From F-theory to brane webs: Non-perturbative effects in type IIB String Theory

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He asked himself
what is a woman standing on
the stairs in the shadow,
listening to distant music,
a symbol of.

J. Joyce $\sim$ The Dead

ABSTRACT

We analyse the flavour sector of $SU(5)$ Grand Unified Theories in F–theory. Two classes of local models are formulated, one with enhancement to E_6 where the masses of the up–type quarks are generated, and one with enhancement to either E_7 or E_8 where the masses for all fermions of the Standard Model are generated. A full rank 3 Yukawa matrix is attained only after the inclusion of non–perturbative effects in the compactification space. By performing a scan over the parameters defining the local models we check whether realistic masses for the fermions may be attained.

Secondly we present two example of the appearance of linear equivalence between cycles in D–brane models. In the first case we show how linear equivalence is tied with kinetic mixing between open and closed string massless $U(1)$'s and discuss potential phenomenological implications for dark matter and unification of gauge couplings. Secondly we show how taking into account the coupling with closed string moduli some of the brane moduli may acquire a mass. We clarify the microscopic origin of this effect and its connection with linear equivalence of cycles, and finally match it with the 4d supergravity description.

Finally we discuss the application of topological string techniques for the computation of the Nekrasov partition function for theories in the Higgs branch. We formulate a general algorithm for the computation of the Nekrasov partition function of the 5d T_N theory in a generic point of the Higgs branch. Afterwards we present a generalisation of the topological vertex applicable to a wide class of non–toric varieties. In both cases we provide some explicit examples of the application of the new rules formulated.

RESUMEN

Se analiza el sector de sabor en teorías $SU(5)$ de Gran Unificación en Teoría F. Se construyen dos clases de modelos locales, una con aumento del grupo gauge a E_6 donde se generan las masas de los quarks de tipo up, y una con aumento del grupo gauge a E_7 o E_8 donde se generan las masas de todos los fermiones del Modelo Estándar. Solamente después de haver incluido efectos no perturbativos en el espacio de compactificación se consigue una matriz de Yukawa de rango 3. Haciendo una búsqueda en los valores de los parámetros que definen los modelos locales se comprueba si es posible conseguir masas realistas para los fermiones.

En segundo lugar se presentan dos ejemplos de cómo la equivalencia lineal entre ciclos aparece en modelos de D–branas. En el primer caso se demuestra cómo la equivalencia lineal está conectada con la mezcla cinética entre $U(1)$'s sin masa de cuerda abierta y cerrada y se discuten implicaciones fenomenológicas para materia oscura y unificación de los acoplos de gauge. Después se demuestra cómo algunos de los módulos de brana reciben masa al tener en cuenta el acoplo con los módulos

de cuerda cerrada. Se aclara el origen microscópico de este efecto y su conexión con la equivalencia lineal de ciclos, comparandolo por último con la descripción en supergravedad en 4d.

Finalmente se discute la aplicación de técnicas de cuerda topológica para el cálculo de funciones de partición de Nekrasov para teoría en la rama de Higgs. Se formula un algoritmo general para el cálculo de la función de partición de Nekrasov de la teoría T_N en 5d en un punto genérico de la rama de Higgs. Después se presenta una generalización del vértice topológico que se puede aplicar a una amplia clase de variedades no tóricas. En ambos casos se presentan algunos ejemplos de la aplicación de las nuevas reglas que se han formulado.

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CONTENTS

1 INTRODUCTION

It is a distressing idea that for any theoretical physicist the least understood force is the one we are most acquainted with, namely gravity. Even if one entire century has passed since the introduction of General Relativity which provides a classical description of gravitational forces, it has not been possible to reconcile with the other pillar on which modern physics is founded, namely Quantum Mechanics. Both theories have been immensely successful, although at diametrically opposite lengths: Quantum Mechanics dwells in the realm of elementary particles at atomic and subatomic distances, whereas General Relativity has its realm in astrophysical distances where gravity dominates. The chief obstacle encountered when trying to apply the canonical quantisation techniques to General Relativity is non renormalisabilty of this theory: this implies that General Relativity is an incomplete theory that may work only at energy scales well below the Planck energy. What lies beyond this energy scale and how General Relativity ought to be completed remains an obscure problem. Given these conspicuous difficulties met in the attempts of reconciling these two fundamental theories it is even surprising that a candidate theory capable of solving all issues exists, this *panacea* being String Theory.

String Theory after its first introduction as a possible candidate for the description of strong interactions subsequently showed its potential as a theory of unification of all forces and matter. It is remarkable that such a simple idea as having the fundamental constituents being tiny strings instead of pointlike particles has such profound consequences: gravitational forces first of all, which are always present and need not to be included by hand, absence of free parameters and many others. And, quite non trivially, String Theory is a theory consistent with quantum mechanics: this implies that String Theory *is* a consistent theory of quantum gravity and, so far, the only one we know sufficiently well. There are some drawbacks of course, one being the fact that, given its formulation as a perturbative theory, String Theory is a consistent theory of quantum gravity in perturbation theory missing the full details of a non–perturbative theory of quantum gravity. While there are strong hints that the theory exists beyond perturbation theory, like the existence of strong–weak dualities, the existence of D–branes and holographic dualities[1], we still are far away from a complete understanding of String Theory beyond its original formulation.

Internal consistency places significant constraints on String Theory, the most celebrated one being that strings will need to propagate on a spacetime with 10 dimensions. The number of consistent theories that it is possible to construct is also limited to five. While no one among these five stands as the "best" String Theory and a choice among them can be made only according to the particular applications one has in mind, it emerged during the final years of the last century that an intricate

1 AdS/CFT in fact provides a non–perturbative formulation of quantum gravity, at least in Anti–de Sitter space. The downside is that a non–perturbative understanding of conformal field theories is necessary and sometimes this is not available.

web of dualities exists among the different String Theories. By employing dualities it has been possible to shed some light in situations that were outside our range of computational possibilities. We already mentioned AdS/CFT as a neat example, and we add to the list strong–weak dualities which have played a crucial rôle in the development of String Theory in the last 20 years. One pivotal actor in this play is clearly the so–called M–theory, which we know how to formulate in terms of eleven dimensional supergravity and branes.

Applications of String Theory during the years have been disparate, and in the following we shall give a brief account of two complementary approaches.

PHENOMENOLOGY & STRING THEORY

The first question that naturally arises after thinking of String Theory as a possible candidate for a fundamental theory of nature is whether it is possible for it to reproduce all phenomena observed in nature. Being strings below the level of elementary particles it is desirable to ask whether String Theory can be a suitable theory of particle physics. In addition to this, being String Theory a theory of quantum gravity, it is natural to study what are the consequences of String Theory for the origin of the Universe with a particular emphasis on the inflationary epoch. The field that tries to address these questions has been called String Phenomenology and in the years it has provided numerous ways of building models of particle physics and inflation. Still, a complete construction capable of providing a realistic description of nature as we observe it is not available.

The main issue that one faces when trying to build a model of particle physics (or inflation) in String Theory is the humongous number of solutions of String Theory. The recipe to cook up a String Theory solution is relatively straightforward: pick up a compactification space (possibly with branes) and then stabilise all moduli. The necessity of choosing a compactification space is forced upon us to reconcile the fact that String Theory is formulated in ten spacetime dimensions with the fact that spacetime with only four macroscopical dimensions are observed. An important consequence of the compactification process is that in the field theory there will appear several massless scalar fields which control the geometry of the internal space. The presence of these massless scalar fields gives severe problems with lack of observation of fifth forces, problems which may be solved by giving a large mass to the scalar fields. The main issue we face is that the amount of arbitrariness in each step necessary in building a String Theory solution is staggering. The choice of a compactification space does not seem to have any guiding principle, and according to the choice of the compactification space the physics of the solution can be completely different, specially if a configuration of branes needs to be added. The stabilisation of moduli is a complicated process which can lead to a whole plethora of solutions (even with the same compactification space) all with a different spectrum of massive particles. Moreover there are still questions to be addressed in this second step, the more urgent ones being whether full moduli stabilisation is achievable, a construction of metastable de Sitter solutions and a good mechanism for the breakdown of supersymmetry. All this combined leads to the label of "lack of predictions" that has been often attributed to String Theory.

While hopes of connecting String Theory with the observed world look shattered we can still take a positive attitude rather than wallowing in a negative one. For an immense number of solutions and relative easiness in building them allows us to study whether some particular patterns arise. In this different light the main aim of String Phenomenology is the search for the prominent features of string models and their potential applications to particle physics and cosmology. An important part of this general programme is the separation of the study of string solutions in many different fragments: while the complete study of a particular model and of its implications is a colossal task the study of the various parts that compose it is quite accessible. This philosophy, often called *bottom–up* approach, has led to important insight in String Phenomenology for already the study of these "atoms" composing a full string solution can give important directions to where good models lie. The price is pay is that no assurance exists that these pieces can be consistently combined in a full jigsaw puzzle.

Following this general philosophy we can try to see if by use of String Theory it is possible to address some of the problems that plague the known models of particle physics. The Standard Model of particle physics is certainly one of the most successful theories ever formulated, and one observation that greatly surprises is the extreme intricacy of its flavour sector. The form of the mass matrices for the fermions present in the Standard Model, both leptons and quarks, and their possible mixings prove to be extremely complicated and call for an explanation. There are several (more or less) convoluted mechanisms in field theory to try to explain these observations, however it is natural to wonder whether these questions may find or not their answer within String Theory, even without resorting to the known mechanisms that are introduced in field theory.

FORMAL ASPECTS OF STRING THEORY

In the immense ocean of String Theory solutions there exist many that can not *clearly* describe the world we observe. Typical indications of this failure are excess of supersymmetry preserved and/or incorrect number of non–compact spacetime dimensions. While these solutions may be thought as useless at first inspection it is remarkable that by studying them it is possible to draw important lessons in quantum field theory, quantum gravity and even mathematics.

Proving that a given Quantum Field Theory is well defined quantum mechanically is in general a difficult enterprise, specially beyond four spacetime dimensions. Available techniques are often perturbative and therefore insufficient to fully address the question, and there even exist cases where no conventional knowledge (namely by means of a Lagrangian) of the theory is available. String Theory is in this sense an extremely powerful tool: the possibility of realising a given field theory within String Theory ensure its existence as a consistent quantum theory. As a bonus String Theory allows us to study additional aspects of these theories unravelling phenomena that would not have been observable otherwise. The most optimistic view in this direction is that String Theory will allow us to obtain a classification of consistent quantum field theories in different dimensions of spacetime, providing a clear separation between the theories embeddable in String Theory and the ones that lie outside in the so–called *swampland*. A curious and unexpected observation is that the coupling to gravity can provide further additional constraints that one would not expect in quantum field theories. Some celebrated examples are the absence of global symmetries and the weak gravity conjecture which can provide efficient tools for testing existence of a given quantum field theory. It is remarkable that in String Theory evasion of these features of quantum gravity is attainable only when gravity is effectively decoupled.

On a related side String Theory has always had a profound connection with mathematics. On the one hand, more or less sophisticated constructions in mathematics are necessary in the study (and also in the construction) of String Theory solutions. But the arrow may be reversed, and in some examples mathematics has drawn some important lessons from String Theory. The discovery of mirror symmetry is surely the most famous example: having the possibility of constructing mirror pairs of Calabi–Yau manifolds has led to the possibility of computation of geometrical invariants of these spaces. Another notorious example is the proof of the monstrous moonshine conjecture, which has been possible by using techniques drawn from String Theory.

PLAN OF THE THESIS

This thesis is divided in three separate parts.

The first two parts of the thesis are devoted to the study of models of particle physics built in String Theory. In the first one, comprising Chapter 2 and Chapter 3, we will discuss the structure of the flavour sector of $SU(5)$ Grand Unified Theories within the framework of F–theory. In Chapter 2 we will give a brief account of GUT models stressing what the chief traits of the unification paradigm. In addition to this we will give a survey of how unification may be attained within type IIB String Theory and its non–perturbative completion F–theory. The first part of Chapter 3 will be devoted to a review of the effective theory governing the dynamics of 7–branes, necessary for all subsequent calculations. The remaining part of Chapter 3 will give a detailed computation of the Yukawa couplings in $SU(5)$ GUT models in two different classes of models: first we will describe the E_6 model, capable of accounting for the mass of the up–type quarks, and afterwards we will move to the E_7 and E_8 models which allow for the computation of the entire mass spectrum of the MSSM. One unique model compatible with both E_7 and E_8 enhancement will be studied in detail and its potential phenomenological consequences will be thoroughly presented. In the second part, comprising Chapter 4, we will give an account of how the concept of linear equivalence between cycles appears in D–brane models. We will present two different examples of this, both with important consequences for the phenomenology of this class of models. We will commence by presenting the relation between linear equivalence and kinetic mixing of abelian gauge bosons. We will give a characterisation of the mixing occurring between the abelian bosons in the open and closed string sectors, which in turn will be translated to the linear equivalence between the cycles wrapped by D–branes. We will also discuss the phenomenological consequences of this general phenomenon, with an emphasis on the possible presence of light particles carrying infinitesimal electrical charge and the effect of kinetic mixing on unification of gauge couplings. Secondly we will discuss the relation between linear equivalence and the stabilisation of open string moduli. We will discuss how, taking into account the coupling with the moduli appearing in the closed string sector, some of the moduli controlling the brane configuration may be stabilised with a high mass. We will elucidate the microscopic origin of this phenomenon, showing that the deformation of D–branes has to go through linearly equivalent cycles in order to avoid a mass for the brane position modulus. The phenomenological consequences of this scenario are also discussed, which include, in addition to a reduction of massless scalars in the effective action, the embedding of chaotic inflation scenarios with a stabiliser field in String Theory.

In the third part, comprising Chapter 5, of the thesis we will move to more formal aspects, not connected with phenomenology. We will study a particular class of superconformal field theories in five dimensional spacetime, the so–called T_N theory, by using topological string techniques. Our main interest will be in the computation of the partition function of these theories in the Higgs branch. We will start by formulating a general algorithm for the computation of the partition function of the T_N theory in a generic Higgs branch and apply this algorithm to obtain the partition function of the 5d E_8 theory which is contained in the Higgs branch of the T_6 theory. We will then formulate a variation of the topological vertex to be applied for the direct computation of the partition function of the T_N theory in the Higgs branch. We will explicitly provide some examples of this newly formulated technique.

Many technical details have been deferred to the appendices. In Appendix A we will collect some details about the exceptional algebras. In Appendix B we will give details concerning the computation of the wavefunctions for all models in Chapter 3. In Appendix C we will present the details of the local geometry close to the Yukawa point for the E_6 model considered in Chapter 3. In Appendix D we will give the general definition of the linear equivalence between cycles. In Appendix E we will give some definitions on generalised geometry necessary in Chapter 4. In Appendix F we will present a reformulation of the supersymmetry conditions for type IIA solutions with fluxes and their relation with symplectic cohomology. In Appendix G we will give the rules for the computation of the topological string partition function by using the topological vertex. Finally, in Appendix H we will collect some details regarding the form of the partition function of an $Sp(N)$ theory.

Part I

YUKAWA TEXTURES IN F–THEORY GUTS

2 GRAND UNIFIED THEORIES & STRING THEORY

In this chapter we will give a rapid overview of the idea of unification of forces and its prime implications. After this introduction to Grand Unified Theories we will look for a possible embedding within String Theory within a particular slice of the string landscape, i.e. D–brane models in type II String Theory, seeing however that this framework does not seem capable of hosting GUT models compatible with experimental observations. This will motivate us to go to a non–perturbative formulation of D–brane models given by F–theory. We will give a short review of F–theory and spell out the main phenomenological features of GUT models in F–theory, seeing how going to F–theory effectively solves the problems plaguing GUT models in perturbative type II scenarios.

2.1 GRAND UNIFIED THEORIES – FIELD THEORY

The idea of unification of forces has permeated physics since Newton formulated his theory of gravitation in the 17th century. After this first example of unification of forces every instance where unification was introduced gave major advances in our understanding of the laws of nature. At the moment the state of the art of the description of particle physics phenomena is given by the Standard Model of particle physics formulated between 1961 and 1967 by Sheldon Glashow, Steven Weinberg and Abdus Salam. In this formulation we find three kind of forces: strong forces (given by an $SU(3)$ Yang–Mills theory), weak forces (an $SU(2)$ Yang–Mills theory) and hypercharge (simply a $U(1)$ theory). Something that the Standard Model fails to explain is the relative strength between these three forces which can be chosen to arbitrary. The main idea for unification of these forces first came in 1974 [1] starting from the observation that the couplings governing the three forces seem to unify at a scale of approximately 10^{15} GeV. This is due to the fact that quantum correction lead to a running of the various couplings according to the renormalisation group equations. In particular writing the couplings in terms of their fine-structure constants $\alpha_i = g_i^2/(4\pi)$ the solution of the RG equations is

$$\frac{1}{\alpha_i(Q^2)} = \frac{1}{\alpha_i(M_W^2)} + \frac{b_i}{4\pi} \log \frac{M_W^2}{Q^2} , \tag{2.1}$$

where we introduced the two scales Q and M_W (the latter being the electroweak scale). The coefficient b_i entering in the solution of the RG equations is the one–loop β function which for a $SU(N)$ gauge theory can be computed in terms of the matter content of the theory

$$b = -\frac{11}{3}N + \frac{2}{3}T(R_f)\, n_f + \frac{1}{3}T(R_s)\, n_s \tag{2.2}$$

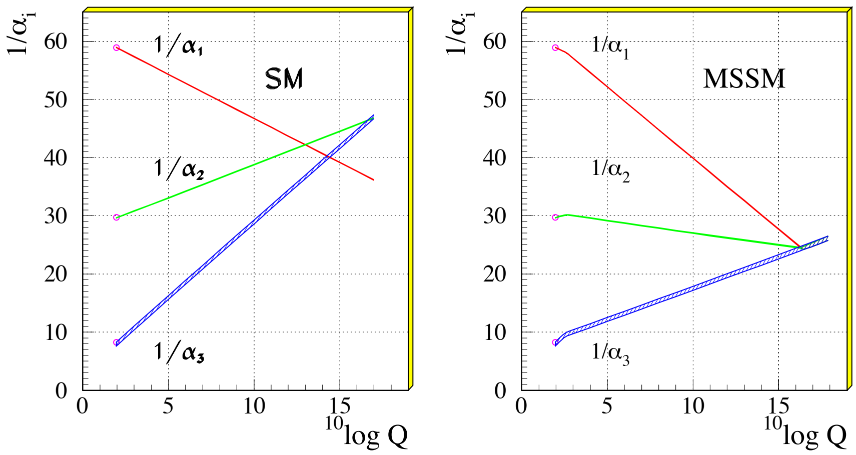

Figure 1: Unification of couplings for the MSSM. Figure taken from [2].

where n_f is the number of left–handed Weyl fermions in the representation R_f of $SU(N)$ and n_s is the number of scalars in the representation R_s of $SU(N)$. Moreover given a representation R of $SU(N)$ we denoted $T(R)$ its quadratic Casimir with the convention that $T(\mathbf{N}) = \frac{1}{2}$. A similar expression may be written for a $U(1)$ theory setting N to zero and taking into account that the quadratic Casimir is $T(q) = q^2$ where q is the charge under the $U(1)$ of a given particle. The result is that for the standard model (namely three generations of quarks and leptons and one Higgs) we have that[1] $(b_1, b_2, b_3) = \left(\frac{41}{10}, -\frac{19}{6}, -7\right)$. The evolution under the RG flow of the Standard Model couplings is quite suggestive for their value seem to converge although perfect unification is not actually achieved. This greatly improves if supersymmetry is added which we shall always assume in the following. In Figure 1 we show the evolution of the Standard model couplings up both in the SM and in MSSM showing how the latter can attain much improved unification.

We shall henceforth take this empirical observation seriously[2], with the reason behind this unification of couplings being the existence of a gauge theory with a simple gauge group G_{GUT} which is broken down to the Standard Model gauge group at a scale M_{GUT}. The gauge group characterising the Grand Unified Theory ought to fulfil three simple criterions:

- G_{GUT} has to contain the Standard Model gauge group as a subgroup;
- G_{GUT} has to have complex representations to allow for a chiral spectrum;

1 Note that we changed the normalisation of the hypercharge generator according to $Y \to \sqrt{\frac{3}{5}}Y$. We shall always use this new normalisation of the hypercharge generator from now on.

2 This amounts to assuming that no new physics other than supersymmetry is present between the electroweak scale and the unification scale, or at least that possible additional physics does not spoil unification of couplings. This is obviously a strong assumption and as we will discuss later in String Theory there are several effects that can spoil perfect unification.

- G_{GUT} should be sufficiently *minimal*[3].

The classical choices for GUT group fulfilling these conditions are $SU(5)$, $SO(10)$ and E_6. While the last two options have nice phenomenological features our prime interest will always be $SU(5)$ GUT theories, also because they may be embedded more easily in String Theory.

Our journey in the realm of $SU(5)$ Grand Unified Theories starts with the embedding of the various matter fields of the Standard Model in $SU(5)$ representations. We commence with the vector fields which need to sit in the adjoint representation of the GUT group. The branching rule for the adjoint representation of $SU(5)$ under $SU(3) \times SU(2) \times U(1)$ is

$$\begin{aligned} SU(5) \quad &\rightarrow \quad SU(3) \times SU(2) \times U(1) \\ \mathbf{24} \quad &\rightarrow \quad (\mathbf{8},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{3},\mathbf{2})_{-\frac{5}{6}} \oplus (\bar{\mathbf{3}},\mathbf{2})_{\frac{5}{6}} \,. \end{aligned} \tag{2.3}$$

We clearly recognise the presence of all the Standard Model gauge bosons (gluons, W_i bosons and hypercharge B) together with some additional bosons usually called X and Y bosons. While these additional vectors will receive mass from the GUT breaking mechanism they are extremely important for they can play the rôle of mediators for processes leading to proton decay. The issue of proton decay is an extremely important one and we will have more to say about this later on after introducing other possible mechanisms leading to it. Matter fields of the Standard Model may also be nicely embedded in $SU(5)$ representations. In particular each generation fits in a pair of $SU(5)$ representations, namely a $\bar{\mathbf{5}}$ (antifundamental representation of $SU(5)$) and a $\mathbf{10}$ (2-index antisymmetric representation). Their branching under the Standard Model gauge group are

$$\begin{aligned} SU(5) \quad &\rightarrow \quad SU(3) \times SU(2) \times U(1) \\ \bar{\mathbf{5}} \quad &\rightarrow \quad (\bar{\mathbf{3}},\mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{1},\mathbf{2})_{-\frac{1}{2}} \,, \\ \mathbf{10} \quad &\rightarrow \quad (\bar{\mathbf{3}},\mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{3},\mathbf{2})_{\frac{1}{6}} \oplus (\mathbf{1},\mathbf{1})_1 \,. \end{aligned} \tag{2.4}$$

Indeed we see that the whole content of a single Standard Model representation is contained in these two $SU(5)$ representations.

The missing ingredient to fully obtain the Standard Model spectrum is the Higgs field. Its embedding in a $SU(5)$ representation is in a $\mathbf{5}$ representation which, in addition to the desired $SU(2)$ doubled contains a $SU(3)$ triplet. These additional triplets, usually called coloured Higgses, are additional mediators for proton decay and therefore a mechanism giving them a sufficiently high mass (usually called doublet–triplet splitting) should be devised. In the supersymmetric case it is necessary to introduce two Higgs fields in the $\mathbf{5} \oplus \bar{\mathbf{5}}$ representation of $SU(5)$ to ensure cancellation of anomalies coming from the Higgs fermionic superpartners[4].

This concludes the discussion of the matter content of a minimal $SU(5)$ Grand Unified Theory. We now turn to the discussion of the main phenomenological features of $SU(5)$ Grand Unified Theories, stressing what are the advantages and disadvantages of this class of models.

3 While this third criterion may seem purely an aesthetic one GUT models with smaller gauge groups fit better experimental data. It is a curious observation that nature seems to favour simplicity.

4 Two Higgs fields forming a vector-like pair under the gauge group are necessary also to give mass to all fermions of the Standard Model in supersymmetric extensions. Anomaly cancellation is nonetheless a more urgent reason for the introduction of a second doublet.

Gauge coupling unification and θ_W

Perfect unification of the gauge couplings has the nice implication that all the different gauge couplings that we observe at the electroweak scale can actually be obtained from a unique coupling. This implies for instance that the value of the weak mixing angle at the unification scale is completely determined and given by

$$\sin^2\theta_W = \frac{3}{8}\,. \tag{2.5}$$

We can follow the evolution of θ_W under the RG flow down to the electroweak scale finding that the value obtained would be $\sin^2\theta_W \sim 0.214$ for a unification scale of $M_{GUT} \simeq 10^{15}$ GeV. This is clearly not compatible with the observed value of $\sin^2\theta_W = 0.2312 \pm 0.0002$ but shows at least that this simple framework can already give a reasonable value for the weak mixing angle.

Proton decay

One of the most prominent features of Grand Unified Theories is the fact that the Lagrangian lacks of baryon number and lepton number global symmetries. Since these symmetries directly imply the stability of the proton in the Standard Model we naturally obtain that the proton is no longer stable in Grand Unified Theories. For example in $SU(5)$ GUTs the proton can decay in the channel $p \to \pi^0\, e^+$ via the exchange of X and Y bosons. Explicit computation shows that the lifetime of the proton scales as $\tau_p \simeq M_X^4/m_p^5$ giving a lifetime of $\tau_p \sim 10^{29}$ years for $M_{GUT} \sim 10^{15}$ GeV. This lifetime is already excluded by the super-Kamiokande experiment which puts a bound on the lifetime of the proton $\tau_p > 10^{33}$ years. Introduction of supersymmetry leads to a larger unification scale and can easily evade the aforementioned bound. Supersymmetry however does not suffice to suppress other possible channels of proton decay[5], for example the ones mediated by the Higgs triplets. This is intimately tied with the doublet–triplet splitting mechanism and as we will discuss later on a natural solution is present in the string theory embedding of $SU(5)$ Grand Unified Theories.

Charge Quantisation

One non trivial experimental observation is that the ratio of the charges of the various particles is always a rational number. From the point of view of the Standard Model this translates to the fact that for all observed particles the ratio of their hypercharges is always a rational number. Group theoretically this would imply that the global structure of the hypercharge group is actually $U(1)$ as opposed to $\mathbb{R}$. While in the context of the Standard Model alone there is no explanation for this fact[6] the embedding of the Standard Model within $SU(5)$ (or any other simple *compact* gauge group) gives as a byproduct automatic quantisation of charges.

5 As a matter of fact the opposite occurs for in supersymmetric models there are additional allowed interactions that violate baryon number if no additional symmetry like R–symmetry is invoked.

6 At this point we are neglecting the existence of gravity, for arguments in quantum gravity (see for instance [3] for a recent account) always imply that the gauge group ought to be compact selecting therefore $U(1)$ rather than $\mathbb{R}$. Non-quantised charges are possible but they require kinetic mixing of massless $U(1)$ gauge fields as we will discuss more in Section 4.1.

Yukawa couplings unification

We shall give a more detailed account of the Yukawa couplings between fermions and the Higgs in $SU(5)$ Grand Unified Theories in the next section, however this is a good place to anticipate that asking for unification implies relation between the various couplings. In particular in $SU(5)$ GUTs there is a unique coupling that is responsible for the masses of down quarks and leptons, which implies that at unification scale these couplings ought to be equal. Running down the couplings to the electroweak scale we find some relations between the masses of down quarks and leptons which are in disagreement with the experimental data for the first and second generations. While it is possible to build more complicated field theory models that can ameliorate the situation string theory models implement some mechanism that can directly solve this issue.

Anomaly cancellation

The cancellation of all gauge anomalies in the Standard Model with its intricate spectrum of fermions looks like a miracle. While this is not a phenomenological advantage of the unification paradigm it is surely nice from a theoretical viewpoint that anomaly cancellation greatly simplifies when a simple gauge group is considered. For example for $SU(5)$ GUTs with fermions only in the **10** and **5** representations cancellation of $SU(5)$ cubic anomalies becomes

$$\chi(\mathbf{10}) = \chi(\mathbf{5}) \tag{2.6}$$

where for any representation we defined the net chirality as $\chi(\mathbf{R}) \equiv n_{\mathbf{R}} - n_{\overline{\mathbf{R}}}$. This naturally holds for $SU(5)$ GUTs and also in supersymmetric extensions where the fermionic superpartners of the Higgs fields are present. Finally it is also curious to note that also cancellation of mixed gravitational and gauge anomalies is much simpler: in fact if the gauge group does not have any abelian factor all mixed gravitational and gauge anomalies are absent[7]. Again in the Standard Model mixed gravitational and $U(1)_Y$ anomalies cancel but this looks like a happy coincidence.

2.2 GRAND UNIFIED THEORIES IN TYPE II STRING THEORY

Having spelled out in some detail the main phenomenological features of Grand Unified Theories we now move to discussing how they may be embedded within String Theory. Among the various corners of the string landscape we choose to focus our attention to type II models with an emphasis on type IIB and its non–perturbative completion F–theory. Most of our discussion can be directly translated to type IIA models with the non–perturbative completion given by M–theory, however given the difficulties in building M–theory solutions suitable for GUT model building we choose not to discuss explicitly this class of models.

7 This happens in 4d where the diagrams giving rise to anomalies have insertions of three currents. In more spacetime dimensions non abelian gauge groups may have mixed gravitational anomalies.

2.2.1 *Magnetised D–brane models*

The most natural way to build 4d solutions of type IIB String Theory is to consider the 10d space to be a (possibly warped) product of 4d Minkowski space and a Calabi–Yau threefold X_3. This leads to a 4d solution with $\mathcal{N}=2$ supersymmetry and non–abelian interactions if X_3 has some singularities in (complex) codimension 2. This however is not at all promising for phenomenology for $\mathcal{N}=2$ supersymmetry does not allow for a chiral spectrum. The missing ingredients to achieve $\mathcal{N}=1$ 4d solutions of type IIB String Theory are D–branes and orientifold planes.

D–branes are some non–perturbative objects in type II String Theory[8] whose presence adds a sector of open strings in the theory. The endpoints of the open strings are forced to lie on the D–branes and their quantisation will give rise to additional degrees of freedom localised on the worldvolume of the D–branes. One interesting feature of D–branes is that their presence induces a partial breaking of supersymmetry therefore allowing us to construct solutions preserving only $\mathcal{N}=1$ 4d supersymmetry. In particular type IIB String Theory has three kind of D–branes that can be used for model building, namely D3, D5 and D7–branes[9]. We shall focus our attention on the latter in view of the embedding in F–theory. It is important to stress that when D–branes are placed in a compact space it is necessary to introduce a quotient on the theory known as orientifold projection. Orientifold projection selects among the states in the string Hilbert space those left invariant by an operation with the form[10] $\Omega\mathcal{R}$: here Ω acts by reversing the orientation of the string and $\mathcal{R}$ is a particular $\mathbb{Z}_2$ involution of the target space. The fixed locus of $\mathcal{R}$ is called orientifold plane, and for the models we shall be interested in it will be necessary to introduce O7–planes and possibly O3–planes. To ensure that some amount of supersymmetry is preserved by the combined system of orientifold planes and D–branes it is necessary to place some restrictions on the locus where they are located: in particular for D7–branes and O7–planes it is necessary for them to wrap a holomorphic divisor of the Calabi–Yau manifold X_3.

The addition of D–branes has the consequence of adding gauge interactions localised on their worldvolume which has profound consequences for model building. In particular a stack of N D–branes carries a $U(N)$ gauge theory on its worldvolume. Other gauge groups are possible if the D–branes wrap cycles left invariant by some orientifold involutions: in these cases we can have either a $SO(N)$ gauge theory or a $USp(N)$ gauge theory (with N even in this case). At this level going beyond classical groups is not possible[11] and this will be our main motive to move to F–theory where exceptional groups are possible.

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8 Curiously while being non–perturbative objects D–branes differ from ordinary solitonic solutions in field–theory. In fact while the latter have a tension going as $\frac{1}{\lambda^2}$ with λ the coupling of the theory D–branes have a tension going as $\frac{1}{g_s}$. This does not apply to all solitonic objects in String Theory, and it is quite curious that the heterotic NS5–brane has a tension going as $\frac{1}{g_s^2}$ even if it can be related via a series of dualities to D–branes in type II String Theory.

9 There are also D9–branes, however solutions with D9–branes are intimately type I solutions.

10 To ensure that the orientifold involution is really a $\mathbb{Z}_2$ action on the string Hilbert space it is sometimes necessary to introduce a factor of $(-1)^{F_L}$ in its definition, with F_L the left moving fermion number. To keep the discussion as general as possible we chose not to introduce it here.

11 There is actually a simple explanation for this fact: open strings have only two endpoints and therefore they can give rise to gauge interactions where the roots of the gauge algebra are charged under only two Cartan generators, something that is compatible only with classical groups. If strings were to have more than two endpoints it would be possible to realise exceptional groups as well, and

Starting with this observations it is quite straightforward to envisage a way to build a $SU(5)$ GUT in type IIB string theory: we may simply consider a stack of 5 D7–branes giving rise to a $U(5)$ gauge theory. At this moment we may worry that the gauge group is not simple, but the abelian factor in $U(5) \simeq [SU(5) \times U(1)]/\mathbb{Z}_5$ generically acquires a large mass because of the following term in the D–brane action

$$S_{D7} \supset \mu_7 \int C_4 \wedge \mathrm{Tr}[F \wedge F] \,, \tag{2.7}$$

which upon compactification to 4d produces term of the form

$$S_{4d} \supset \int_{\mathbb{M}^4} B \wedge F \,, \tag{2.8}$$

with F the field strength of the abelian factor within $U(5)$ and B some particular linear combination of closed string 2-forms coming from the dimensional reduction of the type IIB RR 4-form C_4. This coupling in addition to cancel the anomalies of the $U(1)$ via a Green-Schwarz mechanism gives a large mass to the abelian vector field via a Stuückelberg coupling with the axion field dual to B. This does not mean that the additional $U(1)$ gauge symmetry does not constitute a problem: in fact it will manifest itself as a selection rule in the low energy effective theory which will pose some severe problems in the construction of $SU(5)$ GUT models, problems that will be solved once going to F–theory.

Having understood how to include the gauge sector suitable for an $SU(5)$ GUT model in type IIB D–brane models we now turn to the description of how to introduce matter fields. To do so we shall need to consider the intersection of our stack of D7–branes with either additional D7–branes or with O7–planes: as we will see now in the former case we will obtain matter fields in the **5** representation while in the latter case we will obtain matter in the **10** representation. In the case where a stack of N D7–branes intersects M additional D7–branes there appears a new massless sector in the string spectrum given by strings stretched between the two stacks. In particular the massless strings will be localised on the complex curve Σ that constitutes the intersection of the two stacks. The spectrum we obtain is chiral in 6d[12] (namely on $\mathbb{M}^4 \times \Sigma$) and sitting in the bifundamental representation $(\mathbf{N}, \overline{\mathbf{M}})$ of the gauge group $U(N) \times U(M)$. We shall discuss how chirality may be preserved upon compactification to 4d in a moment. This is clearly sufficient to obtain matter in the fundamental representation of $SU(5)$ if our stack generating GUT interactions intersects some additional $U(1)$ flavour branes. To obtain matter in the antisymmetric representation of $U(N)$ it is necessary to consider the intersection of the stack of D7–branes with the O7– planes. If there is an intersection between a stack of N D–branes and an orientifold plane such that the cycle wrapped by the branes is not left invariant by the orientifold plane we obtain some additional matter localised this time at the intersection between the stack of D–branes and the orientifold plane (which again occurs on a complex curve Σ): these additional massless fields appear because of the open strings stretched between the stack of D–branes and its image under the orientifold projection. Explicit computation shows that these fields sit in

this is exactly what happens in F–theory where there exist string junctions carrying more than two endpoints [4–6].

12 6d chirality will be selected according to the direction of the intersection of the two stack of branes.

the antisymmetric representation of $U(N)$ with charge 2 under the $U(1)$ factor within $U(N)$.

The missing piece to fully realise a GUT model in type IIB String Theory is how to obtain a chiral spectrum in 4d. We have already seen that the spectrum will be chiral in 6d but compactification down to 4d will yield a non–chiral spectrum. In fact the number of left (resp right) handed Weyl fermions in 4d is related to the number of zero modes of the Dirac operator D (resp $D^\dagger$) on the Riemann surface Σ. Therefore we can relate the net chirality in 4d to the index of the Dirac operator defined as follows

$$\mathrm{ind} D \equiv \dim\ker(D) - \dim\ker(D^\dagger)\,. \tag{2.9}$$

Using the Hirzebruch–Riemann–Roch theorem we write this index as the integral of some characteristic classes on the Riemann surface Σ

$$\mathrm{ind} D = \chi(\Sigma,\mathcal{E}) = \int_\Sigma \mathrm{ch}(\mathcal{E})\mathrm{td}(\Sigma) \tag{2.10}$$

where $\mathcal{E}$ is the complex vector bundle to which the fermions couple, $\mathrm{ch}(\mathcal{E})$ is the Chern character of this bundle and $\mathrm{td}(\Sigma)$ is the Todd class of the tangent bundle of Σ. In the situation we have considered so far $\mathcal{E} = K_\Sigma^{\frac{1}{2}}$ which implies that $\mathrm{ind}\, D = 0$ as expected[13]. A more general situation is obtained if some of the branes we are considering carry a magnetic flux on their worldvolume (namely $\langle F\rangle \neq 0$ in the compact space[14]) the situation would radically change for the fermions would couple to this gauge bundle producing an alteration in the Dirac operator. In particular this would imply that the complex vector bundle to which the fermions couple in the Riemann surface is now $\mathcal{E} = K_\Sigma^{\frac{1}{2}} \otimes \mathcal{L}$ where $\mathcal{L}$ is a line bundle whose first Chern class satisfies $c_1(\mathcal{L}) = F$. The resulting Dirac index is now

$$\mathrm{ind} D = \int_\Sigma \mathrm{ch}(K_\Sigma^{\frac{1}{2}} \otimes \mathcal{L})\mathrm{td}(\Sigma) = \int_\Sigma c_1(\mathcal{L})\,. \tag{2.11}$$

Therefore if the magnetisation on the D7–branes induces a topologically non–trivial gauge bundle on the Riemann surface Σ we can obtain chiral fermions in 4d. For the cases we considered in building $SU(5)$ GUT models in type IIB String Theory we can suitably induce magnetisation on the Riemann surfaces hosting the various matter fields by turning on gauge fluxes on the $U(1)$ flavour branes as well as along the $U(1)$ direction within $U(5)$. This is sufficient to obtain a chiral spectrum in 4d, completing therefore the construction of a generic GUT model in type IIB String Theory.

All this discussion seems to suggest that magnetised D7–brane models in type IIB String Theory are a good arena for the realisation GUT models, but as a matter of fact this class of models fail to pass one important phenomenological sanity check

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13 Here we denoted K_Σ the canonical bundle of the Riemann surface and $K_\Sigma^{\frac{1}{2}}$ is a bundle satisfying the condition of squaring to the canonical bundle. The square root of the canonical bundle always exists (this is because the canonical bundle has even degree) but is not unique in most cases. For a genus g Riemann surface we have 2^{2g} different square roots related to the different possible choices of spin structure on the Riemann surface. At the level of characteristic classes the choice of spin structure is irrelevant and all square roots of the canonical bundle are good choices. The reason why $\mathcal{E} = K_\Sigma^{\frac{1}{2}}$ is related to the topological twist which we will discuss in Section 3.1, see [7] for more details.

14 This gauge bundle ought to satisfy some conditions to ensure that supersymmetry is preserved. We shall discuss these conditions in Section 3.1.2 after introducing the effective action on the D7–branes.

that involves generation of masses for the Standard Model fermions. This is a good point to review how masses for fermions are generated in $SU(5)$ GUT models: with supersymmetry all couplings are generated via the following superpotential

$$W = y^{\mathbf{u}}_{ij}\, H_u\, Y^i_{\mathbf{10}_M} Y^j_{\mathbf{10}_M} + y^{\mathbf{d/l}}_{ij}\, H_d\, Y^i_{\mathbf{10}_M} Y^j_{\bar{\mathbf{5}}_M}\,. \tag{2.12}$$

When expressing the various fields in terms of Standard Model representations it is possible to see that the first term generates masses for the up–type quarks and the second one generates masses for leptons as well as down–type quarks. $SU(5)$ invariance is ensured for all these couplings but when embedding GUT models in magnetised D–brane models only couplings invariant under the full $U(5)$ are generated even if the abelian factor inside $U(5)$ acquires a mass. Since the charge of the **10** representation under the $U(1)$ is $+2$ and the charge of the **5** representation is $+1$ we see that the coupling generating the mass for up–type quarks is not invariant under the full $U(5)$ and therefore absent at tree level. It is possible to generate nonetheless a Yukawa coupling responsible for the mass of the up–type quarks when non–perturbative effects breaking the massive $U(1)$ symmetry are taken into account. However it is quite implausible that such effects can generate a coupling of the correct order of magnitude to account for the mass of the top quark in a regime where string perturbation theory can be trusted. This problem will be solved when considering GUT models in F–theory which constitutes the non–perturbative completion of type IIB String Theory.

2.3 BASICS OF F–THEORY

Since a vast portion of this thesis will be devoted to GUT models in F–theory it is worthwhile to offer a brief introduction to F–theory. The first appearance of F–theory [8] dates back to 1996 where it was first observed that the axio–dilaton of type IIB String Theory can be effectively interpreted as the complex structure of an auxiliary torus. Ever since its introduction F–theory has proven a powerful tool for studying string dualities and recently employed for building GUT models following the ideas of [7, 9, 10]. The main drawback of F–theory is that no fundamental theory exists for its description and the most effective way to access it is via its dualities to different string models. In this section we shall describe the three classical ways to access F–theory, namely type IIB String Theory, M–theory and heterotic String Theory. We refer to the reviews [11–13] for additional aspects of F–theory that we will not be able to cover as well as additional references on the subject.

2.3.1 *F–theory via type IIB String Theory*

Type IIB String Theory has a quite peculiar behaviour under weak–strong dualities that makes it somehow unique among the various String Theories. The effective way to describe these dualities is restricting to the subset of massless modes of type IIB String Theory and considering the effective action describing the interactions among them. This effective action has local $\mathcal{N} = 2$ supersymmetry in 10d and its bosonic fields comprise the graviton g_{MN}, a 2–form B_2, a scalar ϕ and some differential forms C_p with $p = 0, 2, 4$ coming from the RR sector. In particular it is possible to pack the two scalar fields present in the spectrum, the dilaton ϕ and the RR axion C_0, in a single complex scalar field defined as $\tau \equiv C_0 + ie^{-\phi}$. It is possible to show that type IIB supergravity is invariant under a $SL(2,\mathbb{R})$ group of transformations that act on the bosonic fields as

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}, \quad \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \longrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix}, \tag{2.13}$$

where $a, b, c, d \in \mathbb{R}$ and moreover $ad - bc = 1$. The string frame metric is left invariant under $SL(2,\mathbb{R})$ transformations and so is the improved field strength for the C_4 field $\tilde{F}_5 \equiv dC_4 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3$. When quantum effect are taken into account the duality group actually reduces to $SL(2,\mathbb{Z})$[15]. The action of $SL(2,\mathbb{Z})$ on τ actually gives transformations that interchange weakly coupled and strongly coupled configurations of type IIB String Theory: for example the transformation

$$\tau \longrightarrow -\frac{1}{\tau}, \tag{2.14}$$

gives for $C_0 = 0$ a transformation that changes the sign of ϕ thus inverting the string coupling g_s. This might give the impression that it is always possible to choose a duality frame where the string coupling constant may be chosen to be "sufficiently" small, but this kind of reasoning turns out to be a bit naïve. One example where it is not possible to find a weakly coupled regime for type IIB String Theory involves configurations where some singularities induce $SL(2,\mathbb{Z})$ monodromies on the axio–dilaton. Given the invariance of type IIB Supergravity under $SL(2,\mathbb{Z})$ transformation these solutions turn out to be perfectly well defined, but as we will see they may lead to some problems in the definition of string perturbation theory which motivate the introduction of F–theory. It is actually simple to construct solutions with $SL(2,\mathbb{Z})$ monodromies by simply considering D7–branes: D7–branes in fact act as magnetic sources for the RR axion C_0

$$dC_0 = \delta_{D7}\,. \tag{2.15}$$

Knowing that the dilaton ϕ is not affected by the presence of the D7–brane and identifying the space transverse to the D7–brane as $\mathbb{C}$ with coordinate z we may write a solution to (2.15) around the position of the D7–brane as

$$\tau = \tau_0 + \frac{1}{2\pi i}\log(z - z_{D7}) + \ldots\,, \tag{2.16}$$

which clearly shows that τ suffers a monodromy when circling the location of the D7–brane

$$z - z_{D7} \;\to\; e^{2\pi i}(z - z_{D7}) \;\Longrightarrow\; \tau \;\to\; \tau + 1\,. \tag{2.17}$$

While this monodromy only affects the real part of τ (and therefore only C_0) by applying a global $SL(2,\mathbb{Z})$ transformation we may reach a situation where the dilaton as well is affected by the monodromy. In particular by applying the following $SL(2,\mathbb{Z})$ transformation

$$M = \begin{pmatrix} p & p \\ q & q \end{pmatrix}, \tag{2.18}$$

15 This can be checked by inspection of the action of a D(-1)–brane which has to be invariant under duality transformations.

$$\begin{tikzcd}
\text{M-theory on } \mathbf{T}^2 \arrow[d, "\mathbf{S}^1 \text{ reduction}"] & \\
\text{Type IIA on } \mathbf{S}^1 \arrow[r, leftrightarrow, "\text{T-duality}"'] & \text{Type IIB on } \mathbf{S}^1
\end{tikzcd}$$

Figure 2: The duality relating M–theory on $\mathbf{T}^2$ and type IIB String Theory on $\mathbf{S}^1$.

we may convert a D7–brane in a (p, q) 7–brane whose monodromy is

$$T_{(p,q)} = \begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix} . \tag{2.19}$$

This argument shows that in type IIB String Theory it is conceivable to consider backgrounds where generic $SL(2, \mathbb{Z})$ monodromies appear when circling around complex codimension one defects. This class of solutions clearly poses some problems with the formulation of type IIB String Theory as a perturbative series in the string coupling g_s for the definition of the string coupling itself is not a globally well defined quantity. The solution of this puzzle came in [8] where it was pointed out that the information regarding the axio–dilaton τ in a generic type IIB solution may be encoded as the complex structure of an auxiliary torus $\mathbf{T}^2$ which will be fibered over the target space of type IIB String Theory. The main advantage of this description is that it automatically allows for a well defined background even in the presence of generic $SL(2, \mathbb{Z})$ monodromies affecting the axio–dilaton, and it also provides a geometrisation for the location of 7–branes. In fact by inspection of (2.16) we see that $\tau \to i\infty$ at the location of the D7–brane, which suggests that the auxiliary torus $\mathbf{T}^2$ develops a singularity at the location of the D7–brane. This extends to more general configurations of 7–branes allowing us to read off the location of 7–branes by simply looking for singularities of the torus fibration providing a geometric picture for the location of 7–branes. This will be sharpened when we considered the duality with M–theory for there is a simple procedure for the identification of singularities of torus fibrations that also yields information on the gauge theory living on the 7–branes. We shall describe this in the next section in full detail, anticipating only that in this more general situation gauge theories with exceptional groups will be possible as well.

2.3.2 *F–theory via M–theory*

The starting point for the description of F–theory using M–theory is the duality between type IIB String Theory on a circle and M–theory on a torus $\mathbf{T}^2$. The duality relating the two theories is a suitable combination of S–dualities and T–dualities which we describe diagrammatically in Figure 2.

The presence of the $\mathbf{T}^2$ on the M–theory side is actually very suggestive, and following the duality carefully to a situation where we recover 10d type IIB String Theory we find that the complex structure of the $\mathbf{T}^2$ is identified with the axio–dilaton τ. More specifically the actual limit which should be taken to recover 10d type IIB String Theory (for a detailed account of the limit see for instance [11]) is the limit where on the M–theory side the $\mathbf{T}^2$ shrinks to zero volume while keeping its complex structure to a finite value. We shall take this as a working definition of generic F–theory solutions: F–theory on an elliptically fibered[16] manifold Y is dual to M–theory on Y in the limit where the elliptic fibre goes to zero volume while retaining a finite value for its complex structure. In the cases we shall be interested in for the construction of F–theory GUT models the condition to have 4d $\mathcal{N} = 1$ supersymmetry forces us to consider compactifications of M–theory down to 3d on an elliptically fibered Calabi–Yau fourfold[17] [38]. We will now describe in detail how elliptic fibrations can be constructed and how the geometry encodes the data of 7–branes.

Elliptic curves

Our main interest will be focused in the case where the elliptic fibration possesses a globally well–defined section, and therefore we shall rely on a mathematical theorem [39] stating that every elliptic fibration can be described as a Weierstraß model. The starting point is to take an elliptic curve as a hypersurface of degree 6 inside $\mathbb{P}^2_{1,2,3}$. Application of the adjunction formula shows that the canonical bundle of such a curve is trivial ensuring therefore that the curve has genus $g = 1$. The generic form of the hypersurface is the so–called Weierstraß form

$$y^2 = x^3 + f\,x\,z^4 + g\,z^6\,, \tag{2.20}$$

where $(x, y, z) \simeq (\lambda^2 x, \lambda^3 y, \lambda z)$ are the homogeneous coordinates of $\mathbb{P}^2_{1,2,3}$. The constants $f, g \in \mathbb{C}$ determine the complex structure of the elliptic curve as we shall describe in more detail in a while. The algebraic description of an elliptic curve is particularly well suited to check whether a given elliptic curve is singular or not: a singularity is present if there at the points on the elliptic curve where all derivatives of the defining polynomial vanish[18]. It is possible to check that for an elliptic curve this occurs if two zeroes of the polynomial $x^3 + f\,x + g$ coalesce, which in turn occurs if the discriminant $\Delta \equiv 4f^3 + 27g^2$ vanishes. To write explicitly the complex structure of the elliptic curve we need to introduce the Jacobi j–function which has the nice feature of being invariant under $SL(2, \mathbb{Z})$ transformations. The explicit definition in terms of ϑ–functions and η–function is

$$j(z) = \frac{1}{8}\frac{\left[\vartheta^8_{10}(0|\tau) + \vartheta^8_{00}(0|\tau) + \vartheta^8_{01}(0|\tau)\right]^3}{\eta^{24}(\tau)}\,. \tag{2.21}$$

16 Here we secretly introduced an important point regarding the duality between F–theory and M–theory, namely that the $\mathbf{T}^2$ fibration ought to have a globally well–defined section allowing for the identification of a point on the $\mathbf{T}^2$ fibre. This unique point on the elliptic fibre is usually identified as the identity element for the group law defined on the elliptic curve. While this is the generic situation considered in the duality between F–theory and M–theory in the recent years there has been a lot of interest in the cases where more than one section exists [14–23] and even in the case where no section exists [24–33].

17 There are other interesting possibilities: one leading to $\mathcal{N} = 1$ in 3d involves compactifications on manifolds with $Spin(7)$ holonomy [34] which might lead in principle to $\mathcal{N} = 0$ theories in 4d. However further studies showed how upon taking the F–theory limit to 4d $\mathcal{N} = 1$ supersymmetry is recovered [35]. Another possibility is to consider other spaces (which are not Calabi–Yau fourfolds) which still lead to $\mathcal{N} = 2$ supersymmetry [36, 37]. However there are not yet known examples that can give F–theory vacua.

18 Since we are dealing with a projective variety the condition for the presence of a singularity ought to be checked in an affine patch. In our case singularities, if present, will always be located in the patch $z = 1$.

With this we can write down explicitly the relation between the parameters f, g and the complex structure τ of the elliptic curve[19]

$$j(\tau) = 1728\,\frac{f^3}{\Delta}\,. \tag{2.22}$$

With this information it is not a difficult task to build an elliptically fibered manifold: given a base B it is possible to build an elliptically fibered manifold Y by simply taking f and g to be holomorphic sections of some suitable bundle $\mathcal{L}$ of B. Compatibility with the projective relations of $\mathbb{P}^2_{1,2,3}$ and asking for Y to be a Calabi–Yau manifold uniquely fixes f and g (as well as the coordinates on $\mathbb{P}^2_{1,2,3}$) to be sections of some suitable powers of the anticanonical bundle of B, more specifically

$$\begin{aligned} &x \in H^0(B, K_B^{-2})\,, \quad y \in H^0(B, K_B^{-3})\,, \quad z \in H^0(B, \mathcal{O})\,, \\ &f \in H^0(B, K_B^{-4})\,, \quad g \in H^0(B, K_B^{-6})\,. \end{aligned} \tag{2.23}$$

From this it descends that the discriminant locus of the elliptic fibration satisfies $\Delta \in H^0(B, K_B^{-12})$: this implies that the 7–branes will always wrap holomorphic divisors and moreover the homology class of the linear combination of divisors wrapped by the 7–branes is fixed and equal to the Poincaré dual of $12c_1(B)$[20]. This completes our brief overview of elliptic curves and elliptic fibrations, and in the next section we shall describe how the singularities on the elliptic curves may be classified and their relation with the gauge theory on the 7–branes.

Singularities of elliptic fibrations

We have already discussed how to identify whether a given elliptic curve is singular or not and how to identify the locus where in a given elliptically fibered manifold the elliptic fibration becomes singular. The missing ingredient in this discussion is how to extract from the elliptic fibration information regarding the gauge theory on the 7–branes. Quite remarkably for singularities of elliptic fibrations in complex codimension 1 there exist a classification formulated by Kodaira [41–43] which allows us to deduce the gauge theory on the 7–branes of a given singularity. We report in Table 1 the result of Kodaira's classification.

We see directly that all groups of the series A, D and E can be realised in F–theory which constitutes a major advance with respect to perturbative type IIB compactifications where exceptional groups were completely absent[21]. The physical picture explaining the reason why a particular singularity corresponds to a given

19 It is possible to find the explicit value of the coupling by inverting the j–function by using hypergeometric functions. We refrain from copying here the explicit inverse form of the j–function, see for example [40] for the explicit form.

20 This condition, namely that $\sum_i [D_i] = 12\,\text{P.D.}[c_1(B)]$ where D_i are the divisors wrapped by the 7–branes is the equivalent in F–theory of the usual tadpole cancellation condition $\sum_i N_i[D_i]+N_i^*[D_i^*] = 4[D_{O7}]$ of type IIB compactifications. In F–theory we see that the 7–brane tadpole is cancelled by the non–trivial curvature of the Kähler manifold B and therefore no analog of orientifold planes is necessary. The other tadpole usually appearing in type IIB compactifications, namely the D3–brane tadpole, need also to be suitably cancelled in F–theory but we choose not to discuss it here, see for instance [11] for an account of it.

21 Note that in the case of an A_n singularity we obtain an $SU(N+1)$ gauge theory as opposed to the $U(N+1)$ gauge theory that we would expect in type IIB. The fate of the abelian factor is a delicate one and it depends on global aspects of the compactification, see for instance [44] for a discussion in this direction.

Fibre type	ord(f)	ord(g)	ord(Δ)	Singularity
I_0	≥ 0	≥ 0	0	smooth
I_n	0	0	n	A_{n-1}
II	≥ 1	1	2	smooth
III	1	≥ 2	3	A_1
IV	≥ 2	2	4	A_2
I_0^*	≥ 2	≥ 3	6	D_4
I_n^*	2	3	$n+6$	D_{n+4}
IV^*	≥ 3	4	8	E_6
III^*	3	≥ 5	9	E_7
II^*	≥ 4	5	10	E_8
non minimal	≥ 4	≥ 6	≥ 12	non canonical

Table 1: Kodaira classification of singular fibres. ord denotes the order of vanishing at the singularity. The case of a non canonical singularity requires additional blow-ups in the base.

gauge theory may be easily explained: given a specific singularity it is possible to perform a series of blow–ups leading to a space which is no longer singular. In this situation the space acquires a set of homologically non–trivial complex curves (which always are Riemann spheres) which as proven by Kodaira intersect according to the extended Dynkin diagram of an ADE group. With this we can understand how the various elements of the gauge algebra can be obtained: the Cartan generators corresponds to the reduction of the C_3 form of M–theory on the Poincaré duals of the additional Riemann spheres present in the resolved geometry and the roots can be obtained by wrapping M2–branes on linear combinations of the Riemann spheres. This fills up completely the gauge algebra, and moreover going back to the singular limit the M2–branes will become massless providing the correct enhancement of the gauge group[22]. It is important to mention that if the base manifold has complex dimension larger than 1 novel phenomena may occur. It is possible to follow a similar strategy to prove that more general gauge groups may be realised in this case leading to the so–called Tate algorithm [45–48]. We choose not to copy the result of Tate algorithm which may be found in the aforementioned references, but simply quote that in this more general situation all gauge algebras may be realised, including B and C series as well as G_2 and F_4.

The increase of the complex dimension of the base manifold B leads also to the interesting possibility of colliding singularities, that is the possibility that in complex codimension 2 a pair of divisors carrying a singular elliptic fibre intersect on a complex curve. Intuition from the study of type IIB models suggests that at this intersection of 7–branes there ought to be localised matter, at least in the case where

22 In fact going in the resolved geometry corresponds to going in the Coulomb branch of the theory on the 7–branes leaving only the vector fields in the Cartan massless.

D7–branes intersect which is the case of colliding A–type singularities. The analysis performed in [49] shows that this indeed happens and also gives a simple algorithm for the deduction of the representation of the localised matter: at the intersection between divisors carrying G_1 and G_2 gauge groups there is an enhancement of the singularity to a gauge group $G_{1\cap 2}$ [23] containing both G_1 and G_2. In order to identify the representation of the localised matter we only need to consider the branching rule of the adjoint representation of $G_{1\cap 2}$ with respect G_1 and G_2

$$\begin{aligned} G_{1\cap 2} &\longrightarrow G_1 \times G_2\,, \\ \mathbf{adj}_{G_{1\cap 2}} &\longrightarrow (\mathbf{adj}_{G_1}, \mathbf{1}) \oplus (\mathbf{adj}_{G_2}, \mathbf{1}) \bigoplus_i (\mathbf{R}_{1,i}, \mathbf{R}_{2,i})\,. \end{aligned} \tag{2.24}$$

The result is that localised matter sits in the $\sum_i (\mathbf{R}_{1,i}, \mathbf{R}_{2,i})$ representation of $G_1 \times G_2$ consistently with the intuition drawn from type IIB.

Having discussed what happens in complex codimension 2 the next step is to go to complex codimension 3. In this case a further enhancement of the singularity is possible and corresponds to a point in the case where the base manifold has complex dimension 3. At these points which corresponds to the intersection of three curves carrying localised matter (we shall henceforth call these curves *matter curves*) a cubic interaction between the fields living on the three matter curves is generated, giving therefore a Yukawa interaction in the 4d theory. The generation of these interactions will be of utmost interest for our discussion in the following. The missing ingredient is, given a specific singularity in complex codimension 3, what kind of Yukawa interactions are generated. Group theory is again sufficient to determine the Yukawa interactions: in this case we need to consider which gauge invariant combinations of the matter fields can be formed out of $\mathbf{adj}^3_{G_{1\cap 2\cap 3}}$.

All of this discussion already gives a path towards the construction of $SU(5)$ GUT models in F–theory. We shall spell out the details for a generic $SU(5)$ model in the following but first we turn to the discussion of the duality between Heterotic String Theory and F–theory.

2.3.3 *F–theory via Heterotic String Theory*

As a matter of fact we shall not need the details of the duality between F–theory and Heterotic String Theory but it sure is worth mentioning the basic details of the duality. The original observation came already in [8] by simple inspection of the moduli spaces of Heterotic String on $\mathbf{T}^2$ and F–theory on an elliptic $K3$ surface. While it is possible to find F–theory duals for both Heterotic String Theories in our discussion we shall focus on Heterotic $E_8 \times E_8$ String Theory[24]. The easiest identification between moduli involves the Heterotic String coupling which in F–theory is identified with the volume of the base of the $K3$ surface (this base is a Riemann sphere which is the only Riemann surface with positive curvature). The remaining moduli on the Heterotic side are the complexified Kähler modulus ρ and complex structure τ of the $\mathbf{T}^2$ as well as the values

23 This assuming that at the intersection no non–minimal singularity appears. If this were the case it would be necessary to blow up the base manifold to reduce the singularity at the intersection to a minimal one.

24 See for instance [50] for the case of $Spin(32)/\mathbb{Z}_2$ Heterotic String Theory.

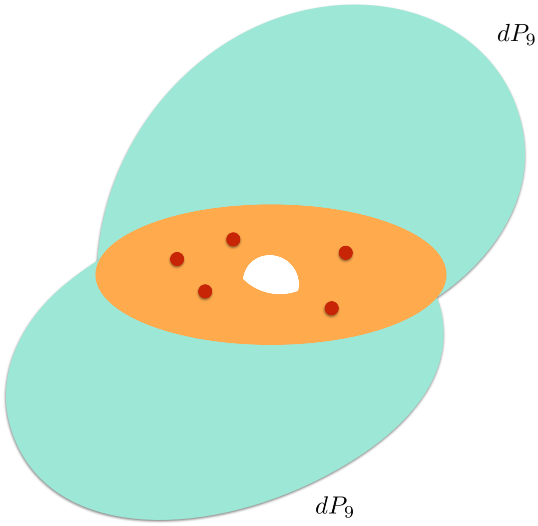

Figure 3: The stable degeneration limit for a K3 surface. The result is a pair of dP_9 surfaces intersecting on a $\mathbf{T}^2$. The marked points on the $\mathbf{T}^2$ will determine the Wilson line moduli on the Heterotic dual.

of the Wilson lines along the cycles of the torus. Together they form the following moduli space

$$\Lambda_{2,18}\backslash O(2,18,\mathbb{R})/(O(2,\mathbb{R})\times O(18,\mathbb{R})) \tag{2.25}$$

where $\Lambda_{2,18}$ takes into account the effect of T–dualities. The map between these moduli and the moduli entering in the definition of the elliptic $K3$ surface is actually a very intricate one which is very well understood only in the cases where either a particular limit for the $K3$ surface is taken (called the *stable degeneration limit* [51, 52]) or the unbroken gauge group on Heterotic side is $E_8\times E_8$ or $E_7\times E_8$. We shall not discuss the latter case (details may be found in [50, 53]) and focus on the former. The result of the stable degeneration limit is to produce a $K3$ surface consisting of a pair of dP_9 surfaces (sometimes called $\frac{1}{2}K3$ surfaces) which intersect along a $\mathbf{T}^2$. This torus is naturally identified with the $\mathbf{T}^2$ appearing in the Heterotic side and leaves open only the identification of the Wilson line moduli. Such information is encoded in the geometry of each dP_9 surface in the following way: a dP_9 surface contains an E_8 singularity which for a generic choice of complex structure moduli will be deformed to a smaller singularity. This will introduce a set of curves in the dP_9 surface which will intersect the $\mathbf{T}^2$ at some specific points which encode the choice of the Wilson lines on the $\mathbf{T}^2$. This suffices to specify the choice of Wilson lines for a single E_8 factor with the other E_8 factor taken into account by the second dP_9 surface. We show in Figure 3 a cartoon explaining the process of the stable degeneration limit.

The duality can be extended to lower dimension suitably fibering it on some base B: the statement becomes that Heterotic String Theory on a Calabi–Yau n–fold elliptically fibered over B is dual to F–theory on a Calabi–Yau $n+1$–fold which is an elliptic $K3$ fibration over B. This has led to many insights on both sides of the duality including the beautiful construction of (semi)stable holomorphic vector bundle for elliptically fibered Calabi–Yau manifolds [54, 55].

2.4 $SU(5)$ GUT MODELS IN F–THEORY

In this section we will spell out the main features of $SU(5)$ GUT models in F–theory, describing how the basic matter fields are embedded and how the couplings are generated. We will also describe the detail of how the GUT group is broken down to the Standard Model gauge group which is a crucial ingredient in the construction in F–theory models and constitutes a great difference between field theory GUT models and their realisation in F–theory.

The starting point is to realise an $SU(5)$ gauge theory in F–theory and to do so we need to consider an F–theory compactification on a Calabi–Yau fourfold that has an A_4 singularity of the elliptic fibration in complex codimension 1 in the base. As reviewed before matter fields appear on complex curves located at further enhancement of singularities in complex codimension 2 in the base, and for the case of an $SU(5)$ GUT model the following matter curves are necessary

- **10 matter curve.** This kind of matter curve is realised where there is an enhancement from A_4 to D_5. The branching rule for $Spin(10)$ to $SU(5) \times U(1)$ is in fact

$$\begin{aligned} Spin(10) &\longrightarrow SU(5) \times U(1) \\ \mathbf{45} &\longrightarrow \mathbf{24}_0 \oplus \mathbf{1}_0 \oplus \mathbf{10}_2 \oplus \overline{\mathbf{10}}_{-2}\,, \end{aligned} \tag{2.26}$$

  showing indeed that the matter fields transform in the **10** representation of $SU(5)$. This kind of local enhancement has a nice interpretation in terms of perturbative type IIB String Theory: the local enhancement to $Spin(10)$ in due to the intersection between the $SU(5)$ stack of D7–branes and the O7–plane, indeed by placing a stack of N D7–branes on top of an O7–plane we get an $SO(2N)$ gauge theory[25].

- **5 matter curve.** In this case the local enhancement of the singularity has to be from A_4 to A_5, which can be checked by inspection of the branching rule of $SU(6)$ to $SU(5) \times U(1)$

$$\begin{aligned} SU(6) &\longrightarrow SU(5) \times U(1) \\ \mathbf{35} &\longrightarrow \mathbf{24}_0 \oplus \mathbf{1}_0 \oplus \mathbf{5}_1 \oplus \overline{\mathbf{5}}_{-1}\,. \end{aligned} \tag{2.27}$$

  Again we can draw intuition from perturbative type IIB String Theory: in this case the local enhancement to $SU(6)$ is due to the intersection between the $SU(5)$ stack with an additional D7–brane and as already discussed matter in the fundamental representation of $SU(5)$ localise at the intersection.

25 Here we are being a bit imprecise about the global structure of the gauge group which will generically be a quotient of $Spin(2N)$ by a discrete subgroup. Since in perturbative type IIB String Theory no spinor representation is allowed it seems more natural to identify the group with $SO(2N)$ whereas in F–theory where spinor representations are possible it seems more natural to identify the group with $Spin(2N)$. In F–theory the global structure of any gauge group can be specified only when a global model is specified and it is encoded in the torsion part of the Mordell–Weyl group of the Calabi–Yau fourfold [56–58].

Our next item on the list are the Yukawa couplings that will appear in the 4d effective action. We know already that two class of couplings are necessary, but we shall also discuss a third one which has some nice phenomenological features

- **$\mathbf{10} \cdot \overline{\mathbf{5}} \cdot \overline{\mathbf{5}}$ coupling.** This coupling is necessary to generate a mass for the down–type quarks and the leptons when one of the $\overline{\mathbf{5}}$ representations is identified with the Higgs field H_d. The local enhancement in this case has to be to D_6 in agreement with the type IIB expectation: in this case the coupling is generated at the intersection of the $SU(5)$ stack, one O7–plane and an additional D7–brane and at this intersection the gauge symmetry would be $SO(12)$.
- **$\mathbf{10} \cdot \mathbf{10} \cdot \mathbf{5}$ coupling.** This coupling is responsible for the generation of a mass for the up–type quarks if the $\mathbf{5}$ representation is identified with the Higgs field H_u. In this case the local enhancement has to be E_6: absence of exceptional groups in perturbative type IIB String Theory is the reason why this coupling was absent in this setting. We see therefore that the presence of exceptional groups in F–theory is a crucial ingredient for the realisation of GUT models.
- **$\mathbf{1} \cdot \mathbf{5} \cdot \overline{\mathbf{5}}$ coupling.** This coupling exists if there is a local enhancement to A_6. While this coupling is not necessary to give mass to the Standard Model fermions it is possible to use it in relation to the solution of the μ–problem in the MSSM as well as to generate masses for the neutrinos. We will give additional details about this in Section 3.4.

Finally we turn to the implementation of GUT breaking mechanisms in F–theory which, as already anticipated, is remarkably different from the usual mechanisms employed in field theory models. The traditional way to break $SU(5)$ down to the Standard Model gauge group has always been via a Higgs mechanism with a Higgs field $\Phi_{\mathbf{24}}$ sitting in the adjoint representation of $SU(5)$ which acquires a vev along the direction of the hypercharge. It is possible to conceive situations where this kind of mechanism can also be implemented in F–theory models, for instance by using either brane deformation moduli or Wilson line moduli, but in both situations generating a potential stabilising open string moduli and triggering GUT breaking constitutes a major issue[26]. There is an additional mechanism (also available in perturbative type IIB String Theory) that we shall employ, which is the so–called hypercharge flux [10, 60]. The strategy is to add a magnetic flux in the hypercharge direction threading the $SU(5)$ stack. We shall now discuss in greater detail the principal points concerning hypercharge flux GUT breaking.

MASSLESS HYPERCHARGE. One main concerning the use of magnetic flux for GUT breaking is the possible generation of a mass for the hypercharge boson due to a coupling with 2–forms coming from the closed string sector. This is not dissimilar from the mechanism generating a mass for the abelian factor of $U(5)$ in perturbative type IIB String Theory, only that in this case this effect is clearly unwelcome. To

26 As a matter of fact it would be desirable to have no brane deformation nor Wilson line moduli for the stack generating the GUT group. The usual choice of a Del Pezzo surface to host the $SU(5)$ gauge theory ensures absence of these moduli together with asymptotic freedom of the theory, which is tied with the possibility of decoupling gravity [7, 59].

see how the hypercharge can become massive let us consider the following coupling appearing in the Chern–Simons action of a stack of D7–branes[27]

$$S_{CS} \supset \int_{D7} C_4 \wedge \mathrm{Tr}[F \wedge F]\,, \tag{2.28}$$

which, upon dimensional reduction for a stack of branes wrapped on the 4–cycle S gives the following term in the 4d effective action

$$\mathrm{Tr}\, T_Y^2 \int_{\mathbb{M}^4} F_Y \wedge c_2^i \int_S \overline{F}_Y \wedge \iota^* \omega_{2,i}\,. \tag{2.29}$$

Here we denoted F_Y the hypercharge boson field strength and with $\overline{F}_Y$ the hypercharge flux threading S, and moreover we reduced C_4 as $C_4 = c_2^i \wedge \omega_{2,i}$ where $\omega_{2,i}$ are a basis of harmonic 2–forms of the Calabi–Yau manifold. We see that a $B \wedge F$ coupling for the hypercharge is generated unless for all $\omega_{2,i}$ we have that

$$\int_S \overline{F}_Y \wedge \iota^* \omega_{2,i} = 0\,. \tag{2.30}$$

This constitutes a topological condition on the hypercharge flux $\overline{F}_Y$ which is equivalent to saying that the hypercharge flux $\overline{F}_Y$ is not the pullback of any 2–form of the Calabi–Yau manifold, or equivalently it is a 2–form trivial in the cohomology of the Calabi–Yau manifold. This is a condition that needs the specification of a complete model and is not easily implemented.

MASSIVE X AND Y BOSONS. While on the one hand we need to keep the hypercharge massless it is also necessary to give a sufficiently large mass to the X and Y bosons. Whatever effect breaks $SU(5)$ down to the Standard Model gauge group usually generates a mass for the X and Y bosons of order $\sim M_{GUT}$ which is more than enough to avoid too rapid proton decay. However the use of the hypercharge flux may generate a chiral spectrum and therefore leave massless X and Y bosons in the spectrum[28]. To avoid this we need to place the following condition on the hypercharge flux[29]

$$\chi(S, \mathcal{L}_Y^{5/6}) = 0\,, \tag{2.31}$$

where we called $\mathcal{L}_Y$ a line bundle whose first Chern class satisfies $c_1(\mathcal{L}) = \overline{F}_Y$ and S is the divisor supporting the GUT group. For any line bundle on a surface we can use Hirzebruch–Riemann–Roch theorem and Noether's formula to write

$$\chi(S, \mathcal{E}) = \int_S \frac{c_1(S)^2 + c_2(S)}{12} + \frac{1}{2} c_1(\mathcal{E}) \left[c_1(\mathcal{E}) + c_1(S)\right]\,. \tag{2.32}$$

27 Since hypercharge flux GUT breaking is possible in perturbative type IIB String Theory we can safely draw our conclusions using the action of D7–branes.

28 A chiral spectrum for the gauge bosons should not come as a surprise. At this level we are not braking supersymmetry and therefore also the X and Y gauginos will have a net chiral spectrum. This justifies the fact that the chiral index for the X and Y bosons may be computed by using an index for a Dirac operator.

29 Note that the hypercharge flux needs to satisfy a quite peculiar quantisation condition with our choice of T_Y.

For the case in which the GUT divisor S is a Del Pezzo surface (and taking into account that $\int_S c_1(\mathcal{L}_Y^{5/6})c_1(S) = 0$ to avoid a mass for the hypercharge) we find that the condition (2.31) becomes

$$\int_S c_1^2(\mathcal{L}_Y^{5/6}) = -2\,. \tag{2.33}$$

ABSENCE OF EXOTICA. Hypercharge flux should be used with sufficient care for in the cases when it starts threading some of the matter curves it may lead to the presence of incomplete GUT multiplets. This is due to the fact that matter fields sitting in the same representation of the GUT group carry different hypercharge and therefore their net chirality in the 4d effective action will be different if hypercharge flux does not integrate to zero on the matter curve. While this should clearly be avoided for all matter curves hosting the Standard Model fermions we can use this effect to directly implement the doublet–triplet splitting mechanism and avoid the presence of massless Higgs triplets. The upshot of this discussion is that the following conditions should be imposed on the hypercharge flux

$$\int_{\Sigma_{\mathbf{10}_M}} \overline{F}_Y = \int_{\Sigma_{\overline{\mathbf{5}}_M}} \overline{F}_Y = 0\,, \quad \int_{\Sigma_{H_u}} \overline{F}_Y \neq 0\,, \quad \int_{\Sigma_{H_d}} \overline{F}_Y \neq 0\,, \tag{2.34}$$

where we called the matter curve hosting the representation $\mathbf{R}$ as $\Sigma_{\mathbf{R}}$.

GAUGE COUPLING UNIFICATION. One curious effect of the use of the hypercharge flux first noted in [61] is that it may be an important effect spoiling gauge coupling unification. This is due to the fact that the gauge kinetic function for a stack of D7–branes on a cycle S will be affected by the presence of magnetic fluxes. More specifically the gauge kinetic functions for the three gauge factors of the Standard Model gauge group become

$$f_{SU(3)} = f_0\,, \quad f_{SU(2)} = f_{U(1)} = f_0 - \frac{\tau}{2}\int_S c_1(\mathcal{L}_Y^2)\,, \tag{2.35}$$

where we called f_0 the tree level gauge kinetic function

$$f_0 = \frac{e^{-\phi}}{2\pi l_s^4}\mathrm{Vol}(S) + i\mu_7 \int_S C_4\,. \tag{2.36}$$

(2.35) implies that in the presence of a hypercharge flux we may obtain a major violation of gauge coupling unification, a situation quite unsatisfactory for the empirical observation that gauge couplings unify at a scale $M_{GUT} \sim 10^{16}$ GeV was our starting point for the study of Grand Unified Theories. The situation concerning gauge coupling unification is however far more complicated for there is a whole plethora of effects that can produce sizeable deviation from the unification observed in the MSSM. One effect may come from the loop of massive modes, like KK modes, winding modes and massive string modes. Moreover it is necessary to consider that the presence of the hypercharge may also induce a mixing with additional $U(1)$ factors as we will discuss in Section 4.1. Finally the scale at which supersymmetry is broken and the mass scale of Higgs triplets and X and Y bosons play an important rôle as well. Summing up a complete picture of gauge coupling unification is still missing in the context of F–theory GUT models due to the complicated interplay between these many effects. It seems reasonable though that unification compatible with the observations at the electroweak scale may be possible, see for instance [62] for a more detailed account.

YUKAWA COUPLINGS UNIFICATION. One important issue with GUT models that we highlighted before is the unification of leptons and down–type quarks Yukawa couplings. This constitutes a major problem especially for the two lighter generations of leptons and down–type quarks. We shall return on this point later on when discussing the explicit form of Yukawa couplings in F–theory GUT models, however we can give a small anticipation of how this problem is solved when passing to F–theory. To do so it is necessary to recall that the $\mathcal{N} = 1$ 4d effective action can be described in terms of a Kähler potential K and a superpotential W. The superpotential is known to receive no contributions from fluxes in F–theory [63] and therefore the Yukawa couplings appearing in it will not be affected by the hypercharge flux leading to the aforementioned unification of couplings for leptons and down–type quarks. The same does not apply to the Kähler potential, and indeed its presence will induce a modification of the kinetic terms for the fermions which will depend on the hypercharge flux and therefore be different for leptons and quarks. After imposing canonical normalisation of the kinetic terms the Yukawa couplings will therefore differ and, as we shall discuss in more detail later on, will suffice to find agreement with the experimental results.

YUKAWA COUPLINGS IN F–THEORY 3

The aim of this chapter is to describe in detail the computation of Yukawa couplings in F–theory GUT models. Since we have seen that Yukawa couplings are actually generated at a single point it will be possible to specify a local model around the Yukawa point for the computation. The dynamics of the system will be encoded in the action living on the worldvolume of 7–branes and we shall see how supersymmetric solutions will be found by asking for the vanishing of 4d F–terms and D–terms. Given a particular solution of the supersymmetry equations it is possible to compute the internal wavefunctions for the zero modes by studying small fluctuations around the background, and after solving for these wavefunctions computation of Yukawa couplings simply boils down to a triple overlap integral between the zero modes. We shall mainly consider two class of local models: in the first class of models on the worldvolume of the 7–branes we will find a local E_6 enhancement at the Yukawa point which allows the computation of the Yukawa couplings for the up–type quarks. The second class of models will have a local E_7 or E_8 enhancement so that all the Yukawa couplings will be generated at a single point (or at least in the proximity of a single point). For both models we will provide a description of the phenomenological implications by performing a scan over the parameters specifying the local model and comparing with the empirical data.

3.1 7–brane effective action

It is surely a remarkable feature that the details concerning the 7–branes effective actions are fully constrained by supersymmetry for we can use the same action even in cases where we do not have an embedding in perturbative type IIB String Theory. This will be of essential importance for us as we will need local models described in terms of gauge theories with exceptional gauge groups where strong coupling effects will be present. The action on the 7–branes will simply be an 8d Yang–Mills theory that can be easily obtained by performing dimensional reduction of the 10d $\mathcal{N}=1$ Yang–Mills theory. The field content in 8d consists of a vector field and scalar field all sitting in the adjoint representation of the gauge group together with their fermionic superpartners. The adjoint scalar field has a nice geometrical interpretation: its profile on the 7–branes controls the configuration of the system in the transverse space. This gives the field content of a $\mathcal{N}=1$ 8d Yang–Mills theory which also enjoys a $U(1)_R$ global symmetry.

The details concerning the 7–brane action were obtained in [7]. To obtain a 4d theory it is necessary to consider compactification on a 4–cycle S which has to be holomorphic to preserve supersymmetry. In the case where we induce a magnetic flux on the 7–branes there will be extra some conditions on the gauge bundle to preserve supersymmetry to be described in the following. Since in general S will be

Field	$\mathcal{N}=1$ superfield	Differential form on S
A_μ	Vector	$\Omega^{(0,0)}(S)\otimes \mathrm{ad}(\mathcal{E})$
$A_{\bar{m}}$	Chiral	$\Omega^{(0,1)}(S)\otimes \mathrm{ad}(\mathcal{E})$
Φ	Chiral	$\Omega^{(2,0)}(S)\otimes \mathrm{ad}(\mathcal{E})$

Table 2: Matter content of 8d super Yang–Mills theory after the topological twist.

a curved manifold in order to preserve at least $\mathcal{N}=1$ supersymmetry in 4d it is necessary to consider the so–called topological twist on the worldvolume of the 7–branes. This amounts to taking into account the embedding of the 7–branes in the compactification space and the effect of the non–trivial normal bundle under which the supercharges transform. To go into more details we need to consider the fact that since S is a Kähler manifold its structure group is $U(2)$. The topological twist consists of a redefinition of the abelian factor of $U(2)$ (whose generator we call J) with the R–symmetry $U(1)_R$ (whose generator we call R) giving the new twisted generator $J_{top}\equiv J+2R$. This replacement ensures that at least one scalar supercharge exists and converts all bosons and fermions on S into differential forms. We summarise the matter content in Table 2 allowing also for the possibility of a non–trivial gauge bundle $\mathcal{E}$.

We find that the matter fields transform as expected from the analysis of the action of D7–branes in perturbative type IIB String Theory. It is also possible to perform a counting of the massless modes present in the 4d effective field theory. The result is that these modes may be suitably counted by the following Dolbeault cohomology groups

$$\underbrace{H^{0,0}_{\bar{\partial}}(S,\mathrm{ad}(\mathcal{E}))}_{A_\mu}\oplus\underbrace{H^{0,1}_{\bar{\partial}}(S,\mathrm{ad}(\mathcal{E}))}_{A_{\bar{m}}}\oplus\underbrace{H^{2,0}_{\bar{\partial}}(S,\mathrm{ad}(\mathcal{E}))}_{\Phi}\ . \tag{3.1}$$

Even in the case where $h^{0,1}(S,\mathrm{ad}(\mathcal{E}))=0$ or $h^{2,0}(S,\mathrm{ad}(\mathcal{E}))=0$ (as in the example of Del Pezzo surfaces) we can still employ a local description in terms of $A_{\bar{m}}$ and Φ in a sufficiently small small neighbourhood of the Yukawa point.

As we already said the scalar field Φ encodes the geometry of the system of 7–branes. We shall now give a brief overview of how (part of) the data encoded in Φ will appear in the geometry in F–theory for the cases of ADE singularities.

3.1.1 *The local geometry of 7–branes*

The purpose of this section is to give a brief glimpse of how the geometry of system of 7–branes appears locally in a ADE singularity. It is important to note that for some particular configurations part of the set of data contained in Φ will not directly appear in the geometry. These "invisible" data have been called *T–brane data* [64], see [65–67] for attempts at describing them within F–theory. Neglecting the possible presence of T–brane data we start by giving a local description of an ADE singularity in terms of an algebraic variety inside $\mathbb{C}^3$. We collect in Table 3 the defining polynomials for the various ADE singularities taking x, y and z as coordinates in $\mathbb{C}^3$.

A_n	$y^2 = x^2 + z^{n+1}$
D_n	$y^2 = x^2 z + z^{n-1}$
E_6	$y^2 = x^3 + z^4$
E_7	$y^2 = x^3 + xz^3$
E_8	$y^2 = x^3 + z^5$

Table 3: Defining polynomials in $\mathbb{C}^3$ for the various ADE singularities.

A_n	$y^2 = x^2 + z^{n+1} + \sum_{k=2}^{n+1} \alpha_k z^{n+1-k}$
D_n	$y^2 = x^2 z + z^{n-1} + \sum_{k=1}^{n-1} \delta_{2k} z^{n-k-1} - 2\gamma_n x$
E_6	$y^2 = x^3 + \frac{z^4}{4} + \epsilon_2 x z^2 + \epsilon_5 x z + \epsilon_6 z^2 + \epsilon_8 x + \epsilon_9 z + \epsilon_{12}$
E_7	$y^2 = -x^3 + 16 x z^3 + \epsilon_2 x^2 z + \epsilon_6 x^2 + \epsilon_8 x z + \epsilon_{10} z^3 + \epsilon_{12} x + \epsilon_{14} z + \epsilon_{18}$
E_8	$y^2 = x^3 - z^5 + \epsilon_2 x z^3 + \epsilon_8 x z^2 + \epsilon_{12} z^3 + \epsilon_{14} x z + \epsilon_{18} z^2 + \epsilon_{20} x + \epsilon_{24} z + \epsilon_{30}$

Table 4: Defining polynomials in $\mathbb{C}^3$ for the various deformed ADE singularities.

The effect of a non trivial profile for Φ is to produce a deformation of the aforementioned singularities and we collect the general form of deformed ADE singularities in Table 4.

The constants α_i for the A series, δ and γ for the D series and ϵ for the E series are simply the Casimir invariants of the corresponding Lie algebra of Φ. We can give a simple description of the Casimir invariants for the case of the A series, in fact we have that

$$\det(z\mathbb{I}_n - \Phi) = z^{n+1} + \sum_{k=2}^{n+1} \alpha_k z^{n+1-k} \tag{3.2}$$

allowing to express the various α's in terms of homogeneous symmetric polynomials of the eigenvalues of Φ. A similar story occurs for the D series where by specifying the value of Φ in the various components of the Cartan algebra of $\mathfrak{so}(2N)$ (we call these values t_i with $i = 1, \ldots, n$) we have that

$$\delta_{2k} = s_k(t_1^2, \ldots, t_n^2)\,, \quad \gamma_n = \prod_{i=1}^{n} t_i\,. \tag{3.3}$$

Note that γ_n is simply the Pfaffian of Φ. The form of the Casimir invariants for the E series can likewise be expressed in terms of homogeneous symmetric polynomials although the expressions here become extremely more complicated, see for instance [68] for the explicit expressions[1].

3.1.2 *BPS configurations*

Having understood how the geometry of the system of 7–branes is encoded in the geometry of a deformed ADE singularity the next step is to specify the conditions

1 We will give an example of a deformation of an E_6 singularity in Appendix C.

that determine whether a given configuration is supersymmetric or not. We will encounter two kinds of conditions, the first kind correspond to the vanishing of F–terms in the 4d effective action and can be derived from a superpotential and the second kind correspond to the vanishing of D–terms in 4d. The superpotential of the configuration of 7–branes has the following form

$$W = m_*^4 \int_S \mathrm{Tr}\left[\Phi \wedge F\right] \,, \tag{3.4}$$

where we introduced the F–theory characteristic scale m_* and $F = dA - iA \wedge A$. Variation of the superpotential with respect to the two fields appearing in it yields the two BPS equations

$$\bar{\partial}_A \Phi = 0 \,, \tag{3.5}$$

$$F^{0,2} = 0 \,, \tag{3.6}$$

where $\bar{\partial}_A = \bar{\partial} - i[A, \cdot]\wedge$ and the superscript in F denotes its Hodge type. These equations which ensure the vanishing of F–terms imply that for supersymmetric configurations the gauge bundle $\mathcal{E}$ endowed on the brane must be a holomorphic bundle and that Φ has to be a holomorphic section of $\Omega^{2,0}(S) \otimes \mathrm{ad}(\mathcal{E})$. Note that these equations simply come from the dimensional reduction of the F–term equations of 10d super Yang–Mills theory with the identification of $\Phi = A_{\bar{z}}$ (here we call z the coordinate normal to the 7–branes in the base manifold) after dropping all dependance on z for the fields. In addition to the F–term equations we need to ensure vanishing of the D–terms which for the case of a system of 7–branes are

$$D = \int_S \omega \wedge F + \frac{1}{2}[\Phi, \Phi^\dagger] \,, \tag{3.7}$$

where we called ω the Kähler form of S. Again it is possible to draw a comparison with supersymmetric solutions of 10d super Yang–Mills: in this case the D–term equations ensure stability (or at least polystability) of the gauge bundle appearing in compactifications of 10d super Yang–Mills on Calabi–Yau manifolds [69, 70]. Construction of stable vector bundles on Calabi–Yau manifolds is a notoriously difficult problem and we shall see in the following that part of these difficulties will appear when looking for solutions of the D–term equations in the case when $[\Phi, \Phi^\dagger] \neq 0$.

In the following when specifying a given model we will always look for solutions of the combined system of F–term and D–term equations. Moreover let us stress two additional important points: by taking the linear approximation of the BPS equations we will be able to obtain the set of differential equations obeyed by the zero modes of a given background which will be our starting point for the computation of Yukawa couplings. In addition to this the information concerning the Yukawa couplings between the various zero modes is already included in the superpotential which includes a cubic coupling

$$W \supset -i m_*^4 \int_S \mathrm{Tr}\left[\Phi \wedge A \wedge A\right] \,. \tag{3.8}$$

This shows how Yukawa couplings may be simply computed by triple overlap of internal wavefunctions, and in the case where these wavefunction localise in a sufficiently small patch around the Yukawa point it will be possible to perform the computation without knowing the details of the topology of S.

Non–perturbative corrections to the superpotential

Since we have discussed the form of the BPS equations for a system of 7–branes it seems appropriate to stop a moment and discuss how non–perturbative physics affects the superpotential. The reason we are interested in the impact of non–perturbative physics in the superpotential is that as we will show more explicitly later on in this chapter the Yukawa matrices computed by dimensional reduction of the term (3.8) have rank one. This feature is quite interesting from a phenomenological point of view: in fact in this situation only one of the generations of leptons and quarks would couple to the Higgs fields and therefore only generation would acquire mass upon electroweak symmetry breaking[2]. This is a clear hint towards a possible generation of hierarchies in the fermionic sector: having only one generation massive would yield a hierarchical spectrum of masses for the fermions if the other masses (or equivalently Yukawa couplings to the Higgs fields) are generated via subleading effects. We shall see in the following that inclusion of non–perturbative effects in the superpotential suffices to generate masses for all generations of fermions and provide directly a hierarchy in the mass spectrum. We will discuss now explicitly how non–perturbative effects will deform the superpotential following [71].

All our reasoning will be in perturbative type IIB String Theory where we have a better control over non–perturbative effects. In the case at hand we will have two possible sources of non–perturbative effects whose backreaction will affect the 7–brane superpotential, namely euclidean D3–brane instantons and gaugino condensation on D7–branes. Since both sources of non–perturbative physics are quite similar we shall be able to treat them on the same ground. Consider the case where on a particular divisor S_{np} there is a stack of N D7–branes undergoing gaugino condensation. This process induces non–perturbative superpotential of the form[3]

$$W_{np} = \mu_3 \mathcal{A}\, e^{-T_{D7}/N}\,, \tag{3.9}$$

where μ_3 is the tension of a D3–brane and $\mathcal{A}$ is some function of the closed string moduli. Finally T_{D7} is the tree level gauge kinetic function of the stack of D7–branes. Similar considerations apply to the case of an euclidean D3–brane instanton where we simply need to consider the case $N = 1$. In (3.9) we have already taken into account the 1–loop effects due to closed string moduli which are included in the function $\mathcal{A}$, however in the presence of additional D7–branes carrying magnetic fluxes there will be additional 1–loop corrections that will alter the gauge kinetic function T_{D7}. In the case where the additional D7–branes wrap a divisor S we find that(3.9) becomes and becomes

$$W_{np} = \mu_3 \mathcal{A}\, e^{-T_{D7}/N} e^{-\frac{1}{8\pi^2 N}\int_S \mathrm{Str}(\log h \wedge F \wedge F)}\,, \tag{3.10}$$

where $h(z)$ is the divisor function of S_{np} and Str denotes the symmetric trace. We see explicitly that these corrections will depend on the position moduli of the D7–branes

2 We are here assuming that all the three generations coming from the $\mathbf{10}_M$ and $\bar{\mathbf{5}}_M$ of $SU(5)$ reside on a single matter curve. With more matter curves and more Yukawa points this does not necessarily happen, but this would require more complicated topologies for the GUT divisor S and moreover would not explain the hierarchy of masses for the fermions of the Standard Model.

3 We are considering the case where the stack of D7–branes carries a $U(N)$ gauge theory. In more general cases we expect N in (3.9) to be replaced with the dual Coxeter number of the gauge group.

wrapping S. Taking a Taylor expansion on S (and taking into account the prescription of [72] for a non–abelian stack of D7–branes on S) we obtain the superpotential[4]

$$W_{np} = \cdots + m_*^4 \frac{\epsilon}{2} \sum_{n\geq 0} \int_S \theta_n \,\mathrm{Str}\,(\Phi_{xy}^n F \wedge F)\,, \tag{3.11}$$

where the dots include all terms not depending on the open string moduli. In (3.11) we defined the following quantities[5][6]

$$\epsilon = \mathcal{A}\, e^{-T_{D7}/N} h_0^{N_{D3}/N}\,, \quad \theta_n = \frac{g_s^{-\frac{n}{2}} \mu_3/N}{(2\pi)^{2+\frac{3n}{2}} m_*^{4+2n}} [\partial_z^n \log(h/h_0)]|_{z=0}\,, \tag{3.12}$$

where $h_0 = \int_S h$ is the average value of h on S and $N_{D3} = \frac{1}{8\pi^2}\int_S F \wedge F$ is the total smeared D3–brane charge induced on S. Among the various terms appearing in the non–perturbative superpotential the leading term is clearly the one containing θ_0 for the other ones will contain additional negative powers of m_* and will be therefore more suppressed. It is important to note however that if θ_0 were to be constant (which would happen in the case when the divisor function h of S_{np} is constant on S) the term containing θ_0 in the superpotential would be topological and therefore would not affect the dynamics of the system. Keeping in mind that to actually generate a superpotential it is necessary to have a non trivial intersection between S_{np} and another divisor S' hosting some 7–branes we find two possible scenarios[7]

- The divisor S' has no intersection with S, which implies that θ_0 is constant;
- The divisor S' intersects S, then in this case θ_0 is not constant on S.

In the former case it would be necessary to go beyond leading order and consider the term containing θ_1 in the non–perturbative superpotential, however it happens that for gauge groups in the D series and the E series (which happens to be the case of interest for us) this term automatically vanishes due to the fact the symmetric trace of the product of an odd number of generators vanishes. The term containing θ_2 would be the leading term in this situation, however the prospect does not look so favourable as in [74] for the case of the $SO(12)$ Yukawa point it was shown that this term alone does not suffice to generate an adequate hierarchy for the masses of fermions.

Therefore in the following we shall consider the second situation and take the term with θ_0 as the leading correction to the superpotential, which now takes the form

$$W = m_*^4 \int_S \mathrm{Tr}(F \wedge \Phi) + \frac{\epsilon}{2}\theta_0 \mathrm{Tr}(F \wedge F)\,. \tag{3.13}$$

Note that due to this correction the BPS equations will now be altered. We shall not concern ourselves very much with this issue for given a solution with $\epsilon = 0$ it

4 Here and in the following Φ_{xy} denotes the components of Φ which is a differential form, namely $\Phi = \Phi_{xy} dx \wedge dy$ where x and y are two local coordinates on S.
5 Here z is a local coordinate such that $z = 0$ corresponds to S.
6 In writing the expression for θ_n we made use of the relation $m_{st}^4 = g_s (2\pi)^3 m_*^4$ derived in [73].
7 This is what we would expect from the result obtained from type IIB String Theory for if S' hosts an O7–plane in a weak coupling limit (if applicable) then the instanton would become an O(1) instanton which indeed has the correct zero mode structure to generate a superpotential. Intersection between S and S_{np} are not allowed because this would invalidate the reasoning that led us to the form of the non–perturbative superpotential.

is possible to find a solution in perturbation theory in ϵ, which is reasonable for ϵ is a small parameter controlling the strength of a non–perturbative effect. We will nonetheless come back to the effect of ϵ corrections to the solutions of BPS equations to show that they do not alter significantly the Yukawa couplings that we will compute. The situation is quite different for the wavefunctions of the zero modes for ϵ corrections will alter such wavefunctions in a significant way and be responsible for an increase in the rank of the Yukawa matrix. Finally one comment about D–terms: we may think at this point that the D–terms for the 7–branes on S will be also altered by the presence of non–perturbative effects, however this is not case as shown in [74]. In any case we would expect a whole plethora of additional corrections to the D–terms (with strength superior to the one of non–perturbative physics), as for instance α' effects due to curvature corrections and the presence of magnetic fluxes. However in the following we will take care of being in a regime with low curvature and diluted magnetic fluxes so that we shall neglect these additional effects.

The T-brane background

A common aspect of our discussion in the following will be the introduction of some particular backgrounds for the adjoint scalar Φ such that $[\Phi, \Phi^\dagger] \neq 0$[8]. By inspection of the D–term equations we see that to obtain a supersymmetric solution it is necessary to turn on some fluxes cancelling the Fayet–Iliopoulos term generated by the non–vanishing commutator between Φ and its adjoint. The particular case where $[\Phi, \Phi^\dagger] \neq 0$ was considered in [64] and dubbed as T–brane background. It is important to note that for a T–brane background there is not a one to one map between the entries of Φ and the Casimir invariants of the gauge algebra $\mathfrak{g}$ and therefore it may occur that part of the data contained in Φ does not directly appear in the form of the elliptic fibration close to the 7–branes, see [65–67] for attempts at understanding how to incorporate these missing data in the geometry.

The solution of the BPS equations in a T–brane background constitutes in general a quite complicated problem and in the following we will follow a two step process introduced in [64] to find a solution: first we specify the form of Φ in a specific gauge called holomorphic gauge. This gives a simple solution to the F–term equations leaving only the D–term equations to be solved. This can be attained by performing a suitable gauge transformation and the result is that the D–term equations will be translated in a set of differential equations for this gauge transformation.

We start by considering how the fields appearing on the worldvolume of the 7–branes transform under a gauge transformation

$$A \longrightarrow g\,A\,g^{-1} + i\,g\,d\,g^{-1}\,, \tag{3.14a}$$

$$\Phi \longrightarrow g\,\Phi\,g^{-1}\,. \tag{3.14b}$$

In writing (3.14) we take g to take value in the gauge group G which is the correct symmetry group of the theory. However for the moment we will consider a more general situation and allow for the more general possibility of $g \in G_{\mathbb{C}}$ where $G_{\mathbb{C}}$ is the complexified gauge group which can be obtained via exponential map from

8 The possibility of having such backgrounds is due to the fact that 7–branes have two transverse coordinates, that is, by expressing Φ as two real scalars $\Phi = \Phi_8 + i\Phi_9$, we have that for this kind of backgrounds $[\Phi_8, \Phi_9] \neq 0$. The spacetime interpretation of this kind of solutions is somehow mysterious as geometric intuition fails.

the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}} \equiv \mathfrak{g} \otimes \mathbb{C}$. Taking $g \in G_{\mathbb{C}}$ does not give a symmetry transformation of the theory as the gauge field ceases to be real and moreover the D–term equations (unlike the superpotential) are not invariant under transformations in $G_{\mathbb{C}}$. The chief advantage of using complexified gauge transformations is that it is possible to reach a gauge, usually dubbed as *holomorphic gauge*, where $A^{0,1} = 0$. The main simplification that justifies the introduction of this particular gauge is that any choice of holomorphic Φ constitutes a solution of the F–term equations.

To explain how to solve the D–term equations it is convenient to consider in detail a specific example that will reappear in all the models we will analyse for the computation of Yukawa couplings. We take Φ to lie in a $\mathfrak{su}(2)$ subalgebra[9] of the gauge algebra $\mathfrak{g}$. To describe Φ it is convenient to use a matrix notation representing it as a 2×2 matrix which we take of the form

$$\Phi = \begin{pmatrix} 0 & m \\ m^2 x & 0 \end{pmatrix}, \tag{3.15}$$

where m is a constant with the units of mass and we introduced local coordinates x and y in a local patch U of the divisor S. Since we are in a holomorphic gauge this choice of Φ automatically yields a solution of the F–term equations, and moreover we have that

$$[\Phi, \Phi^\dagger] = \begin{pmatrix} m^2(1 - m^2|x|^2) & 0 \\ 0 & -m^2(1 - m^2|x|^2) \end{pmatrix} = m^2(1 - m^2|x|^2)\sigma_3\,. \tag{3.16}$$

To solve the D–term equations we need to move away from the holomorphic gauge by performing a gauge transformation $g \in G_{\mathbb{C}}$ to reach a gauge where the gauge fields are real. The gauge transformation we consider has the form

$$g = \exp\left[\frac{f}{2}\sigma_3\right], \tag{3.17}$$

where f is a function yet to be specified. By performing this gauge transformation we can attain a unitary gauge where the background fields have the following form

$$\Phi = \begin{pmatrix} 0 & m\, e^f \\ m^2 x\, e^{-f} & 0 \end{pmatrix}, \qquad A = \frac{i}{2}(\partial f + \bar{\partial} f)\, \sigma_3\,. \tag{3.18}$$

Note that the background is almost entirely specified leaving only the function f as an unknown. However we still have to impose the D–term equations which will become a differential equation for f. To see this we further need to specify the form of the Kähler form on S that appears in the D–term equations. Since we are taking only a local patch U of S and approximating $U \simeq \mathbb{C}^2$ we take ω as

$$\omega = \frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y})\,. \tag{3.19}$$

This implies that, since in the background (3.18) we have that

$$[\Phi, \Phi^\dagger] = m^2(e^f - e^{-f} m^2|x|^2)\, \sigma_3\,, \quad F = -i\partial\bar{\partial} f \sigma_3 \tag{3.20}$$

9 That is the only components of $[\Phi, \Phi^\dagger]$ lie in an $\mathfrak{su}(2)$ algebra.

the D–term equations reduce to the following equation

$$(\partial_x \partial_{\bar{x}} + \partial_y \partial_{\bar{y}})\, f = m^2 (e^{2f} - m^2 |x|^2 e^{-2f})\,. \tag{3.21}$$

This is indeed a quite complicated differential equation but we will see now how it is possible to find an exact solution. First we switch to to polar coordinates $x = re^{i\theta}$ in the x-plane and take the ansatz $f = f(r)$ which gives the simpler equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) f = 4m^2 (e^{2f} - m^2 r^2 e^{-2f})\,. \tag{3.22}$$

We find it convenient to define the function $h(r)$ such that

$$e^{2f(r)} = mre^{2h(r)} \tag{3.23}$$

turning (3.22) into

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}\right) h = 8m^3 r \sinh(2h)\,. \tag{3.24}$$

The last step is to perform the change of variables $s = \frac{8}{3}(mr)^{3/2}$ giving

$$\left(\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds}\right) h = \frac{1}{2}\sinh(2h)\,. \tag{3.25}$$

This equation is a particular instance of the Painlevé III differential equation which already appeared for instance in [75–77] in the physics literature. The solution is known in the mathematics literature [78] and it is unique if we require that the solution extends to the whole r axis without any singularity. For our future purposes it is sufficient to know a local form for the function f around $r = 0$

$$f(r) = \log c + c^2 m^2 x\bar{x} + m^4 (x\bar{x})^2 \left(\frac{c^4}{2} - \frac{1}{4c^2}\right) + \dots\,. \tag{3.26}$$

Here c is a constant that specifies the asymptotic value of f near the origin and absence of singularities in f fixes the value of c to be

$$c = 3^{1/3}\frac{\Gamma\left[\frac{2}{3}\right]}{\Gamma\left[\frac{1}{3}\right]} \sim 0.73\,. \tag{3.27}$$

Nevertheless it is important that in this simplified setting we are neglecting important effects that will alter the form of the D–term equations (and therefore of f) when we move away from the origin of our local patch $U \simeq \mathbb{C}^2$. This implies that fixing c to the particular value (3.27) is by no means necessary and in our upcoming discussion of models for Yukawa couplings we will leave open the possibility of taking c different from this particular value.

This concludes our example of a particular T–brane system, a system that we will meet again in the following. The procedure we described for the solution of D–term equations in a T–brane background applies to other cases as well, see for instance [64, 79] for other examples.

3.2 ZERO MODE WAVEFUNCTIONS AND YUKAWA COUPLINGS

Our main interest will be in the computation of Yukawa couplings between the various zero modes, and in order to perform such computation it is necessary to first understand how the internal wavefunctions for the zero modes are computed. To obtain the set of differential equations obeyed by the zero modes it is sufficient to take the linear approximation to the BPS equations around a given background, namely to expand Φ and A as $\Phi = \langle\Phi\rangle + \varphi$ and $A = \langle A\rangle + a$ and retain only the terms linear in φ and a. The resulting set of differential equations neglecting non–perturbative corrections for the moment is

$$\bar{\partial}_{\langle A\rangle} a = 0\,, \tag{3.28a}$$

$$\bar{\partial}_{\langle A\rangle} \varphi = i[a, \langle\Phi\rangle]\,, \tag{3.28b}$$

$$\omega \wedge \partial_{\langle A\rangle} a = \frac{1}{2}[\langle\bar{\Phi}\rangle, \varphi]\,. \tag{3.28c}$$

We shall discuss in great detail strategies to solve these equations in the following, for the explicit form of the solution is highly dependant on the form of the background values $\langle\Phi\rangle$ and $\langle A\rangle$. At this point it is surely worth mentioning how the appearance of localised zero modes occurs. It is clear that the system (3.28) admits as solutions constant wavefunctions that lie in the commutant of $\langle\Phi\rangle$ and $\langle A\rangle$ but these solutions do not actually localise on any complex curve and would correspond to bulk modes in S. More interesting is the case where $\langle\Phi\rangle$ is chosen to reproduce the case of intersecting branes: in this situation the value of $\langle\Phi\rangle$ would via a Higgs mechanism break the original gauge group G down to the commutant $H_1 \times H_2$ with a restoration of the original gauge symmetry on a codimension 1 sublocus Σ. As we shall see explicitly with specific examples in this situation there exist solutions of (3.28) that localise on Σ, and these localised solutions will be our main interest in the following. In the case of interest for the computation of Yukawa couplings there will appear a more intricate system of matter curves and the original gauge symmetry will be restored only in codimension 2 on S, namely at a single point. Moreover via an appropriate choice of magnetic fluxes it will be possible to ensure that some of the zero modes will localise on an open patch containing the Yukawa point.

It is important to stress that the description of zero modes we are employing is accurate only in regimes where the intersection angles (and flux densities)[10] are small when compared to the string scale, as otherwise there would appear corrections to the D–term equations that would invalidate the description we have given. It is worthwhile to mention that in the opposite regime of large angles it is possible to obtain a different description of the zero modes as topological defects localised on matter curves Σ. The presence of these defects induces a modification of the original BPS equations which we choose not to discuss here, see for example [7] for more details.

Knowledge of the internal wavefunctions suffices for the computation of Yukawa couplings, and we shall give now the resulting expression. To compute the Yukawa

10 The necessity of small values for the intersection angles is directly related to the necessity of small flux densities, in fact in models where a T–dual description in type I String Theory exists the value of the slopes of the branes directly translate to flux densities.

couplings it is actually sufficient already mentioned before to consider the following term in the superpotential

$$W \supset -im_*^4 \int_S \mathrm{Tr}(\Phi \wedge A \wedge A) \tag{3.29}$$

and insert the values of the fluctuations. The reason why we neglect other terms in the superpotential is that they all vanish due to the fact that we are considering zero modes[11]. To obtain an expression for the Yukawa couplings it is convenient to expand all modes in elements of the gauge algebra as follows

$$a = (a_{\bar{x}}^{\alpha} d\bar{x} + a_{\bar{y}}^{\alpha} d\bar{y}) E_{\alpha} + \text{h.c.}\,, \qquad \varphi = \varphi_{xy}^{\alpha}\, dx \wedge dy\, E_{\alpha}\,. \tag{3.30}$$

Inserting the zero modes in (3.29) we obtain

$$\begin{aligned} W \supset &- im_*^4 \int_S \mathrm{Tr}(\varphi \wedge a \wedge a) \\ = &- im_*^4 \int_S dx \wedge dy \wedge d\bar{x} \wedge d\bar{y}\, \mathrm{Tr}([E_\alpha, E_\beta] E_\gamma) \det \begin{pmatrix} a_{\bar{x}}^{\alpha} & a_{\bar{x}}^{\beta} & a_{\bar{x}}^{\gamma} \\ a_{\bar{y}}^{\alpha} & a_{\bar{y}}^{\beta} & a_{\bar{y}}^{\gamma} \\ \varphi_{xy}^{\alpha} & \varphi_{xy}^{\beta} & \varphi_{xy}^{\gamma} \end{pmatrix}. \end{aligned} \tag{3.31}$$

The form of the Yukawa couplings we found suggests that it is convenient to group the the modes in the following form

$$\Psi^{\alpha} = \begin{pmatrix} a_{\bar{x}}^{\alpha} \\ a_{\bar{y}}^{\alpha} \\ \varphi_{xy}^{\alpha} \end{pmatrix}, \tag{3.32}$$

a notation that we shall employ extensively in the following. Using the definition of the structure constants of the algebra $f_{\alpha\beta\gamma} = -i\mathrm{Tr}([E_\alpha, E_\beta]E_\gamma)$ and introducing the volume form on S as $d\mathrm{vol}_S = \frac{1}{(2i)^2} dx \wedge dy \wedge d\bar{x} \wedge d\bar{y}$ we can rewrite the cubic interaction term as follows

$$W_{cubic} = m_*^4 f_{\alpha\beta\gamma} \int_S d\mathrm{vol}_S \det(\Psi^\alpha, \Psi^\beta, \Psi^\gamma)\,. \tag{3.33}$$

This our final form for the Yukawa couplings and the one we shall use in the following. Some comments are due regarding the form of the cubic superpotential: it does not seem at first sight that the Yukawa interactions actually localise at any point on S and therefore this localisation property appears to rely only on the sufficient localisation of the wavefunctions around the triple intersection of the matter curves hosting them. However as we will show in a moment it is possible to rewrite the cubic Yukawa interaction as a complex residue evaluated at the Yukawa point: this demonstrates how the Yukawa interactions are of topological nature (and therefore protected) and moreover that the information of the cubic interaction is fully contained at the Yukawa point and no property of localisation of the wavefunctions is needed for the computation. As an additional consequence this can be used to prove that the terms appearing in the superpotential are altogether independent of magnetic fluxes without any information needed on their explicit form. Secondly we would

11 Massive modes, such as KK modes as well as modes that acquire mass due to the background values of Φ and A would have a quadratic term in the superpotential.

like to stress that there is actually an important difference between the Yukawa interactions appearing in the superpotential and the actual Yukawa couplings: to mark the difference between these two quantities we shall call the former *holomorphic* Yukawa couplings (a name chosen to emphasise the appearance in the superpotential) and the latter *physical* Yukawa couplings. The difference is due to the fact that the actual couplings appearing in the 4d Lagrangian will receive contributions from the Kähler potential as well. In our setup this dependance is encoded directly in the kinetic terms of the fermions which will in general be non–canonical (and depend explicitly on the flux densities) and physical Yukawa couplings will be calculable only upon choosing correct normalisation for the 4d fields (thus inducing a dependance of the physical Yukawa couplings on the flux densities). Finally it is necessary to address the effect of non–perturbative corrections to the superpotential: it was shown in [74] for the θ_0 term how the non–perturbative corrections conspire as to not affect the form of the holomorphic Yukawa couplings which still remains (3.33). What changes in this situation is the profile of the internal wavefunctions which is distorted by the presence of non–perturbative corrections: this effect alone will be responsible for the increase of the rank of the matrix of Yukawa interactions.

3.2.1 *Holomorphic Yukawa couplings as residues*

As anticipated holomorphic Yukawa couplings may be computed as complex residues evaluated at the Yukawa point and we will now prove this statement. Since we are interested in quantities appearing in the superpotential we can again to into the holomorphic gauge without worrying about altering the final results. This produces a drastic simplification in the F–term equations for the fluctuations which now become

$$\bar{\partial}\varphi - i[a, \langle\Phi\rangle] = 0\,, \tag{3.34a}$$
$$\bar{\partial}a = 0\,. \tag{3.34b}$$

In a local patch of S we can find a solution to the second equation by using a version of the Poincaré lemma adapted for complex manifolds, which implies that $a = \bar{\partial}\xi$ for some function ξ in the adjoint of the gauge group. Plugging this in the first equation we can equally find a solution for φ which takes the form

$$\varphi = h - i[\langle\Phi\rangle, \xi]\,, \tag{3.35}$$

where $h \in \Omega^{(2,0)}(S)$ is a holomorphic form in the adjoint of the gauge group. It is remarkable that we have been able to find the general solution for the zero modes, however the set of zero modes we reach in this way includes both bulk modes and localised modes and we would like to select only the latter. Since we know that the zero modes localise on some specific complex curves Σ we expect that all the information of their wavefunction may be contained in a sufficiently small neighbourhood of the respective matter curve Σ. Equivalently we can say that upon excision of the matter curve from S the information of the localised modes ought to disappear, or equivalently become pure gauge. Since under infinitesimal gauge transformations the zero modes transform as

$$a \longrightarrow a + \bar{\partial}_{\langle A\rangle}\chi\,, \tag{3.36a}$$
$$\varphi \longrightarrow \varphi - i[\langle\Phi\rangle, \chi]\,, \tag{3.36b}$$

following the previous criterion we find that a zero mode localised on Σ satisfies

$$\varphi = -i\left[\langle\Phi\rangle, \frac{\eta}{f^n}\right], \tag{3.37}$$

where $f = 0$ is the location of Σ, n is an integer and η is a holomorphic function in the adjoint of the gauge group. (3.37) indeed implies that any localised mode is gauge equivalent to zero away from its matter curve but the gauge transformation becomes singular at the location of the matter curve[12].

At this point it is possible to give a proof of the residue formula for the holomorphic Yukawa couplings. To do so we consider the cubic term in the superpotential and simply plug in the solution for the F–term equations that we have just found

$$W_Y = -im_*^4 \int_S \mathrm{Tr}(\varphi \wedge a \wedge a) = -im_*^4 \int_S \mathrm{Tr}\left[(h - i[\langle\Phi\rangle, \xi]) \wedge \bar{\partial}\xi \wedge \bar{\partial}\xi\right]. \tag{3.38}$$

Some of the terms appearing actually will simply be boundary terms and therefore using localisation of the wavefunctions it is possible to show that they vanish. The surviving terms have the simple form

$$W_Y = m_*^4 f_{abc} \int_S h^a \wedge \bar{\partial}\xi^b \wedge \bar{\partial}\xi^c\,. \tag{3.39}$$

By using the fact that h has to be a holomorphic form it is possible to perform further integrations by parts. The resulting formula is

$$W_Y = m_*^4 f_{abc} \int_{\mathcal{C}} h^c \eta^b \eta^c\,. \tag{3.40}$$

To define η we need to introduce a matrix representation for the adjoint action of $\langle\Phi\rangle$, so that $[\langle\Phi\rangle, \phi_{\mathcal{R}}] = \Psi_{\mathcal{R}}\phi_{\mathcal{R}}$ where $\phi_{\mathcal{R}}$ is a field in given representation $\mathcal{R}$. With this we have that

$$\eta = -i\Psi^{-1}h_{xy}\,. \tag{3.41}$$

The contour $\mathcal{C}$ appearing in (3.40) is a surface surrounding the Yukawa point and containing no other singular point inside it. For example selecting a pair of matter curves Σ_a and Σ_b meeting at the Yukawa point (which pair is chosen is unimportant) defined as the zero locus of two polynomials P_{Σ_a} and P_{Σ_b} the contour around the Yukawa point may be chosen (approximating a local region around the Yukawa point as $\mathbb{C}^2$) $\mathcal{C} = \{(x, y) \in \mathbb{C}^2 : |P_{\Sigma_a}| = \epsilon_a \wedge |P_{\Sigma_b}| = \epsilon_b\}$. We shall take this definition in general in the following, allowing us to write the Yukawa interaction in the superpotential as a complex residue

$$W_Y = m_*^4 \pi^2 f_{abc} \mathrm{Res}(\eta^a \eta^b h^c_{xy})\,. \tag{3.42}$$

To conclude we would like to show how the computation of Yukawa couplings is changed when non–perturbative effects are taken into account. As we already discussed the cubic term that generates the triple interactions is actually unchanged but

12 Note that given the form of $\langle\Phi\rangle$ we can also identify the matter curves for these are the locations on S where the rank of the adjoint action of $\langle\Phi\rangle$ drops by one.

the internal wavefunctions are distorted due to the presence of the non–perturbative effects. The solution of the F–term equations in holomorphic gauge is now[13]

$$a = \bar{\partial}\xi\,, \tag{3.43}$$

$$\varphi = h - i[\langle\Phi\rangle, \xi] - \epsilon\partial\theta_0 \wedge \partial\xi \tag{3.44}$$

One difficulty that arises in this situation is that in obtaining the residue formula it is necessary to invert the relation between ξ and φ and the introduction of non–perturbative effects makes this more involved. The strategy is to take ϵ as a small parameter (a reasonable assumption for ϵ determines the strength of the non–perturbative effects) and solve for ξ in perturbation theory, namely write $\xi = \xi^{(0)} + \epsilon\,\xi^{(1)} + \dots$ allowing us to obtain again a residue formula for the Yukawa interaction which again has the form (3.40) but with η expressed as a Taylor series in ϵ. Up to order $\mathcal{O}(\epsilon)$ we have

$$\eta = -i\Psi^{-1}\left[h_{xy} + i\epsilon\partial_x\theta_0\partial_y(\Psi^{-1}h_{xy}) - i\epsilon\partial_y\theta_0\partial_x(\Psi^{-1}h_{xy})\right] + \mathcal{O}(\epsilon^2)\,. \tag{3.45}$$

This will readily allow us to compute the Yukawa couplings at order $\mathcal{O}(\epsilon)$ by simply computing the corresponding residues.

3.2.2 *Normalised wavefunctions and physical Yukawa couplings*

The next (and final) step in the computation of the Yukawa couplings involves the computation of kinetic terms for the fermions. After this combined knowledge of the holomorphic Yukawa couplings and the normalisation of the kinetic terms will yield the final result for the physical Yukawa couplings. This final step is of extreme importance because it is only after taking into account the correct normalisation of the wavefunctions that dependance on the local flux densities will appear in the Yukawa couplings. The first issue we need to tackle is to find the correct solution for the wavefunctions in a real gauge. To do so a good strategy is to use the solution of the F–term equations found in holomorphic gauge and apply to it a complexified gauge transformation to bring the background fields $\langle\Phi\rangle$ and $\langle A\rangle$ in a real gauge. Invariance of the F–term equations under complexified gauge transformations ensures us that we still have a solution, leaving open only the issue of the D–term equations for the fluctuations. We will discuss at length how to solve these equations when discussing the explicit models we will encounter in the following and focus here on the computation of the kinetic terms assuming already knowledge of the internal wavefunctions. Mere dimensional reduction shows that the factor appearing in front of the kinetic terms is

$$K_{i\bar{j}} = \langle\phi_i|\overline{\phi}_{\bar{j}}\rangle \equiv m_*^4 \int_S \mathrm{dvol}_S \mathrm{Tr}(\phi_i\,\overline{\phi}_{\bar{j}})\,, \tag{3.46}$$

where we called $\{\phi_i\}$ the internal wavefunctions of the zero modes appearing in the 4d effective action. To ensure canonical normalisation for the kinetic terms it is therefore necessary to normalise the fields so that $K_{i\bar{j}} = \delta_{i\bar{j}}$ which fixes the normalisation of the wavefunctions as

$$\phi_i = \mathcal{K}_i{}^j\,\hat{\phi}_j \tag{3.47}$$

13 As a matter of fact there is an additional term in the solution for φ containing $a^\dagger$, however this term disappears in the superpotential (consistently with holomorphicity of the superpotential) against the term in the superpotential containing θ_0 [74] .

where we introduced the vielbein for the appearing in the kinetic terms, that is $\mathcal{K}_i{}^j$ is defined as $\mathcal{K}_i{}^k\, K_{k\bar{l}}\overline{\mathcal{K}}^{l}{}_{\bar{\jmath}} = \delta_{i\bar{\jmath}}$. Finally this allows us to compute the physical Yukawa couplings which simply become

$$\hat{Y}^{ijk} = \mathcal{K}_i{}^l \mathcal{K}_j{}^m \mathcal{K}_k{}^n Y_{lmn}\,, \tag{3.48}$$

where we denoted Y_{lmn} the holomorphic Yukawa couplings. This concludes our survey on the computation of Yukawa couplings, however we would like to make some remark concerning the computation of the kinetic terms for the fermions. As we already discussed one important feature of holomorphic Yukawa couplings is that the computation actually localises at the Yukawa point and therefore a local model obtained by approximating a neighbourhood of the Yukawa point with $\mathbb{C}^2$ can be used for this computation[14]. On the contrary the computation of the kinetic terms does not have any localisation property and therefore it would require a complete knowledge of the wavefunctions over the entire divisor S. In the following we shall always consider the case where the zero modes localise on a sufficiently small neighbourhood of the Yukawa point and approximate this neighbourhood as $\mathbb{C}^2$ with a flat metric. This implies some degree of approximation, like extending the wavefunctions over the entire $\mathbb{C}^2$, neglecting curvature on S and moreover assuming that the wavefunction peaks only close to the Yukawa point[15]. Nevertheless we will use the result for the kinetic terms computed in this approximations as a first estimate and leave open the issue of a computation with the knowledge of the wavefunctions over the entire S.

3.2.3 *Local chirality*

One of the most important consequences of the addition of gauge fluxes on the worldvolume of 7-branes is the generation of a chiral spectrum in the 4d effective theory. We recall that it is possible to compute the net chiral spectrum of the modes localised on a matter curve Σ as an index [7]

$$\chi\left(\Sigma, \mathcal{L}\otimes K_{\Sigma}^{\frac{1}{2}}\right) = \int_{\Sigma} c_1(\mathcal{L})\,, \tag{3.49}$$

where $\mathcal{L}$ is a line bundle on Σ whose first Chern class is equal to the magnetic flux threading the matter curve. Therefore a suitable choice of fluxes can give the correct chiral spectrum in the 4d theory. Moreover, since part of the flux triggers the breaking of the GUT group, fields in different representations of the SM group that are in the same representation of the GUT group may have a different chiral spectrum in 4d. This kind of mechanism allows for a simple implementation of doublet-triplet splitting in F-theory GUTs by imposing the absence of massless Higgs triplets in the 4d theory.

Notice that in our local models we will not be able to compute explicitly the chiral index for the various matter representations because this would require to specify the geometry around a patch containing S_{GUT} and in particular the matter curves Σ. It is however still possible to discuss chirality in our local model by employing the

14 Note that the presence of curvature on S does not affect this computation as the metric on S appears nowhere in the computation.

15 Note that this approximation is likely to be not correct, for instance if the matter curve is a torus $\mathbf{T}^2$ then it is known that zero modes have profiles given by ϑ–functions which have several peaks on the torus [80].

concept of local chirality. This notion introduced in [81] amounts to compute a chiral index for those wavefunctions which are localised around the Yukawa point. To gain a better understanding of how local chirality is formulated it is useful to consider models of magnetised D9-branes which are T-dual to our setting, as in [73]. In order to do so we identify the gauge connection $A_{\bar{z}}$ with Φ where we called z the direction transverse to the 7-branes. All fields do not depend on z and therefore $F_{x\bar{z}} = D_x\Phi$ and $F_{y\bar{z}} = D_y\Phi$ and so on. To formulate local chirality we need the expression of the index of the Dirac operator which for a representation $\mathcal{R}$ is

$$\text{index}_{\mathcal{R}} \not{D} = \frac{1}{48(2\pi)^2} \int \left(\text{Tr}_{\mathcal{R}}\, F \wedge F \wedge F - \frac{1}{8} \text{Tr}_{\mathcal{R}}\, F \wedge \text{Tr} R \wedge R \right) . \tag{3.50}$$

Asking for the existence of a chiral mode in the representation $\mathcal{R}$ amounts to the condition $\mathcal{I}_{\mathcal{R}} < 0$ where $\mathcal{I}_{\mathcal{R}}$ is the integrand in (3.50). Note that since $\mathcal{I}_{\mathcal{R}} = -\mathcal{I}_{\overline{\mathcal{R}}}$ the spectrum in the 4d theory will be chiral if $\mathcal{I}_{\mathcal{R}} \neq 0$. Taking a local patch where we can approximate our configuration by constant fluxes and vanishing curvature we find

$$\begin{aligned} \mathcal{I}_{\mathcal{R}} \equiv \frac{i}{6} \text{Tr}_{\mathcal{R}}\, (F \wedge F \wedge F)_{x\bar{x}y\bar{y}z\bar{z}} = i\, \text{Tr}_{\mathcal{R}} \big(& F_{x\bar{x}} \{F_{y\bar{y}}, F_{z\bar{z}}\} + F_{x\bar{z}} \{F_{y\bar{x}}, F_{z\bar{y}}\} + \\ & F_{x\bar{y}} \{F_{y\bar{z}}, F_{z\bar{x}}\} - \{F_{x\bar{x}}, F_{y\bar{z}}\} F_{z\bar{y}} - \{F_{x\bar{y}}, F_{y\bar{x}}\} F_{z\bar{z}} - \{F_{x\bar{z}}, F_{y\bar{y}}\} F_{z\bar{x}} \big). \end{aligned} \tag{3.51}$$

We will employ this formula in the following to characterise the local chirality of the various models we will analyse.

3.3 THE E_6 MODEL

We start now by giving the details of the local E_6 model considered in [82]. We recall that in the proximity of a point with E_6 enhancement the Yukawa couplings for the up–type quarks are generated and therefore, since our main motive for going from perturbative type IIB String Theory to F–theory involves the generation of the Yukawa coupling for the top quark with the correct order of magnitude it is certainly important to ascertain whether a correct value for the Yukawa of the top quark can be achieved or not.

3.3.1 *Background fields for the E_6 model*

In the following we describe the E_6 local F-theory model which will serve to compute up-type quark Yukawa couplings by specifying the background values for the 7–brane fields Φ and A. One characteristic feature of these models is that we encounter a T–brane configuration for the adjoint field Φ which requires the introduction of non–primitive fluxes. In addition to these fluxes we shall also consider primitive fluxes to break the GUT group and generate chirality for the matter fields.

Matter curves near the E_6 point

As we already explained before in $SU(5)$ F–theory models the coupling $\mathbf{10} \times \mathbf{10} \times \mathbf{5}$ is generated at points where one encounters an enhanced E_6 symmetry on the divisor S hosting the GUT theory. Therefore in our local model we will need a choice for the adjoint scalar Φ that breaks E_6 down to $SU(5)$, and the structure of matter curves in the vicinity of the Yukawa point will be encoded entirely in the form of Φ. Addition of magnetic fluxes will further break the gauge symmetry but for the moment we will not discuss them. In our local model in the vicinity of the Yukawa point we will always take x and y as local coordinates approximating our local patch as $\mathbb{C}^2$.

We start by giving the form of the background value for Φ in the holomorphic gauge. Our choice is

$$\Phi = m(E^+ + m\,xE^-) + \mu^2(bx - y)Q\,, \tag{3.52}$$

where m and μ are constants with dimension of mass and b is a dimensionless parameter. $E^\pm$ and Q are some particular generators of the $\mathfrak{e}_6$ algebra which we define in detail in appendix A. It is important to note that Q lies in the Cartan algebra of $\mathfrak{e}_6$ whereas $E^\pm$ are two roots that do not commute between themselves. This implies that we are indeed in a situation where the background value of Φ is a T–brane background. Moreover since $E^\pm$ together with $P = [E^+, E^-]$ generate a $\mathfrak{su}(2)$ subalgebra and Q commutes with $E^\pm$ we are indeed in a situation identical to the one considered in section 3.1.2 when discussing T–brane backgrounds[16]. We can therefore borrow the result and write directly the form of Φ in a real gauge

$$\langle \Phi_{xy} \rangle = m(e^f E^+ + m\,xe^{-f}E^-) + \mu^2(bx - y)Q\,, \tag{3.53}$$

where f coincides with the function introduced in Section 3.1.2. To solve the D–term equations we need to add a non–primitive flux with gauge connection

$$A = \frac{i}{2}(\partial f + \bar{\partial} f)P\,. \tag{3.54}$$

We now turn to the description of the matter curves for this background. As a matter of fact this can be done directly in holomorphic gauge for the action of complexified gauge transformations does not alter the location of matter curves. We recall that the location of the matter curves is specified by looking for codimension 1 loci on S where additional roots commute with Φ. The matter fields we expect may be identified by decomposing the adjoint of E_6 with respect to an $SU(5) \times SU(2) \times U(1)$ subgroup

$$\mathbf{78} \to (\mathbf{24},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{3})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{10},\mathbf{2})_{-1} \oplus (\overline{\mathbf{10}},\mathbf{2})_1 \oplus (\mathbf{5},\mathbf{1})_2 \oplus (\bar{\mathbf{5}},\mathbf{1})_{-2}\,. \tag{3.55}$$

Therefore we find matter in the **10** and in the **5** representations of $SU(5)$ consistently with our expectation. The roots corresponding to these matter fields are described in appendix A, cf. equations (A.7) and (A.8), and here we only describe the action of Φ on these fields. Note that since the **10** is a doublet of $SU(2) \subset E_6$ it is convenient to use a matrix notation for the action of Φ on these matter fields as already described in section 3.2.1. Employing this notation for the **5** representation as well we find that

$$\Phi_{\mathbf{5}} = 2\mu^2(bx - y)\,, \tag{3.56}$$

$$\Phi_{\mathbf{10}} = \begin{pmatrix} -\mu^2(bx-y) & m \\ m^2 x & -\mu^2(bx-y) \end{pmatrix}. \tag{3.57}$$

16 Note that since Q commutes with $E^\pm$ it will disappear in the D–term equations which will therefore coincide with the ones encountered in Section 3.1.2.

Inspection of the form of $\Phi_{\mathcal{R}}$ for the various representations easily reveals the location of the matter curves $\Sigma_{\mathcal{R}}$ for these are located at $\det\Phi_{\mathcal{R}} = 0$. The location of the matter curves for our model are

$$\Sigma_{\mathbf{5}} := \{\mu^2(bx - y) = 0\}\,, \tag{3.58}$$

$$\Sigma_{\mathbf{10}} := \{\mu^4(bx - y)^2 = m^3 x\}\,. \tag{3.59}$$

Note that we only find a single matter curve hosting fields in the **10** representation of $SU(5)$. Had we chosen a background for Φ without a T–brane configuration we would have encountered two matter curves for the **10**. This constitutes a negative feature for all matter fields in the **10** should localise on a single matter curve to ensure a good structure for the Yukawa matrix. This constitutes a justification for our choice of the background for Φ as without T–branes it would not have been possible to generate a coupling between two copies of the same matter field in the **10** lying in a single matter curve.

Primitive worldvolume fluxes

On top of the flux in (3.18), the above model admits additional contributions to the background worldvolume flux $\langle F\rangle$ if they do not spoil the F-term and D-term conditions. The simplest way to introduce them is to consider primitive $(1,1)$ fluxes $\langle F\rangle$ in the Cartan of E_6, avoiding fluxes in the direction selected by P in the Cartan subalgebra for these do not commute with Φ. Considering such fluxes is important to complete the local F-theory model, not just because they will be generically present, but also because they play an important rôle for the phenomenology of the model. Our main uses of these fluxes is the generation of a chiral spectrum in 4d and GUT breaking. More precisely, let us consider the worldvolume flux

$$\langle F_Q\rangle = i\left[-M(dy\wedge d\bar{y} - dx\wedge d\bar{x}) + N(dx\wedge d\bar{y} + dy\wedge d\bar{x})\right]Q \tag{3.60}$$

where M and N are flux densities near the Yukawa point that we will approximate by constants. It is easy to check that adding such flux will not spoil the equations of motion for any value of M, N, which will be considered as real parameters of the model in the following. In addition to these fluxes we choose to add a flux in the direction of the hypercharge within $SU(5)$ to break the GUT group as explained in Section 2.4. Our parametrisation for such fluxes is the following

$$\langle F_Y\rangle = i\left[\tilde{N}_Y(dy\wedge d\bar{y} - dx\wedge d\bar{x}) + N_Y(dx\wedge d\bar{y} + dy\wedge d\bar{x})\right]Q_Y \tag{3.61}$$

where N_Y, $\tilde{N}_Y$ are local flux densities and

$$Q_Y \;=\; \frac{1}{3}\,(H_2 + H_3 + H_4) - \frac{1}{2}\,(H_5 + H_6) \tag{3.62}$$

is the hypercharge generator[17]. The hypercharge flux at the level of our local models will have an important impact for it will allow us to have no chiral Higgs triplets in the spectrum (at least in the sense of local chirality described in Section 3.2.3).

17 Here H_i denote the Cartan generators of the $\mathfrak{e}_6$ Lie algebra, see Appendix A for more details.

Summary

Let us summarise the details of the E_6 model which we will use to compute up–type Yukawa couplings. If we parametrise the four-cycle S by the complex coordinates x, y, the Higgs background that breaks $E_6 \to SU(5) \times U(1)$ is given by

$$\Phi = m(e^f E^+ + m\, x e^{-f} E^-) + \mu^2(bx - y)Q \tag{3.63}$$

The value of b is completely free in our local model, however we will make in the following the concrete choice

$$b = 1 \tag{3.64}$$

although our discussion can be easily generalised to other values of b.

The worldvolume flux of this model will be given by

$$F = F_{\rm p} + F_{\rm np} \tag{3.65}$$

where $F_{\rm np}$ is the non-primitive flux that is necessary to compensate the contribution of $[\langle\Phi_{xy}\rangle, \langle\bar{\Phi}_{\bar{x}\bar{y}}\rangle]$ to the D-term equation (3.7), and reads

$$F_{\rm np} = -i\partial\bar{\partial} f\, P\,. \tag{3.66}$$

In addition we have that

$$F_{\rm p} = iQ_R(dy \wedge d\bar{y} - dx \wedge d\bar{x}) + iQ_S(dx \wedge d\bar{y} + dy \wedge d\bar{x}) \tag{3.67}$$

is the primitive flux needed to generate chirality and further break the gauge group as $SU(5) \to SU(3) \times SU(2) \times U(1)_Y$. Here we have defined the combinations

$$Q_R = -MQ + \tilde{N}_Y Q_Y\,, \qquad Q_S = NQ + N_Y Q_Y \tag{3.68}$$

with Q_Y the hypercharge generator (3.62) and M, N, N_Y, $\tilde{N}_Y$ real flux densities. For the purpose of computing local chiral indices it is convenient to summarise the whole set of zero modes (in terms of representations of the SM gauge group) by specifying the "eigenvalues" of Q_R and Q_S on the different sectors defined as

$$[Q_R, E_\rho] = q_R\, E_\rho\,, \qquad [Q_S, E_\rho] = q_S\, E_\rho\,. \tag{3.69}$$

We collected these data in Table 5.

As we will see in section 3.3.3, the quantities q_R, q_S enter into the expressions for the internal wavefunctions of the MSSM chiral zero modes. In fact, these charges determine which sectors of those in table 5 have localised zero modes near the Yukawa point. In order to construct a local model with the MSSM chiral spectrum we need to impose that chiral modes only arise from the four first rows of table 5. This imposes some constraints on q_R and q_S which in turn impose some constraints on the allowed values of the flux densities. We can apply the results of Section 3.2.3 to our local model obtaining the following chiral indices[18]

$$\mathcal{I}_{\mathbf{10}} = -2m^4c^4 q_R(\mathbf{10}_i)\,, \quad \mathcal{I}_{\mathbf{5}} = -8\mu^4\, q_S(\mathbf{5}_i)\,. \tag{3.70}$$

18 The expression for the **10** sector is actually approximate for we are neglecting terms of order μ^4. However since we shall take $m^2 \gg \mu^2$ in the following these additional terms will not alter the chirality conditions.

Sector	Root	G_{MSSM}	q_R	q_S
$\mathbf{10}_1$	$(0,\underline{1,1,0},0,0)\oplus\frac{1}{2}(-\sqrt{3},\underline{1,1,-1},-1,-1)$	$(\bar{\mathbf{3}},\mathbf{1})_{\frac{2}{3}}$	$M+\frac{2}{3}\tilde{N}_Y$	$-N+\frac{2}{3}N_Y$
$\mathbf{10}_2$	$(0,\underline{1,0,0},\underline{1,0})\oplus\frac{1}{2}(-\sqrt{3},\underline{1,-1,-1},\underline{1,-1})$	$(\mathbf{3},\mathbf{2})_{-\frac{1}{6}}$	$M-\frac{1}{6}\tilde{N}_Y$	$-N-\frac{1}{6}N_Y$
$\mathbf{10}_3$	$(0,0,0,0,1,1)\oplus\frac{1}{2}(-\sqrt{3},-1,-1,-1,1,1)$	$(\mathbf{1},\mathbf{1})_{-1}$	$M-\tilde{N}_Y$	$-N-N_Y$
$\mathbf{5}_1$	$\frac{1}{2}(\sqrt{3},-1,-1,-1,\underline{1,-1})$	$(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$	$-2M-\frac{1}{2}\tilde{N}_Y$	$2N-\frac{1}{2}N_Y$
$\mathbf{5}_2$	$\frac{1}{2}(\sqrt{3},\underline{1,-1,-1},-1,-1)$	$(\mathbf{3},\mathbf{1})_{\frac{1}{3}}$	$-2M+\frac{1}{3}\tilde{N}_Y$	$2N+\frac{1}{3}N_Y$
$\overline{\mathbf{10}}_1$	$(0,\underline{-1,-1,0},0,0)\oplus\frac{1}{2}(\sqrt{3},\underline{-1,-1,1},1,1)$	$(\mathbf{3},\mathbf{1})_{-\frac{2}{3}}$	$-M-\frac{2}{3}\tilde{N}_Y$	$N-\frac{2}{3}N_Y$
$\overline{\mathbf{10}}_2$	$(0,\underline{-1,0,0},\underline{-1,0})\oplus\frac{1}{2}(\sqrt{3},\underline{-1,1,1},\underline{-1,1})$	$(\bar{\mathbf{3}},\mathbf{2})_{\frac{1}{6}}$	$-M+\frac{1}{6}\tilde{N}_Y$	$N+\frac{1}{6}N_Y$
$\overline{\mathbf{10}}_3$	$(0,0,0,0,-1,-1)\oplus\frac{1}{2}(\sqrt{3},1,1,1,-1,-1)$	$(\mathbf{1},\mathbf{1})_1$	$-M+\tilde{N}_Y$	$N+N_Y$
$\bar{\mathbf{5}}_1$	$\frac{1}{2}(-\sqrt{3},1,1,1,\underline{-1,1})$	$(\mathbf{1},\mathbf{2})_{\frac{1}{2}}$	$2M+\frac{1}{2}\tilde{N}_Y$	$-2N+\frac{1}{2}N_Y$
$\bar{\mathbf{5}}_2$	$\frac{1}{2}(-\sqrt{3},\underline{-1,1,1},1,1)$	$(\bar{\mathbf{3}},\mathbf{1})_{-\frac{1}{3}}$	$2M-\frac{1}{3}\tilde{N}_Y$	$-2N-\frac{1}{3}N_Y$
$\mathbf{X}^+,\mathbf{Y}^+$	$(0,\underline{1,0,0},\underline{-1,0})$	$(\mathbf{3},\mathbf{2})_{\frac{5}{6}}$	$\frac{5}{6}\tilde{N}_Y$	$\frac{5}{6}N_Y$
$\mathbf{X}^-,\mathbf{Y}^-$	$(0,\underline{-1,0,0},\underline{1,0})$	$(\bar{\mathbf{3}},\mathbf{2})_{-\frac{5}{6}}$	$-\frac{5}{6}\tilde{N}_Y$	$-\frac{5}{6}N_Y$

Table 5: Different sectors and charges for the E_6 model.

Let us describe in detail what constraints we can draw from these chiral indices. First of all we would like to avoid the doublet-triplet splitting problem of 4d $SU(5)$ GUT models. Following [10], one can do so by adjusting the fluxes so that the sector of triplets $\mathbf{5}_2$, $\bar{\mathbf{5}}_2$ does not feel any net flux and it is then a non-chiral sector without any localised 4d modes. The condition we need to impose is that $q_S(\mathbf{5}_2)=q_S(\bar{\mathbf{5}}_2)=0$ or in other words that

$$N_Y+6N\;=\;0 \tag{3.71}$$

On the other hand, we would like to have a localised chiral mode in the sector $\mathbf{5}_1$ but not in $\bar{\mathbf{5}}_1$. This amounts to require that $q_S(\mathbf{5}_1)>0$ which, using (3.71) translates into

$$N\;>\;0 \tag{3.72}$$

In addition, we should require that there are localised chiral modes in the sector $\mathbf{10}_i$ but not in $\overline{\mathbf{10}}_i$ for $i=1,2,3$. This can be understood in terms of the condition $q_R(\mathbf{10}_i)>0$ with is achieved by imposing

$$M+q_Y\tilde{N}_Y\;>\;0\quad\text{for}\quad q_Y=\frac{2}{3},-\frac{1}{6},-1\qquad\Rightarrow\qquad -\frac{3}{2}<\frac{\tilde{N}_Y}{M}<6\,. \tag{3.73}$$

Non-perturbative effects

Finally, an essential piece of the model are the non-perturbative effects with source located at a 4-cycle $S_{\text{np}}\subset B$ whose embedding is defined by a holomorphic divisor function $h(x, y, z)$. As discussed in section 3.1.2 such effects will shift the tree-level superpotential to (3.13). The explicit form of θ_0 actually depends on the details of the compactification which are invisible in our local model, however we can choose to parametrise it in the proximity of the Yukawa point as

$$\theta_0 \;=\; i(\theta_{00} + \theta_x x + \theta_y y)\,. \tag{3.74}$$

In general, the presence of such non-perturbative effects will modify the local E_6 model described above, in the sense that the shift in the 7-brane superpotential modifies the F-term equations (3.171) to

$$\bar{\partial}_A \Phi + \epsilon\, \partial\theta_0 \wedge F = 0 \tag{3.75a}$$

$$F^{(0,2)} = 0 \tag{3.75b}$$

and so the background values of F and Φ need to be shifted from the original values in order to satisfy these new equations. We will discuss the form of these non–perturbative corrections to the background values of the 7–brane fields in section 3.3.3. Nevertheless we recall that in in [74] it was shown that such corrections to the background cancel each other out in the computation of holomorphic Yukawa couplings, and so one may still consider (3.63) and (3.65) for such purpose. Using this fact, in the next section we will show that the effect of (3.13) is to generate a hierarchical rank 3 matrix of up-type holomorphic Yukawa couplings.

3.3.2 *Holomorphic Yukawas via residues*

The purpose of this section is to compute the holomorphic piece of the $\mathbf{10} \times \mathbf{10} \times \mathbf{5}$ Yukawa couplings for the E_6 model above, and to show that the effect of the non-perturbative superpotential in (3.13) is to increase the rank of this Yukawa matrix from one to three. We recall that the computation amy be reduced to a residue computation at the Yukawa point as shown in Section 3.2.1.

Holomorphic Yukawas

In order to apply the above residue formula to the E_6 model of Section 3.3 let us first gather the information which is relevant for computing the residue. Clearly, in order to compute the residue we only need to know the details of the model around the Yukawa point $p_{\rm up} = \{x = y = 0\}$, and so the local description of the E_6 model that was given in Section 3.3 is justified. Moreover, from all the parameters that are involved in the local E_6 model only a few of them are relevant for computing (3.42). In fact, as can be deduced from our previous discussion there are basically only two quantities which are relevant in the computation of the residue: the Higgs background $\langle \Phi^{\rm hol} \rangle$ that solves the equations of motion in the holomorphic gauge and in the absence of non-perturbative effects, and the holomorphic function θ_0 that encodes the information of such effects in the vicinity of the Yukawa point. For the reader's convenience we repeat both quantities here:

$$\langle \Phi_{xy}^{\rm hol} \rangle^{(0)} \;=\; m(E^+ + m\, x E^-) + \mu^2 (bx - y) Q \tag{3.76}$$

$$\theta_0 \;=\; i(\theta_{00} + \theta_x x + \theta_y y) \tag{3.77}$$

As discussed in section 3.3 the Higgs vev (3.76) specifies the two matter curves $\Sigma_{\mathbf{5}}$ and $\Sigma_{\mathbf{10}}$ where the chiral modes of the **5**-plets and **10**-plets are localised. For each of these two sectors we need to specify the pair (h, η) that will enter into the residue formula (3.42), and will couple to each other via the structure constants f_{abc} of E_6.

Sector 5

In this case the matter curve is given by $\Sigma_{\mathbf{5}} = \{bx - y = 0\}$ and there the localised zero modes may arise in two possible sectors: along $E_{\mathbf{5}} = \frac{1}{2}(\sqrt{3}, \underline{1, -1, -1, -1, -1})$ and along $E_{\bar{\mathbf{5}}} = \frac{1}{2}(-\sqrt{3}, \underline{-1, 1, 1, 1, 1})$. We will consider the case where, due to the presence of worldvolume fluxes, we have a chiral spectrum and a single zero mode in the sector **5** and none in $\bar{\mathbf{5}}$.[19] The action of (3.76) in this zero mode sector is such that

$$\Psi \;=\; 2\mu^2(bx - y) \tag{3.78}$$

It then only remains to specify the value of h for this sector. While in principle $h = h(x, y)$ may be any holomorphic function in the vicinity of $x = y = 0$ it is always convenient to fix a particular gauge for this matter curve following the philosophy in [63]. By using a suitable gauge transformation we may remove completely the dependance on the coordinate $bx - y$ in h reducing it to be a function of the single coordinate $x + by$. This is a nice feature for this coordinate is a good coordinate on the matter curve $\Sigma_{\mathbf{5}}$ therefore indicating localisation of the mode on its matter curve. We can parametrise different modes on the matter curve as different polynomials in $x + by$, and following the procedure already employed in other instances we will take h to be constant when a single zero mode is present. This implies that

$$h_{\mathbf{5}}/\gamma_{\mathbf{5}} \;=\; 1 \tag{3.79}$$

$$i\eta_{\mathbf{5}}/\gamma_{\mathbf{5}} \;=\; \frac{1}{2\mu^2(bx - y)} - \epsilon\frac{\theta_x + b\theta_y}{4\mu^4(bx - y)^3} + \mathcal{O}(\epsilon^2) \tag{3.80}$$

with $\gamma_{\mathbf{5}}$ a real constant to be computed via wavefunction normalisation in the next section.

Sector 10

In this case the curve is given by $\Sigma_{\mathbf{10}} = \{\mu^4(bx - y)^2 = m^3 x\}$ and the action of Φ on this sector may be represented in terms of a 2 × 2 matrix

$$\Psi = \begin{pmatrix} -\mu^2(bx - y) & m \\ m^2 x & -\mu^2(bx - y) \end{pmatrix} \tag{3.81}$$

as can be read from (3.57). As before we need to specify $h_{\mathbf{10}}$, which now will be an $SU(2)$ doublet of arbitrary holomorphic functions. Again, by performing an appro-

19 We have already spelled the conditions to have the correct chiral spectrum in the matter curve hosting the Higgs field. The number of zero modes localised on this matter curve needs additional data encoded in a global model defined on the entire S and therefore is invisible in our local model. We will take as an assumption that i the model we consider a single zero mode localises on this matter curve. Finally, even if the Higgs triplets are absent in the spectrum, the result for the holomorphic Yukawa couplings may actually be extended to include these modes as well.

priate holomorphic gauge transformation (3.36) we can restrict ourselves to a very particular form for $h_{\mathbf{10}}$ following [64]

$$h_{\mathbf{10}} = \begin{pmatrix} h^+(x,y) \\ h^-(x,y) \end{pmatrix} - i\Psi \begin{pmatrix} \chi^+(x,y) \\ \chi^-(x,y) \end{pmatrix} = \begin{pmatrix} 0 \\ h(bx-y) \end{pmatrix} \tag{3.82}$$

for arbitrary $h^\pm$ and appropriate choices of $\chi^\pm$. While h can be any holomorphic function on the coordinate $bx - y$, under the assumption that we have three zero modes in this sector we can take them to be the monomials $\gamma^i_{\mathbf{10}} m_*^{3-i}(bx-y)^{3-i}$, with $\gamma^i_{\mathbf{10}}$ some normalisation factors to be fixed in the next section. We finally have that

$$h^i_{\mathbf{10}}/\gamma^i_{\mathbf{10}} = \begin{pmatrix} 0 \\ m_*^{3-i}(bx-y)^{3-i} \end{pmatrix} \tag{3.83}$$

$$\begin{aligned} i\eta^i_{\mathbf{10}}/\gamma^i_{\mathbf{10}} &= -\left[\frac{m_*^{3-i}(bx-y)^{3-i}}{\mu^4(bx-y)^2 - m^3 x}\right] \begin{pmatrix} m \\ \mu^2(bx-y) \end{pmatrix} + \mathcal{O}(\epsilon^2) \\ &+ \epsilon \frac{2\mu^4(\theta_x + b\theta_y)(bx-y) + m^3\theta_y}{(\mu^4(bx-y)^2 - m^3x)^3} m_*^{3-i}(bx-y)^{3-i} \begin{pmatrix} 2m\mu^2(bx-y) \\ (m^3x + \mu^4(bx-y)^2) \end{pmatrix} \\ &+ \epsilon \frac{(\theta_x + b\theta_y)}{(\mu^4(bx-y)^2 - m^3x)^2} m_*^{3-i}(bx-y)^{2-i} \begin{pmatrix} 2m\mu^2(bx-y)(6-i) \\ m^3x(3-i) + (4-i)\mu^4(bx-y)^2 \end{pmatrix} \end{aligned} \tag{3.84}$$

which has a rather complicated $\mathcal{O}(\epsilon)$ correction to $\eta^i_{\mathbf{10}}$. Nevertheless, the result that one obtains from applying the residue formula is still quite simple, as we will now see.

$\mathbf{10} \times \mathbf{10} \times \mathbf{5}$ *Yukawas*

Let us now apply the explicit expressions for $(h_{\mathbf{5}}, \eta_{\mathbf{5}})$ and $(h_{\mathbf{10}}, \eta_{\mathbf{10}})$ to the residue formula (3.42) for the Yukawa couplings. An important simplifications arises from the fact that the structure constants of E_6 satisfy

$$\mathrm{Tr}([E_{\mathbf{5}\,i}, E_{\mathbf{10}\,jk}^{\;M}] E_{\mathbf{10}\,lm}^{\;N}) = \epsilon_{ijklm}\epsilon^{MN} \tag{3.85}$$

where i,j,k,l,m are $\mathfrak{su}(5)$ indices and $M, N = \pm$ are $\mathfrak{su}(2)$ indices. As a result the non-trivial contributions to the $\mathbf{10} \times \mathbf{10} \times \mathbf{5}$ Yukawa will be of the form

$$Y = m_*^4 \pi^2 \mathrm{Res}_{(0,0)} \left(\epsilon_{MN} \eta_{\mathbf{5}} \eta^M_{\mathbf{10}} h^N_{\mathbf{10}}\right) = m_* \pi^2 \mathrm{Res}_{(0,0)} \left(\eta_{\mathbf{5}} \eta^+_{\mathbf{10}} h^-_{\mathbf{10}}\right) \tag{3.86}$$

where the contractions of the $SU(5)$ indices have been left implicit. In the first equality we have used that any other contribution will contain a term of the form $\epsilon_{MN}\eta^M_{\mathbf{10}}\eta^N_{\mathbf{10}}$ and so it will vanish identically, and in the second equality we have used that in our solution (3.83) $h^+_{\mathbf{10}} = 0$. Hence, even if (3.84) has a complicated expression only the terms proportional to $E_{\mathbf{10}^+}$ will be relevant when computing up-like Yukawa couplings.

Let us proceed by computing (3.86) explicitly. At zeroth order in ϵ we have a contribution of the form

$$\begin{aligned} Y^{ij}_{\text{tree}} &= m_*^4 \pi^2 \gamma_{\mathbf{5}} \gamma^i_{\mathbf{10}} \gamma^j_{\mathbf{10}} \mathrm{Res}_{(0,0)} \left[\frac{m(m_*(bx-y))^{6-i-j}}{2\mu^2(bx-y)(\mu^4(bx-y)^2 - m^3x)}\right] \\ &= -\frac{m_*^4\pi^2}{2m^2\mu^2} \gamma_{\mathbf{5}} \gamma^i_{\mathbf{10}} \gamma^j_{\mathbf{10}}\, \delta_{i3}\delta_{j3} \end{aligned} \tag{3.87}$$

and so at this level only Y^{33} is non-zero. At order $\mathcal{O}(\epsilon)$ we get a contribution of the form

$$Y_{\rm np}^{ij} \,=\, \epsilon\, \frac{m_*^6\pi^2}{4m^2\mu^4}\,[b\theta_y+\theta_x]\,\gamma_{\mathbf{5}}\gamma_{\mathbf{10}}^i\gamma_{\mathbf{10}}^j\,\delta_{(i+j)4} \tag{3.88}$$

from the $\mathcal{O}(\epsilon)$ correction to $\eta_{\mathbf{5}}$. In fact, one can check that the $\mathcal{O}(\epsilon)$ correction to $\eta_{\mathbf{10}}$ do not contribute to (3.86) and that we are left with the following $\mathbf{10}\times\mathbf{10}\times\mathbf{5}$ Yukawa couplings:

$$Y^{ij} \,=\, \frac{\pi^2\gamma_{\mathbf{5}}}{4\rho_\mu\rho_m}\begin{pmatrix} 0 & 0 & \tilde\epsilon\rho_\mu^{-1}\gamma_{\mathbf{10}}^1\gamma_{\mathbf{10}}^3 \\ 0 & \tilde\epsilon\rho_\mu^{-1}\gamma_{\mathbf{10}}^2\gamma_{\mathbf{10}}^2 & 0 \\ \tilde\epsilon\rho_\mu^{-1}\gamma_{\mathbf{10}}^1\gamma_{\mathbf{10}}^3 & 0 & -2\gamma_{\mathbf{10}}^3\gamma_{\mathbf{10}}^3 \end{pmatrix} + \mathcal{O}(\epsilon^2) \tag{3.89}$$

where we have defined the slope densities

$$\rho_\mu \,=\, \frac{\mu^2}{m_*^2} \qquad\qquad \rho_m \,=\, \frac{m^2}{m_*^2} \tag{3.90}$$

as well as the non-perturbative parameter

$$\tilde\epsilon \,=\, \epsilon\,(\theta_x+b\theta_y) \tag{3.91}$$

As claimed, we obtain a Yukawa matrix such that in the absence of non-perturbative effects has rank one, but when taking them into account increases its rank to three.[20] Note that the eigenvalues of this matrix display a hierarchical structure which, as we will discuss in more detail in Section 3.3.4, has the form $(\mathcal{O}(1),\mathcal{O}(\tilde\epsilon),\mathcal{O}(\tilde\epsilon^2))$.

An interesting feature of this Yukawa matrix it that its entries depend on very few parameters of the model, most notably $\tilde\epsilon$, ρ_μ and $\gamma_{\mathbf{10}}^i$. In fact the last set of parameters can be understood as wavefunction normalisation constants that cannot be determined solely from the analysis of this section. As already anticipated in section 3.2.2 these couplings may be computed by demanding canonical normalisation for their kinetic terms in 4d, and we will now turn to this computation in the next section.

3.3.3 *Zero mode wavefunctions at the E_6 point*

A remarkable aspect of the computations of the last section is that, in order to arrive to the Yukawa matrix (3.89), we did not have to fully solve for the chiral zero mode wavefunctions. Instead, we solved for the F-term equations and used the invariance of the superpotential under complexified gauge transformations. The price to pay for using that trick is that we do not have any physical criterium to fix the constants $\gamma_{\mathbf{5}}$, $\gamma_{\mathbf{10}}^i$ that appear in the Yukawa matrix, because the wavefunctions that we are using are not in a physical gauge. As already stressed before in section 3.2 this is due to the fact that we simply computed the couplings appearing in the superpotential and not the couplings that appear in the Lagrangian. To compute the latter we also need to solve the D–term equations for the zero modes and demand canonical normalisation

20 More precisely, the condition for rank enhancement is that $\tilde\epsilon \neq 0$, which seems to indicate that the pull-back of θ_0 along $\Sigma_{\mathbf{5}}$ must be non-trivial.

for their kinetic terms, fixing therefore the constants $\gamma_{\mathbf{5}}$, $\gamma^i_{\mathbf{10}}$ in terms of the various parameters entering our local model (including local flux densities).

As we will see, solving analytically for the zero modes D-term equations is a rather involved task, mainly because they involve the Painlevé transcendent f found in the solution for the background value of Φ and the fluxes F. While we will not be able to find a general solution for the set of differential equations for the zero modes affected by the T–brane background (namely the zero modes in the **10** representation) we will find a solution for a particular region of parameters of our local model. We expect that our conclusions will hold also away from this particular limit although the normalisation factors will receive corrections that we will not able to compute explicitly. We will first compute these physical wavefunctions in the absence of non-perturbative effects, which will already allow us to compute the normalisation factors $\gamma_{\mathbf{5}}$, $\gamma^i_{\mathbf{10}}$ to a good approximation. Afterwards we will reinstate the presence of non–perturbative corrections and check their impact on the computation of kinetic terms.

Perturbative zero-modes

Without the presence of non–perturbative effects the zero mode equations reduce to (3.28) which we copy here

$$\bar\partial_{\langle A\rangle} a = 0\,, \tag{3.92a}$$

$$\bar\partial_{\langle A\rangle} \varphi = i[a, \langle\Phi\rangle]\,, \tag{3.92b}$$

$$\omega \wedge \partial_{\langle A\rangle} a = \frac{1}{2}[\langle\bar\Phi\rangle, \varphi]\,. \tag{3.92c}$$

We choose to simply give the solution for this set of differential equations here leaving more details on how to find the solution in Appendix B. We start with the zero modes in the **5** representation whose solution for the wavefunctions is an exact one and then turn to the modes in the **10** representation. In this case we will not be able to find an exact solution but we will be able find an approximate one in the limit where the mass parameters entering in the adjoint scalar Φ background obey the condition $\mu^2 \ll m^2$.

Sector 5

To write the solution for this sector it is convenient to pack the various bosonic zero modes $a_{\bar m}$ and φ_{xy} in a single vector

$$\vec\varphi_{\mathbf{5}} = \begin{pmatrix} a_{\bar x} \\ a_{\bar y} \\ \varphi_{xy} \end{pmatrix}. \tag{3.93}$$

Following the strategy outlined in Appendix B it is possible to find a solution for the system of differential equations obeyed by the zero modes and we copy here the result

$$\vec\varphi_{\mathbf{5}} = \gamma_{\mathbf{5}} \begin{pmatrix} i\frac{\zeta_{\mathbf{5}}}{2\mu^2} \\ i\frac{(\zeta_{\mathbf{5}}-\lambda_{\mathbf{5}})}{2\mu^2} \\ 1 \end{pmatrix} \chi_{\mathbf{5}}, \qquad \chi_{\mathbf{5}} = e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar y+y\bar x)+(x-y)(\zeta_{\mathbf{5}}\bar x-(\lambda_{\mathbf{5}}-\zeta_{\mathbf{5}})\bar y)}$$

$$(3.94)$$

with $\lambda_{\mathbf{5}}$ the lowest solution to

$$\lambda_{\mathbf{5}}^3 - (8\mu^4 + (q_R)^2 + (q_S)^2)\lambda_{\mathbf{5}} + 8\mu^4 q_S = 0 \tag{3.95}$$

and $\zeta_{\mathbf{5}} = \frac{\lambda_{\mathbf{5}}(\lambda_{\mathbf{5}} - q_R - q_S)}{2(\lambda_{\mathbf{5}} - q_S)}$. More details may be found in Appendix B. Note that in writing the solution (3.94) we took into account that a single zero mode exists in this sector taking the holomorphic function appearing in the solution written in Appendix B to be constant. If we were to add additional terms in the holomorphic piece of this wavefunction they would only produce subleading corrections to the Yukawa couplings and normalisation factor for this sector justifying therefore our choice to take the holomorphic piece to be constant.

Notice that, because they depend on the hypercharge flux, q_R and q_S take different values for the two subsectors $\mathbf{5}_1$ and $\mathbf{5}_2$ of table 5, and so the same is true for $\lambda_{\mathbf{5}}$, $\zeta_{\mathbf{5}}$. In particular, imposing (3.71) we find that $q_S(\mathbf{5}_2) = 0$ and that the wavefunction for this sector is not localised in the proximity of the Yukawa point in accord with the definition of local chirality that we gave in section 3.2.3.

Sector 10

This sector is more involved because the non–primitive flux enters directly in the differential equations for the zero modes. We will describe now a strategy that will allow us to obtain a solution for some particular region of the parameter space of our local model.

First we note that the zero modes will come as a doublet under the $\mathfrak{su}(2)$ algebra generated by $\{E^+, E^-, P\}$. This suggests the use of the following notation for the zero modes

$$a = \begin{pmatrix} a^+ \\ a^- \end{pmatrix}, \qquad \varphi = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}. \tag{3.96}$$

The strategy we follow is to solve for the zero modes is to solve for the F–term equations (3.28a) and (3.28b) to write a in terms of φ, and then substitute in the D-term equation (3.28c) to find an equation for φ.

It is instructive to first consider the case where the primitive flux F_p in (3.67) is absent. Then solution to the F-term equations is in fact quite similar to the one found previously in the holomorphic gauge and reads

$$a = e^{fP/2}\bar{\partial}\xi \tag{3.97a}$$

$$\varphi = e^{fP/2}\left(h - i\Psi\xi\right) \tag{3.97b}$$

where ξ and h are also doublets with components $\xi^{\pm}$ and $h^{\pm}$ and P and Ψ are the following matrices acting on doublets

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} -\mu^2(x-y) & m \\ m^2 x & -\mu^2(x-y) \end{pmatrix}. \tag{3.98}$$

From (B.18b) we obtain

$$\xi = i\Psi^{-1}\left(e^{-fP/2}\varphi - h\right). \tag{3.99}$$

Finally, the D-term equation for the fluctuations (3.28c) reads

$$\partial_x a_{\bar{x}} + \partial_y a_{\bar{y}} + \frac{1}{2}\partial_x f P a_{\bar{x}} - i e^{-fP/2}\Psi^\dagger e^{fP/2}\varphi = 0 \tag{3.100}$$

which by using (B.18a), and recalling that f only depends on $x, \bar{x}$, we find

$$\partial_x \partial_{\bar{x}}\xi + \partial_y \partial_{\bar{y}}\xi + \partial_x f P \partial_{\bar{x}}\xi - i\Lambda^\dagger (h - i\Psi\xi) = 0 \tag{3.101}$$

where we have defined

$$\Lambda = e^{fP}\Psi e^{-fP} = \begin{pmatrix} -\mu^2(x-y) & me^{2f} \\ m^2 x e^{-2f} & -\mu^2(x-y) \end{pmatrix} \tag{3.102}$$

To proceed it is convenient to make the following change of variables

$$U = e^{-fP/2}\varphi \qquad \Rightarrow \qquad \xi \; = \; i\Psi^{-1}(U - h) \tag{3.103}$$

and express (B.21) entirely in terms of the doublet U

$$\partial_x \partial_{\bar{x}} U + \partial_y \partial_{\bar{y}} U - (\partial_x \Psi)\Psi^{-1}\partial_{\bar{x}} U + (\partial_y \Psi)\Psi^{-1}\partial_{\bar{y}} U + \partial_x f \Psi P \Psi^{-1}\partial_{\bar{x}} U - \Psi\Lambda^\dagger U = 0 \tag{3.104}$$

so that the dependence on h drops completely. However, the D-term equation gives a coupled system of equations for U^+ and U^- that are quite involved to solve. Nevertheless, as discussed in Appendix B in the limit $m \gg \mu$ they decouple and one can prove that there is no localised mode for U^+, which we henceforth set to zero. Moreover, near the Yukawa point $p_{\text{up}} = \{x = y = 0\}$ one can approximate $f = \log c + c^2 m^2 x\bar{x} + \ldots$ and solve analytically for U^-, finding $U^- = \exp(\lambda_{\mathbf{10}} x\bar{x})h$ with $\lambda_{\mathbf{10}}$ the negative solution to $c^2\lambda_{\mathbf{10}}^3 + 4c^4 m^2 \lambda_{\mathbf{10}}^2 - m^4\lambda_{\mathbf{10}} = 0$. At the end one finds the solution

$$\vec{\varphi}^{\,j}_{\mathbf{10}^+} = \gamma^j_{\mathbf{10}} \begin{pmatrix} \frac{i\lambda_{\mathbf{10}}}{m^2} \\ 0 \\ 0 \end{pmatrix} e^{f/2}\chi^j_{\mathbf{10}} \qquad \vec{\varphi}^{\,j}_{\mathbf{10}^-} = \gamma^j_{\mathbf{10}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-f/2}\chi^j_{\mathbf{10}} \tag{3.105}$$

where $e^{f/2} = \sqrt{c}\, e^{m^2c^2 x\bar{x}/2}$ and $\chi^j_{\mathbf{10}} = e^{\lambda_{\mathbf{10}} x\bar{x}}\, g_j(y)$, with g_j holomorphic functions of y and moreover we packed the up components of the doublets in the vector $\vec{\varphi}_{\mathbf{10}^+}$ and the down components in the vector $\vec{\varphi}_{\mathbf{10}^-}$ similarly to the case we considered for the zero modes in the **5** sector.

Switching on the primitive worldvolume fluxes will amount to replace ordinary derivatives $\partial_{x,y}$ with covariant derivative $D_{x,y}$ in the D-term equation and similarly for $\bar{\partial}$ in the F-term equations. Still, in the limit $m \gg \mu$ and near the origin one finds a localised solution for U^- and the wavefunctions read

$$\vec{\varphi}^{\,j}_{\mathbf{10}^+} = \gamma^j_{\mathbf{10}} \begin{pmatrix} \frac{i\lambda_{\mathbf{10}}}{m^2} \\ -\frac{i\lambda_{\mathbf{10}}\zeta_{\mathbf{10}}}{m^2} \\ 0 \end{pmatrix} e^{f/2}\chi^j_{\mathbf{10}} \qquad \vec{\varphi}^{\,j}_{\mathbf{10}^-} = \gamma^j_{\mathbf{10}} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-f/2}\chi^j_{\mathbf{10}} \tag{3.106}$$

where $\lambda_{\mathbf{10}}$ is the negative solution to

$$m^4(\lambda_{\mathbf{10}} - q_R) + \lambda c^2 \left(c^2 m^2 (q_R - \lambda_{\mathbf{10}}) - \lambda_{\mathbf{10}}^2 + q_R^2 + q_S^2\right) = 0 \tag{3.107}$$

and $\zeta_{\mathbf{10}} = -q_S/(\lambda_{\mathbf{10}} - q_R)$. The scalar wavefunctions $\chi_{\mathbf{10}}$ read

$$\chi^j_{\mathbf{10}} = e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+\lambda_{\mathbf{10}}x(\bar{x}-\zeta_{\mathbf{10}}\bar{y})}\, g_j(y+\zeta_{\mathbf{10}}x) \tag{3.108}$$

where g_j holomorphic functions of $y + \zeta_{\mathbf{10}}x$, and $j = 1, 2, 3$ label the different zero mode families. Following [83] we will choose such holomorphic representatives to be

$$g_j = m_*^{3-j}(y + \zeta_{\mathbf{10}}x)^{3-j} \tag{3.109}$$

Finally, notice that within each family the wavefunctions differ for each of the sectors $\mathbf{10}_{1,2,3}$ of table 5 because they have different hypercharges and so q_R and q_S take different values for each. From the results of the previous sections we expect that this difference will only appear in the physical Yukawa couplings via different normalisation factors $\gamma^j_{\mathbf{10}}$, which we now proceed to discuss.

Normalisation factors

We now follow the recipe for the computation of the normalisation factors that we gave in section 3.2.2. We start with the sector $\mathbf{5}$ where only a single zero mode is present leaving only a single constant $\gamma_{\mathbf{5}}$ to be determined. The integral we perform is

$$K_{\mathbf{5}} = m_*^2|\gamma_{\mathbf{5}}|^2||\vec{v}_{\mathbf{5}}|| \int_S \chi^*_{\mathbf{5}}\chi_{\mathbf{5}}\, \mathrm{dvol}_S \tag{3.110}$$

with $\chi_{\mathbf{5}}$ given by (3.94), and $\vec{v}_{\mathbf{5}} = \frac{1}{2\mu^2}(i\zeta_{\mathbf{5}},\, i(\zeta_{\mathbf{5}} - \lambda_{\mathbf{5}}),\, 2\mu^2)^t$. Due to the convergence properties of $\chi_{\mathbf{5}}$ we can compute the above integral by extending the patch in which we define our local model to $\mathbb{C}^2$. We find that the required value for $\gamma_{\mathbf{5}}$ is

$$|\gamma_{\mathbf{5}}|^2 = -\frac{4}{\pi^2}\left(\frac{\mu}{m_*}\right)^4 \frac{(2\zeta_{\mathbf{5}} + q_R)(q_R + 2\zeta_{\mathbf{5}} - 2\lambda_{\mathbf{5}}) + (q_S + \lambda_{\mathbf{5}})^2}{4\mu^4 + \zeta^2_{\mathbf{5}} + (\zeta_{\mathbf{5}} - \lambda_{\mathbf{5}})^2}\,. \tag{3.111}$$

Here $\lambda_{\mathbf{5}}$ and $\zeta_{\mathbf{5}}$ are defined as in (3.94) and so depend on the worldvolume flux densities q_R and q_S, which are given in table 5 for both sectors $\mathbf{5}_1$ and $\mathbf{5}_2$. Hence in general both members of the $\mathbf{5}$-plet have different normalisation factors. In fact, for the sector $\mathbf{5}_2 = (\mathbf{3}, \mathbf{1})_{1/3}$ that could contain a Higgs triplet we find that $\gamma_{5_2} = 0$ after we impose the condition (3.71).[21]

For the zero modes in the $\mathbf{10}$ sector we find the following kinetic matrix

$$\begin{aligned} K^{ij}_{\mathbf{10}} &= m_*^2 \int_S \mathrm{Tr}(\vec{\varphi}^{\,i}_{\mathbf{10}^+}{}^\dagger \vec{\varphi}^{\,j}_{\mathbf{10}^+} + \vec{\varphi}^{\,i}_{\mathbf{10}^-}{}^\dagger \vec{\varphi}^{\,j}_{\mathbf{10}^-})\mathrm{dvol}_S \\ &= m_*^2(\gamma^i_{\mathbf{10}})^*\gamma^j_{\mathbf{10}} \sum_{\kappa=\pm} ||\vec{v}_{\mathbf{10}^\kappa}|| \int_S e^{\kappa f}(\chi^i_{\mathbf{10}})^*\chi^j_{\mathbf{10}}\, \mathrm{dvol}_S \end{aligned} \tag{3.112}$$

with the vectors $\vec{v}_{\mathbf{10}^\pm}$ enter in (3.106) multiplying the scalar wavefunctions $\chi_{\mathbf{10}}$. Because the integrand needs to be invariant under the rotation $(x, y) \to e^{i\alpha}(x, y)$ to have a non-vanishing result we deduce that $K^{ij}_{\mathbf{10}} = 0$ for $i \neq j$, and so we only need

21 For $q_S = 0$ the parameter $\zeta_{\mathbf{5}}$ defined below eq.(3.94) reduces to $\zeta_{\mathbf{5}} = \frac{1}{2}(\lambda_{\mathbf{5}} - q_R)$ which upon substituting in (3.111) shows that $\gamma_{\mathbf{5}_2} = 0$.

to adjust the constants $\gamma^j_{\mathbf{10}}$ in order to have canonical kinetic terms in this sector. In particular we obtain

$$|\gamma^j_{\mathbf{10}}|^2 = -\frac{c}{m_*^2\pi^2(3-j)!}\frac{1}{\frac{1}{2\lambda_{\mathbf{10}}+q_R(1+\zeta^2_{\mathbf{10}})-m^2c^2}+\frac{c^2\lambda^2_{\mathbf{10}}}{m^4}\frac{1}{2\lambda_{\mathbf{10}}+q_R(1+\zeta^2_{\mathbf{10}})+m^2c^2}}\left(\frac{q_R}{m_*^2}\right)^{4-j} \tag{3.113}$$

which not only depend on the family index j, but also on the sectors $\mathbf{10}_{1,2,3}$ of table 5, again via the flux densities q_R and q_S and the quantities $\lambda_{\mathbf{10}}$, $\zeta_{\mathbf{10}}$ that depend on them. Finally, notice that the effects of the non-primitive flux (3.67) in this sector appear through the dependence on the constant c.

Non-perturbative corrections

At this point we turn to the impact of non–perturbative corrections to the zero modes computed so far. It is important to note that the presence of non–perturbative effects will also deform the background values for Φ and A and due to the presence of the T–brane background and the non–primitive fluxes this computation is a particularly involved one. This will backreact on the computation of the $\mathcal{O}(\epsilon)$ in the **10** but, even if we will not compute explicitly these corrections, we will be able to show that they have no effect on both the Yukawa couplings and the kinetic terms. Since we will also show that the normalisation factor $\gamma_{\mathbf{5}}$ is not corrected either at $\mathcal{O}(\epsilon)$ we conclude that the results for the normalisation factors obtained in the previous section are all that we need to compute the physical Yukawa couplings.

Corrections to the background

Following Section 3.3.2, we can solve the equations of motion for the background for $\epsilon \neq 0$ in the holomorphic gauge if we take $\langle A_{0,1}\rangle = 0$ and $\langle\Phi\rangle$

$$\langle\Phi\rangle = \langle\Phi\rangle^{(0)} + \epsilon\,\partial\theta_0 \wedge \langle A^{1,0}\rangle\,, \tag{3.114}$$

Here $\langle\Phi\rangle^{(0)}$ and $\langle A_{1,0}\rangle^{(0)}$ are given by the background at $\epsilon = 0$ and in the holomorphic gauge. Let us first assume that the primitive fluxes (3.67) vanish. Then we have that $\langle\Phi\rangle^{(0)}$ is given by (3.52) and $\langle A_{1,0}\rangle^{(0)} = i\partial f\, P$ and so in the holomorphic gauge

$$\langle\Phi_{xy}\rangle \;=\; m(E^+ + m\,xE^-) + \epsilon\,\theta_y\partial_x f\,P + \mu^2(x-y)Q \tag{3.115}$$

where we have used that $f = f(x,\bar{x})$ and taken the choice (3.64). One may now perform a complexified gauge transformation (3.14) in order to go to a real gauge that satisfies the D-term (3.7) up to $\mathcal{O}(\epsilon^2)$. For this we need generalise the Ansatz (3.17) to

$$g \;=\; e^{\frac{f}{2}P}e^{\frac{\epsilon}{2}(kE^+ + k^*E^-)} \;=\; e^{\frac{f}{2}P} + \frac{\epsilon}{2}(k\,e^{f/2}E^+ + k^*e^{-f/2}E^-) + \mathcal{O}(\epsilon^2) \tag{3.116}$$

with f as above and k a complex function of $x,\bar{x}$. From this transformation we obtain the physical background

$$\begin{aligned}\langle\Phi_{xy}\rangle \;&=\; m(e^fE^+ + m\,xe^{-f}E^-) + \epsilon\left[\theta_y\partial_x f + \frac{m}{2}(mxk - k^*)\right]P + \mu^2(x-y)Q + \mathcal{O}(\epsilon^2)\\ \langle A_{0,1}\rangle \;&=\; -\frac{i}{2}\bar{\partial}fP - i\frac{\epsilon}{2}\left(\bar{\partial}k\,e^fE^+ + \bar{\partial}k^*\,e^{-f}E^-\right) + \mathcal{O}(\epsilon^2)\end{aligned} \tag{3.117}$$

Inserting (3.117) into the D-term equation we recover that f has again to satisfy the Painlevé equation (3.21). The differential equation obeyed by k is more involved, however using that near the origin $f = \log c + m^2 c^2 x\bar{x} + \ldots$ we find the solution

$$k = \bar{\theta}_y \, c^2 m x + \theta_y \, \frac{1-c^2}{2c^2-1} m^2 \bar{x}^2 + \ldots \tag{3.118}$$

where the dots stand for higher powers of $x, \bar{x}$.

Finally, let us restore the presence of primitive fluxes (3.67). As these fluxes commute with all the other elements of the background their presence does not modify the discussion above, and we can add their contribution to the corrected background independently. At the ends one finds

$$\begin{aligned} \langle \Phi_{xy} \rangle &= m(E^+ + m\, x E^-) + \epsilon \left[\theta_y \partial_x f + \frac{m}{2}(mxk - k^*) \right] P \qquad (3.119) \\ &\quad + \mu^2 (x-y) Q + \ \epsilon \left[\theta_y (\bar{x} Q_R - \bar{y} Q_S) + \theta_x (\bar{x} Q_S + \bar{y} Q_R) \right] + \mathcal{O}(\epsilon^2) \\ \langle A_{0,1} \rangle &= \langle A^p_{0,1} \rangle - \frac{i}{2} \bar{\partial} f P - i \frac{\epsilon}{2} \left(\bar{\partial} k \, e^f E^+ + \bar{\partial} k^* \, e^{-f} E^- \right) + \mathcal{O}(\epsilon^2) \qquad (3.120) \end{aligned}$$

where $\langle A^p_{0,1} \rangle$ stands for the potential of the primitive flux (3.67) in a physical gauge. Notice that the $\mathcal{O}(\epsilon)$ corrections to the worldvolume flux lie along the non-commuting generators $E^\pm$, while for the Higgs background they lie along the Cartan of E_6.

In the following we will look for the pair (a, φ) solving the following system of differential equations

$$\bar{\partial}_{\langle A \rangle} a = \mathcal{O}(\epsilon^2) \tag{3.121a}$$

$$\bar{\partial}_{\langle A \rangle} \varphi - i[a, \langle \Phi \rangle] + \epsilon \partial \theta_0 \wedge \partial_{\langle A \rangle} a = \mathcal{O}(\epsilon^2) \tag{3.121b}$$

$$\omega \wedge \partial_{\langle A \rangle} a - \frac{1}{2} [\langle \bar{\Phi} \rangle, \varphi] = \mathcal{O}(\epsilon^2) \tag{3.121c}$$

where $\langle A \rangle$ and $\langle \Phi \rangle$ are respectively specified by (3.119) and (3.120).

Sector 5

The sector **5** is relatively simple due to the fact that its zero modes are not charged under the generators of the $\mathfrak{su}(2)$ algebra $\{E^\pm, P\}$. More precisely, for this sector $\langle A_{0,1} \rangle$ reduces to $\langle A^p_{0,1} \rangle$, and $\langle \Phi_{xy} \rangle$ to the second line of (3.119). As a result, solving the zero mode equations (3.121) for this sector is very similar to the analogous problem for the $SO(12)$ local model of [74]. Hence in the following we simply present the final result, and refer the reader to Section 5.1 of [74] for further details.

The solution to the non-perturbative zero mode equations is given by

$$\vec{\varphi}_{\mathbf{5}} = \gamma_{\mathbf{5}} \begin{pmatrix} i \frac{\zeta_{\mathbf{5}}}{2\mu^2} \\ i \frac{(\zeta_{\mathbf{5}} - \lambda_{\mathbf{5}})}{2\mu^2} \\ 1 \end{pmatrix} \chi^{\text{np}}_{\mathbf{5}}, \qquad \chi^{\text{np}}_{\mathbf{5}} = e^{\frac{q_R}{2}(|x|^2 - |y|^2) - q_S(x\bar{y} + y\bar{x}) + (x-y)(\zeta_{\mathbf{5}}\bar{x} - (\lambda_{\mathbf{5}} - \zeta_{\mathbf{5}})\bar{y}))} (1 + \epsilon \Upsilon_{\mathbf{5}}) \tag{3.122}$$

with $\lambda_{\mathbf{5}}$, $\zeta_{\mathbf{5}}$ defined as in (3.94). The $\mathcal{O}(\epsilon)$ non-perturbative correction is

$$\Upsilon_{\mathbf{5}} = -\frac{1}{4\mu^2} (\zeta_{\mathbf{5}} \bar{x} - (\lambda_{\mathbf{5}} - \zeta_{\mathbf{5}}) \bar{y})^2 (\theta_x + \theta_y) + \frac{\delta_1}{2}(x-y)^2 + \frac{\delta_2}{\zeta_{\mathbf{5}}}(x-y)(\zeta_{\mathbf{5}} y + (\lambda_{\mathbf{5}} - \zeta_{\mathbf{5}}) x) \tag{3.123}$$

where

$$\delta_1 = \frac{2\mu^2}{\lambda_{\mathbf{5}}^2}\{\bar\theta_x(q_R(\zeta_{\mathbf{5}}-\lambda_{\mathbf{5}})+q_S\zeta_{\mathbf{5}})+\bar\theta_y(q_R\zeta_{\mathbf{5}}-q_S(\zeta_{\mathbf{5}}-\lambda_{\mathbf{5}}))\}\,, \tag{3.124}$$

$$\delta_2 = \frac{2\mu^2\zeta_{\mathbf{5}}}{\lambda_{\mathbf{5}}^2}\{\bar\theta_x(q_R+q_S)+\bar\theta_y(q_R-q_S)\}\,. \tag{3.125}$$

As in [74] one can check that the corrections to the norm (3.111) only appear at $\mathcal{O}(\epsilon^2)$, because $\mathcal{O}(\epsilon)$ terms that appear in the integrand of (3.110) are not invariant under the rotation $(x,y)\to e^{i\alpha}(x,y)$.

Sector 10

Similarly to the case of perturbative zero modes, finding the non-perturbative corrections to the wavefunctions of the sector **10** is in general rather involved. Nevertheless, taking the same approximations as in the perturbative case, one may understand how these corrections look like and argue that they will not be relevant for computing the matrix of physical Yukawa couplings.

The first step is to switch off the primitive fluxes and realise that, in the same way that $a=\bar\partial\xi$ and (3.35) solve the F-term equations (3.121a) and (3.121b) in the holomorphic gauge, in the real gauge they are satisfied by

$$a = g\,\bar\partial\xi \tag{3.126a}$$

$$\varphi = g\,(h - i\Psi\xi - \epsilon\partial\theta_0\wedge\partial\xi) \;=\; g\,U\,dx\wedge dy \tag{3.126b}$$

with g given by (3.116) and Ψ given by (3.81). Here a, φ, χ are $SU(2)$ doublets as in eq.(B.18). The same applies to U, which can be expanded in powers of ϵ as

$$U \;=\; U^{(0)}+\epsilon\,U^{(1)}+\,\mathcal{O}(\epsilon^2) \tag{3.127}$$

where $U^{(0)}$ corresponds to solution found for $\epsilon=0$, namely

$$U_-^{(0)} \;=\; e^{\lambda_{\mathbf{10}}x\bar x}h(y) \qquad\qquad U_+^{(0)} \;=\; 0 \tag{3.128}$$

Then one may solve for ξ as

$$\begin{aligned}\xi \;&=\; \xi^{(0)}+i\epsilon\Psi^{-1}\left[U^{(1)}+\partial_x\theta_0\partial_y\xi^{(0)}-\partial_y\theta_0\partial_x\xi^{(0)}\right]+\mathcal{O}(\epsilon^2)\\ \xi^{(0)} \;&=\; i\Psi^{-1}(U^{(0)}-h)\end{aligned} \tag{3.129}$$

and then solve for $U^{(1)}$ by inserting this expression into the D-term equation (3.121c). As in the perturbative case this problem can be easily solved in the limit $\mu\to 0$, obtaining that $U_-^{(1)}=0$. As a result, in this limit we have the structure

$$\xi_+ \;=\; \xi_+^{(0)}+0+\mathcal{O}(\epsilon^2) \qquad\qquad \xi_- \;=\; 0+\epsilon\,\xi_-^{(1)}+\mathcal{O}(\epsilon^2) \tag{3.130}$$

that is, the $\mathcal{O}(\epsilon)$ corrections to ξ are contained in the opposite doublet as the tree-level contribution. The same statement applies to a and φ. Indeed, we have that

$$\varphi_{xy} \;=\; g^{(0)}U^{(0)}+\epsilon(g^{(0)}U^{(1)}+g^{(1)}U^{(0)})+\mathcal{O}(\epsilon^2) \tag{3.131}$$

where we have decomposed $g = g^{(0)} + \epsilon g^{(1)} + \mathcal{O}(\epsilon^2)$ as in (3.116). Then, because $g^{(0)}$ only involves P and $g^{(1)}$ involves $E^{\pm}$ we have

$$\varphi_+ \;=\; 0 + \epsilon\, \varphi_+^{(1)} + \mathcal{O}(\epsilon^2) \qquad\qquad \varphi_- \;=\; \varphi_-^{(0)} + 0 + \mathcal{O}(\epsilon^2) \tag{3.132}$$

Finally, a similar argument shows that $a_+ = a_+^{(0)} + \mathcal{O}(\epsilon^2)$ and $a_- = a_-^{(1)} + \mathcal{O}(\epsilon^2)$ and so the wavefunctions (3.106) have a correction of the form

$$\vec{\varphi}_{\mathbf{10}^+} \;=\; \begin{pmatrix} \bullet \\ \bullet \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ 0 \\ \bullet \end{pmatrix} + \mathcal{O}(\epsilon^2) \qquad \vec{\varphi}_{\mathbf{10}^-} \;=\; \begin{pmatrix} 0 \\ 0 \\ \bullet \end{pmatrix} + \epsilon \begin{pmatrix} \bullet \\ \bullet \\ 0 \end{pmatrix} + \mathcal{O}(\epsilon^2) \tag{3.133}$$

One can check that this structure remains even after we restore the presence of non-primitive fluxes. Then, since the $\mathcal{O}(\epsilon)$ correction vector is orthogonal to the 0^{th}-order solution, it is easy to see that no $\mathcal{O}(\epsilon)$ correction to the normalisation factors $\gamma^j_{\mathbf{10}}$ arises by plugging these corrected wavefunctions into (3.113).

3.3.4 *Physical Yukawas and mass hierarchies*

Summarising the information gathered so far we know that the Yukawa matrix for the up–type quarks is

$$Y_U \;=\; \frac{\pi^2 \gamma_{\mathbf{5}}}{4\rho_\mu \rho_m} \begin{pmatrix} 0 & 0 & \tilde{\epsilon}\rho_\mu^{-1}\gamma_L^1\gamma_R^3 \\ 0 & \tilde{\epsilon}\rho_\mu^{-1}\gamma_L^2\gamma_R^2 & 0 \\ \tilde{\epsilon}\rho_\mu^{-1}\gamma_L^3\gamma_R^1 & 0 & -2\gamma_L^3\gamma_R^3 \end{pmatrix} + \mathcal{O}(\tilde{\epsilon}^2) \tag{3.134}$$

where

$$\rho_\mu \;=\; \frac{\mu^2}{m_*^2} \qquad\qquad \rho_m \;=\; \frac{m^2}{m_*^2} \qquad\qquad \tilde{\epsilon} \;=\; \epsilon\,(\theta_x + \theta_y) \tag{3.135}$$

are all flux-independent parameters. The worldvolume flux dependence (and in particular the hypercharge dependence) is encoded in the normalisation factors $\gamma_{\mathbf{5}}$ and $\gamma^i_{R,L}$, where $\gamma_{\mathbf{5}}$ is given by (3.111) with the values of q_R, q_S for the sector $\mathbf{5}_1$ of table 5. Finally, γ^i_R is given by (3.113) using the values of q_R and q_S in the first row of table 5, and similarly for γ^i_L with the values in the second row.

We would like to see if this structure for Yukawa couplings allows to fit experimental fermion masses. Since our expressions apply at the GUT scale, presumably of order 10^{16} GeV, the data need to be run up to this scale. Table 6 shows the result of doing so for the MSSM quark mass ratios, for different values of $\tan\beta$ as taken from ref.[84]. In the following we will analyse if this spectrum can be accommodated in our scheme.

The top quark Yukawa

The Yukawa for the top quark is given by the 33 entry of (3.134). To analyse its value it is useful to express the quantities ρ_μ and ρ_m as

$$\rho_\mu \;=\; \left(\frac{\mu}{m_*}\right)^2 \;=\; (2\pi)^{3/2} g_s^{1/2} \sigma_\mu \qquad\qquad \rho_m \;=\; \left(\frac{m}{m_*}\right)^2 \;=\; (2\pi)^{3/2} g_s^{1/2} \sigma_m \tag{3.136}$$

$\tan\beta$	10	38	50
m_u/m_c	$2.7 \pm 0.6 \times 10^{-3}$	$2.7 \pm 0.6 \times 10^{-3}$	$2.7 \pm 0.6 \times 10^{-3}$
m_c/m_t	$2.5 \pm 0.2 \times 10^{-3}$	$2.4 \pm 0.2 \times 10^{-3}$	$2.3 \pm 0.2 \times 10^{-3}$
m_d/m_s	$5.1 \pm 0.7 \times 10^{-2}$	$5.1 \pm 0.7 \times 10^{-2}$	$5.1 \pm 0.7 \times 10^{-2}$
m_s/m_b	$1.9 \pm 0.2 \times 10^{-2}$	$1.7 \pm 0.2 \times 10^{-2}$	$1.6 \pm 0.2 \times 10^{-2}$
Y_t	0.48 ± 0.02	0.49 ± 0.02	0.51 ± 0.04
Y_b	0.051 ± 0.002	0.23 ± 0.01	0.37 ± 0.02

Table 6: Running mass ratios of quarks at the unification scale and for different values of *tan* β, as taken from ref.[84]. The Yukawa couplings $Y_{t,b}$ at the unification scale are also shown.

where $\sigma_\mu = (\mu/m_{st})^2$ and $\sigma_m = (m/m_{st})^2$ are the 7-brane intersection slopes measured in units of m_{st}, the scale that in the type IIB limit reduces to the string scale $m_{st} = 2\pi\alpha'$ and which is related to the F-theory scale as $m_{st}^4 = g_s(2\pi)^3 m_*^4$ [74]. We then have that

$$|Y_t| \;=\; (8\pi g_s)^{1/2}\sigma_m c\; \tilde{\gamma}_{\mathbf{5}_1}\tilde{\gamma}_{\mathbf{10}_1}\tilde{\gamma}_{\mathbf{10}_2} \tag{3.137}$$

where

$$\tilde{\gamma}_{\mathbf{5}_1} \;=\; \left(-\frac{(2\zeta_{\mathbf{5}_1} + q_R^{\mathbf{5}_1})(q_R^{\mathbf{5}_1} + 2\zeta_{\mathbf{5}_1} - 2\lambda_{\mathbf{5}_1}) + (q_S^{\mathbf{5}_1} + \lambda_{\mathbf{5}_1})^2}{4\mu^4 + \zeta_{\mathbf{5}_1}^2 + (\zeta_{\mathbf{5}_1} - \lambda_{\mathbf{5}_1})^2}\right)^{1/2} \tag{3.138}$$

$$\tilde{\gamma}_{\mathbf{10}_i} \;=\; \left(-\frac{q_R^{\mathbf{10}_i}}{\frac{m^4}{2\lambda_{\mathbf{10}_i} + q_R^{\mathbf{10}_i}(1+\zeta_{\mathbf{10}_i}^2) - m^2c^2} + \frac{c^2\lambda_{\mathbf{10}_i}^2}{2\lambda_{\mathbf{10}_i} + q_R^{\mathbf{10}_i}(1+\zeta_{\mathbf{10}_i}^2) + m^2c^2}}\right)^{1/2} \tag{3.139}$$

with $q_{R,s}^{\mathbf{10}_i}$, $i = 1, 2$ the values of $q_{R,S}$ in the i^{th} row of table 5, $q_{R,S}^{\mathbf{5}_1}$ the ones in the fourth row, etc.

From (3.137) one may proceed as in [74] and estimate that primitive worldvolume flux densities are of the order

$$M, N \;\simeq\; 0.29\, g_s^{1/2} m_{st}^2 \tag{3.140}$$

with g_s not too small. In fact, the diluted flux approximation is one of the requirements that we need to impose in order to be able to trust the 7-brane effective action that led to the zero mode equations of Section 3.3.3.

Given these restrictions one can see that one may accommodate a realistic value for the Yukawa of the top at the unifications scale. Indeed, if one for instance takes the values (in units of m_{st})

$$M = 0.3\,, \quad N = 0.03\,, \quad \tilde{N}_Y = 0.6\,, \quad N_Y = -0.18\,, \quad m = 0.5\,, \quad \mu = 0.1\,, \tag{3.141}$$

with $g_s = 1$ and c as in (3.27) one obtains

$$Y_t = 0.5 \tag{3.142}$$

in quite good agreement with the values of table 6. We chose for simplicity to take c coinciding with the value in (3.27) leaving open the possibility of a more general value for c for the models to be considered in the next sections. One can also check that the wavefunctions are sufficiently localised in a region where the first two terms of (3.26) are a good approximation for the Painlevé transcendent.

Up-type quarks mass hierarchies

In order to analyse the flavour hierarchies among different U-quarks let us consider the matrix

$$\frac{Y_U}{Y^{33}} = \begin{pmatrix} 0 & 0 & -\frac{1}{2}\tilde{\epsilon}\rho_\mu^{-1}\frac{\gamma_L^1}{\gamma_L^3} \\ 0 & -\frac{1}{2}\tilde{\epsilon}\rho_\mu^{-1}\frac{\gamma_L^2\gamma_R^2}{\gamma_L^3\gamma_R^3} & 0 \\ -\frac{1}{2}\tilde{\epsilon}\rho_\mu^{-1}\frac{\gamma_R^1}{\gamma_L^3} & 0 & 1 \end{pmatrix} + \mathcal{O}(\tilde{\epsilon}^2) \tag{3.143}$$

whose eigenvalues are

$$\begin{aligned} \lambda_1 &= 1 + \mathcal{O}(\epsilon^2) \\ \lambda_2 &= -\epsilon\frac{1}{2\rho_\mu}\frac{\gamma_L^2\gamma_R^2}{\gamma_L^3\gamma_R^3}(\theta_x+\theta_y) + \mathcal{O}(\epsilon^2) \\ \lambda_3 &= \mathcal{O}(\epsilon^2). \end{aligned}$$

where we have used the expression for $\tilde{\epsilon}$ in (3.135). This yields automatically a hierarchy of U-quark masses of the form $(1, \epsilon, \epsilon^2)$ in fact quite similar to the one found in [74] for the D-quarks and leptons. As in there, the quotient of quark masses of different families is rather simple. Namely identifying the first and second eigenvalues with the third and second generations of U-quarks we have

$$\begin{aligned} \frac{m_c}{m_t} &= \frac{1}{2}\left(\frac{q_R^{\mathbf{10}_1}q_R^{\mathbf{10}_2}}{\mu^4}\right)^{1/2}\epsilon\,(\theta_x+\theta_y) \\ &= \frac{1}{2}\frac{M}{\mu^2}\left(1+\frac{2\tilde{N}_Y}{3M}\right)^{1/2}\left(1-\frac{\tilde{N}_Y}{6M}\right)^{1/2}\epsilon\,(\theta_x+\theta_y) \end{aligned} \tag{3.144}$$

where we have used that $q_R^{\mathbf{10}_1} = M + \frac{2}{3}\tilde{N}_Y$ and $q_R^{\mathbf{10}_2} = M - \frac{1}{6}\tilde{N}_Y$. Hence it is quite easy to accommodate the hierarchy between the charm and the top quark with a small non-perturbative parameter ϵ.

3.4 E_7 & E_8 MODELS

So far we have been discussing the case where the Yukawa couplings for up–type quarks and down–type quarks are generated at different points in the GUT divisor S and analysed the former case in detail. While this scenario seems to be the most generic one there are some shortcomings that may be solved if all the Yukawa couplings are generated at a unique point in S. From the point of view of our local models the main advantage of having all Yukawa couplings generated at a single point is that one model is indeed sufficient for computing the whole set of the Yukawa couplings appearing in the MSSM. This allows us to use the same local data defining the model (namely flux densities and brane intersection angles) to compute the various fermion masses without needing this local information at two different points in S. Moreover being able to compute all the couplings between the Higgs fields and the fermions will allow us to compute the CKM matrix and compare the typical values for the Yukawa couplings of the top and bottom quarks (which in turns allow us to find some preferred value for $\tan\beta$). Having all couplings generated at a single point moreover will give us some bonus: in addition to the MSSM fields we find some additional singlets which may be used to generate some interesting couplings with the MSSM fields. Let us mention the two most interesting ones:

- **Mass term for the Higgs fields.**
  It is well known that in the MSSM the value of the μ–term in the superpotential should be of the same order of magnitude of the soft terms to achieve successful electroweak symmetry breaking. The fact that there is no reason a priori for why this should happen is the so–called μ–problem of the MSSM and in the literature there have been some proposals for dynamical generation of the μ–term with the correct order of magnitude[22]. We will be mostly interested in the NMSSM as a solution for the μ–problem for via the aforementioned singlets it is indeed possible to realise this scenario (or a variation thereof) in F–theory GUT models. The main idea is that it is possible to have the following coupling in the superpotential

  $$W \supset \lambda\, S\, H_u\, H_d\,, \tag{3.145}$$

  where λ is a constant and S is the singlet. If S acquires a vev then we generate an effective mass term for the Higgs fields and moreover since the potential felt by S depends on the soft susy breaking parameters we find that the effective μ–term will have the correct order of magnitude. Implementation of the NMSSM in F–theory has been considered in [87] to which we refer for the phenomenological implications.

- **Neutrino masses.**
  Another important use of singlets is the generation of a mass term for the neutrinos which is missing in the MSSM. By identifying the singlets with the right handed neutrinos N_R we have that it is possible to generate a mass term of the form

  $$W \supset \lambda\, H_u\, L\, N_R\,, \tag{3.146}$$

  where L is the lepton superfield and λ is again a constant. One major issue in this scenario is the explanation for why the coupling λ is sufficiently small to account for the low masses of the neutrinos. There have been some proposals in the literature trying to implement different scenarios for neutrinos masses (usually divided in Dirac and Majorana mass terms), we refer to [88] for an account of these scenarios.

22 The most famous ones being the Giudice–Masiero mechanism [85], the Kim–Nilles mechanism [86] and the NMSSM.

Having motivated why we would like to have all couplings generated at a single point we need to understand how this Yukawa point would look like. We need a local enhancement that generates Yukawa couplings for all quarks and leptons and therefore the enhanced gauge group ought to contain both E_6 and $SO(12)$. The possible groups that fulfil this condition are E_7 and E_8, and since these groups both contain $SU(7)$ we are also able to generate the coupling $\mathbf{1} \cdot \mathbf{5} \cdot \bar{\mathbf{5}}$ (which is necessary for the aforementioned couplings with the singlets). The idea for this "unification" of Yukawa points was first pointed out in [89] where E_8 was selected as a more promising avenue based on considerations on neutrino masses and μ–term. However in the following we will not necessarily impose these constraints and therefore leave open the possibility of E_7 as well. We now turn to the classification of models with E_8 and E_7 enhancement and discuss the structure of the Yukawa matrices for these models. The upshot is that only a couple of models will be selected as promising ones with the great advantage that both models will be embeddable in both E_7 and E_8 with the same results for the Yukawa matrices. Among these two models one was selected in [90] with some considerations on neutrino masses and μ–term, however this second model was analysed in more detail in [91] showing that it is not possible to encounter good values for the fermion masses. Therefore after presenting the classification of models we will analyse only one of the two models (the one called model A in [91]) discussing its phenomenological features.

3.4.1 *$SU(5)$ models with E_8 enhancement*

In this section we will present a set of local SU(5) F-theory models that can be described as an E_8 theory higgsed by a T-brane background. Each of these models has the appropriate structure of matter curves so that they can embed the full content of the MSSM chiral spectrum, with only one massive family at tree-level. The remaining families of quark and leptons will become massive due to non-perturbative corrections, but then we find that one may get a hierarchy of masses either of the form $(1, \epsilon, \epsilon^2)$ or of the form $(1, \epsilon^2, \epsilon^2)$. Since ϵ is a very small number that measures the strength of a non-perturbative effect, the latter hierarchical pattern is very unlikely to reproduce empirical data , while the former as we already saw for the case of the E_6 model is adequate to achieve a correct hierarchical spectrum.

Since the hierarchical structure of the Yukawa matrix can be studied by simple inspection of the holomorphic couplings we will be able to simply apply the residue formula discussed in Section 3.2.1 to analyse the full set of models.

T-branes and matter curves

One crucial feature of an F-theory local model with respect to the computation of Yukawa couplings is the profile for the 7-brane Higgs field Φ in the vicinity of the Yukawa point. To have a third family much heavier than the other two naturally selects a T-brane profile for Φ, which then specifies an appropriate local structure of matter curves.

In order to detect the structure of matter curves it proves useful to work with matrix representations of the Higgs field in the algebra $\mathfrak{g}_\perp$ defined such that $\mathfrak{g}_{\mathrm{GUT}} \oplus \mathfrak{g}_\perp$ is a maximal subalgebra of $\mathfrak{g}_p = \mathrm{Lie}(G_p)$. Let us consider the case of interest in this paper, namely $\mathfrak{g}_p = \mathfrak{e}_8$ and $\mathfrak{g}_{\mathrm{GUT}} = \mathfrak{su}_5$. Then we have the well-known maximal decomposition

$$\begin{aligned} \mathfrak{e}_8 &\supset \mathfrak{su}_5^{\mathrm{GUT}} \otimes \mathfrak{su}_5^{\perp} \\ \mathbf{248} &\to (\mathbf{24},\mathbf{1}) \oplus (\mathbf{1},\mathbf{24}) \oplus ((\mathbf{10},\mathbf{5}) \oplus c.c.) \oplus ((\overline{\mathbf{5}},\mathbf{10}) \oplus c.c.) \end{aligned} \tag{3.147}$$

Since the Higgs profile $\langle\Phi\rangle$ belongs to the adjoint of $\mathfrak{e}_8$ and by construction commutes with $\mathfrak{su}_5^{\mathrm{GUT}}$, it will only act non-trivially on the each of the representations $\mathcal{R}$ of $\mathfrak{g}_\perp = \mathfrak{su}_5^\perp$ that appear in (3.147). We will again express the action of Φ on a representation $\mathcal{R}$ as a matrix a matrix $\Phi_\mathcal{R}$ such that $[\langle\Phi\rangle, \mathcal{R}] = \Phi_\mathcal{R}\mathcal{R}$ so that whenever the determinant of $\Phi_\mathcal{R}$ vanishes an element of $\mathcal{R}$ will be commuting with $\langle\Phi\rangle$.

Interestingly, these facts allow to express the structure of matter curves in terms of the spectral surface of the Higgs field, which is defined as[23]

$$P_{\Phi_\mathcal{R}}(x,y,z) = \det(\Phi_\mathcal{R} - z\mathbb{I}) = 0 \tag{3.148}$$

for each of the matrices $\Phi_\mathcal{R}$ associated to $\langle\Phi\rangle$. Following [64], we say that $\Phi_\mathcal{R}$ is reconstructible if its spectral surface is a non singular algebraic variety, and that it is block reconstructible if it has the structure of a block diagonal matrix such that every block is reconstructible. As the property of reconstructibility is independent of the representation $\mathcal{R}$ we then say that the Higgs field is block reconstructible, and in this case the whole information of $\langle\Phi\rangle$ is carried by its spectral surfaces.[24]

Now it is easy to see how the pattern of matter curves can be encoded in the spectral surface (3.148): when the Higgs field is block reconstructible its spectral surfaces will be the product of polynomials whose zero locus is a non-singular algebraic variety, and there will be a one to one correspondence between these varieties and the matter curves in a specific representation. Hence, the presence of several matter curves will induce a splitting of the spectral surface into irreducible polynomials, the number of factors of this splitting matching with the number of matter curves.

In the following we will present a number of local E_8 models whose local spectrum of matter curves can be detected by means of the above considerations. For the sake of simplicity, we will focus on models in which the Higgs field background $\langle\Phi\rangle$ is block reconstructible, since then we can classify our models by the number of matter curves near the Yukawa point.

Catalogue of models

We now proceed to describe several kinds of local E_8 models with only one massive family at tree level. Such models are candidates to yield a realistic hierarchical fermion mass pattern after non-perturbative effects have been taken into account although, as already advertised, this will not always be the case. To find out we will compute the holomorphic Yukawa couplings, which depend on the profile for Φ in the holomorphic gauge [63]. For this purpose we only need to specify $\langle\Phi\rangle$ as a linear combination of holomorphic functions multiplying the E_8 roots, following the notation of appendix A.3. As explained above we may also describe this background as a matrix $\Phi_\mathbf{5}$ acting on the representation $\mathbf{5}$ of $\mathfrak{su}_5^\perp$, which allows to find the local set of $\mathbf{10}$ matter curves

23 The following expression for the spectral surface holds if Φ takes values in a $\mathfrak{u}_n$ subalgebra of $\mathfrak{g}_p$.
24 As shown in [64], $SU(k)$ reconstructible T-branes correspond to spectral covers with monodromy $\mathbb{Z}_k$.

via eq.(3.148). Since we are considering reconstructible backgrounds, the 5×5 matrix $\Phi_{\mathbf{5}}$ will be automatically block diagonal, so we can classify our local models by the different dimension of each of these blocks.

For each model we will describe the local set of matter curves that arise from the profile for Φ and the different possible assignments of the MSSM fields within them. Recall that in the absence of hypercharge flux the matter spectrum is organised in $SU(5)$ multiplets, so for the purpose of computing holomorphic Yukawa couplings we can consider that $SU(5)$ is unbroken. Then to achieve rank one Yukawa matrices at tree-level we need to have three copies of the matter representation $\mathbf{10}_M$ within the same $\mathbf{10}$-curve and three copies of $\bar{\mathbf{5}}_M$ in the same $\mathbf{5}$-curve. Finally the Higgs multiplets $\mathbf{5}_U$ and $\bar{\mathbf{5}}_D$ should be in $\mathbf{5}$-curves different from the one of $\bar{\mathbf{5}}_M$ and such that the couplings $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ and $\mathbf{10}_M \times \bar{\mathbf{5}}_M \times \bar{\mathbf{5}}_D$ are allowed.

For each assignment we will present the structure of holomorphic Yukawa couplings that arise from the superpotential (3.13) with

$$\theta_0 = i(x\theta_x + y\theta_y) \tag{3.149}$$

and (x, y) parametrising the complex coordinates of the 4-cycle S. Note that (3.149) coincides with (3.77) when dropping the constant term in the latter. Since the constant term does not contribute to the Yukawa couplings as discussed in Section 3.1.2 we chose not to introduce it here. We will then discuss whether such structure accommodates favourable hierarchies to fit empirical data. In the next section we will provide a more detailed description of one of the models with such favourable structure, providing all the details that allow to compute its physical Yukawas.

4+1 MODELS Let us first consider a holomorphic background for $\Phi = \Phi_{xy} dx \wedge dy$ of the form

$$\langle \Phi_{xy} \rangle = \lambda(\hat{H}_1 + 2\hat{H}_2 + 3\hat{H}_3 + 4\hat{H}_4) + m(E_1^+ + E_2^+ + E_5^+ + mxE_3^-) \tag{3.150}$$

where the notation and definitions that are used for the E_8 roots are given in appendix A.3. Here $\lambda = \mu^2(bx - y)$ is a holomorphic linear function of (x, y) vanishing at the origin, which is where the Yukawa point p will be located. By acting on the fundamental representation of $\mathfrak{su}_5^\perp$ we obtain the matrix representation

$$\Phi_{\mathbf{5}} = \begin{pmatrix} \lambda & m & 0 & 0 & 0 \\ 0 & \lambda & m & 0 & 0 \\ 0 & 0 & \lambda & m & 0 \\ m^2 x & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & -4\lambda \end{pmatrix}, \tag{3.151}$$

which displays a $4+1$ block structure. The various matter representations and their matter curves are then the following ones

- $\mathbf{10}$ sector

$$\mathbf{10}_a : \lambda^4 = m^5 x\,, \qquad \mathbf{10}_b : \lambda = 0\,,$$

- $\mathbf{5}$ sector

$$\mathbf{5}_a : (3\lambda)^4 = m^5 x\,, \qquad \mathbf{5}_b : \lambda^2(m^5 x + 4\lambda^4) = 0\,,$$

and it is easy to see that all the curves meet at the origin.

This local model has already been considered in [79], where it was found a rank one structure for the holomorphic Yukawas by using the tree-level superpotential (3.4). In the following we would like to extend this result by considering the superpotential (3.13) corrected by non-perturbative effects and providing the resulting holomorphic Yukawas up to order $\mathcal{O}(\epsilon^2)$.

As pointed out in [79] in order to generate an up-type Yukawa coupling $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ it is necessary to assign the representation $\mathbf{10}_M$ to the $\mathbf{10}_a$ curve and $\mathbf{5}_U$ to the curve $\mathbf{5}_b$. Because there are only two $\mathbf{5}$-curves, we will also consider that $\bar{\mathbf{5}}_D$ is also localised in $\mathbf{5}_b$ while the three copies of the $\bar{\mathbf{5}}_M$ are in $\mathbf{5}_a$.

With this setup one can compute the Yukawa matrices via a residue calculation. Schematically we find that

$$Y_U = \begin{pmatrix} 0 & 0 & \epsilon\, y_{13} \\ 0 & \epsilon\, y_{22} & 0 \\ \epsilon\, y_{31} & 0 & y_{33} \end{pmatrix} + \mathcal{O}(\epsilon^2)\,, \tag{3.152}$$

$$Y_{D/L} = \begin{pmatrix} 0 & 0 & \epsilon\, y_{13} \\ 0 & \epsilon\, y_{22} & 0 \\ \epsilon\, y_{31} & 0 & y_{33} \end{pmatrix} + \mathcal{O}(\epsilon^2)\,, \tag{3.153}$$

where y_{ij} are order one numbers We then reproduce the results of [79] in the limit $\epsilon \to 0$, while we see that for $\epsilon \neq 0$ the rank of both matrices is increased to three. Finally, both matrices will have a hierarchy of eigenvalues of the form $(\mathcal{O}(1), \mathcal{O}(\epsilon), \mathcal{O}(\epsilon^2))$ so this model has a Yukawa structure which is favourable to reproduce the empirical data.

Despite this favourable hierarchy, this model has the less attractive feature of having both up and down Higgses $\mathbf{5}_U$, $\bar{\mathbf{5}}_D$ in the same curve. Hence some particular mechanism should be invoked to prevent a large μ-term to be generated. Because of this potential drawback we will not consider this model in the following.

3+2 MODELS We next consider a Higgs background of the form

$$\langle \Phi_{xy} \rangle = -\lambda \left(\frac{2}{3}\hat{H}_1 + \frac{4}{3}\hat{H}_2 + 2\hat{H}_3 + \hat{H}_4 \right) + \tilde{m}(E_1^+ + E_5^+ + \tilde{m} y E_8^-) + m(E_{10}^+ + m x E_{10}^-)\,. \tag{3.154}$$

where again $\lambda = \mu^2(bx - y)$. Its action on the fundamental of $\mathfrak{su}_5^\perp$ is given by

$$\Phi_{\mathbf{5}} = \begin{pmatrix} -\frac{2}{3}\lambda & \tilde{m} & 0 & 0 & 0 \\ 0 & -\frac{2}{3}\lambda & \tilde{m} & 0 & 0 \\ \tilde{m}^2 y & 0 & -\frac{2}{3}\lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & m \\ 0 & 0 & 0 & m^2 x & \lambda \end{pmatrix}, \tag{3.155}$$

showing a $3+2$ block structure. The various matter representations and curves are now

- **10** sector

$$\mathbf{10}_a : -\frac{8}{27}\lambda^3 + \tilde{m}^4 y = 0\,, \qquad \mathbf{10}_b : \lambda^2 - m^3 x = 0\,,$$

- **5** sector

$$\mathbf{5}_a : \tilde{m}^4 y + \frac{64}{27}\lambda^3 = 0\,, \;\; \mathbf{5}_b : m^9 x^3 = \frac{\lambda^2 m^6 x^2}{3} + m^3 x\left(2\lambda\tilde{m}^4 y - \frac{\lambda^4}{27}\right) + \frac{1}{729}\left(\lambda^3 + 27\tilde{m}^4 y\right)^2,$$

$$\mathbf{5}_c : \lambda = 0\,.$$

In this class of models we can assign the three copies of $\mathbf{10}_M$ to either the $\mathbf{10}_a$ or the $\mathbf{10}_b$ sector. If we assign the $\mathbf{10}_M$ to the $\mathbf{10}_a$ sector then we need to assign $\mathbf{5}_U$ to the $\mathbf{5}_a$ sector in order to have $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ Yukawas which are singlets under $SU(5)_\perp$. Nevertheless, an explicit computation shows that the holomorphic up-type Yukawa couplings vanish for this arrangement. This vanishing result is analogous to the one found in [79] for the E_7 model studied in there, with the matter curves involved in the up-type Yukawas having a similar structure. We therefore see that this assignment of chiral matter to curves does not yield realistic Yukawas.

The other possibility in this model is to assign $\mathbf{10}_M$ to the $\mathbf{10}_b$ sector, which requires that $\mathbf{5}_U$ corresponds to the $\mathbf{5}_c$ sector. In addition one has to choose how to assign the representations $\bar{\mathbf{5}}_M$ and $\bar{\mathbf{5}}_D$ to the sectors $\mathbf{5}_a$ and $\mathbf{5}_b$, having two possibilities. The final structure of the Yukawa matrices does however not depend on this choice. In both cases we find that

$$Y_U = \begin{pmatrix} 0 & 0 & \epsilon\, y_{13} \\ 0 & \epsilon\, y_{22} & \epsilon\, y_{23} \\ \epsilon\, y_{31} & \epsilon\, y_{32} & y_{33} \end{pmatrix} + \mathcal{O}(\epsilon^2)\,, \tag{3.156}$$

$$Y_{D/L} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon\, y_{23} \\ 0 & \epsilon\, y_{32} & y_{33} \end{pmatrix} + \mathcal{O}(\epsilon^2)\,, \tag{3.157}$$

where again y_{ij} are order one numbers We see therefore that for this class of models the down-type Yukawa matrix does not have a favourable hierarchical structure.

2+2+1 MODELS We finally consider a Higgs background of the form

$$\langle\Phi_{xy}\rangle = \lambda_1(\hat{H}_1 + 2\hat{H}_2 - 2\hat{H}_4) - \lambda_2(\hat{H}_3 + 2\hat{H}_4) + m(E_1^+ + mxE_1^-) + \tilde{m}(E_2^+ + \tilde{m}yE_2^-) \tag{3.158}$$

whose action on the fundamental of $\mathfrak{su}_5^\perp$ is

$$\Phi_{\mathbf{5}} = \begin{pmatrix} \lambda_1 & m & 0 & 0 & 0 \\ m^2 x & \lambda_1 & 0 & 0 & 0 \\ 0 & 0 & -2\lambda_1 - \lambda_2 & \tilde{m} & 0 \\ 0 & 0 & \tilde{m}^2 y & -2\lambda_1 - \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 2(\lambda_1 + \lambda_2) \end{pmatrix}, \tag{3.159}$$

and where now λ_1 and λ_2 are two different polynomials of x, y which we shall take as $\lambda_1 = \mu_1^2(bx-y)$ and $\lambda_2 = \mu_2^2(bx-y)$. We shall comment later on about the interesting possibility of taking a more general form for λ_1 and λ_2 that will allow us to induce a separation between the two Yukawa points $p_{\rm up}$ and $p_{\rm down}$ leading to some interesting consequences for the mixing of fermions.

The matter representations and matter curves are in this case

- **10** sector

$$\mathbf{10}_a : \lambda_1^2 - m^3 x = 0\,, \qquad \mathbf{10}_b : (2\lambda_1+\lambda_2)^2 - \tilde{m}^3 y = 0\,, \qquad \mathbf{10}_c : \lambda_1+\lambda_2 = 0\,,$$

- **5** sector

$$\mathbf{5}_a : \lambda_1 = 0\,, \quad \mathbf{5}_b : 2\lambda_1 + \lambda_2 = 0\,, \quad \mathbf{5}_c : (3\lambda_1 + 2\lambda_2)^2 - m^3 x = 0\,,$$
$$\mathbf{5}_d : \lambda_2^2 - \tilde{m}^3 y = 0\,, \quad \mathbf{5}_e : (\lambda_1+\lambda_2)^4 - 2(\lambda_1+\lambda_2)^2(m^3x + \tilde{m}^3 y) + (m^3 x - \tilde{m}^3 y)^2 = 0\,.$$

so the amount of matter curves increases considerably with respect to previous models.

In this case we can assign the representation $\mathbf{10}_M$ to either the $\mathbf{10}_a$ or the $\mathbf{10}_b$ sectors. Since both choices end up leading to the same results we will choose the first option, which fixes the $\mathbf{5}_U$ representation within the $\mathbf{5}_a$ sector. The up-type Yukawas then have the following structure

$$Y_U = \begin{pmatrix} 0 & 0 & \epsilon\, y_{13} \\ 0 & \epsilon\, y_{22} & 0 \\ \epsilon\, y_{31} & 0 & y_{33} \end{pmatrix} + \mathcal{O}(\epsilon^2)\,. \tag{3.160}$$

We then find an eigenvalue hierarchy of the form $(\mathcal{O}(1),\ \mathcal{O}(\epsilon), \mathcal{O}(\epsilon^2))$ and therefore a suitable hierarchical structure to fit empirical data.

There are some possibilities now on how to associate the representations $\bar{\mathbf{5}}_M$ and $\bar{\mathbf{5}}_D$ to the remaining matter curves, and this choice affects the down-type Yukawa matrix. We list here the possible choices and the resulting Yukawa matrices:

- Either $\bar{\mathbf{5}}_M$ is associated to $\mathbf{5}_d$ and $\bar{\mathbf{5}}_D$ is associated to $\mathbf{5}_e$ or the other way round. In the first case we find that the down-type Yukawa matrix has the form

$$Y_{D/L} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \epsilon\, y_{23} \\ 0 & \epsilon\, y_{32} & y_{33} \end{pmatrix} + \mathcal{O}(\epsilon^2) \tag{3.161}$$

  whose eigenvalues are $(\mathcal{O}(1),\ \mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon^2))$. Therefore this assignment for the matter fields does not lead to a good hierarchical structure for the down-type Yukawas. On the other hand, if we identify the $\bar{\mathbf{5}}_M$ with $\mathbf{5}_e$, then it is not clear how to perform the analysis due to the fact that the matter curve is singular at the Yukawa point.

- Either $\bar{\mathbf{5}}_M$ is associated to $\mathbf{5}_b$ and $\bar{\mathbf{5}}_D$ to $\mathbf{5}_c$ or the other way round. In both cases we find that the down-type Yukawa matrix has the structure

$$Y_{D/L} = \begin{pmatrix} 0 & 0 & \epsilon\, y_{13} \\ 0 & \epsilon\, y_{22} & 0 \\ \epsilon\, y_{31} & 0 & y_{33} \end{pmatrix} + \mathcal{O}(\epsilon^2)\,. \tag{3.162}$$

that has the favourable eigenvalue hierarchy ($\mathcal{O}(1),\ \mathcal{O}(\epsilon), \mathcal{O}(\epsilon^2)$). Since as far as the MSSM fields are concerned these models coincide with the ones analysed in [91] we will call the model with the assignments $\bar{\mathbf{5}}_M$ to $\mathbf{5}_c$ and $\bar{\mathbf{5}}_D$ to $\mathbf{5}_b$ as model A and the complementary one as model B.

We then see that in the present $2+2+1$ model there are two particular assignments of matter fields that yield a promising hierarchical structure for both up and down-type Yukawas. Both models were analysed in detail in [91] with the result that the model B does not lead to values of Yukawa couplings compatible with the empirical values. We give now another independent argument based on neutrino masses and μ–term consideration for choosing the model A over the model B.

Comments on μ-term and neutrino masses

We already mentioned that one of the most attractive features of the E_8 models is that it is possible to describe the masses of the neutrinos and the μ–term for the MSSM Higgs sector at the same time as the Yukawa couplings [89]. Although in the following we will not consider these couplings any more we can nonetheless see what kind of structure for these couplings the model A and model B have.

For the model A we find that there is a singlet under $SU(5)_{GUT}$ that can give a coupling of the form $\mathbf{1} \times \mathbf{5}_U \times \bar{\mathbf{5}}_M$ and, after breaking $SU(5)_{GUT}$ down to the standard model gauge group this will imply the presence of the following coupling in the superpotential

$$W \supset \lambda\, H_u L\, S\,, \tag{3.163}$$

where we called the singlet S and L the lepton doublet superfield. This coupling, as analysed in [88], corresponds to a Dirac mass for the neutrinos if we identify the singlet S with the right handed neutrino N_R. With this assignment of matter curves however it is not possible to have a renormalisable μ-term for the Higgs fields. It is possible to generate a non-renormalisable μ-term nonetheless if we consider the interactions of the Higgs fields with modes coming from other matter curves. In particular when the fields in the $\mathbf{5}_e$ come in vector-like pairs the following couplings will be allowed in the superpotential

$$W \supset \lambda_1 H_u \tilde{S}\phi + \lambda_2 H_d \tilde{S}\phi^c + \Lambda\, \phi\phi^c\,, \tag{3.164}$$

where we called ϕ any field in the $\mathbf{5}_e$ sector and ϕ^c its conjugate, Λ is a mass term for ϕ and $\tilde{S}$ is a singlet. After integrating out ϕ and ϕ^c using their F-term equations we find in the superpotential the following term

$$W \supset \frac{\lambda_1\lambda_2}{\Lambda}\tilde{S}^2 H_u H_d\,, \tag{3.165}$$

which becomes an effective μ-term for the Higgs fields if the singlet $\tilde{S}$ gets a non-vanishing vev. Note that this kind of non-renormalisable effective μ-term has already been considered in [86] and can provide a solution to the μ-problem in the MSSM.

In the second case, namely in the case we assign the $\bar{\mathbf{5}}_M$ to $\mathbf{5}_b$ and the $\bar{\mathbf{5}}_D$ to $\mathbf{5}_c$, we find that it is possible to have the following coupling in the superpotential

$$W \supset S H_u H_d \tag{3.166}$$

which becomes an effective μ-term if the singlet S gets a non-vanishing vev leading to an NMSSM–like solution of the μ–problem. However for this assignment of matter curves we find the feature that no masses for the neutrinos are possible if they are localised at the intersection of two 7-branes.

3.4.2 *$SU(5)$ models with E_7 enhancement*

As in the case of models with E_8 enhancement one may classify the different embeddings of $SU(5)_{\mathrm{GUT}}$ into E_7 by looking at the pattern of matter curves of the local models, which is in turn specified in terms of the Higgs background $\langle\Phi\rangle$. In this case the maximal subalgebra that commutes with $\mathfrak{su}(5)$ inside $\mathfrak{e}_7$ is $\mathfrak{g}_\perp = \mathfrak{su}_3 \oplus \mathfrak{u}_1$ giving the maximal decomposition of the adjoint representation

$$\begin{array}{rcl} \mathfrak{e}_7 & \supset & \mathfrak{su}_5^{\mathrm{GUT}} \oplus \mathfrak{su}_3 \oplus \mathfrak{u}_1 \\ \mathbf{133} & \to & (\mathbf{24},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{8})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{10},\overline{\mathbf{3}})_{-1} \oplus (\mathbf{5},\overline{\mathbf{3}})_2 \oplus (\mathbf{5},\overline{\mathbf{1}})_{-3} \oplus c.c. \end{array} \tag{3.167}$$

By construction $\langle\Phi\rangle$ commutes with $\mathfrak{su}_5^{\mathrm{GUT}}$, but it acts non-trivially on the representations $\mathcal{R}$ of $\mathfrak{g}_\perp = \mathfrak{su}_3 \otimes \mathfrak{su}_1$ that appear as $(\mathcal{R}_{\mathrm{GUT}}, \mathcal{R})$ in (3.167).

Also in this case one may classify different profiles for $\langle\Phi\rangle$ in terms of the block diagonal structure of the matrices $\Phi_{\mathcal{R}}$ which we assume reconstructible. Because $\mathfrak{g}_\perp$ factorises as $\mathfrak{su}_3 \otimes \mathfrak{u}_1$, we may directly focus on their block diagonal structure within $\mathfrak{su}_3$. In order to discuss the block diagonal structure of the Higgs field it is convenient to choose $\mathcal{R} = \mathbf{3}$, the fundamental representation of $SU(3)$ as the action of the Higgs field on any other representation may be constructed by taking suitable tensor products of the fundamental representation. With this choice the three different possibilities we have are

i) $\Phi_{\mathbf{3}}$ is diagonal

ii) $\Phi_{\mathbf{3}}$ has a $2+1$ block structure

iii) $\Phi_{\mathbf{3}}$ has a single block

Out of these three options the first one represents a $\langle\Phi\rangle$ taking values in the Cartan subalgebra of $\mathfrak{e}_7$, and so it does not correspond to a T-brane background. Option *iii)* was analysed in [79], obtaining that up-type Yukawa couplings identically vanish. Hence, we are left with a splitting of the form *ii)* as the only possibility to obtain realistic hierarchical pattern of Yukawa couplings.

Reconstructible models with the split $2+1$ can be characterised with a profile for $\langle\Phi\rangle$ lying in the subalgebra $\mathfrak{su}_2 \oplus \mathfrak{u}_1 \subset \mathfrak{su}_3 \subset \mathfrak{g}_\perp$. Hence in order to read the spectrum of matter curves one may adapt the above branching rules for the adjoint of $\mathfrak{e}_7$ to the non-maximal decomposition $\mathfrak{su}_5^{\mathrm{GUT}} \oplus \mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1$. We obtain[25]

$$\begin{array}{rcl} \mathfrak{e}_7 & \supset & \mathfrak{su}_5^{\mathrm{GUT}} \oplus \mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1 \\ \mathbf{133} & \to & (\mathbf{24},\mathbf{1})_{0,0} \oplus (\mathbf{1},\mathbf{3})_{0,0} \oplus 2(\mathbf{1},\mathbf{1})_{0,0} \oplus ((\mathbf{1},\mathbf{2})_{-2,1} \oplus c.c.) \\ & & \oplus\ (\mathbf{10},\mathbf{2})_{1,0} \oplus (\mathbf{10},\mathbf{1})_{-1,1} \oplus (\mathbf{5},\mathbf{2})_{0,-1} \oplus (\mathbf{5},\mathbf{1})_{-2,0} \oplus (\mathbf{5},\mathbf{1})_{1,1} \oplus c.c. \end{array} \tag{3.168}$$

and so we have two different kinds of $\mathbf{10}$ matter curves and three kinds of $\mathbf{5}$ matter curves. In order to have a rank one up-type Yukawa matrix we need to identify

25 In writing the decomposition of the $\mathfrak{e}_7$ Lie algebra under $\mathfrak{su}_5^{\mathrm{GUT}} \oplus \mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1$ we choose two particular combinations of the generators of $\mathfrak{u}_1 \oplus \mathfrak{u}_1$.

the matter curve $\mathbf{10}_M$ with $(\mathbf{10},\mathbf{2})_{1,0}$. Hence the curve containing the Higgs up is fixed to be $(\mathbf{5},\mathbf{1})_{-2,0}$, or otherwise the Yukawa coupling $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ cannot be generated. Finally, the remaining two $\mathbf{5}$-curves must host the family representations $\bar{\mathbf{5}}_M$ and down Higgs representation $\bar{\mathbf{5}}_D$, respectively.

To summarise, we find that in order to obtain a hierarchical pattern of Yukawa couplings we only have two possible ways to identify the matter curves with the representations of $SU(5)_{\mathrm{GUT}}$. Namely those are:

i. **Model A**

$$\begin{aligned}(\mathbf{10},\mathbf{2})_{1,0} &= \mathbf{10}_M \\ (\mathbf{5},\mathbf{1})_{-2,0} &= \mathbf{5}_U \\ (\bar{\mathbf{5}},\mathbf{2})_{0,1} &= \bar{\mathbf{5}}_M \\ (\bar{\mathbf{5}},\mathbf{1})_{-1,-1} &= \bar{\mathbf{5}}_D\end{aligned} \tag{3.169}$$

ii. **Model B**

$$\begin{aligned}(\mathbf{10},\mathbf{2})_{1,0} &= \mathbf{10}_M \\ (\mathbf{5},\mathbf{1})_{-2,0} &= \mathbf{5}_U \\ (\bar{\mathbf{5}},\mathbf{1})_{-1,-1} &= \bar{\mathbf{5}}_M \\ (\bar{\mathbf{5}},\mathbf{2})_{0,1} &= \bar{\mathbf{5}}_D\end{aligned} \tag{3.170}$$

Note that as far as the action of the adjoint scalar Φ on the various sector is concerned these models coincide with the ones called model A and model B presented in the previous section. Since we shall focus only on the MSSM sector it is possible to analyse any of these models independently of the embedding in E_7 and E_8. We will now turn to the details of the model A for the analysis of the model B performed in [91] showed that this model does not lead to values of the Yukawa couplings compatible with the empirical values.

3.5 THE MODEL A

3.5.1 *Yukawa hierarchies in the model A*

Let us now consider in more detail the model A highlighted in the previous section. Most of our discussion may be easily translated to the model B given that the difference between the two models amounts to a mild difference in the matter assignment, see [91] for more detail.

Indeed, this local model with either E_7 or E_8 enhancement is specified by choosing a Higgs field Φ and a gauge connection A valued in the algebra $\mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1$. The background chosen has to satisfy the cancellation of F–terms and D–terms in 4d leading to the supersymmetry equations that we copy here for the reader's convenience

$$\bar{\partial}_A \Phi = 0 \tag{3.171a}$$

$$F^{(0,2)} = 0 \tag{3.171b}$$

$$\omega \wedge F + \frac{1}{2}[\Phi, \Phi^\dagger] = 0\,. \tag{3.171c}$$

We already described in detail how a solution of the previous system of differential equations may be found and keeping this in mind we will start by introducing the background value of the Higgs field in holomorphic gauge discussing moreover the structure of the various matter curves. After this we will consider the passage to a real gauge and impose the D-term equations which is in no way different from the example considered in Section 3.1.2. On top of the fluxes needed to cancel the D–terms we will add some primitive fluxes that will be important in the generation of a chiral spectrum. We will conclude this first section with the computation of the Yukawa matrices.

Higgs background

The first element that enters in the definition of our local model is the vacuum expectation value of the Higgs field $\langle\Phi\rangle = \langle\Phi_{xy}\rangle\, dx \wedge dy$ which constitutes the primary source of breaking the exceptional symmetry group down to $SU(5)_{GUT}$. Our choice in holomorphic gauge is the following one[26]

$$\langle\Phi_{xy}\rangle = m\left(E^+ + mxE^-\right) + \mu_1^2\,(ax - y)\,Q_1 + \left[\mu_2^2\,(bx-y) + \kappa\right] Q_2 + \ldots \tag{3.172}$$

Here Q_i and $E^\pm$ are some generators of the E_7 or E_8 algebra whose definition (along with other details involving these Lie algebras) are given in Appendix A.2 for E_7 and A.3 for E_8. In the definition of the Higgs background we introduced the complex constants m, $\mu_{1,2}$ and κ with dimension of mass and $a, b \in \mathbb{C}$ which are dimensionless parameters. The constant κ has a particular rôle in the sense that it controls the separation of the points where the Yukawa couplings for the up and the down–type quarks are generated, as we will now discuss in a moment.

For our purposes it is necessary to consider only the action of an $SU(2) \times U(1) \times U(1)$ group on the various MSSM fields and therefore we will specify the matter embeddings specifying the representation under this particular group. We recall here the assignment for the model A

- **Model A**

$$\mathbf{10}_M : \mathbf{2}_{1,0}\,, \quad \mathbf{5}_U : \mathbf{1}_{-2,0}\,, \quad \bar{\mathbf{5}}_M : \mathbf{2}_{0,1}\,, \quad \bar{\mathbf{5}}_D : \mathbf{1}_{-1,-1}\,. \tag{3.173}$$

This assignment specifies how the Higgs field background (3.172) enters the zero mode equation for each matter fields, and therefore the curves at which they are localised.

26 The dots refer to some additional terms needed for the embedding in E_8. Since these terms do not play any rôle in our forthcoming discussion we do not copy them here, see [90] for additional details.

We can give a matrix representation of the Higgs background (3.172) for the various sectors

$$\Phi|_{\mathbf{10}_M} = \begin{pmatrix} \mu_1^2(ax-y) & m \\ m^2x & \mu_1^2(ax-y) \end{pmatrix}, \quad \Phi|_{\mathbf{5}_U} = -2\mu_1^2(ax-y)\,,$$
$$\Phi|_{\bar{\mathbf{5}}_M} = \begin{pmatrix} \mu_2^2(bx-y)+\kappa & m \\ m^2x & \mu_2^2(bx-y)+\kappa \end{pmatrix}, \quad \Phi|_{\bar{\mathbf{5}}_D} = -\mu_1^2(ax-y) - \mu_2^2(bx-y) - \kappa\,. \tag{3.174}$$

The location of the matter curve hosting the representation $\mathcal{R}_{\mathrm{GUT}}$ is then found by computing $\det\Phi|_{\mathcal{R}_{\mathrm{GUT}}} = 0$ leading to the following matter curves

$$\begin{aligned} &\Sigma_{\mathbf{10}_M} : \mu_1^4(ax-y)^2 - m^3x = 0\,, \quad \Sigma_{\bar{\mathbf{5}}_M} : \left[\mu_2^2(bx-y)+\kappa\right]^2 - m^3x = 0\,, \\ &\Sigma_{\mathbf{5}_U} : \mu_1^2(ax-y) = 0\,, \quad \Sigma_{\bar{\mathbf{5}}_D} : \mu_1^2(ax-y) + \mu_2^2(bx-y) + \kappa = 0\,. \end{aligned} \tag{3.175}$$

Finally, Yukawa couplings for the matter fields are generated at the intersection of these matter curves. In particular the Yukawa coupling $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ of the up-type quarks is generated at the point where the curves $\Sigma_{\mathbf{10}_M}$ and $\Sigma_{\mathbf{5}_U}$ meet whereas the Yukawa coupling $\mathbf{10}_M \times \bar{\mathbf{5}}_M \times \bar{\mathbf{5}}_D$ of the leptons and down-type quarks is generated where the curves $\Sigma_{\mathbf{10}_M}$, $\Sigma_{\bar{\mathbf{5}}_M}$ and $\Sigma_{\bar{\mathbf{5}}_D}$ meet. These two points are

$$\begin{aligned} &Y_U : \Sigma_{\mathbf{10}_M} \cap \Sigma_{\mathbf{5}_U} = \{x = y = 0\} = p_{\mathrm{up}}\,, \\ &Y_{D/L} : \Sigma_{\mathbf{10}_M} \cap \Sigma_{\bar{\mathbf{5}}_D} \cap \Sigma_{\bar{\mathbf{5}}_M} = \{x = x_0,\, y = y_0\} = p_{\mathrm{down}}\,, \end{aligned} \tag{3.176}$$

where

$$x_0 = \frac{\kappa^2\mu_1^4}{m^3(\mu_1^2+\mu_2^2)^2} + \mathcal{O}(\kappa^3)\,, \quad y_0 = \frac{\kappa}{\mu_1^2+\mu_2^2}\left(1 + \frac{\kappa\mu_1^4(a\mu_1^2+b\mu_2^2)}{m^3(\mu_1^2+\mu_2^2)^2}\right) + \mathcal{O}(\kappa^3)\,. \tag{3.177}$$

This shows that the two Yukawa points do not necessarily coincide and that the parameter κ controls the separation between them. Setting $\kappa = 0$ both couplings are generated at the same point while the separation of the two points increases with κ.

The background introduced so far solves the F–term equations in holomorphic gauge. To go in a real gauge we can follow the procedure outlined in Section 3.1.2 obtaining the solution[27]

$$\langle\Phi_{xy}\rangle = m\left(e^f E^+ + mxe^{-f}E^-\right) + \mu_1^2\,(ax-y)\,Q_1 + \left[\mu_2^2\,(bx-y)+\kappa\right]Q_2 + \ldots \tag{3.178}$$

with the flux

$$A = \frac{i}{2}(\partial f + \bar{\partial} f)P\,. \tag{3.179}$$

Here $f(x)$ is the function solving the differential equation ... introduced in section ... and $P = [E^+, E^-]$.

27 For the case of the model A embedded in E_8 there are some additional non–primitive fluxes, however these will not affect us in the following.

Primitive fluxes

While the background fields specified in the previous section are a consistent solution to the equations of motion it is still possible to consider a more general supersymmetric background for the gauge field strength F. In particular one may add an extra flux besides (3.179) that is primitive and commutes with Φ. The most general choice of gauge flux that satisfies these constraints and does not break $SU(5)_{GUT}$ is

$$F_Q = i(dx\wedge d\bar{x} - dy\wedge d\bar{y})\,[M_1 Q_1 + M_2 Q_2] + i(dx\wedge d\bar{y} + dy\wedge d\bar{x})\,[N_1 Q_1 + N_2 Q_2]\ . \tag{3.180}$$

This flux has the main effect of inducing 4d chirality in the matter field spectrum because modes of opposite chirality will feel it differently.

As already explained in Section 2.4 we will implement GUT breaking via hypercharge flux. We assume that the integrals for the hypercharge flux are such that no mass term is generated for the hypercharge gauge boson, a condition that can only be checked in a global realisation of our model. In our local approach we may choose the following parametrisation for this flux

$$F_Y = i\left[\tilde{N}_Y(dy\wedge d\bar{y} - dx\wedge d\bar{x}) + N_Y(dx\wedge d\bar{y} + dy\wedge d\bar{x})\right] Q_Y\,, \tag{3.181}$$

where we defined the hypercharge generator as[28]

$$Q_Y = \frac{1}{3}\,(H_1 + H_2 + H_3) - \frac{1}{2}(H_4 + H_5)\,. \tag{3.182}$$

To summarise, the total primitive flux present in our model is

$$F_p = iQ_R(dy\wedge d\bar{y} - dx\wedge d\bar{x}) + iQ_S(dy\wedge d\bar{x} + dx\wedge d\bar{y})\,, \tag{3.183}$$

where we defined the generators

$$Q_R = -M_1 Q_1 - M_2 Q_2 + \tilde{N}_Y Q_Y\,, \quad Q_S = N_1 Q_1 + N_2 Q_2 + N_Y Q_Y\,. \tag{3.184}$$

These fluxes will enter explicitly in the equations of motion for the physical zero modes and because of this they will enter directly in the expression of the physical Yukawa couplings. We have chosen to summarise how the primitive flux is felt by the various MSSM fields for the case of the model A in Table 7 specifying the two combinations q_R and q_S that will be relevant for the computation in the following sections.

28 Here H_i are the Cartan generators of the $\mathfrak{e}_7$ and $\mathfrak{e}_8$ algebras.

MSSM	Sector	$SU(2)\times U(1)\times U(1)$	$G_{\rm MSSM}$	q_R	q_S
Q	$\mathbf{10}_M$	$\mathbf{2}_{1,0}$	$(\mathbf{3},\mathbf{2})_{-\frac{1}{6}}$	$-\frac{1}{6}\tilde{N}_Y - M_1$	$-\frac{1}{6}N_Y + N_1$
U	$\mathbf{10}_M$	$\mathbf{2}_{1,0}$	$(\bar{\mathbf{3}},\mathbf{1})_{\frac{2}{3}}$	$\frac{2}{3}\tilde{N}_Y - M_1$	$\frac{2}{3}N_Y + N_1$
E	$\mathbf{10}_M$	$\mathbf{2}_{1,0}$	$(\mathbf{1},\mathbf{1})_{-1}$	$-\tilde{N}_Y - M_1$	$-N_Y + N_1$
D	$\bar{\mathbf{5}}_M$	$\mathbf{2}_{0,1}$	$(\bar{\mathbf{3}},\mathbf{1})_{-\frac{1}{3}}$	$-\frac{1}{3}\tilde{N}_Y - M_2$	$-\frac{1}{3}N_Y + N_2$
L	$\bar{\mathbf{5}}_M$	$\mathbf{2}_{0,1}$	$(\mathbf{1},\mathbf{2})_{\frac{1}{2}}$	$\frac{1}{2}\tilde{N}_Y - M_2$	$\frac{1}{2}N_Y + N_2$
H_u	$\mathbf{5}_U$	$\mathbf{1}_{-2,0}$	$(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$	$-\frac{1}{2}\tilde{N}_Y + 2M_1$	$-\frac{1}{2}N_Y - 2N_1$
H_d	$\bar{\mathbf{5}}_D$	$\mathbf{1}_{-1,-1}$	$(\mathbf{1},\mathbf{2})_{\frac{1}{2}}$	$\frac{1}{2}\tilde{N}_Y + M_1 + M_2$	$\frac{1}{2}N_Y - N_1 - N_2$

Table 7: Different sectors and charges for the E_7 model of this section. Here q_R and q_S are the E_7 operators (3.184) evaluated at each different sector. All the multiplets in the table have the same chirality.

Local chirality of matter fields

At this point having specified the data of our local model we can discuss chirality constraints for the various sectors of the theory. Following the discussion in Section 3.2.3 we obtain the following chiral indices[29]

$$\mathcal{I}_{\mathbf{10},\mathbf{2}} = -2m^4c^4 q_R^{(\mathbf{10},\mathbf{2})} \tag{3.185}$$

$$\mathcal{I}_{\bar{\mathbf{5}},\mathbf{2}} = -2m^4c^4 q_R^{(\bar{\mathbf{5}},\mathbf{2})} \tag{3.186}$$

$$\mathcal{I}_{\mathbf{5},\mathbf{1}} = -4\mu_1^4[q_R^{(\mathbf{5},\mathbf{1})}(|a|^2-1) + 2\mathrm{Re}[a]q_S^{(\mathbf{5},\mathbf{1})}] \tag{3.187}$$

$$\begin{aligned}\mathcal{I}_{\bar{\mathbf{5}},\mathbf{1}} = -\Big\{ & q_R^{(\bar{\mathbf{5}},\mathbf{1})}[|a\mu_1^2+b\mu_2^2|^2 - |\mu_1^2+\mu_2^2|^2] \\ & + 2q_S^{(\bar{\mathbf{5}},\mathbf{1})}\mathrm{Re}[(a\mu_1^2+b\mu_2^2)(\mu_1^2+\mu_2^2)]\Big\}\,.\end{aligned} \tag{3.188}$$

The conditions that we need to impose in order to obtain the correct chiral spectrum in 4d are the following ones

$$\begin{aligned} &\mathcal{I}_{\mathcal{R}} < 0\,, \qquad \mathcal{R} = Q,U,E,D,L,H_u,H_d\,, \\ &\mathcal{I}_{\mathcal{R}} = 0\,, \qquad \mathcal{R} = T_u, T_d\,. \end{aligned} \tag{3.189}$$

We choose not to write explicitly the form of these conditions referring to [91] for further details. In the following the only piece of information we will need is that this system has non trivial solutions therefore allowing for a chiral spectrum without Higgs triplets.

Holomorphic Yukawa couplings for the model A

In this section we report the result of the computation of the holomorphic Yukawa couplings for the model A. We focus our attention on the matter curves including

29 In writing $\mathcal{I}_{\mathbf{10},\mathbf{2}}$ and $\mathcal{I}_{\bar{\mathbf{5}},\mathbf{2}}$ we have neglected some terms involving μ_1 and μ_2. We chose to do so because as we will discuss later we shall restrict to the case $\mu_1, \mu_2 \ll m$ implying that these additional terms will give negligible contributions to the local chiral index.

the fields charged under the MSSM gauge group and therefore to the two Yukawa matrices for the couplings $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ and $\mathbf{10}_M \times \bar{\mathbf{5}}_M \times \bar{\mathbf{5}}_D$. The holomorphic functions h_{xy} for the different fields are

$$\begin{aligned} h_{\mathbf{10}_M} &= \gamma_{10,i}\, m_*^{3-i}(ax-y)^{3-i} & h_{\bar{\mathbf{5}}_M} &= \gamma_{5,i}\, m_*^{3-i}(a(x-x_0)-(y-y_0))^{3-i} \\ h_{\mathbf{5}_U} &= \gamma_U & h_{\bar{\mathbf{5}}_D} &= \gamma_D, \end{aligned} \tag{3.190}$$

where (x_0, y_0) corresponds to the coordinates (3.177) of the down-type Yukawa point p_{down}, while recall that p_{up} is located at the origin. Finally, the constants $\gamma_{10,i}$, $\gamma_{5,i}$, γ_U and γ_D are normalisation factors to be computed in the next section and $i = 1, 2, 3$ is a family index. With this form one can compute the functions η in (3.45) which in turn are needed to compute the holomorphic couplings via the residue formula (3.42).

Below we display the Yukawa matrix for the up-type quarks up to first order in the expansion parameter ϵ. For the Yukawa matrix of down quarks and leptons we find an explicit dependance on κ, the parameter controlling the separation between the two Yukawa points. Since the dimensionless combination $\tilde{\kappa} = \kappa/m_*$ will turn out to be very small we chose to retain only the first two orders in $\tilde{\kappa}$ in the Yukawa matrix (dropping also terms of order $\mathcal{O}(\epsilon\tilde{\kappa})$ which are extremely suppressed). Moreover for the Yukawa matrix of down quarks and leptons we also perform an expansion on the parameter $(a-b)$ which we will eventually find to be small as well. Our computations in section 3.5.3 will however be based on the full $(a-b)$ dependence of $Y_{D/L}$, which can be found in [91].

$$Y_U = \frac{\pi^2\, \gamma_U\, \gamma_{10,3}^2}{2\rho_m \rho_\mu} \begin{pmatrix} 0 & 0 & \tilde{\epsilon}\frac{\gamma_{10,1}}{2\rho_\mu \gamma_{10,3}} \\ 0 & \tilde{\epsilon}\frac{\gamma_{10,2}^2}{2\rho_\mu \gamma_{10,3}^2} & 0 \\ \tilde{\epsilon}\frac{\gamma_{10,1}}{2\rho_\mu \gamma_{10,3}} & 0 & 1 \end{pmatrix} + \mathcal{O}(\epsilon^2)\,, \tag{3.191}$$

$$Y_{D/L} = Y_{D/L}^{(0)} + (a-b)\, Y_{D/L}^{(1)} + \mathcal{O}((a-b)^2) \tag{3.192}$$

where

$$Y_{D/L}^{(0)} = -\frac{\pi^2 \gamma_{5,3}\gamma_{10,3}\gamma_D}{(d+1)\rho_\mu \rho_m} \begin{pmatrix} 0 & \tilde{\kappa}\tilde{\epsilon}\frac{2\gamma_{5,2}\gamma_{10,1}}{(d+1)^2\rho_\mu^2 \gamma_{5,3}\gamma_{10,3}} & \frac{\gamma_{10,1}}{(d+1)\rho_\mu \gamma_{10,3}}\left(\frac{2\tilde{\kappa}^2}{(d+1)^2\rho_\mu} - \tilde{\epsilon}\right) \\ \tilde{\kappa}\tilde{\epsilon}\frac{\gamma_{5,1}\gamma_{10,2}}{(d+1)^2\rho_\mu^2 \gamma_{5,3}\gamma_{10,3}} & -\tilde{\epsilon}\frac{\gamma_{5,2}\gamma_{10,2}}{(d+1)\rho_\mu \gamma_{5,3}\gamma_{10,3}} & -\tilde{\kappa}\frac{\gamma_{10,2}}{(d+1)\rho_\mu \gamma_{10,3}} \\ \tilde{\epsilon}\frac{\gamma_{5,1}}{(d+1)\rho_\mu \gamma_{5,3}} & 0 & 1 \end{pmatrix} \tag{3.193}$$

$$Y_{D/L}^{(1)} = -\frac{\pi^2}{(d+1)^3} \begin{pmatrix} 0 & y^{(12)} & y^{(13)} \\ y^{(21)} & y^{(22)} & y^{(23)} \\ y^{(31)} & y^{(32)} & y^{(33)} \end{pmatrix} \tag{3.194}$$

with the entries given by

$$y^{(12)} = -\epsilon\tilde{\kappa}\, \frac{(d-1)\gamma_{5,2}\gamma_{10,1}\gamma_D \theta_y}{(d+1)\rho_\mu^3 \rho_m} \tag{3.195}$$

$$y^{(13)} = \frac{(d-1)\,\gamma_{5,3}\gamma_{10,1}\gamma_D}{2\rho_\mu^2\rho_m}\left[\epsilon\,\theta_y - \tilde{\epsilon}\,\tilde{\kappa}\frac{4d(5d-1)\rho_\mu}{(d^2-1)\,\rho_m^{3/2}}\right] \tag{3.196}$$

$$y^{(21)} = -\epsilon\tilde{\kappa}\,\frac{(d-1)\gamma_{5,1}\gamma_{10,2}\gamma_D\theta_y}{2(d+1)\rho_\mu^3\rho_m} \tag{3.197}$$

$$y^{(22)} = \frac{(d-1)\,\gamma_{5,2}\gamma_{10,2}\gamma_D}{2\rho_\mu^2\rho_m}\left[\epsilon\,\theta_y + \tilde{\epsilon}\tilde{\kappa}\,\frac{18d\rho_\mu}{(d^2-1)\,\rho_m^{3/2}}\right] \tag{3.198}$$

$$y^{(23)} = \frac{3d^2\gamma_{5,3}\gamma_{10,2}\gamma_D}{\rho_m^{5/2}}\left[\tilde{\epsilon} + \tilde{\kappa}^2\frac{2}{d(1+d)\rho_\mu}\right] \tag{3.199}$$

$$y^{(31)} = \frac{(d-1)\gamma_{5,1}\gamma_{10,3}\gamma_D}{2\rho_\mu^2\rho_m}\left[\epsilon\,\theta_y + \tilde{\kappa}\tilde{\epsilon}\frac{4(d-2)\rho_\mu}{(d^2-1)\,\rho_m^{3/2}}\right] \tag{3.200}$$

$$y^{(32)} = -\tilde{\epsilon}\,\frac{3d\gamma_{5,2}\gamma_{10,3}\gamma_D}{\rho_m^{5/2}} \tag{3.201}$$

$$y^{(33)} = -\tilde{\kappa}\,\frac{2d\gamma_{5,3}\gamma_{10,3}\gamma_D}{\rho_m^{5/2}} \tag{3.202}$$

and where we have defined the following quantities

$$d = \frac{\mu_2^2}{\mu_1^2}\,, \quad \rho_\mu = \frac{\mu_1^2}{m_*^2}\,, \quad \rho_m = \frac{m^2}{m_*^2}\,, \quad \tilde{\kappa} = \frac{\kappa}{m_*}\,, \quad \tilde{\epsilon} = \epsilon(\theta_x + a\theta_y)\,. \tag{3.203}$$

3.5.2 *Normalisation factors and physical Yukawas*

So far we have been performing the computation of the Yukawa couplings merely at the holomorphic level, i.e. we have performed the computation of the four dimensional superpotential for the zero modes. To complete the computation and obtain results comparable with measured data it is necessary to compute the kinetic terms of the zero modes and take them to a basis where they are canonically normalised. To compute the kinetic terms it is necessary first to go in a real gauge and solve the zero mode equations in there, which has the effect to induce a dependance on the local flux densities in the kinetic terms.

In this section we will solve the wavefunctions in a real gauge and use this result to obtain the various normalisation factors. In the sectors affected by the T-brane background we will not be able to find an analytical solution. However like in the case of the E_6 model we will be able to find an approximate solution in some regions of the parameter space of our local model. We will first compute the wavefunctions that correspond to the tree-level superpotential and show that no kinetic mixing is present at the level of approximation that we are working. We will then include the non-perturbative corrections and argue that the result does not change.

To summarise, in this section we will compute the normalisation factors for the chiral wavefunctions of the A model. At tree-level and $\mathcal{O}(\epsilon)$ they correspond to kinetic terms with a diagonal structure, a result that is not changed by non-perturbative effects. This implies that we may compute the final result for the physical Yukawa couplings by employing the holomorphic result discussed in the previous section together with the normalisation factors that we are going to derive below.

Perturbative wavefunctions

We recall here the form of the equations obeyed by the zero modes which may be obtained by expanding (3.171) to linear order in the fluctuations

$$\bar{\partial}_{\langle A\rangle} a = 0\,, \tag{3.204a}$$

$$\bar{\partial}_{\langle A\rangle} \varphi = i[a, \langle\Phi\rangle]\,, \tag{3.204b}$$

$$\omega \wedge \partial_{\langle A\rangle} a = \frac{1}{2}[\langle\bar{\Phi}\rangle, \varphi]\,. \tag{3.204c}$$

In (3.204) we choose $\langle\Phi\rangle$ and $\langle A\rangle$ to be background fields in a real gauge. These equations may be solved by using techniques already employed in [73, 74]. To keep the discussion contained we will simply quote the results in this section deferring more details regarding the computation to Appendix B.

Henceforth we are going to use the following notation for the zero modes

$$\vec{\varphi}_\rho = \begin{pmatrix} a^s_{\bar{x}} \\ a^s_{\bar{y}} \\ \varphi^s_{xy} \end{pmatrix} E_{\rho,s} \tag{3.205}$$

where $E_{\rho,s}$ denotes the particular set of roots, labelled by s, corresponding to a given sector ρ. In our models we have that for the sectors unaffected by the T-brane background s takes a single value whereas in the other sectors we have that s takes two values.

Sectors not affected by T-brane

In the sectors not affected by the T-brane background the solution may be computed analytically. In the models we consider we have two sectors that fall in this class and transform as $(\mathbf{5},\mathbf{1})_{-2,0}$ and $(\mathbf{5},\mathbf{1})_{1,1}$ under $SU(5)\times SU(2)\times U(1)\times U(1)$. We recall here that $\mathbf{5}_U := (\mathbf{5},\mathbf{1})_{-2,0}$, and $\bar{\mathbf{5}}_D := (\bar{\mathbf{5}},\mathbf{1})_{-1,-1}$.

The solution for both sectors is the following one

$$\vec{\varphi} = \begin{pmatrix} -\frac{i\zeta}{2\tilde{\mu}_a} \\ \frac{i(\zeta-\lambda)}{2\tilde{\mu}_b} \\ 1 \end{pmatrix} \chi(x,y) \tag{3.206}$$

where

$$\chi(x,y) = e^{\frac{q_R}{2}(x\bar{x}-y\bar{y})-q_S\mathrm{Re}(x\bar{y})+(\tilde{\mu}_a x+\tilde{\mu}_b y)(\zeta_1\bar{x}-\zeta_2\bar{y})}\, f(\zeta_2 x+\zeta_1 y) \tag{3.207}$$

and where we have defined

$$\zeta = \frac{\tilde{\mu}_a\,(4\tilde{\mu}_a\tilde{\mu}_b + \lambda q_S)}{\tilde{\mu}_a q_S + \tilde{\mu}_b\,(\lambda + q_R)}\,, \quad \zeta_1 = \frac{\zeta}{\tilde{\mu}_a}\,, \quad \zeta_2 = \frac{\zeta - \lambda}{\tilde{\mu}_b}\,, \tag{3.208}$$

and λ is the lowest solution to the cubic equation (B.11). The parameters $\tilde{\mu}_a$ and $\tilde{\mu}_b$ are directly related to the ones describing the background Higgs fields and for both sectors are given by

	$(\mathbf{5},\mathbf{1})_{-2,0}$	$(\bar{\mathbf{5}},\mathbf{1})_{-1,-1}$
$\tilde{\mu}_a$	$a\mu_1^2$	$\frac{1}{2}(a\mu_1^2 + b\mu_2^2)$
$\tilde{\mu}_b$	$-\mu_1^2$	$-\frac{1}{2}(\mu_1^2 + \mu_2^2)$

Finally, the function $f(\zeta_2 x + \zeta_1 y)$ is a holomorphic function which can be approximated by a constant if the sector we consider contains an MSSM Higgs.

Sectors affected by T-brane

In the two sectors affected by the T-brane background the equations of motion become more complicated. Here the fields involved in the solution are doublets of $SU(2)$ under the group $SU(5)\times SU(2)\times U(1)\times U(1)$ and therefore we are going to write the solution as

$$\vec{\varphi} = \begin{pmatrix} a_{\bar{x}}^+ \\ a_{\bar{y}}^+ \\ \varphi_{xy}^+ \end{pmatrix} E^+ + \begin{pmatrix} a_{\bar{x}}^- \\ a_{\bar{y}}^- \\ \varphi_{xy}^- \end{pmatrix} E^- = \vec{\varphi}_+ E^+ + \vec{\varphi}_- E^-\,, \tag{3.209}$$

where we denote with a $+$ the upper component of the $SU(2)$ doublet and with a $-$ the lower one. The equations for the zero modes are generally difficult to solve analytically. Nevertheless as discussed in appendix B in the limit $\mu_1, \mu_2, \kappa \ll m$ it is possible to find approximate solutions. The solution for the $\mathbf{10}_M$ sector in real gauge is

$$\vec{\varphi}_{10}^{\,i} = \gamma_{10}^i \begin{pmatrix} \frac{i\lambda_{10}}{m^2} \\ -i\frac{\lambda_{10}\zeta_{10}}{m^2} \\ 0 \end{pmatrix} e^{f/2}\chi_{10}^i E^+ + \gamma_{10}^i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-f/2}\chi_{10}^i E^- \tag{3.210}$$

with λ_{10} the negative solution to the cubic equation (B.32) and $\zeta_{10} = -q_S/(\lambda_{10} - q_R)$. Finally the wavefunctions χ_{10}^i are

$$\chi_{10}^i = e^{\frac{q_R}{2}(|x|^2 - |y|^2) - q_S(x\bar{y} + y\bar{x}) + \lambda_{10}x(\bar{x} - \zeta_{10}\bar{y})} g_{10}^i(y + \zeta_{10}x)\,, \tag{3.211}$$

where g_{10}^i are holomorphic functions of the variable $y + \zeta_{10}x$ and $i = 1,2,3$ is a family index. We choose these holomorphic functions in the following way

$$g_{10}^i(y + \zeta_{10}x) = m_*^{3-i}(y + \zeta_{10}x)^{3-i}\,. \tag{3.212}$$

The other sector affected by the T-brane background is the $(\bar{\mathbf{5}},\mathbf{2})_{0,1}$. We identify this sector with the $\bar{\mathbf{5}}_M$ sector and the solution is

$$\vec{\varphi}_5^{\,i} = \gamma_5^i \begin{pmatrix} \frac{i\lambda_5}{m^2} \\ -i\frac{\lambda_5\zeta_5}{m^2} \\ 0 \end{pmatrix} e^{i\tilde{\psi} + f/2}\chi_5^i(x, y - \nu/a)E^+ + \gamma_5^i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{i\tilde{\psi} - f/2}\chi_5^i(x, y - \nu/a)E^-$$

(3.213)

with $\tilde{\psi}$ defined in (B.38) and $\nu = \kappa/\mu_2^2$. Also, λ_5 is defined as the lowest solution to (B.32) and $\zeta_5 = -q_S/(\lambda_5 - q_R)$. Finally the wavefunctions χ_5^i are

$$\chi_5^i(x,y) = e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+\lambda_5 x(\bar{x}-\zeta_5\bar{y})} g_5^i(y+\zeta_5 x)\,, \tag{3.214}$$

where g_5^i are holomorphic functions of $y+\zeta_5 x$ and $i = 1,2,3$ is a family index. Analogously, the family functions are

$$g_5^i(y+\zeta_5 x) = m_*^{3-i}(y+\zeta_5 x)^{3-i}\,. \tag{3.215}$$

Normalisation factors

With the information regarding the perturbative wavefunctions we can compute the normalisation factors for the various sectors following the prescription given in Section 3.2.2 . We recall here that the kinetic terms in 4d have the form

$$K_\rho^{ij} = \langle \vec{\varphi}_\rho^{\,i} | \vec{\varphi}_\rho^{\,j} \rangle = m_*^4 \int_S \mathrm{Tr}\,(\vec{\varphi}_\rho^{\,i\,\dagger} \cdot \vec{\varphi}_\rho^{\,j})\,\mathrm{dvol}_S \tag{3.216}$$

which follows from direct dimensional reduction.

We find that $K_\rho^{ij} = 0$ for $i \neq j$ and no kinetic mixing is present. Therefore the choice of the normalisation factors $|\gamma_\rho^i|^2 = (K_\rho^{ii})^{-1}$ is sufficient to ensure canonically normalised kinetic terms. The result is

$$|\gamma_{U/D}|^2 = -\frac{4^2}{\pi^2 m_*^4} \frac{(2\mathrm{Re}[\zeta_1\tilde{\mu}_a]+q_R)\,(2\mathrm{Re}[\zeta_2\tilde{\mu}_b]+q_R) + |\zeta_2\tilde{\mu}_a - \zeta_1^*\tilde{\mu}_b^* + q_S|^2}{\zeta_1^2+\zeta_2^2+4} \tag{3.217a}$$

$$|\gamma_{10,j}|^2 = -\frac{c}{m_*^2\pi^2(3-j)!} \frac{1}{\frac{1}{2\mathrm{Re}[\lambda_{10}]+q_R(1+|\zeta_{10}|^2)-|m|^2c^2} + \frac{c^2|\lambda_{10}|^2}{|m|^4}\frac{1}{2\mathrm{Re}[\lambda_{10}]+q_R(1+|\zeta_{10}|^2)+|m|^2c^2}} \left(\frac{q_R}{m_*^2}\right)^{4-j} \tag{3.217b}$$

$$|\gamma_{5,j}|^2 = -\frac{c}{m_*^2\pi^2(3-j)!} \frac{1}{\frac{1}{2\mathrm{Re}[\lambda_5]+q_R(1+|\zeta_5|^2)-|m|^2c^2} + \frac{c^2|\lambda_5|^2}{|m|^4}\frac{1}{2\mathrm{Re}[\lambda_5]+q_R(1+|\zeta_5|^2)+|m|^2c^2}} \left(\frac{q_R}{m_*^2}\right)^{4-j}. \tag{3.217c}$$

Note that the parameters λ and ζ that appear in the various normalisation factors depend on the local flux densities and in particular on the flux hypercharge. This implies that the normalisation factors of the MSSM multiplets sitting in the same GUT multiplet will be different. This is a key feature to obtain realistic mass ratios, as we will see in section 3.5.3.

Non-perturbative corrections to the wavefunctions

So far we have been discussing the kinetic terms of the matter fields neglecting non-perturbative corrections. However, as we are computing Yukawa couplings up to first order in the parameter ϵ, one should consider the expression for the kinetic terms at the same level of approximation. We will discuss now how these effects enter in the computation of the kinetic terms and show that for our model no relevant correction is produced. This implies that the result obtained above may be used in the computation of physical Yukawa matrices.

We recall that the F-term equations of motion corrected at $\mathcal{O}(\epsilon)$ are

$$\begin{aligned}&\bar{\partial}_{\langle A\rangle} a = 0\,,\\&\bar{\partial}_{\langle A\rangle}\varphi = i[a,\langle\Phi\rangle] - \epsilon\partial\theta_0\wedge(\partial_{\langle A\rangle}a + \bar{\partial}_{\langle A\rangle}a^\dagger)\,,\end{aligned} \tag{3.218}$$

which need to be supplemented with the D-term equation (3.204c) which is not affected by non-perturbative corrections [73]. In this section we shall simply show the final result and discuss the impact of non-perturbative corrections on the kinetic terms, deferring the details of the computation to Appendix B.

Sectors not affected by T-brane

In both sectors not affected by the T-brane background the corrections take the same form

$$\vec{\varphi} = \gamma\begin{pmatrix} -\frac{i\zeta}{2\tilde{\mu}_a}\\ \frac{i(\zeta-\lambda)}{2\tilde{\mu}_b}\\ 1\end{pmatrix} e^{\frac{q_R}{2}(x\bar{x}-y\bar{y})-q_S\mathrm{Re}(x\bar{y})+(\mu_a x+\mu_b y)(\zeta_1\bar{x}-\zeta_2\bar{y})}\left[f(\zeta_2 x+\zeta_1 y)+\epsilon B(\zeta_2 x+\zeta_1 y)+\epsilon\Upsilon\right]\,. \tag{3.219}$$

The function Υ that controls the $\mathcal{O}(\epsilon)$ correction is

$$\begin{aligned}\Upsilon =\;& \frac{1}{4}(\zeta_1\bar{x}-\zeta_2\bar{y})^2(\theta_y\mu_a-\theta_x\mu_b)f(\zeta_2 x+\zeta_1 y)+\frac{1}{2}(\zeta_1\bar{x}-\zeta_2\bar{y})(\zeta_2\theta_y-\zeta_1\theta_x)f'(\zeta_2 x+\zeta_1 y)+\\&+\left[\frac{\delta_1}{2}(\zeta_1 x-\zeta_2 y)^2+\delta_2(\zeta_1 x-\zeta_2 y)(\zeta_2 x+\zeta_1 y)\right]f(\zeta_2 x+\zeta_1 y)\,,\end{aligned} \tag{3.220}$$

where

$$\begin{aligned}\delta_1 &= \frac{1}{(\zeta_1^2+\zeta_2^2)^2}\left[\bar{\theta}_x(q_S\zeta_1-q_R\zeta_2)+\bar{\theta}_y(q_R\zeta_1+q_S\zeta_2)\right]\,,\\ \delta_2 &= \frac{1}{(\zeta_1^2+\zeta_2^2)^2}\left[\bar{\theta}_x(q_R\zeta_1+q_S\zeta_2)-\bar{\theta}_y(q_S\zeta_1-q_R\zeta_2)\right]\,.\end{aligned} \tag{3.221}$$

The holomorphic function $B(\zeta_2 x+\zeta_1 y)$ in (3.219) is not determined by the equations of motion, and it may be fixed by asking for regularity of the function ξ that appears in the solution of the F–term equations (3.35) . We shall nevertheless not discuss this point here since it does not affect the result for the kinetic terms.

Having the correction it is now possible to discuss the effect of the correction on the kinetic term. We can use the fact that the integrand has to be invariant under the symmetry $(x,y)\to e^{i\alpha}(x,y)$ to check whether the corrections actually contribute to the kinetic terms. In our case this sector hosts a Higgs field and no correction generated to the kinetic terms.

Sectors affected by T-brane

Similarly to the case of the E_6 model in Section 3.3.3 the structure of the solution for both sectors charged under the T-brane background is

$$\vec{\varphi}_{10^+} = \begin{pmatrix} \bullet \\ \bullet \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ 0 \\ \bullet \end{pmatrix} + \mathcal{O}(\epsilon^2) \qquad \vec{\varphi}_{10^-} = \begin{pmatrix} 0 \\ 0 \\ \bullet \end{pmatrix} + \epsilon \begin{pmatrix} \bullet \\ \bullet \\ 0 \end{pmatrix} + \mathcal{O}(\epsilon^2). \tag{3.222}$$

Following the same arguments as the ones used in the case of the E_6 model it is possible to prove that the presence of these terms does not affect at all the kinetic terms.

3.5.3 *Fitting fermion masses and mixing angles*

Gathering the results of the last two sections one may write the final expression for the physical Yukawa matrices at the GUT scale. In particular, for the model A one obtains the matrices (3.191) and (3.192) with the normalisation factors given by (3.217).As noted above the value of the normalisation factors varies for MSSM field with different hypercharge even if they sit inside the same GUT multiplet, something that we will indicate by adding a superscript to distinguish between them.

Based on these result in this section we explore whether it is possible to find some regions in the parameter space of our models where we may reproduce the realistic values for fermion masses and mixings. We recall that as in the case of the E_6 model of Section 3.3 our calculations are performed at the GUT scale which is usually taken around 10^{16} GeV and therefore to compare the values for the fermion masses it is necessary to follow the values of the fermion masses along the renormalisation group flow. We collect in Table 8 the extrapolation of the fermion masses up to the unification scale taken from [84] in the context of the MSSM. We shall now discuss the comparison between these extrapolated data and the values for the Yukawa couplings that we obtain in our local E_7 models.

Fermion masses

Knowing the Yukawa matrices we can easily extract the values of the fermion masses which depend on the eigenvalues of the matrices. From the Yukawa matrices in (3.191) and (3.192) we see that the eigenvalues are

$$\begin{aligned}
Y_t &= \gamma_U\, \gamma^Q_{10,3}\gamma^U_{10,3}\, Y^U_{33} & \qquad Y_c &= \epsilon\, \gamma_U\, \gamma^Q_{10,2}\gamma^U_{10,2}\, Y^U_{22} \\
Y_b &= \gamma_D \left(\gamma^Q_{10,3}\gamma^D_{5,3}Y^{D/L}_{33} + \epsilon\, \gamma^Q_{10,2}\gamma^D_{5,2}\delta\right) & \qquad Y_s &= \epsilon\, \gamma_D \left(\gamma^Q_{10,2}\gamma^D_{5,2}Y^{D/L}_{22} - \gamma^Q_{10,2}\gamma^D_{5,2}\delta\right) \\
Y_\tau &= \gamma_D \left(\gamma^E_{10,3}\gamma^L_{5,3}Y^{D/L}_{33} + \epsilon\, \gamma^E_{10,2}\gamma^L_{5,2}\delta\right) & \qquad Y_\mu &= \epsilon\, \gamma_D \left(\gamma^E_{10,2}\gamma^L_{5,2}Y^{D/L}_{22} - \gamma^E_{10,2}\gamma^L_{5,2}\delta\right)
\end{aligned} \tag{3.223}$$

while for the first family we have that

$$Y_u,\ Y_d,\ Y_e \sim \mathcal{O}(\epsilon^2) \tag{3.224}$$

$\tan\beta$	10	38	50
m_u/m_c	$2.7 \pm 0.6 \times 10^{-3}$	$2.7 \pm 0.6 \times 10^{-3}$	$2.7 \pm 0.6 \times 10^{-3}$
m_c/m_t	$2.5 \pm 0.2 \times 10^{-3}$	$2.4 \pm 0.2 \times 10^{-3}$	$2.3 \pm 0.2 \times 10^{-3}$
m_d/m_s	$5.1 \pm 0.7 \times 10^{-2}$	$5.1 \pm 0.7 \times 10^{-2}$	$5.1 \pm 0.7 \times 10^{-2}$
m_s/m_b	$1.9 \pm 0.2 \times 10^{-2}$	$1.7 \pm 0.2 \times 10^{-2}$	$1.6 \pm 0.2 \times 10^{-2}$
m_e/m_μ	$4.8 \pm 0.2 \times 10^{-3}$	$4.8 \pm 0.2 \times 10^{-3}$	$4.8 \pm 0.2 \times 10^{-3}$
m_μ/m_τ	$5.9 \pm 0.2 \times 10^{-2}$	$5.4 \pm 0.2 \times 10^{-2}$	$5.0 \pm 0.2 \times 10^{-2}$
Y_τ	0.070 ± 0.003	0.32 ± 0.02	0.51 ± 0.04
Y_b	0.051 ± 0.002	0.23 ± 0.01	0.37 ± 0.02
Y_t	0.48 ± 0.02	0.49 ± 0.02	0.51 ± 0.04

Table 8: Running mass ratios of quarks and leptons at the unification scale from ref.[84].

Here the normalisation factors are those given in the previous section, and we have defined

$$\delta = -\tilde{\kappa}\, \frac{\pi^2 d(a-b)\,[\theta_y(a(d-2)-b(4d+1)) - 3(d+1)\theta_x]}{(d+1)^5 \rho_\mu\, \rho_m^{5/2}} \tag{3.225}$$

Therefore we see that when $a \neq b$ the eigenvalues of the down-type Yukawa matrix are different from the diagonal entries of the matrix. However we note that this correction will be of order $\mathcal{O}(\tilde{\kappa})$, which at the end of this section will be fixed to be $10^{-5}-10^{-6}$ by fixing the value of the quark mixing angles. In this sense we can neglect δ as compared to the contribution coming from the diagonal entries of the down-type Yukawa matrix, as well as any $\tilde{\kappa}$ dependence on these entries. After this it is easy to see manifestly the $(\mathcal{O}(1), \mathcal{O}(\epsilon), \mathcal{O}(\epsilon^2))$ hierarchy between the three families of quarks and leptons. Because the explicit expression for the eigenvalues of the lightest family cannot be computed at the level of approximation that we are working, we turn to discuss the masses for the two heavier families.

Masses for the second family

The strategy that we choose to follow to see if it is possible to fit all fermions masses is to look first at the mass ratios between the second and third families, which do not depend on $\tan\beta$. More specifically we will start by considering the following mass ratios

$$\frac{m_\mu/m_\tau}{m_s/m_b}\,, \qquad \frac{m_c/m_t}{m_s/m_b}\,, \tag{3.226}$$

which, in addition to being independent of $\tan\beta$ do not depend on the parameter ϵ which measures the strength of the non-perturbative effects. From the data in table 8 and the discussion in [74] we aim to reproduce the following values

$$\frac{m_\mu/m_\tau}{m_s/m_b} = 3.3 \pm 1\,, \qquad \frac{m_c/m_t}{m_s/m_b} = 0.13 \pm 0.03\,. \tag{3.227}$$

To complete the discussion of the masses of the second family we can look at an additional mass ratio, namely m_c/m_t. Being able to correctly fix this quantity and (3.227) allows us to obtain correct mass values for the second family of quarks and leptons when the masses of the third family are fitted later on.

We can compute the aforementioned ratios obtaining

$$\frac{m_c}{m_t} = \left|\frac{\tilde{\epsilon}}{2\rho_\mu}\right| \sqrt{q_R^Q\, q_R^U} = \left|\frac{\tilde{\epsilon}\,\tilde{N}_Y}{2\rho_\mu}\right| \sqrt{\left(x-\frac{1}{6}\right)\left(x+\frac{2}{3}\right)} \tag{3.228a}$$

$$\frac{m_s}{m_b} = \frac{Y_s}{Y_b}\sqrt{q_R^Q\, q_R^D} \simeq \frac{\left|\tilde{N}_Y\right| [(d+1)\theta_x + (a+bd)\theta_y]}{(d+1)^2\rho_\mu} \sqrt{\left(x-\frac{1}{6}\right)\left(y-\frac{1}{3}\right)} \tag{3.228b}$$

$$\frac{m_\mu}{m_\tau} = \frac{Y_\mu}{Y_\tau}\sqrt{q_R^Q\, q_R^D} \simeq \frac{\left|\tilde{N}_Y\right| [(d+1)\theta_x + (a+bd)\theta_y]}{(d+1)^2\rho_\mu} \sqrt{(x-1)\left(y+\frac{1}{2}\right)} \tag{3.228c}$$

where we defined

$$x = -\frac{M_1}{\tilde{N}_Y}\,, \quad y = -\frac{M_2}{\tilde{N}_Y}\,, \quad d = \frac{\mu_2^2}{\mu_1^2}\,. \tag{3.229}$$

In writing the final expression for m_s/m_b and m_μ/m_τ we neglected the δ shifts appearing in the expressions for the eigenvalues of the down quark and lepton Yukawa matrix as well as the $\mathcal{O}(\epsilon)$ correction appearing in $Y_{33}^{D/L}$. The reason behind this choice is that these contributions are much smaller when compared to the other terms and therefore will not affect the final results. Once these contributions are neglected the expressions for the ratio of masses become much simpler and depend on a smaller subset of parameters giving therefore more analytical control. Using (3.228) we can compute the ratio of masses (3.226) and the results are

$$\frac{m_\mu/m_\tau}{m_s/m_b} = \sqrt{\frac{(x-1)\left(y-\frac{1}{2}\right)}{\left(x-\frac{1}{6}\right)\left(y-\frac{1}{3}\right)}}\,, \tag{3.230}$$

$$\frac{m_c/m_t}{m_s/m_b} = \frac{(d+1)^2\sqrt{2+3x}(a\theta_y+\theta_x)}{2\sqrt{3y-1}\,[(d+1)\theta_x+(a+bd)\theta_y]}\,, \tag{3.231}$$

Chirality conditions place some constraints in the allowed regions for x and y, and in particular we find that for $\tilde{N}_Y < 0$ we need $x < -2/3$ and $y < 1/2$ and for $\tilde{N}_Y > 0$ we need $x > 1$ and $y > 1/3$. Between the two possibilities we find that it is simpler to fit the empirical data by choosing $\tilde{N}_Y > 0$. Moreover it seems reasonable

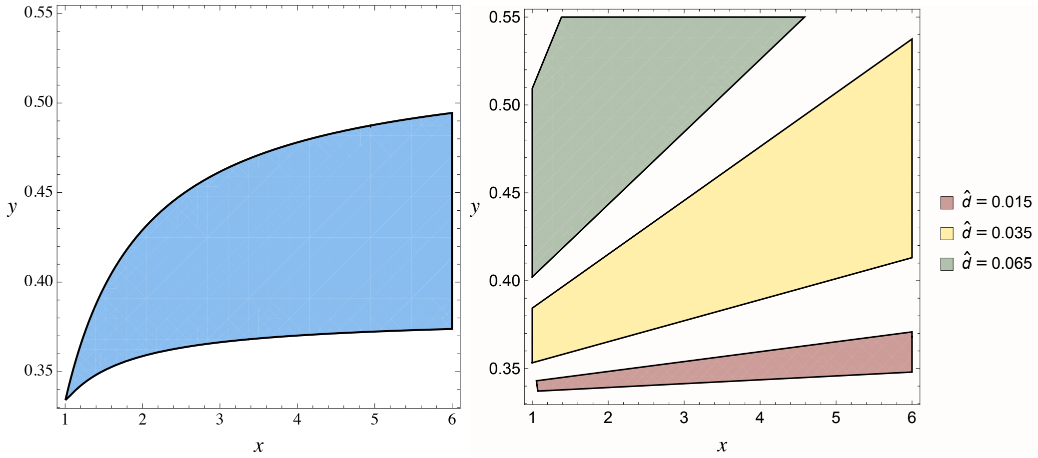

Figure 4: On the left the region in the $x - y$ plane for the ratio of masses (3.230) compatible with the realistic value in (3.227). On the right the region in the $x - y$ plane for the ratio of masses (3.231) compatible with (3.227), for different values of $\hat{d}$.

to take $\theta_x \sim \theta_y$ which implies that both ratios of masses will depend only on three parameters, namely x, y and $\hat{d}$ where

$$\hat{d} = \frac{(d+1)^2(a+1)}{a+1+d(b+1)}\,. \tag{3.232}$$

We show in figure 4 of the x and y parameter space where we find values for the ratios of masses in agreement with the empirical ones. The remaining mass ratio m_c/m_t has also a nice analytical expression in terms of the parameters of our local model

$$\frac{m_c}{m_t} = \frac{\sqrt{\left(x-\frac{1}{6}\right)\left(\frac{2}{3}+x\right)|\tilde{N}_Y|}}{2\mu_1^2}\,\tilde{\epsilon}\,. \tag{3.233}$$

In figure 5 we show in which region of the x and $\tilde{\epsilon}\,|\tilde{N}_Y|/\mu_1^2$ parameter space we are able to find good values for this last ratio of masses.

Yukawa couplings for the third family

Given that in our local model we have been able to find regions where the mass ratios between the second and third families are compatible with the MSSM, all we need to fix now are the masses for the fermions in the third family. We start by looking at the ratio between the mass of the τ-lepton and the b-quark. Such ratio can be expressed in terms of normalisation factors only

$$\frac{Y_\tau}{Y_b} = \frac{\gamma^E_{10,3}\gamma^L_{5,3}}{\gamma^Q_{10,3}\gamma^D_{5,3}}\,, \tag{3.234}$$

but in terms of the model parameters it acquires a rather complicated form, so it is quite hard to describe analytically the region of parameter space that is compatible with the expected value

$$\frac{Y_\tau}{Y_b} = 1.37 \pm 0.1 \pm 0.2\,. \tag{3.235}$$

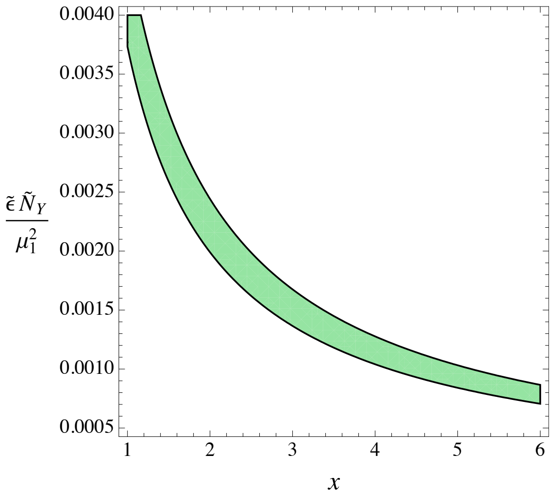

Figure 5: Region in the plane $x - \tilde{\epsilon}\tilde{N}_Y/\mu_1^2$ for the ratio (3.233) to be compatible with the range of values in table 8.

We have therefore performed a numerical scan over the values of the local flux densities which are compatible with the conditions for chirality and doublet-triplet splitting, and with the fermion mass ratios just discussed. More precisely we have chosen the following point in parameter space[30]

$$\begin{aligned}(\rho_m, \rho_\mu, d, c, a, b, \epsilon\,\theta_x, \epsilon\,\theta_y) &= (0.23, 2.5\times 10^{-3}, -0.9, 0.25, -0.4, -0.6, 10^{-4}, 10^{-4})\\ (M_1, M_2, N_1, N_2, \tilde{N}_Y, N_Y) &= (-0.17, -0.0136, -0.14, 0.008, 0.034, 0.1953)\end{aligned} \tag{3.236}$$

and scanned over the allowed values for x and $\tilde{N}_Y$ that do not spoil the constraints above. We show our results in figure 6, which displays a rather large region of these parameters.

Finally we may wish to see whether all constraints for chirality, doublet-triplet splitting and realistic fermion mass ratios may be solved simultaneously. We find that this is true for large regions of the parameter space. To illustrate this fact, in figure 7 we plot regions in the $m - \tilde{N}_Y$ parameter space where all constraints are fulfilled for different values of c. By inspecting the plot we see that regions fulfilling all constraints exist for different values of c which are of the same order as (3.27).

In these regions we can look at the typical value of the b-quark Yukawa to estimate the value of $\tan\beta$ that we typically obtain from our scan. We show in figure 8 the possible values of Y_b and by comparison with the content of the table 8 we obtain an approximated value of $\tan\beta \simeq 10 - 20$.

Quark mixing angles

An additional piece of information that we may extract from the Yukawa matrices involves the quark mixing angles, which are conventionally encoded in the CKM matrix. The definition of the CKM matrix involves a pair of unitary matrices V_U

30 We normalise all local flux densities in units of m_{st}^2 where the string scale m_{st} is related to the typical F-theory scale m_* by $m_{st}^4 = (2\pi)^3 g_s m_*^4$. In all the computations done in this section we take $g_s \sim \mathcal{O}(1)$.

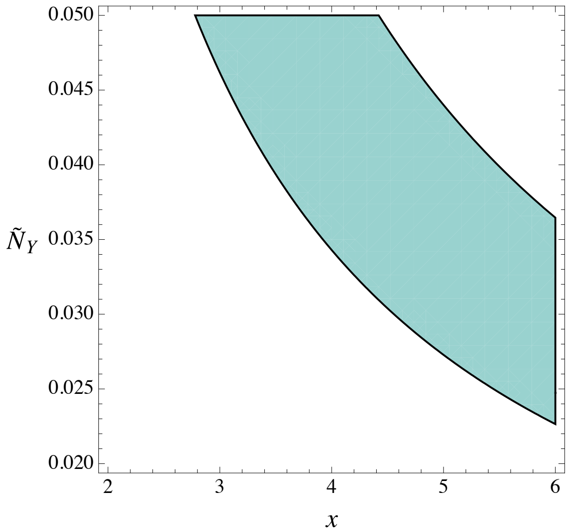

Figure 6: Region in the $x - \tilde{N}_Y$ plane with a ratio Y_τ/Y_b compatible with table 8.

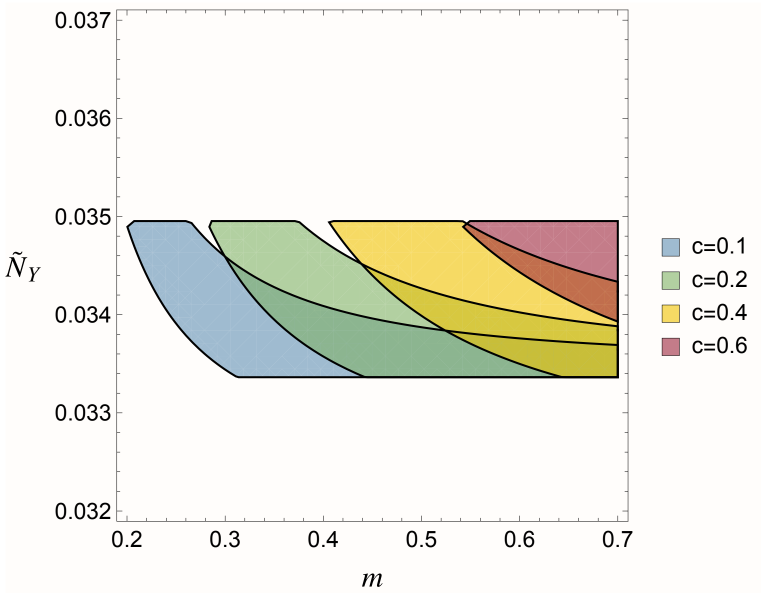

Figure 7: Regions in the m-$\tilde{N}_Y$ plane where all constraints are fulfilled for different values of c.

and V_D which diagonalise the product $YY^\dagger$ of the quark Yukawa matrices. More specifically we have that

$$M_U = V_U Y_U Y_U^\dagger V_U^\dagger \tag{3.237a}$$
$$M_D = V_D Y_D Y_D^\dagger V_D^\dagger \tag{3.237b}$$

with M_U and M_D diagonal. Using this we may define the CKM matrix as

$$V_{CKM} = V_U V_D^\dagger \,. \tag{3.238}$$

We can directly apply this definition to the Yukawa matrices of our model, which are accurate up to $\mathcal{O}(\epsilon^2)$ corrections. We find it convenient to expand the CKM matrix in

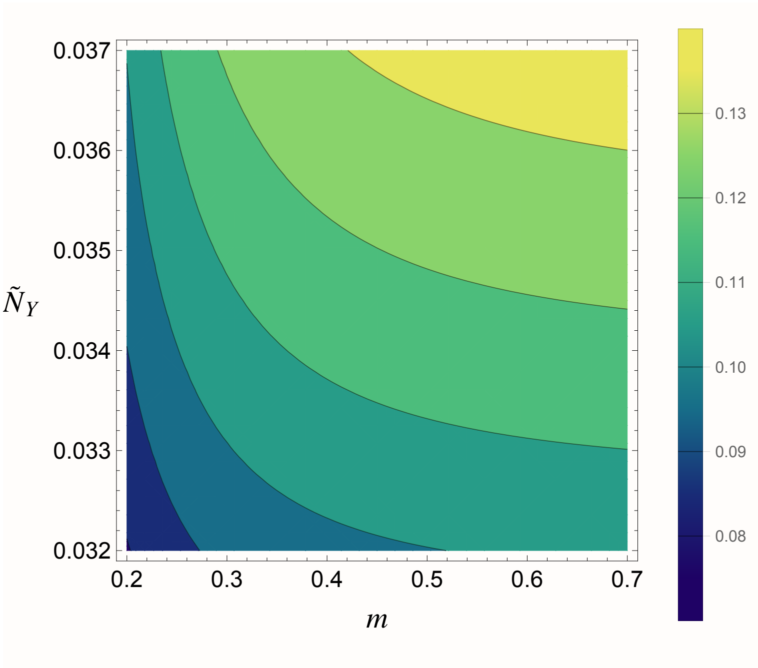

Figure 8: Value of Y_b in the $m - \tilde{N}_Y$ plane with the other parameters fixed.

the parameter $\xi \equiv a - b$. From the above analysis we know that for realistic fermion mass values $|\xi| \sim 0.1$, and so this expansion will quickly converge.

Explicitly we find

$$\hat{V}_U = \begin{pmatrix} 1 & 0 & -\frac{\tilde{\epsilon}\gamma^Q_{10,1}}{2\rho_\mu \gamma^Q_{10,3}} \\ 0 & 1 & 0 \\ \frac{\tilde{\epsilon}^*\gamma^Q_{10,1}}{2\rho^*_\mu \gamma^Q_{10,3}} & 0 & 1 \end{pmatrix} \tag{3.239a}$$

$$\hat{V}_D = \hat{V}_D^{(0)} + \xi \hat{V}_D^{(1)} + \mathcal{O}(\xi^2) \tag{3.239b}$$

$$\hat{V}_D^{(0)} = \begin{pmatrix} 1 & -\frac{i\tilde{\epsilon}\mathrm{Im}[(d+1)\tilde{\kappa}^*\rho_\mu]}{(d+1)|d+1|^2|\rho_\mu|^2\rho_\mu}\frac{\gamma^Q_{10,1}\gamma^Q_{10,2}}{(\gamma^Q_{10,3})^2} & \frac{(d+1)\tilde{\epsilon}\rho_\mu - 2\tilde{\kappa}^2}{(d+1)^2\rho_\mu^2}\frac{\gamma^Q_{10,1}}{\gamma^Q_{10,3}} \\ -\frac{\tilde{\epsilon}^*\tilde{\kappa}^*}{(d^*+1)^2\rho_\mu^{*2}}\frac{\gamma^Q_{10,1}\gamma^D_{10,2}}{(\gamma^Q_{10,3})^2} & 1 - \frac{|\tilde{\kappa}|^2}{2|d+1|^2|\rho_\mu|^2}\frac{(\gamma^Q_{10,2})^2}{(\gamma^Q_{10,3})^2} & \frac{\tilde{\kappa}}{(d+1)\rho_\mu}\frac{\gamma^Q_{10,2}}{\gamma^Q_{10,3}} \\ -\frac{(d^*+1)\tilde{\epsilon}^*\rho^*_\mu - 2\tilde{\kappa}^{*2}}{2(d^*+1)^2\rho_\mu^{*\,2}}\frac{\gamma^Q_{10,1}}{\gamma^D_{10,3}} & -\frac{\tilde{\kappa}^*}{(d^*+1)\rho^*_\mu}\frac{\gamma^Q_{10,2}}{\gamma^Q_{10,3}} & 1 - \frac{|\tilde{\kappa}|^2}{2|d+1|^2|\rho_\mu|^2}\frac{(\gamma^Q_{10,2})^2}{(\gamma^Q_{10,3})^2} \end{pmatrix} \tag{3.239c}$$

$$\hat{V}_D^{(1)} = \begin{pmatrix} 0 & \frac{(d-1)\epsilon\tilde{\kappa}^*\theta_y}{2|d+1|^2(d+1)|\rho_\mu|^2}\frac{\gamma^Q_{10,1}\gamma^Q_{10,2}}{(\gamma^Q_{10,3})^2} & \frac{20d^2\tilde{\kappa}\rho_\mu\tilde{\epsilon} - \epsilon\theta_y\left(d^2-1\right)\rho_m^{3/2}}{2(d+1)^3\rho_\mu\rho_m^{3/2}}\frac{\gamma^Q_{10,1}}{\gamma^Q_{10,3}} \\ 0 & \frac{3d^2\tilde{\epsilon}\tilde{\kappa}^*\rho_\mu}{(d+1)|d+1|^2\rho_m^{3/2}\rho^*_\mu}\frac{(\gamma^Q_{10,2})^2}{(\gamma^Q_{10,3})^2} & -\frac{d\left(3d(d+1)\tilde{\epsilon}\rho_\mu + 4\kappa^2\right)}{(d+1)^3\rho_m^{3/2}}\frac{\gamma^Q_{10,2}}{\gamma^Q_{10,3}} \\ -\frac{20d^{*2}\tilde{\kappa}^*\tilde{\epsilon}^*\rho^*_\mu + \epsilon\,\theta_y\rho_m^{*3/2}(1-d^{*2})}{2(d^*+1)^3\rho^*_\mu\,\rho_m^{*3/2}}\frac{\gamma^Q_{10,1}}{\gamma^D_{10,3}} & \frac{d^*\left(3d^*(d^*+1)\tilde{\epsilon}^*\rho^*_\mu + 4\bar{\kappa}^2\right)}{(d^*+1)^3\rho_m^{*3/2}}\frac{\gamma^Q_{10,2}}{\gamma^Q_{10,3}} & \frac{3\tilde{\kappa}d^{*2}\tilde{\epsilon}^*\rho^*_\mu}{|d+1|^2\left(\bar{d}+1\right)\rho_\mu\rho_m^{*3/2}}\frac{(\gamma^Q_{10,2})^2}{(\gamma^Q_{10,3})^2} \end{pmatrix} \tag{3.239d}$$

One can estimate the effect of $\mathcal{O}(\epsilon^2)$ corrections to the Yukawa matrices and kinetic terms by means of some unknown rotation matrices of the form

$$V_U = R_U \hat{V}_U\,, \qquad V_D = R_D \hat{V}_D\,, \tag{3.240}$$

where

$$R_{U,D} = \begin{pmatrix} 1 & \epsilon^2 \alpha_{U,D} & 0 \\ -\epsilon^2 \alpha_{U,D} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.241}$$

and $\alpha_{U,D}$ are some unknown coefficients. The effect of these rotations is to modify the elements of the CKM matrix involving the first family of quarks but will not affect the mixing between the top and bottom quarks. Such mixing can be measured in terms of the V_{tb} entry of the CKM matrix which at the level of approximation we are working is given by

$$V_{tb} = 1 - \frac{|\tilde{\kappa}|^2 q_{R,Q}}{2|d+1|^2|\rho_\mu|^2}\left[1 + \xi\, \frac{6\tilde{\epsilon}^* d^{*2} \rho_\mu^{*2}}{\left(\bar{d}+1\right) \rho_m^{*3/2} \tilde{\kappa}^*}\right], \tag{3.242}$$

where the factor multiplying ξ takes the value 4×10^{-3} when we substitute the typical values (3.236), and so it corresponds to a negligible term. The experimental value for this entry of the CKM matrix is

$$|V_{tb}|_{exp} \simeq 0.9991 \tag{3.243}$$

and so it can be reproduced by taking $\tilde{\kappa} \sim 10^{-5} - 10^{-6}$. The physical quantity is the distance between these two points in units of the typical scale of S_{GUT}, which we can estimate by looking at the V_{tb} entry of the CKM matrix. In fact we have the following relation[31]

$$\sqrt{1 - |V_{tb}|} \simeq \frac{|\tilde{\kappa}|\sqrt{q_{R,Q}}}{\sqrt{2}|\rho_\mu||d+1|} \propto m_* \left|\frac{a+bd}{d+1} x_0 - y_0\right| \tag{3.244}$$

where (x_0, y_0) are the coordinates of the down Yukawa point, see (3.177). This implies that the separation of the two points directly controls the mixing between the second and third family. In the case $a = b$ this separation is measured along the coordinate $ax - y$ which is precisely the complex coordinate entering in the matter wavefunctions, see (3.190). In this case the whole effect of mixing is due to a mismatch in the wavefunctions bases between the two points as in [92]. It would be however interesting to have an intuitive picture for the general case $a \neq b$. In any case, using the relation between the measured V_{tb} entry of the CKM matrix and the relative distance between the two Yukawa points we can directly estimate the latter and see that it is of the order of $10^{-2} V_{GUT}^{1/4}$. Hence we can see explicitly that the distance between the two points is rather small when compared to the typical size of S_{GUT} as claimed in [83].

31 Here we again discard the $\mathcal{O}(\xi)$ term in the expression for V_{tb} as it is negligible.

Part II

LINEAR EQUIVALENCE

4 LINEAR EQUIVALENCE IN D–BRANE MODELS

In this chapter we move to a different area of String Phenomenology and present two separate instances of how linear equivalence between cycles controls some aspects of the physics of D–branes. Even if these two examples we will present are very different ones they share the fact of being generic in D–brane models and therefore their study will allow us to draw some general lessons on these class of models. We will start in Section 4.1 by presenting the kinetic mixing between massless $U(1)$ gauge bosons and demonstrate that it is intimately tied with the concept of linear equivalence between cycles. Kinetic mixing has some important phenomenological consequences, and we choose to discuss explicitly two of them: first we will present a mechanism for the generation of millicharged particles with potential implications for dark matter and secondly we will discuss the impact of kinetic mixing for unification of gauge couplings. Afterwards in Section 4.2 we will turn to the moduli sector coming from D–branes. We will revisit the usual counting of massless degrees of freedom controlling the configuration of D–branes and show that some of the scalars that were usually thought to be massless acquire a mass via a coupling to the closed string moduli. We will discuss explicitly the microscopic mechanism that is responsible for this mass and match the result with the analysis performed at the level of the effective field theory. The analysis we perform shows that the brane deformation has to go through cycles that linearly equivalent in order to have a massless field. One interesting consequence of the mechanism for moduli stabilisation that we consider is that Wilson line moduli may acquire a mass even before taking into account the effect of worldsheet instantons.

4.1 LINEAR EQUIVALENCE AND $U(1)$ MIXING

The aim of this section is to present a first instance of how the physics of D–branes is sensitive to equivalence relations between cycles that are more refined than the usual homological equivalence. Explicitly we will find that linear equivalence between cycles controls the kinetic mixing between $U(1)$ gauge bosons that come from the open and closed string sector. We give here a rough picture (in type IIA String Theory) of what happens before delving into the details: when D6–brane are added in the compactification it is possible to have some massless $U(1)$ gauge bosons coming from the open string sector if there exist pairs of stack of D6–branes wrapping cycles that are equivalent in homology. While this would suggest that homology is sufficient to characterise the physics of these $U(1)$ bosons it turns out that there exists a mixing with the $U(1)$ bosons coming from the closed string sector if the pairs of stack are not on cycles that are linear equivalent. This is of particular interest for the hypercharge boson is realised from the open string sector and its possible mixing with the additional $U(1)$ bosons appearing in the closed string sector can have potential phenomenological consequences.

Kinetic mixing has already been considered in the literature [93–115], and being this phenomenon a generic one in the string landscape we can draw some general conclusions on this vast class of models. The particular interest in kinetic mixing stems from the fact that it can lead to some detectable signatures in ongoing experiments.

One of the most famous consequences of kinetic mixing is the so-called milli-charged scenario, in which light particles charged under a hidden $U(1)_h$ obtains a small electric charge due to the kinetic mixing of $U(1)_h$ with the hypercharge [116]. Following the recent discussion in [115], one can achieve such effect by considering a string compactification with two different gauge sectors. The first sector contains the SM and in particular the hypercharge $U(1)_Y$, while the other sector contains a hidden gauge group with a second massless $U(1)_h$. When considering the embedding in D–brane models this setup can be obtained by considering a scenario where the visible sector is localised in a set of D–branes wrapping p-cycles in a region of the compactification manifold $\mathcal{M}_6$, and the hidden sector arises from a second D–brane set located in a different region. The fact that the two sets of D-branes are internally separated from each other guarantees that there are no light particles charged under the visible and gauge sectors simultaneously. Moreover, the gauge kinetic mixing χ_{vh} between the visible and hidden $U(1)$ will vanish at tree level. Nevertheless, there will be massive particles charged under $U(1)_v = U(1)_Y$ and $U(1)_h$ which, when integrated out, will generate one-loop corrections to the mixing (see e.g. [101]). Finally, upon diagonalisation of the gauge kinetic terms, light matter charged under $U(1)_h$ will acquire a small hypercharge proportional to χ_{vh} [116].

While this is a rather general scenario, obtaining precise results depends crucially on the computation of threshold corrections to the gauge kinetic mixing, which is technically very difficult beyond simple toroidal orbifold compactifications. Moreover, generically the quantity χ_{vh} will decrease for large separations of the visible and hidden sectors and small values of the string coupling, and so in this regime its value could be quite small.

In the following we will describe a slight variation of this scenario resulting into an alternative mechanism to generate milli-charged particles. As we will see below, these new contributions can be computed by a simple geometric formula describing tree-level local quantities, and so their effect could be more important than the one just described.

RR photon mixing and linear equivalence

The variation comes from assuming the presence of a further $U(1)$ gauge symmetry that does not arise from the D-brane degrees of freedom, but rather from the Kaluza-Klein reduction of the Ramond-Ramond closed string sector of the theory. Such bulk $U(1)$'s, dubbed RR photons in [111], exist for generic choices of the compactification manifold $\mathcal{M}_6$ and are natural sources of hidden $U(1)$ gauge symmetries, because the only particles charged under them are extremely heavy: namely D-branes wrapped on internal cycles and point-like in 4d. Despite their hidden nature, these RR photons will mix kinematically with the D-brane $U(1)$'s [98, 109–111]. However, since there is no light particle charged under them there is naively no measurable effect arising from their mixing with the hypercharge.[1] Let us nevertheless consider the existence of a RR photon $U(1)_b$ living in the bulk and its mixing with the D-brane photons $U(1)_v$ and $U(1)_h$. After normalising the gauge bosons we recover an effective Lagrangian of the form

$$\mathcal{L}_{4d} \supset -\frac{1}{4}\sum_{i=v,h,b} F^{(i)}_{\mu\nu}F^{(i)\,\mu\nu} + \frac{1}{2}\left(\chi_{vb}F^{(v)}_{\mu\nu}F^{(b)\,\mu\nu} + \chi_{hb}F^{(h)}_{\mu\nu}F^{(b)\,\mu\nu}\right) \tag{4.1}$$

where we have assumed that χ_{vh} is negligible, although it can be easily reincorporated. The two mixing terms can be eliminated by performing the change of basis

$$\begin{pmatrix} A'^{(v)} \\ A'^{(h)} \\ A'^{(b)} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1-\chi_{vb}^2-\chi_{hb}^2}{1-\chi_{hb}^2}} & 0 & 0 \\ -\frac{\chi_{vb}\chi_{hb}}{\sqrt{1-\chi_{hb}^2}} & \sqrt{1-\chi_{hb}^2} & 0 \\ -\chi_{vb} & -\chi_{hb} & 1 \end{pmatrix} \begin{pmatrix} A^{(v)} \\ A^{(h)} \\ A^{(b)} \end{pmatrix} \tag{4.2}$$

and so, even if we started with no mixing χ_{vh}, the shift in the D-brane hidden gauge boson $A^{(h)}$ induces a hypercharge on the light matter charged under $U(1)_h$ with a suppression factor $\delta^{\text{eff}}_{vh} = \frac{\chi_{vb}\chi_{hb}}{\sqrt{1-\chi_{vb}^2-\chi_{hb}^2}}$ compared to the SM particles. This effect is to be compared with the factor $\delta_{vh} = \frac{\chi_{vh}}{\sqrt{1-\chi_{vh}^2}}$ that would arise if we only had a non-vanishing mixing between the two D-brane $U(1)$'s, and naively it seems to be more suppressed than the latter. Nevertheless, as we will see the kinetic mixing between closed and open string $U(1)$'s is not a one-loop suppressed effect, and also that δ^{eff}_{vh} is independent of the relative separation between visible and hidden D-brane sectors. Hence it could be a comparable or even stronger effect than the one considered in the standard milli-charge scenario.

Motivated by this observation we would like in the following to study in a systematic way the kinetic mixing between open and closed string abelian gauge bosons in type II compactifications and extend the analysis previously presented in [111]. Rather than (4.1) we will consider the following $U(1)$ action

$$S_{4d,\,U(1)} = -\frac{1}{2\kappa_4^2}\int_{\mathbb{R}^{1,3}} \text{Re}\,(f_{pq})F_2^p \wedge *_4 F_2^q + \text{Im}\,(f_{pq})F_2^p \wedge F_2^q \tag{4.3}$$

where $\kappa_4^2 = \frac{l_s^2}{4\pi}$, and $l_s = 2\pi\sqrt{\alpha'}$ is the string length.[2] This formulation encodes the kinetic mixing and the θ angles of the theory in terms of the gauge kinetic function f_{pq}, a protected quantity in 4d supersymmetric theories. While in the following we will not consider any particular scale of SUSY breaking, we will assume that supersymmetry is at least restored at the compactification scale. This will not only guarantee the stability of our constructions, but also that f_{pq} is a holomorphic function of the 4d chiral fields arising below that scale. It will moreover imply that, fixed the few topological data that describe the scenario we are considering, the kinetic function mixing open and closed string $U(1)$'s will depend in relatively few compactification data.

1 Still, in models with low energy supersymmetry hidden $U(1)$ gauginos may mix with the MSSM neutralinos and this could lead to new signatures at the LHC [117–120].

2 In our conventions all the p-form potentials are dimensionless (except for the Dp-brane gauge field) and the field strengths have dimension 1. By p-form dimension we mean the dimension of its components.

In fact, we will find that the kinetic mixing Re $f_{pq} = 4\pi \frac{\chi_{pq}}{g_p g_q}$ is closely related to the mathematical concept of linear equivalence. In brief, for an open string $U(1)$ to be massless we need to satisfy certain topological conditions regarding the p-cycles wrapped by the D-branes, roughly speaking that a certain linear combination of p-cycles is homologically trivial [111]. Given such combination of p-cycles it is possible to draw a $(p+1)$-chain Σ connecting them. The requirement of linear equivalence can then be formulated by asking that the integrals of certain harmonic bulk $(p+1)$-forms over Σ vanish [121]. This is not a topological condition, in the sense that it depends on the embedding of the p-cycles inside their homology class, but it is a rather simple and robust quantity as it involves integrals of harmonic forms over slices of the internal manifold $\mathcal{M}_6$. As we will see, such integrals over the chain Σ are nothing but the gauge kinetic function mixing open and closed string $U(1)$'s, in agreement with our expectations that these protected quantities should be easier to compute than many other 4d effective couplings, which is a promising starting point to draw model-independent predictions out of them.

4.1.1 *$U(1)$ kinetic mixing for intersecting D6-branes*

In this section we discuss the kinetic mixing between $U(1)$'s in type IIA orientifold compactifications. In particular, we consider models made up of D6-branes wrapping special Lagrangian cycles (sLags) of Calabi-Yau three-folds, and describe the kinetic mixing of open string $U(1)$'s with RR $U(1)$'s. We derive the expression for such mixing by means of the Witten effect, recovering the results of [111] from a different perspective. We then point out the relation between open string $U(1)$'s and the relative cohomology of the compactification manifold, which allows to write down a simple supergravity formula for the kinetic mixing between open and closed string $U(1)$'s. Finally, we discuss the relation between open-closed kinetic mixing and linear equivalence of cycles (see Appendix D for the mathematical definition of linear equivalence).

Type IIA Calabi-Yau orientifolds with D6-branes

Let us consider an orientifold of type IIA string theory on $\mathbb{R}^{1,3} \times \mathcal{M}_6$ with $\mathcal{M}_6$ a Calabi-Yau 3-fold. The orientifold projection is obtained by modding out by the action $\Omega_p(-1)^{F_L}\sigma$ where Ω_p is the worldsheet parity, F_L is the space-time fermion number for the left-movers and σ is an antiholomorphic involution of $\mathcal{M}_6$ acting as $z_i \to \bar{z}_i$ on local coordinates, which introduces O6-planes. Therefore, the action of the involution on the Kähler form J and holomorphic 3-form Ω of $\mathcal{M}_6$ is given by

$$\sigma J = -J, \qquad \sigma\Omega = \bar{\Omega}. \tag{4.4}$$

The supersymmetry conditions for a D6-brane wrapping a 3-cycle π in $\mathcal{M}_6$ are

$$J|_\pi = 0, \qquad \mathrm{Im}\,\Omega|_\pi = 0. \tag{4.5}$$

Since the D6-brane charge lies in the homology group $H_3(\mathcal{M}_6, \mathbb{Z})$, cancellation of the total charge in the compact internal space can be written as

$$\sum_\alpha N_\alpha([\pi_\alpha] + [\pi_\alpha^*]) - 4[\pi_O] = 0 \tag{4.6}$$

where α is an index that runs over the set of branes, N_α is the total number of branes on π_α and $\pi_\alpha^* = \sigma\pi_\alpha$ is the cycle wrapped by the orientifold image of α. Finally, π_O is the fixed locus of the involution σ where the O6-planes lie and the factor -4 is due to the O-plane RR charge which we take to be negative.

The 4d massless spectrum that arises from the closed string sector of the compactification can be computed upon dimensional reduction of the 10d type IIA supergravity action, and is given in terms of harmonic forms on $\mathcal{M}_6$. It is then useful to introduce a basis of harmonic forms of definite parity under the involution σ,

$$\begin{array}{llllll} & & \sigma-\text{even} & & \sigma-\text{odd} \\ 2-\text{forms} & \omega_i & i=1,\dots,h_+^{1,1} & \omega_{\hat{i}} & \hat{i}=1,\dots,h_-^{1,1} \\ 3-\text{forms} & \alpha_I & I=0,\dots,h^{1,2} & \beta^I & I=0,\dots,h^{1,2} \\ 4-\text{forms} & \tilde{\omega}^{\hat{i}} & \hat{i}=1,\dots,h_-^{1,1} & \tilde{\omega}^i & i=1,\dots,h_+^{1,1} \end{array}$$

normalised such that

$$\int_{\mathcal{M}_6} \omega_i \wedge \tilde{\omega}^j = l_s^6 \delta_i^j, \qquad \int_{\mathcal{M}_6} \omega_{\hat{i}} \wedge \tilde{\omega}^{\hat{j}} = l_s^6 \delta_{\hat{i}}^{\hat{j}}, \qquad \int_{\mathcal{M}_6} \alpha_I \wedge \beta^J = l_s^6 \delta_I^J. \tag{4.7}$$

For instance, in order to reduce to 4d the RR forms C_3 and C_5 one can expand them as

$$C_3 = A_1^i \wedge \omega_i + \text{Re}\,(N^I)\alpha_I \tag{4.8}$$
$$C_5 = C_{2,I} \wedge \beta^I + V_{1,i} \wedge \tilde{\omega}^i \tag{4.9}$$

where we have taken into account the intrinsic parity of C_3 and C_5 (respectively even and odd) under the orientifold action. One then obtains that C_3 gives rise to $h_{1,1}^-$ axions Re (N^I) and to $h_{1,1}^+$ gauge bosons A_1^i, while C_5 contains their 4d dual degrees of freedom.

A convenient basis of harmonic p-forms is given by those that have integer cohomology class, that is whose integrals over any p-cycle are integer numbers. In particular we will choose the 2-forms ω_i, $\omega_{\hat{i}}$ such that $[\omega_i] \in H_+^2(\mathcal{M}_6, \mathbb{Z})$, $[\omega_{\hat{i}}] \in H_-^2(\mathcal{M}_6, \mathbb{Z})$.[3] This automatically implies that a D2-brane wrapping a 2-cycle Λ_2 will have integer electric charges under the RR $U(1)$'s. More precisely, a D2-brane wrapping a 2-cycle Λ_2^j whose class is Poincaré dual to $[\tilde{\omega}^j] \in H_-^4(\mathcal{M}_6, \mathbb{Z})$ will have electric charge δ^{ji} under the RR $U(1)$ generated by A_1^i. Another consequence of this choice is that the gauge kinetic mixing for RR $U(1)$'s takes the simple form [122]

$$f_{ij} = -\frac{i}{2l_s^4} \int_{\mathcal{M}_6} J_c \wedge \omega_i \wedge \omega_j = -\frac{i}{2} \mathcal{K}_{ij\hat{k}} T^{\hat{k}} \tag{4.10}$$

where we have defined the complexified Kähler moduli $T^{\hat{k}}$ by[4]

$$J_c \equiv B_2 + iJ = l_s^2 T^{\hat{k}} \omega_{\hat{k}} \tag{4.11}$$

3 In general $H^2(\mathcal{M}_6, \mathbb{Z})$ may not decompose as $H_+^2(\mathcal{M}_6, \mathbb{Z}) \oplus H_-^2(\mathcal{M}_6, \mathbb{Z})$, but the latter may only be a sublattice of the former. In this case one should introduce appropriate factors of 2 in (4.7). To simplify in the following we will assume that (4.7) holds even if $[\omega_i]$, $[\tilde{\omega}^i]$, $[\omega_{\hat{i}}]$, $[\tilde{\omega}^{\hat{i}}]$ all belong to integer cohomology.

4 Here we take the volume of the cycles to be measured via the metric in string frame. If we used Einstein frame metric an additional factor of $e^{\phi/2}$ would appear in front of J.

while

$$\mathcal{K}_{ij\hat{k}} \equiv \frac{1}{l_s^6}\int_{\mathcal{M}_6} \omega_i \wedge \omega_j \wedge \omega_{\hat{k}} \tag{4.12}$$

are the triple intersection numbers, which in this basis are simply integers.

Regarding the open string sector of the compactification, the 4d massless spectrum that arises from a single D6-brane α wrapping a 3-cycle π_α is given by

$$A_1^\alpha = \frac{\pi}{l_s}(A_1^{4d,\alpha} + \theta_\alpha^j \zeta_j) \tag{4.13}$$

$$\phi_\alpha = \phi_\alpha^j X_j \tag{4.14}$$

where $A_1^{4d,\alpha}$ is a 4d gauge vector field (we will henceforth suppress the superscript 4d)[5]. Moreover, θ_α^j are the components of the corresponding Wilson line moduli with

$$\frac{\zeta_j}{2\pi} \in \mathrm{Harm}^1(\pi_\alpha, \mathbb{Z}) \tag{4.15}$$

and ϕ_α^j are the D6-brane position moduli, namely the components of a normal deformation of the brane preserving the sLag conditions (4.5) with

$$X_i \in N(\pi_\alpha) \text{ such that } \mathcal{L}_{X_i} J = \mathcal{L}_{X_i}\mathrm{Im}\Omega = 0 \tag{4.16}$$

where $\mathcal{L}_{X_i}$ is the Lie derivative along X_i. These two scalar fields together form a 4d complex modulus, namely

$$\Phi_\alpha^j = \theta_a^j + \lambda_i^j \phi_\alpha^i \tag{4.17}$$

with λ_i^j a complex matrix relating $\{\zeta_j\}$ and $\{X_i\}$ and defined by

$$\iota_{X_i} J_c|_{\pi_\alpha} = \lambda_i^j \zeta_j \tag{4.18}$$

where J_c is the complexified Kähler form (4.11). It is straightforward to generalise this spectrum to the case of a stack of N_α D6-branes wrapping π_α, so that the 4d gauge group is given by $U(N_\alpha)$ and Φ_α^j transform in its adjoint representation. Finally, 4d chiral multiplets may arise from the transverse intersections of π_α with its orientifold image π_α^* as well as with other 3-cycles wrapped by the remaining D6-branes of the compactification [123–126].

Separating two D6-branes

In order to discuss kinetic mixing between open an closed string $U(1)$'s let us follow [111] and first consider type IIA strings on $\mathbb{R}^{1,3} \times \mathcal{M}_6$, without any orientifold projection, and suppose that we have two D6-branes a and b wrapping the same sLag 3-cycle $\pi_a = \pi_b$. This leads to a gauge group $U(2)$ in 4d, which breaks down to $U(1)_a \times U(1)_b$ when these two 3-cycles are separated[6]. However, only a linear combination of these two $U(1)$'s remains massless at low energies, while the other one becomes massive due to the Stückelberg mechanism.

5 The $\frac{1}{l_s}$ is introduced to keep $A_1^{4d,\alpha}$ and θ_α^j dimensionless and the factor of π for later convenience. The field ϕ_a^j is related to the normal coordinate by $y_a^j = \frac{l_s}{2}\phi_a^j$ so it is also dimensionless.

6 Note that turning on Wilson lines will break the gauge group in the same exact way. We will come back to this in a moment.

Indeed, let us consider the CS action for a single D6-brane wrapping a 3-cycle π_α, which is obtained from a 3-cycle π after a small normal deformation of the form $\phi_\alpha = \phi^j_\alpha X_j$.[7] We have that (see [109–111] for further details)

$$\begin{aligned} S^\alpha_{CS} &\supset \mu_6 \int_{\mathbb{R}^{1,3}\times\pi_\alpha} \left[\mathcal{F}^\alpha_2 \wedge C_5 + \frac{1}{2}\mathcal{F}^\alpha_2 \wedge \mathcal{F}^\alpha_2 \wedge C_3\right] \\ &= \mu_6 \int_{\mathbb{R}^{1,3}\times\pi} e^{\mathcal{L}_{\phi_\alpha}} \left[\mathcal{F}^\alpha_2 \wedge C_5 + \frac{1}{2}\mathcal{F}^\alpha_2 \wedge \mathcal{F}^\alpha_2 \wedge C_3\right] \end{aligned} \tag{4.19}$$

with $\mu_6 = \frac{2\pi}{l_s^7}$ the D6-brane charge, $\mathcal{L}_{\phi_\alpha} = \phi^j_\alpha \mathcal{L}_{X_j}$ the Lie derivative along such deformation and $\mathcal{F}^\alpha_2 = \frac{l_s^2}{2\pi}F^\alpha_2 + B_2$. In the absence of orientifold projection the RR 5-form potential C_5 has the expansion

$$C_5 = C_{2,I} \wedge \beta^I + \tilde{C}^I_2 \wedge \alpha_I + V_{1,i} \wedge \tilde{\omega}^i \tag{4.20}$$

where now $i = 1, \ldots, h^{1,1}$ runs over all harmonic 2-forms in $\mathcal{M}_6$. We now consider two 3-cycles π_a and π_b that are deformations of π and wrap a D6-brane on each of them. The full CS action then contains the following piece

$$S^a_{CS} + S^b_{CS} \supset \frac{\pi}{l_s^6}\left[\int_{\mathbb{R}^{1,3}} (F^a_2 + F^b_2) \wedge C_{2,I} \int_\pi \beta^I + \int_{\mathbb{R}^{1,3}} (F^a_2 + F^b_2) \wedge \tilde{C}^I_2 \int_\pi \alpha_I\right] \tag{4.21}$$

where we have used that the integrals of β^I, α_I only depend on the homology class of the 3-cycle, and in particular that $\int_\pi e^{\mathcal{L}_{\phi_\alpha}}\beta^I = \int_\pi \beta^I$, same for α_I. For a non-trivial $[\pi]$ some of these integrals will be non-vanishing, and so the combination $U(1)_a + U(1)_b$ will develop a BF coupling and therefore a Stückelberg mass, while the orthogonal combination

$$U(1)_{(a-b)} = \frac{1}{2}\left[U(1)_a - U(1)_b\right] \tag{4.22}$$

will remain massless.

We can now read off the kinetic mixing of (4.22) with the RR $U(1)$'s from the remaining terms of $S^a_{CS} + S^b_{CS}$. For this it is useful to consider the following expansion for the RR potentials

$$C_3 = A^i_1 \wedge \omega_i + \ldots \tag{4.23}$$

$$C_5 = \tilde{A}^i_1 \wedge *_6\, \omega_i + \ldots \tag{4.24}$$

where the dots represent terms that do not contain 4d gauge bosons. This new expansion for C_5 is chosen so that $*_4 F^{\mathrm{RR},i}_2 \equiv *_4 dA^i_1 = d\tilde{A}^i_1$. Plugging the expansion (4.24) into (4.19) and projecting into the combination $F^{(a-b)}_2 \equiv \frac{1}{2}[F^a_2 - F^b_2]$ we obtain

$$S^a_{CS} + S^b_{CS} \supset -\frac{\pi}{2l_s^5}\int_{\mathbb{R}^{1,3}} F^{(a-b)}_2 \wedge *F^{\mathrm{RR},i}_2 \int_\pi (\iota_{\phi_a}J - \iota_{\phi_b}J) \wedge \omega_i + \ldots \tag{4.25}$$

where we have only kept terms linear in the deformations ϕ_α. Here we have used that $\mathcal{L}_\phi = d\iota_\phi + \iota_\phi d$ and that $*_6\omega_i = b_i J^2 - J \wedge \omega_i$ with $b_i = (3\int_{\mathcal{M}_6}\omega_i \wedge J^2)/(2\int_{\mathcal{M}_6} J^3)$. Using the definition (4.18) we can recast this result as

$$\mathrm{Re}\, f_{i(a-b)} = \frac{1}{4l_s^3}(\phi^k_a - \phi^k_b)\,\mathrm{Im}\,(\lambda^j_k)\int_\pi \zeta_j \wedge \omega_i \tag{4.26}$$

7 Such normal deformation should be small enough so that π and π_α have the same topology, and in particular the same number of non-trivial 1-cycles.

where the basis of 1-forms $\{\zeta_j\}$ is defined as in (4.15).

The imaginary part of $f_{i(a-b)}$ is obtained from plugging (4.23) into (4.19), which gives

$$S^a_{CS} + S^b_{CS} \supset \frac{\pi}{2l_s^5}\int_{\mathbb{R}^{1,3}} F_2^{(a-b)} \wedge F_2^{\mathrm{RR},i} \int_\pi \left[(\theta^j_a - \theta^j_b)\zeta_j + (\iota_{\phi_a}B - \iota_{\phi_b}B)\right] \wedge \omega_i + \dots \tag{4.27}$$

where we have again integrated by parts and kept terms linear in the deformations. Comparing to the general expression (4.3) we conclude that

$$\mathrm{Im}\ f_{i(a-b)} \ = \ -\frac{1}{4l_s^3}\left[(\theta^j_a - \theta^j_b) + (\phi^k_a - \phi^k_b)\,\mathrm{Re}\,(\lambda^j_k)\right]\int_\pi \zeta_j \wedge \omega_i \tag{4.28}$$

Adding this result to (4.26) we obtain

$$f_{i(a-b)} = -\frac{i}{4l_s^3}(\Phi^j_a - \Phi^j_b)\int_\pi \zeta_j \wedge \omega_i \tag{4.29}$$

which as expected is a holomorphic function of the D6-brane moduli. Notice that the mixing vanishes for $\Phi^j_a = \Phi^j_b$, which corresponds to the case where the two branes are on top of each other and the gauge group enhances to $SU(2)$.

In order to arrive at (4.29) we assumed that π_a and π_b are obtained from deforming the same 3-cycle π. As a result, they are not only in the same homology class but are also homotopic. However, since the vanishing of the Stückelberg mass depends only on the homology of the cycles and not on their homotopy class, one would like to have an expression for the kinetic mixing that applies in the general case. In [111] such a formula was found to be

$$f_{i(a-b)} = -\frac{i}{2l_s^4}\int_\Sigma \left(J_c + \frac{l_s^2}{2\pi}\tilde{F}_2^{(a-b)}\right) \wedge \omega_i \tag{4.30}$$

where Σ is a 4-chain such that $\partial\Sigma = \pi_a - \pi_b$, and $\tilde{F}_2^{(a-b)}$ is such that

$$\int_\Sigma \tilde{F}_2^{(a-b)} \wedge \omega_i \equiv \left[\int_{\pi_a} A_1^a \wedge \omega_i - \int_{\pi_b} A_1^b \wedge \omega_i\right]. \tag{4.31}$$

It can be easily shown that this 4-chain expression reproduces (4.29) for homotopic branes. Moreover, following [111] one can see that the kinetic mixing needs to be of the form (4.30) by performing the M-theory lift of these compactifications. In the next section we will arrive at (4.30) from yet a different viewpoint, without using any M-theory lift.

Notice that the open-closed kinetic mixing vanishes if

$$\int_\pi \omega_i \wedge \zeta_j = \int_{\rho_j} \omega_i = 0 \tag{4.32}$$

where the 2-cycle $\rho_j \subset \pi$ is Poincaré dual to ζ_j. This is true only if none of the 2-cycles of π is non-trivial in $\mathcal{M}_6$. By the results of [121], this is equivalent to saying that the 3-cycles π_a and π_b are linearly equivalent. Hence, in the present case linear equivalence of D-branes translates into a vanishing kinetic mixing with RR photons, as advanced in the introduction. In the following sections we will see how this statement can be generalised to more involved D-brane configurations.

Orientifolding

Let us now include the effect of the orientifold projection. Because in this case C_5 has the expansion (4.9), instead of (4.21) we obtain

$$\begin{aligned} S^a_{CS} + S^{a^*}_{CS} + S^b_{CS} + S^{b^*}_{CS} &\supset \frac{1}{2}\frac{\pi}{l_s^6}\int_{\mathbb{R}^{1,3}} (F_2^a + F_2^b)\wedge C_{2,I}\left[\int_\pi \beta^I - \int_{\pi^*}\beta^I\right] \\ &= \frac{\pi}{l_s^6}\int_{\mathbb{R}^{1,3}} (F_2^a + F_2^b)\wedge C_{2,I}\int_\pi \beta^I \end{aligned} \tag{4.33}$$

where the extra factor of 1/2 arises due to the orientifold projection, and we have used the fact that $F_{\alpha^*} = -F_\alpha$ for the 7d gauge field. As a result, now $U(1)_a + U(1)_b$ will develop a Stückelberg mass if and only if $[\pi] \neq [\pi^*]$.

Let us assume that this is the case and compute the kinetic mixing for the massless $U(1)$ (4.22), for which one can obtain expressions similar to the unorientifolded case. Indeed, we have that

$$S^a_{CS} + S^{a^*}_{CS} + S^b_{CS} + S^{b^*}_{CS} \supset \frac{\pi}{2l_s^5}\int_{\mathbb{R}^{1,3}} F_2^{(a-b)}\wedge F_2^{\mathrm{RR},i}\cdot(\theta_a^j - \theta_b^j)\int_\pi \zeta_j\wedge\omega_i \tag{4.34}$$

from where we can deduce that the kinetic mixing again takes the form (4.29). In terms of a 4-chain formula we would again arrive to (4.30), with the only difference that now Σ is defined by $\partial\Sigma = \pi_a - \pi_b - \pi_a^* + \pi_b^*$.[8]

On the other hand, for $[\pi] = [\pi^*]$ both $U(1)_a$ and $U(1)_b$ remain massless. One should then be able to write the kinetic mixing of each $U(1)_\alpha$ with the RR $U(1)$'s individually. Indeed, one finds that the expression analogous to (4.30) is

$$f_{i\alpha} = -\frac{i}{2l_s^4}\int_{\Sigma_\alpha}\left(J_c + \frac{l_s^2}{2\pi}\tilde{F}_2^\alpha\right)\wedge\omega_i \tag{4.35}$$

where Σ_α is defined in the covering space $\mathcal{M}_6$ and satisfies $\partial\Sigma'_\alpha = \pi_\alpha - \pi^*_\alpha$.

Kinetic mixing via the Witten effect

Let us now describe an alternative derivation for the kinetic mixing formula (4.30), based on the Witten effect [127]. Witten effect amounts to the fact that for $U(1)$ gauge theories with violations of CP magnetic monopoles (if present) acquire an electric charge proportional to the CP breaking term. For theories whose CP violating effect is a θ-term this electric charge can be computed exactly, namely

$$Q^E = -\frac{\theta}{2\pi}e. \tag{4.36}$$

This can be generalised to theories that contain multiple $U(1)$'s and whose action is described by (4.3). The lattice of charges is then [128]

$$\begin{aligned} Q_I^E &= n_I^e - \mathrm{Im}\, f_{IJ}\, n_J^m \\ Q_I^M &= n_I^m \end{aligned} \tag{4.37}$$

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8 In this case the 4-chain sigma can be divided into two pieces as $\Sigma = \Sigma_{ab} - \Sigma^*_{ab}$, where $\partial\Sigma_{ab} = \pi_a - \pi_b$ and Σ^*_{ab} is its orientifold image.

where $I = 1, \ldots, K$ runs over the set of massless $U(1)$'s and $n_e^I \in \mathbb{Z}/2$, $n_m^I \in \mathbb{Z}$ are the charges that appear in the action when we include this particle. In other words, including a particle with charges (n_I^e, n_I^m) amounts to consider the action

$$S = S_{4d,\,U(1)} + \frac{4\pi}{l_s} Q_I^E \int_W A^I + \frac{4\pi}{l_s} \tilde{Q}_J^M \int_W \tilde{A}^J \tag{4.38}$$

where W is the worldline of the 4d particle, $d\tilde{A}^I = *_4 dA^I$ and $\tilde{Q}_J^M = -\mathrm{Re}\ f_{JI} Q_I^M$.

The basic strategy to determine the open-closed $U(1)$ mixing will be to consider the 4d magnetic monopoles of D6-brane $U(1)$'s and compute their electric charges Q_i^E under a closed string $U(1)_i$. Given the above facts, such electric charge should be proportional to the imaginary part of the gauge kinetic function computed in previous sections. Moreover, by looking to the couplings of these monopoles to other closed string $U(1)$ magnetic generators $\tilde{A}_1^i$ we will also be able to obtain the real part of the mixing. As before we will first consider the case of parallel D6-branes without O6-planes and subsequently include the orientifold projection.

Considering again the system of two homotopic D6-branes wrapping π_a and π_b, the 4d monopole with unit charge under $U(1)_{a-b} = \frac{1}{2}\left[U(1)_a - U(1)_b\right]$ is given by a D4-brane wrapping $W \times \Sigma$, where W is the worldline of the monopole in 4d and $\partial\Sigma = \pi_a - \pi_b$. In order to compute the electric charge under the closed string $U(1)$'s we can dimensionally reduce the D4-brane CS action to obtain

$$S_{CS}^{D4} \supset \mu_4 \int_{W\times\Sigma} C_3 \wedge \mathcal{F}^{D4} = \mu_4 \int_{W\times\Sigma} A_1^i \wedge \omega_i \wedge \mathcal{F}^{D4} = \frac{4\pi}{l_s} Q_i^E \int_W A_1^i \tag{4.39}$$

where the electric charges are given by

$$Q_i^E = \frac{1}{2l_s^4} \int_\Sigma \mathcal{F}^{D4} \wedge \omega_i. \tag{4.40}$$

This term is precisely (minus) the imaginary part of the mixing in (4.30). In particular, the field strength in $\mathcal{F}^{D4} = B_2 + \frac{l_s^2}{2\pi} F_2^{D4}$ is such that $F_2^{D4}|_{\pi_\alpha} = F_2^\alpha$, as the monopole must interpolate between the two D6-brane configurations. More precisely, F_2^{D4} is the curvature of a line bundle on Σ such that on its boundary $\partial\Sigma$ it reduces to the line bundles on the corresponding D6-brane. Such line bundle is nothing but the Wilson lines A_1^α, and so one recovers (4.31) by simply identifying $\tilde{F}_2^{(a-b)}$ with F_2^{D4}. Notice that (similarly to the 4-chain Σ) there are many F_2^{D4} that have the appropriate boundary conditions. As we will see in the next section the line bundle extension F_2^{D4} appears naturally in the context of generalised complex geometry.

Let us now consider the coupling of this monopole to the 4d dual vector boson $\tilde{A}_1^i$ that appears in the expansion (4.24) of the RR potential C_5. Looking at the appropriate term on the D4-brane CS action we obtain

$$S_{CS}^{D4} \supset \mu_4 \int_{W\times\Sigma} C_5 = -\mu_4 \int_{W\times\Sigma} \tilde{A}_1^i \wedge J \wedge \omega_i = -\frac{4\pi}{l_s} \mathrm{Re}\ f_{i(a-b)} \int_W \tilde{A}_1^i \tag{4.41}$$

where we have again used that $*_6\omega_i = b_i J^2 - J \wedge \omega_i$ and assumed that $b_i = 0$, which will be automatically satisfied in the orientifold case. We then have that the real part of the mixing is given by

$$\mathrm{Re}\ f_{i(a-b)} = \frac{1}{2l_s^4} \int_\Sigma J \wedge \omega_i \tag{4.42}$$

which as expected reproduces (4.30). Alternatively, we could have interchanged the rôle of the dual vector bosons $A_1^i \leftrightarrow \tilde{A}_1^i$ and applied the Witten effect to obtain this result.

Notice that in this derivation we do not need to assume that the two 3-cycles π_a and π_b are homotopic, nor that they relatively close to a reference 3-cycle π. The only requirement is that they are homologous so the Stückelberg mass vanishes and a 4-chain Σ exists. This then provides an alternative way to derive the expression (4.30) from first principles without having to perform the lift to M-theory. Finally, it is straightforward to extend this derivation to the orientifold case and again obtain (4.30), except that now the 4-chain Σ and the field strength F_2^{D4} should connect the 3-cycles π_α and π_α^*.

General case

It is clear how to generalise these results to arbitrary D6-brane configurations. Indeed, let us consider K stacks of D6-branes, each stack containing N_α D6-branes wrapped on π_α and their corresponding orientifold images on π_α^*, and such that the RR tadpole condition (4.6) is satisfied. We will find a massless $U(1)_X$ for each linear combination

$$\pi_X = \sum_{\alpha=1}^{K} n_{X\alpha} N_\alpha \pi_\alpha, \qquad n_{X\alpha} \in \mathbb{Z} \tag{4.43}$$

such that $[\pi_X] - [\pi_X^*]$ is trivial in $H_3(\mathcal{M}_6, \mathbb{R})$.[9] Thus, the number of massless $U(1)$'s is given by $K - r$ where r is dimension of the vector subspace generated by $[\pi_\alpha] - [\pi_\alpha^*]$ within $H_3^-(\mathcal{M}_6, \mathbb{Z})$. In order to fix the normalisation we pick a basis of $U(1)$'s given by

$$\hat{U}(1)_\alpha = \frac{1}{L_\alpha} \sum_{\beta=1}^{K} n_{\alpha\beta} U(1)_\beta \tag{4.44}$$

with $n_{\alpha\beta} \in \mathbb{Z}$ such that g.c.d$(n_{\alpha 1}, n_{\alpha 2}, \ldots, n_{lK}) = 1$ for all $\alpha = 1, \ldots, K$ and orthogonal, namely $n_{\alpha\gamma} n_{\gamma\beta} = L_\alpha \delta_{\alpha\beta}$.

For a massless $U(1)_X$ we can associate the formal linear combination of 3-cycles (4.43), and we know that there exists a 4-chain Σ_X such that $\partial\Sigma_X = \pi_X - \pi_X^*$. Wrapping a D4-brane on Σ_X corresponds to considering a 4d magnetic monopole of $\hat{U}(1)_X$ which, due to the normalisation (4.44), has magnetic charge $n_X^m = 1$. Dimensionally reducing the CS action for such monopole we will find its charges with respect to the closed string $U(1)$'s from where we can read off the kinetic mixing, namely

$$f_{iX} = \frac{1}{2\, l_s^4} \int_{\Sigma_X} (J - i\mathcal{F}^{D4}) \wedge \omega_i \tag{4.45}$$

where the integral is evaluated in the covering space. This expression is slightly subtle in the sense that it may depend on some discrete choices related to the pair $(\Sigma_X, \mathcal{F}^{D4})$. Such subtleties can be easily removed after a proper understanding of the space of monopoles of the compactification, as we discuss in the following.

9 If $[\pi_X] - [\pi_X^*]$ is trivial in $H_3(\mathcal{M}_6, \mathbb{R})$ but not in $H_3(\mathcal{M}_6, \mathbb{Z})$ then for this $U(1)$ to be massless it must have a component of RR $U(1)$ [111]. Throughout this section we assume that Tor $H_3(\mathcal{M}_6, \mathbb{Z}) = 0$ so that this possibility is not realised, but we will consider it again in section 4.1.2.

Monopoles and relative homology

Besides the general formula (4.45) for the gauge kinetic mixing between open and closed string $U(1)$'s, the previous discussion gives us an overall picture of the set of monopoles that appear in type IIA compactifications with D6-branes. On the one hand, monopoles charged under closed string $U(1)$'s are classified by D4-branes wrapping orientifold-odd 4-cycles, or in other words by the homology group $H_4^-(\mathcal{M}_6, \mathbb{Z})$. On the other hand, open string $U(1)$ monopoles are classified by D4-branes wrapping odd 4-chains Σ_X ending on the D6-branes 3-cycles π_α and their orientifold images π_α^*, whose formal union we will denote as π_{D6}. The appropriate homology group that classifies such 4-chains is the relative homology group $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$, which includes $H_4^-(\mathcal{M}_6, \mathbb{Z})$ as a subgroup. In fact, we can identify $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ as the lattice of integral $U(1)$ magnetic charges, that contains not only monopoles charged under open string $U(1)$'s and closed string $U(1)$'s, but also bound states of those.

Indeed, notice that the formula (4.45) is slightly ambiguous, in the sense that we can have two different 4-chains Σ_X and Σ'_X with the same boundary, and so the expression for the rhs integral could be different for Σ_X and Σ'_X. Let us temporarily simplify this formula by setting $\mathcal{F}^{D4} = B$. Then, if these two chains differ by a trivial 4-cycle (that is if they belong to the same class of $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$) then we have that $\int_{\Sigma_X} J_c \wedge \omega_i = \int_{\Sigma'_X} J_c \wedge \omega_i$ and so we get the same result for the rhs of (4.45) independently of which chain we choose. If on the other hand Σ_X and Σ'_X differ by a 4-cycle Λ_4^j such that $[\Lambda_4^j] \in H_4^-(\mathcal{M}_6, \mathbb{Z})$, then the two integrals will differ by

$$-\frac{i}{2l_s^4} \int_{\Lambda_4^j} J_c \wedge \omega_i \;=\; -\frac{i}{2l_s^4} \int_{\mathcal{M}_6} J_c \wedge \omega_i \wedge \omega_j \;=\; f_{ij} \tag{4.46}$$

where ω_j is Poincaré dual to Λ_4^j and represents a closed string $U(1)_j$, and f_{ij} is the kinetic mixing (4.10) between $U(1)_i$ and $U(1)_j$. The correct way to interprets this fact is that, if a D4-brane wrapping Σ_X corresponds to a 4d monopole with unit charge under $U(1)_X$, then a D4-brane wrapping Σ'_X has unit charge under $U(1)_X$ but also under $U(1)_j$, and so it is equivalent to a bound state of open and closed string $U(1)$ monopoles. Therefore, via the Witten effect it will obtain a electric and magnetic charge under $U(1)_i$ which is not given by the kinetic mixing f_{iX}, but rather by the sum of mixings $f_{iX} + f_{ij}$, see eq.(4.37). In general, it is easy to see that the integral $\int_{\Sigma_X} J_c \wedge \omega_i$ will only depend on the homology class $[\Sigma_X] \in H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ which as stated before is nothing but the lattice of integral $U(1)$ magnetic charges of the 4d effective theory. Hence, in order to properly use eq.(4.45) we first need to take a basis for $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ and identify those 4d monopoles that have unit charge under $U(1)_X$ but no integer charge under the closed string $U(1)$'s, and then apply eq.(4.45) with a 4-chain Σ_X in the corresponding relative homology class.

Let us now restore the full dependence of $\mathcal{F}^{D4}$ in (4.45) and let us see which further source of ambiguity that gives. Even if we keep Σ_X within the same relative homology class there are infinite discrete choices of F_2^{D4} such that the appropriate boundary conditions

$$\int_{\Sigma_X} F_2^{D4} \wedge \omega_i \;=\; \int_{\Sigma_X} \tilde{F}_2^X \wedge \omega_i \;\equiv\; \frac{1}{L_X} \sum_{\beta=1}^{K} n_{X\beta} \int_{\pi_\beta} A_1^\beta \wedge \omega_i \tag{4.47}$$

are satisfied. Indeed, let us consider the case where the 4-chain Σ_X contains a non-trivial 2-cycle Λ_2^j such that $[\Lambda_2^j]$ is also non-trivial in $H_2^+(\mathcal{M}_6, \mathbb{Z})$. By Poincaré duality on Σ_X, one may then consider a 2-form F_2^j on Σ_X which is the curvature of a vanishing line bundle on $\partial\Sigma_X$ and satisfying

$$\int_{\Sigma_X} F_2^j \wedge \gamma = \int_{\Lambda_2^j} \gamma \tag{4.48}$$

for any closed 2-form γ on $\mathcal{M}_6$. Then it is easy to see that if one takes $F_2^{D4} = [F_2^{D4}]_0 + nF_2^j$ with $[F_2^{D4}]_0$ satisfying (4.47), eq.(4.40) reads

$$Q_i^E = \frac{1}{4\pi l_s^2} \int_{\Sigma_X} \tilde{F}_2^X \wedge \omega_i + n\,\delta^{ij} \tag{4.49}$$

where we have assumed that $[\Lambda_2^j]$ is Poincaré dual to $[\tilde{\omega}^j]$ and so $\int_{\Lambda_2^j} \omega_i = \delta^{ij}$. This result is easily interpreted as the fact that the piece of flux $n\,F_2^j$ induces the charge of n D2-branes wrapping Λ_2^j on the D4-brane on Σ_X, so this D4-brane is actually a 4d particle with unit magnetic charge under $\hat{U}(1)_X$ and electric charge n under $U(1)_j$. Therefore comparing (4.37) and (4.49) one concludes that $\mathrm{Im}\, f_{iX} = -\frac{1}{2l_s^4} \int_{\Sigma_X} \left(B + \frac{l_s^2}{2\pi}\tilde{F}_2^X\right) \wedge \omega_i$ and that in eq.(4.45) F_2^{D4} must not induce any non-trivial D2-brane charge.

To summarise, we find that the set of monopoles in a type IIA orientifold compactification is classified by the relative homology group $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$, where π_6 is the formal sum of the D6-brane locations. The dimension of this lattice is the total number of massless $U(1)$'s, open and closed, of the compactification, and so in some sense the space of 4d $U(1)$'s should also be classified by this same relative homology group. This is rather natural if we interpret the whole discussion above from the viewpoint of M-theory. Indeed, lifting the type IIA compactification to M-theory in a 7-dimensional manifold $\mathcal{M}_7$ we have that $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ lifts to the homology group $H_5(\mathcal{M}_7, \mathbb{Z})$, and that the $U(1)$ magnetic monopoles become M5-branes wrapping non-trivial 5-cycles in $\mathcal{M}_7$. The $U(1)$'s themselves are classified by harmonic 2-forms in $\mathcal{M}_7$, hence (assuming no torsion in homology) by the Poincaré dual group $H^2(\mathcal{M}_7, \mathbb{Z})$. Finally, in M-theory the kinetic mixing between $U(1)$'s is given by the simple formula

$$f_{\alpha\beta} = -\frac{2\pi i}{l_M^9} M^I \int_{\mathcal{M}_7} \phi_I \wedge \omega_\alpha \wedge \omega_b \tag{4.50}$$

where l_M is the M-theory characteristic length and ϕ_I, $I = 1, \ldots, b_3(\mathcal{M}_7)$ runs over the harmonic 3-forms of $\mathcal{M}_7$, and M^I are the complex moduli associated to them. Following [111], from this formula one can reproduce not only the gauge kinetic mixing between type IIA closed string $U(1)$'s (4.10), but also the mixing between open and closed string $U(1)$'s (4.45). In the following we will make this last connection more precise, by characterising open string $U(1)$'s by 2-forms on $\mathcal{M}_6$, that instead of representatives of $H^2(\mathcal{M}_6, \mathbb{Z})$ belong to the cohomology $H^2(\mathcal{M}_6 - \pi_{D6}, \mathbb{Z})$, related by Lefschetz duality to the group $H_4(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ classifying the monopoles.

Open-closed $U(1)$ mixing and linear equivalence

The concept of linear equivalence is usually formulated to relate different p-cycles π_p of a d dimensional manifold $\mathcal{M}_d$, being stronger than equivalence in homology. While typically one applies this concept to divisor submanifolds of a complex manifold, one may extend such definition to more general cases following [121] or the discussion in Appendix D.

Indeed, let us consider two p-cycles π_p^a and π_p^b that live in the same homology class of $\mathcal{M}_d$. One can then write down the differential equation

$$d\varpi^{(a-b)} = \delta_{d-p}(\pi_p^a) - \delta_{d-p}(\pi_p^b) \tag{4.51}$$

where $\delta_{p-d}(\pi_p^\alpha)$ is a bump $(d-p)$-form localised on top of the p-cycle π_p^α and transverse to it. Because $[\pi_p^a] = [\pi_p^b]$ we know that ϖ is globally well-defined $(d-p-1)$-form. While there are in principle many solutions to this equation, one may in addition require that

$$d^*\varpi^{(a-b)} = 0 \qquad \text{and} \qquad \int_{\Lambda_{d-p-1}} \varpi^{(a-b)} \in \mathbb{Z} \tag{4.52}$$

which fixes ϖ up to an harmonic representative of the cohomology group $H^{d-p-1}(\mathcal{M}_d, \mathbb{Z})$. From a mathematical viewpoint, this allows to identify ϖ as the connection of a gerbe.[10] From a physical viewpoint we will see that they are natural conditions when we want to relate ϖ with an open string $U(1)$.

Given (4.51) and (4.52), it is easy to see that ϖ admits the following global Hodge decomposition

$$\varpi^{(a-b)} = \omega + d^*H \tag{4.53}$$

where ω is a harmonic $(d-p-1)$-form and H_{d-p} is a globally well-defined $(d-p)$ form. We then say that the two p-cycles π_p^a and π_p^b are *linearly equivalent* if $[\omega] \in H^{d-p-1}(\mathcal{M}_6, \mathbb{Z})$, or in other words if the two components of ϖ are separately quantised.

While the above definition is rather abstract, one may detect linear equivalence in a rather simple way as follows. Given the two p-cycles π_p^a and π_p^b let us construct a $(p+1)$-chain $\Sigma^{(a-b)}$ such that $\partial\Sigma^{(a-b)} = \pi_p^a - \pi_p^b$. Then, from the discussion in Appendix D one can see that

$$\int_{\Sigma^{(a-b)}} \tilde{\omega}_{p+1} = \int_{\mathcal{M}_6} \tilde{\omega}_{p+1} \wedge \varpi^{(a-b)} \mod \mathbb{Z} \tag{4.54}$$

for any closed $(p+1)$-form $\tilde{\omega}_{p+1}$ with integer cohomology class $[\tilde{\omega}_{p+1}] \in H^{p+1}(\mathcal{M}_6, \mathbb{Z})$. Moreover, if $\tilde{\omega}_{p+1}$ is harmonic we have that we can replace $\varpi^{(a-b)} \to \omega$ in the rhs of (4.54). Hence if we take $\tilde{\omega}_{p+1}$ to be harmonic and with integer homology class we have that π_p^a and π_p^b are linearly equivalent if and only if

$$\int_{\Sigma^{(a-b)}} \tilde{\omega}_{p+1} \in \mathbb{Z} \qquad \forall\, \Sigma^{(a-b)} \text{ such that } \partial\Sigma^{(a-b)} = \pi_p^a - \pi_p^b \tag{4.55}$$

Actually, this criterion for linear equivalence can be refined if we restrict the class of chains that enter into eq.(4.55), and such refinement will allow to relate the above definitions with the computation of open-closed $U(1)$ kinetic mixing. For

10 One may actually drop the condition $d^*\varpi^{(a-b)} = 0$, which amounts to take a harmonic representative of the group $H^{d-p-1}(\mathcal{M}_d - \{\pi_p^a \cup \pi_p^b\}, \mathbb{Z})$. We will however maintain it as it simplifies the discussion.

concreteness, let us consider a simple case of interest discussed in the previous sections. Namely, we consider two D6-branes wrapping two homologous 3-cycles π_3^a and π_3^b of $\mathcal{M}_6$. One can then write the differential equation

$$d\varpi_2^{(a-b)} = \delta_3(\pi_3^a) - \delta_3(\pi_3^b) \tag{4.56}$$

which is nothing but (4.51) for the particular case $d = 6$, $p = 3$. Requiring that ϖ_2 is co-closed and quantised as in (4.52) one obtains

$$\varpi_2^{(a-b)} = c^j\omega_j + d^*H \tag{4.57}$$

where $\{\omega_j\}$ is a basis of harmonic 2-forms with integer cohomology class and $c^j \in \mathbb{R}$. This definition of ϖ_2 only fixes the value of c^j mod $\mathbb{Z}$, and so one can always define ϖ_2 such that $c^j \in [0,1)$, $\forall j$. With this choice there is a 4-chain $\Sigma^{(a-b)}$ such that

$$\int_{\Sigma^{(a-b)}} \tilde{\omega} = \int_{\mathcal{M}_6} \tilde{\omega} \wedge \varpi_2^{(a-b)} \tag{4.58}$$

for any closed 4-form $\tilde{\omega}$, and without the need of the mod $\mathbb{Z}$ that appears in eq.(4.54).

We can also see (4.58) as a consequence of Lefschetz duality between the groups $H_4(\mathcal{M}_6, \pi_3^a \cup \pi_3^b, \mathbb{Z})$ and $H^2(\mathcal{M}_6 - \{\pi_3^a \cup \pi_3^b\}, \mathbb{Z})$. Indeed, the 2-forms (4.57) are harmonic representatives of the cohomology group $H^2(\mathcal{M}_6 - \{\pi_3^a \cup \pi_3^b\}, \mathbb{Z})$, which contains $H^2(\mathcal{M}_6, \mathbb{Z})$. Changing the value of the coefficients c^j by an integer number amounts to change the cohomology class by an element of $H^2(\mathcal{M}_6, \mathbb{Z})$, so choosing $c^j \in [0,1)$, $\forall j$ means choosing a particular class in $H^2(\mathcal{M}_6 - \{\pi_3^a \cup \pi_3^b\}, \mathbb{Z})$. Then, restricting the 4-chain Σ^{a-b} to the dual class in $H_4(\mathcal{M}_6, \pi_3^a \cup \pi_3^b, \mathbb{Z})$ allows to write down (4.58) without any mod $\mathbb{Z}$ ambiguity.

As we saw when discussing monopoles, restricting the 4-chain Σ to a particular relative homology class is also needed when computing the open-closed kinetic mixing from the chain integral (4.45). In fact one can see that, if we ignore the contribution of the Wilson lines, in the present case such chain integral reads

$$-\frac{i}{2l_s^4}\int_{\Sigma^{(a-b)}} J_c \wedge \omega_i = -\frac{i}{2l_s^4}\int_{\mathcal{M}_6} J_c \wedge \omega_i \wedge \varpi_2^{(a-b)} = -\frac{i}{2}\mathcal{K}_{ij\hat{k}}T^{\hat{k}}c^j = f_{ij}c^j \tag{4.59}$$

Finally, for $c^j \in [0,1)$ linear equivalence between π_3^a and π_3^b amounts to require that $c^j = 0$, $\forall j$. That is, the equivalence $\pi_3^a \sim \pi_3^b$ corresponds to the vanishing of the kinetic mixing $f_{i(a-b)}$ of $\frac{1}{2}[U(1)_a - U(1)_b]$ with any $U(1)_i$ from the closed string sector.

All this discussion can be easily generalised for the case where we have more than two D6-branes wrapping 3-cycles the same homology class. For each massless $U(1)_X$ of the open string sector given by a linear combination of 3-cycles (4.43) such that $[\pi_X] = 0$ we can define the 2-form ϖ_X by

$$d\varpi_X = \sum_\alpha n_{X\alpha}N_\alpha\delta_3(\pi_\alpha) \tag{4.60}$$

and such that ϖ_2 co-closed and has integer relative homology class. This fixes ϖ_X to be of the form (4.57), where again we impose that $c^j \in [0,1)$. Then we find that the kinetic mixing between $U(1)_X$ and a $U(1)_i$ of the closed string sector is given by

$$f_{iX} = -\frac{i}{2l_s^4}\int_{\mathcal{M}_6} J_c \wedge \omega_i \wedge \varpi_X \tag{4.61}$$

which vanishes if the combination π_X is linearly trivial, since then $c^j = 0, \forall j$. Notice that (4.61) reduces to (4.45) if we replace $\int_{\mathcal{M}_6} \varpi_X \wedge \to \int_{\Sigma_X}$, and then take $\mathcal{F}^{D4} = B|_{\Sigma_X}$ to ignore the Wilson line dependence. Finally, one can generalise this expression to the case of orientifold compactifications by modifying (4.60) in the obvious way and by taking ϖ_X with the appropriate orientifold parity, or more precisely by taking into account that $[\varpi_X] \in H^2_+(\mathcal{M}_6 - \pi_{D6}, \mathbb{Z})$.

To summarise, in the absence of Wilson lines, the vanishing of the gauge kinetic function f_{iX} that mixes open and closed string $U(1)$'s corresponds to the linear equivalence of the two or more cycles that define the open string $U(1)$. The actual expression for this gauge kinetic mixing can either be expressed via a chain integral as in (4.45) or as an integral all over the compactification manifold as in (4.61). The later is suggestive in the sense that it resembles the well-known expression for the closed-closed kinetic mixing (4.10), with the difference that now the open string $U(1)$ is represented by ϖ_X, which is a harmonic representative of $H^2_+(\mathcal{M}_6 - \pi_{D6}, \mathbb{Z})$.

As we have seen, the equality between linear equivalence and vanishing kinetic mixing is no longer true when we take the Wilson line dependence into account, and one needs to define a generalised notion of linear equivalence. Nevertheless, we will see that even in this case one is able to express the open-closed gauge kinetic mixing as an integral of the form (4.61) over the whole compactification manifold. In order to do that, however, we first need to discuss the rôle played by relative cohomology groups in the computation of gauge kinetic functions.

Wilson lines and relative cohomology

Generically, a D6-brane Wilson lines correspond to harmonic one-forms ζ on a 3-cycle π whose one-cycles are trivial in the homology of the ambient space $\mathcal{M}_6$. Because of that, we cannot relate ζ to any closed one-form of $\mathcal{M}_6$. Nevertheless, the Poincaré dual 2-cycle ρ of ζ in π can be nontrivial in $\mathcal{M}_6$ and, if this is the case, such Wilson line will enter in the gauge kinetic functions f_{iX} mixing open and closed string $U(1)$'s. One way to detect this is by looking at the field strength $\mathcal{F}^{D4}$ of the appropriate open string monopole, which will depend on this Wilson line and so will the kinetic mixing via (4.45). Physically, due to the Wilson line ζ one must induce a D2-brane charge along ρ on the D4-brane which is the 4d open string monopole, and this translates into a open-closed θ angle via the Witten effect.

The appropriate way to incorporate this effect into a formula of the form (4.61) is via the use of relative cohomology groups. Let in particular look at the relative cohomology $H^n(\mathcal{M}_6, \pi_{D6})$, where for illustrative purposes we can take $\pi_{D6} = \pi_3^a \cup \pi_3^b$ as the union of two homologous 3-cycles. The groups $H^n(\mathcal{M}_6, \pi_{D6})$ are obtained by considering pairs of forms such that

$$(\alpha_n, \beta_{n-1}) \quad \in \quad \Omega^n(\mathcal{M}_6) \times \Omega^{n-1}(\pi_{D6}) \tag{4.62}$$

and constructing the cohomology in the standard way with the differential

$$d(\alpha_n, \beta_{n-1}) \;=\; (d\alpha_n, \iota^*_{\pi_{D6}}(\alpha_n) - d\beta_{n-1}) \tag{4.63}$$

It is then easy to see that any element of the form $(0, \beta_{n-1})$, with β_{n-1} a non-trivial closed form in π_{D6}, is also nontrivial in $H^n(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ if it cannot be written as the pull-back $\iota^*_{\pi_{D6}}(\gamma_{n-1})$ of a globally well-defined closed form γ_{n-1} of $\mathcal{M}_6$. For instance, for $\pi_{D6} = \pi_3^a \cup \pi_3^b$ with π_3^a, π_3^b homotopic 3-cycles with a harmonic form $\beta_{n-1}^{a,b}$ on each of them, a choice such that $\beta_{n-1}^a \neq \beta_{n-1}^b$ will always correspond to a non-trivial element of $H^n(\mathcal{M}_6, \pi_{D6})$, because there is no closed form γ_{n-1} of $\mathcal{M}_6$ that would give different integrals in the two corresponding $(n-1)$-cycles. In particular, we have that Wilson lines $\theta_a\zeta$ and $\theta_b\zeta$ with $\theta_a \neq \theta_b$ corresponds to a non-trivial element of $H^2(\mathcal{M}_6, \pi_{D6})$. Moreover, by using the definition (4.63) one can see that this Wilson line configuration is equivalent to $(\Upsilon_2, 0)$ where $\Upsilon_2 = d\Theta_1$ is an exact 2-form of $\mathcal{M}_6$ such that $\Theta_1|_{\pi_3^\alpha} = \theta_\alpha\zeta$, $\alpha = a, b$. Finally, it is easy to generalise this construction to the case that there is more than two D6-branes and several Wilson lines on each of them, the final result being that for each open string $U(1)_X$ there is a bulk 2-form $\Upsilon_2^X = d\Theta_1^X$ such that Θ_1^X restricts (up to an exact form) to the corresponding Wilson line on each of the 3-cycles π_3^α that enter into the linear combination (4.43).

Given this setup, it is easy to see that the Wilson line contribution to the gauge kinetic mixing can be written as

$$-\frac{i}{4\pi l_s^2}\int_{\mathcal{M}_6} \Upsilon_2^X \wedge \omega_i \wedge \varpi_X = \frac{i}{4\pi l_s^2}\int_{\mathcal{M}_6} \Theta_1^X \wedge \omega_i \wedge d\varpi_X = -\frac{i}{4\pi l_s^2}\sum_\alpha n_{X\alpha} N_\alpha \int_{\pi_\alpha} A_1^\alpha \wedge \omega_i \tag{4.64}$$

replacing the Wilson line integral (4.31) that defines $\tilde{F}_2^X$, by an integral over the whole manifold $\mathcal{M}_6$ involving the 2-form Υ_2^X. One possible way to interpret this is via the trading of the Wilson line background by a shift in the B-field background, as done in [129]. As a result, the total gauge kinetic mixing can be expressed via the equation (4.61) with the replacement $J_c \to J_c + \frac{l_s^2}{2\pi}\Upsilon_2^X$.

Summary

In this section we have computed the kinetic mixing between open and closed string $U(1)$'s in type IIA compactifications. Via the Witten effect we have obtained the expression

$$f_{iX} = \frac{1}{2\,l_s^4}\int_{\Sigma_X} (J - i\mathcal{F}^{D4}) \wedge \omega_i \tag{4.65}$$

for the mixing between a closed $U(1)_i$ and a open string $U(1)_X$. Here $[\omega_i] \in H^2_+(\mathcal{M}_6, \mathbb{Z})$ and $(\Sigma_X, \mathcal{F}^{D4})$ is a 4-chain and with gauge bundle on it, describing a open string $U(1)$ monopole made up of a D4-brane connecting the D6-branes of the compactification. Notice that this expression depends on very few, local data of the compactification, as opposed to the threshold corrections that induce a mixing between open string $U(1)$'s.

As a byproduct of our discussion we observed that, in the presence of massless open string $U(1)$'s, the lattice of 4d monopoles of a type IIA compactification is no longer given by $H_4^-(\mathcal{M}_6, \mathbb{Z})$, but rather by the relative homology group $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$. This group enlarges $H_4^-(\mathcal{M}_6, \mathbb{Z})$ by adding classes of 4-chains ending on combination of 3-cycles $\pi_{D6} = \cup_\alpha \pi_3^\alpha$ wrapped by D6-branes of the compactification. Replacing $H_4^-(\mathcal{M}_6, \mathbb{Z})$ by $H_4^-(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ is related, by Lefschetz duality, to $H^2_+(\mathcal{M}_6, \mathbb{Z}) \to H^2_+(\mathcal{M}_6 - \pi_{D6}, \mathbb{Z})$, which adds to the closed 2-forms of $\mathcal{M}_6$ those 2-forms that are closed only up to the D6-brane locations π_{D6}. Examples of these new forms are the 2-forms ϖ_X, defined by (4.60) and used to detect linear equivalence between two or more 3-cycles. These new 2-forms can be used to compute open-closed kinetic mixing via the following expression

$$f_{iX} = \frac{1}{2l_s^4}\int_{\mathcal{M}_6}(J - i\mathcal{F}) \wedge \omega_i \wedge \varpi_X \tag{4.66}$$

where $[\varpi_X]$ is a new element of the extension $H^2_+(\mathcal{M}_6 - \pi_{D6}, \mathbb{Z})$ that represents $U(1)_X$, and $\mathcal{F} = B + \frac{l_s^2}{2\pi}\Upsilon_2^X$ where $[\Upsilon_2^X] \in H^2_-(\mathcal{M}_6, \pi_{D6})$ is defined as in the last section. Finally, (4.66) reduces to (4.65) if we replace $\int_{\mathcal{M}_6}\varpi_X \wedge \ \to \ \int_{\Sigma_X}$, and take $\mathcal{F}^{D4} = \mathcal{F}|_{\Sigma_X}$.

Eq. (4.66) is quite suggestive from the viewpoint of M-theory, since by replacing

$$\begin{aligned} \mathcal{M}_6 &\to \hat{\mathcal{M}}_7 \\ J_c + F_2 &\to M^I\phi_I \\ \varpi_X &\to \omega_X \end{aligned} \tag{4.67}$$

with ω_X an harmonic 2-form on $\hat{\mathcal{M}}_7$ one recovers the M-theory expression (4.50) for the kinetic mixing of two $U(1)$'s. All these replacements are standard when lifting a type IIA compactification to M-theory except perhaps the last one, which suggests that a harmonic representative of $H^2_+(\mathcal{M}_6 - \pi_{D6}, \mathbb{Z})$ is related to an harmonic representative of $H^2(\hat{\mathcal{M}}_7, \mathbb{Z})$. This is related to the results of section 4.1.1, where in order to match monopole lattices we concluded that $H_4^-(\mathcal{M}_6, \pi, \mathbb{Z})$ should be identified with $H_5(\hat{\mathcal{M}}_7, \mathbb{Z})$.

4.1.2 *Linear equivalence of magnetised D-branes*

In our analysis of type IIA compactifications we have found that the kinetic mixing between open and closed string $U(1)$'s can be computed by means of a simple chain formula, whose physical meaning can be understood as the Witten effect applied to D-brane $U(1)$ monopoles. Moreover, we found that just like the set of monopoles is classified by a relative homology group, the set of $U(1)$'s is classified by a dual cohomology group. In particular, one is able to characterise the massless open string $U(1)$'s in terms of a pair of bulk 2-forms (ϖ, Υ_2) from which one can reproduce the chain formula for the kinetic mixing. Up to Wilson line contributions, this kinetic mixing will vanish when the set of D6-branes defining the $U(1)$ are linearly equivalent, or in other words when ϖ is orthogonal to any harmonic 2-form in the compactification manifold.

In this section we would like to extend this picture to type IIB compactifications. The novelty of these compactifications is that they contain magnetised branes, and so the objects to consider are bound state of 3, 5 and 7-branes. However, just like in the type IIA case, we will be able to describe open-closed kinetic mixing by using the Witten effect, and to characterise open string $U(1)$'s in terms of bulk forms (ϖ, Υ). Also, the fact that the open-closed mixing vanishes can be translated into a version of linear equivalence adapted to D-brane bound states. Finally, we will derive a supergravity-like formula to compute kinetic mixing between open and closed string $U(1)$'s, and apply it to F-theory GUT models with hypercharge flux breaking [10, 60].

Type IIB orientifolds with O3/O7-planes

Let us then consider type IIB string theory compactified on $\mathbb{R}^{1,3} \times \mathcal{M}_6$, with $\mathcal{M}_6$ a Calabi-Yau manifold, and mod it out by the orientifold action $\Omega_p(-1)^{F_L}\sigma$, where now σ is a holomorphic involution $\mathcal{M}_6$ such that

$$\sigma J = J \ , \qquad \sigma\Omega = -\Omega \tag{4.68}$$

The fixed loci of σ are points and/or complex 4-cycles of $\mathcal{M}_6$, where O3 and O7-planes are respectively located. The cancellation of RR tadpoles imposes that

$$\sum_\alpha N_\alpha([\pi_\alpha] + [\pi_\alpha^*]) - 8[\pi_O] = 0 \tag{4.69}$$

where π_α are the complex 4-cycles wrapped by D7-branes, π_α^* are their orientifold images, and π_O the divisors wrapped by the orientifold planes. Besides (4.69) one needs to impose that the total D5 and D3-brane charges of the compactification vanish.

Dimensionally reducing the 10d type IIB supergravity action one encounters a series of massless fields that arise from the closed string sector of the theory. As before, these are classified by the harmonic forms of $\mathcal{M}_6$ with a definite parity under the action of σ. We now take the following basis of harmonic forms with integer cohomology class

$$\begin{array}{lllll}
 & & \sigma\text{−even} & & \sigma\text{−odd} \\
2-\text{forms} & \omega_i & i = 1, \ldots, h_+^{1,1} & \omega_{\hat{i}} & \hat{i} = 1, \ldots, h_-^{1,1} \\
3-\text{forms} & \alpha_I & I = 0, \ldots, h_+^{1,2} & \tilde{\alpha}_{\hat{I}} & \hat{I} = 0, \ldots, h_-^{1,2} \\
 & \beta^I & I = 0, \ldots, h_+^{1,2} & \tilde{\beta}^{\hat{I}} & \hat{I} = 0, \ldots, h_-^{1,2} \\
4-\text{forms} & \tilde{\omega}^i & i = 1, \ldots, h_+^{1,1} & \tilde{\omega}^{\hat{i}} & \hat{i} = 1, \ldots, h_-^{1,1}
\end{array}$$

and with normalisation

$$\int_{\mathcal{M}_6} \omega_i \wedge \tilde{\omega}^j = l_s^6 \delta_i^j, \quad \int_{\mathcal{M}_6} \omega_{\hat{i}} \wedge \tilde{\omega}^{\hat{j}} = l_s^6 \delta_{\hat{i}}^{\hat{j}}, \quad \int_{\mathcal{M}_6} \alpha_I \wedge \beta^J = l_s^6 \delta_I^J, \quad \int_{\mathcal{M}_6} \tilde{\alpha}_{\hat{I}} \wedge \tilde{\beta}^{\hat{J}} = l_s^6 \delta_{\hat{I}}^{\hat{J}} \tag{4.70}$$

The next step is to expand the RR 4-form potential C_4 in the σ-even harmonic forms ω_i, α_I, β^I, $\tilde{\omega}^i$. Since the field strength of C_4 must satisfy the 10d self-duality condition $\hat{F}_5 = *_{10}\hat{F}_5$, it is convenient to define the following basis of complex harmonic 3-forms

$$\gamma_I \ = \ \alpha_I + i f_{IJ}\beta^J \tag{4.71}$$

where f_{IJ} are function of the complex structure moduli chosen so that γ_I is a $(2,1)$-form. Then it is easy to see that if we expand the 4-form potential as

$$C_4 = \sum_I (A_1^I \wedge \text{Re } \gamma_I - V_1^I \wedge \text{Im } \gamma_I) + \sum_i \left(C_2^i \wedge \omega_i - \text{Re}(T^i)\tilde{\omega}^i\right) \tag{4.72}$$

then 10d self-duality of $\hat{F}_5$ implies that $*_4 dA_1^I = dV_1^I$. We then obtain $h_+^{1,2}$ vector multiplets A_1^I and $h_+^{1,1}$ axions Re (T^i), the other 4d modes being dual degrees of freedom. Finally, using (4.70) it is easy to check that f_{IJ} is precisely the gauge kinetic function for the closed string $U(1)$'s [130–132]. Notice for instance that a D3-brane wrapping a 3-cycle in the Poincaré dual class of α_J will not only have magnetic charge under the closed string $U(1)$ generated by A_1^J, but also an electric charge under A_1^I proportional to Im f_{IJ}, as expected from the Witten effect.

Besides above spectrum there will be 4d massless fields arising from the open string sector of the compactification. In particular, the sector of a single D7-brane wrapping a holomorphic 4-cycle S_α of $\mathcal{M}_6$ is given by

$$A_1^\alpha = \frac{1}{l_s}(\pi A_1^{4d,\alpha} + a_\alpha^j A_j + c.c.) \tag{4.73}$$

$$\phi_\alpha = \Phi_\alpha + c.c. = \Phi_\alpha^m X_m + c.c. \tag{4.74}$$

where A_j are a basis of $H^{(0,1)}(S_\alpha)$ and X_m are holomorphic sections of the normal bundle of S_α, which are in one to one correspondence with the homology group $H^{(2,0)}(S_\alpha)$ by contraction with Ω. As a result, the Wilson line moduli a_α^j and the position moduli Φ_α^j are each complex 4d scalar fields by themselves.[11] One can easily generalise this spectrum to the case of a stack of N_α D7-branes, as well as to include the effect of the orientifold projection. We refer the reader to [98] for further details in this direction.

Separating two D7-branes

For simplicity let us first consider two D7-branes wrapping 4-cycles S_a and S_b of $\mathcal{M}_6$, and threaded respectively by worldvolume fluxes $\bar{F}_a$ and $\bar{F}_b$, leaving the action of the orientifold for later. Like with the D6-branes we assume that S_α, $\alpha = a, b$ lie in the same homology class and that they can both be obtained after normal deformations ϕ_α of a reference 4-cycle S. As before, we can reduce the CS action for these branes and read off the terms that yield Stückelberg masses for the open string $U(1)$'s as well as their kinetic mixing with the closed string sector. The relevant terms of the CS action now are

$$S_{CS}^\alpha \supset \mu_7 \int_{\mathbb{R}^{1,3}\times S_\alpha} P[C_6 \wedge \mathcal{F}_2^\alpha + \frac{1}{2} C_4 \wedge \mathcal{F}_2^\alpha \wedge \mathcal{F}_2^\alpha] \tag{4.75}$$

which upon dimensional reduction yield [98]

$$\begin{aligned} S_{CS}^\alpha \supset\ & \frac{\pi}{l_s^7}\int_{\mathbb{R}^{1,3}} \tilde{C}_2^i \wedge F_2^\alpha \int_{S_\alpha} \tilde{\omega}^i + \frac{1}{2l_s^7}\int_{\mathbb{R}^{1,3}} C_2^i \wedge F_2^\alpha \int_{S_\alpha} \bar{F}_2^\alpha \wedge \omega^i \\ +\ & \frac{1}{2l_s^6}\int_{\mathbb{R}^{1,3}} F_2^\alpha \wedge F_2^{RR,I} \int_{S_\alpha} \text{Re } (a_\alpha \wedge \gamma_I + \iota_{\Phi_\alpha}\gamma_I \wedge \bar{F}_2^\alpha) \\ -\ & \frac{1}{2l_s^6}\int_{\mathbb{R}^{1,3}} F_2^\alpha \wedge * F_2^{RR,I} \int_{S_\alpha} \text{Im } (a_\alpha \wedge \gamma_I + \iota_{\Phi_\alpha}\gamma_I \wedge \bar{F}_2^\alpha) \end{aligned} \tag{4.76}$$

where we have used

$$C_4 = C_2^i \wedge \omega^i + A_1^I \wedge \text{Re } \gamma_I - V_1^I \wedge \text{Im } \gamma_I + \dots \tag{4.77}$$

$$C_6 = \tilde{C}_2^i \wedge \tilde{\omega}^i + \dots \tag{4.78}$$

with ω^i, $\tilde{\omega}^i$ running over all (1,1), (2,2)-forms of $\mathcal{M}_6$. We have also used that $a_\alpha = a_\alpha^j A_j$ is a (0,1)-form and $\bar{F}_2^\alpha$, $\iota_{\Phi_\alpha}\gamma_I$ are (1,1)-forms of S_α. Finally, we omitted any B-field contribution since it can be simply recovered by replacing $\frac{l_s^2}{2\pi}F \to \mathcal{F}$.

11 We normalise the fields Φ_α as $\Phi_\alpha = \frac{y_\alpha}{l_s}$ where y_α are the transverse coordinates to the brane α.

The terms in the first line of (4.76) correspond to the Stückelberg mass for the open string $U(1)$ and the second line gives the kinetic mixing with the closed string sector. Considering both D7-branes a and b, the combination $U(1)_a + U(1)_b$ will be massive due to the first term in (4.76). In order to keep $U(1)_{(a-b)} \equiv \frac{1}{2}[U(1)_a - U(1)_b]$ massless we impose that $[\bar{F}_2^a] = [\bar{F}_2^b] \equiv [\bar{F}_2]$. Notice that the Stückelberg mass for $U(1)_{(a-b)}$ is proportional to

$$\int_S (\bar{F}_2^a - \bar{F}_2^b) \wedge \omega^i \tag{4.79}$$

and so setting both worldvolume flux equal on S prevents this $U(1)$ from getting a mass. There is a subtlety in this statement which is important for the case of F-theory GUTs. Namely, in order to keep $U(1)_{(a-b)}$ massless the class $[\bar{F}_2^a - \bar{F}_2^b]$ should only be zero as an element of $H^2(\mathcal{M}_6)$ and it could be non-trivial in $H^2(S)$ [133]. We will assume that the fluxes are the same also in S and leave the more involved case for the next section.

A computation similar to section 4.1.1 shows that the mixing between $U(1)_{(a-b)}$ and $U(1)_I$ is

$$f_{I(a-b)} = -\frac{i}{4\pi l_s^4}(a_a^j - a_b^j)\int_S A_j \wedge \gamma_I - \frac{i}{4\pi l_s^4}(\Phi_a^m - \Phi_b^m)\int_S \iota_{X_m}\gamma_I \wedge \bar{F}_2 \tag{4.80}$$

Comparing this expression to (4.30) and (4.45) one can guess what the mixing is in the case where the branes are not homotopic, namely

$$f_{I(a-b)} = -\frac{i}{2l_s^5}\int_\Gamma \gamma_I \wedge \tilde{\mathcal{F}} \tag{4.81}$$

where Γ is a 5-chain with $\partial\Gamma = S_a - S_b$ and $\tilde{\mathcal{F}}$ is a 2-form defined on Γ such that $d\tilde{\mathcal{F}} = 0$ and $\tilde{\mathcal{F}}|_{S_\alpha} = \bar{\mathcal{F}}_\alpha$. We implicitly include the contribution coming from the Wilson lines in the boundary $\partial\Gamma$ as in (4.31) (see also the comment below eq.(4.40)). Finally, notice that $\mathcal{F}$ also includes the contribution from the B-field, but that it vanishes in the case that $H = dB = 0$, since then B is an harmonic (1,1)-form and so $\gamma_I \wedge B \equiv 0$.

As before, we can also arrive at (4.81) by computing the electric charge of a 4d $U(1)_{(a-b)}$-monopole with respect to the closed string $U(1)$'s. Indeed, consider a D5-brane wrapped on $W \times \Gamma$ where W is a worldline in $\mathbb{R}^{1,3}$, and with worldvolume flux $\tilde{\mathcal{F}}$ along Γ. This corresponds to a monopole in 4d with unit magnetic charge under $U(1)_{(a-b)}$. The CS action for this objects has a piece of the form

$$S_{CS}^{D5} \supset \mu_5 \int_{W\times\Gamma} C_4 \wedge \tilde{\mathcal{F}} = \mu_5 \int_\Gamma \tilde{\mathcal{F}} \wedge \mathrm{Re}\ \gamma_I \int_W A_1^I - \mu_5 \int_\Gamma \tilde{\mathcal{F}} \wedge \mathrm{Im}\ \gamma_I \int_W V_1^I \tag{4.82}$$

so the induced electric and magnetic charges under $U(1)_I$ are

$$Q_I^E = \frac{1}{2l_s^5}\int_\Gamma \mathrm{Re}\ \gamma_I \wedge \tilde{\mathcal{F}} \qquad \tilde{Q}_I^M = -\frac{1}{2l_s^5}\int_\Gamma \mathrm{Im}\ \gamma_I \wedge \tilde{\mathcal{F}} \tag{4.83}$$

which reproduces (4.81) by virtue of the Witten effect.

The effect of the orientifold projection is also quite similar to the type IIA case. We have to take into account that ω^i and $\tilde{\omega}^i$ in (4.76) only run over σ-even forms of $\mathcal{M}_6$, and we need to include the orientifold image for every D7-brane. This leads to

$$f_{I(a-b)} = -\frac{i}{2l_s^5}\int_\Gamma \gamma_I \wedge \tilde{\mathcal{F}} \tag{4.84}$$

with Γ an orientifold-odd chain such that $\partial\Gamma = S_a - S_b - S_a^* + S_b^*$. Again, the contribution from the B-field vanishes in case that $H = 0$, which we will assume in the following.

Monopoles and generalised cycles

Let us now make a detour and rephrase the previous example in the language of generalised complex geometry. This will allow us to easily extend the above expression for the $U(1)$ mixing to more general situations, and in particular to the case of interest in F-theory GUTs to be analysed in section 4.1.2. In addition, this formalism properly treats D-branes with worldvolume fluxes, and so it allows to generalise the concept of linear equivalence of submanifolds [121] to bound states of D-branes, as well as to derive a supergravity-like formula for its kinetic mixing. Finally, generalised geometry has been shown to be the right framework to include the effect of background fluxes in both type IIA and type IIB $\mathcal{N} = 1$ compactifications, and hence it should be the appropriate tool to understand $U(1)$ kinetic mixing in the context of moduli stabilisation.

In generalised complex geometry a Dp-brane is a pair $(\Sigma, \mathcal{F})$ where Σ is a $(p+1)$-cycle and $\mathcal{F}$ is the worldvolume field strength together with the B-field, $\mathcal{F} = F + B$.[12] Thus, the two D7-branes of the last section correspond to $(S_a, \mathcal{F}_a)$ and $(S_b, \mathcal{F}_b)$ and the condition to have a massless $U(1)$ is that there should be a linear combination of the two that is trivial in generalised homology. Indeed, in the previous example we had

$$(S_a, \mathcal{F}_a) - (S_b, \mathcal{F}_b) = \hat{\partial}(\Gamma, \tilde{\mathcal{F}}) \tag{4.85}$$

where $\hat{\partial}$ is the generalised boundary operator defined in Appendix E, while Γ and $\tilde{\mathcal{F}}$ are the 5-chain and worldvolume flux defined below (4.81). In particular, $\tilde{\mathcal{F}}$ satisfies

$$d\tilde{\mathcal{F}} = 0, \qquad \tilde{\mathcal{F}}|_{S_a} = \mathcal{F}_a, \qquad \tilde{\mathcal{F}}|_{S_b} = \mathcal{F}_b. \tag{4.86}$$

where we have used that $H = 0$. In order to compute the kinetic mixing we look at the CS action of a D5-brane wrapped on $(W \times \Gamma, \mathcal{F})$ which can be written as

$$S_{CS} = \mu_5 \int_{W\times\Gamma} C \wedge e^{\tilde{\mathcal{F}}} \tag{4.87}$$

where C is the RR polyform. Since the closed string $U(1)$'s arise from C_4 it suffices to look at that term. Upon dimensional reduction it yields

$$\begin{aligned} S_{CS} \;&\supset\; \mu_5 \int_\Gamma \text{Re } \gamma_I \wedge e^{\tilde{\mathcal{F}}} \int_W A_1^I \;-\mu_5 \int_\Gamma \text{Im } \gamma_I \wedge e^{\tilde{\mathcal{F}}} \int_W V_1^I \\ &=\; \mu_5\, j_{(\Gamma,\tilde{\mathcal{F}})}(\text{Re } \gamma_I) \int_W A_1^I \;- \mu_5\, j_{(\Gamma,\tilde{\mathcal{F}})}(\text{Im } \gamma_I) \int_W V_1^I \end{aligned} \tag{4.88}$$

where we have introduced the current associated to the generalised chain $(\Gamma, \tilde{\mathcal{F}})$, dubbed $j_{(\Gamma,\tilde{\mathcal{F}})}$, and used the fact that the D5 flux $\tilde{\mathcal{F}}$ is magnetic so it has no component along the time direction W. Thus, the $U(1)_I$ electric and magnetic charges of the D5-monopole are

$$Q_E^I = \frac{1}{2l_s^5}\, j_{(\Gamma,\tilde{\mathcal{F}})}(\text{Re } \gamma_I) \qquad \tilde{Q}_M^I = -\frac{1}{2l_s^5}\, j_{(\Gamma,\tilde{\mathcal{F}})}(\text{Im } \gamma_I) \tag{4.89}$$

12 See Appendix E for some basic definitions of generalised submanifolds and their homology [134].

from where we can read off the kinetic mixing. Applying this formalism to this example is somewhat cumbersome but it shows that the only relevant information for the mixing is in the so-called generalised chain $(\Gamma, \tilde{\mathcal{F}})$, and that this procedure for computing the mixing can be generalised to more involved setups.

We now consider a case which is closely related to the hypercharge $U(1)_Y$ in F-theory GUTs and for which the formalism of generalised geometry turns out to be quite useful. Suppose again that we have two D7-branes wrapped on 4-cycles S_a and S_b, both homotopic to S. This implies that our D7-branes are described by the generalised cycles $(S_a, \mathcal{F}_a)$ and $(S_b, \mathcal{F}_b)$ such that $[S_a] = [S_b] \equiv [S]$. In addition we assume that $[\mathcal{F}_a] = [\mathcal{F}_b]$ in the cohomology of the total space but with $[\mathcal{F}_a]$ and $[\mathcal{F}_b]$ different in $H^2(S)$.[13] In that case we get a massless $U(1)$ so there must exist a generalised chain that connects the D7-branes but it cannot be the same one as before. Indeed, suppose it were $(\Gamma, \tilde{\mathcal{F}})$. The chain Γ connecting the submanifolds is topologically $S \times I$ where I is an interval with coordinate $t \in [0, 1]$ so Γ can be sliced in 4-cycles $S_t \simeq S$. Thus, the 2-form $\tilde{\mathcal{F}}$ on $S \times I$ defines a family of 2-forms $\tilde{\mathcal{F}}_t$ on S_t such that $\tilde{\mathcal{F}}_0 = \mathcal{F}_a$ and $\tilde{\mathcal{F}}_1 = \mathcal{F}_b$. Since $\tilde{\mathcal{F}}$ is continuous and $d\tilde{\mathcal{F}} = 0$ we have that $[\tilde{\mathcal{F}}_0] = [\tilde{\mathcal{F}}_1]$ in $H^2(S)$, contradicting our assumption.

This means that we have to look for another candidate to be the generalised chain associated to the massless $U(1)$. The strategy to find it is to realise that the quantised part of the worldvolume flux $\bar{F}_\alpha$ induces a D5-brane charge that can be related by Poincaré duality to a 2-cycle class $[\Pi_\alpha]$ of $H_2(S_\alpha, \mathbb{Z})$. If instead of two magnetised D7-branes we had two D7-branes on S_a, S_b with $\bar{F}_a = \bar{F}_b = 0$ and two D5-branes on the representative 2-cycles Π_a, Π_b, then we would have the same D-brane charges as in the magnetised system,[14] and it would be simple to connect them by means of a 5-chain Γ and a 3-chain Σ such that $\partial\Sigma = \Pi_a - \Pi_b$. So the only ingredient that we need is a generalised chain that interpolates between a magnetised D7-brane and a D7+D5-brane pair.

Such generalised chain is given by the following equation

$$(S_a, \mathcal{F}_a) = (S_a, 0) + (\Pi_a, 0) + \hat{\partial}\left[-(\Gamma_a, \tilde{\mathcal{F}}_a) + (\Gamma_a, \tilde{\mathcal{F}}_{\Pi_a})\right] \tag{4.90}$$

with $\Pi_a \subset S_a$ a 2-cycle such that $[\Pi_a] = \text{P.D.}\,[F_a]$, Γ_a a 5-chain with $\partial\Gamma_a = S'_a - S_a$ and

$$d\tilde{\mathcal{F}}_a = 0 \qquad \tilde{\mathcal{F}}_a|_{S_a} = \bar{F}_a \qquad \tilde{\mathcal{F}}_a|_{S'_a} = \bar{F}'_a \tag{4.91}$$

$$d\tilde{\mathcal{F}}_{\Pi_a} = \delta^{(3)}_{\Gamma_a}(\Pi_a) \qquad \tilde{\mathcal{F}}_{\Pi_a}|_{S_a} = 0 \qquad \tilde{\mathcal{F}}_{\Pi_a}|_{S'_a} = \bar{F}'_a \tag{4.92}$$

where for simplicity we have removed the presence of the B-field, which anyway will not appear in our final result. This looks rather messy but its interpretation as a physical process is simple. The term $-\hat{\partial}(\Gamma_a, \tilde{\mathcal{F}}_a)$ corresponds to moving the brane from S_a to a reference 4-cycle S'_a keeping the worldvolume flux fixed. The second term $\hat{\partial}(\Gamma_a, \tilde{\mathcal{F}}_{\Pi_a})$ is responsible for moving the brane back to S_a and removing the flux leaving the remnant $(S_a, 0) + (\Pi_a, 0)$ which represents a D7 on S_a and a D5 on Π_a each of them without fluxes. The fact that we have to move the brane back and forth is a technicality that regularises the differential equations, and eventually we will take the limit in which we do not move the brane at all (see the discussion in Appendix E.1).

13 We are also assuming that $\int_{S_a} \mathcal{F}_a \wedge \mathcal{F}_a = \int_{S_b} \mathcal{F}_b \wedge \mathcal{F}_b$.

14 We are ignoring induced D3-brane charges since they become irrelevant in the orientifold case.

Doing the same with the brane b we arrive at

$$(S_a, \mathcal{F}_a) - (S_b, \mathcal{F}_b) = \hat{\partial}\left[(\Gamma, 0) + (\Sigma, 0) - (\Gamma_a, \tilde{\mathcal{F}}_a) + (\Gamma_b, \tilde{\mathcal{F}}_b) + (\Gamma_a, \tilde{\mathcal{F}}_{\Pi_a}) - (\Gamma_b, \tilde{\mathcal{F}}_{\Pi_b})\right] \tag{4.93}$$

with Σ a 3-chain such that $\partial\Sigma = \Pi_a - \Pi_b$. This equation describes a process in which the worldvolume fluxes are turned into D5-branes at both S_a and S_b, and these are connected to each other by means of the 3-chain Σ, while the D7s are connected via Γ. Thus, we have found the generalised chain $(\mathfrak{S}, \mathfrak{F})$ associated with the open string massless $U(1)$.

According to our previous discussion the kinetic mixing with the closed string $U(1)_I$ will be proportional to $j_{(\mathfrak{S},\mathfrak{F})}(\gamma_I)$. One still needs to show that the result is independent of the arbitrary choices made to define $(\mathfrak{S}, \mathfrak{F})$. We relegate the proof to Appendix E.1, in which we also derive the following more convenient expression

$$j_{(\mathfrak{S},\mathfrak{F})}(\gamma_I) = \int_\Sigma \gamma_I + \frac{1}{2\pi l_s}\int_{S_a} \gamma_I \wedge A_{\Pi_a} - \frac{1}{2\pi l_s}\int_{S_b} \gamma_I \wedge A_{\Pi_b}. \tag{4.94}$$

with $dA_{\Pi_a} = 2\pi l_s[\mathcal{F}_a - \delta^2_{S_a}(\Pi_a)]$ and $dA_{\Pi_b} = 2\pi l_s[\mathcal{F}_b - \delta^2_{S_b}(\Pi_b)]$. Again, one can show that this expression is independent of the choice of Π_a and Π_b. Notice that these equations do not fix the harmonic piece of A_{Π_a} and A_{Π_b}. In order to match the result obtained from dimensional reduction we take them to be the Wilson lines in each D7-brane. Introducing the normalisation factor we finally find

$$f_{I(a-b)} = -\frac{i}{2l_s^3}\left[\int_\Sigma \gamma_I + \frac{1}{2\pi l_s}\left(\int_{S_a} \gamma_I \wedge A_{\Pi_a} - \int_{S_b} \gamma_I \wedge A_{\Pi_b}\right)\right]. \tag{4.95}$$

In Appendix E.1 we show that this expression reduces to (4.81) when $[\bar{F}_a] = [\bar{F}_b] \in H^2(S)$.[15] Finally we can easily extend this analysis to the orientifold case, where we have

$$f_{I(a-b)} = -\frac{i}{4l_s^3}\left[\int_\Sigma \gamma_I + \frac{1}{2\pi l_s}\left(\int_{S_a} \gamma_I \wedge A_{\Pi_a} - \int_{S_a^*} \gamma_I \wedge A_{\Pi_a^*} - \int_{S_b} \gamma_I \wedge A_{\Pi_b} + \int_{S_b^*} \gamma_I \wedge A_{\Pi_b^*}\right)\right] \tag{4.96}$$

with Σ a 3-chain such that $\partial\Sigma = \Pi_a - \Pi_b - \Pi_a^* + \Pi_b^*$.

General case and $U(1)$ mixing from supergravity

We can generalise these results to arbitrary configurations of parallel D7-branes with fluxes. Consider K stacks of D7-branes wrapping divisors S_α and carrying magnetic fluxes F_α so we may associate a generalised submanifold $(S_a, \mathcal{F}_\alpha)$ to each of them. We find a massless $U(1)_X$ for every linear combination

$$(S, \mathcal{F})_X = \sum_{\alpha=1}^{K} n_{X\alpha}(S_\alpha, \mathcal{F}_\alpha) \tag{4.97}$$

15 The dependence on A_{Π_α} was overlooked in the expression for the mixing in Appendix B of [111].

such that $(S, \mathcal{F})^-_X \equiv \frac{1}{2}[(S, \mathcal{F})_X - (S, \mathcal{F})^*_X]$ is trivial in generalised homology. Thus, we may associate a chain $(\mathfrak{S}, \mathfrak{F})_X$ such that

$$\hat{\partial}(\mathfrak{S}, \mathfrak{F})_X = (S, \mathcal{F})^-_X. \tag{4.98}$$

Notice that, as our previous example shows explicitly, this chain may be the linear combination of different terms, namely

$$(\mathfrak{S}, \mathfrak{F})_X = \sum_i (\Sigma^{(i)}, \mathcal{F}^{(i)})_X, \tag{4.99}$$

where the $\Sigma^{(i)}_X$ are not necessarily five-dimensional. Following what we did in the type IIA case, we fix the normalisation by taking the following basis of open string $U(1)$'s

$$\hat{U}(1)_\alpha = \frac{1}{L_\alpha} \sum_{\beta=1}^{K} n_{\alpha\beta} U(1)_\beta \tag{4.100}$$

with $n_{\alpha\beta}$ and L_α as in (4.43).

In order to compute the kinetic mixing of a massless $\hat{U}(1)_X$ with the closed string sector we dimensionally reduce the action of a magnetic monopole of $\hat{U}(1)_X$ and look at the relevant couplings in 4d. In the simplest cases the monopoles are given by magnetised D5-branes wrapped on $W \times \Sigma$, with Σ a 5-chain that connects the different D7-branes. However, in the general case these are given by wrapping bound states of different Dp-branes, dubbed D-brane networks, on $W \times (\mathfrak{S}, \mathfrak{F})_X$. By looking at the CS action of such network we find that the kinetic mixing with a closed string $U(1)_I$ is

$$f_{XI} = -\frac{i}{2l_s^4} j_{(\mathfrak{S}, \mathfrak{F})_X}(\gamma_I), \tag{4.101}$$

where γ_I is the harmonic 3-form that yields $U(1)_I$, and the associated current is given by

$$j_{(\mathfrak{S}, \mathfrak{F})_X} = \sum_i j_{(S^{(i)}, \mathcal{F}^{(i)})_X}. \tag{4.102}$$

The current $j_{(\mathfrak{S}, \mathfrak{F})_X}$ corresponds to the Lefschetz dual of the chain $(\mathfrak{S}, \mathfrak{F})_X$ so the equation (4.98) can be written in cohomology as

$$dj_{(\mathfrak{S}, \mathfrak{F})_X} = j_{(S, \mathcal{F})^-_X}, \tag{4.103}$$

which is analogous to (4.51). Since the current $j_{(S,\mathcal{F})^-_X}$ has support only on the D7-branes we find that $j_{(\mathfrak{S}, \mathfrak{F})_X}$ is an element of the generalised cohomology of $\mathcal{M}_6 - S_X$, $H^\bullet(\mathcal{M}_6 - S_X)$, with S_X the union of the divisors wrapped by the D7-branes and their orientifold images. Following the discussion for the D6-branes we define the polyform ϖ_X to be the representative of the class $[j_{(\mathfrak{S}, \mathfrak{F})_X}]$ that is coclosed and integral, namely

$$[\varpi_X] = [j_{(\mathfrak{S}, \mathfrak{F})_X}] \in H^\bullet(\mathcal{M}_6 - S_X), \quad d^*\varpi_X = 0, \quad \int_{\mathcal{M}_6} \langle \varpi_X, \Upsilon \rangle \in \mathbb{Z}, \tag{4.104}$$

where $\langle\cdot,\cdot\rangle$ denotes the Mukai pairing and Υ is any polyform in $H^\bullet(\mathcal{M}_6, S_X, \mathbb{Z})$, meaning that $d\Upsilon = \iota^*_{S_X}\Upsilon = 0$. The Mukai pairing is the natural generalisation of the intersection pairing used in (4.52). Indeed, we may write eq.(4.52) as $\int \varpi \wedge \Upsilon \in \mathbb{Z}$ where $\Upsilon \in H^4(\mathcal{M}_6, \pi_{D6}, \mathbb{Z})$ is the Lefschetz dual of $\Lambda_2 \in H_2(\mathcal{M}_6 - S_a \cup S_b, \mathbb{Z})$.

Using the polyform ϖ_X we may write the mixing (4.101) as an integral over $\mathcal{M}_6$

$$f_{XI} = -\frac{i}{2l_s^4} j_{(\mathfrak{S},\mathfrak{F})_X}(\gamma_I) = -\frac{i}{2l_s^4}\int_{\mathcal{M}_6}\langle\gamma_I, j_{(\mathfrak{S},\mathfrak{F})_X}\rangle = -\frac{i}{2l_s^4}\int_{\mathcal{M}_6}\langle\gamma_I,\varpi_X\rangle \quad (4.105)$$

where we used the fact that $j_{(\mathfrak{S},\mathfrak{F})_X}$ and ϖ_X only differ by an exact polyform which does not contribute to the integral since γ_I is harmonic. Similarly to the case of D6-branes, we say that a linear combination of D-branes are linearly equivalent to zero if this combination is trivial in generalised homology and the corresponding massless $U(1)$ does not mix kinetically with the closed string sector.

An example

Since these expressions are rather abstract, let us briefly illustrate it with the simple example analysed at the beginning of this section, namely two homotopic D7-branes a and b on $S_a \simeq S_b \simeq S$ with $[\mathcal{F}_a] = [\mathcal{F}_b]$ both in the bulk as well as in the cohomology of S so we have that $\hat{\partial}(\Gamma, \tilde{\mathcal{F}}) = (S_a, \mathcal{F}_a) - (S_b, \mathcal{F}_b)$. According to (4.104) we have that the polyform ϖ associated to the massless $U(1) = \frac{1}{2}(U(1)_a - U(1)_b)$ satisfies

$$d\varpi_{(a-b)} = j_{(S_a,\mathcal{F}_a)} - j_{(S_b,\mathcal{F}_b)} \quad (4.106)$$

as well as the last two conditions therein. Because $j_{(S_\alpha,\mathcal{F}_\alpha)}$ contain the D7, D5 and D3-brane sources, they are polyforms and so is $\varpi_{(a-b)}$. More precisely we have that

$$\varpi_{(a-b)} = \varpi_1 + \varpi_3 + \varpi_5 \quad (4.107)$$

where

$$d\varpi_1 = \delta(S_a) - \delta(S_b) \quad (4.108)$$

$$d\varpi_3 = \delta(S_a)\wedge\mathcal{F}_a - \delta(S_b)\wedge\mathcal{F}_b \quad (4.109)$$

$$d\varpi_5 = \delta(S_a)\wedge\frac{1}{2}\mathcal{F}_a^2 - \delta(S_b)\wedge\frac{1}{2}\mathcal{F}_b^2 \quad (4.110)$$

Following our general discussion in subsection 4.1.1, each of these p-forms ϖ_i will have a decomposition of the form (4.53). In particular we have that

$$\varpi_1 = \frac{1}{2\pi} d\left(\ln|h(S_a)| - \ln|h(S_b)|\right) \quad (4.111)$$

where $h(S_\alpha)$ is the divisor function of S_α. Recall that in a generic Calabi-Yau three-fold there will be no harmonic one-forms. Hence ϖ_1 will have no harmonic piece in the decomposition (4.53) and linear equivalence will be trivially satisfied with respect to this piece of ϖ. This corresponds to the well known fact that two homologous divisors are always linearly equivalent in a Calabi-Yau manifold. The same applies to ϖ_5, and so the condition of linearly equivalence and the kinetic mixing only depends on the piece ϖ_3.

Let us now assume that $\hat{\mathcal{F}}$ is a closed 2-form defined over the whole space $\mathcal{M}_6$ and such that $\hat{\mathcal{F}}|_{S_\alpha} = \mathcal{F}_\alpha$, $\alpha = a, b$.[16] We may then write (4.107) as

$$\varpi_{(a-b)} = \varpi_1 \wedge e^{\hat{\mathcal{F}}} \tag{4.112}$$

with ϖ_1 as in (4.111). Applying now (4.105) we obtain

$$f_{I(a-b)} = -\frac{i}{2l_s^6}\int_{\mathcal{M}_6} \gamma_I \wedge \varpi_3 = -\frac{i}{2l_s^6}\int_{\mathcal{M}_6} \gamma_I \wedge \varpi_1 \wedge \hat{\mathcal{F}} \tag{4.113}$$

which can be seen to be equivalent to our previous expression (4.84).

Application to F-theory GUTs

While we have focused the discussion of this section in type IIB orientifold vacua, our main results can be extended to an F-theory setup along the lines of [129]. In particular, they can be easily applied to an F-theory GUT model where the GUT gauge symmetry is broken by the presence of an hypercharge flux. In the present section we will do so, first obtaining the kinetic mixing of the hypercharge with bulk $U(1)$'s in a $SU(5)$ F-theory model. As this derivation is somewhat technical we present the result here

$$f_{YI} = -\frac{5i}{2l_s^4}\left[\int_\Sigma \gamma_I + \frac{1}{2\pi l_s}\int_{S_{GUT}} \gamma_I \wedge A_{GUT}\right] \tag{4.114}$$

where Σ is a 3-chain such that $\partial\Sigma = 6\Pi$, with $[\Pi]$ Poincaré dual to the hypercharge flux class $[\frac{5}{6}F_Y]$, $dA_{GUT} = [\frac{5}{6}F_Y - \delta^2_{S_{GUT}}(\Pi)]$ and γ_I is the 3-form that represents the RR $U(1)_I$ gauge symmetry. As pointed out in the introduction, if a massless $U(1)_h$ of a hidden 7-brane sector mixes with $U(1)_I$ as well this will give rise to light particles with small irrational hypercharges.

Nevertheless, even if the absence of such $U(1)_h$ a non-trivial mixing of the form could be relevant for F-theory GUT models, in the sense that the hypercharge normalisation needed to absorb the kinetic mixing Re f_{YI} will modify the relations between α_1 and α_2, α_3 [135]. This could be an interesting effect in view of the further corrections to the $SU(3) \times SU(2) \times U(1)_Y$ gauge coupling constants that arise from the 7-brane magnetisation [9, 61, 136]. Another potential solution to this problem is to consider the case where the hypercharge has a mass mixing with a RR photon. As pointed out in [111], this may happen if the three-fold base $\mathcal{M}_6$ contains torsional 2-cycles. We will apply our results to this case in the second part of this section.

Hypercharge mixing

Consider 5 parallel 7-branes wrapping divisors S_n in $\mathcal{M}_6$ with $n = 1, \ldots, 5$ and all of them homotopic to S_{GUT} with vanishing B-field. Since the hypercharge generator within $SU(5)$ is given by[17] $Q_Y = \frac{1}{6}\text{diag}(2, 2, 2, -3, -3)$ we will turn on fluxes F_n on all of them such that $F_n = \frac{1}{3}F_Y$ for $n = 1, 2, 3$ and $F_n = -\frac{1}{2}F_Y$ for $n = 4, 5$

16 Such 2-form may be obtained by trading the gauge fields on the D7-branes into the B-field as in [129].

17 Notice that this normalisation does not satisfy the conditions in the last section. However, it is the usual one for the hypercharge.

with $[F_Y]$ trivial in the cohomology of $\mathcal{M}_6$ but non-trivial in $H^2(S_n)$ and such that $\frac{5}{6}[F_Y] \in H^2(S_n, \mathbb{Z})$ [60]. Thus, when we bring the 7-branes back together this will translate into a flux $F_Y Q_Y$ along the hypercharge generator that breaks $SU(5) \to SU(3) \times SU(2) \times U(1)_Y$ and keeps the hypercharge massless.

Given this data we may apply the results of the previous section to compute kinetic mixing with any closed string $U(1)_I$. To each 7-brane we assign the generalised submanifold $(S_n, \mathcal{F}_n)$ and we have to look for the chain $(\mathfrak{S}, \mathfrak{F})_Y$ that satisfies

$$\hat{\partial}(\mathfrak{S}, \mathfrak{F})_Y = 2(S_1, F_1) + 2(S_2, F_2) + 2(S_3, F_3) - 3(S_4, F_4) - 3(S_5, F_5). \tag{4.115}$$

According to eq.(3.23) we then have

$$(S_n, F_n) = (S_n, 0) + a_n(\Pi_n, 0) + \hat{\partial}\left[-(\Gamma_n, \tilde{F}_n) + (\Gamma_n, \tilde{F}_{a_n\Pi_n})\right] \tag{4.116}$$

for all n. Here $\Pi_n \subset S_n$ is any 2-cycle such that $a_n[\Pi_n]$ =P.D.$[F_n]$, with $a_n = \frac{2}{5}$ for $n = 1, 2, 3$ and $a_n = \frac{3}{5}$ for $n = 4, 5$. See also eqs.(3.24-3.25). If we substitute this expression in (4.115) we find that

$$(\mathfrak{S}, \mathfrak{F})_Y = (\Gamma, 0) + (\Sigma, 0) + \sum_{n=1}^{5} n_{Yn}\left[-(\Gamma_n, \tilde{F}_n) + (\Gamma_n, \tilde{F}_{a_n\Pi_n})\right], \tag{4.117}$$

modulo a generalised cycle. Here n_{Yn} are the integers that define the hypercharge. Also, we have defined the 5-chain Γ such that $\partial\Gamma = \sum_n n_{Yn} S_n$ and the 3-chain Σ that satisfies $\partial\Sigma = \sum_n n_{Yn} a_n \Pi_n$. These exist given our initial hypotheses that $U(1)_Y$ is indeed massless.

The kinetic mixing with $U(1)_I$ is given by

$$f_{YI} = -\frac{5i}{2l_s^4}\, j_{(\mathfrak{S},\mathfrak{F})_Y}(\gamma_I) \tag{4.118}$$

where the factor of 5 is due to the normalisation of the hypercharge. The current j is a sum of different contributions, namely

$$j_{(\mathfrak{S},\mathfrak{F})_Y} = j_{(\Gamma,0)} + j_{(\Sigma,0)} + \sum_{n=1}^{5} n_{Yn} j_n. \tag{4.119}$$

where we have defined

$$j_n = j_{(\Gamma_n, \tilde{F}_{a_n\Pi_n})} - j_{(\Gamma_n, \tilde{F}_n)} \tag{4.120}$$

which may be written as

$$j_n(\gamma_I) = \frac{1}{2\pi l_s}\int_{S_n} \gamma_I \wedge A_{\Pi_n} \tag{4.121}$$

with A_{Π_n} a 1-form on S_n that satisfies

$$dA_{\Pi_n} = F_n - \delta^2_{S_n}(a_n\Pi_n) \tag{4.122}$$

for $n = 1, 2, 3$ we then have

$$dA_{\Pi_n} = \frac{2}{5}\left[\frac{5}{6}F_Y - \delta^2_{S_n}(\Pi)\right] \tag{4.123}$$

with Π the Poincaré dual of $\frac{5}{6}F_Y$. For $n = 4, 5$ the corresponding equations reads

$$dA_{\Pi_n} = -\frac{3}{5}\left[\frac{5}{6}F_Y - \delta^2_{S_n}(\Pi)\right] \tag{4.124}$$

Putting everything together we find that

$$f_{YI} = -\frac{5i}{2l_s^4}\left[\int_\Sigma \gamma_I + \frac{1}{2\pi l_s}\sum_{n=1}^{5} n_{Yn}\int_{S_n}\gamma_I \wedge A_{\Pi_n}\right]. \tag{4.125}$$

In order to get the mixing for the F-theory case we have to bring all the 7-branes together so $S_n = S_{GUT}$. This yields

$$f_{YI} = -\frac{5i}{2l_s^4}\left[\int_\Sigma \gamma_I + \frac{1}{2\pi l_s}\int_{S_{GUT}}\gamma_I \wedge A_{GUT}\right] \tag{4.126}$$

with $\partial\Sigma = 6\Pi$ and, using (4.119) and (4.120), we find that $A_{GUT} = \sum_n n_{Yn}A_{\Pi_n}$ satisfies

$$dA_{GUT} = 6\left[F_Y - \frac{6}{5}\,\delta^2_{S_{GUT}}(\Pi)\right]. \tag{4.127}$$

As the RR $U(1)$s do not have any light charged states, one may simply absorb this mixing in a redefinition of the hypercharge coupling constant. Besides the implications for gauge coupling unification mentioned above, in a setup with low scale supersymmetry the gauginos of the hidden sector will be massive and may mix with the MSSM neutralinos which would lead to new signatures at the LHC, if SUSY is found, through different decay patterns.

Implications for torsional hypercharge

Following [111], let us now consider the case in which the hypercharge has a mass mixing with a bulk $U(1)_t$. For this to happen the 2-cycle class $[\Pi]$ Poincaré dual to the hypercharge flux F_Y in S_{GUT} must be a torsional 2-cycle of the three-fold base $\mathcal{M}_6$. This means in particular that we have a non-trivial set of torsional cohomology classes in $\mathcal{M}_6$, due to the identities

$$\mathrm{Tor}H_2(\mathcal{M}_6,\mathbb{Z}) \simeq \mathrm{Tor}H_3(\mathcal{M}_6,\mathbb{Z}) \simeq \mathrm{Tor}H^3(\mathcal{M}_6,\mathbb{Z}) \simeq \mathrm{Tor}H^4(\mathcal{M}_6,\mathbb{Z}) \tag{4.128}$$

Let us in particular assume that these groups are all equal to $\mathbb{Z}_k$. Then we have the set of relations

$$d\omega^{\mathrm{tor}} = k\,\beta^{\mathrm{tor}} \qquad\qquad d\alpha^{\mathrm{tor}} = -k\,\tilde{\omega}^{\mathrm{tor}} \tag{4.129}$$

where α^{tor}, β^{tor} are 3-forms of $\mathcal{M}_6$ which are also eigenforms of the Laplacian, and we have the normalisation

$$\int_{\mathcal{M}_6}\alpha^{\mathrm{tor}}\wedge\beta^{\mathrm{tor}} = \int_{\mathcal{M}_6}\omega^{\mathrm{tor}}\wedge\tilde{\omega}^{\mathrm{tor}} = 1 \tag{4.130}$$

We then expand the RR potential C_4 on these non-harmonic forms and obtain

$$C_4 = A_1^t \wedge \mathrm{Re}\ \gamma^{\mathrm{tor}} - V_1^t \wedge \mathrm{Im}\ \gamma^{\mathrm{tor}} + \mathrm{Re}\ f_{tt}\, C_{2,t} \wedge \omega_i^{\mathrm{tor}} - \mathrm{Re}(T^t)\tilde{\omega}^{\mathrm{tor}} \tag{4.131}$$

where just like for harmonic forms, we consider the following combination

$$\gamma^{\rm tor} = \alpha^{\rm tor} + i f_{tt}\beta^{\rm tor} \tag{4.132}$$

with f_{tt} chosen so that $\gamma^{\rm tor}$ is a (2,1)-form. The 3-form $\gamma^{\rm tor}$ represents the bulk $U(1)_t$ that corresponds to the torsion group (4.128), and f_{tt} its gauge kinetic function. Following [111], it is easy to see that the kinetic mixing between the hypercharge and this $U(1)_t$ will also be given by (4.114) with basically γ_I replaced by $\gamma^{\rm tor}$. More precisely, since $[\Pi]$ is non-trivial in $H_2(\mathcal{M}_6,\mathbb{Z})$ but $k[\Pi]$ is, we can consider a monopole with charge k under the 7-brane $U(1)$, and by analysing its worldvolume theory we get the expression

$$\text{Re } f_{Yt} = \frac{5}{2l_s^4}\frac{1}{k}\left[\int_\Sigma \text{Im } \gamma^{\rm tor} + \frac{1}{2\pi l_s}\int_{S_{GUT}} \text{Im } \gamma^{\rm tor}\wedge A_{GUT}\right] = \frac{5}{2l_s^4}\text{Re } f_{tt}\int_{S_{GUT}} \omega^{\rm tor}\wedge F_Y \tag{4.133}$$

where Σ is a 3-chain ending on k copies of Π, and $dA_{GUT} = [k\frac{5}{6}F_Y - \sum_{i=1}^k \delta^2_{S_{GUT}}(\Pi_i)]$. We could have also obtained this expression by direct dimensional reduction of the Chern-Simons action $\int C_4\wedge F\wedge F$ of the GUT 7-brane, which gives

$$\int_{\mathbb{R}^{1,3}} F_2^Y \wedge (dV_1^t + kC_{2,t})\left[\frac{5}{2l_s^4}\text{Re } f_{tt}\int_{S_{GUT}} \omega^{\rm tor}\wedge F_Y\right] \tag{4.134}$$

Since the hypercharge flux induces a torsion class $[\Pi]$ in $\mathcal{M}_6$ we will have a relation of the form P.D.$[\Pi] = k_Y[\tilde\omega^{\rm tor}]$, where Poincaré duality is performed in $\mathcal{M}_6$. Then, by the results of [111] we have that the hypercharge and the bulk $U(1)_t$ have a mass mixing, and that the massless $U(1)_{\tilde Y}$ generator is given by the linear combination

$$A_1 = \cos\theta \tilde A_Y - \sin\theta \tilde A_t\,, \quad \sin\theta = \frac{k_Y g_Y}{\sqrt{k_Y^2 g_Y^2 + k^2 g_t^2}}\,, \tag{4.135}$$

in terms of the gauge bosons $\tilde A_Y = A_Y g_Y^{-1}$ and $\tilde A_t = A_t g_t^{-1}$ with canonical kinetic term. The Lagrangian in 4d contains the terms

$$\mathcal{L} \supset -\frac{2\pi}{l_s^2}\left[\text{Re } f_{YY}\, F_Y\wedge * F_Y + \text{Re } f_{tt}\, F_t\wedge * F_t + 2\text{Re } f_{Yt}\, F_Y\wedge * F_t\right] \tag{4.136}$$

so we may readily compute the kinetic function of the massless eigenstate by rotating to the mass eigenstate basis, namely

$$\mathcal{L} \supset -\frac{2\pi}{l_s^2}\text{Re } f_{11}\, F_1\wedge * F_1, \tag{4.137}$$

with

$$\text{Re } f_{11} = \cos^2\theta\,\text{Re } f_{YY} - 2\cos\theta\sin\theta\,\text{Re } f_{Yt} + \sin^2\theta\,\text{Re } f_{tt}. \tag{4.138}$$

With this normalisation the field A_1 couples to the charges states with a coupling constant $\cos\theta g_Y$ so we should redefine $A_{\tilde Y} = \cos\theta A_1$ which has a gauge kinetic function

$$\begin{aligned}\text{Re } f_{\tilde Y\tilde Y} &= \text{Re } f_{YY} - 2\tan\theta\,\text{Re } f_{Yt} + \tan^2\theta\,\text{Re } f_{tt} \\ &= \frac{5}{3\alpha_G} - 2\frac{k_Y}{k}\text{ Re } f_{Yt} + \frac{k_Y^2}{k^2\alpha_t^2}\,.\end{aligned} \tag{4.139}$$

This generalises the result obtained in section 5 of [111]. As pointed out there, these corrections may explain the small discrepancy in gauge coupling unification found in standard F-theory GUTs. Our generalisation including the kinetic mixing f_{Yt} provides an even more flexible scheme to fix such discrepancy.

4.2 LINEAR EQUIVALENCE AND MODULI STABILISATION

In this section we will discuss how the concept of linear equivalence appears when discussing the sector of open string moduli. Open string moduli appear when the D–branes added in the compactifications of type II String Theory admit some deformations (either geometric deformations or deformations of the gauge bundle they carry) that preserve supersymmetry. We display the result in Table 9.

	D6-brane	D7-brane
BPS condition	Π_3 special Lagrangian	Π_4 holomorphic
complex moduli	$b_1(\Pi_3)$	$h^{1,0}(\Pi_4) + h^{2,0}(\Pi_4)$

Table 9: BPS conditions and moduli for D6/D7-branes with flat bundles in a Calabi-Yau. $h^{n,0}(\Pi_4)$ counts harmonic $(n,0)$-forms on Π_4 and $b_1(\Pi_3)$ counts harmonic one-forms of Π_3.

Let us review a moment the moduli counting. In Table 9 we take the case of a compactification of type II String Theory on $\mathbb{M}^4 \times \mathcal{M}_6$ where $\mathcal{M}_6$ is a Calabi–Yau threefold. The presence of an orientifold involution will slightly modify the counting, however it will not greatly impact our forthcoming discussion and therefore we shall neglect its effect in the following. We already described the open string moduli sector in Section 4.1.1 for the case of type IIA String Theory with D6–branes and in Section 4.1.2 for the case of type IIB String Theory with D7–branes, however it is good to recall the definitions here. In the case of D6–branes with a flat $U(1)$ bundle supersymmetry imposes that the cycle wrapped by the D6–branes should be a special Lagrangian 3–cycle whose definition we gave in (4.5). For these D–branes the geometric deformations allowed (namely through cycles that preserve the special Lagrangian condition) may be counted by the first Betti number $b_1(\Pi_3)$ of the 3–cycle [137]. Geometric deformations are suitably complexified by the addition of Wilson lines giving $b_1(\Pi_3)$ chiral fields in 4d defined in (4.17). On the other hand in the case of D7–branes (without magnetisation) the branes should wrap a holomorphic 4–cycle S (which implies $\Omega|_S = 0$) with a holomorphic gauge bundle in order not to break supersymmetry. The geometric deformations that preserve holomorphicity of the cycle (and therefore do not break supersymmetry) are counted by $h^{(2,0)}(S)$, and in addition to this there can be Wilson line deformations counted by $h^{(0,1)}(S)$[18]. The main issue is that the counting of massless degrees of freedom in the open string sector is always performed in a frozen closed string background. Taking into account the coupling to a dynamical closed string sector turns out to be crucial for it imposes some additional constraints on the allowed deformations of the brane sector. This therefore implies that the actual number of open string moduli will be lower than the

18 Note that we already gave this counting of moduli in (3.1).

numbers just quoted. Our purpose in the following will be to detail the microscopic mechanism giving rise to this reduction of massless modes in the D–brane sector. Quite interestingly we will observe that for D6–branes the allowed deformations require passing through cycles that are linear equivalent. A similar mechanism works for the case of D7–branes where the coupling to closed string moduli will lift some of Wilson line moduli[19].

This mechanism has some potential interesting consequences. Due to this mechanism some of the open string moduli will be stabilised with a high mass even without taking into account the presence of worldsheet instantons. This is particularly interesting especially for Wilson line moduli which are believed to have a flat potential. From the point of view of 4d effective field theory the open string moduli will always acquire a mass due to a bilinear coupling with the closed string moduli appearing in the superpotential whose schematic form is

$$W = X \cdot \Phi \tag{4.140}$$

where X is a linear combination of closed string moduli (Kähler moduli for the case of D6–branes and complex structure moduli for the case of D7–branes) and Φ is an open string chiral field. This coupling in the superpotential is particularly interesting for, in addition to giving a mass to Φ, it may be used for having an embedding in String Theory of the models of chaotic inflation with a stabiliser field appearing in the supergravity literature [139, 140]. Indeed this direction has been pursued for the case of D6–branes [141, 142] giving an embedding of the models of chaotic inflation with a stabiliser in String Theory.

4.2.1 *D6-brane backreaction in compact spaces*

The backreaction of N D6-branes in flat space is given by

$$\begin{aligned} ds^2 &= e^{2A}\eta_{\mu\nu}dx^\mu dx^\nu + e^{-2A}\delta_{ij}dy^i dy^j && (4.141a)\\ C_7 &= g_s^{-1}(e^{4A}-1)\, dx^0\wedge\cdots\wedge dx^6 && (4.141b)\\ e^\phi &= g_s e^{3A} && (4.141c)\\ e^{-4A} &= 1+\frac{r_6}{r} && (4.141d) \end{aligned}$$

where x^μ, $\mu = 0,\dots 6$ are coordinates parallel to the D6-brane while y^i, $i = 7,8,9$ are transverse to it. In addition r is the radial distance to the D6-brane locus, $g_s = e^{\phi_0}$ the asymptotic value of the string coupling, and $r_6 = \rho_6 g_s N l_6$, with ρ_6 a numerical factor that will not be relevant in the following.

One can rewrite this solution in terms of the Ansatz for type IIA 4d Minkowski vacua

$$ds^2 = e^{2A}ds^2_{\mathbb{R}^{1,3}} + ds^2_{X_6} \tag{4.142}$$

where X_6 is the 6d internal manifold on whose coordinates the warp factor A depends. It is then easy to check that this solution satisfies the supersymmetry conditions

$$d(3A-\phi) = H_{\rm NS} + idJ = 0 \qquad F_0 = \tilde F_4 = \tilde F_6 = 0 \qquad (4.143a)$$

$$d(e^{2A-\phi}{\rm Im}\ \Omega) = 0 \qquad d(e^{4A-\phi}{\rm Re}\ \Omega) = -e^{4A} *_6 F_2 \qquad (4.143b)$$

19 The brane deformation moduli may acquire a mass due to the presence of closed string fluxes, see [138] for the details.

where $*_6$ is the Hodge star operator in the internal space X_6, including the warp factor. The fact that this supergravity background is supersymmetric is an indication that the backreacted D6-brane is BPS and preserves some of the supersymmetry of the initial background (in this case 10d Minkowski flat space). We may now add further D6-branes that also contain the coordinates of $\mathbb{R}^{1,3}$, and which are either parallel to the initial one or intersect it at supersymmetric $SU(3)$ angles in the remaining coordinates [143]. By backreacting them we will obtain a more complicated supergravity solution which will nevertheless satisfy (4.142) and (4.143). Again, this indicates that this system of intersecting D6-branes is mutually BPS and at least $\mathcal{N} = 1$ 4d supersymmetry is preserved by it. Finally, the fact that we can displace transversely the D6-brane locations without spoiling (4.143) corresponds to the fact that there is no force between these mutually BPS D-branes, and that there is a set of flat directions that can interpreted as open string moduli.

In principle, one may expect a similar picture to apply if instead of $\mathbb{R}^{1,9}$ we consider $\mathbb{R}^{1,3} \times \mathcal{M}_6$, where $\mathcal{M}_6$ is a a non-compact Calabi-Yau manifold. Indeed, if we backreact a D6-brane along $\mathbb{R}^{1,3}$ and a submanifold $\Pi_3 \subset \mathcal{M}_6$ we will be sourcing a RR two-form flux which locally satisfies

$$dF_2 \; = \; \delta_3(\Pi_3) \tag{4.144}$$

where δ_3 is delta-function three-form with support on Π_3 and indices transverse to it. Similarly the D6-brane backreaction will source the dilaton, metric and warp factor in such a way that we have an Ansatz of the form (4.142), where X_6 is identical to $\mathcal{M}_6$ in terms of differentiable manifolds but endowed with the backreacted metric. Finally, if Π_3 is a special Lagrangian calibrated by Re Ω, then eqs.(4.143) will be satisfied.

On the other hand, new restrictions may arise when $\mathcal{M}_6$ is a compact manifold with non-trivial topology. First of all, compactness of $\mathcal{M}_6$ implies that the total D6-brane charge needs to cancel, simply because F_2 is globally well-defined. More precisely we have a Bianchi identity of the form

$$dF_2 \; = \; \sum_\alpha \delta(\Pi_3^\alpha) + \delta(\Pi_3^{\alpha^*}) - 4\delta(\Pi_3^{O6}) \tag{4.145}$$

where the index α runs over the D6-branes of a given compactification, α^* over their orientifold images and Π_3^{O6} stand for the O6-plane loci. This equation will have a solution for a globally well-defined F_2 if and only if the following equation in $H_3(\mathcal{M}_6)$ is satisfied

$$\sum_\alpha [\Pi_3^\alpha] + [\Pi_3^{\alpha^*}] - 4[\Pi_3^{O6}] \; = \; 0 \tag{4.146}$$

which is nothing but the RR tadpole condition that we already met in Section 4.1.1. In addition, for the wavefunction of a D0-brane to be well-defined, the integral of F_2 over any two-cycle must be quantised. More precisely, over each two-cycle $\pi_2^a \subset \mathcal{M}_6 - \Pi_3^{D6}$ (with Π_3^{D6} is the sum of all the three-cycles wrapped by D6-branes and O6-planes) we must have

$$n_a \; = \; \frac{1}{l_s} \int_{\pi_2^a} F_2 \, \in \, \mathbb{Z} \tag{4.147}$$

where $l_s = 2\pi\sqrt{\alpha'}$ is the string length. This condition not only applies to those two-cycles that surround a D6-brane but also to the non-trivial two-cycles of $\mathcal{M}_6$ that also belong to $\mathcal{M}_6 - \Pi_3^{D6}$. As we will see, it is due to imposing (4.147) to the latter two-cycles where new restrictions in moduli space appear.

These consistency conditions are of topological nature, but supersymmetry imposes further constraints on F_2. Indeed, notice that one can rewrite the second equation in (4.143b) as

$$F_2 = *_{10} d *_{10} \left(e^{-\phi} \text{Im } \Omega \right) \tag{4.148}$$

which means that F_2, seen as a two-form in the full 10d backreacted space, is co-exact since $e^{-\phi}\text{Im } \Omega$ is globally well-defined. In general, similarly to Hodge decomposition one can split any two-form with legs in the internal six-dimensional space as

$$F_2 = d\alpha_1 + F_2^{\text{harm}} + *_{10} d *_{10} \gamma_3 \tag{4.149}$$

that is into an exact, harmonic and co-exact pieces. Notice that the case of F_2 the three-form γ_3 is fixed by the Bianchi identity (4.145) while the other two components are not. The additional input of supersymmetry is then that $\gamma_3 = e^{-\phi}\text{Im } \Omega$ and $\alpha_1 = F_2^{\text{harm}} = 0$.

On the one hand, that α_1 vanishes is easy to achieve, as one can always adjust the background value of the RR potential C_1 in order to cancel such component. On the other hand, requiring that F_2^{harm} vanishes is non-trivial, due to the quantisation condition (4.147). Indeed, because in general the integral of $d^*\gamma_3$ will be non-vanishing and non-integer over the non-trivial two-cycles of $\mathcal{M}_6$, one needs to include a harmonic piece in F_2 such that the full integral adds up to an integer, as required by consistency. If that is the case, the D6-brane configuration will be non-supersymmetric even if all D6-branes wrap special Lagrangian three-cycles. As we will discuss below the backreacted F_2^{harm} will depend on certain D6-brane locations which means that, at the end of the day, supersymmetry will impose a constraint in the D6-brane moduli space. One then expects that the number of constraints imposed by supersymmetry can be up to $b_2(\mathcal{M}_6) = \dim H_2(\mathcal{M}_6, \mathbb{R})$, which measures the number of independent two-cycles in $\mathcal{M}_6$.[20]

Rather than computing the value of F_2^{harm} for a specific compactification with backreacted D6-branes, let us discuss how does it depend on the D6-brane locations. More precisely, we will show that if (4.148) is satisfied, it cannot be so after changing the location of certain D6-branes. It turns out that the expression (4.148) is not the most suitable one for such analysis, simply because it depends on $*_{10}$, which in turn depends on the warp factor and ultimately on the D6-brane locations. Instead, one can use the equivalent condition formulated in terms of a generalised Dolbeault operator $d^{\mathcal{J}}$, namely [144]

$$F_2 = d^{\mathcal{J}}(e^{-\phi}\text{Re } \Omega)) \tag{4.150}$$

where the definition of $d^{\mathcal{J}}$ is given in Appendix F. For our purposes here it suffices to point out that for the type IIA backgrounds at hand this condition can be rewritten as [145]

$$d(e^{-\phi}\text{Re } \Omega) = -J \wedge F_2 \tag{4.151}$$

20 In fact, in orientifold compactifications F_2 is a two-form odd under the geometric orientifold action, so the maximal number of constraints is actually given by $b_2^-(\mathcal{M}_6) = \dim H_2^-(\mathcal{M}_6, \mathbb{R})$.

which substitutes the second equation in (4.143b). The important point is that the two-form J does not depend on the D6-brane backreaction or their location, and in fact it remains the same as in the unbackreacted space $\mathcal{M}_6$.

Hence, in order to see if the D6-brane backreaction satisfies the supersymmetry equation (4.151), we may consider the Calabi-Yau orientifold $\mathcal{M}_6$ with D6-branes wrapping special Lagrangian three-cycles Π_3^a in it and a quantised two-form flux satisfying (4.145). Let us for instance take a D6-brane wrapping the special Lagrangian Π_3 and displace it to the new special Lagrangian three-cycle Π_3' homotopic to the former. The new backreacted flux F_2' will be given by

$$F_2' = F_2 + \Delta F_2 \tag{4.152}$$

where F_2 and F_2' solve for eq.(4.145) before and after moving the D6-brane, respectively. Both F_2 and F_2' are quantised two-forms, so ΔF_2 is quantised as well, and it satisfies the equation

$$d\Delta F_2 = \delta(\Pi_3') - \delta(\Pi_3) \tag{4.153}$$

We would now like to check whether for some particular D6-brane displacement we have that

$$\int_{\mathcal{M}_6} \Delta(F_2 \wedge J) \wedge \omega_2 = \int_{\mathcal{M}_6} \Delta F_2 \wedge J \wedge \omega_2 \neq 0 \tag{4.154}$$

for some closed two-form ω_2 of $\mathcal{M}_6$, where we have used that J does not depend on the D6-brane location. If the rhs of (4.154) does not vanish for some closed ω_2 it means that $J \wedge F_2$ cannot be written as an exact form either before or after displacing the D6-brane, and that supersymmetry is broken for D6-brane deformations of this sort.

To proceed, one may follow the discussion of our previous section (see also [121]) and use that F_2 is quantised, it satisfies eq.(4.145) and that $J \wedge \omega_2$ is closed to derive the identity

$$\int_{\mathcal{M}_6} \Delta F_2 \wedge J \wedge \omega_2 = \int_{\Sigma_4} J \wedge \omega_2 \tag{4.155}$$

where Σ_4 is a four-chain connecting Π_3 and Π_3'. Notice that if ω_2 is exact and these three-cycles are Lagrangian the integrals will identically vanish, so the only way to obtain a non-vanishing result is if ω_2 contains a harmonic two-form of $\mathcal{M}_6$. In this sense, the integrals (4.155) measure how the harmonic component of ΔF_2 in $\mathcal{M}_6$ changes with the D6-brane location. In the language of [121], we see that the integral vanishes and ΔF_2 has no harmonic piece only if the three-cycles Π_3^a and $\Pi_3^{a\prime}$ are linearly equivalent.

To gain further insight into the condition (4.154) let us take Π_3' to be the infinitesimal deformation of Π_3 by a normal vector X. Using again that $J|_{\Pi_3} = 0$ we have that the chain integral becomes

$$\int_{\Pi_3} \iota_X J \wedge \omega_2 \tag{4.156}$$

By McLean's theorem [137] we know that in order for X to describe a special Lagrangian deformation, $\iota_X J$ must be a harmonic one-form in Π_3 which we can take to be integral. We can then use Poincaré duality to write the above expression as

$$\int_{\pi_2^X} \omega_2 \tag{4.157}$$

where π_2^X is a two-cycle of Π_3 in the Poincaré dual class of $\iota_X J$. We then find that an infinitesimal deformation X of Π_3 violates the supersymmetry condition (4.151) if and only if the integral (4.157) does not vanish. This implies in particular that Π_3 must have a two-cycle π_2 which is non-trivial in the ambient space $\mathcal{M}_6$. Then, deforming the D6-brane location along the direction that corresponds to such two-cycle will break supersymmetry because it modifies the harmonic piece of the backreacted F_2 in the Calabi-Yau metric $\mathcal{M}_6$. Finally, it is easy to check that F_2 is odd under the orientifold involution defined on $\mathcal{M}_6$, and so must be ω_2 and π_2^X in order for the above integrals not to vanish.

Hence, the final picture is that by deforming the location of D6-branes that contain non-trivial odd two-cycles in the compactification space $\mathcal{M}_6$ a harmonic piece will be generated for a quantised flux F_2 satisfying (4.147) and supersymmetry will be broken. From the viewpoint of the fully backreacted supergravity background we will have that the internal metric, warp factor and dilaton will be sourced such that

$$d[e^{-4A} *_6 d(e^{4A-\phi}\text{Re }\Omega)] \;=\; -\sum_\alpha \delta(\Pi_3^\alpha) + \delta(\Pi_3^{\alpha^*}) - 4\delta(\Pi_3^{O6}) \tag{4.158}$$

The backreacted flux F_2 will satisfy a similar Poisson equation, but due to quantisation it will also contain a harmonic piece F_2^{harm} in the 10d decomposition (4.149) which prevents eq.(4.148) to be satisfied. Such component will raise the energy of the compactification via the 4d effective potential computed in [146]

$$\mathcal{V}_{\text{eff}} \;=\; \frac{1}{2}\int_{X_6} d\text{vol}_6 e^{4A}\left[*_6 F_2 + e^{-4A} d(e^{4A-\phi}\text{Re }\Omega)\right]^2 + \ldots \tag{4.159}$$

where we the remaining pieces of the potential are a sum of squares not relevant for the present discussion. More precisely one finds that

$$\mathcal{V}_{\text{eff}} \;=\; \frac{1}{2}\int_{X_6} d\text{vol}_6\, e^{4A} F_2^{\text{harm}} \wedge *_6 F_2^{\text{harm}} \tag{4.160}$$

where F_2^{harm} depends on the D6-brane position as described above. Hence, such would-be open string moduli pick up a mass, even if they preserve the D6-brane special Lagrangian condition. In particular, they will be fixed to those values such that F_2^{harm} vanishes.

4.2.2 *Superpotential analysis*

Let us now show that, from a 4d macroscopic viewpoint, the effective potential (4.160) arises from a superpotential bilinear in open and closed string fields. This can be done by considering the classical D6-brane superpotential[21] [147, 148]

$$\Delta W^{D6}_{\rm clas} = \int_{\Sigma_4} (J_c + F)^2 \tag{4.161}$$

where Σ_4 is a four-chain connecting the three-cycle Π_3 and a homotopic deformation Π_3'. If the deformation is infinitesimal and given by the normal vector X we can write W as [147]

$$\Delta W^{D6}_{\rm clas} = \int_{\Pi_3} (J_c + F) \wedge (\iota_X J_c + A) \tag{4.162}$$

Let us recall from Section 4.1.1 the definition of the relevant moduli in this context. We may now expand the complexified Kähler form as

$$J_c = B + iJ = T_a \omega_2^a \tag{4.163}$$

where $\{\omega_2^a\}$ is a basis of integer harmonic two-forms of $\mathcal{M}_6$, and T_a are the corresponding Kähler moduli. In addition we may define the open string deformation as

$$\Phi_{\rm D6} = \iota_X J_c|_{\Pi_3} + A = (\theta^j + \lambda^j_i \phi^i)\zeta_j = \Phi^j_{\rm D6}\zeta_j \tag{4.164}$$

where $\zeta_j/2\pi$ is a quantised harmonic one-form of Π_3 and (see e.g. [110] for details)

$$X = \phi^j X_j \qquad A = \frac{\pi}{l_s}\theta^j \zeta_j \qquad \iota_{X_i} J_c|_{\Pi_3} = \lambda^j_i \zeta_j \tag{4.165}$$

Because ζ_j is harmonic, the field $\Phi^j_{\rm D6}$ corresponds to a D6-brane deformation that preserves the worldvolume supersymmetry conditions

$$J_c|_{\Pi_3} + F = 0 \qquad {\rm Im}\ \Omega|_{\Pi_3} = 0 \tag{4.166}$$

and so it is typically identified with an open string modulus. However, plugging both expressions into (4.162) we obtain the non-trivial superpotential

$$\Delta W^{D6}_{\rm clas} = m^a_j \Phi^j_{\rm D6} T_a \qquad \text{with} \qquad m^a_j = \int_{\Pi_3} \omega_2^a \wedge \zeta_j \tag{4.167}$$

which, as announced, is a bilinear on the open string Φ^j_{D6} and closed string T_a fields. Notice that the superpotential is only non-trivial if the integer numbers m^a_j are non-vanishing. Recalling the discussion around eq.(4.157), it is easy to see that $m^a_j \neq 0$ if and only if some two-cycle of Π_3 is also non-trivial in the ambient space as an element of $H_2(\mathcal{M}_6, \mathbb{R})$. Precisely when this happens, some of these naive moduli will be stabilised by an F-term scalar potential, in agreement with the results obtained in the previous section.

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21 By classical we mean the superpotential that arises before taking into account worldsheet instantons, most precisely holomorphic disk instantons ending on the D6-brane one-cycles.

In order to see this in some detail let us consider the simple case where Π_3 has just one non-trivial harmonic form ζ and its dual two-cycle π_2 is such that $\int_{\pi_2} \omega_2^a = -\int_{\pi_2} \omega_2^b = 1$, while all the other bulk two-forms integrate to zero. Then we have that the superpotential reads

$$\Delta W_{\rm clas}^{D6} \;=\; \Phi_{D6}(T_a - T_b) \tag{4.168}$$

whose critical points are given by $T_a = T_b$ and $\Phi_{D6} = 0$. Hence, we recover that one open string modulus and one linear combination of closed string moduli are fixed by this superpotential.

Care should be taken when deriving a scalar potential from (4.168) since, as pointed out in [148], the expression (4.161) is only the difference of the superpotential between two D6-brane positions, and not the absolute $W_{\rm clas}^{\rm D6}$. Nevertheless, let us assume that the full superpotential is such that we have a definite-positive, no-scale potential as in [149] and that the D6-brane configuration is such that $W_{\rm clas} = 0$ and we are at a supersymmetric minimum. Then the piece of scalar potential that we obtain from (4.168) reads

$$\mathcal{V} \;=\; e^K \left(K^{\Phi\Phi}|T|^2 + K^{TT}|\Phi_{D6}|^2\right) \tag{4.169}$$

where $T \equiv T_a - T_b$. Hence, we find that the closed and the open string modulus are fixed to the values $T_a = T_b$ and $\Phi_{D6} = 0$ separately.

On the one hand, the potential for the closed string modulus T is easy to interpret. Indeed, whenever $T_a \neq T_b$ the pull-back of J_c on Π_3 will be non-vanishing, and so the supersymmetry conditions (4.166) will not be met. Moreover, the integral of ω_2^a and ω_2^b over $\pi_2 \subset \Pi_3$ will not change if we deform the D6-brane embedding. Hence, a D6-brane wrapping Π_3 will break supersymmetry and have an excess of energy unless $T_a = T_b$. In fact, the piece of the potential that goes like $|T|^2$ can be easily derived from the analysis of D6-brane DBI action in such background, following similar steps as those performed in section 5.2 of [150] for coisotropic D8-branes.

On the other hand, the potential for the open string modulus Φ_{D6} cannot be derived from a DBI analysis. Indeed, since such term arises form the F-term of the closed string modulus T, it will not appear if T is not considered dynamical. But not considering T as dynamical is precisely what is done when we analyse a D6-brane action in a frozen closed string background. Hence, it is only via D6-brane backreaction effects of the bulk that we can understand the nature of this piece of the potential, as done in the previous section.

An interesting byproduct of last section analysis is that it allows to deduce the D6-brane classical superpotential in absolute terms, instead of defining just $\Delta W_{\rm clas}^{\rm D6}$. Indeed, recall that the crucial supersymmetry condition for the above analysis can be rewritten as (4.151), which implies that $J \wedge F_2$ is exact in the cohomology of $\mathcal{M}_6$. In fact, as we will discuss in the next section, one can use the remaining supersymmetry conditions to argue that $J_c \wedge F_2$ must be exact as well. This is satisfied if and only if

$$\int_{\mathcal{M}_6} F_2 \wedge J_c \wedge \omega_2^a = 0 \qquad \forall\, [\omega_2^a] \in H^2(\mathcal{M}_6, \mathbb{R}) \tag{4.170}$$

In terms of a superpotential, this condition can be derived by replacing $\omega_2^a \to J_c = T_a \omega_2^a$ and imposing the F-term condition on each Kähler modulus T_a separately. Hence, we are led to the expression

$$W_{\rm clas}^{\rm D6} \;=\; \int_{\mathcal{M}_6} F_2 \wedge J_c \wedge J_c \;=\; \int_{\Sigma_4^{\rm tot}} J_c^2 \tag{4.171}$$

which is quite familiar from the type IIA literature. Indeed, the first expression for $W_{\rm clas}^{\rm D6}$ is nothing but the standard flux superpotential for type IIA Minkowski vacua [122, 151–154], where now $F_2 = F_2^{\rm D6}$ stands for the flux coming from the backreaction of D6-branes and O6-planes. Of course one can also add to this backreacted flux a quantised background flux $F_2^{\rm bkg}$. The sum of both fluxes will enter the superpotential as $F_2 = F_2^{\rm D6} + F_2^{\rm bkg}$, and we can understand both contributions as the sum $W_{\rm brane} + W_{\rm flux}$ discussed in e.g. [155].

The second expression for $W_{\rm clas}^{\rm D6}$ is obtained by replacing the current $F_2^{\rm D6}$ by a dual four-chain $\Sigma_4^{\rm tot}$ which connects all D6-branes and O6-planes. That such four-chain exists is a direct consequence of the RR tadpole condition (4.146), and by focusing on a single D6-brane one obtains the expression (4.161) on which the analysis os this section is based.

In fact, expression (4.161) is obtained after replacing $J_c \to J_c + F$, where $F = dA$ will contain the Wilson line dependence of the superpotential. In the next section we will discuss how this replacement should arise. In any case, from this superpotential analysis it is clear that certain D6-brane Wilson lines should also be stabilised, since they enter Φ_{D6} as the complexification of the position moduli which get affected by the superpotential (4.167). This may sound surprising, since typically it is assumed that D-brane Wilson lines are free of any scalar potential, and that the D-brane and background configuration is fully independent of them. In the following we will show that this intuition is wrong, and that there is a quite simple microscopic mechanism by which Wilson lines are stabilised.

4.2.3 *Wilson line moduli stabilisation*

An important observation is that the supersymmetry condition (4.151) may not be the only that is spoiled when changing the location of a D6-brane over its naive moduli space. Indeed, a different supersymmetry condition that also turns out to be relevant is

$$\tilde{F}_4 = dC_3 - C_1 \wedge H_{NS} = 0 \tag{4.172}$$

In general the gauge invariant flux $\tilde{F}_4$ satisfies the Bianchi identity

$$d\tilde{F}_4 + F_2 \wedge H_{NS} = j_{\rm D4} \tag{4.173}$$

with $j_{\rm D4}$ a five-form current describing the D4-brane charge carried by the D-branes of the configuration. In the case of compactifications with O6-planes, D-brane BPSness forbids the presence of D4-branes, while D6-branes must carry a vanishing worldvolume flux $\mathcal{F} = B + F$. Hence there is no induced D4-brane charge and so $j_{\rm D4} = 0$.[22] In addition, we have that an independent supersymmetry condition imposes that $H_{NS} = 0$. As a result, we have that for each point of the naive moduli space of special Lagrangian D6-branes we have that the backreacted background satisfies

$$d\tilde{F}_4 = 0 \tag{4.174}$$

which of course also applies for any choice of Wilson lines on such D6-branes. In general, Wilson lines do not enter into the Bianchi identity of any background flux,

22 Coisotropic D8-branes will in general violate this condition, since they do carry induced D4-brane charge [150]. For simplicity we will not consider their presence here.

which typically leads to the intuition that they do not backreact into the closed string background. In the following we will argue that such intuition is wrong by considering the quantisation conditions that $\tilde{F}_4$ must satisfy in compact manifolds.

As discussed in [156] the gauge invariant four-form flux $\tilde{F}_4$ does not satisfy a quantisation condition itself, but we must instead consider the notion of Page charge and take the combination $\tilde{F}_4 + F_2 \wedge B$. Then we have that the proper quantisation condition reads

$$\frac{1}{l_s^3} \int_{\pi_4} \tilde{F}_4 + F_2 \wedge B \in \mathbb{Z} \tag{4.175}$$

over each four-cycle $\pi_4 \subset \mathcal{M}_6 - \Pi_3^{D6}$. Here F_2 is the previous two-form RR flux that arises from the backreaction of D6-branes. As such we have that $d(F_2 \wedge B) = dF_2 \wedge B = 0$ since due to eq.(4.166) the pull-back of the B-field vanishes on each D6-brane. Hence $\tilde{F}_4 + F_2 \wedge B$ is a closed, quantised four-form whose integral over each four-cycle does not change when we move on the naive D6-brane moduli space.

While the sum $\tilde{F}_4 + F_2 \wedge B$ is quantised, both factors may not be so separately. If $F_2 \wedge B$ is not quantised it means that it contains a non-integer harmonic four-form piece, and so $\tilde{F}_4$ must also contain a non-trivial harmonic four-form in order to satisfy (4.175). This means in particular that $\tilde{F}_4 \neq 0$ and so supersymmetry is broken.

Following the same philosophy of section 4.2.1, let us consider the case where $F_2 \wedge B$ is a quantised four-form[23] and let us see whether moving in the naive D6-brane moduli space spoils this condition. As before we consider the case in which we change a D6-brane location from Π_3 to Π_3', and define the corresponding difference of two-form flux ΔF_2 satisfying (4.153). This does not change the B-field at all, and so there is a change in the harmonic component of $F_2 \wedge B$ if

$$\int_{\mathcal{M}_6} \Delta F_2 \wedge B \wedge \omega_2 = \int_{\Sigma_4} B \wedge \omega_2 \neq 0 \tag{4.176}$$

where ω_2 is some harmonic two-form of $\mathcal{M}_6$, and Σ_4 has been defined as in (4.155). Taking now an infinitesimal deformation given by the normal vector X, the change in $F_2 \wedge B$ will be measured by

$$\int_{\Pi_3} \iota_X B \wedge \omega_2 \tag{4.177}$$

which is similar to eq.(4.177) with the replacement $J \to B$. Again, this change will be non-vanishing whenever the three-cycle Π_3 contains a two-cycle π_2^X that is non-trivial in the odd homology of $\mathcal{M}_6$.

So far we have only proven that when deforming certain D6-branes along their special Lagrangian moduli space one can break supersymmetry in two independent ways, by switching on a non-exact component for $J \wedge F_2$ and a non-exact component for $B \wedge F_2$ or equivalently for $\tilde{F}_4$. By holomorphicity in the open string modulus $\Phi_{\text{D6}} = \iota_X J_c + A$, we would expect in these cases there is also a potential for the D6-brane Wilson line moduli.

Indeed, instead of changing the location of the D6-brane three-cycle Π_3 let us perform a change in its Wilson line $\Delta A = A' - A$. By a gauge transformation we can

23 Using large gauge transformations of the B-field, one may simply consider the case where $F_2 \wedge B$ is exact, which is the supersymmetry condition used in the previous section.

transform such change in a shift of the B-field by a exact form $\Delta B = B' - B$ such that

$$\Delta B = d\Theta_1 \qquad \text{and} \qquad \Theta_1|_{\Pi_3} = \Delta A \tag{4.178}$$

Such shift in the B-field by an exact form will not increase the energy of the system via the NS-flux potential $\int |H_{NS}|^2$. However, since F_2 is not closed it may change the harmonic piece of $B \wedge F_2$ and so increase the energy by shifting $\tilde{F}_4$. Indeed, we find that such change is measured by

$$\int_{\mathcal{M}_6} \Delta(F_2 \wedge B) \wedge \omega_2 = \int_{\mathcal{M}_6} F_2 \wedge \Delta B \wedge \omega_2 = -\int_{\mathcal{M}_6} dF_2 \wedge \Delta A \wedge \omega_2 = \int_{\Pi_3} \Delta A \wedge \omega_2 \tag{4.179}$$

and so it will not vanish whenever ΔA is Poincaré dual to a two-cycle of Π_3 non-trivial in $H_2^-(\mathcal{M}_6, \mathbb{R})$, in agreement with our previous results.

To summarise, we have found that certain D6-brane Wilson lines also 'backreact' into a harmonic component for $\tilde{F}_4$, which increases the energy of the system and fixes their value. Notice that this shift of $\tilde{F}_4$ is compatible with all the 10d Bianchi identities and equations of motion. In fact, one can argue that the background value of $\tilde{F}_4$ needs to change as described by looking at the domain wall solution interpolating between different D6-brane configurations.

Indeed, let us first consider the case where the D6-brane location is changed from Π_3 to Π_3'. The domain wall connecting these two configurations will be a D6-brane wrapped on the chain Σ_4 connecting both three-cycles and localised in the 4d coordinate x^3. Now, if (4.176) is true it means that the domain wall D6-brane will be magnetised by the presence of the B-field, and so it will actually be a D6/D4-brane bound state. As such, not only the background value of F_2 will change when we cross this domain wall, but also that of $\tilde{F}_4$. Finally, adding a relative Wilson line between Π_3 and Π_3' will result in a worldvolume flux F threading the D6-brane domain wall, whose total D4-brane charge will be induced by $\mathcal{F} = B + F$. One then obtains again that the D6-brane Wilson lines shift the value of $\tilde{F}_4$, although only if via Poincaré duality on Π_3 they correspond to a non-trivial odd two-cycle of $\mathcal{M}_6$.

It is easy to see this mechanism for stabilising Wilson lines is more general than the type IIA setup that we are discussing, and that it can in principle be applied to other kind of string vacua as well. In the following we will briefly comment on how it allows to stabilise Wilson lines in type IIB vacua with D7-branes.

4.2.4 *Stabilising D7-brane Wilson lines*

Let us consider a type IIB orientifold compactification with O3/O7-planes, and with space-time filling D7-branes wrapping divisors of a Calabi-Yau threefold $\mathcal{W}_6$. Let us in particular consider a D7-brane wrapping a divisor $S_4 \subset \mathcal{W}_6$ such that S_4 contain harmonic one-forms or, in other words, that it contains Wilson line moduli. By Poincaré duality S_4 will contain non-trivial three-cycles and, since it is a complex submanifold, the number of independent odd-cycles must be even. So the minimal setup that we may consider is that S_4 contains two three-cycles $\{\pi_3^1, \pi_3^2\}$.

For our discussion, the key point is whether the three-cycles $\{\pi_3^1, \pi_3^2\}$ are trivial in $\mathcal{W}_6$ or not. If they are non-trivial then one can follow a discussion parallel to the one carried for the D6-brane, and argue that such D7-brane develops a superpotential of the form

$$W = \Phi_{\mathrm{D7}} \cdot X_{\mathrm{closed}} \tag{4.180}$$

where Φ_{D7} is the complexified Wilson line that corresponds to these three-cycles, and X_{closed} is a linear combination of complex structure moduli of $\mathcal{W}_6$.

As before, the obstruction to move on closed string moduli space that arise from (4.180) can be derived by analysing the D-brane supersymmetry conditions, which in this case impose that

$$\Omega|_{S_4} = 0 \tag{4.181}$$

which is equivalent to ask that S_4 is holomorphic. The three-form Ω can be understood as a linear combination of integer three-forms whose coefficients depend on the complex structure moduli of $\mathcal{W}_6$. More precisely we have that the complex structure moduli are (redundantly) defined as

$$z^A = \int_{\gamma_3^A} \Omega \qquad \omega^B = \int_{\gamma_3^B} \Omega \tag{4.182}$$

where $\{\gamma_3^A, \gamma_3^B\}$ is a symplectic basis of integer three-cycles of $\mathcal{W}_6$ with $[\gamma_3^A]\cdot[\gamma_3^B] = \delta^{AB}$.

As we move along the complex structure moduli space the integral of Ω over the non-trivial three-cycles of $\mathcal{W}_6$ changes. Hence, if S_4 contains any of these non-trivial three-cycles we may reach a point in which

$$\int_{\pi_3} \Omega \neq 0 \tag{4.183}$$

so that the four-cycle S_4 is no longer holomorphic. Hence, as already pointed out in [157], the corresponding complex structure deformation should be obstructed.

The Wilson line obstruction can be seen from considering the superpotential

$$W^{\mathrm{D7}} = \int_{\Sigma_5} \Omega \wedge F \tag{4.184}$$

which is the analogue of the D6-brane superpotential of section 4.2.2. Taking Σ_5 the five-chain that connects all D7-branes and O7-planes and using Stokes' theorem we obtain that

$$D_\alpha W^{\mathrm{D7}} = \sum_i \int_{S_4^i} \chi_\alpha \wedge A^i \tag{4.185}$$

where χ_α is a harmonic (2,1)-form of $\mathcal{W}_6$ that represents a complex structure modulus [149, 158], and D_α is the corresponding supergravity covariant derivative. Finally, A^i is a (0,1)-form on S_4^i that represents its Wilson line modulus and which will be stabilised by the scalar potential term $K^{\alpha\alpha}|D_\alpha W^{\mathrm{D7}}|^2$.

Hence, as advanced, we have a superpotential of the form (4.180) with X_{closed} a complex structure moduli and Φ_{open} Wilson line moduli. Just like for D6-branes, this superpotential will be non-trivial only if a topological condition is met, namely that these three-cycles of S_4 are non-trivial also in $\mathcal{W}_6$.

Finally, one can easily extend to this case the microscopic mechanism by which Wilson lines are stabilised. For type IIB flux compactifications we will have that gauge invariant three-form flux

$$\tilde{F}_3 = dC_2 - C_0 H_{\mathrm{NS}} \tag{4.186}$$

is not quantised while the combination of three-forms $\tilde{F}_3 - F_1 \wedge B$ is. Switching on a Wilson line will be equivalent to shift the B-field by the appropriate exact two-form, which will nevertheless contribute to the harmonic piece of $F_1 \wedge B$ when F_1 is non-closed and S_4 contains a non-trivial three-cycle. Hence switching on such Wilson line will result on a shift of the harmonic piece of $\tilde{F}_3$, and this will contribute to the energy of the system via the usual scalar potential induced by background fluxes.

Part III

TOPOLOGICAL STRINGS

5 TOPOLOGICAL STRINGS & 5D SUPERCONFORMAL FIELD THEORIES

In the last chapter of this thesis we would like to turn to a completely different direction. So far we have discussed phenomenological implications of String Theory in the belief that String Theory can provide suitable models of particle physics. But the power of String Theory lies beyond its mere applications to phenomenology and it can help to understand physics even in situations that are not realised in the world we observe. There have been many instances in the literature of how String Theory can be employed to study (for example) quantum field theories in different spacetime dimensions. Our focus here will be on supersymmetric field theory in five spacetime dimensions, and in particular we will always consider the case of theories preserving $\mathcal{N} = 1$ supersymmetry (which is equivalent to $\mathcal{N} = 2$ theories in four spacetime dimensions). The interest in this class of theories comes from the simple observation that a Yang–Mills theory in 5d is not power countable renormalisable and therefore it is not clear whether these theories can exist as quantum theories[1]. A great advance in the understanding these theories came with the seminal paper [159] where it was shown that these theories (at least if preserving some amount of supersymmetry) can exist if they posses a fixed point under the renormalisation group flow in the ultraviolet where the theory enjoys conformal symmetry. To revert the process a non renormalisable theory in the infrared can be recovered by taking a suitable 5d superconformal field theory and deforming it via an irrelevant operator which triggers an RG flow. In the case of super Yang–Mills theory the rôle of this irrelevant operator is played by the Yang–Mills gauge coupling. An important signal of the existence of such UV fixed point is the existence of an enhancement of the flavour symmetry when instantonic particles are taken into account. We will try now to spell out the basics of this enhancement for the case of super Yang–Mills theory with a simple gauge group. In this case we have that the current

$$J = *\mathrm{Tr}(F \wedge F) \tag{5.1}$$

is conserved and gives a global $U(1)_I$ symmetry. States charged under this current (which in 5d are particle states) carry instanton charge and therefore are absent in perturbation theory. The important observation of [159] is that when these states are taken into account the $U(1)_I$ combines with the flavour symmetry manifest in perturbation theory to give a larger global symmetry[2] which is totally recovered at the UV fixed point[3].

1 This does not change when considering theories with supersymmetry.

2 For example for the case of a $SU(2)$ theory with $N_f \leq 7$ fundamental hypermultiplets the global symmetry goes from the perturbative $SO(2N_f) \times U(1)_I$ to E_{N_f+1}.

3 Since the Yang–Mills coupling goes to infinity at the UV fixed point instantonic particles become massless and the global symmetry is no longer broken down to the one observed in perturbation theory.

Our interest in this chapter will be in the interplay between 5d supersymmetric theories and topological string theory. Topological string theory includes only a subsector of the Hilbert space of the full physical string theory and therefore constitutes somehow a simplified model of string theory. Nevertheless due to its computability it has led to many insights even in unexpected directions, providing moreover some beautiful links with mathematics. We will simply cite here some well known reviews present in the literature [160–164] referring to them for an introduction to topological string theory and references. For our purposes topological string theory will be a tool that will allow us to compute protected quantities (in particular partition functions) of 5d supersymmetric theories in cases where mere field theory techniques fail to give an answer. In the following we will be interested in the study of the Higgs branch of a particular superconformal field theory in 5d, the so–called T_N theory, with a focus on the computation of its Nekrasov partition function via topological string techniques.

5.1 PARTITION FUNCTION OF HIGGSED T_N THEORIES

In this section we will study the general form of the Higgs branch of 5d T_N theories and describe how the use of topological strings allows to compute the general form of the Nekrasov partition function of these theories. The great advantage of having a String Theory embedding of these theories is that it allows us to perform the computation even for theories where a Lagrangian description is lacking as it happens to be the case for T_N theories. Needless to say that without the input coming from String Theory this computation would be impossible as the usual localisation techniques applied in field theory [165, 166] do require knowledge of the Lagrangian of the theory. We will see later on that String Theory gives us additional information even in the cases where a Lagrangian is available.

The technique that we will use to perform all computation of partition functions in the following is the so–called topological vertex and its refinement [167–170] which we review in Appendix G. The application of this technique is possible for all these theories are realised as the low energy effective field theory of M–theory on a non–compact toric Calabi–Yau threefold. Dually these theories can be realised also in type IIB String Theory using webs of (p, q) 5-branes [171]. The relation between these two descriptions is remarkably simple: the dual of the toric fan is simply the diagram of 5–branes. In this setup it is possible to embed both ordinary gauge theories (by simply considering the compactification of the worldvolume theory on 5–branes on segments [172, 173]) and more general 5d superconformal field theories. This includes the some 5d class $\mathcal{S}$ theories [174] whose original 4d versions were introduced in [175]. In particular among these theories we will find the 5d T_N theory[4] whose 4d version was originally constructed in [175] via compactification of the 6d (2,0) A_n superconformal field theory on a Riemann sphere with three full punctures. As we mentioned it is possible to embed all theories that we will consider either in M–theory or in type IIB String Theory and we will use both descriptions interchangeably in the following. Let us mention only that the latter has the advantage of making the global flavour symmetry of the theory manifest via the introduction of additional 7-branes [177, 178].

4 See [176] for a review of the T_N theory.

One important point to stress is that the computation via the topological vertex does not simply give the partition function of the 5d theory. In particular there will be present contributions due to additional states that are decoupled from the theory constructed via the appropriate (p, q) 5–branes web [179–182]. Here knowledge of the String Theory embedding, and in particular of the brane web, comes to the rescue as it allows the identification of these additional contributions via the application of a simple rule. We can describe this rule quite simply in a sentence: decoupled factors will appear as strings stretched between parallel external legs in the web diagram. The correct partition function will be recovered after appropriately subtracting all these contributions from the result obtained via the application of the topological vertex. We review the procedure for the cancellation of the decoupled factors in Appendix G. This procedure has been carried out in full detail for the T_N theory obtaining the correct partition function [180, 181]. Let us stress that subtraction of the decoupled factors is crucial for performing checks on the theory like enhancement of the global symmetry at the superconformal fixed point [182–186] and dualities at the level of partition function [187], and without removal of the decoupled factors these checks would not have been successful.

In this section we will be interested in the formulation of an algorithm that allows for the computation of the partition function of a T_N theory on the Higgs branch[5]. While the computations mentioned so far are usually performed in the Coulomb branch of the theory it is possible to access the Higgs branch as well by restricting to some particular subloci in the Coulomb branch moduli space. In particular a direction in the Higgs branch will open whenever one of the hypermultiplets will become massless, and quite remarkably this has a simple counterpart in the web diagram: a Higgs branch direction is opened whenever (part of) a brane in the diagram can be removed [173]. To understand the exact locations in the Coulomb branch where this occurs we can use the results of the computation of the 5d superconformal index [188–190]. In fact this protected quantity that can be computed once the Nekrasov partition function acquires a pole whenever a direction in the Higgs branch opens [191, 192]. This has allowed the authors of [181] to identify the correct sublocus in the Coulomb branch moduli space for some directions in the Higgs branch of the T_N theory. We will extend this result for the general form of the Higgs branch of the T_N theory and this will allow us to formulate a general algorithm for the computation of the partition function of this theory in the Higgs branch.

After the formulation of this procedure we will apply it to a particular Higgs branch of the T_6 theory which gives the 5d version of the E_8 theory of Minahan and Nemeschansky [193]. Having the partition function we will perform several checks on it, and of particular interest is the check of the enhancement of its flavour symmetry to E_8 once the contributions of instantonic particles are taken into account. We will check this by computing the 5d superconformal index of the theory finding the correct enhancement of the global symmetry. Note that this result would not be correctly reproduced if we did not appropriately subtract the contributions of decoupled factors. Similar results were previously obtained in [194] by exploiting the fact that the E_8 theory after turning on some irrelevant deformations becomes equivalent to an $SU(2)$

5 Let us recall that the theory is in the Coulomb branch when the scalars in the vector multiplets acquire a vev. The Higgs branch corresponds to the case where the scalars in the hypermultiplets have a vev. A theory will in general possess also mixed branches and as a matter of fact we will consider this case in the following.

	0	1	2	3	4	5	6	7	8	9
D5-brane	×	×	×	×	×	×				
NS5-brane	×	×	×	×	×		×			
(1,1) 5-brane	×	×	×	×	×	angle				
7-brane	×	×	×	×	×			×	×	×

Table 10: The configuration of 5-branes and 7-branes in webs. The angle in the (x_5, x_6)-plane is related to the charge of a (p, q) 5-brane. For example, the $(1, 1)$ 5-brane corresponds to a diagonal line.

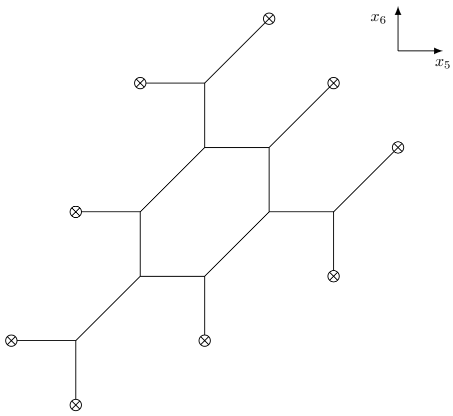

Figure 9: The web diagram for the T_3 theory. Each $\otimes$ represents a 7-brane.

theory with $N_f = 7$ fundamental flavours. We will also perform a comparison with the results of [194] at the level of the partition function finding perfect agreement up to two instantons level.

5.1.1 *Higgs branch of T_N theories*

While there are different ways to engineer 5d T_N theories in string and M-theory the perspective that allows to understand better the Higgs branch of these theories involves webs of (p, q) 5-branes in type IIB string theory [174]. One of the advantages of the use of webs of (p, q) 5-branes, introduced in [172, 173], is that, by terminating the semi-infinite external 5-branes in the diagram on 7-branes, the global symmetry becomes manifest and realised on the 7-branes [177, 178]. The brane configuration of the system is shown in Table 10. The T_N theory can be realised by a web of 5-branes with N external D5-branes, N external NS5-branes and N external $(1, 1)$ 5-branes. As an example we show the web diagram of the T_3 theory in Figure 9 where we can explicitly see $SU(3) \times SU(3) \times SU(3)$ flavour symmetries realised on the 7-branes. In writing a web diagram, our convention is that the horizontal direction represents x_5 and the vertical direction represents x_6, Accordingly, horizontal lines, vertical lines and diagonal lines correspond to D5-branes, NS5-branes and $(1, 1)$ 5-branes respectively.

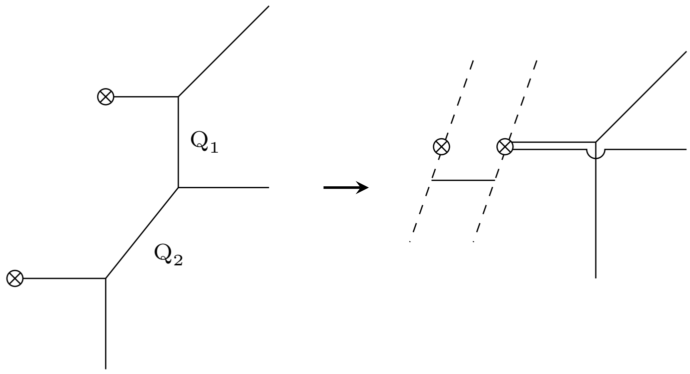

Figure 10: An example of Higgsing by tuning the length of two 5-branes labelled as Q_1, Q_2. The broken lines represent the directions along x^7, x^8, x^9.

Moreover it is easy to realise the Higgs branch of these theories once 7-branes are introduced: after an appropriate tuning of the lengths of 5-branes in the diagrams it is possible to place some of external 5-branes on top of each other, and in this situation the piece of 5-brane hanging between the two 7-branes can move off the diagram as in Figure 10. Stripping off the piece of 5-branes corresponds to giving a large vev to hypermultiplets which in turns induces an RG flow giving a different theory at low energy.

One important constraint on the the Higgsed diagram is preservation of supersymmetry. For instance only a single D5-brane can be connected to an NS5-brane without breaking supersymmetry [174, 195]. Taking into account this and its $SL(2,\mathbb{Z})$ duals in a supersymmetric Higgsed diagram some of the 5-branes will be forced to jump over some other 5-branes like in the example in Figure 10.

All the possible ways to put external 5-branes on 7-branes in a T_N diagram can be nicely represented by a partition of N. A partition of N, namely a set of positive integers $[n_1, n_2, \cdots, n_k]$ such that $\sum_{i=1}^{k} n_i = N$, corresponds to a configuration where n_i 5-branes are put on the same 7-brane. If w_j of the n_i's coincide in the partition, the flavour symmetry is $S[\prod_j U(w_j)]$. For instance the configuration of the external D5-branes of the T_N theory is represented by the partition $[1, \cdots, 1]$ with N 1's and this gives $SU(N)$ flavour symmetry as expected. The Higgsing shown in Figure 10 represents changing the partition $[\ldots, 1, 1, \ldots]$ to $[\ldots, 2, \ldots]$. Quite interestingly these partitions coincide with the ones which appear at a puncture of a Riemann surface characterising class $\mathcal{S}$ theory. Four-dimensional class $\mathcal{S}$ theories are constructed by compactifying a six-dimensional $(2,0)$ theory on a Riemann surface with punctures [175]. 5-brane webs give five-dimensional versions of class $\mathcal{S}$ theories and the partition which classifies the external 5-branes configuration corresponds to the Young diagram at the puncture. Due to this correspondence, we will often call the configuration of external 5-brane ending on 7-branes a puncture.

In the following we will be interested in the computation of the Nekrasov partition function of the T_N theory in the Higgs branch. The computation of the partition function of the T_N theory was performed in [180, 181] using the refined topological vertex [169, 170]. We chose to summarise in Appendix G the techniques necessary for such computation including the identification of the contributions of the decoupled factors.

The computation of the partition functions of the Higgsed 5d T_N theories is more involved. Due to some jumps of 5-branes over other 5-branes Higgsed diagrams are not dual to toric Calabi–Yau threefolds and therefore we cannot directly apply the refined topological vertex formalism. However it is still possible to compute the partition function of the IR theory realised by a Higgsed diagram by first computing the partition function of a UV theory which embeds the IR theory and then applying a suitable tuning condition to the parameters of the UV partition function. For example the Higgsing in Figure 10 requires a tuning

$$Q_1 = \left(\frac{q}{t}\right)^{\frac{1}{2}}, \qquad Q_2 = \left(\frac{q}{t}\right)^{\frac{1}{2}}, \tag{5.2}$$

or

$$Q_1 = \left(\frac{t}{q}\right)^{\frac{1}{2}}, \qquad Q_2 = \left(\frac{t}{q}\right)^{\frac{1}{2}}, \tag{5.3}$$

where the two conditions give equivalent results of the Nekrasov partition function. Here $Q_{1,2} := e^{-L_{1,2}}$ where L is the length of the 5-branes. In the dual description the length is the size of a corresponding two-cycle, q, t are given by $q = e^{-\epsilon_2}, t = e^{\epsilon_1}$ where ϵ_1, ϵ_2 are the chemical potentials associated to the symmetries $SO(2) \times SO(2) \subset SO(4)$ where $SO(4)$ is the little group of the five-dimensional spacetime. An identical tuning condition may be applied for the case where we would like to place on top of each other vertical and diagonal legs.

5.1.2 *The partition function of Higgsed T_N*

We will apply the prescription of the tuning discussed in Section 5.1.1 to formulate an algorithm for the computation of the partition function of the T_N theory in a generic Higgs branch. Afterwards we will apply this to the computation of the partition function of the 5d E_8 which can arise in the infrared limit of a Higgs branch of the T_6 theory [174]. The algorithm consists of the following points

1. We first compute the partition function of a UV theory by the refined topological vertex method. It is important to remove the decoupled factors which are associated with the parallel external legs.
2. For the tuning of putting several parallel external 5-branes on one 7-brane, we impose the condition (5.2) or (5.3) in the case of horizontal 5-branes. The tuning of the Kähler parameters for the two-cycles inside the web diagram is determined by the consistency of the geometry.
3. We parameterise the lengths of internal 5-branes or the Kähler parameters of compact two-cycles by the chemical potentials associated with unbroken gauge symmetries and those of unbroken global symmetries. The unbroken symmetries can be determined by requiring that the tuned two-cycles have no charge under the unbroken symmetries in the Higgs vacuum. Linear combinations of the Cartan generators of the unbroken global symmetries are associated with masses and instanton fugacities in the perturbative regime.
4. After inserting the tuning conditions as well as the new parameterisation, we almost obtain the partition function of the low energy theory in the Higgs branch of the UV theory. However, there can be still some contributions from

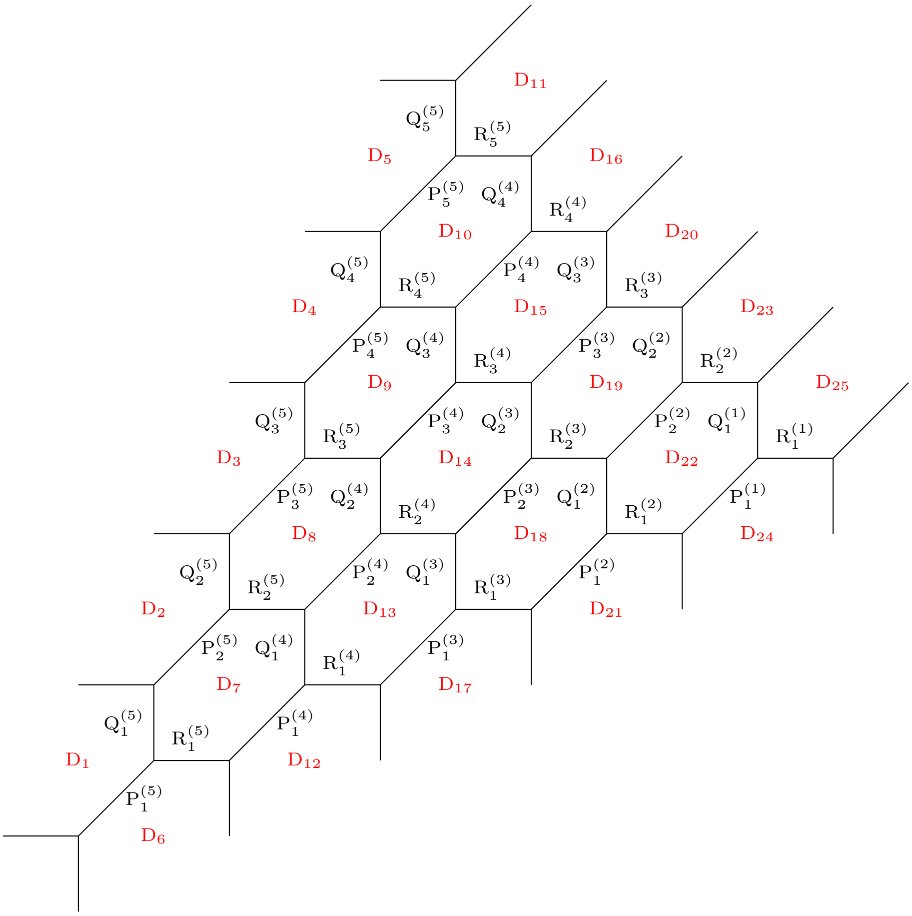

Figure 11: The web diagram for the T_6 theory.

singlet hypermultiplets which we need to remove such contributions. The singlet hypermultiplet factor which depends on some parameters associated with flavour symmetries in the theory may be inferred from the web diagram: their contribution is associated with strings between new parallel external 5-branes which only appear after moving to the Higgs branch[6]. The other singlet hypermultiplet factor which only depends on the Ω–deformation parameters appears in the perturbative part, namely the zeroth order of the instanton fugacities. Once we obtain the perturbative part, we can identify those contributions.

5. After eliminating the singlet hypermultiplet contributions, we finally obtain the partition function of the infrared theory in the Higgs branch.

In this section, we will obtain the partition function of the E_8 theory by following the above steps.

5.1.3 T_6 *partition function*

In this section we review the partition function of the T_6 theory. This theory can be obtained by compactifying M-theory on the blow-up of $\mathbb{C}^3/(\mathbb{Z}_6 \times \mathbb{Z}_6)$ whose toric

6 Note that such a contribution may depend on an instanton fugacity of the theory.

diagram we show in figure 11. In the figure we also show how the fugacities $P_k^{(n)}$, $Q_k^{(n)}$ and $R_k^{(n)}$ are associated to the two cycles present in the geometry. Note that the geometry imposes some conditions on these fugacities

$$Q_k^{(n)} P_k^{(n)} = Q_k^{(n+1)} P_{k+1}^{(n+1)}\,, \quad R_k^{(n+1)} Q_k^{(n+1)} = R_{k+1}^{(n+1)} Q_k^{(n)}\,, \tag{5.4}$$

so that the number of independent Kähler parameters is 25. The partition function of this theory was computed in [180, 181] and here we copy the result

$$Z_{T_6} = (M(t,q)M(q,t))^5\, Z_0\, Z_{inst}\, Z_{dec}^{-1}\,, \tag{5.5}$$

$$M(t,q) = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})^{-1}\,, \tag{5.6}$$

$$\begin{aligned} Z_0 = \prod_{i,j=1}^{\infty} &\left\{ \frac{\left[\prod_{a\le b}(1-e^{-i\lambda_{5;b}+i\tilde{m}_a} q^{i-\frac{1}{2}} t^{j-\frac{1}{2}}) \prod_{b<a}(1-e^{i\lambda_{5;b}-i\tilde{m}_a} q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})\right]}{\prod_{n=1}^{5}\prod_{a<b}(1-e^{i\lambda_{n;a}-i\lambda_{n,b}} q^i t^{j-1})(1-e^{i\lambda_{n;a}-i\lambda_{n,b}} q^{i-1} t^j)} \right\} \\ &\times \prod_{n=2}^{5}\prod_{a\le b}(1-e^{i\lambda_{n;a}-i\lambda_{n-1;b}+i\hat{m}_n} q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})(1-e^{i\lambda_{n-1;b}-i\lambda_{n;a}-i\hat{m}_n} q^{i-\frac{1}{2}} t^{j-\frac{1}{2}})\,, \end{aligned} \tag{5.7}$$

$$\begin{aligned} Z_{inst} = \sum_{\vec{Y}_1,\dots,\vec{Y}_5} &\left\{ \prod_{n=1}^{4} u_n^{|\vec{Y}_n|} \prod_{\alpha=1}^{n}\prod_{s\in Y_{n,\alpha}} \frac{\left[\prod_{\beta=1}^{n+1} 2i\sin\frac{E_{\alpha\beta}-\hat{m}_{n+1}+i\gamma_1}{2}\right]\left[\prod_{\beta=1}^{n-1} 2i\sin\frac{E_{\alpha\beta}+\hat{m}_n+i\gamma_1}{2}\right]}{\prod_{\beta=1}^{n}(2i)^2\sin\frac{E_{\alpha\beta}}{2}\sin\frac{E_{\alpha\beta}+2i\gamma_1}{2}} \right\} \\ &\times \left\{ u_5^{|\vec{Y}_5|} \prod_{\alpha=1}^{5}\prod_{s\in Y_{5,\alpha}} \frac{\left[\prod_{\kappa=1}^{6} 2i\sin\frac{E_{\alpha\emptyset}-\tilde{m}_\kappa+i\gamma_1}{2}\right]\left[\prod_{\beta=1}^{4} 2i\sin\frac{E_{\alpha\beta}+\hat{m}_5+i\gamma_1}{2}\right]}{\prod_{\beta=1}^{5}(2i)^2\sin\frac{E_{\alpha\beta}}{2}\sin\frac{E_{\alpha\beta}+2i\gamma_1}{2}} \right\}\,, \end{aligned} \tag{5.8}$$

$$Z_{dec}^{-1} = \prod_{i,j=1}^{\infty}\prod_{1\le a<b\le 6}\left(1-\left(\prod_{n=a}^{b-1} R_1^{(n)} P_1^{(n)}\right) q^{i-1} t^j\right)\left(1-\left(\prod_{n=a}^{b-1} R_n^{(n)} Q_n^{(n)}\right) q^i t^{j-1}\right)\,. \tag{5.9}$$

In writing the partition function we have used the Coulomb branch moduli $\lambda_{n;k}$ with $1 \le k \le n = 2,\dots,5$ defined as

$$P_k^{(n-1)} Q_k^{(n-1)} = \exp(-i\lambda_{n;k+1} + i\lambda_{n;k})\,, \tag{5.10}$$

and subject to the condition $\sum_{k=1}^{n} \lambda_{n;k} = 0$. Moreover the parameters $\hat{m}_n$ with $n = 2,\dots 5$ are defined as

$$P_k^{(n-1)} = \exp(i\lambda_{n;k} - i\lambda_{n-1;k} + i\hat{m}_n)\,, \tag{5.11}$$

and the parameters $\tilde{m}_k$ with $k = 1,\dots 6$ as

$$P_k^{(5)} Q_k^{(5)} = \exp(-i\tilde{m}_{k+1} + i\tilde{m}_k)\,, \quad P_k^{(5)} = \exp(i\tilde{m}_k - i\lambda_{5;k})\,. \tag{5.12}$$

Moreover the parameters u_k with $k = 1,\dots,5$ are defined as

$$u_k = \sqrt{R_1^{(k)} P_1^{(k)} R_k^{(k)} Q_k^{(k)}}\,. \tag{5.13}$$

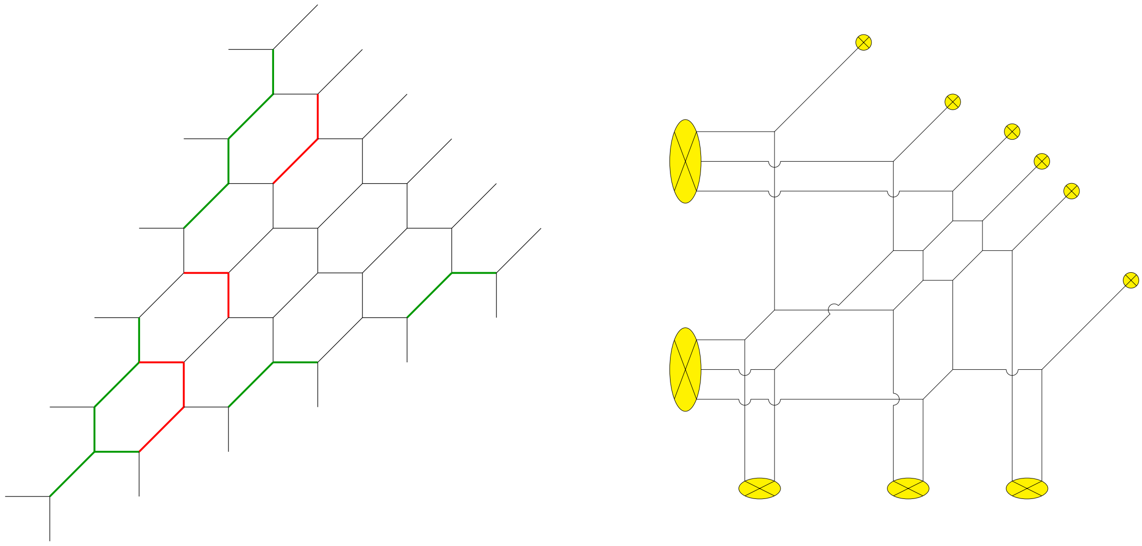

Figure 12: Higgsed T_6 diagram. On the left the original diagram, in green the curves whose Kähler parameters are restricted to engineer the E_8 theory and in red the curves whose Kähler parameters are restricted because of the geometric constraint (5.4). On the right the resulting web diagram after the Higgsing.

The E_8 theory from T_6 theory

It was argued in [174] that it is possible to engineer a theory with an E_8 global symmetry in the Higgs branch of the T_6 theory and we show in figure 12 the web diagram that realises this theory. The resulting theory has a manifest $SU(6)\times SU(3)\times SU(2)$ global symmetry which is believed to enhance to E_8[7]. A similar story happens for a 5d $Sp(1)$ gauge theory with $N_f = 7$ fundamental flavours whose manifest global $SO(14) \times U(1)$ symmetry enhances to E_8 as well at the conformal point [159, 196–198]. The relation between these Lie algebras is shown in figure 13. Furthermore these theories have Coulomb branch and Higgs branch with the same dimensions, namely $\dim_{\mathbb{C}}(\mathcal{M}_C) = 1$ and $\dim_{\mathbb{H}}(\mathcal{M}_H) = 29$. As we will see later the partition function will enjoy an E_8 symmetry providing further evidence for the enhancement of the global symmetry. In order to achieve this diagram from the web diagram of the T_6 theory it is necessary to perform a tuning of the Kähler parameters of some of the curves in the diagram in order to group some of the external 5-branes on a single 7-brane. From figure 12 we see that we need to group the three upper left legs, the three lower left legs, the two leftmost lower legs, the two central lower legs and the two rightmost lower legs. To group the three upper left legs we need to impose

$$Q_5^{(5)} = P_5^{(5)} = Q_4^{(5)} = P_4^{(5)} = \left(\frac{q}{t}\right)^{\frac{1}{2}}\,, \tag{5.14}$$

and to group the three lower left legs the conditions are

$$P_1^{(5)} = Q_1^{(5)} = P_2^{(5)} = Q_2^{(5)} = \left(\frac{q}{t}\right)^{\frac{1}{2}}\,, \tag{5.15}$$

7 As argued in [174] the monodromy given by the system of 11 7-branes is conjugate to the monodromy of the affine E_8 configuration. In particular it is possible to collapse 10 of the 11 7-branes to produce a 7-brane with E_8 gauge symmetry.

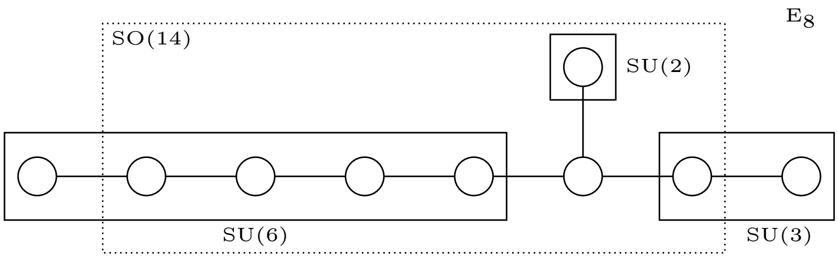

Figure 13: The Dynkin diagram of the affine E_8 Lie algebra. The nodes in the dotted line represent the Dynkin diagram of $SO(14)$. The nodes in the solid lines denote the Dynkin diagram of $SU(6) \times SU(3) \times SU(2)$.

Finally for the leftmost lower legs we impose

$$P_1^{(5)} = R_1^{(5)} = \left(\frac{q}{t}\right)^{\frac{1}{2}} , \tag{5.16}$$

for the central ones we impose

$$P_1^{(3)} = R_1^{(3)} = \left(\frac{q}{t}\right)^{\frac{1}{2}} , \tag{5.17}$$

and finally for the rightmost lower legs we impose

$$P_1^{(1)} = R_1^{(1)} = \left(\frac{q}{t}\right)^{\frac{1}{2}} . \tag{5.18}$$

While these conditions are sufficient to realise the desired pattern for external legs we also need to take into account the geometric constraints of the web diagram (5.4) and in the end some additional Kähler parameters will be restricted. Quite interestingly applying these geometric constraints appears to be equivalent to the propagation of the generalised s-rule presented in [174]. In the end we will have that the geometric constraints (5.4) will imply the following conditions on Kähler parameters

$$Q_4^{(4)} = P_4^{(4)} = Q_1^{(4)} = P_1^{(4)} = R_2^{(5)} = R_3^{(5)} = Q_2^{(4)} = \left(\frac{q}{t}\right)^{\frac{1}{2}} , \tag{5.19}$$

$Sp(1)$ gauge theory parametrisation

In this section we describe how to define the instanton fugacity of the $Sp(1)$ gauge theory analysing the global $SU(6) \times SU(3) \times SU(2)$ symmetry inside E_8. The first step is to determine the unbroken generators of the unbroken flavour symmetry $SU(6) \times SU(3) \times SU(2)$. In the original T_6 theory there are 25 generators, 10 of these generators are associated to compact divisors in the geometry and are parameterised by the Coulomb branch moduli while the remaining 15 are associated to non-compact divisors and realise the $SU(6) \times SU(6) \times SU(6)$ flavour symmetry.

After fixing some Kähler parameters to realise the $Sp(1)$ with 7 flavours gauge theory only a reduced number of generators will be unbroken, namely there will be a single Coulomb branch modulus and the generators of the $SU(6) \times SU(3) \times SU(2)$ flavour symmetry. The unbroken generators are easily identified as the linear combinations of compact and non-compact divisors of the geometry that do not intersect any of curves whose Kähler parameter is restricted. This procedure yields as expected 9 linearly independent generators which we wish to identify with the generators of $SU(6) \times SU(3) \times SU(2)$ and the generator associated to the Coulomb branch modulus. First we label the divisors in the geometry as in figure 11. Naively we would associate the generators of the $SU(6)$ part of the flavour symmetry with the non-compact divisors D_{11}, D_{16}, D_{20}, D_{23}, and D_{25}, the generators of the $SU(3)$ part of the flavour symmetry with the non-compact divisors D_{12} and D_{21}, and the generator of the $SU(2)$ part of the flavour symmetry with the non-compact divisors D_3 while the generator associated with the Coulomb branch modulus with D_{19}. This allows us to identify one of the generators of $SU(6)$ as the linear combination of unbroken generators that contains D_{11} with coefficient 1 but does not contain any of the other flavour generators and the gauge generator. A similar procedure can be applied to the other generators as well allowing the identifications of the generators of the flavour symmetry.

Let us first define the mass parameters $m_i, (i = 1, \cdots, 7)$ as follows,

$$\begin{aligned} Q_3^{(3)} &= e^{i\lambda - im_1}, \quad P_3^{(3)} = e^{-i\lambda + im_2}, \quad R_3^{(4)} = e^{i\lambda + im_3}, \quad P_2^{(3)} = e^{i\lambda + im_4}, \\ Q_1^{(2)} &= e^{i\lambda + im_5}, \quad P_2^{(2)} = e^{-i\lambda - im_6}, \quad R_2^{(2)} = e^{i\lambda - im_7}, \quad R_3^{(3)} = e^{i\tilde{u} - i\lambda}. \end{aligned} \tag{5.20}$$

The dependence of the Coulomb branch modulus λ is determined by the intersection between the compact divisor D_{19} and two-cycles. The two-cycles in (5.20) are the ones which have non-zero intersection number with D_{19}. We also introduced $\tilde{u}$ whose linear combination with $m_i, (i = 1, \cdots, 7)$ eventually becomes a chemical potential for the instanton fugacity of the $Sp(1)$ gauge theory. By using the parameters in (5.20), we find that the fugacities for particles in the canonical simple roots of the flavour symmetry are

$$\begin{aligned} SU(6):& \quad \{e^{im_2 - im_4}, e^{-im_2 - im_3}, e^{im_1 - i\tilde{u}}, e^{-im_6 + im_7}, e^{-im_5 + im_6}\}, \\ SU(3):& \quad \{e^{-im_3 - im_5 - im_6 - i\tilde{u}}, e^{-im_2 - im_4 + im_7 - i\tilde{u}}\}, \\ SU(2):& \quad \{e^{im_1 - im_2 - im_4 - im_5 - im_6 - i\tilde{u}}\}. \end{aligned} \tag{5.21}$$

We would like now the simple roots of $SU(6) \times SU(3) \times SU(2)$ to be understood as roots of E_8 to find a definition for the instanton fugacity. Recalling that the roots of E_8 are

$$\pm(e_i \pm e_j), \tag{5.22}$$

with $i, j = 1, \cdots, 8$ and

$$\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4 \pm e_5 \pm e_6 \pm e_7 \pm e_8), \tag{5.23}$$

with an even number of minus signs, we see that the chemical potentials for the particles in the simple roots of $SU(6) \times SU(3) \times SU(2)$ fit in the E_8 root system if we choose

$$\tilde{u} = \frac{1}{2} m_8 + \frac{1}{2}(m_1 - m_2 - m_3 - m_4 - m_5 - m_6 + m_7). \tag{5.24}$$

Writing the instanton fugacity of the $Sp(1)$ gauge theory as

$$u = e^{\frac{i}{2} m_8} \tag{5.25}$$

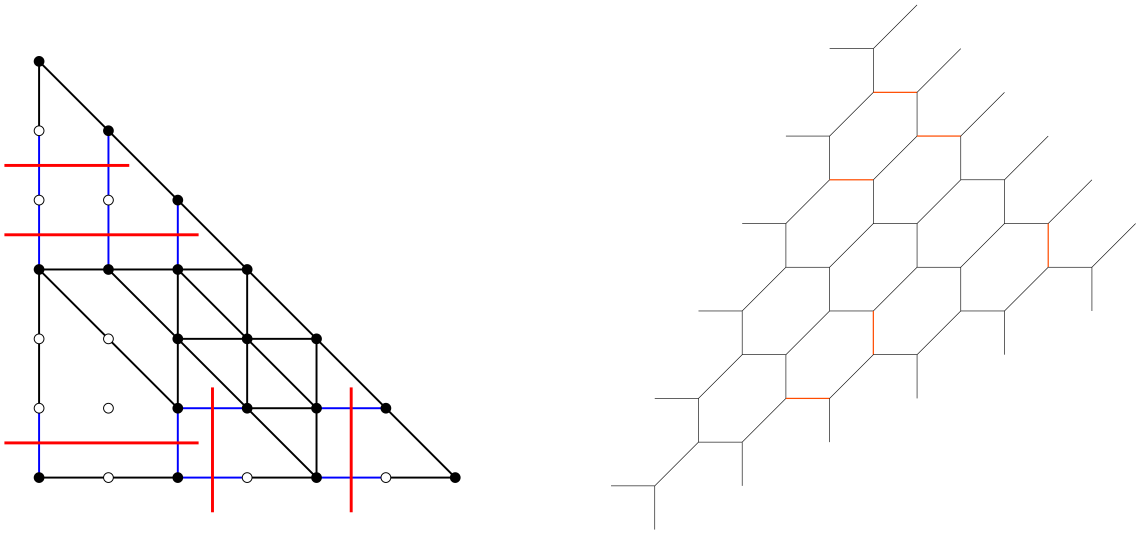

Figure 14: Parallel external legs in the Higgsed T_6 diagram. On the left the identification via the dot diagram, showing in blue parallel pairs of vertical and horizontal legs that can be connected without crossing diagonal lines and in red the corresponding line. On the right the original T_6 diagram with highlighted in orange the legs that become external after Higgsing.

we find that

$$R_3^{(3)} = ue^{-i\lambda+\frac{i}{2}(m_1-m_2-m_3-m_4-m_5-m_6+m_7)} \equiv ue^{-i\lambda+if(m)}\,. \tag{5.26}$$

where for later purposes we have the defined a particular linear combination of masses $f(m)$.

In the perturbative regime of the $Sp(1)$ gauge theory with 7 flavours, the mass parameters are associated with the $SO(14)$ flavour symmetry and the instanton current supplies another $U(1)$ symmetry. However, not all the simple roots of $SO(14)$ inside E_8 as in figure 13 are written by $\pm m_i \pm m_j, (i,j=1,\cdots,7)$ in (5.21). This is because we are in a different Weyl chamber of the E_8 Cartan subalgebra. If we perform a sequence of Weyl reflections, we can write the mass parameters of the particles in the the simple roots of $SO(14)$ inside E_8 as $m_i - m_{i+1}, m_6+m_7, (i=1,\cdots,6)$.

Singlets in the Higgs vacuum

Application of the tuning to the T_6 partition function will not give simply the partition function of $Sp(1)$ gauge theory with $N_f = 7$ fundamental flavours as there will be additional contributions coming from singlet hypermultiplets. Therefore the actual partition function of the E_8 theory will be

$$Z_{E_8} = Z^H_{T_6}/Z_{extra}\,, \tag{5.27}$$

where we called $Z^H_{T_6}$ the T_6 partition function after tuning the Kähler parameters and gathered in Z_{extra} the contributions due to singlet hypermultiplets. In this section, we identify Z_{extra} for the infrared theory in the Higgs branch of the T_6 theory corresponding to figure 11.

We will start by explaining how to identify the singlet hypermultiplets factors that only depend on the Ω–deformation parameters. This kind of singlets originate from M2-branes wrapping two cycles and linear combinations of two cycles whose Kähler parameter is $(q/t)^{\frac{1}{2}}$ and their contributions to the partition function can be understood locally in the diagram. This allows us to split the discussion in six different parts: looking at figure 12 we see that the kind of curves we are interested in appear in the upper left part , in the bottom left part, in the middle top part, in the middle bottom part and in the bottom right part of the diagram. We will now discuss all these contributions separately. In the upper left part the contribution involves the curves $Q_5^{(5)}$, $P_5^{(5)}$, $Q_4^{(5)}$, $P_4^{(5)}$, and the contribution due to singlet hypermultiplets and vector multiplets is

$$Z^{(1)}_{singl} = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})^3 (1 - q^{i+1} t^{j-2})\,. \tag{5.28}$$

In the bottom left part the contribution is a bit more involved, but being careful not to subtract the decoupled factor from parallel diagonal legs that are not external in this case we get the following contribution

$$Z^{(2)}_{singl} = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})^6 (1 - q^{i+1} t^{j-2})\,. \tag{5.29}$$

In the middle left part we have only the curves $Q_2^{(4)}$ and $R_3^{(5)}$. In this case, we need to be careful of subtracting a part of the vector multiplet coming from M2-branes wrapping the two-cycle whose Kähler parameter is $Q_2^{(4)} R_3^{(5)}$. Then, the final contribution is simply

$$Z^{(3)}_{singl} = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})\,. \tag{5.30}$$

Finally we have the contributions in the middle top part (that involves the curves $Q_4^{(4)}$ and $P_4^{(4)}$), in the middle bottom part (that involves the curves $P_1^{(3)}$ and $R_1^{(3)}$) and in the bottom right part (that involves the curves $P_1^{(1)}$ and $R_1^{(1)}$). These contributions are identical and are

$$Z^{(4)}_{singl} = Z^{(5)}_{singl} = Z^{(6)}_{singl} = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})^2\,. \tag{5.31}$$

We are thus able to write the contribution to the partition function coming from decoupled hypermultiplets that only depend on the Ω–deformation parameters

$$Z_{singl} = \prod_{k=1}^{6} Z^{(k)}_{singl} = \prod_{i,j=1}^{\infty} (1 - q^i t^{j-1})^{16} (1 - q^{i+1} t^{j-2})^2\,. \tag{5.32}$$

Next we turn to the discussion of decoupled hypermultiplets that depend on the parameters associated with the flavour symmetry. In [181] this contribution was identified with the perturbative part of the partition function of hypermultiplets and vector multiplets which come from strings stretching between parallel branes that become external after Higgsing. However while in the examples presented in [181] the identification of branes becoming external after Higgsing presented no difficulty in the case of E_8 theory this identification is a bit more subtle because of the propagation of the generalised s-rule inside the diagram, and we propose here a rule to identify new external legs after Higgsing using the dot diagrams introduced in [174]. We identify a new horizontal external leg with a pair of vertical segments in the dot diagram, one external and one internal, that can be connected with a horizontal line without crossing any diagonal line in the dot diagram. A similar identification of parallel external legs works for vertical and diagonal legs in the diagram. Using this procedure we can identify which legs are external for the dot diagram of E_8 theory, and we show in figure 14 the result. In the result of the computation we need to discard the hypermultiplets that only depend on the Ω–deformation parameters as these have already been included in Z_{singl}. Including also the contributions due to the higgsed Cartan part as well as (5.32) we find that the total contribution is

$$\begin{aligned} Z_{extra} = (M(q,t)M(t,q))^{\frac{9}{2}} \prod_{i,j=1}^{\infty} & \left(1-q^{i+1}t^{j-2}\right)^2 \left(1-q^i t^{j-1}\right)^{16} \times \\ & \times (1-ue^{-im_1+im_2+im_4+im_5+im_6+if(m)}q^i t^{j-1})^2 \times \\ & \times (1-ue^{-im_1+im_2+im_4+im_5+im_6+if(m)}q^{i-1} t^{j})^2 \times \\ & \times (1-ue^{-im_1+im_2+im_4+im_5+im_6+if(m)}q^{i-2} t^{j+1}) \times \\ & \times (1-ue^{-im_1+im_2+im_4+im_5+im_6+if(m)}q^{i+1} t^{j-2}) \times \\ & \times (1-ue^{im_2+im_4+im_7+if(m)}q^i t^{j-1})(1-ue^{im_2+im_4+im_7+if(m)}q^{i-1} t^{j}) \times \\ & \times (1-ue^{im_3+im_5+im_6+if(m)}q^{i-1} t^{j})(1-ue^{im_3+im_5+im_6+if(m)}q^{i} t^{j-1}) \times \\ & \times (1-u^2e^{im_2+im_3+im_4+im_5+im_6+im_7+2if(m)}q^{i-1} t^{j}) \times \\ & \times (1-u^2e^{im_2+im_3+im_4+im_5+im_6+im_7+2if(m)}q^{i} t^{j-1}) \,. \end{aligned} \tag{5.33}$$

The partition function of $Sp(1)$ with $N_f = 7$ flavours

Here we write the resulting partition function of the E_8 theory. We recall from the previous section that

$$Z_{E_8} = Z^H_{T_6}/Z_{extra}\,, \tag{5.34}$$

where $Z^H_{T_6}$ is the T_6 partition function after tuning the Kähler parameters and Z_{extra} includes the contributions of singlet hypermultiplets. Before writing the result some comments are needed regarding the instanton summation in (5.7) as the tuning of some Kähler parameters greatly simplifies it. This happens for the tuning of the Kähler parameters will imply the appearance of terms of the form $\sin\left(\frac{E_{\alpha\beta}-\lambda_\alpha+\lambda_\beta}{2}\right)$ giving a zero in the instanton summation whenever $Y_\alpha > Y_\beta$[8]. As in some cases the Young diagram Y_β is trivial this implies that the only possible diagram contributing to the instanton summation is $Y_\alpha = \emptyset$. In the end only 8 Young diagram summations will be non-trivial, and we will call the non-trivial Young diagrams as

$$\begin{aligned} &R_3^{(5)} \to Y_1 \quad R_2^{(4)} \to Y_2 \quad R_3^{(4)} \to Y_3 \quad R_2^{(3)} \to Y_4 \\ &R_3^{(3)} \to Y_5 \quad R_1^{(2)} \to Y_6 \quad R_2^{(2)} \to Y_7 \quad R_1^{(1)} \to Y_8\,. \end{aligned} \tag{5.35}$$

8 We define this inequality as $Y_{\alpha,i} > Y_{\beta,i}$ for each i–th row of Y_α and Y_β

and moreover the result will vanish if $Y_1 > Y_2$ and $Y_8 > Y_6$.

We write the T_6 partition function after tuning the Kähler parameters as

$$Z^H_{T_6} = (M(q,t)M(t,q))^5 Z^H_0 Z^H_{inst}(Z^{//}_{dec} Z^{||}_{dec})^{-1}\,, \tag{5.36}$$

where

$$\begin{aligned}
Z^H_0 = \prod_{i,j=1}^{\infty} & \frac{(1-q^{i+1}t^{j-2})^2(1-q^it^{j-1})^{13}(1-e^{-i\lambda+im_2}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})}{(1-e^{-2i\lambda}q^it^{j-1})(1-e^{-2i\lambda}q^{i-1}t^j)(1-e^{im_5-im_6}q^it^{j-1})(1-e^{im_4-im_2}q^it^{j-1})}\times \\
&\times(1-e^{-i\lambda-im_2}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{i\lambda+im_4}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda+im_4}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})\times \\
&\times(1-e^{i\lambda-im_1}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda-im_1}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda-im_6}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})\times \\
&\times(1-e^{-i\lambda+im_6}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{i\lambda+im_5}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda+im_5}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}}) \\
&\times\frac{(1-ue^{-im_1+im_2+im_4+im_5+im_6+if(m)}q^{i+1}t^{j-2})(1-ue^{-im_1+im_2+im_4+im_5+im_6+if(m)}q^it^{j-1})^2}{(1-ue^{i\lambda+im_2+im_4+im_5+im_6+if(m)}q^{i-\frac{3}{2}}t^{j+\frac{1}{2}})(1-ue^{-i\lambda+im_2+im_4+im_5+im_6+if(m)}q^{i-\frac{3}{2}}t^{j+\frac{1}{2}})} \\
&\times(1-ue^{-im_1+im_2+im_4+im_5+im_6+if(m)}q^{i-1}t^j)^3(1-ue^{im_4+im_5+im_6+if(m)}q^{i-1}t^j) \\
&\times(1-ue^{-im_1+im_2+im_4+im_5+im_6+if(m)}q^{i-2}t^{j+1})(1-ue^{im_2+im_5+im_6+if(m)}q^{i-1}t^j) \\
&\times(1-ue^{im_2+im_4+im_6+if(m)}q^{i-1}t^j)(1-ue^{im_2+im_4+im_5+if(m)}q^{i-1}t^j)\,.
\end{aligned} \tag{5.37}$$

$$\begin{aligned}
1/Z^{//}_{dec} = \prod_{i,j=1}^{\infty} & (1-e^{im_4-im_2}q^it^{j-1})(1-e^{im_3+im_4}q^it^{j-1})(1-e^{im_2+im_3}q^it^{j-1})\times \\
&\times(1-e^{-im_7+im_6}q^it^{j-1})(1-e^{im_5-im_7}q^it^{j-1})(1-e^{im_5-im_6}q^it^{j-1})\times \\
&\times(1-ue^{-im_1+im_3+im_4+if(m)}q^it^{j-1})(1-ue^{-im_1+im_3+im_4+im_5-im_7+if(m)}q^it^{j-1})\times \\
&\times(1-ue^{-im_1+im_2+im_3+if(m)}q^it^{j-1})(1-ue^{-im_1+im_3+im_4-im_7+if(m)+im_6}q^it^{j-1})\times \\
&\times(1-ue^{-im_1+im_5-im_7+if(m)}q^it^{j-1})(1-ue^{-im_1+im_2+im_3-im_7+if(m)+im_6}q^it^{j-1})\times \\
&\times(1-ue^{-im_1+if(m)}q^it^{j-1})(1-ue^{-im_1+im_2+im_3+im_5-im_7+if(m)}q^it^{j-1})\times \\
&\times(1-ue^{-im_1-im_7+if(m)+im_6}q^it^{j-1})\,,
\end{aligned} \tag{5.38}$$

$$\begin{aligned}
1/Z^{||}_{dec} = \prod_{i,j=1}^{\infty} & (1-q^it^{j-1})^3(1-ue^{im_2+im_4-im_7+if(m)}q^{i-1}t^j)(1-ue^{im_2+im_4-im_7+if(m)}q^it^{j-1})\times \\
&\times(1-ue^{im_2+im_3+im_5+im_6+if(m)-im_2}q^{i-1}t^j)(1-ue^{im_3+im_5+im_6+if(m)}q^it^{j-1})\times \\
&\times(1-ue^{im_3+im_5+im_6+if(m)}q^{i-2}t^{j+1})(1-u^2e^{im_2+im_3+im_4+im_5+im_6-im_7+2if(m)}q^{i-1}t^j)\times \\
&\times(1-ue^{im_2+im_4-im_7+if(m)}q^{i-2}t^{j+1})(1-u^2e^{im_2+im_3+im_4+im_5+im_6-im_7+2if(m)}q^it^{j-1})\times \\
&\times(1-ue^{im_2+im_4-im_7+if(m)}q^{i-1}t^j)(1-u^2e^{im_2+im_3+im_4+im_5+im_6-im_7+2if(m)}q^{i-1}t^j)\times \\
&\times(1-ue^{im_3+im_5+im_6+if(m)}q^{i-1}t^j)(1-u^2e^{im_2+im_3+im_4+im_5+im_6-im_7+2if(m)}q^{i-2}t^{j+1})\,,
\end{aligned} \tag{5.39}$$

$$Z_{inst}^{H} = \sum_{Y_1,\dots,Y_8} u_3^{|Y_4|+|Y_5|} Z_L(Y_4,Y_5) Z_M Z_R(Y_4,Y_5) \tag{5.40}$$

$$\begin{aligned} Z_L(Y_4,Y_5) = \prod_{\alpha=2,3} \Bigg[&\prod_{s\in Y_1} \frac{\prod_{\alpha=2,3} 2i\sin\frac{E_{1\alpha}+\lambda_2}{2}}{(2i)^2 \sin\frac{E_{11}}{2}\sin\frac{E_{11}+2i\gamma_1}{2}} \\ &\prod_{s\in Y_\alpha} \frac{\left(2i\sin\frac{E_{\alpha 4}-m_1^L+i\gamma_1}{2} 2i\sin\frac{E_{\alpha\emptyset}-m_2^L+i\gamma_1}{2} 2i\sin\frac{E_{\alpha 5}-m_3^L+i\gamma_1}{2}\right)\left(2i\sin\frac{E_{\alpha 1}-\lambda_2+2i\gamma_1}{2}\right)}{\prod_{\beta=2,3}(2i)^2\sin\frac{E_{\alpha\beta}}{2}\sin\frac{E_{\alpha\beta}+2i\gamma_1}{2}}\Bigg] \\ &u_4^{|Y_2|+|Y_3|} u_5^{|Y_1|} \prod_{\alpha=4,5}\prod_{s\in Y_\alpha}(2i)^2 \sin\frac{E_{\alpha 2}+m_1^L+i\gamma_1}{2}\sin\frac{E_{\alpha 3}+m_1^L+i\gamma_1}{2}\,, \end{aligned} \tag{5.41}$$

$$Z_M = \prod_{\alpha=4,5}\prod_{s\in Y_\alpha} \frac{2i\sin\frac{E_{\alpha\emptyset}-m_1+i\gamma_1}{2}}{2i\sin\frac{E_{\alpha\emptyset}+i\log u-m_2-m_4-m_5-m_6-f(m)+3i\gamma_1}{2}\prod_{\beta=4,5}(2i)^2\sin\frac{E_{\alpha\beta}}{2}\sin\frac{E_{\alpha\beta}+2i\gamma_1}{2}}\,, \tag{5.42}$$

$$\begin{aligned} Z_R(Y_4,Y_5) = \prod_{\alpha=6,7} \Bigg[&\prod_{s\in Y_8} \frac{\prod_{\alpha=6,7} 2i\sin\frac{E_{8\alpha}+\lambda_6}{2}}{(2i)^2 \sin\frac{E_{88}}{2}\sin\frac{E_{88}+2i\gamma_1}{2}} \\ &\prod_{s\in Y_\alpha} \frac{\left(2i\sin\frac{E_{\alpha 4}-m_1^R+i\gamma_1}{2} 2i\sin\frac{E_{\alpha\emptyset}-m_2^R+i\gamma_1}{2} 2i\sin\frac{E_{\alpha 5}-m_3^R+i\gamma_1}{2}\right)\left(2i\sin\frac{E_{\alpha 8}-\lambda_6+2i\gamma_1}{2}\right)}{\prod_{\beta=6,7}(2i)^2\sin\frac{E_{\alpha\beta}}{2}\sin\frac{E_{\alpha\beta}+2i\gamma_1}{2}}\Bigg] \\ &u_2^{|Y_6|+|Y_7|} u_1^{|Y_8|} \prod_{\alpha=4,5}\prod_{s\in Y_\alpha}(2i)^2 \sin\frac{E_{\alpha 6}+m_1^R+i\gamma_1}{2}\sin\frac{E_{\alpha 7}+m_1^R+i\gamma_1}{2} \end{aligned} \tag{5.43}$$

where we defined the parameters

$$\begin{aligned} &\lambda_2 = -\lambda_3 = -\frac{1}{2}(m_2-m_4)\,, \quad \lambda_6 = -\lambda_7 = -\frac{1}{2}(m_6-m_5)\,, \quad \lambda_1 = \lambda_8 = \lambda_\emptyset = 0\,, \quad \lambda_4 = -\lambda_5 = -\lambda\,, \\ &m_1^L = m_3^L = -\frac{1}{2}(m_4+m_2)\,, \quad m_2^L = -i\log u + \frac{1}{2}(m_4+m_2)+m_5+m_6+f(m)-i\gamma_1\,, \\ &m_1^R = m_3^R = -\frac{1}{2}(m_5+m_6)\,, \quad m_2^R = -i\log u + \frac{1}{2}(m_5+m_6)+m_2+m_4+f(m)-i\gamma_1\,, \end{aligned} \tag{5.44}$$

and the instanton fugacities

$$\begin{aligned} &u_5 = e^{i(m_4-m_2)/2}\,, \quad u_4 = u^{1/2}e^{\gamma_1}e^{i[2m_3+m_5+g(m)]/2}\,, \quad u_3 = u^{1/2}e^{-\gamma_1}e^{i[-m_1+f(m)]/2}\,, \\ &u_2 = u^{1/2}e^{\gamma_1}e^{i[m_4+m_5+g(m)]/2}\,, \quad u_1 = e^{-\gamma_1}e^{i(m_5-m_6)/2}\,. \end{aligned}$$

(5.45)

The Z_{inst}^{H} part in (5.36) has a peculiar structure. It is written by gluing $Z_L(Y_4, Y_5)$ and $Z_R(Y_4, Y_5)$ with Z_M. This peculiar structure will be important in the following.

We would like to extract the perturbative part of the partition function, namely we would like to take the limit $\lim_{u\to 0} Z_{E_8}$ and see if this correctly reproduces the perturbative part of $Sp(1)$ gauge theory with $N_f = 7$ fundamental flavours. We start by taking the terms in Z_{inst}^{H} with $Y_4 = Y_5 = \emptyset$ because taking these Young diagrams to be non-trivial only adds terms that vanish in the limit $u \to 0$. Doing this the instanton summation becomes the product of two factors

$$\left(\sum_{Y_1,Y_2,Y_3} Z_L(\emptyset,\emptyset)\right)\left(\sum_{Y_6,Y_7,Y_8} Z_R(\emptyset,\emptyset)\right). \tag{5.46}$$

As for Z_L and Z_R it is actually possible to perform the whole summation by exploiting their relations with a particular Higgs branch of the T_3 theory. The result is particularly simple[9]

$$\left.\left(\sum_{Y_1,Y_2,Y_3} Z_L(\emptyset,\emptyset)\right)\right|_{u=0} = \prod_{i,j=1}^{\infty} \frac{(1-e^{i\lambda+im_3}q^{i-1/2}t^{j-1/2})(1-e^{-i\lambda+im_3}q^{i-1/2}t^{j-1/2})}{(1-e^{im_2+im_3}q^i t^{j-1})(1-e^{im_3+im_4}q^i t^{j-1})}, \tag{5.47}$$

$$\left.\left(\sum_{Y_6,Y_7,Y_8} Z_R(\emptyset,\emptyset)\right)\right|_{u=0} = \prod_{i,j=1}^{\infty} \frac{(1-e^{i\lambda-im_7}q^{i-1/2}t^{j-1/2})(1-e^{-i\lambda-im_7}q^{i-1/2}t^{j-1/2})}{(1-e^{-im_7+im_6}q^i t^{j-1})(1-e^{im_5-im_7}q^i t^{j-1})}. \tag{5.48}$$

We are now able to write the partition function as

$$Z_{E_8} = Z_{pert} Z_{n.p.}\,, \tag{5.49}$$

where

$$\begin{aligned} Z_{pert} = (M(q,t)M(t,q))^{\frac{1}{2}} \prod_{i,j=1}^{\infty} & \frac{(1-e^{-i\lambda+im_2}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda-im_2}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})}{(1-e^{-2i\lambda}q^i t^{j-1})(1-e^{-2i\lambda}q^{i-1}t^j)}\times \\ &\times(1-e^{i\lambda+im_4}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda+im_4}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{i\lambda-im_1}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})\times \\ &\times(1-e^{-i\lambda-im_1}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda-im_6}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda+im_6}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})\times \\ &\times(1-e^{i\lambda+im_5}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda+im_5}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{i\lambda-im_7}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})\times \\ &\times(1-e^{-i\lambda-im_7}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{i\lambda+im_3}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda+im_3}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}}), \end{aligned} \tag{5.50}$$

9 As a matter of fact there is no analytic proof of (5.47) and (5.48). These identities may be checked as a Taylor series in u_2 and u_4.

$$
\begin{aligned}
Z_{n.p.} = Z^H_{inst} \prod_{i,j=1}^{\infty} & \frac{(1-ue^{im_2+im_5+im_6+if(m)}q^{i-1}t^j)(1-ue^{im_4+im_5+im_6+if(m)}q^{i-1}t^j)}{(1-ue^{i\lambda+im_4+im_5+ig(m)}q^{i-\frac{3}{2}}t^{j+\frac{1}{2}})(1-ue^{-i\lambda+im_4+im_5+ig(m)}q^{i-\frac{3}{2}}t^{j+\frac{1}{2}})}\times \\
& \times (1-ue^{-im_1+if(m)}q^i t^{j-1})(1-ue^{-im_1-im_7+if(m)+im_6}q^i t^{j-1})\times \\
& \times (1-ue^{im_1+im_2+im_3+if(m)}q^i t^{j-1})(1-ue^{-im_1+im_5-im_7+if(m)}q^i t^{j-1})\times \\
& \times (1-ue^{im_2+im_4+im_6+if(m)}q^{i-1} t^{j})(1-ue^{-im_1+im_3+im_4+im_5-im_7+if(m)}q^i t^{j-1})\times \\
& \times (1-ue^{im_2+im_4-im_7+if(m)}q^{i-2} t^{j+1})(1-ue^{-im_1+im_2+im_3+im_5-im_7+if(m)}q^i t^{j-1})\times \\
& \times (1-ue^{-im_1+im_2+im_4+im_5+im_6+if(m)}q^{i-1} t^{j})(1-ue^{-im_1+im_3+im_4+if(m)}q^i t^{j-1})\times \\
& \times (1-ue^{im_2+im_4+im_5+if(m)}q^{i-1} t^{j})(1-ue^{-im_1+im_3+im_4-im_7+if(m)+im_6}q^i t^{j-1})\times \\
& \times (1-ue^{im_3+im_5+im_6+if(m)}q^{i-2} t^{j+1})(1-ue^{-im_1+im_2+im_3-im_7+if(m)+im_6}q^i t^{j-1})\times \\
& \times (1-ue^{im_3+im_5+im_6+if(m)}q^{i-1} t^{j})(1-u^2e^{im_2+im_3+im_4+im_5+im_6-im_7+2if(m)}q^{i-1} t^{j})\times \\
& \times (1-ue^{im_2+im_4-im_7+if(m)}q^{i-1} t^{j})(1-u^2e^{im_2+im_3+im_4+im_5+im_6-im_7+2if(m)}q^{i-2} t^{j+1})\times \\
& \times \frac{(1-e^{im_2+im_3}q^i t^{j-1})(1-e^{im_3+im_4}q^i t^{j-1})(1-e^{-im_7-im_6}q^i t^{j-1})(1-e^{im_5-im_7}q^i t^{j-1})}{(1-e^{i\lambda+im_3}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda+im_3}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{i\lambda-im_7}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})(1-e^{-i\lambda-im_7}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}})}\,.
\end{aligned}
\tag{5.51}
$$

With this choice we have that $Z_{n.p.}|_{u=0} = 1$.

Partition function at 1-instanton level

Having successfully reproduced the perturbative part of the partition function of $Sp(1)$ with 7 flavours we would like now to discuss the partition function at 1-instanton level. In order to compute it (and also the partition function at higher instanton level) we will need to take the Young diagrams to be non-trivial and perform the instanton summation for the remaining Young diagrams. We can consider the following quantity

$$
\tilde{Z}_L(Y_4,Y_5) \equiv \frac{\sum_{Y_1,Y_2,Y_3} Z_L(Y_4,Y_5)}{\sum_{Y_1,Y_2,Y_3} Z_L(\emptyset,\emptyset)}\,, \tag{5.52}
$$

and a similar quantity involving $Z_R(Y_4,Y_5)$. Knowing the result of the summation for $Z_L(\emptyset,\emptyset)$ if we are able to compute $\tilde{Z}_L(Y_4,Y_5)$ we automatically have the result of the summation for $Z_L(Y_4,Y_5)$. Explicit computation shows that expressing $\tilde{Z}_L(Y_4,Y_5)$ as a series in the instanton fugacity u_4 the series stops at a finite order. More specifically we expect at level $k=|Y_4|+|Y_5|$ the series terminates at order u_4^k with higher order terms vanishing. We have checked this explicitly up to $k=2$ for higher orders of u_4. We emphasise that the termination of the series happens separately for each choice of Y_4 and Y_5 in the external legs, not only for the sum of all contributions with fixed k. Using this it is possible to compute explicitly the partition function at 1-instanton level and the result matches with field theory one [190]

$$
Z^{Sp(1)}_{k=1} = \frac{1}{32}\left[\frac{\prod_{a=1}^{7} 2i\sin\frac{m_a}{2}}{i^2\sinh\frac{\gamma_1\pm\gamma_2}{2}\sin\frac{i\gamma_1+\lambda}{2}} + \frac{\prod_{a=1}^{7} 2\cos\frac{m_a}{2}}{\sinh\frac{\gamma_1\pm\gamma_2}{2}\cos\frac{i\gamma_1+\lambda}{2}}\right], \tag{5.53}
$$

where we used the notation $\sin(a\pm b) = \sin(a+b)\sin(a-b)$.

2-instanton order and the comparison with field theory result

We would like to understand if Z_{E_8} correctly reproduces the partition function of an $Sp(1)$ gauge theory with 7 fundamental flavours at 2-instanton level, however it is first useful to review how the computation of the instanton partition function is performed in field theory. It is possible to engineer 5d $Sp(N)$ gauge theory with $N_f \leq 7$ in string theory on the worldvolume of N D4-branes in the proximity of N_f D8-branes and an O8-plane. In this system instantons in the 5d gauge theory are D0-branes and as we will discuss later the partition function at k instanton level can be computed as a Witten index in the ADHM quantum mechanics on the worldvolume of k D0-branes. Note that in this system an additional hypermultiplet in the antisymmetric representation of $Sp(N)$ is present which originates from strings stretching between the N D4-branes and the orientifold plane (or the mirror N D4-branes). The presence of the antisymmetric hypermultiplet is important even for the case of $N = 1$ where the antisymmetric representation is trivial for it changes the instanton calculation providing non-perturbative couplings due to small instantons. Even the naive expectation that in the final result for $N = 1$ the contribution due to the antisymmetric representation simply factors out of the partition function is not true for $N_f = 7$ as noted in [190, 194] and the computation performed without including the antisymmetric representation does not give the correct partition function (for instance the superconformal index does not respect the E_8 symmetry). However it is important to note that the computation will contain the contributions of additional states that are present in the string theory realisation but are not present in the field theory, states that can be interpreted as due to strings in the system D0-D8-O8, and once the contributions due to these states are cancelled the 5d partition function is correctly reproduced.

The quantity we would like to discuss is a Witten index Z^k_{QM} for the ADHM quantum mechanics on the worldvolume of k D0-branes

$$Z^k_{QM}(\epsilon_1, \epsilon_2, \alpha_1, z) = \mathrm{Tr}\left[(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-i\epsilon_1(J_1+J_R)} e^{-i\epsilon_2(J_2+J_R)} e^{-i\lambda_i \Pi_i} e^{-izF}\right], \tag{5.54}$$

where Q and its conjugate $Q^\dagger$ are a couple of supercharges, J_1 and J_2 are the Cartan generators of $SO(4)$ symmetry rotating in two orthogonal planes, J_R is the Cartan generator of the $SU(2)_R$ R-symmetry group, λ_i are the Coulomb branch moduli and z generically denote other chemical potentials. Knowing the index $Z_{QM} = \sum_k u^k Z^k_{QM}$ it is possible to compute the instanton part of the 5d partition function as $Z_{inst} = Z_{QM}/Z_{string}$ where Z_{string} contains the contributions of additional states that are present in the string theory realisation but not present in the field theory. We will write its explicit expression later, but first we will discuss how to compute Z^k_{QM}. The result can be expressed as a contour integral in the space of zero modes given by the holonomies of the gauge field and the scalar in the vector multiplet in the ADHM quantum mechanics. Since the gauge group $\hat{G}$ of the ADHM quantum mechanics is compact the holonomies of the vector field actually live in a compact space and the space of zero modes will be the product of r cylinders where r is the rank of $\hat{G}$.

For the case of $Sp(N)$ gauge theories some additional care is needed for $\hat{G} = O(k)$ which is not connected and the k instanton index is the sum of two contributions[10]

$$Z^k_{QM} = \frac{1}{2}(Z^k_+ + Z^k_-) \tag{5.56}$$

where $Z^k_\pm$ is the index for the $O(k)_\pm$ component. We give the relevant definitions in Appendix H.1. The definition of the contour of integration is discussed in [194] and it amounts to selecting some poles according to the Jeffrey–Kirwan prescription. We will not give the definition of the JK residue here referring to [199, 200] for the details and apply a method for the correct identification of the correct poles described in [194]. The rank of $O(2)_+$ is 1 so that the moduli space is a cylinder and we have that

$$\begin{aligned}
Z^2_+ &= \oint_{\mathcal{C}} [d\phi] Z^+_{vec} Z^+_{anti}(m) \prod_{i=1}^{7} Z^+_{fund}(m_i)\,,\\
Z^+_{vec} &= \frac{1}{2^9} \frac{\sinh \gamma_1}{\sinh \frac{\pm\gamma_2+\gamma_1}{2} \sinh \frac{\pm 2\phi \pm\gamma_2+\gamma_1}{2} \sinh \frac{\pm\phi \pm i\lambda+\gamma_1}{2}}\,,\\
Z^+_{anti}(m) &= \frac{\sinh \frac{\pm im-\gamma_2}{2} \sinh \frac{\pm\phi\pm i\lambda - im}{2}}{\sinh \frac{\pm im - \gamma_1}{2} \sinh \frac{\pm 2\phi \pm im - \gamma_1}{2}}\,,\\
Z^+_{fund}(m_i) &= 2 \sinh \frac{\pm\phi + im_i}{2}\,,
\end{aligned} \tag{5.57}$$

where m is the mass of the hypermultiplet in the antisymmetric representation. Moreover the measure of integration is simply $[d\phi] = \frac{1}{2\pi} d\phi$. As we see the integrand has simple poles at the zeroes of the hyperbolic sines with the general form

$$\frac{1}{\sinh \frac{Q\phi + \dots}{2}}\,. \tag{5.58}$$

The contour of integration $\mathcal{C}$ is defined to surround the poles with $Q > 0$, or alternatively we can define the contour of integration as the unit circle in the variable $z = e^\phi$ and substitute $t = e^{-\gamma_1}$ in Z^+_{vec} and $T = e^{-\gamma_1}$ in Z^+_{anti} and taking $t < 1$ and $T > 1$. The two procedures are equivalent for the poles with $Q > 0$ will lay inside the unit circle in z if t is taken sufficiently small and T sufficiently large. In our case the contour $\mathcal{C}$ will surround 10 poles, 6 of which will come from Z^+_{vec} and 4 from Z^+_{anti}, we choose not to write the result of the computation here being it quite long. The

10 This is the correct form for a $Sp(N)$ theory with the discrete θ angle set to zero. If $\theta = \pi$ the result is [179]

$$Z^k_{QM} = \frac{1}{2}(-1)^k (Z^k_+ - Z^k_-)\,. \tag{5.55}$$

Note that the difference between $\theta = 0$ and $\theta = \pi$ is relevant only if there are no fundamental hypermultiplets and therefore it will not affect our computation.

situation is much simpler for Z_-^2 for the rank of $O(2)_-$ is 0 and no integration is needed. The result is

$$\begin{aligned}
Z_-^2 &= Z_{vec}^- Z_{anti}^-(m) \prod_{i=1}^{7} Z_{fund}^-(m_i)\,, \\
Z_{vec}^- &= \frac{1}{32} \frac{\cosh \gamma_1}{\sinh \frac{\pm\gamma_2+\gamma_1}{2} \sinh(\pm\gamma_2+\gamma_1) \sinh(\pm i\lambda+\gamma_1)}\,, \\
Z_{anti}^-(m) &= -\frac{\cosh \frac{\pm im-\gamma_2}{2} \sin(\pm\lambda+m)}{\sinh \frac{im\pm\gamma_1}{2} \sinh(im\pm\gamma_1)}\,, \\
Z_{fund}^-(m_i) &= 2i \sin m_i\,.
\end{aligned} \tag{5.59}$$

The only last piece necessary for the computation of the partition function is the factor Z_{string} that as explained before will cancel from Z_{QM} will cancel the contributions due to additional states present in the string theory realisation of $Sp(1)$ gauge theory. This contribution was computed in [194] and the result for $N_f = 7$ is

$$Z_{string} = \mathrm{PE}\big[f_7(x, y, v, w_i, u)\big]\,, \tag{5.60}$$

where $x = e^{\gamma_1}$, $y = e^{-\gamma_2}$, $v = e^{-im}$, u is the instanton fugacity of $Sp(1)$ gauge theory and $w_i = e^{\frac{i}{2}m_i}$ with $i = 1, \ldots, 7$. In (5.60) we also defined the Plethystic exponential of a function $f(x)$ as

$$\mathrm{PE}[f(x)] = \exp\left[\sum_{n=1}^{\infty} \frac{1}{n} f(x^n)\right]\,. \tag{5.61}$$

Finally in (5.60) f_7 is

$$f_7 = \frac{u\, x^2}{(1-xy)(1-x/y)(1-xv)(1-x/v)} \left[\chi(w_i)^{SO(14)}_{\mathbf{64}} + u\chi(w_i)^{SO(14)}_{\mathbf{14}}\right]\,. \tag{5.62}$$

Knowing this it is possible to extract the instanton partition function of $Sp(1)$ with 7 flavours and one anti-symmetric hypermultiplet at instanton level 2 and check whether there is agreement with the result coming from Z_{E_8}. While it has not been possible so far to check agreement between the two expression because of computational difficulties however it has been possible to check that the two expressions agree in the special limit where all but two masses of the fundamental hypermultiplets are taken to zero. Moreover expanding the two expressions in the fugacity $x = e^{\gamma_1}$ we have found complete agreement between the two expression up to order x^3.

Another check is to see the perturbative flavour symmetry $SO(14)$ at each instanton level. We have checked that the 2-instanton part we obtained is indeed invariant under the Weyl symmetry of $SO(14)$. This is also a non-trivial evidence that our calculation yields the correct result of the 2-instanton part of the E_8 theory. Further check will be discussed in the next section and involves the computation of the superconformal index.

Superconformal index of the E_8 theory

Knowledge of the 5d Nekrasov partition function allows us to perform the computation of the superconformal index which will allow us to verify explicitly the non-perturbative enhancement of the flavour symmetry. The superconformal index for a 5d theory (or equivalently the partition function on $S^1 \times S^4$) is defined as

$$I(\gamma_1, \gamma_2, m_i, u) = \mathrm{Tr}\left[(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-2(j_r+j_R)\gamma_1} e^{-2j_l\gamma_2} e^{-i\sum_i H_i m_i} u^k\right], \quad (5.63)$$

where j_r and j_l are the Cartan generators of $SU(2)_r \times SU(2)_l \subset SO(5)$ with $j_r = \frac{j_1+j_2}{2}$ and $j_l = \frac{j_1-j_2}{2}$, j_R is the Cartan generator of the $SU(2)_R$ R-symmetry group, H_i are the flavour charges and k is the instanton number. The computation of the superconformal index can be performed using localisation techniques and the result is [190]

$$I(\gamma_1, \gamma_2, m_i, u) = \int [d\lambda]_H \,\mathrm{PE}\left[f_{mat}(x, y, e^{i\lambda}, e^{im_i}) + f_{vec}(x, y, e^{i\lambda})\right] \left|I^{inst}(x, y, e^{i\lambda}, e^{im_i}, u)\right|^2, \quad (5.64)$$

where f_{mat} and f_{vec} take into account the perturbative contributions given by hypermultiplets and vector multiplets and they are

$$f_{mat}(x, y, e^{i\lambda}, e^{im_i}) = \frac{x}{(1-xy)(1-x/y)} \sum_{\mathbf{w}\in\mathbf{W}} \sum_{i=1}^{N_f} (e^{-i\mathbf{w}\cdot\lambda - im_i} + e^{i\mathbf{w}\cdot\lambda + im_i}) \quad (5.65)$$

$$f_{vec}(x, y, e^{i\lambda}) = -\frac{xy + x/y}{(1-xy)(1-x/y)} \sum_{\mathbf{R}} e^{-i\mathbf{R}\cdot\lambda} \quad (5.66)$$

where $\mathbf{R}$ is the set of all roots of the Lie algebra of the gauge group and $\mathbf{W}$ is the weight system for the representation of the hypermultiplets. Moreover in (5.64) $[d\lambda]_H$ denotes the the Haar measure of the gauge group which for $Sp(N)$ is equal to

$$[d\lambda]_H = \frac{2^N}{N!}\left[\prod_{i=1}^{N} \frac{d\lambda_i}{2\pi} \sin^2\lambda_i\right] \prod_{i<j}^{N} \left[2\sin\left(\frac{\lambda_i - \lambda_j}{2}\right) 2\sin\left(\frac{\lambda_i + \lambda_j}{2}\right)\right]^2, \quad (5.67)$$

and $|I^{inst}(x, y, e^{i\lambda}, e^{im_i}, u)|^2$ includes the contributions due to instantons and is given by

$$\begin{aligned}\left|I^{inst}(x, y, e^{i\lambda}, e^{im_i}, u)\right|^2 &= I^{inst}_{north}(x, y, e^{i\lambda}, e^{im_i}, u) I^{inst}_{south}(x, y, e^{i\lambda}, e^{im_i}, u) = \\ &= \left[\sum_{k=0}^{\infty} u^{-k} I^k(x, y, e^{-i\lambda}, e^{-im_i})\right]\left[\sum_{k=0}^{\infty} u^k I^k(x, y, e^{i\lambda}, e^{im_i})\right].\end{aligned} \quad (5.68)$$

In (5.68) $I^{inst}_{north}(x, y, e^{i\lambda}, e^{im_i}, u)$ contains the contributions due to anti-instantons localised at the north pole of S^4 and $I^{inst}_{south}(x, y, e^{i\lambda}, e^{im_i}, u)$ contains the contributions of instantons localised at the south pole of S^4. It was noticed in [201] that for the case of an $SU(2)$ gauge group the index can be also computed as

$$I(\gamma_1, \gamma_2, m_i, u) = \int [d\lambda] Z_{Nekra}(\lambda, \gamma_1, \gamma_2, m_i, u) Z_{Nekra}(-\lambda, \gamma_1, \gamma_2, -m_i, u^{-1}), \quad (5.69)$$

where Z_{Nekra} implies the whole partition function obtained by the refined topological vertex after removing decoupled factors. In this case the integration measure does not contain the Haar measure of the gauge group for this is already included in the perturbative contributions due to vector multiplets in the Nekrasov partition function, and so the resulting measure for $SU(2)$ is simply $[d\lambda] = \frac{1}{2\pi} d\lambda$.

We have been able to compute the superconformal index using Z_{E_8} expanding it in the fugacity x up to order x^3 and the result is[11]

$$\begin{aligned} I &= 1 + (1 + \chi^{SO(14)}_{\mathbf{91}} + u\,\chi^{SO(14)}_{\mathbf{64}} + u^{-1}\chi^{SO(14)}_{\overline{\mathbf{64}}} + u^2\chi^{SO(14)}_{\mathbf{14}} + u^{-2}\chi^{SO(14)}_{\overline{\mathbf{14}}})x^2 \\ &+ \chi_{\mathbf{2}}(y)(1 + 1 + \chi^{SO(14)}_{\mathbf{91}} + u\,\chi^{SO(14)}_{\mathbf{64}} + u^{-1}\chi^{SO(14)}_{\overline{\mathbf{64}}} + u^2\chi^{SO(14)}_{\mathbf{14}} + u^{-2}\chi^{SO(14)}_{\overline{\mathbf{14}}})x^3 + \ldots \\ &= 1 + \chi^{E_8}_{\mathbf{248}}\, x^2 + \chi_{\mathbf{2}}(y)(1 + \chi^{E_8}_{\mathbf{248}})x^3 + \ldots \end{aligned} \tag{5.70}$$

which is expected from the branching

$$\begin{aligned} &E_8 \supset SO(14) \times U(1) \\ &\mathbf{248} \to \mathbf{1}_0 + \mathbf{91}_0 + \mathbf{64}_1 + \overline{\mathbf{64}}_{-1} + \mathbf{14}_2 + \overline{\mathbf{14}}_{-2}\,. \end{aligned} \tag{5.71}$$

In (5.70) we have assumed that contributions with higher instanton number will appear in the superconformal index only with higher powers of x. Finally let us mention that we have expanded the partition function Z_{E_8} at order x^4 and found the following contributions to the superconformal index

$$1 + \chi^{SO(14)}_{\mathbf{3080}} + u^2\chi^{SO(14)}_{\overline{\mathbf{1716}}} + u^{-2}\chi^{SO(14)}_{\mathbf{1716}} + \chi_{\mathbf{3}}(y)(1 + \chi^{E_8}_{\mathbf{248}}) \tag{5.72}$$

which again is consistent with the results of [190, 194]. However the complete expression at order x^4 has not been reproduced because part of the expression involves contributions at 3 and 4 instanton number. A similar computation has been performed using the field theory result for the Nekrasov partition function [194] and the same result has been obtained. This provides further evidence for the equality of the partition function at instanton level 2 computed from Z_{E_8} and the field theory result.

5.2 TOPOLOGICAL VERTEX FOR HIGGSED 5D T_N THEORIES

We have seen in the first part of this chapter how it is possible to implement a precise algorithm for the computation of the Nekrasov partition function of 5d the T_N theory in the Higgs branch. While the procedure we outlined is general and works for all the examples checked it is surely a quite intricate one. The main drawback is the necessity of computing first the result of the partition function for the theory in the Coulomb branch and only after this obtain the result in the Higgs branch via a suitable tuning of the Coulomb branch moduli. After this it is also necessary to identify correctly the contribution of singlets in the vacuum and appropriately subtract their contribution to the partition function, a process that can be both tricky and lengthy. In the second part of this chapter we would like to present an alternative method for computing the partition function of the 5d T_N theory which overcomes all these negative features of the previous method. The drawback is that we will be able to formulate a general answer only for the case of the computation done with the topological vertex and obtain only partial results for the case of the refined topological vertex. One important

11 $\chi_{\mathbf{2}}(y) = y + 1/y$ is the character of the fundamental representation of $SU(2)$.

observation is that the web diagram for a theory in the Higgs branch is no longer dual to a toric Calabi–Yau threefold and therefore this would make it impossible to employ directly the topological vertex for this diagram. We will show however that this is not the case and that given a non–toric web diagram it is still possible to directly apply the topological vertex and obtain the correct result for the partition function of the theory. The same applies for some specific examples for the case of the refined topological vertex as well and we will discuss examples where this is possible.

An important point in the computation of the Nekrasov partition function when using the topological vertex and its refinement is the correct identification of the contributions coming from decoupled factors. We described in the previous section how to correctly identify these contributions for the case of toric web diagrams, and an important part of the rule for the topological vertex for non–toric web diagrams is the correct identification of these contributions. We will demonstrate how the rule first described for toric web diagrams still applies in the more general case we are considering, and show moreover that no contributions due to decoupled singlets will be present in the vacuum.

After having completed the formulation of the topological vertex for non–toric diagrams we will apply the newly found rule to some specific examples. In particular we will be able to quickly obtain the result for the partition function of the rank N E_n theories and show that in the case we are considering these theories will coincide with a $Sp(N)$ gauge theory with $N_f = n-1$ fundamental hypermultiplets and one massless antisymmetric hypermultiplet. Quite interestingly the partition function will have a quite peculiar behaviour being a product of N copies of the partition function of an ordinary E_n theory. In particular the unbroken gauge group will be $\bigotimes^n Sp(1)$ with the parent $Sp(N)$ gauge group higgsed by a vev of the antisymmetric hypermultiplet.

5.2.1 *Topological vertex for Higgsed 5d T_N theories*

In this section we discuss how it is possible to define the topological vertex to compute the topological string partition function of the 5d T_N theory in the Higgs branch. We start by showing how the topological vertex can be defined in the simple example of Higgsing parallel external legs of the diagram and then discuss the more general case. Finally we conclude this section by discussing how the decoupled factors that appear in the partition function can be identified in the diagram and appropriately subtracted.

Topological vertex, external legs

Here we will show how it is possible to formulate the topological vertex so that it may be applied directly to non–toric diagrams. We start with the simple example of placing external legs placed on top of each other and deferring the general case to the next section.

The setup we have in mind is the one shown in Figure 15. The topological string partition function for this kind of local diagram can be easily computed for the diagram is toric

$$Z = \sum_{\mu_1,\mu_2} (-1)^{|\mu_1|+|\mu_2|} C_{\lambda_1\mu_1\emptyset}(q) C_{\mu_2\mu_1^t\nu^t}(q) C_{\mu_2^t\lambda_2\emptyset}(q)\,. \tag{5.73}$$

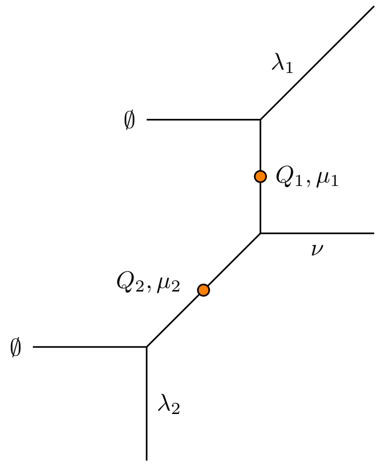

Figure 15: Higgsing of parallel horizontal external legs in a T_N diagram. The orange dots indicate the lines that are shrunk to zero length, and hence we use $Q = 1$ in the computation of the topological string partition function.

The topological vertex is defined as (G.3) with $t = q$. Note that we have set two Kähler parameters $Q_{1,2}$ to 1 since the corresponding two lines are shrunk to zero size. This corresponds to the unrefined version, $q = t$, of the tuning condition (5.2) or (5.3). It is quite straightforward to perform the Young diagram summations on μ_1 and μ_2 using well known identities on Schur functions (G.14) and (G.15), and after these summations the partition function (5.73) becomes

$$\begin{aligned} Z = &\sum_{\eta_1,\eta_2,\kappa_1,\kappa_2,\kappa_3} (-1)^{|\kappa_1|+|\kappa_2|+|\eta_1|+|\eta_2|} q^{\frac{||\lambda_2||^2 - ||\lambda_2^t||^2 + ||\nu^t||^2}{2}} \tilde{Z}_\nu(q) \times \\ &\times s_{\lambda_1^t/\eta_1}(q^{-\rho}) s_{\eta_1^t/\kappa_1^t}(q^{-\nu-\rho}) s_{\kappa_1/\kappa_3}(q^{-\rho}) s_{\kappa_2/\kappa_3}(q^{-\rho}) s_{\eta_2^t/\kappa_2^t}(q^{-\nu^t-\rho}) s_{\lambda_2/\eta_2}(q^{-\rho}) \times \\ &\times \prod_{i,j=1}^{\infty} \frac{(1-q^{i+j-1-\nu_j})(1-q^{i+j-1-\nu_i^t})}{(1-q^{i+j-1})} \, . \end{aligned} \tag{5.74}$$

The last terms in (5.74) are very important at this level, note in fact that if $\nu \neq \emptyset$ the product will always be zero. This greatly simplifies the expression and using the identity (G.17) of Schur functions we can arrive to the simpler expression

$$Z = \sum_{\kappa_3} q^{\frac{||\lambda_2||^2 - ||\lambda_2^t||^2}{2}} s_{\lambda_1^t/\kappa_3}(q^{-\rho}) s_{\lambda_2/\kappa_3}(q^{-\rho}) \prod_{i,j=1}^{\infty} (1-q^{i+j-1}) = C_{\lambda_1\lambda_2\emptyset}(q) \prod_{i,j=1}^{\infty} (1-q^{i+j-1}) \, . \tag{5.75}$$

We see therefore that, up to an infinite product factor, the partition function reduces to a simple topological vertex. Since the infinite product factor will eventually contribute only to decoupled factors in the partition function of a diagram we will cancel it already at this level and propose that the partition function can be computed simply applying the usual topological vertex rule in the Higgsed diagram.

Note that we have carried out the computation with a specific choice of the ordering of the Young diagrams in the topological vertices. A similar computation can

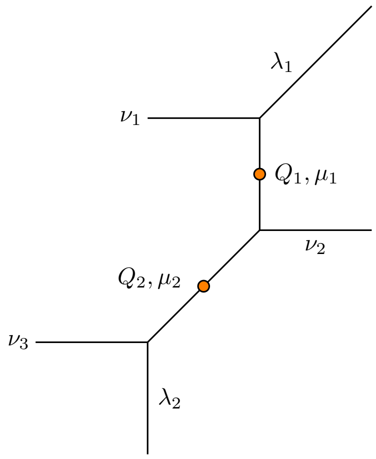

Figure 16: Higgsing of parallel horizontal legs in a T_N diagram. The orange dots indicate the lines that are shrunk to zero length.

be performed for other different choices of orderings in the topological vertices obtaining a result which is always consistent with the cyclic symmetry of the topological vertex.

Topological vertex, general case

The procedure described in the previous subsection was limited to a simple case, namely the case in which the branes that are placed on top of each other are external in the diagram. However, while in order to enter the Higgs branch of the T_N theory placing external branes on top of each other is the starting point, the propagation of the generalised s-rule [174, 195] inside the diagram leads in some cases to a situation in which some internal branes are placed on top of each other, so that it is necessary to have a rule for the topological vertex for this more complicated case. We will follow the same strategy of the previous computation, we show in Figure 16 the diagram we are considering in this case.

The topological string partition function for the local diagram can be computed as usual using the topological vertex

$$Z(\nu_1, \nu_3; \nu_2, \lambda_1, \lambda_2) = \sum_{\mu_1, \mu_2} (-Q_1)^{|\mu_1|}(-Q_2)^{|\mu_2|} C_{\lambda_1^t \mu_1 \nu_1^t}(q) C_{\mu_2 \mu_1^t \nu_2^t}(q) C_{\mu_2^t \lambda_2 \nu_3^t}(q)\,, \tag{5.76}$$

where for the moment we did not impose the tuning condition $Q_1 = Q_2 = 1$. Again it is quite straightforward to perform the summations on the Young diagrams μ_1 and μ_2 giving the following result

$$\begin{aligned}Z(\nu_1,\nu_3;\nu_2,\lambda_1,\lambda_2) = &\sum_{\eta_1,\eta_2,\kappa_1,\kappa_2,\kappa_3} (-Q_1)^{|\eta_1|+|\kappa_2|-|\kappa_3|}(-Q_2)^{|\eta_2|+|\kappa_1|-|\kappa_3|}\\ &\times q^{\frac{||\lambda_2||^2-||\lambda_2^t||^2+||\nu_1^t||^2+||\nu_2^t||^2+||\nu_3^t||^2}{2}} s_{\lambda_1/\eta_1}(q^{-\nu_1^t-\rho})s_{\eta_1^t/\kappa_1^t}(q^{-\nu_2-\rho})\\ &\times s_{\kappa_1/\kappa_3}(q^{-\nu_3^t-\rho})s_{\kappa_2/\kappa_3}(q^{-\nu_1-\rho})s_{\eta_2^t/\kappa_2^t}(q^{-\nu_2^t-\rho})s_{\lambda_2/\eta_2}(q^{-\nu_3-\rho})\\ &\times \tilde{Z}_{\nu_1}(q)\tilde{Z}_{\nu_2}(q)\tilde{Z}_{\nu_3}(q)\\ &\times \prod_{i,j=1}^{\infty}\frac{(1-Q_1q^{i+j-1-\nu_{1,i}-\nu_{2,j}})(1-Q_2q^{i+j-1-\nu_{2,i}^t-\nu_{3,j}^t})}{(1-Q_1Q_2\,q^{i+j-1-\nu_{1,i}-\nu_{3,j}^t})}\,.\end{aligned} \tag{5.77}$$

As in the previous case it is extremely important to carefully look at the last factors in (5.77), and in this case we find that it is important to define properly how to impose the tuning condition $Q_1 = Q_2 = 1$. We start by setting $Q_1 = Q_2 = Q$ and then finally take the limit for Q going to 1. We observe the following behaviour of the infinite product term

$$\begin{aligned}\lim_{Q\to1}\prod_{i,j=1}^{\infty}\frac{(1-Qq^{i+j-1-\nu_{1,i}-\nu_{2,j}})(1-Qq^{i+j-1-\nu_{2,i}^t-\nu_{3,j}^t})}{(1-Q^2\,q^{i+j-1-\nu_{1,i}-\nu_{3,j}^t})} &= \left[\lim_{Q\to1}\left(\frac{1-Q}{1-Q^2}\right)^n\right]F_{\nu_2}^{(\nu_1,\nu_3)}(q)\\ &= \frac{1}{2^n}\,F_{\nu_2}^{(\nu_1,\nu_3)}(q),\end{aligned} \tag{5.78}$$

where n is the number of zero terms in the product $\prod_{i,j=1}^{\infty}(1-Qq^{i+j-1-\nu_{1,i}-\nu_{3,j}^t})$ when we set $Q = 1$. When $\nu_1 = \nu_2^t$ or $\nu_3 = \nu_2^t$, the explicit expression of $F_{\nu_2}^{(\nu_1,\nu_3)}(q)$ is

$$\prod_{i,j=1}^{\infty}(1-Qq^{i+j-1-\nu_{1,i}-\nu_{1,j}^t}) \quad \text{or} \quad \prod_{i,j=1}^{\infty}(1-Qq^{i+j-1-\nu_{3,i}-\nu_{3,j}^t}), \tag{5.79}$$

respectively. It is important to note that (5.78) can be non–zero even if both $\nu_1 = \nu_2^t$ and $\nu_3 = \nu_2^t$ are not satisfied.

As opposed to the intuition from the Higgsed web diagram in Figure 16, neither $\nu_1 = \nu_2^t$ nor $\nu_3 = \nu_2^t$ is required for a non-zero result of (5.77). Even in the case with $\nu_1 = \nu_2^t$ or $\nu_3 = \nu_2^t$, the infinite product factor gives the weight $\frac{1}{2^n}$ as in (5.78). When we simply focus on the case of $\nu_1 = \nu_2^t$ of (5.77), it is possible to use the identity (G.16) and arrive at the following result

$$\begin{aligned}Z(\nu_1,\nu_3;\nu_1^t,\lambda_1,\lambda_2) = &\,q^{\frac{||\lambda_2||^2-||\lambda_2^t||^2+||\nu_1||^2+||\nu_1^t||^2+||\nu_3||^2}{2}}\tilde{Z}_{\nu_1}^2(q)\tilde{Z}_{\nu_3}(q)\\ &\times\frac{1}{2^n}\prod_{i,j=1}^{\infty}(1-q^{i+j-1-\nu_{1,i}-\nu_{1,j}^t})\sum_{\kappa_3}s_{\lambda_1/\kappa_3}(q^{-\nu_3^t-\rho})s_{\lambda_2/\kappa_3}(q^{-\nu_3-\rho})\\ = &\,\frac{1}{2^n}q^{\frac{||\nu_1||^2+||\nu_1^t||^2}{2}}\tilde{Z}_{\nu_1}^2(q)\,C_{\lambda_1^t\lambda_2\nu_3}\prod_{i,j=1}^{\infty}(1-q^{i+j-1-\nu_{1,i}-\nu_{1,j}^t})\,,\end{aligned}$$

(5.80)

where we also used that $\tilde{Z}_\nu(q) = \tilde{Z}_{\nu^t}(q)$. At this point note that

$$\begin{aligned}\prod_{i,j=1}^{\infty}(1-q^{i+j-1-\nu_i-\nu_j^t}) &= \prod_{i,j=1}^{\infty}(1-q^{i+j-1})\prod_{s\in\nu}(1-q^{l_\nu(s)+a_\nu(s)+1})(1-q^{-l_\nu(s)-a_\nu(s)-1}) \\ &= (-1)^{|\nu|}\prod_{i,j=1}^{\infty}(1-q^{i+j-1})\prod_{s\in\nu}(1-q^{l_\nu(s)+a_\nu(s)+1})^2 q^{-l_\nu(s)-a_\nu(s)-1} \\ &= (-1)^{|\nu|}\prod_{i,j=1}^{\infty}(1-q^{i+j-1})\tilde{Z}_\nu^{-2}(q)q^{-\frac{||\nu||^2+||\nu^t||^2}{2}}\,,\end{aligned} \tag{5.81}$$

so that in the end we arrive at the quite simple result

$$Z(\nu_1,\nu_3;\nu_1^t,\lambda_1,\lambda_2) = \frac{1}{2^n}(-1)^{|\nu_1|}\,C_{\lambda_1^t\lambda_2\nu_3}\prod_{i,j=1}^{\infty}(1-q^{i+j-1})\,, \tag{5.82}$$

Therefore, we obtain the single topological vertex as in section 5.2.1 but the weight $\frac{1}{2^n}$ appears in the final expression. Also (5.82) is not exactly equal to (5.77) with $Q_1 = Q_2 = 1$ since we ignore the cases where $\nu_1 \neq \nu_2^t$. The case with $\nu_3 = \nu_2^t$ also yields the same expression with ν_1 exchanged with ν_3 and we again have the weight $\frac{1}{2^n}$.

While these facts may lead to think that it is not possible to define a variation of the vertex rule for the diagram in Figure 16 we will now argue that, after the properly gluing the remaining contributions involving the Young diagrams ν_1 and ν_3, the rule for the computation is rather simple. Since we assume that both ν_1 and ν_3 are non-trivial, those legs should be glued to some other legs in a complete diagram. Hence, in order to compute the partition function of the Higgsed diagram, it is enough if we obtain the same result after performing the Young diagram summations of ν_1, ν_3. In fact, it will turn out that the computation by replacing (5.77) with the single topological vertex as in (5.82) but without $\frac{1}{2^n}$ exactly yields the same result as the original one of (5.77) after summing up the Young diagrams ν_1, ν_3.

To this end we consider the diagram in Figure 17 and compute its local contribution to the topological string partition function. The result is the following one

$$\begin{aligned}Z(\nu_2,\lambda_1,\lambda_2) &= \sum_{\nu_1,\nu_3}(-\tilde{Q})^{|\nu_1|+|\nu_3|}C_{\emptyset\emptyset\nu_1}(q)C_{\emptyset\emptyset\nu_3}(q)Z(\nu_1,\nu_3;\nu_2,\lambda_1,\lambda_2) \\ &= q^{\frac{||\nu_2^t||^2+||\lambda_2||^2-||\lambda_2^t||}{2}}\tilde{Z}_{\nu_2}(q) \\ &\quad \sum_{\nu_1,\nu_3}\frac{1}{2^n}\,(-\tilde{Q})^{|\nu_1|+|\nu_3|}Z_1^{\{\nu_1,\nu_3\}}(\nu_2,\lambda_1,\lambda_2)\,,\end{aligned} \tag{5.83}$$

where we defined

$$\begin{aligned}Z_1^{\{\nu_1,\nu_3\}}(\nu_2,\lambda_1,\lambda_2) = &\sum_{\eta_1,\eta_2,\kappa_1,\kappa_2,\kappa_3}(-1)^{|\eta_1|+|\eta_2|+|\kappa_1|+|\kappa_2|-2|\kappa_3|} \\ &\times s_{\lambda_1/\eta_1}(q^{-\nu_1^t-\rho})s_{\eta_1^t/\kappa_1^t}(q^{-\nu_2-\rho})s_{\kappa_1/\kappa_3}(q^{-\nu_3^t-\rho}) \\ &\times s_{\kappa_2/\kappa_3}(q^{-\nu_1-\rho})s_{\eta_2^t/\kappa_2^t}(q^{-\nu_2^t-\rho})s_{\lambda_2/\eta_2}(q^{-\nu_3-\rho}) \\ &\times q^{\frac{||\nu_1||^2+||\nu_1^t||^2+||\nu_3||^2+||\nu_3^t||^2}{2}}\tilde{Z}_{\nu_1}^2(q)\tilde{Z}_{\nu_3}^2(q)\,F_{\nu_2}^{(\nu_1,\nu_3)}(q)\,,\end{aligned} \tag{5.84}$$

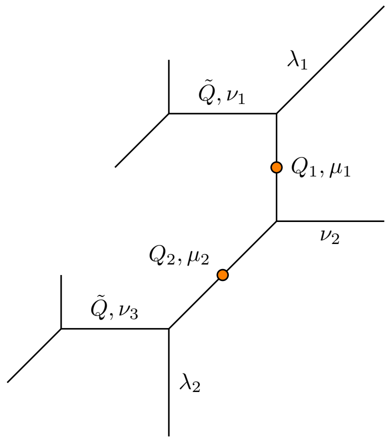

Figure 17: Higgsing of parallel horizontal legs in a T_N diagram with some additional parts of the diagram glued. The orange dots indicate the lines that are shrunk to zero length.

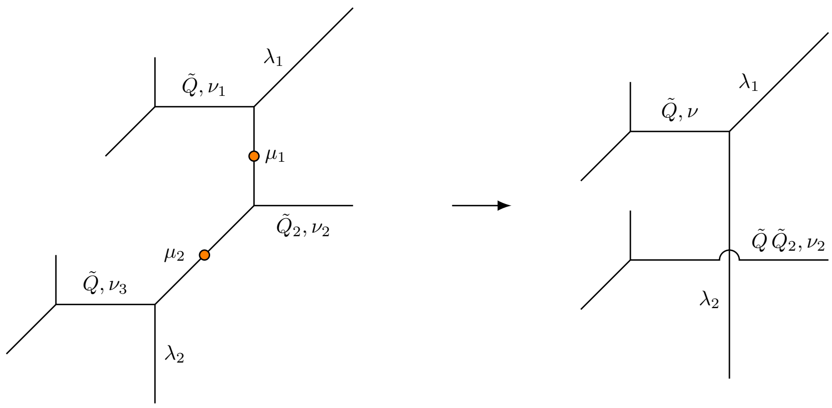

Figure 18: The replacement of the vertical Higgsed strip part with a trivalent vertex after gluing two horizontal legs.

We then compare the result (5.83) with another computation by replacing the vertical strip diagram in Figure 17 with the single topological vertex as in Figure 18. The topological string partition function computed from the local diagram of the right figure in Figure 18 is

$$\begin{aligned} Z'(\nu_2, \lambda_1, \lambda_2) &= \tilde{Q}^{|\nu_2|} C_{\emptyset\emptyset\nu_2^t}(q) \sum_{\nu} (-\tilde{Q})^{|\nu|} C_{\emptyset\emptyset\nu}(q) C_{\lambda_1^t \lambda_2 \nu^t}(q) \\ &= \tilde{Q}^{|\nu_2|} \left[q^{\frac{||\nu_2^t||^2}{2}} \tilde{Z}_{\nu_2}(q) \right] q^{\frac{||\lambda_2||^2 - ||\lambda_2^t||}{2}} \sum_{\nu} (-\tilde{Q})^{|\nu|} Z_2^{\{\nu\}}(\lambda_1, \lambda_2), \end{aligned} \tag{5.85}$$

where we omitted a factor $(-\tilde{Q}_2)^{|\nu_2|}$ since this is not taken into account in (5.83) either. We also defined

$$Z_2^{\{\nu\}}(\lambda_1, \lambda_2) = q^{\frac{||\nu||^2 + ||\nu^t||^2}{2}} \tilde{Z}_\nu^2(q) \sum_{\eta} s_{\lambda_1/\eta}(q^{-\nu^t - \rho}) s_{\lambda_2/\eta}(q^{-\nu - \rho}) \,, \tag{5.86}$$

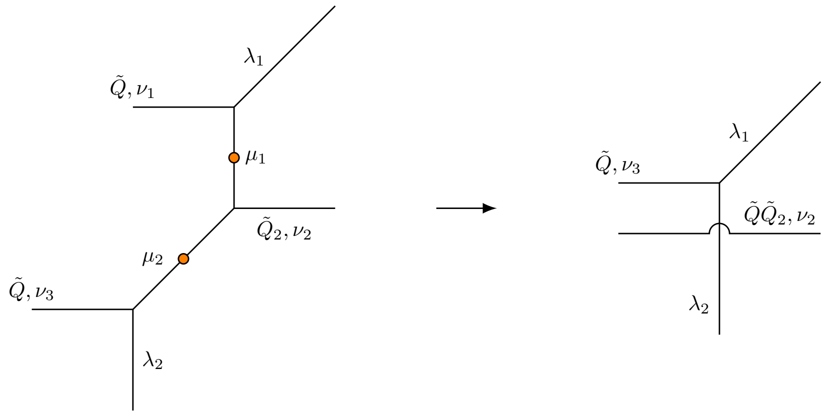

Figure 19: The general procedure for the computation of the topological string partition function in a Higgsed diagram.

We argue that the summation over two Young diagrams ν_1 and ν_3 by (5.84) can be precisely reproduced by the single Young diagram summation of ν by (5.86) up to a irrelevant infinite product. More specifically, we propose that[12][13]

$$\sum_{\substack{\nu_1,\nu_3 \\ |\nu_1|+|\nu_3|=\kappa}} \frac{1}{2^n}(-\tilde{Q})^{|\nu_1|+|\nu_3|} Z_1^{\{\nu_1,\nu_3\}}(\nu_2,\lambda_1,\lambda_2) = \sum_{\substack{\nu \\ |\nu|=\kappa-|\nu_2|}} \tilde{Q}^{|\nu_2|}(-\tilde{Q})^{|\nu|} Z_2^{\{\nu\}}(\lambda_1,\lambda_2)\,. \tag{5.87}$$

Note that the summation in (5.87) is taken over for fixed κ. This relation has been checked for different choices of λ_1, λ_2 and ν_2 and for different values of κ finding always perfect agreement. The appearance of the $\tilde{Q}^{|\nu_2|}$ in (5.87) is consistent with the fact that in Figure 18 in the Higgsed diagram the ν_2 summation is associated with a leg whose Kähler parameter is $\tilde{Q}\tilde{Q}_2$. In other words, $\tilde{Q}^{|\nu_2|}$ factor of (5.87) reproduces the $\tilde{Q}^{|\nu_2|}$ factor appearing in (5.85).

By using (5.87), we have found that[14]

$$Z(\nu_2,\lambda_1,\lambda_2) = Z'(\nu_2,\lambda_1,\lambda_2)\,. \tag{5.88}$$

This implies that after the summation we obtain the same result when we replace the vertical strip part of the diagram in Figure 17 with a trivalent vertex as in Figure 18. This leads to a proposal that we can always perform the replacement of Figure 19 for the computation of the partition function of theories from Higgsed diagrams

12 Actually we observe that something stronger than (5.87) holds. In fact per each Young diagram ν there exist a set of pairs of Young diagrams $\{\nu_1,\nu_3\}$ such that the sum on the left hand side of (5.87) restricted to these pairs correctly reproduces $Z_2^{\{\nu\}}(\lambda_1,\lambda_2)$. In the following explicit examples we will see the occurrence of this phenomenon.

13 In the right hand side of (5.87) we discarded a factor $\prod_{i,j=1}^{\infty}(1-q^{i+j-1})$ which will play no rôle in the following.

14 In the following we shall remove the infinite product term that appears in (5.87) because this is only a contribution due to singlet hypermultiplets.

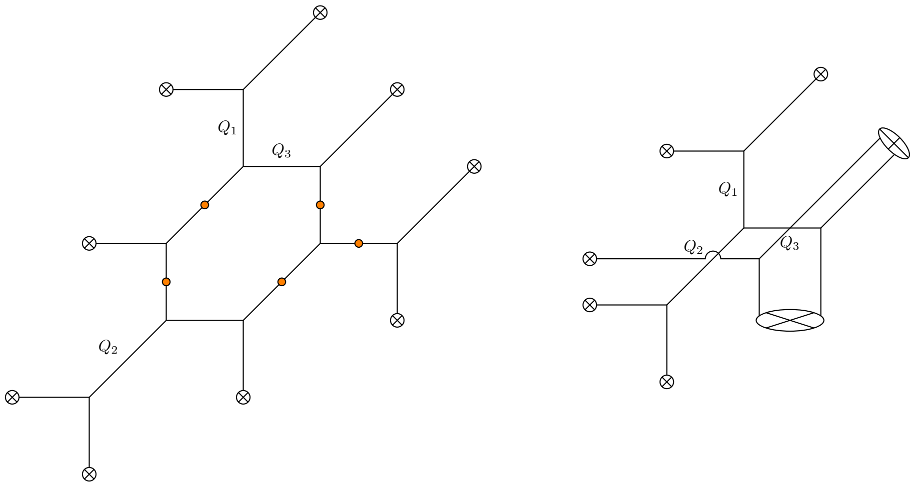

Figure 20: On the left: a particular Higgs branch of a T_3 diagram. On the right: the diagram that allows to compute the topological string partition function with a single gluing.

since both methods give the same result. Therefore, for the Higgsing in Figure 16, we can again use the topological vertex[15]

$$Z(\nu_1, \nu_3; \nu_2, \lambda_1, \lambda_2) \to (-1)^{|\nu_1|}\, C_{\lambda_1^t \lambda_2 \nu_3} \tag{5.90}$$

with $\nu_2^t = \nu_1$. Using this rule which pictorially we illustrate in Figure 19 it is therefore possible to directly compute the topological string partition function applying the topological vertex to the Higgsed diagram.

Note moreover that the vertex (5.90) that replaces the original diagram contains a peculiar factor of the form $(-1)^{|\nu|}$. The presence of this factor is actually perfectly consistent and allows us to compute the topological string partition function in the Higgsed diagram with the usual rules of the topological vertex. We show in Figure 19 how the Kähler parameters are assigned in the case of gluing some legs, and the presence of the additional $(-1)^{|\nu|}$ factor simply allows to compute the topological string partition function introducing the usual factor $(-Q)^{|\nu|}$ where we called Q the total Kähler parameter of the two glued legs. In the example of Figure 19, the Q is $\tilde{Q}\tilde{Q}_2$.

We would also like to discuss an explicit example to further corroborate our conjecture. The diagram we consider is the one in Figure 20. We can first try to compute the result by simply applying the topological vertex to the Higgsed web diagram of the right figure of Figure 20. Since the Higgsed web diagram consists of a

15 Or we can also use

$$Z(\nu_1, \nu_3; \nu_2, \lambda_1, \lambda_2) \to (-1)^{|\nu_3|}\, C_{\lambda_1^t \lambda_2 \nu_1} \tag{5.89}$$

with $\nu_2^t = \nu_3$. The two ways of the replacement give the same result in the end for the computation of a complete diagram.

single T_2 diagram[16] it is possible to compute the topological string partition function performing all Young diagram summations and the result is simply [180, 181, 202]

$$Z^{\text{Higgs } T_3\text{-}1} = \tilde{Z}_{T_2} = \prod_{i,j=1}^{\infty} \frac{(1 - Q_1Q_2Q_3q^{i+j-1}) \prod_{k=1}^{3}(1 - Q_kq^{i+j-1})}{(1 - Q_1Q_2q^{i+j-1})(1 - Q_2Q_3q^{i+j-1})(1 - Q_1Q_3q^{i+j-1})} . \tag{5.91}$$

Let us compare the result with the one computed by the original method using the tuned UV diagram of the left figure of Figure 20. In this example it is also possible to directly use the result of the topological string partition function for a Higgsed T_3 computed in [203] and perform an additional Higgsing in the diagram. The result is simple and can be written in terms of infinite products

$$Z^{\text{Higgs } T_3\text{-}2} = \prod_{i,j=1}^{\infty} \frac{(1 - q^{i+j-1})^2(1 - Q_1Q_2Q_3q^{i+j-1}) \prod_{k=1}^{3}(1 - Q_kq^{i+j-1})}{(1 - Q_1Q_2q^{i+j-1})(1 - Q_2Q_3q^{i+j-1})(1 - Q_1Q_3q^{i+j-1})} . \tag{5.92}$$

We see therefore that up to some irrelevant singlet hypermultiplets the two results perfectly agree providing further evidence for the procedure we described.

Finally one last comment regarding the cyclic symmetry of the topological vertex. While we have done our computation with a very specific choice of ordering of the Young diagrams in the topological vertex it is possible to do the computation with all other possible orderings and the result is always consistent with the cyclic symmetry of the topological vertex (although it is not necessary to appeal to it to do the computation).

Decoupled factors for Higgsed 5d T_N theories

In this section we discuss how to correctly identify the decoupled factors in the Higgsed diagram. It is nice if a simple variant of the usual rule for the identification of these factors can be applied to Higgsed diagrams leading to a very simple procedure for the subtraction of these contributions in the partition function. We will see in this subsection that this is indeed the case. Moreover we will see that a great advantage of using the vertex rule for the Higgsed diagram is that it is no longer necessary to identify and cancel the contributions of singlet hypermultiplets because these contributions are not present at all in the Higgsed diagram. We recall that the rule for the identification of the decoupled factors may be found in Appendix G.

We would like to discuss how the contributions of decoupled factors can be computed in the case of a Higgsed diagram. We will look directly at the example of putting a pair of parallel external 5-branes on top of each other for this already contains all the relevant information. The diagram we consider is in Figure 22, and in particular we will be interested in the contribution of decoupled factors coming from parallel diagonal legs. We see that in the toric diagram (therefore before putting the horizontal 5-branes on top of each other) the only contribution to the decoupled factor in the local part of the diagram we are considering is simply

$$Z_{dec,//} = \prod_{i,j=1}^{\infty} (1 - Q_1Q_2q^{i+j-1})^{-1} . \tag{5.93}$$

16 The trivalent vertex with trivial representations on all the legs gives a trivial contribution.

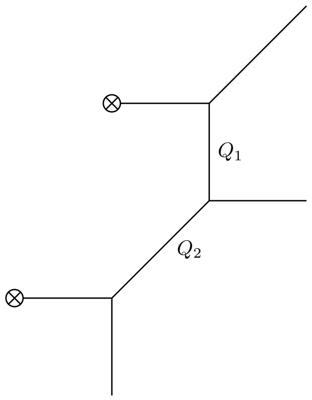

Figure 21: Diagram contributing to decoupled factors in the topological string partition function.

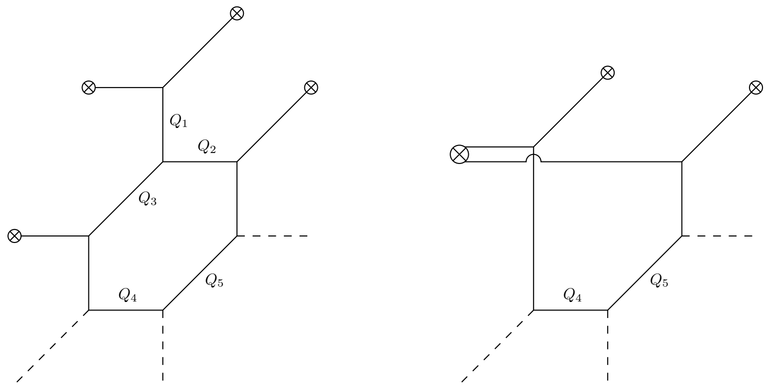

Figure 22: On the left: Diagram with decoupled factors coming from parallel diagonal and horizontal legs. On the right: The diagram after Higgsing corresponding to putting external horizontal legs together.

We would like now to look at the particular case in which we put the two parallel horizontal external legs on top of other, and in order to do this we need to tune the Kähler parameters of the diagram so that $Q_1 = Q_3 = 1$. After entering the Higgs branch the contribution (5.93) will still be present, however the curve with Kähler parameter Q_1Q_2 is no longer present in the diagram and it seems a bit subtle to correctly identify the contribution of such decoupled factor directly in the Higgsed diagram. In fact in the case in which $Q_1 = Q_3 = 1$ because of the geometric relations in the diagram it is true that $Q_2 = Q_4Q_5$ and the contribution to the topological string partition function of the decoupled factor is simply

$$Z_{dec,//} = \prod_{i,j=1}^{\infty}(1 - Q_2 q^{i+j-1})^{-1} = \prod_{i,j=1}^{\infty}(1 - Q_4Q_5 q^{i+j-1})^{-1}\,. \tag{5.94}$$

We see therefore that, while the original curve that contributed to the decoupled factor is no longer present in the Higgsed diagram, it is true that a different curve is still present hosting the same contribution and therefore the contributions due the decoupled factors can be easily identified in the Higgsed diagram as well. The rule we have described is actually sufficient to correctly compute the decoupled factors

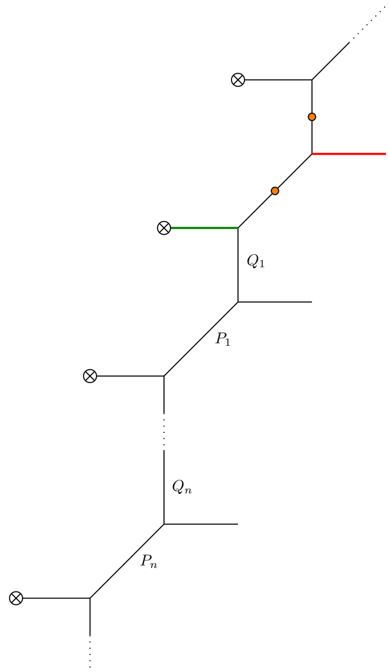

Figure 23: The diagram which illustrates the cancellation between decoupled factors and singlet hypermultiplets. The decoupled factors coming from strings between the green leg and the other external legs cancels the contribution of singlet hypermultiplets coming from strings between the red leg and the external leg after the Higgsing. The orange dots indicates the lines becoming zero size.

in any Higgsed diagram for more complicated cases can always be studied by simple iteration of this rule.

Another fundamental piece in the correct computation of the partition function in the Higgs branch of 5d theories is the correct identification of the contributions of singlet hypermultiplets that we discussed in the first part of this chapter. Remarkably these contributions are totally absent if the partition function is computed using the usual topological vertex rule for Higgsed diagrams[17]. Their absence is due to a cancellation of contributions between a part of the decoupled factors and singlet hypermultiplets and we will show now the basics of this cancellation.

We show in Figure 23 the situation we are considering. Note that after putting the pair of external 5-branes on top of each other the 5-brane coloured in red will become an external brane and therefore there will be contributions due to singlet hypermultiplets in the partition function that originate from curves connecting the new external 5-brane and the 5-branes that were external before entering the Higgs branch. The contributions due to these hypermultiplets in the partition function is easily computed

$$Z_{hyper} = \prod_{i,j=1}^{\infty} (1 - q^{i+j-1})^2 \prod_{k=1}^{n} \left(1 - q^{i+j-1} \prod_{l=1}^{k} Q_l P_l \right) . \tag{5.95}$$

17 We stress that it is crucial to drop the infinite product term in (5.88) in order not to have any contribution of singlet hypermultiplets in the case of Higgsed diagrams.

However the topological string partition function also contains contributions due to the decoupled factors that come from curves connecting the external 5-branes of the original diagram before entering the Higgs branch, and in particular if we consider the contributions that involve only curves connecting the 5-brane coloured in green in Figure 23 we see that this contribution is

$$Z_{dec} = \prod_{i,j=1}^{\infty} (1 - q^{i+j-1})^{-1} \prod_{k=1}^{n} \left(1 - q^{i+j-1} \prod_{l=1}^{k} Q_l P_l\right)^{-1} . \tag{5.96}$$

Therefore we see that the total contribution in the topological string partition function is

$$Z_{hyper} Z_{dec} = \prod_{i,j=1}^{\infty} (1 - q^{i+j-1}) . \tag{5.97}$$

Note that the infinite product term actually coincides with the one appearing in (5.88) and this factor is dropped when computing the topological string partition function for a Higgsed diagram. Therefore we see that contributions of singlet hypermultiplets are not present at all in this kind of computation. Since decoupled hypermultiplets appear only when two legs are placed on top of each other this argument is sufficient to show that the contributions due to these hypermultiplets will be totally absent. In the Higgsed diagram therefore there will be no contributions due to decoupled hypermultiplets whereas some contributions due to decoupled factors will still be present. However these are nothing but the standard decoupled factors that can be easily identified from the Higgsed diagram instead of the original toric diagram, using the prescription described before in this section. Therefore, we need to only remove the contributions of the decoupled factors associated to the Higgsed diagram.

5.2.2 *Examples*

In this section, we apply the technique of the topological vertex obtained in section 5.2.1 to certain Higgsed 5d T_N theories. The analysis in section 5.2.1 shows that we can apply the rule of the topological vertex as well as the decoupled factors directly to Higgsed web diagrams although the web diagram is not dual to a toric Calabi–Yau threefold. Therefore, the computation is greatly simplified because we do not need to compute the topological string partition function for a larger web diagram which yields a UV theory. We will see it by explicitly computing the topological string partition function from web diagrams which realise the rank 2 E_6, E_7 and E_8 theories.

Rank 2 E_6 *theory*

The rank 2 E_6 theory can be realised as an IR theory in a Higgs branch of the T_6 theory. The web diagram is depicted in Figure 24. The full puncture $[1,1,1,1,1,1]$ of the T_6 theory reduces to $[2,2,2]$, which gives an $SU(3)$ flavour symmetry. Hence, we have in total $SU(3) \times SU(3) \times SU(3)$ flavour symmetries which are nicely embedded in E_6. In general, the rank N E_6 theory is obtained by Higgsing each of the full punctures of the T_{3N} theory to $[N,N,N]$. The mass deformation of the rank N

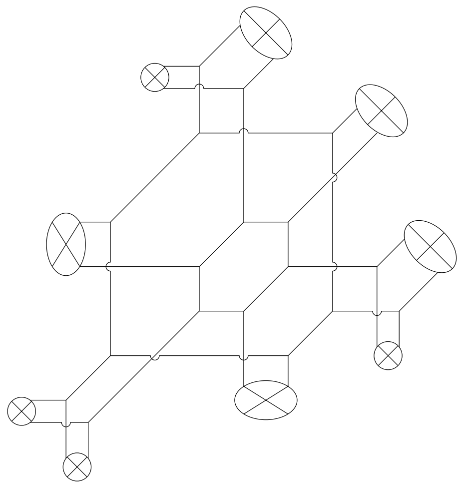

Figure 24: The web diagram of the rank 2 E_6 theory.

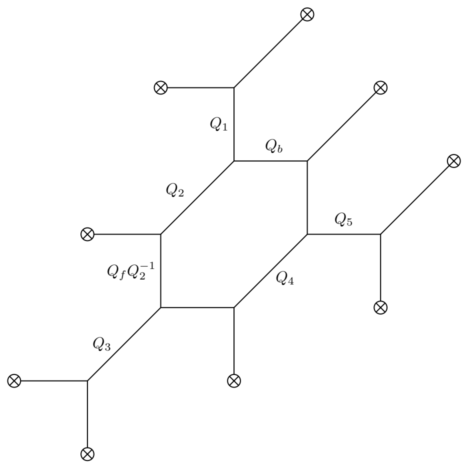

Figure 25: The $[T_3]_1$ web diagram for the rank 1 E_6 theory.

E_6 theory gives an $Sp(N)$ gauge theory with five flavours and one anti-symmetric hypermultiplet.

From the web diagram in Figure 24, we can clearly see that the web diagram of the rank 2 E_6 theory consists of two copies of the web diagram of the rank 1 E_6 theory, which is nothing but the T_3 theory. Let the T_3 diagram with the larger and smaller closed face be $[T_3]_1$, $[T_3]_2$ diagram respectively. The $[T_3]_1$ web diagram is depicted in Figure 25. Since the topological vertex computation for Higgsed web diagrams can be carried out just by the usual rule of the topological vertex for toric web diagrams, the topological string partition function from the rank 2 E_6 web diagram should be given by the product of two topological string partition functions computed from each T_3 web diagram, namely

$$Z_{top}^{\text{rk2 } E_6} = Z_{top}^{[T_3]_1}[P_1] \cdot Z_{top}^{[T_3]_2}[P_2], \tag{5.98}$$

where $Z_{top}^{[T_3]_i}[P_i]$ represents the topological string partition function from the $[T_3]_i$ diagram with a set of parameters and moduli P_i, for $i = 1, 2$. $P_{1,2}$ are related to the lengths of finite length five-branes in the $[T_3]_{1,2}$ web diagram respectively. The relation between P_1 and P_2 may be read off from the web diagram of the rank 2 E_6 theory in Figure 24. Furthermore, the decoupled factor for Higgsed web diagrams can be also directly understood as the decoupled factor associated with strings between parallel external legs of the Higgsed web diagrams. Therefore, the decoupled factor for the rank 2 E_6 diagram should be also the product of the two decoupled factors from each T_3 diagram, namely

$$Z_{dec}^{\mathrm{rk2}\ E_6} = Z_{dec}^{[T_3]_1}\left[P_1\right] \cdot Z_{dec}^{[T_3]_2}\left[P_2\right], \tag{5.99}$$

where $Z_{dec}^{[T_3]_i}\left[P_i\right]$ represents the decoupled factor of the $[T_3]_i$ diagram with a set of parameters P_i, for $i = 1, 2$. Therefore, the partition function of the rank 2 E_6 theory realised by the web diagram should be given by[18]

$$Z^{\mathrm{rk2}\ E_6} = \frac{Z_{top}^{\mathrm{rk2}\ E_6}}{Z_{dec}^{\mathrm{rk2}\ E_6}} = \frac{Z_{top}^{[T_3]_1}\left[P_1\right]}{Z_{dec}^{[T_3]_1}\left[P_1\right]} \cdot \frac{Z_{top}^{[T_3]_2}\left[P_2\right]}{Z_{dec}^{[T_3]_2}\left[P_2\right]}, \tag{5.101}$$

and therefore it is the product of two partition functions of the T_3 theory.

Let us then see how the parameters of the $Sp(2)$ gauge theory arise in the web diagram of Figure 24. For a theory realised by a web diagram a local deformation which does not move semi-infinite 5-branes corresponds to a modulus of the theory while a global deformation which moves semi-infinite 5-branes corresponds to a parameter of the theory. In the case of the rank 2 E_6 diagrams, each size of the two closed faces gives a local deformation. Hence each $[T_3]_{1,2}$ diagram has one modulus, which is in fact a Coulomb branch modulus and we shall call these two moduli $\alpha_i, (i = 1, 2)$ respectively. The rank 2 E_6 web diagram has in general 6 global deformations. Five of them are mass parameters $m_a, (a = 1, \cdots, 5)$ of the five fundamental hypermultiplets, and one of them is the instanton fugacity u or the gauge coupling of the $Sp(2)$ gauge group. One might be tempted to think that there is another parameter which corresponds to the relative distance between the centres of mass of the two closed faces. However the semi-infinite 5-brane ending on the same 7-brane should be grouped together. Hence, such a degree of freedom is frozen. One can also argue that we should have one gauge coupling after tuning off the Coulomb branch moduli. This also implies that the centres of the two closed faces should be at the same position. Hence, the parameter corresponding to the relative distance between the centres of mass of the two closed faces is absent.

When the centres of mass of the two closed faces coincide with each other, all the global deformations are given by the masses of the five flavours and the instanton fugacity of the $Sp(2)$ gauge group. Hence, the theory realised by such a web diagram should correspond to an $Sp(2)$ gauge theory with 5 massive fundamental hypermultiplets and 1 massless anti-symmetric hypermultiplet. Therefore, the web diagram Figure 24 realises a special rank 2 E_6 theory.

18 In this paper, we do not take into account the perturbative contribution from the Cartan part of the vector multiplet. That contribution is not included in the topological vertex computation. The contribution is easily recovered by introducing the factor

$$\prod_{i,j=1}^{\infty}\left(1 - q^{i+j-1}\right)^{-[\mathrm{rank\ G}]}, \tag{5.100}$$

where rank G is the rank of the gauge group G.

For the computation of the topological string partition function from the rank 2 E_6 diagram we need explicit relations between the parameters and the moduli of the theory and the lengths of finite size 5-branes in the web diagram. It turns out that we can use the same parameterisation as that of the T_3 diagram, which is determined in [181], but the only difference is that we use $\alpha_{1,2}$ for the Coulomb branch moduli for the parameterisation of the $[T_3]_{1,2}$ diagrams respectively. Namely, we choose in Figure 25

$$\begin{aligned} &Q_1 = e^{-(m_1-\alpha_1)}, \quad Q_2 = e^{-(\alpha_1-m_2)}, \quad Q_3 = e^{-(-\alpha_1-m_3)}, \quad Q_4 = e^{-(m_4+\alpha_1)}, \\ &Q_5 = e^{-(m_5-\alpha_1)}, \quad Q_f = e^{-2\alpha_1}, \quad Q_b = ue^{-\alpha_1-\frac{1}{2}(-m_1+m_2+m_3+m_4-m_5)}, \end{aligned} \tag{5.102}$$

for the $[T_3]_1$ diagram. The parameterisation of the $[T_3]_2$ diagram is the same as (5.102) with α_1 exchanged with α_2. Here $Q := e^{-L}$ where L is the length of the 5-brane or the size of the corresponding two-cycle in the dual M-theory picture [19]. Hence, the product relation (5.101) can be written more precisely by

$$Z^{\text{rk2 } E_6} = \frac{Z_{top}^{[T_3]_1}\left[\alpha_1, m_1, \cdots, m_5, u\right]}{Z_{dec}^{[T_3]_1}\left[m_1, \cdots, m_5, u\right]} \cdot \frac{Z_{top}^{[T_3]_2}\left[\alpha_2, m_1, \cdots, m_5, u\right]}{Z_{dec}^{[T_3]_2}\left[m_1, \cdots, m_5, u\right]}\,. \tag{5.103}$$

Note that the decoupled factors do not depend on the Coulomb branch moduli.

The physical meaning of the parametrisation is also clear. Strings in the $[T_3]_1$ diagram yield particles with mass given by $\pm\alpha_1 \pm m_a, a = 1, \cdots, 5$ with appropriate signs. On the other hand, strings in the $[T_3]_2$ diagram yield particles with mass given by $\pm\alpha_2 \pm m_a, a = 1, \cdots, 5$ with appropriate signs. Note here that the weights of the fundamental representation of the $Sp(2)$ Lie algebra are given by $\pm e_1, \pm e_2$ where $\{e_1, e_2\}$ are orthonormal basis of $\mathbb{R}^2$. Therefore, the fundamental hypermultiplets related to weights $\pm e_1$ come from strings in the $[T_3]_1$ diagram, and the fundamental hypermultiplets related to weights $\pm e_2$ come from strings in the $[T_3]_2$ diagram. By including both of them, we can form the complete components of the fundamental hypermultiplets of $Sp(2)$.

With the parameterisation (5.102) and similarly that from the $[T_3]_2$ diagram, we can explicitly compute the topological string partition function as well as the decoupled factor for the rank 2 E_6 theory. Due to the relation (5.98), it is enough to compute the topological string partition function from the $[T_3]_{1,2}$ diagrams, which is essentially the topological string partition function from the T_3 diagram. The computation of the topological string partition function for the T_3 diagram was done in [180, 181] and we make use of the result by changing the parameterisation into that

19 In this second part of this chapter we employ a different convention for the parametrisation of the fugacities appearing in the partition function. The convention used in Section 5.1 may be easily recovered by taking all the chemical potentials to be imaginary.

of the $[T_3]_{1,2}$ diagrams. Then the topological string partition function from the $[T_3]_1$ diagram is

$$
\begin{aligned}
Z_{top}^{[T_3]_1} &= Z_0^{[T_3]_1}\cdot Z_1^{[T_3]_1}\cdot Z_{=}^{[T_3]_1}, \qquad (5.104)\\
Z_0^{[T_3]_1} &= \frac{H(e^{\pm\alpha_1-m_1})H(e^{\pm\alpha_1+m_3})\left(\prod_{a=2,4}H(e^{-\alpha_1\pm m_a})\right)}{H(e^{-2\alpha_1})^2},\\
Z_1^{[T_3]_1} &= \sum_{\nu_1,\nu_2,\nu_3} u^{|\nu_1|+|\nu_2|}e^{-\left(|\nu_3|-\frac{1}{2}(|\nu_1|+|\nu_2|)\right)m_5}\\
&\quad \prod_{i=1}^{2}\prod_{s\in\nu_i}\frac{\left(\prod_{a=1}^{3}2\sinh\frac{E_{i0}-m_a}{2}\right)2\sinh\frac{E_{i3}-m_4}{2}}{\prod_{j=1}^{2}\left(2\sinh\frac{E_{ij}}{2}\right)^2}\prod_{s\in\nu_3}\frac{\prod_{i=1}^{2}2\sinh\frac{E_{3i}+m_4}{2}}{\left(2\sinh\frac{E_{33}}{2}\right)^2},\\
Z_{=}^{[T_3]_1} &= \left(H\left(e^{-(m_1-m_2)}\right)H\left(e^{-(m_2-m_3)}\right)H\left(e^{-(m_1-m_3)}\right)\right)^{-1}.
\end{aligned}
$$

We introduced some notations for the simplicity of the expressions

$$
\begin{aligned}
H(Q) &= \prod_{i,j=1}^{\infty}(1-Qq^{i+j-1}) &(5.105)\\
E_{ij}(s) &= \beta_i-\beta_j-\epsilon\left(l_{\nu_i}(s)+a_{\nu_j}(s)+1\right), &(5.106)
\end{aligned}
$$

where $\beta_0=\beta_3=0,\beta_1=-\beta_2=\alpha_1,\nu_0=\emptyset$ and $q=e^{g_s}=e^{\epsilon}$. g_s is the topological string coupling. Also we defined $H(e^{\pm x+y}):=H(e^{x+y})H(e^{-x+y})$.

Similarly, the decoupled factor for the $[T_3]_1$ diagram is

$$
\begin{aligned}
Z_{dec}^{[T_3]_1} &= Z_{dec,=}^{[T_3]_1}\cdot Z_{dec,||}^{[T_3]_1}\cdot Z_{dec,//}^{[T_3]_1}, \qquad (5.107)\\
Z_{dec,=}^{[T_3]_1} &= \left(H\left(e^{-(m_1-m_2)}\right)H\left(e^{-(m_2-m_3)}\right)H\left(e^{-(m_1-m_3)}\right)\right)^{-1},\\
Z_{dec,||}^{[T_3]_1} &= \left(H\left(u\,e^{-\frac{1}{2}(m_1+m_2+m_3+m_4-m_5)}\right)H\left(e^{-(m_5-m_4)}\right)H\left(u\,e^{-\frac{1}{2}(m_1+m_2+m_3-m_4+m_5)}\right)\right)^{-1},\\
Z_{dec,//}^{[T_3]_1} &= \left(H\left(u\,e^{\frac{1}{2}(m_1+m_2+m_3+m_4+m_5)}\right)H\left(e^{-(m_5+m_4)}\right)H\left(u\,e^{\frac{1}{2}(m_1+m_2+m_3-m_4-m_5)}\right)\right)^{-1}.
\end{aligned}
$$

In (5.107) we used the symbols $=$, $||$ and $//$ to denote the contributions to the decoupled factor associated with parallel horizontal, vertical and diagonal legs respectively.

The partition function associated to the $[T_3]_1$ diagram is obtained by dividing $Z_{top}^{[T_3]_1}$ by $Z_{dec}^{[T_3]_1}$, which is

$$\begin{aligned} Z^{[T_3]_1} &= \frac{Z_{top}^{[T_3]_1}}{Z_{dec}^{[T_3]_1}} = Z_{pert}^{[T_3]_1} \cdot Z_{inst}^{[T_3]_1} \\ Z_{pert}^{[T_3]_1} &= \frac{\left(\prod_{a=1,5} H(e^{\pm\alpha_1 - m_a})\right) H(e^{\pm\alpha_1+m_3}) \left(\prod_{a=2,4} H(e^{-\alpha_1 \pm m_a})\right)}{H(e^{-2\alpha_1})^2}, \\ Z_{inst}^{[T_3]_1} &= H\left(u\, e^{-\frac{1}{2}(m_1+m_2+m_3+m_4-m_5)}\right) H\left(u\, e^{-\frac{1}{2}(m_1+m_2+m_3-m_4+m_5)}\right) \\ & \quad H\left(u\, e^{\frac{1}{2}(m_1+m_2+m_3+m_4+m_5)}\right) H\left(u\, e^{\frac{1}{2}(m_1+m_2+m_3-m_4-m_5)}\right) \\ & \quad \frac{H\left(e^{-(m_5 \pm m_4)}\right)}{H(e^{\pm\alpha_1 - m_5})} \sum_{\nu_1,\nu_2,\nu_3} u^{|\nu_1|+|\nu_2|} e^{-\left(|\nu_3| - \frac{1}{2}(|\nu_1|+|\nu_2|)\right) m_5} \\ & \quad \prod_{i=1}^{2} \prod_{s\in\nu_i} \frac{\left(\prod_{a=1}^{3} 2\sinh\frac{E_{i0}-m_a}{2}\right) 2\sinh\frac{E_{i3}-m_4}{2}}{\prod_{j=1}^{2}\left(2\sinh\frac{E_{ij}}{2}\right)^2} \prod_{s\in\nu_3} \frac{\prod_{i=1}^{2} 2\sinh\frac{E_{3i}+m_4}{2}}{\left(2\sinh\frac{E_{33}}{2}\right)^2}. \end{aligned} \tag{5.108}$$

$Z_{pert}^{[T_3]_1}$ is the perturbative part and $Z_{inst}^{[T_3]_1}$ is the instanton part of the partition function. Note that $Z_{inst}^{[T_3]_1}$ does not include a non-trivial term at order $\mathcal{O}(u^0)$ due to the identity [180, 202]

$$\sum_{\nu_3} e^{-|\nu_3| m_5} \prod_{s\in\nu_3} \frac{\prod_{i=1}^{2} 2\sinh\frac{E_{3i}+m_4}{2}}{\left(2\sinh\frac{E_{33}}{2}\right)^2} = \frac{H(e^{\alpha_1 - m_5}) H(e^{-\alpha_1 - m_5})}{H\left(e^{-(m_5-m_4)}\right) H\left(e^{-(m_5+m_4)}\right)}. \tag{5.109}$$

The partition function $Z^{[T_3]_1}$ exactly agrees with the partition function of the $Sp(1)$ gauge theory with five fundamental hypermultiplets and this was checked up to 3-instanton in [181] when we regard u as the instanton fugacity of the $Sp(1)$ gauge theory.

Then, the partition function from the $[T_3]_2$ diagram is essentially the same as that from the $[T_3]_1$ diagram except for the exchange of the Coulomb branch moduli, namely

$$Z^{[T_3]_2}\left[\alpha_2, \{m_{1,2,3,4,5}\}, u\right] = Z^{[T_3]_1}\left[\alpha_2, \{m_{1,2,3,4,5}\}, u\right], \tag{5.110}$$

where the right-hand side of (5.110) is (5.108) with α_2 used instead of α_1. Here, $\{m_{1,2,3,4,5}\}$ means a set of mass parameters $m_1, \cdots, m_5$.

Due to the relation (5.101), we have obtained the partition function of the rank 2 E_6 theory realised by the web diagram in Figure 24,

$$Z^{\text{rk2 } E_6} = Z^{[T_3]_1}\left[\alpha_1, \{m_{1,2,3,4,5}\}, u\right] \cdot Z^{[T_3]_1}\left[\alpha_2, \{m_{1,2,3,4,5}\}, u\right], \tag{5.111}$$

where u is now identified with the instanton fugacity of the $Sp(2)$ gauge theory.

Let us see whether the result (5.111) agrees with the field theory result. The perturbative part is given by

$$Z_{pert}^{\mathrm{rk2}\ E_6} = \frac{\left(\prod_{a=1,5} H(e^{\pm\alpha_1 - m_a})\right) H(e^{\pm\alpha_1+m_3}) \left(\prod_{a=2,4} H(e^{-\alpha_1 \pm m_a})\right)}{H(e^{-2\alpha_1})^2} \frac{\left(\prod_{a=1,5} H(e^{\pm\alpha_2 - m_a})\right) H(e^{\pm\alpha_2+m_3}) \left(\prod_{a=2,4} H(e^{-\alpha_2 \pm m_a})\right)}{H(e^{-2\alpha_2})^2} \tag{5.112}$$

By comparing (5.112) with (H.32) with $N = 2, N_f = 5$ inserted. (5.112) is precisely equal to the perturbative contribution of the $Sp(2)$ gauge theory with five massive fundamental hypermultiplets and one massless anti-symmetric hypermultiplet up to the contribution from the Cartan part of the vector multiplet and also some subtle divergent factors. Note that the factors from the massless anti-symmetric hypermultiplet do not appear in $Z_{pert}^{\mathrm{rk2}\ E_6}$ because they are cancelled by the factors from a part of the vector multiplet.

Let us then turn to the comparison of the 1-instanton part. Since the partition function of the rank 2 E_6 theory is the product of the partition functions from the $[T_3]_{1,2}$ diagrams (5.111), the 1-instanton part is simply given by

$$Z_{1\text{-}inst}^{\mathrm{rk2}\ E_6} = Z^{[T_3]_1}\left[\alpha_1, \{m_{1,2,3,4,5}\}, u\right]|_{\mathcal{O}(u^1)} + Z^{[T_3]_1}\left[\alpha_2, \{m_{1,2,3,4,5}\}, u\right]|_{\mathcal{O}(u^1)}, \tag{5.113}$$

where $|_{\mathcal{O}(u^1)}$ implies taking the term at order $\mathcal{O}(u^1)$. Since the partition function of the T_3 theory agrees with the Nekrasov partition function of the $Sp(1)$ gauge theory with five fundamental hypermultiplet [181], the 1-instanton part of $Z^{[T_3]_1}\left[\alpha_1, \{m_{1,2,3,4,5}\}, u\right]$ is the 1-instanton part of the Nekrasov partition function of the $Sp(1)$ gauge theory with five flavours, which is (H.22) with $N_f = 5$ and $\epsilon_+ = 0$. $\epsilon_+ = 0$ is due to the fact that we use the unrefined topological vertex. Therefore, (5.113) becomes

$$Z_{1\text{-}inst}^{\mathrm{rk2}\ E_6} = \frac{1}{2}\sum_{i=1}^{2}\left\{\frac{\prod_{a=1}^{5} 2\sinh\frac{m_a}{2}}{2\sinh\frac{\pm\epsilon}{2} 2\sinh\frac{\pm\alpha_i}{2}} + \frac{\prod_{a=1}^{N_f} 2\cosh\frac{m_a}{2}}{2\sinh\frac{\pm\epsilon}{2} 2\cosh\frac{\pm\alpha_i}{2}}\right\}. \tag{5.114}$$

This completely agrees with the field theory result of (H.23) with $N = 2$ and ϵ identified with ϵ_-. In (5.114) we introduced the notation $\sinh(\pm x) := \sinh(x)\sinh(-x)$.

Moreover we have checked agreement of the partition function (5.111) of the rank 2 E_6 theory with the partition function of the $Sp(2)$ theory with 5 flavours and one massless anti-symmetric hypermultiplet at 2-instanton in the special limit where one of the masses of the fundamental hypermultiplets is set to zero[20]. The method of how to compute the $Sp(2)$ instanton partition function at the 2-instanton level is summarised at the end of appendix H.1.

Rank 2 E_7 theory

Next example is the rank 2 E_7 theory which is realised by the web diagram in Figure 26. The UV theory is the T_8 theory with three full punctures. Then, we go to a Higgs

20 The reason why we set one mass to zero is not a fundamental issue but a technical issue. Namely our computer program checking the equality did not end in a reasonable time when we chose all the masses are general.

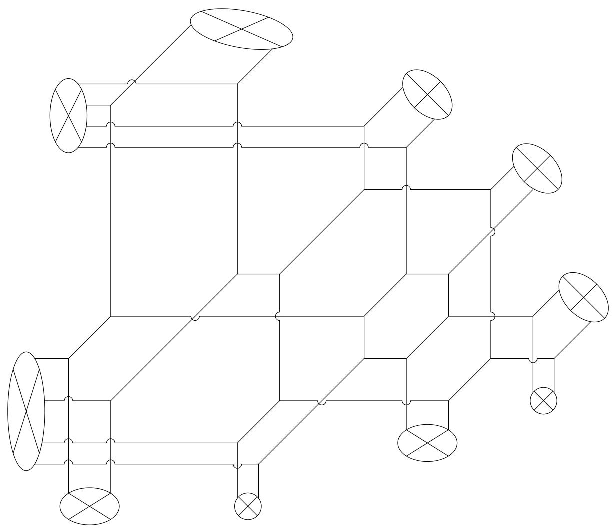

Figure 26: The web diagram for the rank 2 E_7 theory.

branch where one full puncture is reduced to $[4,4]$ and the other full punctures are reduced to $[2,2,2,2]$. The vev of the hypermultiplets induces an RG flow and we obtain the rank 2 E_7 theory at low energies. The puncture with $[4,4]$ gives an $SU(2)$ flavour symmetry and the two punctures with $[2,2,2,2]$ yields $SU(4)\times SU(4)$ flavour symmetries. The total flavour symmetries $SU(2)\times SU(4)\times SU(4)$ can be embedded in E_7. The rank N E_7 theory is realised by Higgsing the three full punctures of the T_{4N} theory down to one $[2N,2N]$ and two $[N,N,N,N]$. The mass deformation of the rank N E_7 theory is the $Sp(N)$ gauge theory with six fundamental hypermultiplets and one anti-symmetric hypermultiplet.

Again, we can see that the web diagram of the rank 2 E_7 theory is composed by two copies of the web diagram of the rank 1 E_7 theory. The web diagram of the rank 1 E_7 theory is shown in Figure 27. We will call one copy of the rank 1 E_7 web diagram with the larger closed face as $[E_7]_1$ web diagram, and another copy with the smaller closed face as $[E_7]_2$ web diagram. The topological vertex formalism of Higgsed web diagrams again implies that the topological string partition function from the rank 2 E_7 diagram is given by

$$Z_{top}^{\text{rk2 } E_7} = Z_{top}^{[E_7]_1}\left[P_1\right]\cdot Z_{top}^{[E_7]_2}\left[P_2\right], \tag{5.115}$$

where $Z_{top}^{[E_7]_i}\left[P_i\right]$ represents the topological string partition function from the $[E_7]_i$ diagram with a set of parameters and moduli P_i, for $i=1,2$. The decoupled factor of the rank 2 E_7 diagram is also written by a product

$$Z_{dec}^{\text{rk2 } E_7} = Z_{dec}^{[E_7]_1}\left[P_1\right]\cdot Z_{dec}^{[E_7]_2}\left[P_2\right], \tag{5.116}$$

where $Z_{dec}^{[E_7]_i}\left[P_i\right]$ represents the decoupled factor of the $[E_7]_i$ diagram with a set of parameters P_i, for $i=1,2$. Therefore, the partition function of the rank 2 E_7 theory realised by the web diagram Figure 26 can be obtained by

$$Z^{\text{rk2 } E_7} = \frac{Z_{top}^{\text{rk2 } E_7}}{Z_{dec}^{\text{rk2 } E_7}} = \frac{Z_{top}^{[E_7]_1}\left[P_1\right]}{Z_{dec}^{[E_7]_1}\left[P_1\right]}\cdot\frac{Z_{top}^{[E_7]_2}\left[P_2\right]}{Z_{dec}^{[E_7]_2}\left[P_2\right]}. \tag{5.117}$$

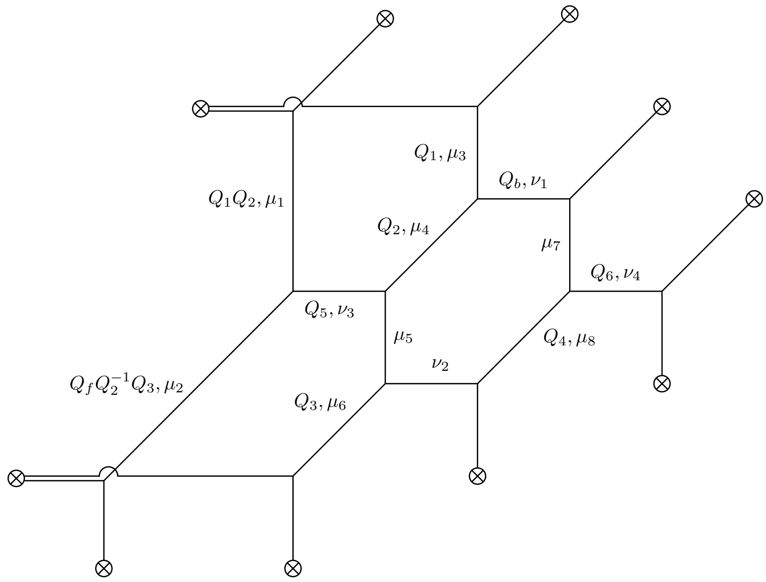

Figure 27: The $[E_7]_1$ web diagram for the rank 1 E_7 theory.

Hence, in order to compute the partition function of the rank 2 E_7 theory, it is enough to compute the partition function from the $[E_7]_1$ web diagram and then multiply it by the same function with different arguments.

As in the case of the parameterisation of the rank 2 E_6 theory, the parameterisation of the rank 2 E_7 theory is determined by making use of the parameterisation of the rank 1 E_7 theory. The relation between the gauge theory parameters and the lengths of five-branes for the rank 1 E_7 theory was determined in [181]. The only difference between the parameterisation of the $[E_7]_1$ diagram and that of the $[E_7]_2$ diagram is whether we use the Coulomb branch modulus α_1 or α_2. Here, α_1 is related to the size of the larger closed face and α_2 is related to the size of the smaller closed face. More precisely, we use the parameterisation

$$\begin{aligned}
&Q_1 = e^{-(m_1-\alpha_1)}, \quad Q_2 = e^{-(\alpha_1-m_2)}, \quad Q_3 = e^{-(-\alpha_1-m_3)}, \quad Q_4 = e^{-(m_4+\alpha_1)}, \\
&Q_5 = e^{-(m_5-\alpha_1)}, \quad Q_6 = e^{-(m_6-\alpha_1)}, \quad Q_f = e^{-2\alpha_1}, \quad Q_b = ue^{-\alpha_1-\frac{1}{2}(-m_1+m_2+m_3+m_4-m_5-m_6)}.
\end{aligned} \tag{5.118}$$

for the $[E_7]_1$ web diagram. The correspondence between Q and 5-branes is depicted in Figure 27. u is the instanton fugacity of the $Sp(2)$ gauge theory. The parameterisation of the $[E_7]_2$ diagram is the same as (5.118) except that α_1 is exchanged with α_2. We also choose the parameterisation such that the center of mass of the larger closed face coincides with the center of mass of the smaller closed face. The theory realised by the web diagram should be the $Sp(2)$ gauge theory with six massive fundamental hypermultiplets and one massless anti-symmetric hypermultiplet.

With the parameterisation of (5.118) and similarly that from the $[E_7]_2$ web diagram, we perform the explicit computation of the topological string partition function from the rank 2 E_7 web diagram by making use of the technique developed in section 5.2.1. Due to the product structure (5.117), we first compute the topological string partition function from the $[E_7]_1$ diagram. The refined version of the computation was essentially done in section 6.3 of [181], but we repeat the computation here since the discussion of the decoupled factor from the rank 1 E_7 diagram was unclear in [181]. The application of the topological vertex to the web diagram Figure 27 gives the topological string partition function

$$
\begin{aligned}
Z^{[E_7]_1}_{top} = & \sum_{\nu_1,\cdots,\nu_4,\mu_1,\cdots,\mu_8} (-Q_b)^{|\nu_1|}(-Q_bQ_2Q_5^{-1})^{|\nu_2|}(-Q_5)^{|\nu_3|}(-Q_6)^{|\nu_4|}(-Q_1Q_2)^{|\mu_1|}(-Q_fQ_2^{-1}Q_3)^{|\mu_2|} \\
& (-Q_1)^{|\mu_3|}(-Q_2)^{|\mu_4|}(-Q_fQ_2^{-1})^{|\mu_5|}(-Q_3)^{|\mu_6|}(-Q_fQ_4^{-1})^{|\mu_7|}(-Q_4)^{|\mu_8|} \\
& C_{\emptyset\mu_1\emptyset}(q)C_{\mu_2\mu_1^t\nu_3^t}(q)C_{\mu_2^t\emptyset\emptyset}(q)C_{\emptyset\mu_3^t\emptyset}(q)C_{\mu_4^t\mu_3\nu_1^t}(q)C_{\mu_4\mu_5\nu_3}(q)C_{\mu_6\mu_5^t\nu_2^t}(q)C_{\mu_6^t\emptyset\emptyset}(q) \\
& C_{\emptyset\mu_7\nu_1}(q)C_{\mu_8^t\mu_7^t\nu_4^t}(q)C_{\mu_8\emptyset\nu_2}(q)C_{\emptyset\emptyset\nu_4}(q). \qquad (5.119)
\end{aligned}
$$

The straightforward computation by using the formulae in appendix G.1 gives

$$
Z^{[E_7]_1}_{top} = Z^{[E_7]_1}_0 \cdot Z^{[E_7]_1}_1 \cdot Z^{[E_7]_1}_{=}, \qquad (5.120)
$$

$$
Z^{[E_7]_1}_0 = \frac{H(e^{\pm\alpha_1-m_1})H(e^{\pm\alpha_1+m_3})\left(\prod_{a=2,4}H(e^{-\alpha_1\pm m_a})\right)}{H\left(e^{-2\alpha_1}\right)^2},
$$

$$
\begin{aligned}
Z^{[E_7]_1}_1 = & \sum_{\nu_1,\nu_2,\nu_3,\nu_4} u^{|\nu_1|+|\nu_2|}e^{-\left(|\nu_3|-\frac{1}{2}(|\nu_1|+|\nu_2|)\right)m_5}e^{-\left(|\nu_4|-\frac{1}{2}(|\nu_1|+|\nu_2|)\right)m_6} \\
& \prod_{i=1}^{2}\prod_{s\in\nu_i}\frac{\left(\prod_{a=1,3}2\sinh\frac{E_{i0}-m_a}{2}\right)2\sinh\frac{E_{i3}-m_2}{2}2\sinh\frac{E_{i4}-m_4}{2}}{\prod_{j=1}^{2}\left(2\sinh\frac{E_{ij}}{2}\right)^2} \\
& \prod_{s\in\nu_3}\frac{\prod_{i=1}^{2}2\sinh\frac{E_{3i}+m_2}{2}}{\left(2\sinh\frac{E_{33}}{2}\right)^2}\prod_{s\in\nu_4}\frac{\prod_{i=1}^{2}2\sinh\frac{E_{4i}+m_4}{2}}{\left(2\sinh\frac{E_{44}}{2}\right)^2},
\end{aligned}
$$

$$
Z^{[E_7]_1}_{=} = H\left(e^{-(m_1-m_3)}\right)^{-2}. \qquad (5.121)
$$

We defined $\beta_0=\beta_3=\beta_4=0, \beta_1=-\beta_2=\alpha_1, \nu_0=\emptyset$.

The decoupled factor can be also directly read off from the Higgsed diagram. Namely it is associated to the contribution from strings between the parallel external legs of the Higgsed diagram, which is, in this case, the $[E_7]_1$ web diagram in Figure 27. The decoupled factor from the $[E_7]_1$ diagram is given by

$$
Z^{[E_7]_1}_{dec} = Z^{[E_7]_1}_{dec,=} \cdot Z^{[E_7]_1}_{dec,||} \cdot Z^{[E_7]_1}_{dec,//}, \qquad (5.122)
$$

where

$$
\begin{aligned}
Z^{[E_7]_1}_{dec,=} &= H\left(Q_1Q_3Q_f^{-1}\right)^{-2} \\
&= H\left(e^{-(m_1-m_3)}\right)^{-2},
\end{aligned}
$$

$$
\begin{aligned}
Z^{[E_7]_1}_{dec,||} = & H\left(Q_2Q_5\right)^{-1}H\left(Q_1Q_b\right)^{-1}H\left(Q_fQ_4^{-1}Q_6\right)^{-1}H\left(Q_1Q_2Q_5Q_b\right)^{-1} \\
& H\left(Q_1Q_4^{-1}Q_6Q_fQ_b\right)^{-1}H\left(Q_1Q_2Q_4^{-1}Q_5Q_6Q_fQ_b\right)^{-1} \\
= & H\left(e^{-(m_5-m_2)}\right)^{-1}H\left(ue^{-\frac{1}{2}(m_1+m_2+m_3+m_4-m_5-m_6)}\right)^{-1}H\left(e^{-(m_6-m_4)}\right)^{-1} \\
& H\left(ue^{-\frac{1}{2}(m_1-m_2+m_3+m_4+m_5-m_6)}\right)^{-1}H\left(ue^{-\frac{1}{2}(m_1+m_2+m_3-m_4-m_5+m_6)}\right)^{-1} \\
& H\left(ue^{-\frac{1}{2}(m_1-m_2+m_3-m_4-m_5-m_6)}\right)^{-1}
\end{aligned}
$$

$$\begin{aligned}
Z^{[E_7]_1}_{dec,//} &= H\left(Q_f Q_2^{-1} Q_5\right)^{-1} H\left(Q_2 Q_3 Q_4^{-1} Q_b\right)^{-1} H\left(Q_4 Q_6\right)^{-1} H\left(Q_3 Q_4^{-1} Q_5 Q_f Q_b\right)^{-1} \\
&\quad H\left(Q_2 Q_3 Q_6 Q_b\right)^{-1} H\left(Q_3 Q_5 Q_6 Q_f Q_b\right)^{-1} \\
&= H\left(e^{-(m_5+m_2)}\right)^{-1} H\left(ue^{\frac{1}{2}(m_1+m_2+m_3+m_4+m_5+m_6)}\right)^{-1} H\left(e^{-(m_6+m_4)}\right)^{-1} \\
&\quad H\left(ue^{\frac{1}{2}(m_1-m_2+m_3+m_4-m_5+m_6)}\right)^{-1} H\left(ue^{\frac{1}{2}(m_1+m_2+m_3-m_4+m_5-m_6)}\right)^{-1} \\
&\quad H\left(ue^{\frac{1}{2}(m_1-m_2+m_3-m_4-m_5-m_6)}\right)^{-1}
\end{aligned}$$

Again, we used the symbols $=, ||$ and $//$ to denote the contributions that come from strings between the parallel horizontal, vertical and diagonal legs respectively.

By combining the topological string partition function (5.120) with the decoupled factor (5.122), we obtain the partition function associated to the $[E_7]_1$ web diagram

$$Z^{[E_7]_1} = \frac{Z^{[E_7]_1}_{top}}{Z^{[E_7]_1}_{dec}} = Z^{[E_7]_1}_{pert} \cdot Z^{[E_7]_1}_{inst} \tag{5.123}$$

$$Z^{[E_7]_1}_{pert} = \frac{\left(\prod_{a=1,5,6} H(e^{\pm\alpha_1 - m_a})\right) H(e^{\pm\alpha_1+m_3}) \left(\prod_{a=2,4} H(e^{-\alpha_1 \pm m_a})\right)}{H\left(e^{-2\alpha_1}\right)^2},$$

$$\begin{aligned}
Z^{[E_7]_1}_{inst} &= H\left(ue^{-\frac{1}{2}(m_1+m_2+m_3+m_4-m_5-m_6)}\right) H\left(ue^{-\frac{1}{2}(m_1-m_2+m_3+m_4+m_5-m_6)}\right) \\
&\quad H\left(ue^{-\frac{1}{2}(m_1+m_2+m_3-m_4-m_5+m_6)}\right) H\left(ue^{-\frac{1}{2}(m_1-m_2+m_3-m_4-m_5-m_6)}\right) \\
&\quad H\left(ue^{\frac{1}{2}(m_1+m_2+m_3+m_4+m_5+m_6)}\right) H\left(ue^{\frac{1}{2}(m_1-m_2+m_3+m_4-m_5+m_6)}\right) \\
&\quad H\left(ue^{\frac{1}{2}(m_1+m_2+m_3-m_4+m_5-m_6)}\right) H\left(ue^{\frac{1}{2}(m_1-m_2+m_3-m_4-m_5-m_6)}\right) \\
&\quad \frac{H\left(e^{-(m_5\pm m_2)}\right) H\left(e^{-(m_6\pm m_4)}\right)}{H(e^{\pm\alpha_1-m_5}) H(e^{\pm\alpha_1-m_6})} \\
&\quad \sum_{\nu_1,\nu_2,\nu_3,\nu_4} u^{|\nu_1|+|\nu_2|} e^{-\left(|\nu_3|-\frac{1}{2}(|\nu_1|+|\nu_2|)\right)m_5} e^{-\left(|\nu_4|-\frac{1}{2}(|\nu_1|+|\nu_2|)\right)m_6} \\
&\quad \prod_{i=1}^{2}\prod_{s\in\nu_i} \frac{\left(\prod_{a=1,3} 2\sinh\frac{E_{i0}-m_a}{2}\right) 2\sinh\frac{E_{i3}-m_2}{2} 2\sinh\frac{E_{i4}-m_4}{2}}{\prod_{j=1}^{2}\left(2\sinh\frac{E_{ij}}{2}\right)^2} \\
&\quad \prod_{s\in\nu_3} \frac{\prod_{i=1}^{2} 2\sinh\frac{E_{3i}+m_2}{2}}{\left(2\sinh\frac{E_{33}}{2}\right)^2} \prod_{s\in\nu_4} \frac{\prod_{i=1}^{2} 2\sinh\frac{E_{4i}+m_4}{2}}{\left(2\sinh\frac{E_{44}}{2}\right)^2},
\end{aligned}$$

The instanton part $Z^{[E_7]_1}_{inst}$ of the partition function starts from 1 at order $\mathcal{O}(u^0)$ due to the identity (5.109). The expression (5.123) exactly agrees with the partition function of the rank 1 E_7 theory obtained in [181]. The computation presented here is simplified compared to the computation in [181] in two respects. First, the partition function has four Young diagram summations from the first whereas the partition function of the UV T_4 theory has six Young diagram summations which reduces to the four Young diagram summations after the Higgsing. Second, we do not need to eliminate the decoupled factor in two steps where we first eliminate the decoupled factor of the UV diagram and then eliminate the singlet hypermultiplet in the Higgsed vacuum. In this formalism, we just remove the decoupled factor associated to the IR diagram of Figure 27.

The partition function associated to the $[E_7]_2$ web diagram is simply given by the same function as (5.123) with the Coulomb branch moduli exchanged

$$Z^{[E_7]_2}[\alpha_2, \{m_{1,2,3,4,5,6}\}, u] = Z^{[E_7]_1}[\alpha_2, \{m_{1,2,3,4,5,6}\}, u]. \tag{5.124}$$

After obtaining the partition function associated to the $[E_7]_1$ diagram, it is easy to obtain the partition function of the rank 2 E_7 theory realised by the web diagram in Figure 26 due to the product relation (5.117),

$$Z^{\mathrm{rk2}\ E_7} = Z^{[E_7]_1}[\alpha_1, \{m_{1,2,3,4,5,6}\}, u] \cdot Z^{[E_7]_1}[\alpha_2, \{m_{1,2,3,4,5,6}\}, u]. \tag{5.125}$$

The perturbative part of the partition function (5.125) is

$$\begin{aligned} Z^{\mathrm{rk2}\ E_7}_{pert} \quad = \quad & \frac{\left(\prod_{a=1,5,6} H(e^{\pm\alpha_1 - m_a})\right) H(e^{\pm\alpha_1 + m_3}) \left(\prod_{a=2,4} H(e^{-\alpha_1 \pm m_a})\right)}{H\left(e^{-2\alpha_1}\right)^2} \\ & \frac{\left(\prod_{a=1,5,6} H(e^{\pm\alpha_2 - m_a})\right) H(e^{\pm\alpha_2 + m_3}) \left(\prod_{a=2,4} H(e^{-\alpha_2 \pm m_a})\right)}{H\left(e^{-2\alpha_2}\right)^2}. \end{aligned}$$

By the comparison with (H.32). this is precisely the perturbative contribution from six fundamental hypermultiplets with mass $m_{1,2,3,4,5,6}$, one massless anti-symmetric hypermultiplet, and $Sp(2)$ vector multiplet.

Also the 1-instanton part of the partition function from the $[E_7]_1$ web diagram should be the 1-instanton contribution of the $Sp(1)$ gauge theory with six flavours since (5.123) agrees with the partition function of the rank 1 E_7 theory. Therefore, its contribution is written by (H.22) with $\epsilon_+ = 0, \epsilon_- = \epsilon$ and $m = 0$. Hence the 1-instanton part of $Z^{\mathrm{rk2}\ E_7}_{inst}$ is given by the sum of the 1-instanton part of the partition function from the $[E_7]_1$ diagram and that from the $[E_7]_2$ diagram,

$$Z^{\mathrm{rk2}\ E_7}_{1\text{-}inst} = \frac{1}{2}\sum_{i=1}^{2}\left\{\frac{\prod_{a=1}^{6} 2\sinh\frac{m_a}{2}}{2\sinh\frac{\pm\epsilon}{2} 2\sinh\frac{\pm\alpha_i}{2}} + \frac{\prod_{a=1}^{6} 2\cosh\frac{m_a}{2}}{2\sinh\frac{\pm\epsilon}{2} 2\cosh\frac{\pm\alpha_i}{2}}\right\}. \tag{5.126}$$

This result completely agrees with the field theory computation of the 1-instanton part (H.23) of the partition function of the $Sp(2)$ gauge theory with six fundamental hypermultiplets and one massless anti-symmetric hypermultiplet by setting $N = 2, \epsilon_- = \epsilon$.

Moreover we have checked agreement of the partition function of the rank 2 E_7 theory with the partition function of a $Sp(2)$ theory with 6 flavours and one massless anti-symmetric hypermultiplet at 2-instanton in the special limit where one of the masses of the fundamental hypermultiplets is set to zero[21].

Rank 2 E_8 theory

The last example we would like to discuss is the rank 2 E_8 which can be realised by the web diagram in Figure 28. In this case the UV theory is the T_{12} theory and in

21 Again, the reason why we set one mass to zero is the technical issue.

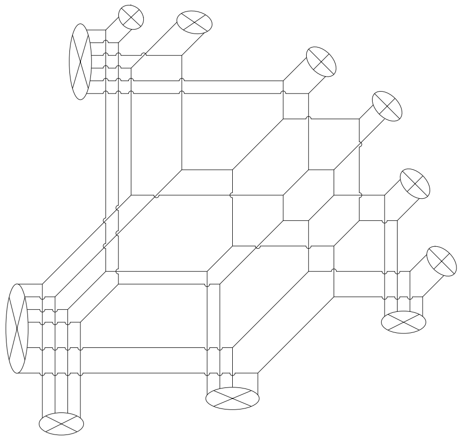

Figure 28: The web diagram for the rank 2 E_8 theory.

order to realise rank 2 E_8 theory it is necessary to go into the Higgs branch to have the original three full punctures reduced to $[6,6]$, $[4,4,4]$ and $[2,2,2,2,2,2]$ In this situation the flavour symmetry of the theory is $SU(2)\times SU(3)\times SU(6)$ which can be embedded in E_8. It is possible to generalise this construction to realise rank N E_8 theory by going into the Higgs branch of T_{6N} theory and reducing the original three full punctures to $[3N,3N]$, $[2N,2N,2N]$ and $[N,N,N,N,N,N]$. We will show by computing explicitly its topological string partition function that rank 2 E_8 theory for a specific choice of parameters (to be discussed later) is an $Sp(2)$ gauge theory with 7 fundamental hypermultiplets and one massless antisymmetric hypermultiplet. This leads us to conjecture that the same happens for rank N E_8 theory, namely that for some specific choice of parameters rank N E_8 theory is a $Sp(N)$ gauge theory with 7 fundamental hypermultiplets and one massless antisymmetric hypermultiplet.

Like in the previous examples discussed in this section it is possible to see that the diagram of rank 2 E_8 theory is made of two copies of a E_8 theory whose diagram is displayed in Figure 29. We shall call $[E_8]_1$ the external copy of the rank 1 E_8 diagram and $[E_8]_2$ the internal copy. The fact that the diagram of the rank 2 E_8 theory is made of two copies of a rank 1 E_8 diagram implies that the topological string partition function has the following structure

$$Z_{top}^{\mathrm{rk}2\, E_8}=Z_{top}^{[E_8]_1}[P_1]\cdot Z_{top}^{[E_8]_2}[P_2] \tag{5.127}$$

where $Z_{top}^{[E_8]_i}[P_i]$ is the topological string partition function of the rank 1 E_8 theory with the set of parameters and moduli $[P_i]$. Note however that, like in the previous examples, the two set of parameters $[P_1]$ and $[P_2]$ are not independent but there are some simple relations between the two (which we will discuss later) implied by the structure of the web diagram. Moreover the particular structure of the web diagram of the rank 2 E_8 theory implies that also the decoupled factor of the theory has a product structure

$$Z_{dec}^{\mathrm{rk}2\ E_8}=Z_{dec}^{[E_8]_1}\left[P_1\right]\cdot Z_{dec}^{[E_8]_2}\left[P_2\right], \tag{5.128}$$

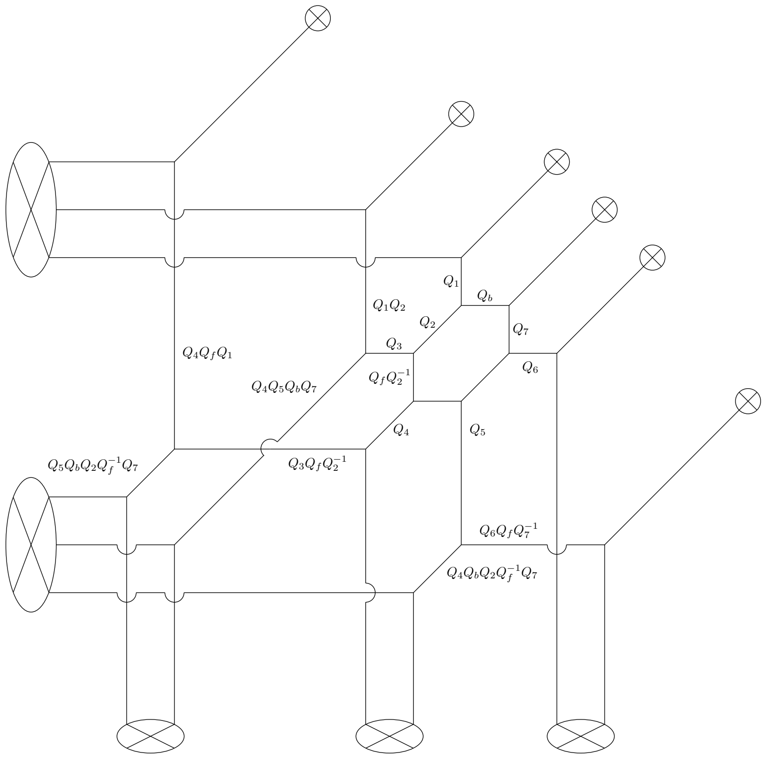

Figure 29: The web diagram for the rank 1 E_8 theory.

where $Z_{dec}^{[E_8]_i}\,[P_i]$ is the decoupled factor of a single rank 1 E_8 theory with parameters $[P_i]$. Summing up we find that the partition function of the rank 2 E_8 theory is

$$Z^{\text{rk2 } E_8} = \frac{Z_{top}^{\text{rk2 } E_8}}{Z_{dec}^{\text{rk2 } E_8}} = \frac{Z_{top}^{[E_8]_1}\,[P_1]}{Z_{dec}^{[E_8]_1}\,[P_1]} \cdot \frac{Z_{top}^{[E_8]_2}\,[P_2]}{Z_{dec}^{[E_8]_2}\,[P_2]}\,. \tag{5.129}$$

Because of this particular structure we will focus on the computation of the partition function of a single rank 1 E_8 theory and then reconstruct the full rank 2 E_8 theory partition function by multiplying two copies of such partition function with a different set of parameters.

In the following we will choose the parameters of the web diagram of the rank 2 E_8 theory so that the centres of mass of the two closed faces present in the diagram coincide: as we will show later with this particular choice the partition function of the rank 2 E_8 theory for a specific choice of parameters coincides with the partition function of a $Sp(2)$ gauge theory with 7 fundamental hypermultiplets and a massless antisymmetric hypermultiplet. This also implies that the parameters defining the $[E_8]_2$ diagram can be obtained by the parameters of the $[E_8]_1$ diagram by simply replacing the Coulomb branch modulus α_1 with the Coulomb branch modulus α_2. Because of this we will simply give the parametrisation for the $[E_8]_1$ diagram which was determined in Section 5.1.3

$$\begin{aligned} &Q_1 = e^{\alpha_1 - m_1}\,,\ Q_2 = e^{-\alpha_1 + m_2}\,,\ Q_3 = e^{\alpha_1 + m_3}\,,\ Q_4 = e^{\alpha_1 + m_4}\,,\\ &Q_5 e^{\alpha_1 + m_5}\,,\ Q_6 = e^{\alpha_1 - m_7}\,,\ Q_7 = e^{-\alpha_1 + m_6}\,,\ Q_b = u\, e^{-\alpha_1 + f(m)}\,, \end{aligned} \tag{5.130}$$

where for sake of simplicity we defined $f(m) = \frac{1}{2}(m_1 - m_2 - m_3 - m_4 - m_5 - m_6 + m_7)$. In this parametrisation u will be identified with the instanton fugacity of the $Sp(2)$ gauge theory.

We now will discuss the computation of the partition function of the rank 1 E_8 theory which will be the first step for the computation of the partition function of the rank 2 E_8 theory. The topological string partition function for the $[E_8]_1$ diagram can be computed using the rules described in section 5.2.1 it is given by the following expression

$$
\begin{aligned}
Z_{top}^{[E_8]_1} = \sum_{\lambda_i,\nu_i,\mu_i} & (-Q_1)^{|\mu_1|}(-Q_f Q_2^{-1})^{|\mu_2|}(-Q_4 Q_f Q_1)^{|\mu_3|}(-Q_1 Q_2)^{|\mu_4|}(-Q_7)^{|\mu_5|}(-Q_5)^{|\mu_6|} \\
& (-Q_2)^{|\lambda_1|}(-Q_4)^{|\lambda_2|}(-\tilde{Q}_b Q_5)^{|\lambda_3|}(-Q_f Q_4 Q_2^{-1}\tilde{Q}_b Q_5)^{|\lambda_4|}(-Q_f Q_7^{-1})^{|\lambda_5|}(-\tilde{Q}_b Q_4)^{|\lambda_6|} \\
& (-Q_b)^{|\nu_1|}(-\tilde{Q}_b)^{|\nu_2|}(-Q_3)^{|\nu_3|}(-Q_3 Q_f Q_2^{-1})^{|\nu_4|}(-Q_6)^{|\nu_5|}(-Q_6 Q_f Q_7^{-1})^{|\nu_6|} \\
& C_{\emptyset\mu_3\emptyset}(q) C_{\lambda_3\mu_3^t\nu_4}(q) C_{\lambda_3^t\emptyset\emptyset}(q) C_{\emptyset\mu_4\emptyset}(q) C_{\lambda_4\mu_4^t\nu_3}(q) C_{\lambda_4^t\emptyset\emptyset}(q) C_{\emptyset\mu_1\emptyset}(q) C_{\lambda_1\mu_1^t\nu_1}(q) C_{\lambda_1^t\mu_2\nu_3^t}(q) \\
& C_{\lambda_2\mu_2^t\nu_2}(q) C_{\lambda_2^t\emptyset\nu_4^t}(q) C_{\emptyset\mu_5\nu_1^t}(q) C_{\lambda_5\mu_5^t\nu_5}(q) C_{\lambda_5^t\mu_6\nu_2^5}(q) C_{\lambda_6\mu_6^t\nu_6}(q) C_{\lambda_6^t\emptyset\emptyset}(q) C_{\emptyset\emptyset\nu_5^t}(q) C_{\emptyset\emptyset\nu_6^t}(q)\,.
\end{aligned}
\tag{5.131}
$$

This expression can be greatly simplified using the rules described in appendix G.1 and the result is the following

$$
Z_{top}^{[E_8]_1} = Z_0^{[E_8]_1} \cdot Z_1^{[E_8]_1} \cdot Z_{=}^{[E_8]_1}\,, \tag{5.132}
$$

$$
\begin{aligned}
Z_0^{[E_8]_1} = & \frac{H(e^{\pm\alpha_1 - m_1})H(e^{-\alpha_1 \pm m_2})H(e^{\pm\alpha_1 + m_4})H(e^{\pm\alpha_1 + m_5})H(e^{-\alpha_1 \pm m_6})}{H(e^{-2\alpha_1})^2 H(e^{m_4 - m_2})H(e^{m_5 - m_6})} \\
& \frac{H(u\, e^{m_2+m_5+m_6+f(m)})H(u\, e^{m_2+m_4+m_6+f(m)})\prod_{k=2,6} H(u\, e^{m_k+m_4+m_5+f(m)})}{H(u\, e^{\pm\alpha_1+m_2+m_4+m_5+m_6+f(m)})}
\end{aligned}
\tag{5.133}
$$

$$
\begin{aligned}
Z_1^{[E_8]_1} = \sum_{\nu_i} & \left(u\, e^{2m_3+f(m)+m_2+m_5+m_6}\right)^{\frac{|\nu_3|+|\nu_4|}{2}} \left(u\, e^{-2m_7+f(m)+m_2+m_4+m_6}\right)^{\frac{|\nu_5|+|\nu_6|}{2}} \left(u\, e^{f(m)-m_1}\right)^{\frac{|\nu_1|+|\nu_2|}{2}} \\
& \hat{Z}(\nu_3,\nu_4)\hat{Z}(\nu_5,\nu_6) \prod_{i=1,2}\prod_{s\in\nu_i} \frac{\prod_{k=1,2,4,5,6} 2\sinh\frac{E_{i\emptyset}-m_k}{2}}{2\sinh\frac{E_{i\emptyset}+f(m)-m_2-m_4-m_5-m_6+\log u}{2}\prod_{j=1,2}\left(2\sinh\frac{E_{ij}}{2}\right)^2}
\end{aligned}
\tag{5.134}
$$

$$
Z_{=}^{[E_8]_1} = H(u\, e^{m_2+m_4+m_5+m_6-m_1+f(m)})^{-2}\,, \tag{5.135}
$$

where $\beta_1 = -\beta_2 = \alpha_1$ and moreover we defined

$$
\begin{aligned}
\hat{Z}(\nu_i,\nu_j) = & \prod_{s\in\nu_i} \frac{2\sinh\frac{E_{i\emptyset}-\tilde{m}_1^{ij}}{2}\, 2\sinh\frac{E_{i2}-\tilde{m}_2^{ij}}{2}\, 2\sinh\frac{E_{i3}-\tilde{m}_3^{ij}}{2}}{\left(2\sinh\frac{E_{ii}}{2}\right)^2 2\sinh\frac{E_{ij}}{2}} \\
& \prod_{s\in\nu_j} e^{\frac{\beta_i-\beta_j}{2}} \frac{2\sinh\frac{E_{j\emptyset}-\tilde{m}_1^{ij}}{2}\, 2\sinh\frac{E_{j2}-\tilde{m}_2^{ij}}{2}\, 2\sinh\frac{E_{j3}-\tilde{m}_3^{ij}}{2}}{\left(2\sinh\frac{E_{jj}}{2}\right)^2 2\sinh\frac{E_{ji}}{2}}\,,
\end{aligned}
\tag{5.136}
$$

with

$$
\begin{aligned}
&\beta_3 = -\beta_4 = \frac{1}{2}(m_2 - m_4)\,, \quad \beta_5 = -\beta_6 = \frac{1}{2}(m_7 - m_5)\,, \\
&\tilde{m}_1^{34} = \log u + m_5 + m_6 + \frac{1}{2}(m_2 + m_4) + f(m)\,, \quad \tilde{m}_2^{34} = \tilde{m}_3^{34} = -\frac{1}{2}(m_2 + m_4)\,, \\
&\tilde{m}_1^{56} = \log u + m_2 + m_4 + \frac{1}{2}(m_5 + m_6) + f(m)\,, \quad \tilde{m}_2^{34} = \tilde{m}_3^{34} = -\frac{1}{2}(m_5 + m_6)\,.
\end{aligned}
\tag{5.137}
$$

It is also important to correctly take into account the decoupled factors which can be computed from the web diagram using the rule described in section 5.2.1

$$
Z_{dec}^{[E_8]_1} = Z_{dec,=}^{[E_8]_1} \cdot Z_{dec,||}^{[E_8]_1} \cdot Z_{dec,//}^{[E_8]_1} \tag{5.138}
$$

$$
Z_{dec,=}^{[E_8]_1} = H(u\,e^{m_2+m_4+m_5+m_6-m_1+f(m)})^{-2} H(u\,e^{m_2+m_4+m_5+m_6+f(m)-m_1})^{-1}\,, \tag{5.139}
$$

$$
\begin{aligned}
Z_{dec,//}^{[E_8]_1} =& H(e^{m_2+m_3})^{-1} H(u\,e^{f(m)-m_1})^{-1} H(e^{m_6-m_7})^{-1} H(u\,e^{m_2+m_3+f(m)-m_1})^{-1} \\
& H(u\,e^{m_2+m_3+m_6+f(m)-m_1-m_7})^{-1} H(u\,e^{m_6+f(m)-m_1-m_7})^{-1} H(e^{m_5-m_6})^{-1} \\
& H(e^{m_5-m_7})^{-1} H(u\,e^{m_5+f(m)-m_1-m_7})^{-1} H(u\,e^{m_2+m_3+m_5+f(m)-m_1-m_7})^{-1} \\
& H(u\,e^{m_3+m_4+m_5+f(m)-m_1-m_7})^{-1} H(u\,e^{m_3+m_4+f(m)-m_1})^{-1} \\
& H(u\,e^{m_3+m_4+m_6+f(m)-m_1-m_7})^{-1} H(e^{m_3+m_4})^{-1} H(e^{m_4-m_2})^{-1}\,,
\end{aligned}
\tag{5.140}
$$

$$
Z_{dec,||}^{[E_8]_1} = H(u\,e^{m_3+m_5+m_6+f(m)})^{-2} H(u\,e^{m_2+m_4+f(m)-m_7})^{-2} H(u^2\,e^{m_2+m_3+m_4+m_5+m_6+f(m)-m_7})^{-2} \tag{5.141}
$$

Note that, in the formalism in section 5.2.1, we can simply take into account the decoupled factors from the Higgsed diagram without considering the singlet hypermultiplet contribution in the Higgsed vacuum. This is one of the advantage of the prescription of using the topological vertex rule for Higgsed diagrams instead of UV diagrams.

By combining the topological string partition function (5.132) with the decoupled factor (5.138), we obtain the partition function associated to the $[E_8]_1$ web diagram

$$
Z^{[E_8]_1} = \frac{Z_{top}^{[E_8]_1}}{Z_{dec}^{[E_8]_1}} = Z_{pert}^{[E_8]_1} Z_{inst}^{[E_8]_1} \tag{5.142}
$$

$$
Z_{pert}^{[E_8]_1} = \frac{\left(\prod_{a=2,6} H(e^{-\alpha_1 \pm m_a})\right)\left(\prod_{a=3,4,5} H(e^{\pm\alpha_1 + m_a})\right)\left(\prod_{a=1,7} H(e^{\pm\alpha_1 - m_a})\right)}{H(e^{-2\alpha_1})^2}
$$

(5.143)

$$
\begin{aligned}
Z_{inst}^{[E_8]_1} =& \frac{H(u\,e^{m_2+m_5+m_6+f(m)})H(u\,e^{m_2+m_4+m_6+f(m)})\prod_{k=2,6} H(u\,e^{m_k+m_4+m_5+f(m)})}{H(e^{\pm\alpha_1+m_3})H(e^{\pm\alpha_1-m_7})H(u\,e^{\pm\alpha_1+m_2+m_4+m_5+m_6+f(m)})} \\
& H(u\,e^{m_2+m_4+m_5+m_6+f(m)-m_1})H(e^{m_6-m_7})H(u\,e^{m_2+m_3+f(m)-m_1}) \\
& H(u\,e^{m_2+m_3+m_6+f(m)-m_1-m_7})H(u\,e^{m_6+f(m)-m_1-m_7})H(e^{m_5-m_7}) \\
& H(u\,e^{m_5+f(m)-m_1-m_7})H(u\,e^{m_2+m_3+m_5+f(m)-m_1-m_7})H(u\,e^{f(m)-m_1}) \\
& H(u\,e^{m_3+m_4+m_5+f(m)-m_1-m_7})H(u\,e^{m_3+m_4+f(m)-m_1})H(e^{m_2+m_3}) \\
& H(u\,e^{m_3+m_4+m_6+f(m)-m_1-m_7})H(e^{m_3+m_4})H(u\,e^{m_3+m_5+m_6+f(m)})^2 \\
& H(u\,e^{m_2+m_4+f(m)-m_7})^2 H(u^2\,e^{m_2+m_3+m_4+m_5+m_6+f(m)-m_7})^2 \\
& \sum_{\nu_i}\left(u\,e^{2m_3+f(m)+m_2+m_5+m_6}\right)^{\frac{|\nu_3|+|\nu_4|}{2}}\left(u\,e^{-2m_7+f(m)+m_2+m_4+m_6}\right)^{\frac{|\nu_5|+|\nu_6|}{2}}\left(u\,e^{f(m)-m_1}\right)^{\frac{|\nu_1|+|\nu_2|}{2}} \\
& \hat{Z}(\nu_3,\nu_4)\hat{Z}(\nu_5,\nu_6)\prod_{i=1,2}\prod_{s\in\nu_i}\frac{\prod_{k=1,2,4,5,6} 2\sinh\frac{E_{i\emptyset}+m_k}{2}}{2\sinh\frac{E_{i\emptyset}+f(m)-m_2-m_4-m_5-m_6+\log u}{2}\prod_{j=1,2}\left(2\sinh\frac{E_{ij}}{2}\right)^2}\,.
\end{aligned}
\tag{5.144}
$$

The instanton part $Z_{inst}^{[E_8]_1}$ is appropriately chosen to be 1 at order $\mathcal{O}(u^0)$. Note that (5.142) exactly agrees with the partition function of the rank 1 E_8 theory which was computed in Section 5.1.

After obtaining the partition function of the rank 1 E_8 theory it is straightforward to obtain the partition function of the rank 2 E_8 theory for this is simply given by the product

$$
Z^{\text{rk2 }E_8} = Z^{[E_8]_1}[\alpha_1,\{m_{1,2,3,4,5,6,7}\},u]\cdot Z^{[E_8]_1}[\alpha_2,\{m_{1,2,3,4,5,6,7}\},u]\,. \tag{5.145}
$$

It follows that the perturbative part is

$$
\begin{aligned}
Z_{pert}^{\text{rk2 }E_8} =& \frac{\left(\prod_{a=2,6} H(e^{-\alpha_1\pm m_a})\right)\left(\prod_{a=3,4,5} H(e^{\pm\alpha_1+m_a})\right)\left(\prod_{a=1,7} H(e^{\pm\alpha_1-m_a})\right)}{H(e^{-2\alpha_1})^2} \\
& \frac{\left(\prod_{a=2,6} H(e^{-\alpha_2\pm m_a})\right)\left(\prod_{a=3,4,5} H(e^{\pm\alpha_2+m_a})\right)\left(\prod_{a=1,7} H(e^{\pm\alpha_2-m_a})\right)}{H(e^{-2\alpha_2})^2}\,.
\end{aligned}
\tag{5.146}
$$

By comparing this result with (H.32), one can see that this is exactly the perturbative contribution from seven fundamental hypermultiplets with masses given by $m_{1,2,3,4,5,6,7}$, one massless antisymmetric hypermultiplet and the $Sp(2)$ vector multiplet. Like in the previous examples discussed in this section we see that the contribution of the anti-symmetric hypermultiplet does not appear in (5.146) because it is cancelled by some factors in the $Sp(2)$ vector multiplet contribution.

It is also possible to compute the instanton contributions to the partition function of the rank 2 E_8 theory. At 1-instanton level the result is particularly simple because it is simply the sum of the 1-instanton part of the $[E_8]_1$ diagram and the

1-instanton part of the $[E_8]_2$ diagram. The 1-instanton part of each diagram agrees with the expression (H.22) of the partition function of an $Sp(1)$ gauge theory with $N_f = 7$ fundamental flavours

$$Z^{\text{rk2 } E_8}_{1\text{-}inst} = \frac{1}{2}\sum_{i=1}^{2}\left\{ \frac{\prod_{a=1}^{7} 2\sinh\frac{m_a}{2}}{2\sinh\frac{\pm\epsilon}{2} 2\sinh\frac{\pm\alpha_i}{2}} + \frac{\prod_{a=1}^{7} 2\cosh\frac{m_a}{2}}{2\sinh\frac{\pm\epsilon}{2} 2\cosh\frac{\pm\alpha_i}{2}} \right\}, \tag{5.147}$$

which agrees with the expression (H.23) of the 1-instanton part of $Sp(2)$ with seven fundamental hypermultiplets and one massless antisymmetric hypermultiplets computed on the field theory side.

Moreover we have checked agreement of the partition function of the rank 2 E_8 theory with the partition function of a $Sp(2)$ theory with 7 flavours and one massless anti-symmetric hypermultiplet at 2-instanton in the special limit where one of the masses of the fundamental hypermultiplets is set to zero[22].

Rank N $E_{6,7,8}$ theories

The generalisation to the partition functions of the rank N E_6, E_7, E_8 theories is straightforward. The web diagrams of the rank N $E_{6,7,8}$ theories are simply the superposition of N copies of the web diagram of the rank 1 $E_{6,7,8}$ theories respectively [174]. Therefore, the topological vertex formalism on the rank N $E_{6,7,8}$ diagrams shows that it can be written by the product

$$Z^{\text{rkN}E_{N_f+1}} = \prod_{i=1}^{N} \frac{Z^{[E_{N_f+1}]_i}_{top}[P_i]}{Z^{[E_{N_f+1}]_i}_{dec}[P_i]}, \tag{5.148}$$

where $Z^{[E_{N_f+1}]_i}_{top}[P_i]$ and $Z^{[E_{N_f+1}]_i}_{dec}[P_i]$ represent the topological string partition function and the decoupled factor computed from the i-th copy of the web diagram of the rank N E_{N_f+1} theory. In this subsection, N_f always mean either $N_f = 5, 6$, or 7. We also consider special web diagrams such that the centres of mass of all the closed faces in the diagram coincide with each other. In this particular situation the web diagram realises an $Sp(N)$ gauge theory with N_f fundamental hypermultiplets and with a massless anti-symmetric hypermultiplet. Moreover, we essentially choose the same parameterisation as (5.102), (5.118) and (5.130) for the first copy of the web diagram of the rank N E_6, E_7, E_8 theories respectively, The parameterisation of the i-th copy is simply obtained by exchanging α_1 with α_i. The product (5.148) is more precisely given by

$$Z^{\text{rkN } E_{N_f+1}} = \prod_{i=1}^{N} \frac{Z^{[E_{N_f+1}]_i}_{top}[\alpha_i, \{m_{1,\cdots,N_f}\}, u]}{Z^{[E_{N_f+1}]_i}_{dec}[\{m_{1,\cdots,N_f}\}, u]}, \tag{5.149}$$

where each factor $Z^{[E_{N_f+1}]_i}_{top}[\alpha_i, \{m_{1,\cdots,N_f}\}, u]/Z^{[E_{N_f+1}]_i}_{dec}[\{m_{1,\cdots,N_f}\}, u]$ for $N_f = 5, 6, 7$ is essentially given by (5.108), (5.123) and (5.142) respectively with α_1 exchanged with α_i.

One can easily see that (5.149) correctly realises the perturbative part and the 1-instanton part of the partition function of the $Sp(N)$ gauge theory with N_f

22 Again, the reason why we set one mass to zero is the technical issue.

fundamental hypermultiplets and one massless anti-symmetric hypermultiplet. The perturbative part of (5.149) is given by

$$Z_{pert}^{\text{rkN } E_{N_f+1}} = \prod_{i=1}^{N} Z_{pert}^{[E_{N_f+1}]_i}[\alpha_i, \{m_{1,\cdots,N_f+1}\}]. \tag{5.150}$$

This indeed agrees with the perturbative partition function (H.32) of the $Sp(N)$ gauge theory with N_f flavours and a massless anti-symmetric hypermultiplet.

Also, the 1-instanton part of (5.149) is given by

$$Z_{1\text{-}inst}^{\text{rkN } E_{N_f+1}}| = \sum_{i=1}^{N} Z_{inst}^{[E_{N_f+1}]_i}[\alpha_i, \{m_{1,\cdots,N_f}\}, u]|_{\mathcal{O}(u)} \tag{5.151}$$

Since the 1-instanton part of the partition function $Z_{inst}^{[E_{N_f+1}]_i}$ reproduces the 1-instanton part of the partition function of the $Sp(1)$ gauge theory with N_f flavours and a massless anti-symmetric hypermultiplet with the Coulomb branch modulus α_i, (5.151) precisely agrees with (H.23).

We have seen that the partition function of the $Sp(N)$ gauge theory with N_f fundamental hypermultiplets and a massless anti-symmetric hypermultiplet always shows the product structure. A physical reason why the factorisation happens may be that we are studying an IR theory in a Higgs branch of the $Sp(N)$ gauge theory with a non-vanishing vev for a "0" weight of the anti-symmetric hypermultiplet[23]. When it acquires a vev along the "0" weight of the anti-symmetric representation of $Sp(N)$, then the Cartan parts remain massless but some (but not all) of the root components of the adjoint representation of the $Sp(N)$ vector multiplet become massive. In fact the $Sp(N)$ gauge group is broken to $Sp(1)^N$ by the vev.

The factorisation of the partition function of the $Sp(N)$ gauge theories with massless anti–symmetric hypermultiplet is consistent with the fact that the Seiberg–Witten curves of the theories also factorise [204, 205]. Although the product structure of the Seiberg–Witten curves imply that the prepotential is written by the sum of N copies of $Sp(1)$ prepotential, our analysis shows that the full Nekrasov partition function itself also factorises.

5.2.3 *Towards refined topological vertex for Higgsed 5d T_N theories*

In this section, we extend the analysis in section 5.2.1 to the refined topological vertex for Higgsed 5d T_N theories. In some special cases, we can derive new refined topological vertex that can be directly applied to Higgsed web diagrams. In all the computation of the refined topological string partition function, we will choose the horizontal directions as the preferred directions.

Refined topological vertex, external horizontal legs

We will start by considering the case in which the external legs that we put on top of each other are horizontal. We show in Figure 30 the diagram we consider. The

23 The anti-symmetric hypermultiplet can acquire a non-vanishing vev because it is massless.

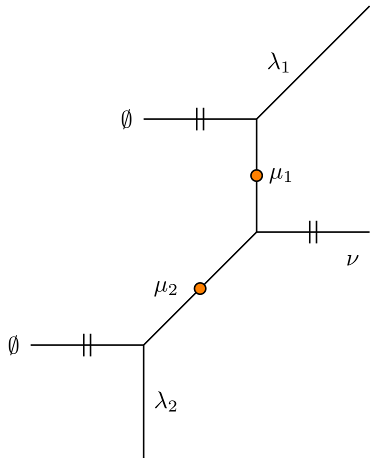

Figure 30: Higgsing of parallel horizontal legs in a T_N diagram. The orange dots indicate the curves that are shrunk to zero length and the double lines the preferred directions.

local part of the topological string partition function can be easily computed using the refined topological vertex and the result is

$$Z = \sum_{\mu_1,\mu_2} \left(-\left(\frac{q}{t}\right)^{\frac{1}{2}}\right)^{|\mu_1|+|\mu_2|} C_{\lambda_1\mu_1\emptyset}(q,t) C_{\mu_2\mu_1^t\nu}(t,q) C_{\mu_2^t\lambda_2\emptyset}(q,t)\,. \tag{5.152}$$

Here we used the tuning condition (5.2). It is possible to perform also in this case the Young diagrams summations over μ_1 and μ_2 arriving at the following expression

$$\begin{aligned} Z = &\sum_{\eta_i,\xi_i} (-1)^{|\xi_1|+|\xi_2|} \left(\frac{q}{t}\right)^{\frac{|\lambda_1|-|\lambda_2|-|\xi_1|-|\xi_2|}{2}} q^{-\frac{||\lambda_2^t||^2+||\nu||^2}{2}} t^{\frac{||\lambda_2||^2}{2}} \tilde{Z}_\nu(t,q) \\ &s_{\lambda_1^t/\eta_1}(q^{-\rho}) s_{\lambda_2/\eta_3}(t^{-\rho}) s_{\eta_1^t/\xi_1}(q^{-\rho}t^{-\nu^t}) s_{\eta_2^t/\xi_1^t}(t^{-\rho}) s_{\eta_2^t/\xi_2}(q^{-\rho}) s_{\eta_3^t/\xi_2^t}(t^{-\rho}q^{-\nu}) \\ &\prod_{i,j=1}^{\infty} \frac{(1-q^i t^{j-1-\nu_i^t})(1-q^{i-\nu_j}t^{j-1})}{(1-q^{i+1}t^{j-2})}\,. \end{aligned} \tag{5.153}$$

In this case looking at the infinite product factor we see that the result will be zero unless $\nu = \emptyset$, and so in the following we will focus in this case. In this situation using (G.17) it is possible to perform some additional simplifications and arrive at the following simple result

$$Z = q^{-\frac{||\lambda_2^t||^2}{2}} t^{\frac{||\lambda_2||^2}{2}} \left(\frac{q}{t}\right)^{|\lambda_1|+|\lambda_2|} \sum_\kappa \left(\frac{t}{q}\right)^{\frac{3|\kappa|+|\lambda_1|-|\lambda_2|}{2}} s_{\lambda_1^t/\kappa}(q^{-\rho}) s_{\lambda_2/\kappa}(t^{-\rho}) \prod_{i,j=1}^{\infty} \frac{(1-q^i t^{j-1})^2}{(1-q^{i+1}t^{j-2})}\,. \tag{5.154}$$

Motivated by the form of the result we will define the new vertex

$$\tilde{C}_{\lambda\mu\nu}(q,t) = q^{-\frac{||\mu^t||^2}{2}} t^{\frac{||\mu||^2+||\nu||^2}{2}} \tilde{Z}_\nu(q,t) \sum_\eta \left(\frac{t}{q}\right)^{\frac{3|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}(q^{-\rho}t^{-\nu}) s_{\mu/\eta}(t^{-\rho}q^{-\nu^t})$$

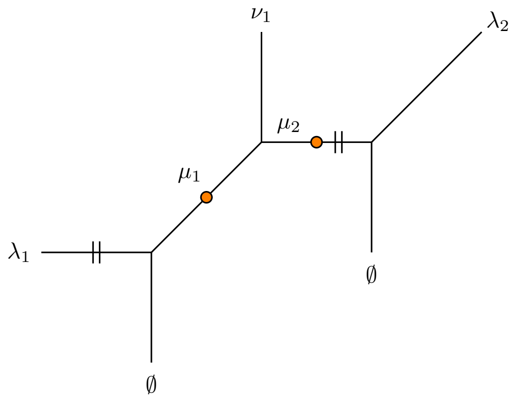

Figure 31: Higgsing of parallel vertical legs in a T_N diagram. The orange dots indicate the curves that are shrunk to zero length and the double lines the preferred directions.

(5.155)

which allows us to rewrite (5.154) as

$$\tilde{Z} = \left(\frac{q}{t}\right)^{|\lambda_1|+|\lambda_2|} \tilde{C}_{\lambda_1\lambda_2\emptyset}(q,t)\,, \tag{5.156}$$

where we also dropped the infinite product terms which only contribute to the decoupled factor. Therefore we see that in the case of putting a pair of parallel horizontal external legs on top of each other we can use the new refined topological vertex (5.156) to compute the partition function for a Higgsed diagram.

Refined topological vertex, external vertical and diagonal legs

We then consider a Higgs branch realised by a tuning that places parallel external NS5-branes on top of each other. Since the refined topological vertex is not symmetric under the exchange between the three Young diagrams, one needs to work on this case separately.

The corresponding local diagram is depicted in Figure 31. The refined topological string partition function for the local diagram is

$$Z(\lambda_1,\nu_1,\lambda_2) = \sum_{\mu_1,\mu_2}\left(-\left(\frac{t}{q}\right)^{\frac{1}{2}}\right)^{|\mu_1|+|\mu_2|} C_{\mu_1\emptyset\lambda_1}(q,t)C_{\mu_1^t\nu_1\mu_2}(t,q)C_{\lambda_2\emptyset\mu_2^t}(q,t). \tag{5.157}$$

Here we used the tuning condition (5.3). It is possible to sum over the Young diagram of μ_1 in (5.157),

$$\begin{aligned} Z(\lambda_1,\nu_1,\lambda_2) \quad = \quad & \sum_{\mu_2,\eta}\left(-\left(\frac{t}{q}\right)^{\frac{1}{2}}\right)^{|\mu_2|} q^{\frac{1}{2}(||\nu_1||^2+||\mu_2||^2)}t^{\frac{1}{2}(||\lambda_1||-||\nu_1^t||^2+||\mu_2^t||^2)}\tilde{Z}_{\lambda_1}(q,t)\tilde{Z}_{\mu_2}(t,q)\tilde{Z}_{\mu_2^t}(q,t) \\ & s_{\nu_1/\eta}(t^{-\mu_2^t+\frac{1}{2}}q^{-\rho-\frac{1}{2}})s_{\lambda_2^t}(t^{-\mu_2^t+\frac{1}{2}}q^{-\rho-\frac{1}{2}})s_{\eta^t}(-t^{-\lambda_1+\frac{1}{2}}q^{-\rho-\frac{1}{2}}) \\ & \prod_{i,j=1}^{\infty}\left(1-q^{i-\mu_{2,j}-1}t^{j-\lambda_{1,i}}\right). \end{aligned} \tag{5.158}$$

The last term of (5.158) indicates that the μ_2 summation of (5.158) is bounded by the Young diagram λ_1. More precisely, (5.158) is zero unless $\lambda_{1,i} \leq \mu_{2,i}^t$ for each i. In the μ_2 summation, we can proceed further for a special case where $\mu_2 = \lambda_1^t$. When $\mu_2 = \lambda_1^t$, (5.158) further reduces to

$$Z(\lambda_1, \emptyset, \lambda_2)|_{\mu_2=\lambda_1^t} = \left(\frac{t}{q}\right)^{|\lambda|} C_{\lambda_2\emptyset\lambda_1}(q,t) \prod_{i,j=1}^{\infty} \left(1 - q^{i-1}t^j\right), \tag{5.159}$$

where $|_{\mu_2=\lambda_1^t}$ indicates extracting the term of $\mu_2 = \lambda_1^t$ in the μ_2 summation. In this case, ν_1 is restricted to $\emptyset$.

For general Young diagrams of ν_1, λ_2, it is difficult to get an analytic expression after performing the summations of μ_2 and η. However, we can still perform the summation in a special case where $\nu_1 = \emptyset, \lambda_2 = \emptyset$. In fact in this particular situation we find that

$$\frac{\sum_{\mu_2}(-Q)^{|\mu_2|}q^{\frac{1}{2}||\mu_2||^2}t^{\frac{1}{2}||\mu_2^t||^2}\tilde{Z}_{\mu_2}(t,q)\tilde{Z}_{\mu_2^t}(q,t)\prod_{i,j=1}^{\infty}\left(1-q^{i-\mu_{2,j}-1}t^{j-\lambda_{1,i}}\right)}{\sum_{\mu_2}(-Q)^{|\mu_2|}q^{\frac{1}{2}||\mu_2||^2}t^{\frac{1}{2}||\mu_2^t||^2}\tilde{Z}_{\mu_2}(t,q)\tilde{Z}_{\mu_2^t}(q,t)\prod_{i,j=1}^{\infty}\left(1-q^{i-\mu_{2,j}-1}t^{j}\right)} = \left(\frac{t}{q}\right)^{\frac{|\lambda_1|}{2}} Q^{|\lambda_1|}. \tag{5.160}$$

This identity has been checked order by order in Q up to $|\mu_2| = 4$. The denominator of the left-hand side of (5.160) can be evaluated exactly,

$$\begin{aligned}&\sum_{\mu_2}(-Q)^{|\mu_2|}q^{\frac{1}{2}||\mu_2||^2}t^{\frac{1}{2}||\mu_2^t||^2}\tilde{Z}_{\mu_2}(t,q)\tilde{Z}_{\mu_2^t}(q,t)\prod_{i,j=1}^{\infty}\left(1-q^{i-\mu_{2,j}-1}t^{j}\right)\\ &= \prod_{i,j=1}^{\infty}\left(1-Q\left(\frac{t}{q}\right)^{\frac{1}{2}}q^{i-1}t^j\right)\left(1-\left(\frac{t}{q}\right)^{\frac{1}{2}}q^{i-\frac{1}{2}}t^{j-\frac{1}{2}}\right)\left(1-Qq^{i-\frac{1}{2}}t^{j-\frac{1}{2}}\right)\end{aligned} \tag{5.161}$$

By using (5.158), (5.160) and (5.161), we obtain

$$Z(\lambda_1, \emptyset, \emptyset) = \left(\frac{t}{q}\right)^{|\lambda_1|} C_{\emptyset\emptyset\lambda_1}(q,t) \prod_{i,j=1}^{\infty} \frac{(1-q^{i-1}t^j)^2}{(1-q^{i-2}t^{j+1})}. \tag{5.162}$$

The three factors at the end of (5.162) may be a part of the decoupled factor. Hence, we can use a different new refined topological vertex

$$\tilde{Z}(\lambda_1, \emptyset, \emptyset) = \left(\frac{t}{q}\right)^{|\lambda_1|} C_{\emptyset\emptyset\lambda_1}(q,t), \tag{5.163}$$

when $\nu_1 = \lambda_2 = \emptyset$ in Figure 31.

Note that (5.162) is obtained by setting $\lambda_2 = \emptyset$ in (5.158) up to decoupled factors. This is not a coincidence. Since the numerator of (5.160) starts from $\mathcal{O}(Q^{|\lambda_1|})$, the coefficient at this order is dictated by the Young diagram $\mu_2 = \lambda_1^t$. Therefore, (5.162) agrees with (5.158) when $\lambda_2 = \emptyset$ up to the decoupled factor.

It is also possible to consider a Higgs branch realised by placing external (1,1)-branes on top of each other. The diagram is shown in Figure 32. In this case the local part of the partition function is

$$Z(\lambda_1, \nu_1, \lambda_2) = \sum_{\mu_1,\mu_2}\left(-\left(\frac{q}{t}\right)^{\frac{1}{2}}\right)^{|\mu_1|+|\mu_2|} C_{\emptyset\mu_1\lambda_1}(q,t)C_{\nu_1\mu_1^t\mu_2}(t,q)C_{\emptyset\lambda_2\mu_2^t}(q,t), \tag{5.164}$$

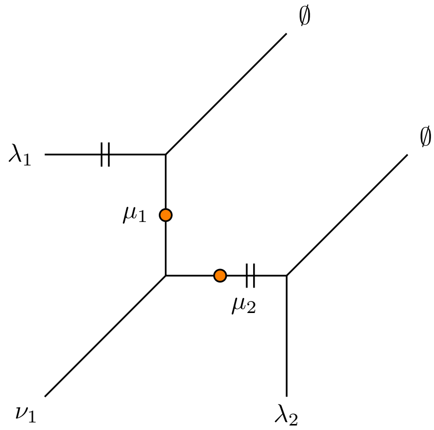

Figure 32: Higgsing of parallel diagonal legs in a T_N diagram. The orange dots indicate the curves that are shrunk to zero length and the double lines the preferred directions.

where we have used the tuning condition (5.2). It is possible to perform the μ_1 summation in (5.164) and the result is

$$Z(\lambda_1,\nu_1,\lambda_2) = \sum_{\mu_2,\eta}\left(-\left(\frac{q}{t}\right)^{\frac{1}{2}}\right)^{|\mu_2|} t^{\frac{1}{2}(||\lambda_2||^2+||\mu_2^t||^2+||\lambda_1||^2)} q^{\frac{1}{2}(||\mu_2||^2-||\lambda_2^t||^2)} \tilde{Z}_{\lambda_1}(q,t)\tilde{Z}_{\mu_2}(t,q)\tilde{Z}_{\mu_2^t}(q,t)$$
$$s_{\nu_1/\eta}(t^{-\rho-\frac{1}{2}}q^{-\mu_2+\frac{1}{2}})s_{\lambda_2}(t^{-\rho+\frac{1}{2}}q^{-\mu_2^t-\frac{1}{2}})s_{\eta^t}(q^{-\lambda_1^t+\frac{1}{2}}t^{-\rho-\frac{1}{2}})$$
$$\prod_{i,j=1}^{\infty}(1-t^{i-\mu_{2,j}^t-1}q^{j-\lambda_{1,i}^t})\,. \tag{5.165}$$

Note that like in the previous case there is a bound on the μ_2 summation which in this case is $\lambda_{1,i} \leq \mu_{2,i}$. Again it is difficult to obtain an analytic expression after performing the μ_2 and η summations for general λ_2 and ν_1, however in the special case $\lambda_2 = \nu_1 = \emptyset$ there are some great simplifications. In particular we find that

$$\frac{\sum_{\mu_2}(-Q)^{|\mu_2|}t^{\frac{1}{2}||\mu_2^t||^2}q^{\frac{1}{2}||\mu_2||^2}\tilde{Z}_{\mu_2}(t,q)\tilde{Z}_{\mu_2^t}(q,t)\prod_{i,j=1}^{\infty}(1-t^{i-\mu_{2,j}^t-1}q^{j-\lambda_{1,i}^t})}{\sum_{\mu_2}(-Q)^{|\mu_2|}t^{\frac{1}{2}||\mu_2^t||^2}q^{\frac{1}{2}||\mu_2||^2}\tilde{Z}_{\mu_2}(t,q)\tilde{Z}_{\mu_2^t}(q,t)\prod_{i,j=1}^{\infty}(1-t^{i-\mu_{2,j}^t-1}q^{j})} = \left(\frac{q}{t}\right)^{\frac{|\lambda_1|}{2}} Q^{|\lambda_1|}\,. \tag{5.166}$$

Therefore like in the case of placing parallel vertical legs on top of each other we find that, up to some decoupled factors, it is possible to write the local contribution to the partition function using a new vertex

$$\tilde{Z}(\lambda_1,\emptyset,\emptyset) = \left(\frac{q}{t}\right)^{|\lambda_1|} C_{\emptyset\emptyset\lambda_1}(q,t)\,. \tag{5.167}$$

Part IV

CONCLUSION & APPENDICES

6 CONCLUSIONS - CONCLUSIONES

CONCLUSIONS

In this thesis we have discussed the flavour structure of $SU(5)$ GUT models in F–theory. We have shown how, despite the fact that the fermion mass matrix has rank 1, inclusion of non–perturbative effects increases its rank leading to a full rank 3 matrix.

For the case of the E_6 model we have provided a local model for the masses of the up–type quarks. Explicit computation of Yukawa couplings via residues shows that after inclusion of non–perturbative effects the Yukawa matrix has a good hierarchical structure of the form $(\mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon), \mathcal{O}(1))$ in its eigenvalues. To obtain the value of the physical couplings we have computed the wavefunctions in real gauge. The presence of non–primitive fluxes, necessary to ensure supersymmetry of the background in the presence of T–branes, makes this task a difficult one, however for a specific region in parameter space we have obtained a solution for the entire set of wavefunctions. This has allowed us to compute the physical masses of fermions and, performing a scan over the parameters defining our local model, we have shown that values of fermion masses compatible with the empirical values are possible. This analysis shows that, unlike in type II D–brane models, in F–theory GUT models a realistic value for the top quark mass is attainable.

Afterwards we considered a more general class of models showing a local enhancement to either E_7 or E_8. These models have the advantage of generating the whole set of Yukawa couplings at a unique point. We performed a scan over the different embeddings of matter fields showing that only a pair of models exist that show a good hierarchy of form $(\mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon), \mathcal{O}(1))$ in the Yukawa matrices. In other cases which we chose to discard the eigenvalues have the form $(\mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon^2), \mathcal{O}(1))$ which does not seem promising for comparison with empirical data. One great advantage is that both these models can be embedded in both E_7 and E_8 and the resulting fermion masses are independent of the embedding. We performed a thorough analysis of one of the models showing that it possesses wide regions in parameter space where we attain compatibility with empirical values. Knowledge of the full spectrum of the MSSM has allowed us to find an interval of preferred values for $\tan\beta$ which should be $\tan\beta \sim 10 - 20$ in our setup. Finally we showed how a small separation of the $SO(12)$ and E_6 Yukawa points has important consequences for the CKM matrix, allowing us to fix this separation to be two orders of magnitude lower than the typical size of the GUT divisor.

It would be desirable to generalise the computations presented in this thesis. The natural way to proceed would be developing techniques for the computation of Yukawa couplings in global F–theory models. This would have important consequences, including a full study of the chirality conditions for these models with an emphasis on the effect of the hypercharge flux. In addition to this it would be desirable to develop the computation of the Yukawa couplings between MSSM charged fields and singlets. The presence of these singlets can be extremely important for the phenomenology of neutrinos and electroweak symmetry breaking and viability of these various scenarios can only be tested after proper estimation of the couplings with the visible sector.

Additionally we have studied the effect of linear equivalence in D–brane models. The first impact of linear equivalence in D–brane models appears in the kinetic mixing of open and closed string $U(1)$'s. We have fully characterised this coupling computing it via direct dimensional reduction and via Witten effect. We have also shown how the kinetic mixing is intimately related to linear equivalence of cycles, showing that if the stack of branes giving a massless $U(1)$ lie on linear equivalent cycles then no mixing with the closed string sector is present. This parallelism with linear equivalence has been generalised to include models where D–branes carry magnetic fluxes on their worldvolume. We have also discussed two possible implications of kinetic mixing. First, due to the kinetic mixing with the closed string sector there may appear light particles with infinitesimal electric charge. In the scenario we proposed the closed string $U(1)$ via kinetic mixing plays the rôle of a messenger with a hidden sector. This effect ought to be compared with the additional source of mixing with the hidden sector due to massive modes charged under both the visible and the hidden sectors. Secondly we discussed the implications of the mixing of hypercharge in GUT models. This effect has potential implications for unification of the gauge couplings, for the presence of a mixing with the closed string sector may induce distortions in the usual unification relations between the couplings of the Standard Model.

The natural continuation would be the computation of the kinetic mixing among the various open string $U(1)$'s. This is in general a difficult question for this effect is only induced as a one loop effect and therefore the answer is known only in some simplified setups. Knowing a formula for the mixing in general solutions of type II String Theory would be an important piece in the study of the phenomenology of D–brane models, for mixing with a hidden sector can have potential implications for dark matter and mediation of supersymmetry breaking.

The concept of linear equivalence has also appeared when studying the moduli sector of D–branes. The conditions for the existence of massless fields in the open string sector are usually derived in a frozen closed string background. Reinstating a dynamical closed string background has profound consequences for the brane moduli sector as some fields may acquire a mass due to coupling with the closed string moduli. We have elucidated the microscopic mechanism leading to this phenomenon finding agreement with the analysis performed at the level of the effective action. In particular our result shows a novel mechanism that may give mass to Wilson line moduli that were thought to receive mass only via coupling to worldsheet instantons.

There are some interesting directions for further study on the stabilisation mechanism that we discussed. First of all it would be interesting to perform a study of the moduli stabilisation for the entire set of moduli of a given compactification to see whether full moduli stabilisation is attainable or not. In addition to this it would be interesting to see if the mechanism we discussed can appear in other String Theory models and study its implications for moduli stabilisation and inflationary models.

Finally we applied topological string techniques to the study of the Higgs branch of the 5d T_N theory. This theory has no known Lagrangian description and without the use of topological strings it would be impossible to perform such computations. We formulated an algorithm for the computation of the partition function in the Higgs branch after finding the general set of conditions for having a flat direction for a hypermultiplet. We discussed in detail the rôle played by singlet hypermultiplets in the vacuum and correct identification of their contributions in the partition function. Finally we applied this algorithm to the case of the E_8 theory obtaining the result for the partition function. Explicit computation of the superconformal index shows that the perturbative global symmetry enhances to E_8 when instantonic particles are taken into account. Moreover we compared our results with the ones obtained via field theory techniques finding agreement between the two. After this we developed a new formulation of the topological vertex that allows its application directly to web diagrams that are not dual to toric varieties. This gives a great advantage on the computational side with the respect to the previously discussed algorithm but is not applicable in general in the case of the refined topological vertex. We formulated in detail the prescription for the correct identification of decoupled factors and singlet hypermultiplets showing that the latter are absent when applying the topological vertex to non–toric varieties. This technique has been applied to the case of rank N E_n theories in 5d obtaining the full partition function which shows a factorisation property. The interpretation in terms of a gauge theory is that of a $Sp(N)$ gauge theory with $n-1$ fundamental hypermultiplets and one massless antisymmetric hypermultiplet. The latter acquires a non–vanishing vev leaving only $Sp(1)^n$ as the unbroken gauge group.

The techniques we developed may be employed in some interesting examples. A prominent one is given by the computation of the partition function in the presence of topological defects. A simple generalisation of our prescription may be applied to obtain the resulting partition function. Finally one further generalisation of the topological vertex is in cases when orientifold planes are present. Brane webs in the presence of orientifold planes can give rise to new superconformal field theories in 5d and have only recently been considered in the literature. However computations performed in these situations heavily draw from field theory localisation techniques and therefore the use of topological string techniques would be a major advance in the study of these theories.

CONCLUSIONES

En esta tesis se ha discutido la estructura del sector de sabor de modelos de Gran Unificación $SU(5)$ en Teoría F. Se he demostrado cómo, aunque la matriz de masa de los fermiones tenga rango 1, la inclusion de efectos no perturbativos aumenta el rango de esta matriz a 3. Para el caso del modelo E_6 se ha aportado un modelo local para las masas de los quarks de tipo up. El calculo explícito de los acoplos de Yukawa a través de residuos demuestra que después de la introducción de efectos no perturbativos la matriz de Yukawa tiene una buena estructura jerárquica de la forma $(\mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon), \mathcal{O}(1))$ en sus autovalores. Para conseguir los valores de los acoplos físicos se han calculado las funciones de onda en el gauge real. La presencia de flujos no primitivos, necesaria para asegurar supersimetría de la solución en presencia de T-branas, hace esta tarea complicada, sin embargo para una región específica del espacio de parámetros se pudo encontrar una solución para todas las funciones de onda. Con este resultado se pudo calcular las masas para todos los fermiones y, efectuando una búsqueda en el espacio de parámetros que definen el modelo, se ha demostrado la posibilidad de alcanzar valores de masas para los fermiones compatibles con los valores empíricos. Este análisis demuestra que, contrariamente a lo que ocurre en modelos de D–branas en TeorŠa de Cuerdas de tipo II, en modelos de Gran Unificación en Teoría F es posible lograr un valor realista para la masa del quark top.

Después se ha considerado una clase más general de modelos que poseen un aumento local del grupo gauge a E_7 o E_8. Estos modelos tienen la ventaja de generar todos los acoplos de Yukawa en un único punto. Se han analizados las varias posibilidades de asignación de los campos de materia demostrando que solamente dos modelos tienen una buena jerarquía de la forma $(\mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon), \mathcal{O}(1))$ en la matriz de Yukawa. En los casos que se descartaron los autovalores tienen la forma $(\mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon^2), \mathcal{O}(1))$, una estructura que no parece prometedora en comparación con los datos empíricos. Una gran ventaja de los dos modelos que tienen una jerarquía correcta es que admiten una incorporación en tanto en E_7 como en E_8 y la masas de los fermiones son iguales en los dos casos. Se ha efectuado un análisis detallado de uno de los dos modelos demostrando que existen amplias regiones en el espacio de parámetros donde se encuentra compatibilidad con los valores empíricos. El conocimiento de todas las masas de los fermiones del MSSM ha permitido fijar un interval de valores favorecido para $\tan\beta$ que debería ser $\tan\beta \sim 10 - 20$ en este modelo. Finalmente se ha demostrado como una pequeña separación de los puntos de Yukawa $SO(12)$ y E_6 tiene consecuencias importantes para la matriz CKM. Esto permite determinar la separación entre estos dos puntos que debe de ser dos órdenes de magnitud menor que el tamaño típico del divisor GUT.

Sería recomendable generalizar los cálculos presentados en esta tesis. La manera natural de proseguir sería el desarrollo de técnicas para el calculo de acoplos de Yukawa en modelos globales de Teoría F. Esto tendría consecuencias importantes, como por ejemplo el estudio detallado de condiciones de quiralidad para estos modelos, con un énfasis en el efecto del flujo de hipercarga. Además de esto sería recomendable desarrollar el cálculo de los acoplos de Yukawa entre los campos del MSSM y los singletes. La presencia de estos singletes puede ser extremadamente importante para la fenomenología de los neutrinos y la ruptura de la simetría electrodébil y se puede analizar la factibilidad de estos escenarios solamente después de haber estimado propiamente los acoplos con el sector visible.

Además de esto se ha estudiado el efecto de la equivalencia lineal en modelos de D–branas. El primer efecto de la equivalencia lineal en modelos de D–branas aparece en la mezcla cinética entre los $U(1)$'s de cuerda abierta y cuerda cerrada. Se ha caracterizado esta mezcla calculándola directamente a través de reducción dimensional y a través del efecto Witten. Se ha demostrado que la mezcla cinética está relacionada con la equivalencia lineal de ciclos en el sentido que si las pilas de branas que dan un $U(1)$ sin masa están en ciclos equivalentes lineales no ocurre ninguna mezcla cinética con el sector de cuerda cerrada. Se ha generalizado este paralelismo con la equivalencia lineal de ciclos para incluir modelos donde las D–branas llevan flujos magnéticos en su volumen. Se han discutido también dos posibles implicaciones de la mezcla cinética. Primero, debido a la mezcla cinética con el sector de cuerda cerrada existe la posibilidad de tener partículas de carga eléctrica infinitesimal. En este escenario que se ha propuesto el $U(1)$ de cuerda cerrada juega el papel de un mensajero con un sector oculto. Este efecto debe de ser comparado con la fuente adicional de mezcla con el sector oculto debida a la existencia de partículas masivas cargadas tanto bajo el sector visible como bajo el sector oculto. En segundo lugar se ha discutido las posibles implicaciones de la mezcla de hipercarga en modelos de Gran Unificación. Esto tiene potenciales implicaciones para la unificación de los acoplos de gauge, porque la presencia de una mezcla con el sector de cuerda cerrada puede inducir distorsiones en las relaciones habituales de unificación entre los acoplos de gauge del Modelo Estándar.

La continuación natural sería el cálculo de la mezcla cinética entre los $U(1)$'s de cuerda abierta. Ésta es una pregunta difícil en general debido al hecho que este efecto se produce solo a 1-loop y por lo tanto la respuesta se conoce solo en casos simples. Tener una expresión para la mezcla en soluciones generales de teoría de cuerdas de tipo II sería una pieza importante en el estudio de la fenomenología de los modelos de D–branas porque la mezcla con un sector oculto tiene importantes consecuencias para materia oscura y mediación de ruptura de supersimetría.

La equivalencia lineal hizo su aparición también en el estudio del sector de modulos de cuerda abierta. Las condiciones para la existencia de escalares sin masa en el sector de cuerda abierta se suelen calcular en un fondo de cuerda cerrada congelado. La reincorporación de un fondo de cuerda cerrada dinámico tiene consecuencias profundas para los modulos de las branas porque algunos campos reciben masa debido a un acoplo con los modulos de cuerda cerrada. Se ha elucidado el mecanismo microscópico que causa este fenómeno encontrando acuerdo con el análisis hecho a nivel de teoría efectiva. En particular el resultado obtenido demuestra un novedoso mecanismo que puede dar una masa a los modulos de línea de Wilson, contrariamente a la idea que estos modulos reciban masa solamente por la presencia de efectos de instantones de cuerda abierta.

Hay algunas posibilidades interesantes para estudios adicionales sobre el mecanismo de estabilización que se ha discutido. Primero sería interesante hacer un estudio de la estabilización de todos los modulos de una solución para comprobar si es posible lograr estabilización de todos los modulos. Además de esto sería interesante ver si este mecanismo puede aparecer en otros modelos de Teoría de Cuerdas y estudiar las consecuencias para estabilización de modulos y cosmología.

Finalmente se han aplicado técnicas de cuerda topológica al estudio de la rama de Higgs de la teoría T_N en 5d. De esta teoría no se conoce una descripción en término de un Lagrangiano y sería imposible hacer este tipo de cálculos si no se utilizaran técnicas de cuerda topológica. Se ha formulado un algoritmo para el cálculo de la función de partición en la rama de Higgs después de encontrar las condiciones generales para tener un hipermultiplete sin masa. Se ha discutido en detalle la presencia de hipermultipletes que son singletes en el vacío y la correcta identificación de sus contribuciones en la función de partición. Por último se ha aplicado este algoritmo al caso de la teoría E_8 obteniendo la función de partición. El cálculo del índice superconforme demuestra que la simetría global perturbativa se aumenta a E_8 al tener en cuenta partículas instantónicas. Además de esto se ha comparado el resultado conseguido con el resultado calculado utilizando técnicas de teoría de campos encontrando acuerdo entre los dos. Después se ha desarrollado una nueva formulación del vertice topológico que se puede aplicar a diagramas de branas que no son duales a variedades tóricas. Esto da una gran ventaja desde el punto de vista computacional comparado con el algoritmo discutido anteriormente pero no se puede aplicar en general al caso del vértice topológico refinado. Se ha formulado en detalle como identificar correctamente los factores desacoplados y los hipermultipletes que son singletes demostrando que los últimos no están presentes al aplicar la nueva formulación del vértice topológico. Se ha aplicado esta técnica al caso de las teorías E_n de rango N en 5d consiguiendo la función de partición y demostrando que esta tiene una propiedad de factorización. La teoría de gauge tiene un grupo gauge $Sp(N)$, $n-1$ hipermultipletes en la fundamental del grupo gauge y un hipermultiplete en la antisimétrica del grupo gauge. Este último adquiere un valor esperado en el vacío no nulo dejando solamente $Sp(1)^n$ como simetría de gauge no rota.

Las técnicas que se han desarrollado se pueden emplear en algunos ejemplos interesantes. Un ejemplo prometedor es el cálculo de la función de partición en presencia de defectos topológicos. Una generalización simple de la prescripción que se ha formulado permitiría hacer este tipo de cálculo. Por último sería importante llegar a una formulación del vértice topológico en presencia de planos de orientifold. La presencia de planos de orientifold permite construir nuevas teorías superconformes en 5d y esta posibilidad se ha considerado solo recientemente en la literatura. Sin embargo los cálculos hechos en estos casos siempre necesitan técnicas de localización en teoría de campos y por lo tanto el empleo de técnicas de cuerda topológica constituiría un gran avance en el estudio de estas teorías.

A EXCEPTIONAL ALGEBRAS

In this appendix we collect some details regarding the exceptional algebras.

A.1 E_6 ALGEBRA

The Lie algebra of E_6 has 78 generators. In the Weyl–Cartan basis we can decompose the generators as $\{Q_\alpha\} = \{H_i, E_\rho\}$, where H_i generate the Cartan subalgebra of E_6 and E_ρ correspond to the roots of E_6. More precisely we have the usual relation

$$[H_i, E_\rho] \;=\; \rho_i E_\rho \tag{A.1}$$

where ρ_i is the i-th component of the root ρ. The 72 non-trivial roots are given by

$$(0, \underline{\pm 1, \pm 1, 0, 0, 0}) \tag{A.2}$$

where we should consider all possible permutation of the underlined vector entries, and

$$\frac{1}{2}(\pm\sqrt{3}, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1) \qquad \text{with even number of } +'\text{s} \tag{A.3}$$

In Section 3.3 we need the definitions of the following Cartan generators

$$\begin{aligned} P &= \frac{1}{2}(\sqrt{3}H_1 + H_2 + H_3 + H_4 + H_5 + H_6) &\text{(A.4)}\\ Q &= \frac{1}{2}\Big(\frac{5}{\sqrt{3}}H_1 - H_2 - H_3 - H_4 - H_5 - H_6\Big) &\text{(A.5)}\end{aligned}$$

as well as the following roots

$$E^{\pm} = \pm\frac{1}{2}(\sqrt{3}, 1, 1, 1, 1, 1) \tag{A.6}$$

Finally we can give the roots corresponding to the various matter fields. Under the decomposition $E_6 \to SU(5) \times SU(2) \times U(1)$ we have that the roots transforming in the $(\mathbf{10}, \mathbf{2})_{-1} \oplus$ h.c. are

$$\begin{array}{rlrl} E_{\mathbf{10}^+} &= (0, \underline{1, 1, 0, 0, 0}) & E_{\mathbf{10}^-} &= \tfrac{1}{2}(-\sqrt{3}, \underline{1, 1, -1, -1, -1}) \\ E_{\overline{\mathbf{10}}^+} &= -(0, \underline{1, 1, 0, 0, 0}) & E_{\overline{\mathbf{10}}^-} &= -\tfrac{1}{2}(-\sqrt{3}, \underline{1, 1, -1, -1, -1}) \end{array} \tag{A.7}$$

and the roots in the $(\mathbf{5}, \mathbf{1})_2 \oplus$ h.c. are

$$E_{\mathbf{5}_2} = \frac{1}{2}(\sqrt{3}, \underline{1, -1, -1, -1, -1})\,, \quad E_{\bar{\mathbf{5}}_{-2}} = -\frac{1}{2}(\sqrt{3}, \underline{1, -1, -1, -1, -1})\,. \tag{A.8}$$

A.2 E_7 ALGEBRA

The Lie algebra of E_7 has 133 generators Q_α. We will always work in the Weyl-Cartan basis, where such generators are split in the 7 generators of the Cartan subalgebra $H_i, i = 1...7$ and 126 roots E_ρ. In this basis he commutation rules among Cartan and roots are the following

$$[H_i, E_\rho] = \rho_i E_\rho \tag{A.9}$$

The 126 roots of $\mathfrak{e}_7$ take the following form:

$$\left(\underline{\pm1, \pm1, 0, 0, 0, 0}, 0\right) \tag{A.10}$$

$$2\left(\pm1, \pm1, \pm1, \pm1, \pm1, \pm1, \pm\sqrt{2}\right) \tag{A.11}$$

$$\left(0, 0, 0, 0, 0, 0, \pm\sqrt{2}\right) \tag{A.12}$$

where in (A.11) we consider only charge vectors in which an even number of $+1$ appear.

We are interested in the decomposition $E_7 \to SU(5) \times SU(2) \times U(1)^2$ under which the adjoint of E_7 decomposes as

$$\begin{aligned} \mathfrak{e}_7 &\supset \mathfrak{su}_5^{\text{GUT}} \oplus \mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1 \\ \mathbf{133} &\to (\mathbf{24}, \mathbf{1})_{0,0} \oplus (\mathbf{1}, \mathbf{3})_{0,0} \oplus 2(\mathbf{1}, \mathbf{1})_{0,0} \oplus ((\mathbf{1}, \mathbf{2})_{-2,1} \oplus c.c.) \\ &\quad \oplus (\mathbf{10}, \mathbf{2})_{1,0} \oplus (\mathbf{10}, \mathbf{1})_{-1,1} \oplus (\mathbf{5}, \mathbf{2})_{0,-1} \oplus (\mathbf{5}, \mathbf{1})_{-2,0} \oplus (\mathbf{5}, \mathbf{1})_{1,1} \oplus c.c. \end{aligned} \tag{A.13}$$

The generators of $\mathfrak{su}(2)$ are the roots

$$E^+ := E_{\frac{1}{2}(1,1,1,1,1,1,\sqrt{2})} \tag{A.14a}$$

$$E^- := E_{-\frac{1}{2}(1,1,1,1,1,1,\sqrt{2})} \tag{A.14b}$$

together with the Cartan generator.

$$P := [E^+, E^-] = \frac{1}{2}(H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + \sqrt{2}H_7) \tag{A.15}$$

so that $\{E^+, E^-, P\}$ generates a $\mathfrak{su}(2)$ subalgebra of $\mathfrak{e}_7$. The generators of the two Abelian factors in $\mathfrak{su}_5^{\text{GUT}} \oplus \mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1$ can be taken as

$$Q_1 = -\frac{1}{2}\left(H_1 + H_2 + H_3 + H_4 + H_5 - H_6 - 2\sqrt{2}H_7\right)$$

$$Q_2 = -\frac{1}{2}\left(2H_6 - \sqrt{2}H_7\right)$$

With this assignment for the roots of the $SU(2)$ subgroup and the generators of the two $U(1)$s, we can also identify how all the other roots of E_7 split into representations of $SU(5) \times SU(2) \times U(1) \times U(1)$. We collect the results in Table 11.

A.3 E_8 ALGEBRA

The Lie algebra of E_8 consists of 248 generators Q_α. We will work in the Cartan-Weyl basis $\{H_i, E_\rho\}$ of $\mathfrak{e}_8$ and where the generators H_i with $i = 1, \ldots, 8$ form a basis of the

E_7 generator	$SU(5)\times SU(2)$	Q_1, Q_2 charges
$(\underline{+1,-1,0,0,0},0,0)\oplus H_1,H_2,H_3,H_4$	$(\mathbf{24},\mathbf{1})$	(0,0)
Q_1, Q_2 cartans	$2(\mathbf{1},\mathbf{1})$	(0,0)
ρ_+,ρ_-,P	$(\mathbf{1},\mathbf{3})$	(0,0)
$(0,0,0,0,0,0,\sqrt{2})$ and $\frac{1}{2}(-1,-1,-1,-1,-1,-1,\sqrt{2})$	$(\mathbf{1},\mathbf{2})$	(-2,1)
$\frac{1}{2}(1,1,1,1,1,1,-\sqrt{2})$ and $(0,0,0,0,0,0,-\sqrt{2})$	$(\mathbf{1},\mathbf{2})$	(2,-1)
$(\underline{1,0,0,0,0},1,0)$ and $\frac{1}{2}(\underline{1,-1,-1,-1,-1},1,-\sqrt{2})$	$(\mathbf{5},\mathbf{2})$	(0,-1)
$\frac{1}{2}(\underline{1,-1,-1,-1,-1},1,\sqrt{2})$	$(\mathbf{5},\mathbf{1})$	(-2,0)
$(\underline{1,0,0,0,0},-1,0)$	$(\mathbf{5},\mathbf{1})$	(1,1)
$(\underline{1,1,0,0,0},0,0)$ and $\frac{1}{2}(\underline{1,1,-1,-1,-1},-1,-\sqrt{2})$	$(\mathbf{10},\mathbf{2})$	(1,0)
$\frac{1}{2}(\underline{-1,-1,-1,1,1},-1,\sqrt{2})$	$(\mathbf{10},\mathbf{1})$	(-1,1)

Table 11: Roots of E_7 and their charges under the subgroup $SU(5)\times SU(2)\times U(1)^2$.

Cartan subalgebra and the remaining 240 roots are chosen to satisfy the following commutation relations

$$[H_i, E_\rho] = \rho_i E_\rho \,. \tag{A.16}$$

This allows to represent the roots with a vector of charges under the Cartan subalgebra and for the case of $\mathfrak{e}_8$ the roots are

$$(\underline{\pm 1,\pm 1,0,0,0,0,0,0})\,,\quad \left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}\right) \text{ with even } + \,. \tag{A.17}$$

For our purposes we need to decompose the E_8 Lie algebra as $E_8 \to SU(5)_{GUT}\times SU(5)_\perp$. In particular the branching rule for the adjoint representation of E_8 is the following

$$\mathbf{248} \to (\mathbf{24},\mathbf{1})\oplus(\mathbf{1},\mathbf{24})\oplus((\mathbf{10},\mathbf{5})\oplus c.c.)\oplus((\overline{\mathbf{5}},\mathbf{10})\oplus c.c.)\,. \tag{A.18}$$

We identify the roots in the adjoint representation of $SU(5)_{GUT}$ as

$$(0,0,0,\underline{1,-1,0,0,0})\,, \tag{A.19}$$

which together with the Cartan generators:

$$\tilde{H}_1 = H_4 - H_5\,,\quad \tilde{H}_2 = H_5 - H_6\,,\quad \tilde{H}_3 = H_6 - H_7\,,\quad \tilde{H}_4 = H_7 - H_8\,. \tag{A.20}$$

give the adjoint representation of $SU(5)_{GUT}$. The adjoint of $SU(5)_\perp$ is consists of the following roots

$$(\underline{\pm 1,\pm 1,0},0,0,0,0,0) + \left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) + \left(\underline{\frac{1}{2},-\frac{1}{2},-\frac{1}{2}},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$$
$$+\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) + \left(\underline{\frac{1}{2},\frac{1}{2},-\frac{1}{2}},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)\,,$$

(A.21)

and Cartan generators

$$\hat{H}_1 = H_2 - H_3\,, \quad \hat{H}_2 = \frac{1}{2}(H_1 - H_2 + H_3 - H_\perp)\,, \quad \hat{H}_3 = \frac{1}{2}(H_1 - H_2 - H_3 + H_\perp)\,, \quad \hat{H}_4 = H_2 + H_3\,, \tag{A.22}$$

where $H_\perp = \sum_{i=4}^{8} H_i$. We will label the roots of the adjoint of $SU(5)_\perp$ as follows

$$\begin{aligned}
E_1^\pm &= \pm(0,-1,1,0,0,0,0,0)\,,\\
E_2^\pm &= \pm\left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)\,,\\
E_3^\pm &= \pm(-1,0,1,0,0,0,0,0)\,,\\
E_4^\pm &= \pm(-1,-1,0,0,0,0,0,0)\,,\\
E_5^\pm &= \pm\left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\,,\\
E_6^\pm &= \pm(-1,1,0,0,0,0,0,0)\,,\\
E_7^\pm &= \pm(-1,0,-1,0,0,0,0,0)\,,\\
E_8^\pm &= \pm\left(-\frac{1}{2},-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)\,,\\
E_9^\pm &= \pm\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)\,,\\
E_{10}^\pm &= \pm(0,-1,-1,0,0,0,0,0)\,.
\end{aligned} \tag{A.23}$$

In Section 3.5 we will need two particular linear combinations of these generators

$$Q_1 = \hat{H}_1 + 2\hat{H}_2 + 2\hat{H}_3 + 2\hat{H}_4\,, \quad Q_2 = \hat{H}_3 + 2\hat{H}_4\,. \tag{A.24}$$

The other representations can also be identified. The roots in the representation $(\mathbf{10},\mathbf{5})$ are the following ones

$$\mu_5 = \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\underline{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\right)\,, \tag{A.25a}$$

$$\mu_5 - \alpha_1 = \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\underline{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\right)\,, \tag{A.25b}$$

$$\mu_5 - \alpha_1 - \alpha_2 = (0,0,0,\underline{1,1,0,0,0})\,, \tag{A.25c}$$

$$\mu_5 - \alpha_1 - \alpha_2 - \alpha_3 = \left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\underline{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\right)\,, \tag{A.25d}$$

$$\mu_5 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 = \left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\underline{\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}}\right)\,, \tag{A.25e}$$

where we identified the five $\mathbf{10}$ representations of $SU(5)_{GUT}$ with their weight under $SU(5)_\perp$. We called the highest weight of $SU(5)_\perp$ in the fundamental representation

μ_5 and the simple roots α_i of $SU(5)_\perp$. We can apply the same procedure to the representation $(\bar{\mathbf{5}}, \mathbf{10})$ and the result is

$$\mu_{10} = (1,0,0,\underline{-1,0,0,0,0})\,, \tag{A.26a}$$

$$\mu_{10} - \alpha_2 = \left(\frac{1}{2},\frac{1}{2},-\frac{1}{2},\underline{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}\right)\,, \tag{A.26b}$$

$$\mu_{10} - \alpha_1 - \alpha_2 = \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},\underline{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}\right)\,, \tag{A.26c}$$

$$\mu_{10} - \alpha_2 - \alpha_3 = (0,1,0,\underline{-1,0,0,0,0})\,, \tag{A.26d}$$

$$\mu_{10} - \alpha_1 - \alpha_2 - \alpha_3 = (0,0,1,\underline{-1,0,0,0,0})\,, \tag{A.26e}$$

$$\mu_{10} - \alpha_2 - \alpha_3 - \alpha_4 = (0,0,-1,\underline{-1,0,0,0,0})\,, \tag{A.26f}$$

$$\mu_{10} - \alpha_1 - 2\alpha_2 - \alpha_3 = \left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},\underline{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}\right)\,, \tag{A.26g}$$

$$\mu_{10} - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 = (0,-1,0,\underline{-1,0,0,0,0})\,, \tag{A.26h}$$

$$\mu_{10} - \alpha_1 - 2\alpha_2 - \alpha_3 - \alpha_4 = \left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\underline{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}}\right)\,, \tag{A.26i}$$

$$\mu_{10} - \alpha_1 - 2\alpha_2 - 2\alpha_3 - \alpha_4 = (-1,0,0,\underline{-1,0,0,0,0})\,, \tag{A.26j}$$

where we called μ_{10} the highest weight of the antisymmetric representation of $SU(5)_\perp$.

WAVEFUNCTIONS IN REAL GAUGE B

When computing the zero mode wavefunctions in real gauge we find that there is a great difference in the computation according to if the sector we are considering is charged or not under the T-brane background. Because of this we shall separate the discussion starting with sectors not affected by the T-brane background.

B.1 SECTORS NOT AFFECTED BY THE T-BRANE BACKGROUND

In these sectors which do not feel the effect of the non-commutativity of the background Higgs field it is possible to solve exactly for the wavefunctions using the techniques already employed in [73, 74]. The F-term and D-term equations may be compactly rewritten as a Dirac-like equation

$$\begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & -D_{\bar{z}} & D_{\bar{y}} \\ -D_y & D_{\bar{z}} & 0 & -D_{\bar{x}} \\ -D_z & -D_{\bar{y}} & D_{\bar{x}} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \\ \vec{\varphi}_U \\ \\ \end{pmatrix} = 0 \tag{B.1}$$

where we defined the covariant derivatives

$$D_x = \partial_x + \frac{1}{2}(q_R\bar{x} - q_S\bar{y}) \qquad D_y = \partial_y - \frac{1}{2}(q_R\bar{y} + q_S\bar{x}) \qquad D_z = 2i(\tilde{\mu}_a^2\bar{x} - \tilde{\mu}_b^2\bar{y}) \tag{B.2}$$

and $D_{\bar{m}}$ are their conjugate. In writing the covariant derivatives we took the following gauge connection

$$A = \frac{i}{2}Q_R(yd\bar{y} - \bar{y}dy - xd\bar{x} + \bar{x}dx) + \frac{i}{2}Q_S(xd\bar{y} - \bar{y}dx + yd\bar{x} - \bar{x}dy) - \frac{i}{2}m^2c^2P(xd\bar{x} - \bar{x}dx)\,, \tag{B.3}$$

which gives the flux[1]

$$F = iQ_R(dy \wedge d\bar{y} - dx \wedge d\bar{x}) + iQ_S(dx \wedge d\bar{y} + dy \wedge d\bar{x}) + im^2c^2P dx \wedge d\bar{x}\,. \tag{B.4}$$

Note that for the E_6 model of Section 3.3 we have that $\tilde{\mu}_a^2 = -\tilde{\mu}_b^2 = -\mu^2$, while for the model A of Section 3.5 the expressions for these two quantities are given in Section 3.5.2. Moreover the explicit form of the generators Q_R and Q_S may be found in Section 3.3 for the E_6 model and in Section 3.5 for the model A, see Table 5 and 7. Note that the explicit form of these quantities will not matter in the following and therefore we shall be able to treat all sectors on the same footing. To solve the system of differential equations we start by noticing that (B.1) may be written as

$$\mathbf{D}_A\Psi = 0\,, \tag{B.5}$$

1 Note that in the sectors we are considering the action of P is trivial. We kept it in the definition of the fluxes for sake of completeness.

which reminds of a Dirac equation. To solve (B.5) it is convenient to take its modulo square for it is possible to decompose the operator $\mathbf{D}_A^\dagger \mathbf{D}_A$ as

$$\mathbf{D}_A^\dagger \mathbf{D}_A = -\Delta \mathbf{1}_4 + \mathbf{M}\,, \tag{B.6}$$

where the Laplacian Δ is defined as $\Delta = \{D_x, D_{\bar{x}}\} + \{D_y, D_{\bar{y}}\} + \{D_z, D_{\bar{z}}\}$ and the matrix $\mathbf{M}$ will depend on the worldvolume fluxes and intersection slopes. Whenever the flux matrix $\mathbf{M}$ and the Laplacian commute (for instance this happens in the case of constant fluxes and abelian Higgs) it is possible to diagonalise simultaneously these two operators. We will start by diagonalising the operator $\mathbf{M}$ and then use its eigenmodes to solve the complete set of equations. For the sectors we are considering the flux matrix has the form

$$\mathbf{M_5} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -q_R^{\mathbf{5}} & q_S^{\mathbf{5}} & -2i\tilde{\mu}_a^2 \\ 0 & q_S^{\mathbf{5}} & q_R^{\mathbf{5}} & 2i\tilde{\mu}_b^2 \\ 0 & \tilde{2}i\mu_a^2 & -2i\tilde{\mu}_b^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{m_5} \end{pmatrix}. \tag{B.7}$$

Diagonalising the matrix we find that the general solution has the form

$$\vec{\varphi} = \begin{pmatrix} -\frac{i\zeta}{2\tilde{\mu}_a} \\ \frac{i(\zeta-\lambda)}{2\tilde{\mu}_b} \\ 1 \end{pmatrix} \chi(x,y)\,. \tag{B.8}$$

Finally we can solve for the scalar wavefunction $\chi(x,y)$ finding the solution

$$\chi(x,y) = e^{\frac{q_R}{2}(x\bar{x}-y\bar{y}) - q_S \mathrm{Re}(x\bar{y}) + (\mu_a x + \mu_b y)(\zeta_1 \bar{x} - \zeta_2 \bar{y})}\, f(\zeta_2 x + \zeta_1 y)\,. \tag{B.9}$$

In writing the solution we have defined

$$\zeta = \frac{\tilde{\mu}_a\,(4\tilde{\mu}_a\tilde{\mu}_b + \lambda q_S)}{\tilde{\mu}_a q_S + \tilde{\mu}_b\,(\lambda + q_R)}\,, \quad \zeta_1 = \frac{\zeta}{\tilde{\mu}_a}\,, \quad \zeta_2 = \frac{\zeta - \lambda}{\tilde{\mu}_b}\,, \tag{B.10}$$

and λ is defined as the lowest solution of the cubic equation

$$-\lambda^3 + 4\lambda\mu_a^2 + 4\lambda\mu_b^2 + \lambda q_R^2 - 4\mu_a^2 q_R + 4\mu_b^2 q_R + \lambda q_S^2 + 8\mu_a\mu_b q_S = 0\,. \tag{B.11}$$

This general solution applies to any sector whose matter curve goes through the origin. The effect of a non-zero separation (which affects only the $\bar{\mathbf{5}}_{-1,-1}$ sector of the model A of Section 3.5) can be easily taken into account by performing a shift in the (x,y) plane

$$x \to x - x_0\,, \quad y \to y - y_0\,. \tag{B.12}$$

However by simply performing the shift in the scalar wavefunction χ we would obtain a solution for a shifted gauge field A. This may be easily remedied by a suitable gauge transformation

$$A(x - x_0, y - y_0) = A(x,y) + d\psi\,, \tag{B.13}$$

with

$$\psi = \frac{i}{2}Q_R(y_0\bar{y}-\bar{y}_0 y-x_0\bar{x}+x_0 x)+\frac{i}{2}Q_S(x_0\bar{y}-\bar{y}_0 x+y_0\bar{x}-\bar{x}_0 y)-\frac{i}{2}m^2c^2P(x_0\bar{x}-\bar{x}_0 x). \tag{B.14}$$

Therefore the general shifted solution may be written as

$$\vec{\varphi} = \begin{pmatrix} -\frac{i\zeta}{2\tilde{\mu}_a} \\ \frac{i(\zeta-\lambda)}{2\tilde{\mu}_b} \\ 1 \end{pmatrix} e^{-i\psi}\chi(x-x_0, y-y_0)\,. \tag{B.15}$$

B.2 SECTORS AFFECTED BY THE T-BRANE BACKGROUND

The presence of the the T-brane background greatly affects the sectors charged under it and in particular as we are now going to show it turns out prohibitive to find a simple solution to the zero modes equations of motion. However in particular region in the space of parameters, more precisely when the diagonal terms in the Higgs background are negligible compared to the off-diagonal ones, great simplifications occur in the zero-mode equations and a solution may be easily obtained.

Since the action of the adjoint Higgs on all matter fields affected by the T–brane background will be identical up to the elements in the diagonal entries (which we are neglecting) we shall be able to discuss all these matter fields at the same time.

The general form of the wavefunctions for the sectors charged under the T-brane is the following one

$$\begin{pmatrix} a_{\bar{x}} \\ a_{\bar{y}} \\ \varphi_{xy} \end{pmatrix} = \vec{\varphi}_{10^+}E_1^+ + \vec{\varphi}_{10^-}E_1^-\,. \tag{B.16}$$

The zero-mode equations take the same form of (B.1) when written in terms of

$$a = \begin{pmatrix} a^+ \\ a^- \end{pmatrix}, \qquad \varphi = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}. \tag{B.17}$$

We will start by looking for a general solution of the F-term equations and eventually impose the D-term equations on this solution. While the first step may be done for a general choice of the parameters entering in the Higgs background the latter turns out to be feasible if we restrict to the particular case in which the diagonal terms in the Higgs background are negligible as opposed to the off-diagonal ones.

For sake of notational simplicity we will consider the case in which the primitive fluxes are vanishing and reinstate them at the end of the computation. Then the general solution to the F-terms is

$$a = e^{fP/2}\bar{\partial}\xi \tag{B.18a}$$

$$\varphi = e^{fP/2}\left(h - i\Psi\xi\right) \tag{B.18b}$$

where ξ and h are both doublets whose components we denote as $\xi^{\pm}$ and $h^{\pm}$ and P and Ψ when acting on doublets may be represented as

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \tilde{\mu}^2 F(x,y) & m \\ m^2 x & \tilde{\mu}^2 f(x,y) \end{pmatrix}. \tag{B.19}$$

The explicit form of $\tilde{\mu}^2 F(x,y)$ is different according to which matter field we are considering but it will be unimportant in the upcoming discussion as we will choose these terms to be negligible.

We may now solve (B.18) for ξ obtaining

$$\xi = i\Psi^{-1}\left(e^{-fP/2}\varphi - h\right), \tag{B.20}$$

and plug this solution in the D-term equations for the zero-modes which therefore become an equation in ξ and h

$$\partial_x\partial_{\bar{x}}\xi + \partial_y\partial_{\bar{y}}\xi + \partial_x f P \partial_{\bar{x}}\xi - i\Lambda^{\dagger}\left(h - i\Psi\xi\right) = 0\,. \tag{B.21}$$

Note that in writing (B.21) we have used that the function f does not depend on $(y,\bar{y})$ and we have defined

$$\Lambda = e^{fP}\Psi e^{-fP} = \begin{pmatrix} \tilde{\mu}^2 F(x,y) & m e^{2f} \\ m^2 x e^{-2f} & \tilde{\mu}^2 F(x,y) \end{pmatrix}. \tag{B.22}$$

While (B.21) depends on both ξ and h it is possible to write it as an equation for one single doublet U defined as

$$U = e^{-fP/2}\varphi\,, \qquad \to \qquad \xi = i\Psi^{-1}(U - h)\,. \tag{B.23}$$

When written in terms of U (B.21) becomes

$$\partial_x\partial_{\bar{x}}U + \partial_y\partial_{\bar{y}}U - (\partial_x\Psi)\Psi^{-1}\partial_{\bar{x}}U + (\partial_y\Psi)\Psi^{-1}\partial_{\bar{y}}U + \partial_x f\Psi P\Psi^{-1}\partial_{\bar{x}}U - \Psi\Lambda^{\dagger}U = 0\,. \tag{B.24}$$

This system is in general a quite involved one to solve, specially for the appearance of the non–primitive fluxes. A great simplification occurs if we take the limit $\tilde{\mu}^2 \ll m^2$ for (B.24) will reduce to a pair of independent differential equations for U^+ and U^-.

$$\partial_x\partial_{\bar{x}}U^+ + \partial_y\partial_{\bar{y}}U^+ + -\partial_x f\partial_{\bar{x}}U^+ - m^2 e^{2f}U^+ = 0\,, \tag{B.25a}$$

$$\partial_x\partial_{\bar{x}}U^- + \partial_y\partial_{\bar{y}}U^- - \frac{1}{x}\partial_{\bar{x}}U^- + \partial_x f\partial_{\bar{x}}U^- - m^4 e^{-2f}x\bar{x}U^- = 0\,. \tag{B.25b}$$

Since the equations for U^+ and U^- are now independent we will solve them separately. We start with U^+ which will not admit a neighbourhood solution and then move to U^-.

Solution for U^+

Using the known asymptotic form of the Painlevé transcendent in a neighbourhood of the origin we find that the equation for U^+ becomes

$$\partial_x \partial_{\bar{x}} U^+ + \partial_y \partial_{\bar{y}} U^+ - m^2 c^2 \bar{x} \partial_{\bar{x}} U^+ - m^2 c^2 (1 + 2m^2 c^2 x\bar{x}) U^+ = 0\,. \tag{B.26}$$

We can take the function U^+ to be a function of $r = \sqrt{x\bar{x}}$ times a holomorphic function of y

$$U^+ = g(y) G(r)\,. \tag{B.27}$$

Using this form we see that the equation for U^+ becomes an equation for $G(r)$

$$G''(r) + \frac{1}{r} G'(r) - c^2 m^2 G'(r) - 4c^2 m^2 \left(2c^2 m^2 r^2 + 1\right) G(r) = 0\,. \tag{B.28}$$

Eq.(B.28) has a regular singular point at $r = 0$ and it can be shown easily that at this point there is an analytic solution and a solution with a logarithmic singularity that diverges and must be discarded. Up to normalisation the analytic solution has a series expansion

$$G(r) = 1 + m^2 c^2 r^2 + \ldots\,. \tag{B.29}$$

This function is not localised at $r = 0$. It is possible to check that addition of fluxes does not improve the situation and therefore no localised solution exists. Therefore in the following we set $U^+ = 0$

Solution for U^-

The equations for U^- once we take into account the asymptotics of the Painlevé transcendent has the following form

$$\partial_x \partial_{\bar{x}} U^- + \partial_y \partial_{\bar{y}} U^- - \frac{1}{x} \partial_{\bar{x}} U^- + (q_x + m^2 c^2 \bar{x}) \partial_{\bar{x}} U^- + q_y \partial_{\bar{y}} U^- - m^4 c^{-2} x\bar{x} U^- = 0\,. \tag{B.30}$$

Note that we reinstated the fluxes at this level with the definitions $q_x = q_R \bar{x} - q_S \bar{y}$ and $q_y = -(q_R \bar{y} + q_S \bar{x})$. The solution to this equation is quite simple

$$U^- = e^{\lambda_{\mathbf{10}} x(\bar{x} - \zeta_{\mathbf{10}} \bar{y})} g_j(y + \zeta_{\mathbf{10}} x)\,, \tag{B.31}$$

where $g_j(y + \zeta_{\mathbf{10}} x)$ are holomorphic family functions, $\lambda_{\mathbf{10}}$ is the lowest root of the polynomial

$$m^4 (\lambda_{\mathbf{10}} - q_R) + \lambda c^2 \left(c^2 m^2 (q_R - \lambda_{\mathbf{10}}) - \lambda_{\mathbf{10}}^2 + q_R^2 + q_S^2\right) = 0 \tag{B.32}$$

and

$$\zeta_{\mathbf{10}} = -\frac{q_S}{(\lambda_{\mathbf{10}} - q_R)}\,. \tag{B.33}$$

Knowing U we can readily obtain the physical wavefunctions

$$\vec{\varphi}^{\,j}_{\mathbf{10}^+} = \gamma^j_{\mathbf{10}} \vec{v}_+ e^{f/2} \chi^j_{\mathbf{10}} \qquad \vec{\varphi}^{\,j}_{\mathbf{10}^-} = \gamma^j_{\mathbf{10}} \vec{v}_- e^{-f/2} \chi^j_{\mathbf{10}} \tag{B.34}$$

with

$$\vec{v}_{\mathbf{10}^+} = \begin{pmatrix} \frac{i\lambda_{\mathbf{10}}}{m^2} \\ -\frac{i\lambda_{\mathbf{10}}\zeta_{\mathbf{10}}}{m^2} \\ 0 \end{pmatrix} \qquad \vec{v}_{\mathbf{10}^-} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{B.35}$$

The scalar wavefunctions $\chi_{\mathbf{10}}$ read

$$\chi^j_{\mathbf{10}} = e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+\lambda_{\mathbf{10}}x(\bar{x}-\zeta_{\mathbf{10}}\bar{y})}\, g_j(y+\zeta_{\mathbf{10}}x) \tag{B.36}$$

where $g_j = m_*^{3-j}(y+\zeta_{\mathbf{10}}x)^{3-j}$ for $i = 1,2,3$.

Note that in neglecting the diagonal terms in the Higgs background we may also discard the effect of the separation of the Yukawa points appearing in the model A. If however we consider the case $\kappa, \mu_2 \ll m$ with $\kappa/\mu_2^2 = \nu$ finite we find that the down Yukawa point is located at $(x_0, y_0) = (0, \nu/2)$. We may follow the same strategy as in the previous section and obtain the solution by simply performing a shift and the result is

$$\vec{\varphi}^i = \gamma^i \begin{pmatrix} \frac{i\lambda}{m^2} \\ -i\frac{\lambda\zeta}{m^2} \\ 0 \end{pmatrix} e^{i\tilde{\psi}+f/2}\chi^i(x, y-\nu/2)E^+ + \gamma^i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{i\tilde{\psi}-f/2}\chi^i(x, y-\nu/2)E^- \tag{B.37}$$

where $\tilde{\psi}$ is

$$\tilde{\psi} = \frac{i}{2}Q_R(\nu\bar{y}/2 - \bar{\nu}y/2) + \frac{i}{2}Q_S(\nu\bar{x}/2 - \bar{\nu}x/2)\,, \tag{B.38}$$

and the definitions of χ, ζ and λ are unchanged.

C ELLIPTIC FIBRATION FOR THE E_6 SINGULARITY

In the main text a local description of the GUT divisor has been used without any reference to its embedding into a three-fold used for the compactification. In this appendix using deformation of ADE singularities we will be able to have a local description of the geometry of the elliptic fibration around the E_6 point and have a further check of the location of the matter curves. We start recalling that the general form of an unfolded E_6 singularity is

$$Y^2 = X^3 + X(\epsilon_2 z^2 + \epsilon_5 z + \epsilon_8) + \left(\frac{z^4}{4} + \epsilon_6 z^2 + \epsilon_9 z + \epsilon_{12}\right) . \tag{C.1}$$

Here $X, Y \in \mathbb{C}^2$ are coordinates in the elliptic fibre[1] and z is a local coordinate in the base manifold. The Casimir invariants of E_6 whose explicit expression can be found in the appendices of [206], will be determined by a particular choice of Higgs background on the GUT divisor. It is convenient to define

$$f = \epsilon_2 z^2 + \epsilon_5 z + \epsilon_8 , \quad g = \frac{z^4}{4} + \epsilon_6 z^2 + \epsilon_9 z + \epsilon_{12} . \tag{C.2}$$

Now inspecting the equation defining the elliptic fibre we can see that it will be singular whenever

$$\Delta = 27g^2 + 4f^3 = 0 . \tag{C.3}$$

1 Here we are describing the elliptic curve in an affine patch so that X and Y are inhomogeneous coordinates. However it is easy to go to the usual Weierstra form of the elliptic fibre taking the projective closure of (C.1) in $\mathbb{P}_{1,2,3}$. If we call the homogenous coordinates of $\mathbb{P}_{1,2,3}$ (u, v, w) then we have $X = vu^{-2}$ and $Y = wu^{-3}$ in the affine patch $\mathbb{P}_{1,2,3} \smallsetminus \mathcal{Z}(u)$.

If we specialise to the Higgs background presented in the main text we find that the Casimir invariants have the following expression

$$\epsilon_2 = \frac{1}{6}\left(-3m^3x - 5\mu^4(x-y)^2\right) \tag{C.4a}$$

$$\epsilon_5 = -\frac{8}{81}\mu^6(x-y)^3\left(15m^3x + \mu^4(x-y)^2\right) \tag{C.4b}$$

$$\epsilon_6 = \frac{1}{1944}\left[81m^9x^3 - 135\mu^4m^6x^2(x-y)^2 - 1125\mu^8m^3x(x-y)^4 + 155\mu^{12}(x-y)^6\right] \tag{C.4c}$$

$$\epsilon_8 = \frac{1}{34992}\big[-729m^{12}x^4 + 4860\mu^4m^9x^3(x-y)^2 - 15390\mu^8m^6x^2(x-y)^4 - 5460\mu^{12}m^3x(x-y)^6 + 335\mu^{16}(x-y)^8\big] \tag{C.4d}$$

$$\epsilon_9 = \frac{2\mu^6(x-y)^3}{19683}\big(1215m^9x^3 - 4941\mu^4m^6x^2(x-y)^2 - 675\mu^8m^3x(x-y)^4 + 305\mu^{12}(x-y)^6\big) \tag{C.4e}$$

$$\epsilon_{12} = \frac{1}{5668704}\big[6561m^{18}x^6 - 65610\mu^4m^{15}x^5(x-y)^2 + 317115\mu^8m^{12}x^4(x-y)^4 - 536220\mu^{12}m^9x^3(x-y)^6 - 289305\mu^{16}m^6x^2(x-y)^8 + 27846\mu^{20}m^3x(x-y)^{10} + 15325\mu^{24}(x-y)^{12}\big] \tag{C.4f}$$

In order to analyse the singularity it is convenient to define a shifted variable $z' = z - \frac{1}{27}\mu^2(x-y)\left(9m^3x + 7\mu^4(x-y)^2\right)$. In terms of z' the discriminant takes the form:

$$\Delta = -\frac{1}{8}z'^5\left[\mu^2(x-y)\left(m^3x - \mu^4(x-y)^2\right)^4\right] + \mathcal{O}\left(z'^6\right) . \tag{C.5}$$

Thus we can conclude that the fibre will be singular at $z' = 0$, and moreover the singularity will enhance at the loci $x = y$ and $m^3x = \mu^4(x-y)^2$. It is quite simple to analyse these singularities using Kodaira classification

	ord(f)	ord(g)	ord(Δ)	Singularity
$z' = 0$	0	0	5	A_4
$z' = 0$ $x = y$	0	0	6	A_5
$z' = 0$ $m^3x = \mu^4(x-y)^2$	2	3	7	D_5
$z' = 0$ $x = y = 0$	∞	4	8	E_6

As a further check of the structure of the fibre over the discriminant locus we can try to resolve the singularity and analyse the intersection pattern of its components. It is first convenient to pass from the Weierstra form to the Tate form of the fibration. This can be achieved using the following change of variables

$$(X,Y) \to \left(X + \frac{1}{12}\left(m^3x - \mu^4(x-y)^2\right)^2 - \frac{2}{3}\mu^2z'(x-y), Y + \frac{1}{2}X\left(m^3x - \mu^4(x-y)^2\right) - \frac{z'^2}{2}\right) .$$

(C.6)

This change of variables gives the following elliptic fibre:

$$Y^2 + a_1XY + a_3Y = X^3 + a_2X^2 + a_4X + a_6\,, \tag{C.7}$$

where:

$$a_1 = m^3x - \mu^4(x-y)^2\,, \quad a_2 = 2\mu^2 z'(y-x)\,, \quad a_3 = -z'^2\,, \quad a_4 = a_6 = 0\,. \tag{C.8}$$

We note that since $a_6 = 0$ our fibration is a case of the so-called $U(1)$-restricted Tate model which was introduced in [207].[2] An explicit resolution of this class of fibrations was given in [209] using toric methods: of particular interest is the Yukawa point where the extended Dynkin diagram of E_6 does not appear in any possible toric resolution.[3,4] The fact that we can not recover the extended Dynkin diagram of E_6 matches a distinctive feature of T-brane backgrounds. In fact if the complex structure of the Calabi-Yau hypersurface is tuned to avoid monodromy like in [214] it is possible to find a resolution of the singularities that lead to the extended Dynkin diagram of E_6.

2 These kind of models admit a global section in addition to the usual section of the elliptic fibration and this introduces additional massless $U(1)$ generators in the spectrum. However this is an artefact of the choice of minimal E_6 singularity in (C.1). If we had added a term z^5 then a_6 would no longer be zero, however this contribution can not be captured in our local approach [208].

3 The resolution of singularities in the context of $SU(5)$ models and the appearance of non-Kodaira fibres has also been studied in [210–212]. For a systematic analysis of the resolution of singularities of Tate models and the appearance of exotic fibres see [213].

4 There are six different resolutions of (C.7) that come from different triangulations of the toric ambient space. The actual number of triangulations of the toric ambient space is larger but some triangulations become equivalent once we restrict to the Calabi-Yau hypersurface.

LINEAR EQUIVALENCE OF p-CYCLES D

In this appendix we will review the definition of linear equivalence as given in [121] for general submanifolds. We will start by reviewing the concept of p-gerbe with connection which will enter directly in the definition of linear equivalence and then give the definition of linear equivalence of submanifolds.

D.1 FROM BUNDLES TO p-GERBES

Roughly speaking a p-gerbe is a generalisation in higher dimension of a line bundle. In order to motivate its definition we start by giving three equivalent characterisations of a line bundle $\mathcal{L}$ on a manifold X:

- A cohomology class in $H^2(X, \mathbb{Z})$,
- A real codimension 2 submanifold M of X,
- An element in the Čech cohomology group $\check{H}^1(X, U(1))$.

The cohomology class characterising $\mathcal{L}$ is its first Chern class $c_1(\mathcal{L})$ while the real codimension 2 submanifold is the Poincaré dual of $c_1(\mathcal{L})$. Finally the element in $\check{H}^1(X, U(1))$ specifies the transition functions of the bundle, namely taking an open cover $\{\mathcal{U}_\alpha\}$ of X such that $\mathcal{L}$ is trivial over each set $\mathcal{U}_\alpha$ given $g \in \check{H}^1(X, U(1))$ we get the functions

$$g_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to U(1) , \tag{D.1}$$

such that $g_{\alpha\beta}(x) g_{\beta\alpha}(x) = 1$ in $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ and furthermore

$$g_{\alpha\beta}(x) g_{\beta\gamma}(x) g_{\gamma\alpha}(x) = 1 , \quad \forall x \in \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma . \tag{D.2}$$

The natural generalisation of the previous characterisations of a line bundle are the following ones:

- A cohomology class in $H^{p+2}(X, \mathbb{Z})$,
- A real codimension $p + 2$ submanifold M of X,
- An element in the Čech cohomology group $\check{H}^{p+1}(X, U(1))$.

We will take one of the three equivalent characterisations as a definition of a p-gerbe. We now will endow p-gerbes with a connection in a way similar to how we endow line bundles with a connection and relate the curvature of this connection to the cohomology class in $H^{p+2}(X, \mathbb{Z})$.

D.1.1 *Connections on p-gerbes*

It is again useful to start recalling how a connection is built on a line bundle. Given a open cover $\{\mathcal{U}_\alpha\}$ of X a connection a connection on a line bundle $\mathcal{L}$ with transitions functions $g \in \check{H}^1(X, U(1))$ is a set of 1-forms A_α defined on $\mathcal{U}_\alpha$ that on double intersections $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ satisfy

$$i(A_\beta - A_\alpha) = g^{-1}_{\alpha\beta} dg_{\alpha\beta} \,. \tag{D.3}$$

In particular since $d(g^{-1}_{\alpha\beta} dg_{\alpha\beta}) = 0$ we have that there is a global closed 2-form (the curvature of the bundle) satisfying

$$F|_{\mathcal{U}_\alpha} = dA_\alpha \,. \tag{D.4}$$

We can now adapt this procedure to the case of a p-gerbe. Let us call $C^q(\mathcal{U}, \mathcal{F})$ the set of q Čech cochains with values in a sheaf $\mathcal{F}$ for the open covering $\mathcal{U}$. Then given the transition functions of the p-gerbe $\mathcal{G}$ we can build the element $\varpi^{(1)} \in C^p(\mathcal{U}, \Omega^1(X))$ satisfying

$$(\delta \varpi^{(1)})_{\alpha_1 \ldots \alpha_p} = (g^{-1} dg)_{\alpha_1 \ldots \alpha_p} \,, \tag{D.5}$$

where $g \in \check{H}^{p+1}(X, U(1))$ are the transition functions of the gerbe $\mathcal{G}$. We can iteratively arrive at the definition of the curvature of the connection

$$(d\varpi^{(q)})_{\alpha_1 \ldots \alpha_{p-q}} = (\delta \varpi^{(q+1)})_{\alpha_1 \ldots \alpha_{p-q}} \,, \tag{D.6}$$

where at each stage we have $\varpi^{(q)} \in C^{p-q+1}(\mathcal{U}, \Omega^q(X))$. This can be repeated until we arrive at $\varpi^{(p+1)}$ which is an element of $C^0(\mathcal{U}, \Omega^{p+1}(X))$, we define $G/2\pi = d\varpi^{(p+1)}$ to be the curvature of the p-gerbe. It is a globally defined and closed $p+2$ form whose class $[G]/2\pi \in H^{p+2}(X, \mathbb{R})$ is the image of the characteristic class of the gerbe $\mathcal{G}$ under the natural inclusion $i : H^{p+2}(X, \mathbb{Z}) \to H^{p+2}(X, \mathbb{R})$.

We can give a direct construction of the connection on a p-gerbe associated to a codimension $p+2$ submanifold and this will be very important in the definition of linear equivalence of submanifolds. Given a codimension $p+2$ submanifold M we will denote its Poincaré dual $p+2$-form as $\delta(M)$. We now would like to find a $p+1$ ϖ form that will be the connection on our p-gerbe, this will be a 0 Čech cochain that is defined by the following differential equations

$$d\varpi^{(p+1)}_\alpha = \delta(M)|_{\mathcal{U}_\alpha} \,. \tag{D.7}$$

This does not specify completely the connection for the addition of a closed form to it does not alter the previous differential equation. We can partially fix this ambiguity asking for the two conditions

$$d^* \varpi^{(p+1)} = 0 \,, \qquad \int_{\Lambda_{p+1}} \varpi^{(p+1)} \in \mathbb{Z} \,, \tag{D.8}$$

where $\Lambda_{p+1} \in H_{p+1}(X \setminus M, \mathbb{Z})$. These conditions will still be satisfied if we add to ϖ an integral harmonic form but this ambiguity is not important in characterising the connection on the p-gerbe. Once these conditions are imposed we see that the connection has the following Hodge decomposition

$$\varpi^{(p+1)} = \omega + d^* H \,, \tag{D.9}$$

where ω the harmonic part of the connection and $dd^* H = \delta(M)$. We next characterise the data specifying the connection. As we already have the notion of curvature

$G/2\pi \in H^{p+2}(X,\mathbb{Z})$ of a p-gerbe we only need to adapt the definition of holonomy which will take values in $H^{p+1}(X,\mathbb{R})/H^{p+1}(X,\mathbb{Z})$. We characterise the holonomy of a p-gerbe as follows: take a non trivial $p+1$ cycle Σ_{p+1} and the define the holonomy of the connection around it as

$$\mathrm{hol}(\varpi^{(p+1)}, \Sigma_{p+1}) = \exp\left(2\pi i \int_{\Sigma_{p+1}} \omega\right) , \tag{D.10}$$

Choosing a basis of non trivial $p+1$ cycles we get a complete characterisation of the holonomy of the connection. Since the holonomy and curvature data completely specify the connection we have that a connection on a gerbe is trivial if both its holonomy and its curvature are zero.

D.2 LINEAR EQUIVALENCE OF SUBMANIFOLDS

We now give the definition of linear equivalence of submanifolds. Appearance of gerbes in the definition of linear equivalence is quite natural for every gerbe we have an associated submanifold. We say that two submanifolds M and N are linearly equivalent if the gerbe $\mathcal{G}_M\mathcal{G}_N^{-1}$ has a trivial connection.[1]. We can easily extend this definition to linear combinations of submanifolds as it is usually done in the case of divisors: two linear combinations of submanifolds $M = \sum_i a_i M_i$ and $N = \sum_i b_i N_i$ are linearly equivalent if the connection on the gerbe $\prod_i \mathcal{G}_{M_i}^{a_i} \prod_j \mathcal{G}_{N_j}^{-b_j}$ has a trivial connection.

We now would like to characterise when two submanifolds of codimension $p+2$ are linearly equivalent, the more general case of a linear combination can be similarly understood. The p-gerbe $\mathcal{G}_M\mathcal{G}_N^{-1}$ has a connection satisfying the following differential equation

$$d\varpi_\alpha = 2\pi\left[\delta(M) - \delta(N)\right]|_{\mathcal{U}_\alpha} . \tag{D.11}$$

A first condition that we need to impose on the connection on the gerbe in order to be trivial is that its curvature is zero, this happens when the two submanifolds are in the same homology class so that the right hand side of (D.11) is an exact form and the connection ϖ is globally well defined. All we need to check is that the bundle has trivial holonomy and this happens if the harmonic part of the connection ϖ is form with integer periods. If we impose the conditions (D.8) we see that linear equivalence amounts to the following equivalent conditions on the connection on the gerbe $\mathcal{G}_M\mathcal{G}_N^{-1}$

$$\omega \in H^{p+1}(X,\mathbb{Z}) \quad \Leftrightarrow \quad d^*H \in H_c^{p+1}(U,\mathbb{Z}) \tag{D.12}$$

where $U \equiv X \setminus (M \cup N)$.[2] We can moreover give following [121] a further characterisation of linear equivalence of submanifolds. We focus again on the case of two

1 Given a gerbe $\mathcal{G}$ we define its dual $\mathcal{G}^{-1}$ to be the gerbe whose transition functions are the inverse of the ones of the gerbe $\mathcal{G}$ Moreover we can introduce a product in the space of p-gerbes defining the product of two p-gerbes $\mathcal{M}$ and $\mathcal{N}$ with transitions functions $g_\mathcal{M}$ and $g_\mathcal{N}$ to be the p-gerbe with transition functions $g_\mathcal{M} g_\mathcal{N}$. We shall call the product gerbe simply $\mathcal{MN}$. Note that the notion of dual and product agree with the known ones in the case of line bundles.

2 We need to remove M and N from X because the form d^*H has poles on these submanifolds.

submanifolds but this can be easily generalised to a linear combination of submanifolds. We will show the following: two submanifolds M and N of codimension $p+2$ are linearly equivalent if and only if their homology classes agree and moreover

$$\int_\Gamma \theta \in \mathbb{Z}\,, \tag{D.13}$$

where $\partial\Gamma = M - N$ and θ is an harmonic form with integral cohomology class. This is quite easy to show: let us start by integrating a general form with compact support on Γ

$$\int_\Gamma \alpha = \int_U \alpha \wedge \gamma \tag{D.14}$$

where we called γ the Poincaré dual of Γ, we write its Hodge decomposition as $\gamma = A + dB + d^*C$. We can see that the coclosed part of γ agrees with the coexact part of our connection on the gerbe $\mathcal{G}_M\mathcal{G}_N^{-1}$, in fact if we integrate an exact form on Γ we get

$$\int_\Gamma d\beta = \int_{\partial\Gamma} \beta = \int_M \beta - \int_N \beta\,, \tag{D.15}$$

but we also have that

$$\int_\Gamma d\beta = \int_U d\beta \wedge \gamma = \int_U \beta \wedge d\gamma = \int_U \beta \wedge dd^*C\,, \tag{D.16}$$

so that in the end we get that $dd^*C = \delta(M) - \delta(N) = dd^*H$. Moreover we have that Γ is an integral cycle and if we integrate a closed form with compact support and integral cohomology class on it we get

$$\int_\Gamma \phi = \int_U \phi \wedge (A + dB + d^*C) = \int_U \phi \wedge (A + d^*H) \in \mathbb{Z}\,, \tag{D.17}$$

that implies that d^*H is an integral form if and only if $\int_U \phi \wedge A$ is integral. Since the compactly supported closed form with integral class ϕ is cohomologous to an integral harmonic form θ defined on the whole manifold X we have that

$$\int_U \phi \wedge A = \int_X \theta \wedge A = \int_\Gamma \theta\,, \tag{D.18}$$

which implies that

$$\int_U \phi \wedge A \in \mathbb{Z} \iff \int_\Gamma \theta \in \mathbb{Z}\,. \tag{D.19}$$

GENERALISED HOMOLOGY

E

In this appendix we give the basic definitions of the generalised homology introduced in [134]. We refer the reader to [134] for a detailed discussion on the subject.

The RR charges of D-branes in the presence of a NSNS H-flux are classes in twisted K-theory but it is generally very difficult to compute them in concrete examples. On the other hand, generalised homology captures some aspects of the RR charges beyond usual homology and is much easier to work with.

In the context of generalised complex geometry a Dp-brane can be described as a submanifold Σ carrying a gauge bundle $\mathcal{F} = F + B$ which is a generalised submanifold $(\Sigma, \mathcal{F})$.[1] In other to define a homology theory for these objects we need to define chains and a boundary operator that squares to zero.

Consider a Dp-brane wrapping a $(p+1)$-submanifold with a $U(1)$ gauge field that may have a Dirac monopole on a $(p-2)$-submanifold $\Pi \subset \Sigma$ which must satisfy $\partial\Pi \subset \partial\Sigma$. The field strength $\mathcal{F}_\Pi$ thus satisfies

$$d\mathcal{F}_\Pi = H|_\Sigma + \delta_\Sigma(\Pi) \tag{E.1}$$

in the presence of a H-flux. Since $\mathcal{F}_\Pi$ should be globally well-defined we have that

$$\text{P.D.}[H|_\Sigma] + [\Pi] = 0. \tag{E.2}$$

Thus, the pair $(\Sigma, \mathcal{F}_\Pi)$ is a generalised submanifold and a generalised chain is defined to be a formal sum of these pairs. We restrict the sum to contain only even or odd dimensional submanifolds as suggested by Type IIB and IIA string theories respectively.

We define the generalised boundary operator in such a way that that its action is dual to the action of the H-twisted exterior derivative on forms. Therefore, inspired in the CS action for a D-brane, we associate a current $j_{(\Sigma,\mathcal{F}_\Pi)}$ to each chain by

$$j_{(\Sigma,\mathcal{F}_\Pi)}(C) \equiv \int_\Sigma C|_\Sigma \wedge e^{\mathcal{F}_\Pi} \tag{E.3}$$

where C is an arbitrary polyform of definite parity. The derivative $d_H = d + H\wedge$ acts on the current in the following way

$$(d_H j_{(\Sigma,\mathcal{F}_\Pi)})(C) = \int_\Sigma d_H C|_\Sigma \wedge e^{\mathcal{F}_\Pi} = \int_{\partial\Sigma} C|_\Sigma \wedge e^{\mathcal{F}_\Pi|_{\partial\Sigma}} - \int_\Pi C|_\Pi \wedge e^{\mathcal{F}_\Pi|_\Pi} \tag{E.4}$$

where we used Stokes' theorem. Thus,

$$d_H j_{(\Sigma,\mathcal{F}_\Pi)} = j_{(\partial\Sigma,\mathcal{F}_\Pi|_{\partial\Sigma})} - j_{(\Pi,\mathcal{F}_\Pi|_\Pi)} \tag{E.5}$$

so we define the generalised boundary operator $\hat{\partial}$ by imposing

$$d_H j_{(\Sigma,\mathcal{F}_\Pi)} = j_{\hat{\partial}(\Sigma,\mathcal{F}_\Pi)} \tag{E.6}$$

1 We restrict ourselves to abelian D-branes.

which leads to

$$\hat{\partial}(\Sigma, \mathcal{F}_\Pi) \equiv (\partial\Sigma, \mathcal{F}_\Pi|_{\partial\Sigma}) - (\Pi, \mathcal{F}_\Pi|_\Pi). \tag{E.7}$$

One can check that $\hat{\partial}^2 = 0$ which allows to define the generalised homology as $\mathrm{Ker}(\hat{\partial})/\mathrm{Im}(\hat{\partial})$.

In order to preserve RR gauge invariance D-branes can only wrap generalised chains that are closed [134]. Then for any two generalised cycles that are in the same homology class there is a physical process that connects them although it may not be energetically favorable. In order to define the energy of a generalised cycle one can introduce a generalised calibration [134]. Of particular relevance for our discussion is the effect of dissolving D(p-2)-branes in Dp-branes which is nicely captured in this formalism since both situations correspond to different representatives of the same homology class as is shown in the main text.

E.1 DETAILS ON THE COMPUTATION OF $j_{(\mathfrak{S},\mathfrak{F})}$

Here we derive the equation (4.94) starting with the generalised chain $(\mathfrak{S}, \mathfrak{F})$ and show it has all the desired properties.

First, one can check that a change in S'_a, $\mathcal{F}'_a$ or B_a corresponds to choosing a different generalised chain $(\mathfrak{S}', \mathfrak{F}') = (\mathfrak{S}, \mathfrak{F}) + \hat{\partial}(\mathfrak{s}, \mathfrak{f})$ so the associated current is $j_{(\mathfrak{S}',\mathfrak{F}')} = j_{(\mathfrak{S},\mathfrak{F})} + d\, j_{(\mathfrak{s},\mathfrak{f})}$. Since γ_I is harmonic we have that $d\, j_{(\mathfrak{s},\mathfrak{f})}(\gamma_I) = 0$ which shows that our result is independent of all these choices.

Now we take the limit where B_a and B_b go to zero that yields a simpler and more transparent expression which is manifestly independent of S'_a, $\mathcal{F}'_a$ or B_a. Let us focus on the contribution due to $\hat{\partial}[-(B_a, \tilde{\mathcal{F}}_a) + (B_a, \tilde{\mathcal{F}}_{\Pi_a})]$, namely

$$j_{(\mathfrak{S},\mathfrak{F})}(\gamma_I) \supset \int_B \gamma_I \wedge (\tilde{\mathcal{F}}_\Pi - \tilde{\mathcal{F}}). \tag{E.8}$$

where we dropped the subscript a to simplify the notation. Let us define $\mathcal{H} = \tilde{\mathcal{F}}_\Pi - \tilde{\mathcal{F}}$ which satisfies

$$d\mathcal{H} = \delta^3_B(\Pi), \qquad \mathcal{H}|_S = -\mathcal{F}, \qquad \mathcal{H}|_{S'} = 0. \tag{E.9}$$

We have that $B = S \times I_L$ with I_L an interval of length L with coordinate $t \in [0, L]$ and a slicing of B in S_t with $S_0 = S$ and $S_L = S'$. Since the integral above does not depend on B it can not depend on L so we may take the limit $L \to 0$ to make it manifestly independent of B. We define

$$\gamma_I = \gamma_I|_{S_t} + \tilde{\gamma}_I \tag{E.10}$$

$$\mathcal{H} = \mathcal{H}|_{S_t} + \tilde{\mathcal{H}}. \tag{E.11}$$

The equation for $\mathcal{H}$ translates into

$$d_S(\mathcal{H}|_{S_t}) + d_{I_L}(\mathcal{H}|_{S_t}) + d_S\tilde{\mathcal{H}} = \delta^2_{S_0}(\Pi) \wedge \delta(t)dt, \qquad \mathcal{H}|_{S_0} = -\mathcal{F}, \qquad \mathcal{H}|_{S_L} = 0 \tag{E.12}$$

where we used the fact that $d = d_{I_L} + d_S$ with d_{I_L} and d_S the exterior derivatives on I_L and S_t respectively. From the boundary conditions for $\mathcal{H}$ we find that

$$\lim_{L\to 0} d_S(\mathcal{H}|_{S_t}) = 0, \qquad \lim_{L\to 0} d_{I_L}(\mathcal{H}|_{S_t}) = \mathcal{F} \wedge \delta(t)dt. \tag{E.13}$$

Thus, the differential equation for $L \to 0$ is

$$d_S \tilde{\mathcal{H}} = (\delta^2_{S_0}(\Pi) - \mathcal{F}) \wedge \delta(t) dt \tag{E.14}$$

so we necessarily have that

$$\lim_{L \to 0} \tilde{\mathcal{H}} = -\frac{1}{2\pi l_s} A_\Pi \wedge \delta(t) dt \tag{E.15}$$

with A_Π a 1-form in S that satisfies

$$\frac{1}{2\pi l_s} d_S A_\Pi = \mathcal{F} - \delta^2_S(\Pi). \tag{E.16}$$

Going back to the integral (E.8) we find

$$\int_B \gamma_I \wedge (\tilde{\mathcal{F}}_\Pi - \tilde{\mathcal{F}}) = \int_{S \times I_L} (\gamma_I|_{S_t} \wedge \tilde{\mathcal{H}} + \tilde{\gamma}_I \wedge \mathcal{H}|_{S_t}) \tag{E.17}$$

where only the first term contributes in the limit $L \to 0$ since it contains a delta function unlike the second one. Therefore,

$$\int_B \gamma_I \wedge (\tilde{\mathcal{F}}_\Pi - \tilde{\mathcal{F}}) = -\frac{1}{2\pi l_s} \int_{S \times I_L} \gamma_I|_S \wedge \tilde{A}_\Pi \wedge \delta(t) dt = \frac{1}{2\pi l_s} \int_S \gamma_I \wedge \tilde{A}_\Pi. \tag{E.18}$$

Notice there is minus sign due to the orientation of S in $\partial B = S' - S$.

Using this result we may write

$$j_{(\mathfrak{S},\mathfrak{F})}(\gamma_I) = \int_\Sigma \gamma_I + \frac{1}{2\pi l_s} \left(\int_{S_a} \gamma_I \wedge A_{\Pi_a} - \int_{S_b} \gamma_I \wedge A_{\Pi_b} \right). \tag{E.19}$$

The only thing left is to show that (E.19) is independent of the choice of Π_a and Π_b. Consider $\Pi'_a = \Pi_a + \partial\sigma_a$ and $\Pi'_b = \Pi_b + \partial\sigma_b$ which also changes Σ into $\Sigma' = \Sigma + \sigma_a - \sigma_b$. One can readily show that the formula above is independent of σ_a and σ_b. Finally, a change $\Sigma' = \Sigma + \pi$ with π a closed 3-cycle in $\mathcal{M}_6$ does change the expression above but just by integer number which can be interpreted as a redefinition of the $U(1)$ sector.

As a further check of this result let us derive eq.(4.83) starting from $j_{(\mathfrak{S},\mathfrak{F})}(\gamma_I)$ when $[F_a] = [F_b] \in H^2(S)$. More explicitly, we show that

$$\int_\Gamma \gamma \wedge \tilde{\mathcal{F}} = \int_\Sigma \gamma + \int_{B_a} \gamma \wedge H_a - \int_{B_b} \gamma \wedge H_b. \tag{E.20}$$

Since nothing depends on B_a and B_b we choose $B_a = -B_b = -\Gamma$. Also, since $[F_a] = [F_b] \in H^2(S)$ we have that we may take $\Sigma \subset \Gamma$ so

$$\int_\Sigma \gamma + \int_{B_a} \gamma \wedge H_a - \int_{B_b} \gamma \wedge H_b = \int_\Gamma \gamma \wedge (\delta^2_\Gamma(\Sigma) - H_a + H_b). \tag{E.21}$$

The quantity $\mathcal{Q} \equiv \delta^2_\Gamma(\Sigma) + H_a - H_b$ satisfies the equation

$$d\mathcal{Q} = \delta^3_{S_b}(\Pi_b) - \delta^3_{S_a}(\Pi_a) + d\delta^2_\Gamma(\Sigma) = 0 \tag{E.22}$$

where we used $d\delta^2_\Gamma(\Sigma) = \delta^3_\Gamma(\partial\Sigma) = \delta^3_{S_a}(\Pi_a) - \delta^3_{S_b}(\Pi_b)$. The boundary conditions are $\mathcal{Q}|_{S_a} = \mathcal{F}_a$ and $\mathcal{Q}|_{S_b} = \mathcal{F}_b$ so $\mathcal{Q} = \tilde{\mathcal{F}}$.

F GENERALISED DOLBEAULT OPERATOR AND $\mathcal{N} = 1$ SUPERSYMMETRY

In this appendix we write down the $\mathcal{N} = 1$ supersymmetry equations for a 4d Minkowski compactification of Type II string theories in terms of a generalised Dolbeault operator following [144]. We also rederive the results of section 4.2.1 in a way that can be applied to more general situations. We refer the reader to [144, 145, 215] more more details. We follow the conventions in [145].

We consider the following ansatz for the metric in the 10 space $\mathbb{R}^{1,3} \times X_6$,

$$ds^2 = e^{2A} ds^2_{\mathbb{R}^{1,3}} + ds^2_{X_6} \tag{F.1}$$

where the warp factor depends generically on the coordinates y^m on X_6. We take as independent degrees of freedom the RR field strengths with legs only on X_6, namely, F_0, F_2 and F_4, that we arrange into a polyform F. In the presence of D-branes and O-planes, this polyform satisfies the Bianchi identity

$$dF = -j \tag{F.2}$$

with j the corresponding current. More explicitly, for a single D-brane on Σ_α with gauge flux F_α we have $j_\alpha = \delta(\Sigma_\alpha) \wedge e^{-F_\alpha}$. Also, the H-field should satisfy its own Bianchi identity $dH = 0$.

This kind of compactifications are completely determined by two complex polyforms $\mathcal{Z}$ and T of opposite parity that define an $SU(3)\times SU(3)$-structure. Generically, only one of them is integrable, which we take to be $\mathcal{Z}$, and it defines an integrable generalised complex structure $\mathcal{J}$ on X_6. The explicit expressions of the polyforms $\mathcal{Z}$ and T for the cases of type IIA with O6-planes and type IIB with O3/O7-planes are

$$\begin{aligned} \text{IIA}: \qquad & \mathcal{Z} = e^{3A-\phi} e^{iJ+B}\,, \qquad T = e^{-\phi}\,\Omega \wedge e^B\,, \\ \text{IIB}: \qquad & \mathcal{Z} = e^{-\phi}\,\Omega \wedge e^B\,, \qquad T = e^{3A-\phi} e^{iJ+B}\,. \end{aligned} \tag{F.3}$$

Given these definitions, the supersymmetry conditions can be written as [144, 145]

$$d\mathcal{Z} = 0, \qquad d(e^{2A}\text{Im }T) = 0, \qquad d\,\text{Re }T = -\mathcal{J}\cdot F. \tag{F.4}$$

Notice that none of these conditions involves the metric or B-field explicitly, which are encoded in the polyforms $\mathcal{Z}$ and T that characterise the internal manifold.

In order to preserve supersymmetry, the (backreacted) sources must be calibrated [216, 217]. This means that they should wrap generalised complex submanifolds, i.e. $\mathcal{J}\cdot j = 0$ and also that $\langle \text{Im }T, j\rangle = 0$. It has been shown [218] that the supersymmetry equations F.4 together with the calibration condition for the sources and the Bianchi identities imply that the whole set of equations of motion are satisfied. However, as stressed in the main text, one should also impose that the field strengths are quantised to have a consistent supersymmetric solution.

In the following we shall give a different derivation of the results of section 4.2.1. The key point for this derivation is that, using the equations above, we can write the RR field strength as

$$F = -d^{\mathcal{J}}\,\mathrm{Re}\;T \tag{F.5}$$

where we defined the operator $d^{\mathcal{J}} = [d, \mathcal{J}]$ that satisfies $(d^{\mathcal{J}})^2 = 0$. This is the generalisation of $d^c = -i(\partial - \bar{\partial})$ in complex geometry, where ∂ is the Dolbeault differential. Thus, introducing $d^{\mathcal{J}}$ is equivalent to defining a generalised Dolbeault operator.

F.1 SYMPLECTIC COHOMOLOGIES

Here we arrive at the main result in section 4.2.1 in terms of eq.(F.5) and of certain cohomology groups that can be defined in a symplectic manifold [215] . This is useful since it shows that the reasoning is more general and only depends on the integrable generalised complex structure on X_6, which behaves nicely when deforming the sources. For Type IIA with D6-branes and O6-planes it corresponds to a symplectic structure so we will focus on this case.

We give the relevant definitions and results without proof since they can be found in [215] (see also [219]). Let (X_6, J) be a compact symplectic manifold, then the operator $d^{\mathcal{J}}$ is given by $\delta = [d, J^{-1}\llcorner]$ where $\llcorner$ denotes index contraction. One can then define the following cohomology groups

$$H^k_{d+\delta}(X_6) = \frac{\ker(d+\delta)\cap\Omega^k(X_6)}{\mathrm{im}\,d\delta\cap\Omega^k(X_6)}, \qquad H^k_{d\delta}(X_6) = \frac{\ker(d\delta)\cap\Omega^k(X_6)}{(\mathrm{im}\,d+\mathrm{im}\,\delta)\cap\Omega^k(X_6)} \tag{F.6}$$

where $\Omega^k(X_6)$ is the space of k-forms on X_6. The elements in the first group are forms α_k such that $d\alpha_k = \delta\alpha_k = 0$ and $\alpha_k \neq d\delta\beta_k$ for all β_k. Regarding the second group, it consists of forms α_k such that $d\delta\alpha_k = 0$ and $\alpha_k \neq d\beta_{k-1}$, $\alpha_k \neq \delta\beta_{k+1}$ for every β_{k-1} and β_{k+1}. One can prove that there exists a non-degenerate pairing between $H^k_{d+\delta}(X_6)$ and $H^{6-k}_{d\delta}(X_6)$ given by

$$(\alpha_k, \beta_{6-k}) = \int_{X_6} \alpha_k \wedge \beta_{6-k}. \tag{F.7}$$

Furthermore, there is an isomorphism between $H^k_{d+\delta}(X_6)$ and $H^{6-k}_{d+\delta}(X_6)$ given by $(J\wedge)^{3-k}$. Finally, if X_6 satisfies the $d\delta$-lemma (which we assume), there is yet another isomorphism between $H^k_{d+\delta}(X_6)$ and the usual de Rham cohomology $H^k_{dR}(X_6)$.

Now we have all the necessary ingredients to arrive at the result in the main text. Equation (F.5) says that F_2 (which is the only non-vanishing RR field strength) is the trivial class in $H^2_{d\delta}(X_6)$ since $d\delta F_2 = 0$ and $F_2 = -\delta(\mathrm{Re}\;T)$. Thus, given the pairing with $H^4_{d+\delta}(X_6)$, we may write

$$\int_{X_6} F_2 \wedge Q_4 = 0 \qquad \forall\, Q_4 \in H^4_{d+\delta}(X_6). \tag{F.8}$$

Using the isomorphisms quoted above we can rewrite this condition in terms of the de Rham cohomology, namely

$$\int_{X_6} F_2 \wedge J \wedge \omega_2 = 0 \qquad \forall\, \omega_2 \in H^2_{dR}(X_6). \tag{F.9}$$

Deforming the location of the D6-branes produces a ΔF_2 that must satisfy the equation above to preserve supersymmetry, which is precisely the same as eq.(4.154) in the main text. The rest of the argument involving the quantisation condition of F_2 is the same so we do not repeat it here.

G THE REFINED TOPOLOGICAL VERTEX

In this appendix we give a brief overview of the rules for the topological vertex and its refinement that have been used in Chapter 5.

G.1 REFINED TOPOLOGICAL VERTEX

The topological string partition function can be defined as an exponential of a generating function of the Gromov–Witten invariants of a Calabi–Yau threefold X,

$$Z_{top} = \exp\left(\sum_{g=0}^{\infty} \mathcal{F}_g g_s^{2g-2}\right), \tag{G.1}$$

where g_s is the topological string coupling and $\mathcal{F}_g$ is the genus g topological string amplitude

$$\mathcal{F}_g = \sum_{C \in H_2(X,\mathbb{Z})} N_C^g \, Q_C. \tag{G.2}$$

Q_C is defined by $Q_C := e^{-\int_C J}$ where J is the Kähler form of X , and N_C^g is the genus g Gromov–Witten invariant. Although it is difficult to compute the higher genus topological string amplitudes directly, the M-theory interpretation of the topological string amplitude enables us to write down the all genus answer [220, 221], Moreover, if the Calabi–Yau threefold is toric, the refined topological vertex [169, 170] can compute the refined version of the all genus topological string partition function up to constant map contributions in a diagrammatic way. We call such a partition function the refined topological string partition function.

In order to apply the refined topological vertex to a toric Calabi–Yau threefold, we first choose the preferred direction in the toric diagram. In this paper, we choose the horizontal direction as the preferred direction unless stated. For other lines, we assign two parameters t and q. At each trivalent vertex, one leg is in the preferred direction, t is assigned to another leg, and q is assigned to the other leg. For each internal line we assign a Young diagram with an orientation and for each external leg we assign a trivial diagram.

We then assign the refined topological vertex for each trivalent vertex shown in Figure 33

$$C_{\lambda\mu\nu}(t,q) := t^{-\frac{||\mu^t||^2}{2}} q^{\frac{||\mu||^2+||\nu||^2}{2}} \tilde{Z}_\nu(t,q) \sum_\eta \left(\frac{q}{t}\right)^{\frac{|\eta|+|\lambda|-|\mu|}{2}} s_{\lambda^t/\eta}\left(t^{-\rho}q^{-\nu}\right) s_{\mu/\eta}\left(t^{-\nu^t}q^{-\rho}\right), \tag{G.3}$$

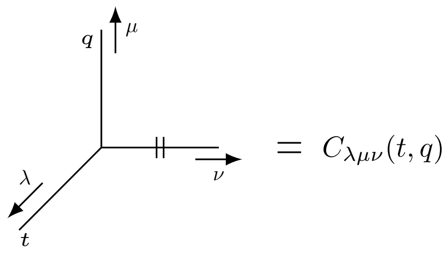

Figure 33: A trivalent vertex with a particular assignment of Young diagrams, the preferred direction and t, q. $||$ represents an line in the preferred direction.

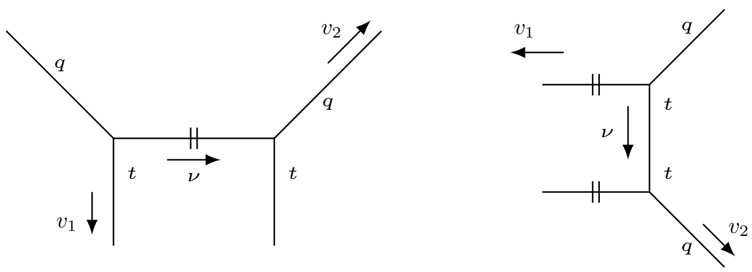

Figure 34: Two examples of a propagator connecting two vertices. The left figure shows a propagator in the preferred direction. The right figure shows a propagator in the non-preferred direction.

where $|\lambda| := \sum_i \lambda_i$ and $||\lambda||^2 := \sum_i \lambda_i^2$. λ_i stands for the height of i-th column[1]. $s_{\lambda/\eta}(x)$ is the skew Schur function defined as (G.12), and $\rho := -i + \frac{1}{2}, (i = 1, 2, \cdots)$. $\tilde{Z}_\nu(t,q)$ is defined as

$$\tilde{Z}_\nu(t,q) := \prod_{(i,j)\in\nu} \left(1 - q^{l_\nu(i,j)} t^{a_\nu(i,j)+1}\right)^{-1}, \tag{G.4}$$

where $l_\nu(i,j) := \nu_i - j$ is the leg length and $a_\nu(i,j) := \nu_j^t - i$ is the arm length of the Young diagram ν.

For each propagator with a Young diagram ν in Figure 34, which connects two vertices, we assign the Kähler parameter $Q = e^{-\int_C J}$ associated to the size of the two-cycle C corresponding to the internal line and also the framing factor

$$(-Q)^{|\nu|} f_\nu(t,q)^n \quad \text{or} \quad (-Q)^{|\nu|} \tilde{f}_\nu(t,q)^n \tag{G.5}$$

$f_\nu(t,q)$ is the framing factor for the preferred direction

$$f_\nu(t,q) := (-1)^{|\nu|} t^{\frac{||\nu^t||^2}{2}} q^{-\frac{||\nu||^2}{2}} \tag{G.6}$$

$\tilde{f}_\nu(t,q)$ is the framing factor for the non-preferred direction

$$\tilde{f}_\nu(t,q) := (-1)^{|\nu|} q^{\frac{||\nu^t||^2}{2}} t^{-\frac{||\nu||^2}{2}} \left(\frac{q}{t}\right)^{\frac{|\nu|}{2}}. \tag{G.7}$$

1 Our convention of writing a Young diagram is that the height of a column is equal to or higher than the height of its right column.

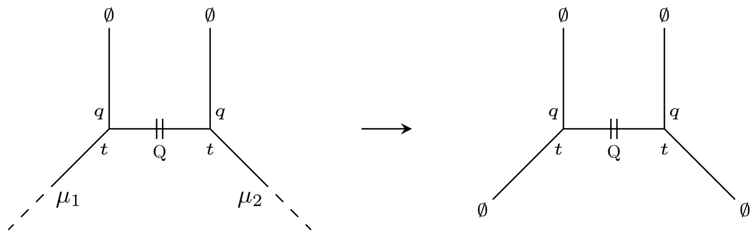

Figure 35: A diagram with parallel external legs. The right-hand side stands for a diagram whose refined topological string partition function reproduce the decoupled factor of the theory from the diagram in the left.

n is defined as $n = \det(v_1, v_2)$ where v_1, v_2 are vectors depicted in Figure 34.

The refined topological string partition function up to the constant map contributions can be computed by combining the vertices, propagators and then sum up all the Young diagrams.

In fact, the toric diagrams in the M-theory compactification are dual to webs of (p, q) 5-branes [171]. In the dual picture, the five-dimensional theory is effectively realised on the worldvolume theory on the 5-branes compactified on segments.

The unrefined topological string partition function can be obtained is simply by setting $t = q$. Therefore, the rule of the topological vertex is given by setting $t = q$ in (G.3) and (G.5), which is essentially the same rule obtained in [167, 168].

G.2 DECOUPLED FACTOR

The refined topological string partition itself does not reproduce the Nekrasov partition function of the five-dimensional theory realised by the M-theory compactification on the toric Calabi–Yau threefold X, but we need to eliminate some factors which come from BPS states with no gauge charge [179–182]. In the dual brane picture, those BPS states come from strings between parallel external legs. We call such a factor as decoupled factor. The decoupled factor can be obtained by the refined topological string partition function of a strip diagram with parallel external legs.

Let us consider a simple example of Figure 35. Left figure in Figure 35 represents a diagram with parallel external vertical legs. The refined topological string partition function from the diagram contains the decoupled factor associated to the parallel external legs. The contribution can be extracted by computing the refined topological string partition function on the right figure in Figure 35, which is the toric diagram of a local Calabi–Yau threefold $\mathcal{O}(-2) \oplus \mathcal{O}$ fibered over $\mathbb{P}^1$. The factor can be explicitly written as

$$Z_{dec,||} = \prod_{i,j=1}^{\infty} \left(1 - Qq^i t^{j-1}\right)^{-1}. \tag{G.8}$$

Eq. (G.8) may be seen as a part of the perturbative partition function of a vector multiplet. Namely, it does not contain the full spin content of a vector multiplet. From the perspective of 5d BPS states labeled by (j_l, j_r), which are quantum numbers under $SU(2)_l \times SU(2)_r \subset SO(4)$, a vector multiplet may be written as $\left[2(0,0) + \left(\frac{1}{2}, 0\right)\right] \otimes$

$\left(0, \frac{1}{2}\right)$. Then the contribution (G.8) comes from components of the tensor product between a half–hypermultiplet, *i.e.* $\left[2(0,0) + \left(\frac{1}{2}, 0\right)\right]$, and a lower component of $\left(0, \frac{1}{2}\right)$. The non–full spin content appears since the moduli space of the $\mathbb{P}^1$ inside $\mathcal{O}(-2) \oplus \mathcal{O}$ fibered over $\mathbb{P}^1$ is non–compact. Therefore, as for removing the decoupled factor, we may also say that we remove the contribution of components which do not fill the full spin content of the vector multiplet or recover the invariance under the exchange between q and t [179, 180, 182].

After removing the decoupled factor, we can obtain the 5d Nekrasov partition function of the theory realised by a web diagram,

$$Z_{Nek} = \frac{Z_{top}}{Z_{dec}}, \tag{G.9}$$

with appropriate parametrisation.

G.3 SCHUR FUNCTIONS

In this appendix we will recall some facts about Schur functions that will be used in Section 5.2. Recall that Schur functions $s_\mu(x)$ are certain symmetric polynomials in n variables x_i indexed by a Young diagram μ. Among the different ways of expressing these functions we will choose to use the first Jacobi–Trudi formula to relate them to complete homogeneous symmetric polynomials

$$s_\mu(x) = \det\left[h_{\mu_i + j - i}\right], \quad 1 \le i, j \le n\,, \tag{G.10}$$

where the complete homogeneous symmetric polynomials of degree k in n variables are defined as

$$h_k(x) = \sum_{1 \le i_1 \le i_2 \le \cdots \le i_k \le n} \prod_{j=1}^{k} x_{i_j}\,. \tag{G.11}$$

It is also possible to define a generalisation of Schur functions (called skew Schur functions) in terms of two partitions λ and μ such that λ interlaces μ as

$$s_{\mu/\lambda}(x) = \det\left[h_{\mu_i - \lambda_j + j - i}\right], \quad 1 \le i, j \le n\,. \tag{G.12}$$

Since Schur functions form an orthonormal basis of symmetric polynomials it is possible to express skew Schur functions in terms of Schur functions using the Littlewood-Richardson coefficients $c^\lambda_{\mu\nu}$

$$s_{\mu/\lambda}(x) = \sum_\nu c^\lambda_{\mu\nu}\, s_\nu(x)\,. \tag{G.13}$$

In the computations of topological string partition functions it is often necessary to use the following identities of Schur functions

$$\sum_\mu s_\mu(x) s_\mu(y) = \prod_{i,j=1}^{\infty} (1 - x_i y_j)^{-1}\,, \tag{G.14}$$

$$\sum_\mu s_{\mu^t}(x) s_\mu(y) = \prod_{i,j=1}^{\infty} (1 + x_i y_j)\,. \tag{G.15}$$

Moreover in the main text we will also use two additional identities: the first one is the following one[2]

$$\sum_{\nu}(-1)^{|\nu|}s_{\mu/\nu}(x)s_{\nu^t/\lambda^t}(x) = (-1)^{|\mu|}\delta_{\mu\lambda}\,, \tag{G.16}$$

and the second one which readily follows from (G.16) is the following one

$$\sum_{\eta,\kappa}(-1)^{|\eta|+|\kappa|}s_{\lambda/\eta}(x)s_{\eta^t/\kappa}(x)s_{\kappa^t/\xi}(x) = s_{\lambda/\xi}(x)\,. \tag{G.17}$$

2 See for instance [222] for a proof of this identity.

H $Sp(N)$ NEKRASOV PARTITION FUNCTIONS

In section 5.2.2, we computed the partition functions of rank 2 E_6, E_7, E_8 theories by directly applying the new rules of the topological vertex to the web diagrams. The mass deformation of the rank 2 E_6, E_7, E_8 theories triggers an RG flow to the $Sp(2)$ gauge theories with $N_f = 5, 6, 7$ fundamental and $N_A = 1$ anti-symmetric hypermultiplets respectively. In this appendix, we recall the results of the Nekrasov partition functions of the $Sp(N)$ gauge theories with N_f flavours and an anti-symmetric hypermultiplet.

H.1 $Sp(N)$ INSTANTON PARTITION FUNCTIONS

The 5d $Sp(N)$ gauge theories with N_f flavours and one anti-symmetric hypermultiplet are realised on the worldvolume theory on N D4-branes with N_f D8-branes and one O8-plane. The Witten index of the ADHM quantum mechanics on k D0-branes in the system gives the k instanton partition function up to a factor which is a contribution of D0-branes moving on the worldvolume of the D8-branes and the O8-plane [194]. The index of the ADHM quantum mechanics is given by the Witten index with some fugacities associated to symmetries of the theory

$$Z^k_{QM}\left(\epsilon_1, \epsilon_2, \alpha_i, z\right) = \mathrm{Tr}\left[(-1)^F e^{-\beta\{Q,Q^\dagger\}} e^{-\epsilon_1(J_1+J_2)} e^{-\epsilon_2(J_2+J_R)} e^{-\alpha_i \Pi_i} e^{-m_a F'_a}\right]. \quad \text{(H.1)}$$

$Q, Q^\dagger$ are supercharges that commute with the fugacities. F is the Fermion number operator, J_1, J_2 are the Cartan generators of the spacetime rotation $SO(4)$, J_R is the Cartan generator of the $SU(2)$ R-symmetry, Π_i are the Cartan generators of the gauge group, and F'_a are the Cartan generators of the flavour symmetry. $\epsilon_1, \epsilon_2, \alpha_i, m_a$ are chemical potentials associated to the symmetries. In particular, α_i are the Coulomb branch moduli, m_a are the mass parameters. In comparison with the refined topological string partition function, we choose $q = e^{-\epsilon_2}, t = e^{\epsilon_1}$. The instanton partition function is then basically given by $Z_{QM} = \sum_k Z^k_{QM} u^k$ up to the factor from the D0-D8-O8 bound states. Here u is the instanton fugacity of the gauge group.

We will simply quote the final result of (H.1) for the $Sp(N)$ gauge theories with N_f flavours and one anti-symmetric hypermultiplet. The dual gauge group on the k D0-branes for the $Sp(N)$ gauge theory is $O(k)$, which have two components. Hence, we have two contributions $Z^k_\pm$ from $O(k)_\pm$ at each instanton order, and the end result from k-instanton is written by [190, 194]

$$Z^k_{QM} = \frac{Z^k_+ + Z^k_-}{2}. \quad \text{(H.2)}$$

$Z^k_\pm$ consists of the contribution from the vector multiplet $Z^\pm_{vec}$, the fundamental hypermultiplet $Z^\pm_{fund}$ and the anti-symmetric hypermultiplet $Z^\pm_{anti}$

$$Z^k_\pm = \frac{1}{|W|}\oint \prod_I \frac{d\phi_I}{2\pi i} Z^\pm_{vec}\, Z^\pm_{fund}\, Z^\pm_{anti}. \tag{H.3}$$

The integration region is a unit circle. I runs from 1 to n for $O(k)_+$ with $k = 2n, 2n+1$. As for $O(k)_-$, I runs from 1 to n for $k = 2n+1$, and 1 to $n-1$ for $k = 2n$. $|W|$ is the order of the Weyl group of $O(k)_\pm$,

$$|W|^{\chi=0}_+ = \frac{1}{2^{n-1}n!}, \quad |W|^{\chi=1}_+ = \frac{1}{2^{n-1}(n-1)!}, \quad |W|^{\chi=1}_- = \frac{1}{2^{n-1}n!}, \quad |W|^{\chi=0}_- = \frac{1}{2^n n!}, \tag{H.4}$$

where χ is defined as $k = 2n+\chi$, $(\chi = 0,1)$. The integrands from the $O(k)_+$ component is

$$\begin{aligned} Z^+_{vec} &= \prod_{I<J}^n 2\sinh\frac{\pm\phi_I\pm\phi_J}{2}\left(\frac{\prod_I^n 2\sinh\frac{\pm\phi_I}{2}}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}\prod_{i=1}^N 2\sinh\frac{\pm\alpha_i+\epsilon_+}{2}}\prod_{I=1}^n\frac{2\sinh\frac{\pm\phi_I+2\epsilon_+}{2}}{2\sinh\frac{\pm\phi_I\pm\epsilon_-+\epsilon_+}{2}}\right)^\chi \\ &\prod_{I=1}^n\frac{2\sinh\epsilon_+}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}2\sinh\frac{\pm2\phi_I\pm\epsilon_-+\epsilon_+}{2}\prod_{i=1}^N 2\sinh\frac{\pm\phi_I\pm\alpha_i+\epsilon_+}{2}}\prod_{I<J}^n\frac{2\sinh\frac{\pm\phi_I\pm\phi_j+2\epsilon_+}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm\epsilon_-+\epsilon_+}{2}} \end{aligned} \tag{H.5}$$

$$\begin{aligned} Z^+_{anti} &= \left(\frac{\prod_{i=1}^N 2\sinh\frac{m\pm\alpha_i}{2}}{2\sinh\frac{m\pm\epsilon_+}{2}}\prod_{I=1}^n\frac{2\sinh\frac{\pm\phi_I\pm m-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm m-\epsilon_-}{2}}\right)^\chi\prod_{I=1}^n\frac{2\sinh\frac{\pm m-\epsilon_-}{2}\prod_{i=1}^N 2\sinh\frac{\pm\phi_I\pm\alpha_i-m}{2}}{2\sinh\frac{\pm m-\epsilon_+}{2}2\sinh\frac{\pm2\phi_I\pm m-\epsilon_+}{2}} \\ &\prod_{I<J}^n\frac{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_+}{2}} \end{aligned} \tag{H.6}$$

$$Z^+_{fund} = \prod_{a=1}^{N_f}\left\{\left(2\sinh\frac{m_a}{2}\right)^\chi\prod_{I=1}^n 2\sinh\frac{\pm\phi_I+m_a}{2}\right\}. \tag{H.7}$$

Here we defined $2\sinh(a\pm b) = 2\sinh(a)\,2\sinh(b)$. We also defined $\epsilon_+ = \frac{\epsilon_1+\epsilon_2}{2}, \epsilon_- = \frac{\epsilon_1-\epsilon_2}{2}$. On the other hand, the integrands from the $O(k)_-$ component depend on whether k is even or odd. When $k = 2n+1$, we have

$$\begin{aligned} Z^-_{vec} &= \prod_{I<J}^n 2\sinh\frac{\pm\phi_I\pm\phi_J}{2}\left(\frac{\prod_I^n 2\cosh\frac{\pm\phi_I}{2}}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}\prod_{i=1}^N 2\cosh\frac{\pm\alpha_i+\epsilon_+}{2}}\prod_{I=1}^n\frac{2\cosh\frac{\pm\phi_I+2\epsilon_+}{2}}{2\cosh\frac{\pm\phi_I\pm\epsilon_-+\epsilon_+}{2}}\right) \\ &\prod_{I=1}^n\frac{2\sinh\epsilon_+}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}2\sinh\frac{\pm2\phi_I\pm\epsilon_-+\epsilon_+}{2}\prod_{i=1}^N 2\sinh\frac{\pm\phi_I\pm\alpha_i+\epsilon_+}{2}}\prod_{I<J}^n\frac{2\sinh\frac{\pm\phi_I\pm\phi_j+2\epsilon_+}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm\epsilon_-+\epsilon_+}{2}} \end{aligned} \tag{H.8}$$

$$\begin{aligned} Z^-_{anti} &= \frac{\prod_{i=1}^N 2\cosh\frac{m\pm\alpha_i}{2}}{2\sinh\frac{m\pm\epsilon_+}{2}}\prod_{I=1}^n\frac{2\cosh\frac{\pm\phi_I\pm m-\epsilon_-}{2}}{2\cosh\frac{\pm\phi_I\pm m-\epsilon_-}{2}}\,\frac{2\sinh\frac{\pm m-\epsilon_-}{2}\prod_{i=1}^N 2\sinh\frac{\pm\phi_I\pm\alpha_i-m}{2}}{2\sinh\frac{\pm m-\epsilon_+}{2}2\sinh\frac{\pm2\phi_I\pm m-\epsilon_+}{2}} \\ &\prod_{I<J}^n\frac{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_+}{2}} \end{aligned} \tag{H.9}$$

$$Z^{-}_{fund} = \prod_{a=1}^{N_f}\left(2\cosh\frac{m_a}{2}\prod_{I=1}^{n}2\sinh\frac{\pm\phi_I+m_a}{2}\right). \tag{H.10}$$

When $k=2n$, we have

$$\begin{aligned} Z^{-}_{vec} &= \prod_{I<J}^{n}2\sinh\frac{\pm\phi_I\pm\phi_J}{2}\prod_{I}^{n-1}2\sinh(\pm\phi_I) \\ &\quad \frac{2\cosh(\epsilon_+)}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}2\sinh(\pm\epsilon_-+\epsilon_+)\prod_{i=1}^{N}2\sinh(\pm\alpha_i+\epsilon_+)}\prod_{I=1}^{n-1}\frac{2\sinh(\pm\phi_I+2\epsilon_+)}{2\sinh(\pm\phi_I\pm\epsilon_-+\epsilon_+)} \\ &\quad \prod_{I=1}^{n-1}\frac{2\sinh\epsilon_+}{2\sinh\frac{\pm\epsilon_-+\epsilon_+}{2}2\sinh\frac{\pm2\phi_I\pm\epsilon_-+\epsilon_+}{2}\prod_{i=1}^{N}2\sinh\frac{\pm\phi_I\pm\alpha_i+\epsilon_+}{2}}\prod_{I<J}^{n-1}\frac{2\sinh\frac{\pm\phi_I\pm\phi_j+2\epsilon_+}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm\epsilon_-+\epsilon_+}{2}} \end{aligned} \tag{H.11}$$

$$\begin{aligned} Z^{-}_{anti} &= \frac{2\cosh\frac{\pm m-\epsilon_-}{2}\prod_{i=1}^{N}2\sinh(m\pm\alpha_i)}{2\sinh\frac{m\pm\epsilon_+}{2}2\sinh(m\pm\epsilon_+)} \\ &\quad \prod_{I=1}^{n-1}\frac{2\sinh(\pm\phi_I\pm m-\epsilon_-)}{2\sinh(\pm\phi_I\pm m-\epsilon_-)}\frac{2\sinh\frac{\pm m-\epsilon_-}{2}\prod_{i=1}^{N}2\sinh\frac{\pm\phi_I\pm\alpha_i-m}{2}}{2\sinh\frac{\pm m-\epsilon_+}{2}2\sinh\frac{\pm2\phi_I\pm m-\epsilon_+}{2}}\prod_{I<J}^{n-1}\frac{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_-}{2}}{2\sinh\frac{\pm\phi_I\pm\phi_J\pm m-\epsilon_+}{2}} \end{aligned} \tag{H.12}$$

$$Z^{-}_{fund} = \prod_{a=1}^{N_f}\left(2\sinh m_a\prod_{I=1}^{n-1}2\sinh\frac{\pm\phi_I+m_a}{2}\right). \tag{H.13}$$

Note that we included the Haar measure into the integrands.

The summation Z_{QM} is not exactly the instanton partition function of the $Sp(N)$ gauge theories but we need to factor out the contribution from the D0-D8-O8 bound states. The contribution is written by the Plethystic exponential

$$Z_{\text{D0-D8-O8}} = PE[f_{N_f}(x,y,v,w_i,u)] = \exp\left[\sum_{n=1}^{\infty}\frac{f_{N_f}(x^n,y^n,v^n,w_i^n,u^n)}{n}\right], \tag{H.14}$$

where f_{N_f} is [194]

$$f_0 = -\frac{t^2}{(1-tu)\left(1-\frac{t}{u}\right)(1-tv)\left(1-\frac{t}{v}\right)}q, \tag{H.15}$$

$$f_{N_f} = -\frac{t^2}{(1-tu)\left(1-\frac{t}{u}\right)(1-tv)\left(1-\frac{t}{v}\right)}q\chi(y_i)^{SO(2N_f)}_{2^{N_f-1}} \quad \text{for } 1\le N_f\le 5, \tag{H.16}$$

$$f_7 = -\frac{t^2}{(1-tu)\left(1-\frac{t}{u}\right)(1-tv)\left(1-\frac{t}{v}\right)}\left(q\chi(y_i)^{SO(12)}_{32}+q^2\right), \tag{H.17}$$

$$f_8 = -\frac{t^2}{(1-tu)\left(1-\frac{t}{u}\right)(1-tv)\left(1-\frac{t}{v}\right)}\left(q\chi(y_i)^{SO(14)}_{64}+q^2\chi(y_i)^{SO(14)}_{14}\right), \tag{H.18}$$

We defined $x=e^{-\epsilon_+}, y=e^{-\epsilon_-}, v=e^{-m}, w_i=e^{\frac{m_a}{2}}$ for $a=1,\cdots,N_f$. $\chi(w_i)^{SO(2N_f)}_{2^{N_f-1}}$ is the character of the positive chirality spinor representation of $SO(2N_f)$. The explicit expression is

$$\chi(w_i)^{SO(2N_f)}_{2^{N_f-1}} = \frac{1}{2}\prod_{a=1}^{N_f}2\sinh\frac{m_a}{2}+\frac{1}{2}\prod_{a=1}^{N_f}2\cosh\frac{m_a}{2}. \tag{H.19}$$

The instanton partition functions of the $Sp(N)$ gauge theories are then given by

$$Z_{inst}^{Sp(N)} = \frac{Z_{QM}}{Z_{\text{D0-D8-O8}}}. \tag{H.20}$$

Let us explicitly write down the 1-instanton part of (H.20). At the 1-instanton order, we do not have the contour integral in (H.3) and it is straightforward to obtain

$$\begin{aligned} Z_{1\text{-}inst}^{Sp(N)} &= \frac{1}{2}\Bigg\{\frac{\prod_{a=1}^{N_f} 2\sinh\frac{m_a}{2}}{2\sinh\frac{\epsilon_+\pm\epsilon_-}{2}\prod_{i=1}^{N} 2\sinh\frac{\pm\alpha_i+\epsilon_+}{2}}\frac{\prod_{i=1}^{N} 2\sinh\frac{m\pm\alpha_i}{2} - \prod_{i=1}^{N} 2\sinh\frac{\pm\alpha_i+\epsilon_+}{2}}{2\sinh\frac{m\pm\epsilon_+}{2}} \\ &\quad + \frac{\prod_{a=1}^{N_f} 2\cosh\frac{m_a}{2}}{2\sinh\frac{\epsilon_+\pm\epsilon_-}{2}\prod_{i=1}^{N} 2\cosh\frac{\pm\alpha_i+\epsilon_+}{2}}\frac{\prod_{i=1}^{N} 2\cosh\frac{m\pm\alpha_i}{2} - \prod_{i=1}^{N} 2\cosh\frac{\pm\alpha_i+\epsilon_+}{2}}{2\sinh\frac{m\pm\epsilon_+}{2}}\Bigg\}. \end{aligned} \tag{H.21}$$

When, $N = 1$, then (H.21) reduces to

$$Z_{1\text{-}inst}^{Sp(1)} = \frac{1}{2}\left\{\frac{\prod_{a=1}^{N_f} 2\sinh\frac{m_a}{2}}{2\sinh\frac{\epsilon_+\pm\epsilon_-}{2}2\sinh\frac{\pm\alpha+\epsilon_+}{2}} + \frac{\prod_{a=1}^{N_f} 2\cosh\frac{m_a}{2}}{2\sinh\frac{\epsilon_+\pm\epsilon_-}{2}2\cosh\frac{\pm\alpha+\epsilon_+}{2}}\right\}. \tag{H.22}$$

In the special cases with $m = 0, \epsilon_+ = 0$, the 1-instanton part of the $Sp(N)$ gauge theory, (H.21), can be written by the N copies of the 1-instanton part of the $Sp(1)$ gauge theory, (H.22),

$$Z_{1\text{-}inst}^{Sp(N)}|_{m=\epsilon_+=0} = \frac{1}{2}\sum_{i=1}^{N}\left\{\frac{\prod_{a=1}^{N_f} 2\sinh\frac{m_a}{2}}{2\sinh\frac{\pm\epsilon_-}{2}2\sinh\frac{\pm\alpha_i}{2}} + \frac{\prod_{a=1}^{N_f} 2\cosh\frac{m_a}{2}}{2\sinh\frac{\pm\epsilon_-}{2}2\cosh\frac{\pm\alpha_i}{2}}\right\}. \tag{H.23}$$

One can show (H.23) for arbitrary N by induction.

From the 2-instanton order, we need to carefully evaluate the contour integral in (H.3) [194]. In this case it is necessary to evaluate the contour integrals only for the $O(2)_+$ component. In particular we find that the integrand has simple poles at the zeroes of the hyperbolic sines in the denominator with the general form

$$\frac{1}{\sinh\frac{Q\phi+\ldots}{2}}, \tag{H.24}$$

where Q is an integer. The prescription of [194] to compute Z_+^2 is to take the sum of the residues of the integrand at the poles with $Q > 0$. Equivalently we can take the contour of integration to be the unit circle in the variable $z = e^{\phi}$ and replace $t = e^{-\epsilon_+}$ in Z_{vec}^+ and $T = e^{-\epsilon_+}$ in Z_{anti}^+ and taking $t \ll 1$ and $T \gg 1$. These two procedures are equivalent because if t is taken sufficiently small and T sufficiently large only the poles with $Q > 0$ will lay inside the unit circle in z. We find that for an $Sp(N)$ gauge theory with one anti-symmetric hypermultiplet and N_f fundamental flavours there are $8 + 2N$ poles, 4 coming from Z_{anti}^+ and $4 + 2N$ coming from Z_{vec}^+. In particular the poles are located at

$$\begin{aligned} &\phi_1 = -\frac{1}{2}(\epsilon_+ + \epsilon_-)\,, \quad \phi_2 = -\frac{1}{2}(\epsilon_+ + \epsilon_-) + i\pi\,, \\ &\phi_3 = -\frac{1}{2}(\epsilon_+ - \epsilon_-)\,, \quad \phi_4 = -\frac{1}{2}(\epsilon_+ - \epsilon_-) + i\pi\,, \\ &\phi_5 = \frac{1}{2}(\epsilon_+ - m)\,, \quad \phi_6 = \frac{1}{2}(\epsilon_+ - m) + i\pi\,, \\ &\phi_7 = \frac{1}{2}(\epsilon_+ + m)\,, \quad \phi_8 = \frac{1}{2}(\epsilon_+ + m) + i\pi\,, \\ &\phi_{8+i} = -\epsilon_+ - \alpha_i\,, \quad \phi_{8+N+i} = -\epsilon_+ + \alpha_i\,, \quad i = 1, \ldots, N\,. \end{aligned} \tag{H.25}$$

By evaluating the residues at the poles. we obtain the $Sp(N)$ partition function from the $O(2)_+$ component. The $Sp(N)$ partition function from the $O(2)_-$ component is obtained straightforwardly since it does not involve the contour integrals.

H.2 PERTURBATIVE PARTITION FUNCTIONS

We summarise the results of the perturbative contribution to the partition function.

The perturbative contribution from a vector multiplet of a gauge group G is [190]

$$\prod_{i,j=1}^{\infty}\prod_{r}\left(2\sinh\frac{(i-1)\epsilon_1+(j-1)\epsilon_2+r\cdot\alpha}{2}\right)^{\frac{1}{2}}\left(2\sinh\frac{i\epsilon_1+j\epsilon_2+r\cdot\alpha}{2}\right)^{\frac{1}{2}}, \tag{H.26}$$

where r represents all the root vectors of the Lie algebra of $\mathfrak{g}$. α is a vector of the Coulomb branch moduli. On the other hand, the perturbative contribution from a hypermultiplet in a representation R and a mass m is [190]

$$\prod_{i,j=1}^{\infty}\prod_{w\in R}\left(2\sinh\frac{\left(i-\frac{1}{2}\right)\epsilon_1+\left(j-\frac{1}{2}\right)\epsilon_2+w\cdot\alpha-m}{2}\right)^{-1}, \tag{H.27}$$

where w is all the weight vectors of the representation R. To obtain (H.26) and (H.27) on the field theory side, constant divergent factors are factored out. Therefore, we will ignore any constant prefactor in the perturbative partition functions.

Then, we move on to the unrefined case of $\epsilon_1=-\epsilon_2=\epsilon$ and obtain the expression of the perturbative partition function which is appropriate for the comparison with the perturbative contribution in the topological string partition function. We also focus on the $Sp(N)$ gauge theory with five fundamental hypermultiplets with mass $m_a, a=1,\cdots N_f$ and an anti-symmetric hypermultiplet with mass m. The vector multiplet contribution can be written as

$$\begin{aligned} Z^{vec}_{pert} &= \prod_{i,j=1}^{\infty}\Big[\left(1-q^{i+j-1}\right)^{-N} \\ &\prod_{1\le k<l\le N}\left(1-e^{\alpha_k+\alpha_l}q^{i+j-1}\right)^{-1}\left(1-e^{-\alpha_k+\alpha_l}q^{i+j-1}\right)^{-1}\left(1-e^{\alpha_k-\alpha_l}q^{i+j-1}\right)^{-1}\left(1-e^{-\alpha_k-\alpha_l}q^{i+j-1}\right)^{-1} \\ &\prod_{1\le k\le N}\left(1-e^{2\alpha_k}q^{i+j-1}\right)^{-1}\left(1-e^{-2\alpha_k}q^{i+j-1}\right)^{-1}\Big] \end{aligned} \tag{H.28}$$

where the explicit expressions of the root vectors $r=\pm e_k\pm e_l, \pm 2e_k$ with $1\le k<l\le N$ are used. The first line of (H.28) is the contribution from the Cartan part of the $Sp(N)$ vector multiplet. To obtain the expression (H.28), we used analytic continuation

$$\prod_{i,j=1}^{\infty}\left(1-Qt^{i-1}q^{-j+1}\right)^{\frac{1}{2}}=\exp\left(\sum_{k=1}^{\infty}\frac{Q^k}{2k}\frac{1}{(1-t^k)}\frac{q^k}{(1-q^k)}\right)=\prod_{i,j=1}^{\infty}\left(1-Qt^{i-1}q^{j}\right)^{-\frac{1}{2}},$$

$$\prod_{i,j=1}^{\infty}\left(1-Qt^{i}q^{-j}\right)^{\frac{1}{2}}=\exp\left(\sum_{k=1}^{\infty}\frac{Q^k}{2k}\frac{t^k}{(1-t^k)}\frac{1}{(1-q^k)}\right)=\prod_{i,j=1}^{\infty}\left(1-Qt^{i}q^{j-1}\right)^{-\frac{1}{2}},$$

and then setting $t = q$. Similarly, the perturbative contribution from the fundamental hypermultiplets is

$$Z_{pert}^{fund} = \prod_{i,j=1}^{\infty} \prod_{a=1}^{N_f} \prod_{k=1}^{N} \left(e^{-m_a}\right) \left(1 - e^{\alpha_k - m_a} q^{i+j-1}\right) \left(1 - e^{-\alpha_k - m_a} q^{i+j-1}\right). \quad \text{(H.29)}$$

We used the fact that the weights of the fundamental representation are $\pm e_k, (k = 1, \cdots, N)$, and also analytic continuation

$$\prod_{i,j=1}^{\infty} \left(1 - Qt^{i-\frac{1}{2}} q^{-j+\frac{1}{2}}\right)^{-1} = \exp\left(-\sum_{k=1}^{\infty} \frac{Q^k}{k} \frac{t^{\frac{k}{2}}}{(1-t^k)} \frac{q^{\frac{k}{2}}}{(1-q^k)}\right) = \prod_{i,j=1}^{\infty} \left(1 - Qt^{i-\frac{1}{2}} q^{j-\frac{1}{2}}\right), \quad \text{(H.30)}$$

and then setting $t = q$. Note that we have an overall divergent factor in (H.29). In the comparison with the perturbative contribution from the topological string partition function, we ignore the factor. The perturbative contribution from the anti-symmetric hypermultiplet is

$$\begin{aligned} Z_{pert}^{anti\text{-}sym} = \prod_{i,j=1}^{\infty} \prod_{1 \le k < l \le N} & \left(-e^{-\frac{5}{2}m}\right) \left(1 - e^{\alpha_k + \alpha_l - m} q^{i+j-1}\right) \left(1 - e^{-\alpha_k + \alpha_l - m} q^{i+j-1}\right) \\ & \left(1 - e^{\alpha_k - \alpha_l - m} q^{i+j-1}\right) \left(1 - e^{-\alpha_k - \alpha_l - m} q^{i+j-1}\right) \\ & \left(1 - e^{-m} q^{i+j-1}\right), \end{aligned} \quad \text{(H.31)}$$

where we used the fact that the weights of the anti-symmetric representation are $\pm e_k \pm e_l$, $(1 \le k < l \le N)$ and 0, and also the analytic continuation (H.30) again. Hence, the perturbative partition function of the $Sp(N)$ gauge theory with N_f fundamental hypermultiplets and one anti-symmetric hypermultiplet is simply given by the product of (H.28), (H.29) and (H.31).

When we further concentrate on the case where mass of the anti-symmetric hypermultiplet is zero, *i.e.* $m = 0$, there is a cancellation between the contribution of the anti-symmetric hypermultiplet and that of a part of the vector multiplet. In fact, the perturbative partition function of the anti-symmetric hypermultiplet disappears up to the divergent factor. More explicitly, the perturbative partition function becomes

$$Z_{pert} = \prod_{i,j=1}^{\infty} \left[\left(1 - q^{i+j-1}\right)^{-N+1} \prod_{k=1}^{N_f} \frac{\prod_{a=1}^{N_f} \left(1 - e^{\alpha_k - m_a} q^{i+j-1}\right) \left(1 - e^{-\alpha_k - m_a} q^{i+j-1}\right)}{\left(1 - e^{2\alpha_k} q^{i+j-1}\right) \left(1 - e^{-2\alpha_k} q^{i+j-1}\right)} \right], \quad \text{(H.32)}$$

up to the divergent factors. We compare (H.32) with the perturbative part of the partition function obtained from the topological vertex computation for rank N E_6, E_7, E_8 theories.

There is another subtle point when one tries to compare (H.32) with the topological string result. By using the identity

$$\prod_{i,j=1}^{\infty} \left(1 - Qq^{i+j-1}\right) = \prod_{n=1}^{\infty} (1 - Qq^n)^n, \quad \text{(H.33)}$$

one can obtain

$$
\begin{aligned}
\prod_{n=1}^{\infty}(1-Qq^n)^n &= \prod_{n=1}^{\infty}(-Q)^n q^{n^2}\left(1-Q^{-1}q^{-n}\right)^n \\
&= \prod_{n=1}^{N}(-Q)^n q^{n^2}\left(1-Q^{-1}q^{n}\right)^n \\
&= \prod_{n=1}^{N}(-Q)^n \left(1-Q^{-1}q^{n}\right)^n . \qquad \text{(H.34)}
\end{aligned}
$$

From the first line to the second line of (H.34), we used analytic continuation

$$\prod_{n=1}^{\infty}\left(1-Q^{-1}q^{-n}\right)^n = \exp\left(-\sum_{k=1}^{\infty}\frac{Q^k}{k}\frac{q^k}{(1-q^k)^2}\right) = \prod_{n=1}^{\infty}\left(1-Q^{-1}q^{n}\right)^n . \qquad \text{(H.35)}$$

From the second line to the third line of (H.34) we used the zeta function regularisation $\zeta(-2)=0$. Therefore, as for the perturbative factor, the factor with Q is basically the same as the factor with Q^{-1} up to the regularisation and also the divergent factor. In the comparison between (H.32) and the perturbative contribution from the topological string partition function, we will neglect the subtlety regarding the regularisation, the divergent factor and also the analytic continuation.

BIBLIOGRAPHY

[1] H. Georgi and S. L. Glashow. "Unity of All Elementary Particle Forces." In: *Phys. Rev. Lett.* 32 (1974), pp. 438–441.

[2] D. I. Kazakov. "Beyond the standard model: In search of supersymmetry." In: *2000 European School of high-energy physics, Caramulo, Portugal, 20 Aug-2 Sep 2000: Proceedings*. 2000, pp. 125–199. arXiv: hep-ph/0012288 [hep-ph].

[3] Tom Banks and Nathan Seiberg. "Symmetries and Strings in Field Theory and Gravity." In: *Phys. Rev.* D83 (2011), p. 084019. arXiv: 1011.5120 [hep-th].

[4] Matthias R. Gaberdiel and Barton Zwiebach. "Exceptional groups from open strings." In: *Nucl. Phys.* B518 (1998), pp. 151–172. arXiv: hep-th/9709013 [hep-th].

[5] Matthias R. Gaberdiel, Tamas Hauer, and Barton Zwiebach. "Open string-string junction transitions." In: *Nucl. Phys.* B525 (1998), pp. 117–145. arXiv: hep-th/9801205 [hep-th].

[6] Oliver DeWolfe and Barton Zwiebach. "String junctions for arbitrary Lie algebra representations." In: *Nucl. Phys.* B541 (1999), pp. 509–565. arXiv: hep-th/9804210 [hep-th].

[7] Chris Beasley, Jonathan J. Heckman, and Cumrun Vafa. "GUTs and Exceptional Branes in F-theory - I." In: *JHEP* 01 (2009), p. 058. arXiv: 0802.3391 [hep-th].

[8] Cumrun Vafa. "Evidence for F theory." In: *Nucl. Phys.* B469 (1996), pp. 403–418. arXiv: hep-th/9602022 [hep-th].

[9] Ron Donagi and Martijn Wijnholt. "Model Building with F-Theory." In: *Adv. Theor. Math. Phys.* 15.5 (2011), pp. 1237–1317. arXiv: 0802.2969 [hep-th].

[10] Chris Beasley, Jonathan J. Heckman, and Cumrun Vafa. "GUTs and Exceptional Branes in F-theory - II: Experimental Predictions." In: *JHEP* 01 (2009), p. 059. arXiv: 0806.0102 [hep-th].

[11] Frederik Denef. "Les Houches Lectures on Constructing String Vacua." In: *String theory and the real world: From particle physics to astrophysics. Proceedings, Summer School in Theoretical Physics, 87th Session, Les Houches, France, July 2-27, 2007*. 2008, pp. 483–610. arXiv: 0803.1194 [hep-th].

[12] Timo Weigand. "Lectures on F-theory compactifications and model building." In: *Class. Quant. Grav.* 27 (2010), p. 214004. arXiv: 1009.3497 [hep-th].

[13] Anshuman Maharana and Eran Palti. "Models of Particle Physics from Type IIB String Theory and F-theory: A Review." In: *Int. J. Mod. Phys.* A28 (2013), p. 1330005. arXiv: 1212.0555 [hep-th].

[14] David R. Morrison and Daniel S. Park. "F-Theory and the Mordell-Weil Group of Elliptically-Fibered Calabi-Yau Threefolds." In: *JHEP* 10 (2012), p. 128. arXiv: 1208.2695 [hep-th].

[15] Jan Borchmann, Christoph Mayrhofer, Eran Palti, and Timo Weigand. "Elliptic fibrations for $SU(5) \times U(1) \times U(1)$ F-theory vacua." In: *Phys. Rev.* D88.4 (2013), p. 046005. arXiv: 1303.5054 [hep-th].

[16] Mirjam Cvetic, Denis Klevers, and Hernan Piragua. "F-Theory Compactifications with Multiple U(1)-Factors: Constructing Elliptic Fibrations with Rational Sections." In: *JHEP* 06 (2013), p. 067. arXiv: 1303.6970 [hep-th].

[17] Volker Braun, Thomas W. Grimm, and Jan Keitel. "Geometric Engineering in Toric F-Theory and GUTs with U(1) Gauge Factors." In: *JHEP* 12 (2013), p. 069. arXiv: 1306.0577 [hep-th].

[18] Mirjam Cvetic, Antonella Grassi, Denis Klevers, and Hernan Piragua. "Chiral Four-Dimensional F-Theory Compactifications With SU(5) and Multiple U(1)-Factors." In: *JHEP* 04 (2014), p. 010. arXiv: 1306.3987 [hep-th].

[19] Jan Borchmann, Christoph Mayrhofer, Eran Palti, and Timo Weigand. "SU(5) Tops with Multiple U(1)s in F-theory." In: *Nucl. Phys.* B882 (2014), pp. 1–69. arXiv: 1307.2902 [hep-th].

[20] Mirjam Cvetic, Denis Klevers, and Hernan Piragua. "F-Theory Compactifications with Multiple U(1)-Factors: Addendum." In: *JHEP* 12 (2013), p. 056. arXiv: 1307.6425 [hep-th].

[21] Mirjam Cvetic, Denis Klevers, Hernan Piragua, and Peng Song. "Elliptic fibrations with rank three Mordell-Weil group: F-theory with U(1) x U(1) x U(1) gauge symmetry." In: *JHEP* 03 (2014), p. 021. arXiv: 1310.0463 [hep-th].

[22] Mirjam Cvetic, Denis Klevers, Hernan Piragua, and Washington Taylor. "General U(1)×U(1) F-theory compactifications and beyond: geometry of unHiggsings and novel matter structure." In: *JHEP* 11 (2015), p. 204. arXiv: 1507.05954 [hep-th].

[23] Mirjam Cvetic, Antonella Grassi, Denis Klevers, Maximilian Poretschkin, and Peng Song. "Origin of Abelian Gauge Symmetries in Heterotic/F-theory Duality." In: *JHEP* 04 (2016), p. 041. arXiv: 1511.08208 [hep-th].

[24] Edward Witten. "Nonperturbative superpotentials in string theory." In: *Nucl. Phys.* B474 (1996), pp. 343–360. arXiv: hep-th/9604030 [hep-th].

[25] Volker Braun and David R. Morrison. "F-theory on Genus-One Fibrations." In: *JHEP* 08 (2014), p. 132. arXiv: 1401.7844 [hep-th].

[26] David R. Morrison and Washington Taylor. "Sections, multisections, and U(1) fields in F-theory." In: (2014). arXiv: 1404.1527 [hep-th].

[27] Lara B. Anderson, IŨaki García-Etxebarria, Thomas W. Grimm, and Jan Keitel. "Physics of F-theory compactifications without section." In: *JHEP* 12 (2014), p. 156. arXiv: 1406.5180 [hep-th].

[28] Denis Klevers, Damian Kaloni Mayorga Pena, Paul-Konstantin Oehlmann, Hernan Piragua, and Jonas Reuter. "F-Theory on all Toric Hypersurface Fibrations and its Higgs Branches." In: *JHEP* 01 (2015), p. 142. arXiv: 1408.4808 [hep-th].

[29] IŨaki García-Etxebarria, Thomas W. Grimm, and Jan Keitel. "Yukawas and discrete symmetries in F-theory compactifications without section." In: *JHEP* 11 (2014), p. 125. arXiv: 1408.6448 [hep-th].

[30] Christoph Mayrhofer, Eran Palti, Oskar Till, and Timo Weigand. "Discrete Gauge Symmetries by Higgsing in four-dimensional F-Theory Compactifications." In: *JHEP* 12 (2014), p. 068. arXiv: 1408.6831 [hep-th].

[31] Christoph Mayrhofer, Eran Palti, Oskar Till, and Timo Weigand. "On Discrete Symmetries and Torsion Homology in F-Theory." In: *JHEP* 06 (2015), p. 029. arXiv: 1410.7814 [hep-th].

[32] Mirjam Cvetic, Ron Donagi, Denis Klevers, Hernan Piragua, and Maximilian Poretschkin. "F-theory vacua with $\mathbb{Z}_3$ gauge symmetry." In: *Nucl. Phys.* B898 (2015), pp. 736–750. arXiv: 1502.06953 [hep-th].

[33] Ling Lin, Christoph Mayrhofer, Oskar Till, and Timo Weigand. "Fluxes in F-theory Compactifications on Genus-One Fibrations." In: *JHEP* 01 (2016), p. 098. arXiv: 1508.00162 [hep-th].

[34] Federico Bonetti, Thomas W. Grimm, and Tom G. Pugh. "Non-Supersymmetric F-Theory Compactifications on Spin(7) Manifolds." In: *JHEP* 01 (2014), p. 112. arXiv: 1307.5858 [hep-th].

[35] Federico Bonetti, Thomas W. Grimm, Eran Palti, and Tom G. Pugh. "F-Theory on Spin(7) Manifolds: Weak-Coupling Limit." In: *JHEP* 02 (2014), p. 076. arXiv: 1309.2287 [hep-th].

[36] C. S. Shahbazi. "M-theory on non-Kähler eight-manifolds." In: *JHEP* 09 (2015), p. 178. arXiv: 1503.00733 [hep-th].

[37] C. S. Shahbazi. "A class of non-geometric M-theory compactification backgrounds." In: (2015). arXiv: 1508.01750 [hep-th].

[38] Katrin Becker and Melanie Becker. "M theory on eight manifolds." In: *Nucl. Phys.* B477 (1996), pp. 155–167. arXiv: hep-th/9605053 [hep-th].

[39] Pierre Deligne. "Courbes elliptiques." In: 3rd ed. Vol. 476. 10. An optional note. The address: Lecture Notes in Math., Springer, July 1975.

[40] Mirjam Cvetic, Inaki Garcia Etxebarria, and James Halverson. "Three Looks at Instantons in F-theory – New Insights from Anomaly Inflow, String Junctions and Heterotic Duality." In: *JHEP* 11 (2011), p. 101. arXiv: 1107.2388 [hep-th].

[41] K. Kodaira. "On the Structure of Compact Complex Analytic Surfaces, I." In: *American Journal of Mathematics* 86.4 (1964), p. 751.

[42] K. Kodaira. "On the Structure of Compact Complex Analytic Surfaces, II." In: *American Journal of Mathematics* 88.3 (1966), p. 682.

[43] André Néron. "Modèles minimaux des variétés abéliennes sur les corps locaux et globaux." In: *Publications Mathématiques de L'Institut des Hautes Scientifiques* 21.1 (1964), pp. 5–125.

[44] Andreas P. Braun, Andres Collinucci, and Roberto Valandro. "The fate of U(1)'s at strong coupling in F-theory." In: *JHEP* 07 (2014), p. 028. arXiv: 1402.4054 [hep-th].

[45] J. Tate. "Algorithm for determining the type of a singular fiber in an elliptic pencil." In: *Modular Functions of One Variable IV*. Springer Science + Business Media, 1975, pp. 33–52.

[46] M. Bershadsky, Kenneth A. Intriligator, S. Kachru, David R. Morrison, V. Sadov, and Cumrun Vafa. "Geometric singularities and enhanced gauge symmetries." In: *Nucl. Phys.* B481 (1996), pp. 215–252. arXiv: hep-th/9605200 [hep-th].

[47] Sheldon Katz, David R. Morrison, Sakura Schafer-Nameki, and James Sully. "Tate's algorithm and F-theory." In: *JHEP* 08 (2011), p. 094. arXiv: 1106.3854 [hep-th].

[48] Craig Lawrie and Damiano Sacco. "Tate's algorithm for F-theory GUTs with two U(1)s." In: *JHEP* 03 (2015), p. 055. arXiv: 1412.4125 [hep-th].

[49] Sheldon H. Katz and Cumrun Vafa. "Matter from geometry." In: *Nucl.Phys.* B497 (1997), pp. 146–154. arXiv: hep-th/9606086 [hep-th].

[50] Andreas Malmendier and David R. Morrison. "K3 surfaces, modular forms, and non-geometric heterotic compactifications." In: *Lett. Math. Phys.* 105.8 (2015), pp. 1085–1118. arXiv: 1406.4873 [hep-th].

[51] David R. Morrison and Cumrun Vafa. "Compactifications of F theory on Calabi-Yau threefolds. 1." In: *Nucl. Phys.* B473 (1996), pp. 74–92. arXiv: hep-th/9602114 [hep-th].

[52] David R. Morrison and Cumrun Vafa. "Compactifications of F theory on Calabi-Yau threefolds. 2." In: *Nucl. Phys.* B476 (1996), pp. 437–469. arXiv: hep-th/9603161 [hep-th].

[53] Jie Gu and Hans Jockers. "Nongeometric F-theoryÐheterotic duality." In: *Phys. Rev.* D91 (2015), p. 086007. arXiv: 1412.5739 [hep-th].

[54] Robert Friedman, John Morgan, and Edward Witten. "Vector bundles and F theory." In: *Commun. Math. Phys.* 187 (1997), pp. 679–743. arXiv: hep-th/9701162 [hep-th].

[55] Robert Friedman, John W. Morgan, and Edward Witten. "Vector bundles over elliptic fibrations." In: (1997). arXiv: alg-geom/9709029 [alg-geom].

[56] Paul S. Aspinwall and David R. Morrison. "Nonsimply connected gauge groups and rational points on elliptic curves." In: *JHEP* 07 (1998), p. 012. arXiv: hep-th/9805206 [hep-th].

[57] Paul S. Aspinwall, Sheldon H. Katz, and David R. Morrison. "Lie groups, Calabi-Yau threefolds, and F theory." In: *Adv. Theor. Math. Phys.* 4 (2000), pp. 95–126. arXiv: hep-th/0002012 [hep-th].

[58] Christoph Mayrhofer, David R. Morrison, Oskar Till, and Timo Weigand. "Mordell-Weil Torsion and the Global Structure of Gauge Groups in F-theory." In: *JHEP* 10 (2014), p. 16. arXiv: 1405.3656 [hep-th].

[59] Clay Cordova. "Decoupling Gravity in F-Theory." In: *Adv. Theor. Math. Phys.* 15.3 (2011), pp. 689–740. arXiv: 0910.2955 [hep-th].

[60] Ron Donagi and Martijn Wijnholt. "Breaking GUT Groups in F-Theory." In: *Adv. Theor. Math. Phys.* 15.6 (2011), pp. 1523–1603. arXiv: 0808.2223 [hep-th].

[61] Ralph Blumenhagen. "Gauge Coupling Unification in F-Theory Grand Unified Theories." In: *Phys. Rev. Lett.* 102 (2009), p. 071601. arXiv: 0812.0248 [hep-th].

[62] Arthur Hebecker and James Unwin. "Precision Unification and Proton Decay in F-Theory GUTs with High Scale Supersymmetry." In: *JHEP* 09 (2014), p. 125. arXiv: 1405.2930 [hep-th].

[63] Sergio Cecotti, Miranda C. N. Cheng, Jonathan J. Heckman, and Cumrun Vafa. "Yukawa Couplings in F-theory and Non-Commutative Geometry." In: (2009). arXiv: 0910.0477 [hep-th].

[64] Sergio Cecotti, Clay Cordova, Jonathan J. Heckman, and Cumrun Vafa. "T-Branes and Monodromy." In: *JHEP* 07 (2011), p. 030. arXiv: 1010.5780 [hep-th].

[65] Lara B. Anderson, Jonathan J. Heckman, and Sheldon Katz. "T-Branes and Geometry." In: *JHEP* 05 (2014), p. 080. arXiv: 1310.1931 [hep-th].

[66] Andres Collinucci and Raffaele Savelli. "T-branes as branes within branes." In: *JHEP* 09 (2015), p. 161. arXiv: 1410.4178 [hep-th].

[67] Andres Collinucci, Simone Giacomelli, Raffaele Savelli, and Roberto Valandro. "T-branes through 3d mirror symmetry." In: (2016). arXiv: 1603.00062 [hep-th].

[68] Sheldon Katz and David R. Morrison. "Gorenstein Threefold Singularities with Small Resolutions via Invariant Theory for Weyl Groups." In: (1992). eprint: arXiv:alg-geom/9202002.

[69] S. K. Donaldson. "Anti Self-Dual Yang-Mills Connections Over Complex Algebraic Surfaces and Stable Vector Bundles." In: *Proceedings of the London Mathematical Society* s3-50.1 (1985), pp. 1–26.

[70] K. Uhlenbeck and S. T. Yau. "On the existence of hermitian-yang-mills connections in stable vector bundles." In: *Communications on Pure and Applied Mathematics* 39.S1 (1986), S257–S293.

[71] Fernando Marchesano and Luca Martucci. "Non-perturbative effects on seven-brane Yukawa couplings." In: *Phys. Rev. Lett.* 104 (2010), p. 231601. arXiv: 0910.5496 [hep-th].

[72] Robert C. Myers. "Dielectric branes." In: *JHEP* 12 (1999), p. 022. arXiv: hep-th/9910053 [hep-th].

[73] Luis Aparicio, Anamaria Font, Luis E. Ibanez, and Fernando Marchesano. "Flux and Instanton Effects in Local F-theory Models and Hierarchical Fermion Masses." In: *JHEP* 08 (2011), p. 152. arXiv: 1104.2609 [hep-th].

[74] Anamaria Font, Luis E. Ibanez, Fernando Marchesano, and Diego Regalado. "Non-perturbative effects and Yukawa hierarchies in F-theory SU(5) Unification." In: *JHEP* 03 (2013). [Erratum: JHEP07,036(2013)], p. 140. arXiv: 1211.6529 [hep-th].

[75] Sergio Cecotti and Cumrun Vafa. "Topological antitopological fusion." In: *Nucl. Phys.* B367 (1991), pp. 359–461.

[76] Vladimir Fateev, Dmitri Fradkin, Sergei L. Lukyanov, Alexander B. Zamolodchikov, and Alexei B. Zamolodchikov. "Expectation values of descendent fields in the sine-Gordon model." In: *Nucl. Phys.* B540 (1999), pp. 587–609. arXiv: hep-th/9807236 [hep-th].

[77] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. "Wall-crossing, Hitchin Systems, and the WKB Approximation." In: (2009). arXiv: 0907.3987 [hep-th].

[78] Barry M. McCoy, Craig A. Tracy, and Tai Tsun Wu. "Painleve Functions of the Third Kind." In: *J. Math. Phys.* 18 (1977), p. 1058.

[79] Chan-Chi Chiou, Alon E. Faraggi, Radu Tatar, and William Walters. "T-branes and Yukawa Couplings." In: *JHEP* 05 (2011), p. 023. arXiv: 1101.2455 [hep-th].

[80] D. Cremades, L. E. Ibanez, and F. Marchesano. "Computing Yukawa couplings from magnetized extra dimensions." In: *JHEP* 05 (2004), p. 079. arXiv: hep-th/0404229 [hep-th].

[81] Eran Palti. "Wavefunctions and the Point of E_8 in F-theory." In: *JHEP* 07 (2012), p. 065. arXiv: 1203.4490 [hep-th].

[82] Anamaria Font, Fernando Marchesano, Diego Regalado, and Gianluca Zoccarato. "Up-type quark masses in SU(5) F-theory models." In: *JHEP* 11 (2013), p. 125. arXiv: 1307.8089 [hep-th].

[83] Jonathan J. Heckman and Cumrun Vafa. "Flavor Hierarchy From F-theory." In: *Nucl. Phys.* B837 (2010), pp. 137–151. arXiv: 0811.2417 [hep-th].

[84] Graham Ross and Mario Serna. "Unification and fermion mass structure." In: *Phys. Lett.* B664 (2008), pp. 97–102. arXiv: 0704.1248 [hep-ph].

[85] G. F. Giudice and A. Masiero. "A Natural Solution to the mu Problem in Supergravity Theories." In: *Phys. Lett.* B206 (1988), pp. 480–484.

[86] Jihn E. Kim and Hans Peter Nilles. "The mu Problem and the Strong CP Problem." In: *Phys. Lett.* B138 (1984), p. 150.

[87] L. Aparicio, P. G. Camara, D. G. Cerdeno, L. E. Ibanez, and I. Valenzuela. "The NMSSM with F-theory unified boundary conditions." In: *JHEP* 02 (2013), p. 084. arXiv: 1212.4808 [hep-ph].

[88] Vincent Bouchard, Jonathan J. Heckman, Jihye Seo, and Cumrun Vafa. "F-theory and Neutrinos: Kaluza-Klein Dilution of Flavor Hierarchy." In: *JHEP* 01 (2010), p. 061. arXiv: 0904.1419 [hep-ph].

[89] Jonathan J. Heckman, Alireza Tavanfar, and Cumrun Vafa. "The Point of E(8) in F-theory GUTs." In: *JHEP* 08 (2010), p. 040. arXiv: 0906.0581 [hep-th].

[90] Fernando Marchesano, Diego Regalado, and Gianluca Zoccarato. "Yukawa hierarchies at the point of E_8 in F-theory." In: *JHEP* 04 (2015), p. 179. arXiv: 1503.02683 [hep-th].

[91] Federico Carta, Fernando Marchesano, and Gianluca Zoccarato. "Fitting fermion masses and mixings in F-theory GUTs." In: *JHEP* 03 (2016), p. 126. arXiv: 1512.04846 [hep-th].

[92] Lisa Randall and David Simmons-Duffin. "Quark and Lepton Flavor Physics from F-Theory." In: (2009). arXiv: 0904.1584 [hep-ph].

[93] Chuichiro Hattori, Masahisa Matsuda, Takeo Matsuoka, and Daizo Mochinaga. "The String unification of gauge couplings and gauge kinetic mixings." In: *Prog. Theor. Phys.* 90 (1993), pp. 895–910. arXiv: hep-ph/9307305 [hep-ph].

[94] Keith R. Dienes, Christopher F. Kolda, and John March-Russell. “Kinetic mixing and the supersymmetric gauge hierarchy.” In: *Nucl. Phys.* B492 (1997), pp. 104–118. arXiv: hep-ph/9610479 [hep-ph].

[95] Andre Lukas and K. S. Stelle. “Heterotic anomaly cancellation in five-dimensions.” In: *JHEP* 01 (2000), p. 010. arXiv: hep-th/9911156 [hep-th].

[96] D. Lust and S. Stieberger. “Gauge threshold corrections in intersecting brane world models.” In: *Fortsch. Phys.* 55 (2007), pp. 427–465. arXiv: hep-th/0302221 [hep-th].

[97] S. A. Abel and B. W. Schofield. “Brane anti-brane kinetic mixing, millicharged particles and SUSY breaking.” In: *Nucl. Phys.* B685 (2004), pp. 150–170. arXiv: hep-th/0311051 [hep-th].

[98] Hans Jockers and Jan Louis. “The Effective action of D7-branes in N = 1 Calabi-Yau orientifolds.” In: *Nucl. Phys.* B705 (2005), pp. 167–211. arXiv: hep-th/0409098 [hep-th].

[99] Ralph Blumenhagen, Gabriele Honecker, and Timo Weigand. “Loop-corrected compactifications of the heterotic string with line bundles.” In: *JHEP* 06 (2005), p. 020. arXiv: hep-th/0504232 [hep-th].

[100] Steven A. Abel, Joerg Jaeckel, Valentin V. Khoze, and Andreas Ringwald. “Illuminating the Hidden Sector of String Theory by Shining Light through a Magnetic Field.” In: *Phys. Lett.* B666 (2008), pp. 66–70. arXiv: hep-ph/0608248 [hep-ph].

[101] S. A. Abel, M. D. Goodsell, J. Jaeckel, V. V. Khoze, and A. Ringwald. “Kinetic Mixing of the Photon with Hidden U(1)s in String Phenomenology.” In: *JHEP* 07 (2008), p. 124. arXiv: 0803.1449 [hep-ph].

[102] Thomas W. Grimm, Tae-Won Ha, Albrecht Klemm, and Denis Klevers. “The D5-brane effective action and superpotential in N=1 compactifications.” In: *Nucl. Phys.* B816 (2009), pp. 139–184. arXiv: 0811.2996 [hep-th].

[103] Mark Goodsell, Joerg Jaeckel, Javier Redondo, and Andreas Ringwald. “Naturally Light Hidden Photons in LARGE Volume String Compactifications.” In: *JHEP* 11 (2009), p. 027. arXiv: 0909.0515 [hep-ph].

[104] Florian Gmeiner and Gabriele Honecker. “Complete Gauge Threshold Corrections for Intersecting Fractional D6-Branes: The Z6 and Z6’ Standard Models.” In: *Nucl. Phys.* B829 (2010), pp. 225–297. arXiv: 0910.0843 [hep-th].

[105] Mark Goodsell and Andreas Ringwald. “Light Hidden-Sector U(1)s in String Compactifications.” In: *Fortsch. Phys.* 58 (2010), pp. 716–720. arXiv: 1002.1840 [hep-th].

[106] Mathew Bullimore, Joseph P. Conlon, and Lukas T. Witkowski. “Kinetic mixing of U(1)s for local string models.” In: *JHEP* 11 (2010), p. 142. arXiv: 1009.2380 [hep-th].

[107] Michele Cicoli, Mark Goodsell, Joerg Jaeckel, and Andreas Ringwald. “Testing String Vacua in the Lab: From a Hidden CMB to Dark Forces in Flux Compactifications.” In: *JHEP* 07 (2011), p. 114. arXiv: 1103.3705 [hep-th].

[108] M. Williams, C. P. Burgess, Anshuman Maharana, and F. Quevedo. “New Constraints (and Motivations) for Abelian Gauge Bosons in the MeV-TeV Mass Range.” In: *JHEP* 08 (2011), p. 106. arXiv: 1103.4556 [hep-ph].

[109] Thomas W. Grimm and Daniel Vieira Lopes. "The N=1 effective actions of D-branes in Type IIA and IIB orientifolds." In: *Nucl. Phys.* B855 (2012), pp. 639–694. arXiv: 1104.2328 [hep-th].

[110] Max Kerstan and Timo Weigand. "The Effective action of D6-branes in N=1 type IIA orientifolds." In: *JHEP* 06 (2011), p. 105. arXiv: 1104.2329 [hep-th].

[111] Pablo G. Camara, Luis E. Ibanez, and Fernando Marchesano. "RR photons." In: *JHEP* 09 (2011), p. 110. arXiv: 1106.0060 [hep-th].

[112] Gabriele Honecker. "Kaehler metrics and gauge kinetic functions for intersecting D6-branes on toroidal orbifolds - The complete perturbative story." In: *Fortsch. Phys.* 60 (2012), pp. 243–326. arXiv: 1109.3192 [hep-th].

[113] Mark Goodsell, Saul Ramos-Sanchez, and Andreas Ringwald. "Kinetic Mixing of U(1)s in Heterotic Orbifolds." In: *JHEP* 01 (2012), p. 021. arXiv: 1110.6901 [hep-th].

[114] Gabriele Honecker, Martin Ripka, and Wieland Staessens. "The Importance of Being Rigid: D6-Brane Model Building on $T^6/Z_2 x Z_6'$ with Discrete Torsion." In: *Nucl. Phys.* B868 (2013), pp. 156–222. arXiv: 1209.3010 [hep-th].

[115] Gary Shiu, Pablo Soler, and Fang Ye. "Milli-Charged Dark Matter in Quantum Gravity and String Theory." In: *Phys. Rev. Lett.* 110.24 (2013), p. 241304. arXiv: 1302.5471 [hep-th].

[116] Bob Holdom. "Two U(1)'s and Epsilon Charge Shifts." In: *Phys. Lett.* B166 (1986), p. 196.

[117] Daniel Feldman, Boris Kors, and Pran Nath. "Extra-weakly Interacting Dark Matter." In: *Phys. Rev.* D75 (2007), p. 023503. arXiv: hep-ph/0610133 [hep-ph].

[118] Daniel Feldman, Zuowei Liu, Pran Nath, and Brent D. Nelson. "Explaining PAMELA and WMAP data through Coannihilations in Extended SUGRA with Collider Implications." In: *Phys. Rev.* D80 (2009), p. 075001. arXiv: 0907.5392 [hep-ph].

[119] A. Ibarra, A. Ringwald, and C. Weniger. "Hidden gauginos of an unbroken U(1): Cosmological constraints and phenomenological prospects." In: *JCAP* 0901 (2009), p. 003. arXiv: 0809.3196 [hep-ph].

[120] Asimina Arvanitaki, Nathaniel Craig, Savas Dimopoulos, Sergei Dubovsky, and John March-Russell. "String Photini at the LHC." In: *Phys. Rev.* D81 (2010), p. 075018. arXiv: 0909.5440 [hep-ph].

[121] Nigel J. Hitchin. "Lectures on special Lagrangian submanifolds." In: (1999). arXiv: math/9907034 [math].

[122] Thomas W. Grimm and Jan Louis. "The Effective action of type IIA Calabi-Yau orientifolds." In: *Nucl. Phys.* B718 (2005), pp. 153–202. arXiv: hep-th/0412277 [hep-th].

[123] Luis E. Ibanez and Angel M. Uranga. *String theory and particle physics: An introduction to string phenomenology.* Cambridge University Press, 2012.

[124] Ralph Blumenhagen, Mirjam Cvetic, Paul Langacker, and Gary Shiu. "Toward realistic intersecting D-brane models." In: *Ann. Rev. Nucl. Part. Sci.* 55 (2005), pp. 71–139. arXiv: hep-th/0502005 [hep-th].

[125] Ralph Blumenhagen, Boris Kors, Dieter Lust, and Stephan Stieberger. "Four-dimensional String Compactifications with D-Branes, Orientifolds and Fluxes." In: *Phys. Rept.* 445 (2007), pp. 1–193. arXiv: hep-th/0610327 [hep-th].

[126] Fernando Marchesano. "Progress in D-brane model building." In: *Fortsch. Phys.* 55 (2007), pp. 491–518. arXiv: hep-th/0702094 [HEP-TH].

[127] Edward Witten. "Dyons of Charge e theta/2 pi." In: *Phys. Lett.* B86 (1979), pp. 283–287.

[128] Felix Brummer, Joerg Jaeckel, and Valentin V. Khoze. "Magnetic Mixing: Electric Minicharges from Magnetic Monopoles." In: *JHEP* 06 (2009), p. 037. arXiv: 0905.0633 [hep-ph].

[129] Michael R. Douglas, Daniel S. Park, and Christian Schnell. "The Cremmer-Scherk Mechanism in F-theory Compactifications on K3 Manifolds." In: *JHEP* 05 (2014), p. 135. arXiv: 1403.1595 [hep-th].

[130] B. de Wit, P. G. Lauwers, and Antoine Van Proeyen. "Lagrangians of N=2 Supergravity - Matter Systems." In: *Nucl. Phys.* B255 (1985), p. 569.

[131] L. Andrianopoli, M. Bertolini, Anna Ceresole, R. D'Auria, S. Ferrara, P. Fre, and T. Magri. "N=2 supergravity and N=2 superYang-Mills theory on general scalar manifolds: Symplectic covariance, gaugings and the momentum map." In: *J. Geom. Phys.* 23 (1997), pp. 111–189. arXiv: hep-th/9605032 [hep-th].

[132] Thomas W. Grimm and Jan Louis. "The Effective action of N = 1 Calabi-Yau orientifolds." In: *Nucl. Phys.* B699 (2004), pp. 387–426. arXiv: hep-th/0403067 [hep-th].

[133] Matthew Buican, Dmitry Malyshev, David R. Morrison, Herman Verlinde, and Martijn Wijnholt. "D-branes at Singularities, Compactification, and Hypercharge." In: *JHEP* 01 (2007), p. 107. arXiv: hep-th/0610007 [hep-th].

[134] Jarah Evslin and Luca Martucci. "D-brane networks in flux vacua, generalized cycles and calibrations." In: *JHEP* 07 (2007), p. 040. arXiv: hep-th/0703129 [HEP-TH].

[135] Javier Redondo. "The Low energy frontier: Probes with photons." In: *Proceedings, 43rd Rencontres de Moriond on Electroweak Interactions and Unified Theories*. 2008. arXiv: 0805.3112 [hep-ph].

[136] Joseph P. Conlon and Eran Palti. "On Gauge Threshold Corrections for Local IIB/F-theory GUTs." In: *Phys. Rev.* D80 (2009), p. 106004. arXiv: 0907.1362 [hep-th].

[137] Robert C. Mclean. "Deformations Of Calibrated Submanifolds." In: *Commun. Analy. Geom* 6 (1996), pp. 705–747.

[138] Jaume Gomis, Fernando Marchesano, and David Mateos. "An Open string landscape." In: *JHEP* 11 (2005), p. 021. arXiv: hep-th/0506179 [hep-th].

[139] M. Kawasaki, Masahide Yamaguchi, and T. Yanagida. "Natural chaotic inflation in supergravity." In: *Phys. Rev. Lett.* 85 (2000), pp. 3572–3575. arXiv: hep-ph/0004243 [hep-ph].

[140] Renata Kallosh, Andrei Linde, and Tomas Rube. "General inflaton potentials in supergravity." In: *Phys. Rev.* D83 (2011), p. 043507. arXiv: 1011.5945 [hep-th].

[141] Dagoberto Escobar, Aitor Landete, Fernando Marchesano, and Diego Regalado. “Large field inflation from D-branes.” In: *Phys. Rev.* D93.8 (2016), p. 081301. arXiv: 1505.07871 [hep-th].

[142] Dagoberto Escobar, Aitor Landete, Fernando Marchesano, and Diego Regalado. “D6-branes and axion monodromy inflation.” In: *JHEP* 03 (2016), p. 113. arXiv: 1511.08820 [hep-th].

[143] Micha Berkooz, Michael R. Douglas, and Robert G. Leigh. “Branes intersecting at angles.” In: *Nucl. Phys.* B480 (1996), pp. 265–278. arXiv: hep-th/9606139 [hep-th].

[144] Alessandro Tomasiello. “Reformulating supersymmetry with a generalized Dolbeault operator.” In: *JHEP* 02 (2008), p. 010. arXiv: 0704.2613 [hep-th].

[145] Luca Martucci. “On moduli and effective theory of N=1 warped flux compactifications.” In: *JHEP* 05 (2009), p. 027. arXiv: 0902.4031 [hep-th].

[146] Dieter Lust, Fernando Marchesano, Luca Martucci, and Dimitrios Tsimpis. “Generalized non-supersymmetric flux vacua.” In: *JHEP* 11 (2008), p. 021. arXiv: 0807.4540 [hep-th].

[147] R. P. Thomas. “Moment maps, monodromy and mirror manifolds.” In: *Symplectic geometry and mirror symmetry. Proceedings, 4th KIAS Annual International Conference, Seoul, South Korea, August 14-18, 2000.* 2001, pp. 467–498. arXiv: math/0104196 [math-dg].

[148] Luca Martucci. “D-branes on general N=1 backgrounds: Superpotentials and D-terms.” In: *JHEP* 06 (2006), p. 033. arXiv: hep-th/0602129 [hep-th].

[149] Steven B. Giddings, Shamit Kachru, and Joseph Polchinski. “Hierarchies from fluxes in string compactifications.” In: *Phys. Rev.* D66 (2002), p. 106006. arXiv: hep-th/0105097 [hep-th].

[150] A. Font, L. E. Ibanez, and F. Marchesano. “Coisotropic D8-branes and model-building.” In: *JHEP* 09 (2006), p. 080. arXiv: hep-th/0607219 [hep-th].

[151] Tomasz R. Taylor and Cumrun Vafa. “R R flux on Calabi-Yau and partial supersymmetry breaking.” In: *Phys. Lett.* B474 (2000), pp. 130–137. arXiv: hep-th/9912152 [hep-th].

[152] Sergei Gukov. “Solitons, superpotentials and calibrations.” In: *Nucl. Phys.* B574 (2000), pp. 169–188. arXiv: hep-th/9911011 [hep-th].

[153] Jean-Pierre Derendinger, Costas Kounnas, P. Marios Petropoulos, and Fabio Zwirner. “Superpotentials in IIA compactifications with general fluxes.” In: *Nucl. Phys.* B715 (2005), pp. 211–233. arXiv: hep-th/0411276 [hep-th].

[154] Shamit Kachru and Amir-Kian Kashani-Poor. “Moduli potentials in type IIa compactifications with RR and NS flux.” In: *JHEP* 03 (2005), p. 066. arXiv: hep-th/0411279 [hep-th].

[155] Wolfgang Lerche. “Special geometry and mirror symmetry for open string backgrounds with N = 1 supersymmetry.” In: *ICTP Spring School on Superstring Theory and Related Topics Trieste, Italy, March 31-April 8, 2003.* 2003. arXiv: hep-th/0312326 [hep-th].

[156] Donald Marolf. "Chern-Simons terms and the three notions of charge." In: *Quantization, gauge theory, and strings. Proceedings, International Conference dedicated to the memory of Professor Efim Fradkin, Moscow, Russia, June 5-10, 2000. Vol. 1+2.* 2000, pp. 312–320. arXiv: hep-th/0006117 [hep-th].

[157] Hans Jockers and Jan Louis. "The Effective action of D7-branes in N = 1 Calabi-Yau orientifolds." In: *Nucl. Phys.* B705 (2005), pp. 167–211. arXiv: hep-th/0409098 [hep-th].

[158] Philip Candelas and Xenia de la Ossa. "Moduli Space of Calabi-Yau Manifolds." In: *Nucl. Phys.* B355 (1991), pp. 455–481.

[159] Nathan Seiberg. "Five-dimensional SUSY field theories, nontrivial fixed points and string dynamics." In: *Phys.Lett.* B388 (1996), pp. 753–760. arXiv: hep-th/9608111 [hep-th].

[160] Marcos Marino. "Chern-Simons theory and topological strings." In: *Rev. Mod. Phys.* 77 (2005), pp. 675–720. arXiv: hep-th/0406005 [hep-th].

[161] Marcos Marino. "Les Houches lectures on matrix models and topological strings." In: 2004. arXiv: hep-th/0410165 [hep-th].

[162] Andrew Neitzke and Cumrun Vafa. "Topological strings and their physical applications." In: (2004). arXiv: hep-th/0410178 [hep-th].

[163] Marcel Vonk. "A Mini-course on topological strings." In: (2005). arXiv: hep-th/0504147 [hep-th].

[164] Murad Alim. "Lectures on Mirror Symmetry and Topological String Theory." In: (2012). arXiv: 1207.0496 [hep-th].

[165] Nikita A. Nekrasov. "Seiberg-Witten prepotential from instanton counting." In: *Adv.Theor.Math.Phys.* 7 (2004), pp. 831–864. arXiv: hep-th/0206161 [hep-th].

[166] Nikita Nekrasov and Andrei Okounkov. "Seiberg-Witten theory and random partitions." In: (2003). arXiv: hep-th/0306238 [hep-th].

[167] Amer Iqbal. "All genus topological string amplitudes and five-brane webs as Feynman diagrams." In: (2002). arXiv: hep-th/0207114 [hep-th].

[168] Mina Aganagic, Albrecht Klemm, Marcos Marino, and Cumrun Vafa. "The Topological vertex." In: *Commun.Math.Phys.* 254 (2005), pp. 425–478. arXiv: hep-th/0305132 [hep-th].

[169] Hidetoshi Awata and Hiroaki Kanno. "Instanton counting, Macdonald functions and the moduli space of D-branes." In: *JHEP* 0505 (2005), p. 039. arXiv: hep-th/0502061 [hep-th].

[170] Amer Iqbal, Can Kozcaz, and Cumrun Vafa. "The Refined topological vertex." In: *JHEP* 0910 (2009), p. 069. arXiv: hep-th/0701156 [hep-th].

[171] Naichung Conan Leung and Cumrun Vafa. "Branes and toric geometry." In: *Adv.Theor.Math.Phys.* 2 (1998), pp. 91–118. arXiv: hep-th/9711013 [hep-th].

[172] Ofer Aharony and Amihay Hanany. "Branes, superpotentials and superconformal fixed points." In: *Nucl.Phys.* B504 (1997), pp. 239–271. arXiv: hep-th/9704170 [hep-th].

[173] Ofer Aharony, Amihay Hanany, and Barak Kol. "Webs of (p,q) five-branes, five-dimensional field theories and grid diagrams." In: *JHEP* 9801 (1998), p. 002. arXiv: hep-th/9710116 [hep-th].

[174] Francesco Benini, Sergio Benvenuti, and Yuji Tachikawa. "Webs of five-branes and N=2 superconformal field theories." In: *JHEP* 0909 (2009), p. 052. arXiv: 0906.0359 [hep-th].

[175] Davide Gaiotto. "N=2 dualities." In: *JHEP* 1208 (2012), p. 034. arXiv: 0904.2715 [hep-th].

[176] Yuji Tachikawa. "A review of the T_N theory and its cousins." In: *PTEP* 2015.11 (2015), 11B102. arXiv: 1504.01481 [hep-th].

[177] Oliver DeWolfe, Amihay Hanany, Amer Iqbal, and Emanuel Katz. "Five-branes, seven-branes and five-dimensional E(n) field theories." In: *JHEP* 9903 (1999), p. 006. arXiv: hep-th/9902179 [hep-th].

[178] Yasuhiko Yamada and Sung-Kil Yang. "Affine seven-brane backgrounds and five-dimensional E(N) theories on S**1." In: *Nucl.Phys.* B566 (2000), pp. 642–660. arXiv: hep-th/9907134 [hep-th].

[179] Oren Bergman, Diego Rodríguez-Gómez, and Gabi Zafrir. "Discrete θ and the 5d superconformal index." In: *JHEP* 1401 (2014), p. 079. arXiv: 1310.2150 [hep-th].

[180] Ling Bao, Vladimir Mitev, Elli Pomoni, Masato Taki, and Futoshi Yagi. "Non-Lagrangian Theories from Brane Junctions." In: *JHEP* 1401 (2014), p. 175. arXiv: 1310.3841 [hep-th].

[181] Hirotaka Hayashi, Hee-Cheol Kim, and Takahiro Nishinaka. "Topological strings and 5d T_N partition functions." In: *JHEP* 1406 (2014), p. 014. arXiv: 1310.3854 [hep-th].

[182] Oren Bergman, Diego Rodríguez-Gómez, and Gabi Zafrir. "5-Brane Webs, Symmetry Enhancement, and Duality in 5d Supersymmetric Gauge Theory." In: *JHEP* 1403 (2014), p. 112. arXiv: 1311.4199 [hep-th].

[183] Masato Taki. "Notes on Enhancement of Flavor Symmetry and 5d Superconformal Index." In: (2013). arXiv: 1310.7509 [hep-th].

[184] Gabi Zafrir. "Duality and enhancement of symmetry in 5d gauge theories." In: *JHEP* 1412 (2014), p. 116. arXiv: 1408.4040 [hep-th].

[185] Oren Bergman and Gabi Zafrir. "Lifting 4d dualities to 5d." In: *JHEP* 1504 (2015), p. 141. arXiv: 1410.2806 [hep-th].

[186] Oren Bergman and Gabi Zafrir. "5d fixed points from brane webs and O7-planes." In: *JHEP* 12 (2015), p. 163. arXiv: 1507.03860 [hep-th].

[187] Masato Taki. "Seiberg Duality, 5d SCFTs and Nekrasov Partition Functions." In: (2014). arXiv: 1401.7200 [hep-th].

[188] Justin Kinney, Juan Martin Maldacena, Shiraz Minwalla, and Suvrat Raju. "An Index for 4 dimensional super conformal theories." In: *Commun. Math. Phys.* 275 (2007), pp. 209–254. arXiv: hep-th/0510251 [hep-th].

[189] Jyotirmoy Bhattacharya, Sayantani Bhattacharyya, Shiraz Minwalla, and Suvrat Raju. "Indices for Superconformal Field Theories in 3,5 and 6 Dimensions." In: *JHEP* 02 (2008), p. 064. arXiv: 0801.1435 [hep-th].

[190] Hee-Cheol Kim, Sung-Soo Kim, and Kimyeong Lee. "5-dim Superconformal Index with Enhanced En Global Symmetry." In: *JHEP* 1210 (2012), p. 142. arXiv: 1206.6781 [hep-th].

[191] Davide Gaiotto and Shlomo S. Razamat. "Exceptional Indices." In: *JHEP* 1205 (2012), p. 145. arXiv: 1203.5517 [hep-th].

[192] Davide Gaiotto, Leonardo Rastelli, and Shlomo S. Razamat. "Bootstrapping the superconformal index with surface defects." In: *JHEP* 1301 (2013), p. 022. arXiv: 1207.3577 [hep-th].

[193] Joseph A. Minahan and Dennis Nemeschansky. "Superconformal fixed points with E(n) global symmetry." In: *Nucl. Phys.* B489 (1997), pp. 24–46. arXiv: hep-th/9610076 [hep-th].

[194] Chiung Hwang, Joonho Kim, Seok Kim, and Jaemo Park. "General instanton counting and 5d SCFT." In: (2014). arXiv: 1406.6793 [hep-th].

[195] Amihay Hanany and Edward Witten. "Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics." In: *Nucl.Phys.* B492 (1997), pp. 152–190. arXiv: hep-th/9611230 [hep-th].

[196] David R. Morrison and Nathan Seiberg. "Extremal transitions and five-dimensional supersymmetric field theories." In: *Nucl.Phys.* B483 (1997), pp. 229–247. arXiv: hep-th/9609070 [hep-th].

[197] Michael R. Douglas, Sheldon H. Katz, and Cumrun Vafa. "Small instantons, Del Pezzo surfaces and type I-prime theory." In: *Nucl.Phys.* B497 (1997), pp. 155–172. arXiv: hep-th/9609071 [hep-th].

[198] Kenneth A. Intriligator, David R. Morrison, and Nathan Seiberg. "Five-dimensional supersymmetric gauge theories and degenerations of Calabi-Yau spaces." In: *Nucl.Phys.* B497 (1997), pp. 56–100. arXiv: hep-th/9702198 [hep-th].

[199] Francesco Benini, Richard Eager, Kentaro Hori, and Yuji Tachikawa. "Elliptic genera of two-dimensional N=2 gauge theories with rank-one gauge groups." In: *Lett. Math. Phys.* 104 (2014), pp. 465–493. arXiv: 1305.0533 [hep-th].

[200] Francesco Benini, Richard Eager, Kentaro Hori, and Yuji Tachikawa. "Elliptic Genera of 2d $\mathcal{N} = 2$ Gauge Theories." In: *Commun. Math. Phys.* 333.3 (2015), pp. 1241–1286. arXiv: 1308.4896 [hep-th].

[201] Amer Iqbal and Cumrun Vafa. "BPS Degeneracies and Superconformal Index in Diverse Dimensions." In: *Phys.Rev.* D90.10 (2014), p. 105031. arXiv: 1210.3605 [hep-th].

[202] Can Kozcaz, Sara Pasquetti, and Niclas Wyllard. "A & B model approaches to surface operators and Toda theories." In: *JHEP* 1008 (2010), p. 042. arXiv: 1004.2025 [hep-th].

[203] Hirotaka Hayashi and Gianluca Zoccarato. "Exact partition functions of Higgsed 5d T_N theories." In: *JHEP* 1501 (2015), p. 093. arXiv: 1409.0571 [hep-th].

[204] Michael R. Douglas, David A. Lowe, and John H. Schwarz. "Probing F theory with multiple branes." In: *Phys.Lett.* B394 (1997), pp. 297–301. arXiv: hep-th/9612062 [hep-th].

[205] Sung-Soo Kim and Futoshi Yagi. "5d E_n Seiberg-Witten curve via toric-like diagram." In: (2014). arXiv: 1411.7903 [hep-th].

[206] Sheldon Katz and David R. Morrison. "Gorenstein Threefold Singularities with Small Resolutions via Invariant Theory for Weyl Groups." In: (1992). eprint: arXiv:alg-geom/9202002.

[207] Thomas W. Grimm and Timo Weigand. "On Abelian Gauge Symmetries and Proton Decay in Global F-theory GUTs." In: *Phys. Rev.* D82 (2010), p. 086009. arXiv: 1006.0226 [hep-th].

[208] Hirotaka Hayashi, Teruhiko Kawano, Yoichi Tsuchiya, and Taizan Watari. "Flavor Structure in F-theory Compactifications." In: *JHEP* 08 (2010), p. 036. arXiv: 0910.2762 [hep-th].

[209] Sven Krause, Christoph Mayrhofer, and Timo Weigand. "G_4 flux, chiral matter and singularity resolution in F-theory compactifications." In: *Nucl. Phys.* B858 (2012), pp. 1–47. arXiv: 1109.3454 [hep-th].

[210] Mboyo Esole and Shing-Tung Yau. "Small resolutions of SU(5)-models in F-theory." In: *Adv. Theor. Math. Phys.* 17.6 (2013), pp. 1195–1253. arXiv: 1107.0733 [hep-th].

[211] Joseph Marsano and Sakura Schafer-Nameki. "Yukawas, G-flux, and Spectral Covers from Resolved Calabi-Yau's." In: *JHEP* 11 (2011), p. 098. arXiv: 1108.1794 [hep-th].

[212] Hirotaka Hayashi, Craig Lawrie, and Sakura Schafer-Nameki. "Phases, Flops and F-theory: SU(5) Gauge Theories." In: *JHEP* 10 (2013), p. 046. arXiv: 1304.1678 [hep-th].

[213] Craig Lawrie and Sakura SchŁfer-Nameki. "The Tate Form on Steroids: Resolution and Higher Codimension Fibers." In: *JHEP* 04 (2013), p. 061. arXiv: 1212.2949 [hep-th].

[214] Andreas P. Braun and Taizan Watari. "On Singular Fibres in F-Theory." In: *JHEP* 07 (2013), p. 031. arXiv: 1301.5814 [hep-th].

[215] Li-Sheng Tseng and Shing-Tung Yau. "Cohomology and Hodge Theory on Symplectic Manifolds: I." In: (2009). eprint: arXiv:0909.5418.

[216] Paul Koerber. "Stable D-branes, calibrations and generalized Calabi-Yau geometry." In: *JHEP* 08 (2005), p. 099. arXiv: hep-th/0506154 [hep-th].

[217] Luca Martucci and Paul Smyth. "Supersymmetric D-branes and calibrations on general N=1 backgrounds." In: *JHEP* 11 (2005), p. 048. arXiv: hep-th/0507099 [hep-th].

[218] Paul Koerber and Dimitrios Tsimpis. "Supersymmetric sources, integrability and generalized-structure compactifications." In: *JHEP* 08 (2007), p. 082. arXiv: 0706.1244 [hep-th].

[219] Gil R. Cavalcanti. "New aspects of the ddc-lemma." In: (2005). eprint: arXiv:math/0501406.

[220] Rajesh Gopakumar and Cumrun Vafa. "M theory and topological strings. 1." In: (1998). arXiv: hep-th/9809187 [hep-th].

[221] Rajesh Gopakumar and Cumrun Vafa. "M theory and topological strings. 2." In: (1998). arXiv: hep-th/9812127 [hep-th].

[222] Jian Zhou. "A Conjecture on Hodge Integrals." In: (2003). arXiv: math.AG/0310282 [math.AG].