Universidad Autónoma de Madrid — Departamento de Matemáticas

Tesis Doctoral

K-spectral sets, operator tuples and related function theory

Daniel Estévez Sánchez

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Director: Dmitry Yakubovich Lazarev

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… and thanks for all the fish.

Douglas Adams
The Hitchhiker's Guide to the Galaxy

a Concepción

Introducción: resumen y conclusiones

Antes de pasar a una descripción de los resultados principales de esta tesis, haremos un breve resumen de algunas nociones relevantes en la Teoría de Operadores.

Teorema Espectral para operadores normales y teoría de modelos de Nagy y Foias para contracciones en un espacio de Hilbert

Uno de los resultados más importantes en la teoría espectral de operadores lineales en un espacio de Hilbert es el teorema espectral para operadores normales, en especial el modelo de integral directa de von Neumann. Si N es un operador normal en un espacio de Hilbert separable y denotamos su espectro como $\sigma(N)$, entonces N es unitariamente equivalente al operador M_z de multiplicación por la variable z actuando en un espacio H que viene dado como integral directa

$$H = \int_{\sigma(N)}^{\oplus} H(z)\, d\mu(z).$$

Recordamos que la equivalencia unitaria de N y M_z significa que hay un operador unitario U tal que $N = UM_zU^*$, así que N y M_z son esencialmente el mismo operador desde el punto de vista de la teoría de operadores. En el caso particular en el que el espectro de N es simple, es decir, cuando $H(z)$ tiene dimensión 1 para todo $z \in \sigma(N)$, entonces H es el espacio $L^2(\mu)$.

Este teorema es una generalización infinito-dimensional del teorema bien conocido que afirma que toda matriz normal puede ser diagonalizada por medio de una base ortonormal. Muchas preguntas sobre el operador N pueden ser formuladas en términos del operador M_z, que tiene una expresión más sencilla y es más fácil de entender en términos del espacio de funciones H, que es un espacio L^2 de Lebesgue de funciones con valores vectoriales con respecto a la medida μ. De este modo, en cierto sentido, el estudio de operadores normales se reduce al estudio de las funciones medibles.

El teorema espectral tiene muchas consecuencias diversas: una construcción de un cálculo funcional $L^\infty(\mu)$ (y por tanto, en particular, la definición de raíces, logaritmos de N, etc.), una descripción de todos los operadores que conmutan con N, de subespacios invariantes e hiperinvariantes, una prueba transparente de teoremas ergódicos para un operador unitario, etc. Es una base para el estudio de procesos estocásticos y se usa en problemas clásicos de momentos, teoría de la dispersión, mecánica cuántica...

No existe una teoría tan extensa para cualquier otra clase de operadores. Quizá la teoría espectral más exitosa para operadores no normales sea la teoría espectral de contracciones en un espacio de Hilbert de Sz.-Nagy y Foias. Recordamos que se dice que T es una contracción si $\|T\| \le 1$. La teoría de Sz.-Nagy y Foias usa la dilatación isométrica para construir un modelo unitariamente equivalente para una contracción T en términos del operador M_z actuando en un cierto espacio H^2 de Hardy de funciones analíticas en el disco unidad $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Formulemos este modelo de manera más precisa en el caso especial en el que $T^{*n} \to 0$ en la topología fuerte cuando $n \to \infty$. En este caso el modelo resulta más sencillo de escribir.

Si U es un espacio de Hilbert, denotamos por $H^2(U)$ el espacio H^2 de Hardy de funciones definidas en $\mathbb{D}$ con valores en U. La forma más fácil de escribir este espacio es como el espacio de series de potencias de cuadrados sumables con coeficientes vectoriales en U. El espacio modelo de T se escribe como $\mathcal{M} = H^2(U) \ominus \Theta_T H^2(Y)$. Aquí, Θ_T es la función característica de T, que es una función analítica y acotada definida en $\mathbb{D}$ y cuyos valores son operadores que van de Y a U. La función Θ_T es interna, lo que significa que sus valores frontera son isométricos en casi todo punto de $\mathbb{T} = \partial\mathbb{D}$. El operador T es unitariamente equivalente a la compresión al espacio modelo $\mathcal{M}$ del operador M_z de multiplicación por la variable z en $H^2(U)$. Esta compresión es el operador $P_{\mathcal{M}} M_z|\mathcal{M}$. En el caso en el que T tiene defectos finitos (esto es, si $I - TT^*$ e $I - T^*T$ tienen rango finito), los espacios U e Y son finito-dimensionales, por lo que Θ_T es una función con valores matriciales.

Empleando el modelo analítico, diversas preguntas sobre la contracción T pueden formularse en términos de funciones analíticas y la función característica Θ_T. Por ejemplo, existe una relación entre los subespacios invariantes de T y ciertas factorizaciones de Θ_T. También es posible usar el modelo para definir un cálculo funcional para T para una clase amplia de funciones analíticas y acotadas en $\mathbb{D}$. En el caso en el que T no tiene parte unitaria, esta clase es todo el espacio $H^\infty(\mathbb{D})$ de funciones analíticas y acotadas en $\mathbb{D}$. Referimos al lector al libro de Sz.-Nagy y Foias [SNFBK10] para más información sobre la teoría espectral para contracciones.

Desigualdad de von Neumann, conjuntos K-espectrales y nociones relacionadas

Una de las consecuencias de este cálculo funcional $H^\infty(\mathbb{D})$ es la desigualdad de von Neumann. Si T es una contracción, entonces

$$\|p(T)\| \leq \max_{|z|\leq 1} |p(z)|,$$

para cualquier polinomio p con coeficientes complejos. Cuando pasamos de la teoría de un solo operador a la teoría de varios operadores que conmutan, es interesante considerar el análogo natural de la desigualdad de von Neumann. Si $T_1, \dots, T_n$ es una tupla de contracciones que conmutan, decimos que satisfacen la desigualdad de von Neuman si

$$\|p(T_1, \dots, T_n)\| \leq \max_{|z_1|\leq 1, \dots, |z_n|\leq 1} |p(z_1, \dots, z_n)| \tag{1}$$

se cumple para cualquier polinomio complejo p en n variables. En lo que sigue usaremos la notación

$$\|p\|_\infty := \max_{|z_1|\leq 1, \dots, |z_n|\leq 1} |p(z_1, \dots, z_n)|.$$

Como ya hemos comentado, en el caso $n = 1$, la desigualdad de von Neumann se cumple para toda contracción T, como consecuencia de la teoría de Sz.-Nagy y Foias. En particular, es muy fácil probar esto usando la existencia de una dilatación unitaria, la cual fue descubierta por Sz.-Nagy en [SN53]. Sin embargo, von Neumann probó originalmente la desigualdad en [vN51], antes de los resultados de Sz.-Nagy y Foias, por lo que tuvo que usar otras técnicas. Definiremos la dilatación unitaria y mostraremos cómo puede usarse para probar la desigualdad de von Neumann más adelante en esta introducción.

Ando resolvió el caso $n = 2$ en [And63] probando que cualquier par de contracciones que conmutan tiene una dilatación unitaria. Como en el caso de un solo operador, una consecuencia sencilla de esto es que (1) se cumple para todo par de contracciones que conmutan T_1, T_2.

Sin embargo, el caso $n \geq 3$ es distinto. Kaijser y Varopoulos [Var74] y Crabb y Davie [CD75] encontraron de manera independiente ejemplos de tres matrices contractivas que conmutan y que no cumplen la desigualdad de von Neumann (1). Por tanto, la desigualdad de von Neumann no se cumple en general para tres o más contracciones que conmutan. Aun así, hay casos especiales y familias de contracciones para las que la desigualdad sí se cumple. Por ejemplo, es trivial ver que se cumple cuando $T_1, \ldots, T_n$ son de la forma $T_j = \varphi_j(S_1, S_2)$, donde S_1, S_2 son dos contracciones fijas que conmutan y φ_j son funciones analíticas en $\overline{\mathbb{D}}$ y tales que $|\varphi_j| \leq 1$ en $\mathbb{D}$. También mencionamos los resultados positivos de Lotto [Lot94], Grinshpan, Kaliuzhnyi-Verbovetski, Vinnikov and Woerdeman [GKVVW09] y Hartz [Har15].

Todavía resulta misterioso qué es lo que hace que la teoría de tres o más operadores que conmutan sea fundamentalmente distinta de la teoría de pares de operadores que conmutan. Una de las preguntas abiertas más importantes es si la desigualdad de von Neumann se cumple con una constante. En otras palabras, si exsite una constante finita C que depende solo de n tal que

$$\|p(T_1, \ldots, T_n)\| \leq C\|p\|_\infty \tag{2}$$

para toda tupla de contracciones que conmutan $T_1, \ldots, T_n$ y todo polinomio p. La respuesta se desconoce incluso en el caso más simple $n = 3$. Probablemente la impresión más extendida entre la comunidad es que no existe tal constante finita C, incluso para $n = 3$.

Una noción relacionada con la desigualdad de von Neumann es la de conjunto K-espectral. Si X es un subconjunto compacto de $\mathbb{C}$, T es un operador con espectro contenido en X y $K \geq 1$, decimos que X es un conjunto K-espectral para T si la desigualdad

$$\|f(T)\| \leq K \sup_{z \in X} |f(z)| \tag{3}$$

se cumple para cualquier función racional f sin polos en X. Esto puede verse como una generalización de la desigualdad de von Neumann para dominios distintos de $\overline{\mathbb{D}}$. De hecho, la desigualdad de von Neumann para T es equivalente a la condición de que $\overline{\mathbb{D}}$ sea un conjunto 1-espectral para T. En esta noción, consideramos funciones racionales en lugar de solamente polinomios porque X puede no ser simplemente conexo.

El álgebra de Agler es el algebra de todas las funciones analíticas f en $\mathbb{D}^n$ tales que

$$\sup_{\|T_1\|<1, \ldots, \|T_n\|<1} \|f(T_1, \ldots, T_n)\| < \infty,$$

donde el supremo recorre todas las tuplas de n contracciones estrictas que conmutan $T_1, \ldots, T_n$. La desigualdad (2) se cumple para todas las tuplas de contracciones que conmutan si y solo si el álgebra de Agler es la misma álgebra que $H^\infty(\mathbb{D}^n)$ (con una norma equivalente). Obsérvese que el álgebra de Agler está contenida en $H^\infty(\mathbb{D}^n)$, así que la pregunta es si esta inclusión es propia o no.

Para intentar probar que (2) es falso, se debe encontrar una serie de ejemplos que constan de una terna de contracciones que conmutan T_1, T_2, T_3 y un polinomio p tales que los cocientes

$$\frac{\|p(T_1, T_2, T_3)\|}{\|p\|_\infty}$$

tiendan a infinito. Hay diversos artículos en la literatura que intentan encontrar ejemplos en los que este cociente sea tan grande como sea posible. Sin embargo, hasta ahora el mejor ejemplo conocido, por Grinshpan, Kaliuzhnyi-Verbovetskyi y Wordeman [GKVW13], da un cociente de solo alrededor de 1,23. Esto está todavía muy lejos de intentar probar que (2) no se cumple para ninguna constante C.

La dilatación unitaria de una o varias contracciones que conmutan

Aquí definimos y estudiamos las dilataciones unitarias, que son un concepto estrechamente relacionado con la desigualdad de von Neumann. Si $T_1, \ldots, T_n$ son contracciones que conmutan en un espacio de Hilbert H, decimos que $U_1, \ldots, U_n$ es una dilatación unitaria de $T_1, \ldots, T_n$ si $U_1, \ldots, U_n$ son operadores unitarios que conmutan y que actúan en un espacio de Hilbert más grande $\mathcal{K} \supset H$ y

$$T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} = P_H U_1^{k_1} U_2^{k_2} \cdots U_n^{k_n} | H$$

se cumple para cualquier elección de enteros no negativos $k_1, \ldots, k_n$. Aquí P_H denota la proyección ortogonal de $\mathcal{K}$ sobre H. Si $U_1, \ldots, U_n$ son isometrías en lugar de unitarios, hablamos de una dilatación isométrica. Si la tupla $(U_1, \ldots, U_n)$ es una dilatación de $(T_1, \ldots, T_n)$, se dice que $(T_1, \ldots, T_n)$ es una *compresión* de $(U_1, \ldots, U_n)$. Se conoce que $(T_1, \ldots, T_n)$ es una compresión de $(U_1, \ldots, U_n)$ si y solo si los operadores $U_1, \ldots, U_n$ tienen la estructura

$$U_j = \begin{bmatrix} * & 0 & 0 \\ * & T_j & 0 \\ * & * & * \end{bmatrix}, \tag{4}$$

$j = 1, \ldots, n$, con respecto a una descomposición ortogonal $\mathcal{K} = \mathcal{K}_1 \oplus H \oplus \mathcal{K}_2$. Obsérvese que en este caso tanto $\mathcal{K}_2$ como $H \oplus \mathcal{K}_2$ son invariantes para U_j.

Si $T_1, \ldots, T_n$ tienen una dilatación unitaria, entonces es muy fácil probar que satisfacen la desigualdad de von Neumann. En efecto,

$$\|p(T_1, \ldots, T_n)\| = \|P_H p(U_1, \ldots, U_n)|H\| \leq \|p(U_1, \ldots, U_n)\| \leq \|p\|_\infty,$$

donde la última igualdad se obtiene del cálculo funcional para operadores unitarios.

Como ya hemos mencionado, Sz.-Nagy probó en [SN53] que cualquier contracción tiene una dilatación unitaria y Ando probó en [And63] que cualquier par de contracciones que conmutan tiene una dilatación unitaria. Parrott encontró en [Par70] el primer ejemplo de tres contracciones que conmutan y que no tienen dilatación unitaria. Casualmente, sus tres contracciones sí satisfacen la desigualdad de von Neumann. Por tanto, la existencia de una dilatación unitaria implica la desigualdad de von Neumann, pero el recíproco en general es falso.

Para formular un enunciado de tipo recíproco que sea cierto, se necesita considerar desigualdades de von Neumann con valores matriciales. Si $p = [p_{jk}]$ es un polinomio complejo en n variables cuyos valores son matrices $s \times s$, podemos definir el operador $p(T_1, \dots, T_n)$ como el operador que actúa en H^s, la suma directa de s copias de H, y viene dado en forma de bloques como $p(T_1, \dots, T_n) = [p_{jk}(T_1, \dots, T_n)]$. Entonces, podemos estudiar si

$$\|p(T_1, \dots, T_n)\| \leq \max_{|z_1|\leq 1, \dots, |z_n|\leq 1} \|p(z_1, \dots, z_n)\|$$

se cumple para todo $s \geq 1$ y todo polinomio p cuyos valores son matrices $s \times s$. Aquí, $\|p(z_1, \dots, z_n)\|$ es la norma de la matriz $p(z_1, \dots, z_n)$ vista como un operador actuando en el espacio de Hilbert $\mathbb{C}^s$. Esta desigualdad se llama desigualdad de von Neumann con valores matriciales.

Es posible aplicar un teorema de Arveson sobre la extensión de aplicaciones completamente positivas para probar que $T_1, \dots, T_n$ satisfacen la desigualdad de von Neumann con valores matriciales si y solo si $T_1, \dots, T_n$ tienen una dilatación unitaria. Véase, por ejemplo, [AM02, Corollary 14.16] o [Pau02, Corollary 7.7].

Una aproximación posible al estudio de si la desigualdad de von Neumann se cumple con una constante, como en (2), consiste en restringirse al estudio de clases particulares de contracciones. Fijamos n y denotamos por $\mathfrak{C}$ el conjunto de todas las n-tuplas de contracciones que conmutan. Si X es una subfamilia de $\mathfrak{C}$, definimos

$$\|p\|_X = \sup\{\|p(T_1, \dots, T_n)\| : (T_1, \dots, T_n) \in X\}.$$

Resulta de interés encontrar conjuntos X grandes tales que $\|p\|_X = \|p\|_\infty$ para todo p, y conjuntos X pequeños tales que $\|p\|_X = \|p\|_{\mathfrak{C}}$ para todo p.

Drury probó en [Dru83] que si N es el conjunto de todas las n-tuplas de matrices (finito-dimensionales) nilpotentes que conmutan, entonces $\|p\|_N = \|p\|_{\mathfrak{C}}$ para todo p. En otras palabras, si uno desea estudiar si (2) se cumple, entonces solamente es necesario comprobar con matrices nilpotentes.

Estructura general de la tesis

La tesis está dividida en dos partes. En la Parte I probamos varios teoremas sobre conjuntos completamente K-espectrales y la semejanza a una contracción. Empleamos álgebras de Banach, una extensión de las técnicas de separación de singularidades de Havin, Nersessian y Ortega-Cerdà [HN01, Hav04, HNOC07] y algunos argumentos de teoría de operadores que están relacionados con aplicaciones completamente acotadas. Nuestros teoremas pueden aplicarse al estudio de la desigualdad de von Neumann con una constante en caso en el que las contracciones son de la forma $T_j = \varphi_j(T)$, $j = 1, \dots, n$, donde $\sigma(T) \subset \mathbb{D}$ y $|\varphi_j| \leq 1$ en $\mathbb{D}$, pero T no es necesariamente una contracción. La Parte II es un desarrollo de algunas ideas no publicadas de Vinnikov y Yakubovich. Se introducen varios objetos novedosos en teoría de operadores ("pools" y "estructuras separadas"). Se muestra que estos objetos son naturales en el contexto de la teoría de tuplas de operadores que conmutan.

Resumen de la Parte I

Los resultados principales de la Parte I tratan sobre el concepto de conjunto completamente K-espectral. Esto es una versión con valores matriciales del concepto de conjunto K-espectral que hemos introducido anteriormente. Si X es un subconjunto compacto de $\mathbb{C}$ y T es un operador con espectro contenido en X, decimos que X es un conjunto completamente K-espectral para T si la desigualdad

$$\|f(T)\| \leq K \sup_{z \in X} \|f(z)\| \tag{5}$$

se cumple para toda función racional f cuyos valores son matrices $s \times s$ y sus polos están fuera de X y para todo $s \in \mathbb{N}$. La constante K debe ser independiente de s y de f. Aquí, el operador $f(T)$ está definido en forma de bloques por $f(T) = [f_{jk}(T)]$, donde $f = [f_{jk}]$. Esto está bien definido porque los polos de f se encuentran fuera del espectro de T. Obsérvese que (5) es un análogo de la desigualdad de von Neumann con valores matriciales y con una constante para un dominio distinto de $\overline{\mathbb{D}}$.

Como en el caso de la desigualdad de von Neumann con valores matriciales, la desigualdad (5) es equivalente a un cierto resultado de dilatación. Decimos que un operador T que actúa en H tiene una *dilatación racional* a ∂X si existe un operador normal N que actúa en un espacio de Hilbert más grande $\mathcal{K} \supset H$, que tiene espectro contenido en ∂X y es tal que

$$f(T) = P_H f(N)|H$$

se cumple para toda función racional f sin polos en X. Se conoce que T es semejante a un operador que tiene una dilatación normal a ∂X si y solo si X es un conjunto completamente K-espectral para T, para alguna constante K.

En la Parte I consideramos un dominio finitamente conexo $\Omega \subset \mathbb{C}$ y una colección de funciones analíticas $\varphi_1, \dots, \varphi_n : \overline{\Omega} \to \overline{\mathbb{D}}$. Imponemos unas ciertas condiciones geométricas y de regularidad a estas funciones. Si estas condiciones se cumplen, decimos que la tupla $(\varphi_1, \dots, \varphi_n)$ es admisible. La más importante de estas condiciones es que cada una de las funciones φ_k debe mandar cierto arco de $\partial\Omega$ biyectivamente sobre un arco de $\partial\mathbb{D}$. Denotamos este arco como J_k. La unión de los arcos J_k, $k = 1, \dots, n$, debe cubrir toda la frontera $\partial\Omega$.

Un ejemplo importante de una de una colección de funciones $\varphi_1, \dots, \varphi_n$ que satisfacen estas condiciones es el siguiente. Supongamos que

$$\Omega = \Omega_1 \cap \Omega_2 \cap \cdots \cap \Omega_n, \tag{6}$$

donde Ω_k son dominios de Jordan que se intersecan transversalmente y que tienen una frontera regular. Entonces las aplicaciones de Riemann $\varphi_k : \Omega_k \to \mathbb{D}$ son un ejemplo de funciones admisibles. Los arcos J_k son $J_k = \partial\Omega_k \cap \partial\Omega$ en este caso.

Los resultados principales de la Parte I son de la siguiente forma: *Si T es un operador tal que $\|\varphi_k(T)\| \leq 1$ para $k = 1, \dots, n$, entonces $\overline{\Omega}$ es un conjunto completamente K-espectral para T.* Probamos varios teoremas de esta forma. Las diferencias entre estos teoremas son algunas condiciones técnicas, como por ejemplo si se permite que el espectro de T toque $\partial\Omega$ o no. Llamamos a las funciones $\varphi_1, \dots, \varphi_n$ *funciones test*, porque pueden usarse como un test de K-espectralidad completa para T, comprobando si $\|\varphi_k(T)\| \leq 1$.

El Capítulo 1 está dedicado a una de las herramientas clave que usamos para probar estos resultados: una modificación de las técnicas de separación de singularidaddes de Havin, Nersessian y Ortega-Cerdà. Si Ω es como en (6), Havin, Nersessian y Ortega-Cerdà probaron que, bajo ciertas condiciones de regularidad, toda función $f \in H^\infty(\Omega)$ puede escribirse como $f = f_1 + f_2 + \cdots + f_n$, donde $f_j \in H^\infty(\Omega_j)$. Aquí tratamos un problema relacionado: si toda función $f \in H^\infty(\Omega)$ puede escribirse como

$$f = g_1 \circ \varphi_1 + g_2 \circ \varphi_2 + \cdots + g_n \circ \varphi_n, \tag{7}$$

con $g_j \in H^\infty(\mathbb{D})$. Si las funciones φ_j son univalentes, entonces estos dos problemas son equivalentes, pues puede ponerse $g_j = f_j \circ \varphi_j^{-1}$. Cuando φ_j no son univalentes, entonces no es posible en general escribir f como en (7).

Sin embargo, probamos que es posible encontrar aplicaciones lineales $f \mapsto g_j = g_j(f) \in H^\infty(\mathbb{D})$ tales que el operador lineal

$$f \mapsto f - (g_1(f) \circ \varphi_1 + g_2(f) \circ \varphi_2 + \cdots + g_n(f) \circ \varphi_n)$$

es compacto. Aplicando algunos resultados de teoría Fredholm, esto resulta ser suficiente para las aplicaciones a Teoría de Operadores que tenemos en mente.

En el Capítulo 2 aplicamos los resultados del Capítulo 1 al estudio de álgebras de funciones en curvas analíticas dentro del polidisco $\mathbb{D}^n$. Si consideramos la aplicación vectorial $\Phi = (\varphi_1, \ldots, \varphi_n) : \Omega \to \mathbb{D}^n$, su imagen $\mathcal{V} = \Phi(\Omega)$ es una curva analítica dentro del polidisco. Damos una caracterización del álgebra $H^\infty(\mathcal{V})$ de funciones analíticas y acotadas en $\mathcal{V}$ y usamos esta caracterización para probar que toda función $f \in \mathcal{V}$ se puede extender a una función F en el álgebra de Agler de $\mathbb{D}^n$, de modo que $F|\mathcal{V} = f$.

Estos son dos de los resultados principales de este capítulo:

Teorema (Theorem 2.1, página 23)**.** *Si $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ es admisible e inyectiva y Φ' no se anula en Ω, entonces*

$$H^\infty(\Omega) = \left\{ \sum_{j=1}^{l} \prod_{k=1}^{n} f_{j,k}(\varphi_k(z)) : l \in \mathbb{N}, f_{j,k} \in H^\infty(\mathbb{D}) \right\}.$$

La palabra *admisible* aquí se refiere a ciertas condiciones geométricas que hemos impuesto a Φ en el Capítulo 1.

Teorema (Theorem 2.4, página 23)**.** *Si $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ es admisible, entonces para toda $f \in H^\infty(\mathcal{V})$ existe una $F \in \mathcal{SA}(\mathbb{D}^n)$ tal que $F|\mathcal{V} = f$ y $\|F\|_{\mathcal{SA}(\mathbb{D}^n)} \leq C\|f\|_{H^\infty(\mathcal{V})}$, para alguna constante C que no depende de f.*

Aquí $\mathcal{SA}(\mathbb{D}^n)$ denota el álgebra de Agler de $\mathbb{D}^n$. Recordamos que su norma viene dada por

$$\|F\|_{\mathcal{SA}(\mathbb{D}^n)} = \sup_{\|T_1\|, \ldots, \|T_n\| \leq 1} \|F(T_1, \ldots, T_n)\|.$$

Hay muchos resultados en la literatura sobre la extensión de funciones que pertenecen a H^∞ de una curva analítica a una función en $H^\infty(\mathbb{D}^n)$ (o H^∞ de algún otro dominio en $\mathbb{C}^n$). Sin embargo, la extensión al álgebra de Agler es algo novedoso que no ha sido estudiado previamente.

El capítulo 3 está dedicado al enunciado y la prueba de nuestros resultados sobre conjuntos completamente K-espectrales. Aquí enunciamos algunos de los resultados principales de este capítulo.

Teorema (Theorem 3.1, página 39). *Sean $\Omega_1, \ldots, \Omega_n$ abiertos en $\widehat{\mathbb{C}}$ tales que la frontera de cada conjunto Ω_k, $k = 1, \ldots, n$, es una unión finita y disjunta de curvas de Jordan. También asumimos que las fronteras de los conjuntos Ω_k, $k = 1, \ldots, n$, son regulares Ahlfors y rectificables y que se intersecan transversalmente. Sea $\Omega = \Omega_1 \cap \cdots \cap \Omega_n$. Supongamos que $\sigma(T) \subset \overline{\Omega}$ y que una constante $K \geq 1$ está dada. Entonces existe una constante K' tal que*

(i) si cada uno de los conjuntos $\overline{\Omega}_j$, $j = 1, \ldots, n$, es K-espectral para T, entonces $\overline{\Omega}$ es también un conjunto K'-espectral para T; y

(ii) si cada uno de los conjuntos $\overline{\Omega}_j$, $j = 1, \ldots, n$, es completamente K-espectral para T, entonces $\overline{\Omega}$ es un conjunto completamente K'-espectral para T.

En ambos casos, K' depende solo de los conjuntos $\Omega_1, \ldots, \Omega_n$ y la constante K, pero no depende del operador T.

Este teorema es una generalización de un teorema de Badea, Beckermann y Crouzeix en [BBC09] que trata el caso en el que Ω_j son discos en la esfera de Riemann. En ese teorema, la constante que ellos obtienen depende solamente de n, el número de discos.

Teorema (Theorem 3.6, página 43). *Supongamos que $\Phi = (\varphi_1, \ldots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ es admisible y analítica en un entorno de $\overline{\Omega}$, donde Ω es un dominio de Jordan. Si $\sigma(T) \subset \overline{\Omega}$ y $\varphi_j(T)$ es semejante a una contracción para $j = 1, \ldots, n$, (es decir, $\|S_j \varphi_j(T) S_j^{-1}\| \leq 1$, para algún S_j), entonces $\overline{\Omega}$ es un conjunto completamente K-espectral para T, donde K depende de Ω, Φ, $\|S_j\| \cdot \|S_j^{-1}\|$ y T. Si además Φ es inyectiva y Φ' no se anula en $\overline{\Omega}$, entonces K puede tomarse que dependa solo de Ω, Φ y $\|S_j\| \cdot \|S_j^{-1}\|$, pero no de T.*

La definición de "función admisible" se dará en el Capítulo 1. Esencialmente, es una función que satisface ciertas condiciones geométricas y de regularidad, como hemos explicado anteriormente. En el Capítulo 3 daremos algunos resultados similares que tratan el caso de dominios no simplemente conexos.

En el Capítulo 4 damos una aplicación de alguntos de nuestros resultados sobre conjuntos completamente K-espectrales a operadores con espectro delgado. Consideramos operadores T cuyo espectro está contenido en una curva $\Gamma \subset \mathbb{C}$ suficientemente suave. Probamos que si la resolvente de T satisface ciertas condiciones de crecimiento, entonces T es semejante a un operador normal. Éste es uno de los resultados principales de este capítulo:

Teorema (Theorem 4.1, página 73). *Sea $\Gamma \subset \mathbb{C}$ una curva de Jordan de clase $C^{1+\alpha}$ y Ω el dominio que encierra. Sea $T \in \mathcal{B}(H)$ un operador con $\sigma(T) \subset \Gamma$. Supongamos que*

$$\|(T-\lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, \Gamma)}, \qquad \lambda \in U \setminus \overline{\Omega},$$

para algún conjunto abierto U que contenga $\partial\Omega$ y que

$$\|(T-\lambda)^{-1}\| \leq \frac{C}{\operatorname{dist}(\lambda, \Gamma)}, \qquad \lambda \in \Omega,$$

para alguna constante $C > 0$. Entonces T es semejante a un operador normal.

Este teorema está relacionado con un resultado de Stampfli en [Sta69] que afirma que si $\|(T-\lambda)^{-1}\| \leq \operatorname{dist}(\lambda, \Gamma)^{-1}$ para λ en un entorno de Γ, entonces T es normal. En la prueba de este teorema usamos el Teorema 3.3, que es un teorema sobre conjuntos completamente K-espectrales incluido en el Capítulo 3. El Teorema 3.3 es una generalización a dominios no convexos de un teorema de Putinar y Sandberg sobre conjuntos completamente K-espectrales convexos y el rango numérico. También puede verse este teorema como una generalización de las ρ-contracciones para un dominio distinto de $\mathbb{D}$.

En el Capítulo 4 también usamos nuestras técnicas para generalizar teoremas de van Casteren [vC80, vC83] y Naboko [Nab84] que trataban el caso $\Gamma = \mathbb{T}$ al caso en el que Γ es una curva de Jordan general de clase $C^{1+\alpha}$. Una de nuestras herramientas principales es un cálculo funcional basado en la fórmula de Cauchy-Green y la continuación pseudoanalítica. Este cálculo fue definido originalmente por Dynkin en [Dyn72]. Su cálculo nos ayuda a pasar entre operadores con espectro en una curva Γ y operadores con espectro en $\mathbb{T}$, manteniendo el control en las estimaciones de sus resolventes.

Resumen de la Parte II

La Parte II está dedicada a la definición y el estudio de lo que llamamos estructuras separadas. Relacionamos estas estructuras con la teoría de Livšic y Vinnikov de tuplas de operadores no autoadjuntos que conmutan, que fue desarrollada en una serie de artículos de Livšic en los años 60. El libro [LKMV95] está dedicado a una exposición sistemática de esta teoría. Como explicaremos más tarde, Livšic y Vinnikov definen una construcción llamada *"vessel" de operadores*. Uno de los objetos importantes asociados a un vessel es su curva discriminante, que es una curva algebraica.

En la literatura de la Teoría de Operadores hay por lo menos dos lugares donde aparecen las curvas algebraicas. El primero es la teoría de Livšic y Vinnikov. El segundo son los trabajos de Xia [Xia87a, Xia87b, Xia96] y Yakubovich [Yak98a, Yak98b] sobre operadores subnormales de tipo finito, donde se define cierto tipo de curva discriminante para el operador subnormal. La Parte II ofrece un marco común para estas dos teorías, las cuales parecían no tener conexión. Explicaremos cómo las construcciones de Xia y Yakubovich pueden ser escritas en términos de una estructura separada. Una relación entre estructuras separadas y vessels se obtiene por medio de la compresión generalizada, que definiremos más adelante.

En el Capítulo 5 repasamos la teoría de Livšic y Vinnikov. Esta teoría estudia tuplas de operadores $A_1, \dots, A_n$ que conmutan y son tales que sus partes imaginarias $(A_j - A_j^*)/(2i)$ tienen rango finito para todo $j = 1, \dots, n$. Observamos que en el caso particular en el que todos los operadores son disipativos, es decir, cuando

$$\operatorname{Im}(A_j) = \frac{A_j - A_j^*}{2i} \geq 0,$$

se puede aplicar la transformada de Cayley

$$T_j = (A_j - iI)(A_j + iI)^{-1}$$

para obtener una tupla $(T_1, \dots, T_n)$ de contracciones que conmutan.

Obsérvese que para estudiar la desigualdad de von Neumann basta considerar operadores T_j obtenidos de este modo, pues de hecho es suficiente estudiar operadores finito-dimensionales.

Como $A_j - A_j^*$ tienen rango finito, es posible encontrar un espacio de Hilbert E de dimensión finita, un operador $\Phi : H \to E$ (aquí H es el espacio donde actúan los operadores A_j) y matrices autoadjuntas σ_k, $k = 1, \ldots, n$, que actúan en E y tales que

$$\frac{1}{i}(A_k - A_k^*) = \Phi^* \sigma_k \Phi.$$

La teoría de Livšic y Vinnikov también usa matrices auxiliares autoadjuntas γ_{kj}^{in} y γ_{kj}^{out}, $j, k = 1, \ldots, n$, que actúan en E y que se llaman "gyrations". En cierto sentido, las matrices σ_j, γ_{kj}^{in} y γ_{kj}^{out} codifican la interacción entre los operadores A_j. La colección de los operadores A_j, la aplicación Φ y todas las matrices auxiliares se denomina *"vessel" de operadores*. Cada vessel tiene asociada una variedad discriminante, que es una variedad algebraica en $\mathbb{C}^n$. Esto da una conexión entre la Teoría de Operadores y la Geometría Algebraica. La teoría de Livšic y Vinnikov da resultados más definitivos para pares de operadores que conmutan A_1, A_2 que para n-tuplas. En este caso, la variedad discriminante es una curva algebraica en $\mathbb{C}^2$ que se llama curva discriminante. Es el conjunto de ceros del polinomio

$$p(z_1, z_2) = \det(z_1 \sigma_2 - z_2 \sigma_1 + \gamma_{12}^{\text{in}})$$

(esta fórmula se llama una representación determinantal de la curva algebraica).

Empleando la desingularización, la curva discriminante puede considerarse una unión disjunta de superficies de Riemann. Decimos que la curva es *separada* si cada una de estas superficies de Riemann se parte en dos componentes conexas cuando eliminamos el conjunto de sus puntos reales. Si la curva es separada, entonces podemos definir sus dos mitades.

En algunos casos, la curva discriminante de un vessel es separada (véase, por ejemplo [SV05]). Si esto ocurre, las dos mitades de la curva discriminante juegan un papel análogo al disco unidad y su complementario en el caso de la teoría de una sola contracción. De este modo, se puede pensar que el marco adecuado para estudiar varios operadores que conmutan es el de las variedades algebraicas.

Livšic y Vinnikov probaron que el espectro conjunto de la tupla $(A_1, \ldots, A_n)$ se contiene en su curva discriminante. Por otro lado, resulta fácil ver que el espectro conjunto de la tupla $(\varphi_1(T), \ldots, \varphi_n(T))$ que aparece en la Parte I se contiene en la variedad compleja de dimensión uno $\Phi(\mathbb{D})$. Esto muestra que hay una fuerte conexión entre la dos partes de esta tesis. En la Sección 3.8 describimos esta conexión. Esta sección es material nuevo que no apareció en el artículo [DEY15], en el que se basa el Capítulo 3. Probamos que basta estudiar la desigualdad de von Neumann para tuplas de contracciones que conmutan $(T_1, \ldots, T_n)$ donde $T_j = B_j(T)$, $j = 1, \ldots, n$, la matriz T es diagonalizable y con radio espectral menor que 1 (esto es, $\sigma(T) \subset \mathbb{D}$) y B_j son productos de Blaschke finitos. De hecho, es posible escoger B_j de modo que la aplicación $\Phi = (B_1, \ldots, B_n) : \overline{\mathbb{D}}^n \to \overline{\mathbb{D}}$ sea inyectiva y Φ' no se anule. Esto nos permite aplicar nuestros resultados del Capítulo 3.

Se puede pensar que la teoría de Sz.-Nagy y Foias para contracciones tiene dos partes. La primera es la teoría espectral de isometrías, que tienen un modelo analítico construido sobre los espacios H^2 de Hardy del disco unidad. La segunda parte es la relación entre contracciones e isometrías: uno puede pasar entre ellas usando compresiones y dilataciones. Esto permite construir un modelo analítico de una contracción usando el espacio H^2.

En el estudio de las tuplas de operadores que conmutan, la idea clave de la teoría de Livšic y Vinnikov es usar curvas algebraicas complejas en lugar del plano complejo. Los vessels pueden verse como un análogo de las contracciones, pero Livšic y Vinnikov no dan un homólogo de la teoría espectral de isometrías.

Los Capítulos 6 y 7 son un intento de construir este homólogo. En el Capítulo 6 comenzamos considerando una nueva estructura que llamamos *"pool" de operadores*. Su definición recuerda a la de un vessel de operadores. Un pool está formado por los siguientes objetos: un par de operadores autoadjuntos A_1, A_2 que conmutan en un espacio de Hilbert $\mathcal{K}$, un espacio de Hilbert finito dimensional auxiliar M, un operador $\Phi : \mathcal{K} \to M$ y matrices autoadjuntas $\sigma_1, \sigma_2, \gamma$ que satisfacen la llamada relación de tres términos:

$$\sigma_2 \Phi A_1 - \sigma_1 \Phi A_2 + \gamma \Phi = 0.$$

Una relación de esta forma también aparece en la teoría de vessels de operadores. La curva discriminante del pool se define como el conjunto de ceros del polinomio

$$p(z_1, z_2) = \det(z_1 \sigma_2 - z_2 \sigma_1 + \gamma),$$

de manera similar a la teoría de Livšic y Vinnikov. Obsérvese que un pool de operadores se construye alrededor de operadores autoadjuntos A_1, A_2, mientras que los operadores A_1, A_2 en un vessel son no autoadjuntos normalmente. Por tanto, aunque los pools de operadores y los vessels podrían parecer similares a primera vista, son conceptos bastante distintos.

Después introducimos nuestro objeto principal de estudio, al cual llamamos *estructuras separadas*. Éstas están formadas por un par de operadores autoadjuntos que conmutan A_1, A_2, que actúan en un espacio de Hilbert $\mathcal{K} = H_- \oplus H_+$ y tales que $P_{H_-} A_j P_{H_+}$, $j = 1, 2$, tienen rango finito. Una forma equivalente de escribir estas condiciones es que A_1, A_2 tengan la estructura

$$A_j = \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}, \qquad j = 1, 2,$$

con respecto a una decomposición ortogonal

$$\mathcal{K} = H_{0,-} \oplus M_- \oplus M_+ \oplus H_{0,+},$$

donde M_- y M_+ son finito dimensionales. Entonces, se puede poner $H_- = H_{0,-} \oplus M_-$ y $H_+ = M_+ \oplus H_{0,+}$. Obsérvese que esta estructura es similar a la estructura de una compresión dada en (4), pero aquí ninguno de los subespacios involucrados tiene que ser invariante para A_1 o A_2.

Demostramos que es posible construir un pool a partir de una estructura separada. Esto nos permite asociar una curva discriminante a la estructura separada. De forma más precisa, primero definimos el espacio canal $M = M_- \oplus M_+$. Este espacio tiene dimesión finita y contiene los rangos de $P_{H_-}A_jP_{H_+}$ y $P_{H_+}A_jP_{H_-}$ para $j = 1, 2$. Después probamos que si ponemos $\Phi = P_M$ la proyección ortogonal de $\mathcal{K}$ sobre M y definimos σ_j y γ mediante

$$\begin{aligned} \sigma_j\Phi &= -i(P_{H_+}A_j - A_jP_{H_+}), \qquad j = 1, 2, \\ \gamma\Phi &= i(A_1P_{H_+}A_2 - A_2P_{H_+}A_1), \end{aligned} \tag{8}$$

entonces $A_1, A_2, \sigma_1, \sigma_2, \gamma$ satisfacen la relación de tres términos y por tanto forman un pool de operadores. De este modo, las estructuras separadas son un ejemplo particular de pools de operadores, pero tienen una estructura adicional (dada por la descomposición $\mathcal{K} = H_- \oplus H_+$) en comparación con los pools de operadores en general.

Como ya hemos mencionado, las estructuras separadas están relacionadas con los trabajos de Xia y Yakubovich sobre operadores subnormales de tipo finito. Recordamos que un operador S en un espacio de Hilbert H se llama subnormal si es una restricción de un operador normal que actúa en un espacio de Hilbert más grande. Yakubovich (siguiendo a Xia) consideró operadores subnormales de tipo finito, definidos por la condición de que el auto-conmutador $S^*S - SS^*$ tiene que ser de rango finito. Si S es un operador subnormal puro de tipo finito y N es su extensión normal mínima, entonces hay una manera natural de incluir los operadores $A_1 = \operatorname{Re} N$ y $A_2 = \operatorname{Im} N$ en una estructura separada. La curva discriminante construida a partir de esta estructura separada coincide con la curva discriminante para el operador subnormal, tal cual fue definida por Xia y Yakubovich. Algunas de las definiciones y resultados sobre estructuras separadas están inspiradas por estos trabajos anteriores.

Uno de nuestros resultados principales sobre estructuras separadas es que, bajo ciertas condiciones no restrictivas, la curva discriminante es separada. En este caso, las matrices auxiliares $\sigma_1, \sigma_2, \gamma$ pueden usarse para recuperar la estructura separada completa por medio de una función meromorfa con valores de proyección, definida en la curva discriminante. Esto implica otro de nuestros resultados principales sobre estructuras separadas: *la estructura separada (que incluye objetos infinito-dimensionales) está determinada de manera única por los datos finito-dimensionales $\sigma_1, \sigma_2, \gamma$.*

En el Capítulo 7, damos la definición de una noción de compresión generalizada. Esta noción se define primero para una aplicación lineal A que actúa en un espacio vectorial $\mathcal{K}$. Si $G \subset H \subset \mathcal{K}$ son subespacios, bajo ciertas condiciones podemos definir la compresión generalizada $\widetilde{A}$ de A al cociente H/G mediante

$$\widetilde{A}(h + G) = A(h - g) + G,$$

donde $g \in G$ es tal que $A(h - g) \in H$. Esta noción generaliza la compresión clásica. Recordamos de (4) que un operador A que actúa en $\mathcal{K}$ tiene una compresión clásica a un subespacio $H \ominus G$, donde $G \subset H \subset \mathcal{K}$, si y solo si tanto G como H son invariantes para A. En el contexto de la compresión generalizada, no es necesario que G o H sean invariantes para A.

Probamos que en muchos casos podemos aplicar esta compresión generalizada a una estructura separada y obtener un vessel de operadores. Dadas dos estructuras separadas ω y $\widehat{\omega}$ para los mismos operadores A_1, A_2, escribimos $\mathcal{K} = H_- \oplus H_+$ para la descomposición asociada a ω y $\mathcal{K} = \widehat{H}_- \oplus \widehat{H}_+$ para la descomposción asociada a $\widehat{\omega}$. Si $\widehat{H}_+ \subset H_+$ y algunas condiciones no restrictivas se cumplen, podemos definir la compresión de estas estructuras al espacio $R = H_+/\widehat{H}_+$. Esta compresión es un vessel.

Las matrices auxiliares de este vessel pueden escribirse en términos de las matrices $\sigma_1, \sigma_2, \gamma$ definidas en (8) de una manera sencilla. Además, la curva discriminante de este vessel es la misma que la curva discriminante de la estructura separada. Esto nos permite obtener un vessel comprimiendo una estructura separada, lo cual puede verse como un primer paso hacia una teoría más completa de compresiones y dilataciones de tuplas de varios operadores que conmutan.

La relación entre la dos partes de esta tesis es el uso de curvas analíticas o algebraicas para estudiar problemas en Teoría de Operadores. Especialmente, el uso de curvas algebraicas para construir modelos de operadores. Puede considerarse que la curva analítica $\mathcal{V} = \Phi(\Omega)$ que aparece en la Parte I juega un papel semejante al de una de las dos mitades de la curva discriminante en la Parte II.

List of articles and preprints

Articles and preprints this thesis is based on

- M. A. Dritschel, D. Estévez and D. Yakubovich, *Traces of analytic uniform algebras on subvarieties and test collections*, J. London Math. Soc. (2017), published online, DOI 10.1112/jlms.12003. Chapters 1 and 2 in this thesis.
- M. A. Dritschel, D. Estévez and D. Yakubovich, *Tests for complete K-spectral sets*, accepted in J. Funct. Anal.; preprint available at `arXiv:15010.08350`. Chapter 3 in this thesis.
- M. A. Dritschel, D. Estévez and D. Yakubovich, *Similarity to normals for operators with spectra on a curve and linear resolvent growth*, in preparation. Chapter 4 in this thesis.

Other articles by the author which do not enter this thesis

- D. Estévez, D. Yakubovich, *Decay rate estimations for linear quadratic optimal regulators*, Linear Algebra Appl. **439** (2013), no. 11, 3332–3358, DOI 10.1016/j.-laa.2013.08.030. MR3119856
- D. Estévez, *Explicit traces of functions from Sobolev spaces and quasi-optimal linear interpolators*, Math. Ineq. Appl. **20** (2017), no. 2, 441–457, DOI 10.7153/mia-20-30.

Introduction: summary and conclusions

Before passing to a description of main results of the thesis, we will make a brief overview of some relevant notions from Operator Theory.

Spectral Theorem for normal operators and model theory by Nagy and Foias for Hilbert space contractions

One of the most important achievements in the spectral theory of linear operators on a Hilbert space is the spectral theorem for normal operators, especially the von Neumann's direct integral model. If N is a normal operator on a separable Hilbert space and we denote its spectrum by $\sigma(N)$, then N is unitarily equivalent to the operator M_z of multiplication by the variable z acting on a space H that is given as a direct integral

$$H = \int_{\sigma(N)}^{\oplus} H(z)\, d\mu(z).$$

We recall that the unitary equivalence of N and M_z means that there is a unitary operator U such that $N = UM_zU^*$, so essentially N and M_z are the same from an operator theoretic point of view. In the particular case when the spectrum of N is simple, that is, when $H(z)$ has dimension 1 for all $z \in \sigma(N)$, then H is the space $L^2(\mu)$.

This theorem is an infinite-dimensional generalization of the well known theorem which says that every normal matrix can be diagonalized using an orthonormal basis. Many questions about the operator N can be formulated in terms of the operator M_z, which has a simpler expression and is easier to understand in terms of the function space H, which is just a Lebesgue vector-valued L^2 space with respect to the measure μ. Thus, in a certain sense, the study of normal operators reduces to the study of measurable functions.

The spectral theorem has many diverse consequences: a construction of an $L^\infty(\mu)$ functional calculus (and so, in particular, the definition of roots, logarithms of N, etc.), a description of all operators commuting with N, of invariant and hyperinvariant subspaces, a transparent proof of ergodic theorems for a unitary operator, and so on. It is a basis for the study of stochastic processes, and it is used in classical moment problems, scattering theory, control theory, quantum mechanics, etc.

There is no such a far-reaching spectral theory for any other class of operators. Perhaps the most successful spectral theory for non-normal operators is the spectral theory of Hilbert space contractions of Sz.-Nagy and Foias. Recall that T is called a contraction if $\|T\| \leq 1$. The theory of Sz.-Nagy and Foias uses the isometric dilation to give a unitarily equivalent model for a contraction T in terms of the operator M_z acting on a certain Hardy H^2 space of functions analytic in the unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

Let us formulate this model more precisely in the special case when $T^{*n} \to 0$ strongly as $n \to \infty$. In this case the model is easier to write.

If U is a Hilbert space, we denote by $H^2(U)$ the Hardy H^2 space of functions defined on $\mathbb{D}$ with values in U. The easiest way to write this space is as the space of square summable power series with vector coefficients in U. The model space of T is written as $\mathcal{M} = H^2(U) \ominus \Theta_T H^2(Y)$. Here, Θ_T is the characteristic function of T, which is a bounded analytic function defined on $\mathbb{D}$ and whose values are linear operators that take Y into U. The function Θ_T is inner, which means that its boundary values are isometric almost everywhere on $\mathbb{T} = \partial\mathbb{D}$. The operator T is unitarily equivalent to the compression to the model space $\mathcal{M}$ of the operator M_z of multiplication by the variable z acting on $H^2(U)$. This compression is the operator $P_{\mathcal{M}} M_z | \mathcal{M}$. In the case when T has finite defects (that is, $I - T^*T$ and $I - TT^*$ are of finite rank), the spaces U and Y are finite dimensional, so Θ_T is a matrix-valued function.

Using the analytic model, several questions about the contraction T can be stated in terms of analytic functions and the characteristic function Θ_T. For instance, there is a relation between the invariant subspaces of T and certain factorizations of Θ_T. It is also possible to use the model to define a functional calculus for T for a wide class of bounded analytic functions on $\mathbb{D}$. In the case when T has no unitary part, this class is the whole space $H^\infty(\mathbb{D})$ of bounded analytic functions on $\mathbb{D}$. We refer the reader to the book by Sz.-Nagy and Foias [SNFBK10] for more information about the spectral theory of contractions.

Von Neumann inequality, K-spectral sets and related notions

One of the consequences of this $H^\infty(\mathbb{D})$ functional calculus is von Neumann's inequality. If T is a contraction, then

$$\|p(T)\| \leq \max_{|z|\leq 1} |p(z)|,$$

for every complex polynomial p. When passing from the theory of a single operator to the theory of several commuting operators, it is interesting to consider the natural analogue of von Neumann's inequality. If $T_1, \dots, T_n$ is a tuple of commuting contractions, we say that they satisfy von Neumann's inequality if

$$\|p(T_1, \dots, T_n)\| \leq \max_{|z_1|\leq 1, \dots, |z_n|\leq 1} |p(z_1, \dots, z_n)| \tag{1}$$

holds for every complex polynomial p in n variables. In the sequel we will use the notation

$$\|p\|_\infty := \max_{|z_1|\leq 1, \dots, |z_n|\leq 1} |p(z_1, \dots, z_n)|.$$

As we have already remarked, in the case $n = 1$, von Neumann's inequality holds for every contraction T, as a consequence of the theory of Sz.-Nagy and Foias. In particular, it is very easy to prove it using the existence of a unitary dilation, which was discovered by Sz.-Nagy in [SN53]. However, von Neumann originally proved the inequality in [vN51], before the results of Sz.-Nagy and Foias, so he had to use different techniques. We will define the unitary dilation and show how it can be used to prove von Neumann's inequality later in this introduction.

Ando solved the case $n = 2$ in [And63] by proving that every pair of commuting contractions has a unitary dilation. As in the case of a single operator, an easy consequence of this is that (1) holds for every pair of commuting contractions T_1, T_2.

However, the case $n \geq 3$ is different. Kaijser and Varopoulos [Var74] and Crabb and Davie [CD75] found independently examples of three commuting contractive matrices that do not satisfy von Neumann's inequality (1). Therefore, von Neumann's inequality does not hold in general for three or more commuting contractions. Nevertheless, there are special cases and families of contractions for which the inequality holds. For instance, it is trivial to see that it holds when $T_1, \dots, T_n$ are of the form $T_j = \varphi_j(S_1, S_2)$, for S_1, S_2 two fixed commuting contractions and φ_j functions analytic in $\overline{\mathbb{D}}$ and such that $|\varphi_j| \leq 1$ in $\mathbb{D}$. Also, we mention the positive results by Lotto [Lot94], Grinspan, Kaliuzhnyi-Verbovetskyi, Vinnikov and Woerdeman [GKVVW09], and Hartz [Har15].

It still remains mysterious, what makes the theory of three or more commuting operators fundamentally different from the theory of pairs of commuting operators. One of the important open questions is whether von Neumann's inequality holds with a constant. In other words, whether there is a finite constant C depending only on n such that

$$\|p(T_1, \dots, T_n)\| \leq C\|p\|_\infty \tag{2}$$

for all tuples of commuting contractions $T_1, \dots, T_n$ and all polynomials p. The answer is even unknown in the simplest case $n = 3$. Probably the most widespread impression among the community is that there is no such finite constant C, even for $n = 3$.

A notion related to von Neumann's inequality is that of K-spectral sets. If X is a compact subset of $\mathbb{C}$, T is an operator with spectrum contained in X, and $K \geq 1$, we say that X is a K-spectral set for T if the inequality

$$\|f(T)\| \leq K \sup_{z \in X} |f(z)| \tag{3}$$

holds for every rational function f with poles off X. This can be seen as a generalization of von Neumann's inequality to domains different from $\overline{\mathbb{D}}$. In fact, von Neumann's inequality for T is equivalent to the condition that $\overline{\mathbb{D}}$ is a 1-spectral set for T. Here we consider rational functions instead of only polynomials because X may be non-simply connected.

The Agler algebra is the algebra of all analytic functions f on $\mathbb{D}^n$ such that

$$\sup_{\|T_1\|<1, \dots, \|T_n\|<1} \|f(T_1, \dots, T_n)\| < \infty,$$

where the supremum runs over all tuples of n commuting strict contractions $T_1, \dots, T_n$. The inequality (2) holds for all tuples of commuting contractions if and only if the Agler algebra is the same algebra as $H^\infty(\mathbb{D}^n)$ (with an equivalent norm). Note that the Agler algebra is contained in $H^\infty(\mathbb{D}^n)$, so the question is whether this inclusion is proper or not.

To try to disproof (2), one has to find a series of examples consisting of a triple of commuting contractions T_1, T_2, T_3 and a polynomial p such that the quotients

$$\frac{\|p(T_1, T_2, T_3)\|}{\|p\|_\infty}$$

tend to infinity. There are several articles in the literature which try to find examples such that this quotient is as large as possible. However, to date, the best example known, by Grinshpan, Kaliuzhnyi-Verbovetskyi and Woerdeman [GKVW13], gives a quotient of only around 1.23. This is still very far from trying to prove that (2) does not hold with any constant C.

The unitary dilation of one or several commuting contractions

Here we define and discuss unitary dilations, a concept closely related to von Neumann's inequality. If $T_1, \dots, T_n$ are commuting contractions acting on a Hilbert space H, we say that $U_1, \dots, U_n$ is a unitary dilation of $T_1, \dots, T_n$ if $U_1, \dots, U_n$ are commuting unitaries acting on a larger Hilbert space $\mathcal{K} \supset H$ and

$$T_1^{k_1} T_2^{k_2} \cdots T_n^{k_n} = P_H U_1^{k_1} U_2^{k_2} \cdots U_n^{k_n} | H$$

holds for every choice of non-negative integers $k_1, \dots, k_n$. Here P_H denotes the orthogonal projection of $\mathcal{K}$ onto H. If $U_1, \dots, U_n$ are isometries instead of unitaries, we speak of an isometric dilation. If the tuple $(U_1, \dots, U_n)$ is a dilation of $(T_1, \dots, T_n)$, it is said that $(T_1, \dots, T_n)$ is *a compression of* $(U_1, \dots, U_n)$. It is known that $(T_1, \dots, T_n)$ is a compression of $(U_1, \dots, U_n)$ if and only if the operators $U_1, \dots, U_n$ have the structure

$$U_j = \begin{bmatrix} * & 0 & 0 \\ * & T_j & 0 \\ * & * & * \end{bmatrix}, \tag{4}$$

$j = 1, \dots, n$, according to an orthogonal decomposition $\mathcal{K} = \mathcal{K}_1 \oplus H \oplus \mathcal{K}_2$. Note that in this case, both $\mathcal{K}_2$ and $H \oplus \mathcal{K}_2$ are invariant for U_j.

If $T_1, \dots, T_n$ have a unitary dilation, then it is very easy to show that they satisfy von Neumann's inequality. Indeed,

$$\|p(T_1, \dots, T_n)\| = \|P_H p(U_1, \dots, U_n) | H\| \leq \|p(U_1, \dots, U_n)\| \leq \|p\|_\infty,$$

where the last inequality comes from the functional calculus for unitary operators.

As we have already mentioned, Sz.-Nagy proved in [SN53] that every contraction has a unitary dilation and Ando proved in [And63] that every pair of commuting contractions has a unitary dilation. Parrott found in [Par70] the first example of three commuting contractions which do not have a unitary dilation. Incidentally, his three contractions do satisfy von Neumann's inequality. Thus, the existence of a unitary dilation implies von Neumann's inequality, but the converse is false in general.

To formulate a kind of the converse, which is true, one needs to consider matrix-valued von Neumann's inequalities. If $p = [p_{jk}]$ is an $s \times s$ matrix-valued complex polynomial in n variables, we can define the operator $p(T_1, \dots, T_n)$ as the operator acting on H^s, the direct sum of s copies of H, given in block form by $p(T_1, \dots, T_n) = [p_{jk}(T_1, \dots, T_n)]$. Then, one can study whether

$$\|p(T_1, \dots, T_n)\| \leq \max_{|z_1| \leq 1, \dots, |z_n| \leq 1} \|p(z_1, \dots, z_n)\|$$

holds for every $s \geq 1$ and every $s \times s$ matrix-valued polynomial p. Here, $\|p(z_1, \dots, z_n)\|$ is the norm of the matrix $p(z_1, \dots, z_n)$ as an operator acting on the Hilbert space $\mathbb{C}^s$. This inequality above is called the matrix-valued von Neumann's inequality.

A theorem of Arveson on the extension of completely positive maps can be applied to show that $T_1, \dots, T_n$ satisfy the matrix-valued von Neumman's inequality if and only if $T_1, \dots, T_n$ have a unitary dilation. See, for instance, [AM02, Corollary 14.16] or [Pau02, Corollary 7.7].

One possible approach to study whether von Neumann's inequality holds with a constant, as in (2), is to restrict the study to particular classes of contractions. We fix n and denote by $\mathfrak{C}$ the set of all n-tuples of commuting contractions. If X is a subfamily of $\mathfrak{C}$, we define

$$\|p\|_X = \sup\{\|p(T_1, \dots, T_n)\| : (T_1, \dots, T_n) \in X\}.$$

It is of particular interest to find large sets X such that $\|p\|_X = \|p\|_\infty$, for all p, and small sets X such that $\|p\|_X = \|p\|_{\mathfrak{C}}$, for all p.

Drury proved in [Dru83] that if N is the set of all n-tuples of commuting nilpotent (finite-dimensional) contractive matrices, then $\|p\|_N = \|p\|_{\mathfrak{C}}$ for all p. In other words, if one wishes to study whether (2) holds, then it is only necessary to look at nilpotent matrices.

A general outline of the thesis

The thesis is divided into two parts. In Part I we prove several theorems about complete K-spectral sets and the similarity to a contraction. Here we use Banach algebras, an extension of the techniques of separation of singularities by Havin, Nersessian and Ortega-Cerdà [HN01, Hav04, HNOC07] and some operator-theoretic arguments related to completely bounded maps. Our theorems can be applied to the study of von Neumann's inequality with a constant in the case when the contractions are of the form $T_j = \varphi_j(T)$, $j = 1, \dots, n$, where $\sigma(T) \subset \mathbb{D}$ and $|\varphi_j| \leq 1$ on $\mathbb{D}$, but T is not necessarily a contraction. Part II is a development of some unpublished ideas of Vinnikov and Yakubovich. It introduces several new operator-theoric objects ("pools" and "separating structures"). It is shown that these objects are natural in the context of the theory of tuples of commuting operators

Outline of Part I

The main results of Part I deal with the concept of complete K-spectral sets. This is a matrix-valued version of the concept of K-spectral sets that we have introduced above. If X is a compact subset of $\mathbb{C}$ and T is an operator with spectrum contained in X, we say that X is a complete K-spectral set for T if the inequality

$$\|f(T)\| \leq K \sup_{z \in X} \|f(z)\| \tag{5}$$

holds for every $s \times s$ matrix-valued rational function f with poles off X and every $s \in \mathbb{N}$. The constant K must be independent of s and f. Here, the operator $f(T)$ is defined in block form as $f(T) = [f_{jk}(T)]$, where $f = [f_{jk}]$. This is well defined because the poles of f are outside the spectrum of T. Note that (5) is an analogue of the matrix-valued von Neumann's inequality with a constant for a domain different from $\overline{\mathbb{D}}$.

As in the case of the the matrix-valued von Neumann's inequality, the inequality (5) is equivalent to a certain dilation result. We say that an operator T acting on H has a *rational dilation* to ∂X if there is a normal operator N acting on a larger Hilbert space $\mathcal{K} \supset H$, having spectrum contained in ∂X and such that

$$f(T) = P_H f(N)|H$$

for every rational function f with poles off X. It is known that T is similar to an operator which has a normal dilation to ∂X if and only if X is a complete K-spectral set for T, for some constant K.

In Part I we consider a finitely connected domain $\Omega \subset \mathbb{C}$ and a collection of analytic functions $\varphi_1, \dots, \varphi_n : \overline{\Omega} \to \overline{\mathbb{D}}$. We impose certain geometric and regularity conditions on these functions; if these conditions are met, we say that the tuple $(\varphi_1, \dots, \varphi_n)$ is admissible. The most important of these conditions is that each of the functions φ_k should map some subarc of $\partial\Omega$ bijectively onto an arc of $\partial\mathbb{D}$. We denote this arc by J_k. The union of the arcs J_k, $k = 1, \dots, n$, should cover all the boundary $\partial\Omega$.

An important example of a collection of functions $\varphi_1, \dots, \varphi_n$ satisfying these conditions is the following. Assume that

$$\Omega = \Omega_1 \cap \Omega_2 \cap \cdots \cap \Omega_n, \tag{6}$$

where Ω_k are Jordan domains that intersect transversely and have a regular boundary. Then the Riemann maps $\varphi_k : \Omega_k \to \mathbb{D}$ are an example of admissible functions. The arcs J_k are $J_k = \partial\Omega_k \cap \partial\Omega$ in this case.

The main results of Part I are of the following form: *If T is an operator such that $\|\varphi_k(T)\| \leq 1$ for $k = 1, \dots, n$, then $\overline{\Omega}$ is a complete K-spectral set for T.* We prove several theorems along these lines. These theorems differ in some technical conditions, such as whether the spectrum of T is allowed to touch $\partial\Omega$ or should be completely contained inside Ω and whether the constant K is allowed to depend on T or not. We call the functions $\varphi_1, \dots, \varphi_n$ *test functions*, because they can be used as a test of complete K-spectrality for T, by checking if $\|\varphi_k(T)\| \leq 1$.

Chapter 1 is devoted to one of the key tools that is used to prove these results: a modification of the techniques of separation of singularities by Havin, Nersessian and Ortega-Cerdà. If Ω is as in (6), Havin, Nersessian and Ortega-Cerdà proved that, under some regularity conditions, every function $f \in H^\infty(\Omega)$ can be written as $f = f_1 + f_2 + \cdots + f_n$, where $f_j \in H^\infty(\Omega_j)$. Here we treat a related problem: whether every $f \in H^\infty(\Omega)$ can be written as

$$f = g_1 \circ \varphi_1 + g_2 \circ \varphi_2 + \cdots + g_n \circ \varphi_n, \tag{7}$$

with $g_j \in H^\infty(\mathbb{D})$. If the functions φ_j are all univalent, then these two problems are equivalent, as one can put $g_j = f_j \circ \varphi_j^{-1}$. When φ_j are not univalent, it is not possible in general to write f as in (7).

However, we prove that it is possible to find linear mappings $f \mapsto g_j = g_j(f) \in H^\infty(\mathbb{D})$ such that the linear operator

$$f \mapsto f - (g_1(f) \circ \varphi_1 + g_2(f) \circ \varphi_2 + \cdots + g_n(f) \circ \varphi_n)$$

is compact. Applying some Fredholm theory, this turns out to be enough for our intended applications to Operator Theory.

In Chapter 2 we apply the results of Chapter 1 to the study of algebras of analytic functions on analytic curves inside the polydisc $\mathbb{D}^n$. If we consider the vector map $\Phi = (\varphi_1, \dots, \varphi_n) : \Omega \to \mathbb{D}^n$, its image $\mathcal{V} = \Phi(\Omega)$ is an analytic curve inside the polydisc. We give a characterization of the algebra $H^\infty(\mathcal{V})$ of bounded analytic functions on $\mathcal{V}$ and use this characterization to show that every function $f \in \mathcal{V}$ can be extended to a function F in the Agler algebra of $\mathbb{D}^n$, so that $F|\mathcal{V} = f$.

These are two of the main results of this chapter:

Theorem (Theorem 2.1, page 23). *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible and injective and Φ' does not vanish on Ω, then*

$$H^\infty(\Omega) = \left\{ \sum_{j=1}^{l} \prod_{k=1}^{n} f_{j,k}(\varphi_k(z)) : l \in \mathbb{N}, f_{j,k} \in H^\infty(\mathbb{D}) \right\}.$$

The word *admissible* here refers to certain geometric conditions we have imposed on Φ in Chapter 1.

Theorem (Theorem 2.4, page 23). *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible, for every $f \in H^\infty(\mathcal{V})$ there is an $F \in \mathcal{SA}(\mathbb{D}^n)$ such that $F|\mathcal{V} = f$ and $\|F\|_{\mathcal{SA}(\mathbb{D}^n)} \le C\|f\|_{H^\infty(\mathcal{V})}$, for some constant C independent of f.*

Here $\mathcal{SA}(\mathbb{D}^n)$ denotes the Agler algebra of $\mathbb{D}^n$. Recall that its norm is given by

$$\|F\|_{\mathcal{SA}(\mathbb{D}^n)} = \sup_{\|T_1\|,\dots,\|T_n\| \le 1} \|F(T_1, \dots, T_n)\|.$$

There are many results in the literature about the extension of functions in H^∞ of an analytic curve to a function in $H^\infty(\mathbb{D}^n)$ (or H^∞ of some other domain in $\mathbb{C}^n$). However, the extension to the Agler algebra is something new that has not been studied before.

Chapter 3 is devoted to the statement and proof of the results about complete K-spectral sets. Here we state some of the main results of this chapter.

Theorem (Theorem 3.1, page 39). *Let $\Omega_1, \dots, \Omega_n$ be open sets in $\widehat{\mathbb{C}}$ such that the boundary of each set Ω_k, $k = 1, \dots, n$, is a finite disjoint union of Jordan curves. We also assume that the boundaries of the sets Ω_k, $k = 1, \dots, n$, are Ahlfors regular and rectifiable, and intersect transversally. Put $\Omega = \Omega_1 \cap \cdots \cap \Omega_n$. Suppose that $\sigma(T) \subset \overline{\Omega}$ and a constant $K \ge 1$ is given. Then there is a constant K' such that*

(i) *if each of the sets $\overline{\Omega}_j$, $j = 1, \dots, n$, is K-spectral for T, then $\overline{\Omega}$ is also K'-spectral set for T; and*

(ii) *if each of the sets $\overline{\Omega}_j$, $j = 1, \dots, n$, is complete K-spectral for T, then $\overline{\Omega}$ is a complete K'-spectral set for T.*

In both cases, K' depends only on the sets $\Omega_1, \dots, \Omega_n$ and the constant K, but not on the operator T.

This theorem is a generalization of a theorem of Badea, Beckermann and Crouzeix in [BBC09] that treats the case when Ω_j are discs in the Riemann sphere. In that theorem, the constant they obtain depends only on n, the number of discs.

Theorem (Theorem 3.6, page 43). *Assume that $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible and analytic in a neighbourhood of $\overline{\Omega}$, where Ω is a Jordan domain. If $\sigma(T) \subset \overline{\Omega}$ and $\varphi_j(T)$ is similar to a contraction for $j = 1, \dots, n$, (i.e., $\|S_j \varphi_j(T) S_j^{-1}\| \leq 1$, for some S_j), then $\overline{\Omega}$ is a complete K-spectral set for T, where K depends on Ω, Φ, $\|S_j\| \cdot \|S_j^{-1}\|$ and T. If moreover Φ is injective and Φ' does not vanish on $\overline{\Omega}$, then K can be taken to depend only on Ω, Φ and $\|S_j\| \cdot \|S_j^{-1}\|$, but not on T.*

The definition of "admissible function" will be given in Chapter 1. Essentially, it is a function that satisfies certain geometric and regularity conditions, as we have explained above. In Chapter 3 we will give some similar results, which treat the case of a non-simply connected domains.

In Chapter 4 we give an application of some of our results about complete K-spectral sets to operators with thin spectrum. We consider operators T whose spectrum is contained in a sufficiently smooth curve $\Gamma \subset \mathbb{C}$. We prove that if the resolvent of T satisfies certain growth conditions, then T is similar to a normal operator. Here is one of the main results this chapter:

Theorem (Theorem 4.1, page 73). *Let $\Gamma \subset \mathbb{C}$ be a $C^{1+\alpha}$ Jordan curve, and Ω the domain it bounds. Let $T \in \mathcal{B}(H)$ an operator with $\sigma(T) \subset \Gamma$. Assume that*

$$\|(T - \lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda, \Gamma)}, \qquad \lambda \in U \setminus \overline{\Omega},$$

for some open set U containing $\partial\Omega$, and

$$\|(T - \lambda)^{-1}\| \leq \frac{C}{\operatorname{dist}(\lambda, \Gamma)}, \qquad \lambda \in \Omega,$$

for some constant $C > 0$. Then T is similar to a normal operator.

This theorem is related to a result of Stampfli in [Sta69], which states that if $\|(T - \lambda)^{-1}\| \leq \operatorname{dist}(\lambda, \Gamma)^{-1}$ for λ in a neighbourhood of Γ, then T is normal. In the proof of this theorem we use Theorem 3.3, which is a theorem about complete K-spectral sets included in Chapter 3. Theorem 3.3 is a generalization to non-convex domains of a theorem of Putinar and Sandberg about convex complete K-spectral sets and the numerical range. It can also be seen as a generalization of ρ-contractions to a domain different from $\mathbb{D}$.

In Chapter 4 we also use our techniques to generalize theorems of van Casteren [vC80, vC83] and Naboko [Nab84] that treated the case $\Gamma = \mathbb{T}$ to the case where Γ is a general $C^{1+\alpha}$ Jordan curve. One of our main tools is a functional calculus based on the Cauchy-Green formula and the pseudoanalytic continuation. This calculus was originally defined by Dynkin in [Dyn72]. His calculus allows us to pass back and forth between operators with spectrum in a curve Γ and operators with spectrum in $\mathbb{T}$, maintaining the control on the estimates of their resolvents.

Outline of Part II

Part II is devoted to a definition and a study of what we call separating structures. We relate them with the theory of Livšic and Vinnikov of tuples of commuting nonselfadjoint operators, which was developed in a series of papers by Livšic in the 60s. The book [LKMV95] is devoted to a systematic exposition of this theory. As we will explain below, Livšic and Vinnikov define an operator theoretic construction called *vessel*. One of the important objects associated with a vessel is its discriminant curve, which is an algebraic cruve.

In the Operator Theory literature, there are at least two places where algebraic curves appear. The first one is the theory of Livšic and Vinnikov. The second are the works of Xia [Xia87a, Xia87b, Xia96] and Yakubovich [Yak98a, Yak98b] about subnormal operators of finite type, where some type of discriminant curve for the subnormal operator is defined. Part II provides a common framework for these two theories, which seemed to have no connection. We will explain how the constructions of Xia and Yakubovich can be written in terms of a separating structure. A relation between separating structures and vessels is obtained by means of the generalized compression, which will be defined below.

In Chapter 5, we review the Livšic and Vinnikov theory. It studies tuples of commuting operators $A_1, \ldots, A_n$ such that their imaginary parts $(A_j - A_j^*)/(2i)$ have finite rank for all $j = 1, \ldots, n$. We remark that in the particular case when all the operators A_j are dissipative, meaning that

$$\operatorname{Im}(A_j) = \frac{A_j - A_j^*}{2i} \geq 0,$$

one can apply the Cayley transform

$$T_j = (A_j - iI)(A_j + iI)^{-1}$$

to obtain a tuple $(T_1, \ldots, T_n)$ of commuting contractions.

Note that to study von Neumann's inequality it is enough to consider operators T_j obtained in this way, because it is in fact sufficient to study finite-dimensional operators.

Since $A_j - A_j^*$ have finite rank, it is possible to choose a finite dimensional Hilbert space E, an operator $\Phi : H \to E$ (here H is the Hilbert space where A_j act) and selfadjoint matrices σ_k, $k = 1, \ldots, n$, acting on E such that

$$\frac{1}{i}(A_k - A_k^*) = \Phi^* \sigma_k \Phi.$$

The theory of Livšic and Vinnikov also uses auxiliary selfadjoint matrices γ_{kj}^{in} and γ_{kj}^{out}, $j, k = 1, \ldots, n$, acting on E, which are called gyrations. In some sense, the matrices σ_j, γ_{kj}^{in} and γ_{kj}^{out} encode the interplay between the operators A_j. The collection of the operators A_j, the map Φ and all the auxiliary matrices is called an *operator vessel*. Each vessel has an associated discriminant variety, which is an algebraic variety in $\mathbb{C}^n$. This gives a connection between Operator Theory and Algebraic Geometry. The theory of Livšic and Vinnikov gives more definite results for pairs of commuting operators A_1, A_2

than for n-tuples. In this case, the discriminant variety is an algebraic curve in $\mathbb{C}^2$, which is called the discriminant curve. It is the zero set of the polynomial

$$p(z_1, z_2) = \det(z_1\sigma_2 - z_2\sigma_1 + \gamma_{12}^{\mathrm{in}})$$

(this formula is called a determinantal representation of the algebraic curve).

By means of the desingularization, the discriminant curve can be regarded as a disjoint union of Riemann surfaces. We say that the curve is *separated* if each of these Riemann surfaces splits into two connected components when we remove its set of real points. If the curve is separated, we can define its two halves.

In some cases, the discriminant curve of a vessel is separated (see for instance [SV05]). If this happens, the two halves of the discriminant curve play a role analogous to the unit disk and its complement in the case of the theory of a single contraction. Thus, one may think that the correct setting to study several commuting operators is that of algebraic varieties.

Livšic and Vinnikov prove that the joint spectrum of the tuple $(A_1, \dots, A_n)$ is contained in its discriminant curve. On the other hand, it is easy to see that the joint spectrum of the tuple $(\varphi_1(T), \dots, \varphi_n(T))$, which appears in Part I, is contained in the one-dimensional complex variety $\Phi(\mathbb{D})$. This shows that there is a strong connection between the two parts of the thesis. In Section 3.8, we describe this connection. This section is new material that did not appear in the article [DEY15], on which Chapter 3 is based. We show that it is enough to study von Neumann's inequality for tuples of commuting contractions $(T_1, \dots, T_n)$ where $T_j = B_j(T)$, $j = 1, \dots, n$, the matrix T is diagonalizable with spectral radius smaller than 1 (this is, $\sigma(T) \subset \mathbb{D}$), and B_j are finite Blaschke products. In fact, it is possible to choose B_j in such a way that the map $\Phi = (B_1, \dots, B_n) : \overline{\mathbb{D}}^n \to \overline{\mathbb{D}}$ is injective and Φ' does not vanish. This allows us to apply our results in Chapter 3.

One can think of the Sz.-Nagy and Foias theory of contractions as having two parts. The first one is the spectral theory of isometries, which have an analytic model built upon the Hardy H^2 space of the unit disk. The second part is the relation between contractions and isometries: one can pass back and forth between them using compressions and dilations. This allows one to construct an analytic model for contractions using the H^2 space.

In the study of tuples of commuting operators, the key idea of the theory of Livšic and Vinnikov is to use complex algebraic curves instead of the complex plane. Vessels can be seen as an analogue of contractions for tuples of operators, but Livšic and Vinnikov do not give a counterpart of the spectral theory of isometries.

Chapters 6 and 7 are an attempt to build this counterpart. In Chapter 6, we start by considering a new structure that we call *operator pool*. Its definition resembles that of an operator vessel. A pool is formed by the following objects: a pair of commuting selfadjoint operators A_1, A_2 acting on a Hilbert space $\mathcal{K}$, a finite dimensional auxiliary Hilbert space M, an operator $\Phi : \mathcal{K} \to M$, and selfadjoint matrices $\sigma_1, \sigma_2, \gamma$ which satisfy the so called three-term relationship:

$$\sigma_2\Phi A_1 - \sigma_1\Phi A_2 + \gamma\Phi = 0.$$

A relation of this form also appears in the theory of operator vessels. The discriminant curve of the pool is defined as the set of zeros of the polynomial

$$p(z_1, z_2) = \det(z_1\sigma_2 - z_2\sigma_1 + \gamma),$$

similarly to the theory of Livšic and Vinnikov. Note that an operator pool is built around selfadjoint operators A_1, A_2, while the operators A_1, A_2 in a vessel are usually non-selfadjoint. Thus, although operator pools and vessels might seem similar at first glance, they are rather different concepts.

Then we introduce our main object of study, which we call *separating structures*. These are formed by a pair of commuting selfadjoint operators A_1, A_2 acting on a Hilbert space $\mathcal{K} = H_- \oplus H_+$ and such that $P_{H_-}A_jP_{H_+}$, $j = 1, 2$, have finite rank. One equivalent way to write this conditions is that A_1, A_2 have the structure

$$A_j = \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix}, \qquad j = 1, 2,$$

according to an orthogonal decomposition

$$\mathcal{K} = H_{0,-} \oplus M_- \oplus M_+ \oplus H_{0,+},$$

where M_- and M_+ are finite dimensional. Then we can put $H_- = H_{0,-} \oplus M_-$ and $H_+ = M_+ \oplus H_{0,+}$. Note that this structure is similar to the structure of a compression given in (4), but here none of the subspaces involved has to be invariant for A_1 or A_2.

We show that it is possible to construct a pool from a separating structure in a canonical way. This allows us to assign a discriminant curve to the separating structure. More precisely, we first define the channel space $M = M_- \oplus M_+$. This space M is finite dimensional and it contains the ranges of $P_{H_-}A_jP_{H_+}$ and $P_{H_+}A_jP_{H_-}$ for $j = 1, 2$. Then we show that if we put $\Phi = P_M$ the orthogonal projection of $\mathcal{K}$ onto M and we define σ_j and γ by

$$\begin{aligned} \sigma_j\Phi &= -i(P_{H_+}A_j - A_jP_{H_+}), \qquad j = 1, 2, \\ \gamma\Phi &= i(A_1P_{H_+}A_2 - A_2P_{H_+}A_1), \end{aligned} \tag{8}$$

then $A_1, A_2, \sigma_1, \sigma_2, \gamma$ satisfy the three-term relationship and hence form an operator pool. In this way, separating structures are a particular example of operator pools, but they have much more additional structure (given by the decomposition $\mathcal{K} = H_- \oplus H_-$) in comparison with operator pools in general.

As we have already mentioned, separating structures are related to the work by Xia and Yakubovich about subnormal operators of finite type. We recall that an operator S on a Hilbert space H is called subnormal if it is a restriction of a normal operator, acting on a larger Hilbert space. Yakubovich (after Xia) considered subnormal operators of finite type, defined by the condition that the self-commutator $S^*S - SS^*$ has to be of finite rank. If S is a pure subnormal operator of finite type and N is its minimal normal extension, then there is a way natural way to embed the selfadjoint operators $A_1 = \operatorname{Re} N$ and $A_2 = \operatorname{Im} N$ into a separating struture. The discriminant curve constructed from this separating structure coincides with the discriminant curve for the subnormal operator, as defined by Xia and Yakubovich. Some of the definitions and results about separating structures are inspired by these earlier works.

One of our main results regarding separating structures is that, under some non-restrictive conditions, the discriminant curve is separated. In this case, the auxiliary matrices $\sigma_1, \sigma_2, \gamma$ can be used to recover the whole separating structure by using a meromorphic projection valued function, defined on the discriminant curve. This implies another of our main results about separating structures: *the separating structure (which involves infinite-dimensional objects) is uniquely determined by the finite-dimensional data* $\sigma_1, \sigma_2, \gamma$.

In Chapter 7, we give the definition of a notion of generalized compression. This notion is first defined for a linear map A acting on a vector space $\mathcal{K}$. If $G \subset H \subset \mathcal{K}$ are subspaces, under certain algebraic conditions we can define the generalized compression $\widetilde{A}$ of A to the quotient H/G by

$$\widetilde{A}(h+G) = A(h-g) + G,$$

where $g \in G$ is such that $A(h-g) \in H$. This notion generalizes the classical compression. Recall from (4) that an operator A acting on $\mathcal{K}$ has a classical compression to a subspace $H \ominus G$, where $G \subset H \subset \mathcal{K}$, if and only if both G and H are invariant for A. In the setting of the generalized compression, it is not necessary that G or H are invariant for A.

We show that in many cases we can apply this generalized compression to a separating structure and obtain an operator vessel. Given two separating structures ω and $\widehat{\omega}$ for the same operators A_1, A_2, we write $\mathcal{K} = H_- \oplus H_+$ for the decomposition associated to ω and $\mathcal{K} = \widehat{H}_- \oplus \widehat{H}_+$ for the decomposition associated to $\widehat{\omega}$. If $\widehat{H}_+ \subset H_+$ and some non-restrictive conditions hold, we can define the compression of these structures to the space $R = H_+/\widehat{H}_+$. This compression is a vessel.

The auxiliary matrices for this vessel can be written in terms of the matrices $\sigma_1, \sigma_2, \gamma$ defined in (8) in a simple way. Moreover, the discriminant curve of this vessel is the same as the discriminant curve of the separating structure. This allows us to obtain a vessel by compressing a separating structure, which can be seen as a first step towards a more complete theory of compressions and dilations of tuples of several commuting operators.

The relation between the two parts of this thesis is the use of analytic or algebraic curves to study problems in Operator Theory. Especially, the use of algebraic curves to construct operator models. The analytic curve $\mathcal{V} = \Phi(\Omega)$ that appears in Part I can be thought of as playing a similar role to one of the halves of the discriminant curve in Part II.

Part I.

Test collections

1. Separation of singularities

This chapter is based on joint work with Michael Dritschel and Dmitry Yakubovich. The results of this chapter are contained in the article [DEY17].

1.1. Separation of singularities of bounded analytic functions

In this section we will give a review of the results by Havin, Nersessian and Ortega-Cerdà about the separation of singularities of bounded analytic functions. We will use later a modification of some of their arguments. Here we will reproduce some of the main results of [HN01]. Some other papers that deal with separation of singularities and extend the results of [HN01] are [Hav04, HNOC07].

Let Ω be an open set in $\mathbb{C}$ and take k_1, k_2 a pair of relatively closed subsets of Ω. We put $k = k_1 \cup k_2$. Aronszajn proved the following result in his thesis [Aro35].

Theorem 1.1 (Aronszajn [Aro35])**.** *If f is analytic in $\Omega \setminus k$, there exist functions f_j analytic in $\Omega \setminus k_j$, $j = 1, 2$, such that*

$$f = f_1 + f_2 \qquad \textit{in } \Omega \setminus k.$$

We can think that f has singularities on k, while each of the functions f_j only has singularities on its corresponding set k_j. We say that the singularities of f have been separated. An interesting result related to this theorem is that, in general, it is not possible to produce f_1, f_2 from f by means of a linear operator. This was proved in [MH71].

Now assume that f is analytic and bounded in $\Omega \setminus k$, in other words, that f belongs to $H^\infty(\Omega \setminus k)$. A natural question is whether one can take the functions f_1, f_2 in Theorem 1.1 to be bounded. There are simple examples that show that this is not possible in general. Here we give one of them.

Example 1.2. Let Ω be an open set in $\mathbb{C}$ containing the origin. Put $k_1 = \{z \in \mathbb{R} : z \leq 0\} \cap \Omega$, $k_2 = \{z \in \mathbb{R} : z \geq 0\} \cap \Omega$. We define f in $\Omega \setminus \mathbb{R}$ by $f(z) = 0$ when $\operatorname{Im} z > 0$ and $f(z) = -2\pi i$ when $\operatorname{Im} z < 0$. Then the singularities of f can be separated by putting

$$\begin{aligned} f_1(z) &= \log|z| + i \arg z, \qquad & z \in \Omega \setminus k_1,\ -\pi < \arg z < \pi \\ f_2(z) &= -\log|z| - i \arg_1 z, \qquad & z \in \Omega \setminus k_2,\ 0 < \arg_1 z < 2\pi \end{aligned}$$

These functions satisfy $f = f_1 + f_2$ in $\Omega \setminus \mathbb{R}$, but they are unbounded.

Assume that $f = \widetilde{f_1} + \widetilde{f_2}$ in $\Omega \setminus \mathbb{R}$, for $f_j \in H^\infty(\Omega \setminus k_j)$. Then $f_1 - \widetilde{f_1} = \widetilde{f_2} - f_2$ in $\Omega \setminus \mathbb{R}$. Since the left hand side of this equality is analytic in $\Omega \setminus k_1$ and the right hand side is analytic in $\Omega \setminus k_2$, there is a function h analytic in $\Omega \setminus \{0\}$ that coincides with $f_1 - \widetilde{f}_1$ in $\Omega \setminus k_1$ and coincides with $\widetilde{f}_2 - f_2$ in $\Omega \setminus k_2$.

Since $\widetilde{f}_j$ is bounded and $|f_j(z)|$ grows as $|\log|z||$ when $z \to 0$, we see that $|h(z)|$ also grows as $|\log|z||$ when $z \to 0$. Since the origin is an isolated singularity for h, this implies that h is bounded near 0. Hence, $f_1 = h + \widetilde{f}_1$ must be bounded near 0, which is a contradiction.

Despite these examples, there are many cases in which it is possible to perform bounded separation of singularities. This means that it is possible to find functions $f_j \in H^\infty(\Omega \setminus k_j)$ such that $f = f_1 + f_2$. The problem in the example above is that the rays k_1 and k_2 do not meet transversally at the origin. In general, some sort of transversality condition is needed to have bounded separation of singularities. In the papers [HN01, Hav04, HNOC07], Havin, Nersessian and Ortega-Cerdà study several situations under which separation of singularities is possible.

An important remark when studying the problem of separation of singularities is that it is always possible to perform a localization of the problem. This is done with the Vitushkin localization operator. Assume that $f \in L^\infty(\mathbb{C})$ and that $\overline{\partial} f$ is compactly supported. Recall that $\overline{\partial} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$ and that $\overline{\partial} f(z) = 0$ if and only if f satisfies the Cauchy-Riemann equations at z. Thus, such an f belongs to $H^\infty(\Omega \setminus K)$, where $K = \operatorname{supp} \overline{\partial} f$. Given $\alpha \in C_c^\infty(\mathbb{C})$ (this means that α is of class C^∞ and has compact support), we put

$$V_\alpha(f) = \big(\alpha \overline{\partial} f\big) * \frac{1}{\pi z}.$$

Here the convolution is done in the sense of distributions. Then $V_\alpha(f) \in L^\infty(\mathbb{C})$ and $\overline{\partial} V_\alpha(f) = \alpha \overline{\partial} f$ (see [HN01, Lemma 3.2]). Thus, $V_\alpha(f)$ is analytic and bounded outside $K \cap \operatorname{supp} \alpha$.

Vitushkin localization operator can be used to perform bounded separation of singularities by means of a partition of unity. See [HN01, Theorem 3.1]. Here we will use it to prove a simple lemma that we will need later.

Lemma 1.3. *Let Ω be a domain with rectifiable boundary and $f \in H^\infty(\Omega)$. Let $\eta \in C^\infty(\partial\Omega)$. Then the function*

$$z \mapsto \frac{1}{2\pi i} \int_{\partial\Omega} \frac{\eta(w) f(w)\, dw}{w - z} \tag{1.1}$$

belongs to H^∞.

Proof. We extend f to $\mathbb{C}$ by defining it to be zero outside of Ω. Since

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(w)\, dw}{w - z}, \qquad z \notin \partial\Omega,$$

we see that $\overline{\partial} f = \frac{1}{2i} f\mu$, where μ denotes the arc length measure of $\partial\Omega$ (see [HN01, Section 3.1], for instance). Therefore, the function defined by (1.1) is precisely $V_\eta(f)$, which belongs to L^∞. □

We will now state the key theorem of [HN01]. First, we need to introduce some notation.

Definition 1.4. A compact set $K \subset \mathbb{C}$ is called *regular* if the following conditions are satisfied:

(a) $K \subset \Gamma := \Gamma_1 \cup \cdots \cup \Gamma_N$, where Γ_j are simple Jordan rectifiable arcs such that $\Gamma_j \setminus \{0\}$ are mutually disjoint, $j = 1, \ldots, N$;

(b) The one-dimensional Hausdorff measure of $K \cap r\mathbb{D}$ is $O(r)$ as $r \to 0$.

If $l \subset \mathbb{T}$ is a compact arc, we define

$$A(l) = \{0\} \cup \{\zeta \in \mathbb{C} : \zeta \neq 0,\ \zeta/|\zeta| \in l\}, \qquad S(l) = A(l) \cap \overline{\mathbb{D}}.$$

Theorem 1.5 (Havin, Nersessian [HN01, Theorem 4.1])**.** *Let l_1, l_2 be disjoint compact arcs of $\mathbb{T}$, $\mathbb{T} \setminus (l_1 \cup l_2) = \lambda_1 \cup \lambda_2$, λ_j being disjoint open arcs. Suppose*

$$\max(|\lambda_1|, |\lambda_2|) > \min(|l_1|, |l_2|),$$

say,

$$|\lambda_1| > |l_1|,$$

so that $\theta l_1 \subset \lambda_1$, $\theta S(l_1) \subset S(\lambda_1)$ for a $\theta \in \mathbb{T}$. Let K_1, K_2 be regular compact sets such that $K_j \subset S(\widetilde{l_j}) \cup \{0\}$, $j = 1, 2$, where $\widetilde{l_j}$ is a compact arc of $\mathbb{T}$ such that $\widetilde{l_j}$ is contained in the interior of $A(l_j)$. Put $K = K_1 \cup K_2$ and let $f \in H^\infty(\mathbb{C} \setminus K)$. Then there exist functions $f_j \in H^\infty(\mathbb{C} \setminus (K_j \cup \theta K_1))$, $j = 1, 2$ such that $f = f_1 + f_2$ in $\mathbb{C} \setminus (K \cup \theta K_1)$.

We refer the reader to the article by Havin and Nersessian for the proof of this theorem. It follows form the proof that the functions f_j can be produced by $f_j = F_j(f)$, where F_j is a bounded linear operator from $H^\infty(\mathbb{C} \setminus K)$ into $H^\infty(\mathbb{C} \setminus (K_j \cup \theta K_1))$.

The key idea of this theorem is that, while it may not be possible to separate f into functions f_j having singularities only in K_j, it becomes possible if one also allows f_j to have singularities in the rotated set θK_1. This is sufficient for many applications. For instance, if one is interested in functions which are of class H^∞ in some domain Ω, then it is possible to arrange the rotation so that θK_1 lies outside of Ω.

We will now state an application of Theorem 1.5 given by Havin and Nersessian in [HN01, Example 4.1]. We will also give some more generalizations. Let us first introduce the idea of transverse intersection.

By an *open circular sector with vertex z_0* we mean a set in $\mathbb{C}$ of the form

$$\{z \in \mathbb{C} : 0 < |z - z_0| < r,\ \alpha < \arg z < \beta\},$$

where $r > 0$ and $\alpha, \beta \in \mathbb{R}$, $0 < \beta - \alpha < 2\pi$. The aperture of such a circular sector is the number $\beta - \alpha$.

We denote by $B(z, r)$ the open disk of centre $z \in \mathbb{C}$ and radius $r > 0$.

Definition 1.6. If $\Omega_1, \Omega_2 \subset \widehat{\mathbb{C}}$ are two open sets in the Riemann sphere, and $\infty \neq z_0 \in \partial\Omega_1 \cap \partial\Omega_2$ is a point in the intersection of their boundaries, we say that the boundaries of Ω_1 and Ω_2 *intersect transversally at z_0* if one can find five pairwise disjoint open circular sectors $S_0, S_1^l, S_1^r, S_2^l, S_2^r$ with vertex z_0, having the same aperture, and such that the following conditions are satisfied:

- S_0 does not intersect $\overline{\Omega}_1 \cup \overline{\Omega}_2$.
- $B(z_0, \varepsilon) \cap \partial\Omega_j \subset S_j^l \cup S_j^r \cup \{z_0\}$ for $j = 1, 2$ and some $\varepsilon > 0$.
- For every $\delta > 0$, $B(z_0, \delta) \cap \Omega_1 \cap \Omega_2$ is not empty.

In the case when $\infty \in \partial\Omega_1 \cap \partial\Omega_2$, we say that the boundaries of Ω_1 and Ω_2 intersect transversally at ∞ if the boundaries of $\psi(\Omega_1)$ and $\psi(\Omega_2)$ intersect transversally at 0, where $\psi(z) = 1/z$. We say that the boundaries of Ω_1 and Ω_2 intersect transversally if they intersect transversally at every point of $\partial\Omega_1 \cap \partial\Omega_2$.

Note that the third condition in the definition of a transversal intersection implies that $\overline{\Omega_1 \cap \Omega_2} = \overline{\Omega}_1 \cap \overline{\Omega}_2$. We also remark that if Ω_1 and Ω_2 intersect transversally, then $\partial\Omega_1 \cap \partial\Omega_2$ is a finite set.

Proposition 1.7 (Havin, Nersessian [HN01, Example 4.1])**.** *Let Ω_1, Ω_2 be Jordan domains in $\mathbb{C}$ with piecewise C^1 boundaries which intersect transversally and such that $\Omega = \Omega_1 \cap \Omega_2$ is a Jordan domain. Then there are bounded linear operators $G_j : H^\infty(\Omega) \to H^\infty(\Omega_j)$, $j = 1, 2$, such that $f = G_1(f) + G_2(f)$ for all $f \in H^\infty(\Omega)$.*

The proof of this proposition amounts to first localizing the problem to each of the points of the intersection of $\partial\Omega_1$ and $\partial\Omega_2$ and then using Theorem 1.5. Using the same kind of arguments, one can also prove the following.

Proposition 1.8. *Let $\Omega_1, \dots, \Omega_n$ be open sets in $\widehat{\mathbb{C}}$ such that the boundary of each set Ω_k, $k = 1, \dots, n$, is a finite disjoint union of Jordan curves. We also assume that the boundaries of the sets Ω_k, $k = 1, \dots, n$, are Ahlfors regular and rectifiable, and intersect transversally. Put $\Omega = \Omega_1 \cap \dots \cap \Omega_n$. Then there are bounded linear operators $G_j : H^\infty(\Omega) \to H^\infty(\Omega_j)$ such that $f = G_1(f) + \cdots G_n(f)$ for every $f \in H^\infty(\Omega)$.*

Moreover, if $f \in A(\overline{\Omega})$, then $G_j(f) \in A(\overline{\Omega}_j)$.

The assertion that $f \in A(\overline{\Omega})$ implies $G_j(f) \in A(\overline{\Omega}_j)$ is not contained in the article by Havin and Nersessian [HN01], however one can follow their arguments and see that the continuity of f is preserved in the construction of $G_j(f)$ that Havin and Nersessian do. See the proof of Theorem 1.11 (page 9) for a similar argument.

For the convenience of the reader, we recall the definition of Ahlfors regularity, which was required in the proposition above.

Definition 1.9. A curve $\Gamma \subset \mathbb{C}$ is called *Ahlfors regular* if $|B(z, \varepsilon) \cap \Gamma| \leq C\varepsilon$, for every $\varepsilon > 0$ and every $z \in \Gamma$, where C is a constant independent of ε and z. Here $|\cdot|$ denotes the arc-length measure and $B(z, \varepsilon)$ is the open disk of radius ε and center z.

1.2. Separation of singularities with the composition

In this section, we treat a problem which is related to Proposition 1.8. Assume that Ω is a domain in $\mathbb{C}$ and that we are given analytic functions $\varphi_1, \dots, \varphi_n : \Omega \to \mathbb{D}$. We would like to be able to write each $f \in H^\infty(\Omega)$ as

$$f = g_1 \circ \varphi_1 + \cdots + g_n \circ \varphi_n, \tag{1.2}$$

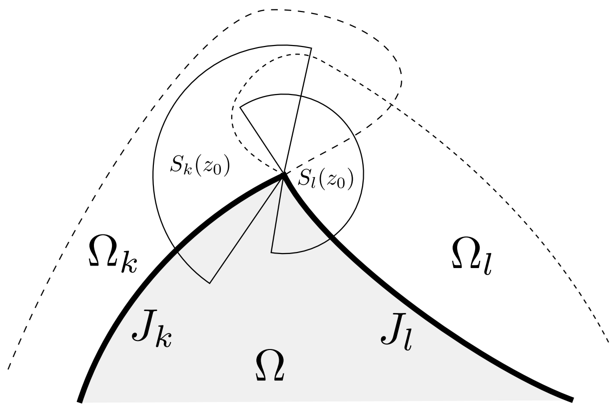

Figure 1.1.: The sets involved in the definition of an admissible function

where $g_j \in H^\infty(\mathbb{D})$, $j = 1, \dots, n$.

The relation of this problem with Proposition 1.8 is as follows. Suppose that the sets $\Omega_1, \dots, \Omega_n$ and Ω are as in the statement of Proposition 1.8. We also assume that Ω_j are simply connected. Let $\varphi_j : \Omega_j \to \mathbb{D}$ be the Riemann map. Then we can put $g_j = G_j(f) \circ \varphi_j^{-1}$, so that g_j belong to $H^\infty(\mathbb{D})$ and (1.2) is satisfied. However, this trick only works if φ_j are univalent.

In general, a necessary condition to be able to write every $f \in H^\infty(\Omega)$ as in (1.2) is that the map $\Phi = (\varphi_1, \dots, \varphi_n) : \Omega \to \mathbb{D}^n$ has to be injective. If Φ glues two points z_1, z_2 (meaning that $\Phi(z_1) = \Phi(z_2)$) then every f as in (1.2) will also glue z_1 and z_2. However, it is not sufficient that Φ is injective. To see this, assume that $n = 2$ and that there are distinct points z_1, z_2, z_3, z_4 in Ω such that $\varphi_1(z_1) = \varphi_1(z_2)$, $\varphi_1(z_3) = \varphi_1(z_4)$, $\varphi_2(z_1) = \varphi_2(z_3)$, $\varphi_2(z_2) = \varphi_2(z_4)$ (this can happen even if Φ is injective). Then every function f as in (1.2) will satisfy $f(z_1) + f(z_4) = f(z_2) + f(z_3)$.

The main goal of this section is to show that we are able to construct bounded linear operators F_j such that the difference

$$f - (F_1(f) \circ \varphi_1 + \cdots + F_n(f) \circ \varphi_n) \tag{1.3}$$

defines a compact operator acting on $H^\infty(\Omega)$. We have argued that in general we cannot arrange for the difference (1.3) to be zero. It will be enough for our aplications that this difference is a compact operator, as we will then be able to apply some results from Fredholm theory.

To state the main theorem of this section, we first need to define *admissible functions*, which are the class of functions $\Phi = (\varphi_1, \dots, \varphi_n) : \Omega \to \mathbb{D}^n$ that we will consider.

Definition 1.10. Let Ω be a domain whose boundary is a disjoint finite union of piecewise analytic Jordan curves such that the interior angles of the "corners" of $\partial\Omega$ are in $(0, \pi]$. We say that a function $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is *admissible* if $\varphi_k \in A(\overline{\Omega})$, for $k = 1, \dots, n$, and there is a collection of sets $\{J_k\}_{k=1}^n$, where each J_k is a finite union of disjoint closed analytic subarcs of $\partial\Omega$, and a constant α, $0 < \alpha \leq 1$, such that the following conditions are satisfied (see Figure 1.1):

(a) $\bigcup_{k=1}^{n} J_k = \partial\Omega$.

(b) $|\varphi_k| = 1$ in J_k, for $k = 1, \dots, n$.

(c) For each $k = 1, \dots, n$, there exists an open set $\Omega_k \supset \Omega$ such that the interior of J_k relative to $\partial\Omega$ is contained in Ω_k, φ_k is defined in $\overline{\Omega}_k$, $\varphi_k \in A(\overline{\Omega}_k)$, and φ_k' is of class Hölder α in Ω_k; i.e.,

$$|\varphi_k'(\zeta) - \varphi_k'(z)| \le C|\zeta - z|^\alpha, \qquad \zeta, z \in \Omega_k.$$

(d) If z_0 is an endpoint of one of the arcs comprising J_k, then there exists an open circular sector $S_k(z_0)$ with vertex on z_0 and such that $S_k(z_0) \subset \Omega_k$ and $J_k \cap \mathbb{D}_\varepsilon(z_0) \subset S_k(z_0) \cup \{z_0\}$, for some $\varepsilon > 0$. If z_0 is a common endpoint of both one of the arcs comprising J_k and one of the arcs comprising J_l, $k \ne l$, then we require $(S_k(z_0) \cap S_l(z_0)) \setminus \overline{\Omega}$ to be nonempty.

(e) $|\varphi_k'| \ge C > 0$ in J_k, for $k = 1, \dots, n$.

(f) For each $k = 1, \dots, n$, $\varphi_k | J_k$ is injective and $\varphi_k(J_k) \cap \varphi_k(\partial\Omega \setminus J_k) = \emptyset$.

The hypothesis that φ_k' is of class Hölder α in Ω_k can be weakened a little by instead requiring that φ_k' be of class Hölder α only in a relative neighbourhood of J_k in $\overline{\Omega}_k$.

It follows from the above hypotheses that if z_0 is an endpoint of one of the arcs comprising J_k, then φ_k is conformal at z_0. Since $\varphi_k(\overline{\Omega}) \subset \overline{\mathbb{D}}$, and φ_k preserves angles, the interior angle of $\partial\Omega$ at z_0 must be less than or equal to π. This justifies the assumption on the angles at the corners of $\partial\Omega$. This is an important restriction on the class of domains which our methods do not permit us to relax.

By the Schwarz reflection principle and condition (b), one can always find sets Ω_k as in (c) by continuing φ_k analytically across J_k. In general, these sets Ω_k do not intersect in a way that permits the construction of the open circular sectors required in (d). However, if all the interior angles of the corners of $\partial\Omega$ are greater than $2\pi/3$, then it is easy to see that Schwarz reflection produces sets Ω_k which contain such open circular sectors.

Additionally, if φ_k is defined only in $\overline{\Omega}$, φ_k' is Hölder α on Ω and $|\varphi_k'| \ge C > 0$ in J_k, then the extension of φ_k to Ω_k by Schwarz reflection also satisfies that φ_k' is of class Hölder α.

It is easy to check from the definition of an admissible function Φ that Φ' vanishes at most in a finite number of points in $\overline{\Omega}$ and that there is a finite set $X \subset \Omega$ such that the restriction of Φ to $\overline{\Omega} \setminus X$ is injective (i.e., Φ identifies or "glues" at most a finite number of points of $\overline{\Omega}$).

The motivation for our definition of admissible function comes from the case when $\Omega = \Omega_1 \cap \dots \cap \Omega_n$, where Ω_j are Jordan domains and $\varphi_j : \overline{\Omega}_j \to \overline{\mathbb{D}}$ are Riemann maps. See Figure 1.2 for a drawing of the case $n = 2$. In this case $J_k = \partial\Omega \cap \partial\Omega_k$ and we see that $|\varphi_k| = 1$ in J_k, because φ_k maps $\partial\Omega_k$ onto $\mathbb{T}$. In this situation, as we have remarked above, we can use Proposition 1.8 and put $F_k(f) = G_k(f) \circ \varphi_k^{-1}$. Then the difference (1.3) is identically zero. This section aims to extend these kinds of arguments to the case when the φ_k are not univalent but still have some of the properties of Riemann maps (namely that they map some arcs of the boundary bijectively onto subarcs of $\mathbb{T}$).

Now we can state the main result of this section.

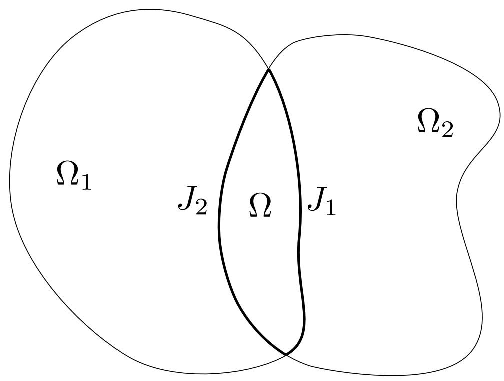

Figure 1.2.: The case $\Omega = \Omega_1 \cap \Omega_2$

Theorem 1.11. *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible, then there exist bounded linear operators $F_k : H^\infty(\Omega) \to H^\infty(\mathbb{D})$, $k = 1, \ldots, n$, such that the operator defined by*

$$f \mapsto f - \sum_{k=1}^{n} F_k(f) \circ \varphi_k, \qquad f \in H^\infty(\Omega),$$

is compact in $H^\infty(\Omega)$ and its range is contained in $A(\overline{\Omega})$. Moreover, F_k maps $A(\overline{\Omega})$ into $A(\overline{\mathbb{D}})$, for $k = 1, \ldots, n$.

The rest of this section is devoted to the proof of this theorem. We will first prove some lemmas that are related to weakly singular integral operators.

Definition 1.12. We say that a domain $\Omega \subset \mathbb{C}$ satisfies the inner chord-arc condition if there is a constant $C > 0$ depending only on Ω such that for every $\zeta, z \in \overline{\Omega}$ there is a piecewise smooth curve $\gamma(\zeta, z)$ which joins ζ and z, is contained in Ω except for its endpoints, and whose length is smaller or equal than $C|\zeta - z|$.

Lemma 1.13. *Let $U \subset \mathbb{C}$ be a domain satisfying the inner chord-arc condition and $\varphi \in A(\overline{U})$ with φ' of class Hölder α, $0 < \alpha \le 1$ in U (so that φ' extends to $\overline{U}$ by continuity). Let $K \subset \overline{U}$ be compact and $\Omega \subset U$ be a domain. Assume that $\varphi(\zeta) \neq \varphi(z)$ if $\zeta \in K$ and $z \in \overline{\Omega} \setminus \{\zeta\}$, and that φ' does not vanish in K.*

Then, the function

$$G(\zeta, z) = \frac{\varphi'(\zeta)}{\varphi(\zeta) - \varphi(z)} - \frac{1}{\zeta - z}$$

satisfies

$$|G(\zeta, z)| \le C|\zeta - z|^{\alpha - 1}, \qquad \zeta \in K, \ z \in \overline{\Omega} \setminus \{\zeta\}.$$

Proof. Let us first check that

$$\left| \frac{\varphi(\zeta) - \varphi(z)}{\zeta - z} \right| \ge C_1 > 0, \qquad \zeta \in K, \ z \in \overline{\Omega} \setminus \{\zeta\}. \tag{1.4}$$

To see this, put

$$h(\zeta, z) = \begin{cases} \frac{\varphi(\zeta) - \varphi(z)}{\zeta - z}, & \text{if } \zeta \in K, z \in \overline{\Omega} \setminus \{\zeta\}, \\ \varphi'(\zeta), & \text{if } \zeta = z \in K. \end{cases}$$

Since h is continuous on the compact set $K\times\overline{\Omega}$ and does not vanish, we get $|h| \geq C_1 > 0$, which implies (1.4).

If $\zeta \in K$ and $z \in \overline{\Omega}\setminus\{\zeta\}$, let $\gamma(\zeta, z)$ be an arc joining ζ and z, contained in U except for its endpoints, and whose length is comparable to $|\zeta - z|$. Then

$$
\begin{aligned}
|\varphi(z) - \varphi(\zeta) - \varphi'(\zeta)(z-\zeta)| = \left| \int_{\gamma(\zeta,z)} \big(\varphi'(u) - \varphi'(\zeta)\big)\, du \right| &\leq C_2 \int_{\gamma(\zeta,z)} |u-\zeta|^{\alpha} |du| \\
&\leq C_3 |z-\zeta|^{\alpha+1}.
\end{aligned}
\tag{1.5}
$$

Using (1.4) and (1.5), we get with $C = C_3/C_1$,

$$
|G(\zeta, z)| = \frac{|\varphi(z) - \varphi(\zeta) - \varphi'(\zeta)(z-\zeta)|}{|\varphi(\zeta) - \varphi(z)||\zeta - z|} \leq C|\zeta - z|^{\alpha-1},
$$

which proves the lemma. □

The following lemma on the compactness of weakly singular integral operators may be well know to specialists. It appears throughout the literature in different forms. The one given here is similar to that in [Kre14, Theorem 2.22].

Lemma 1.14. *Let $\Omega \subset \mathbb{C}$ be bounded domain, $K \subset \mathbb{C}$ a compact piecewise smooth curve, and $G(\zeta, z)$ continuous in $(K \times \overline{\Omega}) \setminus \{(\zeta, \zeta) : \zeta \in K\}$ with $|G(\zeta, z)| \leq C|\zeta - z|^{-\beta}$ for some $\beta < 1$ and every $\zeta \in K$, $z \in \overline{\Omega} \setminus \{\zeta\}$.*

Then the operator

$$
(T\psi)(z) = \int_K G(\zeta, z)\psi(\zeta) d\zeta \tag{1.6}
$$

defines a compact operator $T : L^\infty(K) \to C(\overline{\Omega})$.

Proof. Let $h : \mathbb{R} \to \mathbb{R}$ be continuous with $h(t) = 0$ for $t \leq 1/2$, $h(t) = 1$ for $t \geq 1$, and $0 \leq h \leq 1$ on $\mathbb{R}$. Put $G_n(\zeta, z) = h(n|\zeta - z|)G(\zeta, z)$, so that G_n is continuous in the compact $K \times \overline{\Omega}$, hence uniformly continuous. Let T_n be the operator defined by (1.6) with G_n instead of G. Let us check that $G_n : L^\infty(K) \to C(\overline{\Omega})$ is compact.

First note that for $z \in \overline{\Omega}$ and $\psi \in L^\infty(K)$

$$
|(T_n\psi)(z)| \leq \|\psi\|_\infty |K| \cdot \max_{K\times\overline{\Omega}} |G_n|,
$$

where $|K|$ denotes the length of K. Moreover, if $z_1, z_2 \in \overline{\Omega}$,

$$
\begin{aligned}
|(T_n\psi)(z_1) - (T_n\psi)(z_2)| &\leq \int_K |G_n(\zeta, z_1) - G_n(\zeta, z_2)||\psi(\zeta)||d\zeta| \\
&\leq \|\psi\|_\infty |K| \omega(G_n, |z_1 - z_2|),
\end{aligned}
$$

where $\omega(G_n, \cdot)$ is the modulus of continuity of G_n. This shows that $T_n\psi$ is uniformly continuous on $\overline{\Omega}$. It also shows that T_n maps the unit ball of $L^\infty(K)$ into a uniformly bounded and equicontinuous family in $C(\overline{\Omega})$. Therefore, T_n is compact by the Arzelà-Ascoli theorem.

To see that T is compact, it is enough to show that $T_n \to T$ in the operator norm.

$$\begin{aligned} |(T\psi)(z) - (T_n\psi)(z)| &\le \|\psi\|_\infty \int_K |G(\zeta, z) - G_n(\zeta, z)||d\zeta| \\ &\le C\|\psi\|_\infty \int_{K\cap B_{1/n}(z)} |\zeta - z|^{-\beta}|d\zeta|. \end{aligned}$$

Since $\beta < 1$ and K is piecewise smooth, it is easy to see that the last integral tends to zero uniformly in $z \in \overline{\Omega}$. This finishes the proof. □

If $\Gamma \subset \mathbb{C}$ is a piecewise smooth closed Jordan arc $\psi \in L^\infty(\Gamma)$, and φ and φ' are defined and continuous on Γ, we define the modified Cauchy integral

$$\mathcal{C}_\Gamma^\varphi(\psi)(z) = \int_\Gamma \frac{\varphi'(\zeta)}{\varphi(\zeta) - z}\psi(\zeta)\, d\zeta.$$

The function $\mathcal{C}_\Gamma^\varphi(\psi)$ is analytic in $\mathbb{C} \setminus \overline{\varphi(\Gamma)}$. We write $\mathcal{C}_\Gamma(\psi)$ for the usual Cauchy transform (i.e., when $\varphi(z) = z$).

Lemma 1.15. *Under the hypotheses of Theorem 1.11, if* Γ *is a piecewise smooth closed arc contained in* $\overline{\Omega}_k$*, then the operator defined by*

$$\psi \mapsto \mathcal{C}_\Gamma^{\varphi_k}(\psi) \circ \varphi_k - \mathcal{C}_\Gamma(\psi) \tag{1.7}$$

maps $L^\infty(\Gamma)$ *into* $A(\overline{\Omega})$ *and is compact.*

Proof. We compute

$$\mathcal{C}_\Gamma^{\varphi_k}(\psi) \circ \varphi_k - \mathcal{C}_\Gamma(\psi) = \int_\Gamma \left[\frac{\varphi_k'(\zeta)}{\varphi_k(\zeta) - \varphi_k(z)} - \frac{1}{\zeta - z}\right]\psi(\zeta)\, d\zeta. \tag{1.8}$$

Using Lemma 1.13 with $U = \Omega_k$, we have

$$\left|\frac{\varphi_k'(\zeta)}{\varphi_k(\zeta) - \varphi_k(z)} - \frac{1}{\zeta - z}\right| \le C|\zeta - z|^{\alpha - 1}, \qquad \zeta \in \Gamma,\ z \in \overline{\Omega} \setminus \{\zeta\}. \tag{1.9}$$

By Lemma 1.14, we see that the operator defined by (1.7) is compact from $L^\infty(\Gamma)$ to $C(\overline{\Omega})$. Since its image clearly consists of analytic functions in Ω, the lemma follows. □

Lemma 1.16. *Under the hypotheses of Theorem 1.11, let* $\widehat{J}_k$ *be a closed arc contained in the interior of* J_k *relative to* $\partial\Omega$*. If* $\psi \in L^\infty(\widehat{J}_k)$ *and* $\mathcal{C}_{\widehat{J}_k}(\psi) \in H^\infty(\mathbb{C} \setminus \widehat{J}_k)$*, then the modified Cauchy integral* $\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi)$ *belongs to* $H^\infty(\mathbb{C} \setminus \varphi_k(\widehat{J}_k))$*.*

Proof. We must verify that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi)$ is bounded in $\mathbb{C} \setminus \varphi_k(\widehat{J}_k)$. It is enough to check that it is bounded in $\varphi_k(\Omega_k) \setminus \varphi_k(\widehat{J}_k)$ as $\varphi_k(\Omega_k)$ is an open set containing $\varphi_k(\widehat{J}_k)$.

By Lemma 1.15, the function

$$\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi)(\varphi_k(z)) - \mathcal{C}_{\widehat{J}_k}(\psi)(z)$$

continues to a function in $A(\overline{\Omega}_k)$. In particular, it is bounded in $\Omega_k \setminus \widehat{J}_k$. Since $\mathcal{C}_{\widehat{J}_k}(\psi)$ is bounded in $\mathbb{C} \setminus \widehat{J}_k$, it follows that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi) \circ \varphi_k$ is bounded in $\Omega_k \setminus \widehat{J}_k$, or equivalently $\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi)$ is bounded in $\varphi_k(\Omega_k \setminus \widehat{J}_k)$. Since $\varphi_k(\Omega_k) \setminus \varphi_k(\widehat{J}_k) \subset \varphi_k(\Omega_k \setminus \widehat{J}_k)$, we conclude that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k}(\psi) \in H^\infty(\mathbb{C} \setminus \varphi_k(\widehat{J}_k))$. □

We will need the following lemma, which is well known from the classical theory of Cauchy integrals. See, for instance, [Gak90, Chapter I, Section 5.1].

Lemma 1.17. *Let Γ be a piecewise smooth Jordan curve and Ω the region interior to it. If ψ is of class Hölder α on Γ, $0 < \alpha < 1$, then $\mathcal{C}_\Gamma(\psi)$ is of class Hölder α in $\overline{\Omega}$.*

Now we can give the proof of Theorem 1.11.

Proof of Theorem 1.11. Let us first justify that it is enough to prove the theorem for the case when each of the sets J_k is a single arc and these arcs intersect only at their endpoints. Write $J_k = \Gamma_{k,1} \cup \cdots \cup \Gamma_{k,r_k}$, where $\Gamma_{k,j}$ are disjoint arcs, and put $\psi_{k,j} = \varphi_k$, for $j = 1, \ldots, r_k$. Now pass to smaller arcs $\widetilde{\Gamma}_{k,j} \subset \Gamma_{k,j}$ such that the arcs $\widetilde{\Gamma}_{k,j}$ intersect only at endpoints but still cover all $\partial\Omega$. The functions $\psi_{k,j}$ and sets $\Gamma_{k,j}$ form an admissible family. Assume that the conclusion of Theorem 1.11 is true for this family, and let $F_{k,j}$ be the linear operators associated to each of the functions $\psi_{k,j}$. Putting $F_k = F_{k,1} + \cdots + F_{k,r_k}$ and recalling that $\psi_{k,j} = \varphi_k$, we see that the conclusion of Theorem 1.11 is true for the family $\{\varphi_k\}$ as well.

We give the proof for a simply connected domain Ω. This case has the advantage that $\partial\Omega$ is a single Jordan curve, so the notation for numbering the arcs $J_k \subset \partial\Omega$ is easier. The proof for a multiply connected domain Ω is essentially the same, except that the notation for the arcs J_k is a bit more complex.

Let us assume that the arcs $J_1, \ldots, J_n$ are numbered in a cyclic order, i.e., in such a way that J_k intersects J_{k-1} and J_{k+1} (here and henceforth we consider subindices modulo n). Let $z_k \in \partial\Omega$ be the common endpoint of J_k and J_{k+1}, $k = 1, \ldots, n$.

Let V_k be a small disk centered at z_k (its radius is determined later). Choose functions $\eta_1, \ldots, \eta_n, \nu_1, \ldots, \nu_n \in C^\infty(\partial\Omega)$ such that $0 \le \eta_k \le 1$, $0 \le \nu_k \le 1$ on $\partial\Omega$, $\eta_1 + \cdots + \eta_n + \nu_1 + \cdots + \nu_n = 1$, $\operatorname{supp} \nu_k \subset V_k \cap \partial\Omega$, and η_k is supported on the interior of J_k relative to $\partial\Omega$.

Put $J_k^+ = J_k \cap V_k$, $J_k^- = J_k \cap V_{k-1}$ and let R_k be a rigid rotation around the point z_k such that $J_k^R \stackrel{\text{def}}{=} R_k J_k^+$ is contained in $[(S_k(z_k) \cap S_{k+1}(z_k)) \setminus \overline{\Omega}] \cup \{z_k\}$ (see condition (d) in Definition 1.10). Figure 1.3 is a picture of the relevant geometric objects.

For $f \in H^\infty(\Omega)$, define

$$F_k(f) = \mathcal{C}_{J_k}^{\varphi_k}(f) - \mathcal{C}_{J_k^R}^{\varphi_k}((\nu_k f) \circ R_k^{-1}) + \mathcal{C}_{J_{k-1}^R}^{\varphi_k}((\nu_{k-1} f) \circ R_{k-1}^{-1}). \tag{1.10}$$

Let us first check that $F_k(f) \in H^\infty(\mathbb{D})$. To do this, put

$$\begin{aligned} G_k^+(f) &= \mathcal{C}_{J_k^+}^{\varphi_k}(\nu_k f) - \mathcal{C}_{J_k^R}^{\varphi_k}((\nu_k f) \circ R_k^{-1}), \\ G_k^-(f) &= \mathcal{C}_{J_{k+1}^-}^{\varphi_{k+1}}(\nu_k f) + \mathcal{C}_{J_k^R}^{\varphi_{k+1}}((\nu_k f) \circ R_k^{-1}). \end{aligned} \tag{1.11}$$

Then we can write $F_k(f)$ as

$$F_k(f) = \mathcal{C}_{J_k}^{\varphi_k}(\eta_k f) + G_k^+(f) + G_{k-1}^-(f),$$

because

$$\mathcal{C}_{J_k}^{\varphi_k}(f) = \mathcal{C}_{J_k}^{\varphi_k}(\eta_k f) + \mathcal{C}_{J_k^+}^{\varphi_k}(\nu_k f) + \mathcal{C}_{J_k^-}^{\varphi_k}(\nu_{k-1} f).$$

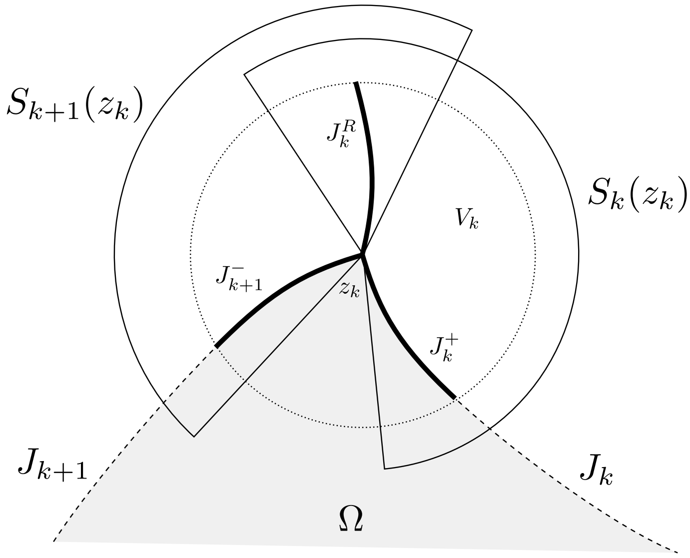

Figure 1.3.: Geometric picture of the proof of Theorem 1.11

Since $f \in H^\infty(\Omega)$, and η_k is supported on a closed arc contained in the interior of J_k, it is easy to see that $\mathcal{C}_{J_k}(\eta_k f)$ belongs to $H^\infty(\mathbb{C} \setminus J_k)$. Lemma 1.16 allows us to conclude that $\mathcal{C}_{J_k}^{\varphi_k}(\eta_k f)$ belongs to $H^\infty(\mathbb{C} \setminus \varphi_k(J_k))$. As $\varphi_k(J_k) \subset \mathbb{T}$, this implies that $\mathcal{C}_{J_k}^{\varphi_k}(\eta_k f) \in H^\infty(\mathbb{D})$.

Since $|\varphi_k| < 1$ in Ω and $|\varphi_k| = 1$ in J_k, by the Schwarz reflection principle we can assume that

$$|\varphi_k| > 1 \text{ in } \Omega_k \setminus \Omega$$

just by making Ω_k smaller if necessary (i.e., replacing Ω_k by $U_k \cap \Omega_k$, where U_k is some open set containing $J_k \cup \Omega$).

The following claim is justified below.

Claim 1. *$G_k^+(f) \in H^\infty(\mathbb{C} \setminus \varphi_k(J_k^+ \cup J_k^R))$ and $G_k^-(f) \in H^\infty(\mathbb{C} \setminus \varphi_{k+1}(J_{k+1}^- \cup J_k^R))$.*

Since $|\varphi_k| = 1$ in J_k, $\varphi_k(J_k^+ \cup J_k^R) \cap \mathbb{D} = \emptyset$, and so by the claim, $G_k^+(f)$ and $G_k^-(f)$ belong to $H^\infty(\Omega)$. It follows that $F_k(f) \in H^\infty(\mathbb{D})$ for every $f \in H^\infty(\Omega)$. Moreover, it is clear from the proof of these lemmas that F_k maps $H^\infty(\Omega)$ into $H^\infty(\mathbb{D})$ and is bounded.

We next show that the linear map

$$f \mapsto f - \sum_{k=1}^{n} F_k(f) \circ \varphi_k,$$

is a compact operator on $H^\infty(\Omega)$, whose range is contained in $A(\overline{\Omega})$. A simple calculation using

$$f = \sum_{k=1}^{n} \mathcal{C}_{J_k}(f)$$

gives

$$f - \sum_{k=1}^{n} F_k(f) \circ \varphi_k = \sum_{k=1}^{n} A_k(f) - B_k(\nu_k f), \tag{1.12}$$

where

$$\begin{aligned} A_k(\psi) &= \mathcal{C}_{J_k}(\psi) - \mathcal{C}^{\varphi_k}_{J_k}(\psi) \circ \varphi_k, \\ B_k(\psi) &= \mathcal{C}^{\varphi_{k+1}}_{J_k^R}(\psi \circ R_k^{-1}) \circ \varphi_{k+1} - \mathcal{C}^{\varphi_k}_{J_k^R}(\psi \circ R_k^{-1}) \circ \varphi_k. \end{aligned} \tag{1.13}$$

By Lemma 1.15, the operator A_k is compact from $L^\infty(\partial\Omega)$ into $A(\overline{\Omega})$. To see that B_k has the same property, write

$$B_k(\psi) = \big[\mathcal{C}^{\varphi_{k+1}}_{J_k^R}(\psi \circ R_k^{-1}) \circ \varphi_{k+1} - \mathcal{C}_{J_k^R}(\psi \circ R_k^{-1})\big] + \big[\mathcal{C}_{J_k^R}(\psi \circ R_k^{-1}) - \mathcal{C}^{\varphi_k}_{J_k^R}(\psi \circ R_k^{-1}) \circ \varphi_k\big],$$

and apply Lemma 1.15 to each of the two terms in brackets.

It remains to prove that the operators F_k map $A(\overline{\Omega})$ into $A(\overline{\mathbb{D}})$. It is enough to check that if f is analytic on some open neighbourhood of $\overline{\Omega}$, then $F_k(f) \in C(\overline{\mathbb{D}})$, as the space of functions analytic on $\overline{\Omega}$ is dense in $A(\overline{\Omega})$ and F_k is bounded.

By (1.10) and properties of the modified Cauchy integral, $F_k(f)$ is continuous on $\overline{\mathbb{D}} \setminus \varphi_k(J_k)$. Next check that $F_k(f)$ extends by continuity to $\varphi_k(J_k)$. Since φ_k' does not vanish on J_k, there exists a continuous local inverse of φ_k on each point of $\varphi_k(J_k)$. This implies that it is enough to verify that $F_k(f) \circ \varphi_k$ is continuous in $\overline{\Omega}$. Put

$$\begin{aligned} \widetilde{F}_k(f) &= \mathcal{C}_{J_k}(\eta_k f) + \widetilde{G}_k^+(f) + \widetilde{G}_{k-1}^-(f), \\ \widetilde{G}_k^+(f) &= \mathcal{C}_{J_k^+}(\nu_k f) - \mathcal{C}_{J_k^R}((\nu_k f) \circ R_k^{-1}), \\ \widetilde{G}_k^-(f) &= \mathcal{C}_{J_{k+1}^-}(\nu_k f) + \mathcal{C}_{J_k^R}((\nu_k f) \circ R_k^{-1}), \end{aligned}$$

i.e., replace the modified Cauchy integrals in the formulas for F_k, G_k^- and G_k^+ by regular Cauchy integrals to get $\widetilde{F}_k$, $\widetilde{G}_k^-$ and $\widetilde{G}_k^+$. Arguing as above for the operators A_k and B_k, we see that $f \mapsto F_k(f) \circ \varphi_k - \widetilde{F}_k(f)$ defines a compact operator whose range is contained in $C(\overline{\Omega})$. Thus it is enough to show that $\widetilde{F}_k(f) \in C(\overline{\Omega})$.

By Lemma 1.17, it is easy to see that $\mathcal{C}_{J_k}(\eta_k f) \in C(\overline{\Omega})$. We have

$$\widetilde{G}_k^+(f) + \widetilde{G}_k^-(f) = \mathcal{C}_{\partial\Omega \cap V_k}(\nu_k f).$$

Also by Lemma 1.17, the right hand side of this equality belongs to $C(\overline{\Omega})$. Therefore, it suffices to check that $\widetilde{G}_k^-(f) \in C(\overline{\Omega})$.

Now $\widetilde{G}_k^-(f) = \mathcal{C}_{J_{k+1}^- \cup J_k^R}(\widetilde{f})$, where $\widetilde{f}(z) = (\nu_k f)(z)$ for $z \in J_{k+1}^-$, and $\widetilde{f}(z) = (\nu_k f)(R_k^{-1}(z))$ for $z \in J_k^R$. Since f is analytic in a neighbourhood of $\overline{\Omega}$, $\widetilde{f}$ is Lipschitz in $J_{k-1}^- \cup J_k^R$, and since $\widetilde{f}$ vanishes identically near the endpoints of $J_{k-1}^- \cup J_k^R$, Lemma 1.17 implies that $\widetilde{G}_k^-(f) \in C(\overline{\Omega})$. This finishes the proof of the theorem. □

Proof of Claim 1. We use the same techniques as those used in [HN01] to prove Theorem 4.1 to show that $g_k^- \overset{\text{def}}{=} G_k^-(f) \in H^\infty(\mathbb{C} \setminus \varphi_{k+1}(J_{k+1}^- \cup J_k^R))$. Similar reasoning can be applied to $G_k^+(f)$.

Let

$$h_k^+ = \mathcal{C}_{J_k^+}(\nu_k f), \qquad h_k^- = \mathcal{C}_{J_{k+1}^-}(\nu_k f),$$

so that $h_k^- + h_k^+ = \mathcal{C}_{V_k \cap \partial\Omega}(\nu_k f)$, which because $f \in H^\infty(\Omega)$, belongs to $H^\infty(\mathbb{C} \setminus (V_k \cap \partial\Omega))$. Theorem 4.1 in [HN01] applies, and so $h_k^- + h_k^+ \circ R_k^{-1}$ belongs to $H^\infty(\mathbb{C} \setminus (J_k^- \cup J_k^R))$.

We next prove that g_k^- is bounded in $\mathbb{C} \setminus \varphi_{k+1}(J_{k+1}^- \cup J_k^R)$. It is clearly analytic in this set. Let S_{k+1}^- be an open circular sector with vertex on z_k, such that $J_{k+1}^- \cup J_k^R \subset S_{k+1}^- \cup \{z_k\}$ and $S_{k+1}^- \subset \Omega_{k+1}$. This circular sector can be chosen by shrinking one of the circular sectors which appear in condition (d) in the Definition 1.10. We first show that g_k^- is bounded in $\varphi_{k+1}(S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R))$. To do this, observe that by a change of variables in the integral defining the Cauchy transform,

$$h_k^+ \circ R_k^{-1} = \mathcal{C}_{J_k^R}((\nu_k f) \circ R_k^{-1}).$$

Now compute

$$\begin{aligned} g_k^- \circ \varphi_{k+1} - (h_k^- + h_k^+ \circ R_k^{-1}) = {} & \big[\mathcal{C}_{J_{k+1}^-}^{\varphi_{k+1}}(\nu_k f) \circ \varphi_{k+1} - \mathcal{C}_{J_{k+1}^-}(\nu_k f)\big] \\ & + \big[\mathcal{C}_{J_k^R}^{\varphi_{k+1}}((\nu_k f) \circ R_k^{-1}) \circ \varphi_{k+1} - \mathcal{C}_{J_k^R}((\nu_k f) \circ R_k^{-1})\big]. \end{aligned}$$

A similar argument to the one used in Lemma 1.16 shows that each of the expressions in square brackets is bounded in $S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R)$. Therefore, $g_k^- \circ \varphi_{k+1}$ is also bounded in this set as $h_k^- + h_k^+ \circ R_k^{-1}$ is bounded there. It follows that g_k^- is bounded in $\varphi_{k+1}(S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R))$.

It remains to prove that g_k^- is bounded in $\mathbb{C} \setminus \varphi_{k+1}(S_{k+1}^-)$. If S is an open circular sector, we say that its straight edges are the two line segments which form a part of its boundary. Put $w_k = \varphi_{k+1}(z_k)$. Choose two open circular sectors S and S' with vertex w_k having the following properties (see Figure 1.4):

- $\overline{S} \cap \overline{S'} = \{w_k\}$.
- $\varphi_{k+1}(J_{k+1}^- \cup J_k^R) \subset S' \cup \{w_k\}$.
- $\mathbb{D}_\varepsilon(w_k) \setminus \varphi_{k+1}(S_{k+1}^-) \subset S \cup \{w_k\}$ for some $\varepsilon > 0$.
- The straight edges of S are contained in $\varphi_{k+1}(S_{k+1}^-) \cup \{w_k\}$.

Such circular sectors can be chosen by shrinking V_k if necessary, using the fact that φ_{k+1} is conformal at z_k.

It is enough to show that g_k^- is bounded in S, because $g_k^-(z)$ is clearly uniformly bounded when z is away from $\varphi_{k+1}(J_{k+1}^- \cup J_k^R)$, and we have already seen that g_k^- is bounded in $\varphi_{k+1}(S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R))$. This is done by using a weak form of the Phragmén-Lindelöf principle, in the same manner as in [HN01]. Since g_k^- is bounded in the straight edges of S except at the vertex w_k (the straight edges are contained in $\varphi_{k+1}(S_{k+1}^- \setminus (J_{k+1}^- \cup J_k^R))$, except for w_k), it suffices to show that g_k^- is $O(|z - w_k|^{-1})$ as $z \to w_k$, $z \in S$.

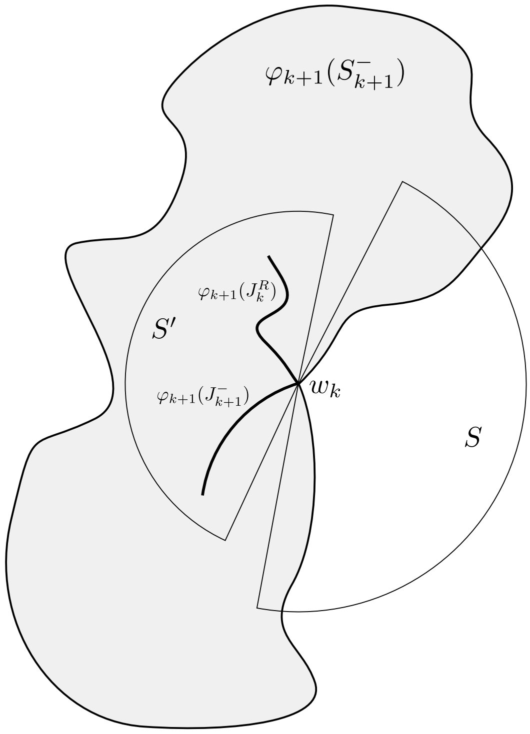

Figure 1.4.: The circular sectors S and S'

First, estimate

$$\begin{aligned}|(z-w_k)g_k^-(z)| &\le \int_{J^-_{k+1}} \frac{|z-w_k|}{|\varphi_{k+1}(\zeta)-z|}|(\nu_k f)(\zeta)\varphi'_{k+1}(\zeta)|\,|d\zeta| \\ &\quad + \int_{J^R_k} \frac{|z-w_k|}{|\varphi_{k+1}(\zeta)-z|}|(\nu_k f)(\zeta)\varphi'_{k+1}(\zeta)|\,|d\zeta| \\ &\le \|f\|_\infty \int_{J^-_{k+1}\cup J^R_k} \frac{|z-w_k|}{|\varphi_{k+1}(\zeta)-z|}|\varphi'_{k+1}(\zeta)|\,|d\zeta|.\end{aligned}$$

We claim that $a(\zeta,z) \stackrel{\text{def}}{=} |z-w_k|/|\varphi_{k+1}(\zeta)-z|$ is uniformly bounded for $z \in S$ and $\zeta \in J^-_k \cup J^R_k$, which follows from the observation that $\varphi_{k+1}(\zeta) \in S'$ and $z \in S$, so that $a(\zeta,z) \le C$ due to the geometry of the cones S and S'. The last integral is therefore uniformly bounded, so g_k^- is $O(|z-w_k|^{-1})$ and we conclude that g_k^- belongs to $H^\infty(\mathbb{C}\setminus\varphi_{k+1}(J_{k+1}\cup J^R_k))$.

To see that g_k^+ belongs to $H^\infty(\mathbb{C}\setminus\varphi_k(J^+_k\cup J^R_k))$, use similar reasoning with $h_k^+ - h_k^+\circ R_k^{-1}$ instead of $h_k^- + h_k^+\circ R_k^{-1}$, an appropriate open circular sector S_k^+ for S^-_{k+1}, and φ_k in place of φ_{k+1}. □

The rest of this section deals with a version of Theorem 1.11 for families of admissible functions that depend on some parameter continuously. This will be used in some of our applications to operator theory in Chapter 3. For some technical difficulties that appear there, we will have to apply an appropriate small perturbation to our admissible function and show that the results we know for the perturbed admissible function pass well to the limit as the perturbation tends to the original admissible function. The result we will need in Chapter 3 is the following lemma.

Lemma 1.18. *Let $\Phi_\varepsilon = \left(\varphi_1^\varepsilon, \dots, \varphi_n^\varepsilon\right) : \overline{\Omega} \to \overline{\mathbb{D}}^n$, $0 \le \varepsilon \le \varepsilon_0$ be a collection of functions. Assume that Ψ_ε is admissible for every ε, and, moreover, that one can choose sets Ω_k in Definition 1.10 so as not to depend on ε. Assume that $\varphi_k^\varepsilon \in C^{1+\alpha}(\Omega_k)$, with $0 < \alpha < 1$, and that the mapping $\varepsilon \mapsto \varphi_k^\varepsilon$ is continuous from $[0, \varepsilon_0]$ to $C^{1+\alpha}(\Omega_k)$.*

Then there exist bounded linear operators $F_k^\varepsilon : A(\overline{\Omega}) \to A(\overline{\mathbb{D}})$, such that for

$$L_\varepsilon(f) = \sum F_k^\varepsilon(f) \circ \varphi_k^\varepsilon,$$

and $0 \le \varepsilon \le \varepsilon_0$, $L_\varepsilon - I$ is a compact operator on $A(\overline{\Omega})$, the mapping $\varepsilon \mapsto L_\varepsilon$ is norm continuous, and $\|F_k^\varepsilon\| \le C$ for $k = 1, \dots, n$, where C is a constant independent of k and ε.

The proof of Lemma 1.18 uses the following technical fact:

Lemma 1.19. *Let Ω be a bounded domain satisfying the inner chord-arc condition, $\{\varphi_\varepsilon\}_{0 \le \varepsilon \le \varepsilon_0} \subset A(\overline{\Omega})$ with φ_ε' of class Hölder α in Ω and such that the mapping $\varepsilon \mapsto \varphi_\varepsilon$ is continuous from $[0, \varepsilon_0]$ to $C^{1+\alpha}(\Omega)$. Let $K \subset \overline{\Omega}$ be compact. Assume that $\varphi_\varepsilon(\zeta) \ne \varphi_\varepsilon(z)$ if $\zeta \in K$, $z \in \overline{\Omega} \setminus \{\zeta\}$ and $0 \le \varepsilon \le \varepsilon_0$. Assume also that for each $0 \le \varepsilon \le 1$, φ_ε' does not vanish in K. Then for $\zeta \in K$, $z \in \overline{\Omega} \setminus \{\zeta\}$, and $\varepsilon, \delta \in [0, \varepsilon_0]$,*

$$\left| \frac{\varphi_\varepsilon'(\zeta)}{\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)} - \frac{\varphi_\delta'(\zeta)}{\varphi_\delta(\zeta) - \varphi_\delta(z)} \right| \le C \|\varphi_\varepsilon' - \varphi_\delta'\|_{C^\alpha} |\zeta - z|^{\alpha - 1}.$$

Proof. First check that

$$\left| \frac{\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)}{\zeta - z} \right| \ge C > 0, \qquad \zeta \in K, \ z \in \overline{\Omega} \setminus \{\zeta\}, \ 0 \le \varepsilon \le \varepsilon_0. \tag{1.14}$$

To see this, put

$$h(\zeta, z, \varepsilon) = \begin{cases} \frac{\varphi_\varepsilon(\zeta) - \varphi_\varepsilon(z)}{\zeta - z}, & \text{if } \zeta \in K, z \in \overline{\Omega} \setminus \{\zeta\}, \\ \varphi_\varepsilon'(\zeta), & \text{if } \zeta = z \in K, \end{cases}$$

which is continuous on the compact set $K \times \overline{\Omega} \times [0, \varepsilon_0]$. As h does not vanish, $|h| \ge C > 0$, implying (1.14).

Take $u \in \Omega$. Since $|\psi(u) - \psi(\zeta)| \le \|\psi\|_{C^\alpha} |u - \zeta|^\alpha$ and $|\psi(u)| \le \|\psi\|_{C^\alpha}$,

$$\begin{aligned}
&|\varphi_\varepsilon'(\zeta)\varphi_\delta'(u) - \varphi_\delta'(\zeta)\varphi_\varepsilon'(u)| \\
&\quad = \Big| [\varphi_\varepsilon'(u) - \varphi_\varepsilon'(\zeta)][\varphi_\delta'(\zeta) - \varphi_\varepsilon'(\zeta)] + \varphi_\varepsilon'(\zeta)[\varphi_\varepsilon'(u) - \varphi_\varepsilon'(\zeta) + \varphi_\delta'(\zeta) - \varphi_\delta'(u)] \Big| \\
&\quad \le \Big| \varphi_\varepsilon'(u) - \varphi_\varepsilon'(\zeta) \Big| \Big| \varphi_\delta'(\zeta) - \varphi_\varepsilon'(\zeta) \Big| + \Big| \varphi_\varepsilon'(\zeta) \Big| \Big| (\varphi_\varepsilon' - \varphi_\delta')(u) - (\varphi_\varepsilon' - \varphi_\delta')(\zeta) \Big| \\
&\quad \le \|\varphi_\varepsilon'\|_{C^\alpha} |u - \zeta|^\alpha \|\varphi_\varepsilon' - \varphi_\delta'\|_{C^\alpha} + \|\varphi_\varepsilon'\|_{C^\alpha} \|\varphi_\varepsilon' - \varphi_\delta'\|_{C^\alpha} |u - \zeta|^\alpha.
\end{aligned}$$

But

$$\varphi_\varepsilon'(\zeta)[\varphi_\delta(\zeta)-\varphi_\delta(z)]-\varphi_\delta'(\zeta)[\varphi_\varepsilon(\zeta)-\varphi_\varepsilon(z)] = \int_{\gamma(z,\zeta)} \big(\varphi_\varepsilon'(\zeta)\varphi_\delta'(u)-\varphi_\delta'(\zeta)\varphi_\varepsilon'(u)\big)\,du,$$

where $\gamma(z,\zeta)$ is an arc joining z and ζ, contained in Ω except for its endpoints, and whose length is comparable to $|z-\zeta|$. Therefore

$$\begin{aligned}\Big|\varphi_\varepsilon'(\zeta)[\varphi_\delta(\zeta)-\varphi_\delta(z)]-\varphi_\delta'(\zeta)[\varphi_\varepsilon(\zeta)-\varphi_\varepsilon(z)]\Big| &\le C\|\varphi_\varepsilon'-\varphi_\delta'\|_{C^\alpha}\int_{\gamma(z,\zeta)}|u-\zeta|^\alpha\,|du| \\ &\le C\|\varphi_\varepsilon'-\varphi_\delta'\|_{C^\alpha}|z-\zeta|^{\alpha+1}.\end{aligned}$$

Combining this with (1.14),

$$\begin{aligned}\left|\frac{\varphi_\varepsilon'(\zeta)}{\varphi_\varepsilon(\zeta)-\varphi_\varepsilon(z)}-\frac{\varphi_\delta'(\zeta)}{\varphi_\delta(\zeta)-\varphi_\delta(z)}\right| &= \left|\frac{\varphi_\varepsilon'(\zeta)[\varphi_\delta(\zeta)-\varphi_\delta(z)]-\varphi_\delta'(\zeta)[\varphi_\varepsilon(\zeta)-\varphi_\varepsilon(z)]}{[\varphi_\varepsilon(\zeta)-\varphi_\varepsilon(z)][\varphi_\delta(\zeta)-\varphi_\delta(z)]}\right| \\ &\le C\|\varphi_\varepsilon'-\varphi_\delta'\|_{C^\alpha}\frac{|z-\zeta|^{\alpha+1}}{|\varphi_\varepsilon(\zeta)-\varphi_\varepsilon(z)||\varphi_\delta(\zeta)-\varphi_\delta(z)|} \\ &\le C\|\varphi_\varepsilon'-\varphi_\delta'\|_{C^\alpha}|z-\zeta|^{\alpha-1}.\end{aligned}$$

□

Proof of Lemma 1.18. The construction of the functions η_k and ν_k used in the proof of Theorem 1.11 depends solely on the geometry of Ω, and not on the functions φ_k. So define F_k^ε by equation (1.10), replacing φ_k with φ_k^ε. Then $L_\varepsilon - I$ is compact by the proof of Theorem 1.11. We also define A_k^ε and B_k^ε by equation (1.13), with φ_k^ε instead of φ_k. Then by (1.12),

$$L_\varepsilon(f)-L_\delta(f)=\sum_{k=1}^n (A_k^\delta - A_k^\varepsilon)(f)+(B_k^\delta-B_k^\varepsilon)(\nu_k f).$$

Note that

$$\begin{aligned}(A_k^\delta - A_k^\varepsilon)(f)(z) &= \mathcal{C}_{J_k}^{\varphi_k^\varepsilon}(f)(z)-\mathcal{C}_{J_k}^{\varphi_k^\delta}(f)(z) \\ &= \int_{J_k}\Big(\frac{(\varphi_k^\varepsilon)'(\zeta)}{\varphi_k^\varepsilon(\zeta)-\varphi_k^\varepsilon(z)}-\frac{(\varphi_k^\delta)'(\zeta)}{\varphi_k^\delta(\zeta)-\varphi_k^\delta(z)}\Big)f(\zeta)\,d\zeta.\end{aligned}$$

Using Lemma 1.19 and the fact that

$$\int_{J_k}|z-\zeta|^{\alpha-1}\,|d\zeta|<\infty,$$

we have $\|A_k^\delta - A_k^\varepsilon\| \le C\|(\varphi_k^\varepsilon)'-(\varphi_k^\delta)'\|_{C^\alpha}$. Also, $\|B_k^\delta - B_k^\varepsilon\| \le C\|(\varphi_k^\varepsilon)'-(\varphi_k^\delta)'\|_{C^\alpha}$ by similar reasoning. These inequalities imply that L_ε depends continuously on ε in the norm topology.

To see that $\|F_k^\varepsilon\| \le C$, with C independent of ε, one must examine the proofs of Theorem 1.11 and Lemma 1.16 to check that the constants that bound the operators which appear there can be taken to be independent of ε. First, we give the details concerning the proof of Lemma 1.16.

Instead of (1.9), we require the inequality

$$\left|\frac{(\varphi_k^\varepsilon)'(\zeta)}{\varphi_k^\varepsilon(\zeta)-\varphi_k(z)}-\frac{1}{\zeta-z}\right| \le C|\zeta-z|^{\alpha-1}, \qquad \zeta\in J_k,\ z\in\overline{\Omega}_k\setminus\{\zeta\},\ 0\le\varepsilon\le\varepsilon_0. \quad (1.15)$$

Here C should be a constant independent of ε, so we cannot simply apply Lemma 1.13. To prove this inequality, apply Lemma 1.13 to φ_k^0 and get (1.15) for $\varepsilon=0$, and then use Lemma 1.19 to obtain

$$\left|\frac{(\varphi_k^\varepsilon)'(\zeta)}{\varphi_k^\varepsilon(\zeta)-\varphi_k^\varepsilon(z)}-\frac{(\varphi_k^0)'(\zeta)}{\varphi_k^0(\zeta)-\varphi_k^0(z)}\right| \le C|\zeta-z|^{\alpha-1}, \quad \zeta\in J_k,\ z\in\overline{\Omega}_k\setminus\{\zeta\},\ 0\le\varepsilon\le\varepsilon_0,$$

where C is independent of ε. Then (1.15) follows from the triangle inequality.

Let $\widehat{J}_k$ and ψ be as in the statement of Lemma 1.16. We verify that $\|\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)\|_\infty \le C$, with C independent of ε. By the proof of Lemma 1.16, $\varphi_k^\varepsilon(\Omega_k)$ is an open set containing $\varphi_k^\varepsilon(\widehat{J}_k)$. Moreover, it follows from the continuity of φ_k^ε in ε and the compactness of the interval $[0,\varepsilon_0]$ that the distance from $\varphi_k^\varepsilon(\widehat{J}_k)$ to the boundary of $\varphi_k^\varepsilon(\Omega_k)$ is bounded below by a positive constant independent of ε. Therefore, it suffices to show that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)$ is bounded in $\varphi_k^\varepsilon(\Omega_k)\setminus\varphi_k^\varepsilon(\widehat{J}_k)$ by a constant independent of $\varepsilon_k^\varepsilon$, because when the distance from some point z to $\varphi_k^\varepsilon(\widehat{J}_k)$ is greater than a constant, $\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)(z)$ is readily bounded by a constant independent of z and ε.

To show that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)$ is bounded in $\varphi_k^\varepsilon(\Omega_k)\setminus\varphi_k^\varepsilon(\widehat{J}_k)$, we prove as in Lemma 1.16 that $\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)\circ\varphi_k^\varepsilon$ is bounded in $\Omega_k\setminus\widehat{J}_k$. Write (1.8) for φ_k^ε instead of φ_k and $\widehat{J}_k$ instead of Γ, and then use (1.15) to obtain

$$\left|\mathcal{C}_{\widehat{J}_k}^{\varphi_k^\varepsilon}(\psi)(\varphi_k(z))-\mathcal{C}_{\widehat{J}_k}(\psi)(z)\right| \le C\|\psi\|_\infty\int_{\widehat{J}_k}|\zeta-z|^{\alpha-1}\,d\zeta \le C\|\psi\|_\infty,$$

where C is independent of ε. Since $\mathcal{C}_{\widehat{J}_k}(\psi)\in H^\infty(\mathbb{C}\setminus\widehat{J}_k)$, we get the required bound.

It remains to check that the H^∞ norms in Claim 1 (see the proof of Theorem 1.11) can be bounded by a constant independent of ε. We can apply methods similar to the ones that we have used for Lemma 1.16. Define $(G_k^-)^\varepsilon$ as in (1.11), replacing φ_k with φ_k^ε. Put $g_k^\varepsilon=(G_k^-)^\varepsilon(f)$. This is in $H^\infty(\mathbb{C}\setminus\varphi_{k+1}^\varepsilon(J_{k+1}^-\cup J_k^R))$ by Claim 1. We want to show that g_k^ε is bounded by a constant independent of ε.

Define h_k^+ and h_k^- as in the proof of Claim 1 (these functions do not depend on φ_k). Compute

$$\begin{aligned} g_k^\varepsilon\circ\varphi_{k+1}^\varepsilon-(h_k^-+h_k^+\circ R_k^{-1}) = {}& \\ & [\mathcal{C}_{J_{k+1}^-}^{\varphi_{k+1}^\varepsilon}(\nu_k f)\circ\varphi_{k+1}^\varepsilon-\mathcal{C}_{J_{k+1}^-}(\nu_k f)] \\ & +[\mathcal{C}_{J_k^R}^{\varphi_{k+1}^\varepsilon}((\nu_k f)\circ R_k^{-1})\circ\varphi_{k+1}^\varepsilon-\mathcal{C}_{J_k^R}((\nu_k f)\circ R_k^{-1})]. \end{aligned}$$

Arguing as before and using (1.15), each of the two terms in brackets is bounded by a constant independent of φ. Since $h_k^-+h_k^+\circ R_k^{-1}\in H^\infty(\mathbb{C}\setminus(J_k^-\cup J_k^R))$, $g_k^\varepsilon\circ\varphi_{k+1}^\varepsilon$ is uniformly bounded in $S^-_{k+1} \setminus (J^-_{k+1} \cup J^R_k)$, and so g^ε_k is uniformly bounded in $\varphi^\varepsilon_{k+1}(S^-_{k+1} \setminus (J^-_{k+1} \cup J^R_k))$.

Now choose open circular sectors S_ε and S'_ε with vertex on $w^\varepsilon_k = \varphi^\varepsilon_{k+1}(z_k)$ such that $\overline{S_\varepsilon} \cap \overline{S'_\varepsilon} = \{w^\varepsilon_k\}$, and satisfying the following conditions (see Figure 1.4 in the proof of Theorem 1.11):

- $\varphi^\varepsilon_{k+1}(J^-_{k+1} \cup J^R_k) \subset S'_\varepsilon \cup \{w^\varepsilon_k\}$.
- $\mathbb{D}_{\varepsilon_0}(w^\varepsilon_k) \setminus \varphi^\varepsilon_{k+1}(S^-_{k+1}) \subset S_\varepsilon \cup \{w^\varepsilon_k\}$, for some $\varepsilon_0 > 0$.
- The straight edges of S_ε are contained in $\varphi^\varepsilon_{k+1}(S^-_{k+1}) \cup \{w^\varepsilon_k\}$.

This can be done by the continuity of $\varphi^\varepsilon_{k+1}$ in ε and by shrinking V_k if necessary.

To show that g^ε_k is bounded in S_ε, we use the Phragmén-Lindelöf principle as in the proof of Claim 1. There, we proved that g^ε_k is $O(|z - w^\varepsilon_k|^{-1})$ as $z \to w^\varepsilon_k$. Thus, g^ε_k is bounded in S_ε by the supremum of $|g^\varepsilon_k|$ on the straight edges of S_ε. Since these straight edges are contained in $\varphi^\varepsilon_{k+1}(S_{k+1})$ and we had a bound for φ^ε_k which is uniform in ε on this set, there is a bound on S_ε which is also uniform in ε.

Clearly, $g^\varepsilon_k(z)$ is uniformly bounded in ε and z when the distance from z to the set $\varphi^\varepsilon_{k+1}(J^-_{k+1} \cup J^R_k)$ is greater than a positive constant. Also, g^ε_k is uniformly bounded on $U_\varepsilon \setminus \varphi^\varepsilon_{k+1}(J^-_{k+1} \cup J^R_k)$, where U_ε is some open set containing $\varphi^\varepsilon_{k+1}(J^-_{k+1} \cup J^R_k)$ and such that the distance from ∂U_ε to $\varphi^\varepsilon_{k+1}(J^-_{k+1} \cup J^R_k)$ is bounded below by a positive constant independent of ε. This finishes the proof. $\square$

2. Algebras of functions on analytic varieties

This chapter is based on joint work with Michael Dritschel and Dmitry Yakubovich. The results of this chapter are contained in the article [DEY17].

2.1. Introduction and statement of main results

In this chapter, we apply the results of Chapter 1 to the problem of extending a bounded analytic function from a subvariety of the polydisk $\mathbb{D}^n$ to a bounded analytic function on the polydisk, as well as a related problem of the generation of algebras.

Let $\Omega \subset \mathbb{C}$ be a domain and $\Phi : \Omega \to \mathbb{D}^n$ be an analytic function. Its image $\mathcal{V} = \Phi(\Omega)$ is an analytic variety inside $\mathbb{D}^n$ (which may have singular points). We say that a complex function f defined on $\mathcal{V}$ is analytic if, for every point $p \in \mathcal{V}$, there is a neighbourhood U of p in $\mathbb{C}^n$ and an analytic function F on U such that $f|(\mathcal{V} \cap U) = F|(\mathcal{V} \cap U)$. We define $H^\infty(\mathcal{V})$ to be the Banach algebra of bounded analytic functions on $\mathcal{V}$, equipped with the supremum norm.

A fundamental question is whether it is possible to extend a function in $H^\infty(\mathcal{V})$ to a function in $H^\infty(\mathbb{D}^n)$, the Banach algebra of bounded analytic functions on $\mathbb{D}^n$, also equipped with the supremum norm. Since the restriction map $H^\infty(\mathbb{D}^n) \to H^\infty(\mathcal{V})$ is a contractive homomorphism, this question asks whether the image of this homomorphism, $H^\infty(\mathbb{D}^n)|\mathcal{V}$, is all of $H^\infty(\mathcal{V})$.

Denote by Φ^* the pullback by Φ; that is, the map $\Phi^* : H^\infty(\mathbb{D}^n) \to H^\infty(\Omega)$ defined by $\Phi^*(f) = f \circ \Phi$. If this map is onto, i.e, if $\Phi^* H^\infty(\mathbb{D}^n) = H^\infty(\Omega)$, then every function in $H^\infty(\mathcal{V})$ can be extended to a function in $H^\infty(\mathbb{D}^n)$, because if $f \in H^\infty(\mathcal{V})$, then $f \circ \Phi \in H^\infty(\Omega)$. When Φ^* is onto, we can find an $F \in H^\infty(\mathbb{D}^n)$ such that $f \circ \Phi = \Phi^* F = F \circ \Phi$. This equality implies that $F|\mathcal{V} = f$, so F extends f to $H^\infty(\mathbb{D}^n)$.

In this chapter we show that one has $\Phi^* H^\infty(\mathbb{D}^n) = H^\infty(\Omega)$ when Ω and Φ are admissible domains and functions (see Definition 1.10) such that Φ is injective and Φ' does not vanish (in this case, $\mathcal{V}$ is an analytic variety). If we drop the assumptions that Φ is injective and Φ' does not vanish, then we show that $\Phi^* H^\infty(\mathbb{D}^n)$ is a finite codimensional subalgebra of $H^\infty(\Omega)$. It is easy to see that one cannot get the whole $H^\infty(\Omega)$ algebra in this case. As will be seen however, even under these weaker assumptions, every function in $H^\infty(\mathcal{V})$ can be extended to a function in $H^\infty(\mathbb{D}^n)$.

We also consider other algebras of functions on $\mathbb{D}^n$ besides $H^\infty(\mathbb{D}^n)$. One of these algebras is $\mathcal{SA}(\mathbb{D}^n)$, the Agler algebra of $\mathbb{D}^n$. Recall that it is defined as the Banach algebra of functions analytic on $\mathbb{D}^n$ such that the norm

$$\|f\|_{\mathcal{SA}(\mathbb{D}^n)} \stackrel{\text{def}}{=} \sup_{\substack{\|T_j\| \le 1 \\ \sigma(T_j) \subset \mathbb{D}}} \|f(T_1, \ldots, T_n)\|$$

is finite. Here the supremum is taken over all tuples $(T_1, \ldots, T_n)$ of commuting contractions on a Hilbert space such that the spectra $\sigma(T_j)$ are contained in $\mathbb{D}$ ($f(T_1, \ldots, T_n)$ is well defined for such tuples). Clearly, $\mathcal{SA}(\mathbb{D}^n)$ is a subset of $H^\infty(\mathbb{D}^n)$ and $\|f\|_{H^\infty(\mathbb{D}^n)} \leq \|f\|_{\mathcal{SA}(\mathbb{D}^n)}$. For $n = 1, 2$, we have the equality $\mathcal{SA}(\mathbb{D}^n) = H^\infty(\mathbb{D}^n)$, and the norms coincide. However, for $n \geq 3$, the norms do not coincide. Also, if $n \geq 3$, it is currently unknown whether or not $\mathcal{SA}(\mathbb{D}^n)$ is a proper subset of $H^\infty(\mathbb{D}^n)$. The unit ball of the Agler algebra is known as the Schur-Agler class. It turns out that it is the proper analog of the unit ball in $H^\infty(\mathbb{D})$ (the so called Schur class) when studying the Pick interpolation problem in $\mathbb{D}^n$. The Schur-Agler class also has important applications in operator theory and function theory.

We can ask whether every function in $H^\infty(\mathcal{V})$ can be extended to a function in $\mathcal{SA}(\mathbb{D}^n)$ and whether $\Phi^*\mathcal{SA}(\mathbb{D}^n) = H^\infty(\Omega)$. We show that for our admissible functions Φ, the answer to the first question is affirmative, and the answer to the second question is also affirmative if Φ is injective and Φ' does not vanish.

Another interesting algebra is $H^\infty(\mathcal{K}_\Psi)$. This algebra, extensively studied in [DM07], is associated with a collection of test functions Ψ. It turns out that if $\Phi = (\varphi_1, \ldots, \varphi_n) : \Omega \to \mathbb{D}^n$ is injective, then $\{\varphi_1, \ldots, \varphi_n\}$ is a collection of test functions, which we also denote by Φ, and $H^\infty(\mathcal{K}_\Phi) = \Phi^*\mathcal{SA}(\mathbb{D}^n)$. Therefore, the question of whether $H^\infty(\mathcal{K}_\Phi) = H^\infty(\Omega)$, is a reformulation of the question from the previous paragraph.

If Ω is a nice domain (say with piecewise smooth boundary), and Φ extends by continuity to $\overline{\Omega}$, then we can also consider the algebra $A(\overline{\Omega})$ of functions analytic in Ω and continuous in $\overline{\Omega}$ instead of $H^\infty(\Omega)$. The set $\overline{\mathcal{V}} = \Phi(\overline{\Omega})$ is a bordered analytic variety, and we can consider the algebra $A(\overline{\mathcal{V}})$ of functions analytic in $\mathcal{V}$ and continuous in $\overline{\mathcal{V}}$. The extension problem can also be formulated for these algebras. One can ask whether every function in $A(\overline{\mathcal{V}})$ extends to a function in $A(\overline{\mathbb{D}}^n)$, the algebra of functions analytic in $\mathbb{D}^n$ and continuous in $\overline{\mathbb{D}}^n$, or to $\mathcal{SA}_A(\mathbb{D}^n) \stackrel{\text{def}}{=} \mathcal{SA}(\mathbb{D}^n) \cap A(\overline{\mathbb{D}^n})$. Our methods apply to this problem, and so many of our results have two versions: one for algebras of type H^∞, another for algebras of functions continuous up to the boundary.

Another important algebra for us is $\mathcal{H}_\Phi$, the (not necessarily closed) subalgebra of $H^\infty(\Omega)$ generated by functions of the form $f \circ \varphi_k$, with $f \in H^\infty(\mathbb{D})$, and $k = 1, \ldots, n$:

$$\mathcal{H}_\Phi = \left\{ \sum_{j=1}^{l} \prod_{k=1}^{n} f_{j,k}(\varphi_k(z)) : l \in \mathbb{N}, f_{j,k} \in H^\infty(\mathbb{D}) \right\}.$$

We have the following algebra inclusions:

$$\mathcal{H}_\Phi \subset \Phi^*\mathcal{SA}(\mathbb{D}^n) \subset \Phi^*H^\infty(\mathbb{D}^n) \subset \Phi^*H^\infty(\mathcal{V}) \subset H^\infty(\Omega). \tag{2.1}$$

The first inclusion follows from the observation that any function on $\mathbb{D}^n$ of the form $f(z_k)$, with $f \in H^\infty(\mathbb{D})$, belongs to $\mathcal{SA}(\mathbb{D}^n)$ by the von Neumann inequality, as do sums of products of such functions since $\mathcal{SA}(\mathbb{D}^n)$ is an algebra. The inclusion $\Phi^*H^\infty(\mathbb{D}^n) \subset \Phi^*H^\infty(\mathcal{V})$ holds since if $F \in H^\infty(\mathbb{D}^n)$, then $F|\mathcal{V} \in H^\infty(\mathcal{V})$ and $\Phi^*F = \Phi^*(F|\mathcal{V})$.

We define $\mathcal{A}_\Phi$ to be the (not necessarily closed) subalgebra of $A(\overline{\Omega})$ generated by functions of the form $f \circ \varphi_k$ with $f \in A(\overline{\mathbb{D}})$ and $k = 1, \ldots, n$:

$$\mathcal{A}_\Phi = \left\{ \sum_{j=1}^{l} \prod_{k=1}^{n} f_{j,k}(\varphi_k(z)) : l \in \mathbb{N}, f_{j,k} \in A(\overline{\mathbb{D}}) \right\}.$$

We likewise have the inclusions

$$\mathcal{A}_\Phi \subset \Phi^*\mathcal{SA}_A(\mathbb{D}^n) \subset \Phi^*A(\overline{\mathbb{D}}^n) \subset \Phi^*A(\overline{\mathcal{V}}) \subset A(\overline{\Omega}). \tag{2.2}$$

The main results of this chapter are the following theorems.

Theorem 2.1. *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible and injective and Φ' does not vanish on Ω, then $\mathcal{H}_\Phi = H^\infty(\Omega)$ and $\mathcal{A}_\Phi = A(\overline{\Omega})$.*

It follows that in this case all the algebras in (2.1) coincide, as do all those in (2.2).

Some of the conditions that we are imposing on Φ are easily seen to in fact be necessary for the equality $\Phi^*H^\infty(\mathbb{D}^n) = H^\infty(\Omega)$, which is weaker than $\Phi^*\mathcal{SA}(\mathbb{D}^n) = H^\infty(\Omega)$, to hold. For instance, if Φ is not injective, then no function in $\Phi^*H^\infty(\mathbb{D}^n)$ is injective, so this set cannot be all of $H^\infty(\Omega)$. Similarly, if $\Phi'(z_0) = 0$ for some $z_0 \in \Omega$, then we have $f'(z_0) = 0$ for every $f \in \Phi^*H^\infty(\mathbb{D}^n)$, which again implies $\Phi^*H^\infty(\mathbb{D}^n) \neq H^\infty(\Omega)$. Finally, if there is a point $z_0 \in \partial\Omega$ such that $|\varphi_k(z_0)| < 1$ for all functions φ_k, then every function in $\Phi^*H^\infty(\mathbb{D}^n)$ is continuous at z_0, so once again $\Phi^*H^\infty(\mathbb{D}^n) \neq H^\infty(\Omega)$. It is also easy to show that $\Phi^*A(\overline{\mathbb{D}}^n) \neq A(\overline{\Omega})$ in this case as well. This serves as additional motivation for conditions (a) and (b) in Definition 1.10.

In the case when Φ is not injective or Φ' vanishes at some points, we prove the following result, which according to the remarks above is the best that we can hope.

Lemma 2.2. *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible, then $\mathcal{H}_\Phi$ is a closed subalgebra of finite codimension in $H^\infty(\Omega)$, and $\mathcal{A}_\Phi$ is a closed subalgebra of finite codimension in $A(\overline{\Omega})$.*

In fact, we prove below that $\mathcal{H}_\Phi$ is also weak*-closed in $H^\infty(\Omega)$ (see Section 2.3).

Regarding the algebras $H^\infty(\mathcal{V})$ and $A(\overline{\mathcal{V}})$ of functions defined on the analytic curve $\mathcal{V}$, we prove the following result.

Theorem 2.3. *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible, then $\Phi^*H^\infty(\mathcal{V}) = \mathcal{H}_\Phi$ and $\Phi^*A(\overline{\mathcal{V}}) = \mathcal{A}_\Phi$.*

In this case the first four algebras in (2.1) and the first four in (2.2) coincide, while the last inclusions can be proper, though $\Phi^*H^\infty(\mathcal{V})$ happens to be weak*-closed in $H^\infty(\Omega)$, while $\Phi^*A(\overline{\mathcal{V}})$ is norm closed in $A(\overline{\Omega})$, and both have finite codimension.

This theorem allows us to prove a result on the extension of functions in $\mathcal{V}$ to the Agler algebra.

Theorem 2.4. *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible, for every $f \in H^\infty(\mathcal{V})$ there is an $F \in \mathcal{SA}(\mathbb{D}^n)$ such that $F|\mathcal{V} = f$ and $\|F\|_{\mathcal{SA}(\mathbb{D}^n)} \leq C\|f\|_{H^\infty(\mathcal{V})}$, for some constant C independent of f. Additionally, if $f \in A(\overline{\mathcal{V}})$, then F can be taken to belong to $\mathcal{SA}_A(\mathbb{D}^n)$.*

There is some relationship between our setting and the algebra generation problem. Given any finite family $\Phi = \{\varphi_k\} \subset A(\overline{\Omega})$, one can also consider the algebras $\overline{A}_\Phi$, the smallest *norm closed* subalgebra of $A(\overline{\Omega})$ containing Φ, and $\overline{H}^\infty_\Phi$, the weak*-closed subalgebra of $H^\infty(\Omega)$ generated by the family Φ.

Proposition 2.5. *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible, then $\mathcal{A}_\Phi = \overline{A}_\Phi$ and $\mathcal{H}_\Phi = \overline{H}^\infty_\Phi$.*

Proof. It is clear that in general, $\mathcal{A}_\Phi \subset \overline{A}_\Phi$ and $\mathcal{H}_\Phi \subset \overline{H}^\infty_\Phi$. By Lemma 2.2, $\mathcal{A}_\Phi$ is closed in norm, and by Lemma 2.15, $\mathcal{H}_\Phi$ is weak*-closed. The equalities now follow. □

Theorem 2.1 then implies corresponding results about the generation of algebras.

Corollary 2.6. *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible and injective and Φ' does not vanish in Ω, then $\overline{A}_\Phi = A(\overline{\Omega})$ and $\overline{H}^\infty_\Phi = H^\infty(\Omega)$.*

The assertions that $\mathcal{A}_\Phi = A(\overline{\Omega})$ and $\mathcal{H}_\Phi = H^\infty(\Omega)$ are much stronger than just the fact that Φ generates algebras $A(\overline{\Omega})$ and $H^\infty(\Omega)$ (in the weak* sense, in the last case). For instance, as was mentioned, the equalities $\mathcal{A}_\Phi = A(\overline{\Omega})$ and $\mathcal{H}_\Phi = H^\infty(\Omega)$ are impossible if there is a point $z_0 \in \partial\Omega$ such that $\max_k |\varphi_k(z_0)| < 1$, whereas Φ still can still generate algebras $A(\overline{\Omega})$ and $H^\infty(\Omega)$ in this case. (Notice that for any nonzero constants $\{\lambda_k\}$, the families $\{\varphi_k\}$ and $\{\lambda_k\varphi_k\}$ generate the same closed subalgebras of $A(\overline{\Omega})$ and $H^\infty(\Omega)$.) In the applications to operator theory that we consider in this thesis, algebra generation does not suffice, and the assertions that $\mathcal{A}_\Phi = A(\overline{\Omega})$ and $\mathcal{H}_\Phi = H^\infty(\Omega)$ and so Theorem 1.11 play an important role.

Brief review of previous results about algebra generation and extension

The study of generators of algebras of the type $A(\overline{\Omega})$ dates back to Wermer [Wer58], where he considered pairs of functions as generators of the algebra $A(K)$ for a compact subset K of a Riemann surface. Bishop worked on the same problem independently in [Bis58], using a different approach. In these two articles, sufficient conditions are given for generation of the whole algebra, or of a finite codimensional subalgebra. Several later works are devoted to giving weaker sufficient conditions; see [Blu74] and [SW74]. In [ST03, MS05], the H^p closure rather than of the uniform closure of the algebra is considered, although [ST03] does also give a result for the disk algebra. Even in the simple case of $A(\overline{\mathbb{D}})$, necessary and sufficient conditions for a pair of functions to generate the whole algebra are still unknown (see [HN94, Problem 2.32]).

In most articles on algebra generation, it is assumed that the derivatives of the generators are continuous up to the boundary. In our setting, as we remarked after Definition 1.10, it is only necessary that each function φ_k' be Hölder continuous near the arc J_k. In this sense, it seems that Corollary 2.6 is a new result. We also stress that our results concern the algebra $\mathcal{A}_\Phi$, which is *a priori* a non-closed algebra smaller than $\overline{A}_\Phi$, the smallest closed algebra containing $\{\varphi_1, \dots, \varphi_n\}$. In our applications to operator theory in Chapter 3, it is essential that the theorems we have stated in this section are proved for $\mathcal{A}_\Phi$ rather than its closure.

The case when $\Omega = \mathbb{D}$ is important. Then $\mathcal{V} = \Phi(\mathbb{D})$ is called an analytic disk inside the polydisk. It is a particular kind of hyperbolic analytic curve, the theory of which has been treated extensively in the literature. The function theory of these curves and its relation with finite codimensional subalgebras of holomorphic functions was studied by Agler and McCarthy in [AM07]. A classification of the finite codimensional subalgebras of a function algebra which is related to the one that we use in the proof of Theorem 2.1 was given by Gamelin in [Gam68].

The problem of extension of a bounded analytic function defined on an analytic curve $\mathcal{V} \subset \mathbb{D}^n$ to the polydisk $\mathbb{D}^n$ dates back to Rudin and to Stout (see [Sto75]), and was also treated by Polyakov in [Pol83], and more generally by Polyakov and Khenkin in [PK90]. The book [HL84] by Henkin and Leiterer also treats the extension of bounded analytic functions defined on subvarieties in a fairly general context. In these works, the subvariety $\mathcal{V}$ is assumed to be extendable to a neighbourhood of $\overline{\mathbb{D}}^n$, which means that there is a larger analytic subvariety $\widetilde{\mathcal{V}}$ of a neighbourhood of $\overline{\mathbb{D}}^n$ such that $\mathcal{V} = \widetilde{\mathcal{V}} \cap \mathbb{D}^n$. This is in contrast to a function Φ meeting our requirements (a)–(f) "in general position", in which case the variety $\mathcal{V} = \Phi(\Omega)$ does not extend to a larger analytic variety $\widetilde{\mathcal{V}}$.

The works of Amar and Charpentier [AC80] and of Chee [Che76, Che83] do deal with the setting when this extension of $\mathcal{V}$ may be absent. In [AC80], extensions by bounded analytic functions to bidisks are considered, whereas the papers [Che76, Che83] concern the case of an analytic variety $\mathcal{V}$ of codimension 1 in a polydisk $\mathbb{D}^n$ and therefore can be compared with our results only for $n = 2$. Theorem 1.1 in [Che83] implies that in our setting, for the case of $n = 2$, every $f \in H^\infty(\mathcal{V})$ can be extended to an $F \in H^\infty(\mathbb{D}^2)$ such that $F|\mathcal{V} = f$.

For the case of $\mathcal{V} = \Phi(\mathbb{D})$, where $\Phi : \mathbb{D} \to \mathbb{D}^n$ extends to a neighbourhood of $\overline{\mathbb{D}}$, a necessary and sufficient condition for the property of analytic bounded extension was given by Stout in [Sto75], in which case, at least one φ_k must be a finite Blaschke product.

If $\widetilde{\mathcal{V}}$ is an analytic curve in a neighbourhood of $\overline{\mathbb{D}}^n$ such that there is a biholomorphic map $\widetilde{\Phi}$ of a domain $G \subset \mathbb{C}$ onto $\widetilde{\mathcal{V}}$ and $\mathcal{V} = \widetilde{\mathcal{V}} \cap \mathbb{D}^n$, the set $\Omega = \widetilde{\Phi}^{-1}(\mathcal{V})$ is connected and $\Phi = \widetilde{\Phi}|\Omega$, then typically all the above conditions (a)–(e) on Φ are satisfied, whereas (f) is an additional requirement. In this case, if Ω is simply connected and $\widehat{\Phi} = \Phi \circ \eta : \mathbb{D} \to \mathcal{V}$, where η is a Riemann mapping of $\mathbb{D}$ onto Ω, then $\widehat{\Phi}$ does not continue analytically to a larger disk unless Ω has analytic boundary. In other words, there are cases when $\mathcal{V}$ has an extension to a larger subvariety whereas Φ does not extend.

Bounded extensions to an analytic polyhedron W in $\mathbb{C}^n$ from a subvariety $\mathcal{V}$ of arbitrary codimension were studied by Adachi, Andersson and Cho in [AAC99]. It was assumed there that $\mathcal{V}$ is continuable to a neighbourhood of W. Notice that polydisks are particular cases of analytic polyhedra.

The property of the bounded extension of H^∞ functions does not hold in general, and one can find several counterexamples in the literature, see [Ale69, DM97, DM01, Maz00].

There are also many papers in the literature that deal with bounded extensions in the context of strictly pseudoconvex domains or domains with smooth boundary (the polydisk does not belong to these classes). See Diederich and Mazzilli [DM97, DM01] and the recent paper by Alexandre and Mazzilli [AM15]. Holomorphic extensions have also been studied extensively in different contexts of L^p norms; we refer to Chee [Che83, Che87] for the case of the polydisk. See also the review [Ada01] by Adachi, the paper [AM15] and references therein for a more complete information.

The above-mentioned papers use diverse techniques from several complex variables, such as the Cousin problem and integral representations for holomorphic functions. In another group of papers, interesting results around the problem of bounded continuation are obtained using the tools of operator theory and the theory of linear systems. Agler and McCarthy [AM03] treat the bounded extension property with preservation of norms for the bidisk $\mathbb{D}^2$. See also [GHW08] for partial results for the case of tridisks and general polydisks. It seems that very few varieties $\mathcal{V}$ have this norm preserving extension property.

In [Kne10], Knese studies the existence of bounded extensions from distinguished subvarieties of $\mathbb{D}^2$ (without preservation of norms). His approach is based on certain representations of two-variable transfer functions and permits him to give concrete estimates of the constants.

The same problem can also be studied for the ball $\mathcal{B}^n$ instead of the polydisk. In [APV03], Alpay, Putinar and Vinnikov use reproducing kernel Hilbert space techniques to show that a bounded analytic function defined on an analytic disk in $\mathcal{B}^n$ can be extended to $\mathcal{B}^n$. Indeed they show that it can be extended to an element of the multiplier algebra of the Drury-Arveson space $H^2(\mathcal{B}^n)$; this algebra is properly contained in $H^\infty(\mathcal{B}^n)$. See also [DHS15] for further examples and counterexamples, and the relationship with the complete Nevanlinna-Pick property.

The extension problem is also treated in [KMS13], where it appears as a consequence of isomorphism of certain multiplier algebras of analytic varieties. Some problems considered there resemble those we consider on the pullback by Φ.

Our approach differs from the approaches described above, in that it relies on techniques inspired by the Havin-Nersessian work, certain compactness arguments, which show that some subalgebras of H^∞ have finite codimension, and the study of maximal ideals and derivations in H^∞. An important aspect distinguishing it from earlier results, is that we can prove continuation to $\mathcal{SA}(\mathbb{D}^n)$. If $\mathcal{SA}(\mathbb{D}^n)$ is strictly contained in $H^\infty(\mathbb{D}^n)$ (it is not known whether this is true), then our results are stronger. Indeed, we prove even more: it follows from the proof of Theorem 2.4 that there is closed subspace of finite codimension in $H^\infty(\mathcal{V})$ such that every function in this space can be extended to a function of the form $F_1(z_1) + \cdots + F_n(z_n)$, where $F_j \in H^\infty(\mathbb{D})$ for $j = 1, \ldots, n$.

2.2. Banach algebra structure of $H^\infty(\Omega)$

In this section we prove Theorem 2.1. The proof of this theorem uses Banach algebra techniques, so first we have to review a few results about the Banach algebra structure of $H^\infty(\Omega)$. First we recall some general notions about Banach algebras and introduce some results by Gorin [Gor69] that we will use later.

Let $\mathcal{A}$ be a commutative unital Banach algebra over $\mathbb{C}$. A *complex homomorphism* of $\mathcal{A}$ is a bounded linear functional $\psi : \mathcal{A} \to \mathbb{C}$ which is also multiplicative:

$$\psi(ab) = \psi(a)\psi(b) \qquad a, b \in \mathcal{A},$$

and such that $\psi \neq 0$. Note that this implies that $\psi(1) = 1$. We denote by $\mathfrak{M}(\mathcal{A})$ the space of all complex homomorphisms of $\mathcal{A}$. Since $\mathfrak{M}(\mathcal{A})$ is a subset of the dual space $\mathcal{A}^*$, it can be endowed with the weak* topology it inherits from $\mathcal{A}^*$. This is the natural topology to consider in $\mathfrak{M}(\mathcal{A})$ and it makes it a compact Hausdorff space.

The space $\mathfrak{M}(\mathcal{A})$ is also called the maximal ideal space of $\mathcal{A}$ as there is a one-to-one correspondence between the maximal ideals of $\mathcal{A}$ and the complex homomorphisms of $\mathcal{A}$: the kernel of each complex homomorphism is a maximal ideal, and given a maximal ideal J, the linear functional ψ defined by $\psi|J \equiv 0$ and $\psi(1) = 1$ is a complex homomorphism.

The main application of the maximal ideal space to the study of commutative Banach algebras is the Gelfand representation. This is the function $\Psi : \mathcal{A} \to C(\mathfrak{M}(\mathcal{A}))$ defined by $\Psi(a)(\psi) = \psi(a)$. This representation allows one to view the Banach algebra $\mathcal{A}$ as an algebra of continuous functions on a compact Hausdorff space, and in fact it is an injective representation when $\mathcal{A}$ is semisimple.

Given $\psi \in \mathfrak{M}(\mathcal{A})$, a linear functional $\eta \in \mathcal{A}^*$ is called a *derivation* at ψ if

$$\eta(ab) = \eta(a)\psi(b) + \psi(a)\eta(b), \qquad a, b \in \mathcal{A}.$$

Note that it follows from this definition that $\eta(1) = 0$, so

In the sequel, we will use the following lemmas from Gorin [Gor69].

Lemma 2.7 (Gorin [Gor69]). *Let $\mathcal{B}$ be a proper subalgebra of a commutative complex algebra $\mathcal{A}$. Then there is a subalgebra $\mathcal{A}_0 \subset \mathcal{A}$ of codimension one such that $\mathcal{B} \subset \mathcal{A}_0$.*

Lemma 2.8 (Gorin [Gor69]). *Let $\mathcal{A}_0$ be a closed unital subalgebra of codimension one in a complex Banach algebra $\mathcal{A}$. Then $\mathcal{A}_0$ has one of the possible two forms:*

(a) $\mathcal{A}_0 = \{a \in \mathcal{A} : \psi_1(a) = \psi_2(b)\}$, for some particular $\psi_1, \psi_2 \in \mathfrak{M}(\mathcal{A})$, $\psi_1 \neq \psi_2$;

(b) $\mathcal{A}_0 = \ker \eta$, where $\eta \neq 0$ is a derivation at some $\psi \in \mathfrak{M}(\mathcal{A})$.

The algebras that appear in conditions (a) and (b) of this lemma are clearly closed unital subalgebras of codimension one, for every choice of ψ_1, ψ_2 and η. What this lemma says is that every subalgebra of codimension one must have one of these two forms.

Now we will specialize our discussion to the Banach algebra $H^\infty(\Omega)$, for Ω a multiply-connected domain whose boundary is a disjoint union of Jordan curves. We refer to [Hof62, Chapter 10] for a detailed discussion of the Banach algebra structure of $H^\infty(\mathbb{D})$. The properties of $H^\infty(\Omega)$ for a finitely connected domain Ω are similar.

We denote by $\mathbf{z}$ the identity function on Ω, i.e., $\mathbf{z}(z) = z$. For any complex homomorphism $\psi \in \mathfrak{M}(H^\infty(\Omega))$, either $\psi(\mathbf{z}) \in \Omega$ or $\psi(\mathbf{z}) \in \partial\Omega$. If $\psi(\mathbf{z}) = z_0 \in \Omega$, then $\psi(f) = f(z_0)$ for every $f \in H^\infty(\Omega)$. If $\psi(\mathbf{z}) = z_0 \in \partial\Omega$, then we can assert that $\psi(f) = f(z_0)$ for every $f \in H^\infty(\Omega)$ that extends by continuity to z_0 (the proof for $\Omega = \mathbb{D}$, given in [Hof62], easily adapts to any finitely connected domain).

It is easy to see that if η is a derivation at ψ with $\psi(\mathbf{z}) = z_0 \in \Omega$, then $\eta(f) = \eta(\mathbf{z})f'(z_0)$ for every $f \in H^\infty(\Omega)$ (one must check first that $\eta(1) = 0$ and then write $f = f(z_0) + f'(z_0)(\mathbf{z} - z_0) + (\mathbf{z} - z_0)^2 g$ with $g \in H^\infty(\Omega)$). Derivations at ψ with $\psi(\mathbf{z}) \in \partial\Omega$ have the following somewhat similar property.

Lemma 2.9. *Let $f \in H^\infty(\Omega)$ be continuous at $z_0 \in \partial\Omega$ with $(f - f(z_0))/(\mathbf{z} - z_0) \in H^\infty(\Omega)$. If η is a derivation in $H^\infty(\Omega)$ at $\psi \in \mathfrak{M}(H^\infty(\Omega))$ with $\psi(\mathbf{z}) = z_0$, then*

$$\eta(f) = \eta(\mathbf{z})\psi\Big(\frac{f - f(z_0)}{\mathbf{z} - z_0}\Big).$$

Proof. We just compute

$$\begin{aligned}\eta(f) &= \eta(f - f(z_0)) = \eta\Big((\mathbf{z} - z_0)\frac{f - f(z_0)}{\mathbf{z} - z_0}\Big)\\ &= \psi(\mathbf{z} - z_0)\eta\Big(\frac{f - f(z_0)}{\mathbf{z} - z_0}\Big) + \eta(\mathbf{z} - z_0)\psi\Big(\frac{f - f(z_0)}{\mathbf{z} - z_0}\Big) = \eta(\mathbf{z})\psi\Big(\frac{f - f(z_0)}{\mathbf{z} - z_0}\Big).\end{aligned}$$

$\square$

The two following lemmas will be the key tools in the proof of Theorem 2.1.

Lemma 2.10. *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible and $\psi_1 \neq \psi_2$ are in $\mathfrak{M}(H^\infty(\Omega))$ and satisfy $\psi_1(f) = \psi_2(f)$ for every $f \in \mathcal{H}_\Phi$, then $\psi_1(\mathbf{z}), \psi_2(\mathbf{z}) \in \Omega$, $\psi_1(\mathbf{z}) \neq \psi_2(\mathbf{z})$ and $\Phi(\psi_1(\mathbf{z})) = \Phi(\psi_2(\mathbf{z}))$. The same is true if $H^\infty(\Omega)$ is replaced by $A(\overline{\Omega})$ and $\mathcal{H}_\Phi$ is replaced by $\mathcal{A}_\Phi$.*

Proof. Since by assumption $\varphi_k \in A(\overline{\Omega})$, we have $\psi_j(\varphi_k) = \varphi_k(\psi_j(\mathbf{z}))$ for $j = 1, 2$, $k = 1, \dots, n$. Therefore, $\Phi(\psi_1(\mathbf{z})) = \Phi(\psi_2(\mathbf{z}))$, because the functions φ_k belong to $\mathcal{H}_\Phi$. Let $z_j = \psi_j(\mathbf{z}) \in \overline{\Omega}$. If $z_1 \in \partial\Omega$, then by condition (f), $z_2 = z_1$, and hence $\psi_1(f) = \psi_2(f)$ for all $f \in A(\overline{\Omega})$. Take an $f \in H^\infty(\Omega)$ and put $g = \sum_{k=1}^N F_k(f) \circ \varphi_k$, where F_k are as in Theorem 1.11. Then $f - g \in A(\overline{\Omega})$ and $g \in \mathcal{H}_\Phi$. Therefore, we have $\psi_1(f - g) = \psi_2(f - g)$, and also $\psi_1(g) = \psi_2(g)$. It follows that $\psi_1(f) = \psi_2(f)$, so that $\psi_1 = \psi_2$, because f was arbitrary. This contradicts our assumption. Hence $\psi_j(\mathbf{z}) \in \Omega$ for $j = 1, 2$ and, since $\psi_1 \neq \psi_2$, it must happen that $\psi_1(\mathbf{z}) \neq \psi_2(\mathbf{z})$. The reasoning for $A(\overline{\Omega})$ is the same. □

Lemma 2.11. *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible and $\eta \neq 0$ is a derivation of $H^\infty(\Omega)$ at $\psi \in \mathfrak{M}(H^\infty(\Omega))$ such that $\eta(f) = 0$ for every $f \in \mathcal{H}_\Phi$, then $\psi(\mathbf{z}) \in \Omega$ and $\Phi'(\psi(\mathbf{z})) = 0$. The same is true if $H^\infty(\Omega)$ is replaced by $A(\overline{\Omega})$ and $\mathcal{H}_\Phi$ is replaced by $\mathcal{A}_\Phi$.*

Proof. We consider two cases according to whether $\psi(\mathbf{z})$ belongs to Ω or to $\partial\Omega$. The case when $\psi(\mathbf{z}) \in \Omega$ is clear: since $\varphi_k \in \mathcal{H}_\Phi$, we have $0 = \eta(\varphi_k) = \varphi_k'(\psi(\mathbf{z}))$, and so $\Phi'(\psi(\mathbf{z})) = 0$.

Now we show that the case $\psi(\mathbf{z}) \in \partial\Omega$ cannot happen. Here we also distinguish two cases according to whether $\eta(\mathbf{z})$ is zero or not. If $\eta(\mathbf{z}) \neq 0$, then we take $k \in \{1, \dots, n\}$ such that $\psi(\mathbf{z}) \in J_k$. Since φ_k is derivable at $\psi(\mathbf{z})$, we have $\eta(\varphi_k) = \eta(\mathbf{z})\varphi_k'(\psi(\mathbf{z}))$ by Lemma 2.9. Therefore, $\varphi_k'(\psi(\mathbf{z})) = 0$, because $\varphi_k \in \mathcal{H}_\Phi$. This contradicts condition (e) in Definition 1.10.

In the case when $\eta(\mathbf{z}) = 0$, we get $\eta(f) = 0$ for every f analytic on some neighbourhood of $\overline{\Omega}$. This implies $\eta(f) = 0$ for every $f \in A(\overline{\Omega})$, because functions analytic on $\overline{\Omega}$ are dense in $A(\overline{\Omega})$. Now take $f \in H^\infty(\Omega)$ and put $g = \sum_{k=1}^n F_k(f) \circ \varphi_k$, where F_k are as in Theorem 1.11. Then $f - g \in A(\overline{\Omega})$ and $g \in \mathcal{H}_\Phi$. This implies that $0 = \eta(g) = \eta(g) + \eta(f - g) = \eta(f)$. Therefore, $\eta = 0$, a contradiction.

The proof for $A(\overline{\Omega})$ follows similar steps, and is indeed even easier. □

The only other tool that we need to proof Theorem 2.1 is Lemma 2.2, which was stated above and which we prove now. It is a simple consequence of the Fredholm theory.

Proof of Lemma 2.2. By Theorem 1.11 and the standard theory of Fredholm operators, the range of the operator $f \mapsto \sum F_k(f) \circ \psi_k$, $f \in H^\infty(\Omega)$, is a closed subspace of finite codimension in $H^\infty(\Omega)$. Since $\mathcal{H}_\Phi$ contains this range, we get that $\mathcal{H}_\Phi$ is a closed subalgebra of finite codimension in $H^\infty(\Omega)$. To obtain the analogous result for $\mathcal{A}_\Phi$, we just consider the restriction of the operator $f \mapsto \sum F_k(f) \circ \psi_k$ to $A(\overline{\Omega})$. □

We are now ready to give the proof of Theorem 2.1.

Proof of Theorem 2.1. We first show that $\mathcal{H}_\Phi = H^\infty(\Omega)$. By Lemma 2.2, $\mathcal{H}_\Phi$ is a closed unital subalgebra of $H^\infty(\Omega)$ of finite codimension. Let us assume by way of contradiction that $\mathcal{H}_\Phi \neq H^\infty(\Omega)$. Lemmas 2.7 and 2.8 imply that there is a closed unital subalgebra $\mathcal{A}_0$ of codimension one in $H^\infty(\Omega)$ such that $\mathcal{H}_\Phi \subset \mathcal{A}_0$. Moreover, $\mathcal{A}_0$ must have one of the following two possible forms:

(a) $\mathcal{A}_0 = \{f \in H^\infty : \psi_1(f) = \psi_2(f)\}$, for some $\psi_1, \psi_2 \in \mathfrak{M}(H^\infty(\Omega))$, $\psi_1 \neq \psi_2$.

(b) $\mathcal{A}_0 = \ker \eta$, where $\eta \neq 0$ is a derivation at some $\psi \in \mathfrak{M}(H^\infty(\Omega))$.

We show that each of these two cases leads to a contradiction. In the case (a), Lemma 2.10 shows that $\psi_1(\mathbf{z}) \neq \psi_2(\mathbf{z}) \in \Omega$, yet $\Phi(\psi_1(\mathbf{z})) = \Phi(\psi_2(\mathbf{z}))$. Since Φ is injective, we get a contradiction.

In the case (b), Lemma 2.11 shows that $\psi(\mathbf{z}) \in \Omega$ and $\Phi'(\psi(\mathbf{z})) = 0$. This is a contradiction, because Φ' does not vanish in Ω.

The proof of the equality $\mathcal{A}_\Phi = A(\overline{\Omega})$ is identical. $\square$

A few comments about the proof of Theorem 2.1 are in order. The first is about the classification of the derivations of $H^\infty(\Omega)$. We treat the case $\Omega = \mathbb{D}$, since the case of a finitely connected domain Ω is similar. We have already described the derivations at points $\psi \in \mathfrak{M}(H^\infty(\mathbb{D}))$ such that $\psi(\mathbf{z}) \in \mathbb{D}$ and given some properties about those derivations such that $\psi(\mathbf{z}) \in \mathbb{T}$. The first question is whether there exists any such (non-zero) derivations "supported on $\mathbb{T}$", and whether the case of a derivation η such that $\eta(\mathbf{z}) = 0$ but $\eta \neq 0$ that appeared in the proof of Lemma 2.11 can really happen.

It is important to remark the existence of analytic disks inside each of the fibres of $\mathfrak{M}(H^\infty(\mathbb{D}))$ that project into $\mathbb{T}$ under the map $\psi \mapsto \psi(\mathbf{z})$ (once again, we refer the reader to [Hof62]). Thus there are maps of the form $\lambda \mapsto \Psi_\lambda : \mathbb{D} \to \mathfrak{M}(H^\infty(\mathbb{D}))$ such that for every $\lambda \in \mathbb{D}$ the point $\Psi_\lambda \in \mathfrak{M}(H^\infty(\mathbb{D}))$ lies in the same fibre (i.e., $\Psi_\lambda(\mathbf{z})$ is constant in λ) and such that the map $f(\lambda) \mapsto \Psi_\lambda(f)$ is an algebra homomorphism of $H^\infty(\mathbb{D})$ onto $H^\infty(\mathbb{D})$. This map Ψ endows its image $\mathcal{D}$ in $\mathfrak{M}(H^\infty(\mathbb{D}))$ with an analytic structure. The complex derivative according to the analytic structure of $\mathcal{D}$ gives a (non-zero) derivation at each of the points in $\mathcal{D}$ (explicitly, these are maps $f \mapsto (d/d\lambda)|_{\lambda=\lambda_0}\Psi_\lambda(f))$. Clearly, if η is one of these derivations, then $\eta(\mathbf{z}) = 0$, because $\mathbf{z}$ is constant on each of the fibres over $\mathbb{T}$, and hence in $\mathcal{D}$. However, we do not know whether these derivations are (up to a constant multiple) the only ones that exist over points of $\mathcal{D}$, or whether there exist (non-zero) derivations on points which do not belong to such analytic disks. It seems that there is not much information about the classification of the derivations of $H^\infty(\mathbb{D})$ in the literature.

Another comment is that one could use the results of Section 2.3 to simplify somewhat the proof of the Theorem 2.1. If we know that the algebra $\mathcal{A}_0$ is weak*-closed, then we only need to consider weak*-continuous complex homomorphisms and derivations. In the next section we show that it is indeed the case that $\mathcal{H}_\Phi$, and hence $\mathcal{A}_0$, is weak*-closed.

2.3. Weak*-closedness of $\mathcal{H}_\Phi$

In this section we prove that the algebra $\mathcal{H}_\Phi$ is weak*-closed in $H^\infty(\Omega)$. First recall a well known result about weak*-continuity of adjoint operators, the proof of which is elementary and so is omitted.

Lemma 2.12. *If X is a Banach space and $T : X \to X$ is a bounded operator, then its adjoint $T^* : X^* \to X^*$ is continuous in the weak*-topology of X^*.*

The operator T is called the predual of T^*. Thus any operator with a predual is weak*-continuous, a condition applying to many integral operators on L^∞.

Lemma 2.13. *Let $T : L^\infty(\partial\Omega) \to L^\infty(\partial\Omega)$ be defined by*

$$(Tf)(z) = \int_{\partial\Omega} G(\zeta, z) f(\zeta)\, d\zeta,$$

where $G : \partial\Omega \times \partial\Omega \to \mathbb{C}$ is a measurable function satisfying

$$\int_{\partial\Omega} |G(\zeta, z)|\, |d\zeta| \leq C$$

for every $z \in \partial\Omega$. Then the operator S defined by

$$(Sg)(\zeta) = \int_{\partial\Omega} G(\zeta, z) g(z)\, dz$$

is a bounded operator $S : L^1(\partial\Omega) \to L^1(\partial\Omega)$ and satisfies $S^* = T$.

Proof. Fubini's Theorem shows that

$$\left| \int_{\partial\Omega} (Sg)(\zeta) f(\zeta)\, d\zeta \right| \leq C \|g\|_1 \|f\|_\infty,$$

and so S is bounded on $L^1(\partial\Omega)$. Another application of Fubini's Theorem gives $S^* = T$. □

Recall from (1.12) and the proof of Theorem 1.11 that the operator

$$K(f) = f - \sum_{k=1}^{n} F_k(f) \circ \varphi_k,$$

is a weakly singular integral operator of the form

$$K(f)(z) \mapsto \int_{\partial\Omega} G(\zeta, z) f(\zeta)\, d\zeta,$$

where the function G is continuous outside the diagonal $\{\zeta = z\}$ and $|G(\zeta, z)| \leq C|\zeta - z|^{-\beta}$, for some $\beta < 1$. Also, for each $\zeta \in \partial\Omega$, $G(\zeta, z)$ is analytic in $z \in \Omega$. Thus, K is compact from $L^\infty(\Omega)$ to $H^\infty(\Omega)$. By Lemma 2.13, the operator K has a predual, so is weak*-continuous.

Lemma 2.14. *For every $\varepsilon > 0$, there is an operator $K_\varepsilon : L^\infty(\partial\Omega) \to L^\infty(\partial\Omega)$ of finite rank which has a predual and such that $\|K_\varepsilon - K\| < \varepsilon$.*

Proof. Fix $\varepsilon > 0$. Since G is continuous outside $\{\zeta = z\}$, there exist $\alpha_j \in L^\infty(\partial\Omega)$ and $\beta_j \in L^1(\partial\Omega)$ such that

$$\int_{\partial\Omega} \left| G(\zeta, z) - \sum_{j=1}^{N} \alpha_j(z)\beta_j(\zeta) \right| d\zeta < \varepsilon/2$$

for every $z \in \partial\Omega$. This implies that the finite rank operator K_ε defined by

$$K_\varepsilon(\psi)(z) = \int_{\partial\Omega} \sum_{j=1}^{N} \alpha_j(z)\beta_j(\zeta)\psi(\zeta)\, d\zeta$$

satisfies $\|K_\varepsilon - K\| \le \varepsilon/2$. Clearly, by Lemma 2.13, the operator K_ε has a predual. $\square$

Lemma 2.15. *Let $L(f) = \sum_{k=1}^{n} F_k(f) \circ \varphi_k$ be the operator $L : H^\infty(\Omega) \to H^\infty(\Omega)$ of Theorem 1.11. Then the range of L is weak*-closed in $H^\infty(\Omega)$.*

Proof. We have $L = (I - K)|H^\infty(\Omega)$. By the preceding lemma with $\varepsilon = 1$, there is a finite rank operator K_1 such that $\|K_1 - K\| < 1$. Put $M = H^\infty(\Omega) + K_1(L^\infty(\partial\Omega))$. Then, since $H^\infty(\Omega)$ is weak*-closed in $L^\infty(\partial\Omega)$, M is a weak*-closed subset of $L^\infty(\Omega)$ such that $H^\infty(\Omega)$ has finite codimension in M. Define $\Delta = I - (K - K_1)$. Note that $K_1(L^\infty(\partial\Omega)) \subset M$ and $K(L^\infty(\partial\Omega)) \subset H^\infty(\Omega) \subset M$. Since

$$\Delta^{-1} = \sum_{j=0}^{\infty} (K - K_1)^j,$$

this series being convergent in operator norm, we also have $\Delta^{-1}M \subset M$.

Now observe that

$$L(H^\infty(\Omega)) = (I - K)H^\infty(\Omega) = \Delta(I - \Delta^{-1}K_1)H^\infty(\Omega).$$

Put $X = (I - \Delta^{-1}K_1)H^\infty(\Omega)$ and note that $\ker K_1 \cap H^\infty(\Omega) \subset X$. Since $\ker K_1 \cap H^\infty(\Omega)$ is weak*-closed and has finite codimension in M, and $X \subset M$, it follows that X is weak* closed.

It remains to show that ΔX is weak*-closed. It is enough to check that Δ^{-1} is weak*-continuous. Since K and K_1 have preduals, it follows that Δ has a predual. Therefore, Δ^{-1} also has a predual, and so it is weak*-continuous. $\square$

Finally, we can show that $\mathcal{H}_\Phi$ is weak*-closed in $H^\infty(\Omega)$. The argument is similar to the proof of Lemma 2.2.

Lemma 2.16. *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible, then $\mathcal{H}_\Phi$ is weak*-closed in $H^\infty(\Omega)$.*

Proof. We have already seen in the proof of Lemma 2.2 that the range of the operator $f \mapsto \sum F_k(f) \circ \varphi_k$, $f \in H^\infty(\Omega)$, has finite codimension in $H^\infty(\Omega)$. By the preceding lemma, the range is also weak*-closed. Since $\mathcal{H}_\Phi$ contains this range, we get that $\mathcal{H}_\Phi$ is weak*-closed. $\square$

2.4. Glued subalgebras

In this section we introduce an algebraic tool that we will use to study the algebras of functions on the analytic variety $\mathcal{V}$, with the objective of proving Theorems 2.3 and 2.4, which were stated above. We study finite codimensional subalgebras of a certain form, which we call glued subalgebras. These subalgebras will be used in the next section, because they appear in the classification of finite codimensional subalgebras of a uniform algebra given by Gamelin [Gam68].

The goal of this section is to characterize the maximal ideal space and the derivations of finite codimensional subalgebras of a unital commutative Banach algebra. The arguments used are purely algebraic and similar results hold for arbitrary unital commutative complex algebras. In the algebraic setting, one should replace the maximal ideal space by the set of all (unital) homomorphisms of the algebra into the complex field and disregard every reference made to the topology, such as closed subspaces and continuity of homomorphisms and derivations.

Let $\mathcal{A}$ be a commutative unital Banach algebra. A *glued subalgebra* of $\mathcal{A}$ is understood to be a (unital) subalgebra of the form

$$\mathcal{B} = \{f \in \mathcal{A} : \alpha_j(f) = \beta_j(f),\ j = 1, \dots, r\}, \tag{2.3}$$

where $\alpha_j, \beta_j \in \mathfrak{M}(A)$ and $\alpha_j \neq \beta_j$ for $j = 1, \dots, r$. We define the set of points of $\mathcal{A}$ glued in $\mathcal{B}$ as

$$G(\mathcal{A}, \mathcal{B}) = \{\alpha_j : j = 1, \dots, r\} \cup \{\beta_j : j = 1, \dots, r\} \subset \mathfrak{M}(\mathcal{A}).$$

Our first goal is to characterize the space $\mathfrak{M}(\mathcal{B})$ in terms of $\mathfrak{M}(\mathcal{A})$. Since $\mathcal{B}$ is a subalgebra of $\mathcal{A}$, there is a map $i^* : \mathfrak{M}(\mathcal{A}) \to \mathfrak{M}(\mathcal{B})$ which sends each complex homomorphism $\psi \in \mathfrak{M}(\mathcal{A})$ to its restriction $\psi|\mathcal{B} \in \mathfrak{M}(\mathcal{B})$. We first show that i^* is onto. To do this, we need to use the so called "lying over lemma", which applies to integral ring extensions.

Recall that if R is a subring of some ring S, then S is called *integral* over R if for every $\alpha \in S$ there is a monic polynomial $p \in R[x]$ such that $p(\alpha) = 0$. It is well known that if $\mathcal{B}$ is a finite codimensional subalgebra of some algebra $\mathcal{A}$, then $\mathcal{A}$ is integral over $\mathcal{B}$.

The following "lying over lemma" or Cohen-Seidenberg theorem is a standard result from commutative algebra. It was originally proved in [CS46].

Lemma 2.17 (Lying over lemma). *If S is integral over R and P is a prime ideal in R, then there is a prime ideal Q in S such that $P = Q \cap R$ (we say that Q is* lying *over P). If Q is a prime ideal in S lying over P, then Q is maximal if and only if P is maximal.*

Lemma 2.18. *If $\mathcal{B}$ is a finite codimensional closed unital subalgebra of a commutative unital Banach algebra $\mathcal{A}$, then $i^* : \mathfrak{M}(\mathcal{A}) \to \mathfrak{M}(\mathcal{B})$ is onto.*

Proof. Let $\psi_{\mathcal{B}} \in \mathfrak{M}(\mathcal{B})$ and put $P = \ker \psi_{\mathcal{B}}$. Then P is a maximal ideal in $\mathcal{B}$. By the lying over lemma, there is a maximal ideal Q in $\mathcal{A}$ such that $Q \cap \mathcal{B} = P$. Since P is closed and has finite codimension in $\mathcal{A}$ and $Q \supset P$, it follows that Q is closed. Every maximal ideal in a Banach algebra has codimension one, so Q has codimension one in $\mathcal{A}$. Therefore, there exists a unique $\psi_{\mathcal{A}} \in \mathfrak{M}(\mathcal{A})$ such that $\ker \psi_{\mathcal{A}} = Q$. The equality $Q \cap \mathcal{B} = P$ implies $i^*(\psi_{\mathcal{A}}) = \psi_{\mathcal{B}}$. $\square$

Note that since $\mathfrak{M}(A)$ is compact and i^* is continuous, it follows that i^* is topologically a quotient map.

In the purely algebraic setting, it is no longer true that every maximal ideal has codimension one. However, one can still show that Q has codimension one in $\mathcal{A}$. Indeed, note that P has finite codimension in $\mathcal{B}$. Therefore P has finite codimension in $\mathcal{A}$. Since $P \subset Q$, it follows that Q has finite codimension in $\mathcal{A}$. Now, $\mathcal{A}/Q$ is a field, because Q is a maximal ideal in $\mathcal{A}$. Also, $\mathcal{A}/Q$ is a finite dimensional vector space over $\mathbb{C}$. Since finite field extensions are algebraic and $\mathbb{C}$ is algebraically closed, it follows that $\mathcal{A}/Q$ is isomorphic to $\mathbb{C}$, so that Q has codimension one in $\mathcal{A}$.

We can also give an alterantive proof of Lemma 2.18 for the particular case when $\mathcal{B}$ is a glued subalgebra of $\mathcal{A}$. This alternative proof does not use the lying over lemma. In a certain way, it is a more constructive proof.

Alternative proof of Lemma 2.18 when $\mathcal{B}$ is a glued subalgebra of $\mathcal{A}$. Take an arbitrary $\psi_{\mathcal{B}} \in \mathfrak{M}(\mathcal{B})$. We have to show that there is some $\psi_{\mathcal{A}} \in \mathfrak{M}(\mathcal{A})$ such that $\psi_{\mathcal{A}}|\mathcal{B} = \psi_{\mathcal{B}}$. If $\psi_{\mathcal{B}} \in i^*(G(\mathcal{A},\mathcal{B}))$, then we have finished, so assume that $\psi_{\mathcal{B}} \notin i^*(G(\mathcal{A},B))$.

Applying Lemma 2.19 below to the Banach algebra $\mathcal{B}$, we can take a function $f \in \mathcal{B}$ such that $\psi_{\mathcal{B}}(f) = 1$ and $\psi(f) = 0$ for every $\psi \in G(\mathcal{A},\mathcal{B})$. We define $\psi_{\mathcal{A}}$ by $\psi_{\mathcal{A}}(g) = \psi_{\mathcal{B}}(fg)$ for every $g \in \mathcal{A}$. Note that this is well defined, because since $\psi(fg) = 0$ for every $\psi \in G(\mathcal{A},\mathcal{B})$, it follows that $fg \in \mathcal{B}$. If $g \in \mathcal{B}$, then $\psi_{\mathcal{A}}(g) = \psi_{\mathcal{B}}(fg) = \psi_{\mathcal{B}}(f)\psi_{\mathcal{B}}(g) = \psi_{\mathcal{B}}(g)$, so $\psi_{\mathcal{A}}|\mathcal{B} = \psi_{\mathcal{B}}$. Also, $\psi_{\mathcal{A}}$ is clearly linear and continuous. To show that it is multiplicative, we take $g, h \in \mathcal{A}$ and compute

$$\psi_{\mathcal{A}}(gh) = \psi_{\mathcal{B}}(fgh) = \psi_{\mathcal{B}}(fgh)\psi_{\mathcal{B}}(f) = \psi_{\mathcal{B}}(fg)\psi_{\mathcal{B}}(fh) = \psi_{\mathcal{A}}(g)\psi_{\mathcal{A}}(h).$$

Therefore, $\psi_{\mathcal{A}} \in \mathfrak{M}(\mathcal{A})$, as we wanted to show. $\square$

The following is a kind of "interpolation" lemma. It will be very useful in the rest of this section.

Lemma 2.19. *Let $\mathcal{A}$ be a commutative unital Banach algebra. If $\psi_0, \ldots, \psi_s$ are distinct points in $\mathfrak{M}(\mathcal{A})$, then there is some $f \in \mathcal{A}$ such that $\psi_j(f) = 0$ for $j = 1, \ldots, s$, but $\psi_0(f) = 1$.*

Proof. Fix $j \in \{1, \ldots, s\}$. There is some $f_j \in \mathcal{A}$ such that $\psi_0(f_j) \neq 0$ and $\psi_j(f_j) = 0$, for if this were not the case, then $\ker \psi_0 \subset \ker \psi_j$, which would imply that $\ker \psi_0 = \ker \psi_j$ since both kernels have codimension one in $\mathcal{A}$. Hence we would have $\psi_0 = \psi_j$, a contradiction.

For $f_1, \ldots, f_s$ chosen in this way,

$$f = \prod_{j=1}^{s} \frac{f_j}{\psi_0(f_j)}.$$

has the required properties. $\square$

Lemma 2.20. *If $\mathcal{B}$ is a glued subalgebra of $\mathcal{A}$ and $\psi_{\mathcal{B}} \in \mathfrak{M}(\mathcal{B})$, then either we have $(i^*)^{-1}(\{\psi_{\mathcal{B}}\}) \subset G(\mathcal{A},\mathcal{B})$ or $(i^*)^{-1}(\{\psi_{\mathcal{B}}\}) = \{\psi_{\mathcal{A}}\}$, for some $\psi_{\mathcal{A}} \notin G(\mathcal{A},\mathcal{B})$.*

Proof. Assume that we have distinct $\psi_{\mathcal{A}}, \widetilde{\psi}_{\mathcal{A}} \in (i^*)^{-1}(\{\psi_{\mathcal{B}}\})$ with $\psi_{\mathcal{A}} \notin G(\mathcal{A},\mathcal{B})$. By Lemma 2.19, there is an $f \in \mathcal{A}$ such that $\psi_{\mathcal{A}}(f) = 1$ and $\psi(f) = 0$ for $\psi \in G(\mathcal{A},\mathcal{B}) \cup \{\widetilde{\psi}_{\mathcal{A}}\}$, as $\psi_{\mathcal{A}} \notin G(\mathcal{A},\mathcal{B}) \cup \{\widetilde{\psi}_{\mathcal{A}}\}$ by hypothesis. Then $f \in \mathcal{B}$, because $\alpha_j(f) = \beta_j(f) = 0$, for $j = 1,\ldots,r$, and

$$1 = \psi_{\mathcal{A}}(f) = \psi_{\mathcal{B}}(f) = \widetilde{\psi}_{\mathcal{A}}(f) = 0,$$

because $\psi_{\mathcal{A}}|\mathcal{B} = \psi_{\mathcal{B}} = \widetilde{\psi}_{\mathcal{A}}|\mathcal{B}$. This is a contradiction. □

We next describe the derivations of $\mathcal{B}$ in terms of the derivations of $\mathcal{A}$. This requires the following well-known characterization of derivations: *A linear functional η on $\mathcal{A}$ is a derivation at $\psi \in \mathfrak{M}(\mathcal{A})$ if and only if $\eta(1) = 0$ and $\eta(fg) = 0$ whenever $f, g \in \mathcal{A}$ and $\psi(f) = \psi(g) = 0$.*

Lemma 2.21. *Let $\mathcal{B}$ be a glued subalgebra of $\mathcal{A}$, and $\eta_{\mathcal{B}}$ a derivation of $\mathcal{B}$ at a point $\psi_{\mathcal{B}} \in \mathfrak{M}(\mathcal{B})$. Put*

$$\{\psi^1_{\mathcal{A}},\ldots,\psi^s_{\mathcal{A}}\} = (i^*)^{-1}(\{\psi_{\mathcal{B}}\}) \subset \mathfrak{M}(\mathcal{A})$$

(*this set is finite by Lemma 2.20*). *Then there exist unique derivations $\eta^1_{\mathcal{A}},\ldots,\eta^s_{\mathcal{A}}$ of $\mathcal{A}$ at the points $\psi^1_{\mathcal{A}},\ldots,\psi^s_{\mathcal{A}}$ respectively such that*

$$\eta_{\mathcal{B}} = (\eta^1_{\mathcal{A}} + \ldots + \eta^s_{\mathcal{A}})|\mathcal{B}.$$

Proof. For each $k = 1,\ldots,s$, use Lemma 2.19 to obtain an $f_k \in \mathcal{A}$ such that $\psi^j_{\mathcal{A}}(f_k) = \delta_{jk}$ and $\psi(f_k) = 0$ for $\psi \in G(\mathcal{A},\mathcal{B}) \setminus \{\psi^1_{\mathcal{A}},\ldots,\psi^s_{\mathcal{A}}\}$. Put $g_k = 2f_k - f_k^2$. Then g_k also satisfies $\psi^j_{\mathcal{A}}(g_k) = \delta_{jk}$ and $\psi(g_k) = 0$ for $\psi \in G(\mathcal{A},\mathcal{B}) \setminus \{\psi^1_{\mathcal{A}},\ldots,\psi^s_{\mathcal{A}}\}$.

Define $\eta^j_{\mathcal{A}}$ by

$$\eta^j_{\mathcal{A}}(f) = \eta_{\mathcal{B}}(g_j^2(f - \psi^j_{\mathcal{A}}(f))), \qquad f \in \mathcal{A}.$$

Since $\psi(g_j^2(f - \psi^j_{\mathcal{A}}(f))) = 0$ for every $\psi \in G(\mathcal{A},\mathcal{B})$, $g_j^2(f - \psi^j_{\mathcal{A}}(f)) \in \mathcal{B}$ and so $\eta^j_{\mathcal{A}}$ is well defined.

We claim that $\eta^j_{\mathcal{A}}$ is a derivation of $\mathcal{A}$ at $\psi^j_{\mathcal{A}}$. It is clearly linear and $\eta^j_{\mathcal{A}}(1) = 0$. Take $f, g \in \mathcal{A}$ with $\psi^j_{\mathcal{A}}(f) = \psi^j_{\mathcal{A}}(g) = 0$. Then,

$$\eta^j_{\mathcal{A}}(fg) = \eta_{\mathcal{B}}(g_j^2 fg) = 0,$$

because $g_j f$ and $g_j g$ belong both to $\mathcal{B}$ and $\psi_{\mathcal{B}}(g_j f) = \psi_{\mathcal{B}}(g_j g) = 0$ as $\psi(g_j f) = \psi(g_j g) = 0$ for every $\psi \in G(\mathcal{A},\mathcal{B}) \cup \{\psi^1_{\mathcal{A}},\ldots \psi^s_{\mathcal{A}}\}$. It follows that $\eta^j_{\mathcal{A}}$ is a derivation of $\mathcal{A}$ at $\psi^j_{\mathcal{A}}$.

Now we check that $(\eta^1_{\mathcal{A}} + \cdots + \eta^s_{\mathcal{A}})|\mathcal{B} = \eta_{\mathcal{B}}$. For this, put

$$g_0 = g_1^2 + \cdots + g_s^2.$$

Note that $\psi^1_{\mathcal{A}}(g_0) = \cdots = \psi^s_{\mathcal{A}}(g_0) = 1$ and $\psi(g_0) = 0$ for $\psi \in G(\mathcal{A},\mathcal{B}) \setminus \{\psi^1_{\mathcal{A}},\ldots,\psi^s_{\mathcal{A}}\}$. If α_j, β_j are as in (2.3), then $i^*(\alpha_j) = i^*(\beta_j)$. Therefore, either α_j, β_j both belong to $(i^*)^{-1}(\{\psi_{\mathcal{B}}\}) = \{\psi^1_{\mathcal{A}},\ldots,\psi^s_{\mathcal{A}}\}$ or they both belong to $G(\mathcal{A},\mathcal{B}) \setminus \{\psi^1_{\mathcal{A}},\ldots,\psi^s_{\mathcal{A}}\}$. Hence,

$\alpha_j(g_0) = \beta_j(g_0)$, because $\alpha_j(g_0)$ and $\beta_j(g_0)$ are both 1 or both 0. Therefore, $g_0 \in \mathcal{B}$. Also, $\psi_{\mathcal{B}}(g_0) = \psi^1_{\mathcal{A}}(g_0) = 1$.

Take any $f \in \mathcal{B}$. Then

$$\sum_{j=1}^{s} \eta^j_{\mathcal{A}}(f) = \sum_{j=1}^{s} \eta_{\mathcal{B}}(g_j^2(f - \psi^j_{\mathcal{A}}(f))) = \eta_{\mathcal{B}}(g_0(f - \psi_{\mathcal{B}}(f))) = \eta_{\mathcal{B}}(f - \psi_{\mathcal{B}}(f)) = \eta_{\mathcal{B}}(f),$$

because $\psi_{\mathcal{B}}(f - \psi_{\mathcal{B}}(f)) = 0$, and $\psi_{\mathcal{B}}(g_0) = 1$. This shows that $(\eta^1_{\mathcal{A}} + \cdots + \eta^s_{\mathcal{A}})|\mathcal{B} = \eta_{\mathcal{B}}$.

To prove uniqueness, assume that $\widetilde{\eta}^1_{\mathcal{A}}, \ldots, \widetilde{\eta}^s_{\mathcal{A}}$ are derivations of $\mathcal{A}$ at $\psi^1_{\mathcal{A}}, \ldots, \psi^s_{\mathcal{A}}$ respectively and such that $(\widetilde{\eta}^1_{\mathcal{A}} + \cdots + \widetilde{\eta}^s_{\mathcal{A}})|\mathcal{B} = \eta_{\mathcal{B}}$.

Since $\psi^j_{\mathcal{A}}(g_j) = \psi^j_{\mathcal{A}}(f_j) = 1$,

$$\widetilde{\eta}^j_{\mathcal{A}}(g_j^2) = 2\widetilde{\eta}^j_{\mathcal{A}}(g_j)\psi^j_{\mathcal{A}}(g_j) = 2\widetilde{\eta}^j_{\mathcal{A}}(g_j) = 2\widetilde{\eta}^j_{\mathcal{A}}(2f_j - f_j^2) = 4\widetilde{\eta}^j_{\mathcal{A}}(f_j) - 4\widetilde{\eta}^j_{\mathcal{A}}(f_j)\psi^j_{\mathcal{A}}(f_j) = 0,$$

and so $\widetilde{\eta}^j_{\mathcal{A}}(g_j^2) = 0$. Thus for any $f \in \mathcal{A}$,

$$\widetilde{\eta}^j_{\mathcal{A}}(f) = \widetilde{\eta}^j_{\mathcal{A}}(g_j^2 f) = \widetilde{\eta}^j_{\mathcal{A}}(g_j^2(f - \psi^j_{\mathcal{A}}(f))),$$

and if $j \neq k$, then

$$\widetilde{\eta}^k_{\mathcal{A}}(g_j^2(f - \psi^j_{\mathcal{A}}(f))) = 0,$$

because $\psi^k_{\mathcal{A}}(g_j) = \psi^k_{\mathcal{A}}(g_j(f - \psi^j_{\mathcal{A}}(f))) = 0$. Also, since $\psi(g_j^2(f - \psi_j(f))) = 0$ for every $\psi \in G(\mathcal{A}, \mathcal{B})$, $g_j^2(f - \psi_j(f)) \in \mathcal{B}$. Hence,

$$\widetilde{\eta}^j_{\mathcal{A}}(f) = \sum_{k=1}^{s} \widetilde{\eta}^k_{\mathcal{A}}(g_j^2(f - \psi^j_{\mathcal{A}}(f))) = \eta_{\mathcal{B}}(g_j^2(f - \psi^j_{\mathcal{A}}(f))) = \eta^j_{\mathcal{A}}(f).$$

This shows that $\widetilde{\eta}^j_{\mathcal{A}} = \eta^j_{\mathcal{A}}$ for $j = 1, \ldots, s$. □

Remark. If we denote by $\mathrm{Der}_{\psi}(\mathcal{A})$ the linear space of derivations of $\mathcal{A}$ at $\psi \in \mathfrak{M}(\mathcal{A})$, then the lemma above shows that

$$\mathrm{Der}_{\psi_{\mathcal{B}}}(\mathcal{B}) \cong \bigoplus_{\psi \in (i^*)^{-1}(\{\psi_{\mathcal{B}}\})} \mathrm{Der}_{\psi}(\mathcal{A}).$$

2.5. Algebras in analytic varieties

In this section we study the algebras $H^\infty(\mathcal{V})$ and $A(\mathcal{V})$ of functions on the analytic curve $\mathcal{V}$. The goal of this section is to prove Theorems 2.3 and 2.4. Theorem 2.3 gives the relation of the algebras $H^\infty(\mathcal{V})$ and $A(\mathcal{V})$ with the algebras $\mathcal{H}_\Phi$ and $\mathcal{A}_\Phi$ that where introduced above. Theorem 2.4 shows that every function in $H^\infty(\mathcal{V})$ can be extended to a function in the Agler algebra of $\mathbb{D}^n$.

We use results by Gamelin [Gam68] on finite codimensional subalgebras of uniform algebras. In particular, we need to use his concept of a *θ-subalgebra*.

Definition 2.22. Let $\mathcal{A}$ be a commutative unital Banach algebra and $\theta \in \mathfrak{M}(\mathcal{A})$. A θ-*subalgebra* of $\mathcal{A}$ is a subalgebra $\mathcal{B}$ such that there is a chain $\mathcal{B} = \mathcal{B}_0 \subset \mathcal{B}_1 \subset \cdots \subset \mathcal{B}_n = \mathcal{A}$ where $\mathcal{B}_k$ is the kernel of some derivation in $\mathcal{B}_{k+1}$ at the point θ (the restriction map $i^* : \mathfrak{M}(\mathcal{B}_{k+1}) \to \mathfrak{M}(\mathcal{B}_k)$ is a bijection, so the maximal ideal spaces of all the algebras $\mathcal{B}_k$ can be viewed as being the same).

The idea behind this definition is that a θ-subalgebra is the subalgebra obtained by taking only the functions whose derivatives at θ satisfy certain linear conditions. Here it is possible to take higher order derivatives. We show this idea through some examples.

Example 2.23. We put $\mathcal{A} = H^\infty(\mathbb{D})$ and define θ by $\theta(f) = f(0)$. Given $s \in \mathbb{N}$, the subalgebra
$$\{f \in H^\infty(\mathbb{D}) : f'(0) = f''(0) = \cdots = f^{(s)}(0) = 0\}$$
is a θ-subalgebra of $H^\infty(\mathbb{D})$.

Indeed, it is possible to write a chain of subalgebras $\mathcal{B}_k$, $k = 0, \dots, s$, as in the definition of θ-subalgebra. We put
$$\mathcal{B}_k = \{f \in H^\infty(\mathbb{D}) : f'(0) = \cdots = f^{(s-k)}(0) = 0\}$$
for $k < s$ and $\mathcal{B}_s = H^\infty(\mathbb{D})$. Then we see that $\eta_k(f) = f^{(k-s)}(0)$ is a derivation of $\mathcal{B}_{k+1}$ at θ and $\mathcal{B}_k = \ker \eta_k$.

Example 2.24. We put $\mathcal{A} = \{f \in H^\infty(\mathbb{D}) : f(a) = f(b)\}$, where a, b are two distinct points in $\mathbb{D}$. Note that $\mathcal{A}$ is a glued subalgebra of $H^\infty(\mathbb{D})$. The maximal ideal space of $\mathcal{A}$ is like the maximal ideal of $H^\infty(\mathbb{D})$ but with the points correponding to evaluations at a and b identified. We denote by θ this "glued" point: $\theta(f) = f(a) = f(b)$, for $f \in \mathcal{A}$.

Given $\alpha, \beta \in \mathbb{C}$ with $(\alpha, \beta) \neq (0, 0)$, the subalgebra
$$\{f \in \mathcal{A} : \alpha f'(a) + \beta f'(b) = 0\}$$
is a θ-subalgebra of $\mathcal{A}$, since $\eta(f) = \alpha f'(a) + \beta f'(b)$ is a derivation of $\mathcal{A}$ at θ by Lemma 2.21.

The main result of Gamelin about the classification of finite-codimensional subalgebras is the following theorem.

Theorem 2.25 (Gamelin [Gam68, Theorem 9.8])**.** *Let $\mathcal{A}$ be a function algebra (this means that $\mathcal{A}$ is a closed unital subalgebra of $C(X)$ for some compact space X and such that $\mathcal{A}$ separates the points of X). Let $\mathcal{B}$ be a closed unital subalgebra of $\mathcal{A}$ of finite codimension. Then there exists $\mathcal{A}_0$ a glued subalgebra of $\mathcal{A}$, a finite collection of points $\theta_1, \dots, \theta_r \in \mathfrak{M}(\mathcal{A}_0)$ and θ_j-subalgebras $\mathcal{B}_j$ of $\mathcal{B}_0$ such that $\mathcal{B} = \mathcal{B}_1 \cap \cdots \cap \mathcal{B}_r$.*

It is clear that every subalgebra obtained in this manner is a finite codimensional subalgrebra. For instance, the intersection of the θ-subalgebras of the two examples above is a finite-codimensional subalgebra of $H^\infty(\mathbb{D})$. What the result by Gamelin says is that every finite-codimensional subalgebra has to be obtained in this way.

Note that every semisimple Banach algebra $\mathcal{A}$ is isomorphic to a subalgebra of $C(\mathcal{M}(\mathcal{A}))$ via the Gelfand transform. Therefore, Gamelin's theorem can be applied in this case (recall that a commutative algebra is semisimple if the intersection of all its maximal ideals is the zero ideal).

We will also use the following lemma by Gamelin.

Lemma 2.26 (Gamelin [Gam68, Lemma 9.3])**.** *Let $\mathcal{B}$ be a θ-subalgebra of $\mathcal{A}$ of codimension k. If $f_1, \dots, f_{2^k} \in \mathcal{A}$ are 2^k functions such that $\theta(f_j) = 0$ for all j, then their product $f = f_1 f_2 \cdots f_{2^k}$ belongs to $\mathcal{B}$.*

Lemma 2.27. *Assume that $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible. There is a finite set $X \subset \Omega$ and $N \in \mathbb{N}$ such that if $f \in H^\infty(\Omega)$ has a zero of order N at each point of X, then $f \in \mathcal{H}_\Phi$. The same is true if one replaces $H^\infty(\Omega)$ by $A(\overline{\Omega})$ and $\mathcal{H}_\Phi$ by $\mathcal{A}_\Phi$.*

Proof. By Theorem 2.25, there exists a glued subalgebra H_0 of $H^\infty(\Omega)$ and a finite family of θ_j-subalgebras H_j of H_0 such that

$$\mathcal{H}_\Phi = H_1 \cap \cdots \cap H_r.$$

(Here $\theta_j \in \mathfrak{M}(H_0)$.)

Put $G = G(H^\infty(\Omega), H_0)$. By Lemma 2.10, $\psi(\mathbf{z}) \in \Omega$ for every $\psi \in G$. Consider the map $i^* : \mathfrak{M}(H^\infty(\Omega)) \to \mathfrak{M}(H_0)$ and put $Y = (i^*)^{-1}(\{\theta_1, \dots, \theta_r\})$. By Lemma 2.20, Y is a finite set.

If $\psi \in Y$, then $\psi(\mathbf{z}) \in \Omega$, since either $\psi \in G$ or ψ is the unique preimage of some θ_j. In the latter case, since $\mathcal{H}_\Phi \subset H_j$, there is some derivation η of H_0 at θ_j such that $\eta|\mathcal{H}_\Phi = 0$ but $\eta \neq 0$. By Lemma 2.21, η extends to a derivation $\widehat{\eta}$ of $H^\infty(\Omega)$ at ψ. By Lemma 2.11, $\psi(\mathbf{z}) \in \Omega$, because $\widehat{\eta}|\mathcal{H}_\Phi = 0$.

We claim that $X = \{\psi(\mathbf{z}) : \psi \in G \cup Y\} \subset \Omega$ is the desired set. Note that X is finite. Also, if $f \in H^\infty(\Omega)$ vanishes on X then $f \in H_0$, because $\psi(f) = 0$ for every $\psi \in G$.

Let k_j be the codimension of H_j in H_0. Assume that $f \in H^\infty(\Omega)$ has a zero of order 2^{k_j} at every point of X. Then f can be factored as $f = f_1 \cdots f_{2^{k_j}}$, where each of the 2^{k_j} functions belongs to $H^\infty(\Omega)$ and vanishes on X. This implies $f \in H_j$ by lemma 2.26. Thus, the lemma holds with $N = \max_j 2^{k_j}$.

The proof for $A(\overline{\Omega})$ is similar. $\square$

Now we can give the proofs of Theorems 2.3 and 2.4

Proof of Theorem 2.3. We only show that $\Phi^* H^\infty(\mathcal{V}) = \mathcal{H}_\Phi$ as the proof of $\Phi^* A(\overline{\Omega}) = \mathcal{A}_\Phi$ is identical. The inclusion $\mathcal{H}_\Phi \subset \Phi^* H^\infty(\mathcal{V})$ follows from (2.1). We examine the reverse inclusion.

Let $n \in \mathbb{N}$ be the integer and $X = \{z_1, \dots, z_r\} \subset \Omega$ the finite set from Lemma 2.27. Put $w_j = \Phi(z_j) \in \mathcal{V}$, $j = 1, \dots, r$. Take $F \in H^\infty(\mathcal{V})$ and observe that F extends to an analytic function, which we also denote by F, on a neighbourhood in $\mathbb{C}^n$ of each of the points w_j, $j = 1, \dots, r$. Let $P \in \mathbb{C}[z_1, \dots, z_n]$ be a polynomial such that

$$D^\alpha P(z_j) = D^\alpha F(z_j), \qquad 0 \leq |\alpha| \leq N, \ j = 1, \dots, r.$$

Then $\Phi^*(F - P) = F \circ \Phi - P \circ \Phi$ has a zero of order N at each point of X, so $\Phi^*(F - P) \in \mathcal{H}_\Phi$. Also, $\Phi^* P \in \mathcal{H}_\Phi$, because it is a polynomial in $\varphi_1, \dots, \varphi_n$. It follows that $\Phi^* F \in \mathcal{H}_\Phi$. $\square$

Proof of Theorem 2.4. By Lemma 2.2, $\mathcal{H}_\Phi$ is a closed subspace of $H^\infty(\Omega)$. Define an operator $L : \mathcal{H}_\Phi \to \mathcal{H}_\Phi$ by $Lf = \sum_{k=1}^n F_k(f) \circ \varphi_k$, where $F_1, \dots, F_n$ are the operators that appear in Theorem 1.11. Since $I - L$ is compact, by the Fredholm theory there are bounded operators $R, P : \mathcal{H}_\Phi \to \mathcal{H}_\Phi$ such that P has finite rank and $I = LR + P$. The operator P can be written as

$$Pf = \sum_{k=1}^{r} \alpha_k(f) g_k, \qquad f \in H^\infty(\Omega),$$

for some $g_k \in \mathcal{H}_\Phi$ and $\alpha_k \in \mathcal{H}_\Phi^*$. The functions g_k can be expressed as

$$g_k = \sum_{j=1}^{l} \prod_{i=1}^{n} f_{j,i,k} \circ \varphi_i$$

(so as to have the same number of multiplicands and terms in these sums, we can take some of the $f_{j,i,k}$ equal to 0 or 1).

Take an $f \in H^\infty(\mathcal{V})$. By Theorem 2.3, $\Phi^* f \in \mathcal{H}_\Phi$. Put

$$F(z_1, \dots, z_n) = \sum_{k=1}^{n} F_k(R\Phi^* f)(z_k) + \sum_{k=1}^{r} \sum_{j=1}^{l} \alpha_k(\Phi^* f) \prod_{i=1}^{n} f_{j,i,k}(z_i).$$

Then $\Phi^* F = F \circ \Phi = LR\Phi^* f + P\Phi^* f = \Phi^* f$, so $F|\mathcal{V} = f$.

If $g \in H^\infty(\mathbb{D})$, then $\|g(z_k)\|_{\mathcal{SA}(\mathbb{D}^n)} = \|g\|_\infty$, so

$$\begin{aligned} \|F\|_{\mathcal{SA}(\mathbb{D}^n)} &\leq \sum_{k=1}^{n} \|F_k(R\Phi^* f)\|_\infty + \sum_{k=1}^{r} \sum_{j=1}^{l} |\alpha_k(\Phi^* f)| \cdot \prod_{i=1}^{n} \|f_{j,i,k}\|_\infty \\ &\leq C \|\Phi^* f\|_{H^\infty(\mathbb{D})} = C \|f\|_{H^\infty(\mathcal{V})}. \end{aligned}$$

□

3. Complete K-spectral sets

This chapter is based on joint work with Michael Dritschel and Dmitry Yakubovich. The results of this chapter are contained in the article [DEY15].

3.1. Introduction

Let T be an operator on a Hilbert space H and Ω a bounded subset of $\mathbb{C}$ containing the spectrum $\sigma(T)$. We recall that, given a constant $K \geq 1$, the closure $\overline{\Omega}$ of Ω is said to be *a complete K-spectral set for T* if the matrix von Neumann inequality

$$\|p(T)\|_{\mathcal{B}(H \otimes \mathbb{C}^s)} \leq K \max_{z \in \overline{\Omega}} \|p(z)\|_{\mathcal{B}(\mathbb{C}^s)} \tag{3.1}$$

holds for any square $s \times s$ rational matrix function $p(z)$ of any size s and with poles off of $\overline{\Omega}$; here $\mathcal{B}(H)$ denotes the space of linear operators on H. The set $\overline{\Omega}$ is called a *K-spectral set for T* if (3.1) holds for $s = 1$. By a well-known theorem of Arveson [Arv69], $\overline{\Omega}$ is a complete K-spectral set for T for some $K \geq 1$ if and only if T is similar to an operator, which has a normal dilation N with $\sigma(N) \subset \partial\Omega$; the importance of complete K-spectral sets is due to this result.

Let us state some of the main results of this chapter.

Theorem 3.1. *Let $\Omega_1, \dots, \Omega_n$ be open sets in $\widehat{\mathbb{C}}$ such that the boundary of each set Ω_k, $k = 1, \dots, n$, is a finite disjoint union of Jordan curves. We also assume that the boundaries of the sets Ω_k, $k = 1, \dots, n$, are Ahlfors regular and rectifiable, and intersect transversally (see Definition 1.6). Put $\Omega = \Omega_1 \cap \dots \cap \Omega_n$. Suppose that $T \in \mathcal{B}(H)$, and $\sigma(T) \subset \overline{\Omega}$ and a constant $K \geq 1$ is given. There is a constant K' such that*

(i) if each of the sets $\overline{\Omega}_j$, $j = 1, \dots, n$, is K-spectral for T, then $\overline{\Omega}$ is also K'-spectral set for T; and

(ii) if each of the sets $\overline{\Omega}_j$, $j = 1, \dots, n$, is complete K-spectral for T, then $\overline{\Omega}$ is a complete K'-spectral set for T.

In both cases, K' depends only on the sets $\Omega_1, \dots, \Omega_n$ and the constant K, but not on the operator T.

The Ahlfors regularity condition has been given in Definition 1.9 (page 6). As will be seen from the proof, the Ahlfors regularity condition can be weakened, by requiring that it hold only in some neighbourhoods of the intersection points of the boundary curves $\partial\Omega_j$.

The results of Theorem 3.1 can be viewed as a generalization of the so called surgery of K-spectral sets. The articles [Lew90, Sta86, Sta90] are devoted to this topic. In the case when the sets that one is dealing with are Jordan domains and their boundaries intersect transversally, the results of these articles can be obtained as a particular case of Theorem 3.1.

In [BBC09], Badea, Beckermann and Crouzeix prove that the intersection of complete spectral sets which are disks on the Riemann sphere is a complete K'-spectral set (see Theorem 3.10 below for the precise statement). Theorem 3.1 is a generalization of this result in two ways. Firstly, it allows for the sets $\overline{\Omega}_j$ to be complete K-spectral sets instead of complete spectral sets. Secondly, it allows for the sets Ω_j to be open sets with some conditions on the boundary, rather than just disks. The points of [BBC09] that are not covered by Theorem 3.1 is that there they do not need transversality and obtain a value of K which is an explicit universal constant depending only on the number of disks. We remark that in our results we do not have an explicit control on the constant K.

The particular case $K = 1$ is important. Let us say that a domain Ω has the rational dilation property if whenever $\overline{\Omega}$ is a 1-spectral set for T, then $\overline{\Omega}$ is also a complete 1-spectral set for T. It follows from the fact that every contraction has a unitary dilation that $\mathbb{D}$ has the rational dilation property. In [Agl85], Agler proved that every annulus has the rational dilation property. When Ω is a domain with two or more holes, it does not have the rational dilation property in general. Dritschel and McCullough in [DM05] and Agler, Harland and Raphael in [AHR08] found independently examples of domains with two holes which do not have the rational dilation property. See also the article [Pic10] by Pickering, where he shows that every symmetric domain with two or more holes does not have the rational dilation property.

In the next theorem, we deal with open sets satisfying a certain regularity condition. If $\Omega \subset \widehat{\mathbb{C}}$ is an open set with $\infty \notin \partial\Omega$ and $R > 0$, we say that Ω satisfies the *exterior disk condition with radius* R if for every $\lambda \in \partial\Omega$ there is $\mu \in \mathbb{C}$ such that the open disk $B(\mu, R)$ *touches* Ω *at* λ; that is $|\lambda - \mu| = R$ and $B(\mu, R) \cap \Omega = \emptyset$.

In order to simplify the geometrical arguments, we will also assume that Ω satisfies the following technical condition.

Condition 3.2. There exists a finite collection of closed arcs $\{\gamma_k\}_{k=1}^N \subset \partial\Omega$ which cover $\partial\Omega$ and intersect at most in their endpoints, radii R_k, $k = 1, \dots, N$, and maps $\mu_k : \gamma_k \to \mathbb{C}$, such that for every $\lambda \in \gamma_k$, the disk $B(\mu_k(\lambda), R_k)$ touches Ω at λ and the intersection $\bigcap_{\lambda\in\gamma_k} B(\mu_k(\lambda), R_k)$ is not empty. We also assume that if γ_k and γ_l intersect at their common endpoint z_0, then they do so transversally: that is, there are disjoint circular sectors S_k and S_l with vertex z_0 such that $\gamma_k \subset S_k \cup \{z_0\}$ and $\gamma_l \subset S_l \cup \{z_0\}$.

If $\partial\Omega$ is piecewise C^2 and the exterior angles at its corners are nonzero, then Ω clearly satisfies Condition 3.2. Moreover, it is possible to prove that if $\partial\Omega$ is a finite disjoint union of Jordan curves and Ω satisfies the exterior disk condition and an interior cone condition, then Ω also satisfies Condition 3.2. In particular, the exterior disk condition is formally weaker than Condition 3.2.

Theorem 3.3. *Let T be a bounded linear operator and $\Omega \subset \widehat{\mathbb{C}}$ an open set whose boundary is a finite disjoint union of Jordan curves. Assume that $\infty \notin \partial\Omega$, that Ω satisfies Condition 3.2 and that $\sigma(T) \subset \overline{\Omega}$. Furthermore, assume that for every*

$k = 1, \dots, N$ and every $\lambda \in \gamma_k$ we have $\|(T - \mu_k(\lambda)I)^{-1}\| \le R_k^{-1}$. Then $\overline{\Omega}$ is a complete K-spectral set for some $K > 0$.

It is easy to see that the hypotheses are satisfied (for any $R_k > 0$) if Ω is a convex Jordan domain and the numerical range of T is contained in $\overline{\Omega}$. This case was first proved by Delyon and Delyon in [DD99]. Theorem 3.3 will be deduced from the Delyon-Delyon result and from Theorem 3.1. Putinar and Sandberg gave a different proof of the Delyon-Delyon result in [PS05] by constructing a so called normal skew-dilation. These articles consider only K-spectral sets instead of complete K-spectral sets. However, the arguments used both in [DD99] and [PS05] imply the existence of a normal operator N on a larger Hilbert space $K \supset H$ and having $\sigma(N) \subset \partial\Omega$, and a bounded linear map $\Xi : C(\partial\Omega) \to C(\partial\Omega)$ such that

$$f(T) = P_H(\Xi(f)(N))|H, \qquad f \in \mathrm{Rat}(\overline{\Omega}).$$

It follows from Lemma 3.17 below that the map Ξ is completely bounded (see also Crouzeix [Cro07]). Therefore, (3.1) implies that $\overline{\Omega}$ is a complete K-spectral set for T, and so that under the assumptions of the Delyon-Delyon theorem, T is similar to an operator having a normal dilation to $\partial\Omega$.

It is also known that Theorem 3.3 is valid if Ω is the unit disk. In fact, by results of Sz.-Nagy and Foias, if the hypotheses hold in this case, then T is a ρ-contraction for some $\rho < \infty$ and hence is similar to a contraction. Therefore Theorem 3.3 can be considered as a generalization of both of the above mentioned results. We refer to Section 3.4 for a further discussion and some consequences of this result.

We will deduce the first part of Theorem 3.1 from results of Havin, Nersessian and Cerdà, namely Proposition 1.8 (see page 6).

The proof of the second part of Theorem 3.1 will also use Lemma 3.17. This lemma says, basically, that if the range of a bounded linear map is commutative, then the map is automatically completely bounded. The particular maps we will be considering have commutative ranges, so this lemma will be important in our proofs. The arguments by Havin, Nersessian and Cerdà will also be key here, because they allow us to deal with commutative algebras of functions (to which Lemma 3.17 can be applied) instead of noncommutative algebras of operators.

Suppose now that Φ is a collection of functions mapping into $\mathbb{D}$, such that each of them is analytic on (its own) neighbourhood of Ω. The rest of this chapter is devoted to finding sufficient conditions for complete K-spectrality of the form

$$\begin{aligned} &\exists K' : \forall \varphi \in \Phi \ \overline{\mathbb{D}} \text{ is a complete } K'\text{-spectral set for } \varphi(T) \\ &\implies \exists K : \overline{\Omega} \text{ is a complete } K\text{-spectral set for } T. \end{aligned} \tag{3.2}$$

Notice that for any $\varphi \in \Phi$, $\varphi(T)$ is defined by the Cauchy-Riesz functional calculus. Our conditions concern the set Ω and the family Φ (namely the conditions we consider in the previous chapter), but we do not impose extra conditions on T.

In particular, a special case of (3.2) is that

$$\forall \varphi \in \Phi \ \|\varphi(T)\| \le 1 \implies \exists K : \overline{\Omega} \text{ is a complete } K\text{-spectral set for } T. \tag{3.3}$$

In this case, Φ will be called a test collection (a more precise definition of this notion will be given in the next section). As we will show, many known sufficient conditions for complete K-spectrality are easily formulated in the form (3.3) or in the form (3.2) for specific test collections. Indeed, Theorems 3.1 and 3.3 can also be given this form if one uses appropriate Riemann mappings for the test functions (see Section 3.3 below).

As we will show, the study of implication (3.2) can be related to the problems of algebra generation that we have studied in Chapter 1.

3.2. Test collections

We denote by M_s the C^*-algebra of complex $s \times s$ matrices. If S is a (not necessarily closed) linear subspace of a C^*-algebra $\mathcal{A}$, we denote by $S \otimes M_s$ the tensor product equipped with the norm inherited from $\mathcal{A} \otimes M_s$, which has a unique C^* norm. One can view $S \otimes M_s$ as the space of $s \times s$ matrices with entries in S. The simplest way to norm this is to represent $\mathcal{A}$ faithfully as a subspace of $\mathcal{B}(H)$ and then to take the natural norm of $s \times s$ operator matrices. If B is another C^*-algebra and $\varphi : S \to B$ is a linear map, we can form the map $\varphi \otimes \mathrm{id}_s : S \otimes M_s \to B \otimes M_s$, which amounts to applying φ entrywise to $s \times s$ matrices over S. The completely bounded norm of φ is then defined as

$$\|\varphi\|_{\mathrm{cb}} = \sup_{s \geq 1} \|\varphi \otimes \mathrm{id}_s\|.$$

If a compact set $X \subset \mathbb{C}$ is a complete K-spectral set for a bounded linear operator T and $\mathrm{Rat}(X)$, the algebra of rational functions with poles off of X, is dense in $A(X)$, then the functional calculus for T extends continuously to $f \in A(X)$, and we say that such a T admits a continuous $A(X)$-calculus. Note that there are various sorts of geometric conditions on X guaranteeing that $\mathrm{Rat}(X)$ is dense in $A(X)$ (see, for instance, [Con91, Chapter V, Theorem 19.2] for one such). In particular, it suffices for X to be finitely connected (see [Con91, Chapter V, Corollary 19.3]). In what follows, we only consider finitely connected domains.

Here we give the definitions of the several kinds of test collections that we consider in this thesis. As a convenient notation, for $\lambda \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, define $p_\lambda(z) = (z - \lambda)^{-1}$ if $\lambda \neq \infty$, and $p_\infty(z) = z$.

Definition 3.4. Assume that $\Omega \subset \widehat{\mathbb{C}}$ is some finitely connected set. A *pole set* for Ω is a finite set $\Lambda \subset \widehat{\mathbb{C}} \setminus \overline{\Omega}$ that intersects each connected component of $\widehat{\mathbb{C}} \setminus \overline{\Omega}$. If $T \in \mathcal{B}(H)$ and $\sigma(T) \subset \overline{\Omega}$, the Λ-*pole size* of T is defined as $\max_{\lambda \in \Lambda} \|p_\lambda(T)\|$. We denote the Λ-*pole size* of T by $S_\Lambda(T)$. In the setting of this article, Ω will be an (open or closed) k-connected domain and we usually choose pole sets of minimal cardinality; that is, having k elements, one in each connected component of $\widehat{\mathbb{C}} \setminus \overline{\Omega}$.

Definitions 3.5. Let Φ be a collection of functions mapping Ω into $\mathbb{D}$ and analytic in neighbourhoods of Ω. Fix a pole set Λ for Ω. We say that Φ is a

(i) *uniform test collection over* Ω if the implication (3.3) holds, where the constant K depends only Ω and Φ (and not on T);

(ii) *quasi-uniform test collection over* Ω if (3.3) holds, where K depends on Ω, Φ and $S_\Lambda(T)$;

(*iii*) *non-uniform test collection over* Ω if (3.3) holds, where K can depend on Ω, Φ and the operator T;

(*iv*) *uniform strong test collection over* Ω if (3.2) holds, where K depends only on Ω, Φ and K' (but not on T);

(*v*) *quasi-uniform strong test collection over* Ω if (3.2) holds, where K depends on Ω, Φ, K' and $S_\Lambda(T)$;

(*vi*) *non-uniform strong test collection over* Ω if (3.2) holds, where K depends on Ω, Φ, K', and also may depend on T.

To summarize, there is the basic notion of a *test collection*, which roughly means that whenever $\varphi(T)$ is a contraction for every φ in the collection, then T has $\overline{\Omega}$ as a K-spectral set. To this, one can add the adjectives *uniform*, *quasi-uniform* and *non-uniform*, which mean respectively that K does not depend on T, that K depends only on $S_\Lambda(T)$, and that that K may depend on T. Finally, the term *strong* indicates that we can replace the condition $\|\varphi(T)\| \le 1$ by the weaker condition that $\overline{\mathbb{D}}$ is a complete K'-spectral set for $\varphi(T)$ for all $\varphi \in \Phi$.

An operator R has $\overline{\mathbb{D}}$ as a complete 1-spectral set if and only if R is a contraction. In this case, R has $\overline{\mathbb{D}}$ as a complete K'-spectral set for all $K' > 1$. Therefore, each strong test collection is a test collection.

Also note that when $\Phi = \{\varphi\}$ consists of a single element, the *strong* part comes for free, since if $\varphi(T)$ has $\overline{\mathbb{D}}$ as a complete K-spectral set for some K, then there is some invertible operator S such that $S\varphi(T)S^{-1} = \varphi(STS^{-1})$ is a contraction, and so we can reason with STS^{-1} instead of T.

In most cases, Ω will be an open domain or the closure of an open domain. Given a domain Ω, the notions of a test collection over Ω and over $\overline{\Omega}$ might seem very similar, but as we will see below, the condition that $\sigma(T) \subset \overline{\Omega}$, as opposed to the stronger condition $\sigma(T) \subset \Omega$, represents an additional technical challenge in some arguments.

Finally, the notion of a non-uniform test collection over an open set Ω is trivial, since if $\sigma(T) \subset \Omega$, then $\overline{\Omega}$ is a complete K-spectral set for T, where K depends on Ω and T. This was first proved for $\Omega = \mathbb{D}$ by Rota [Rot60], and follows in general from the Herrero-Voiculescu theorem (see [Pau02, Theorem 9.13]).

Recall the definition of an admissible function given in Definition 1.10 (page 7). Given an admissible function $\Phi = (\varphi_1, \dots, \varphi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$, we will denote the set of functions $\{\varphi_1, \dots, \varphi_n\}$ by the same letter Φ.

Theorem 3.6. *Assume that $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible and analytic in an open neighbourhood of $\overline{\Omega}$, where Ω is a Jordan domain. Then Φ is a quasi-uniform strong test collection in $\overline{\Omega}$. If, moreover, Φ is injective and Φ' does not vanish on Ω, then Φ is a uniform strong test collection over $\overline{\Omega}$.*

This means that if $T \in \mathcal{B}(H)$ satisfies $\sigma(T) \subset \overline{\Omega}$, Λ is an arbitrary fixed pole set for Ω, and $\varphi_k(T)$ have $\overline{\mathbb{D}}$ as a complete K'-spectral set for $k = 1, \dots, n$, then T has $\overline{\Omega}$ as a complete K-spectral set, with $K = K(\Omega, \Phi, K', S_\Lambda(T))$. If Φ is injective and Φ' does not vanish on Ω, then one can even choose K independently of T.

Theorem 3.7. *Let $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ be admissible and Λ an arbitrary fixed pole set for Ω. Given $T \in \mathcal{B}(H)$, assume that there are operators $C_1, \dots, C_n \in \mathcal{B}(H)$ such that $\overline{\mathbb{D}}$ is a complete K'-spectral set for every C_k, $k = 1, \dots, n$. Furthermore, assume that whenever $f \in \operatorname{Rat}(\overline{\Omega})$ can be written as*

$$f(z) = \sum_{k=1}^{n} f_k(\varphi_k(z)), \qquad f_k \in A(\overline{\mathbb{D}}), \tag{3.4}$$

we have

$$f(T) = \sum_{k=1}^{n} f_k(C_k). \tag{3.5}$$

Then $\overline{\Omega}$ is a complete K-spectral set for T with K depending only on Ω, Φ and $S_\Lambda(T)$. If moreover, Φ is injective and Φ' does not vanish on Ω, then one can choose K independently of T.

A posteriori, since $\overline{\Omega}$ is a complete K-spectral for T, the operators $\varphi_k(T)$ are defined by the $A(\overline{\Omega})$ calculus for T. The hypotheses of the theorem imply that $C_k = \varphi_k(T)$, so the operators C_k are uniquely defined. However, *a priori*, the operators $\varphi_k(T)$ are not defined by any reasonable functional calculus, so the theorem cannot be stated in terms of these operators.

If $\sigma(T) \subset \Omega$, then it is an easy consequence of the Cauchy-Riesz functional calculus that $C_k = \varphi_k(T)$ satisfy the hypotheses of this theorem. Therefore, this proves the following corollary.

Corollary 3.8. *If $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is admissible, then Φ is a quasi-uniform strong test-collection over Ω. If moreover Φ is injective and Φ' does not vanish on Ω, then Φ is a uniform strong test collection over Ω.*

Remark. The main differences between Theorems 3.6 and 3.7 is that Theorem 3.6 assumes that Ω is simply connected and Theorem 3.7 does not. On the other hand, Theorem 3.7 requires the existence of some operators C_k which behave in an informal sense like $\varphi_k(T)$ (the formal condition is that (3.4) implies (3.5)). As it will be clear from the proofs of these theorems, the case when $\sigma(T) \subset \Omega$ is easy to handle, while the case when $\sigma(T)$ contains part of the boundary of Ω presents some technical difficulties. Theorems 3.6 and 3.7 represent two different ways of sorting out these difficulties. In Theorem 3.6, we will use the existence of a certain family $\{\psi_\varepsilon\}_{0\le\varepsilon\le\varepsilon_0}$ of univalent functions on Ω to pass from the operator T to operators $\psi_\varepsilon(T)$, whose spectra are contained inside Ω. In Theorem 3.7 we postulate some kind of functional calculus for T.

We remark that it follows from the proofs of our theorems that similar results hold if one replaces complete K-spectral sets by (not necessarily complete) K-spectral sets. For instance, in Theorem 3.7, if C_k have $\overline{\mathbb{D}}$ as a K'-spectral set, then $\overline{\Omega}$ is K-spectral for T.

Theorems 3.1 and 3.3, which were stated in the Introduction, can be reformulated in terms of test collections. Theorem 3.1 shows that if $\varphi_k : \Omega_k \to \mathbb{D}$ are Riemann conformal maps, then $\{\varphi_1, \dots, \varphi_s\}$ is a uniform strong test collection for $\overline{\Omega}$. In Theorem 3.3, we can put $\varphi_{k,\lambda}(z) = R(z - \mu_k(\lambda))^{-1}$. Then $\{\varphi_{k,\lambda} : k = 1, \dots, N,\ \lambda \in \gamma_k\}$ is a uniform test collection over $\overline{\Omega}$.

In Theorem 3.6, it is easy to see that when Φ is not injective or Φ' has zeros, then Φ can only be a quasi-uniform strong test collection in $\overline{\Omega}$ (i.e., one cannot remove the adjectives "quasi-uniform" or "non-uniform"). For instance, if $\Phi(z_1) = \Phi(z_2)$ for distinct points $z_1, z_2 \in \Omega$, then we can take an operator T acting on $\mathbb{C}^2$ and having z_1 and z_2 as eigenvalues, with associated eigenvectors v_1 and v_2. For every k, we have $\varphi_k(T) = \varphi_k(z_1)I$, which is a contraction. If the angle between v_1 and v_2 is very small, then $\|T\|$ will be very large, so there is no constant K independent of T such that $\overline{\Omega}$ is K-spectral for T.

Similarly, if $\Phi'(z_0) = 0$ for some $z_0 \in \Omega$, we can take an operator T such that $T \neq z_0 I$ and $(T - z_0 I)^2 = 0$. For every $n \geq 1$, we put $T_n = n(T - z_0 I) + z_0 I$. Then it is easy to check that for every n and every k, we have $\varphi_k(T_n) = \varphi_k(z_0)I$, which is a contraction. However, $\|T_n\| \to \infty$ as $n \to \infty$. This implies that there is no constant K independent of n such that $\overline{\Omega}$ is K-spectral for T_n, for every n.

Example 3.9. To illustrate the phenomenon described in the last paragraph, we construct a domain Ω and an admissible function $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^3$ such that Φ' vanishes at some point $z_0 \in \Omega$. Choose a small $\varepsilon > 0$ and put $z_1 = 0$, $z_2 = \varepsilon$, $z_3 = \varepsilon/2 + i\sqrt{3}\varepsilon/2$, so that z_1, z_2, z_3 are the vertices of a equilateral triangle of side length ε. Let z_0 be the center of this triangle.

Let D_j be the disk of radius 1 and center z_j. We put $\Omega = D_1 \cap D_2 \cap D_3$. We can divide the boundary of Ω in three arcs J_k by putting $J_k = (\partial\Omega) \cap (\partial D_k)$, for $k = 1, 2, 3$. Since ε is small, it is easy to see that the length of each arc J_k is close to $2\pi/3$.

Let $\varphi_1(z) = (z - z_0)^2/(1 - \overline{z_0}z)^2$. Then φ_1 maps D_1 onto $\mathbb{D}$, and it maps J_1 bijectively onto some arc of $\mathbb{T}$. For $k = 2, 3$, let η_k be the orientation-preserving rigid motion taking z_k to z_1 and J_k to J_1 (so that it maps $\overline{D}_k$ onto $\overline{D}_1$). Note that $\eta_k(z_0) = z_0$. We define $\varphi_k = \varphi_1 \circ \eta_k$, for $k = 2, 3$. We see that $\varphi_k'(z_0) = 0$ for $k = 1, 2, 3$. It is easy to check that $\Phi = (\varphi_1, \varphi_2, \varphi_3)$ is admissible in $\overline{\Omega}$, because, for every k, φ_k is analytic on a neighbourhood of $\overline{\Omega}$ and takes J_k bijectively onto some arc of $\mathbb{T}$.

It is worthwhile mentioning that the condition in the definition of an admissible family of functions requiring the interior angle of a corner of the domain Ω to be in $(0, \pi]$ can be relaxed in the results stated above if one instead requires that the corner is not in the spectrum of the operator T under consideration. This is seen by altering Ω, removing the intersection with a small enough disk about the corner. The complement of the disk will be a complete spectral set for T and the new corners created will satisfy the condition that the interior angles are in $(0, \pi]$. Since the disk can be made arbitrarily small, it essentially has no effect on the statements given above, other than that there is now a dependence on the choice of T through this additional requirement on the spectrum.

Part of the inspiration for our definition of test collections comes from [DM07]. There, such a notion is defined abstractly as a (possibly infinite) collection of complex valued functions on a set with the property that at any given point in the set, the supremum over the test functions evaluated at the point is strictly less than 1 and functions separate the points of the set. In such cases as when the set X is contained in $\mathbb{C}^n$, the boundary of X corresponds to points where some test function is equal to 1.

A test collection in this context is used to define the dual notion of admissible kernels, and from these a normed function algebra is constructed, with the functions in the test collection in the unit ball of the algebra. The realization theorem then states that unital representations of the algebra which send the functions in the test collection to strict contractions are (completely) contractive. In the case that the set where we define the test collection is a bounded set $\Omega \subset \mathbb{C}$, this is reminiscent of the test collection being a uniform test collection. In the general setting of [DM07], the algebra obtained may not be equal to $A(\overline{\Omega})$, which is the issue being addressed in this chapter.

3.3. Some examples of test collections from the literature

Here we interpret the known criteria for being a complete K-spectral set in terms of our notion of a test collection and its variants. For a good recent review of different aspects of K-spectral sets and complete K-spectral sets, the reader is referred to [BB14].

Intersection of disks

A set $D \subset \mathbb{C}$ will be called a *closed disk* in the Riemann sphere $\widehat{\mathbb{C}}$ if it has of one of the following three forms:

$$\{z \in \widehat{\mathbb{C}} : |z-a| \leq r\}, \quad \{z \in \widehat{\mathbb{C}} : |z-a| \geq r\}, \quad \{z \in \widehat{\mathbb{C}} : \operatorname{Re} \alpha(z-a) \geq 0\},$$

i.e., it is either the interior of a disk in $\mathbb{C}$, the exterior of a disk, or a half-plane.

Theorem 3.10 (Badea, Beckermann, Crouzeix [BBC09])**.** *Let $\{D_k\}_{k=1}^n$ be a finite collection of closed disks in $\widehat{\mathbb{C}}$ and $\{\varphi_k\}_{k=1}^n$ be fractional linear transformations taking D_k onto $\overline{\mathbb{D}}$. Then $\{\varphi_k\}_{k=1}^n$ is a uniform test collection for $\bigcap_{k=1}^n D_k$.*

Nice n-holed domains

We say that an open bounded set $\Omega \subset \widehat{\mathbb{C}}$ is an *n-holed domain* if its boundary $\partial\Omega$ consists of $n+1$ disjoint Jordan curves. Given an n-holed domain Ω, we will denote by $\{U_k\}_{k=0}^n$ the connected components of $\widehat{\mathbb{C}} \setminus \overline{\Omega}$, with U_0 the unbounded component. Let $X_k = \widehat{\mathbb{C}} \setminus U_k$.

Theorem 3.11 (Douglas, Paulsen [DP86])**.** *Let Ω be an n-holed domain, and define $\{X_k\}_{k=0}^n$ as above. Assume that each X_k has an analytic boundary, so that there exist analytic homeomorphisms $\varphi_k : X_k \to \overline{\mathbb{D}}$, for $k = 0, \dots, n$. Then $\{\varphi_k\}_{k=0}^n$ is a uniform strong test collection in $\overline{\Omega}$.*

This theorem can also be found in Paulsen's book [Pau02, Chapter 11].

Convex domains and the numerical range

For $T \in \mathcal{B}(H)$, the *numerical range* is defined as the set

$$W(T) = \{\langle Tx, x\rangle : \|x\| = 1\}.$$

It is well-know that this set is convex, and so its closure can be written as the intersection of a (generally infinite) collection of closed half planes $\{H_\alpha\}$. Let φ_α be a linear fractional transformation taking H_α onto $\overline{\mathbb{D}}$. It is easy to check that $W(T) \subset H_\alpha$ if and only if $\|\varphi_\alpha(T)\| \leq 1$. As we have already commented, it follows from the arguments in [DD99] and [PS05] that every compact convex set containing $W(T)$ is a complete K-spectral set for T. This result can be rewritten in terms of test collections as follows.

Theorem 3.12. *Let Ω be a convex domain in $\mathbb{C}$ and let $\{H_\alpha\}$ be a collection of closed half-planes such that $\overline{\Omega} = \bigcap H_\alpha$. Let φ_α be a fractional linear transform taking H_α onto $\overline{\mathbb{D}}$. Then $\{\varphi_\alpha\}$ is a uniform test collection in $\overline{\Omega}$.*

Remark. If Ω is a smooth bounded convex set, we denote by C_Ω the optimal constant K such that $\overline{\Omega}$ is a (complete) K-spectral set for T whenever $\overline{W(T)} \subset \Omega$. The constant $Q = \sup_\Omega C_\Omega$ is know as Crouzeix constant. It is conjuntured that $Q = 2$. In a recent preprint by Crouzeix and Palencia [CP17], it is proved that $Q \leq 1 + \sqrt{2}$.

ρ-contractions

If $\rho > 0$, we say that an operator $T \in \mathcal{B}(H)$ is a ρ-contraction if T has an unitary ρ-dilation. This is a unitary operator U acting on a larger Hilbert space $K \supset H$ and such that

$$T^n = \rho P_H U^n | H, \qquad n \geq 1.$$

Alternatively, one can ask that $\sigma(T) \subset \overline{\mathbb{D}}$ and that the operator-valued Poisson kernel of T

$$K_{r,t}(T) = (I - re^{it}T^*)^{-1} + (I - re^{-it}T)^{-1} - I, \qquad 0 < r < 1,\ t \in \mathbb{R}$$

satisfies

$$K_{r,t}(T) + (\rho - 1)I \geq 0, \qquad 0 < r < 1, t \in \mathbb{R}. \tag{3.6}$$

The class of ρ contractions becomes larger as ρ increases, as (3.6) clearly shows, and $\rho = 1$ corresponds to the usual contractions, while $\rho = 2$ corresponds to $W(T) \subseteq \mathbb{D}$.

If $1 < \rho < 2$, then T being a ρ-contraction is also equivalent to the condition that

$$\|\mu I - T\| \leq |\mu| + 1, \qquad \frac{\rho - 1}{2 - \rho} \leq |\mu| < \infty. \tag{3.7}$$

(See, for instance, [SNFBK10, Chapter I].) If $a \in \mathbb{T}$, we denote by $D_a(\rho)$ the closed disk of radius $1 + (\rho - 1)/(2 - \rho)$ whose boundary is tangent to $\mathbb{T}$ at a and which contains $\overline{\mathbb{D}}$. Let $\varphi_{a,\rho}$ be a linear fractional transformation taking $D_a(\rho)$ onto $\overline{\mathbb{D}}$. Then (3.7) is equivalent to the condition that $\varphi_{a,\rho}(T)$ is a contraction for every $a \in \mathbb{T}$.

Similarly, if $\rho > 2$, T is a ρ contraction if and only if

$$\|(\mu I - T)^{-1}\| \leq \frac{1}{|\mu| - 1}, \qquad 1 \leq |\mu| \leq \frac{\rho - 1}{\rho - 2}. \tag{3.8}$$

For these values of ρ, denote by $D_a(\rho)$ the complement of the open disk of radius $(\rho - 1)/(\rho - 2) - 1$, which is tangent to $\mathbb{T}$ at $a \in \mathbb{T}$ and does not contain $\mathbb{D}$. Let $\varphi_{a,\rho}$ be a linear fractional transformation which takes $D_a(\rho)$ onto $\overline{\mathbb{D}}$. Then (3.8) is equivalent to the condition that $\varphi_{a,\rho}(T)$ is a contraction for every $a \in \mathbb{T}$.

For the case $\rho = 2$, let $D_a(2)$ be the closed half-plane which is tangent to $\mathbb{T}$ at $a \in \mathbb{T}$ and which contains $\overline{\mathbb{D}}$ and let $\varphi_{a,2}$ be a fractional linear transformation taking $D_a(2)$ onto $\overline{\mathbb{D}}$. Then it follows from the above comments regarding the numerical range that T is a 2-contraction if and only if $\varphi_{a,2}(T)$ is a contraction for every $a \in \mathbb{T}$.

It is also known [SNFBK10] that every ρ-contraction is similar to a contraction. We summarize in terms of test collections as follows.

Theorem 3.13. *For $\rho > 1$, let $\Phi_\rho = \{\varphi_{a,\rho}\}_{a\in\mathbb{T}}$, where $\varphi_{a,\rho}$ is defined as above. Then Φ_ρ is a uniform test collection over $\overline{\mathbb{D}}$.*

Inner functions

Recall that a Blaschke product is a function of the form

$$B(z) = e^{i\theta} z^k \prod_{j=1}^{N} b_{\lambda_j}(z),$$

where

$$b_\lambda(z) = \frac{\overline{\lambda}}{|\lambda|} \cdot \frac{\lambda - z}{1 - \overline{\lambda} z},$$

is a disk automorphism, N may be either a finite number or ∞ (in which case its zeros $\lambda_j \in \mathbb{D}$ satisfy the Blaschke condition $\sum_{j=1}^{\infty}(1 - |\lambda_j|) < \infty$). The Blaschke product is called finite if N is finite.

Theorem 3.14 (Mascioni, [Mas94])**.** *Let φ be a finite Blaschke product. Then the one element set $\{\varphi\}$ is a non-uniform strong test collection over $\overline{\mathbb{D}}$.*

We cannot say that the one element set $\{\varphi\}$ is a uniform test collection in $\overline{\mathbb{D}}$. For example, take $\varphi(z) = z^2$, which is a finite Blaschke product. Then the operators T_n on $\mathbb{C}^2$ defined by the matrices $T_n = \begin{pmatrix} 0 & n \\ 0 & 0 \end{pmatrix}$ satisfy $\varphi(T_n) = 0$, but we have $\|T_n\| = n$. Hence, $\overline{\mathbb{D}}$ can be a K-spectral set for T_n only if $K \geq n$.

In some of the theorems given above, the conclusion is that some family of functions is a strong test collection, whereas in others the conclusion is just that the family is a test collection. Indeed, we do not know whether one can replace "test collection" by "strong test collection" in Theorems 3.10 and 3.12. The proofs of these theorems involve some kind of operator valued Poisson kernel which turn out to be positive when $\varphi(T)$ is a contraction, but they do not seem to work well if $\varphi(T)$ simply has $\overline{\mathbb{D}}$ as a K'-spectral set. Similarly, we do not know whether one can replace "test collection" by "strong test collection" in Theorem 3.13.

Theorem 3.14 has been generalized by Stessin [Ste99] and Kazas and Kelley [KK06] to several classes of infinite Blaschke products. These generalizations give examples of test functions on a set Ω which is neither an open domain nor its closure. We restate here Stessin's theorem in the language of test collections.

Theorem 3.15 (Stessin, [Ste99])**.** *Let φ be a Blaschke product whose zeros $\{\lambda_j\}_{j=1}^{\infty}$ satisfy $\sum(1 - |\lambda_j|^2)^{1/2} < \infty$. Let $\mathcal{P}$ be the set of poles of φ and put $\Omega = \overline{\mathbb{D}} \setminus \overline{\mathcal{P}}$. Then, the one element set $\{\varphi\}$ is a non-uniform strong test collection over Ω.*

Lemniscates

Another recent result that can be put into the terminology of test collections is one that concerns lemniscates.

Theorem 3.16 (Nevanlinna, [Nev12]). *Let p be a monic polynomial, $R > 0$, and denote by γ_R the set $\{z \in \mathbb{C} : |p(z)| = R\}$. Assume that no critical point of p lies on γ_R. Let $\Omega = \{z \in \mathbb{C} : |p(z)| < R\}$ and $\varphi = p/R$. Then the single-element set $\{\varphi\}$ is a non-uniform test collection over $\overline{\Omega}$.*

Remark. If Ω is a complete K-spectral set for an operator T, one can speak about constructing a concrete Sz.-Nagy-Foias like model of T in Ω. In the simplest case, this model will be the compression of the multiplication operator $f \mapsto zf$ on the space $H^2(\Omega, U) \ominus \Theta H^2(\Omega, Y)$, where U, Y are auxiliary Hilbert spaces and the function $\Theta \in H^\infty(\Omega, \mathcal{B}(Y, U))$ is an analogue of the characteristic function. As it was shown in [Yak03], there are important cases when the function Θ can be calculated explicitly. This is also true for some of the above examples. If T is a ρ-contraction, there is an explicit formula for its similarity to a contraction. See, for instance, [OA75].

Such an explicit similarity transform is also available when $\|B(T)\| \leq 1$ for a finite Blaschke product B. Indeed, let

$$B(z) = \prod_{k=1} b_k(z),$$

where $b_k(z) = (z - \lambda_k)/(1 - \bar{\lambda}_k z)$ are Blaschke factors, $|\lambda_k| < 1$. The functions

$$s_k(z) = \frac{(1 - |\lambda_k|^2)^{1/2}}{1 - \bar{\lambda}_k z} \prod_{j=1}^{k-1} b_j(z), \qquad k = 1, \dots, n,$$

form an orthonormal basis of the model space $H^2 \ominus BH^2$, whose reproducing kernel is $\big(1 - \overline{B(w)}B(z)\big)/(1 - \bar{w}z)$. For z, w in a neighbourhood of the closed unit disk, this gives

$$1 - \overline{B(w)}B(z) = (1 - \bar{w}z) \sum_{k=1}^{n} \overline{s_k(w)} s_k(z),$$

This then implies that for any $h \in H$,

$$\sum_{k=1}^{n} \|s_k(T)h\|^2 - \sum_{k=1}^{n} \|s_k(T)Th\|^2 = \|h\|^2 - \|B(T)h\|^2 \geq 0,$$

and therefore $\|h\|_*^2 := \sum_{k=1}^{n} \|s_k(T)h\|^2$ defines a Hilbert space norm on H for which T is a contraction. Since $s_1(T)$ is an invertible operator, this norm is equivalent to the original.

As shown in [Yak03], there are ways of calculating the characteristic function Θ of an operator T explicitly without knowing an explicit form of the similarity transform converting T into an operator for which Ω is a complete spectral set with constant 1. As also explained in this work, one obtains additional cases where explicit formulas are available by admitting a larger class of characteristic functions (which are then no longer unique). Apart from these examples, we do not know either an explicit form of the similarity transform of the above type, nor explicit characteristic functions.

3.4. Proofs of Theorems 3.1 and 3.3

In what follows, we will use the following lemma. It is a special case of a well known principle in the theory of C^*-algebras that says that whenever the range of a linear map is commutative, the complete boundedness of this map comes for free.

Lemma 3.17. *Let $T : A(\overline{\Omega}_1) \to A(\overline{\Omega}_2)$ be a bounded operator, and $\alpha \in A(\overline{\Omega})^*$ a bounded linear functional. Then T and α are completely bounded, and $\|T\|_{cb} = \|T\|$ and $\|\alpha\|_{cb} = \|\alpha\|$.*

Proof. If A and B are C^*-algebras, with B commutative, S a (not necessarily closed) linear subspace of A and $\varphi : S \to B$ is a bounded linear map, then it is well known that φ is completely bounded and $\|\varphi\|_{\mathrm{cb}} = \|\varphi\|$ (see, for instance, [Pau02, Theorem 3.9] or [Loe75, Lemma 1]). The lemma then follows from the fact that $A(\overline{\Omega})$ is a subspace of the commutative C^*-algebra $C(\partial\Omega)$ and the norm in $A(\overline{\Omega})$ coincides with the norm that it inherits as a subspace of $C(\partial\Omega)$. □

Proof of Theorem 3.1. We start by proving (ii) with $n = 2$.

By Proposition 1.8, there are bounded operators $G_k : A(\overline{\Omega}) \to A(\overline{\Omega}_k)$, $k = 1, 2$, such that $f = G_1(f) + G_2(f)$, for every $f \in A(\overline{\Omega})$. Take $f \in \mathrm{Rat}(\overline{\Omega}_1 \cap \overline{\Omega}_2) \otimes M_s$, an $s \times s$ matrix-valued rational function with poles off $\overline{\Omega}_1 \cap \overline{\Omega}_2$. We first want to check that

$$f(T) = [(G_1 \otimes \mathrm{id}_s)(f)](T) + [(G_2 \otimes \mathrm{id}_s)(f)](T). \tag{3.9}$$

Note that the operators $[(G_k \otimes \mathrm{id}_s)(f)](T)$ are defined by the $A(\overline{\Omega}_k)$ calculus for T, which is well defined because each $\overline{\Omega}_k$ is a complete K-spectral set for T.

The function f can be decomposed as $f = f_1 + f_2$, with $f_j \in \mathrm{Rat}(\overline{\Omega}_j) \otimes M_s$, because any pole a of f satisfies $a \in \widehat{\mathbb{C}} \setminus \overline{\Omega}_j$ for either $j = 1$ or $j = 2$. Put $g_k = (G_k \otimes \mathrm{id}_s)(f)$, $k = 1, 2$. We have

$$f_1 - g_1 = g_2 - f_2, \quad \text{in } \overline{\Omega}_1 \cap \overline{\Omega}_2.$$

The left hand side of this equality belongs to $A(\overline{\Omega}_1) \otimes M_s$ and the right hand side belongs to $A(\overline{\Omega}_2) \otimes M_s$. Thus this equation defines a function h in $A(\overline{\Omega}_1 \cup \overline{\Omega}_2) \otimes M_s$ by $h = f_1 - g_1$ in $\overline{\Omega}_1$ and $h = g_2 - f_2$ in $\overline{\Omega}_2$. Let $\{h_n\}_{n=1}^\infty \subset \mathrm{Rat}(\overline{\Omega}_1 \cap \overline{\Omega}_2) \otimes M_s$ be rational functions such that $h_n \to h$ uniformly in $\overline{\Omega}_1 \cup \overline{\Omega}_2$. Since $h_n \to f_1 - g_1$ uniformly in $\overline{\Omega}_1$, we have that $h_n(T) \to f_1(T) - g_1(T)$ in operator norm. On the other hand, since $h_n \to g_2 - f_2$ uniformly in $\overline{\Omega}_2$, we have that $h_n(T) \to g_2(T) - f_2(T)$ in operator norm. Hence $f_1(T) - g_1(T) = g_2(T) - f_2(T)$. This proves (3.9), because $f_1(T) + f_2(T) = f(T)$ by the rational functional calculus.

Now we estimate

$$\|f(T)\| \le \sum_{k=1}^{2} \|[(G_k \otimes \mathrm{id}_s)(f)](T)\| \le K \left(\sum_{k=1}^{2} \|G_k\|_{\mathrm{cb}} \right) \|f\|_{A(\overline{\Omega}) \otimes M_s}.$$

By Lemma 3.17, $\overline{\Omega}$ is a complete K'-spectral set for T, with $K' = K(\|G_1\| + \|G_2\|)$.

Now suppose that $n > 2$ and that the sets $\Omega_1, \dots, \Omega_n$ satisfy the hypotheses of the theorem. Then the transversality conditions imply that $\Omega_1 \cap \Omega_2, \Omega_3, \dots, \Omega_n$ also satisfy the hypotheses of the theorem. This enables us to apply induction in n.

The proof of (i) is by the same argument, and indeed is somewhat simpler, since one only has to deal with scalar analytic functions and there is no need to invoke Lemma 3.17. □

Remark. The proof relies essentially on the Havin-Nersessian separation of singularities: given some domains Ω, Ω_1, Ω_2 with good geometry such that $\Omega = \Omega_1 \cap \Omega_2$, any function f in $H^\infty(\Omega)$ admits a decomposition $f = f_1 + f_2$, $f_j = G_j(f) \in H^\infty(\Omega_j)$. However, there are cases when the Havin-Nersessian separation fails, but nevertheless the following assertion is still true: if $\overline{\Omega}_j$ is a (1-)spectral set for T, then $\overline{\Omega}$ is a complete K-spectral set for T. This holds, for instance, if Ω_1 and Ω_2 are open half-planes such that $\Omega_1 \cup \Omega_2 = \mathbb{C}$. In this case, Ω_j are simply connected, and so they are spectral sets for T if and only if they are complete spectral sets.

This assertion follows, for instance, from [Cro07]. However, there is no Havin-Nersessian separation in this case. This is easy to show using the kind of arguments that appeared in Example 1.2. It is easy to modify this example to likewise produce bounded simply connected domains Ω_1 and Ω_2 with the same properties.

Now we pass to the proof of Theorem 3.3. We need some preliminaries. Recall that we say that $D \subset \widehat{\mathbb{C}}$ is a closed disk in $\widehat{\mathbb{C}}$ if it has of one of the following three forms:

$$\{z \in \widehat{\mathbb{C}} : |z-a| \le r\}, \quad \{z \in \widehat{\mathbb{C}} : |z-a| \ge r\}, \quad \{z \in \widehat{\mathbb{C}} : \operatorname{Re} \alpha(z-a) \ge 0\},$$

i.e., it is either the interior of a disk in $\mathbb{C}$, the exterior of a disk, or a half-plane. We will refer to disks $\{z \in \widehat{\mathbb{C}} : |z-a| \le r\}$ as "genuine" disks. Next, suppose T is a Hilbert space operator and D has one of the above three forms. We say that D is *a good disk for T* if the following condition holds (depending on the case):

- If $D = \{z \in \widehat{\mathbb{C}} : |z-a| \le r\}$, we require that $\|T-a\| \le r$;
- If $D = \{z \in \widehat{\mathbb{C}} : |z-a| \ge r\}$, we require that $a \notin \sigma(T)$ and $\|(T-a)^{-1}\| \le r^{-1}$;
- If $D = \{z \in \widehat{\mathbb{C}} : \operatorname{Re} \alpha(z-a) \ge 0\}$, we require that $\operatorname{Re}\big(\alpha(T-a)\big) \ge 0$.

Lemma 3.18. *Let T be a Hilbert space operator.*

(*i*) *Suppose $\psi : \mathbb{C} \to \mathbb{C}$ is a Möbius map, and that $\sigma(T)$ does not contain the pole of ψ, so that the operator $\psi(T)$ is bounded. Then, given a closed disk D in the Riemann sphere, D is good for T if and only if its image $\psi(D)$ is good for $\psi(T)$.*

(*ii*) *Whenever $D_1 \subset D_2$ are two Riemann sphere disks such that D_1 is good for T, the disk D_2 is also good for T.*

Proof. We will show that D is good for T if and only if $\sigma(T) \subset D$ and $\psi(T)$ is a contraction, where ψ is a Möbius transform taking D onto $\overline{\mathbb{D}}$. Part (*i*) clearly follows from this property. First note that if φ is a disk automorphism, then T is a contraction if and only if $\varphi(T)$ is a contraction. Since every two Möbius maps taking D onto $\overline{\mathbb{D}}$ differ by composition on the right with a disk automorphism, we see that $\psi(T)$ is a contraction for every Möbius map ψ taking D onto $\overline{\mathbb{D}}$ if and only if $\psi(T)$ is a contraction for some particular choice of such Möbius map.

Now we examine the three kinds of disks separately. If $D = \{z : |z-a| \le r\}$, then we can take $\psi(z) = (z-a)/r$ as a Möbius map taking D onto $\overline{\mathbb{D}}$. The disk D is good for T precisely when $\|\psi(T)\| \le 1$. Similarly, a disk $D = \{z : |z-a| \ge r\}$ is good for T if and only if $\psi(T)$ is well-defined and is a contraction, where now we put $\psi(z) = r/(z-a)$, which is a Möbius map taking D onto $\overline{\mathbb{D}}$.

In the last case, when $D = \{z : \operatorname{Re} \alpha(z-\alpha) \ge 0\}$ is a half-plane, D is good for T if and only if $\mathbb{C}_+ = \{z : \operatorname{Re} z \ge 0\}$ is good for $\alpha(T-a)$. Hence it suffices to consider only the case when $D = \mathbb{C}_+$. Using the standard fact that $\operatorname{Re} T \ge 0$ if and only if

$$\|(I+T)x\|^2 \ge \|(I-T)x\|^2, \quad \forall x,$$

we see that $\operatorname{Re} T \ge 0$ if and only if $\psi(T)$ is a contraction, where now ψ is a Möbius map that takes $\mathbb{C}_+$ onto $\mathbb{D}$, given by $\psi(z) = (1-z)/(1+z)$.

To prove (ii), we can use (i) to reduce first to the case when $D_1 = \overline{\mathbb{D}}$. In this case, T is a contraction. Let ψ be a Möbius transform taking D_2 onto $\overline{\mathbb{D}}$. Then $|\psi| \le 1$ in $\overline{\mathbb{D}}$, so $\psi(T)$ is a contraction by von Neumann's inequality. It follows that D_2 is good for T, since $\sigma(T) \subset D_1 \subset D_2$. □

Proof of Theorem 3.3. By Condition 3.2, there are closed arcs $\gamma_1, \dots, \gamma_N$ satisfying the hypotheses listed there. We are going to construct domains $\Omega_1, \dots, \Omega_N$, whose closures are complete K-spectral for T, with Ω their intersection. Then we will apply Theorem 3.1 to deduce that $\overline{\Omega}$ is also complete K'-spectral for T, for some K'.

Fix $k \in \{1, \dots, N\}$, and choose some point $z_k \in \bigcap_{\lambda \in \gamma_k} B(\mu_k(\lambda), R_k)$. Put $\varphi_k(z) = (z - z_k)^{-1}$. Now take some $\lambda \in \gamma_k$. Since $z_k \in B(\mu_k(\lambda), R_k)$, it follows that the closed disk

$$D_\lambda^k = \varphi_k\big(\mathbb{C} \setminus B(\mu_k(\lambda), R_k)\big)$$

is genuine. Since $\lambda \in \partial B(\mu_k(\lambda), R_k)$, we have $\varphi_k(\lambda) \in \partial D_\lambda^k$. Let ℓ_λ^k be the straight line tangent to ∂D_λ^k at $\varphi_k(\lambda)$ and let Π_λ^k be the closed half plane bordered by ℓ_λ^k that contains D_λ^k. Consider the (possibly unbounded) closed convex sets

$$G_k = \bigcap_{\lambda \in \gamma_k} \Pi_\lambda^k.$$

Since $\Omega \subset \widehat{\mathbb{C}} \setminus B(\mu_k(\lambda), R_k)$, we have $\varphi_k(\Omega) \subset D_\lambda^k \subset \Pi_\lambda^k$, for any $\lambda \in \gamma_k$. Therefore $\varphi_k(\Omega) \subset G_k$. By Lemma 3.18, the disk D_λ^k and the half plane Π_λ^k are good for $\varphi_k(T)$. It follows that $W(\varphi_k(T)) \subset G_k$. By the Delyon-Delyon theorem [DD99], G_k is a complete K-spectral set for $\varphi_k(T)$ (see Theorem 3.12).

Next, we consider the Jordan domains $\Omega_k = \operatorname{int}(\varphi_k^{-1}(G_k))$ in the Riemann sphere $\widehat{\mathbb{C}}$. Each Ω_k contains Ω, and its closure is a complete K-spectral for T. By construction, $\varphi_k(\gamma_k) \subset \partial G_k$. Hence, $\gamma_k \subset \partial \Omega_k$. We wish to apply Theorem 3.1 to the intersection of the sets Ω_k, $k = 1, \dots, N$. It may happen however that the boundaries of these sets do not intersect transversally. Nevertheless, it is possible to choose larger Jordan domains $\widetilde{\Omega}_k \supset \Omega_k$ whose boundaries do intersect transversally, and such that $\gamma_k \subset \partial \widetilde{\Omega}_k$.

To prove this, it suffices to choose the sets $\widetilde{\Omega}_k$ in such a way that they intersect transversally at the endpoints of the arcs γ_k, as it is otherwise easy to ensure transversality at any other intersection points. So suppose λ is a common endpoint of two arcs

γ_k and γ_l. By construction, the open disk $\Delta_k = \varphi_k^{-1}(\widehat{\mathbb{C}} \setminus \Pi_\lambda^k)$ has the point λ on its boundary and does not intersect Ω_k, and similarly for the disk Δ_l. Therefore, there is an open circular sector S with vertex λ that does not intersect $\overline{\Omega}_k \cup \overline{\Omega}_l$. Since γ_k and γ_l intersect transversally, we can find disjoint open circular sectors S_k^+ and S_l^+ which are also disjoint with S and such that $\gamma_k \cap B(\lambda, \varepsilon) \subset S_k^+ \cup \{\lambda\}$ and $\gamma_l \cap B(\lambda, \varepsilon) \subset S_l^+ \cup \{\lambda\}$ for some $\varepsilon > 0$. Now observe that we can choose disjoint open circular sectors S_k^- and S_l^- which are also disjoint from S, S_k^+, S_l^+, and then the larger sets $\widetilde{\Omega}_k \supset \Omega_k$ and $\widetilde{\Omega}_l \supset \Omega_l$ to satisfy $(\partial\widetilde{\Omega}_k \setminus \gamma_k) \cap B(\lambda, \varepsilon) \subset S_k^-$ and $(\partial\widetilde{\Omega}_l \setminus \gamma_l) \cap B(\lambda, \varepsilon) \subset S_l^-$. Consequently, $\widetilde{\Omega}_k$ and $\widetilde{\Omega}_l$ intersect transversally at λ.

Put
$$\widetilde{\Omega} = \widetilde{\Omega}_1 \cap \cdots \cap \widetilde{\Omega}_N.$$
By Theorem 3.1, the closure of $\widetilde{\Omega}$ is a complete K'-spectral set for T, for some K'. By construction, each point λ of $\partial\Omega$ has a neighbourhood $B(\lambda, \varepsilon)$ such that $B(\lambda, \varepsilon) \cap \partial\Omega = B(\lambda, \varepsilon) \cap \partial\widetilde{\Omega}$. Therefore $\widetilde{\Omega} \setminus \Omega$ is at a positive distance from Ω. Since $\Omega \subset \widetilde{\Omega}$ and $\sigma(T) \subset \overline{\Omega}$, it follows that $\overline{\Omega}$ also is a complete K''-spectral set for T. ☐

We recall that a Hilbert space operator T is *hyponormal* if $T^*T \geq TT^*$. In this case, the equality $\|(T - \lambda)^{-1}\| = 1/\operatorname{dist}(\lambda, \sigma(T))$ holds for all $\lambda \notin \sigma(T)$; see, for instance the book by Martin and Putinar [MP89, Proposition 1.2]. Consequently, we get the following corollary to Theorem 3.3.

Corollary 3.19. *Let T be hyponormal and let $\Omega \subset \widehat{\mathbb{C}}$ be an open set satisfying the hypotheses of Theorem 3.3* (*in particular, the exterior disc condition*) *and such that $\sigma(T) \subset \overline{\Omega}$. Then $\overline{\Omega}$ is a complete K-spectral set for T.*

So in other words, in this situation, T can be dilated to an operator S which is similar to a normal operator and satisfies $\sigma(S) \subset \partial\Omega$. It is interesting to compare Corollary 3.19 with Putinar's result [Put84] that every hyponormal operator T is subscalar and can in fact can be represented as a restriction of a scalar operator L of order 2 (in the sense of Colojoara-Foias) to an invariant subspace. Thus, if $T = L|H$, where H is an invariant subspace of $L \in B(K)$, then L is a dilation of T of a special kind. On the other hand, the spectrum of a scalar operator L, as constructed by Putinar, contains a neighbourhood of $\sigma(T)$. By contrast, in Corollary 3.19, if $\sigma(T)$ is a closed Jordan domain satisfying the exterior disk condition, the dilation S of T is a scalar operator of order 0 and its spectrum is contained in the spectrum of T (and even in its boundary).

Generally speaking, the conditions of Corollary 3.19 do not imply that $\overline{\Omega}$ is a (1-)spectral set for T; this is seen from any of the examples by Wadhwa [Wad73] and Hartman [Har82], where one can put $\overline{\Omega} = \sigma(T)$ (it is an annulus for the Hartman's example and a disjoint union of an annulus and a disc for Wadhwa's example). On the other hand, consider the hyponormal operator from Clancey's example [Cla70]; let us call it B. Its spectrum is a compact subset of $\mathbb{C}$ of positive area, whose interior is empty. It is proved in [Cla70] that $\sigma(B)$ is not a 1-spectral set for B. A modification of Clancey's arguments also shows that it is not even K-spectral for any K. Indeed, by applying [Pau02, Exercise 9.11], one gets that if $\sigma(B)$ were K-spectral for B, then B would be similar to a normal operator. Since B is hyponormal, [SW76, Corollary 1] would then give that B is normal, which is not true. So the equality $\|(T - \lambda)^{-1}\| =$

$1/\operatorname{dist}\big(\lambda, \sigma(T)\big)$ ($\lambda \notin \sigma(T)$) in general does not imply that $\sigma(T)$ is a K-spectral set for T.

We also refer to [Put97, Theorem 4] for a result on subscalarity of operators with a power-like estimate for the resolvent.

3.5. Auxiliary lemmas

In this section we state and prove the lemmas that are needed in the proof of the Theorems 3.6 and 3.7.

Lemma 3.20. *Let X be a closed subspace of finite codimension r in a Banach space V and Y a (not necessarily closed) subspace of V such that $X + Y = V$. Then there exist vectors $g_1, \dots, g_r \in Y$ such that $Z = \operatorname{span}\{g_1, \dots, g_r\}$ is a complement of X; that is, $V = X \dotplus Z$, and there are functionals $\alpha_1, \dots, \alpha_r \in V^*$ such that*

$$G(f) \overset{def}{=} f - \sum \alpha_k(f) g_k$$

is the projection of V onto X parallel to Z.

Proof. Let $\pi : V \to V/X$ be the natural projection onto the quotient. Then $\pi(Y) = \pi(X + Y) = V/X$. We can therefore choose vectors $g_1, \dots, g_r \in Y$ such that the set $\{\pi(g_1), \dots, \pi(g_r)\}$ is a basis of V/X. It follows that $Z = \operatorname{span}\{g_1, \dots, g_r\}$ is a complement of X in V. The existence of the functionals $\alpha_1, \dots, \alpha_r$ is now clear. □

Note that, since $X + Y$ is always closed, the hypotheses of the lemma in particular hold in the case when Y is a dense subspace of V.

The next lemma roughly says that to prove von Neumann's inequality with a constant, it is enough to prove it only for rational functions in a space of finite codimension.

Lemma 3.21. *Let $T \in \mathcal{B}(H)$ be such that for all $s \geq 1$,*

$$\|f(T)\| \leq C\|f\|_{A(\overline{\Omega})\otimes M_s}, \qquad \forall f \in (X \cap \operatorname{Rat}(\overline{\Omega})) \otimes M_s, \tag{3.10}$$

where X is some closed subspace of finite codimension in $A(\overline{\Omega})$. Then $\overline{\Omega}$ is a complete K-spectral set for T, where K depends only on X, C and $S_\Lambda(T)$, where Λ is an arbitrary pole set for Ω.

Proof. Fix a pole set Λ for Ω. Denote by $\operatorname{Rat}_\Lambda$ the set of rational functions with poles in Λ. Note that $\operatorname{Rat}_\Lambda$ is dense in $A(\overline{\Omega})$. Hence, we apply Lemma 3.20 with $V = A(\overline{\Omega})$, $Y = \operatorname{Rat}_\Lambda$ to obtain functions $g_1, \dots, g_r \in \operatorname{Rat}_\Lambda$, functionals $\alpha_1, \dots, \alpha_r \in A(\overline{\Omega})^*$, and an operator $G : A(\overline{\Omega}) \to A(\overline{\Omega})$ as in the statement of that lemma.

We can write

$$g_k(z) = c_0 + \sum_{\lambda\in\Lambda} \sum_{j=1}^{N} c_{\lambda,j,k} p_\lambda(z)^j$$

for suitable coefficients $c_{\lambda,j,k}$. (Recall that $p_\lambda(z) = (z-\lambda)^{-1}$ for $\lambda \neq \infty$, and $p_\infty(z) = z$.) This shows that for $k = 1 \dots, r$, $\|g_k(T)\| \leq K'$, where K' is a constant depending only on $g_1, \dots, g_r$ and $S_\Lambda(T)$, but not on T.

Let $f \in \mathrm{Rat}(\overline{\Omega}) \otimes M_s$. By Lemma 3.17, G and $\alpha_1, \ldots, \alpha_r$ are completely bounded. Also, G maps Rat_Λ into $X \cap \mathrm{Rat}(\overline{\Omega})$, so by (3.10),

$$\begin{aligned}
\|f(T)\| &= \Big\| [(G \otimes \mathrm{id}_s)(f)](T) + \sum_k g_k(T) \otimes [(\alpha_k \otimes \mathrm{id}_s)(f)] \Big\| \\
&\le \|[(G \otimes \mathrm{id}_s)(f)](T)\| + \sum_{k=1}^{n} \|g_k(T) \otimes [(\alpha_k \otimes \mathrm{id}_s)(f)]\| \\
&\le C \|(G \otimes \mathrm{id}_s)(f)\|_{A(\overline{\Omega}) \otimes M_s} + \sum_{k=1}^{r} K' \|\alpha_k \otimes \mathrm{id}_s\| \cdot \|f\|_{A(\overline{\Omega}) \otimes M_s} \\
&\le C \|G\|_{\mathrm{cb}} \|f\|_{A(\overline{\Omega}) \otimes M_s} + \sum_{k=1}^{r} K' \|\alpha_k\|_{\mathrm{cb}} \|f\|_{A(\overline{\Omega}) \otimes M_s},
\end{aligned}$$

and the result follows. □

Definition 3.22. Given a domain $\Omega \subset \mathbb{C}$, a *shrinking* for Ω is a collection $\{\psi_\varepsilon\}_{0 \le \varepsilon \le \varepsilon_0}$ of univalent analytic functions in some open set $U \supset \overline{\Omega}$, such that ψ_0 is the identity map on U, $\psi_\varepsilon(\overline{\Omega}) \subset \Omega$ for $\varepsilon > 0$, and the map $\varepsilon \mapsto \psi_\varepsilon$ is continuous in the topology of uniform convergence on compact subsets of U.

If Ω is star-shaped with respect to $a \in \mathbb{C}$, then it admits a shrinking; namely, $\psi_\varepsilon(z) = (1-\varepsilon)(z-a) + a$. The next lemma says that any admissible Jordan domain admits a shrinking.

Lemma 3.23. *Let Ω be a Jordan domain with piecewise C^2 smooth boundary composed of closed C^2 arcs $\{J_k\}_{k=1}^n$. If the angles between these arcs are non-zero, then Ω admits a shrinking.*

Proof. Denote by $z_1, \ldots, z_n \in \partial\Omega$ the endpoints of the arcs $J_1, \ldots, J_n$, so that z_k is a common endpoint of J_{k-1} and J_k (we assume that the numbering of these arcs is counterclockwise and cyclic modulo n). The points $z_1, \ldots, z_n$ will be referred to as the *corners* of $\partial\Omega$. First we construct a function $\mu \in A(\overline{\Omega})$ such that $\mu \ne 0$ on $\partial\Omega$ and for any $z \in \overline{\Omega}$, $\mu(z)$ points strictly inside Ω, by which we mean that there is $\sigma = \sigma(z) > 0$ such that the interval $[z, z + \sigma\mu(z)]$ is contained in $\overline{\Omega}$ and is not tangent to $\partial\Omega$ at z. If $z = z_k$, we require this interval to be non-tangential to both J_{k-1} and J_k at z. We denote by $\rho(z) \in \mathbb{C}$ the unit inner normal vector to the boundary. It is defined for points $z \in \partial\Omega$ which are not corners.

Let $\eta : \overline{\Omega} \to \overline{\mathbb{D}}$ be a Riemann conformal map, and put

$$\nu(z) = -\frac{\eta(z)}{\eta'(z)}.$$

Then ν is continuous on $\overline{\Omega} \setminus \{z_1, \ldots, z_n\}$; moreover, $\nu(z_0)$ points strictly inside Ω for any non-corner point $z_0 \in \partial\Omega$ and, in fact, $\rho(z_0) = c(z_0)\nu(z_0)$ for some $c(z_0) > 0$. Indeed, if $z(t)$ $(0 \le t \le T)$ is a counterclockwise parametrization of $\partial\Omega$, which is smooth at z_0, $z(t_0) = z_0$, $z'(t_0) = b$, then $\eta'(z_0)b = ic\eta(z_0)$ for some $c > 0$, so that $\rho(z_0) = ib = c\nu(z_0)$. The function $\mu(z)$ will be, in a sense, a small correction of $\nu(z)$, which mostly affects neighbourhoods of the corner points.

Denote by $R_{z,\theta} = \{w \in \mathbb{C} : \arg(w - z) = \theta\}$ the ray starting at z with angle θ. We assume that the rays $R_{z_k,\theta_k-\beta_k}$, $R_{z_k,\theta_k+\beta_k}$ are correspondingly tangent to $\partial\Omega$ at z_k to the arcs J_{k-1}, J_k, where $0 < \beta_k < \pi$, and $\theta_k \in [0, 2\pi)$ is such that the ray R_{z_k,θ_k} points strictly inside Ω. Theorem 3.9 in [Pom92] implies that for $z \in \overline{\Omega}$,

$$\eta(z) = \eta(z_k) + u_k(z)(z - z_k)^{a_k}, \tag{3.11}$$
$$\eta'(z) = v_k(z)a_k(z - z_k)^{a_k-1}, \tag{3.12}$$

where $u_k(z)$, $v_k(z)$ have finite non-zero limits as $z \to z_k$, and $a_k = \frac{\pi}{2\beta_k} \in (\frac{1}{2}, +\infty)$. (We use the principal branch of the logarithm in the definition of powers.) For small $\sigma > 0$, put $\tau_{k,\sigma} := z_k - \sigma e^{i\theta_k} \notin \overline{\Omega}$, and set

$$\mu_\sigma(z) = \Pi_\sigma(z)\nu(z) = -\Pi_\sigma(z)\frac{\eta(z)}{\eta'(z)},$$

where

$$\Pi_\sigma(z) = \prod_{k=1}^{n} \left(\frac{z - \tau_{k,\sigma}}{z - z_k}\right)^{1-a_k}.$$

Since the intervals $[z_k, \tau_{k,\sigma}]$ are outside Ω, the function Π_σ is well-defined and analytic in Ω.

We assert that for sufficiently small $\sigma > 0$, $\mu(z) = \mu_\sigma(z)$ will satisfy all the necessary requirements. To begin with, it follows from (3.11) and (3.12) that for any fixed (small) $\sigma > 0$ and any k, $\mu_\sigma(z)$ has a finite non-zero limit as $z \to z_k$, $z \in \overline{\Omega}$. Hence μ_σ continues to a function in $A(\overline{\Omega})$ such that $\mu_\sigma \neq 0$ on $\partial\Omega$.

Fix some small positive δ such that for all k, $2\delta < \beta_k < 2\pi - 2\delta$. Easy geometric arguments show that there is some $\sigma_0 > 0$ such that for any k, any $z \in J_{k-1}$ such that $|z - z_k| < \sigma_0$ and any $\sigma \in (0, \sigma_0)$, either $\arg \Pi_\sigma(z) \in (-\delta, \beta_k - \frac{\pi}{2} + \delta)$ if $\beta_k \in [\frac{\pi}{2}, \pi)$, or $\arg \Pi_\sigma(z) \in (\beta_k - \frac{\pi}{2} + \delta, \delta)$ if $\beta_k \in (0, \frac{\pi}{2})$. One has symmetric estimates for $\arg \Pi_\sigma(z)$ if $z \in J_k$, $|z - z_k| < \sigma_0$. Since $\Pi_\sigma(z) \to 1$ uniformly on $\partial\Omega \setminus \cup_k B_{\sigma_0}(z_k)$, it follows that for any $z \in \partial\Omega$, $z \neq z_1, \dots, z_n$ when $\sigma \in (0, \sigma_0)$ is sufficiently small,

$$-\frac{\pi}{2} + \delta \leq \arg \frac{\mu_\sigma(z)}{\rho(z)} = \arg \Pi_\sigma(z) \leq \frac{\pi}{2} - \delta.$$

For such fixed σ, $\mu(z) := \mu_\sigma(z)$ satisfies all the requirements.

By Mergelyan's theorem [Rud87], there is a sequence of polynomials μ_m, such that $\mu_m \to \mu$ uniformly on $\overline{\Omega}$. For a sufficiently large m, put $\tilde{\mu}(z) = \mu_m(z)$. Then the polynomial $\tilde{\mu}$ also satisfies all the requirements on μ.

We assert that $\psi_\varepsilon(z) = z + \varepsilon\tilde{\mu}(z)$ defines a shrinking of Ω. Indeed, for small $\varepsilon > 0$, $\psi_\varepsilon(\partial\Omega)$ is a Jordan curve, contained in Ω. An application of the argument principle shows that for these values of ε, ψ_ε maps Ω univalently onto the interior of the curve $\psi_\varepsilon(\partial\Omega)$. There exists a Jordan domain Ω' such that $\overline{\Omega} \subset \Omega'$ and the boundary of Ω' consists of C^2 smooth arcs $J'_1, \dots, J'_n$, which are close to the arcs $J_1, \dots, J_n$ in C^1 metric. The domain Ω' can be chosen in such a way that $\tilde{\mu} \neq 0$ on $\partial\Omega'$ and $\tilde{\mu}(z)$ points strictly inside Ω' for all $z \in \partial\Omega'$. By the same argument, the functions ψ_ε are univalent on Ω' for $0 \leq \varepsilon \leq \varepsilon_0$, where $\varepsilon_0 > 0$. Therefore the family $\{\psi_\varepsilon\}$ of functions, defined on Ω', is a shrinking of Ω. □

The following lemma improves upon the results of Lemma 3.21 by imposing certain constraints on $\varphi_k(T)$ and $\mathcal{A}_\Phi$. Recall that the algebra $\mathcal{A}_\Phi$ was defined in page 22.

Lemma 3.24. *Let $\Phi \subset A(\overline{\Omega})$ be a collection of functions taking Ω into $\mathbb{D}$. If, in addition to the hypotheses of Lemma 3.21, we have that for every $\varphi \in \Phi$, $\overline{\mathbb{D}}$ is a (not necessarily complete) K'-spectral set for $\varphi(T)$, then for all $s \geq 1$,*

$$\|f(T)\| \leq K\|f\|_{A(\overline{\Omega})\otimes M_s}, \qquad \forall f \in (X + \mathcal{A}_\Phi) \otimes M_s, \tag{3.13}$$

where K depends only on X, Φ, C and K', but not on T. In the case when $X + \mathcal{A}_\Phi = A(\overline{\Omega})$, then $\overline{\Omega}$ is a complete K-spectral set for T, with $K = K(X, \Phi, C, K')$.

Note that the operators $\varphi(T)$ and $f(T)$ are defined by the $A(\overline{\Omega})$-functional calculus for T because by Lemma 3.21, $\overline{\Omega}$ is a complete K-spectral for T for some K. On the other hand, and in contrast to the situation in most of this chapter, here the complete K'-spectrality of $\overline{\mathbb{D}}$ for $\varphi(T)$ is not needed. K'-spectrality suffices. The reason for this is that all the functions that appear in the proof of this lemma are scalar-valued rather than matrix-valued.

Proof of Lemma 3.24. First we apply Lemma 3.20 with $V = X + \mathcal{A}_\Phi$, and $Y = \mathcal{A}_\Phi$ to obtain functions $g_1, \dots, g_r \in \mathcal{A}_\Phi$, functionals $\alpha_1, \dots, \alpha_r \in A(\overline{\Omega})^*$, and an operator G as in the statement of that lemma.

By Lemma 3.21, $\overline{\Omega}$ is a complete K-spectral set for T (with K depending on T). It follows that T has a continuous $A(\overline{\Omega})$-functional calculus, and so the operators $g_k(T)$ are well defined. Let us show that there is some constant C' depending only on $g_1, \dots, g_r$ (and not on T) such that $\|g_k(T)\| \leq C'$, for $k = 1, \dots, r$. Since $g_k \in \mathcal{A}_\Phi$, we can write

$$g_k(z) = \sum_{j=1}^{N} f_{j,1}^k(\varphi_1(z)) \cdot \dots \cdot f_{j,n}^k(\varphi_n(z)),$$

where $f_{j,l}^k \in A(\overline{\mathbb{D}})$. (Because there are a finite number of functions g_k, the same N will do every k.) By the properties of the $A(\overline{\Omega})$-functional calculus for T we see that for $k = 1, \dots, r$,

$$g_k(T) = \sum_{j=1}^{N} f_{j,1}^k(\varphi_1(T)) \cdot \dots \cdot f_{j,n}^k(\varphi_n(T)).$$

Using the fact that $\overline{\mathbb{D}}$ is a K'-spectral set for $\varphi_k(T)$, we get

$$\begin{aligned}\|g_k(T)\| &\leq \sum_{j=1}^{N} \|f_{j,1}^k(\varphi_1(T))\| \cdots \|f_{j,n}^k(\varphi_n(T))\| \\ &\leq \sum_{j=1}^{N} (K')^n \|f_{j,1}^k\|_{A(\overline{\mathbb{D}})} \cdots \|f_{j,n}^k\|_{A(\overline{\mathbb{D}})}.\end{aligned}$$

This shows that for $k = 1, \dots, n$, $\|g_k(T)\| \leq C'$, with C' independent of T.

Finally, we proceed as in the proof of Lemma 3.21. Take $f \in (X + \mathcal{A}_\Phi) \otimes M_s$ and estimate

$$\begin{aligned} \|f(T)\| &\le \|[(G \otimes \mathrm{id}_s)(f)](T)\| + \sum_{k=1}^{r} \|g_k(T) \otimes [(\alpha_k \otimes \mathrm{id}_s)(f)]\| \\ &\le C\|G\|_{\mathrm{cb}}\|f\|_{A(\overline{\Omega})\otimes M_s} + \sum_{k=1}^{r} C'\|\alpha_k\|_{\mathrm{cb}}\|f\|_{A(\overline{\Omega})\otimes M_s}. \end{aligned}$$

Apply Lemma 3.17 to get (3.13). The remaining part of the lemma now follows. □

We will also need a lemma that allows one to pass to the limit in a family of inequalities of the form (3.10) depending on some parameter ε. The subspaces which play the role of X will be given by the kernels of finite rank operators Σ_ε.

Lemma 3.25. *Let $\{T_\varepsilon\}_{0\le\varepsilon\le\varepsilon_0} \subset \mathcal{B}(H)$ be a family of operators with $\sigma(T_\varepsilon) \subset \overline{\Omega}$ for $0 \le \varepsilon \le \varepsilon_0$, and $\{\Sigma_\varepsilon\}_{0\le\varepsilon<\varepsilon_0} \subset \mathcal{B}(A(\overline{\Omega}), \mathbb{C}^r)$. Assume that the maps $\varepsilon \mapsto T_\varepsilon$ and $\varepsilon \mapsto \Sigma_\varepsilon$ are continuous in the norm topology. Assume also that Σ_0 is surjective and that for all $s \ge 1$ and for all $\varepsilon \in (0, \varepsilon_0]$,*

$$\|f(T_\varepsilon)\| \le C\|f\|_{A(\overline{\Omega})\otimes M_s}, \qquad \forall f \in (\ker \Sigma_\varepsilon \cap \mathrm{Rat}(\overline{\Omega})) \otimes M_s, \tag{3.14}$$

where C is a constant independent of ε. Then (3.14) also holds with $\varepsilon = 0$.

Proof. Since Σ_0 is surjective, $X = \ker \Sigma_0$ has codimension r in $A(\overline{\Omega})$. We apply Lemma 3.20 with $Y = \mathrm{Rat}(\overline{\Omega})$ to obtain functions $g_1, \dots, g_r \in \mathrm{Rat}(\overline{\Omega})$, a subspace $Z = \mathrm{span}\{g_1, \dots, g_r\}$, functionals $\alpha_1, \dots, \alpha_r \in A(\overline{\Omega})^*$ and an operator G as in the statement of that lemma.

Consider the restrictions $\Sigma_\varepsilon|Z : Z \to \mathbb{C}^r$. The operator $\Sigma_0|Z$ is invertible, therefore, $\Sigma_\varepsilon|Z$ is invertible for ε sufficiently small. Put $P_\varepsilon = (\Sigma_\varepsilon|Z)^{-1}\Sigma_\varepsilon$. Thus $P_\varepsilon : A(\overline{\Omega}) \to Z$ and $P_\varepsilon^2 = P_\varepsilon$. Indeed, P_ε is the projection onto Z parallel to $\ker \Sigma_\varepsilon$. Define $\alpha_k^\varepsilon \in (A(\overline{\Omega}))^*$ by $\alpha_k^\varepsilon(f) = \alpha_k(P_\varepsilon f)$, and check that $G_\varepsilon(f) \overset{\mathrm{def}}{=} f - \sum \alpha_k^\varepsilon(f) g_k$ is in $\ker \Sigma_\varepsilon$ for every $f \in A(\overline{\Omega})$. We compute

$$\begin{aligned} P_\varepsilon G_\varepsilon(f) &= P_\varepsilon f - \sum_{k=1}^{r} \alpha_k^\varepsilon(f) P_\varepsilon g_k \\ &= P_\varepsilon^2 f - \sum_{k=1}^{r} \alpha_k(P_\varepsilon f) P_\varepsilon g_k = P_\varepsilon G(P_\varepsilon f) = 0, \end{aligned}$$

because $P_\varepsilon f \in Z$ and $\ker G = Z$. It follows that $G_\varepsilon(f)$ is in $\ker P_\varepsilon = \ker \Sigma_\varepsilon$.

Since T_ε depends continuously on ε, there is some constant K independent of ε such that $\|g_k(T_\varepsilon)\| \le K$ for small ε and $k = 1, \dots, r$. Take $f \in \mathrm{Rat}(\overline{\Omega}) \otimes M_s$ and estimate

$$\begin{aligned} \|f(T_\varepsilon)\| &= \left\| [(G_\varepsilon \otimes \mathrm{id}_s)(f)](T_\varepsilon) + \sum_{k=1}^{r} g_k(T_\varepsilon) \otimes [(\alpha_k^\varepsilon \otimes \mathrm{id}_s)(f)] \right\| \\ &\le C\|(G_\varepsilon \otimes \mathrm{id}_s)(f)\|_{A(\overline{\Omega})\otimes M_s} + \sum_{k=1}^{r} K\|(\alpha_k^\varepsilon \otimes \mathrm{id}_s)(f)\|_{A(\overline{\Omega})\otimes M_s}. \end{aligned}$$

Since G_ε and α_k^ε depend continuously on ε, we can let $\varepsilon \to 0$ to obtain

$$\|f(T_0)\| \leq C\|(G \otimes \mathrm{id}_s)(f)\|_{A(\overline{\Omega})\otimes M_s} + \sum_{k=1}^{r} K\|(\alpha_k \otimes \mathrm{id}_s)(f)\|_{A(\overline{\Omega})\otimes M_s}.$$

The proof concludes by noting that if $f \in \ker \Sigma_0 \otimes M_s$, then $(G \otimes \mathrm{id}_s)(f) = f$ and $(\alpha_k \otimes \mathrm{id}_s)(f) = 0$ for $k = 1, \dots, r$. $\square$

The next lemma constructs a family of admissible functions Φ_ε which work well with the operators $\psi_\varepsilon(T)$, where $\{\psi_\varepsilon\}$ is a shrinking for Ω.

Lemma 3.26. *Let Ω be a Jordan domain with a shrinking $\{\psi_\varepsilon\}_{0\leq\varepsilon\leq\varepsilon_0}$, and let $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ be admissible. Let $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \overline{\Omega}$ and such that $\overline{\mathbb{D}}$ is a complete K-spectral set for $\varphi_k(T)$, for $k = 1, \dots, n$. Then there is some $0 < \delta \leq \varepsilon_0$ and a family of admissible functions $\{\Phi_\varepsilon\}_{0\leq\varepsilon\leq\delta}$ over Ω with $\Phi_\varepsilon = (\varphi_1^\varepsilon, \dots, \varphi_n^\varepsilon)$ and $\Phi_0 = \Phi$, such that each φ_k^ε is analytic in some neighbourhood U_k of $\Omega \cup J_k$, the map $\varepsilon \mapsto \varphi_k^\varepsilon$ is continuous from $[0, \delta]$ to $C^\infty(U_k)$, and $\overline{\mathbb{D}}$ is a complete K-spectral set for $\varphi_k^\varepsilon(\psi_\varepsilon(T))$.*

Proof. We construct admissible functions $\Phi_\varepsilon = (\varphi_1^\varepsilon, \dots, \varphi_n^\varepsilon)$ satisfying the statement of the lemma by choosing φ_k^ε to have the form $\varphi_k^\varepsilon = \eta_k^\varepsilon \circ \varphi_k \circ \psi_\varepsilon^{-1}$, where $\eta_k^\varepsilon \in A(\overline{\mathbb{D}})$ and $\|\eta_k^\varepsilon\|_{A(\overline{\mathbb{D}})} \leq 1$. Because $\varphi_k^\varepsilon(\psi_\varepsilon(T)) = \eta_k^\varepsilon(\varphi_k(T))$, this will guarantee that $\varphi_k^\varepsilon(\psi_\varepsilon(T))$ has $\overline{\mathbb{D}}$ as a complete K-spectral set. The construction of η_k^ε is geometric.

First, continue analytically the arcs $J_k \subset \partial\Omega$ to larger arcs $\widetilde{J}_k$ such that φ_k and ψ_ε are analytic in a neighbourhood of $\widetilde{J}_k$ (recall that φ_k and ψ_ε are analytic in a neighbourhood of $\overline{\Omega}$). In this proof, we only deal with closed arcs. Assume that $\widetilde{J}_k$ are small enough that each $\varphi_k|\widetilde{J}_k$ is still one to one. Put $\Gamma_k^\varepsilon = \varphi_k(\psi_\varepsilon^{-1}(J_k))$ and $\widetilde{\Gamma}_k^\varepsilon = \varphi_k(\psi_\varepsilon^{-1}(\widetilde{J}_k))$. Since $\widetilde{\Gamma}_k^0 = \varphi_k(\widetilde{J}_k)$ is an arc of $\mathbb{T}$, it follows by continuity that for small ε, there exists $\widetilde{I}_k^\varepsilon$ an arc of $\mathbb{T}$, and a function $a_k^\varepsilon : \widetilde{I}_k^\varepsilon \to \mathbb{R}_+$ such that $\widetilde{\Gamma}_k^\varepsilon = \{a_k^\varepsilon(\zeta)\zeta : \zeta \in \widetilde{I}_k^\varepsilon\}$. Also, $a_k^\varepsilon \geq 1$ in $\widetilde{I}_k^\varepsilon$ and $a_k^0 = 1$ in $\widetilde{I}_k^0$. The functions a_k^ε are assumed to be defined for $0 \leq \varepsilon \leq \delta$. Let I_k^ε be the sub-arc of $\widetilde{I}_k^\varepsilon$ such that $\Gamma_k^\varepsilon = \{a_k^\varepsilon(\zeta)\zeta : \zeta \in I_k^\varepsilon\}$.

Next find functions $b_k^\varepsilon : \mathbb{T} \to \mathbb{R}_+$, $0 \leq \varepsilon \leq \delta$, such that $b_k^\varepsilon \in C^\infty(\mathbb{T})$ for each ε, the map $\varepsilon \mapsto b_k^\varepsilon$ is continuous from $[0, \delta]$ to $C^\infty(\mathbb{T})$, $b_k^\varepsilon = a_k^\varepsilon$ in I_k^ε, $b_k^\varepsilon \geq 1$ in $\mathbb{T}$, and if D_k^ε is the interior domain of the Jordan curve $\{b_k^\varepsilon(\zeta)\zeta : \zeta \in \mathbb{T}\}$, then $\varphi_k(\psi_\varepsilon^{-1}(\Omega)) \subset \overline{D_k^\varepsilon}$. These are first constructed in a local manner and then a partition of unity argument is employed. This construction is done as follows.

For each k, define the following closed subsets of $\mathbb{T} \times [0, \delta]$:

$$V_k = \bigcup_{0\leq\varepsilon\leq\delta} (I_k^\varepsilon \times \{\varepsilon\}), \qquad \widetilde{V}_k = \bigcup_{0\leq\varepsilon\leq\delta} (\widetilde{I}_k^\varepsilon \times \{\varepsilon\}).$$

(These are closed because I_k^ε and $\widetilde{I}_k^\varepsilon$ depend continuously on ε.) Next, for every point $p = (\zeta, \varepsilon) \in \mathbb{T}\times[0, \delta]$ and every k, construct a function $c_k^p : W_p \to \mathbb{R}_+$, where W_p is some neighbourhood of p in $\mathbb{T} \times [0, \delta]$. If $\zeta \in I_k^\varepsilon$, choose W_p small enough so that $W_p \subset \widetilde{V}_k$ and put $c_k^p(\zeta', \varepsilon') = a_k^{\varepsilon'}(\zeta')$. Note that if $(\zeta', \varepsilon') \in W_p$ and $r\zeta' \in \partial\varphi_k(\psi_{\varepsilon'}^{-1}(\Omega))$, then $r = c_k^p(\zeta', \varepsilon')$. If $\zeta \notin I_k^\varepsilon$, then choose W_p small enough so that W_p does not intersect V_k, and then choose as c_k^p some C^∞ function satisfying the property that if $(\zeta', \varepsilon') \in W_p$ and $r\zeta' \in \partial\varphi_k(\psi_{\varepsilon'}^{-1}(\Omega))$, then $r \leq c_k^p(\zeta', \varepsilon')$. We also require $c_k^p \geq 1$ in all W_p.

By compactness, choose a finite subfamily $\{W_{p_j}\}$ of $\{W_p\}$, which still covers $\mathbb{T}\times[0,\delta]$. Let $\{\tau_{p_j}\}$ be a C^∞ partition of unity in $\mathbb{T}\times[0,\delta]$ subordinate to the cover $\{W_{p_j}\}$ and put

$$b_k^\varepsilon(\zeta) = \sum_{p_j} \tau_{p_j}(\zeta,\varepsilon) c_k^{p_j}(\zeta,\varepsilon).$$

It is easy to see that b_k^ε satisfies the required conditions because the functions c_k^p satisfy them in a local manner.

Let D_k^ε be defined as above and let η_k^ε be the Riemann map from D_k^ε onto $\mathbb{D}$ such that $\eta_k^\varepsilon(0) = 0$ and $(\eta_k^\varepsilon)'(0) > 0$. This exists since $\mathbb{D} \subset D_k^\varepsilon$. Clearly, $\eta_k^\varepsilon \in A(\overline{\mathbb{D}})$ and $\|\eta_k^\varepsilon\|_{A(\overline{\mathbb{D}})} \le 1$.

We prove that $\varphi_k^\varepsilon = \eta_k^\varepsilon \circ \varphi_k \circ \psi_\varepsilon^{-1}$ depend continuously on ε. Put $\beta = \max_{k,\varepsilon,\zeta} b_k^\varepsilon(\zeta)$. Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a C^∞ function such that $\gamma(r) = 0$ in a neighbourhood of 0, $\gamma(r) = r$ on (σ,∞) for some $\sigma \in (0,1)$ and $\gamma'(r) < \beta/(\beta-1)$ for all r. For each $\varepsilon \in [0,\delta]$, put

$$h_k^\varepsilon(r\zeta) = \rho_k^\varepsilon(r)\zeta, \qquad \rho_k^\varepsilon(r) = r - \left(1 - \frac{1}{b_k^\varepsilon(\zeta)}\right)\gamma(r), \qquad r \ge 0,\ \zeta \in \mathbb{T}. \tag{3.15}$$

The condition $\gamma'(r) < \beta/(\beta-1)$ implies that $(\rho_k^\varepsilon)' > 0$. Thus, (3.15) defines maps $h_k^\varepsilon : \mathbb{C} \to \mathbb{C}$ which are diffeomorphisms from $\overline{D_k^\varepsilon}$ to $\overline{\mathbb{D}}$ and depend continuously on ε. By [BG14, Corollary 9.4], the maps $\varepsilon \mapsto \eta_k^\varepsilon \circ (h_k^\varepsilon)^{-1}$ are continuous from $[0,\delta]$ to $C^\infty(\overline{\mathbb{D}})$. Hence, the maps $\varepsilon \mapsto \varphi_k^\varepsilon$ are continuous from $[0,\delta]$ to $C^\infty(\overline{\Omega})$.

Since by construction $|\varphi_k^\varepsilon| = 1$ in $\widetilde{J}_k$, the Schwartz reflection principle implies that each φ_k^ε is analytic in some neighbourhood U_k of $\Omega \cup J_k$ and that the map $\varepsilon \mapsto \varphi_k^\varepsilon$ is continuous from $[0,\delta]$ to $C^\infty(U_k)$. As $\Phi_0 = \Phi$ is admissible, by continuity the maps Φ_ε must also be admissible for sufficiently small ε. This finishes the proof. $\square$

The following is a continuous (ε-dependent) version of the right regularization for Fredholm operators of index 0.

Lemma 3.27. *Let V be a Banach space, and $\{L_\varepsilon\}_{0\le\varepsilon\le\varepsilon_0} \subset \mathcal{B}(V)$ be such that the map $\varepsilon \mapsto L_\varepsilon$ is continuous in the norm topology and $L_0 - I$ is compact. Then there is a finite rank operator $P \in \mathcal{B}(V)$, some $0 < \delta \le \varepsilon_0$ and operators $\{R_\varepsilon\}_{0\le\varepsilon\le\delta}, \{S_\varepsilon\}_{0\le\varepsilon\le\delta} \subset \mathcal{B}(V)$ such that the maps $\varepsilon \mapsto R_\varepsilon$ and $\varepsilon \mapsto S_\varepsilon$, $S_0 = I$, are continuous in the norm topology, and*

$$L_\varepsilon R_\varepsilon = I + PS_\varepsilon$$

holds for $0 \le \varepsilon \le \delta$.

Proof. Since $L_0 - I$ is compact, it is well know that there is a finite rank operator P and an operator R_0 such that $LR_0 = I + P$. Let $B_\varepsilon = I + (L_\varepsilon - L_0)R_0$. Then there is some $\delta > 0$ such that B_ε is invertible for $0 \le \varepsilon \le \delta$. We have $L_\varepsilon R_0 B_\varepsilon^{-1} = I + PB_\varepsilon^{-1}$, so the lemma holds with $R_\varepsilon = R_0 B_\varepsilon^{-1}$ and $S_\varepsilon = B_\varepsilon^{-1}$. $\square$

Lemma 3.28. *Let $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ be admissible. Assume that there are operators $T \in \mathcal{B}(H)$ and $C_1,\dots,C_n \in \mathcal{B}(H)$ such that $\overline{\mathbb{D}}$ is a complete K'-spectral set for every C_k, $k = 1,\dots,n$. Assume that if $f \in \operatorname{Rat}(\overline{\Omega})$ can be written as in* (3.4)*, then* (3.5) *holds* *(see the statement of Theorem 3.7). Then $\overline{\Omega}$ is a complete K-spectral set for T for some K depending on Ω, Φ, K' and $S_\Lambda(T)$. Furthermore,*

$$\|f(T)\| \le C\|f\|_{A(\overline{\Omega})\otimes M_s}, \qquad \forall f \in (\mathcal{A}_\Phi \cap \mathrm{Rat}(\overline{\Omega})) \otimes M_s, \tag{3.16}$$

where C is a constant depending only on Ω, Φ and K', and not on T.

The main point of (3.16) is that, under the hypotheses of this Lemma, $\mathcal{A}_\Phi$ is a closed subspace of finite codimension in $A(\overline{\Omega})$. Thus, (3.16) shows that, in a space of finite codimension, the inequality $\|f(T)\| \le C\|f\|$ holds with a constant independent of T.

Proof of Lemma 3.28. Use Theorem 1.11 to obtain operators F_k as in the statement of the theorem. Denote by $L \in \mathcal{B}(A(\overline{\Omega}))$ the operator defined by $L(f) = \sum F_k(f) \circ \varphi_k$. Since $I - L$ is compact, there exist an operator R and a finite rank operator P such that $LR = I + P$. The space $X = \ker P$ has finite codimension in $A(\overline{\Omega})$ and does not depend on T. We will now check that (3.10) holds for some constant C independent of T.

Take $f \in (X \cap \mathrm{Rat}(\overline{\Omega})) \otimes M_s$ and put $g = (R \otimes \mathrm{id}_s)f$. Then $(L \otimes \mathrm{id}_s)g = f$, and so by (3.5),

$$f(T) = \sum_{k=1}^{n} [(F_k \otimes \mathrm{id}_s)(g)](C_k).$$

Since $\overline{\mathbb{D}}$ is complete K'-spectral for C_k,

$$\begin{aligned}\|f(T)\| &\le \sum_{k=1}^{n} K'\|F_k\|_{\mathrm{cb}}\|g\|_{A(\overline{\Omega})\otimes M_s} \le \sum_{k=1}^{n} K'\|F_k\|_{\mathrm{cb}}\|R\|_{\mathrm{cb}}\|f\|_{A(\overline{\Omega})\otimes M_s} \\ &= \sum_{k=1}^{n} K'\|F_k\| \cdot \|R\| \cdot \|f\|_{A(\overline{\Omega})\otimes M_s},\end{aligned}$$

where the last equality uses Lemma 3.17. Thus (3.10) holds with $C = \sum K'\|F_k\|\cdot\|R\| < \infty$. Apply Lemma 3.24 to get (3.16). The remaining part of the lemma follows from Lemma 3.21. □

3.6. Proofs of Theorems 3.7 and 3.6

We first give the proof of Theorem 3.7, as it is simpler than that of Theorem 3.6 and both proofs follow the same general idea.

Proof of Theorem 3.7. The first part of Theorem 3.7 is already contained in the statement of Lemma 3.28. For the case when Φ is injective and Φ' does not vanish, use Theorem 2.1 (page 23). Then (3.16) implies that $\overline{\Omega}$ is a complete K-spectral set for T, with K independent of T. □

To prove Theorem 3.6, in the case when $\sigma(T) \subset \Omega$, one can argue as in the proof of Theorem 3.7, putting $C_k = \varphi_k(T)$ and using the Cauchy-Riesz functional calculus for T to get (3.5). However, such a direct proof will not work in the general case. The idea then is to apply a shrinking $\{\psi_\varepsilon\}$ for Ω to obtain operators $T_\varepsilon = \psi_\varepsilon(T)$ which have $\sigma(T_\varepsilon) \subset \Omega$, so that the above argument is again valid. The difficulties reside in constructing admissible functions $\Phi_\varepsilon = (\varphi_1^\varepsilon, \dots, \varphi_n^\varepsilon)$ adapted to T_ε, in the sense that each $\varphi_k^\varepsilon(T_\varepsilon)$ has $\overline{\mathbb{D}}$ as a complete K'-spectral set, as well as in passing to the limit as ε tends to 0.

Proof of Theorem 3.6. Let $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \overline{\Omega}$ and such that for $k = 1, \dots, n$, $\overline{\mathbb{D}}$ is a complete K'-spectral set for $\varphi_k(T)$. We must prove that $\overline{\Omega}$ is a complete K-spectral set for T with K depending on $\overline{\Omega}$, Φ, K' and $S_\Lambda(T)$.

By Lemma 3.23, there is a shrinking $\{\psi_\varepsilon\}$ for Ω. Apply Lemma 3.26 to obtain a collection of admissible functions $\{\Phi_\varepsilon\}_{0\le\varepsilon\le\varepsilon_0}$ such that $\overline{\mathbb{D}}$ is a complete K'-spectral set for $\varphi_k^\varepsilon(\psi_\varepsilon(T))$, and such that the maps $\varepsilon \mapsto \varphi_k^\varepsilon$ are continuous from $[0, \varepsilon_0]$ to $C^\infty(U_k)$, where U_k is a neighbourhood of J_k. Then use Lemma 1.18 to get operators L_ε, where

$$L_\varepsilon(f) = \sum F_k^\varepsilon(f) \circ \varphi_k^\varepsilon.$$

Since $L_0 - I$ is compact, Lemma 3.27 (with $V = A(\overline{\Omega})$) yields operators $P, R_\varepsilon, S_\varepsilon : A(\overline{\Omega}) \to A(\overline{\Omega})$, where $\varepsilon \in [0, \delta]$, with the properties stated in the lemma.

Next we wish to apply Lemma 3.25. To this end, fix $Q : \operatorname{ran} P \to \mathbb{C}^r$ an isomorphism, where r is the rank of P, put $\Sigma_\varepsilon = QPS_\varepsilon$, so that $\Sigma_\varepsilon : A(\overline{\Omega}) \to \mathbb{C}^r$, Σ_ε depends continuously on ε in the norm topology and $\Sigma_0 = QPS_0 = QP$ is surjective, and set $T_\varepsilon = \psi_\varepsilon(T)$. Note that T_ε depends continuously on ε in the norm topology because ψ_ε depends continuously on ε in the topology of uniform convergence on compact subsets of U, where U is some open neighbourhood of $\sigma(T)$.

It is necessary to check that (3.14) holds. For this, take $f \in (\ker \Sigma_\varepsilon \cap \operatorname{Rat}(\overline{\Omega})) \otimes M_s$, put $g = (R_\varepsilon \otimes \mathrm{id}_s)f$ and note that $f = (L_\varepsilon \otimes \mathrm{id}_s)g$. Since $\sigma(T_\varepsilon) \subset \Omega$, an application of the Cauchy-Riesz functional calculus gives

$$f(T_\varepsilon) = \sum_{k=1}^n [(F_k^\varepsilon \otimes \mathrm{id}_s)(g)](\varphi_k^\varepsilon(T_\varepsilon)).$$

Therefore, by Lemma 3.17, and since $\|F_k^\varepsilon\| \le C$ (coming from Lemma 1.18),

$$\|f(T_\varepsilon)\| \le \sum_{k=1}^n K' \|F_k^\varepsilon\|_{\mathrm{cb}} \|g\|_{A(\overline{\Omega})\otimes M_s} \le \sum_{k=1}^n K' C \|R_\varepsilon\| \|f\|_{A(\overline{\Omega})\otimes M_s}.$$

Since R_ε depends continuously on ε, (3.14) holds, as desired.

Apply Lemma 3.25 to obtain for all $s \ge 1$,

$$\|f(T)\| \le C' \|f\|_{A(\overline{\Omega})\otimes M_s}, \qquad \forall f \in (\ker \Sigma_0 \cap \operatorname{Rat}(\overline{\Omega})) \otimes M_s.$$

By Lemma 3.21, this yields that $\overline{\Omega}$ is a complete K-spectral set for T, with K depending on Ω, Φ and $S_\Lambda(T)$. Therefore, Φ is a quasi-uniform strong test collection.

In the case that Φ is injective and Φ' does not vanish on Ω, Theorem 2.1 and Lemma 3.24 together imply that $\overline{\Omega}$ is a complete K-spectral set for T, with K independent of T. □

3.7. Weakly admissible functions

In this section we will expand the class of functions under consideration to a wider class that we call weakly admissible functions. The main goal of this class is to replace condition (f) in the definition of an admissible function (see Definition 1.10, page 7) by a weaker separation condition. In particular, a collection of functions which includes inner functions (i.e., functions with modulus 1 in all $\partial\Omega$) may be weakly admissible, though not admissible, except in trivial cases.

Let $\zeta \in \partial\Omega$. A *right neighbourhood* of ζ in $\partial\Omega$ is understood to be the image $\gamma([0,\varepsilon))$, where the function $\gamma : [0,\varepsilon) \to \partial\Omega$ is continuous and injective, $\gamma(0) = \zeta$, and as t increases $\gamma(t)$ follows the positive orientation of $\partial\Omega$. Define the *left neighbourhoods* of ζ in a similar manner.

If $\Psi \subset A(\overline{\Omega})$ is a collection of functions taking Ω into $\mathbb{D}$ and $\zeta \in \partial\Omega$, set

$$\Psi_\zeta^+ = \{\psi \in \Psi : |\psi| = 1 \text{ in some right neighbourhood of } \zeta\},$$

and

$$\Psi_\zeta^- = \{\psi \in \Psi : |\psi| = 1 \text{ in some left neighbourhood of } \zeta\}.$$

Definition 3.29. Let Ω be a domain whose boundary is a disjoint finite union of piecewise analytic Jordan curves such that the interior angles of the corners of $\partial\Omega$ are in $(0,\pi]$. Then $\Psi = (\psi_1,\ldots,\psi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$, $\psi_k \in A(\overline{\Omega})$ for $k = 1,\ldots,n$, is *weakly admissible* if for $\Gamma_k = \{\zeta \in \partial\Omega : |\psi_k(\zeta)| = 1\}$ in place of J_k and a constant α, $0 < \alpha \leq 1$, it is the case that conditions (a)–(e) for an admissible function hold, and additionally:

(f′) $\quad \forall\zeta \in \partial\Omega,\ \forall z \in \partial\Omega, z \neq \zeta,\ \exists\psi \in \Psi_\zeta^+ : \psi(\zeta) \neq \psi(z).$

(g′) $\quad \forall\zeta \in \partial\Omega,\ \forall z \in \partial\Omega, z \neq \zeta,\ \exists\psi \in \Psi_\zeta^- : \psi(\zeta) \neq \psi(z).$

In fact, it is easy to see that conditions (a) and (b) follow formally from conditions (c)–(e), (f′) and (g′).

Lemma 3.30. *Let $\Psi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ be a weakly admissible function. Then there is an admissible function $\Phi : \overline{\Omega} \to \overline{\mathbb{D}}^m$, $\Phi = (\varphi_1,\ldots,\varphi_m)$, such that its components φ_k are of the form*

$$\varphi_k = (h_{1,k} \circ \psi_1) \cdot \cdots \cdot (h_{n,k} \circ \psi_n),$$

where $h_{j,k} : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ and $h_{j,k} \in A(\overline{\mathbb{D}})$.

Proof. First, fix some $\zeta \in \partial\Omega$. For each $\psi \in \Psi_\zeta^+$, put $P_\psi = \psi^{-1}(\{\psi(\zeta)\})$, which is a finite set of points of $\partial\Omega$. By condition (f′) in the definition of a weakly admissible function, $\bigcap_{\psi\in\Psi_\zeta^+} P_\psi = \{\zeta\}$. Let J_ζ^+ be the closure of a sufficiently small right neighbourhood of ζ. For $\psi \in \Psi_\zeta^+$, put $Q_\psi = \psi^{-1}(\psi(J_\zeta^+))$. If J_ζ^+ is small enough, then each set Q_ψ is a union of disjoint right neighbourhoods of each of the points in P_ψ. Since $\bigcap_{\psi\in\Psi_\zeta^+} P_\psi = \{\zeta\}$, it can then be assumed that $\bigcap_{\psi\in\Psi_\zeta^+} Q_\psi = J_\zeta^+$.

Next, for each $\psi \in \Psi_\zeta^+$, construct a function $h_\psi^+ \in A(\overline{\mathbb{D}})$ such that the function

$$\psi_\zeta^+ = \prod_{\psi\in\Psi_\zeta^+} h_\psi^+ \circ \psi$$

associated with J_ζ^+ satisfies $|\psi_\zeta^+| = 1$ in J_ζ^+ and $|\psi_\zeta^+| < 1$ in $\partial\Omega \setminus J_\psi^+$. This is done as follows.

Take $\psi \in \Psi_\zeta^+$. Choose a function h_ψ^+ satisfying the following conditions:

- $|h_\psi^+| = 1$ in $\psi(J_\zeta^+)$ and $|h_\psi^+| < 1$ in $\partial\Omega \setminus \psi(J_\zeta^+)$;
- h_ψ^+ maps $\psi(J_\zeta^+)$ bijectively onto a small arc of $\mathbb{T}$;
- h_ψ^+ is analytic on some open set $U \supset \mathbb{D}$ such that the interior of $\psi(J_\zeta^+)$ relative to $\mathbb{T}$ is contained in U, and $(h_\psi^+)'$ is Hölder α in U;
- $|(h_\psi^+)'| \geq C > 0$ in $\psi(J_\zeta^+)$;
- If ζ is an endpoint of the set $\{w \in \partial\Omega : |\psi(w)| = 1\}$ and $S(\zeta)$ is the sector that appears on condition (d) in the definition of an admissible function (for $\varphi_k = \psi$), then $\psi(S_k(\zeta)) \subset U$.

Then $|h_\psi^+ \circ \psi| = 1$ in Q_ψ, and $|h_\psi^+ \circ \psi| < 1$ in $\partial\Omega \setminus Q_\psi$. Since $|\psi_\zeta^+(z)| = 1$ only when $|h_\psi^+(\psi(z))| = 1$ for every $\psi \in \Psi_\zeta^+$ (that is, when $z \in \bigcap_{\psi\in\Psi_\zeta^+} Q_\psi = J_\zeta^+$), we get that $|\psi_\zeta^+| = 1$ in J_ζ^+ and $|\psi_\zeta^+| < 1$ in $\partial\Omega \setminus J_\zeta^+$. Also, since $h_\psi^+(\psi(J_\zeta^+))$ is a small arc of $\mathbb{T}$, it follows that ψ_ζ^+ maps J_ζ^+ bijectively onto some arc of $\mathbb{T}$.

Similarly, construct an arc J_ζ^- which is the closure of a small left neighbourhood of ζ, and a corresponding function ψ_ζ^-. By compactness, we can choose a finite set of points $\zeta_1, \dots, \zeta_r$ such that $J_{\zeta_k}^- \cup J_{\zeta_k}^+$, $k = 1, \dots, r$, cover all $\partial\Omega$. Rename the functions $\psi_{\zeta_1}^-, \psi_{\zeta_1}^+, \dots, \psi_{\zeta_r}^-, \psi_{\zeta_r}^+$ as $\varphi_1, \dots, \varphi_m$ and the corresponding arcs $J_{\zeta_1}^-, J_{\zeta_1}^+, \dots, J_{\zeta_r}^-, J_{\zeta_r}^+$ as $J_1, \dots, J_m$. Functions $\varphi_1, \dots, \varphi_m$ now satisfy condition (f) in the definition of an admissible family, because if, for instance, $J_k = J_\zeta^+$, then $\varphi_k = \psi_\zeta^+$ sends J_ζ^+ bijectively onto an arc of $\mathbb{T}$ and $|\varphi_k| < 1$ on $\partial\Omega \setminus J_\zeta^+$.

The functions φ_k satisfy conditions (c)–(e) because the functions ψ_k satisfy these conditions, and the functions h_ψ^+, h_ψ^- satisfy similar regularity conditions which have been given above. It follows that $\Phi = (\varphi_1, \dots, \varphi_m)$ is admissible. $\square$

Theorem 3.31. *Let $\Psi : \overline{\Omega} \to \overline{\mathbb{D}}^n$ be a weakly admissible function. Then Ψ is a quasi-uniform test collection over Ω. Moreover, if $\Psi_0 \subset A(\overline{\Omega})$ is a collection of functions taking Ω into $\mathbb{D}$ with the property that $\Psi \subset \Psi_0$, $\Psi_0 : \overline{\Omega} \to \overline{\mathbb{D}}^m$ is injective and Ψ_0' does not vanish on Ω, then Ψ_0 is a uniform test collection over Ω.*

Proof. Suppose that $T \in \mathcal{B}(H)$, $\sigma(T) \subset \Omega$ and $\psi_k(T)$ are contractions for $k = 1, \dots, n$. Let Φ be the admissible function obtained from Ψ using Lemma 3.30. Put $C_k = \varphi_k(T)$, $k = 1, \dots, m$. Check that C_k is a contraction for all k.

Since $\varphi_k = (h_{1,k} \circ \psi_1) \cdot \ldots \cdot (h_{n,k} \circ \psi_n)$,

$$C_k = \varphi_k(T) = h_{1,k}(\psi_1(T)) \cdot \cdots \cdot h_{n,k}(\psi_n(T)).$$

Each $\psi_j(T)$ is a contraction, and $\|h_{j,k}\|_{A(\overline{\mathbb{D}})} \leq 1$, so $h_{j,k}(\varphi_j(T))$ is also a contraction for all j and k. It follows that, being a product of contractions, $\psi_k(T)$ is a contraction.

Because $\sigma(T) \subset \Omega$, the hypotheses of Theorem 3.7 are satisfied by the Cauchy-Riesz functional calculus for T. Therefore, Lemma 3.28 applies, and so $\overline{\Omega}$ is a complete K-spectral set for T with $K = K(\Omega, \Psi, S_\Lambda(T))$, for an arbitrary pole set Λ for Ω. In other words, Ψ is a quasi-uniform test collection over Ω.

Now assume that Ψ_0 is as in the statement of the theorem. Form an admissible function Φ_0 from Φ by adding to Φ all the functions in Ψ_0 together with their associated arcs J_k which are defined to be the empty set. Then Φ_0 is injective and Φ_0' does not vanish, because Ψ_0 already had these properties. Therefore, Φ_0 is a uniform strong test collection over Ω by Corollary 3.8. If $\psi(T)$ is a contraction for every $\psi \in \Psi_0$, then $\varphi(T)$ is also a contraction for every $\varphi \in \Phi_0$. Hence, Ψ_0 is a uniform test collection over Ω. □

Unfortunately, the methods of the above proof cannot be used to show that Ψ is a strong test collection over Ω. If the operators $\psi_k(T)$, $k = 1, \ldots, n$, are contractions, then it follows that $\varphi_k(T)$, $k = 1, \ldots, m$, is a product of contractions and therefore a contraction. However, if $\psi_k(T)$, $k = 1, \ldots, n$, just have $\overline{\mathbb{D}}$ as a complete K-spectral set for some K, then we only get that $\varphi_k(T)$ is a product of operators which have $\overline{\mathbb{D}}$ as a complete K-spectral set. In general, it is false that an operator which is the product of two commuting operators which are similar to a contraction is itself similar to a contraction [Pis98]. Therefore, one cannot prove by this method that $\varphi_k(T)$ has $\overline{\mathbb{D}}$ as a complete K'-spectral set for some K'. Indeed, we do not know whether it is true, under the hypotheses of the theorem, that Ψ is a strong test collection over Ω.

Corollary 3.32. *Let Ω be a finitely connected domain with analytic boundary and let $\psi_1, \ldots, \psi_n : \overline{\Omega} \to \overline{\mathbb{D}}$ be inner (i.e., $|\psi_j| = 1$ in $\partial\Omega$ for $j = 1, \ldots, n$). Assume that the map $\Psi = (\psi_1, \ldots, \psi_n) : \overline{\Omega} \to \overline{\mathbb{D}}^n$ is injective. Then Ψ is a quasi-uniform test-collection over Ω. If, moreover, Ψ' does not vanish in Ω, then Ψ is a uniform test collection over Ω.*

Proof. Since $\psi_1, \ldots, \psi_n$ are inner, $\Psi_\zeta^- = \Psi_\zeta^+ = \Psi$ for all $\zeta \in \partial\Omega$. Therefore, the conditions (f′) and (g′) in the definition of a weakly admissible are equivalent to the condition that $\Psi|\partial\Omega$ is injective. Since $\psi_1, \ldots, \psi_n$ are inner and $\partial\Omega$ is analytic, $\psi_1, \ldots, \psi_n$ can be extended analytically across $\partial\Omega$. Hence, Ψ is a weakly admissible function. To finish the proof, apply Theorem 3.31 with $\Psi_0 = \Psi$. □

On a general finitely connected domain Ω with analytic boundary, one can always choose three inner functions ψ_1, ψ_2, ψ_3 such that the map $\Psi = (\psi_1, \psi_2, \psi_3) : \overline{\Omega} \to \overline{\mathbb{D}}^3$ is injective and Ψ' does not vanish in Ω. Hence, such Ψ is a uniform test collection according to Corollary 3.32. See [Sto66, Theorem IV.1] and [Fed90, §3] for two different proofs of the existence of such a Ψ. It is also known that when Ω is doubly connected then the same can be done using only two inner functions ψ_1, ψ_2. However, for a domain Ω of connectivity greater or equal than 3, a pair of inner functions ψ_1, ψ_2 will never be enough under the constraint that Ψ is injective (see [Rud69, Fed90]).

3.8. Blaschke products and von Neumann's inequality

The contents of this section are new. They were not included in [DEY15] and their first appearance is in this thesis. Here we explore the relation between von Neumann's inequality and tuples of contractions $(T_1, \dots, T_n)$ which are of the form $T_j = B_j(T)$, where B_j are finite Blaschke products.

We fix $n \in \mathbb{N}$ and denote by $\mathscr{B}$ the set of all tuples $\Phi = (\varphi_1, \dots, \varphi_n)$ such that φ_j is a finite Blaschke product for $j = 1, \dots, n$ and Φ is injective in $\overline{\mathbb{D}}$ and Φ' does not vanish on $\mathbb{D}$. Note that if $\Phi \in \mathscr{B}$ then $\Phi(\mathbb{D})$ is a distinguished variety in the polydisk $\mathbb{D}^n$ (this means that the variety only touches the boundary of the polydisk at the distinguished boundary, which is the set $\mathbb{T}^n$).

Recall that if $p \in \mathbb{C}[z_1, \dots, z_n]$, the norm of p in the Agler algebra is defined as

$$\|p\|_{\mathcal{SA}(\mathbb{D}^n)} = \sup\{\|p(T_1, \dots, T_n)\| : (T_1, \dots, T_n) \text{ commuting contractions}\}.$$

If $\Phi = (\varphi_1, \dots, \varphi_n)$ and T is an operator, we denote by $\Phi(T)$ the tuple of commuting operators $(\varphi_1(T), \dots, \varphi_n(T))$. The main result of this section is the following theorem.

Theorem 3.33. *Suppose that $n \geq 3$ and $p \in \mathbb{C}[z_1, \dots, z_n]$. Then*

$$\begin{aligned}\|p\|_{\mathcal{SA}(\mathbb{D}^n)} = \sup\{\|p(\Phi(T))\| :&\Phi = (\varphi_1, \dots, \varphi_n) \in \mathscr{B},\ k \geq 1,\ T \in \mathcal{B}(\mathbb{C}^k),\\ &\sigma(T) \subset \mathbb{D},\ \|\varphi_j(T)\| \leq 1\}.\end{aligned}$$

Moreover, in the supremum above it is also possible to include the condition that all the eigenvalues of T are different.

In other words, when computing $\|p\|_{\mathcal{SA}(\mathbb{D}^n)}$, it is enough to look at tuples of commuting contractions $(T_1, \dots, T_n)$ which are of the form $\Phi(T)$, for $\Phi \in \mathscr{B}$ and T a matrix with $\sigma(T) \subset \mathbb{D}$.

One of the tools used in the proof of this theorem is the following lemma, which is a restatement of a theorem of Agler, McCarthy and Young in [AMY13].

Lemma 3.34 ([AMY13, Theorem 6.1])**.** *Let $p \in \mathbb{C}[z_1, \dots, z_n]$. Then*

$$\|p\|_{\mathcal{SA}(\mathbb{D}^n)} = \sup_{(T_1, \dots, T_n) \in \Delta} \|p(T_1, \dots, T_n)\|,$$

where Δ is the collection of all tuples of commuting contractive matrices $(T_1, \dots, T_n)$ which are of the form

$$T_j = V \begin{pmatrix} \lambda_{1,j} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_{r,j} \end{pmatrix} V^{-1}, \tag{3.17}$$

where $\lambda_{k,j}$ are points in $\mathbb{D}$ which are all distinct and V is an invertible $r \times r$ matrix.

In other words, to compute $\|p\|_{\mathcal{SA}(\mathbb{D}^n)}$ it is enough to look at diagonalizable matrices.

Note that since $\|p(T_1, \dots, T_n)\|$ depends continuously on $(T_1, \dots, T_n)$, to study the norm $\|p\|_{\mathcal{SA}(\mathbb{D}^n)}$ it is enought to look at a set of commuting contractions which is dense in the set of all commuting contractions. The following question appears naturally when trying to prove Lemma 3.34: is Δ dense in the set of all commuting contractive matrices? This question is equivalent to the following: can every tuple of commuting matrices be perturbed to a tuple of commuting diagonalizable matrices?

The answer to this question is "no" in general. Every pair of commuting matrices can be perturbed to a pair of commuting diagonalizable matrices (see [MT55, Theorem 5]). For triples of matrices, if the matrices are of small enough size then it is possible to perturb them to commuting diagonalizable matrices. However, there are counterexamples with matrices of large size that show that it is not possible to do this in general. More precisely, if we consider triples of $s \times s$ commuting matrices, then perturbation is possible for $s \leq 11$ and not possible in general if $s \geq 29$. For tuples of four matrices, there are already counterexamples of size 4×4 in which perturbation is not possible. We refer the reader to the article [HO15] and references therein.

The technique of perturbing a tuple of commuting contractions to a commuting diagonalizable tuple has being used before by Lotto and Steger to give a diagonalizable counterexample of von Neumann's inequality in [LS94]. As follows from the above-mentioned results, these techniques cannot be applied to arbitrary matrices.

However, it is possible to prove Lemma 3.34 using other means. The proof that Agler, McCarthy and Young give in [AMY13] is based on Agler's theory of realizations.

Once we know that we can restrict our study of von Neumann's inequality to diagonalizable matrices, we can assume that the operators $T_1, \dots, T_n$ are as in (3.17). We will use Pick's interpolation theorem to try to find a matrix T and Blaschke products φ_j such that $T_j = \varphi_j(T)$. We recall the statement of Pick's theorem.

Theorem 3.35 (Pick's interpoation theorem)**.** *Let $w_1, \dots, w_r, \lambda_1, \dots, \lambda_r$ be points in $\mathbb{D}$. There is an analytic function $\varphi : \mathbb{D} \to \mathbb{D}$ such that $\varphi(w_j) = \lambda_j$ for all j if and only if the Pick matrix*

$$\left(\frac{1 - \overline{\lambda}_k \lambda_j}{1 - \overline{w}_k w_j} \right)_{j,k=1}^{r}$$

is positive semi-definite. If such a φ exists, it is possible to choose it so that it is a finite Blaschke product.

Put $w_k = s\omega^k$, $k = 0, \dots, r-1$, where $s < 1$ is close to 1 and $\omega = e^{\frac{2\pi i}{r}}$. Then it is easy to see that the Pick interpolation problems

$$\varphi_j(w_k) = \lambda_{k,j} \tag{3.18}$$

have a solution, since the correspoding Pick matrices are strictly positive definite. Indeed, the terms on the diagonal of the Pick matrices are positive and very large and the off-diagonal terms are controlled. Therefore, we can choose finite Blaschke products φ_j satisfying (3.18).

The only problem is that it is possible that the map $\Phi = (\varphi_1, \dots, \varphi_n)$ is not injective on $\overline{\mathbb{D}}$ or its derivative vanishes on some points of $\mathbb{D}$. We will show that it is possible to do a small perturbation to the points $\lambda_{k,j}$ such that the finite Blaschke products φ_j that we get from solving (3.18) satisfy these two required conditions (namely that Φ is injective on $\overline{\mathbb{D}}$ and Φ' does not vanish on $\mathbb{D}$).

Clearly it is enough to solve this problem for $n = 3$, because if $(\varphi_1, \varphi_2, \varphi_3)$ is already injective on $\overline{\mathbb{D}}$ and its derivative does not vanish on $\mathbb{D}$, then the remaining finite Blaschke products $\varphi_4, \dots, \varphi_n$ can be chosen in any way so that they satisfy (3.18) and then Φ will still be injective on $\overline{\mathbb{D}}$ and its derivative will not vanish on $\mathbb{D}$.

Lemma 3.36. *By doing a small perturbation of the points $\lambda_{k,j}$, $j = 1, 2$, it is possible to obtain finite Blaschke products φ_1, φ_2 which solve* (3.18) *and such that the set*

$$G = \{z \in \overline{\mathbb{D}} : \exists w \in \overline{\mathbb{D}},\ w \neq z : \varphi_j(z) = \varphi_j(w),\ j = 1, 2\}$$

is finite.

Proof. Let B_1, B_2 be finite Blaschke products such that $B_j(w_k) = \lambda_{k,j}$. We define the set of *singular points* of B_j as the set of points $z \in \overline{\mathbb{D}}$ such that there is some $w \in \mathbb{D}$ such that $B_j'(w) = 0$ and $B_j(z) = B_j(w)$. The set of singular points of B_j is finite. Recall that the derivative of a Blaschke product does not vanish on $\mathbb{T}$, so the singular points belong to $\mathbb{D}$.

First we want to replace B_2 by $B_2 \circ \psi$, where $\psi \in \operatorname{Aut}(\mathbb{D})$ is a suitable disk automorphism which is close to the identity and such that the sets of singular points of B_1 and $B_2 \circ \psi$ are disjoint. To do this, note that the disk automorphism ψ depends on three real parameters. Indeed $\operatorname{Aut}(\mathbb{D})$ is a real manifold of dimension 3. We will use this remark in several steps of the proof. Let z_0 be a singular point of B_1. This point z_0 is also a singular point of $B_2 \circ \psi$ if and only if $\psi^{-1}(z_0)$ is a singular point of B_2. Since a disk automorphism that takes a fixed point $a \in \mathbb{D}$ to a fixed point $b \in \mathbb{D}$ is uniquely determined up to a rotation, the set of automorphisms ψ such that B_1 and $B_2 \circ \psi$ have common singular points is a finite union of one-dimensional submanifolds of $\operatorname{Aut}(\mathbb{D})$. Therefore, it is possible to choose ψ arbitrarily close to the identity so that B_1 and $B_2 \circ \psi$ have no singular points in common. We rename $B_2 \circ \psi$ as B_2 and assume henceforth that B_1 and B_2 have no singular points in common.

Now, given $z_0 \in \overline{\mathbb{D}}$, we assign it to B_1 or B_2 according to the following rule: if z_0 is not a singular point of B_1, we assign it to B_1; otherwise, we assign it to B_2 (and in this case it cannot be a singular point of B_2, because B_1 and B_2 have no singular points in common).

Let $z_0 \in \overline{\mathbb{D}}$, and assume it is assigned to B_j according to the rule above. We can choose a disk $B(z_0, r_{z_0})$ of centre $z_0 \in \mathbb{C}$ and radius r_{z_0}, and analytic functions $\eta_1, \dots, \eta_k$ in $B(z_0, r_{z_0})$ such that they are biholomorphic, $\eta_1(z_0), \dots, \eta_k(z_0)$ are all the distinct solutions of the $B_j(\zeta) = B_j(z_0)$, and

$$B_j(\eta_k(z)) = B_j(z), \qquad z \in B(z_0, r_{z_0}). \tag{3.19}$$

By compactness we pass to a finite subcollection of points $z_1, \dots, z_s$ such that the discs $B(z_j, \frac{1}{2} r_{z_j})$ cover all $\overline{\mathbb{D}}$. We separate this finite subcollection of points into two groups according to their assignment to B_1 or to B_2. We denote by $\zeta_1, \dots, \zeta_l$ the points z_j which are assigned to B_1 and by $\omega_1, \dots, \omega_t$ the points z_j which are assigned to B_2.

We want to choose $\psi_1, \psi_2 \in \operatorname{Aut}(\mathbb{D})$ which are close to the identity and such that:

(a) For each point ζ_j, with associated functions $\eta_1, \dots, \eta_r$ as in (3.19),

$$(B_2 \circ \psi_2)(\zeta_j) \neq (B_2 \circ \psi_2 \circ \psi_1^{-1} \circ \eta_k \circ \psi_1)(\zeta_j).$$

(b) For each point ζ_j, with associated functions $\eta_1, \dots, \eta_r$ as in (3.19),

$$(B_1 \circ \psi_1)(\omega_j) \neq (B_1 \circ \psi_1 \circ \psi_2^{-1} \circ \eta_k \circ \psi_2)(\omega_j).$$

The set of automorphisms (ψ_1, ψ_2) that fail to satisfy the conditions above is a finite union of submanifolds of $\mathrm{Aut}(\mathbb{D}) \times \mathrm{Aut}(\mathbb{D})$ of dimension 5 or less. Therefore, it is possible to choose ψ_1, ψ_2 close to the identity satisfying the required properties.

Let us now show that in each disk $B(z_j, \frac{1}{2}r_{z_j})$ there is only a finite number of points ξ such that

$$(B_1 \circ \psi_1, B_2 \circ \psi_2)(\xi) = (B_1 \circ \psi_1, B_2 \circ \psi_2)(w) \tag{3.20}$$

for some $w \neq \xi$. We assume without loss of generality that $z_j = \zeta_j$ (i.e., that z_j is assigned to B_1).If $B_1(\psi_1(z)) = B_1(\psi_1(w))$, then $\psi_1(w) = \eta_k(\psi_1(z))$ for some k. We consider the functions

$$z \mapsto (B_2 \circ \psi_2)(z) - (B_2 \circ \psi_2 \circ \psi_1^{-1} \circ \eta_k \circ \psi_1)(z), \qquad z \in B(\zeta_j, \frac{1}{2}r_{\zeta_j}). \tag{3.21}$$

These functions do not vanish at $z = \zeta_j$, so they have a finite number of zeros inside $B(\zeta_j, \frac{1}{2}r_{\zeta_j})$. Note that (3.20) implies that $\psi_1(w) = \eta_k(\psi_1)(z)$ for some k. Hence ζ is a zero of the function defined by (3.21). This shows that (3.20) can only happen for a finite collection of points $\zeta \in B(\zeta_j, \frac{1}{2}r_{z_j})$. We put $\varphi_j = B_j \circ \psi_j$. The finiteness of the set G follows from the fact that the discs $B(z_j, \frac{1}{2}r_{z_j})$ cover all $\overline{\mathbb{D}}$. □

Lemma 3.37. *Assume that φ_1, φ_2 are as in the statement of Lemma 3.36. By doing a small perturbation of the points $\lambda_{j,3}$, it is possible to find a finite Blaschke product φ_3 which solves* (3.18) *and such that $(\varphi_1, \varphi_2, \varphi_3)$ is injective in $\overline{\mathbb{D}}$ and its derivative does not vanish on $\mathbb{D}$.*

Proof. Let B_3 be a finite Blashcke product which solves the Pick problem $B_3(w_j) = \lambda_{j,3}$. We denote by $g_1, \dots, g_r$ the points of G and by $h_1, \dots, h_s$ the points where (φ_1', φ_2') vanishes.

We want to find $\psi \in \mathrm{Aut}(\mathbb{D})$ a suitable disk automorphism which is close to the identity and so that $(\varphi_1, \varphi_2, B_3 \circ \psi)$ is injective in $\overline{\mathbb{D}}$ and its derivative does not vanish on $\mathbb{D}$. We require that

$$\begin{aligned} &B_3(\psi(g_j)) \neq B_3(\psi(g_k)), \\ &B_3'(\psi(h_j)) \neq 0. \end{aligned}$$

The set of $\psi \in \mathrm{Aut}(\mathbb{D})$ which do not satisfy these conditions is a finite union of submanifolds of dimension 2 or less in $\mathrm{Aut}(\mathbb{D})$ (recall that $\mathrm{Aut}(\mathbb{D})$ is a real manifold of dimension 3). Therefore, it is possible to choose ψ close to the identity satisfying the required conditions.

We put $\varphi_3 = B_3 \circ \psi$. □

Proof of Theorem 3.33. Let $p \in \mathbb{C}[z_1, \dots, z_n]$. By Lemma 3.34,

$$\|p\|_{\mathcal{SA}(\mathbb{D}^n)} = \sup_{(T_1,\dots,T_n)\in\Delta} \|p(T_1, \dots, T_n)\|.$$

We now show that if $(T_1, \dots, T_n) \in \Delta$, then there is some $\Phi \in \mathscr{B}$ and some matrix T with $\sigma(T) \subset \mathbb{D}$ and such that all its eigenvalues are different and $\Phi(T)$ is arbitrarily close to $(T_1, \dots, T_n)$. This is enough to prove the theorem.

We write T_j as in (3.17) and consider the Pick interpolation problems (3.18). By Lemmas 3.36 and 3.37, doing a small perturbation of the points $\lambda_{k,j}$ $j = 1, 2, 3$, we can choose finite Blaschke products $\varphi_1, \dots, \varphi_n$ which solve these problems and such that if $\Phi = (\varphi_1, \dots, \varphi_n)$, then Φ is injective in $\overline{\mathbb{D}}$ and Φ' does not vanish on $\mathbb{D}$. Hence, $\Phi \in \mathscr{B}$. We put

$$T = V \begin{pmatrix} w_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & w_n \end{pmatrix} V^{-1}.$$

Here w_j are chosen as above: $w_j = s\omega$, where $s < 1$ is close to 1 and $\omega = e^{\frac{2\pi i}{r}}$. Then $\sigma(T) \subset \mathbb{D}$ and $\Phi(T)$ is close to $(T_1, \dots, T_n)$. $\square$

Let us now give the relation between Theorem 3.33 and our results about test collections. Fix $\Phi = (\varphi_1, \dots, \varphi_n) \in \mathscr{B}$ and take any matrix T such that $\sigma(T) \subset \mathbb{D}$ and $\varphi_j(T)$ are contractions. Any $\Phi \in \mathscr{B}$ satisfies the hypothesis of Corollary 3.32. Applying this corollary, we obtain that $\overline{\mathbb{D}}$ is a complete K_Φ-set for T, where K_Φ is a constant that depends on Φ but not on T. Hence,

$$\|p(\Phi(T))\| = \|(p \circ \Phi)(T)\| \le K_\Phi \|p \circ \Phi\|_{H^\infty(\mathbb{D})} \le K_\Phi \|p\|_{H^\infty(\mathbb{D}^n)}.$$

This shows how complete K-spectral sets can be used to study von Neumann's inequality. In particular, we can take K_Φ above to be the optimal constant of K-spectrality: the infimum of the constants K such that $\overline{\mathbb{D}}$ is a complete K-spectral set for every T such that $\sigma(T) \subset \mathbb{D}$ and $\varphi_j(T)$ are contractions. According to Theorem 3.33, we see that

$$\|p\|_{\mathcal{SA}(\mathbb{D}^n)} \le \|p\|_{H^\infty(\mathbb{D})} \sup_{\Phi \in \mathscr{B}} K_\Phi.$$

We do not know whether the supremum of K_Φ is finite or not, because the constant that one would get from the proof of Corollary 3.32 is far from being optimal.

Now we will give a relation between von Neumann's inequality and the extension of functions on analytic curves to functions in the Agler algebra of $\mathbb{D}^n$. Recall that Theorem 2.4 is a result of this type. We fix n and define

$$C_{\mathrm{VN}} = \sup\{\|p\|_{\mathcal{SA}(\mathbb{D}^n)} : p \in \mathbb{C}[z_1, \dots, z_n],\ \|p\|_{H^\infty(\mathbb{D}^n)} \le 1\}.$$

Note that C_{VN} is the best contant with which von Neumann's inequality holds for tuples of n commuting contractions. If von Neumann's inequality does not hold with a constant, then $C_{\mathrm{VN}} = +\infty$.

If $\mathcal{V}$ is an analytic variety inside $\mathbb{D}^n$, we put

$$C(\mathcal{V}) = \sup\{\|f\|_{\mathcal{SA}(\mathbb{D}^n)} : \|g\|_{H^\infty(\mathbb{D}^n)} \le 1,\ f|\mathcal{V} = g|\mathcal{V}\}.$$

In other words, $C(\mathcal{V})$ is the optimal constant C such that given $g \in H^\infty(\mathbb{D}^n)$ we can find an $f \in \mathcal{SA}(\mathbb{D}^n)$ which coincides with g on $\mathcal{V}$ and such that $\|f\|_{\mathcal{SA}(\mathbb{D}^n)} \le C\|g\|_{H^\infty(\mathbb{D}^n)}$. Clearly,

$$C(\mathcal{V}) \le C_{\mathrm{VN}},$$

because $\|f\|_{\mathcal{SA}(\mathbb{D}^n)} \leq C_{\mathrm{VN}}\|f\|_{H^\infty(\mathbb{D}^n)}$ for every $f \in H^\infty(\mathbb{D}^n)$ (if we understand that $\|f\|_{\mathcal{SA}(\mathbb{D}^n)} = +\infty$ when $f \in H^\infty(\mathbb{D}^n) \setminus \mathcal{SA}(\mathbb{D}^n)$).

Now assume that $\Phi = (\varphi_1, \dots, \varphi_n) \in \mathscr{B}$ and T is a matrix such that $\sigma(T) \subset \mathbb{D}$ and $\varphi_j(T)$ are contractions. If $p \in \mathbb{C}[z_1, \dots, z_n]$, given $\varepsilon > 0$ we can find $f \in \mathcal{SA}(\mathbb{D}^n)$ such that $p = f$ on $\mathcal{V} = \Phi(\mathbb{D})$ and $\|f\|_{\mathcal{SA}(\mathbb{D}^n)} \leq (C(\mathcal{V}) + \varepsilon)\|p\|_{H^\infty(\mathbb{D}^n)}$. Hence,

$$\|p(\Phi(T))\| = \|f(\Phi(T))\| \leq \|f\|_{\mathcal{SA}(\mathbb{D}^n)} \leq (C(\mathcal{V}) + \varepsilon)\|p\|_{H^\infty(\mathbb{D}^n)}.$$

By taking the supremum over $\Phi \in \mathscr{B}$ and $p \in \mathbb{C}[z_1, \dots, z_n]$ we obtain

$$C_{\mathrm{VN}} \leq \sup_{\Phi \in \mathscr{B}} C(\Phi(\mathbb{D})).$$

This proves the following result.

Proposition 3.38.

$$C_{VN} = \sup_{\Phi \in \mathscr{B}} C(\Phi(\mathbb{D}))$$

Note that if $\Phi \in \mathscr{B}$, then $\Phi(\mathbb{D})$ is a distinguished variety without singular points (recall that a distinguished variety $\mathcal{V}$ is a variety which touches the boundary of $\mathbb{D}^n$ only at the distinguished boundary, which is the set $\mathbb{T}^n$). Therefore, this proposition gives a relation between von Neumann's inequality and the extension of analytic functions on distinguished varieties to functions on the Agler algebra.

4. An application: operators with thin spectrum

This chapter is based on joint work with Michael Dritschel and Dmitry Yakubovich. The results of this chapter are contained in an article which is in preparation.

4.1. Introduction

In this chapter we apply some of the results of Chapter 3 to the study of operators with thin spectrum. This means that the spectrum is contained inside a smooth curve. We study conditions for similarity to a normal operator involving resolvent growth estimates.

In [Sta69], Stampfli proved that if $\Gamma \subset \mathbb{C}$ is a smooth curve, $T \in \mathcal{B}(H)$ is a bounded operator in a Hilbert space with $\sigma(T) \subset \Gamma$, and U is a neighbourhood of Γ such that $\|(T-\lambda)^{-1}\| \leq \operatorname{dist}(\lambda,\Gamma)^{-1}$ for all $\lambda \in U \setminus \Gamma$, then T is a normal operator. Theorems of this type were first proved by Nieminen [Nie62] for the case $\Gamma = \mathbb{R}$ and by Donoghue [Don63] for the case when Γ is a circle. If Γ is not smooth, a result of this kind might no longer be true. See, for instance, the counterexample in [Sta65]. Many other conditions for an operator to be normal have been studied in the literature. See [Ber69, Ber70a, Ber70b] for a series of articles by Berberian about this topic. In [CG76, CG77], Cambell and Gelar study operators T for which T^*T and $T+T^*$ commute, showing, for instance, that if $\sigma(T)$ is a subset of a vertical line or $\mathbb{R}$, then T is normal. Djordjević gives in [Djo07] several conditions for an operator to be normal using the Moore-Penrose inverse. Gheondea studies operators which are the product of two normal operators in [Ghe09]. See also [MNS11] and references therein.

One of the results of this chapter is the following theorem, which is in some sense related to Stampfli's result.

Theorem 4.1. *Let $\Gamma \subset \mathbb{C}$ be a $C^{1+\alpha}$ Jordan curve, and Ω the domain it bounds. Let $T \in \mathcal{B}(H)$ an operator with $\sigma(T) \subset \Gamma$. Assume that*

$$\|(T-\lambda)^{-1}\| \leq \frac{1}{\operatorname{dist}(\lambda,\Gamma)}, \qquad \lambda \in U \setminus \overline{\Omega},$$

for some open set U containing $\partial\Omega$, and

$$\|(T-\lambda)^{-1}\| \leq \frac{C}{\operatorname{dist}(\lambda,\Gamma)}, \qquad \lambda \in \Omega,$$

for some constant $C > 0$. Then T is similar to a normal operator, i.e., $T = SNS^{-1}$, for some normal operator N and some invertible operator S.

In this theorem, we assume that a resolvent estimate with constant 1 is satisfied outside Ω and an estimate with constant C is satisfied inside Ω. Replacing T by $R = (T - z_0)^{-1}$, for some fixed $z_0 \in \Omega$, we obtain an analogous result where the estimate with constant 1 is assumed inside the domain and the estimate with constant C is assumed outside the domain. In fact, it follows from Lemma 3.18 (page 51) that if $\|(T - \lambda)^{-1}\| \le \operatorname{dist}(\lambda, \Gamma)^{-1}$, then $\|(R - \mu)^{-1}\| \le \operatorname{dist}(\mu, \widetilde{\Gamma})^{-1}$, where $\mu = (\lambda - z_0)^{-1}$ and $\widetilde{\Gamma}$ is the image of Γ under the map $z \mapsto (z - z_0)^{-1}$. Writing the resolvent of R in terms of the resolvent of T, it is also easy to obtain an estimate for R with a constant $C' > 1$ outside the domain bounded by $\widetilde{\Gamma}$.

The proof of Theorem 4.1 will use Theorem 3.3 (page 40). In fact, it is an easy corollary of this theorem and Lemma 4.6, which is stated below. In Section 4.6 we will give several examples related to Theorem 4.1.

It is worthy to note that there are operators T with $\sigma(T) \subset \Gamma$ which are not similar to a normal operator and satisfy $\|(T - \lambda)^{-1}\| \le C \operatorname{dist}(\lambda, \Gamma)^{-1}$ for all $\lambda \in \mathbb{C} \setminus \Gamma$. See the paper Markus [Mar64], and also the paper by Benamara and Nikolski [BN99, Section 3.2] for a general result in this direction. In a related article, Nikolski and Treil give a counterexample in [NT02] where T is a rank one perturbation of a unitary operator and $\sigma(T)$ is contained in the unit circle $\mathbb{T}$.

There are several works devoted to studying sufficient conditions for an operator to be similar to a unitary in terms of estimates of its resolvent. In [vC80], van Casteren proves the following theorem.

Theorem VC1. *Let $T \in \mathcal{B}(H)$ be an operator with $\sigma(T) \subset \mathbb{T}$. Assume that T satisfies the resolvent estimate*

$$\|(T - \lambda)^{-1}\| \le C(1 - |\lambda|)^{-1}, \qquad |\lambda| < 1$$

and

$$\|T^n\| \le C, \qquad n \ge 0.$$

Then T is similar to a unitary.

An operator satisfying the last condition in this theorem is said to be power bounded. Van Casteren improved this result in [vC83], giving the following theorem.

Theorem VC2. *Let $T \in \mathcal{B}(H)$ be an operator with $\sigma(T) \subset \mathbb{T}$. Assume that T satisfies the resolvent estimate*

$$\|(T - \lambda)^{-1}\| \le C(1 - |\lambda|)^{-1}, \qquad |\lambda| < 1$$

and

$$\int_{|\lambda|=r} \|(T - \lambda)^{-1}x\|^2 \, |d\lambda| \le \frac{C\|x\|^2}{r - 1}, \qquad 1 < r < 2, \ x \in H,$$

$$\int_{|\lambda|=r} \|(T^* - \lambda)^{-1}x\|^2 \, |d\lambda| \le \frac{C\|x\|^2}{r - 1}, \qquad 1 < r < 2, \ x \in H.$$

Then T is similar to a unitary.

By writing the power series for the resolvent, one can check that every power bounded operator satisfies the last two conditions in this theorem.

The following related result was proved independently by Naboko in [Nab84].

Theorem N. *Let $T \in \mathcal{B}(H)$ be an operator with $\sigma(T) \subset \mathbb{T}$. Assume that T satisfies the resolvent conditions*

$$\int_{|\lambda|=r} \|(T-\lambda)^{-1}x\|^2 \, |d\lambda| \leq \frac{C\|x\|^2}{r-1}, \qquad 1 < r < 2, \; x \in H,$$

and

$$\int_{|\lambda|=r} \|(T^*-\lambda)^{-1}x\|^2 \, |d\lambda| \leq \frac{C\|x\|^2}{1-r}, \qquad r < 1, \; x \in H.$$

Then T is similar to a unitary.

The conditions in Theorems VC2 and N are not comparable, so none of these two theorems is stronger than the other one.

We remark that in the theorems above it is possible to replace T by T^*, T^{-1} or T^{*-1}, thus obtaining another set of sufficient conditions. For related results and conditions, we refer the reader to [Nik02, Section 1.5.6].

In this chapter we only concern ourselves with operators having *thin* spectrum, in other words, we always assume that the spectrum of T lies on some curve in $\mathbb{C}$. For operators having a *thick* spectrum, in general there is no hope of obtaining criteria for similarity to a normal operator solely in terms of estimates of its resolvent operator. As an example, one can consider the unilateral shift S, whose spectrum is $\overline{\mathbb{D}}$, and whose resolvent satisfies $\|(S-\lambda)^{-1}\| = (|\lambda|-1)^{-1}$, for all $|\lambda| > 1$.

However, some extra conditions can be imposed. In [BN99], Benamara and Nikolski show that a contraction T with finite defects is similar to a normal operator if and only if $\sigma(T) \neq \overline{\mathbb{D}}$ and $\|(T-\lambda)^{-1}\| \leq C \operatorname{dist}(\lambda, \sigma(T))^{-1}$ for all $\lambda \in \mathbb{C} \setminus \sigma(T)$. Note that for such a contraction $\sigma(T) \cap \mathbb{D}$ is always a Blaschke sequence, so indeed $\sigma(T)$ is also thin in some sense. In fact, Benamara and Nikolski prove that the resolvent estimate forces $\sigma(T) \cap \mathbb{D}$ to be quite sparse (more precisely, it satisfies the Δ-Carleson condition). Kupin studied in [Kup01] contractions with infinite defects. He proves that if the spectrum of a contraction T is not all $\overline{\mathbb{D}}$ and it satisfies $\|(T-\lambda)^{-1}\| \leq C \operatorname{dist}(\lambda, \sigma(T))^{-1}$ for all $\lambda \in \sigma(T)$ and the so called Uniform Trace Boundedness condition, then it is similar to a normal operator. This Uniform Trace Boundedness condition also appears in [VK01] as a condition for a dissipative integral operator to be similar to a normal operator. Kupin also uses the Uniform Trace Boundedness condition in [Kup03] to give conditions for an operator similar to a contraction to be similar to a normal operator.

On the other hand, Kupin and Treil show in [KT01] that if T is a contraction with $\sigma(T) \neq \overline{\mathbb{D}}$ and $\|(T-\lambda)^{-1}\| \leq C \operatorname{dist}(\lambda, \sigma(T))^{-1}$ but one only assumes that $I - T^*T$ is trace class (instead of finite rank), then T need not be similar to a normal operator. This solves a conjecture in [BN99].

Resolvent conditions for similarity to other classes of operators, such as selfadjoint operators or isometries, have also been studied in the literature. Fadeev gives conditions in [Fad89] for similarity to an isometry in the case when $\dim \ker(T^* - \lambda I) = 1$ for all $\lambda \in \mathbb{D}$.

Faddeev and Shterenberg use in [FS02] a version of Theorem N above for selfadjoint operators to study differential operators of the form $A = -\frac{\operatorname{sign} x}{|x|^\alpha p(x)} \frac{d^2}{dx^2}$, where p is a positive function which is bounded above and below. Naboko and Tretter also use Theorem N in their article [NT98]. They give conditions for similarity to a selfadjoint operator for Volterra perturbations of the operator of multiplication by x on $L^2[0,1]$. These are operators of the form

$$(Af)(x) = xf(x) + \int_0^x \varphi(x)\psi(s)f(s)\,ds,$$

where $\varphi\psi = 0$.

In [Mal01], Malamud gives necessary and sufficient conditions for a a triangular operator A on $L^2([0,1], d\mu)$ of the form

$$(Af)(x) = \alpha(x)f(x) + i\int_x^1 K(x,t)f(t)\,d\mu(t)$$

to be similar to a selfadjoint operator. His conditions involve resolvent estimates such as $\|V^{1/2}(A-\lambda)^{-1}\| \le C|\operatorname{Im}\lambda|^{-1/2}$, where $V = |\operatorname{Im} A|$.

4.2. Dynkin's functional calculus

A key tool in this chapter will be the functional calculus defined by Dynkin using the Cauchy-Green formula, which appeared in [Dyn72]. Before defining this calculus, we need to set down some definitions and notation.

Let $\Gamma \subset \mathbb{C}$ be a Jordan curve of class $C^{1+\alpha}$, $0 < \alpha < 1$. This means that Γ is the image of $\mathbb{T}$ under a bijective map $\psi : \mathbb{T} \to \Gamma$ such that $\psi \in C^1(\mathbb{T})$, ψ' does not vanish and ψ' is Hölder α, i.e,

$$|\psi'(z) - \psi'(w)| \le C|z-w|^\alpha, \qquad z, w \in \mathbb{T}.$$

A function $f : \Gamma \to \mathbb{C}$ is said to belong to $C^{1+\alpha}(\Gamma)$ if $f \circ \psi \in C^1(\mathbb{T})$ and $(f\circ\psi)'$ is Hölder α. The norm in $C^{1+\alpha}(\Gamma)$ is defined as

$$\|f\|_{C^{1+\alpha}(\Gamma)} = \|f\circ\psi\|_{C(\mathbb{T})} + \|(f\circ\psi)'\|_{C(\mathbb{T})} + \|(f\circ\psi)'\|_\alpha,$$

where

$$\|g\|_\alpha = \sup_{z,w\in\mathbb{T}, z\ne w} \frac{|g(z)-g(w)|}{|z-w|^\alpha}.$$

The definition of this norm depends on the choice of the parametrization ψ, but different choices yield equivalent norms.

Let $T \in \mathcal{B}(H)$ be an operator with $\sigma(T) \subset \Gamma$, where Γ is a Jordan curve of class $C^{1+\alpha}$. We assume that T satisfies the following resolvent growth condition:

$$\|(T-\lambda)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda,\Gamma)}, \qquad \lambda \in \mathbb{C}\setminus\Gamma. \tag{4.1}$$

Following Dynkin [Dyn72], we will now define a $C^{1+\alpha}(\Gamma)$ functional calculus for T. We remark that Dynkin defines his calculus also for other function algebras instead of

$C^{1+\alpha}$ and operators satisfying a resolvent estimate different from (4.1). However, we will only treat the case that we need to use in the sequel.

First, let us recall the notion of pseudoanalytic extension. If $f \in C^{1+\alpha}(\Gamma)$, by [Dyn76, Theorem 2] there is a function $F \in C^1(\mathbb{C})$ such that $F|\Gamma = f$ and

$$\left|\frac{\partial F}{\partial \overline{z}}(z)\right| \leq C\|f\|_{C^{1+\alpha}(\Gamma)} \operatorname{dist}(z,\Gamma)^{\alpha}. \tag{4.2}$$

Here, $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ and C is a constant depending only on Γ. Every such function F which extends f and satisfies (4.2) is called a pseudoanlytic extension of f.

Dynkin uses the pseudoanalytic extension F to define the operator $f(T)$ by means of the Cauchy-Green integral formula. We define

$$f(T) = \frac{1}{2\pi i}\int_{\partial D} F(\lambda)(\lambda - T)^{-1}\, d\lambda - \frac{1}{\pi}\iint_D \frac{\partial F}{\partial \overline{z}}(\lambda)(\lambda - T)^{-1}\, dA(\lambda).$$

Here D is a domain with smooth boundary such that $\Gamma \subset D$. The estimate (4.2) for F and the resolvent estimate (4.1) for T show that the second integral is well defined. It is possible to prove that this definition does not depend on the particular choice of D or the pseudoanlytic extension F.

This calculus has the usual properties of a functional calculus: it is continuous from $C^{1+\alpha}(\Gamma)$ to $\mathcal{B}(H)$, is linear and multiplicative, and coincides with the natural definition of $f(T)$ if f is rational. It also satisfies the spectral mapping property: $\sigma(f(T)) = f(\sigma(T))$.

4.3. Passing from Γ to $\mathbb{T}$

In this section we explain how to use Dynkin's functional calculus to pass from an operator T with $\sigma(T) \subset \Gamma$ to an operator A with $\sigma(A) \subset \mathbb{T}$. The main result of this section is Theorem 4.3, which relates the estimates for the resolvents of T and A. In this way, resolvent growth conditions for T imply equivalent conditions for A and viceversa. Therefore, this result will be a key tool.

In the following Lemma, we prove some regularity conditions for a certain function $\eta : \Gamma \to \mathbb{T}$. This function η will be important, as we will then construct the operator A as $A = \eta(T)$ using Dynkin's calculus.

Lemma 4.2. *Let Γ be a Jordan curve of class $C^{1+\alpha}$ and $\eta \in C^{1+\alpha}(\Gamma)$ a function such that $\eta(\Gamma) = \mathbb{T}$ and $\eta^{-1} : \mathbb{T} \to \Gamma$ exists and is differentiable. Let us also denote by η its pseudoanalytic extension to $\mathbb{C}$. Then there is a neighbourhood U of Γ such that $\eta : U \to \eta(U)$ is a C^1 diffeomorphism, $\eta(U)$ is a neighbourhood of $\mathbb{T}$, and η is bi-Lipschitz in U, i.e.,*

$$c|z-w| \leq |\eta(z) - \eta(w)| \leq C|z-w|, \qquad z, w \in U.$$

Proof. Since $\partial\eta/\partial\overline{z} \equiv 0$ on Γ, the condition that $\eta^{-1} : \mathbb{T} \to \Gamma$ is differentiable implies that the differential of η is non-singular on Γ. Therefore, for each point $x \in \Gamma$, there is an open ball $B(x, r(x))$ of centre x and radius $r(x)$ such that η is bi-Lipschitz on $B(x, r(x))$. By a compactness argument, we see that η is Lipschitz on some neighbourhood of Γ.

Since $\eta|\Gamma$ is injective, we see that

$$\delta(\varepsilon) := \inf_{\substack{|x-y|\geq\varepsilon \\ x,y\in\Gamma}} |\eta(x)-\eta(y)| > 0$$

for every $\varepsilon > 0$. It follows that there is some function φ such that $\varphi(\varepsilon) > 0$ for all $\varepsilon > 0$ and

$$\widetilde{\delta}(\varepsilon) := \inf_{\substack{|x-y|\geq\varepsilon \\ \operatorname{dist}(x,\Gamma)\leq\varphi(\varepsilon),\ \operatorname{dist}(y,\Gamma)\leq\varphi(\varepsilon)}} |\eta(x)-\eta(y)| > 0$$

for every $\varepsilon > 0$.

We pass to a finite collection of centres $\{x_j\}$ on Γ such that the balls $B(x_j, r(x_j)/2)$ cover Γ and put $\varepsilon_0 = \min r(x_j)/2$. Let us check that η is bi-Lipschitz on the open set

$$W = \Big(\bigcup B\big(x_j, \frac{r(x_j)}{2}\big)\Big) \cap \{x \in \mathbb{C} : \operatorname{dist}(x,\Gamma) < \varphi(\varepsilon_0)\}.$$

Given points $x, y \in W$, then either $|x-y| < \varepsilon_0$, so that x, y both belong to the same ball $B(x_k, r(x_k)/2)$, where we already know that η is bi-Lipschitz, or $|x-y| \geq \varepsilon_0$. In this latter case,

$$|\eta(x)-\eta(y)| \geq \widetilde{\delta}(\varepsilon_0) \geq \widetilde{\delta}(\varepsilon_0)\varepsilon_0^{-1}|x-y|.$$

The injectivity of η follows from the bi-Lipschitz property, and the fact that we may choose $U \subset W$ so that η is a C^1 diffeomorphism of U is true because the differential of η is non-singular in some neighbourhood of Γ. Finally, $\eta(U)$ is a neighbourhood of $\mathbb{T}$ because $\eta(\Gamma) = \mathbb{T}$ and η is an open mapping, since it is bi-Lipschitz. $\square$

The next theorem relates the resolvent estimates for T and $\eta(T)$. We remark that, since η is bi-Lipschitz, $\operatorname{dist}(\lambda,\Gamma)$ and $\operatorname{dist}(\eta(\lambda),\mathbb{T}))$ are comparable.

Theorem 4.3. *Let Γ, η and U be as in Lemma 4.2 and $T \in \mathcal{B}(H)$ an operator satisfying the resolvent estimate* (4.1)*. Let the operator $\eta(T)$ be defined by the $C^{1+\alpha}$-calculus for T. Then $\sigma(\eta(T)) \subset \mathbb{T}$ and*

$$C^{-1}\|(T-\lambda)^{-1}x\| \leq \|(\eta(T)-\eta(\lambda))^{-1}x\| \leq C\|(T-\lambda)^{-1}x\|, \qquad \lambda \in U\setminus\Gamma,\ x \in H$$

where $C \geq 1$ is a constant depending on Γ, η, T but not on λ or x.

Proof. The fact that $\sigma(\eta(T)) \subset \mathbb{T}$ follows from the spectral mapping theorem for Dynkin's calculus.

For $\lambda \in U \setminus \Gamma$, we consider the functions

$$\varphi_\lambda(z) = \frac{\eta(z)-\eta(\lambda)}{z-\lambda}, \qquad \psi_\lambda(z) = \frac{z-\lambda}{\eta(z)-\eta(\lambda)}. \tag{4.3}$$

The functions $\varphi_\lambda, \psi_\lambda$ belong to $C^{1+\alpha}(\Gamma)$, so the operators $\varphi_\lambda(T)$ and $\psi_\lambda(T)$ are defined. In fact,

$$\varphi_\lambda(T) = (\eta(T)-\eta(\lambda))(T-\lambda)^{-1}, \qquad \psi_\lambda(T) = (T-\lambda)(\eta(T)-\eta(\lambda))^{-1}.$$

Thus, it is clear that it is enough to show that

$$\|\varphi_\lambda(T)\| \le C_0, \qquad \|\psi_\lambda(T)\| \le C_0,$$

for C_0 independent of λ.

First note that (4.3) defines the functions φ_λ and ψ_λ in $U \setminus \{\lambda\}$ and

$$|\varphi_\lambda(z)| \le C_1, \qquad |\psi_\lambda(z)| \le C_1, \qquad z \in U \setminus \{\lambda\},$$

since η is bi-Lipschitz. Let D be a domain with smooth boundary such that $\Gamma \subset D \subset \overline{D} \subset U$ and $\varepsilon > 0$ to be chosen later.

We first estimate $\|\varphi_\lambda(T)\|$. We have

$$\begin{aligned}\varphi_\lambda(T) =& \frac{1}{2\pi i}\int_{\partial D} \varphi_\lambda(z)(z-T)^{-1}\,dz - \frac{1}{2\pi i}\int_{\partial B(\lambda,\varepsilon)} \varphi_\lambda(z)(z-T)^{-1}\,dz \\ &- \frac{1}{\pi}\iint_{D\setminus B(\lambda,\varepsilon)} \frac{\partial\varphi_\lambda}{\partial\bar z}(z)(z-T)^{-1}\,dA(z).\end{aligned}$$

(Here we assume that $\lambda \in D$ and that ε is chosen small enough so that $\overline{B(\lambda,\varepsilon)} \subset D$. The case $\lambda \notin D$ is similar.)

Let us estimate these three terms separately. To estimate the second term, note that for $\varepsilon \le \operatorname{dist}(\lambda,\Gamma)$,

$$\int_{\partial B(\lambda,\varepsilon)} |\varphi_\lambda(z)|\|(z-T)^{-1}\|\,|dz| \le C_2\varepsilon(\operatorname{dist}(\lambda,\Gamma)-\varepsilon)^{-1}.$$

By letting $\varepsilon \to 0$, we see that we can completely ignore this term. The norm of the first term can be estimated by

$$\frac{1}{2\pi}\int_{\partial U} |\varphi_\lambda(z)|\|(z-T)^{-1}\|\,|dz| \le C_3\operatorname{dist}(\partial U,\Gamma)^{-1}\operatorname{length}(\partial U).$$

Now we estimate the third term, using again the fact that η is bi-Lipschitz:

$$\begin{aligned}\iint_{D\setminus B(\lambda,\varepsilon)} \left|\frac{\partial\varphi_\lambda}{\partial\bar z}(z)\right| \|(z-T)^{-1}\|\,dA(z) &\le \iint_D \frac{1}{|z-\lambda|}\left|\frac{\partial\eta}{\partial\bar z}(z)\right| \|(T-\lambda)^{-1}\|\,dA(z) \\ &\le C_4\iint_D |z-\lambda|^{-1}\operatorname{dist}(z,\Gamma)^{\alpha-1}\,dA(z) \\ &\le C_5\iint_D |\eta(z)-\eta(\lambda)|^{-1}\operatorname{dist}(\eta(z),\mathbb{T})^{\alpha-1}\,dA(z) \\ &\le C_6\iint_{\eta(D)} |\zeta-\eta(\lambda)|^{-1}\operatorname{dist}(\zeta,\mathbb{T})^{\alpha-1}\,dA(\zeta) \\ &\le C_6\int_{a\le|\zeta|\le b} |\zeta-\eta(\lambda)|^{-1}|1-|\zeta||^{\alpha-1}\,dA(\zeta).\end{aligned}$$

Here we have used the fact that $\operatorname{dist}(z,\Gamma)$ and $\operatorname{dist}(\eta(z),\mathbb{T})$ are comparable, which is true because η is bi-Lipschitz. Also, we have performed the change of variables $\zeta = \eta(z)$ and chosen $a < b$ such that the set $\eta(D)$ is contained in the annulus $a \le |\zeta| \le b$. By

Lemma 4.4 below, the last term in this chain of inequalities is smaller than a constant independent of λ. This shows that $\|\varphi_\lambda(T)\| \le C_0$, with C independent of λ.

The proof that $\|\psi_\lambda(T)\| \le C_0$ is very similar. Here one has to use that

$$\left|\frac{\partial \psi_\lambda}{\partial \bar{z}}(z)\right| = \frac{|z-\lambda|}{|\eta(z)-\eta(\lambda)|^2}\left|\frac{\partial \eta}{\partial \bar{z}}(z)\right| \le C_7|\eta(z)-\eta(\lambda)|^{-1}\left|\frac{\partial \eta}{\partial \bar{z}}(z)\right|.$$

The remaining estimates can then be done in the same way. □

Lemma 4.4. *Let $0 < a < 1 < b$ and $a \le |w| \le b$ and $\beta > -1$. Then*

$$\iint_{a\le|z|\le b} |z-w|^{-1}|1-|z||^\beta \, dA(z) \le C,$$

where C is independent of w.

Proof. Performing a rotation, we may assume that w is real and positive, so that $a \le w \le b$. By passing to polar coordinates and using the estimate

$$|re^{i\theta} - w|^{-1} \le C_0|r + i\theta - w|^{-1},$$

valid for $a \le r \le b$, we see that the integral we have to estimate is less than a constant times

$$\iint_{[a,b]\times[-\pi,\pi]} |\zeta - w|^{-1}|1-\operatorname{Re}\zeta|^\beta \, dA(\zeta).$$

Now assume that $a \le w \le 1$. The case $1 \le w \le b$ is similar. We estimate the integral above by first dividing the integration region into two pieces. Put $t = (w+1)/2$. The integral above equals

$$\begin{aligned}
&\iint_{[a,t]\times[-\pi,\pi]} |\zeta - w|^{-1}|1-\operatorname{Re}\zeta|^\beta \, dA(\zeta) + \iint_{[t,b]\times[-\pi,\pi]} |\zeta - w|^{-1}|1-\operatorname{Re}\zeta|^\beta \, dA(\zeta) \\
&\le \iint_{[a,t]\times[-\pi,\pi]} |\zeta - w|^{-1}|w-\operatorname{Re}\zeta|^\beta \, dA(\zeta) + \iint_{[t,b]\times[-\pi,\pi]} |\zeta - 1|^{-1}|1-\operatorname{Re}\zeta|^\beta \, dA(\zeta) \\
&\le \iint_{[a-1+w,t-1+w]\times[-\pi,\pi]} |\zeta' - 1|^{-1}|1-\operatorname{Re}\zeta'|^\beta \, dA(\zeta') \\
&\qquad + \iint_{[t,b]\times[-\pi,\pi]} |\zeta - 1|^{-1}|1-\operatorname{Re}\zeta|^\beta \, dA(\zeta) \\
&\le 2\iint_{[a,2b]\times[-\pi,\pi]} |\zeta - 1|^{-1}|1-\operatorname{Re}\zeta|^\beta \, dA(\zeta).
\end{aligned}$$

Here we have also performed the change of variables $\zeta' = \zeta - 1 + w$ and used that $a \le a - 1 + w$ and $t - 1 + w \le 2b$. The last integral is easily seen to be finite by a change to polar coordinates: $\zeta = 1 + re^{i\theta}$. □

4.4. The proof of Theorem 4.1

In the proof of Theorem 4.1, we will need the following auxiliary lemma.

Lemma 4.5. *In the hypotheses of Theorem 4.3, if $\eta(T)$ is similar to a unitary operator, then T is is similar to a normal operator.*

We remark that, to prove this lemma, it seems tempting to argue that if $A = \eta(T)$ is similar to unitary, then $\eta^{-1}(A)$ is defined (for instance by the usual L^∞-calculus for normal operators) and $\eta^{-1}(A)$ is similar to a normal operator. However, it is not clear a priori why $\eta^{-1}(A) = T$.

Proof of Lemma 4.5. Replacing T by STS^{-1}, where S is an invertible operator such that $S\eta(T)S^{-1} = \eta(STS^{-1})$ is unitary, we may assume that $\eta(T)$ is unitary. Note that $\eta^{-1} \in C^{1+\alpha}(\mathbb{T})$. We can choose a sequence of rational functions $\{r_n\}_{n=1}^\infty$ with poles off $\mathbb{T}$ such that r_n tend to η^{-1} in $C^{1+\alpha/2}(\mathbb{T})$. Then $r_n \circ \eta$ tends to the identity function in $C^{1+\alpha/2}(\Gamma)$. By continuity of the $C^{1+\alpha/2}(\Gamma)$-calculus for T, we see that $(r_n \circ \eta)(T)$ tend to T in operator norm. As Dynkin's calculus for T is a homomorphism, we have $(r_n \circ \eta)(T) = r_n(\eta(T))$. Since $r_n(\eta(T))$ are normal operators, it follows that T is also normal. $\square$

Theorem 4.1 is a straightforward consequence of the following lemma and Theorem 3.3.

Lemma 4.6. *Let $\Gamma \subset \mathbb{C}$ be a $C^{1+\alpha}$ Jordan curve, and Ω the domain it bounds. Let $T \in \mathcal{B}(H)$ an operator with $\sigma(T) \subset \Gamma$. Assume that $\overline{\Omega}$ is a K-spectral set for T and*

$$\|(T-\lambda)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda,\Gamma)}, \qquad \lambda \in \Omega,$$

for some constant $C > 0$. Then T is similar to a normal operator.

Proof. Let $\eta : \overline{\Omega} \to \overline{\mathbb{D}}$ be the Riemann map. Since $\partial\Omega$ is of class $C^{1+\alpha}$, then $\eta \in \mathbb{C}^{1+\alpha}(\partial\Omega)$ (see, for instance, [Pom92, Theorem 3.6]). Following the steps in the proof of Lemma 4.2, we see that η satisfies all the conclusions of that Lemma.

The operator $\eta(T)$ is power bounded, because $|\eta^n| \le 1$ in $\overline{\Omega}$ for all $n \ge 0$, and $\overline{\Omega}$ is K-spectral for T. By Theorem 4.3, and the fact that $\operatorname{dist}(\lambda, \partial\Omega)$ and $\operatorname{dist}(\eta(\lambda), \mathbb{T})$ are comparable, we get

$$\|(\eta(T)-\lambda)^{-1}\| \le \frac{C}{1-|\lambda|}, \qquad |\lambda| < 1.$$

Applying Theorem VC1 we get that $\eta(T)$ is similar to a unitary. By Lemma 4.5, it follows that T is similar to a normal operator. $\square$

Proof of Theorem 4.1. Theorem 3.3 implies that $\overline{\Omega}$ is a complete K-spectral set for T. It suffices to apply Lemma 4.6. $\square$

4.5. Mean-squares type resolvent estimates

In this section we give criteria for similarity to a normal operator analogous to the results by van Casteren [vC83] and Naboko [Nab84], in a context of $C^{1+\alpha}$ Jordan curves. The first ingredient that we need in our generalization of these results is a substitute of the curves $r\mathbb{T}$, that tend in some sense to $\mathbb{T}$ as $r \to 1$. To this end, we give the following definition.

Let $\Gamma \subset \mathbb{C}$ be a Jordan curve and Ω the region it bounds. We say that a family of curves $\{\gamma_s\}_{0<s<1}$ *tends nicely* to Γ from the outside if $\gamma_s \subset \mathbb{C} \setminus \Omega$ for all $0 < s < 1$ and the following conditions are satisfied for some constant $C \geq 1$:

(a) For all $0 < s < 1$, $C^{-1}s \leq \operatorname{dist}(x, \Gamma) \leq Cs$, for all $x \in \gamma_s$.

(b) For every $0 < s < 1$, $x \in \gamma_s$, and $r > 0$, $\operatorname{length}(\gamma_s \cap B(x, r)) \leq Cr$.

Condition (b) states that the curves γ_s satisfy the Ahlfors-David condition with a uniform constant. This condition was first studied in the papers [Ahl35, Dav84]. We say that the family $\{\gamma_s\}_{0<s<1}$ tends to Γ from the inside if $\gamma_s \subset \Omega$ for all $0 < s < 1$ and conditions (a) and (b) are satisfied.

In our setting, Γ will be of class $C^{1+\alpha}$. In this case it is clear that there exist a family of curves which tends nicely to Γ from the outside and another family of curves which tends nicely to Γ from the inside.

The first thing we need to prove is that the mean-squares type resolvent estimates that we are going to consider do not depend on the concrete choice of the family of curves $\{\gamma_s\}$ which tends nicely to Γ.

We will need a lemma concerning Smirnov spaces. Recall that the Smirnov space $E^2(\Omega, H)$ of H-valued function on a (nice) domain Ω is defined as the $L^2(\partial\Omega)$-closure of the H-valued rational functions with poles off $\overline{\Omega}$. The following lemma dates back to David and his theorem about the boundedness of certain singular integral operators on Ahlfors regular curves. In particular, it follows from the results in [Dav84, Proposition 6].

Lemma 4.7. *Let Ω_1, Ω_2 be Jordan domains with Ahlfors regular boundaries such that $\overline{\Omega}_2 \subset \Omega_1$. If H is a Hilbert space and $f \in E^2(\Omega_1, H)$, then $f|\Omega_2 \in E^2(\Omega_2, H)$ and*

$$\|f|\Omega_2\|_{E^2(\Omega_2,H)} \leq C\|f\|_{E^2(\Omega_1,H)},$$

for some constant C depending only on the Ahlfors constants for $\partial\Omega_1$ and $\partial\Omega_2$.

Lemma 4.8. *Let Γ be a Jordan curve, $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \Gamma$, and $\{\gamma_s\}_{0<s<1}$, $\{\widetilde{\gamma}_s\}_{0<s<1}$ two families of curves which both tend nicely to Γ from the inside (or both from the outside). If*

$$\int_{\gamma_s} \|(T-\lambda)^{-1}x\|^2 \, |d\lambda| \leq \frac{C\|x\|^2}{s}, \qquad x \in H, 0 < s < 1,$$

for some constant C independent of x and s, then

$$\int_{\widetilde{\gamma}_s} \|(T-\lambda)^{-1}x\|^2 \, |d\lambda| \leq \frac{C'\|x\|^2}{s}, \qquad x \in H, 0 < s < 1,$$

for some constant C' independent of x and s.

Proof. First assume that $\{\gamma_s\}$ and $\{\widetilde{\gamma}_s\}$ tend nicely to Γ from the inside. We denote by Ω_s the domain bounded by γ_s and by $\widetilde{\Omega}_s$ the domain bounded by $\widetilde{\gamma}_s$. Take $0 < s < 1$. Then there is some constant $c > 0$ independent of s such that we can choose a $t \geq cs$ which satisfies that the closure of $\widetilde{\Omega}_s$ is contained in Ω_t. This can be done because the distance between each point in γ_s and Γ is comparable to s (and similarly for $\widetilde{\gamma}_s$).

We apply Lemma 4.7 to the function $f(z) = (T - z)^{-1}x$ and the domains $\Omega_1 = \Omega_s$, $\Omega_2 = \widetilde{\Omega}_t$. We obtain

$$\begin{aligned}\int_{\widetilde{\gamma}_s} \|(T-\lambda)^{-1}x\|^2\,|d\lambda| &= \|f|\widetilde{\Omega}_s\|_{E^2(\widetilde{\Omega}_s,H)} \leq K\|f\|_{E^2(\Omega_t,H)} \\ &= K\int_{\gamma_t} \|(T-\lambda)^{-1}x\|^2\,|d\lambda| \leq \frac{KC\|x\|^2}{t} \leq \frac{KC\|x\|^2}{cs}.\end{aligned}$$

If $\{\gamma_s\}$ and $\{\widetilde{\gamma}_s\}$ tend nicely to Γ from the outside, we choose a point z_0 inside the domain bounded by Γ and apply an inversion: $z \mapsto (z - z_0)^{-1}$. Since

$$((T-z_0)^{-1} - (\lambda - z_0)^{-1})^{-1} = (\lambda - z_0)(T - z_0)(T-\lambda)^{-1},$$

the estimates for the resolvent of T imply equivalent estimates for the resolvent of $(T - z_0)^{-1}$ and viceversa. Thus, the case when the family of curves tends from the outside follows from the case when the family of curves tends from the inside. □

It is well known that, in the context of $\mathbb{T}$, a resolvent estimate of mean-square type implies a pointwise resolvent estimate such as (4.1). The proof of this fact uses the usual pointwise estimate for H^2 function in the disk, which involves the norm of the reproducing kernel. The following lemma is a generalization of this.

Lemma 4.9. *Let $\Gamma \subset \mathbb{C}$ be a Jordan curve of class $C^{1+\alpha}$, Ω the region it bounds and $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \Gamma$.*

If

$$\int_{\gamma_s} \|(T-\lambda)^{-1}x\|^2\,|d\lambda| \leq \frac{C\|x\|^2}{s}, \qquad x \in H,\ 0 < s < 1, \tag{4.4}$$

for some constant C independent of x and s and some family of curves $\{\gamma_s\}$ which tends nicely to Γ from the inside (outside), then

$$\|(T-\lambda)^{-1}\| \leq \frac{C'}{\operatorname{dist}(\lambda,\Gamma)}, \qquad \lambda \in \Omega\ (\lambda \in \mathbb{C}\setminus\overline{\Omega}),$$

for some constant C' independent of λ.

Proof. Assume that $\{\gamma_s\}$ tends to nicely to Γ from the inside. Let η be a function as in the statement of Lemma 4.2, and U the neighbourhood of Γ that appears in that lemma. We fix $\lambda \in \Omega \cap U$. First note that $t = \operatorname{dist}(\lambda, \Gamma)$ is comparable to $\operatorname{dist}(\eta(\lambda), \mathbb{T})$, because η is bi-Lipschitz. We put $r = 1 - \operatorname{dist}(\eta(\lambda, \mathbb{T}))/2$.

Now we consider the Jordan curve $\Lambda = \eta^{-1}(r\mathbb{T})$. This is a Jordan curve inside Ω and $\operatorname{dist}(z, \Gamma)$ is comparable to t for every $z \in \Lambda$. Therefore, it is possible to choose $0 < s < 1$ such that $t \leq C_1 s$, and Λ is inside the region bounded by γ_s.

We fix $x \in H$ and put $f(z) = (\eta(T) - z)^{-1}x$, and $g(z) = f(z/r)$. By the usual pointwise estimate for a function in the Hardy H^2 space of the disk

$$\begin{aligned}\|g(z)\| &\le (1-|z|^2)^{-1/2}\|g\|_{E^2(\mathbb{D},H)} = (1-|z|^2)^{-1/2}\|f\|_{E^2(r\mathbb{D},H)}\\ &\le (1-|z|^2)^{-1/2}\|f\|_{E^2(W,H)},\end{aligned}$$

where W is the domain bounded by $\eta(\gamma_s)$ and the last inequality comes from Lemma 4.7. Now,

$$\begin{aligned}\|f\|^2_{E^2(W,H)} &= \int_{\eta(\gamma_s)} \|(\eta(T)-z)^{-1}x\|^2\,|dz| \le C_1 \int_{\gamma_s} \|(\eta(T)-\eta(w))^{-1}x\|^2\,|dw|\\ &\le C_2 \int_{\gamma_s} \|(T-w)^{-1}x\|^2\,|dw| \le \frac{C_3\|x\|^2}{s}.\end{aligned}$$

Here we have used Theorem 4.3 and (4.4).

Hence, we see that

$$\|(\eta(T) - z/r)^{-1}x\|^2 \le C_3(1-|z|^2)^{-1}s^{-1}\|x\|^2.$$

Putting $z = r\eta(\lambda)$ and noting that the inequality above is valid for every $x \in H$, we get that

$$\|(\eta(T)-\eta(\lambda))^{-1}\|^2 \le C_3(1-|r\eta(\lambda)|^2)^{-1}s^{-1} \le C_4 t^{-2}.$$

Applying Theorem 4.3 again, we get

$$\|(T-\lambda)^{-1}\| \le \frac{C'}{t} = \frac{C'}{\operatorname{dist}(\lambda,\Gamma)}.$$

The case when $\{\gamma_s\}$ tends nicely to Γ from the outside is proved by applying an inversion $z \mapsto (z-z_0)^{-1}$, as in the proof of Lemma 4.8. □

Now we can prove generalizations of Theorems VC2 and N. Theorem 4.10 below is a generalization of Theorem VC2 and Theorem 4.12 is a generalization of Theorem N. Their proofs are very similar, since in both cases we use the tools we have developed to pass to $\mathbb{T}$ and then we apply van Casteren's or Naboko's theorem. It is worthy to note that, as it happens in the original theorems, the sets of conditions in these two theorems are not comparable, so no theorem is stronger than the other one.

Theorem 4.10 (A van Casteren-type theorem for curves)**.** *Let $\Gamma \subset \mathbb{C}$ be a Jordan curve of class $C^{1+\alpha}$, Ω the region it bounds and $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \Gamma$. Let $\{\gamma_s\}_{0<s<1}$ be a family of curves which tends nicely to Γ from the outside. Then T is similar to a normal operator if and only if the following three conditions are satisfied.*

$$\|(T-\lambda)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda,\Gamma)}, \qquad \lambda \in \Omega,$$

$$\int_{\gamma_s} \|(T-\lambda)^{-1}x\|^2\,|d\lambda| \le \frac{C\|x\|^2}{s}, \qquad x \in H,\ 0<s<1,$$

$$\int_{\gamma_s} \|(T^*-\overline{\lambda})^{-1}x\|^2\,|d\lambda| \le \frac{C\|x\|^2}{s}, \qquad x \in H,\ 0<s<1.$$

Proof. First assume that T satisfies the three conditions on its resolvent. Let η be a function as in the statement of Lemma 4.2, and U as in that lemma. We can assume that $\gamma_s \subset U$ for every $0 < s < 1$. First note that by Lemma 4.9, the operator T satisfies the resolvent estimate (4.1), so $\eta(T)$ is defined by the $C^{1+\alpha}(\Gamma)$ calculus for T. By Theorem 4.3 and the fact that $\operatorname{dist}(\lambda, \Gamma)$ and $\operatorname{dist}(\eta(\lambda), \mathbb{T})$ are comparable, we get that

$$\|(\eta(T) - \lambda)^{-1}\| \le \frac{C_1}{1 - |\lambda|}, \qquad |\lambda| < 1.$$

We also get that

$$\int_{\eta(\gamma_s)} \|(\eta(T) - \lambda)^{-1}x\|^2 \, |d\lambda| \le \frac{C_2\|x\|^2}{s}, \qquad x \in H,\ 0 < s < 1,$$

by making a change of variables $\lambda = \eta(\mu)$ and applying Theorem 4.3.

The family of curves $\{\eta(\gamma_s)\}_{0<s<1}$ tends nicely to $\mathbb{T}$ from the outside, because $\eta : U \to \eta(U)$ is a C^1 diffeomorphism and bi-Lipschitz. Therefore, by applying Lemma 4.8 to the family $\widetilde{\gamma_s} = (1 + s)\mathbb{T}$, we see that

$$\int_{|\lambda|=r} \|(\eta(T) - \lambda)^{-1}x\|^2 \, |d\lambda| \le \frac{C_3\|x\|^2}{r - 1}, \qquad x \in H,\ 1 < r < 2.$$

A similar reasoning with T^* in place of T and $\widetilde{\eta}(z) = \overline{\eta(\overline{\zeta})}$ in place of η shows that

$$\int_{|\lambda|=r} \|(\eta(T)^* - \lambda)^{-1}x\|^2 \, |d\lambda| \le \frac{C_4\|x\|^2}{r - 1}, \qquad x \in H,\ 1 < r < 2,$$

because $\widetilde{\eta}(T^*) = \eta(T)^*$.

Now we can apply Theorem VC2 to deduce that $\eta(T)$ is similar to a unitary. By Lemma 4.5, then T is similar to a normal operator.

Conversely, assume that T is similar to a normal operator. Replacing T by STS^{-1} we can assume that T is normal. Clearly the first condition on the resolvent of T holds, because $\|(T - \lambda)^{-1}\| \le \operatorname{dist}(\lambda, \Gamma)^{-1}$ for a normal operator T.

The operator $\eta(T)$ is unitary, so it satisfies

$$\int_{r\mathbb{T}} \|(\eta(T) - \lambda)^{-1}x\|^2 \, |d\lambda| \le \frac{C_5\|x\|^2}{r - 1}, \qquad 1 < r < 2,\ x \in H.$$

(To show this, one can write the power series of the resolvent and use the fact that $\|\eta(T)^n\| = 1$ for all $n \ge 0$.)

We also have that

$$C^{-1}\|(T - \lambda)^{-1}x\| \le \|(\eta(T) - \eta(\lambda))^{-1}x\| \le C\|(T - \lambda)^{-1}x\|, \qquad \lambda \in U \setminus \Gamma,\ x \in H,$$

for some constant $C > 0$ and some neighbourhood U of Γ. In fact, we can apply Theorem 4.3, although since T is normal we may also use a simpler argument.

Therefore, we get that

$$\int_{\eta^{-1}(r\mathbb{T})} \|(T - \lambda)^{-1}x\|^2 \, |d\lambda| \le \frac{C_6\|x\|^2}{r - 1}, \qquad 1 < r < 2,\ x \in H.$$

The family of curves $\widetilde{\gamma}_s = \eta^{-1}((1+s)\mathbb{T})$ tends nicely to Γ. Applying Lemma 4.8, we get that T satisfies the second condition in the statement of this theorem. To see that it also satisfy the third condition, we use a similar reasoning with T^* instead of T and $\widetilde{\eta}$ instead of η. $\square$

Sz.-Nagy proved in [SN47] that an operator T is similar to a unitary if and only if $\|T^n\| \le C$ for all $n \in \mathbb{Z}$. It is easy to use this result to show that if $\sigma(T) \subset \Gamma$, the resolvent estimate (4.1) holds and $\|\eta^n(T)\| \le C$ for all $n \in \mathbb{Z}$, then T is similar to a normal operator.

The following corollary is a generalization of Theorem VC1. Note that here we only assume that $\|\eta(T)^n\| \le C$ for all $n \ge 0$.

Corollary 4.11. *Let $\Gamma \subset \mathbb{C}$ be a Jordan curve of class $C^{1+\alpha}$, and $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \Gamma$. Let $\{\gamma_s\}_{0<s<1}$ a family of curves which tends nicely to Γ from the outside. Assume that*

$$\|(T-\lambda)^{-1}\| \le \frac{C}{\operatorname{dist}(\lambda, \Gamma)}, \qquad \lambda \in \mathbb{C} \setminus \Gamma.$$

Let $\eta : \Gamma \to \mathbb{T}$ be a function as in the statement of Lemma 4.2. The operator $\eta(T)$ is defined by the $C^{1+\alpha}$-calculus. If $\eta(T)$ is power bounded, then T is similar to a normal operator.

Proof. The proof is similar to the proof of Theorem 4.10, but Theorem VC1 is used instead of Theorem VC2. $\square$

Theorem 4.12 (A Naboko-type theorem for curves)**.** *Let $\Gamma \subset \mathbb{C}$ be a Jordan curve of class $C^{1+\alpha}$, Ω the region it bounds and $T \in \mathcal{B}(H)$ with $\sigma(T) \subset \Gamma$. Let $\{\gamma_s\}_{0<s<1}$ a family of curves which tends nicely to Γ from the outside and $\{\widetilde{\gamma}_s\}$ a family of curves which tends nicely to Γ from the inside. Then T is similar to a normal operator if and only if the following two conditions are satisfied:*

$$\int_{\gamma_s} \|(T-\lambda)^{-1}x\|^2\,|d\lambda| \le \frac{C\|x\|^2}{s}, \qquad x \in H,\ 0<s<1,$$

$$\int_{\widetilde{\gamma}_s} \|(T^*-\overline{\lambda})^{-1}x\|^2\,|d\lambda| \le \frac{C\|x\|^2}{s}, \qquad x \in H,\ 0<s<1.$$

Proof. The proof of this theorem is similar to the proof of Theorem 4.10. If T satisfies the two conditions in the statement of this theorem, instead of using van Casteren's theorem, here we will use Naboko's result to show that $\eta(T)$ is similar to a unitary. We only give a sketch of the proof.

First, Lemma 4.9 implies that T satisfies the resolvent estimate (4.1). Then we can choose a function η as in Lemma 4.2 and the operator $\eta(T)$ is well defined. Using Theorem 4.3 and Lemma 4.8, we get

$$\int_{|\lambda|=r} \|(\eta(T)-\lambda)^{-1}x\|^2\,|d\lambda| \le \frac{C_1\|x\|^2}{r-1}, \qquad x \in H,\ 1<r<2$$

and

$$\int_{|\lambda|=r} \|(\eta(T)^*-\lambda)^{-1}x\|^2\,|d\lambda| \le \frac{C_2\|x\|^2}{1-r}, \qquad x \in H,\ 0<r<1.$$

Applying Theorem N, we see that $\eta(T)$ is similar to a unitary. It follows that $\mathbb{T}$ is similar to a normal operator by Lemma 4.5.

The converse direction is proved as in Theorem 4.12. $\square$

4.6. Some examples

The conditions for a contraction T to be similar to a unitary are well known. If $T \in \mathcal{B}(H)$ is a contraction, the defect operators $D_T = (I-T^*T)^{\frac{1}{2}}$ and $D_{T^*} = (I-TT^*)^{\frac{1}{2}}$ are well defined. The defect spaces are $\mathfrak{D}_T = \overline{D_T H}$ and $\mathfrak{D}_{T^*} = \overline{D_T^* H}$. For $\lambda \in \mathbb{D}$, the characteristic function $\Theta_T(\lambda) : \mathfrak{D}_T \to \mathfrak{D}_{T^*}$ is defined by

$$\Theta_T(\lambda) = [-T + \lambda D_{T^*}(I - \lambda T^*)^{-1} D_T] | \mathfrak{D}_T.$$

Sz.-Nagy and Foias proved in [SNF65] that T is similar to a unitary if and only if $\Theta_T(\lambda)$ is invertible for all $\lambda \in \mathbb{D}$ and

$$\sup_{\lambda \in \mathbb{D}} \|\Theta_T(\lambda)^{-1}\| < \infty.$$

This result can also be found in the book by Sz.-Nagy and Foias [SNF67, Chapitre IX].

We can choose any purely contractive function Θ satisfying this condition and use the Sz.-Nagy Foias model to construct a completely non unitary contraction T such that $\Theta_T = \Theta$. Such a contraction T is non-unitary and similar to a unitary. We have

$$\|(T-\lambda)^{-1}\| \leq \frac{C}{1-|\lambda|}, \qquad |\lambda| < 1.$$

Since T is also a contraction,

$$\|(T-\lambda)^{-1}\| \leq \frac{1}{|\lambda|-1}, \qquad |\lambda| > 1,$$

by von Neumann's inequality. Thus, such an operators T gives an example of an operator which satisfies the hypotheses of Theorem 4.1 and is non-normal. Recall that, by Stampfli's theorem stated in the introduction, if T satisfies

$$\|(T-\lambda)^{-1}\| \leq \frac{1}{||\lambda|-1|}, \qquad |\lambda| \neq 1,$$

then T must be normal.

The class of ρ-contractions can also give examples of this type. If $\rho > 0$, an operator $T \in \mathcal{B}(H)$ is called a ρ-contraction if there is a larger Hilbert space $K \supset H$ and a unitary $U \in \mathcal{B}(K)$ such that

$$T^n = \rho P_H U^n | H, \qquad n = 1, 2, \ldots,$$

where P_H denotes the orthogonal projection onto H. The classes $\mathcal{C}_\rho$ of ρ-contractions are increasing with ρ, and the class $\mathcal{C}_1$ coincides with the class of contractions.

If T is a ρ-contraction with $\rho \geq 2$, then

$$\|(T-\lambda)^{-1}\| \leq \frac{1}{|\lambda|-1}, \qquad 1 < |\lambda| < \frac{\rho-1}{\rho-2}.$$

(Here $\frac{\rho-1}{\rho-2} = +\infty$ if $\rho = 2$). Therefore, any ρ-contraction which is similar to a unitary also satisfies the hypotheses of Theorem 4.1. Also note that, if T is a contraction, then one may take the set $U = \mathbb{C}$ in the hypotheses of Theorem 4.1. However, for a ρ-contraction with $\rho > 2$, in general one needs to take a smaller set U.

It is natural to ask whether there is an example of a ρ-contraction which is not a contraction and which is similar to a unitary. Here we will give such an example with $\rho = 2$.

Proposition 4.13. *Assume that $\alpha, \beta > 0$, $\max(\alpha, \beta) > 1$, and $\alpha^2 + \beta^2 < 2$. Let T be the bilateral weighted shift T on $\ell^2(\mathbb{Z})$ with weights $\dots, 1, 1, \boxed{\alpha}, \beta, 1, 1, \dots$ defined by*

$$T(\dots, x_{-1}, \boxed{x_0}, x_1, x_2, \dots) = (\dots, x_{-2}, \boxed{x_{-1}}, \alpha x_0, \beta x_1, x_2, \dots).$$

(Here $\boxed{}$ marks the 0-th component). Then T is a 2-contraction which is similar to a unitary, but it is not a contraction.

Proof. Clearly, $\|T\| = \max(\alpha, \beta, 1) > 1$. Since $\alpha, \beta > 0$, the operator T is similar to the unitary bilateral shift U on $\ell^2(\mathbb{Z})$ with all weights equal to 1. It remains to see that T is a 2-contraction.

Recall that T is a 2-contraction if and only if

$$\operatorname{Re}(\theta T) \le I, \qquad |\theta| = 1.$$

Since θT is unitarily equivalent to T when $|\theta| = 1$, it is enough to check this inequality for $\theta = 1$.

We put $A = 2 \operatorname{Re} T$. We have to check that $\sigma(A) \cap (2, +\infty) = \emptyset$. Since A is a finite rank perturbation of $U + U^*$ and $\sigma(U + U^*) = [-2, 2]$, it suffices to check that A has no eigenvalues in $(2, +\infty)$.

Assume that $x = (x_n)_{n \in \mathbb{Z}}$ is a non-zero vector in $\ell^2(\mathbb{Z})$ that satisfies $(A - \lambda)x = 0$ for some $\lambda > 2$. This means that

$$x_n - \lambda x_{n+1} + x_{n+2} = 0, \quad |n| \ge 2, \tag{4.5}$$

$$x_{-1} - \lambda x_0 + \alpha x_1 = 0, \tag{4.6}$$

$$\alpha x_0 - \lambda x_1 + \beta x_2 = 0, \tag{4.7}$$

$$\beta x_1 - \lambda x_2 + \alpha x_3 = 0. \tag{4.8}$$

Put

$$u_\pm = u_\pm(\lambda) = \frac{\lambda \pm \sqrt{\lambda^2 - 4}}{2}.$$

Then (4.5) and $x \in \ell^2(\mathbb{Z})$ imply that

$$\begin{aligned} x_n &= a u_-^n, \qquad n \ge 2, \\ x_n &= b u_+^n, \qquad n \le 0, \end{aligned}$$

for some non-zero a, b.

We consider the quotients $y_n = \frac{x_{n+1}}{x_n}$. Then

$$y_n = u_-, \qquad n \geq 2,$$
$$y_n = u_+, \qquad n \leq -1.$$

If

$$Fx_n - \lambda x_{n+1} + Gx_{n+2} = 0,$$

then y_n is obtained from y_{n+1} by applying the Möbius transformation $z \mapsto \frac{F}{-Gz+\lambda}$, which can be encoded by the 2×2 matrix $\left(\begin{smallmatrix} F & 0 \\ -G & \lambda \end{smallmatrix}\right)$. Since the composition of Möbius transformations reduces to multiplying the corresponding 2×2 matrices, equations (4.6)–(4.8) yield

$$u_+(\lambda) = y_{-1} = \frac{M_{11}(\lambda)y_2 + M_{12}(\lambda)}{M_{21}(\lambda)y_2 + M_{22}(\lambda)} = \frac{M_{11}(\lambda)u_-(\lambda) + M_{12}(\lambda)}{M_{21}(\lambda)u_-(\lambda) + M_{22}(\lambda)},$$

where

$$\begin{pmatrix} M_{11}(\lambda) & M_{12}(\lambda) \\ M_{21}(\lambda) & M_{22}(\lambda) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\alpha & \lambda \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ -\beta & \lambda \end{pmatrix} \begin{pmatrix} 0 & \beta \\ -1 & \lambda \end{pmatrix}.$$

Putting

$$\begin{aligned} f(\lambda) &= u_+(\lambda)\big(M_{21}(\lambda)u_-(\lambda) + M_{22}(\lambda)\big) - \big(M_{11}(\lambda)u_-(\lambda) + M_{12}(\lambda)\big) = \\ &= \lambda^3 u_+(\lambda) + \lambda^2[u_+(\lambda)u_-(\lambda) - 1] + \lambda[u_-(\lambda) - (\alpha^2+\beta^2)u_+(\lambda)] \\ &\quad + \beta^2[1 - u_+(\lambda)u_-(\lambda)], \end{aligned}$$

we see that $f(\lambda) = 0$. The same conclusion holds if $y_n = \infty$ for some n. However, a straightforward computation shows that if $\alpha^2 + \beta^2 < 2$, then $f(2) > 0$ and $f'(t) > 0$ for $t > 2$. Therefore, $f(\lambda) > 0$, which is a contradiction. $\square$

We can also give examples of non-normal operators which satisfy the hypotheses of Theorem 4.1 for a Jordan domain $\Omega \neq \mathbb{D}$. Let A be a non-unitary contraction which is similar to a unitary. We denote by $\varphi : \Omega \to \mathbb{D}$ the Riemann mapping and we put $\psi = \varphi^{-1}$. The operator $T = \varphi(A)$ is well defined and non-normal. If $\lambda \in \mathbb{C} \setminus \overline{\Omega}$, then by von Neumann's inequality

$$\|(T-\lambda)^{-1}\| \leq \|(\varphi - \lambda)^{-1}\|_{H^\infty(\mathbb{D})} = \operatorname{dist}(\lambda, \Omega)^{-1}.$$

If $\lambda \in \Omega$, the inequality

$$\|(T-\lambda)^{-1}\| \leq C \operatorname{dist}(\lambda, \Omega)^{-1}$$

follows from the fact that T is similar to a normal operator. We see that this operator T satisfies the hypotheses of Theorem 4.1. However, it does not seem so easy to perform in a similar way a Riemann mapping transplantation of the example in Proposition 4.13 to a general Jordan domain Ω .

We remark that in [AT97], Ando and Takahashi proved that if an operator T is polynomially bounded and there exists an injective operator X and a unitary operator W whose spectral measure is singular with respect to the Lebesgue measure on $\mathbb{T}$, then T is similar to a unitary. Moreover, if such T is also a ρ-contraction for some $\rho > 0$, then T is itself unitary. Note that in Proposition 4.13 above, the operator T is similar to the bilateral shift in $L^2(\mathbb{T})$, whose spectral measure is not singular.

Part II.

Separating structures

5. Operator vessels

This chapter contains an introductory exposition of the Livšic-Vinnikov theory of commuting non-selfadjoint operators. The main idea of this theory is, roughly speaking, to embed the tuple of operators into a richer structure, which, in particular, includes some auxiliary matrices. These matrices characterize, in a certain sense, the interplay of the operators. This structure is called an operator vessel. The auxiliary matrices can be used to assign an algebraic curve to the vessel, thus giving a connection with Algebraic Geometry. It is possible to understand vessels in terms of a control system, so terms from Control Theory are used to name the auxiliary matrices and other objects related to vessels.

The exposition of this chapter is organized in the following way. First we treat the theory in the case of n-tuples of operators. Then we specify to the case $n = 2$, which has a richer theory in some respects.

The main monograph about this theory is the book by Livšic, Kravitsky, Marcus and Vinnikov [LKMV95]. Some interesting expository papers are [Vin98, BV03].

5.1. Vessels of several commuting operators

We start by giving the definition of an operator vessel. After that, we will give some motivation for this definition.

Definition 5.1 (Operator vessel)**.** Suppose that we are given H a Hilbert space, E a finite-dimensional Hilbert space, $\Phi \in \mathcal{B}(H, E)$, a tuple of commuting operators $A_k \in \mathcal{B}(H)$, $k = 1, \dots, n$, selfadjoint operators $\sigma_k \in \mathcal{B}(E)$, and selfadjoint operators $\gamma_{kj}^{\text{in}}, \gamma_{kj}^{\text{out}} \in \mathcal{B}(E)$ satisfying $\gamma_{kj}^{\text{in}} = -\gamma_{jk}^{\text{in}}$ and $\gamma_{kj}^{\text{out}} = -\gamma_{jk}^{\text{out}}$. We say that the tuple $\mathcal{V} = (A_k; H, \Phi, E; \sigma_k, \gamma_{kj}^{\text{in}}, \gamma_{kj}^{\text{out}})$ is a (commutative) vessel if the following conditions are satisfied:

$$\frac{1}{i}(A_k - A_k^*) = \Phi^* \sigma_k \Phi, \tag{5.1}$$

$$\sigma_k \Phi A_j^* - \sigma_j \Phi A_k^* = \gamma_{kj}^{\text{in}} \Phi, \tag{5.2}$$

$$\gamma_{kj}^{\text{out}} = \gamma_{kj}^{\text{in}} + i(\sigma_k \Phi \Phi^* \sigma_j - \sigma_j \Phi \Phi^* \sigma_k), \tag{5.3}$$

$$\sigma_k \Phi A_j - \sigma_j \Phi A_k = \gamma_{kj}^{\text{out}} \Phi. \tag{5.4}$$

The space H is called the *inner space*, and E is called the *outer space*. The operator Φ is called the *window*. The operators σ_k are called the *rates* of the vessel and γ_{kj}^{in} and γ_{kj}^{out} are called *gyrations*. Note that since they act on the finite-dimensional space E, one many regard these as matrices.

Conditions (5.3) are called the *linkage conditions*. It is easy to show that conditions (5.1)–(5.3) imply conditions (5.4), and also that conditions (5.1), (5.3), (5.4) imply conditions (5.2). Therefore, using the linkage conditions, the gyrations γ_{kj}^{out} can be defined in terms of γ_{kj}^{in} or vice versa.

Note that the conditions (5.1) imply that $\operatorname{Im} A_k = (A_k - A_k^*)/(2i)$ have finite rank. We will now show that every tuple $(A_1, \ldots, A_n)$ of commuting operators such that $\operatorname{Im} A_k$ have finite rank can be embedded in a vessel. This means that it is possible to find E, Φ and the rates and gyrations as above such the conditions in the definition of vessel are satisfied.

To do so, we put

$$E = \bigvee_{k=1}^{n} (A_k - A_k^*)H$$

and $\Phi = P_E$, the orthogonal projection of H onto E. We define the rates by

$$\sigma_k = \frac{1}{i}(A_k - A_k^*)|E.$$

Clearly, σ_k are selfadjoint operators on E and satisfy conditions (5.1).

To define the gyrations first we note that

$$\frac{1}{i}(A_k A_j^* - A_j A_k^*) = \frac{1}{i}[(A_k - A_k^*)A_j^* - (A_j - A_j^*)A_k^*], \tag{5.5}$$

and

$$\frac{1}{i}(A_j^* A_k - A_k^* A_j) = \frac{1}{i}[(A_k - A_k^*)A_j - (A_j - A_j^*)A_k]. \tag{5.6}$$

The ranges of the operators defined by the right hand sides of these equalities are clearly contained in E. The left hand sides of these inequalities clearly define selfadjoint operators. Hence, it is possible to choose selfadjoint operators γ_{kj}^{in} and γ_{kj}^{out} acting on E and such that

$$\begin{aligned} \frac{1}{i}(A_k A_j^* - A_j A_k^*) &= \Phi^* \gamma_{kj}^{\text{in}} \Phi, \\ \frac{1}{i}(A_j^* A_k - A_k^* A_j) &= \Phi^* \gamma_{kj}^{\text{out}} \Phi. \end{aligned} \tag{5.7}$$

Now (5.5) and (5.6) and the definition of the rates σ_k show that conditions (5.2) and (5.4) are satisfied. We also note that $\gamma_{kj}^{\text{in}} = -\gamma_{jk}^{\text{in}}$ and $\gamma_{kj}^{\text{out}} = -\gamma_{jk}^{\text{out}}$.

To check that conditions (5.3) hold, we first subtract conditions (5.2) and (5.4) to obtain

$$(\gamma_{kj}^{\text{in}} - \gamma_{kj}^{\text{out}})\Phi = \sigma_k \Phi (A_j^* - A_j) - \sigma_j \Phi (A_k^* - A_k).$$

Using the definition of the rates σ_k, we get

$$(\gamma_{kj}^{\text{in}} - \gamma_{kj}^{\text{out}})\Phi = -i\sigma_k \Phi\Phi^* \sigma_j \Phi + i\sigma_j \Phi\Phi^* \sigma_k \Phi.$$

Cancelling the factor Φ on the right of both sides of this equality and rearranging terms, we obtain (5.3).

A vessel is said to be *strict* if $\Phi H = E$ and $\cap_k \ker \sigma_k = 0$. This means, in some sense, that the outer space E is as small as possible. We have shown that every tuple $(A_1, \ldots, A_n)$ of commutative operators such that $\operatorname{Im} A_k$ have finite rank can be embedded in a strict vessel in a canonical way. Of course, there are many other ways of embedding the tuple in a vessel, which need not be strict.

There is an interpretation of vessels in terms of a kind of control system with several temporal variables. This system interpretation helps to explain some of the terminology. To each vessel we assign the dynamical system

$$\begin{aligned} i\frac{\partial f}{\partial t_k} + A_k f &= \Phi^* \sigma_k u, \\ v &= u - i\Phi f. \end{aligned} \tag{5.8}$$

Here, f, u and v are functions of n variables $t_1, \dots, t_n$, which one can think of as n independent temporal variables or as one temporal variable and $n-1$ variables representing space. In this latter case, (5.8) models a continuum of interacting temporal systems distributed in space.

The function f takes values in H and is the state of the system. The functions u and v take values in E and are the input and output of the system. This explains the terms inner space for H and outer space for E. The system is like a box, whose (inner) state is a vector from H and where we feed in an input and measure an output which are vectors in E.

There are two possible interpretations of this system. The first one is the *vector field interpretation*: given an input vector field $u(t)$ on $\mathbb{R}^n$, and an initial state $f_0 \in H$, find, if possible, vector fields $f(t)$ and $v(t)$ on $\mathbb{R}^n$ satisfying (5.8) and $f(0) = f_0$. The second one is the *curve interpretation*: given a piecewise smooth curve L on $\mathbb{R}^n$ parametrized by $(t_1(\tau), \dots, t_n(\tau))$, $\tau \in \mathbb{R}$, an input vector field $u(\tau)$ along L and an initial state $f_0 \in H$, find functions of the parameter $f = f(\tau)$ and $v = v(\tau)$ satisfying the system

$$\begin{aligned} i\frac{df}{d\tau} + \sum_{k=1}^{n} \frac{\partial t_k}{\partial \tau} A_k f &= \Phi^* \sum_{k=1}^{n} \frac{\partial t_k}{\partial \tau} \sigma_k u, \\ v &= u - i\Phi f, \end{aligned} \tag{5.9}$$

and $f(0) = f_0$. This system is obtained by restricting (5.8) to the curve L and writing everything as a function of the parameter τ.

In general, the system (5.8) is overdetermined, and will not be consistent. Given an input vector field $u(t)$ on $\mathbb{R}^n$, we will say that the system obtained is *consistent* if the vector field interpretation has a solution for any $f_0 \in H$. This is equivalent to the following condition using the curve interpretation: for any initial condition $f_0 \in H$, and any parametrized curve $(t_1(\tau), \dots, t_n(\tau))$ such that $(t_1(0), \dots, t_n(0)) = 0$, if $f(\tau)$ and $v(\tau)$ are the solutions of (5.9), then the values $f(1)$ and $v(1)$ depend only on the initial condition f_0 and the point $p = (t_1(1), \dots, t_n(1))$, but not on the curve in question which joins 0 and p.

The necessary and sufficient conditions for the system to be consistent are the compatibility conditions which arise from the equality of the mixed partial derivatives:

$$\frac{\partial^2 f}{\partial t_j \partial t_k} = \frac{\partial^2 f}{\partial t_k \partial t_j}$$

(see [LKMV95, Theorem 3.2.1]). When the system is given the zero input (i.e., $u \equiv 0$), we see that the system is consistent because the operators A_k commute. However, for an arbitrary input u, the system is not going to be consistent in general. Let us deduce what are the conditions on u for the equality of the mixed partial derivatives.

We compute

$$\frac{\partial^2 f}{\partial t_k \partial t_j} = \frac{\partial}{\partial t_k}(iA_j f - i\Phi^* \sigma_j u) = iA_j(iA_k f - i\Phi^* \sigma_k u) - i\Phi^* \sigma_j \frac{\partial u}{\partial t_k}.$$

Hence, since A_k commute, we see that the equality of the mixed partials is equivalent to the conditions

$$\left(\Phi^* \sigma_j \frac{\partial}{\partial t_k} - \Phi^* \sigma_k \frac{\partial}{\partial t_j} + iA_j \Phi^* \sigma_k - iA_k \Phi^* \sigma_j\right) u = 0, \qquad j,k = 1, \ldots, n. \tag{5.10}$$

We now rewrite these conditions (5.10) introducing the gyrations of the vessel. Taking adjoints in (5.2), we see that (5.10) rewrites as

$$\Phi^* \left(\sigma_j \frac{\partial}{\partial t_k} - \sigma_k \frac{\partial}{\partial t_j} + i\gamma_{kj}^{\text{in}}\right) u = 0, \qquad j,k = 1, \ldots, n. \tag{5.11}$$

Hence, we see that an input vector field u makes the system (5.8) compatible if and only if u satisfies the compatibility conditions (5.11). This explains the label "in" in the gyrations γ_{kj}^{in}.

Now let us see that if the input u satisfies (5.11), then the output v satisfies similar compatibility conditions, with γ_{kj}^{out} in place of γ_{kj}^{in}. Since $u = v + i\Phi f$, we get

$$\Phi^* \left(\sigma_j \frac{\partial}{\partial t_k} - \sigma_k \frac{\partial}{\partial t_j} + i\gamma_{kj}^{\text{in}}\right) (v + i\Phi f) = 0. \tag{5.12}$$

Now we compute

$$\begin{aligned}
\Phi^* \Big(\sigma_j \frac{\partial}{\partial t_k} &- \sigma_k \frac{\partial}{\partial t_j} + i\gamma_{kj}^{\text{in}}\Big)(i\Phi f) = \\
&= \Phi^* \sigma_j \Phi(\Phi^* \sigma_k u - A_k f) - \Phi^* \sigma_k \Phi(\Phi^* \sigma_j u - A_j f) - \Phi^* \gamma_{kj}^{\text{in}} \Phi f \\
&= i\Phi^* \gamma_{kj}^{\text{out}} u - i\Phi^* \gamma_{kj}^{\text{in}} u + \Phi^* \gamma_{kj}^{\text{out}} \Phi f - \Phi^* \gamma_{kj}^{\text{in}} \Phi f \\
&= i\Phi^* (\gamma_{kj}^{\text{out}} - \gamma_{kj}^{\text{in}}) v.
\end{aligned}$$

Here we have used the system (5.8) in the first equality, the vessel conditions (5.3) and (5.4) in the second equality and the identity $v = u - i\Phi f$ in the third equality. Hence, we get from (5.12) that the output of the system satisfies the compatibility conditions

$$\Phi^* \left(\sigma_j \frac{\partial}{\partial t_k} - \sigma_k \frac{\partial}{\partial t_j} + i\gamma_{kj}^{\text{out}}\right) v = 0, \qquad j,k = 1, \ldots, n. \tag{5.13}$$

If we drop Φ^* in (5.11), we obtain the conditions

$$\left(\sigma_j \frac{\partial}{\partial t_k} - \sigma_k \frac{\partial}{\partial t_j} + i\gamma_{kj}^{\text{in}}\right) u = 0, \qquad j,k = 1, \ldots, n. \tag{5.14}$$

These conditions are sufficient for the input u to make the system compatible, and are also necessary when Φ^* is injective. The important aspect of these conditions is that they are written entirely in terms of operators on E, so they can be checked using matrices. If the input u satisfies (5.14), arguing as above we see that the output v satisfies the compatibility conditions

$$\left(\sigma_j \frac{\partial}{\partial t_k} - \sigma_k \frac{\partial}{\partial t_j} + i\gamma_{kj}^{\text{out}}\right) v = 0, \qquad j, k = 1, \ldots, n. \tag{5.15}$$

With this in mind, the system theoretical interpretation of the vessel $\mathcal{V}$ is the system (5.8) together with the compatibility conditions (5.11) and (5.13) (or (5.14) and (5.15)) at the input and output respectively. Indeed, it is more usual to take (5.14) and (5.15) as the compatibility conditions. They are referred to as the *input compatibility condition* and the *output compatibility condition.*

Even if the system is not consistent, we can always consider the curve interpretation with $L = L_\xi$ the straight line given by $(\xi_1\tau, \ldots, \xi_n\tau)$, where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ is a fixed direction and $\tau \in \mathbb{R}$. Then the system (5.9) that we obtain is

$$\begin{aligned} i\frac{df}{d\tau} + \xi A f &= \Phi^* \xi\sigma u, \\ v &= u - i\Phi f, \end{aligned} \tag{5.16}$$

where we abuse notation a bit and write

$$\xi A = \sum_{k=1}^{n} \xi_k A_k, \qquad \xi\sigma = \sum_{k=1}^{n} \xi_k \sigma_k.$$

We will now prove that this system satisfies the law of conservation of energy

$$\frac{d}{d\tau}\langle f, f\rangle = \langle \xi\sigma u, u\rangle - \langle \xi\sigma v, v\rangle. \tag{5.17}$$

We interpret the (indefinite) quadratic form given by $\xi\sigma$ as the energy at the input and output, so that (5.17) just says that the variation of the internal energy of the system $\|f\|^2$ amounts just to the energy added at the input and the energy extracted at the output. When speaking about the system in n-variables (5.8), we say that the energy is conserved along any direction ξ, when one takes into account the energy added at the input and the energy extracted at the output, both measured with the indefinite quadratic form $\xi\sigma$.

To prove (5.17), we first use the system (5.16) to obtain

$$\frac{d}{d\tau}\langle f, f\rangle = 2\operatorname{Re}\langle \frac{df}{d\tau}, f\rangle = 2\operatorname{Re}\langle i\xi A f - i\Phi^*\xi\sigma u, f\rangle.$$

Since $iA_k = iA_k^* - \Phi^*\sigma_k\Phi$, we see that $i\xi A = i\xi A^* - \Phi^*\xi\sigma\Phi$. Using this and $i\Phi f = u - v$, we get

$$2\operatorname{Re}\langle iAf, f\rangle = \operatorname{Re}\langle i\xi Af + i\xi A^* f - \Phi^*\xi\sigma\Phi f, f\rangle = -\langle \xi\sigma\Phi f, \Phi f\rangle = -\langle \xi\sigma(u - v), u - v\rangle,$$

where the second equality holds because $\xi A + \xi A^*$ is selfadjoint. Also,

$$2\operatorname{Re}\langle -i\Phi^*\xi\sigma u, f\rangle = 2\operatorname{Re}\langle \xi\sigma u, i\Phi f\rangle = 2\operatorname{Re}\langle \sigma u, u - v\rangle.$$

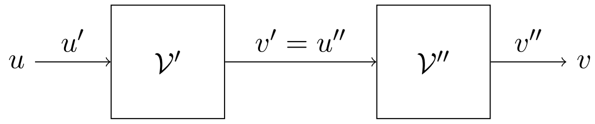

Figure 5.1.: Coupling of vessels $\mathcal{V}'$ and $\mathcal{V}''$ as the cascade connection of their corresponding systems

Putting together these two equalities, we finally get

$$\begin{aligned}\frac{d}{d\tau}\langle f, f\rangle &= 2\operatorname{Re}\langle i\xi Af - i\Phi^*\xi\sigma u, f\rangle = -\langle \xi\sigma(u-v), u-v\rangle + 2\operatorname{Re}\langle \xi\sigma u, u-v\rangle \\ &= \langle \xi\sigma u, u\rangle - \langle \xi\sigma v, v\rangle.\end{aligned}$$

There are two fundamental operations on vessels: the *coupling* and the *decomposition*. The coupling allows one to build a "larger" vessel out of two vessels. The decomposition is the inverse operation, which allows one to break up a vessel into two vessels such that their coupling is the original vessel. These operations are easier to motivate and understand using the system interpretation.

Let $\mathcal{V}'$ and $\mathcal{V}''$ be two vessels. We will denote all the objects of $\mathcal{V}'$ with a $'$ symbol and all the objects of $\mathcal{V}''$ with a symbol $''$. The idea to form a larger vessel $\mathcal{V}$ which is the coupling of $\mathcal{V}'$ and $\mathcal{V}''$ is to cascade connect the corresponding systems of $\mathcal{V}'$ and $\mathcal{V}''$. This means to feed the input of the system of $\mathcal{V}''$ with the output of the system of $\mathcal{V}'$. Of course to do this it is necessary that both vessels have the same outer space $E = E' = E''$. The input of the coupled system is the input of the system of $\mathcal{V}'$, and the output of coupled system is the output of the system of $\mathcal{V}''$. That is, we set

$$u = u', \qquad u'' = v', \qquad v = v'. \tag{5.18}$$

See Figure 5.1 for a graphical representation of the cascade connection of the systems corresponding to $\mathcal{V}'$ and $\mathcal{V}''$.

Since the rates σ_k are used to measure the energy in the space E, as we have explained above, it is also natural to assume that $\sigma_k' = \sigma_k''$. We will denote the rates by $\sigma_k = \sigma_k' = \sigma_k''$, as these will also be the rates of the coupled vessel $\mathcal{V}$.

We rewrite the systems of $\mathcal{V}'$ and $\mathcal{V}''$ using (5.18). We get the system

$$\begin{aligned} i\frac{\partial f'}{\partial t_k} + A_k' f' &= \Phi'^*\sigma_k u, \\ i\frac{\partial f''}{\partial t_k} + A_k'' f'' &= \Phi''^*\sigma_k(u - i\Phi' f'), \\ v &= u - i\Phi' f' - i\Phi'' f''. \end{aligned} \tag{5.19}$$

We put $H = H' \oplus H''$, $f = (f', f'')$, $\Phi = \Phi' \oplus \Phi''$, and define the operators A_k in H by

$$A_k = \begin{bmatrix} A_k' & 0 \\ i\Phi''^*\sigma_k\Phi' & A_k'' \end{bmatrix}. \tag{5.20}$$

Then we see that (5.19) rewrites as

$$\begin{aligned} i\frac{\partial f}{\partial t_k} + A_k f &= \Phi^* \sigma_k u, \\ v &= u - i\Phi f. \end{aligned} \tag{5.21}$$

This is precisely the system we would associate to a vessel with operators A_k, window Φ and rates σ_k.

There are two things we have to check to justify that this construction makes sense. The first is that it is not clear why the operators A_k defined in (5.20) commute. The second is whether the dynamical system (5.21) we have obtained is compatible. In our discussion, we have ignored the input and output compatibility conditions for the systems of $\mathcal{V}'$ and $\mathcal{V}''$. Taking these into account will help us motivate the definition of the gyrations of the coupled vessel. Moreover, it turns out that these two problems are very related.

Let us first address the second one: what happens with the compatibility conditions. Recall that when the input $u = u'$ of the system of $\mathcal{V}'$ satisfies the compatibility condition

$$\left(\sigma_j \frac{\partial}{\partial t_k} - \sigma_k \frac{\partial}{\partial t_j} + i\gamma_{kj}^{\mathrm{in}\prime}\right) u = 0, \qquad j,k = 1,\dots,n,$$

then its output v' satisfies the compatibility condition

$$\left(\sigma_j \frac{\partial}{\partial t_k} - \sigma_k \frac{\partial}{\partial t_j} + i\gamma_{kj}^{\mathrm{out}\prime}\right) v' = 0, \qquad j,k = 1,\dots,n.$$

Similarly, when the input $u'' = v'$ of the system of $\mathcal{V}''$ satisfies the compatibility condition

$$\left(\sigma_j \frac{\partial}{\partial t_k} - \sigma_k \frac{\partial}{\partial t_j} + i\gamma_{kj}^{\mathrm{in}\prime\prime}\right) v' = 0, \qquad j,k = 1,\dots,n,$$

then its output $v = v''$ satisfies the compatibility condition

$$\left(\sigma_j \frac{\partial}{\partial t_k} - \sigma_k \frac{\partial}{\partial t_j} + i\gamma_{kj}^{\mathrm{out}\prime\prime}\right) v = 0, \qquad j,k = 1,\dots,n.$$

The cascade connection only makes sense if, whenever the system of $\mathcal{V}'$ is fed a compatible input, the output of the system of $\mathcal{V}'$ is compatible as an input of the system of $\mathcal{V}''$. Looking at the compatibility conditions above, we see that this forces

$$\gamma_{kj}^{\mathrm{out}\prime} = \gamma_{kj}^{\mathrm{in}\prime\prime}, \qquad j,k = 1,\dots,n, \tag{5.22}$$

so that the output compatibility conditions of $\mathcal{V}'$ and the input compatibility conditions of $\mathcal{V}''$. The conditions (5.22) are called *matching conditions*. They are the necessary and sufficient conditions to form the coupling of two vessels.

So far, we have seen that the cascade connection of the systems of $\mathcal{V}'$ and $\mathcal{V}''$ makes sense if and only if the matching conditions (5.22) hold. Now we will address the question of whether the operators A_k defined in (5.20) commute. We see that A_k commute if and only if

$$\Phi''^* \sigma_k \Phi' A'_j + A''_k \Phi''^* \sigma_j \Phi' = \Phi''^* \sigma_j \Phi' A'_k + A''_j \Phi''^* \sigma_k \Phi', \qquad j,k = 1,\dots,n.$$

Using the vessel conditions for the gyrations, we can rewrite this as

$$\Phi''^{*}\gamma_{kj}^{\mathrm{out}'}\Phi = \Phi''^{*}\gamma_{kj}^{\mathrm{in}''}\Phi, \qquad j,k = 1,\ldots,n.$$

Hence, if the matching conditions are satisfied, then the operators A_k commute. Moreover, if Φ^* is injective (and therefore Φ is surjective), the matching conditions are also necessary to make A_k commute.

It remains to define the gyrations $\gamma_{kj}^{\mathrm{in}}$ and $\gamma_{kj}^{\mathrm{out}}$ for the coupled vessel and to check that all the conditions in the definition of a vessel are satisfied. Again, the interpretation of coupling as cascade connection of two systems motivates the definition of the gyrations: we should put $\gamma_{kj}^{\mathrm{in}} = \gamma_{kj}^{\mathrm{in}'}$ and $\gamma_{kj}^{\mathrm{out}} = \gamma_{kj}^{\mathrm{out}''}$, because the input of the cascade connected system is the input of the first system and the output of the cascade connected system is the output of the second system.

Now, the linkage conditions (5.3) are satisfied because

$$\begin{aligned}
\gamma_{kj}^{\mathrm{out}} &= \gamma_{kj}^{\mathrm{out}''} = \gamma_{kj}^{\mathrm{in}''} + i(\sigma_k\Phi''\Phi''^{*}\sigma_j - \sigma_j\Phi''\Phi''^{*}\sigma_k)\\
&= \gamma_{kj}^{\mathrm{out}'} + i(\sigma_k\Phi''\Phi''^{*}\sigma_j - \sigma_j\Phi''\Phi''^{*}\sigma_k)\\
&= \gamma_{kj}^{\mathrm{in}'} + i(\sigma_k\Phi'\Phi'^{*}\sigma_j - \sigma_j\Phi'\Phi'^{*}\sigma_k) + i(\sigma_k\Phi''\Phi''^{*}\sigma_j - \sigma_j\Phi''\Phi''^{*}\sigma_k)\\
&= \gamma_{kj}^{\mathrm{in}} + i(\sigma_k\Phi\Phi^{*}\sigma_j - \sigma_j\Phi\Phi^{*}\sigma_k).
\end{aligned}$$

Here we have used the matching conditions (5.22) and the fact that the vessels $\mathcal{V}'$ and $\mathcal{V}''$ satisfy the linkage conditions.

A similar calculation using conditions (5.4) for $\mathcal{V}'$ and $\mathcal{V}''$ and the linkage conditions for $\mathcal{V}''$ shows that $\mathcal{V}$ satisfies conditions (5.4). Using the linkage conditions for $\mathcal{V}$, we also obtain conditions (5.2) for $\mathcal{V}$. Conditions (5.1) for $\mathcal{V}$ are also easy to obtain. Thus, we have proved the following.

Proposition 5.2 (Coupling of vessels)**.** *Let*

$$\mathcal{V}' = (A_k'; H', \Phi', E; \sigma_k, \gamma_{kj}^{in'}, \gamma_{kj}^{out'})$$

and

$$\mathcal{V}'' = (A_k''; H'', \Phi'', E; \sigma_k, \gamma_{kj}^{in''}, \gamma_{kj}^{out''})$$

be two vessels with the same rates and outer space and such that $\gamma_{kj}^{out''} = \gamma_{kj}^{in'}$. *Then*

$$\mathcal{V} = (A_k; H' \oplus H'', \Phi' \oplus \Phi'', E; \sigma_k, \gamma_{kj}^{in'}, \gamma_{kj}^{out''})$$

is also a vessel, where the operators A_k are defined by (5.20)*. The vessel $\mathcal{V}$ is called the coupling of $\mathcal{V}'$ and $\mathcal{V}''$.*

The inverse procedure of coupling is called *decomposition*. Given a vessel $\mathcal{V}$, a decomposition of $\mathcal{V}$ is a way to break up $\mathcal{V}$ into two vessels $\mathcal{V}'$ and $\mathcal{V}''$ such that $\mathcal{V}$ is the coupling of $\mathcal{V}'$ and $\mathcal{V}''$. If we observe (5.20), we see that H' is invariant for the operators A_k'. It turns out that this is the only thing that is needed to form the decomposition.

Proposition 5.3 (Decomposition of a vessel)**.** *Let*

$$\mathcal{V} = (A_k; H, \Phi, E; \sigma_k, \gamma_{kj}^{in}, \gamma_{kj}^{out})$$

be a vessel and H' a subspace of H that is invariant for all the operators A_k. Put $H'' = H \ominus H'$, $\Phi' = \Phi|H'$, $\Phi'' = \Phi|H''$, $A'_k = A_k|H'$, $A''_k = P_{H''}A_k|H''$, $\gamma_{kj}^{in'} = \gamma_{kj}^{in}$, $\gamma_{kj}^{out''} = \gamma_{kj}^{out}$ and define $\gamma_{kj}^{out''}$ and $\gamma_{kj}^{in'}$ by

$$\begin{aligned}\gamma_{kj}^{out'} &= \gamma_{kj}^{in} + i(\sigma_k \Phi' \Phi'^* \sigma_j - \sigma_j \Phi' \Phi'^* \sigma_k),\\ \gamma_{kj}^{in''} &= \gamma_{kj}^{out} - i(\sigma_k \Phi'' \Phi''^* \sigma_j - \sigma_j \Phi'' \Phi''^* \sigma_k).\end{aligned}$$

Then

$$\mathcal{V}' = (A'_k; H', \Phi', E; \sigma_k, \gamma_{kj}^{in'}, \gamma_{kj}^{out'})$$

and

$$\mathcal{V}'' = (A''_k; H'', \Phi'', E; \sigma_k, \gamma_{kj}^{in''}, \gamma_{kj}^{out''})$$

are vessels. Moreover, $\gamma_{kj}^{out''} = \gamma_{kj}^{in'}$ and $\mathcal{V}$ is the coupling of $\mathcal{V}'$ and $\mathcal{V}''$.

The proof of this proposition amounts to doing similar calculations to the above ones, so we omit it. The coupling of vessels allows one to construct vessels out of simpler vessels (for instance, vessels where H has small dimension). The decomposition allows one to understand a vessel in terms of its parts, which are simpler vessels. These two procedures are, in this sense, very similar to the dilation and compression of operators.

Another construction that can be done with vessels is the *adjoint vessel*. If

$$\mathcal{V} = (A_k; H, \Phi, E; \sigma_k, \gamma_{kj}^{\text{in}}, \gamma_{kj}^{\text{out}})$$

is a vessel, then it is easy to check that

$$\mathcal{V}^* = (A_k^*; H, -\Phi, E; -\sigma_k, -\gamma_{kj}^{\text{out}}, -\gamma_{kj}^{\text{in}})$$

is also a vessel, which is called the adjoint vessel. Note that the input and the output of the system corresponding to the adjoint vessel are interchanged.

A useful application of the system interpretation of a vessel is the definition of its so called *complete characteristic function*. Once again, we consider a vessel $\mathcal{V}$, and we restrict its associated dynamical system to a straight line $L = L_\xi$, $\xi \in \mathbb{R}^n$, obtaining the system (5.16). The complete characteristic function $S(\xi, z)$ is, as a function of z, the transfer function of this system. To make sense of this transfer function and to compute it, we assume that the input, output and state of the system are all monochromatic waves of the same complex frequency $z \in \mathbb{C}$:

$$u(\tau) = u_0 e^{iz\tau}, \qquad v(\tau) = v_0 e^{iz\tau}, \qquad f(\tau) = f_0 e^{iz\tau},$$

for some vectors $u_0, v_0 \in E$, $f_0 \in H$.

Plugging these expressions into the system we get

$$\begin{aligned}-zf_0 + \xi A f_0 &= \Phi^* \xi \sigma u_0,\\ v_0 &= u_0 - i\Phi f_0,\end{aligned}$$

so we see that

$$v_0 = u_0 - i\Phi(\xi A - zI)^{-1}\Phi^*\xi\sigma u_0.$$

This means that the transfer function of the system is

$$S(\xi, z) = I - i\Phi(\xi A - zI)^{-1}\Phi^*\xi\sigma, \qquad \xi \in \mathbb{R}^n,\ z \in \mathbb{C} \setminus \sigma(\xi A). \tag{5.23}$$

so that $v_0 = S(\xi, z)u_0$. This is the complete characteristic function. It is usually defined for $\xi \in \mathbb{C}^n$ (instead of $\xi \in \mathbb{R}^n$). This no longer has a clear interpretation in terms of the system, but can be thought of as an analytic continuation of the case $\xi \in \mathbb{R}^n$.

For fixed $\xi \in \mathbb{C}^n$, the complete characteristic function is a $\mathcal{B}(E)$-valued analytic function on $\widehat{\mathbb{C}} \setminus \sigma(\xi A)$. It allows one to use tools from the theory of analytic functions to study the vessel. For instance, using the definition of coupling as cascade connection of systems, it is trivial to check that if the vessels $\mathcal{V}'$ and $\mathcal{V}''$ have complete characteristic functions $S'(\xi, z)$ and $S''(\xi, z)$, then the complete characteristic function of their coupling is $S''(\xi, z)S'(\xi, z)$. Indeed, for $\xi \in \mathbb{R}^n$ one just uses the fact that the transfer function of the cascade connection is the composition of the transfer functions of the two systems. Then this can be extended to $\xi \in \mathbb{C}^n$ by analytic continuation.

Conversely, if a vessel $\mathcal{V}$ is decomposed into vessels $\mathcal{V}'$ and $\mathcal{V}''$, then its characteristic function $S(\xi, z)$ factors as $S(\xi, z) = S''(\xi, z)S'(\xi, z)$. Since every subspace which is joint invariant for the operators A_k gives a decomposition of $\mathcal{V}$, this gives a link between joint invariant subspaces and the factorizations of $S(\xi, z)$.

Another important fact about the complete characteristic function is that, in a certain sense, it contains all the information about the vessel. To make this precise, we need to introduce a natural notion of isomorphism for vessels. The *principal subspace* of a vessel $\mathcal{V}$ is the subspace

$$\widehat{H} = \bigvee_{k_1,\ldots,k_n \geq 0} A_1^{k_1} \cdots A_n^{k_n} \Phi^* E = \bigvee_{k_1,\ldots,k_n \geq 0} A_1^{k_1 *} \cdots A_n^{k_n *} \Phi^* E.$$

Here the second inequality is not obvious but it can be proved using (5.1) (see [LKMV95, Lemma 3.4.2]). A vessel is called *irreducible* if its principal subspace $\widehat{H}$ is all of H.

Two vessels

$$\mathcal{V} = (A_k; H, \Phi, E; \sigma_k, \gamma_{kj}^{\text{in}}, \gamma_{kj}^{\text{out}})$$

and

$$\mathcal{V}' = (A'_k; H', \Phi', E; \sigma_k, \gamma_{kj}^{\text{in}}, \gamma_{kj}^{\text{out}})$$

with the same rates and gyrations are called *unitarily equivalent* if there exists a unitary $U : H \to H'$ such that $A'_k = UA_kU^*$ and $\Phi' = \Phi U^*$. These two vessels are called *unitarily equivalent on their principal subspaces* if

$$(A_k|\widehat{H}; \widehat{H}, \Phi|\widehat{H}, E; \sigma_k, \gamma_{kj}^{\text{in}}, \gamma_{kj}^{\text{out}})$$

and

$$(A'_k|\widehat{H}'; \widehat{H}', \Phi|\widehat{H}', E; \sigma_k, \gamma_{kj}^{\text{in}}, \gamma_{kj}^{\text{out}})$$

are unitarily equivalent, where $\widehat{H}$ and $\widehat{H}'$ are the principal subspaces of $\mathcal{V}$ and $\mathcal{V}'$ respectively.

Proposition 5.4. *Let $\mathcal{V}$ and $\mathcal{V}'$ be two vessels as above with the same rates and gyrations. Fix $\xi \in \mathbb{C}^n$ and assume that $\det \xi\sigma \neq 0$. Then $\mathcal{V}$ and $\mathcal{V}'$ are unitarily equivalent on their principal subspaces if and only if $S(\xi, z) = S'(\xi, z)$ for all z in a neighbourhood of ∞.*

For the proof of this proposition, see [LKMV95, Theorem 3.4.4].

We will now define the *discriminant varieties* of a vessel. These are algebraic varities in $\mathbb{C}^n$ that allow one to study vessels using tools from algebraic geometry. Among other things, it is possible to use the discriminant varieties to to give a generalization of the Cayley-Hamilton Theorem. We define the input discriminant ideal $\mathcal{J}^{\text{in}}$ as the ideal in $\mathbb{C}[z_1, \dots, z_n]$ generated by the polynomials of the form

$$\det\left(\sum_{j,k=1}^{n} \Gamma^{jk}(z_j\sigma_k - z_k\sigma_j + \gamma_{jk}^{\text{in}})\right),$$

where $\Gamma^{jk} = -\Gamma^{kj}$ are arbitrary operators on the outer space E. Similarly, the output discriminant ideal $\mathcal{J}^{\text{out}}$ is defined by replacing the input gyrations γ_{jk}^{in} by the output gyrations γ_{jk}^{out}. The input and output discriminant varieties, D^{in} and D^{out} respectively, are defined as the algebraic varieties in $\mathbb{C}^n$ associated with these ideals. This means that

$$D^{\text{in}} = \{z \in \mathbb{C}^n : p(z) = 0, \forall p \in \mathcal{J}^{\text{in}}\}, \qquad D^{\text{out}} = \{z \in \mathbb{C}^n : p(z) = 0, \forall p \in \mathcal{J}^{\text{out}}\}.$$

We can also define, for $z \in \mathbb{C}^n$, the following subspaces of E:

$$E^{\text{in}}(z) = \bigcap_{j,k=1}^{n} \ker(z_j\sigma_k - z_k\sigma_j + \gamma_{jk}^{\text{in}}), \qquad E^{\text{out}}(z) = \bigcap_{j,k=1}^{n} \ker(z_j\sigma_k - z_k\sigma_j + \gamma_{jk}^{\text{out}}).$$

Their connection with the discriminant varieties is that $z \in D^{\text{in}}$ if and only if $E^{\text{in}}(z) \neq 0$, and analogously for the output discriminant variety (see [LKMV95, Proposition 4.1.3]).

Now we can give the statement of the generalized Cayley-Hamilton theorem. For the proof, see [LKMV95, Theorem 4.1.2].

Theorem 5.5 (Generalized Cayley-Hamilton)**.** *Let $\mathcal{V} = (A_k; H, \Phi, E; \sigma_k, \gamma_{kj}^{in}, \gamma_{kj}^{out})$ be an irreducible vessel, and $p^{in}(z) \in \mathcal{J}^{in}$, $p^{out}(z) \in \mathcal{J}^{out}$ arbitrary polynomials in the input and output discriminant ideals of the vessel. Then the operators $A_1, \dots, A_n$ satisfy the algebraic equations*

$$p^{in}(A_1^*, \dots, A_n^*) = 0, \qquad p^{out}(A_1, \dots, A_n) = 0.$$

In the next section we will give the statement of this theorem for vessels of two operators and show how the classical Cayley-Hamilton theorem can be derived from it.

The next lemma gives a relation between the tuples of commuting non-selfadjoint operators studied by the theory of Livšic and Vinnikov and tuples of commuting contractions. An operator A is called *dissipative* if $A - A^* \geq 0$.

Lemma 5.6. *Let $A \in \mathcal{B}(H)$ be a dissipative operator such that $A - A^*$ has finite rank. Then the operator T defined by $T = (A - iI)(A + iI)^{-1}$ is a contraction with finite defects. Conversely, if T is a contraction such that $1 \notin \sigma(T)$ and either $D_T = (I - T^*T)^{1/2}$, or $D_{T^*} = (I - TT^*)^{1/2}$ has finite rank, then the operator A defined by $A = i(I + T)(I - T)^{-1}$ is dissipative and $A - A^*$ has finite rank.*

Proof. It is well known that if A is dissipative then $T = (A - iI)(A + iI)^{-1}$ is a contraction and conversely if T is a contraction with $1 \notin \sigma(T)$ then $A = i(I + T)(I - T)^{-1}$ is dissipative. Moreover, these two transformations are the inverse of each other. See, for instance [SNF67, Chapter IV, Section 4].

Let us now relate the defect operators of T with the operator $A - A^*$. We have

$$I - T^*T = I - (A^* - iI)^{-1}(A^* + iI)(A - iI)(A + iI)^{-1}.$$

Since

$$(A^* + iI)(A - iI) = (A^* - iI)(A + iI) + 2i(A - A^*),$$

we see that

$$I - T^*T = -2i(A^* - iI)^{-1}(A - A^*)(A + iI)^{-1}.$$

A similar calculation yields

$$I - TT^* = -2i(A + iI)^{-1}(A - A^*)(A^* - iI)^{-1}.$$

If $A - A^*$ has finite rank, we see that both $I - T^*T$ and $I - TT^*$ have finite rank, so T has finite defects. Conversely, if either $I - T^*T$ or $I - TT^*$ has finite rank, then we see that $A - A^*$ also has finite rank. Note that it also follows that both $I - T^*T$ and $I - TT^*$ must have finite rank in this case. □

Assume that $\mathcal{V}$ is a vessel such that all the operators A_k are dissipative. The lemma above implies that if we define $T_k = (A_k - iI)(A_k + iI)^{-1}$, then $(T_1, \dots, T_n)$ is a tuple of commuting contractions with finite defects. Conversely, if $(T_1, \dots, T_n)$ is a tuple of commuting contractions with finite defects and such that $1 \notin \sigma(T_k)$, $k = 1, \dots, n$, then the operators $A_k = i(I + T_k)(I - T_k)^{-1}$ commute and $A_k - A_k^*$ has finite rank for $k = 1, \dots, n$. It follows that the tuple $(A_1, \dots, A_n)$ can be embedded into a vessel using the procedure described at the beginning of this section.

When studying von Neumann's inequality, it is enough to study tuples of commuting strictly contractive (finite-dimensional) matrices. Therefore, it is enough to study von Neumman's inequality for tuples $(T_1, \dots, T_n)$ which arise from a vessel by means of this transformation.

5.2. Vessels of two commuting operators and the discriminant curve

For a vessel of two operators all the results of the preceding section can be applied. Moreover, one can make some simplifications that allow one to develop the theory further.

First note that since $\gamma_{12}^{\text{in}} = -\gamma_{21}^{\text{in}}$, there is essentially only one input gyration. We will write $\gamma^{\text{in}} = \gamma_{12}^{\text{in}}$. Analogously, there is essentially only one output gyration and we will write $\gamma^{\text{out}} = \gamma_{12}^{\text{out}}$. Also, the input discriminant ideal $\mathcal{J}^{\text{in}}$ is principal, which means that it is generated by a single polynomial. This happens because

$$\det\left(\Gamma^{12}(z_1\sigma_2 - z_2\sigma_1 + \gamma_{12}^{\text{in}}) + \Gamma^{21}(z_2\sigma_1 - z_1\sigma_2 + \gamma_{21}^{\text{in}})\right) = \det(2\Gamma^{12})\det(z_1\sigma_2 - z_2\sigma_1 + \gamma^{\text{in}}).$$

Hence, the polynomial $\det(z_1\sigma_2 - z_2\sigma_1 + \gamma^{\text{in}})$ generates the ideal $\mathcal{J}^{\text{in}}$. This polynomial is called the input discriminant polynomial. Similarly, the output discriminant ideal is generated by the polynomial $\det(z_1\sigma_2 - z_2\sigma_1 + \gamma^{\text{out}})$, which is called the output discriminant polynomial.

In fact, it turns out that the input and output discriminant polynomials are equal:

$$\det(z_1\sigma_2 - z_2\sigma_1 + \gamma^{\text{in}}) = \det(z_1\sigma_2 - z_2\sigma_1 + \gamma^{\text{out}}),$$

see [LKMV95, Corollary 4.2.2]. We will denote by $\Delta(z_1, z_2)$ this polynomial, and we will call it the *discriminant polynomial*. This implies that the input and output discriminant varieties coincide, so we will write

$$D = D^{\text{in}} = D^{\text{out}}.$$

The variety D is either an algebraic curve in $\mathbb{C}^2$ or all of $\mathbb{C}^2$ (this second case is considered to be degenerate), so it will usually be called the *discriminant curve* of the vessel.

For vessels of more than two operators, the input and output discriminant varieties are distinct in general. However, they may only differ by a finite number of isolated points. It is believed that these points may be related to some pathologies involving commuting tuples of more than two operators, such as the failure of von Neumann's inequality or the non-existence of a dilation. A discussion of this fact is included in [LKMV95, Section 7.2].

With these observations, the generalized Cayley-Hamilton theorem becomes the following Theorem.

Theorem 5.7 (Generalized Cayley-Hamilton)**.** *Let $\mathcal{V} = (A_1, A_2; H, \Phi, E; \sigma_1, \sigma_2, \gamma^{in}, \gamma^{out})$ be an irreducible two operator vessel, and $\Delta(z_1, z_2)$ its discriminant polynomial. Then the operators A_1, A_2 satisfy the algebraic equations*

$$\Delta(A_1, A_2) = 0, \qquad \Delta(A_1^*, A_2^*) = 0.$$

Example 5.8. Here we show how to derive the classical Cayley-Hamilton theorem from this theorem. Let A be an operator on a finite-dimensional Hilbert space H of dimension m. Recall that the classical Cayley-Hamilton theorem states that if p is the characteristic polynomial of A:

$$p(z) = \det(zI - A), \tag{5.24}$$

then $p(A) = 0$. To write rewrite this, we cannot replace z by A in the right hand side of (5.24), as this is not correct. We would obtain $\det(A - A)$, which is indeed 0, but this is not at all the reason why the classical Cayley-Hamilton theorem is true. We need to introduce some notation to do this correctly.

We use the following notation, inspired by the Kronecker product or tensor product. If $C = [c_{jk}]$ is an $s \times s$ matrix and $D = [d_{jk}]$ is a $t \times t$ matrix, we denote by $C \otimes D$ the block matrix $[c_{jk}D]$. We define the determinant $\det(C \otimes D)$ as the $t \times t$ matrix $p(D)$, where $p(z) = \det(zC)$. In other words, we apply the formula for the determinant of an $s \times s$ matrix to the block matrix $C \otimes D$ (which has $s \times s$ blocks). We extend this notation to expressions of the form $\det(C_1 \otimes D_1 + \cdots + C_r \otimes D_r)$. Here the matrices C_j must have the same size $s \times s$, and the matrices D_j must have the same size $t \times t$ and commute. In this case, the determinant is defined as $p(D_1, \dots, D_n)$ where $p(z_1, \dots, z_n) = \det(z_1 C_1 + \cdots + z_n C_n)$.

With this notation, the classical Caylely-Hamilton theorem can be rewritten as the identity

$$\det(I \otimes A - A \otimes I) = 0. \tag{5.25}$$

Now we construct a vessel which includes the operator A and show how to derive (5.25) from Theorem 5.7. We put $A_1 = A$, $A_2 = iI$ and embed these two operators in a vessel following the procedure that we have explained above. The space E is defined by

$$E = \frac{1}{i}(A_1 - A_1^*)H + \frac{1}{i}(A_2 - A_2^*)H = H.$$

Here the second equality comes from $A_2 - A_2^* = 2iI$. Hence, $\Phi = P_E = I$. The rates are defined by

$$\sigma_1 = \frac{1}{i}(A - A^*), \qquad \sigma_2 = 2I,$$

and the gyrations can be computed as

$$\gamma^{\text{in}} = \gamma_{12}^{\text{in}} = \frac{1}{i}(A_1 A_2^* - A_2 A_1^*) = -(A + A^*),$$
$$\gamma^{\text{out}} = \gamma_{12}^{\text{out}} = \frac{1}{i}(A_2^* A_1 - A_1^* A_2) = -(A + A^*).$$

Hence, the discriminant polynomial is

$$\Delta(z_1, z_2) = \det\big(z_1 2I + z_2 i(A - A^*) - (A + A^*)\big).$$

Therefore, the conclusion of the generalized Cayley-Hamilton theorem corresponds to

$$0 = \Delta(A_1, A_2) = \det\big(2I \otimes A - I(A - A^*) \otimes I - (A + A^*) \otimes I\big) = 2^m \det(I \otimes A - A \otimes I),$$

so we get (5.25).

An important tool in the study of two operator vessels is the joint characteristic function. We have seen that one can define a complete characteristic function $S(\xi, z)$, $\xi \in \mathbb{C}^n$, $z \in \mathbb{C}$ for vessels of n operators. In the case of a two operator vessel, the complete characteristic function has the form

$$S(\xi_1, \xi_2, z) = I - i\Phi(\xi_1 A_1 + \xi_2 A_2 - zI)^{-1}\Phi^*(\xi_1\sigma_1 + \xi_2\sigma_2).$$

This is a function of three complex variables, but because of homogeneity, it can be thought of as a function of two independent complex variables. We have also seen that there is a relation between the factorizations of the complete characteristic function and the joint invariant subspaces of the operators in the vessel. However, functions of two complex variables do not have a good factorization theory.

A better alternative is the *joint characteristic function* $\widehat{S}(z)$. If $z = (z_1, z_2) \in D$ is a point $\widehat{S}(z)$ on the discriminant curve, then the operator

$$S(\xi_1, \xi_2, \xi_1 z_1 + \xi_2 z_2)|E^{\mathrm{in}}(z)$$

does not depend on the election of $(\xi_1, \xi_2) \in \mathbb{C}^2$ as long as

$$\xi_1 z_1 + \xi_2 z_2 \notin \sigma(\xi_1 A_1 + \xi_2 A_2). \tag{5.26}$$

Moreover, this operator maps $E^{\mathrm{in}}(z)$ into $E^{\mathrm{out}}(z)$. A proof of these facts can be seen in [LKMV95, Theorem 4.3.1]. We will also give later a proof based on the system theoretical interpretation. Hence, we can define the joint characteristic function

$$\widehat{S}(z) : E^{\mathrm{in}}(z) \to E^{\mathrm{out}}(z),$$

for all $z = (z_1, z_2) \in D$ for which there exists $(\xi_1, \xi_2) \in \mathbb{C}^2$ such that (5.26) holds, by

$$\widehat{S}(z) = S(\xi_1, \xi_2, \xi_1 z_1 + \xi_2 z_2)|E^{\mathrm{in}}(z).$$

One can think of $E^{\mathrm{in}}(z)$ and $E^{\mathrm{out}}(z)$ as vector bundles on the algebraic curve D. More precisely, they are vector bundles on the desingularization of D (see Section 6.2 for an introduction to the desingularization of an algebraic curve in a slightly different context). Hence, $\widehat{S}$ can be interpreted as a bundle map on an algebraic curve, so it is a function of one independent complex variable, and admits a good factorization theory.

The joint characteristic function has also an interpretation as a transfer function of the associated system. We take a double frequency $\lambda = (\lambda_1, \lambda_2) \in \mathbb{C}^2$ and assume that the input, state and output of the system are waves with this frequency:

$$u(t_1, t_2) = u_0 e^{i\lambda_1 t_1 + i\lambda_2 t_2}, \quad f(t_1, t_2) = f_0 e^{i\lambda_1 t_1 + i\lambda_2 t_2}, \quad v(t_1, t_2) = v_0 e^{i\lambda_1 t_1 + i\lambda_2 t_2}.$$

Then the input and output compatibility conditions (5.14) and (5.15) are

$$(\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma^{\mathrm{in}}) u_0 = 0, \qquad (\lambda_1 \sigma_2 - \lambda_2 \sigma_1 + \gamma^{\mathrm{out}}) v_0 = 0.$$

This means that $\lambda \in D$, $u_0 \in E^{\mathrm{in}}(\lambda)$ and $v_0 \in E^{\mathrm{out}}(\lambda)$. If we integrate the system along any temporal straight line $(\xi_1, \xi_2) \cdot \mathbb{R}$, and apply the results obtained above for the transfer function of a vessel, we get

$$v_0 = S(\xi_1, \xi_2; \xi_1 \lambda_1 + \xi_2 \lambda_2) u_0.$$

This proves that $S(\xi_1, \xi_2; \xi_1 z_1 + \xi_2 z_2)|E^{\mathrm{in}}(z)$ does not depend on ξ, that it maps $E^{\mathrm{in}}(\lambda)$ into $E^{\mathrm{out}}(\lambda)$ and that

$$v_0 = \widehat{S}(\lambda) u_0.$$

Finally, another important fact of the joint characteristic function is that the complete characteristic function can be recovered from it by the so called *restoration formula*. Assume that $\dim E = m$ and that the discriminant curve D has degree m (this is a non-degeneracy condition). Fix a line $\xi_1 y_1 + \xi_2 y_2 = z$ in $\mathbb{C}^2$ and assume that the line intersects D in m different points $p_1, \dots, p_m$. Then the space E decomposes in direct sum as

$$E = E(p_1) \dotplus \cdots \dotplus E(p_m).$$

Let $P(p_j, \xi_1, \xi_2, z)$ be the projection onto $E(p_j)$ according to this decomposition. The restoration formula allows one to recover the complete characteristic function by

$$S(\xi_1, \xi_2, z) = \sum_{j=1}^{m} \widehat{S}(p_j) P(p_j, \xi_1, \xi_2, z).$$

A proof of these facts can be found in [LKMV95, Section 10.3]. Thus, the joint characteristic function essentially contains all the information about the vessel.

6. Operator pools and separating structures

Chapters 6 and 7 are based on unpublished joint work in progress with Victor Vinnikov and Dmitry Yakubovich. The main goal of this project is to construct a structure that allows vessels to be dilated. We will call this new structure a *separating structure*.

In this chapter, we first define another new structure, which we call *operator pool*. This structure is formed by a pair of selfadjoint operators A_1 and A_2 and auxiliary matrices $\sigma_1, \sigma_2, \gamma$ which satisfy a relation which resembles the relations satisfied by a vessel. First, we give the definition of a pool and its basic properties. This motivates the definition of its discriminant curve. It is defined using a determinantal representation in a similar way to how it is done for vessels.

Then we pass to the subject of separating structures. First we introduce *affine separating structures*. These are separating structures for which no orthogonality conditions are imposed. Treating them first allows us to see which properties of separating structures are consequences of the linear space structure only. Then we introduce the *mosaic function*, which is an analytic function whose values are parallel projections on a finite dimensional space. In a certain sense, it codifies the information about the separating structure and it is used in the construction of a model for the structure using vector-valued analytic functions.

Next, we define *orthogonal separating structures*, which is a particular class of affine separating structures in which we impose certain additional orthogonality conditions. These are the structures which we will call *separating structures* in the sequel. We show how an orthogonal separating structure can be embedded in a pool. In particular, a discriminant curve is associated with a separating structure.

Then we show how the discriminant curve of a separating structure can be divided into two halves. This is a decomposition into a disjoint union $\widehat{X} = \widehat{X}_+ \cup \widehat{X}_\mathbb{R} \cup \widehat{X}_-$, where $\widehat{X}_\mathbb{R}$ is the set of real points of the curve $\widehat{X}$ and $\widehat{X}_+$ and $\widehat{X}_-$ are the two halves of the curve. Using the two halves and a projection valued meromorphic function on $\widehat{X}$, the mosaic function can be recovered by means of the *restoration formula*. A restoration formula playing a similar role has already appeared in the context of vessels.

6.1. Operator pools

In this section we give the definition and basic properties of an operator pool.

Definition 6.1. Let K be a Hilbert space, M a finite-dimensional Hilbert space, $\Phi : K \to M$ an operator, and A_1, A_2 two commuting selfadjoint operators on K. The tuple

$$\mathcal{P} = (A_1, A_2; K, \Phi, M; \sigma_1, \sigma_2, \gamma)$$

is called an *operator pool* if σ_j, γ are selfadjoint operators on M such that the following three-term relationship holds:

$$\sigma_2\Phi A_1 - \sigma_1\Phi A_2 + \gamma\Phi = 0. \tag{6.1}$$

The operators σ_j are called *rates* and the operator γ is called *gyration*, as in the case of vessels (see Chapter 5). We define the *principal subspace* of the pool $\mathcal{P}$ as

$$\widehat{K} = \bigvee_{k_1,k_2\geq 0} A_1^{k_1}A_2^{k_2}\Phi^* M. \tag{6.2}$$

We say that the pool is *irreducible* if $K = \widehat{K}$. Typically, we will be considering irreducible pools.

We say that a direction $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$, is nondegenerate if $\xi_1\overline{\xi_2} \notin \mathbb{R}$. We will denote by Ξ the set of all nondegenerate directions:

$$\Xi = \{\xi = (\xi_1, \xi_2) : \xi_1\overline{\xi_2} \notin \mathbb{R}\}.$$

For every fixed nondegenerate direction $\xi = (\xi_1, \xi_2) \in \Xi$, the operator

$$N_\xi = \xi_1 A_1 + \xi_2 A_2 \tag{6.3}$$

is normal and (6.1) is equivalent to

$$\alpha_\xi^*\Phi N_\xi + \alpha_\xi\Phi N_\xi^* + \gamma_\xi\Phi = 0, \tag{6.4}$$

where

$$\alpha_\xi = i(\xi_1\sigma_1 + \xi_2\sigma_2), \qquad \gamma_\xi = -2\,\mathrm{Im}(\overline{\xi_1}\xi_2)\gamma. \tag{6.5}$$

Now, we will construct a functional model for the pool $\mathcal{P}$ using an L^2 space of M-valued functions. Let E_ξ be the (projection-valued) spectral measure of N_ξ. It is easy to see that

$$\widehat{K} = \bigvee_{k_1,k_2\geq 0} N_\xi^{k_1}N_\xi^{*k_2}\Phi^* M = \bigvee_{\substack{\Omega\subset\mathbb{C} \\ \Omega \text{ Borel}}} E_\xi(\Omega)\Phi^* M. \tag{6.6}$$

Here, the first equality is a direct consequence of (6.2) and (6.3). The second equality is true because the polynomials in z and $\overline{z}$ are uniformly dense in $C(\sigma(N_\xi))$, and the operator $E_\xi(\Omega)$ is strong limit of operators $g_n(N_\xi)$, where $g_n \in C(\sigma(N_\xi))$, by the properties of the spectral measure. The uniform density of the polynomials in z and $\overline{z}$ in $C(\sigma(N_\xi))$ is a consequence of the Stone-Weierstrass theorem (see, for instance, [Rud91, Theorem 5.7]).

We consider the non-negative matrix-valued measure e_ξ given by

$$e_\xi(\Omega) = \Phi E_\xi(\Omega)\Phi^* \in \mathcal{B}(M), \qquad \Omega \subset \mathbb{C}. \tag{6.7}$$

Next, we define the space $L^2(e_\xi)$ of Borel functions $\mathbb{C} \to M$ with the scalar product

$$\langle f, g\rangle_{L^2(e_\xi)} = \int_{\mathbb{C}} \langle de_\xi(u)f(u), g(u)\rangle.$$

After factoring by the set $\{f : \|f\|_{L^2(e_\xi)} = 0\}$, it becomes a Hilbert space. We have $f = 0$ in $L^2(e_\xi)$ if and only if $de_\xi(u)f(u) = 0$ a.e. $u \in \mathbb{C}$.

Recall that for every bounded Borel function g on $\mathbb{C}$, we can define the operator $g(N_\xi)$ by means of the spectral functional calculus. This will allow us to construct a unitary $W_\xi : \widehat{K} \to L^2(e_\xi)$.

Proposition 6.2. *If $\mathcal{P}$ is an irreducible pool, the operator W_ξ given by*

$$W_\xi g(N_\xi)\Phi^* m = g(\cdot)m$$

for $m \in M$ and g an arbitrary Borel function is well defined and extends by continuity to a unitary $W_\xi : \widehat{K} \to L^2(e_\xi)$. It also satisfies

$$(W_\xi N_\xi W_\xi^* h)(u) = uh(u), \qquad (W_\xi N_\xi^* W_\xi^* h)(u) = \overline{u}h(u) \tag{6.8}$$

and

$$\Phi W_\xi^* h = \int_{\mathbb{C}} de_\xi(u)h(u), \tag{6.9}$$

for every $h \in L^2(de_\xi)$.

Proof. First compute, for g, h Borel functions and $m, n \in M$,

$$\begin{aligned}\langle g(N_\xi)\Phi^* m, h(N_\xi)\Phi^* n\rangle &= \int_{\mathbb{C}} (\overline{h}g)(u)\langle dE_\xi(u)\Phi^* m, \Phi^* n\rangle = \int_{\mathbb{C}} (\overline{h}g)(u)\langle de_\xi(u)m, n\rangle \\ &= \langle g(\cdot)m, h(\cdot)n\rangle_{L^2(de_\xi)} \\ &= \langle W_\xi g(N_\xi)\Phi^* m, W_\xi h(N_\xi)\Phi^* n\rangle_{L^2(de_\xi)}.\end{aligned}$$

Using (6.6), since $\{g(\cdot)m : m \in M,\ g \text{ bounded Borel}\}$ spans $L^2(e_\xi)$, we see that W_ξ continues to a unitary.

To prove equations (6.8) and (6.9), observe that they are trivial for $h = g(\cdot)m$, with $m \in M$ and g bounded Borel, so they are also true for a general $h \in L^2(e_\xi)$ by continuity. □

The following Proposition will play an important role in the next section, because it will motivate the definition of the discriminant curve of the pool.

Proposition 6.3. *The following relation holds:*

$$(u\alpha_\xi^* + \overline{u}\alpha_\xi + \gamma_\xi)de_\xi(u) \equiv 0. \tag{6.10}$$

Proof. Multiply (6.4) by $W_\xi^* h$ on the right and use (6.8) and (6.9) to obtain

$$\int_{\mathbb{C}} (u\alpha_\xi^* + \overline{u}\alpha_\xi + \gamma_\xi)de_\xi(u)h(u) = 0.$$

Since this is true for every $h \in L^2(e_\xi)$, the Proposition follows. □

6.2. The discriminant curve

In this section we define the discriminant curve associated with an operator pool. We also introduce all the notation that we will use in the sequel.

The affine algebraic curve

$$X_{\mathrm{aff}} = \{(x_1, x_2) \in \mathbb{C}^2 : \det(x_1\sigma_2 - x_2\sigma_1 + \gamma) = 0\} \tag{6.11}$$

is called the *discriminant curve* of the pool. The discriminant curve is a real algebraic curve, equipped with the involution $*$ which sends $p = (x_1, x_2)$ to $p^* = (\overline{x_1}, \overline{x_2})$. The real part of the curve is

$$X_{\mathrm{aff},\mathbb{R}} = \{p \in X_{\mathrm{aff}} : p = p^*\} = X_{\mathrm{aff}} \cap \mathbb{R}^2.$$

For a nondegenerate direction $\xi = (\xi_1, \xi_2) \in \Xi$, we introduce the coordinates (z_ξ, w_ξ) in $\mathbb{C}^2$. For any $p = (x_1, x_2) \in \mathbb{C}^2$, we put

$$z_\xi(p) = \xi_1 x_1 + \xi_2 x_2, \qquad w_\xi(p) = \overline{\xi_1} x_1 + \overline{\xi_2} x_2.$$

Then we see that in these coordinates, the equation of X_{aff} rewrites as

$$\det(z_\xi \alpha_\xi^* + w_\xi \alpha_\xi + \gamma_\xi) = 0 \tag{6.12}$$

(see (6.5)). The involution $*$ can be written in these coordinates as $(z_\xi, w_\xi)^* = (\overline{w_\xi}, \overline{z_\xi})$. Using this and Proposition 6.3, we see that

$$\operatorname{supp} e_\xi \subset z_\xi(X_{\mathrm{aff},\mathbb{R}}). \tag{6.13}$$

Moreover, if the pool $\mathcal{P}$ is irreducible, we also have

$$\sigma(N_\xi) = \operatorname{supp} E_\xi = \operatorname{supp} e_\xi,$$

so we get

$$\sigma(N_\xi) \subset z_\xi(X_{\mathrm{aff},\mathbb{R}}).$$

We will always assume that X_{aff} is a curve of full degree $\dim M$. It is easy to see that this happens if and only if

$$\det(x_1\sigma_2 - x_2\sigma_1) \not\equiv 0. \tag{6.14}$$

In this case, α_ξ is invertible for a general direction $\xi \in \mathbb{C}^2$ (just put $x_2 = -\xi_1$, $x_1 = \xi_2$ in (6.14)). This non-degeneracy condition (6.14) also appears in the theory of vessels.

Whenever α_ξ is invertible, we define the operators

$$\Sigma_\xi = -\alpha_\xi^{-1}\alpha_\xi^*, \qquad D_\xi = -\alpha_\xi^{-1}\gamma_\xi. \tag{6.15}$$

The equation of X_{aff} can be rewritten as

$$\det(z_\xi \Sigma_\xi + D_\xi - w_\xi) = 0$$

(here and in the sequel we write λ instead of λI). This means that $(z_\xi, w_\xi) \in X_{\text{aff}}$ if and only if $w_\xi \in \sigma(z_\xi \Sigma_\xi + D_\xi)$. We use this to define a projection-valued function Q_ξ on X_{aff}. If $p \in X_{\text{aff}}$, we put

$$Q_\xi(p) = \Pi_{w_\xi(p)}(z_\xi(p)\Sigma_\xi + D_\xi) \in \mathcal{B}(M),$$

that is, $Q_\xi(p)$ is the Riesz projection of the matrix $z_\xi(p)\Sigma_\xi + D_\xi$ associated to the eigenvalue $w_\xi(p)$. By the properties of the Riesz projections, we get, for every $z_0 \in \mathbb{C}$, the direct sum decomposition

$$M = \sum_{\substack{p \in X_{\text{aff}} \\ z_\xi(p) = z_0}} Q_\xi(p) M. \tag{6.16}$$

We will also consider the projectivization X of the affine curve X_{aff}. We use projective coordinates $(\zeta_1 : \zeta_2 : \zeta_3)$ in $\mathbb{CP}^2$ and embed $\mathbb{C}^2$ in $\mathbb{CP}^2$ by

$$x_1 = \frac{\zeta_1}{\zeta_3}, \qquad x_2 = \frac{\zeta_2}{\zeta_3}.$$

The line $\zeta_3 = 0$ is the line at infinity. It will play an important role in the sequel. Since X_{aff} has degree $\dim M$, the projective curve X is

$$X = \{(\zeta_1 : \zeta_2 : \zeta_3) \in \mathbb{CP}^2 : \det(\zeta_1\sigma_2 - \zeta_2\sigma_1 + \zeta_3\gamma) = 0\}.$$

The involution $*$ extends to $\mathbb{CP}^2$ by $(\zeta_1 : \zeta_2 : \zeta_3)^* = (\overline{\zeta_1} : \overline{\zeta_2} : \overline{\zeta_3})$, the curve X is a real projective curve, and its real part $X_\mathbb{R}$ is the set of points of X fixed by the involution.

If $\xi = (\xi_1, \xi_2) \in \Xi$ is a nondegenerate direction, we define the projective coordinates in $\mathbb{CP}^2$

$$\eta_{\xi,1} = \xi_1\zeta_1 + \xi_2\zeta_2, \qquad \eta_{\xi,2} = \overline{\xi_1}\zeta_1 + \overline{\xi_2}\zeta_2, \qquad \eta_{\xi,3} = \zeta_3. \tag{6.17}$$

In these coordinates, the equation of X is

$$\det(\eta_{\xi,1}\alpha_\xi^* + \eta_{\xi,2}\alpha_\xi + \eta_{\xi,3}\gamma_\xi) = 0.$$

The functions z_ξ and w_ξ extend to meromorphic functions on $\mathbb{CP}^2$ by

$$z_\xi = \frac{\eta_{\xi,1}}{\eta_{\xi,3}}, \qquad w_\xi = \frac{\eta_{\xi,2}}{\eta_{\xi,3}}. \tag{6.18}$$

We define the points at infinity of X by

$$X_\infty = \{p \in X : \zeta_3(p) = 0\}.$$

By the Fundamental Theorem of Algebra (or Bézout's Theorem about the number of intersections of two projective curves), X_∞ is a set of $\dim M$ points counting multiplicities. Indeed, for a general direction $\xi \in \mathbb{C}^2$, we can rewrite the equation of X as

$$\det(\eta_{\xi,1}\Sigma_\xi + \eta_{\xi,3}D_\xi - \eta_{\xi,2}) = 0.$$

From this, the following proposition follows.

Proposition 6.4. *A point $p \in \mathbb{CP}^2$ belongs to X_∞ if and only if $\zeta_3(p) = 0$ and $(w_\xi/z_\xi)(p) \in \sigma(\Sigma_\xi)$.*

Now we will construct the blow-up (or desingularization) $\widehat{X}$ of X. Assume that the polynomial $\det(x_1\sigma_2 - x_2\sigma_1 + \gamma)$ decomposes into irreducible factors over $\mathbb{C}[x_1, x_2]$ as

$$\det(x_1\sigma_2 - x_2\sigma_1 + \gamma) = \prod_{j=1}^{J} p_j(x_1, x_2)^{m_j},$$

where all the polynomials $p_j(x_1, x_2)$ are distinct. Then we say that X has J components, $p_j(x_1, x_2) = 0$ is the (affine) equation of the j-th component X_j, and m_j is the multiplicity of X_j. An affine point $p \in X_{\mathrm{aff}}$ will be called regular if

$$\frac{\partial}{\partial x_k}\bigg|_p \left(\prod_{j=1}^{J} p_j(x_1, x_2)\right) \neq 0$$

for $k = 1$ or $k = 2$. The set of regular points, which will be denoted by X_0, has a natural Riemann surface structure, and $X \setminus X_0$ is a finite collection of points. The blow-up $\widehat{X}$ is a compact Riemann surface with J connected components $\widehat{X}_j$ and a surjective continuous map $\pi_X : \widehat{X} \to X$ such that the restriction $\pi_X|\pi_X^{-1}(X_0)$ is an isomorphism of Riemann surfaces. (Most authors require Riemann surfaces to be connected. Here we do not assume this, so for us, a Riemann surface will be a finite union of connected Riemann surfaces). The blow-up can be constructed by gluing a finite number of points to X_0 (see, for instance, [Mir95, Section III.2]). We put $\widehat{X}_0 = \pi_X^{-1}(X_0)$ and observe that $\widehat{X} \setminus \widehat{X}_0$ is finite.

Example 6.5. We include here a fairly trivial example to help clarify the notation. Suppose that the equation of X_{aff} is

$$(x_1x_2 - 1)^2(x_1x_2 - 2) = 0.$$

Then X has two components. The first component X_1 has the affine equation $x_1x_2 - 1 = 0$ and multiplicity 2, and the second component X_2 has the affine equation $x_1x_2 - 2 = 0$ and multiplicity 1.

All the points of X_{aff} are regular. The Riemann surface corresponding to the component X_1 is a Riemann sphere. Likewise, the Riemann surface which corresponds to X_2 is also a Riemann sphere. Therefore, $\widehat{X}$ is the union of two disjoint Riemann spheres, $\widehat{X}_1$ and $\widehat{X}_2$. The component $\widehat{X}_1$ has multiplicity 2 and $\widehat{X}_2$ has multiplicity 1.

The meromorphic functions z_ξ, w_ξ on $\mathbb{CP}^2$ induce meromorphic functions on $\widehat{X}$, which we will denote by the same letters:

$$z_\xi(p) = z_\xi(\pi_X(p)), \qquad w_\xi(p) = w_\xi(\pi_X(p)), \qquad p \in \widehat{X}.$$

The involution $*$ maps X_0 onto X_0, so it induces an antianalytic involution in $\widehat{X}$ (which we will also call $*$) by $\pi_X(p^*) = \pi_X(p)^*$ for $p \in X_0$, and then extending $*$ to all of $\widehat{X}$ by continuity. The real part of $\widehat{X}$, denoted by $\widehat{X}_{\mathbb{R}}$, is the set of points fixed by the involution. By definition, $\pi_X(\widehat{X}_\mathbb{R} \cap \widehat{X}_0) = X_\mathbb{R} \cap X_0$. However, the set $X_\mathbb{R}$ might be larger that $\pi_X(\widehat{X}_\mathbb{R})$ (although it will only differ by a finite number of points). Indeed, if $p \in X_\mathbb{R}$, then $*$ permutes the points in the fibre $\pi_X^{-1}(\{p\})$, but these points are not necessarily fixed by $*$ if the fibre has more than one point.

We define the points at infinity of $\widehat{X}$ by $\widehat{X}_\infty = \pi_X^{-1}(X_\infty)$. Note that every connected component $\widehat{X}_j$ contains points of $\widehat{X}_\infty$ (indeed degree(p_j) points).

The function Q_ξ induces a projection-valued meromorphic function on $\widehat{X}$ (which we will also denote by Q_ξ) defined by $Q_\xi(p) = Q_\xi(\pi_X(p))$ for $p \in \widehat{X}_0$.

6.3. Affine separating structures

In this section we give the definition and basic properties of affine separating structures. These are separating structures in which no orthogonality assumptions are made. This allows us to see what properties of a separating structure are just a consequence of the linear space structure.

Definition 6.6. Let K be a Hilbert space with a direct sum decomposition

$$K = H_{0,-} \dotplus M_- \dotplus M_+ \dotplus H_{0,+} \tag{6.19}$$

(not necessarily orthogonal). Assume that the *channel space*

$$M = M_- + M_+$$

is finite dimensional. We denote

$$H_- = H_{0,-} + M_-, \qquad H_+ = H_{0,+} + M_+.$$

A bounded operator N on K together with this decomposition of K is called an *affine separating structure* if

$$NH_{0,-} \subset H_-, \quad NH_- \subset H_- + M_+, \quad NH_+ \subset H_+ + M_-, \quad NH_{0,+} \subset H_+. \tag{6.20}$$

According to the decomposition (6.19), we can write

$$N = \begin{bmatrix} * & \widetilde{R_{-2}} & 0 & 0 \\ \widetilde{T_{-1}} & \Lambda_{-1} & R_{-1} & 0 \\ 0 & T_0 & \Lambda_0 & \widetilde{R_0} \\ 0 & 0 & \widetilde{T_1} & * \end{bmatrix}. \tag{6.21}$$

The decomposition (6.19) also induces the dual decomposition

$$K = H'_{0,-} \dotplus M'_- \dotplus M'_+ \dotplus H'_{0,+}. \tag{6.22}$$

Here we put

$$H'_{0,-} = (M_- + M_+ + H_{0,+})^\perp$$

and analogously for the other subspaces:

$$M'_- = (H_{0,-} + M_+ + H_{0,+})^\perp, \qquad M'_+ = (H_{0,-} + M_- + H_{0,+})^\perp,$$
$$H'_{0,+} = (H_{0,-} + M_- + M_+)^\perp.$$

We can make the duality identifications $H'_{0,-} \cong (H_{0,-})^*$, and so on. We put

$$M' = M'_- \dotplus M'_+.$$

We denote by $P_{H_{0,-}}$, P_{M_-}, etc. the parallel projections corresponding to the summands in (6.19). The projections corresponding to (6.22) are $P_{H'_{0,-}} = P^*_{H_{0,-}}$, etc. We also define the channel operators

$$P_M = P_{M_-} + P_{M_+}, \qquad P_{M'} = P^*_M = P_{M'_-} + P_{M'_+},$$

the parallel projections

$$P_- = P_{H_-}, \qquad P_+ = P_{H_+},$$

and the corresponding parallel projections for the dual decomposition.

We define s, the compression of N to M:

$$s = P_M N|M. \tag{6.23}$$

We also define the operator $\alpha : M \to M$ by

$$P_+ N - N P_+ = \alpha P_M. \tag{6.24}$$

It is easy to check that α is well defined and

$$\alpha = \begin{bmatrix} 0 & -R_{-1} \\ T_0 & 0 \end{bmatrix}. \tag{6.25}$$

Example 6.7. This example concerns the relation between separating structures and subnormal operators, and we will return to it several times later. A *subnormal operator* $S \in \mathcal{B}(H)$ is, by definition, an operator having an extension to a normal operator $N \in \mathcal{B}(K)$, with $K \supset H$. Recall that this means that $N|H = S$. We say that S is *pure* if no nontrivial subspace of H reduces S to a normal operator. The normal extension N is called minimal if $K = \bigvee_{n \geq 0} N^{*n} H$. Every subnormal operator has a minimal normal extension. The subnormal operator S is said to be of *finite type* if its self-commutator $C = S^*S - SS^*$ has finite rank. See [Con91] for a treatment of the theory of subnormal operators.

A pure subnormal operator of finite type S and its minimal normal extension N give rise to a separating structure in the following way. The operator N and the space K of the structure will be the minimal normal extension N and the space on which it acts. We put

$$H_+ = H, \qquad M_+ = CH, \qquad H_{0,+} = H \ominus CH, \qquad H_- = K \ominus H.$$

Note that M_+ has finite dimension because S is of finite type.

The operator N has the structure

$$N = \begin{bmatrix} S'^* & 0 \\ X & S \end{bmatrix} \tag{6.26}$$

according to the decomposition $K = (K \ominus H) \oplus H$. The operator S' is pure subnormal and is called the dual of S. Using the fact that N is normal, we get the equalities

$$XX^* = S^*S - SS^* = C, \tag{6.27}$$
$$X^*X = S'^*S' - S'S'^* = C', \tag{6.28}$$

where C' is the self-commutator of S'. We note that

$$X(K \ominus H) = X\overline{X^*H} = CH = M_+.$$

Here the first equality comes from the fact that $\ker X = (K \ominus H) \ominus \overline{X^*H}$. We see that

$$C'(K \ominus H) = X^*M_+,$$

so that C' has finite rank and S' is pure subnormal of finite type.

We can define $M_- = C'(K \ominus H)$ and $H_{0,-} = (K \ominus H) \ominus M_-$. Now we have to check that conditions (6.20) are satisfied. We have

$$NH_- \subset S'^*(K \ominus H) + X(K \ominus H) \subset H_- + M_+.$$

The inclusion $NH_+ \subset H_+ + M_-$ is trivial. Indeed, $NH_+ \subset H_+$, which shows that $NH_{0,+} \subset H_+$ is also trivial. It remains to show that $NH_{0,-} \subset H_-$.

We have

$$\ker X^* = H \ominus X(K \ominus H) = H \ominus M_+ = H_{0,+}. \tag{6.29}$$

This implies $M_- = X^*M_+ = X^*H_+$. Hence, $\ker X = H_{0,-}$. It follows that

$$NH_{0,-} \subset S'^*H_{0,-} + XH_{0,-} = S'^*H_{0,-} \subset H_-.$$

Hence, we see that $\omega = (K, N, H_{0,-}, M_-, M_+, H_{0,+})$ is a separating structure. Moreover, the decomposition $K = H_{0,-} \oplus M_- \oplus M_+ \oplus H_{0,+}$ is orthogonal. Thus, ω is a particular kind of separating structure, called orthogonal separating structure, which will be introduced in the next section.

From what we have done, it also follows that X maps M_- onto M_+ and X^* maps M_+ onto M_-. Hence, the dimensions of M_- and M_+ coincide and $T_0 = P_{M_+}N|M_- = X|M_-$ is an isomorphism from M_- to M_+. This will play a role later. Also, note that the operator R_{-1} from (6.21) is 0.

Another important fact about subnormal operators is that $M_+ = CH$ is invariant for S^* (this is easy to prove; see [Con91, Section II.3, exercises 6 and 7]). The operator $(S^*|M_+)^*$ is just the operator Λ_0 in (6.21). The pair of operators (C, Λ_0) determines the subnormal operator S. The associated algebraic curve constructed in [Yak98a] is given in terms of this pair (there, the operator Λ_0 is denoted just by Λ).

6.4. The mosaic function

In this section we define the mosaic function $\nu(z)$ of an affine separating structure and prove some of its basic properties. This is a function whose values are parallel projections on M. It is a way to encode the information about the separating structure and to construct a model for the structure in terms of analytic functions.

Definition 6.8. The $\mathcal{B}(M)$-valued analytic function V by

$$\nu(z) = P_M(N - z)^{-1}P_+(N - z)|M, \qquad z \notin \sigma(N). \tag{6.30}$$

is called the *mosaic function.*

Definition 6.9. The *almost diagonalizing transform* is the operator V defined by

$$(Vx)(z) = P_M(N - z)^{-1}x, \qquad x \in K,\ z \notin \sigma(N). \tag{6.31}$$

It takes vectors $x \in K$ to M-valued analytic functions on $\mathbb{C} \setminus \sigma(N)$.

An affine separating structure is called *pure* if

$$K = \bigvee_{z \notin \sigma(N^*)} (N^* - z)^{-1}M.$$

It is easy to see that a structure is pure if and only if its associated almost diagonalizing transform V is injective. In this case, the transform V gives an analytic model for the structure.

If $\omega = (K, N, H_{0,-}, M_-, M_+, H_{0,+})$ is an affine separating structure, then

$$\omega^* = (K, N^*, H'_{0,-}, M'_-, M'_+, H'_{0,+}) \tag{6.32}$$

is also an affine separating structure, called the dual structure. We denote by ν_* and V_* the mosaic and the almost diagonalizing transform assigned to the dual structure.

The following theorem lists the main properties of the almost diagonalizing transform and the mosaic.

Theorem 6.10. *Suppose that ω is an affine separating structure. Denote by VK the image of the almost diagonalizing transform V, endowed with the Hilbert space structure inherited from K (i.e., the unique structure that makes $V : K \ominus \ker V \to VK$ a unitary). Then the following statements are true:*

(i) VK is a reproducing kernel Hilbert space of M-valued holomorphic functions on $\Omega = \widehat{\mathbb{C}} \setminus \sigma(N)$ which vanish at ∞.

(ii) ν is a holomorphic projection-valued function on Ω. The operator P_ν defined by $(P_\nu f)(z) = \nu(z)f(z)$ for $f \in VK$ is a parallel projection on VK.

(iii) V almost diagonalizes N:

$$(VNx)(z) = z(Vx)(z) - [z(Vx)(z)]|_{z=\infty}.$$

(iv) V transforms the resolvent operator $(N-z)^{-1}$ into the operator $f(u) \mapsto (f(u) - f(z))/(u-z)$:

$$(V(N-z)^{-1}x)(u) = \frac{(Vx)(u) - (Vx)(z)}{u-z}.$$

(v) V transforms P_+ into the operator P_ν, i.e., $VP_+ = P_\nu V$.

(vi) We have the following formula for the mosaic:

$$\nu(z) = P_M(N-z)^{-1}\alpha + P_{M_+}.$$

In particular, $\nu(\infty) = P_{M_+}$ and

$$\nu(z)m = (V\alpha m)(z) + P_{M_+}m, \qquad m \in M.$$

(vii) We have

$$(V(N-z)^{-1}\alpha m)(u) = \frac{\nu(u)-\nu(z)}{u-z}m, \qquad m \in M.$$

Proof. Statement (i) is clear from the definition of the transform V. Next, observe that

$$P_+(N-z)P_M = P_+(N-z)(I - P_{H_{0,+}}) = P_+(N-z) - (N-z)P_{H_{0,+}}.$$

Hence,

$$\begin{aligned} P_M(N-z)^{-1}P_+(N-z)P_M(N-z)^{-1} &= P_M(N-z)^{-1}P_+ - P_M P_{H_{0,+}}(N-z)^{-1} \\ &= P_M(N-z)^{-1}P_+. \end{aligned} \tag{6.33}$$

This gives (v). Also, multiplying by $P_+(N-z)|M$ on the right, we get $\nu^2(z) = \nu(z)$. This implies that ν is projection-valued. The operator P_ν is bounded by (v), and hence, it is a parallel projection. This gives (ii).

To prove (iii) we observe that

$$(VNx)(z) = P_M(N-z)^{-1}Nx = P_Mx + z(Vx)(z).$$

It is easy to check from the definition that $[z(Vx)(z)]|_{z=\infty} = -P_Mx$.

Part (iv) is obtained directly from the definition of V. To check (vi), we compute

$$\begin{aligned} \nu(z) &= P_M(N-z)^{-1}P_+(N-z)|M \\ &= P_M(N-z)^{-1}[P_+(N-z) - (N-z)P_+]|M + P_M(N-z)^{-1}(N-z)P_+|M \\ &= P_M(N-z)^{-1}\alpha + P_{M_+}. \end{aligned}$$

The remaining statements in (vi) are obvious and (vii) is a direct consequence of (iv) and (vii). $\square$

Now we will do a brief geometric study of the mosaic ν. Define, for $z \notin \sigma(N)$, the following two subspaces of M:

$$\widetilde{F}(z) = P_M(N-z)^{-1}H_+, \qquad \widetilde{G}(z) = P_M(N-z)^{-1}H_-. \tag{6.34}$$

Proposition 6.11. *The space M decomposes in a direct sum as*

$$M = \widetilde{F}(z) \dotplus \widetilde{G}(z)$$

for every $z \notin \sigma(N)$. The operator $\nu(z)$ is the projection onto $\widetilde{F}(z)$ parallel to $\widetilde{G}(z)$.

Proof. If $m \in M$, put $h_- = P_-(N-z)m$, and $h_+ = P_+(N-z)m$. Then $h_- \in H_-$ and $h_+ \in H_+$. It follows that $m_- = P_M(N-z)^{-1}h_- \in \widetilde{G}(z)$, and $m_+ = P_M(N-z)^{-1}h_+ \in \widetilde{F}(z)$. Moreover, $m_- + m_+ = m$. This shows that $M = \widetilde{F}(z) + \widetilde{G}(z)$.

We must check that $\widetilde{F}(z) \cap \widetilde{G}(z) = 0$. To do this, take $m \in \widetilde{F}(z)$ and write $m = P_M(N-z)^{-1}h_+$, where $h_+ \in H_+$. Multiplying (6.33) by h_+ on the right and using formula (6.30) for the mosaic $\nu(z)$, we get $\nu(z)m = m$. Similarly, we see that if $m \in \widetilde{G}(z)$, then $\nu(z)m = 0$. This shows that $\widetilde{F}(z) \cap \widetilde{G}(z) = 0$ and that $\nu(z)$ is the projection onto $\widetilde{F}(z)$ parallel to $\widetilde{G}(z)$. □

An alternative proof of this Proposition can be given by observing that

$$\widetilde{F}(z) = \{(Vh_-)(z) : h_- \in H_-\}, \qquad \widetilde{G}(z) = \{(Vh_+)(z) : h_+ \in H_+\}.$$

Then it is enough to use Theorem 6.10 (v).

Here and in the sequel we will use the notation 1 for the identity matrix on a finite-dimensional Hilbert space. If we put

$$E_0^2(\nu) = \{v \in VK : v(z) \in \nu(z)M = \widetilde{F}(z)\}, \tag{6.35}$$

$$E_0^2(1-\nu) = \{v \in VK : v(z) \in (1-\nu(z))M = \widetilde{G}(z)\}, \tag{6.36}$$

then Theorem 6.10 (v) implies that

$$VH_- = E_0^2(1-\nu), \qquad VH_+ = E_0^2(\nu).$$

Example 6.12. A mosaic function $\mu(z)$ for a subnormal operator S appears in the article [Yak98a] by Yakubovich. This function is holomorphic on $\widehat{\mathbb{C}} \setminus \sigma(N)$ and its values are parallel projections on M_+. In the notation of separating structures, it is defined by

$$\mu(z) = \alpha P_M(N-z)^{-1}|M_+,$$

where N is the minimal normal extension of S (see Example 6.7). Using Theorem 6.10 (vi), we see that

$$\alpha\nu(z) = \mu(z)\alpha, \tag{6.37}$$

because $R_{-1} = 0$ in the case of a subnormal operator.

The spaces $\mu(z)M_+$ and $(1-\mu(z))M_+$ play an important role in [Yak98b]. Using (6.37), we see that

$$\mu(z)M_+ = \alpha\widetilde{F}(z), \qquad (1-\mu(z))M_+ = \alpha\widetilde{G}(z).$$

Also, an almost diagonalizing transform $\widetilde{U}$ appears in [Yak98b]. It is defined by

$$(\widetilde{U}x)(z) = P_+NP_-(N-z)^{-1}x,$$

and plays a similar role to V. For instance, $\widetilde{U}$ transforms the projection P_+ into the operator of multiplication by $\mu(z)$ (c.f. Theorem 6.10 (v)). We see that

$$(\widetilde{U}x)(z) = \alpha(Vx)(z).$$

Hence, the operator α, which in the subnormal case maps M onto M_+, can be used to pass from many of our constructions to the analogue constructions in [Yak98a, Yak98b].

The following two Lemmas are not used later but could be helpful to keep in mind. To interpret these Lemmas, one should know that in the case considered in the following sections, the operator α typically will be invertible.

Lemma 6.13. *Assume that ω is pure. Then $\nu(z)m$ is constant if and only if $\alpha m = 0$.*

Proof. If $\nu(z)m$ is constant, then Theorem 6.10 (vii) shows that $V(N-z)^{-1}\alpha m \equiv 0$, so that $\alpha m = 0$, because ω is pure, and hence, V is injective. Conversely, if $\alpha m = 0$, Theorem 6.10 (vi) shows that $\nu(z)m = P_{M_+}m$. □

Lemma 6.14. *There is the following relation between the mosaic ν of the structure ω and the mosaic ν_* of the dual structure ω^*:*

$$(1 - \nu_*^*(\overline{z}))\alpha = \alpha\nu(z).$$

Proof. Recall that the dual structure ω^* is defined by (6.32). We will denote by α_* the operator defined by (6.24) with the dual structure ω^* in place of ω, i.e., the operator $\alpha_* : M' \to M'$ defined by

$$P'_+N^* - N^*P'_+ = \alpha_* P_{M'}.$$

We see that $\alpha_* = -\alpha^*$.

By Theorem 6.10 (vi),

$$\nu_*(z) = P_{M'_+} - P_{M'}(N^* - z)^{-1}\alpha^*.$$

We get

$$\begin{aligned}\nu_*^*(\overline{z})\alpha + \alpha\nu(z) &= P_{M_+}\alpha - \alpha(N-z)^{-1}P_M\alpha + \alpha P_{M+} + \alpha P_M(N-z)^{-1}\alpha\\ &= P_{M_+}\alpha + \alpha P_{M_+} = \alpha.\end{aligned}$$

Here the last equality can be seen by (6.25). This proves the Lemma. □

6.5. Orthogonal separating structures

Now we introduce the concept of an orthogonal separating structure. This is an affine seperating structure where we also impose some orthogonality conditions. Orthogonal separating structures will be called separating structures later.

Definition 6.15. We say that an affine separating structure ω is orthogonal if N is normal and the decomposition of K is orthogonal:

$$K = H_{0,-} \oplus M_- \oplus M_+ \oplus H_{0,+}. \tag{6.38}$$

Given an orthogonal decomposition as in (6.38), two commuting selfadjoint operators A_1, A_2 on K satisfying

$$A_j H_{0,-} \subset H_-, \quad A_j H_- \subset H_- \oplus M_+, \quad A_j H_+ \subset H_+ \oplus M_-, \quad A_j H_{0,+} \subset H_-, \tag{6.39}$$

for $j = 1, 2$, and a direction $\xi = (\xi_1, \xi_2) \in \mathbb{C}^2$, the operator $N_\xi = \xi_1 A_1 + \xi_2 A_2$ gives rise to an orthogonal separating structure ω_ξ with respect to (6.38). Conversely, given an operator N forming an orthogonal separating structure, one can put $A_1 = \operatorname{Re} N$, $A_2 = \operatorname{Im} N$ and form the family of structures $\{\omega_\xi : \xi \in \mathbb{C}^2\}$ as above.

It will be convenient to think of orthogonal separating structures in this way, as a family of structures $\{\omega_\xi : \xi \in \mathbb{C}^2\}$ generated by two commuting selfadjoint operators A_1, A_2.

For each direction $\xi \in \mathbb{C}^2$, we obtain a separating structure ω_ξ with associated operator $N_\xi = \xi_1 A_1 + \xi_2 A_2$. Hence, we can apply the results of the preceding section to the structure ω_ξ. We will mark with the subscript $_\xi$ the objects of the preceding section constructed for the operator N_ξ. Therefore, we will write V_ξ, ν_ξ, $\Lambda_{-1\xi}$, $\Lambda_{0\xi}$, $R_{-1\xi}$, $T_{0\xi}$, etc.

Example 6.16. Here we introduce a simple class of orthogonal separating structures, which will be used to illustrate our constructions. We will return to this example when we define the compression of separating structures in the next chapter.

We put $K = L^2(\mathbb{T})$ and

$$\begin{aligned} H_+ &= H^2(\mathbb{D}) = \{f \in L^2(\mathbb{T}) : \langle f, e^{int} \rangle = 0,\ n < 0\}, \\ H_- &= H^2_0(\mathbb{C} \setminus \mathbb{D}) = \{f \in L^2(\mathbb{T}) : \langle f, e^{int} \rangle = 0,\ n \geq 0\}, \end{aligned}$$

so that $K = H_- \oplus H_+$. We choose rational functions f_1, f_2 with poles off $\mathbb{T}$ and such that $f_j(\mathbb{T}) \subset \mathbb{R}$ for $j = 1, 2$. We define A_1, A_2 as the operators of multiplication by f_1 and f_2 on $K = L^2(\mathbb{T})$. These are bounded commuting selfadjoint operators. Since $P_{H_-} A_j P_{H_+}$ have finite rank, these two operators can be embedded into an orthogonal separating structure (with $N = A_1 + iA_2$) by defining M_+ and M_- appropriately. In fact, we can always put

$$M_+ = P_{H_+} A_1 H_- + P_{H_+} A_2 H_-, \qquad M_- = P_{H_-} A_1 H_+ + P_{H_-} A_2 H_+. \tag{6.40}$$

Let B be the minimal finite Blaschke product such that Bf_1 and Bf_2 have no poles in $\mathbb{D}$. The minimality condition here means that if $(Bf_j)(z) = 0$ for some $z \in \mathbb{D}$ and some $j = 1, 2$, then $B(z) \neq 0$.

We define

$$\begin{aligned} H_{0,-} &= B^{-1} H^2_0(\mathbb{C} \setminus \mathbb{D}), \\ M_- &= H^2_0(\mathbb{C} \setminus \mathbb{D}) \ominus B^{-1} H^2_0(\mathbb{C} \setminus \mathbb{D}), \\ M_+ &= H^2(\mathbb{D}) \ominus BH^2(\mathbb{D}), \\ H_{0,+} &= BH^2(\mathbb{D}). \end{aligned}$$

Note that $f_j(\overline{z}^{-1}) = \overline{f_j(z)}$, because $f_j(\mathbb{T}) \subset \mathbb{R}$. This implies that z is a pole of f_j of order r if and only if $\overline{z}^{-1}$ is a pole of f_j of order r. Using this, it is easy to check that (6.39) are satisfied.

Moreover, (6.40) holds, so M_- and M_+ are the minimal subspaces so that (6.39) holds when A_1, A_2 and $K = H_- \oplus H_+$ have been defined as above. Also note that M_- and M_+ have the same dimension. This a typical nondegeneracy condition for separating structures.

In the next Theorem, we relate the orthogonal separating structure $\{\omega_\xi\}$ with the notion of pool given in Section 6.1.

Theorem 6.17. *If $\{\omega_\xi\}$ is an orthogonal separating structure, we can construct a pool*

$$\mathcal{P} = (A_1, A_2; K, \Phi, M; \sigma_1, \sigma_2, \gamma),$$

defined by $\Phi = P_M$ and

$$\begin{aligned} \sigma_j P_M &= -i(P_+ A_j - A_j P_+), \qquad j = 1, 2, \\ \gamma P_M &= i(A_1 P_+ A_2 - A_2 P_+ A_1). \end{aligned}$$

The operators α_ξ defined in (6.5) *coincide with the operators α_ξ defined by using* (6.24) *for $N = N_\xi$. Moreover, the operator γ_ξ defined in* (6.5) *can be computed as*

$$\gamma_\xi = -(\alpha_\xi^* s_\xi + \alpha_\xi s_\xi^*), \tag{6.41}$$

where $s_\xi = P_M N_\xi | M$ (see (6.23)*).*

Proof. Conditions (6.39) imply that σ_j are well defined. If $T = i(A_1 P_+ A_2 - A_2 P_+ A_1)$, then (6.39) implies that $K \ominus M \subset \ker T$. Since T is selfadjoint, then $M \supset TK$, so γ is also well defined. The operators σ_j and γ are clearly self adjoint. We compute

$$\sigma_2 P_M A_1 - \sigma_1 P_M A_2 = -i(P_+ A_2 - A_2 P_+) A_1 + i(P_+ A_1 - A_1 P_+) A_2 = -\gamma P_M,$$

so $\mathcal{P}$ is a pool.

A simple computation yields that the operator α_ξ defined by (6.5) coincides with the operator α_ξ appearing in (6.24). To check (6.41), just restrict (6.4) to M. $\square$

Example 6.18. As we already commented in Example 6.7, the separating structure generated by a subnormal operator S is orthogonal. Now we will compute the discriminant curve of its associated pool, according to Theorem 6.17. We fix $\xi = (1, i)$ so that the operator N_ξ is just the minimal normal extension N. Thus, we will omit the subscript ξ in the rest of this example.

We have,

$$\alpha = \begin{bmatrix} 0 & 0 \\ T_0 & 0 \end{bmatrix}, \qquad s = \begin{bmatrix} \Lambda_{-1} & 0 \\ T_0 & \Lambda_0 \end{bmatrix}.$$

Using formula (6.41) for γ, we see that the defining polynomial of the discriminant curve (6.12) is

$$\begin{aligned}
\det(z\alpha^* + w\alpha + \gamma) &= \det\left(\begin{bmatrix} -T_0^*T_0 & zT_0^* - T_0^*\Lambda_0 \\ wT_0 - T_0\Lambda_{-1}^* & -T_0T_0^* \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} T_0^* & 0 \\ 0 & T_0 \end{bmatrix}\begin{bmatrix} -T_0 & z - \Lambda_0 \\ w - \Lambda_{-1}^* & -T_0^* \end{bmatrix}\right) \\
&= |\det T_0|^2 \det\left(\begin{bmatrix} -T_0 & z - \Lambda_0 \\ w - \Lambda_{-1}^* & -T_0^* \end{bmatrix}\right) \\
&= |\det T_0|^2 \det T_0 \det(-T_0^* + (w - \Lambda_{-1}^*)T_0^{-1}(z - \Lambda_0)) \\
&= |\det T_0|^2 \det(-T_0T_0^* + T_0(w - \Lambda_{-1}^*)T_0^{-1}(z - \Lambda_0)) \\
&= -|\det T_0|^2 \det(C - (w - \Lambda_0^*)(z - \Lambda_0)).
\end{aligned}$$

Here we have used $T_0T_0^* = C$, which comes from (6.27) and $T_0\Lambda_{-1}^* = \Lambda_0 T_0$, which is obtained using the fact that N is normal (and $R_{-1} = 0$).

The equation for the discriminant curve associated to the subnormal operator S in [Yak98a, Yak98b] was precisely

$$\det(C - (w - \Lambda_0^*)(z - \Lambda_0)) = 0.$$

Therefore, this shows that the discriminant curve of the corresponding pool is the same curve.

The next Proposition relates the concepts of purity of a separating structure and irreducibility of a pool.

Proposition 6.19. *Let $\{\omega_\xi\}$ be an orthogonal separating structure and $\mathcal{P}$ its associated pool, according to Theorem 6.17. Assume that the discriminant curve X_{aff} of $\mathcal{P}$ is not all of $\mathbb{C}^2$. Then the following statements are equivalent:*

(i) The separating structure ω_ξ is pure for some nondegenerate direction $\xi \in \Xi$.

(ii) The separating structure ω_ξ is pure for every nondegenerate direction $\xi \in \Xi$.

(iii) The pool $\mathcal{P}$ is irreducible.

Proof. First we show (i) $\Rightarrow$ (iii), so we assume that ω_ξ is pure for a certain direction $\xi \in \Xi$. This means that the set

$$K_0 = \{(N_\xi^* - w)^{-1}m : m \in M, w \notin \sigma(N^*)\}$$

spans K. Since the function $(\overline{z} - w)^{-1}$, $w \notin \sigma(N^*)$ can be approximated uniformly in $\sigma(N)$ by polynomials in z and $\overline{z}$ by the Stone-Weierstrass theorem, it follows that every member of the set K_0 can approximated by some $p(N_\xi, N_\xi^*)m$, where p is a polynomial in two variables. By (6.6), we see that $K_0 \subset \widehat{K}$, where $\widehat{K}$ is the principal subspace of $\mathcal{P}$ (see (6.2)). This means that $\widehat{K}$ must also span K, so that $\widehat{K} = K$ and the pool $\mathcal{P}$ is irreducible.

Now we show (iii) ⇒ (ii), so we assume that $\mathcal{P}$ is irreducible and take an arbitrary direction $\xi \in \Xi$. The set

$$K_1 = \{p(N_\xi, N_\xi^*)m : m \in M, p \in \mathbb{C}[z, \overline{z}]\}$$

spans K. Since the discriminant curve X is not all of $\mathbb{C}^2$ and the spectrum of N_ξ lies in $z_\xi(X_{\mathrm{aff},\mathbb{R}})$, we see that $\sigma(N_\xi)$ has area 0. By the Hartogs-Rosenthal theorem (see [Con91, Theorem V.3.6]), every continuous function on $\sigma(N_\xi)$ can be uniformly approximated by rational functions with poles outside $\sigma(N_\xi)$. This applies to any polynomial $p(z, \overline{z})$. Since every rational function with poles outside $\sigma(N_\xi)$ can be approximated uniformly in $\sigma(N_\xi)$ by a linear combination of functions of the form $(z - w_k)^{-1}$, $z_k \notin \sigma(N_\xi)$, we see that the vector $p(N_\xi, N_\xi^*)m$ can be approximated by a linear combination of vectors $(N_\xi - w_k)^{-1}m$. This shows that if

$$K_2 = \bigvee_{w \notin \sigma(N_\xi)} (N_\xi - w)^{-1}M,$$

then $K_1 \subset K_2$. Hence, we get $K_2 = K$. This implies that $\omega_{\overline{\xi}}$ is pure, because $N_{\overline{\xi}}^* = N_\xi$. Therefore, we get (ii), because $\xi \in \Xi$ was arbitrary.

The remaining implication (ii) ⇒ (i) is trivial. □

Example 6.20. Consider a pure subnormal operator of finite type S, the separating structure it generates according to Example 6.7, and its associated pool $\mathcal{P}$. We will show that the purity of S implies the irreducibility of $\mathcal{P}$.

Put $G_+ = P_+(K \ominus \widehat{K})$. Since $M_+ \subset \widehat{K}$, we have $G_+ \subset H_{0,+}$. Using this, it is easy to check that G_+ is invariant for N and N^*. We have $N|G_+ = S|G_+$, because $N|H_+ = S$. Moreover, (6.26) and (6.29) imply that $N^*|H_{0,+} = S^*|H_{0,+}$. Hence, $N^*|G_+ = S^*|G_+$. It follows that G_+ reduces S and $S|G_+$ is a normal operator. Since S is pure, $G_+ = 0$.

Similarly, one can prove that $G_- = P_-(K \ominus \widehat{K}) = 0$. One has to follow the reasoning above interchanging N and N^* and using the pure subnormal operator $S' = N^*|H_-$ instead of S. Since we have $G_- = G_+ = 0$, we get $K \ominus \widehat{K} = 0$, so the pool $\mathcal{P}$ is irreducible.

Now we will relate the analytic model for the separating structure, constructed in terms of the almost diagonalizing transform V_ξ and the L^2 model for the pool, constructed in terms of the transform W_ξ, by means of the Cauchy operators. First, observe that there exists a positive scalar measure ρ_ξ and a matrix-valued ρ_ξ-measurable function $\mathcal{E}_\xi$ such that

$$de_\xi(z) = \mathcal{E}_\xi(z) d\rho_\xi(z),$$

The Cauchy operators are defined by

$$(\mathcal{K}_\xi f)(z) = \int_{\mathbb{C}} \frac{f(u)}{u - z} \, d\rho_\xi(u),$$

$$(\overline{\mathcal{K}}_\xi f)(z) = \int_{\mathbb{C}} \frac{f(u)}{\overline{u} - \overline{z}} \, d\rho_\xi(u).$$

We will also denote by $\mathcal{E}_\xi$ the operator of multiplication by $\mathcal{E}_\xi(z)$ in $L^2(de_\xi)$.

Proposition 6.21. *We have the following relation between the transforms of an orthogonal separating structure and its associated pool:*

$$V_\xi = \mathcal{K}_\xi \mathcal{E}_\xi W_\xi$$

Proof. For $m \in M$, g bounded Borel and $x = g(N_\xi)m$, we have

$$(V_\xi x)(z) = P_M(N_\xi - z)^{-1} g(N_\xi)m = P_M \int_{\mathbb{C}} \frac{g(u)}{u-z} dE_\xi(u)m = (\mathcal{K}_\xi \mathcal{E}_\xi W_\xi x)(z).$$

The Proposition follows by density. □

Proposition 6.22. *The mosaic ν_ξ has the following integral representation:*

$$\nu_\xi(z) = P_{M_+} + \int_{\mathbb{C}} \frac{de_\xi(u)\alpha_\xi}{u-z}.$$

Proof. It suffices to observe that

$$P_M(N_\xi - z)^{-1}|M = \int_{\mathbb{C}} \frac{de_\xi(u)}{u-z}$$

and to use Theorem 6.10 (vi). □

The next Lemma is a bit technical but it will be very useful later. Recall that Q_ξ was the projection-valued function defined on the discriminant curve (see Section 6.2).

Lemma 6.23. *If α_ξ is invertible, we have*

$$\alpha_\xi \nu_\xi(z) \alpha_\xi^{-1}(z\alpha_\xi^* + \gamma_\xi) = (z\alpha_\xi^* + \gamma_\xi)\nu_\xi(z), \tag{6.42}$$

Therefore $\nu_\xi(z)$ commutes both with $\alpha_\xi^{-1}(z\alpha_\xi^* + \gamma_\xi)$ and with $Q_\xi(p)$, for those $p \in X_{aff}$ such that $z_\xi(p) = z$.

Proof. To prove (6.42), using Proposition 6.22, we have to check that

$$\alpha_\xi P_{M_+} \alpha_\xi^{-1}(z\alpha_\xi^* + \gamma_\xi) + \int_{\mathbb{C}} \frac{\alpha_\xi de_\xi(u)(z\alpha_\xi^* + \gamma_\xi)}{u-z} = (z\alpha_\xi^* + \gamma_\xi)P_{M_+} + \int_{\mathbb{C}} \frac{(z\alpha_\xi^* + \gamma_\xi)de_\xi(u)\alpha_\xi}{u-z}.$$

Now we use the identities $\alpha_\xi P_{M_+} \alpha_\xi^{-1} = P_{M_-}$, and $P_{M_-}\alpha_\xi^* = \alpha_\xi^* P_{M_+}$, and rearrange terms to see that the equation above is equivalent to

$$P_{M_-}\gamma_\xi - \gamma_\xi P_{M_+} = \int_{\mathbb{C}} \frac{(z\alpha_\xi^* + \gamma_\xi)de_\xi(u)\alpha_\xi - \alpha_\xi de_\xi(u)(z\alpha_\xi^* + \gamma_\xi)}{u-z}. \tag{6.43}$$

Using (6.10) and the relation obtained from it by taking adjoints, we get

$$\gamma_\xi de_\xi(u) = -(u\alpha_\xi^* + \overline{u}\alpha_\xi)de_\xi(u), \qquad de_\xi(u)\gamma_\xi = -de_\xi(u)(u\alpha_\xi^* + \overline{u}\alpha_\xi).$$

Substituting these identities into the right hand side of (6.43), we see that it equals

$$\int_{\mathbb{C}} \alpha_\xi de_\xi(u)\alpha_\xi^* - \alpha_\xi^* de_\xi(u)\alpha_\xi = \alpha_\xi\alpha_\xi^* - \alpha_\xi^*\alpha_\xi.$$

Hence we just need to check that

$$P_{M_-}\gamma_\xi - \gamma_\xi P_{M_+} = \alpha_\xi\alpha_\xi^* - \alpha_\xi^*\alpha_\xi.$$

This is an easy computation with matrices, using (6.41), (6.21) and (6.25).

From (6.42), it is obvious that $\nu_\xi(z)$ and $\alpha_\xi^{-1}(\alpha_\xi^* z+\gamma_\xi)$ commute. The fact that $\nu_\xi(z)$ and $Q_\xi(p)$ commute is a consequence of the definition of $Q_\xi(p)$ and the properties of the Riesz-Dunford calculus. □

Proposition 6.24. *If α_ξ is invertible, we have*

$$P_M(N_\xi - z)^{-1}(N_\xi^* - \overline{w})^{-1}P_M = (\gamma_\xi + z\alpha_\xi^* + \overline{w}\alpha_\xi)^{-1}(P_M - \alpha_\xi\nu(z)\alpha_\xi^{-1}P_M - \nu_\xi^*(w)P_M),$$

for every pair (z, w) such that $z, w \notin \sigma(N_\xi)$ and $(z, \overline{w}) \notin X_{aff}$.

Proof. First rewrite (6.4) as

$$\alpha_\xi^* P_M(N_\xi - z) + \alpha P_M(N_\xi^* - \overline{w}) + (\gamma_\xi + z\alpha_\xi^* + \overline{w}\alpha_\xi)P_M = 0.$$

Multiplying by $(N_\xi - z)^{-1}(N_\xi^* - \overline{w})^{-1}P_M$ on the right and rearranging terms, we get

$$\begin{aligned} &P_M(N_\xi - z)^{-1}(N_\xi^* - \overline{w})^{-1}P_M = \\ &\quad - (\gamma_\xi + z\alpha_\xi^* + \overline{w}\alpha_\xi)^{-1}[\alpha_\xi^* P_M(N_\xi^* - \overline{w})^{-1}P_M + \alpha_\xi P_M(N_\xi - z)^{-1}P_M]. \end{aligned} \tag{6.44}$$

By Theorem 6.10 (vi),

$$\alpha_\xi^* P_M(N_\xi^* - \overline{w})^{-1}P_M = \nu_\xi^*(w)P_M - P_{M_+}.$$

Also,

$$\alpha_\xi P_M(N_\xi - z)^{-1}P_M = \alpha_\xi(\nu_\xi(z) - P_{M_+})\alpha_\xi^{-1}P_M = \alpha_\xi\nu_\xi(z)\alpha_\xi^{-1}P_M - P_{M_-},$$

because $\alpha_\xi P_{M_+} = P_{M_-}\alpha_\xi$. The Proposition follows by substituting these two equalities into (6.44). □

This Proposition implies that the data $\alpha_\xi, \gamma_\xi, \nu_\xi$ completely determines the separating structure $\{\omega_\xi\}$ whenever the structure is pure. Indeed, we see from the Proposition that the inner product

$$\langle (N_\xi^* - \overline{w})^{-1}m, (N_\xi^* - \overline{z})^{-1}m'\rangle \qquad m, m' \in M,\ z, w \notin \sigma(N_\xi),$$

depends only on $\alpha_\xi, \gamma_\xi, \nu_\xi$ and m, m', z, w.

Therefore, given two pure structures $\{\omega_\xi\}$ and $\{\widetilde{\omega}_\xi\}$ with the same data $\alpha_\xi, \gamma_\xi, \nu_\xi$, the operator Z defined by

$$Z(N_\xi^* - \overline{w})^{-1}m = (\widetilde{N}_\xi^* - \overline{w})^{-1}m$$

continues to a unitary operator. By the Riesz-Dunford functional calculus, we see that

$$ZN_\xi^* x = \widetilde{N}_\xi^* Zx,$$

for vector of the form $x = (N_\xi^* - \overline{w})^{-1} m$, and hence for every $x \in K$ by density. This shows that N_ξ and $\widetilde{N}_\xi$ are unitarily equivalent.

Below we will see that in many cases, the mosaic function ν_ξ can be computed from the matrices α_ξ and γ_ξ alone by the discriminant curve (which, of course, is defined solely in terms of α_ξ and γ_ξ) and a "restoration formula".

6.6. The halves of the discriminant curve and restoration formula

In this section we define the two *halves* of the discriminant curve $\widehat{X}$. This is a partition of $\widehat{X} \setminus \widehat{X}_{\mathbb{R}}$ into two open sets $\widehat{X}_-$, $\widehat{X}_+$ such that $*(\widehat{X}_-) = \widehat{X}_+$, and such that we can recover the mosaic ν_ξ by the restoration formula

$$\nu_\xi(z) = \sum_{\substack{p \in \widehat{X}_+ \\ z_\xi(p) = z}} Q_\xi(p),$$

where Q_ξ is the projection-valued meromorphic function on $\widehat{X}$ that was defined in Section 6.2.

First recall the definition of Σ_ξ from (6.15). By (6.25),

$$\Sigma_\xi = \begin{bmatrix} \Sigma_\xi^- & 0 \\ 0 & \Sigma_\xi^+ \end{bmatrix}, \qquad \text{where } \Sigma_\xi^- = T_{0\xi}^{-1} R_{-1\xi}^*, \text{ and } \Sigma_\xi^+ = R_{-1\xi}^{-1} T_{0\xi}^*.$$

Hence, $\sigma(\Sigma_\xi) = \sigma(\Sigma_\xi^-) \cup \sigma(\Sigma_\xi^+)$ and the map $\lambda \mapsto \overline{\lambda}^{-1}$ interchanges $\sigma(\Sigma_\xi^-)$ and $\sigma(\Sigma_\xi^+)$.

We will assume from now on that

$$\sigma(\Sigma_\xi^-) \cap \sigma(\Sigma_\xi^+) = \emptyset. \tag{S}$$

This happens, for instance, if X_∞ is a set of $\dim M$ different points, because then all the eigenvalues of Σ_ξ are distinct by Proposition 6.4.

We define the meromorphic function λ_ξ on $\widehat{X}$ by

$$\lambda_\xi(p) = \left(\frac{w_\xi}{z_\xi}\right)(p), \qquad p \in \widehat{X}.$$

If (S) holds, we can partition $\widehat{X}_\infty = \widehat{X}_\infty^- \cup \widehat{X}_\infty^+$, where we put

$$\widehat{X}_\infty^\pm = \{p \in \widehat{X}_\infty : \lambda_\xi(p) \in \sigma(\Sigma_\xi^\pm)\}$$

(recall that if $p \in \widehat{X}_\infty$, then $\lambda_\xi(p) \in \sigma(\Sigma_\xi)$ by Proposition 6.4).

We also define

$$\Gamma_\xi = z_\xi(\widehat{X}_{\mathbb{R}}).$$

The next proposition is a list of properties of the discriminant curve $\widehat{X}$ and the functions ν_ξ and Q_ξ. These properties will be used in the sequel. Some of this properties have already been proved above.

Proposition 6.25. *The following statements are true:*

(a) Q_ξ is meromorphic and projection-valued on $\widehat{X}$;

(b) ν_ξ is holomorphic and projection-valued on $\widehat{\mathbb{C}} \setminus \sigma(N_\xi)$;

(c) ν_ξ is holomorphic and projection-valued on $\widehat{\mathbb{C}} \setminus (\Gamma_\xi \cup F)$, where F is a finite subset of $\widehat{\mathbb{C}}$;

(d) $\widehat{X}_{\mathbb{R}} \cap \widehat{X}_\infty = \emptyset$;

(e) If $\widehat{X}_j$ is a connected component of $\widehat{X}$, then $\widehat{X}_j \cap \widehat{X}_\infty \neq \emptyset$.

Proof. Statement (a) was proved in Section 6.2. Statement (b) was proved in Theorem 6.10. By Proposition 6.22, ν_ξ is holomorphic outside the support of e_ξ. Using (6.13) and the fact that $X_{\mathrm{aff},\mathbb{R}}$ and $\widehat{X}_{\mathbb{R}}$ differ by a finite number of points, we get (c).

To prove (d), observe that Σ_ξ cannot have eigenvalues λ with $|\lambda| = 1$, because such an eigenvalue will be fixed by the map $\lambda \mapsto \overline{\lambda}^{-1}$, but this map interchanges the disjoint sets $\sigma(\Sigma_\xi^-)$ and $\sigma(\Sigma_\xi^+)$. If $p \in \widehat{X}_{\mathbb{R}}$, then $|\lambda_\xi(p)| = 1$, because $z_\xi(p) = \overline{w_\xi}(p)$. Hence, $p \notin \widehat{X}_\infty$, because if $p \in \widehat{X}_\infty$, then $\lambda_\xi(p) \in \sigma(\Sigma_\xi)$ by Proposition 6.4, but Σ_ξ has no eigenvalues of modulus 1.

Statement (e) follows from the fact that any connected component $\widehat{X}_j$ of $\widehat{X}$ has a corresponding component X_j in $\mathbb{CP}^2$ which must intersect the line at infinity. □

Now we give a Lemma which relates the behaviour of $\nu_\xi(z_\xi(p))$ and $Q_\xi(p)$ for p near $\widehat{X}_\infty$.

Lemma 6.26. *Define*

$$\varphi_+(p) = [1 - \nu_\xi(z_\xi(p))]Q_\xi(p), \qquad p \in \widehat{X} \setminus z_\xi^{-1}(\sigma(N_\xi))$$
$$\varphi_-(p) = \nu_\xi(z_\xi(p))Q_\xi(p), \qquad p \in \widehat{X} \setminus z_\xi^{-1}(\sigma(N_\xi)).$$

If $p_0 \in \widehat{X}_\infty^+$, then φ_+ vanishes in a neighbourhood of p_0. If $p_0 \in \widehat{X}_\infty^-$, then φ_- vanishes in a neighbourhood of p_0.

Proof. Let $p_0 \in \widehat{X}_\infty$. First we show that $z_\xi(p_0) = \infty$. We denote by $(\zeta_1 : \zeta_2 : \zeta_3)$ the projective coordinates in $\mathbb{CP}^2$ of the point $q_0 = \pi_X(p_0) \in X_\infty$. We define $\eta_{\xi,j}$, $j = 1, 2, 3$, as in (6.17). By Proposition 6.4, $\zeta_3 = 0$ and $\lambda_\xi(q_0) \in \sigma(\Sigma_\xi)$, so, in particular, $\lambda_\xi(q_0) \neq \infty$. Note that $\zeta_3 = \eta_{\xi,3} = 0$. Also, $\eta_{\xi,1}$ and $\eta_{\xi,2}$ cannot be both zero. Since $\lambda_\xi(q_0) = (w_\xi/z_\xi)(q_0) = \eta_{\xi,2}/\eta_{\xi,1}$ by (6.18), we have $\eta_{\xi,1} \neq 0$. Therefore, $z_\xi(p_0) = z_\xi(q_0) = \eta_{\xi,1}/\zeta_3 = \infty$.

Recall that if $p \in \widehat{X}$ is such that $z_\xi(p) \neq 0$, then $\lambda_\xi(p) \in \sigma(\Sigma_\xi + z_\xi(p)^{-1}D_\xi)$. Let $V \subset \mathbb{C}$ be an open disk with centre $\lambda_\xi(p_0)$ and such that $\overline{V}$ does not contain any other eigenvalue of Σ_ξ.

If q is a point in $\widehat{X}$ and $\lambda_\xi(q) \in V$, let Δ_q be a small circle in $\mathbb{C}$ around $\lambda_\xi(q)$ which is positively oriented, is contained entirely inside V, and is such that no other eigenvalue of $\Sigma_\xi + z_\xi(q)^{-1}D_\xi$ lies inside Δ_q. Recall that if such point q satisfies $q \in \widehat{X}_0$ and $z_\xi(q) \neq 0$, then

$$Q_\xi(q) = \Pi_{\lambda_\xi(q)}(\Sigma_\xi + z_\xi(q)^{-1}D_\xi) = \frac{1}{2\pi i}\int_{\Delta_q} \left(\lambda - \Sigma_\xi - z_\xi(q)^{-1}D_\xi\right)^{-1} d\lambda.$$

Put

$$L(z) = \{q \in \widehat{X} \ : \ z_\xi(q) = z, \ \lambda_\xi(q) \in V\}.$$

For $z \in \widehat{\mathbb{C}}$ near ∞, no eigenvalue of the matrix $\Sigma_\xi + z^{-1}D_\xi$ lies on ∂V, by continuity of the spectrum. Hence, for a general $z \in \widehat{\mathbb{C}}$ near ∞,

$$\sum_{q\in L(z)} Q_\xi(q) = \frac{1}{2\pi i}\int_{\partial V} \left(\lambda - \Sigma_\xi - z^{-1}D_\xi\right)^{-1} d\lambda.$$

because the eigenvalues of $\Sigma_\xi + z^{-1}D_\xi$ which are contained in V are precisely the values of $\lambda_\xi(q)$ for $q \in L(z)$.

Since

$$\Pi_{\lambda_\xi(p_0)}(\Sigma_\xi) = \frac{1}{2\pi i}\int_{\partial V} (\lambda - \Sigma_\xi)^{-1} d\lambda,$$

because $z_\xi(p_0) = \infty$, it follows that

$$\sum_{q\in L(z)} Q_\xi(q) \xrightarrow[z\to\infty]{} \Pi_{\lambda_\xi(p_0)}(\Sigma_\xi). \tag{6.45}$$

Assume that $p_0 \in \widehat{X}^+_\infty$. This means that $\lambda_\xi(p_0) \in \sigma(\Sigma^+_\xi)$. Put

$$\psi(z) = (1 - \nu_\xi(z))\left(\sum_{q\in L(z)} Q_\xi(q)\right).$$

Since $Q_\xi(q)$ and $\nu_\xi(z_\xi(q))$ commute for $q \in \widehat{X}_0 \setminus z_\xi^{-1}(\sigma(N_\xi))$ (see Lemma 6.23), $\psi(z)$ is projection-valued and meromorphic in a punctured neighbourhood of ∞. Also, by (6.45), we have $\psi(\infty) = 0$, because $\nu_\xi(\infty) = P_{M_+}$ and $P_{M_-}\Pi_{\lambda_\xi(p_0)}(\Sigma_\xi) = 0$. Since $\psi(z)$ is continuous at $z = \infty$ and projection-valued, it follows that $\psi(z)$ vanishes in a neighbourhood of ∞.

The proof of the Lemma for the case when $p_0 \in \widehat{X}^+_\infty$ concludes by observing that

$$\varphi_+(p) = \psi(z_\xi(p))Q_\xi(p)$$

for a general p near p_0. The case where $p_0 \in \widehat{X}^-_\infty$ is treated in a similar way. □

For the proof of the next lemma, we will need to use the following version of the Privalov-Plemelj jump formula. Let $\Gamma : [0, 1] \to \mathbb{C}$ be a parametrized piecewise smooth curve. We also denote by Γ its image $\Gamma([0, 1])$. This curve has a well defined tangent except at a finite number of points. Let $\theta(s)$ be the angle between the tangent vector $\Gamma'(s)$ at $\Gamma(s)$ and the vector $(1, 0)$.

Let F be a function defined on $\mathbb{C} \setminus \Gamma$ and fix a point $z_0 = \Gamma(s_0) \in \Gamma$. Put $\theta_0 = \theta(s_0)$. Let

$$z_{\pm,\varepsilon} = z_0 \pm \varepsilon i e^{i\theta_0}$$

be the two points $z_{-,\varepsilon}$ and $z_{+,\varepsilon}$ that are on the line normal to Γ at z_0 and are at a distance ε of z_0. If the limit

$$\lim_{\varepsilon \to 0} F(z_{+,\varepsilon}) - F(z_{-,\varepsilon})$$

exists, we call that number the jump of F at z_0 and we denote it by $\operatorname{Jump} F(z_0)$.

Now we consider μ a finite complex Borel measure on Γ and its Cauchy integral

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \frac{d\mu(u)}{u - z}, \qquad z \in \mathbb{C} \setminus \Gamma.$$

The function $F(z)$ has nontangential boundary values from each side at almost every point of Γ. Indeed, F belongs to the Smirnov class $E^p(\mathbb{C} \setminus \Gamma)$ for every $p < 1$. The jump of F at $z_0 \in \Gamma$ is precisely the difference of these two boundary values.

Denote by $\frac{d\mu}{|dz|}$ the Radon-Nikodym derivative of the absolutely continuous part of μ with respect to the arc-length measure on Γ, and put

$$\frac{d\mu}{dz}(z_0) = e^{-i\theta_0} \frac{d\mu}{|dz|}(z_0).$$

Then the Privalov-Plemelj jump formula states that

$$\operatorname{Jump} F(z_0) = \frac{d\mu}{dz}(z_0).$$

for almost every $z_0 \in \Gamma$ (with respect to arc-length measure on Γ).

The jump formula for the general case where Γ is a rectifiable curve is due to Privalov. This result was published originally in [Pri50] in Russian. A German translation of this work can be found in [Pri56]. Unfortunately, there is no english translation of this book. The statement of Privalov's result can be found in the Encyclopaedia of mathematics [88, Cauchy integral]. In the special case where $\Gamma = \mathbb{T}$, one can give a simpler proof of the jump formula using Fatou's theorem (see [CMR06, Section 2.4]). An introduction to the Smirnov class E^p can be found in [Dur70, Chapter 10].

Another fact that we will need is Privalov's uniqueness theorem. This states that if f is holomorphic on a connected open set Ω bounded by a piecewise smooth curve and has zero nontangential boundary values on a subset of positive measure of $\partial\Omega$, then f is identically zero. It is easy to deduce that this is also true for holomorphic functions on a Riemann surface. A proof of this theorem for the case when $\Omega = \mathbb{D}$ can be found in [Koo98, Section III.D]. The case when $\partial\Omega$ is piecewise smooth is obtained from the case $\Omega = \mathbb{D}$ by applying a Riemann mapping.

Recall that if $p_0 \in \widehat{X}_\infty$, then one of the functions φ_+, φ_- defined in Lemma 6.26 is identically zero in a neighbourhood of p_0, according to whether $p_0 \in \widehat{X}_\infty^+$ or $p_0 \in \widehat{X}_\infty^-$. If U is an open connected set contained in $\widehat{X} \setminus \widehat{X}_\mathbb{R}$ and such that $p_0 \in U$, we want to show that φ_+ (or φ_-) is identically zero in U. The function Q_ξ is meromorphic on $\widehat{X}$, but ν_ξ is analytic only outside of $\sigma(N_\xi)$. Therefore, a priori the functions φ_+ and φ_-

can have singularities at the points p such that $z_\xi(p) \in \sigma(N_\xi)$. Therefore, we need to use a continuation argument involving the jump formula to show that φ_+ (or φ_-) is identically zero in U.

Lemma 6.27. *Let U be an open connected set in $\widehat{X} \setminus \widehat{X}_{\mathbb{R}}$. If $U \cap \widehat{X}^+_\infty \neq \emptyset$, then*

$$\nu_\xi(z_\xi(p))Q_\xi(p) = Q_\xi(p), \qquad p \in U \setminus z_\xi^{-1}(\Gamma_\xi).$$

If $U \cap \widehat{X}^-_\infty \neq \emptyset$, then

$$\nu_\xi(z_\xi(p))Q_\xi(p) = 0, \qquad p \in U \setminus z_\xi^{-1}(\Gamma_\xi).$$

Proof. We will give the proof for the case $U \cap \widehat{X}^+_\infty \neq \emptyset$. The other case is symmetric. Take $p_0 \in U \cap \widehat{X}^+_\infty$, and define $\varphi_+(p)$ as in Lemma 6.26. We know that $\varphi_+ \equiv 0$ near p_0. We will use a continuation argument to show that $\varphi_+ \equiv 0$ on all of $U \setminus z_\xi^{-1}(\Gamma_\xi)$.

Fix an arbitrary point $p \in U \setminus z_\xi^{-1}(\Gamma_\xi)$. Then we can make a finite list $\Omega_0, \dots, \Omega_k$, where Ω_j are connected components of $U \setminus z_\xi^{-1}(\Gamma_\xi)$, the boundaries $\partial\Omega_j$ and $\partial\Omega_{j+1}$ have a common arc Γ_j contained in U, the point p_0 lies in Ω_0, and the point p lies in Ω_k.

Since φ_+ is identically zero on a neighbourhood of p_0, it is identically zero on all Ω_0. Let us prove by induction that φ_+ is identically zero on Ω_j for all $j = 0, 1, \dots, k$, so in particular $\varphi_+(p) = 0$. Assume that φ_+ is identically zero on Ω_j. If we can show that φ_+ has zero nontangential limit from Ω_{j+1} at almost every $q \in \Gamma_j$, the common arc of $\partial\Omega_j$ and $\partial\Omega_{j+1}$ inside U, then we will have $\varphi_+ \equiv 0$ in Ω_{j+1} by the Privalov's uniqueness theorem. This will prove the inductive step, and the base case, namely that φ_+ is indentically zero on Ω_0 has already being proved.

Since we are interested only on what happens a.e. on Γ_j, we can take a $q_0 \in \Gamma_j$ such that $q_0 \in \widehat{X}_0$ and $dz_\xi(q_0) \neq 0$ (because $\widehat{X} \setminus \widehat{X}_0$ is finite, and $dz_\xi(q) = 0$ only for a finite number of points $q \in \widehat{X}$). This second condition implies that $z = z_\xi(q)$ gives a local coordinate near q_0. We fix this point q_0, put $z_0 = z_\xi(q_0)$, and write everything using this coordinate z. We must study

$$\varphi_+(z) = [1 - \nu_\xi(z)]Q_\xi(z).$$

Since $\varphi_+(z)$ is identically zero for z on one side of Γ (the side corresponding to $z_\xi(\Omega_j)$), to see that the nontangential limit from the other side at z_0 is zero, it is enough to show that $\operatorname{Jump} \varphi_+(z_0) = 0$.

The function $Q_\xi(z)$ is continuous at z_0. Also, the nontangential boundary value of $\nu_\xi(z)$ exists a.e. on Γ, because by Proposition 6.22, $\nu_\xi(z) - P_{M_+}$ is the Cauchy integral of a finite Borel measure. We assume that $\nu_\xi(z)$ has nontangential boundary values at z_0. Hence,

$$\operatorname{Jump} \varphi_+(z_0) = \operatorname{Jump} F(z_0),$$

where

$$F(z) = -(\nu_\xi(z) - P_{M_+})Q_\xi(z_0).$$

The function $F(z)$ is the Cauchy integral of the measure

$$-2\pi i \, de_\xi \, \alpha_\xi Q_\xi(z_0).$$

By the Privalov-Plemelj jump formula,

$$\operatorname{Jump} F(z_0) = -2\pi i \frac{de_\xi}{dz}(z_0)\alpha_\xi Q_\xi(z_0).$$

We have $w_\xi(q_0) \neq \overline{z_\xi}(q_0)$, because $U \cap \widehat{X}_{\mathbb{R}} = \emptyset$ by the hypotheses of the lemma. Therefore, we can define a function ψ analytic in a neighbourhood of $\sigma(z_\xi(q_0)\Sigma_\xi + D_\xi)$ such that $\psi(u) = (u - \overline{z_\xi}(q_0))^{-1}$ in a small neighbourhood of $w_\xi(q_0) \in \sigma(z_\xi(q_0)\Sigma_\xi + D_\xi)$ and $\psi(u) = 0$ outside of this neighbourhood. Consider the matrix

$$\Psi = \psi(z_\xi(q_0)\Sigma_\xi + D_\xi).$$

Then we get by the Riesz-Dunford calculus that

$$Q_\xi(q_0) = (z_\xi(q_0)\Sigma_\xi + D_\xi - \overline{z_\xi}(q_0))\Psi.$$

This implies that

$$\operatorname{Jump} F(z_0) = -2\pi i \frac{de_\xi}{dz}(z_0)\alpha_\xi(z_0\Sigma_\xi + D_\xi - \overline{z_0})\Psi = 2\pi i \frac{de_\xi}{dz}(z_0)(z_0\alpha_\xi^* + \overline{z_0}\alpha_\xi + \gamma_\xi)\Psi.$$

By the relation obtained by taking adjoints in (6.10), we see that

$$\frac{de_\xi}{dz}(z_0)(z_0\alpha_\xi^* + \overline{z_0}\alpha_\xi + \gamma_\xi) = 0,$$

which implies that $\operatorname{Jump} F(z_0) = 0$. This argument is valid for almost every point $q_0 \in \Gamma_j$ (recall that $z_0 = z_\xi(q_0)$). This finishes the proof, because it shows that φ_+ has zero jump at almost every point of Γ_j, and therefore zero boundary value from Ω_{j+1} at almost every point of Γ_j. Hence, φ_+ must be identically zero on Ω_{j+1}, which proves the inductive step. $\square$

Using this Lemma, now we can define $\widehat{X}_-$ and $\widehat{X}_+$, and prove the restoration formula for the mosaic ν_ξ.

Theorem 6.28. *Assume that $\xi \in \Xi$ is a nondegenerate direction such that α_ξ is invertible. Suppose that* (S) *holds. Then there exists a partition of $\widehat{X} \setminus \widehat{X}_{\mathbb{R}}$ into two* halves *$\widehat{X}_-$ and $\widehat{X}_+$ with the following properties:*

(a) Each half is the union of some of the connected components of $\widehat{X} \setminus \widehat{X}_{\mathbb{R}}$.

(b) The two halves are conjugate to each other: $*(\widehat{X}_-) = \widehat{X}_+$, and $*(\widehat{X}_+) = \widehat{X}_-$.

(c) If a connected component $\widehat{X}_j$ of $\widehat{X}$ intersects $\widehat{X}_{\mathbb{R}}$, then it intersects both halves $\widehat{X}_-$ and $\widehat{X}_+$.

(d) If a component $\widehat{X}_j$ does not intersect $\widehat{X}_{\mathbb{R}}$, then it is contained either in $\widehat{X}_-$ or in $\widehat{X}_+$. Moreover, $*(\widehat{X}_j)$ is a different component of $\widehat{X}$.

(e) The restoration formula holds:

$$\nu_\xi(z) = \sum_{\substack{p\in\widehat{X}_+ \\ z_\xi(p)=z}} Q_\xi(p), \qquad z \in \widehat{\mathbb{C}} \setminus \Gamma_\xi, \tag{6.46}$$

where $\Gamma_\xi = z_\xi(\widehat{X}_\mathbb{R})$.

Proof. The theory of real algebraic curves shows that there are two possibilities for the (connected) Riemann surface of an irreducible real algebraic curve in $\mathbb{C}^2$: either the set of points not fixed by the involution induced by complex conjugation is connected, or it consists of precisely two connected components, which are interchanged by the involution. In the second case, we say that the surface is separated. See [GH81, Section 3] for an exposition of the topological properties of real algebraic curves.

We define $\widehat{X}_-$ as the union of the connected components of $\widehat{X} \setminus \widehat{X}_\mathbb{R}$ which intersect $\widehat{X}_\infty^-$, and similarly for $\widehat{X}_+$. By Lemma 6.27, $\widehat{X}_-$ and $\widehat{X}_+$ are disjoint (note that $Q_\xi(p)$ cannot vanish).

Now we observe that every connected component of $\widehat{X} \setminus \widehat{X}_\mathbb{R}$ must intersect $\widehat{X}_\infty$, and hence $\widehat{X} = \widehat{X}_- \cup \widehat{X}_\mathbb{R} \cup \widehat{X}_+$. Indeed, assume that U is a connected component of $\widehat{X} \setminus \widehat{X}_\mathbb{R}$. Let $\widehat{X}_j$ be the connected component of $\widehat{X}$ which contains U.

There are two possible cases: either $\widehat{X}_j$ contains no real points, or $\widehat{X}_j$ contains real points, and hence, it is fixed by the involution $*$ (because $*$ permutes the components of $\widehat{X}$ and some points of $\widehat{X}_j$ are fixed by $*$). In the first case, we have $U = \widehat{X}_j$. Since $\widehat{X}_j$ contains points of $\widehat{X}_\infty$ by Proposition 6.25, we see that U intersects $\widehat{X}_\infty$.

In the second case, $\widehat{X}_j$ is the Riemann surface of an irreducible real algebraic curve. The surface $\widehat{X}_j$ must be separated, and U must be one of the connected components of $\widehat{X}_j \setminus \widehat{X}_\mathbb{R}$. Therefore, $\widehat{X}_j \setminus \widehat{X}_\mathbb{R} = U \cup *(U)$. Note that $\widehat{X}_\mathbb{R}$ does not intersect $\widehat{X}_\infty$. Since $\widehat{X}_j$ contains points of $\widehat{X}_\infty$, either U or $*(U)$ must intersect $\widehat{X}_\infty$. If U intersects $\widehat{X}_\infty$, we are done. If $*(U)$ intersects $\widehat{X}_\infty$, since the involution $*$ maps $\widehat{X}_\infty$ onto $\widehat{X}_\infty$, U must intersect $\widehat{X}_\infty$ as well.

Therefore, we see that $\widehat{X}_-$ and $\widehat{X}_+$ form a partition of $\widehat{X} \setminus \widehat{X}_\mathbb{R}$. Property (a) holds by construction, and it is also clear that (b) is true. Properties (c) and (d) are obtained using the fact that the involution $*$ interchanges $\widehat{X}_\infty^-$ and $\widehat{X}_\infty^+$.

To obtain the restoration formula, we use (6.16) to get the equation

$$\sum_{\substack{p\in\widehat{X} \\ z_\xi(p)=z}} Q_\xi(p) = I_M.$$

Then we multiply this equation on the left by $\nu_\xi(z)$ and use Lemma 6.27. □

In the theorem above, we choose a nondegenerate direction $\xi \in \Xi$ to construct the two halves of the curve. Clearly, properties (a) through (d) depend only on the topological properties of the Riemann surface $\widehat{X}$ and its real part $\widehat{X}_\mathbb{R}$. However, a priori it is not clear whether the construction of $\widehat{X}_+$ and $\widehat{X}_-$ depends on the choice of ξ or not. We will show that it does not, meaning that for every choice of $\xi \in \Xi$, the procedure used above to define $\widehat{X}_+$ and $\widehat{X}_-$ produces the same result.

Proposition 6.29. *Assume that* (S) *holds for some nondegenerate direction* $\xi \in \Xi$ *such that* α_ξ *is invertible. Then* (S) *holds for every* $\xi \in \Xi$ *such that* α_ξ *is invertible. Moreover, the construction of* $\widehat{X}_+$ *and* $\widehat{X}_-$ *done in Theorem 6.28 does not depend on the choice of* ξ.

Proof. Recall that, given ξ, each connected component U of $\widehat{X} \setminus \widehat{X}_\mathbb{R}$ is assigned to either $\widehat{X}_+$ or $\widehat{X}_-$ according to the following procedure: we take a point $p \in U \cap \widehat{X}_\infty$. Then, either $\lambda_\xi(p) \in \sigma(\Sigma_\xi^+)$ or $\lambda_\xi(p) \in \sigma(\Sigma_\xi^-)$. In the first case we include U into $\widehat{X}_+$ and in the second case we include U into $\widehat{X}_-$. Both cases cannot happen simultaneosly because of condition (S), and we have proved that this procedure does not depend on the choice of p, so this procedure makes sense.

We will now rewrite this procedure in terms of the selfadjoint matrices σ_1, σ_2, which do not depend on ξ. Recall from the construction of the discriminant curve that a point $p \in \mathbb{CP}^2$ with homogeneous coordinates $(\zeta_1 : \zeta_2 : \zeta_3)$ is in X_∞ if and only if $\zeta_3 = 0$ and

$$\det(\zeta_1\sigma_2 - \zeta_2\sigma_1) = 0. \tag{6.47}$$

The matrices σ_j, $j = 1, 2$, have the structure

$$\sigma_j = \begin{bmatrix} 0 & \delta_j \\ \delta_j^* & 0 \end{bmatrix}$$

according to the decomposition $M = M_- \oplus M_+$. To see this, note that σ_j are selfadjoint and $P_{M_+}\sigma_j P_{M_+} = 0$ and $P_{M_-}\sigma_j P_{M_-} = 0$ by the definition of σ_j in Theorem 6.17, so σ_j have the required structure. Therefore, (6.47) is equivalent to either

$$\det(\zeta_1\delta_2 - \zeta_2\delta_1) = 0$$

or

$$\det(\zeta_1\delta_2^* - \zeta_2\delta_1^*) = 0$$

Using this notation, the operators Σ_ξ^+ and Σ_ξ^- can be written as

$$\begin{aligned} \Sigma_\xi^+ &= -(\xi_1\delta_1 + \xi_2\delta_2)^{-1}(\overline{\xi}_1\delta_1 + \overline{\xi}_2\delta_2) \\ \Sigma_\xi^- &= -(\xi_1\delta_1^* + \xi_2\delta_2^*)^{-1}(\overline{\xi}_1\delta_1^* + \overline{\xi}_2\delta_2^*). \end{aligned}$$

For a point $p \in \widehat{X}_\infty$ with homogeneous coordinates $(\zeta_1 : \zeta_2 : 0)$, we have

$$\lambda_\xi(p) = \frac{\overline{\xi}_1\zeta_1 + \overline{\xi}_2\zeta_2}{\xi_1\zeta_1 + \xi_2\zeta_2}.$$

Using this, we see that $\lambda_\xi(p) \in \sigma(\Sigma_\xi^+)$ if and only if

$$\det[(\xi_1\zeta_1 + \xi_2\zeta_2)(\overline{\xi}_1\delta_1 + \overline{\xi}_2\delta_2) - (\overline{\xi}_1\zeta_1 + \overline{\xi}_2\zeta_2)(\xi_1\delta_1 + \xi_2\delta_2)] = 0.$$

The determinant on the left hand side of this equality is equal to

$$(-2i \operatorname{Im} \overline{\xi}_1\xi_2)^{\dim M} \det(\zeta_1\delta_2 - \zeta_2\delta_1).$$

Therefore, we see that $\lambda_\xi(p) \in \sigma(\Sigma_\xi^+)$ if and only if $\det(\zeta_1\delta_2 - \zeta_2\delta_1) = 0$. A similar argument replacing δ_1, δ_2 by δ_1^*, δ_2^* shows that $\lambda_\xi(p) \in \sigma(\Sigma_\xi^-)$ if and only if $\det(\zeta_1\delta_2^* - \zeta_2\delta_1^*) = 0$

This means that the criterion for whether the point p belongs to either $\widehat{X}_\infty^+$ or $\widehat{X}_\infty^-$ can be written in terms of the matrices σ_1, σ_2, which do not depend on ξ. Moreover, it follows from the reasoning above that an equivalent way to write condition (S) is that, for each point $p \in X_\infty$ with homogeneous coordinates $(\zeta_1 : \zeta_2 : 0)$, the determinants $\det(\zeta_1\delta_2 - \zeta_2\delta_1)$ and $\det(\zeta_1\delta_2^* - \zeta_2\delta_1^*)$ do not vanish simultaneously. This shows that if condition (S) is satisfied for some $\xi \in \Xi$ such that α_ξ is invertible, then this condition is satisfied for every such ξ. □

Example 6.30. In the case of the separating structure generated by a subnormal operator S, the operator α_ξ has the form

$$\alpha_\xi = \frac{1}{2}\begin{bmatrix} 0 & -(\xi_1 + i\xi_2)T_0^* \\ (\xi_1 - i\xi_2)T_0 & 0 \end{bmatrix},$$

where $T_0 = X|M_-$ (see Example 6.7). Hence, the operator Σ_ξ is

$$\Sigma_\xi = -\alpha_\xi^{-1}\alpha_\xi^* = \begin{bmatrix} \dfrac{\overline{\xi_1} - i\overline{\xi_2}}{\xi_1 - i\xi_2} I_{M_-} & 0 \\ 0 & \dfrac{\overline{\xi_1} + i\overline{\xi_2}}{\xi_1 + i\xi_2} I_{M_+}. \end{bmatrix}$$

Therefore, each of the spectra $\sigma(\Sigma_\xi^-)$ and $\sigma(\Sigma_\xi^+)$ has only one point. Moreover, we see that these two points are different if and only if the direction ξ is nondegenerate, i.e., if $\operatorname{Im} \xi_1\overline{\xi_2} \neq 0$. This means that we can carry out the construction given above to define the halves $\widehat{X}_-$ and $\widehat{X}_+$, as long as we choose a general direction ξ.

A necessary remark is that Theorem 6.28 does not prove that the discriminant curve $\widehat{X}$ is separated in the sense that each component $\widehat{X}$ is divided in two connected components when we remove its real points. Here, it may happen that some components have no real points and so, belong either to $\widehat{X}_-$ or to $\widehat{X}_+$, and there is a conjugate component in the other half of the curve $\widehat{X}$. These components are in some sense degenerate. However, this partition into halves $\widehat{X}_-$ and $\widehat{X}_+$ should be good enough to allow the development of the theory. In the context of subnormal operators in [Yak98a], it happened that the only degenerate components that appeared were those of degree one.

Another remark is that the restoration formula (6.46) imposes a strong condition on the spectrum of N_ξ. Using the restoration formula, we see that $\nu_\xi(z)$ is discontinuous at Γ_ξ except for a finite number of points. Since we know that $\nu_\xi(z)$ is holomorphic outside $\operatorname{supp} e_\xi$, we get that $\Gamma_\xi \subset \operatorname{supp} e_\xi \subset \sigma(N_\xi)$. We also know that e_ξ is supported in Γ_ξ and perhaps a finite number of additional points. This implies that if the pool associated with the separating structure is irreducible, then $\sigma(N_\xi)$ is precisely the set Γ_ξ, which is a finite union of smooth closed curves, and perhaps a finite number of isolated points.

7. Generalized compression

In this chapter we introduce a new notion of generalized compression. This notion can be used to compress a separating structure and obtain a vessel. First, we define the generalized compression for a linear map acting on a vector space. We also give a slightly different definition which is more convenient when working with bounded operators on Hilbert spaces.

Then we give the definition of the compression of separating structures and find the conditions so that a separating structure can be compressed. Finally, we show that the compression of a separating structure produces a vessel. The matrices $\sigma_1, \sigma_2, \gamma^{\text{out}}$ for the vessel coincide with the matrices $\sigma_1, \sigma_2, \gamma$ of the separating structure. In particular the discriminant curves of the separating structure and the vessel are the same.

7.1. Compression of linear maps

We start by giving the definition of generalized compression for linear maps on a vector space.

Definition 7.1. Let $K \supset H \supset G$ be vector spaces and $A : K \to K$ a linear map which satisfies the two following conditions:

$$AG \cap H \subset G, \tag{C1}$$

and

$$AH \subset AG + H. \tag{C2}$$

Then we define the compression $\widetilde{A} : H/G \to H/G$ by the following procedure. Given a vector $h \in H$, using (C2), we can find a $g \in G$ such that $h' = A(h - g) \in H$. Then we define

$$\widetilde{A}(h + G) = h' + G.$$

To check that this is well defined, we must see that if $h \in G$, then $h' \in G$, but this is a consequence of (C1). Hence, the compression $\widetilde{A}$ is a linear map on the quotient space H/G. Note that the spaces G and H need not be invariant under A.

In the context of Hilbert spaces, i.e., when K, H, G are Hilbert spaces and $A \in \mathcal{B}(K)$, we will usually require stronger conditions that guarantee that the compression $\widetilde{A}$ is bounded.

Proposition 7.2. *Assume that $K \supset H \supset G$ are Hilbert spaces and $A : K \to K$ is a bounded linear map which satisfies* (C1) *and* (C2), *so that its compression $\widetilde{A}$ is defined. Suppose that*

$$\overline{AG} \cap H \subset G, \tag{C1*}$$

holds and that if $L = \overline{AG} \cap (K \ominus G)$, *then*

$$L \dotplus H \text{ is a direct sum} \tag{C3}$$

(note that $L \cap H = 0$ *by* (C1*)*). Then* $\widetilde{A}$ *is bounded.*

Proof. Let

$$P : L \dotplus H \to H \tag{7.1}$$

be the parallel projection onto H according to this direct sum decomposition. We see that (C2) implies that $AH \subset L + H$, and that the compression $\widetilde{A}$ is

$$\widetilde{A}(h + G) = PAh + G, \qquad h \in H.$$

Hence, $\widetilde{A}$ is bounded and $\|\widetilde{A}\| \le \|P\|\|A\|$. □

Using the notation in the proof of this proposition, we see that if we identify the quotient space H/G with the space $R = H \ominus G$, then

$$\widetilde{A} = P_R PA|R. \tag{7.2}$$

Example 7.3. Let us see that the classical notion of a compression can be written as a generalized compression. Assume that $K = H_1 \oplus H_2 \oplus H_3$ and that A has the structure

$$A = \begin{bmatrix} * & 0 & 0 \\ * & A_0 & 0 \\ * & * & * \end{bmatrix}$$

according to this decomposition, so that A is a dilation of A_0 and A_0 is its classical compression. Then we put $G = H_3$, $H = H_2 \oplus H_3$. We have $AG \subset G$, so that (C1*) holds. Also, $AH \subset H$, and (C2) holds. Moreover, $L = 0$, which implies that (C3) holds and $P = I_H$ (see (7.1)). We identify the quotient H/G with the space $R = H \ominus G = H_2$. Now, (7.2) shows that $\widetilde{A} = P_{H_2}A|H_2 = A_0$. Hence, in this setting, the generalized compression coincides with the classical compression.

Lemma 7.4. *Assume that* A_1 *and* A_2 *are commuting linear maps on* K *satisfying the conditions in Definition 7.1 and let* $\widetilde{A}_1$ *and* $\widetilde{A}_2$ *be their respective compressions. Assume that* $A_1 + A_2$ *also satisfies the conditions in Definition 7.1, so its compression* $\widetilde{A_1 + A_2}$ *is defined. If*

$$(A_1G + A_2G) \cap H \subset G, \tag{7.3}$$

then $\widetilde{A_1 + A_2} = \widetilde{A}_1 + \widetilde{A}_2$.

Proof. Take $x \in H$. Then there are $g_0, g_1, g_2 \in G$ such that

$$\begin{aligned} y_0 &:= (A_1 + A_2)(x - g_0) \in H, \\ y_1 &:= A_1(x - g_1) \in H, \\ y_2 &:= A_2(x - g_2) \in H. \end{aligned}$$

We have $\widetilde{A_1 + A_2}(x + G) = y_0 + G$, $\widetilde{A}_1(x + G) = y_1 + G$ and $\widetilde{A}_2(x + G) = y_2 + G$, so it suffices to check that $y_0 - y_1 - y_2 \in G$. Now, $y_0 - y_1 - y_2 = A_1(g_1 - g_0) + A_2(g_2 - g_0)$. The vector on the left hand side of this equality is in H and the vector on the right hand side belongs to $A_1G + A_2G$, so by (7.3) we get $y_0 - y_1 - y_2 \in G$. □

Since we are interested in compressing the operators A_1, A_2 in a separating structure to obtain operators $\widetilde{A}_1$, $\widetilde{A}_2$ forming a (commutative) pool, we should know when the compressions of two commuting operators also commute.

Lemma 7.5. *Assume that A_1 and A_2 are commuting linear maps on K satisfying the conditions in Definition 7.1 and let $\widetilde{A}_1$ and $\widetilde{A}_2$ be their respective compressions. Assume that*

$$(A_1G + A_2G + A_1A_2G) \cap H \subset G, \tag{7.4}$$

holds. Then the compressions $\widetilde{A}_1$ and $\widetilde{A}_2$ commute. If moreover

$$A_1A_2H \subset A_1A_2G + H \tag{7.5}$$

so that the compression $\widetilde{A_1A_2}$ of A_1A_2 is defined, then $\widetilde{A_1A_2} = \widetilde{A}_1\widetilde{A}_2$.

Proof. Let us first check that $\widetilde{A}_1$ and $\widetilde{A}_2$ commute. Take an $x \in H$. Then there are vectors $g, g', l, l' \in G$ such that

$$\begin{aligned} y &:= A_1(x-g) \in H, \qquad & y' &:= A_2(x-g') \in H, \\ z &:= A_2(y-l) \in H, \qquad & z' &:= A_1(y'-l') \in H. \end{aligned}$$

By definition of the compression, $\widetilde{A}_2\widetilde{A}_1(x+G) = z + G$ and $\widetilde{A}_1\widetilde{A}_2(x+G) = z' + G$. We must check that $z - z' \in G$. We compute

$$z - z' = A_2(A_1(x-g) - l) - A_1(A_2(x-g') - l') = A_1A_2(g'-g) + A_1l' - A_2l.$$

The vector on the right hand side of this equation is in $A_1G + A_2G + A_1A_2G$ and the vector on the left hand side is in H. It suffices to use (7.4) to see that $z - z' \in G$.

Now assume that (7.5) holds. Take $x \in H$. Then there is $g_0 \in G$ such that $z_0 = A_1A_2(x-g_0) \in H$. We have $\widetilde{A_1A_2}(x+G) = z_0 + G$. Let us check that $z_0 - z \in G$, which shows that $\widetilde{A_1A_2} = \widetilde{A}_1\widetilde{A}_2$. To this end, we compute

$$z_0 - z = A_1A_2(x-g_0) - A_2(A_1(x-g) - l) = A_1A_2(g-g_0) + A_2l.$$

The left hand side of this equality belongs to H and the right hand side belongs to $A_2G + A_1A_2G$. By (7.4) we have that $z_0 - z \in G$. $\square$

Note that to show the equality $\widetilde{A_1A_2} = \widetilde{A}_1\widetilde{A}_2$ we have not used the full hypothesis (7.4). It is also possible to show that $\widetilde{A_1A_2} = \widetilde{A}_1\widetilde{A}_2$ (so in particular $\widetilde{A}_1$ and $\widetilde{A}_2$ commute) if one assumes (7.5) and either

$$(A_1G + A_1A_2G) \cap H \subset G$$

or

$$(A_2G + A_1A_2G) \cap H \subset G.$$

7.2. Compression of separating structures

Now we pass to the compression of separating structures. Recall that a tuple

$$\omega = (K, A_1, A_2, H_{0,-}, M_-, M_+, H_{0,+})$$

is called an orthogonal separating structure if A_1 and A_2 are selfadjoint operators on K,

$$K = H_{0,-} \oplus M_- \oplus M_+ \oplus H_{0,+},$$

and

$$\begin{aligned} &A_j H_{0,-} \subset H_-, && A_j H_- \subset H_- + M_+, \\ &A_j H_+ \subset M_- + H_+, && A_j H_{0,+} \subset H_+, \end{aligned}$$

for $j = 1, 2$, where

$$H_- = H_{0,-} + M_-, \qquad H_+ = M_+ + H_{0,+}.$$

Suppose that A_1 and A_2 are two selfadjoint operators on K which are included in two orthogonal separating structures ω and $\widehat{\omega}$, so that:

$$\begin{aligned} \omega &= (K, A_1, A_2, H_{0,-}, M_-, M_+, H_{0,+}), \\ \widehat{\omega} &= (K, A_1, A_2, \widehat{H}_{0,-}, \widehat{M}_-, \widehat{M}_+, \widehat{H}_{0,+}). \end{aligned}$$

We write $\widehat{\omega} \prec \omega$ if

$$H_- \subset \widehat{H}_-, \qquad H_+ \supset \widehat{H}_+ \tag{7.6}$$

(note that $H_- \subset \widehat{H}_-$ if and only if $H_+ \supset \widehat{H}_+$).

Observe that conditions (7.6) and those involved in the definition of the separating structures ω and $\widehat{\omega}$ remain invariant if we exchange the subscripts $_+$ and $_-$, remove the hat $\widehat{\ }$ from those spaces which had it, and add it to those spaces which did not have it. This kind of symmetry will be called *hat-symmetry* and it will be useful later.

Assume that $\widehat{\omega} \prec \omega$. We will now define the compression of these two structures. We start by defining the operators $\beta_j : M_+ \to M_-$ and $\widehat{\beta}_j : \widehat{M}_+ \to \widehat{M}_-$ by

$$\beta_j = P_{M_-} A_j | M_+, \qquad \widehat{\beta}_j = P_{\widehat{M}_-} A_j | \widehat{M}_+, \qquad j = 1, 2. \tag{7.7}$$

Note that their adjoints are

$$\beta_j^* = P_{M_+} A_j | M_-, \qquad \widehat{\beta}_j^* = P_{\widehat{M}_+} A_j | \widehat{M}_-,$$

Using the formula given in Theorem 6.17 for the rates σ_j of the pool $\mathcal{P}$ generated by ω and the rates $\widehat{\sigma}_j$ of the pool $\widehat{\mathcal{P}}$ generated by $\widehat{\omega}$, we see that

$$\sigma_j = \begin{bmatrix} 0 & i\beta_j \\ -i\beta_j^* & 0 \end{bmatrix}, \qquad \widehat{\sigma}_j = \begin{bmatrix} 0 & i\widehat{\beta}_j \\ -i\widehat{\beta}_j^* & 0 \end{bmatrix}, \tag{7.8}$$

according to the decompositions $M = M_- \oplus M_+$ and $\widehat{M} = \widehat{M}_- \oplus \widehat{M}_+$.

Hence, if we assume the non-degeneracy condition (6.14) for both σ_1, σ_2 and $\widehat{\sigma}_1, \widehat{\sigma}_2$, replacing A_1 and A_2 by $t_1A_1 + t_2A_2$ and $t_3A_1 + t_4A_2$, where $t_k \in \mathbb{R}$ and $t_1t_4 - t_2t_3 \neq 0$, we may assume that

$$\beta_j \text{ and } \widehat{\beta}_j \text{ are invertible, for } j = 1, 2. \tag{Iβ}$$

We will do so hereafter.

Definition 7.6. We define the space

$$R = \widehat{H}_- \ominus H_- = H_+ \ominus \widehat{H}_+ = H_+ \cap \widehat{H}_-. \tag{7.9}$$

This space will be the compression space when we compress A_1 and A_2 to either $\widehat{H}_-/H_-$ or $H_+/\widehat{H}_+$, because it can be identified with both quotients.

Now we define the operators $\tau_- : \widehat{M}_- \to M_-$ and $\tau_+ : \widehat{M}_+ \to M_+$ by

$$\tau_- = P_{M_-}|\widehat{M}_-, \qquad \tau_+ = P_{M_+}|\widehat{M}_+. \tag{7.10}$$

Note that the adjoints of these operators are

$$\tau_-^* = P_{\widehat{M}_-}|M_-, \qquad \tau_+^* = P_{\widehat{M}_+}|M_+.$$

The following theorem relates these two operators with the possibility of compressing the operators A_1 and A_2 using the generalized compression as defined above.

Theorem 7.7. *Let $\omega, \widehat{\omega}$ be orthogonal separating structures. Assume that $\widehat{\omega} \prec \omega$ and that* (Iβ) *holds. Put $H = H_+$ and $G = \widehat{H}_+$. The operators A_1 and A_2 satisfy the conditions* (C1*), (C2), *and* (C3) *needed for the construction of their compressions to $H_+/\widehat{H}_+$ if and only if τ_- is invertible.*

Similarly, the operators A_1, A_2 can be compressed to $\widehat{H}_-/H_-$ if and only if τ_+ is invertible.

Before proving this theorem, we need to introduce a technical lemma.

Lemma 7.8. *The following relation holds:*

$$(P_{M_-} - I)\widehat{M}_- \subset R.$$

Proof. First, $(P_{M_-} - I)\widehat{M}_- \subset \widehat{H}_-$, because $M_- \subset H_- \subset \widehat{H}_-$ and $\widehat{M}_- \subset \widehat{H}_-$. Second,

$$P_{H_-}(P_{M_-} - I)|\widehat{M}_- = (P_{M_-} - P_{H_-})|\widehat{M}_- = -P_{H_{0,-}}|\widehat{M}_-.$$

Since $\widehat{\beta}_1$ is invertible,

$$\widehat{M}_- = \widehat{\beta}_1\widehat{M}_+ = P_{\widehat{M}_-}A_1\widehat{M}_+.$$

Using $P_{H_{0,-}}P_{\widehat{H}_+} = 0$ and $A_1\widehat{M}_+ \subset \widehat{M}_- \oplus \widehat{H}_+$, we get

$$\begin{aligned} P_{H_{0,-}}\widehat{M}_- &= P_{H_{0,-}}P_{\widehat{M}_-}A_1\widehat{M}_+ = P_{H_{0,-}}(P_{\widehat{M}_-} + P_{\widehat{H}_+})A_1\widehat{M}_+ \\ &= P_{H_{0,-}}A_1\widehat{M}_+ \subset P_{H_{0,-}}A_1H_+ = 0. \end{aligned}$$

This implies the relation

$$P_{H_{0,-}} P_{\widehat{M}_-} = 0, \tag{7.11}$$

which will be useful later.

We see that $P_{H_-}(P_{M_-} - I)|\widehat{M}_- = 0$. This finishes the proof, because $R = \widehat{H}_- \ominus H_-$. □

Now we can give the proof of Theorem 7.7.

Proof of Theorem 7.7. First assume that τ_- is invertible. We have

$$\overline{A_j \widehat{H}_+} \cap H_+ \subset (\widehat{M}_- + \widehat{H}_+) \cap H_+ = (\widehat{M}_- \cap H_+) + \widehat{H}_+,$$

because $\widehat{H}_+ \subset H_+$. Now we check that

$$\widehat{M}_- \cap H_+ = 0, \tag{7.12}$$

so that we get condition (C1*). Assume that $x \in \widehat{M}_- \cap H_+$. Then $\tau_- x = P_{M_-} x = 0$, so that $x = 0$, because τ_- is invertible.

To prove condition (C2), we first check that

$$M_- \subset H_+ + \widehat{M}_-. \tag{7.13}$$

Take any $m_- \in M_-$. Since τ_- is invertible, $m_- = \tau_- \widehat{m}_-$ for some $\widehat{m}_- \in \widehat{M}_-$. Then, by Lemma 7.8,

$$m_- - \widehat{m}_- = (P_{M_-} - I)\widehat{m}_- \in R \subset H_+.$$

Hence, $m_- \in H_+ + \widehat{M}_-$.

Now we see that

$$A_j H_+ \subset H_+ + M_- \subset H_+ + \widehat{M}_- = A_j \widehat{H}_+ + H_+.$$

Here the last inequality comes from the fact that

$$A_j \widehat{H}_+ + \widehat{H}_+ = \widehat{M}_- + \widehat{H}_+, \tag{7.14}$$

which is true because $\widehat{\beta}_j$ is onto.

Condition (C3) holds because

$$L = \overline{A_j \widehat{H}_+} \cap \widehat{H}_- \subset \widehat{M}_-,$$

so that L is finite-dimensional and therefore the sum $L \dotplus H_+$ is always direct. Hence, A_j can be compressed to $H_+/\widehat{H}_+$.

Let us now assume that A_j can be compressed to $H_+/\widehat{H}_+$ and prove that τ_- is invertible. By (C1*) for A_j in place of A,

$$A_j \widehat{H}_+ \cap H_+ \subset \widehat{H}_+.$$

Since (7.14) holds, we have $\widehat{M}_- \cap H_+ \subset \widehat{H}_+$. This implies (7.12), and from this it follows that τ_- is injective. We also have

$$M_- \subset A_j H_+ + H_+ \subset A_j \widehat{H}_+ + H_+ = \widehat{M}_- + H_+.$$

Here the first inclusion comes from the relation obtained by removing all the hats $\widehat{\ }$ in (7.14) (the relation obtained is true because β_j is onto), the second inclusion comes from (C2) for A_j instead of A, and the last equality uses again (7.14). Hence, we have (7.13), and from this it follows that τ_- is onto. This proves the first statement of the theorem.

To prove the second statement, we apply hat-symmetry to see that A_1, A_2 can be compressed to $\widehat{H}_-/H_-$ if and only if τ_+^* (which is the hat-symmetric of τ_-) is invertible. □

Lemma 7.9. *The following relations hold:*

$$\beta_j \tau_+ = \tau_- \widehat{\beta}_j, \qquad j = 1, 2.$$

Proof. Since $\widehat{M}_+ \subset \widehat{H}_+ \subset H_+$,

$$A_j|\widehat{M}_+ = A_j P_{H_{0,+}}|\widehat{M}_+ + A_j P_{M_+}|\widehat{M}_+.$$

Hence,

$$P_{M_-} A_j|\widehat{M}_+ = P_{M_-} A_j P_{H_{0,+}}|\widehat{M}_+ + P_{M_-} A_j P_{M_+}|\widehat{M}_+ = \beta_j \tau_+,$$

because $P_{M_-} A_j P_{H_{0,+}} = 0$.

Also, since $A_j \widehat{M}_+ \subset \widehat{H}_+ + \widehat{M}_-$,

$$P_{M_-} A_j|\widehat{M}_+ = P_{M_-} P_{\widehat{H}_+} A_j|\widehat{M}_+ + P_{M_-} P_{\widehat{M}_-} A_j|\widehat{M}_+ = \tau_- \widehat{\beta}_j,$$

because $\widehat{H}_+ \subset H_+$ implies $P_{M_-} P_{\widehat{H}_+} = 0$. We have obtained the desired equality. □

Since we assume that β_j and $\widehat{\beta}_j$ are invertible (see (Iβ) on page 141), Lemma 7.9 implies that τ_+ is invertible if and only if τ_- is invertible. By Theorem 7.7, we see that A_1 and A_2 can be compressed to $H_+/\widehat{H}_+$ if and only if they can be compressed to $\widehat{H}_-/H_-$, and that this happens whenever both τ_- and τ_+ are invertible. From now on, we will assume (Iβ) and

$$\tau_- \text{ and } \tau_+ \text{ are invertible.} \tag{Iτ}$$

Example 7.10. This example is a continuation of Example 6.16 (page 122). We have seen that the operators A_1, A_2 of multiplication by the rational functions f_1, f_2 form an orthogonal separating structure with respect to the decomposition $K = H_- \oplus H_+$, where $K = L^2(\mathbb{T})$, $H_- = H_0^2(\mathbb{C} \setminus \mathbb{D})$ and $H_+ = H^2(\mathbb{D})$.

Let Θ be an inner function in $H^2(\mathbb{D})$. We put $\widehat{H}_- = \Theta H_0^2(\mathbb{C} \setminus \mathbb{D})$, $\widehat{H}_+ = \Theta H^2(\mathbb{D})$. Then A_1, A_2 also form an orthogonal separating structure with respect to the decomposition $K = \widehat{H}_- \oplus \widehat{H}_+$. In fact, we can define the Blaschke product B as above and define $\widehat{H}_{0,-}$, $\widehat{M}_-$, $\widehat{M}_+$ and $\widehat{H}_{0,+}$ as follows. For the convenience of the reader, we also repeat the definitions of $H_{0,-}$, M_+, M_+ and $H_{0,+}$, which were given in Example 6.16.

$$\begin{aligned} H_{0,-} &= B^{-1}H_0^2(\mathbb{C}\setminus\mathbb{D}), & \widehat{H}_{0,-} &= \Theta B^{-1}H_0^2(\mathbb{C}\setminus\mathbb{D}),\\ M_- &= H_0^2(\mathbb{C}\setminus\mathbb{D})\ominus B^{-1}H_0^2(\mathbb{C}\setminus\mathbb{D}), & \widehat{M}_- &= \Theta H_0^2(\mathbb{C}\setminus\mathbb{D})\ominus \Theta B^{-1}H_0^2(\mathbb{C}\setminus\mathbb{D}),\\ M_+ &= H^2(\mathbb{D})\ominus BH^2(\mathbb{D}), & \widehat{M}_+ &= \Theta H^2(\mathbb{D})\ominus \Theta BH^2(\mathbb{D}),\\ H_{0,+} &= BH^2(\mathbb{D}), & \widehat{H}_{0,+} &= \Theta BH^2(\mathbb{D}). \end{aligned}$$

Condition (7.6) is satisfied, because $\Theta H^2(\mathbb{D})\subset H^2(\mathbb{D})$. The conditions (I$\beta$) and (I$\tau$) are necessary to define the compression of this separating structure to the space $R = H^2(\mathbb{D})\ominus\Theta H^2(\mathbb{D})$. Let us discuss their meaning.

Since M_- and M_+ have the same dimension, to check that β_j is invertible it suffices to show that $\ker\beta_j = 0$. To the contrary, assume that $g\in M_+ = H^2(\mathbb{D})\ominus BH^2(\mathbb{D})$, $g\neq 0$, and $\beta_j g = 0$. This implies that $f_j g\in H^2(\mathbb{D})$. This can only happen if f_j/B has poles in $\mathbb{D}$. Therefore, β_1 and β_2 are invertible if and only if f_1 and f_2 have precisely the same poles in $\mathbb{D}$ counting multiplicity. We can assume that this is the case by replacing f_1 and f_2 by $t_1f_1+t_2f_2$ and $t_3f_1+t_4f_4$, where $t_k\in\mathbb{R}$, if necessary. In a similar manner one can check that $\widehat{\beta}_1$ and $\widehat{\beta}_2$ are invertible.

Let us now check condition (Iτ). Since M_+ and $\widehat{M}_+$ have the same dimension, it is enough to check that $\ker\tau_+ = 0$. Assume that there is $g\in\widehat{M}_+$, $g\neq 0$, such that $\tau_+g = 0$. This implies $g\in BH^2(\mathbb{D})$. We write $g = \Theta h$, with $h\in H^2(\mathbb{D})\ominus BH^2(\mathbb{D})$. We see that $\Theta h\in BH^2(\mathbb{D})$ can only happen if Θ and B have common zeros in $\mathbb{D}$. Therefore, (Iτ) is equivalent to the condition that Θ and B have no common zeros in $\mathbb{D}$.

We will assume that these conditions above are satisfied, so that (Iβ) and (Iτ) hold. In this case, the compression of the separating structure is well defined. Now we will show that, for the linear span, we have

$$\operatorname{Lin}\{A_1^jA_2^kg : g\in G,\ j,k\geq 0\}\cap H\subset G \tag{7.15}$$

and

$$p(A_1,A_2)H\subset p(A_1,A_2)G + H \tag{7.16}$$

for all $p\in\mathbb{C}[z_1,z_2]$. By applying Lemma 7.5 this implies that $\widetilde{A}_1$ and $\widetilde{A}_2$ commute. Moreover, we can use Lemmas 7.4 and 7.5 to show that (7.15) and (7.16) imply that, for every polynomial $p\in\mathbb{C}[z_1,z_2]$, the compression of $p(A_1,A_2)$ to $H^2(\mathbb{D})\ominus\Theta H^2(\mathbb{D})$ is defined and it equals $p(\widetilde{A}_1,\widetilde{A}_2)$.

First we show (7.15). Here, $G=\Theta H^2(\mathbb{D})$ and $H = H^2(\mathbb{D})$. Thus, we take functions $g_1,\ldots,g_n\in\Theta H^2(\mathbb{D})$ and assume that $\varphi := f_1^{j_1}f_2^{k_1}g_1+\cdots f_1^{j_n}f_2^{k_n}g_n\in H^2(\mathbb{D})$ for some non-negative integers $j_1,\ldots,j_n$ and $k_1,\ldots,k_n$. We must show that $\varphi\in\Theta H^2(\mathbb{D})$. To this end, we write $g_r = \Theta h_r$, with $h_r\in H^2(\mathbb{D})$, for $r=1,\ldots,n$. We put $\psi=\varphi/\Theta = f_1^{j_1}f_2^{k_1}h_1+\cdots f_1^{j_n}f_2^{k_n}h_n$. Note that $B^m\psi\in H^2(\mathbb{D})$ for some $m\geq 0$, because Bf_1 and Bf_2 have no poles in $\mathbb{D}$. Therefore, we have $\psi\in(1/\Theta)H^2(\mathbb{D})\cap(1/B^m)H^2(\mathbb{D})$. Since Θ and B have no zeros in common, $(1/\Theta)H^2(\mathbb{D})\cap(1/B^m)H^2(\mathbb{D})\subset H^2(\mathbb{D})$, so we get $\psi\in H^2(\mathbb{D})$, which implies $\varphi=\Theta\psi\in\Theta H^2(\mathbb{D})$.

Next we prove (7.16). Fix any polynomial p. There is an $m \in \mathbb{N}$ such that $B^m p(f_1, f_2)$ has no poles in $\mathbb{D}$. It is possible to find functions $g, h \in H^\infty(\mathbb{D})$ such that $1 = \Theta g + B^m h$. To see this, one can use Carleson's corona theorem (see [Car62]) and the fact that B has a finite number of zeros in $\mathbb{D}$ and Θ does not vanish at the zeros of B. In this simple setting is it also possible to use a more elementary argument interpolation argument at the zeros of B instead of the corona theorem. Now take an arbitrary $f \in H^2(\mathbb{D})$. Then $p(f_1, f_2)f = p(f_1, f_2)\Theta gf + p(f_1, f_2)B^m hf$. We see that $p(f_1, f_2)\Theta gf$ belongs to $p(A_1, A_2)G$. Since $p(f_1, f_2)B^n$ has no poles in $\mathbb{D}$, then $p(f_1, f_2)B^n hf$ belongs to $H = H^2(\mathbb{D})$, so we get $p(A_1, A_2)f = p(f_1, f_2)f \in p(A_1, A_2)G + H$, as we wanted to show.

We conclude that if f_1 and f_2 have precisely the same poles in $\mathbb{D}$ counting multiplicities and Θ does not vanish at any of the poles of f_1 and f_2 in $\mathbb{D}$, then the compression of the separating structure to $H^2(\mathbb{D}) \ominus \Theta H^2(\mathbb{D})$ is well defined and the compression operators commute. Moreover, for every $p \in \mathbb{C}[z_1, z_2]$ the compression of $p(A_1, A_2)$ is defined and equals $p(\widetilde{A}_1, \widetilde{A}_2)$.

Remark. The spectral behaviour of the compression operators $\widetilde{A}_1$ and $\widetilde{A}_2$ is usally very different from the spectral behaviour of the operators A_1 and A_2, which are selfadjoint. To illustrate this, let us find the spectra of some operators related to the previous example. Since A_1 and A_2 are commuting selfadjoint operators, the operator $N = A_1 + iA_2$ is normal. We put $\eta = f_1 + if_2$, so N is the operator of multiplication by the rational function η on $L^2(\mathbb{T})$. Its spectrum $\sigma(N)$ is the curve $\eta(\mathbb{T})$. The generalized compression of N is $\widetilde{N} = \widetilde{A}_1 + i\widetilde{A}_2$. We claim that $\widetilde{N}$ is the operator $\eta(\mathcal{M}_\Theta)$, where $\mathcal{M}_\Theta = P_\Theta M_z | K_\Theta$ is the so-called model operator, $K_\Theta = H^2(\mathbb{D}) \ominus \Theta H^2(\mathbb{D})$ denotes the model space and P_Θ is the orthogonal projection onto K_Θ.

To see this, first note that if η is analytic in $\mathbb{D}$ then G and H are invariant for N so its generalized compression coincides with the classical compression, which is $\eta(\mathcal{M}_\Theta) = P_\Theta M_\eta | K_\Theta$. If $1/\eta$ is analytic in $\mathbb{D}$, then $\widetilde{N^{-1}} = (1/\eta)(\mathcal{M}_\Theta)$. By Lemma 7.5, $\widetilde{N}\widetilde{N^{-1}} = \widetilde{I} = I$, so $\widetilde{N^{-1}} = \widetilde{N}^{-1}$. It follows that $\widetilde{N} = \eta(M_\Theta)$. For a general rational function η, we write $\eta = \eta_1/\eta_2$ where η_1, η_2 are analytic in $\mathbb{D}$ and apply again Lemma 7.5 and the two previous cases to show that $\widetilde{N} = \eta(M_\Theta)$.

By the Livšic-Moeller theorem (see [Nik86, Lecture III]) and the spectral mapping theorem, the spectrum of $\widetilde{N} = \eta(M_\Theta)$ is the set $\eta(\sigma(\Theta))$, where $\sigma(\Theta)$ denotes the spectrum of the inner function Θ. This means that $\sigma(\Theta)$ is the set of points $\lambda \in \overline{\mathbb{D}}$ such that $1/\Theta$ cannot be continued analytically to a neighbourhood of λ in $\mathbb{C}$. Note that $\sigma(\Theta) \cap \mathbb{D}$ coincides with the set of zeros of Θ in $\mathbb{D}$. Therefore, the spectrum of $\widetilde{N}$ is the union of $\eta(Z)$ and $\eta(\Delta)$, where Z is a Blaschke sequence and Δ is a subset of $\mathbb{T}$. Similarly, we can see that $\widetilde{A}_1$ and $\widetilde{A}_2$ are the operators $f_1(\mathcal{M}_\Theta)$ and $f_2(\mathcal{M}_\Theta)$ respectively and thus obtain that their spectrums are of the form $f_j(Z) \cup f_j(\Delta)$, for $j = 1, 2$ respectively. Compare this with [LKMV95, Lemma 6.1.1], which says that if A has compact imaginary part then the non-real spectrum of A is at most countable and may have only real limit points. Here the operators $\widetilde{A}_1$ and $\widetilde{A}_2$ have imaginary parts of finite rank, since they can be embedded in a vessel according to Theorem 7.13 below.

Example 7.10 above is a simple model that illustrates the compression of a separating structure. A more general model could be obtained by replacing the complex plane and the unit disc and its complement by a separated algebraic curve and its two halves. The paper [AV02] by Alpay and Vinnikov contains some of the tools that could be used to do these function-theoretic constructions on a Riemann surface.

Now we are ready to give a formula for the compression of A_1 and A_2 to $H_+/\widehat{H}_+$.

Lemma 7.11. *If we identify R with the quotient space $H_+/\widehat{H}_+$, then the compression of A_j to $H_+/\widehat{H}_+$ has the form*

$$\widetilde{A}_j = P_R(A_j - \tau_-^{-1}\beta_j P_{M_+})|R.$$

Proof. We will write the proof for $\widetilde{A}_1$. The same argument applies to $\widetilde{A}_2$. Let $h \in R$. Then, by definition of the compression, there is a $g \in \widehat{H}_+$ such that $A_1(h-g) \in H_+$, and $\widetilde{A}_1 h = P_R A_1(h-g)$. Now, since $A_1 g \in \widehat{H}_+ + \widehat{M}_-$, and $R = H_+ \cap \widehat{H}_-$, we have $P_R A_1 g = P_R P_{\widehat{M}_-} A_1 g$. Since $g \in \widehat{H}_+$, we have $P_{\widehat{M}_-} A_1 g = P_{\widehat{M}_-} A_1 P_{\widehat{M}_+} g = \widehat{\beta}_1 P_{\widehat{M}_+} g$. Hence,

$$P_R A_1 g = P_R \widehat{\beta}_1 P_{\widehat{M}_+} g = P_R \tau_-^{-1} \tau_- \widehat{\beta}_1 P_{\widehat{M}_+} g = P_R \tau_-^{-1} \beta_1 \tau_+ P_{\widehat{M}_+} g.$$

By applying hat-symmetry in (7.11), we get $P_{\widehat{H}_{0,+}} P_{M_+} = 0$, which by taking adjoints becomes $P_{M_+} P_{\widehat{H}_{0,+}} = 0$. This shows that

$$\tau_+ P_{\widehat{M}_+} | \widehat{H}_+ = P_{M_+}(P_{\widehat{M}_+} + P_{\widehat{H}_{0,+}})|\widehat{H}_+ = P_{M_+}|\widehat{H}_+.$$

Since $g \in \widehat{H}_+$, we see that

$$P_R A_1 g = P_R \tau_-^{-1} \beta_1 P_{M_+} g = P_R \tau_-^{-1} P_{M_-} A_1 P_{M_+} g = P_R \tau_-^{-1} P_{M_-} A_1 g.$$

Here the last equality holds because $g \in H_+$ and $P_{M_-} A_1 P_{H_{0,+}} = 0$. The condition $A_1(h-g) \in H_+$ implies $P_{M_-} A_1 g = P_{M_-} A_1 h$. Hence,

$$P_R A_1 g = P_R \tau_-^{-1} P_{M_-} A_1 h = P_R \tau_-^{-1} \beta_1 P_{M_+} h,$$

where the last equality is true because $h \in H_+$. This proves the Lemma, because $\widetilde{A}_1 h = P_R A_1(h-g)$. □

We have two different options to construct the compression of A_j to R. We can either do the compression to the quotient space $H_+/\widehat{H}_+$ or to the quotient space $\widehat{H}_-/H_-$. Both of these spaces are identified with R, but the compression produces different operators. We denote by $\widetilde{A}_j$ the compression of A_j to $H_+/\widehat{H}_+$. The surprising fact is that the compression of A_j to $\widehat{H}_-/H_-$ is just the adjoint $\widetilde{A}_j^*$.

Proposition 7.12. *For $j = 1, 2$, let $\widetilde{A}_j$ denote the compression of A_j to the quotient space $H_+/\widehat{H}_+$, which we identify with R. Then the compression of A_j to the quotient space $\widehat{H}_-/H_-$, also identified with R, is $\widetilde{A}_j^*$.*

We also have the following formula for $\widetilde{A}_j^*$:

$$\widetilde{A}_j^* = P_R(A_j - \beta_j^* \tau_-^{-1*} P_{\widehat{M}_-})|R$$

(Here $\tau_+^{-1*} = (\tau_+^{-1})^*$).

Proof. By applying hat-symmetry to Lemma 7.11, we see that the formula for the compression of A_j to $\widehat{H}_-/H_-$ is

$$P_R(A_j - \tau_+^{-1*}\widehat{\beta}_j^* P_{\widehat{M}_-})|R.$$

Note that R is hat-symmetric to itself, τ_- is hat-symmetric to τ_+^*, and β_j is hat-symmetric to $\widehat{\beta}_j^*$.

Now we compute the adjoint of the second part in the formula for $\widetilde{A}_j$ given in Lemma 7.11, using Lemma 7.9:

$$[P_R\tau_-^{-1}\beta_j P_{M_+}|R]^* = P_R\beta_j^*\tau_-^{-1*}P_{\widehat{M}_-}|R = P_R\tau_+^{-1*}\widehat{\beta}_j^* P_{\widehat{M}_-}|R.$$

The first part now follows, because $(P_R A_j|R)^* = P_R A_j|R$. The formula for $\widetilde{A}_j^*$ has been obtained throughout the proof. $\square$

The following is the main result of this section.

Theorem 7.13. *Suppose that A_1, A_2 are two selfadjoint operators on K which are included in two separating structures ω and $\widehat{\omega}$ such that $\widehat{\omega} \prec \omega$, as in* (7.6). *Assume that* (Iβ) *and* (Iτ) *hold (see pages 141 and 143). Define R from* (7.9) *and let $\widetilde{A}_j$ be the compression of A_j to R, considered as the quotient $H_+/\widehat{H}_+$, for $j = 1, 2$. Assume that the compression operators $\widetilde{A}_j$ commute. Let $\sigma_1, \sigma_2, \gamma$ be the matrices that appear in the three-term relationship* (6.1) *for the pool $\mathcal{P}$ associated with ω according to Theorem 6.17, and let $\widehat{\sigma}_1, \widehat{\sigma}_2, \widehat{\gamma}$ be the corresponding matrices for $\widehat{\omega}$. Define $\widetilde{\Phi} : R \to M = M_- \oplus M_+$ by*

$$\widetilde{\Phi} = \begin{bmatrix} -\tau_-^{-1*}P_{\widehat{M}_-}|R \\ P_{M_+}|R \end{bmatrix}. \tag{7.17}$$

Then, the following tuples are commutative vessels:

(a) $(\widetilde{A}_1^*, \widetilde{A}_2^*; R, \widetilde{\Phi}, M; \sigma_j, \gamma^{in} = \gamma - i(\sigma_1\widetilde{\Phi}\widetilde{\Phi}^*\sigma_2 - \sigma_2\widetilde{\Phi}\widetilde{\Phi}^*\sigma_1), \gamma^{out} = \gamma)$.

(b) $(\widetilde{A}_1, \widetilde{A}_2; R, -\widetilde{\Phi}, M; -\sigma_j, \gamma^{in} = -\gamma, \gamma^{out} = -\gamma + i(\sigma_1\widetilde{\Phi}\widetilde{\Phi}^*\sigma_2 - \sigma_2\widetilde{\Phi}\widetilde{\Phi}^*\sigma_1))$.

It is worthy to mention that Lemma 7.5 gives a sufficient condition for the compressions $\widetilde{A}_1$ and $\widetilde{A}_2$ to commute, which is required in this theorem.

We will break the proof of this theorem into several lemmas. The first step is to compute the rates and the gyrations of a vessel in which the compressions $\widetilde{A}_1^*$ and $\widetilde{A}_2^*$ can be included. The next lemma motivates the definition of the window operator $\widetilde{\Phi}$ and shows that the rates of the vessel will coincide with the rates σ_j.

Lemma 7.14. *Under the hypotheses of Theorem 7.13, we have*

$$\frac{1}{i}(\widetilde{A}_j^* - \widetilde{A}_j) = \widetilde{\Phi}^*\sigma_j\widetilde{\Phi}, \qquad j = 1, 2,$$

where σ_j are given by (7.8).

Proof. Using the formulas for $\widetilde{A}_j$ and $\widetilde{A}_j^*$ given in Lemma 7.11 and Proposition 7.12, we see that

$$\widetilde{A}_j^* - \widetilde{A}_j = P_R(-\beta_j^* \tau_-^{-1*} P_{\widehat{M}_-} + \tau_-^{-1}\beta_j P_{M_+})|R. \tag{7.18}$$

Now we compute

$$\begin{aligned}\widetilde{\Phi}^* \sigma_j \widetilde{\Phi} &= \begin{bmatrix} -P_R\tau_-^{-1} & P_R \end{bmatrix} \begin{bmatrix} 0 & i\beta_j \\ -i\beta_j^* & 0 \end{bmatrix} \begin{bmatrix} -\tau_-^{-1*} P_{\widehat{M}_-}|R \\ P_{M_+}|R. \end{bmatrix} \\ &= P_R(i\beta_j^* \tau_-^{-1*} P_{\widehat{M}_-} - i\tau_-^{-1}\beta_j P_{M_+})|R = \frac{1}{i}(\widetilde{A}_j^* - \widetilde{A}_j). \end{aligned}$$ □

The next two lemmas are calculations needed to compute the gyrations of the vessel.

Lemma 7.15. *Under the hypothesis of Theorem 7.13, the following equality holds for $j = 1, 2$:*

$$P_{\widehat{M}_-}(A_j\tau_-^* - \beta_j^*)|M_- = (\tau_-^* P_{M_-} A_j - P_{\widehat{M}_-} A_j P_{\widehat{H}_{0,-}})|M_-.$$

Proof. Since $M_- \subset \widehat{H}_-$, we have

$$P_{\widehat{M}_-} A_j|M_- = P_{\widehat{M}_-} A_j(P_{\widehat{M}_-} + P_{\widehat{H}_{0,-}})|M_- = (P_{\widehat{M}_-} A_j \tau_-^* + P_{\widehat{M}_-} A_j P_{\widehat{H}_{0,-}})|M_-.$$

Since $A_j M_- \subset H_- + M_+$, using the relation obtained by taking adjoints in (7.11), we get

$$P_{\widehat{M}_-} A_j|M_- = P_{\widehat{M}_-}(P_{M_-} + P_{M_+})A_j|M_- = (\tau_-^* P_{M_-} A_j + P_{\widehat{M}_-}\beta_j^*)|M_-.$$

Hence, we see that

$$(P_{\widehat{M}_-} A_j \tau_-^* + P_{\widehat{M}_-} A_j P_{\widehat{H}_{0,-}})|M_- = (\tau_-^* P_{M_-} A_j + P_{\widehat{M}_-}\beta_j^*)|M_-.$$

The lemma now follows by rearranging terms. □

Lemma 7.16. *Under the hypothesis of Theorem 7.13, the following two relations hold for $j = 1, 2$:*

$$P_{M_+}\widetilde{A}_j^* = P_{M_+} A_j \widetilde{\Phi} + P_{M_+} A_j P_{H_{0,+}}|R. \tag{7.19}$$

$$-\tau_-^{-1*} P_{\widehat{M}_-}\widetilde{A}_j^* = P_{M_-} A_j \widetilde{\Phi} - \tau_-^{-1*} P_{\widehat{M}_-} A_j P_{\widehat{H}_{0,-}}(I - \tau_-^{-1*} P_{\widehat{M}_-})|R. \tag{7.20}$$

Proof. First we prove (7.19). Take a fixed $r \in R$ and put $m_- = \tau_-^{-1*} P_{\widehat{M}_-} r \in M_-$. We note that $r - m_- \in \widehat{H}_{0,-}$, because $r - m_- \in \widehat{H}_-$ (recall that $R \subset \widehat{H}_-$ and $M_- \subset H_- \subset \widehat{H}_-$), and

$$P_{\widehat{M}_-}(r - m_-) = P_{\widehat{M}_-} r - \tau_-^* m_- = 0.$$

It follows that $A(r - m_-) \in \widehat{H}_-$. By the definition of the compression, we see that

$$\widetilde{A}_j^* r + H_- = A(r - m_-) + H_-,$$

because $\widetilde{A}_j^*$ is the compression of A_j to the quotient $\widehat{H}_-/H_-$.

Since $P_{M_+}|H_- = 0$, the operator P_{M_+} is well defined in the quotient $\widehat{H}_-/H_-$. This implies that

$$P_{M_+}\widetilde{A}_j^* r = P_{M_+}A(r - m_-).$$

Therefore, we see that

$$P_{M_+}\widetilde{A}_j^* = P_{M_+}A_j|R - P_{M_+}A_j\tau_-^{-1*}P_{\widehat{M}_-}|R,$$

because $r \in R$ was arbitrary. Writing

$$P_{M_+}A_j|R = P_{M_+}A_jP_{M_+}|R + P_{M_+}A_jP_{H_{0,+}}|R,$$

which is true because $R \subset H_+$, and using the definition of $\widetilde{\Phi}$ given in (7.17), we get (7.19).

To prove (7.20), we first apply hat-symmetry in (7.19) to obtain

$$P_{\widehat{M}_-}\widetilde{A}_j = (P_{\widehat{M}_-}A_jP_{\widehat{M}_-} - P_{\widehat{M}_-}A_j\tau_+^{-1}P_{M_+} + P_{\widehat{M}_-}A_jP_{\widehat{H}_{0,-}})|R.$$

(Note that hat-symmetry interchanges the operators $\widetilde{A}_j$ and $\widetilde{A}_j^*$). Since

$$P_{\widehat{M}_-}A_j\tau_+^{-1} = \widehat{\beta}_j\tau_+^{-1} = \tau_-^{-1}\beta_j$$

by Lemma 7.9, we get

$$P_{\widehat{M}_-}\widetilde{A}_j = (P_{\widehat{M}_-}A_jP_{\widehat{M}_-} - \tau_-^{-1}\beta_jP_{M_+} + P_{\widehat{M}_-}A_jP_{\widehat{H}_{0,-}})|R.. \tag{7.21}$$

Now we will use (7.18) to compute $P_{\widehat{M}_-}(\widetilde{A}_j^* - \widetilde{A})$. We have

$$P_{\widehat{M}_-}P_R|M_+ = P_{\widehat{M}_-}P_{\widehat{H}_-}|M_+ = P_{\widehat{M}_-}|M_+,$$

because $R = \widehat{H}_- \cap H_+$ and $M_+ \subset H_+$. Similarly,

$$P_{\widehat{M}_-}P_R|\widehat{M}_- = P_{\widehat{M}_-}P_{H_+}|\widehat{M}_- = P_{\widehat{M}_-}(I - P_{M_-})|\widehat{M}_- = (I - \tau_-^*\tau_-)|\widehat{M}_-.$$

Here, the second equality is true by the relation obtained by taking adjoints in (7.11). Using these last two identities in (7.18), we see that

$$\begin{aligned} P_{\widehat{M}_-}(\widetilde{A}_j^* - \widetilde{A}_j) &= (-P_{\widehat{M}_-}\beta_j^*\tau_-^{-1*}P_{\widehat{M}_-} + (I - \tau_-^*\tau_-)\tau_-^{-1}\beta_jP_{M_+})|R \\ &= (-P_{\widehat{M}_-}\beta_j^*\tau_-^{-1*}P_{\widehat{M}_-} + \tau_-^{-1}\beta_jP_{M_+} - \tau_-^*\beta_jP_{M_+})|R. \end{aligned} \tag{7.22}$$

By (7.21) and (7.22),

$$\begin{aligned} P_{\widehat{M}_-}\widetilde{A}_j^* &= P_{\widehat{M}_-}\widetilde{A}_j + P_{\widehat{M}_-}(\widetilde{A}_j^* - \widetilde{A}_j) \\ &= (P_{\widehat{M}_-}A_jP_{\widehat{M}_-} + P_{\widehat{M}_-}A_jP_{\widehat{H}_{0,-}} - P_{\widehat{M}_-}\beta_j^*\tau_-^{-1*}P_{\widehat{M}_-} - \tau_-^*\beta_jP_{M_+})|R \\ &= (P_{\widehat{M}_-}(A_j\tau_-^* - \beta^*)\tau_-^{-1*}P_{\widehat{M}_-} - \tau_-^*\beta_jP_{M_+} + P_{\widehat{M}_-}A_jP_{\widehat{H}_{0,-}})|R. \end{aligned}$$

In this last equality, we have just rearranged terms. Using Lemma 7.15, we see that the last expression equals

$$(\tau_-^*P_{M_-}A_j\tau_-^{-1*}P_{\widehat{M}_-} - P_{\widehat{M}_-}AP_{\widehat{H}_{0,-}}\tau_-^{-1*}P_{\widehat{M}_-} - \tau_-^*\beta_jP_{M_+} + P_{\widehat{M}_-}A_jP_{\widehat{H}_{0,-}})|R.$$

Multiplying by $-\tau_-^{-1*}$ on the left and using (7.17), we get (7.20). □

The next lemma shows that (5.4) is satisfied for $\widetilde{A}_1^*$ and $\widetilde{A}_2^*$ in place of A_1 and A_2, and with $\gamma^{\text{out}} = \gamma_{12}^{\text{out}} = \gamma$.

Lemma 7.17. *Under the hypothesis of Theorem 7.13, the compressions* $\widetilde{A}_1^*$ *and* $\widetilde{A}_2^*$ *satisfy the three-term relationship*

$$\sigma_2 \widetilde{\Phi} \widetilde{A}_1^* - \sigma_1 \widetilde{\Phi} \widetilde{A}_2^* + \gamma \widetilde{\Phi} = 0,$$

where $\sigma_1, \sigma_2, \gamma$ *are the matrices that appear in the three-term relationship* (6.1) *for the pool* $\mathcal{P}$ *associated with the separating structure* ω *according to Theorem 6.17.*

Proof. Multiplying the three-term relationship (6.1) for $\mathcal{P}$ by P_{M_-} on the left and $\widetilde{\Phi}$ on the right, using (7.8) and $\Phi\widetilde{\Phi} = \widetilde{\Phi}$ (which is true because $\Phi = P_M$), we get

$$i\beta_2 P_{M_+} A_1 \widetilde{\Phi} - i\beta_1 P_{M_+} A_2 \widetilde{\Phi} + P_{M_-} \gamma \widetilde{\Phi} = 0.$$

Using (7.19), we have

$$i\beta_2 P_{M_+} \widetilde{A}_1^* - i\beta_1 P_{M_+} \widetilde{A}_2^* + P_{M_-} \gamma \widetilde{\Phi} - i\beta_2 P_{M_+} A_1 P_{H_{0,+}} | R + i\beta_1 P_{M_+} A_2 P_{H_{0,+}} | R = 0. \quad (7.23)$$

Now we will show that since A_1 and A_2 commute, the last two terms in the left hand side of the preceding equality cancel. Since (6.39) holds, we can write A_1 and A_2 as tridiagonal matrices according to the decomposition $K = H_{0,-} \oplus M_- \oplus M_+ \oplus H_{0,+}$, in a way similar to (6.21). Indeed,

$$A_j = \begin{bmatrix} * & * & 0 & 0 \\ * & * & \beta_j & 0 \\ 0 & * & * & P_{M_+} A_j P_{H_{0,+}} \\ 0 & 0 & * & * \end{bmatrix}, \qquad j = 1, 2.$$

Multiplying the second row of A_1 by the fourth column of A_2, we obtain the operator $\beta_1 P_{M_+} A_2 P_{H_{0,+}}$. Symetrically, multiplying the second row of A_2 by the fourth row of A_1, we obtain $\beta_2 P_{M_+} A_1 P_{H_{0,+}}$. Since $A_1 A_2 = A_2 A_1$, we must have

$$\beta_1 P_{M_+} A_2 P_{H_{0,+}} = \beta_2 P_{M_+} A_1 P_{H_{0,+}}. \quad (7.24)$$

By (7.23), this implies that

$$i\beta_2 P_{M_+} \widetilde{A}_1^* - i\beta_1 P_{M_+} \widetilde{A}_2^* + P_{M_-} \gamma \widetilde{\Phi} = 0.$$

Using (7.8) again and (7.17), we get

$$P_{M_-} (\sigma_2 \widetilde{\Phi} \widetilde{A}_1^* - \sigma_1 \widetilde{\Phi} \widetilde{A}_2^* + \gamma \widetilde{\Phi}) = 0. \quad (7.25)$$

Now we multiply (6.1) by P_{M_+} on the left and $\widetilde{\Phi}$ on the right. We get

$$-i\beta_2^* P_{M_-} A_1 \widetilde{\Phi} + i\beta_1^* P_{M_-} A_2 \widetilde{\Phi} + P_{M_+} \gamma \widetilde{\Phi} = 0.$$

Using (7.20), this rewrites as

$$\begin{aligned}
& i\beta_2^* P_{M_-}\tau_-^{-1*}P_{\widehat{M}_-}\widetilde{A}_1^* - i\beta_1^*\tau_-^{-1*}P_{\widehat{M}_-}\widetilde{A}_2^* + P_{M_+}\gamma\widetilde{\Phi} \\
& - i\beta_2^*\tau_-^{-1*}P_{\widehat{M}_-}A_1 P_{\widehat{H}_{0,-}}(I-\tau_-^{-1*}P_{\widehat{M}_-})|R \\
& + i\beta_1^*\tau_-^{-1*}P_{\widehat{M}_-}A_1 P_{\widehat{H}_{0,-}}(I-\tau_-^{-1*}P_{\widehat{M}_-})|R = 0.
\end{aligned}$$

Now we will see that the last two terms in the left hand side of this equality cancel. Using Lemma 7.9, we have

$$\begin{aligned}
& \beta_2^*\tau_-^{-1*}P_{\widehat{M}_-}A_1 P_{\widehat{H}_{0,-}} - \beta_1^*\tau_-^{-1*}P_{\widehat{M}_-}A_1 P_{\widehat{H}_{0,-}} \\
& \qquad = \tau_+^{-1*}\left(\widehat{\beta}_2^* P_{\widehat{M}_-}A_1 P_{\widehat{H}_{0,-}} - \widehat{\beta}_1^* P_{\widehat{M}_-}A_2 P_{\widehat{H}_{0,-}}\right).
\end{aligned}$$

By applying hat-symmetry to (7.24), we see that the expression in brackets is zero.

Therefore, we have

$$i\beta_2^* P_{M_-}\tau_-^{-1*}P_{\widehat{M}_-}\widetilde{A}_1^* - i\beta_1^*\tau_-^{-1*}P_{\widehat{M}_-}\widetilde{A}_2^* + P_{M_+}\gamma\widetilde{\Phi} = 0.$$

Using (7.8) and (7.17), this yields

$$P_{M_+}(\sigma_2\widetilde{\Phi}\widetilde{A}_1^* - \sigma_1\widetilde{\Phi}\widetilde{A}_2^* + \gamma\widetilde{\Phi}) = 0. \tag{7.26}$$

The Lemma now follows from (7.25) and (7.26). □

The proof of Theorem 7.13 only consists in putting together all the lemmas above.

Proof of Theorem 7.13. The fact that (a) is a vessel is just a consequence of the preceding Lemma 7.14 and Lemma 7.17. Then (b) is just the adjoint vessel of (a). □

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