sqrt{n}(G_\mu(x) - G_{\hat{\mu}}(x))\,\mathrm{d}x, \\
\mathbb{I}_n^r(t) &:= \int_0^t \sqrt{n}(F_n(x) - F(x))\,\mathrm{d}x, \\
h_n^r(t) &:= \int_0^t \sqrt{n}(F(x) - G_\mu(x))\,\mathrm{d}x.
\end{aligned} \tag{27}$$

If $X \notin \mathcal{H}$, there exists $t_0 > 0$ such that $\int_0^{t_0}(F(x) - G_\mu(x))\,\mathrm{d}x > 0$. By continuity, there exist $\epsilon, \delta > 0$ such that

$$\int_0^t (F(x) - G_\mu(x))\,\mathrm{d}x > \epsilon, \qquad \text{for all } t \in I := (t_0, t_0 + \delta). \tag{28}$$

From (26), (27) and (28), we have

$$\begin{aligned}
T_n &\geq \int_I h_n^r - \int_I |\mathbb{G}_n^r| - \int_I |\mathbb{I}_n^r| \\
&\geq \epsilon\delta\sqrt{n} - \int_I |\mathbb{G}_n^r| - \int_I |\mathbb{I}_n^r|.
\end{aligned} \tag{29}$$

Therefore, we set $c := \epsilon\delta$. It is easy to check that there exists a point μ_t between

μ and $\hat{\mu}$ such that

$$\begin{aligned}\int_I |\mathbb{G}_n^r| &= \int_I \sqrt{n}\left|\hat{\mu}(1-e^{-t/\hat{\mu}}) - \mu(1-e^{-t/\mu})\right| \,\mathrm{d}t \\ &= \sqrt{n}|\hat{\mu}-\mu| \int_I \left[1-e^{-t/\mu_t}(1+t/\mu_t)\right] \,\mathrm{d}t \\ &\le \delta\sqrt{n}|\hat{\mu}-\mu|.\end{aligned} \tag{30}$$

Hence, we set $a := \delta$. Finally, using Fubini's theorem

$$\begin{aligned}\int_I |\mathbb{I}_n^r| &\le \int_I \int_0^t \sqrt{n}|F_n(x)-F(x)| \,\mathrm{d}x \,\mathrm{d}t \\ &\le \delta \int_0^{t_0+\delta} \sqrt{n}|F_n(x)-F(x)| \,\mathrm{d}x \\ &\le \delta(t_0+\delta)\sqrt{n}\|F_n - F\|_\infty.\end{aligned} \tag{31}$$

Setting $b := \delta(t_0+\delta)$, the conclusion of the lemma follows by (29), (30) and (31). □

Lemma 5. *The functional $\theta(X)$ defined in (4) satisfies*

$$|\theta(X) - \mathrm{E}X^2/2| \le \mu^2. \tag{32}$$

In particular,

$$\left|\frac{T_n}{\sqrt{n}} - \frac{1}{2n}\sum_{i=1}^n X_i^2\right| \le \hat{\mu}^2. \tag{33}$$

Proof. We first have

$$\begin{aligned}\theta(X) &\le \int_0^\infty \int_t^\infty (1-F(x)) \,\mathrm{d}x \,\mathrm{d}t + \int_0^\infty \int_t^\infty (1-G_\mu(x)) \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_0^\infty x(1-F(x)) \,\mathrm{d}x + \int_0^\infty x(1-G_\mu(x)) \,\mathrm{d}x \\ &= \frac{\mathrm{E}X^2}{2} + \mu^2.\end{aligned}$$

To check the other inequality in (32), it is enough to use the fact that $a \le a_+$. Finally, (33) is a direct consequence of (32) since $T_n/\sqrt{n}$ is the empirical counterpart of $\theta(X)$. □

Proof of Theorem 2. Using Lemma 4 and its notation, we have that $T_n \geq \sqrt{n}C_n$, where $C_n := c - a\,|\mu - \hat{\mu}| - b\,\|F_n - F\|_\infty$. By the strong law of large numbers and Glivenko-Cantelli theorem, we have that $C_n \to c$ a.s. and part (a) holds.

Finally, from (33), we obtain

$$T_n \geq \sqrt{n}\left(\frac{1}{2n}\sum_{i=1}^{n} X_i^2 - \hat{\mu}^2\right).$$

Therefore, part (b) follows from the strong law of large numbers. □

Proof of Theorem 3. Following the same notation as in Lemma 4, we select $T := c > 0$, and for $0 \leq t < T$, we have

$$\begin{aligned} \mathrm{P}\left(T_n \leq t\sqrt{n}\right) &\leq \mathrm{P}\left(A_n + B_n \geq (T-t)\sqrt{n}\right) \\ &= \mathrm{P}\left(|\mu - \hat{\mu}| \geq \frac{T-t}{2a}\right) + \mathrm{P}\left(\|F_n - F\|_\infty \geq \frac{T-t}{2b}\right). \end{aligned}$$

Finally, the conclusion follows by the Dvoretsky-Kiefer-Wolfowitz inequality (see Massart (1990) [15]). □

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References

[1] Araujo, A. and Giné, E. (1980). *The Central Limit Theorem for Real and Banach Valued Random Variables.* Wiley.

[2] Barlow, R.E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing: Probability Models*. Holt, Rinehart and Winston.

[3] del Barrio, E., Giné, E. and Matrán, C. (1999). Central limit theorems for the Wasserstein distance between the empirical and the true distributions. *The Annals of Probability*, 27, 1009–1071.

[4] Berrendero, J.R. and Cárcamo, J. (2009). Characterizations of exponentiality within the HNBUE class and related tests. *Journal of Statistical Planning and Inference*, 139, 2399–2406.

[5] Dembo, A. and Zeitouni, O. (1998). *Large Deviations Techniques and Applications*. 2nd Edition. Springer.

[6] Denuit, M., Goderniaux, A.-C. and Scaillet, O. (2007). A Kolmogorov-Smirnov-type test for shortfall dominance against parametric alternatives. *Technometrics*, 49, 88–98.

[7] Getman, K.V., Flaccomio, E., Broos, P.S., Grosso, N., Tsujimoto, M., Townsley, L., Garmire, G.P., Kastner, J., Li, J., Harnden, F.R., Wolk, S., Murray, S.S., Lada, C.J., Muench, A.A., McCaughrean, M.J., Meeus, G., Damiani, F., Micela, G., Sciortino, S., Bally, J., Hillenbrand, L.A., Herbst, W., Preibisch, T. and Feigelson, E.D. (2005a). Chandra Orion Ultradeep Project: Observations and source lists. *The Astrophysical Journal Supplement Series*, 160, 319–352.

[8] Getman, K.V., Feigelson, E.D., Grosso, N., McCaughrean, M.J., Micela, G., Broos, P. Garmire, G. and Townsley, L. (2005b). Membership of the Orion

Nebula population from the Chandra Orion Ultradeep Project. *The Astrophysical Journal Supplement Series*, 160, 353–378.

[9] Grafakos, L. (2008). *Classical Fourier Analysis.* Springer.

[10] Gill, R. (1983). Large sample behaviour of the product-limit estimator on the whole line. *The Annals of Statistics*, 11, 49–58.

[11] Kallemberg, O. (1997). *Foundations of Modern Probability.* Springer.

[12] Lai, C.-D. and Xie, M. (2006). *Stochastic Ageing and Dependence for Reliability.* Springer.

[13] Ledoux, M. and Talagrand, M. (2002). *Probability in Banach Spaces.* Springer.

[14] Lee, E.T. and Wang, J.W. (2013). *Statistical Methods for Survival Data Analysis.* Wiley.

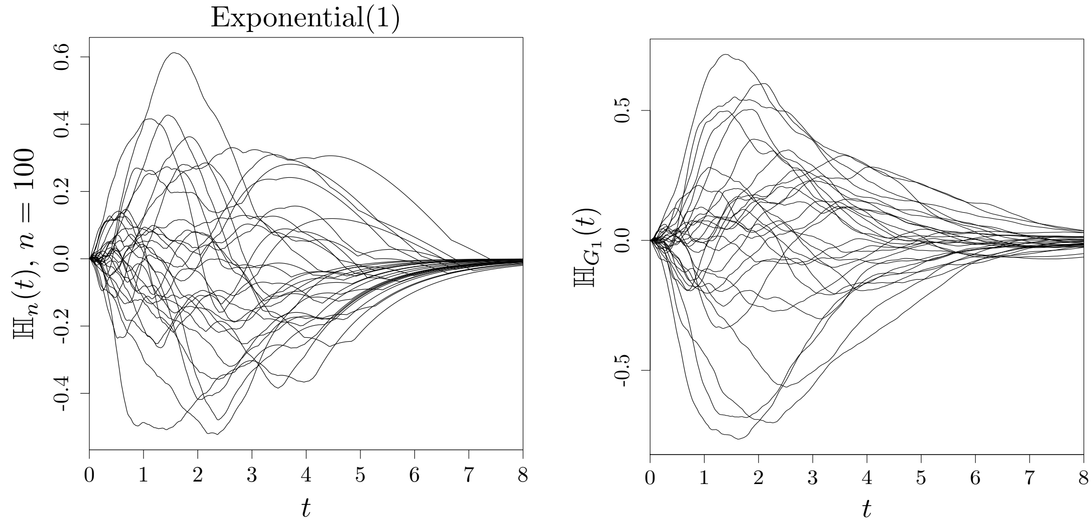
[15] Massart, P. (1990). The tight constant in the Dvoretsky-Kiefer-Wolfowitz inequality. *The Annals of Probability*, 18, 1269–1283.

[16] Müller, A. and Stoyan, D. (2002). *Comparison Methods for Stochastic Models and Risks.* Wiley.

[17] Nikitin, Y. (1995). *Asymptotic Efficiency of Nonparametric Tests.* Cambridge University Press.

[18] Rausand, M. and Høyland, A. (2004). *System Reliability Theory: Models, Statistical Methods, and Applications.* Wiley.

[19] Scaillet, O. (2005). A Kolmogorov-Smirnov type test for positive quadrant dependence. *The Canadian Journal of Statistics*, 33, 415–427.



(a) (b)

Figure 1: 30 trajectories of the processes (a) $\mathbb{H}_n$, with $n = 100$, and (b) the corresponding limit $\mathbb{H}_{G_1}$.

[20] Seguro, J.V. and Lambert, T.W. (2000). Modern estimation of the parameters of the Weibull wind speed distribution for wind energy analysis. *Journal of Wind Engineering and Industrial Aerodynamics*, 85, 75–84.

[21] Shaked, M. and Shanthikumar, J.G. (2006). *Stochastic Orders*. Springer.

[22] van der Vaart, A. (1998). *Asymptotic Statistics*. Cambridge University Press.

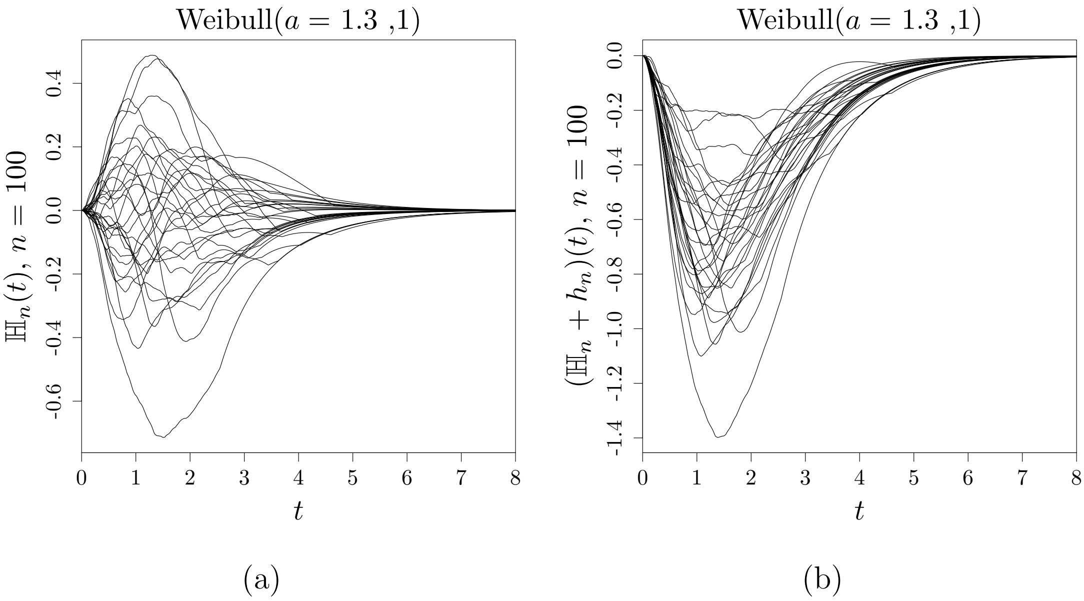


Figure 2: 30 trajectories of the processes (a) $\mathbb{H}_n$ and (b) $\mathbb{H}_n + h_n$, with $n = 100$, for the HNBUE distribution Weibull ($a = 1.3$, 1).

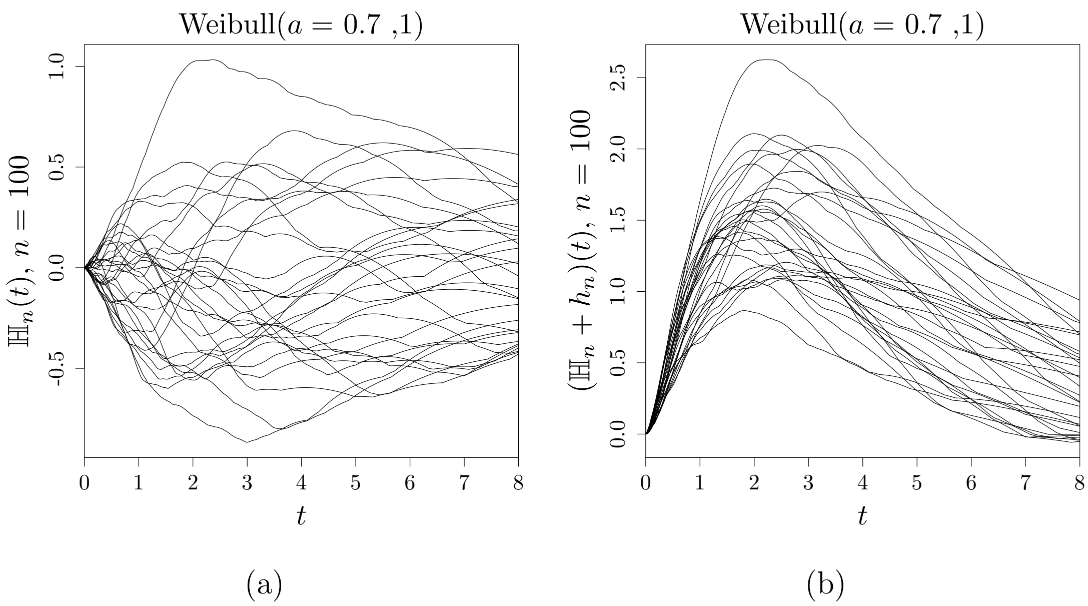


Figure 3: 30 trajectories of the processes (a) $\mathbb{H}_n$ and (b) $\mathbb{H}_n + h_n$, with $n = 100$, for the non-HNBUE distribution Weibull ($a = 0.7$, 1).

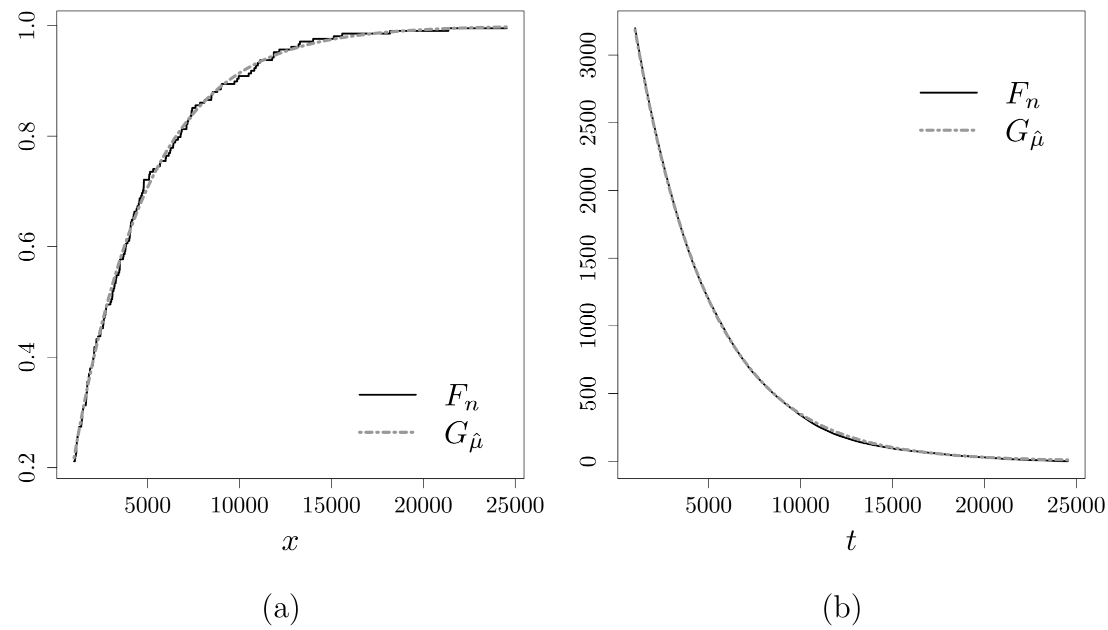
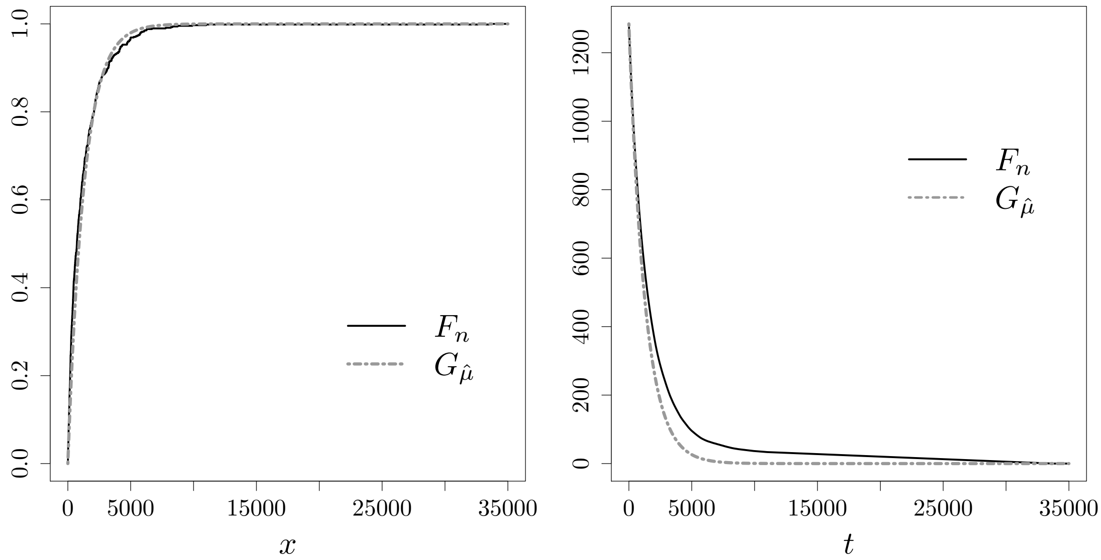
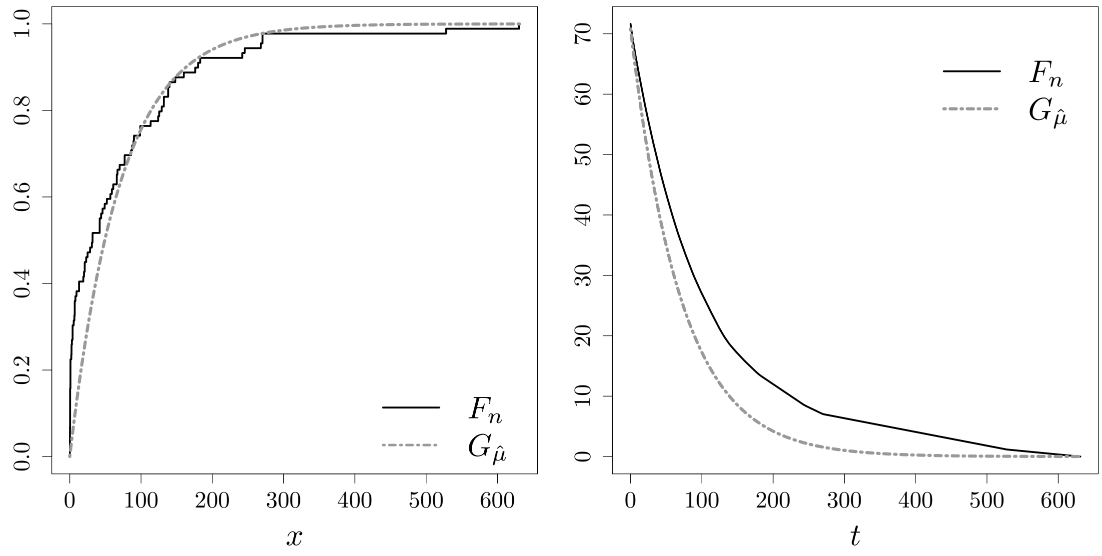


Figure 4: Plots for inter-arrival times of photons in the COUP 263 data: (a) empirical and exponential $\exp(\hat{\mu})$ distribution functions; (b) the corresponding reverse integrated survival functions $\int_t^\infty (1 - \hat{F}(x))\,\mathrm{d}x$, for $\hat{F} = F_n$ and $\hat{F} = G_{\hat{\mu}}$.



(a) (b)

Figure 5: Plots for inter-arrival times of photons in the COUP 551 data: (a) empirical and exponential $\exp(\hat{\mu})$ distribution functions; (b) the corresponding reverse integrated survival functions $\int_t^{\infty}(1 - \hat{F}(x))\,\mathrm{d}x$, for $\hat{F} = F_n$ and $\hat{F} = G_{\hat{\mu}}$.



(a) (b)

Figure 6: Plots for the times between failures of a compressor in a Norwegian process plant: (a) empirical and exponential $\exp(\hat{\mu})$ distribution functions; (b) the corresponding reverse integrated survival functions $\int_t^\infty (1 - \hat{F}(x))\,\mathrm{d}x$, for $\hat{F} = F_n$ and $\hat{F} = G_{\hat{\mu}}$.

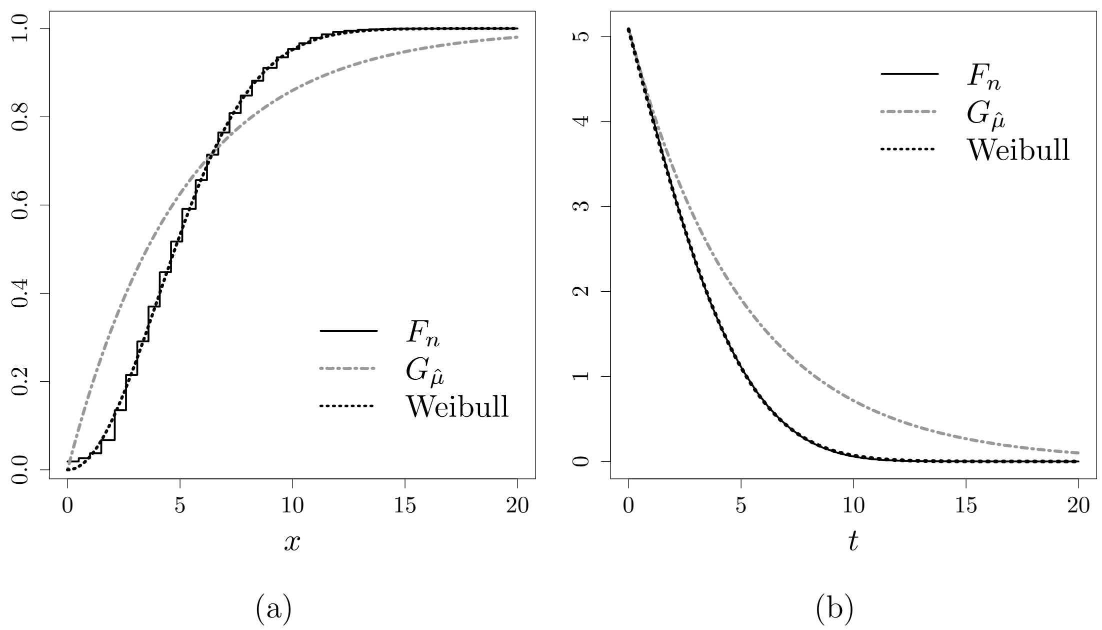


Figure 7: Plots for the wind speeds in the Eyre Peninsula: (a) empirical, exponential $\exp(\hat{\mu})$ and Weibull (with m.l.e. parameters) distribution functions; (b) the reverse integrated survival functions $\int_t^\infty (1 - \hat{F}(x))\,\mathrm{d}x$ corresponding to the estimated distribution functions $\hat{F}$ plotted in (a).