Non-Abelian Black Holes in String Theory

Memoria de Tesis Doctoral realizada por

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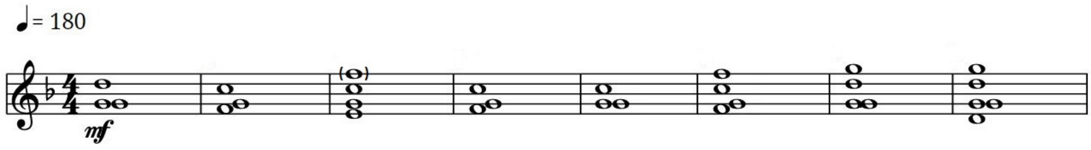

Ritchie Blackmore

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List of Publications

The following articles, some of which are unrelated to the content of this thesis, were published by the candidate during the realization of this work:

1 P. A. Cano, P. Meessen, T. Ortín and P. F. Ramírez, "Non-Abelian black holes in string theory," arXiv:1704.01134 [hep-th].

2 P. A. Cano, T. Ortín and P. F. Ramírez, "A gravitating Yang-Mills instanton," arXiv:1704.00504 [hep-th].

3 P. F. Ramírez, "Non-Abelian bubbles in microstate geometries", JHEP (2016) 2016:152 doi:10.1007/JHEP11(2016)152 [arXiv:1608.01330 [hep-th]].

4 T. Ortín and P. F. Ramírez, "A non-Abelian Black Ring", Phys. Lett. B 760 (2016) 475 [arXiv:1605.00005 [hep-th]].

5 T. Ortín and P. F. Ramírez, "Three Lectures on the FGK Formalism and Beyond", School and Conference Theoretical Frontiers in Black Holes and Cosmology, Springer Proc. Phys. 176 (2016) 1.

6 P. Bueno, P. A. Cano, A. O. Lasso and P. F. Ramírez, "f(Lovelock) theories of gravity", JHEP 1604 (2016) 028 [arXiv:1602.07310 [hep-th]].

7 P. Meessen, T. Ortín and P. F. Ramírez, "Non-Abelian, supersymmetric black holes and strings in 5 dimensions", JHEP 1603 (2016) 112 [arXiv:1512.07131 [hep-th]].

8 P. Bueno, P. Meessen, T. Ortín and P. F. Ramírez, "Resolution of SU(2) monopole singularities by oxidation", Phys. Lett. B. 746 (2015) 109113 [arXiv:1503.01044 [hep-th]].

9 P. Bueno, P. Meessen, T. Ortín and P. F. Ramírez, "N = 2 Einstein-Yang-Mills' static two-center solutions", JHEP 1412 (2014) 093. [arXiv:1410.4160 [hep-th]].

10 P. Bueno and P. F. Ramírez, "Higher-curvature corrections to holographic entanglement entropy in geometries with hyperscaling violation", JHEP 1412 (2014) 078. [arXiv: 1408.6380 [hep-th]].

Abstract

This thesis is devoted to the study of the interaction of non-Abelian Yang-Mills fields with the gravitational field in the context of String Theory and its low energy effective supergravity description. While this type of interactions has been considered for decades within several theoretical frameworks, limited progress has been made, especially when compared to the knowledge we have of the interaction of Abelian fields and gravity. The main reason for that is the complexity of this sort of systems; the differential equations that govern the dynamics of both, gravitational and non-Abelian Yang-Mills fields, are highly non-linear and their resolution represents a formidable problem.

The complexity of these systems, however, can be reduced through the restriction to supersymmetric solutions. This type of solutions, which include extremal black holes, have very special properties. Nevertheless, a great deal of information can be acquired from them. In particular, not only we can learn about the properties of the classical interaction between the corresponding fields and gravity, but it is also possible to glimpse the quantum nature of certain gravitational systems. The "three-charge" Abelian black hole constitutes the most popular example. This system of String Theory can be understood both as a supersymmetric solution of $\mathcal{N} = 1$ five dimensional supergravity and as a quantum ensamble in which gravity plays no role. One of the greatest achievements of this theory is precisely the computation of the entropy of this black hole from these two perspectives with identical result.

The main result of this thesis is the construction of "three-charge" non-Abelian black holes in supergravity and its interpretation in String Theory. In its turn, this allows for the microscopic computation of its entropy, which at the same time implies the resolution of the non-Abelian hair puzzle in this type of black holes.

Another prominent result of this thesis is the development of a solution generating technique that allows for the construction of many other families of non-Abelian gravitating solutions in supergravity. Their interpretation in terms of fundamental objects of String Theory has only just started.

Contents

1

Introduction

1.1 Black Holes

Few concepts attract as much interest as that of a black hole: a region of spacetime from which not even light can escape. Any process inside the black hole cannot have any influence on the outer region, while external observers can never find out about these events unless they choose to fall in. Certainly this idea, together with associated notions as the event horizon or the always mysterious singularity, is not only inherently beautiful, but also has been proven to be extraordinarily powerful in the advance of modern physics.

Probably the most important fact about black holes is that, despite their extremely exotic physical properties, they actually seem to exist in nature. They have not been observed directly yet, although numerous indirect observations have provided a large body of evidence[1]. Among those, the most impressive are the direct detection of gravitational waves from merging black holes by the LIGO collaboration [1–3], or the motion of almost 100 stars orbiting *Sagittarius A**, what is believed to be a supermassive black hole of $4.3 \times 10^6\ M_\odot$ located near the center of our galaxy [44, 95, 100]. Experimental observations are going to improve both qualitatively and quantitatively in the forthcoming decades. ESA's Laser Interferometer Space Antenna, the Event Horizon Telescope or the already mentioned LIGO collaboration, among others, might be able to perform precision tests to explore the near-horizon region of black holes.

We are thus witnessing the birth of a new era in gravitational physics in which theory will confront experiments in an unprecedented scenario. And it is precisely in that scenario in which novel phenomena might take place. According to the theory of General Relativity, the event horizon has no local significance. For a big enough black hole, the Equivalence Principle dictates that an infalling observer would not experience any particular gravitational effect when crossing its horizon. On the other hand, the absence of a complete theory of quantum gravity undermines any prediction one could do about the physics in that regime. To make further progress, we shall be precise about *what a black hole is*[2,3].

[1] See for instance [160, 185] for general references about astrophysical evidences for the existence of black holes.

[2] Actually, we can be precise about what we mean by black holes in a particular context. The *true nature* of these physical objects remains, of course, far from understood.

[3] Many of the contents in this section are based on [86, 117, 151, 170, 174, 210, 211].

1.1.1 Definitions

In order to make a rigorous definition of the classical notion of black hole, one must first elaborate what it means to *escape* from a region of spacetime. The concept of asymptotically flat manifold introduced by Penrose [175] is useful for that purpose, and requires prior definitions.

In terms of the conformal structure of spacetime, infinity can be treated as the ordinary boundary of a *finite* conformal region. A manifold $\mathcal{M}$ with Lorentzian metric $\mathbf{g}$, i.e. a spacetime $(\mathcal{M}, \mathbf{g})$, is *asymptotically simple* if there exists a new spacetime $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ with boundary $\partial\tilde{\mathcal{M}}$ such that

- $\mathcal{M}$ can be embedded in $\tilde{\mathcal{M}} \setminus \partial\tilde{\mathcal{M}}$ with pullback metric $\tilde{\mathbf{g}}^* = \Omega^2 \mathbf{g}$.
- $\Omega_{|\mathcal{M}} > 0$, $\Omega_{|\partial\tilde{\mathcal{M}}} = 0$ and $\partial_\mu \Omega_{|\partial\tilde{\mathcal{M}}} \neq 0$.
- All null geodesics in $\mathcal{M}$ begin and end at $\partial\tilde{\mathcal{M}}$.

$(\mathcal{M}, \tilde{\mathbf{g}}^*)$ is said to be a *conformal compactification* of the original spacetime[4] and $\tilde{\mathcal{M}}$ is called the *conformal Penrose space*. Minkowski space or generic spaces containing bound objects that have not collapsed such as planets or stars are asymptotically simple. Penrose proved that asymptotically simple spaces satisfying Einstein's vacuum equations (without cosmological constant) have the global structure of Minkowski and do not allow for black holes, since those contain null geodesics that do not have endpoints in $\partial\tilde{\mathcal{M}}$. To include those into consideration, we need to introduce a more general concept.

A spacetime $(\mathcal{M}, \mathbf{g})$ is said to be *weakly asymptotically simple* if there exists an asymptotically simple space $(\tilde{\mathcal{M}}, \tilde{\mathbf{g}})$ such that for a neighborhood $\tilde{U}$ of $\partial\tilde{\mathcal{M}}$, the space $\tilde{U} \cap \tilde{\mathcal{M}}$ is isometric to a subset of $\mathcal{M}$. The basic idea is that a weakly asymptotically simple space can be converted into an asymptotically simple space by "cutting out" some inner regions and "patching" smoothly the resulting "holes". A weakly asymptotically simple space is *asymptotically flat* if its metric $\mathbf{g}$ is a solution of Einstein's vacuum equations in the neighborhood $\tilde{U}$ of the boundary. In an asymptotically flat space there is a region in which, at leading order, Minkowski geometry is recovered.

We can formally define a *black hole* as the region $\mathcal{B}$ of an asymptotically flat spacetime such that

$$\mathcal{B} \equiv \mathcal{M} \setminus J^-(\mathcal{J}^+)\,, \tag{1.1}$$

where $J^-(\mathcal{J}^+)$ denotes the chronological past of $\mathcal{J}^+$; that is, the set of points in $\mathcal{M}$ traversed by a future directed causal curve connecting it with the subset of the boundary $\partial\tilde{\mathcal{M}}$ where null geodesics can end, known as the future null infinity $\mathcal{J}^+$. In other words, $J^-(\mathcal{J}^+)$ represents the region of spacetime casually connected to asymptotic infinity. The boundary of $J^-(\mathcal{J}^+)$, which of course coincides with that of $\mathcal{B}$, is called *event horizon*.

Notice that complete knowledge about the history of spacetime is required to determine the location of a putative event horizon, which possesses no intrinsically local significance. Then, this cumbersome definition is hard to exploit when analyzing general spacetimes. For most practical purposes, the rigidity theorems developed by Carter and Hawking [59, 117] are of crucial importance. Once again, we need some definitions before.

[4] Even if the third condition does not apply.

An asymptotically flat spacetime is *stationary* if there exists a one parameter group of isometries with an associated Killing vector field k^α that in the asymptotic region becomes a unit timelike vector field[5]. When there exists a family of spacelike hypersurfaces orthogonal to the Killing vector field, the spacetime is also *static*.

The rigidity theorems state that the event horizon of any stationary black hole must be a Killing horizon, i.e. a null hypersurface whose null generators are given by the orbits of a Killing vector l^α. For static black holes this vector coincides with k^α at the event horizon. On the other hand, if the horizon is rotating, there exists another Killing vector field m^α such that $l^\alpha = k^\alpha + \omega_h m^\alpha$, where ω_h is the angular velocity of the horizon. The domain of outer communication, i.e. the complement region to the black hole ($\bar{\mathcal{B}}$), can then be argued to have an axial symmetry generated by m^α. Therefore, the rather abstract original definition is related to a more useful concept in a wide variety of cases of interest.

1.1.2 Conserved quantities

The concept of energy and its associated law of conservation has played a prominent role in most physical theories. The modern approach that results from Noether's (first) theorem [167] states that the law of conservation of energy is a mathematical consequence of the fact that the laws of physics do not change with time. In a special relativistic theory of fields, the associated energy-momentum tensor $T_{\mu\nu}$ satisfies the equation

$$\partial_\mu T^\mu{}_\nu = 0\,. \tag{1.2}$$

This leads to a law of conservation for the quantity $E = \int_\Sigma T_{\mu\nu} n^\mu t^\nu$, where t^μ is the Killing vector associated to time translations and Σ is a spacelike surface with unit normal vector n^μ.

However it is well-known that in the framework of General Relativity, or other generally covariant metric-based theories of gravity, there is no appropriate notion of energy density. Still, there is an energy-momentum tensor characterizing matter and its local energy density associated by a given observer remains well defined. This can be easily seen. Notice that general covariance transforms the above equation (1.2) into

$$\nabla_\mu T^\mu{}_\nu = 0\,. \tag{1.3}$$

By virtue of the Equivalence Principle the connection can be made trivial locally, recovering the special relativistic expression and its associated conservation law. But (1.3) does not in general lead to a global conservation law. This should not be surprising. One would expect the appearance of some "gravitational energy" contribution to the total energy, but $T_{\mu\nu}$ contains information only about the matter content. Precisely, this is the physical reason why we can only define an energy density in a preferred system of coordinates that makes the "gravitational field" disappear locally. The situation gets even more puzzling when we take into consideration that there is no meaningful notion of gravitational energy density. Indeed, the gravitational energy-momentum tensor that can be constructed using Noether's theorem is not unique [170]. We could try to gain some insight by comparison

[5]The Killing vector field cannot be assumed to be timelike everywhere, however, since this would forbid the existence of ergoregions.

with Newton's theory, where gravitational energy density is proportional to the gradient of the potential squared. However, that would imply the construction of the candidate tensor using just the metric and their first derivatives, which cannot be done unless a privileged coordinate system is defined[6].

The absence of a fully general-covariant gravitational energy-momentum tensor seems to unavoidably imply that gravitational energy cannot be localized; only the total energy of spacetime can be well defined. Although there is not a unique manner to characterize global conserved quantities in general, in the case of asymptotically flat spacetimes the different approaches produce the same result [7]. Probably the most popular constructions are the Arnowitt-Deser-Misner (ADM) mass [6] and the Komar mass [140], which is defined for stationary configurations. The Komar mass formula can be deduced from physical principles, [210], although a more straightforward derivation is obtained by considering the asymptotic expansion of the gravitational field around a background metric, followed by the construction of the Noether current associated with the background timelike Killing vector. This results in the expression [163]

$$M = -\frac{1}{16\pi G_N^{(d)}}\frac{d-2}{d-3}\int_{\partial\Sigma} d^{d-2}\Sigma_{\mu\nu}\nabla^{\mu}k^{\nu}\,, \tag{1.4}$$

being k^{μ} the timelike Killing vector and Σ a spacelike hypersurface extending to infinity. We can immediately apply this formula to the simple case of Schwarzschild-Tangerlini solution,

$$ds^2 = \left(1-\frac{\omega}{r^{d-3}}\right)dt^2 - \left(1-\frac{\omega}{r^{d-3}}\right)^{-1}dr^2 - r^2 d\Omega^2_{(d-2)}\,, \tag{1.5}$$

to obtain

$$M = \frac{(d-2)\omega_{(d-2)}}{16\pi G_N^{(d)}}\omega\,, \qquad \text{with} \;\; \omega_{(d-2)} \equiv \frac{2\pi^{\frac{d-1}{2}}}{\Gamma\left(\frac{d-1}{2}\right)}\,. \tag{1.6}$$

This expression provides a simple way of obtaining the mass of any asymptotically flat spacetime that we will use through the text. Indeed, we can just preform the asymptotic expansion of the corresponding metric and identify the coefficient ω from the g_{tt} metric component, with the understanding that t and r are respectively a coordinate adapted to the timelike Killing vector and a coordinate with constant value at the horizon.

A similar discussion with analogous conclusion can be raised about the definition of angular momentum; a value can be assigned only for the global angular momentum. We will skip a deep analysis and directly give a practical computational method. When more than four dimensions are considered there is the possibility of rotation in several independent planes. This can be seen from the fact that the Cartan subgroup of $SO(d-1)$ is $U(1)^{\lfloor\frac{d-1}{2}\rfloor}$. This basically means that, among all possible spatial rotations in d-dimensional Minkowski space, there are $\lfloor\frac{d-1}{2}\rfloor$ which commute and are, therefore, independent. The different angular momenta are the Noether charges associated with these Killing vectors of Minkowski space, considered as the background metric in the weak field asymptotic expansion, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

[6] Another possibility is to consider the decomposition of the metric into a background and a dynamical part, as it is done in the linearized approach.

In order to find their values, it is convenient to use a coordinate system in which these independent rotations take place in manifestly independent planes. The background metric can be written as

$$ds_b^2 = dt^2 - dr^2 - r^2 \sum_{i=1}^{\lfloor \frac{d-1}{2} \rfloor} \left(d\mu_i^2 + \mu_i^2 d\phi_i^2 \right) + dz^2 \delta_0^{d \bmod 2} \,, \tag{1.7}$$

where the last term is only present when d is even and μ_i are the "direction cosines", satisfying $\sum_i \mu_i^2 + \delta_0^{d \bmod 2} z^2/r^2 = 1$. Euclidean coordinates on the independent rotation planes are recovered with the identification $(x_i, y_i) = (r\mu_i \cos\phi_i, r\mu_i \sin\phi_i)$. The associated angular momenta are[7] [163]

$$J_i = - \lim_{r\to\infty} \frac{\omega_{(d-2)} h_{t\phi_i} r^{(d-3)}}{8\pi G_N^{(d)} \mu_i^2} \,. \tag{1.8}$$

1.1.3 Laws of classical black hole mechanics

We discussed above that the event horizon of stationary black holes is a Killing horizon, whose normal Killing vector we denote as k^μ. Along the horizon we have $g_{\mu\nu} k^\mu k^\nu = 0$, so it is clear that the vector[8] $\nabla^\mu \left(k^\nu k_\nu \right)$, if it does not vanish, must be normal to the Killing horizon. In turns this means that it is proportional to k^μ,

$$\nabla^\mu \left(k^\nu k_\nu \right) |_{\mathcal{H}} = -2\kappa k^\mu |_{\mathcal{H}} \,. \tag{1.9}$$

The scalar κ is known as *surface gravity*, because it corresponds to the asymptotic force per unit mass that would have to be exerted to hold a point like particle at the horizon. When $\kappa \neq 0$ the Killing horizon is *bifurcate* and, when $\kappa = 0$, it is a *degenerate Killing horizon.*

The *zeroth law of black hole mechanics* states that the surface gravity is constant over the horizon [10]. This observation constitutes a first analogy between a black hole and a thermodynamical system in equilibrium, which has the same temperature at any point. But, in fact, the temperature of a classical black hole is absolute zero. It cannot be in equilibrium with a thermal bath, as it absorbs radiation but does not emit any. So at this stage this analogy could be seen as a mere coincidence. However, the scenario gets more confusing as one goes deeper.

The *first law of black hole mechanics* provides a relation between the changes in the mass, area, angular momenta and other conserved charges,

$$\delta M = \frac{\kappa}{8\pi G_N^{(d)}} \delta A_{\mathcal{H}} + \omega_h \delta J + \ldots \tag{1.10}$$

Additional terms containing information about the matter content may be included. For example if the black hole is electrically charged the term $(\phi^h \delta Q)$ must be included, ϕ^h

[7] Notice that for this expression to be well defined $h_{t\phi_i}$ can be at most of order $r^{-(d-3)}$.

[8] It is evident that in this expression the covariant derivative can be replaced by a simple partial derivative, we just use this notation to emphasize that the combination is a tensor.

being the value of the electrostatic potential at the horizon and Q the electric charge[9]. The validity of this law depends only on general properties of diffeomorphism invariant theories [129].

The *second law of black hole mechanics* indicates that the area of the event horizon of a black hole spacetime does not decrease with time

$$\delta A_{\mathcal{H}} \geq 0\,. \tag{1.11}$$

Although this fact might seem obvious from the definition of classical black holes, the rigorous proof is subtle [114]. This relation can actually be applied to non-stationary solutions, such as those describing the merging of black holes, in which case the horizon can only be defined by means of the rather abstract expression (1.1).

Inspired by these ideas a *third law of black hole mechanics* was conjectured, indicating that it should be impossible to reduce the surface gravity of a black hole to zero value by a finite sequence of operations, no matter how idealized [10].

These relations suggest that there are two quantities in black holes, the surface gravity and the horizon area, that behave like the temperature and entropy of the system in some aspects. Still, the identifications $\kappa \sim T$ and $A_{\mathcal{H}} \sim S$ result rather odd: in thermodynamical systems, having finite temperature means radiating energy while entropy usually scales with the volume, not the area. In a parallel line of research, few months before the laws of classical black holes were stated, Bekenstein suggested that in order to prevent a violation of the second law of thermodynamics, black holes should have a well-defined entropy proportional to the event horizon area [16]. He provided an extraordinarily simple and beautiful argument based on the area increase of a black hole's horizon as it captures a beam of thermal radiation, noticing that it is of the order of the value of the entropy of the beam. Bekenstein's arguments supported the identification $A_{\mathcal{H}} \sim S$ as a consequence of the second law of thermodynamics, which in turn implies that $\kappa \sim T$ via the first law of thermodynamics and (1.10). But, how can a black hole have non vanishing temperature? And what are the microscopic degrees of freedom responsible of having a non vanishing entropy?

1.1.4 Hawking radiation and the Information Paradox

In 1974 Hawking published a stunning result: black holes radiate as perfect black bodies with temperature

$$T = \frac{\hbar\kappa}{2\pi}\,. \tag{1.12}$$

The result was found performing a semiclassical analysis with quantum fields propagating in a fixed background geometry describing the gravitational collapse of a black hole [115, 116]. The presence of an event horizon produces the Hawking radiation, although the backreaction of the radiation on the geometry was not considered. The original derivation considered that the quantum field is in its vacuum state before the collapse and computed the particle content of the field at infinity at late times. The result has been generalized to include arbitrary regular initial states of the quantum field and it has been checked that

[9] Also it has been shown that dipole terms make appearance in higher dimensional theories [66, 79], with the Chern-Simons terms in the action playing a crucial role.

the density matrix describing asymptotic radiation completely agrees with black body emission [209].

It has been proposed that the physical process giving raise to Hawking radiation is the creation of Schwinger pairs in strong background fields [86, 188], rather than quantum tunneling across the horizon. Schwinger carried out his original work for electric fields and observed that the background field can give energy to a virtual pair of particles, separating them. This effect causes the spontaneous discharge of charged bodies in vacuum and, interestingly, it was signaled as responsible of the discharge of Reissner-Nordström black holes before Hawking's discovery [149]. In the case of the gravitational field, all kinds of particles are produced as a consequence of the universal coupling of gravity to all forms of energy.

The discovery of Hawking radiation removed the last theoretical obstruction to a complete identification of black holes as thermodynamical systems. At the same time it made precise the identification of entropy and area,

$$S = \frac{A_{\mathcal{H}}}{4\hbar G_N^{(d)}} \,. \tag{1.13}$$

As black holes radiate, they lose mass and their entropy decreases. This indicates that the second law of black hole mechanics is purely classical and does not hold when quantum effects are taken into account. However it can be proven that the total entropy, corresponding to the black hole and the outgoing radiation, never decreases, giving rise to the concept of generalized second law of thermodynamics [17].

These results seem to indicate that stationary black holes are thermal states of the quantum gravitational field and the laws of classical black hole mechanics are simply the result of the application of the ordinary laws of thermodynamics to this type of systems. There are still, however, two unresolved key issues in the area of black hole thermodynamics that lie at the very heart of quantum gravitational physics and should be explained by any candidate theory of quantum gravity:

- **The black hole information paradox**. The Schwinger pairs that appear at the vicinity of the horizon are entangled. This means that as the evaporation process goes on, the outgoing radiation is entangled with the interior of the black hole. In a simple approximation in which the particles created have two possible states the magnitude of this effect can be quantified to yield an entanglement entropy of the order
$$S_{\text{entanglement}} \sim N \ln 2 \,, \tag{1.14}$$
Where N is the number of pairs that have been created. When the black hole evaporates completely the final state has a huge associated entanglement entropy, but there is nothing left for these particles to be entangled with. This means that it is not possible to describe the final system as a product state; a density matrix must be used instead. Now one can consider the complete evolution history of a mass distribution which initially is in a pure (product) state, collapses into a black hole and completely evaporates, resulting in a mixed state. This sort of evolution cannot be described by the action of a unitary operator on a Hilbert space. That is the basis of the famous problem known as the black hole information paradox, since it seems that the information about the original state has been lost. There are three different approaches to attack this problem:

1. In the first approach it is postulated that a complete theory of quantum gravity must preserve unitarity. In this scenario the information paradox is just an illusion caused by the fact that black hole evaporation is a process that the semiclassical analysis captures only rudimentarily. In the full quantum computation of the gravitational collapse and the subsequent evaporation, Hawking radiation would carry the information about the original state. The absorption and radiation of matter and energy by a black hole is not different to any standard scattering experiment. Nowadays, this is the most popular manner to approach the paradox.
2. The second possibility is to consider that, in contrast to other physical phenomena, the theory of quantum gravity is non-unitary. The information of the original state that produced the black hole is simply lost, since Hawking radiation carries no information as the black hole evaporates until completely disappearing.
3. The third option suggests that the evaporation process stops at some point leaving a remnant with which the outgoing radiation is entangled. Hawking's computation requires that the region surrounding the event horizon has low curvature (in Planck's units), so the possible effects of quantum gravity can be neglected. It is clear that this condition will eventually be violated as the black hole becomes smaller and one can speculate with the possibility that quantum gravity effects prevent black holes from disappearing. Notice that such a remnant can be entangled with the outgoing radiation with $S_{\text{entanglement}} \sim N \ln 2$ only if the number of possible states of the object is at least of order N. Thus it seems that such remnants would have unbounded degeneracy even though their mass and size is, by definition, bounded by the scale at which quantum gravity effects become relevant. While this does not constitute a violation of quantum mechanics *per se*, still differs from usual expectations of any physical system.

It is important to emphasize that, contrary to a quite extended belief, Hawking's paradox is a deep problem and the evaporation process cannot describe unitary evolution by the inclusion of subleading corrections. Actually Mathur has recently shown that this would be possible only when the evolution process is altered by order one corrections, see [151]. That is suggesting a striking result: a unitary quantum theory of gravity might modify physical processes drastically at the event horizon scale.

- **Microscopic origin of entropy**. Birkhoff's theorem [38] established that the only stationary, spherically symmetric solution of Einsteins vacuum equation $R_{\mu\nu} = 0$ is Schwarzschild's. Another solution of these equations is the Kerr black hole, which preserves axial symmetry and is characterized by its mass M and its angular momentum J. Also when other fields are coupled to gravity it is natural that the associated conserved charges[10], collectively denoted as Q, label stationary solutions, as is the case of Reissner-Nordström.

  Beyond that, there must necessarily be other black hole solutions describing non-stationary systems, which include the gravitational collapse of stars or the evolution

[10]Scalar fields do not have a conserved charge, while Abelian vector fields conserve their monopole momentum.

of perturbations on any of the stationary solutions already mentioned. Since there are many states in which a star can be or the variety of conceivable perturbations is huge, one might hope that these give rise to many different black holes labeled by the same conserved charges. On the contrary, the analysis of perturbations of gravitational collapse in General Relativity shows that, independently of the initial state and the peculiarities of the perturbations, these always decay and the final configuration is fully described by the conserved quantities M, J and Q. All the higher multipole momenta of the gravitational and matter fields are radiated away. More generally, it has been shown that this conclusion is valid beyond the perturbative regime. These results are known as *uniqueness theorems* [18,58,127,128], giving rise to the *no-hair conjecture* that states that stationary black holes cannot have other characteristics different from M, J and Q.

In standard statistical thermodynamics the entropy is related to the number of microscopic states, or *microstates*, of a system in equilibrium described by a reduced set of macroscopic variables. In black hole physics, those variables are the global conserved quantities M, J and Q, and one should be able to construct a huge number of microstates reflecting the degeneracy of such system. However, the uniqueness theorems prevent that construction within the framework of General Relativity and, probably, within more general classical field theories of gravity. We can get an idea of the magnitude of that problem considering that the number of microstates of a standard astrophysical black hole is of the order of $10^{10^{80}}$, while we only are capable of constructing one classical solution describing this system. This is the essence of the black hole entropy problem, that constitutes, by far, the largest computational discrepancy in the history of theoretical physics. The solution of this tremendous problem needs the use of quantum gravity tools.

1.2 Supergravity and String Theory

The previous section closed with an exposition of the information and entropy problems that appear when one tries to do quantum mechanics in classical black hole backgrounds. String Theory is, probably, the only consistent theory of quantum gravity that can be used to face these problems at present. Actually, one of its major successes has been the microscopic computation of the entropy of certain families of simple black holes. In this thesis we perform this identification for a special solution containing non-Abelian fields. On the other hand, the details associated to the information paradox remain unclear. Several proposals have been made and we will make some comments about this topic in chapter 6 in the context of the *Fuzzball proposal*.

The aim of this section is to describe the theories of supergravity that we will study in forthcoming chapters and to characterize their supersymmetric solutions. These are $\mathcal{N}=1$, $d=5$ Super-Einstein-Yang-Mills (SEYM) theories and, in chapter 2, their lower dimensional counterpart $\mathcal{N}=2$, $d=4$ SEYM. Some theories of supergravity describe low energy limits of String Theory. One of the main results of this thesis is precisely the explicit identification of some SEYM theories as such: they can be obtained from the compactification of $\mathcal{N} = 1$, $d = 10$ heterotic supergravity on T^5, the low energy limit of Heterotic String Theory, see chapters 7 and 8.

We follow a top-bottom approach, starting with an extremely brief description of the highlights of String Theory and following with a general characterization of its low energy supergravity approximation. Afterwards, we introduce Kaluza-Klein compactification as a technical tool which will be used through the text. The lower dimensional theories of supergravity obtained from this dimensional reduction are then presented and we then show how SEYM theories are constructed as gauged supergravities. We continue with an overview description of the method that we use to obtain supersymmetric solutions of general theories. We finish this introductory chapter with a discussion about black holes in String Theory.

1.2.1 Perturbative string theory

String theory is a presumably consistent theory of quantum gravity that still lacks a fully satisfactory formulation. The standard approach to define it has a perturbative nature and is based on the quantization of the dynamics of a relativistic string propagating in a given background. This worldsheet formulation turns out to be very powerful, since it allows not only for the calculation of the perturbative spectrum but also for the description of some of the non-perturbative states: the D-branes. However, the extended objects that appear in string theory, which are generally called p-branes being p its spacelike dimensionality, are related among each other by *dualities* so it would be desirable to have a formulation of the theory in which all the fundamental objects are treated in the same way. Unfortunately, we only know how to quantize particles and strings and so far we have had to content ourselves with a worldsheet formulation that, nevertheless, has been proven to be extraordinarily effective, specially when complemented with the insights that the low energy effective actions of supergravity provide. For instance, much of the information we have learned about non-perturbative objects has been acquired in this manner.

A free (super)string propagating in Minkowski space is described by the Ramond-Neveu-Schwarz action [165, 184]

$$S_{RNS} = -\frac{1}{4\pi\alpha'}\int_{\Sigma} d^2\xi \left[\eta^{ij}\partial_i X^\mu \partial_j X_\mu - i\bar{\psi}^\mu \partial\!\!\!/ \psi_\mu\right] , \tag{1.15}$$

that has to be supplemented with the equations of motion of the worldsheet metric γ and the gravitino ϕ fields, that have been previously eliminated from the action using symmetries [105, 177]. The boundary term that results from the variation of the above action for open strings is not trivial. In order to eliminate it, boundary conditions must be imposed. For the bosonic fields X^μ there are two possibilities:

- Neumann boundary conditions, that preserve Poincaré invariance in the target target space. They impose that no momentum flow beyond the end of the strings,

  $$n^i \partial_i X^\mu|_{\partial\Sigma} = 0 , \tag{1.16}$$

  where n^i is a unit vector normal to the boundary of the worldsheet $\partial\Sigma$.

- Dirichlet boundary conditions. These break Poincaré invariance by requiring that the endpoint of the string has a fixed position in some directions. They imply momentum is flowing through the endpoints.

$$t^i \partial_i X^\mu|_{\partial\Sigma} = 0\,, \qquad \rightarrow \qquad X^\mu|_{\partial\Sigma} = c^\mu\,. \tag{1.17}$$

being t^i a unit tangential vector to the boundary. These conditions explicitly break translation invariance and only allow the string's endpoints to move on $(p+1)-$ dimensional timelike hypersurfaces. These hypersurfaces correspond to the world-volume swept by D-branes. This is precisely the way in which Dp-branes, non-perturbative fundamental objects, are captured by the perturbative formulation.

Boundary conditions have to be imposed on the fermionic fields ψ^μ as well. In this case the left- and right-moving components $\psi^\mu_\pm$ must be identified at the endpoints $\xi^1 = 0, 2\pi l$. Once again, there are two possibilities

- Ramond (R) boundary conditions,

$$\psi^\mu_+(\xi^1=0) = \psi^\mu_-(\xi^1=0)\,, \qquad \psi^\mu_+(\xi^1=2\pi l) = \psi^\mu_-(\xi^1=2\pi l)\,. \tag{1.18}$$

- Neveu-Schwarz (NS) boundary conditions,

$$\psi^\mu_+(\xi^1=0) = \psi^\mu_-(\xi^1=0)\,, \qquad \psi^\mu_+(\xi^1=2\pi l) = -\psi^\mu_-(\xi^1=2\pi l)\,. \tag{1.19}$$

For closed superstrings there is no boundary but we can choose between four different possibilities (two for left- and two for right-moving componentes) for the periodicity of the fermions,

- Ramond (R) periodic conditions,

$$\psi^\mu_\pm(\xi^1=0) = \psi^\mu_\pm(\xi^1=2\pi l)\,. \tag{1.20}$$

- Neveu-Schwarz (NS) antiperiodic conditions,

$$\psi^\mu_\pm(\xi^1=0) = -\psi^\mu_\pm(\xi^1=2\pi l)\,. \tag{1.21}$$

The quantization of the superstring action proceeds in the canonical manner, solving the equations of motion taking into consideration the boundary and periodicity conditions (if applicable) and promoting Poisson brackets to commutators and superspace variables to operators. See [14,177] for detailed and careful expositions. Notice that imposing different boundary and periodicity conditions we are actually realizing the quantization on different backgrounds with distinct degrees of freedom. This is why, in the perturbative analysis, one distinguishes between different string theories. However these, in principle, different theories have been found to be connected through the action of non-trivial relations known as dualities, reflecting the extraordinary beauty and depth of String Theory. We will make some comments about dualities in the next subsection.

Superstring theories can only be quantized preserving Poincaré invariance in $d = 10$ dimensions. Self consistency and absence of tachyons and negative norme states is only possible for very precise combinations of periodicity and boundary conditions through the *GSO projection*. As a result, 5 different theories are usually distinguished: type I, type IIA, type IIB, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. Type I and the heterotics have $\mathcal{N} = 1$ spacetime supersymmetry while type II have $\mathcal{N} = 2$, which means they are invariant under 16 and 32 independent supersymmetry transformations respectively. The type IIA has non-chiral fermions, while the type IIB is chiral. The gauge groups $SO(32)$ and $E_8 \times E_8$ in the two heterotic theories correspond to the anomaly-free gauge groups that arise from the compactification of 16 extra spacetime dimensions that are only accessible to bosonic string excitations.

Before moving to the next subsection, let us include a brief comment about the nature of strings interactions. Strings can basically split and join. To compute an amplitude one has to evaluate the path integral summing over all possible *classical paths*,

$$Z = \int \mathcal{D}X\mathcal{D}\psi\mathcal{D}\gamma\mathcal{D}\phi e^{-S_{RNS}-S_E} , \tag{1.22}$$

where, besides the Ramond-Neveu-Schwarz action, the following topological term that does not modify the classical equations of motion is included

$$S_E = -\frac{\phi_0}{4\pi}\int d^2\xi\sqrt{|\gamma|}R(\gamma) , \tag{1.23}$$

This term is simply the vacuum expectation value of the *dilaton*, a massless state present in all consistent string theories, times the *Euler characteristic* of the worldsheet[11]. Then, the path integral can be decomposed into a sum of path integrals over worldsheets Σ_t with different topologies t,

$$Z = \sum_t \left(e^{\phi_0}\right)^{-\chi(t)} \int_{\{\Sigma_t\}} \mathcal{D}X\mathcal{D}\psi\mathcal{D}\gamma\mathcal{D}\phi e^{-S_{RNS,\Sigma_t}} . \tag{1.24}$$

This expression can be understood as a perturbative series expansion in the string coupling constant $g_s \equiv e^{\phi_0}$. Each possible worldsheet topology enters in the expansion at order $-\chi(t)$ in the coupling constant. For a fixed topology, the reduced path integral receives larger contributions from lower energy excitation states, and one can think of it as another perturbative expansion with parameter α'. It can be described as an expansion in "stringiness" about the point-particle limit [14]. Therefore perturbative string theory can be understood as a simultaneous double expansion in two different parameters.

1.2.2 Low energy effective actions and dualities

For each string theory we can construct an effective field theory action[12] capturing its low-energy dynamics. This is done by taking the $\alpha' \to 0$ limit, which in practice means we are neglecting the string length to recover a field theory. Since the mass of the string modes is proportional to $1/\sqrt{\alpha'}$, only the massless modes of each string theory are relevant in this limit. The proper way to find this field theory involves the construction of the action that reproduces the string amplitudes for the massless modes in the $\alpha' \to 0$ limit. In principle the effective action is given by an expansion in α', but only the terms of lowest order are usually considered.

[11] This is given by $\chi = 2 - 2g - b - c$, where g is the genus, b is the number of boundaries and c is the number of crosscaps (which is zero for oriented surfaces).

[12] Actually for type IIB the best we can do is to construct an effective *pseudoaction* that must be complemented with a self-duality restriction for the 5-form field strength.

Surprisingly, the low energy effective actions constructed in this manner can be obtained straightforward using supersymmetry arguments[13]. Actually, the fields associated to the massless modes of the type II theories correspond to the only two supergravity multiplets (chiral and non-chiral) of $\mathcal{N}=2$ in ten dimensions. On the other hand, there is only one possible supergravity multiplet with $\mathcal{N}=1$, which gives the effective action of the heterotic and type I theories when it is coupled to vector multiplets with the appropriate gauge group. We should point out that these theories of supergravity were found before the advent of string theory.

Theory	NSNS	RR	Vectors	Chiral fermions	Non-chiral fermions
Type I	$g_{\mu\nu}$, ϕ	C_2	A^I_μ	ψ_μ, λ	
Type IIA	$g_{\mu\nu}$, $B_{\mu\nu}$, ϕ	C_1, C_3			ψ_μ, λ
Type IIB	$g_{\mu\nu}$, $B_{\mu\nu}$, ϕ	C_0, C_2, C_4		ψ^i_μ, λ^i	
Heterotic	$g_{\mu\nu}$, $B_{\mu\nu}$, ϕ		A^I_μ	ψ_μ, λ	

Table 1.1: Fields associated to the massless modes of the string theories

The fields of the effective theories are given in table 1.1. We will not pay much attention to the fermionic fields, and, from now on, we will only describe the bosonic content and set to zero all fermions for simplicity. This is always a consistent truncation in theories of supergravity[14]. The NSNS fields include the metric $g_{\mu\nu}$, the dilaton ϕ and the Kalb-Ramond 2-form $B_{\mu\nu}$, except for the type I in which the latter is absent (although there is a 2-form in the RR sector that plays a similar role in the action). These fields are sometimes called the common sector, as they occur in all theories. They appear in the action as follows[15]

$$S_{cs} = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x\sqrt{|g|}e^{-2\phi}\left[R - 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{2\cdot 3!}H_{\mu\nu\rho}H^{\mu\nu\rho}\right], \tag{1.25}$$

where $H = dB$. This expression for the field strength gets modified in heterotic theories when Yang-Mills fields are included, as we will shortly see. The action has been presented in the *string frame*, which refers to the fact that the Ricci scalar appears multiplied by the exponential of the dilaton. This factor can be understood as being associated to the genus-0 term in the quantum expansion of the string's worldsheet. An appropriate conformal transformation, $g_{\mu\nu} = e^{\frac{\phi}{2}}g_{E\mu\nu}$, can eliminate this factor and take us to the *Einstein frame*. We also define a *modified Einstein frame*, which is obtained with the conformal transformation $g_{\mu\nu} = e^{\frac{\phi-\phi_0}{2}}\tilde{g}_{E\mu\nu}$, and guarantees the transformed metric is asymptotically flat if the original string metric is.

The most remarkable fact about the action (1.25) is that string theories contains General Relativity in their low energy limit. The consistent quantization of string theory not only requires the existence of gravity, but also indicates that it is described by Einstein's theory at low energies.

[13]Perhaps not so surprisingly, as supersymmetry imposes strong constraints in the possible theories that can be constructed.

[14]We refer to [14, 170] for information about the fermions in these theories.

[15]In type I supergravity the coupling between fields is different, see (1.28).

Let us present the remaining terms that complement the above action. For type II theories we have to include the RR sector. In the type IIA theory the additional fields enter as

$$S_{IIA} = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x\sqrt{|g|}\left[-\frac{1}{4}\left(G_2\right)^2 - \frac{1}{2\cdot 4!}\left(G_4\right)^2 - \frac{1}{144}\frac{1}{\sqrt{|g|}}\epsilon\partial C_3\partial C_3 B\right], \quad (1.26)$$

with $G_2 = dC_1$, $G_4 = dC_3 - H\wedge C_1$ and $H = dB$. We have omitted antisymmetrized contracted indices for notational simplicity. The type IIB supergravity pseudoaction includes the terms

$$S_{IIB} = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x\sqrt{|g|}\left[\frac{1}{2}\left(G_1\right)^2 + \frac{1}{2\cdot 3!}\left(G_3\right)^2 + \frac{1}{4\cdot 5!}\left(G_5\right)^2 - \frac{1}{192}\frac{1}{\sqrt{|g|}}\epsilon\partial C_4\partial C_2 B\right], \quad (1.27)$$

and is supplemented with the selfduality condition $\star G_5 = G_5$, with the field strengths defined as $G_1 = dC_0$, $G_3 = dC_2 - C_0 H$ and $G_5 = dC_4 - \frac{1}{2}C_2\wedge H + \frac{1}{2}B\wedge dC_2$.

We write the complete bosonic action of the heterotic supergravities, since they play a prominent role in this thesis,

$$S_h = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x\sqrt{|g|}e^{-2\phi}\left[R - 4\partial_\mu\phi\partial^\mu\phi + \frac{1}{2\cdot 3!}H_{\mu\nu\rho}H^{\mu\nu\rho} - \alpha' F^I{}_{\mu\nu}F^{I\,\mu\nu}\right]. \quad (1.28)$$

The field strengths are given by $F^I = dA^I + \frac{1}{2}f^I{}_{JK}A^J\wedge A^K$ and $H = dB + 2\alpha'\omega_{CS}$, where ω_{CS} is the Chern-Simons 3-form defined as $\omega_{CS} = F^I\wedge A^I - \frac{1}{3!}f_{IJK}A^I\wedge A^J\wedge A^K$. The gauge group of the vector multiplets can be $SO(32)$ or $E_8\times E_8$, with structure constants generically written as $f_{IJ}{}^K$.

The last effective action is that of type I, which is

$$S_I = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x\sqrt{|g|}\left\{e^{-2\phi}\left[R - 4\partial_\mu\phi\partial^\mu\phi\right] + \frac{1}{2\cdot 3!}\left(G_3\right)^2 - \alpha' e^{-\phi}F^I{}_{\mu\nu}F^{I\,\mu\nu}\right\}. \quad (1.29)$$

Type I supergravity has the same field content as the heterotic, although there are differences in the way the dilaton couples to the field strengths. This reflects that those effective terms arise from worldsheets with different topologies in each theory.

As we have already mentioned, the distinct perturbative string theories are believed to correspond to different limits of a single underlying theory. Such great expectations are supported by the dualities that relate them, like type IIA/IIB T-duality, type IIB S-duality, heterotic/type I duality or the 11-dimensional strong-coupling limit of type IIA, among others. Only the first of these was known from the worldsheet description because T-duality is the only one that has a perturbative nature. The rest of them were conjectured after some specific relations between the effective supergravity actions were observed. Let us briefly describe these dualities.

We can start with the observation that $\mathcal{N} = 1$, 11-dimensional supergravity [67], whose interpretation in the context of string theory remained unclear for decades, gives type IIA supergravity when compactified on a circle [36]. The technical details about the compactification of field theories of gravity on a circle are compiled in next section. The most relevant factor for our discussion is that the lower dimensional dilaton emerges as the Kaluza-Klein scalar of the compactification, which measures the size of the compact dimension. Since in perturbative string theory the dilaton gives the string coupling constant, one can suggest that the strong coupling limit of type IIA string theory is an 11-dimensional theory. It turned out that this vague idea was extraordinarily deep, as one could also argue about the higher dimensional origin of fundamental objects of type IIA [4]. The existence of *M theory* has been conjectured [214]. It is not a string theory and its low energy limit is given by 11-dimensional supergravity, but its complete formulation is still missing.

T-duality is associated to compactifications of apparently different theories on circles of different radii. The best known example relates type IIA and type IIB compactified on a circle of different radii $R_A = l_s^2/R_B$. The spectra of the two reductions describes the same fields and interactions, although the higher dimensional origin of those naturally differs for each of the possible *oxidations*. One can then introduce a set of relations known as *Buscher rules* that transform directly then 10-dimensional fields of type IIA and IIB supergravities compactified on a circle. Another example of T-duality is found between the two heterotic strings, $SO(32)$ and $E_8 \times E_8$ compactified on circles of dual radii. It is clear that further compactification enlarges the number of possible dualities. In the first place because there are more directions in which one can perform T-dualities. But new dualities also emerge because the lower dimensional fields can be Hodge-dualized and this can increase the number of fields that can be rotated into each other.

S-duality is a strong-weak coupling duality and it is necessarily non-perturbative in String Theory. The above presented relation between M theory and type IIA is an example of this class of dualities. Another interesting and illustrative case is provided by type IIB selfduality. The (pseudo-)action of type IIB supergravity has been presented here in a frame that is well suited to study T-duality to type IIA but obscures the existence of a symmetry under $SL(2,\mathbb{R})$ transformations acting on some combinations of the fields. In particular, one can define a complex scalar that parametrizes the coset space $SL(2,\mathbb{R})/SO(2)$ using the dilaton ϕ and the RR scalar C_0. Some of the $SL(2,\mathbb{R})$ transformations involve an inversion of the dilaton, so they involve a weak/strong coupling transformation. In type IIB superstring theory, S-duality[16] implies the existence of D-branes as fundamental objects dual to previously known perturbative string states. This conclusion can be extended to type IIA using T-duality arguments. Therefore, consistency requires that open strings should also be considered in type II theories. Actually, we will show in section 1.2.7 how the entropy of a very special black hole can be reinterpreted in type IIB theory by counting how many open strings can be attached between two stacks of D-branes.

Before finishing this section we would like to include a comment about heterotic/type I duality. Type I supergravity is obtained as a consistent truncation of type IIB that reduces the supersymmetry. In string language this is achieved by introducing an O9-plane, and consistency requires 16 D9-branes are also included. The S dual of this construction is the heterotic $SO(32)$ superstring, that is, therefore, interpreted as the S-dual of the

[16] In this case the group is broken to $SL(2,\mathbb{Z})$ due to charge quantization.

type I theory. At the level of the effective supergravity actions, this duality involves the transformations

$$g_{I\,\mu\nu} = e^{-(\phi_h - \phi_{h\infty})} g_{h\,\mu\nu}\,, \qquad \phi_I = -\phi_h\,, \qquad C_2 = e^{-\phi_{h\infty}}\,, \qquad A^I_{I,\mu} = e^{\frac{\phi_{h\infty}}{2}} A^I_{h,\mu}\,. \quad (1.30)$$

We will make use of these relations in chapter 8 to interpret non-Abelian black holes in terms of collections of intersecting D-branes.

1.2.3 Kaluza-Klein compactification (à la Scherk-Schwarz)

The effective field theories described in the previous section are defined in higher dimensional spacetimes. If they are to be considered candidates to describe real world phenomena, it is necessary to make contact with the 4-dimensional spacetime experience. We now describe the Scherk-Schwarz formalism that provides a systematic procedure to perform a dimensional reduction at the action level [187]. It can be seen as a refinement of the original Kaluza-Klein compactification that makes use of the Vielbein formalism[17]. We only consider here compactification on a circle or products of circles. Other internal spaces such as Calabi-Yau manifolds (and other general manifolds with exceptional holonomy) have been extensively used in the literature, as they are specially attractive for phenomenological purposes. Since we are more interested in studying geometrical properties of bosonic configurations, we prefer to consider toroidal compactifications because in that case it is possible to use an explicit form for the metric and other fields in the decomposition. Notably this simple dimensional reduction is very interesting and powerful, as toroidal compactifications of the heterotic string are claimed to be dual to type IIA on $K3$ [125].

We shall ignore all dynamics in the internal space. In field theory, this implies that the higher dimensional fields are decomposed in a Fourier expansion and we only consider the zero mode in the compactified theory. In some particular configurations this truncation to the zero mode modifies important properties of the initial solution. One interesting example is found in the compactification of the BPST instanton on $\mathbb{R}^3 \times S^1$, known as the caloron solution [109]. The field configuration depends on the compact coordinate and its dimensional reduction involves a zero mode truncation, a modification that does not preserve the original equations of motion. Remarkably it is possible to dimensionally reduce the original, non-truncated BPST instanton on $\mathbb{R}^4$ exploiting spherical symmetry to obtain a *colored monopole* in $\mathbb{R}^3$, as we show in Chapter 3. This relation plays a crucial role in our construction of solutions with non-Abelian fields.

The Scherk-Schwarz formalism starts with a convenient choice of Vielbein basis exploiting the fact that $\hat{d}$-dimensional Lorentz invariance is broken to $d = (\hat{d}-1)$-dimensional Lorentz invariance times an internal $U(1)$ isometry[18]. We choose an upper-triangular Vielbein basis of the form

$$(\hat{e}_{\hat{\mu}}{}^{\hat{a}}) = \begin{pmatrix} e_\mu{}^a & kV_\mu \\ 0 & k \end{pmatrix}, \qquad (\hat{e}_{\hat{a}}{}^{\hat{\mu}}) = \begin{pmatrix} e_a{}^\mu & -V_a \\ 0 & k^{-1} \end{pmatrix}, \qquad (1.31)$$

[17] In particular this means that the Scherk-Schwarz formalism for Kaluza-Klein compactification can be applied to fermions in curved spacetimes.

[18] In this section all $\hat{d}$-dimensional objects carry a hat, whereas $d = (\hat{d}-1)$-dimensional ones do not. The $\hat{d}$-dimensional indices split as follows: $\hat{\mu} = (\mu, \underline{z})$ (curved) and $\hat{a} = (a, z)$ (tangent-space indices). We take the periodicity of z to be $2\pi l$.

with $V_a \equiv e_a{}^{\mu} V_\mu$. As we are about to see, in d dimensions V_μ and k become dynamical fields that transform as a vector and a scalar respectively. It is straightforward to compute the *anholonomy coefficients* $\hat{\Omega}_{\hat{a}\hat{b}}{}^{\hat{c}}$ associated to this frame using their definition,

$$\left[\hat{e}_{\hat{a}}, \hat{e}_{\hat{b}}\right] = -2\hat{\Omega}_{\hat{a}\hat{b}}{}^{\hat{c}}\hat{e}_{\hat{c}}\,, \qquad \rightarrow \qquad \hat{\Omega}_{\hat{a}\hat{b}}{}^{\hat{c}} = \hat{e}_{\hat{a}}{}^{\hat{\mu}}\hat{e}_{\hat{b}}{}^{\hat{\nu}}\partial_{[\hat{\mu}}\hat{e}_{\hat{\nu}]}{}^{\hat{c}}\,. \tag{1.32}$$

The non-vanishing components are

$$\hat{\Omega}_{abc} = \Omega_{abc}\,, \qquad \hat{\Omega}_{abz} = -\frac{1}{2}kG_{ab}\,, \qquad \hat{\Omega}_{azz} = -\hat{\Omega}_{zaz} = -\frac{1}{2}\partial_a lnk\,, \tag{1.33}$$

Here G_{ab} is the field strength of the Kaluza-Klein vector V_a with tangent space indices. The non-vanishing components of the spin connection, $\hat{\omega}_{\hat{a}\hat{b}}{}^{\hat{c}} = (\Omega_{\hat{b}}{}^{\hat{c}}{}_{\hat{a}} - \Omega_{\hat{a}\hat{b}}{}^{\hat{c}} - \Omega^{\hat{c}}{}_{\hat{a}\hat{b}})$are, then

$$\hat{\omega}_{abc} = \omega_{abc}\,, \qquad \hat{\omega}_{abz} = \frac{1}{2}kG_{ab}\,, \qquad \hat{\omega}_{zbc} = -\frac{1}{2}kG_{ab}\,, \qquad \hat{\omega}_{zbz} = -\partial_b lnk\,. \tag{1.34}$$

We now recall the Palatini's identity, which allows us to remove the derivatives of the spin connection in the Einstein-Hilbert action,

$$\int d^dx\sqrt{|g|}KR = \int d^dx\sqrt{|g|}K[-\omega_b{}^{ba}\omega_c{}^c{}_a - \omega_a{}^{bc}\omega_{bc}{}^a + 2\omega_b{}^{ba}(\partial_a lnK)]\,. \tag{1.35}$$

We can use this identity forward and backwards after substituting the spin connection, to obtain the dimensional reduction of the Einstein-Hilbert action

$$S = \frac{1}{16\pi G_N^{(\hat{d})}}\int d^{\hat{d}}\hat{x}\sqrt{|\hat{g}|}\hat{R} = \frac{2\pi l}{16\pi G_N^{(\hat{d})}}\int d^dx\sqrt{|g|}k[R - \frac{1}{4}k^2G^2]\,. \tag{1.36}$$

We can make explicit the dynamical nature of the Kaluza-Klein scalar with a conformal transformation to the modified Einstein frame, $g_{\mu\nu} = (k/k_\infty)^{-\frac{2}{d-2}}\,\tilde{g}_{E\mu\nu}$, to get[19]

$$S = \frac{2\pi l k_\infty}{16\pi G_N^{(\hat{d})}}\int d^dx\sqrt{|\tilde{g}|}\left[\tilde{R}_E + \frac{d-1}{d-2}\left(\partial \ln k\right)^2 - \frac{k_\infty^{-\frac{2}{d-2}}}{4}k^{2\frac{d-1}{d-2}}G^2\right]\,. \tag{1.37}$$

In the modified Einstein frame the dimensionally reduced space is asymptotically flat and therefore this is the appropriate frame to define the global conserved charges. From the prefactor that appears in the reduced action we conclude that Newton's gravitational constant in d dimensions is given by $G_N^{(d)} = G_N^{(\hat{d})}/R_z$, where $R_z = lk_\infty$ is the asymptotic radius of the compact direction.

We have exposed here in great detail how to dimensionally reduce the metric. We will be more succinct with the rest of fields of interest in this thesis, which are scalars

[19] If the reader feels uncomfortable with the presence of the factor k_∞ in the lagrangian, this can be eliminated by a field redefinition $\tilde{k} = k/k_\infty$ and $\tilde{V_\mu} = k_\infty V_\mu$.

and p-forms. Actually, the reduction of scalars is completely trivial so there is no need to further consideration.

The vector representation of $SO(1,\hat{d}-1)$ gives a scalar and a vector of $SO(1,d-1)$. The scalar emerges from the spontaneous breaking of the $\hat{d}$-dimensional gauge transformations that depend on the internal coordinate. Notice that gauge transformations cannot introduce a dependence of the fields on this coordinate in order to preserve the Kaluza-Klein ansatz. The correct decomposition of a vector $\hat{A}_{\hat{\mu}}$ is easily seen to be[20],

$$\hat{A}_{\mu} = A_{\mu} + lV_{\mu}\,, \qquad\qquad \hat{A}_{\underline{z}} = l\,. \tag{1.38}$$

It is convenient to include here the decomposition of the field strength $\hat{F} = d\hat{A}$,

$$\hat{F}_{\mu\nu} = 2\partial_{[\mu}A_{\nu]} + 2l\partial_{[\mu}V_{\nu]}\,, \qquad\qquad \hat{F}_{\mu\underline{z}} = k^{-1}\partial_{\mu}l\,. \tag{1.39}$$

The dimensional reduction of Maxwell and Chern-Simons terms is then straightforward.

More general p-forms $\hat{C}^{(p)}_{\hat{\mu}_1\ldots\hat{\mu}_p}$ are reduced in a similar fashion, giving rise to a p-form $C^{(p)}_{\mu_1\ldots\mu_p}$ (as long as $d \geq p$) and a $(p-1)$-form $C^{(p-1)}_{\mu_1\ldots\mu_{(p-1)}}$ in d dimensions. The explicit decomposition of these forms can generally be written as

$$\hat{C}^{(p)}_{\mu_1\ldots\mu_p} = C^{(p)}_{\mu_1\ldots\mu_p} + pV_{[\mu_1}\hat{C}^{(p-1)}_{\mu_2\ldots\mu_p]}\,, \qquad\qquad \hat{C}^{(p)}_{\mu_1\ldots\mu_{(p-1)}\underline{z}} = C^{(p-1)}_{\mu_1\ldots\mu_{(p-1)}}\,, \tag{1.40}$$

although one can always make convenient field redefinitions.

1.2.4 Five dimensional Supergravity

Most of the work carried out in this thesis can be interpreted in the context of 4- and 5-dimensional supergravity coupled to non-Abelian matter multiplets. These theories are presented here as they were originally obtained, i.e. as the result of gauging global isometries of the scalar manifold of the original (Abelian) matter coupled supergravities. The 4- and 5-dimensional theories that we consider[21] are related by dimensional reduction. In this section we describe the ungauged 5-dimensional theory, which can be obtained compactifying 10-dimensional supergravities. Its 4-dimensional counterpart is described in Chapter 2.

The field content of matter coupled $\mathcal{N}=1$, $d=5$ supergravity is given by a supergravity multiplet and vector multiplets[22]. The supergravity multiplet contains a graviton $e^a{}_\mu$, a graviphoton A_μ and a gravitino given by a pair of symplectic-Majorana spinors $\psi^i{}_\mu$ (eight real components in total). Each vector multiplet, labeled as $x = 1,\ldots,n_V$, contains a vector $A^x{}_\mu$, a scalar ϕ^x and a gaugino λ^{xi}. The most general symmetry of the equations of motion is necessarily a subgroup of $GL(n_V+1)$ that rotates the graviphoton with the rest of vector fields. For this reason it is convenient to introduce a notation that considers all vectors collectively. We can label all vectors with indices $I,J = 0,\ldots,n_V$ such that

[20] Based on invariance under local reparametrizations of the internal circle.

[21] We work with supergravity theories with 8 supercharges. In the 4-dimensional case the models are specified by a cubic prepotential.

[22] We ignore hypermultiplets through the whole text, which is always a consistent truncation.

$$A^I{}_\mu = (A_\mu, A^x_\mu)\,. \tag{1.41}$$

The scalars parametrize a so-called real special manifold with σ-model metric $g_{xy}(\phi)$. Real special geometry arises as the combination of the Riemannian character of the σ-model with the $GL(n_V+1)$ structure that controls how scalars couple to vectors via the kinetic matrix $a_{IJ}(\phi)$. The most convenient approach to tackle this problem is through the definition of (n_V+1) functions of the scalars $h^I(\phi)$ transforming as vectors under $GL(n_V+1)$. Then, these parametrize a (n_V+1)-dimensional space with metric a_{IJ} in which the real special manifold is embedded, as given by the following constraint

$$C_{IJK}h^I h^J h^K = 1\,, \tag{1.42}$$

for some real constant symmetric tensor C_{IJK}. The metric g_{xy} is then given by the pullback of a_{IJ}, which is fixed by the real special structure itself. Actually the real special geometry is completely determined by the value of C_{IJK}; given this tensor one can find $a_{IJ}(\phi)$ and $g_{xy}(\phi)$ for a parametrization $h^I(\phi)$ as follows

$$h_I = C_{IJK}h^J h^K\,, \qquad a_{IJ} = -2C_{IJK}h^K + 3h_I h_J\,, \qquad g_{xy} = 3a_{IJ}\frac{\partial h^I}{\partial\phi^x}\frac{\partial h^J}{\partial\phi^y}\,. \tag{1.43}$$

The bosonic part of the action is

$$S = \int d^5x\sqrt{g}\left\{R + \frac{1}{2}g_{xy}\partial_\mu\phi^x\partial^\mu\phi^y - \frac{1}{4}a_{IJ}F^I{}_{\mu\nu}F^{J\,\mu\nu} + \frac{\varepsilon^{\mu\nu\rho\sigma\lambda}}{12\sqrt{3}\sqrt{g}}C_{IJK}F^I{}_{\mu\nu}F^J{}_{\rho\sigma}A^K{}_\lambda\right\}\,. \tag{1.44}$$

The supersymmetry transformations for vanishing fermions, which is the only case we consider, are

$$\begin{aligned}
\delta_\epsilon\psi^i_\mu &= D_\mu\epsilon^i - \frac{1}{8\sqrt{3}}h_I F^{I\,\alpha\beta}\left(\gamma_{\mu\alpha\beta} - 4g_{\mu\alpha}\gamma_\beta\right)\epsilon^i\,,\\
\delta_\epsilon\lambda^{xi} &= \frac{1}{2}\left(\not{\partial}\phi^x - \frac{\sqrt{3}}{2}h_{I\,,x}\not{F}^I\right)\epsilon^i\,.
\end{aligned} \tag{1.45}$$

We include the equations of motion for completeness,

$$\begin{aligned}
0 &= G_{\mu\nu} - \frac{1}{2}a_{IJ}\left(F^I{}_\mu{}^\rho F^J{}_{\nu\rho} - \frac{1}{4}g_{\mu\nu}F^{I\,\rho\sigma}F^J{}_{\rho\sigma}\right) + \frac{1}{2}g_{xy}\left(\partial_\mu\phi^x\partial_\nu\phi^y - \frac{1}{2}g_{\mu\nu}\partial_\rho\phi^x\partial^\rho\phi^y\right)\,,\\
0 &= \nabla^2\phi^x + \frac{1}{4}g^{xy}\partial_y a_{IJ}F^{I\,\rho\sigma}F^J{}_{\rho\sigma}\,,\\
0 &= \nabla_\nu\left(a_{IJ}F^{J\,\nu\mu}\right) + \frac{1}{4\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\lambda}}{\sqrt{g}}F^I{}_{\sigma\rho}F^J{}_{\sigma\lambda}\,.
\end{aligned} \tag{1.46}$$

Pure $\mathcal{N}=1$, $d=5$ supergravity has no matter multiplets, i.e. $n_V=0$. This theory is characterized by a trivial constant symmetric tensor with only one component,

$C_{000} = 1$, so there are no scalars and we have $a_{00} = 1$ and $h^0 = 1$. We usually will work in theories with $n_V \neq 0$ in the $ST[2, n_V + 1]$ *model*. The name of this model is taken from the 4-dimensional theory that results from dimensional reduction, which describes the supergravity multiplet coupled to $(n_V + 1)$ vector multiplets. The $ST[2, n_V + 1]$ model is characterized by the following non-vanishing components of the constant symmetric tensor

$$C_{0\alpha\beta} = \frac{1}{6}\eta_{\alpha\beta} = \frac{1}{6}\mathrm{diag}(1, -1, \ldots, -1)\,, \qquad \text{with } \alpha\,, \beta = 1, \ldots, n_V\,. \tag{1.47}$$

For the particular value $n_V = 2$ we recover the *STU model*, that is usually presented as the model with $C_{IJK} = \frac{1}{6}|\varepsilon_{IJK}|$. It is trivial to check that this can be recast in the form of the $ST[2, 3]$ model through the field redefinition

$$A^{\pm}{}_{STU} = \frac{1}{\sqrt{2}}\left(A^1{}_{ST} \pm A^2{}_{ST}\right)\,, \qquad\qquad h^{\pm}_{STU} = \frac{1}{\sqrt{2}}\left(h^1_{ST} \pm h^2_{ST}\right)\,. \tag{1.48}$$

1.2.5 Gauging isometries of the scalar manifold

The global symmetries of generic $\mathcal{N} = 1$, $d = 5$ supergravities are given by the product

$$G = G_V \times SU(2)_R\,. \tag{1.49}$$

The second factor corresponds to the *R-symmetry group* of the theory. This group acts on the indices i, j carried by the fermionic fields that label the pair of symplectic-Majorana spinors. The term G_V represents the group of transformations on the vector multiplets that preserve the real special structure of the theory. The non-Abelian theories on which we work are obtained gauging a subgroup of G_V [106]. Therefore, it is worth reviewing here how this is done.

In the first place, we need to understand how the isometries of a general non-linear σ-model can be gauged. Consider the following term of the action

$$S_\phi = \frac{1}{2}\int d^5x\sqrt{g}\,[g^{\mu\nu}g_{xy}(\phi)\partial_\mu\phi^x\partial_\nu\phi^y]\,. \tag{1.50}$$

While $g_{xy}(\phi)$ has been presented as a "metric" on the scalar manifold, it does not transforms as such under transformations of the scalars[23]. Actually, under general infinitesimal redefinitions of the scalars $\delta\phi^z = \epsilon^z(\phi)$ it transforms as a set of functions, i.e.

$$\delta_\epsilon g_{xy}(\phi) = \epsilon^z\partial_z g_{xy}(\phi)\,. \tag{1.51}$$

The variation of the action can be computed to be

$$\delta_\epsilon S_\phi = \frac{1}{2}\int d^5x\sqrt{g}\,[g^{\mu\nu}\mathcal{L}_\epsilon g_{xy}(\phi)\partial_\mu\phi^x\partial_\nu\phi^y]\,, \tag{1.52}$$

[23] Nonetheless the use of the word "metric" is well motivated, as we are about to see.

where $\mathcal{L}_\epsilon g_{xy} = \epsilon^z \partial_z g_{xy} + 2 g_{z(x}\partial_{\nu)}\epsilon^z$. That is, $\mathcal{L}_\epsilon g_{xy}$ is the standard Lie derivative of a symmetric $(0,2)$-type tensor. Thus, the σ-model action is left invariant (up to total derivatives) under infinitesimal redefinitions of the scalars only if those are generated by a Killing vector of the scalar metric, preserving the Riemannian structure of the real special manifold.

We denote the Killing vectors of the scalar manifold as $k_A(\phi)$, which generally satisfy the Lie algebra

$$[k_A, k_B] = -f_{AB}{}^C k_C\,, \tag{1.53}$$

where $f_{AB}{}^C$ are the structure constants of the algebra. Then the possible infinitesimal transformations that leave the action invariant are of the form

$$\delta_\alpha \phi^z = \alpha^A k_A{}^z\,. \tag{1.54}$$

for α^A a set of infinitesimally small constant parameters.

We can promote these infinitesimal parameters to local spacetime functions $\alpha^A(x)$. It is then clear that the scalar reparametrization they generate via (1.54) is not a symmetry of the action above, as now $\delta_\alpha \partial_\mu \phi^x = \left(\partial_\mu \alpha^A\right) k_A{}^x + \alpha^A \partial_z k_A{}^x \partial_\mu \phi^z$ and the first term cannot be canceled. The process of deforming the σ-model action such that it becomes invariant under these local transformations is referred as *gauging* the theory.

This can be achieved by a modification of the derivative operator on the scalars, which do not transform covariantly as we just discussed. The standard construction of a covariant derivative $\mathfrak{D}_\mu \phi^x$ requires the use of additional fields, the gauge vectors $A^I{}_\mu$, that compensate the terms $\partial_\mu \alpha^A$. The theories that we consider already contain some vector fields[24] with Abelian gauge symmetry

$$\delta_\Lambda A^I{}_\mu = -\partial_\mu \Lambda^I\,, \tag{1.55}$$

for any spacetime functions $\Lambda^I(x)$. The gauging process identifies a subset of these functions, let us say Λ^A, with the local parameters α^A. This identification avoids the introduction of additional fields or in the theory. In this manner the couplings will change but the degrees of freedom remain the same, which is the reason why this method is well-suited for supersymmetric theories.

The covariant derivative of a generic object Φ that under the action of (1.54) behaves as

$$\delta_\alpha \Phi = \alpha^A \delta_A \Phi\,, \tag{1.56}$$

is defined as

$$\mathfrak{D}_\mu \Phi = \nabla_\mu \Phi + g A^A{}_\mu \delta_A \Phi\,, \tag{1.57}$$

where the first term is spacetime and target-space covariant. In the case at hand, the covariant derivative acting on the scalars that substitutes the original partial derivative is

[24]The vector field sector at the action has to be invariant under the transformations generated by the group that it is being gauged, and the vectors must transform in the adjoint representation, $\partial_\alpha A^A{}_\mu = \alpha^B f_{BC}{}^A A^C{}_\mu$ for α^A constants.

$$\mathfrak{D}_\mu \phi^x = \partial_\mu \phi^x + g A^A{}_\mu k_A{}^x \,. \tag{1.58}$$

We can also define a covariant derivative for objects with adjoint indices,

$$\mathfrak{D}_\mu h^A = \partial_\mu h^A + g f_{BC}{}^A A^B{}_\mu h^C \,, \qquad \mathfrak{D}_\mu h_A = \partial_\mu h_A + g f_{AB}{}^C A^B{}_\mu h_C \,. \tag{1.59}$$

The gauge fields transform in the usual manner

$$\delta_\alpha A^A{}_\mu = -\frac{1}{g}\mathfrak{D}_\mu \alpha^A = -\frac{1}{g}\left(\partial_\mu \alpha^A + g f_{BC}{}^A A^B{}_\mu \alpha^C\right) \,. \tag{1.60}$$

And the gauged σ-model action is

$$S_\phi = \frac{1}{2}\int d^5x \sqrt{g}\,[g^{\mu\nu} g_{xy}(\phi) \mathfrak{D}_\mu \phi^x \mathfrak{D}_\nu \phi^y] \,. \tag{1.61}$$

Since the scalars couple to other fields, the symmetries of a σ-model might not always be symmetries of the complete theory. Only when the Killing vectors $k_A{}^x$ respect the complete real special structure we can gauge the isometries they generate. In a nutshell, this requires that the functions of the scalars $h^B(\phi)$ remain invariant up to $GL(n_V + 1)$ rotations with matrices T_A, and that these rotations themselves do not modify the symmetric constant tensor C_{ABC}. The first condition can be written as

$$\mathcal{L}_{k_A} h^B \equiv k_A{}^x \partial_x h^B = T_A{}^B{}_C h^C \,, \qquad \text{with } [T_A, T_B] = f_{AB}{}^C T_C \,, \tag{1.62}$$

so the matrices T_A are just the structure constants, $T_A{}^B{}_C = f_{AC}{}^B$. The second condition reads

$$\delta_{k_E} C_{ABC} = -3 T_E{}^D{}_{(A} C_{BC)D} = 0 \,. \tag{1.63}$$

In this work, we consider the gauging of a $SU(2)$ subgroup of the group of isometries of the scalar manifold. The coset space that the scalars parametrize in the $ST[2, n_V + 1]$ model is

$$SO(1,1) \times \frac{SO(1, n_V - 1)}{SO(n_V - 1)} \,, \tag{1.64}$$

and therefore we need at least 4 vector multiplets for $SU(2)$ to be an isometry of the scalar manifold. In some cases we will find convenient to consider 5 vector multiplets, so the theory can be understand as a non-Abelian extension of the STU model which already contains 2 Abelian vector multiplets.

1.2.6 Unbroken Supersymmetry

The solutions of the equations of motion of general theories break most of their symmetries, if not all. In metric theories of gravity, the infinite-dimensional group of general coordinate transformations cannot be preserved by any solution, as a metric can be invariant only under the action of a finite-dimensional group of isometries generated infinitesimally by a set of Killing vectors. In this section we are more interested on solutions preserving some of the (also infinite) local supersymmetry transformations that leave supergravity theories invariant. We refer to them as supersymmetric or BPS. A supersymmetric field configuration, not necessarily a solution, is given by a collection of bosonic (B) and fermionic (F) fields schematically satisfying

$$\delta_\epsilon B \sim \epsilon F = 0\,, \tag{1.65}$$

$$\delta_\epsilon F \sim \partial\epsilon + B\epsilon = 0\,, \tag{1.66}$$

for some infinitesimal supersymmetry generator $\epsilon(x)$. Since we consider only bosonic solutions, the first set of equations is automatically satisfied. The second set, on the other hand, is not trivial and is known as the *Killing spinor equations* (KSE). The set of Killing spinors $\epsilon(x)$ that solve these equations generate a finite-dimensional subgroup of the infinite-dimensional group of superspace reparametrizations.

The KSE turn out to be very powerful. Supersymmetric field configurations depend on a reduced number of independent functions, as the KSE impose relations between the different fields. Beyond that, the equations of motion are not all independent when working with supersymmetric configurations. There are relations among them, known as *Killing spinor identities* [131], whose derivation is illustrative. The remaining of this subsection is devoted to giving a generic description of this derivation and to exposing *Tod's program*, which is a method to characterize the supersymmetric solutions of any theory.

The action of a supergravity theory is invariant under arbitrary local supersymmetry transformations. Up total derivatives, we have

$$\delta_\epsilon S = \int d^d x\,[S_{,b}\delta_\epsilon B + S_{,f}\delta_\epsilon F] = 0\,. \tag{1.67}$$

In that expression $S_{,(b,f)}$ represents the variation of the Lagrangian with respect to bosonic and fermionic fields respectively, or, in other words, the equations of motion. Then we take a second functional derivative of the integrand with respect to fermionic fields,

$$[S_{,bf}\delta_\epsilon B + S_{,b}(\delta_\epsilon B)_{,f} + S_{,ff}\delta_\epsilon F + S_{,f}(\delta_\epsilon F)_{,f}] = 0\,. \tag{1.68}$$

The terms $\delta_\epsilon B$ and $S_{,f}$ are odd in the fermion fields, so they vanish automatically for purely bosonic configurations. Besides, we only consider supersymmetric configurations so $\delta_\epsilon F = 0$. Then, only one term in that expression survives:

$$S_{,b}(\partial_\epsilon B)_{,f}|_{F=0} = 0\,. \tag{1.69}$$

Thus we obtain the Killing spinor identities, that consist in a sum of terms with coefficients that contain the equations of motion. These relations can be used to reduce the number of equations of motion that have to be solved when looking for supersymmetric solutions.

We know give a systematic procedure introduced by Tod [201], in a formulation due to Gauntlet and collaborators [92, 94], that makes use of this tools to characterize supersymmetric solutions of a theory:

1. Assume there exists a Killing spinor ϵ that solves the KSE (1.66).

2. Construct all possible spinor bilinears of the form $\bar{\epsilon}\gamma^{\mu_1\cdots\mu_n}\epsilon$. Using Fierz identities, it is possible to rewrite the product of tensor bilinears as different products of other bilinears.

3. The KSE are of the form $\mathcal{O}\epsilon + B\epsilon = 0$, where $\mathcal{O}$ is some operator and the other terms are linear on the Killing spinor. Acting with this operator on the bilinears constructed in the previous step and making use of the KSE, we obtain several tensor equations for the bilinears solely.

4. These equations for the bilinears always indicate the existence of a Killing vector of the spacetime metric, with all the fields invariant under the isometry associated. From this moment one can introduce adapted coordinates and conveniently decompose the fields. The tensor equations yield relations among the decomposed fields, so supersymmetric configurations depend on a reduced number of independent functions.

5. Derive the Killing spinor identities to identify the minimal set of independent equations of motion that have to be solved using supersymmetric configurations. One then obtains simpler equations for the few independent functions.

1.2.7 Black holes in String Theory

Many classical black hole solutions to the equations of motion of different supergravities are known. Among those, some families of supersymmetric solutions, which are extremal black holes, are very well understood. In this section we give an extremely brief presentation of a very special family that can be interpreted from the string theory perspective; the 5-dimensional *three-charge black hole*. Our goal here is just to describe rudimentarily how string theory matches the classical entropy of a black hole with a degeneracy of microstates. As we will see along this thesis, our non-Abelian black holes can be understood as an extension of this family of solutions. We follow a *bottom-up* approach, starting with the 5-dimensional solution of the STU model of supergravity. The three-charge black hole is given by the static metric

$$ds^2_{(5)} = (Z_0 Z_+ Z_-)^{-\frac{2}{3}}\, dt^2 - (Z_0 Z_+ Z_-)^{\frac{1}{3}} \left[d\rho^2 + \rho^2 d\Omega^2_{(3)}\right] , \tag{1.70}$$

three vector fields

$$A_0 = -Z_0^{-1} dt\,, \qquad A_+ = -Z_+^{-1} dt\,, \qquad A_- = -Z_-^{-1} dt\,, \tag{1.71}$$

and two scalars that can be parametrized as $\phi_+ = Z_+/Z_0$ and $\phi_- = Z_-/Z_0$. The solution is completely specified in terms of Z_0, Z_+ and Z_-, which are harmonic functions in $\mathbb{E}^3$ and can be taken of the form

$$Z_0 = 1 + \frac{c_0 N_0}{\rho^2}\,, \qquad Z_+ = 1 + \frac{c_+ N_+}{\rho^2}\,, \qquad Z_- = 1 + \frac{c_- N_-}{\rho^2}\,. \tag{1.72}$$

Here $c_{0,\pm}$ are constants whose precise value is not important for our current discussion, while $N_{0,\pm}$ are natural numbers. The value $\rho = 0$ defines a null hypersurface that is an event horizon. The entropy of the black hole can be readily computed using (1.13), which yields

$$S = 2\pi\sqrt{N_0 N_+ N_-}\,. \tag{1.73}$$

It is certainly remarkable that the entropy is given by a set of integers, which already suggests that a microscopic interpretation in terms of degeneracy of states might be possible. This is a generic feature of supersymmetric black holes.

This solution can be straighforwardly embedded in 11-dimensional supergravity compactified on $T^2\times T^2\times T^2$ as follows

$$ds^2_{(11)} = ds^2_{(5)} - \left(\frac{Z_+Z_-}{Z_0^2}\right)^{\frac{1}{3}} d\vec{\tau}_1^{\,2} - \left(\frac{Z_0Z_-}{Z_+^2}\right)^{\frac{1}{3}} d\vec{\tau}_2^{\,2} - \left(\frac{Z_0Z_+}{Z_-^2}\right)^{\frac{1}{3}} d\vec{\tau}_3^{\,2}\,, \tag{1.74}$$

$$C_3 = A_0\wedge\omega_0 + A_+\wedge\omega_+ + A_-\wedge\omega_-\,, \tag{1.75}$$

where $d\vec{\tau}_i^{\,2}$ are the metrics of three sets of T^2, $d\vec{\tau}_i^{\,2} = dx_{i_1}^2 + dx_{i_2}^2$, and ω_i are the corresponding volume form $\omega_i = dx_{i_1}\wedge dx_{i_2}$. This metric corresponds to the low energy, classical description of three orthogonal stacks of M2 branes, each wrapping one of the two-tori and smeared along the other two at $\rho = 0$. For example, a set of N_0 M2 branes is wrapping the two-torus parametrized by $\vec{\tau}_1$ and is smeared along $\vec{\tau}_2$ and $\vec{\tau}_3$, and so on. From the 5-dimensional perspective these are perceived as sources of mass and electric charge.

The string theory solution can be expressed in different supergravity theories upon use of dualities. For instance we can consider the following chain:

1. Dimensional reduction along x_{-_2}.
2. T-dualities along x_{+_1}, x_{+_2} and x_{-_1}.

In the first step, the N_- M2 branes become fundamental strings F1 of type IIA theory. On the other hand, the remaining two sets of M2 branes become D2 branes. After the second step we end up with the following configuration in type IIB theory

(N_0) D5	t	x_{0_1}	x_{0_2}	x_{+_1}	x_{+_2}	x_{-_1}
(N_+) D1	t	-	-	-	-	x_{-_1}
(N_-) P	t	-	-	-	-	x_{-_1}

In this table the symbol $-$ denotes smearing, while the coordinates indicate extension of the object along this direction. The computation of the supergravity solution in this frame is a bit lengthy but straightforward,

$$\begin{aligned} ds^2_{10} &= Z_0^{-1/2}Z_+^{-1/2}Z_-^{-1}dt^2 - Z_0^{1/2}Z_+^{1/2}ds_4^2 - Z_+^{1/2}Z_0^{-1/2}(d\vec{\tau}_1^{\,2}+d\vec{\tau}_2^{\,2}) \\ &\quad - Z_0^{-1/2}Z_+^{-1/2}Z_-\left(dx_{-_1}+A_-\right)^2\,, \end{aligned} \tag{1.76}$$

$$\phi = \frac{1}{2}\log\left(\frac{Z_+}{Z_0}\right)\,, \qquad B=0\,, \qquad C_0=0\,, \qquad C_4=0\,, \tag{1.77}$$

$$F_3 = -\left(\frac{Z_0^5}{Z_+^3Z_-^2}\right)^{1/4}\star_5 dA_0 - dA_+\wedge\left(dx_{-_1}-A_-\right)\,. \tag{1.78}$$

The near-horizon geometry obtained in the $\rho \to 0$ limit yields the metric of $AdS_3 \times S^3 \times T^4$. String theory on this background is dual to a 2-dimensional CFT and it is most remarkable that the black hole entropy can be recovered by counting the many ways in which states of this CFT theory can carry this amount of momentum.

It is possible to get an idea of how this computation works as follows. First, assume that there exist two regimes in which the same string theory system can be described. In the first, the supergravity limit, it is a large classical black hole. In the second, it is described as open strings ending on D-branes and gravity is negligible. The existence of these two limits as good descriptions requires $N_{0,\pm} \gg 1$, $N_- \gg N_+ N_0$ and $g_s \ll 1$. The combination $g_s N_{0,\pm}$ indicates how important gravity effects are and we can move from the first regime to the second by varying g_s, so that $g_s N_{0,\pm} \gg 1$ or $g_s N_{0,\pm} \ll 1$, respectively, while maintaining a small value of the string coupling. Notice that the entropy of these supersymmetric black holes does not depend on g_s. We just computed the entropy in the first regime. Let us summarize how it can be computed in the second.

The condition $N_- \gg N_+ N_0$ means that the size of the compact direction x_{-1}, denoted by R, is much larger than that of the four-torus. In this limit the theories on the D1 and D5 branes are well approximated by a (1+1) theory on this circle. The momentum P along the circle receives its dominant contribution from open strings attached to these branes, that in this limit look like point particles connecting coincident branes. Since the string coupling is small we are basically led to a theory of free particles on a circle, so their wavefunction is of the form

$$\psi(x_{-1}) = \sum_n e^{-\frac{2\pi n}{R} x_{-1}} \,, \tag{1.79}$$

Now observe that having N D branes wrapping a circle is equivalent to having one D brane with winding number N. As the endpoints of the strings are attached to the D branes their wavefunction need not be periodic in R, but only in NR. That is, the string has to go N times around the circle to be back at its initial position. In the case at hand, a string stretching between a D1 and a D5 finds the system periodic in $N_0 N_+ R$[25]. Therefore, standard arguments about the quantization of momentum along a circle imply that it is quantized in units of $1/N_0 N_+ R$, instead of simply $1/R$. We can then ask the following question: how many configurations of open strings can we construct such that the total momentum is given by N_-/R? The answer is found by counting the integer partitions of the number $N_0 N_+ N_-$.

The function defined as

$$Z = \prod_{n=1}^{\infty} \frac{1}{1 - x^n} = 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots \,, \tag{1.80}$$

is called the partition function, a polynomial where each coefficient counts the number of integer partitions of the corresponding order[26]. It seems impossible to compute directly the coefficient of this function for order $N_0 N_+ N_-$, but we can make use of the tools of

[25] This of course applies when N_0 and N_+ are coprime. Otherwise there would be corrections to the value of the entropy that we compute, but those corrections would be generally subdominant.

[26] Recall that $\sum_{m=0}^{\infty} (x^n)^m = \frac{1}{1-x^n}$ and that Euler proved the partition function can be written as the product $Z = (1 + x + x^2 + \dots)(1 + x^2 + x^4 + \dots)\dots(1 + x^k + x^{2k} + \dots)\dots$

statistical mechanics. The partition function (1.80) coincides with that of a canonical ensemble upon the identification,

$$x = e^{-\beta}\,, \qquad\qquad \text{with } \beta = \frac{1}{T}\,, \tag{1.81}$$

provided the energy levels of the microstates are integer numbers. In the canonical ensemble the entropy is given by

$$S = \log Z + \beta < n >\,, \qquad\qquad < n >= -\frac{\partial}{\partial\beta}\log Z\,. \tag{1.82}$$

Given a value of temperature, the average energy $< n >$ and the entropy can be computed from Z. In our problem we start with a given value of $< n >= N_0 N_+ N_-$ and then we compute the entropy. We first notice that

$$\log Z = \sum_{m=1}^{\infty} \frac{x^m}{m(1-x^m)}\,. \tag{1.83}$$

This expression is hard to evaluate, so we take the high temperature limit $\beta \ll 1$, $x = 1 - \beta + \mathcal{O}(\beta^2)$ in which case

$$\log Z \simeq \sum_{m=1}^{\infty} \frac{1}{\beta m^2} + \mathcal{O}(\beta^0) = \frac{\zeta(2)}{\beta} + \mathcal{O}(\beta^0)\,, \tag{1.84}$$

where $\zeta(x)$ is Riemann's zeta function, in particular $\zeta(2) = \pi^2/6$. Computing $< n >$ and inverting the relation we get $\beta^2 = \pi^2/(6N_0N_+N_-)$, which justifies the high temperature approximation is valid. The entropy is

$$S = 2\pi\sqrt{\frac{N_0 N_+ N_-}{6}}\,, \tag{1.85}$$

which differs with the supergravity computation by a factor of $\sqrt{6}$.

Actually, what we have computed is the entropy associated to strings stretching between the D1 and the D5 branes with only one bosonic degree of freedom. But we have neglected the compact T^4 in this discussion. This compact space has an effect: there are four bosonic degrees of freedom and four fermionic, as the system is supersymmetric. For c bosonic degrees of freedom, the proper partition function and the entropy are

$$Z_{b_c} = \left(\prod_{n=1}^{\infty} \frac{1}{1-x^n}\right)^c\,, \qquad\qquad S_{b_c} = 2\pi\sqrt{\frac{cN_0 N_+ N_-}{6}}\,. \tag{1.86}$$

On the other hand, fermions can only have occupation number 0 or 1. This implies that d fermionic degrees of freedom have

$$Z_{f_c} = \left(\prod_{n=1}^{\infty} 1 + x^n\right)^d\,, \qquad\qquad S_{f_c} = 2\pi\sqrt{\frac{dN_0 N_+ N_-}{12}}\,. \tag{1.87}$$

Finally, in a supersymmetric system $d = c$ and we get

$$Z_{b_c} = \left(\prod_{n=1}^{\infty} \frac{1 + x^n}{1 - x^n} \right)^c \,, \qquad\qquad S = 2\pi \sqrt{\frac{c N_0 N_+ N_-}{4}} \,. \tag{1.88}$$

Therefore, we conclude that the microscopic entropy of the D1D5P system is

$$S = 2\pi \sqrt{N_0 N_+ N_-} \,. \tag{1.89}$$

The result coincides with the value obtained in the supergravity regime, which constitutes a major achievement of string theory.

2

$\mathcal{N} = 2$ Einstein-Yang-Mills' static two-center solutions

This chapter is based on

Pablo Bueno, Patrick Meessen, Tomas Ortín and Pedro F. Ramírez
"$\mathcal{N} = 2$ Einstein-Yang-Mills' static two-center solutions",
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Contrary to what one might think, multi-black hole solutions need not be related to supersymmetry or, like in the case of Kastor and Traschen's solution in Ref. [133], fake-supersymmetry. Proof of this is given by various solutions *e.g.* the ones presented in Refs. [25] and [63]. The benefit of using supersymmetry, however, is that after a few decades' worth of investigations there are workable recipes for creating supersymmetric solutions, which greatly facilitates the construction and study of multi-black hole solutions.

The construction is particularly straightforward in ungauged $\mathcal{N} = 2$, $d = 4$ supergravity coupled to vector multiplets where there are clear-cut rules for a supersymmetric multi-object solution to give rise to a well-defined multi-black hole solution [22,65,72,111, 126,147,173,176]: i) positive mass of the constituents, ii) the near-horizon limit has to have definite entropy, iii) the 2^{nd} law of thermodynamics must hold in the coalescence of constituents, and iv) the Denef constraints [72] must be satisfied. Depending on the charges the latter may constrain the distance between the constituents but it always implies the absence of NUT charge.

The oft forgotten case of ungauged $\mathcal{N} = 2$, $d = 4$ supergravity coupled to non-Abelian vector multiplets, which we refer to as $\mathcal{N} = 2$ Einstein-Yang-Mills, is similar to the Abelian case in that there is a well-defined recipe for constructing supersymmetric solutions [122, 123]. However, the construction of supersymmetric solutions is greatly hindered not only by the fact that not every Abelian theory can be non-Abelianized, but doubly more so by the fact that the supersymmetric recipe requires the use of solutions of the (non-Abelian) Bogomol'nyi equation on $\mathbb{R}^3$ [42]. Our lack of knowledge of the space of all solutions to this equation is a serious limitation to the application of the supersymmetric recipe: there exists a vast literature on single monopole solutions, *i.e.* regular single-center solutions to the Bogomol'nyi equation (see *e.g.* Refs. [200, 212]). Depending on the chosen $\mathcal{N} = 2$, $d = 4$ model, these can be used to construct globally regular supergravity solutions known as *global monopoles*. A lot less is known about the singular solutions to the Bogomol'nyi equation which are the ones which give rise to black holes with different degrees of non-Abelian hair [122,123,154]. Finally, even less is known about multi-center solutions to the Bogomol'nyi equation. These are the ones we need in order to apply the supersymmetric recipe to the construction of multi-center supergravity solutions, with centers that correspond to global monopoles or black holes.

Luckily enough, some explicit solutions are known.[1] In this chapter we are going to use the solutions of the SU(2) Bogomol'nyi equation found by Cherkis and Durcan [62] and Blair and Cherkis [41] (which we will generalize by adding *Protogenov hair* [154]). These solutions describe an 't Hooft-Polyakov (-Protogenov) monopole in the presence of an arbitrary number of Dirac monopoles embedded in SU(2), all having charge opposite to the one carried by the former. These solutions can (in principle) give rise to supergravity solutions describing black holes in the presence of a global monopole. The construction of these solutions is, at the same time, our main goal and our main result.

Before we start constructing multi-black hole solutions, however, it is worth reviewing briefly some of the previous work on solutions of YM theories coupled to gravity[2]. Most of the previous work on this topic was focused on pure Einstein-Yang-Mills (EYM) theories, (the minimal non-Abelian extension of the Einstein-Maxwell theory), ignoring the possible existence of unbroken supersymmetry which is, however, one of our main concerns here.

Soon after the discovery of the 't Hooft-Polyakov monopole [119, 179] several groups found solutions to the pure EYM theory [218] whose SU(2) gauge field is that of the Wu–Yang SU(2) monopole [216]. The metric of all these solutions is that of the (dS or AdS) non-extremal Reissner-Nordström black hole and the singularity in the gauge field (generically expected for static YM solutions [73]) is covered by an event horizon.

This coincidence of the metrics is due to the relation between the Wu–Yang SU(2) monopole and the non-Abelian embedding of the Dirac monopole Eq. (A.15): they are related by a singular gauge transformation and therefore give rise to exactly the same energy-momentum tensor as it is gauge invariant whether the gauge transformation is singular or not. For this reason, these solutions have been regarded as not truly non-Abelian, even though there are potentially measurable differences, see *e.g.* Refs. [53, 113].

Finding less trivial ("genuinely or essentially non-Abelian") solutions proved much more difficult and a *non-Abelian baldness theorem* stating that the only black-hole solutions of the EYM SU(2) theory with a regular horizon and non-vanishing magnetic charge had to be non-Abelian embeddings of the Reissner–Nordström solution was proven in [89]. This theorem was subsequently generalized to prove the absence of regular monopole or dyon solutions to the EYM theory in Refs. [40, 83].

An "essentially non-Abelian" solution, globally regular [195] to EYM theory had already been found: the Bartnik-McKinnon particle [11]. The Bartnik-McKinnon particle and its black hole-type generalizations [206], are in fact families of unstable solutions indexed by a discrete parameter and evade the non-Abelian baldness theorem by being bald, *i.e.* they have no asymptotic charge. It is worth pointing out that even though these solutions are only known numerically, they have been proven to exist [193].

The classification of the possible EYM solutions for the gauge group SU(2) [194] suggests that one has to add more fields to the theory in order to get "essentially non-Abelian" black-hole or gravitating monopole solutions with non-vanishing charges. Investigations of solutions to the EYM theory coupled to a Higgs field in the adjoint representation [143]

[1] Finite-energy, multi-center solutions of the Yang-Mills or Yang-Mills-Higgs system which do not satisfy the Bogomol'nyi equation like those in Refs. [134, 136, 138] are also known.

[2] For more comprehensive reviews see *e.g.* Refs. [208].

in the BPS-limit, a theory that is closer to the one we are going to study than EYM, shows that a globally well-defined 't Hooft-Polyakov monopole exists and furthermore the existence of other Bartnik-McKinnon-like solutions.

As far as 4-dimensional supergravity is concerned we have the (supersymmetric) Harvey-Liu [112] and the Chamseddine-Volkov [60] regular gravitating monopole solutions to gauged $\mathcal{N} = 4$, $d = 4$ supergravity; in $\mathcal{N} = 2$, $d = 4$ theories there are analytical solutions describing global monopole solutions and non-Abelian black hole solutions with and without asymptotic magnetic charge. Needless to say, all the solutions mentioned in this little historical *exposé* describe the fields corresponding to a single object. To our knowledge, there are no known, essentially non-Abelian multi-object analytic[3] solutions and this article intends to fill this gap by constructing static solutions describing the interplay between an 't Hooft-Polyakov monopole and a Dirac monopole of opposite charge in two generic classes of gauged $\mathcal{N} = 2$, $d = 4$ models.

As we stressed in the introduction, in the theories we have called $\mathcal{N} = 2$, $d = 4$ SEYM the gauge group does not contain any part of the R-symmetry group. Indeed, in general (ungauged) $\mathcal{N} = 2$, $d = 4$ theories, the global symmetry group G can be written as

$$\mathrm{G} = \mathrm{G}_{\mathrm{V}} \times \mathrm{G}_{\mathrm{hyper}} \times \mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_{\mathrm{R}}\,, \tag{2.1}$$

where G_{V} and $\mathrm{G}_{\mathrm{hyper}}$ stand for the isometry groups of the special and quaternionic Kähler manifolds respectively. When a (necessarily non-Abelian) subgroup of G_{V} is gauged (as in $\mathcal{N} = 2$, $d = 4$ SEYM theories) the scalar potential is positive semidefinite, which allows for asymptotically De-Sitter and asymptotically flat solutions (such as the ones we construct in this paper). This is in contradistinction to theories in which a subgroup of $\mathrm{SU}(2)_{\mathrm{R}}$ (or the complete $\mathrm{SU}(2)_{\mathrm{R}}$) is gauged via Fayet-Iliopoulos terms[4] in whose case the scalar potential becomes negative definite, the solutions thus being asymptotically anti-De Sitter. Lately, an intense effort has been devoted to the construction of black-hole solutions of theories with Abelian gaugings (that is, theories in which a subgroup $\mathrm{U}(1) \in \mathrm{SU}(2)_{\mathrm{R}}$ has been gauged); see, for instance, Refs. [48, 104, 107, 120, 139, 202] and references therein. The case in which the full $\mathrm{SU}(2)_{\mathrm{R}}$ has been gauged remains as unexplored as challenging, even though the general form of the timelike supersymmetric solutions of this theory has been given in Ref. [156].

This chapter is organized as follows: in section 2.1 we review the theories we are going to work with ($\mathcal{N} = 2$, $d = 4$ Super-Einstein-Yang-Mills theories) and the recipe for constructing timelike supersymmetric solutions (black holes, in particular). In section 2.2 we apply that recipe to construct single, static supersymmetric black-hole and monopole solutions of two particular examples of SU(2)-gauged $\mathcal{N} = 2$, $d = 4$ SEYM: the $\overline{\mathbb{CP}}^3$ model (quadratic) (2.2.2) and the ST[2, 4] model (cubic) (2.2.3). We use as seeds for these solutions the single-center solutions of the Bogomol'nyi equations reviewed in section A.5. In section 2.3 we construct multi-black-hole solutions for the same models using the multi-center solutions of the Bogomol'nyi equations reviewed in section 2.3.1. Our conclusions are contained in section 7.2. In the Appendices we review a particularly interesting single-

[3] Numerical, multi-center solutions have been found previously, though. See, *e.g.* Refs. [135, 137]. Some of those solutions can be embedded in $\mathcal{N} = 1, d = 4$ supergravity. However, representing massive objects, they can never be supersymmetric in that theory. The embedding in higher-$\mathcal{N}$ supergravities is much more difficult (if possible at all). We thank J. Kunz for pointing these works to us.

[4] The overall $\mathrm{U}(1)_R$ group cannot be gauged in this way. The Abelian gaugings discussed in the literature deal with a subgroup $\mathrm{U}(1) \in \mathrm{SU}(2)_{\mathrm{R}}$.

center solution of the SU(2) Bogomol'nyi equations which appears in different guises: as a "Lorentzian meron" (Appendix A.1), as the Wu-Yang monopole (Appendix A.2) or as a solution of the Skyrme model (Appendix A.3). A higher-charge generalization of this solution is reviewed in Appendix A.4.

2.1 $\mathcal{N}=2$, $d=4$ SEYM and its supersymmetric black-hole solutions (SBHSs)

In this section we are going to introduce the class of theories that we have called $\mathcal{N}=2$, $d=4$ SEYM theories and we are going to review the recipe to construct all their timelike supersymmetric solutions, presented in Ref. [123]. We shall be extremely brief. The interested reader can find more details in Refs. [85, 122, 168]; our conventions are those of Refs. [122, 123, 168].

2.1.1 The theory

$\mathcal{N}=2$, $d=4$ SEYM theories can be seen as the simplest $\mathcal{N}=2$ supersymmetrization of the Einstein-Yang-Mills (EYM) theories. They are nothing but theories of $\mathcal{N}=2$, $d=4$ supergravity coupled to n vector multiplets in which a (necessarily non-Abelian)[5] subgroup of the isometry group of the (Special Kähler) scalar manifold has been gauged using some of the vector fields of the theory as gauge fields[6].

We will only be concerned with the bosonic sector of the theory, which consists on the metric $g_{\mu\nu}$, the vector fields $A^{\Lambda}{}_{\mu}$ ($\Lambda=0,1,\cdots,n$) and the complex scalars Z^i ($i=1,\cdots,n$). The action of the bosonic sector reads

$$
\begin{aligned}
S[g_{\mu\nu}, A^{\Lambda}{}_{\mu}, Z^i] \quad = \quad & \int d^4x\sqrt{|g|}\,\big[R + 2\mathcal{G}_{ij^*}\mathfrak{D}_\mu Z^i \mathfrak{D}^\mu Z^{*\,j^*} + 2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} \\
& -2\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu} - V(Z,Z^*)\big]\,.
\end{aligned}
\tag{2.2}
$$

In this expression, $\mathcal{G}_{ij^*}$ is the Kähler metric, $\mathfrak{D}_\mu Z^i$ is the gauge-covariant derivative

$$\mathfrak{D}_\mu Z^i = \partial_\mu Z^i + gA^{\Lambda}{}_{\mu}k_{\Lambda}{}^{i}\,, \tag{2.3}$$

$F^{\Lambda}{}_{\mu\nu}$ is the vector field strength

$$F^{\Lambda}{}_{\mu\nu} = 2\partial_{[\mu}A^{\Lambda}{}_{\nu]} - gf_{\Sigma\Gamma}{}^{\Lambda}A^{\Sigma}{}_{\mu}A^{\Gamma}{}_{\nu}\,, \tag{2.4}$$

$\mathcal{N}_{\Lambda\Sigma}$ is the period matrix and, finally, $V(Z,Z^*)$ is the scalar potential

$$V(Z,Z^*) = -\tfrac{1}{4}g^2\Im\mathfrak{m}\mathcal{N}^{\Lambda\Sigma}\mathcal{P}_{\Lambda}\mathcal{P}_{\Sigma}\,. \tag{2.5}$$

Since the imaginary part of the period matrix is negative definite, the scalar potential is positive semidefinite, which leads to asymptotically-flat or -De Sitter solutions.

[5] The theory becomes identical to the ungauged one when the gauge group is Abelian.

[6] A global symmetry group can be gauged if it acts on the vector fields in the adjoint representation. Furthermore, it is required to be a symmetry of the prepotential; see *e.g.* ref. [123] for more details.

In the above equations, $k_\Lambda{}^i(Z)$ are the holomorphic Killing vectors of the isometries that have been gauged[7] and $\mathcal{P}_\Lambda(Z,Z^*)$ the corresponding momentum maps, which are related to the Killing vectors and to the Kähler potential $\mathcal{K}$ by

$$i\mathcal{P}_\Lambda = k_\Lambda{}^i\partial_i\mathcal{K} - \lambda_\Lambda\,, \tag{2.6}$$

$$k_{\Lambda\, i^*} = i\partial_{i^*}\mathcal{P}_\Lambda\,, \tag{2.7}$$

for some holomorphic functions $\lambda_\Lambda(Z)$. Furthermore, the holomorphic Killing vectors and the generators T_Λ of the gauge group satisfy the Lie algebras

$$[k_\Lambda, k_\Sigma] = -f_{\Lambda\Sigma}{}^\Gamma k_\Gamma\,, \qquad [T_\Lambda, T_\Sigma] = +f_{\Lambda\Sigma}{}^\Gamma T_\Gamma\,. \tag{2.8}$$

For the gauge group SU(2), which is the only one we are going to consider, we use lowercase indices[8] $a,b,c=1,2,3$ and the structure constants are $f_{ab}{}^c = -\varepsilon_{abc}$, so

$$[k_a, k_b] = +\varepsilon_{abc}k_c\,, \qquad [T_a, T_b] = -\varepsilon_{abc}T_c\,. \tag{2.9}$$

We will use the fundamental representation, in which the generators are proportional to the standard Pauli matrices[9] σ^a

$$T_a \equiv +\tfrac{i}{2}\sigma^a\,, \quad \Rightarrow \quad \mathrm{Tr}(T_aT_b) = -\tfrac{1}{2}\delta_{ab}\,. \tag{2.11}$$

The equations of motion of the theory can be written in the following form:

$$\begin{aligned} G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_{(\mu}Z^i\mathfrak{D}_{\nu)}Z^{*\,j^*} - \tfrac{1}{2}g_{\mu\nu}\mathfrak{D}_\rho Z^i\mathfrak{D}^\rho Z^{*\,j^*}]& \\ +4\mathcal{M}_{MN}\mathcal{F}^M{}_\mu{}^\rho\mathcal{F}^N{}_{\nu\rho} + \tfrac{1}{2}g_{\mu\nu}V(Z,Z^*) &= 0, \end{aligned} \tag{2.12}$$

$$\mathfrak{D}^2Z^i + \partial^i G_{\Lambda\,\mu\nu}\star F^{\Lambda\,\mu\nu} + \tfrac{1}{2}\partial^i V(Z,Z^*) = 0, \tag{2.13}$$

$$\mathfrak{D}_\nu \star G_\Lambda{}^{\nu\mu} + \tfrac{1}{4}g\left(k_{\Lambda\,i^*}\mathfrak{D}_\mu Z^{*i^*} + k^*_{\Lambda\,i}\mathfrak{D}_\mu Z^i\right) = 0\,, \tag{2.14}$$

where $G_{\Lambda\,\mu\nu}$ is the dual vector field strength

$$G_\Lambda \equiv \mathfrak{Re}\mathcal{N}_{\Lambda\Sigma}F^\Sigma + \mathfrak{Im}\mathcal{N}_{\Lambda\Sigma}\star F^\Sigma\,, \tag{2.15}$$

$\mathcal{F}^M{}_{\mu\nu}$ is the symplectic vector of vector field strengths

$$(\mathcal{F}^M) \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}, \tag{2.16}$$

[7] The employed notation associates a Killing vector to each value of the index Λ in order to avoid the introduction of yet another class of indices and the embedding tensor (See *e.g.* the reviews [203]); it is understood that not all the k_Λ need to be non-vanishing.

[8] These will be a certain subset of those represented by $\Lambda, \Sigma, \ldots$.

[9] These are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^a\sigma^b = \delta^{ab} + i\varepsilon^{abc}\sigma^c\,. \tag{2.10}$$

$\mathcal{M}_{MN}$ is the symmetric $2(n+1)\times 2(n+1)$ matrix defined by

$$(\mathcal{M}_{MN}) \equiv \begin{pmatrix} \Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma} + R_{\Lambda\Gamma}\Im\mathfrak{m}\mathcal{N}^{-1|\Gamma\Omega}R_{\Omega\Sigma} & -R_{\Lambda\Gamma}\Im\mathfrak{m}\mathcal{N}^{-1|\Gamma\Sigma} \\ & \\ -\Im\mathfrak{m}\mathcal{N}^{-1|\Lambda\Omega}R_{\Omega\Sigma} & \Im\mathfrak{m}\mathcal{N}^{-1|\Lambda\Sigma} \end{pmatrix}, \tag{2.17}$$

and

$$\mathfrak{D}_\nu \star G_\Lambda{}^{\nu\mu} = \partial_\nu \star G_\Lambda{}^{\nu\mu} + g f_{\Lambda\Sigma}{}^{\Gamma} A^{\Sigma}{}_{\nu} \star G_\Lambda{}^{\nu\mu}\,. \tag{2.18}$$

Most of the literature and earlier work on non-Abelian black-hole and monopole solutions has been carried out in the context of the Einstein-Yang-Mills (EYM) and Einstein-Yang-Mills-Higgs (EYMH) theories. Before closing this introduction, it is worth discussing the relation between those and the theories we are considering here. The main differences of the latter w.r.t. the former are the complexification of the Higgs field and the presence of a non-trivial period matrix. A further difference is the possibility of having more general scalar manifolds, which is reflected in the expressions of the gauge-covariant derivatives of the scalar fields. Solutions to the $\mathcal{N}=2$, $d=4$ SEYM theory have a chance of being also solutions of the EYMH theory if they have covariantly-constant scalars with identical phases (*e.g.* all of them purely imaginary). Then, if the scalar potential vanishes on the solutions, they also have a chance of being solutions to the EYM system as well; as we are going to see, some of the solutions found in Refs. [122, 123] are also solutions of the EYM theory and have the same metric as the EYM solutions of Refs. [53, 218].

2.1.2 The recipe to construct SBHSs of $\mathcal{N}=2$, $d=4$ SEYM

To construct timelike supersymmetric solutions of the $\mathcal{N}=2$, $d=4$ SEYM theory, it suffices to follow this recipe [122, 123] to find the elementary building blocks of the solutions, which are the $2(n+1)$ time-independent functions $(\mathcal{I}^M) = \begin{pmatrix}\mathcal{I}^\Lambda \\ \mathcal{I}_\Lambda\end{pmatrix}$:

1. Take a solution of the Bogomol'nyi equations

$$\tilde{F}^{\Lambda}{}_{\underline{mn}} = -\tfrac{1}{\sqrt{2}}\varepsilon_{mnp}\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^\Lambda, \tag{2.19}$$

for a gauge field $\tilde{A}^{\Lambda}{}_{\underline{m}}$ ($\underline{m}=1,2,3$ labels the 3 spatial coordinates) and a real "Higgs" field $\mathcal{I}^\Lambda$. $\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^\Lambda$ is the covariant derivative in the adjoint representation with gauge field $\tilde{A}^{\Lambda}{}_{\underline{m}}$. Observe that this equation has to be solved in the gauged (non-Abelian) and ungauged (Abelian) directions. The integrability condition in the Abelian directions is the familiar requirement that the $\mathcal{I}^\Lambda$ be harmonic functions on $\mathbb{R}^3$.

2. Find the functions $\mathcal{I}_\Lambda$ by solving these equations:

$$\tilde{\mathfrak{D}}_{\underline{m}}\tilde{\mathfrak{D}}_{\underline{m}}\mathcal{I}_\Lambda = \tfrac{1}{2}g^2\left[f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\,\mathcal{I}^\Sigma\mathcal{I}^\Delta\right]\mathcal{I}_\Omega\,. \tag{2.20}$$

In the non-Abelian directions these equations can, in many cases, be solved by taking $\mathcal{I}_\Lambda \propto \mathcal{I}^\Lambda$, but currently we only know how to generate non-trivial solutions to them in the cases where the gauge doublet $(\tilde{A}^\Lambda, \mathcal{I}^\Lambda)$ describes a non-Abelian Wu-Yang monopole; Observe that $\mathcal{I}_\Lambda = 0$ is always a solution, but the physical fields may be singular in some models.

In the Abelian directions, the $\mathcal{I}_\Lambda$ are just independent harmonic functions on $\mathbb{R}^3$.

3. Given the functions $\mathcal{I}^M$, we must find the 1-form on $\mathbb{R}^3$ $\omega_{\underline{m}}$ by solving the following equation:
$$\partial_{[\underline{m}}\omega_{\underline{n}]} = \varepsilon_{mnp}\mathcal{I}_M\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^M = \varepsilon_{mnp}\left(\mathcal{I}_\Lambda\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^\Lambda - \mathcal{I}^\Lambda\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}_\Lambda\right) . \tag{2.21}$$
The integrability conditions of this equation impose constraints on the integration constants of the functions $\mathcal{I}^M$ in exactly the same manner as in the ungauged case [13, 72].
In the case of static solutions, *i.e.* when $\omega = 0$, the above equation becomes a constraint on the integration constants of the functions $\mathcal{I}^M$ that will have to be solved. Observe, however, that this constraint is independent of the specific $\mathcal{N}=2$, $d=4$ model and only depends on the choice of gauge group; possible restrictions on the solution to said constraint can come from the desired behaviour of the physical fields in the full solution.

4. To reconstruct the physical fields from the functions $\mathcal{I}^M$ we need to solve the stabilization equations, a.k.a. *Freudenthal duality equations*, which give the components of the Freudenthal dual[10] $\tilde{\mathcal{I}}^M(\mathcal{I})$ in terms of the functions $\mathcal{I}^M$ [87]; These relations completely characterize the model of $\mathcal{N}=2$, $d=4$ supergravity.
Equivalently, the $\tilde{\mathcal{I}}$ can be derived from a homogeneous function of degree 2 $W(\mathcal{I})$ called the *Hesse potential* as [13, 157, 162]
$$\tilde{\mathcal{I}}_M \;=\; \tfrac{1}{2}\frac{\partial W}{\partial \mathcal{I}^M} \quad\longrightarrow\quad W(\mathcal{I}) \;=\; \tilde{\mathcal{I}}_M\mathcal{I}^M . \tag{2.22}$$

5. The metric takes the form
$$ds^2 = e^{2U}(dt+\omega)^2 - e^{-2U}dx^m dx^m , \tag{2.23}$$
where $\omega = \omega_{\underline{m}}dx^m$ is the above spatial 1-form and the metric function e^{-2U} is given by
$$e^{-2U} = \tilde{\mathcal{I}}_M(\mathcal{I})\mathcal{I}^M = W(\mathcal{I}) . \tag{2.24}$$

6. The scalar fields are given by
$$Z^i = \frac{\tilde{\mathcal{I}}^i + i\mathcal{I}^i}{\tilde{\mathcal{I}}^0 + i\mathcal{I}^0} . \tag{2.25}$$

7. The components of the vector fields are given by
$$A^\Lambda{}_t \;=\; -\tfrac{1}{\sqrt{2}}e^{2U}\tilde{\mathcal{I}}^\Lambda , \tag{2.26}$$
$$A^\Lambda{}_{\underline{m}} \;=\; \tilde{A}^\Lambda{}_{\underline{m}} + \omega_{\underline{m}}\; A^\Lambda{}_t . \tag{2.27}$$

After having gone through the steps of the recipe, one ends up with a supersymmetric solution to a chosen $\mathcal{N}=2$, $d=4$ EYM theory and what remains to be done is to analyze the constraints coming from imposing appropriate regularity conditions such as the absence of naked singularities.

[10] In Refs. [122, 123, 155] the components of the Freudenthal dual are denoted by $\mathcal{R}^M$.

2.2 Static, single-SBHSs of $\mathrm{SU}(2)$ $\mathcal{N}=2$, $d=4$ SEYM and pure EYM

Following the recipe given in section 5.1, we are going to construct static, single-center SBHSs of SU(2) $\mathcal{N}=2$, $d=4$ SEYM. Some of the solutions will simultaneously solve the equations of motion of the EYM and EYMH theories.

The first step consists in finding a solution $\tilde{A}^{\Lambda}{}_{\underline{m}}, \mathcal{I}^{\Lambda}$ of the SU(2) Bogomol'nyi equations in $\mathbb{R}^3$ Eqs. (2.19).

2.2.1 Single-center solutions of the $\mathrm{SU}(2)$ Bogomol'nyi equations in $\mathbb{R}^3$

Before we search for solutions of the Bogomol'nyi equations it is worth reviewing the origin and meaning of those equations in the context of the SU(2) Yang-Mills-Higgs theory (in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit in which the Higgs potential vanishes).

The $\mathrm{SU}(2)$ Yang-Mills-Higgs system

With the normalization in Eq. (2.11) and writing $F \equiv F^a T_a, \Phi \equiv \Phi^a T_a$, the action of the YMH theory in our conventions reads

$$S_{\mathrm{YMH}} = -2\int d^4x \, \mathrm{Tr}\left\{\tfrac{1}{2}\mathfrak{D}_\mu\Phi\mathfrak{D}^\mu\Phi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right\}\,, \tag{2.28}$$

and the corresponding equations of motion are

$$\mathfrak{D}_\mu F^{\mu\nu} = g[\Phi, \mathfrak{D}^\nu\Phi]\,, \tag{2.29}$$

$$\mathfrak{D}^2\Phi = 0\,. \tag{2.30}$$

For static configurations $F_{t\underline{m}} = \mathfrak{D}_t\Phi = 0$, the action can be written, up to a total derivative, in the manifestly positive form

$$S_{\mathrm{YMH}} = -2\int d^4x \, \mathrm{Tr}\left\{-\tfrac{1}{4}\left(F_{\underline{mn}} \mp \varepsilon_{mnp}\mathfrak{D}_{\underline{p}}\Phi\right)\left(F_{\underline{mn}} \mp \varepsilon_{mnp}\mathfrak{D}_{\underline{p}}\Phi\right)\right\}\,, \tag{2.31}$$

which leads to the conclusion that static field configurations satisfying the first-order *Bogomol'nyi equations* [42]

$$F_{\underline{mn}} = \pm\varepsilon_{mnp}\mathfrak{D}_{\underline{p}}\Phi\,, \tag{2.32}$$

extremize the action Eq. (2.28) and are solutions of the full Yang-Mills-Higgs equations. Indeed, if we act with $\mathfrak{D}_{\underline{m}}$ on both sides of the equation and use the Ricci identity and the Bogomol'nyi equation we get the Yang-Mills equation:

$$\mathfrak{D}_{\underline{m}}F_{\underline{mn}} = \mp\varepsilon_{nmp}\mathfrak{D}_{\underline{m}}\mathfrak{D}_{\underline{p}}\Phi = \mp\tfrac{1}{2}g\varepsilon_{nmp}[F_{\underline{mp}}, \Phi] = -g[\mathfrak{D}_{\underline{n}}\Phi, \Phi]\,. \tag{2.33}$$

If, instead, we act with $\varepsilon_{pmn}\mathfrak{D}_{\underline{p}}$ and use the Bianchi identity, we get the Higgs equation:

$$0 = \varepsilon_{pmn}\mathfrak{D}_{\underline{p}}F_{\underline{mn}} = \pm\mathfrak{D}_{\underline{p}}\mathfrak{D}_{\underline{p}}\Phi\,. \tag{2.34}$$

Observe that the source of the Yang-Mills field, the *Higgs current* $g[\Phi, \mathfrak{D}\Phi]$, not only vanishes when the Higgs field is covariantly constant $\mathfrak{D}\Phi = 0$ but also when Φ and $\mathfrak{D}\Phi$ are parallel in $\mathfrak{su}(2)$.

Eqs. (A.38) are identical to the ones that arise in $\mathcal{N}=2$, $d=4$ SEYM theory, (2.19) upon the identification of the vector fields and

$$\tfrac{1}{\sqrt{2}}\mathcal{I}^a = \mp\Phi^a\,. \tag{2.35}$$

The hedgehog ansatz

In order to construct static, single-center black-hole-type solutions, it is natural to look for spherically symmetric solutions of Eqs. (A.38). Substituting the *hedgehog ansatz*

$$\mp\,\Phi^a = \delta^a{}_m f(r)x^m\,, \qquad A^a{}_{\underline{m}} = -\varepsilon^a{}_{mn}x^n h(r) \tag{2.36}$$

in the Bogomol'nyi Eqs. (A.38) we get an equivalent system of differential equations for $f(r)$ and $h(r)$:

$$\begin{aligned} r\partial_r h + 2h - f(1+gr^2h) &= 0\,,\\ r\partial_r(h+f) - gr^2h(h+f) &= 0\,. \end{aligned} \tag{2.37}$$

After Prasad and Sommerfield [181] found the solution describing the 't Hooft-Polyakov monopole in the BPS limit, Protogenov [182] classified all spherically symmetric solutions to the SU(2) Bogomol'nyi equations: the ones that can be used to generate BH-like spacetimes are a 2-parameter family $(f_{\mu,s}, h_{\mu,s})$ plus a 1-parameter family (f_λ, h_λ) given by

$$\begin{aligned} rf_{\mu,s} &= \frac{1}{gr}\left[1-\mu r\coth(\mu r+s)\right], & rh_{\mu,s} &= \frac{1}{gr}\left[\frac{\mu r}{\sinh(\mu r+s)}-1\right],\\ rf_\lambda &= \frac{1}{gr}\left[\frac{1}{1+\lambda^2 r}\right], & rh_\lambda &= -rf_\lambda\,. \end{aligned} \tag{2.38}$$

The parameter s is known in the black-hole context as the *Protogenov hair* parameter [154]. The BPS 't Hooft-Polyakov monopole [181] is the only globally regular solution of this family (which explains why it is the only one usually considered in the monopole literature[11]) and corresponds to $s=0$. In the $s\to\infty$ limit we get

$$-\,rf_{\mu,\infty} = \frac{\mu}{g} - \frac{1}{gr}, \qquad rh_{\mu,\infty} = -\frac{1}{gr}\,, \tag{2.39}$$

which, for $\mu=0$, coincides with the Wu-Yang monopole [216] given in Eq. (A.15), and is a solution of the pure Yang-Mills theory. This is possible because the Higgs current $g[\Phi,\mathfrak{D}\Phi]$ vanishes even though Φ is neither zero nor covariantly constant[12]. With a non-trivial Higgs field, though, we can assign a well-defined monopole charge to it: for any μ and s

$$\frac{1}{4\pi}\int_{S^2_\infty}\mathrm{Tr}(\hat{\Phi}F) = \frac{1}{g}\,, \qquad \hat{\Phi} \equiv \frac{\Phi}{\sqrt{|\mathrm{Tr}(\Phi^2)|}}\,. \tag{2.40}$$

The same field configuration can be seen as a Lorentzian meron (see Appendix A.1) and as a solution to the Skyrme model (see Appendix A.3), and, crucially, it is related

[11] After coupling the system to gravity, the singularities of the other solutions may become "harmless" if they can be covered by regular event horizons.

[12] Actually, the only field configuration in this ansatz with a vanishing Higgs current is this one.

to the SU(2)-embedded Dirac monopole by a singular gauge transformation (see Appendix A.2). Since the metric is oblivious to gauge transformations, singular or not, the Wu-Yang monopole gives rise to solutions whose metric is identical to that of Abelian case.[13] The same applies to the higher-charge generalizations of the Lorentzian meron/Wu-Yang monopole reviewed in Appendix A.4.

If fact, this mechanism can be used to generate Wu-Yang monopoles of higher charge from the well-known Dirac monopole solutions of charge higher than 1 embedded in SU(2), as reviewed in Appendix A.4. The metric cannot see the difference between the non-Abelian and the Abelian fields given in Eq. (A.42).

The 1-parameter family is singular for all values of the parameter λ, which also appears in black-hole solutions as hair. The magnetic charge measured at spatial infinity vanishes according to the above definition. However, it can be argued that these solutions do describe a magnetic monopole placed at the origin whose charge is screened: the entropy of black hole associated to this field has the same form as that of the black hole associated to the Wu-Yang monopole. Observe that, for $\lambda = 0$, the solution is identical to the Wu-Yang monopole with $\mu = 0$, Eqs. (A.42).

The Protogenov trick

As it turns out, many regular monopole solutions can be deformed by adding a parameter s to the argument μr, generating a family of solutions that contains the original one ($s = 0$) and, typically, a new and simpler solution in the $s \to \infty$ limit. We will refer to this procedure as *the Protogenov trick* and it can be justified as follows: let us consider, for instance, the 't Hooft-Polyakov monopole. Since the Bogomol'nyi equation is polynomial, having elementary functions such as hyperbolic functions in the solution means that they must cancel amongst themselves and that only their derivatives contribute to the polynomial part of the solution. This means that one should be able to deform the dependency of the elementary functions introducing a shift s of the radial coordinate and still solve the Bogomol'nyi equations.

Of course, the cancellations necessary for having a regular solution will not work out anymore (assuming they did work for $s = 0$) and one will end up with a family of singular solutions. We will use this trick later.

2.2.2 Embedding in the SU(2)-gauged $\overline{\mathbb{CP}}^3$ model

The $\overline{\mathbb{CP}}^3$ model

As we already explained, the $\overline{\mathbb{CP}}^n$ models have n vector supermultiplets and are defined by the quadratic prepotentials

$$\mathcal{F} = -\tfrac{i}{4}\eta_{\Lambda\Sigma}\mathcal{X}^\Lambda\mathcal{X}^\Sigma\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+-\cdots-)\,. \tag{2.41}$$

The n physical scalar fields can be defined as

$$Z^i \equiv \mathcal{X}^i/\mathcal{X}^0\,, \tag{2.42}$$

[13] Of course there are measurable differences between these two situations, see *e.g.* Refs. [53, 113].

and they parametrize the symmetric space $\mathrm{U}(1,n)/(\mathrm{U}(1)\times\mathrm{U}(n))$. It is convenient to define $Z^0 \equiv 1$, $Z^\Lambda \equiv \mathcal{X}^\Lambda/\mathcal{X}^0$ and $Z_\Lambda \equiv \eta_{\Lambda\Sigma}Z^\Sigma$. In the $\mathcal{X}^0=1$ gauge, the Kähler potential and the Kähler metric are given by

$$\mathcal{K} = -\log\left(Z^{*\Lambda}Z_\Lambda\right), \quad \mathcal{G}_{ij^*} = -e^{\mathcal{K}}\left(\eta_{ij^*} - e^{\mathcal{K}}Z^*_i Z_{j^*}\right), \quad \Rightarrow \quad 0 \leq \sum_i |Z^i|^2 < 1\,. \tag{2.43}$$

The above metric is the standard (Bergman) metric for the $\mathrm{U}(1,n)/(\mathrm{U}(1)\times\mathrm{U}(n))$ symmetric spaces [37]. The covariantly holomorphic symplectic section $\mathcal{V}$ and the period matrix $\mathcal{N}_{\Lambda\Sigma}$ are given by

$$\mathcal{V} = e^{\mathcal{K}/2}\begin{pmatrix} Z^\Lambda \\ \\ -\frac{i}{2}Z_\Lambda \end{pmatrix}, \qquad \mathcal{N}_{\Lambda\Sigma} = \tfrac{i}{2}\left[\eta_{\Lambda\Sigma} - 2\frac{Z_\Lambda Z_\Sigma}{Z^\Gamma Z_\Gamma}\right]. \tag{2.44}$$

The isometry subgroup $\mathrm{SU}(1,n)$ acts linearly, in the fundamental representation, on the coordinates $\mathcal{X}^\Lambda$

$$\mathcal{X}'^\Lambda = \Lambda^\Lambda{}_\Sigma\mathcal{X}^\Sigma\,, \quad \text{with} \quad \Lambda^\dagger\eta\Lambda = \eta\,, \quad \text{and} \quad \det\Lambda = 1\,. \tag{2.45}$$

This linear action induces a non-linear action on the special coordinates:

$$Z'^\Lambda = \frac{\Lambda^\Lambda{}_\Sigma Z^\Sigma}{\Lambda^0{}_\Sigma Z^\Sigma}\,. \tag{2.46}$$

The Kähler potential is invariant under these transformations up to Kähler transformations $\mathcal{K}' = \mathcal{K} + f + f^*$ with

$$f(Z) = \log\left(\Lambda^0{}_\Sigma Z^\Sigma\right). \tag{2.47}$$

The $n(n+2)$ infinitesimal generators T_m of $\mathfrak{su}(1,n)$ in the fundamental representation are defined by

$$\Lambda^\Lambda{}_\Sigma \sim \delta^\Lambda{}_\Sigma + \alpha^m\, T_m{}^\Lambda{}_\Sigma\,, \quad \text{with} \quad \eta T^\dagger_m\eta = -T_m\,, \quad \text{and} \quad T_m{}^\Lambda{}_\Lambda = 0\,. \tag{2.48}$$

Substituting this definition into Eq. (2.46) we find an expression for the holomorphic Killing vectors[14].

$$Z'^\Lambda = Z^\Lambda + \alpha^m k_m{}^\Lambda(Z)\,, \qquad k_m{}^\Lambda(Z) = T_m{}^\Lambda{}_\Sigma\, Z^\Sigma - T_m{}^0{}_\Omega\, Z^\Omega Z^\Lambda\,, \tag{2.49}$$

and, from this expression, we also find explicit expressions for the holomorphic functions $\lambda_m(Z)$ and the momentum maps

$$\lambda_m = T_m{}^0{}_\Sigma Z^\Sigma\,, \qquad \mathcal{P}_m = ie^{\mathcal{K}}T_m{}^\Lambda{}_\Sigma Z^\Sigma Z^*_\Lambda = ie^{\mathcal{K}}\eta_{\Lambda\Omega}T_m{}^\Lambda{}_\Sigma Z^\Sigma Z^{*\Omega}\,. \tag{2.50}$$

Although the theory is invariant under the whole $\mathrm{SU}(1,n)$ group, the prepotential is invariant only under the subgroup of $\mathrm{SU}(1,n)$ with real matrices, $\mathrm{SO}(1,n)$, which is the largest group that we could eventually gauge. However, the requirements that the vectors must transform in the adjoint representation restricts the possibilities to either $\mathrm{SO}(1,2)$ or $\mathrm{SO}(3)$ (if $n\geq 2$ or $n\geq 3$, respectively); we are going to consider the latter case embedded into the minimal model admitting this gauge group, namely $\overline{\mathbb{CP}}^3$.

[14]The $k_m{}^0(Z)$ component vanishes identically, as it must, but it is convenient to keep it.

In this model, the adjoint indices $a, b, c, \ldots$ and the fundamental indices $i, j, k, \ldots$ take the same values $1, 2, 3$ and we will only use the latter. The infinitesimal transformations of the scalars are

$$\delta_\alpha Z^i = \alpha^j T_j{}^i{}_k Z^k \,, \quad \text{where} T_j{}^i{}_k = f_{jk}{}^i = -\epsilon_{jki} \,, \tag{2.51}$$

and the momentum maps, holomorphic Killing vectors etc. take the values

$$\mathcal{P}_i = -ie^{\mathcal{K}} \epsilon_{ijk} Z^j Z^{*\,k} \,, \qquad k_i{}^j = \epsilon_{ijk} Z^k \,, \qquad \lambda_i = 0 \,. \tag{2.52}$$

This means that the gauge-covariant derivative of the scalars is just that of a complex adjoint SO(3) scalar

$$\mathfrak{D}_\mu Z^i = \partial_\mu Z^i - g\epsilon_{ijk} A^j{}_\mu Z^k, \tag{2.53}$$

and that the scalar potential takes the form

$$V(Z, Z^*) \;=\; -\tfrac{1}{2} g^2 e^{\mathcal{K}} \epsilon_{ijk} \epsilon_{imn} Z^j Z^{*k^*} Z^m Z^{*n^*} \;=\; \tfrac{1}{2} g^2 \left| \vec{Z} \times \vec{Z}^* \right|^2 \,. \tag{2.54}$$

The solutions

To construct the solutions of this model[15] we just have to follow the recipe spelled out in section 5.1. We will only consider static solutions (so $\omega = 0$ and $\tilde{A}^\Lambda{}_{\underline{m}} = A^\Lambda{}_{\underline{m}}$). First of all, we need a solution of the Bogomol'nyi Eqs. (2.19). These equations split into an Abelian part (the 0th component) and the non-Abelian part (the $i = 1, 2, 3$ components):

$$F^0{}_{\underline{mn}} \;=\; -\tfrac{1}{\sqrt{2}} \epsilon_{mnp} \partial_{\underline{p}} \mathcal{I}^0 \,, \tag{2.55}$$

$$F^i{}_{\underline{mn}} \;=\; -\tfrac{1}{\sqrt{2}} \epsilon_{mnp} \mathfrak{D}_{\underline{p}} \mathcal{I}^i \,. \tag{2.56}$$

The Abelian equation is solved by

$$\mathcal{I}^0 = A^0 + \frac{p^0/\sqrt{2}}{r} \,, \tag{2.57}$$

where A^0 is an integration constant and p^0 is the normalized Abelian magnetic charge. The non-Abelian set of equations can be solved making the identification Eq. (2.35) and using Protogenov's solutions Eqs. (2.38).

The second step in the recipe (finding solutions $\mathcal{I}_\Lambda$ to Eqs. (2.20)) will be solved, for the sake of simplicity, by choosing another harmonic function in the Abelian direction and vanishing functions in the rest:

$$\mathcal{I}_0 = A_0 + \frac{q_0/\sqrt{2}}{r} \,, \qquad \mathcal{I}_i = 0 \,. \tag{2.58}$$

The third point in the recipe, combined with the staticity of the solutions implies the following constraint on the integration constants:

$$A^0 q_0 - A_0 p^0 = 0 \,. \tag{2.59}$$

[15] All these solutions have already been presented in Refs. [122, 123, 154]. We review them here for pedagogical reasons and also for the sake of making easier the comparison with the solutions of other models.

Another constraint will arise from the normalization of the metric at infinity. The solution is completely determined and, now, we only have to write the physical fields and make, if necessary, sensible choices of the values of the physical parameters to make the solutions regular.

In order to write the physical fields we need the solutions of the Freudenthal duality equations of this model. These are given by (see, *e.g.* Ref. [45])

$$(\tilde{\mathcal{I}}^M) = \begin{pmatrix} \tilde{\mathcal{I}}^\Lambda \\ \tilde{\mathcal{I}}_\Lambda \end{pmatrix} = \begin{pmatrix} -2\eta^{\Lambda\Sigma}\mathcal{I}_\Sigma \\ \frac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Sigma \end{pmatrix}, \quad \Rightarrow \quad e^{-2U} = \tfrac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Lambda\mathcal{I}^\Sigma + 2\eta^{\Lambda\Sigma}\mathcal{I}_\Lambda\mathcal{I}_\Sigma\,, \tag{2.60}$$

and the metric function and the physical scalars are given by

$$e^{-2U} = \tfrac{1}{2}(\mathcal{I}^0)^2 + 2(\mathcal{I}_0)^2 - (rf)^2, \tag{2.61}$$

$$Z^i = \frac{\sqrt{2}rf}{\mathcal{I}^0 + 2i\mathcal{I}_0}\delta^i{}_m\frac{x^m}{r}. \tag{2.62}$$

At least one of the two functions $\mathcal{I}^0, \mathcal{I}_0$ must be different from zero for the metric function to be positive. Then, there are two possible cases, depending on the vanishing of the Abelian charges p^0, q_0:

I. $p^0 = q_0 = 0$ The only regular solution is the one with $s = 0$ (the 't Hooft-Polyakov monopole). In this solution, the integration constants satisfy the normalization condition

$$\tfrac{1}{2}(A^0)^2 + 2(A_0)^2 = 1 + (\mu/g)^2\,. \tag{2.63}$$

They are also related to the asymptotic values of the scalars. These cannot be constant, in general, because the scalars transform under local SU(2) transformations, but they are covariantly constant and their asymptotic values are determined by a single gauge-invariant complex parameter that we call Z_∞:[16]

$$Z^i \sim Z_\infty\delta^i{}_m\frac{x^m}{r}\,, \qquad Z_\infty \equiv \frac{\mu/g}{1+(\mu/g)^2}\left(\tfrac{1}{\sqrt{2}}A^0 - \sqrt{2}iA_0\right)\,, \qquad 0 \leq |Z_\infty|^2 < 1\,. \tag{2.64}$$

These expressions lead to the following identification of the integration constant μ in terms of the physical parameters:

$$\mu^2 = \frac{|Z_\infty|^2}{1-|Z_\infty|^2}g^2\,, \tag{2.65}$$

and to the following expression for the mass of the solution

$$M_{\text{monopole}} = \sqrt{\frac{|Z_\infty|^2}{1-|Z_\infty|^2}}\frac{1}{g}\,. \tag{2.66}$$

This asymptotically flat solution has no singularities nor horizons (one finds a Minkowski spacetime in the $r \to 0$ limit, hence zero entropy and temperature). Globally-regular solutions of this kind [60,112] are sometimes called *global monopoles*.

[16] Observe that the scalar potential of this theory, Eq. (2.54), vanishes at infinity for those solutions, which is why they are asymptotically flat.

Observe that a solution of the ungauged theory with

$$H^0 = A^0\,, \qquad H_0 = A_0\,, \qquad H^1 = A^1 + \frac{\sqrt{2}}{gr}\,, \tag{2.67}$$

in which the non-Abelian monopole is replaced by an Abelian monopole with the same charge, would have the same asymptotic behavior but it would mean having a naked singularity at some value of $r > 0$.

II. $p^0 q_0 \neq 0$ [17] Solving Eq. (2.59) the metric can be written in the form

$$e^{-2U} = \frac{1}{1-|Z_\infty|^2}H^2 - (rf)^2, \tag{2.68}$$

$$Z^i = \frac{2\beta}{p^0 + 2iq_0}\frac{rf}{H}\delta^i{}_m\frac{x^m}{r}\,, \tag{2.69}$$

where H is the harmonic function

$$H \equiv 1 + \frac{\beta}{r}\,, \qquad \beta^2 = (1-|Z_\infty|^2)W_{\rm RN}(\mathcal{Q})/2\,, \qquad W_{\rm RN}(\mathcal{Q}) \equiv \tfrac{1}{2}(p^0)^2 + 2(q_0)^2\,, \tag{2.70}$$

and the integration constant μ is still given by Eq. (2.65). We have expressed all the constants (except for Protogenov's hair parameter s and λ) in terms of physical constants. Observe that the isolated solution f_* has $\mu = 0$ and corresponds to $Z_\infty = 0$. These identifications allow us to compute the mass and entropy of all the possible solutions in terms of the physical parameters. We get a completely general mass formula and two formulae for the entropy, one for the $s \neq 0$ solutions and another one for the $s = 0$ and the isolated solutions (which corresponds to $Z_\infty = 0$):

$$M = \sqrt{\tfrac{1}{2}\frac{W_{RN}(\mathcal{Q})}{1-|Z_\infty|^2}} + M_{\rm monopole}, \tag{2.71}$$

$$S/\pi = \tfrac{1}{2}W_{\rm RN}(\mathcal{Q}) - \frac{1}{g^2}\,, \quad \text{for} \quad s \neq 0 \;\text{ and }\; Z_\infty = 0, \tag{2.72}$$

$$S/\pi = \tfrac{1}{2}W_{\rm RN}(\mathcal{Q}), \quad \text{for} \quad s = 0\,, \tag{2.73}$$

where $M_{\rm monopole}$ is given by Eq. (2.66).

The entropy is moduli-independent as in the ungauged case and both the entropy and the mass are independent of the hair parameters s and λ.

Observe that the charge of the BPS 't Hooft-Polyakov monopole $s = 0$ does not contribute to the entropy which suggests that it must be associated to a pure state in the quantum theory. The non-Abelian field of the isolated solution does not contribute to the mass at infinity ($M_{\rm monopole} = 0$ for $Z_\infty = 0$) but there is a magnetic-charge contribution to the entropy, which suggests that there really is a magnetic charge but it is screened at infinity. Further support for this interpretation comes

[17] It is easier to work with both charges non-vanishing. The results will still be valid when we set one of them to zero.

from the near-horizon limit of the scalars, which is the covariantly-constant function of the charges

$$Z^i_{\rm h} = \frac{1/g}{\frac{1}{2}p^0 + iq_0}\ \delta^i{}_m \frac{x^m}{r}. \tag{2.74}$$

even for the isolated case, when no magnetic charge is observed at infinity.

In the case of the 1-parameter (λ) family, neither the mass nor the entropy depend on λ.

Some of the solutions in this family can also be seen as solutions of the pure EYM theory. They are identical to those obtained in Refs. [53, 218]. As discussed at the end of section C.2, we need to tune the parameters of the solutions so as to get covariantly constant scalars which do not contribute to the energy-momentum tensor This is only possible for the $s \to \infty$ solutions (Wu–Yang monopoles) for which rf is a harmonic function. In that case

$$Z^i = Z\ \delta^i{}_m \frac{x^m}{r}\,, \qquad Z = \frac{1/g}{\frac{1}{2}p^0 + iq_0} = Z_\infty\,. \tag{2.75}$$

The metric is identical to that of a Reissner-Nordström black hole. These solutions were called *black hedgehogs* in Ref. [123] and *black merons* in Ref. [53] because the gauge field of the Wu–Yang monopole can also be understood as Lorentzian meron solution.

A closely related solution with non-covariantly constant scalars was obtained in a different context in Ref. [132].

2.2.3 Embedding in SU(2)-gauged ST$[2,n]$ models

The $ST[2,n]$ models

The $ST[2,n]$ models are cubic models with $n_V = n+1$ vector supermultiplets and as many complex scalars and, as all other cubic models, they can be embedded in type II String Theory compactified Calabi-Yau 3-folds and then uplifted to M-theory. They can also be obtained from corresponding models of $N=1$, $d=5$ supergravity compactified on S^1.

A generic cubic model is defined by the prepotential

$$\mathcal{F} \;=\; -\frac{1}{3!}\, d_{ijk}\, \frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0}\,, \tag{2.76}$$

where d is completely symmetric in its indices; the $ST[2,n]$ models are characterized by d-tensors with non-vanishing components $d_{1\alpha\beta} = \eta_{\alpha\beta}$ where $(\eta_{\alpha\beta}) = \mathrm{diag}(+-\cdots-)$ and where the indices α, β take n values between 2 and $n+1$.

The scalar $Z^1 = \mathcal{X}^1/\mathcal{X}^0$ plays a special role and parametrizes a $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ coset space. For this and other reasons, it is called axidilaton and we will denote it by τ. The other n scalars parametrize a $\mathrm{SO}(2,n)/(\mathrm{SO}(2)\times\mathrm{SO}(n))$ coset space and will be denoted by $Z^\alpha = \mathcal{X}^\alpha/\mathcal{X}^0$ ($\alpha = 2,\cdots,n$). The Kähler metric and 1-form connection are the products of those of the two spaces.

Using this notation and using the gauge $\mathcal{X}^0 = 1$, the canonical symplectic section Ω, the Kähler potential $\mathcal{K}$ and the components of Kähler 1-form $\mathcal{Q}_i$ and of the Kähler metric

$\mathcal{G}_{ij^*}$ are given by

$$
\Omega = \begin{pmatrix} 1 \\ \tau \\ Z^\alpha \\ \frac{1}{2}\tau\eta_{\alpha\beta}Z^\alpha Z^\beta \\ -\frac{1}{2}\eta_{\alpha\beta}Z^\alpha Z^\beta \\ -\tau\eta_{\alpha\beta}Z^\beta \end{pmatrix}, \quad e^{-\mathcal{K}} = 4\Im\mathfrak{m}\,\tau\ \eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\ \Im\mathfrak{m}\,Z^\beta,
$$

$$
\mathcal{Q}_\tau = \frac{1}{4\Im\mathfrak{m}\,\tau}, \qquad \mathcal{Q}_\alpha = \frac{\eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\beta}{2\eta_{\gamma\delta}\Im\mathfrak{m}\,Z^\gamma\ \Im\mathfrak{m}\,Z^\delta},
$$

$$
\mathcal{G}_{\tau\tau^*} = \frac{1}{4(\Im\mathfrak{m}\,\tau)^2}, \qquad \mathcal{G}_{\alpha\beta^*} = \frac{\eta_{\alpha\gamma}\Im\mathfrak{m}\,Z^\gamma\ \eta_{\beta\delta}\Im\mathfrak{m}\,Z^\delta}{[\eta_{\epsilon\varphi}\Im\mathfrak{m}\,Z^\epsilon\ \Im\mathfrak{m}\,Z^\varphi]^2} - \frac{\eta_{\alpha\beta}}{2\eta_{\epsilon\varphi}\Im\mathfrak{m}\,Z^\epsilon\ \Im\mathfrak{m}\,Z^\varphi}. \tag{2.77}
$$

The reality of the Kähler potential constrains the values of the scalars. The model has two branches characterized by

$$
\Im\mathfrak{m}\,\tau > 0\,, \qquad \eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\ \Im\mathfrak{m}\,Z^\beta > 0\,, \tag{2.78}
$$

and

$$
\Im\mathfrak{m}\,\tau\ < 0\,, \qquad \eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\ \Im\mathfrak{m}\,Z^\beta < 0\,, \tag{2.79}
$$

that will be distinguished where required by $+$ and $-$ indices, respectively.

Only the subgroup $\mathrm{SO}(1,n)\subset\mathrm{SO}(2,n)$ acts linearly (in the fundamental representation) on the special coordinates Z^α and the group SO(3) acts in the adjoint (for instance) on the coordinates $\alpha=3,4,5$ if $n\geq 4$. We take $n=4$ for simplicity and denote the $\alpha=3,4,5$ indices by $a,b,\dots=1,2,3$. For the sake of simplicity we will write Z^a instead of Z^{a+2} for Z^3,Z^4,Z^5 etc. The generators and structure constants of $\mathfrak{so}(3)$ and their action on the scalars are the same as in the $\overline{\mathbb{CP}}^3$ model with obvious changes of notation:

$$
(T_a)^b{}_c = f_{ac}{}^b = -\varepsilon_{acb}\,, \qquad \delta_\alpha Z^a = \alpha^b (T_b)^a{}_c Z^c = -\epsilon_{abc}\alpha^b Z^c = \alpha^b k_b{}^a(Z)\,, \tag{2.80}
$$

(τ and Z^2 are inert) so the holomorphic Killing vectors and the momentum maps are

$$
k_a{}^b(Z) = \epsilon_{abc}Z^c\,, \qquad \mathcal{P}_a = -\tfrac{i}{2}\frac{\epsilon_{abc}Z^b Z^{*\,c^*}}{\eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\ \Im\mathfrak{m}\,Z^\beta}\,. \tag{2.81}
$$

The scalar potential has a structure similar to that of the $\overline{\mathbb{CP}}^3$ model, but more complicated. We will not give it here since it is not needed anyway.

The solutions

To find solutions to this non-Abelian model we just need to follow the recipe. First, we find the functions $\mathcal{I}^\Lambda$ and the spatial components of the vector fields $A^\Lambda{}_{\underline{m}}$ by solving the Bogomol'nyi equations

$$
F^\Lambda{}_{\underline{mn}} = -\tfrac{1}{\sqrt{2}}\epsilon_{mnp}\partial_{\underline{p}}\mathcal{I}^\Lambda\,, \qquad I=0,1,2, \tag{2.82}
$$

$$
F^{a+2}{}_{\underline{mn}} = -\tfrac{1}{\sqrt{2}}\epsilon_{mnp}\mathfrak{D}_{\underline{p}}\mathcal{I}^{a+2}\,, \qquad a=1,2,3, \tag{2.83}
$$

(we will suppress the +2 in the non-Abelian indices in most places). The Abelian equations are solved by harmonic functions and the non-Abelian ones by making the identification Eq. (2.35) with the Higgs field and using Protogenov's solutions Eqs. (2.38), as we did in the $\overline{\mathbb{CP}}^3$ model.

Next, we have to find the functions $\mathcal{I}_\Lambda$ by solving Eqs. (2.20). In the Abelian directions $\Lambda = 0, 1, 2$ we can simply choose harmonic functions and in the non-Abelian ones we take $\mathcal{I}_a = 0$. This choice gives non-singular solutions, as we are going to see. We will also set some of the harmonic functions to zero for simplicity.

The Hesse potential defined in Eq. (2.22) can be found from Shmakova's solution of the stabilization (or Freudenthal duality) equations for cubic models [189]; it can be written as

$$\mathsf{W}(\mathcal{I}) = 2\sqrt{J_4(\mathcal{I})}\,, \tag{2.84}$$

with the quartic invariant $J_4(\mathcal{I})$ given by

$$J_4(\mathcal{I}) \equiv (\mathcal{I}^\alpha\mathcal{I}^\beta\eta_{\alpha\beta} + 2\mathcal{I}^0\mathcal{I}_1)(\mathcal{I}_\alpha\mathcal{I}_\beta\eta^{\alpha\beta} - 2\mathcal{I}^1\mathcal{I}_0) - (\mathcal{I}^0\mathcal{I}_0 - \mathcal{I}^1\mathcal{I}_1 + \mathcal{I}^\alpha\mathcal{I}_\alpha)^2\,. \tag{2.85}$$

This potential does not vanish for the choice $\mathcal{I}_a = 0$, as we advanced and it will remain non-singular if we set $\mathcal{I}^0 = \mathcal{I}_1 = \mathcal{I}_2 = 0$. In other words: the only non-trivial components of $\mathcal{I}^M$ are $\mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^{a+2}, \mathcal{I}_0$. With this choice the metric function is given by

$$e^{-2U} = \mathsf{W}(\mathcal{I}) = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0\,\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta} = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0[(\mathcal{I}^2)^2 - \mathcal{I}^a\mathcal{I}^a]}\,. \tag{2.86}$$

As instructed by the recipe in section (5.1), we can calculate the $\tilde{\mathcal{I}}$ from Eq. (2.22), which for our choice of non-trivial components of $\mathcal{I}^M$ means that $\tilde{\mathcal{I}}^i = 0$ $(i = 1, \cdots, 5)$; this implies that all the scalars are purely imaginary and given by

$$Z^i = i\frac{\mathcal{I}^i}{\tilde{\mathcal{I}}^0}\,, \qquad \text{where} \quad \tilde{\mathcal{I}}^0 = \frac{2\mathcal{I}^1\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta}{\mathsf{W}(\mathcal{I})}\,. \tag{2.87}$$

It is convenient to write all of them in terms of $\tau = Z^1$

$$Z^\alpha = \frac{\mathcal{I}^\alpha}{\mathcal{I}^1}\tau\,, \qquad \tau = i\frac{e^{-2U}}{2\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta}\,. \tag{2.88}$$

In the two (+ and −) branches of the model corresponding, respectively, to the upper and lower signs $\pm\Im\mathfrak{m}\,\tau_{(\pm)} > 0$ and, since $e^{-2U} > 0$, we must choose the functions $\mathcal{I}^\alpha_{(\pm)}$ so that

$$\pm\,\eta_{\alpha\beta}\mathcal{I}^\alpha_{(\pm)}\mathcal{I}^\beta_{(\pm)} = \pm\left[(\mathcal{I}^2_{(\pm)})^2 - \mathcal{I}^a_{(\pm)}\mathcal{I}^a_{(\pm)}\right] > 0\,. \tag{2.89}$$

In order for $\mathsf{W}(\mathcal{I})$ to be real the $\mathcal{I}_{(\pm)\,0}$ and $\mathcal{I}^1_{(\pm)}$ must be chosen so as to satisfy

$$\pm\,\mathcal{I}^1_{(\pm)}\mathcal{I}_{(\pm)\,0} < 0\,. \tag{2.90}$$

(We will suppress the ± subindices in what follows, to simplify the notation, except where this may lead to confusion.)

Observe that with our choice of non-vanishing components of $\mathcal{I}^M$ the r.h.s. of Eq. (2.21) vanishes automatically, whence the staticity condition $\omega = 0$ does not impose any constraint.

According to the preceding discussions, the non-vanishing components of $\mathcal{I}^M$ will be assumed to take the form

$$\begin{aligned} \mathcal{I}^1 &= A^1 + \frac{p^1/\sqrt{2}}{r}\,, \quad \mathcal{I}^2 = A^2 + \frac{p^2/\sqrt{2}}{r}\,, \quad \mathcal{I}^a = \sqrt{2}\,\delta^a{}_m x^m f(r)\,, \\ \mathcal{I}_0 &= A_0 + \frac{q_0/\sqrt{2}}{r}\,, \end{aligned} \tag{2.91}$$

where $f(r)$ is $f_{\mu,s}$ or f_λ in Eqs. (2.38), p^1, p^2, q_0 are magnetic and electric charges and A^1, A^2, A_0 are integration constants to be determined in terms of the asymptotic values of the scalars and the metric. These constants must have the same sign as the corresponding charges

$$\mathrm{sign}(A^{1,2}) = \mathrm{sign}(p^{1,2})\,, \qquad \mathrm{sign}(A_0) = \mathrm{sign}(q_0)\,, \tag{2.92}$$

as the functions $\mathcal{I}^1$, $\mathcal{I}^2$ and $\mathcal{I}_0$ are required to have no zeroes on the interval $r \in (0, +\infty)$ in order to avoid naked singularities there. Then, the above constraint on the signs of $\mathcal{I}^1$ and $\mathcal{I}_0$ translates into the following constraints on the signs of the charges in the two branches:

$$\mathrm{sign}(p^1)\mathrm{sign}(q_0) = \mp 1\,. \tag{2.93}$$

Defining as in the $\overline{\mathbb{CP}}^3$ case the asymptotic value Z_∞ of the adjoint scalars by

$$Z^a_\infty \equiv Z_\infty\, \delta^a{}_m \frac{x^m}{r}\,, \tag{2.94}$$

and imposing the normalization of the metric at infinity it is not hard to express the integration constants μ, A^1, A^2, A_0 in terms of the moduli (the asymptotic values of the scalars $\Im\mathfrak{m}\tau_\infty, \Im\mathfrak{m}\, Z^2_\infty$ and $\Im\mathfrak{m}\, Z_\infty$) and the coupling constant g

$$\begin{aligned} A^1 &= \frac{\mathrm{sign}(p^1)|\Im\mathfrak{m}\tau_\infty|}{\sqrt{2}\chi_\infty}\,, \\ A^2 &= \frac{\mathrm{sign}(p^2)|\Im\mathfrak{m}\, Z^2_\infty|}{\sqrt{2}\chi_\infty}\,, \\ \mu &= \frac{g|\Im\mathfrak{m}\, Z_\infty|}{2\chi_\infty}\,, \\ A_0 &= \tfrac{1}{2\sqrt{2}}\mathrm{sign}(q_0)\chi_\infty\,, \end{aligned} \tag{2.95}$$

where we have defined the combination (real in both branches of the theory)

$$\chi_\infty \equiv \sqrt{\Im\mathfrak{m}\tau_\infty\,[(\Im\mathfrak{m}\, Z^2_\infty)^2 - (\Im\mathfrak{m}\, Z_\infty)^2]}\,. \tag{2.96}$$

The mass of the solutions in terms of the moduli and the charges is

$$M = \tfrac{1}{4}\frac{\chi_\infty}{|\Im\mathfrak{m}\tau_\infty|}|p^1| + \frac{1}{2\chi_\infty}|q_0| \pm \tfrac{1}{2}\frac{|\Im\mathfrak{m}\tau_\infty \Im\mathfrak{m}\, Z^2_\infty|}{\chi_\infty}|p^2| \pm \frac{|\Im\mathfrak{m}\tau_\infty \Im\mathfrak{m}\, Z_\infty|}{\chi_\infty}\frac{1}{g}\,. \tag{2.97}$$

In the above expressions we have used two consistency conditions:

$$\mathrm{sign}(\Im\mathfrak{m}\, Z_\infty) = \mp\mathrm{sign}(p^1)\,, \qquad \mathrm{sign}(\Im\mathfrak{m}\, Z^2_\infty) = \pm\mathrm{sign}(p^1)\mathrm{sign}(p^2)\,. \tag{2.98}$$

These expressions for the integration constants and the mass are valid both for the 2- and 1-parameter families, the latter being recovered by setting $\Im\mathfrak{m}\, Z_\infty = 0$ everywhere. The contribution of the monopole charge $1/g$ to the mass disappears because it is screened.

Observe that the positivity of the mass is not guaranteed in the $-$ branch for arbitrary values of the charges and moduli: it has to be imposed by hand.

Let us now study the behavior of the solution in the near-horizon limit $r \to 0$. For $f_{\mu,s\neq 0}$ and f_λ the metric function behaves as

$$e^{-2U} \sim \sqrt{-2p^1 q_0\left[(p^2)^2 - (2/g)^2\right]}\,\frac{1}{r^2}\,, \tag{2.99}$$

which corresponds to a regular horizon in both branches. The solutions will describe regular black holes if the charges and moduli are such that $M > 0$. Observe that in the $-$ branch it is possible to chose those such that $M = 0$ with a non-vanishing entropy.

In the $f_{\mu,s=0}$ case with $p^2 \neq 0$ the solution is only well defined in the $+$ branch because there is no $1/r$ contribution from the monopole in the $r \to 0$ limit and it is impossible to satisfy the inequality $-\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta > 0$ in that limit. In this case (the $+$ branch with $p^2 \neq 0$) we have

$$e^{-2U} \sim \sqrt{-2p^1 q_0 (p^2)^2}\,\frac{1}{r^2}\,, \tag{2.100}$$

which corresponds to a regular horizon.

In the $f_{\mu,s=0}$ case with $p^2 = 0$ there are two possibilities:

1. We can set $p^1 = q_0 = 0$. Then, in the $r \to 0$ limit, e^{-2U} is the moduli-dependent constant $2\sqrt{-2A^1 A_0 (A^2)^2}$. There is neither horizon nor singularity and the solution, which is a global monopole, belongs to the $+$ branch (this also guarantees that the mass is positive).

2. We can keep both $p^1 \neq 0$ and $q_0 \neq 0$, setting $A^2 = 0$ and profit from the fact that, in this limit $\Phi^a\Phi^a$ goes to zero as r^2. The solution is only well defined in the $-$ branch. The metric function takes the constant value

$$e^{-2U} \sim \sqrt{+p^1 q_0 \frac{\mu^4}{g^2}}\,, \tag{2.101}$$

We have, as far as the metric is concerned, a global monopole solution (as long as $M > 0$), but since we need two Abelian charges switched on, namely p^1 and q_0, the scalar fields and the gauge fields are singular at $r = 0$. As before, it is possible to tune the moduli and charges so that $M = 0$.

The near-horizon limits of the scalars are, in the $f_{\mu,s\neq 0}$ and f_λ cases

$$\begin{aligned}
\Im\mathfrak{m}\tau_{\rm h} &= \frac{\sqrt{-2p^1 q_0\left[(p^2)^2 - (2/g)^2\right]}}{2\left[(p^2)^2 - (2/g)^2\right]}\,,\\
\Im\mathfrak{m}\, Z^2_{\rm h} &= \frac{p^2}{p^1}\Im\mathfrak{m}\tau_{\rm h}\,,\\
\Im\mathfrak{m}\, Z^a_{\rm h} &= \frac{2\Im\mathfrak{m}\tau_{\rm h}}{g p^1}\delta^a{}_m \frac{x^m}{r}\,,
\end{aligned} \tag{2.102}$$

and, in the $f_{\mu,s=0}$ case with $p^2 \neq 0$, we get the same results up to the contribution of the monopole which disappears (formally, $1/g = 0$).

2.2.4 Embedding in pure SU(2) EYM

The scalars can only be trivialized for the Wu-Yang monopole $s = \infty$. In that case, it is easy to construct a double-extremal black hole with constant scalars and the metric is, as usual, Reissner-Nordström's.

2.3 Multi-center SBHSs

To construct multi-center SBHSs we can use the same recipe as in the single-center case but we need multi-center solutions of the Bogomol'nyi equations. We start by discussing these.

2.3.1 Multi-center solutions of the SU(2) Bogomol'nyi equations on $\mathbb{R}^3$

In the Abelian case, the multicenter solutions of the Bogomol'nyi equations are associated to harmonic functions with isolated point-like singularities. They are the seed solutions of the multi-black-hole solutions of the Einstein-Maxwell theory [65, 111, 126, 147, 173, 176] and $\mathcal{N}=2$, $d=4$ supergravities [13, 15, 22, 72]. In the non-Abelian case, the hedgehog ansatz is clearly inappropriate and more sophisticated methods need to be used. Only a few explicit solutions are known, even though solutions describing several BPS objects in equilibrium are, on general grounds, expected to exist. For instance, there is no explicit solution describing two BPS 't Hooft-Polyakov monopoles in equilibrium (see however Ref. [172]).

Perhaps not surprisingly, the only general families of explicit solutions available involve an arbitrary number of Wu-Yang or Dirac monopoles embedded in SU(2). The simplest of these only involve Wu-Yang monopoles and formally, it can be obtained from solutions describing Dirac monopoles embedded in SU(2) via singular gauge transformations [180], generalizing the constructions reviewed in Appendices A.2 (minimal charge) and A.4 (higher charge). As we have explained at length in the preceding sections, the metric is completely oblivious to these gauge transformations and takes the same form as in the Abelian cases. We will not study such solutions in this section.

In Refs. [62], using the Nahm equations [164], Cherkis and Durcan found new solutions describing one or two, charge 1, Wu-Yang monopoles embedded in SU(2) in the background of a single BPS 't Hooft-Polyakov monopole.[18] We are going to use the first of them to construct multi-center solutions of the $\overline{\mathbb{CP}}^3$ and $ST[2,4]$ models of $\mathcal{N}=2$, $d=4$ SEYM. Let us review the Cherkis-Durcan solution first: take the BPS 't Hooft-Polyakov monopole to be located at $x^n = x^n_0$ and the Wu-Yang monopole at $x^m = x^m_1$. We define

[18] In Ref. [41] Blair and Cherkis generated a solution describing an arbitrary number of charge 1 Wu-Yang monopoles in the presence of an 't Hooft-Polyakov monopole; one can easily generalize this solution to one describing an arbitrary number of charge $n(>0)$ Wu-Yang monopoles in the background of an 't Hooft-Polyakov monopole, by coalescing n charge 1 Wu-Yang monopoles. Needless to say, the Protogenov trick works as expected. For the sake of simplicity of exposition, we will not consider this more general solution in this article.

the coordinates relative to each of those centers and the relative position by

$$r^m \equiv x^m - x_0^m\,, \qquad u^m \equiv x^m - x_1^m\,, \qquad d^m \equiv u^m - r^m = x_0^m - x_1^m\,, \tag{2.103}$$

and their norms by respectively, r, u and d. The Higgs field and gauge potential of this solution (adapted to our conventions) are given by [62]

$$\pm\Phi^a = \frac{1}{g}\delta^a{}_m \left\{ \left[\frac{1}{r} - \left(\mu + \frac{1}{u}\right)\frac{K}{L}\right]\frac{r^m}{r} + \frac{2r}{uL}\left(\delta^{mn} - \frac{r^m r^n}{r^2}\right) d^n \right\}\,, \tag{2.104}$$

$$\begin{aligned} A^a &= -\frac{1}{g}\left[\frac{1}{r} - \frac{\mu \mathrm{D} + 2d + 2u}{\mathrm{L}}\right] \frac{\varepsilon^a{}_{mn} r^m dx^n}{r} + 2\frac{\mathrm{K}}{\mathrm{L}}\frac{\varepsilon_{npq} d^n u^p dx^q}{u\mathrm{D}}\,\delta^a{}_m \frac{r^m}{r} \\ &\quad - \frac{2r}{u\mathrm{L}}\,\delta^a{}_m \left(\delta^{mn} - \frac{r^m r^n}{r^2}\right)\varepsilon_{npq} u^p dx^q\,, \end{aligned} \tag{2.105}$$

where the functions K, L, D of u and r are defined by

$$K \equiv \left[(u+d)^2 + r^2\right]\cosh \mu r + 2r(u+d)\sinh \mu r\,, \tag{2.106}$$

$$L \equiv \left[(u+d)^2 + r^2\right]\sinh \mu r + 2r(u+d)\cosh \mu r\,, \tag{2.107}$$

$$\mathrm{D} = 2\left(ud + u^m d^m\right) = (d+u)^2 - r^2\,. \tag{2.108}$$

The function D is clearly zero along the direction[19] $u^m/u = -d^m/d$ signaling the possible presence of a Dirac string in Eq. (2.105); that this is however not the case is demonstrated in Ref. [41].

In the models that we are going to study, the Higgs field enters the metric in the combination $\Phi^a\Phi^a$, which takes the value

$$\Phi^a\Phi^a = \frac{1}{g^2}\left\{\left[\frac{1}{r} - \left(\mu + \frac{1}{u}\right)\frac{K}{L}\right]^2 + \frac{4|\vec{r}\times\vec{d}|^2}{u^2 L^2}\right\}\,. \tag{2.109}$$

To better understand this solution one will consider several limits:

1. The limit in which we take the BPS 't Hooft-Polyakov anti-monopole infinitely far away, keeping the Dirac monopole at x_1^m: in this limit $d \to \infty$, $r^m \sim -d^m$ while u remains finite. The Higgs and gauge fields take the form

$$\pm\Phi^a \sim -\frac{1}{g}\delta^a{}_m\left(\mu + \frac{1}{u}\right)\frac{d^m}{d}\,, \tag{2.110}$$

$$A^a \sim -\frac{1}{g}\left(1 + \frac{d^m}{d}\frac{u^m}{u}\right)^{-1}\varepsilon_{mnp}\frac{d^m}{d}\frac{u^m}{u}d\frac{u^p}{u}\,. \tag{2.111}$$

The gauge field should be compared with the embedding of a Dirac monopole with a string in the direction $-d^m$ into the direction $\delta^a{}_m d^m T^a$ of the gauge group, Eqs. (A.11) and (A.17) with $s^m = -d^m$.

[19] This is the half of the line that joins $r=0$ to $u=0$ that stretches from the Dirac monopole $u=0$ to infinity in the direction opposite to the 't Hooft-Polyakov monopole at $r=0$

2. The limit in which we take the Dirac monopole infinitely away, keeping the BPS 't Hooft-Polyakov anti-monopole at x_0^m: In this limit $d \to \infty$, $u^m \sim d^m$ while r remains finite. The Higgs and gauge fields become those of a single BPS 't Hooft-Polyakov anti-monopole at x_0^m.

3. In the limit in which we are infinitely far away from both monopoles ($r \to \infty$, $u \to \infty$), which remain at a finite relative distance, the Higgs and gauge fields take the form

$$\pm\Phi^a = -\left[\frac{\mu}{g} + \mathcal{O}(|x|^{-2})\right]\delta^a{}_m \frac{x^m}{|x|}\,, \tag{2.112}$$

$$A^a = -\frac{1}{g}\varepsilon^a{}_{mn}\frac{x^m dx^n}{|x|^2} + \frac{1}{2g}\delta^a{}_m \frac{x^m}{|x|}\left(\frac{\varepsilon_{npq} d^n x^p dx^q}{|x|^2}\right)\,. \tag{2.113}$$

   The first term in the gauge potential is identical to that of a Wu-Yang anti-monopole (compare with Eq. (A.2)). This is also the asymptotic behavior of the BPS 't Hooft-Polyakov monopole. The Higgs field is asymptotically covariantly constant and, in particular

$$\Phi^a\Phi^a \sim \frac{\mu^2}{g^2} + \mathcal{O}(\frac{1}{|x|^2})\,. \tag{2.114}$$

4. The limit in which we approach the center of the BPS 't Hooft-Polyakov anti-monopole $r^m \to 0$, $u^m \to d^m$

$$\Phi^a\Phi^a \sim \frac{1}{4g^2 d^2(1+\mu d)^2} + \mathcal{O}(r)\,. \tag{2.115}$$

   This limit is finite and only vanishes when the Dirac monopole is taken to infinity $d \to \infty$.

   For finite values of d, Eq. (2.109) says that $\Phi^a\Phi^a$ can only vanish along the line that stretches from $r=0$ to $u=0$ so $\vec{r}\times\vec{d}=0$. Substituting $r^m = \alpha d^m$ in $\Phi^a\Phi^a$ we get a function of α and of the parameter μd. Plotting the functions of α for different values of μd we find that they have a single zero, which is also a local minimum. At this minimum the second derivative does not vanish, and therefore, there, $\Phi^a\Phi^a \sim \mathcal{O}(r^2)$, as in the single-monopole case. However, the value of this second derivative depends on the direction.

5. The limit in which we approach the singularity of the Dirac monopole $u^m \to 0$, $r^m \to -d^m$

$$\Phi^a\Phi^a \to \frac{1}{g^2}\left\{\frac{1}{u^2} + \left(\frac{1}{d} - \mu\right)\frac{1}{u}\right\} + \mathcal{O}(1)\,. \tag{2.116}$$

Growing Protogenov hair

As we have argued in section (2.2.1) we can add a Protogenov hair parameter s to the Cherkis & Durcan solution by simply replacing the argument μr of the hyperbolic sines and cosines in the functions K and L by the shifted on $\mu r + s$. We do not need to write explicitly the solution, but we do need to reconsider the different limits studied for the $s = 0$ case:

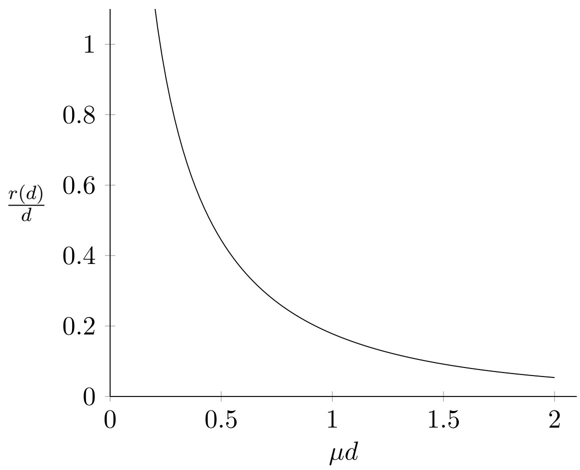

Figure 2.1: The zeros of the Higgs density as measured by r as a function of the dimensionless separation μd.

1. In the limit in which we take the BPS 't Hooft-Polyakov-Protogenov anti-monopole infinitely away, keeping the Dirac monopole at x_1^m the Higgs and gauge fields become, to leading order, those of the Dirac monopole with the Dirac string in the direction $-d^m$, as in the $s=0$ case (See Eqs. (2.110) and (2.105)).

2. In the limit in which we take the Dirac monopole infinitely away, keeping the BPS 't Hooft-Polyakov-Protogenov anti-monopole at x_0^m the Higgs and gauge fields become those of a single BPS 't Hooft-Polyakov-Protogenov anti-monopole at $x^m = x_0^m$ (the first two equations (2.38)).

3. In the limit in which we are infinitely far away from both monopoles ($r \to \infty$, $u \to \infty$), which remain at a finite relative distance, the Higgs and gauge fields take the same form as in the $s=0$ case, Eqs. (2.112-2.114).

4. The limit in which we approach the singularity of the BPS 't Hooft-Polyakov-Protogenov anti-monopole $r^m \to 0$, $u^m \to d^m$ (for $s \neq 0$)

$$\pm\Phi^a \sim \frac{1}{g}\delta^a{}_m \left[\frac{1}{r} - \left(\mu + \frac{1}{d}\right)\coth s + \mathcal{O}(r)\right]\frac{r^m}{r}\,, \tag{2.117}$$

$$\Rightarrow \Phi^a\Phi^a \sim \frac{1}{g^2 r^2} + \mathcal{O}\left(\frac{1}{r}\right)\,, \tag{2.118}$$

which is similar to the behaviour near the Dirac monopole as in Eq. (2.116) (with u replaced by r).

5. The limit in which we approach the singularity of the Dirac monopole $u^m \to 0$, $r^m \to -d^m$ we have the same behavior as in the $s=0$ case Eq. (2.116).

The solutions with Protogenov hair have another limit, namely the one in which $s \to \infty$; this case will be studied separately.

The $s \to \infty$ limit solution

In this limit we get a solution that describes the same Dirac monopole together with a ($\mu \neq 0$) Wu-Yang anti-monopole:[20]

$$\pm\Phi^a = \frac{1}{g}\delta^a{}_m \left[-\mu + \frac{1}{r} - \frac{1}{u}\right]\frac{r^m}{r}\,, \tag{2.119}$$

$$A^a = \frac{1}{g}\frac{\varepsilon^a{}_{mn} r^m dx^n}{r^2} + \frac{1}{g}\frac{\varepsilon_{npq} d^n u^p du^q}{u(ud + u^r d^r)}\delta^a{}_m \frac{r^m}{r}\,. \tag{2.120}$$

This solution is a particular example of a more general family describing an arbitrary number of Dirac monopoles in the background of a Wu-Yang anti-monopole. These solutions can be obtained from a solution describing only Dirac monopoles embedded in SU(2) via a singular gauge transformation that only removes the Dirac string of one of them, which becomes the Wu-Yang anti-monopole. The general family of solutions can be written in the form:

$$\Phi = \Phi_{\rm WY} + HU\,, \qquad A = A_{\rm WY} + CU\,, \tag{2.121}$$

where U is the SU(2) (and $\mathfrak{su}(2)$) matrix defined in Eq. (A.1) and where $\Phi_{\rm WY}$ and $A_{\rm WY}$ are the Higgs and Yang-Mills fields of a Wu-Yang monopole, given, respectively, by

$$\mp\Phi_{\rm WY} = \frac{1}{2g}\left[-\mu + \frac{1}{r}\right]U\,, \tag{2.122}$$

and by Eq. (A.2) and where H is a function and C a 1-form on $\mathbb{R}^3$. If we substitute into the Bogomol'nyi equations (A.38) and use, on the one hand, that they are satisfied by the pair $A_{\rm WY}, \Phi_{\rm WY}$, and, on the other hand, that U is covariantly constant with the connection $A_{\rm WY}$ we arrive at the Dirac monopole equation

$$dC = \star_{(3)} dH\,. \tag{2.123}$$

The integrability condition of this equation is $d\star_{(3)} dH = 0$ so H is any harmonic function. We can choose it to have isolated poles at the points $x^m = x^m_i$ $i = 1, \cdots, N$

$$H = \sum_i \frac{p_i}{2u_i}\,, \qquad u^m_i \equiv x^m - x^m_i\,, \tag{2.124}$$

in which case C is the 1-form potential of N Dirac monopoles with charges p_i which can be constructed by summing over the potentials of each individual monopole:

$$C = \sum C_i\,, \qquad dC_i = \star_{(3)} d\frac{p_i}{2u_i}\,. \tag{2.125}$$

The expression for each of the C_i is of the form Eq. (A.11) where we can, in principle, choose the direction s^m_i of each Dirac string independently:

[20] One can see fairly easily that in the limiting solution one can, as far as the Bogomol'nyi equations are concerned, allow for μ to be negative; for finite values of s this is impossible.

$$C_i = \frac{p_i}{2}\left(1 - \frac{s_i^m}{s_i}\frac{u_i^m}{u_i}\right)^{-1} \varepsilon_{mnp}\frac{s_i^m}{s_i}\frac{u_i^m}{u_i} d\frac{u_i^p}{u_i}\,, \qquad \text{(no sum over } i\text{)}. \tag{2.126}$$

This solution of the Yang-Mills-Higgs system shares two important properties with the original Wu-Yang monopole and which are related to the fact that they are related to Abelian embeddings by singular gauge transformations:

1. Both Φ and $D\Phi$ are proportional to U:

$$\Phi = \left(-\frac{\mu}{2g} + \frac{1}{2gr} + H\right)U\,, \qquad D\Phi = d\left(-\frac{\mu}{2g} + \frac{1}{2gr} + H\right)U\,, \tag{2.127}$$

   and, therefore, commute with each other, so the Higgs current vanishes and the gauge field is, by itself, a solution of the pure Yang-Mills theory.

2. The gauge field strength is also proportional to U, the coefficient being the field strength of an Abelian gauge field:

$$F(A) = d(B+C)U\,, \tag{2.128}$$

   which implies that the energy-momentum tensors are related as in the single-center case.

These solutions can be generalized even further, by allowing the the charge of the "original" Wu-Yang monopole at $r=0$ to be n/g (that is: using the generalization of the Wu-Yang monopole due to Bais [8] which is studied in Appendix A.4). If we now substitute into the Bogomol'nyi equations (A.38) the ansatz

$$\Phi = \Phi_{(n)} + HU_{(n)}\,, \qquad A = A_{(n)} + CU_{(n)}\,, \tag{2.129}$$

where $U_{(n)}, A_{(n)}$ and $\Phi_{(n)}$ are given, respectively, in Eqs. (A.28),(A.29) and (A.34), H is a function and C a 1-form on $\mathbb{R}^3$, and use that they are satisfied by the pair $A_{(n)}, \Phi_{(n)}$ and that $U_{(n)}$ is covariantly constant with the connection $A_{(n)}$, we arrive again at the Dirac monopole equation (2.123).

Since all these solutions are related to Abelian embeddings, they contribute to the black-hole solutions as the Abelian solutions. We will not consider them in what follows.

2.3.2 Embedding in the SU(2)-gauged $\overline{\mathbb{CP}}^3$ model

We can use the Cherkis & Durcan solution of the SU(2) Bogomol'nyi equations reviewed in the previous section as a seed solution for a multicenter solution of $\mathcal{N}=2$, $d=4$ SEYM, adding the same harmonic functions as in the single-center case $(\mathcal{I}^0, \mathcal{I}_0)$ or a generalization with poles at the locations of the monopoles $r=0$[21] and $u=0$. More explicitly, we take

$$\begin{aligned}
\mathcal{I}^0 &= A^0 + \frac{p_r^0/\sqrt{2}}{r} + \frac{p_u^0/\sqrt{2}}{u}\,,\\
\mathcal{I}_0 &= A_0 + \frac{q_{r,0}/\sqrt{2}}{r} + \frac{q_{u,0}/\sqrt{2}}{u}\,,\\
\mathcal{I}^i &= \mp\sqrt{2}\Phi^i(r,u)\,,\\
\mathcal{I}_i &= 0\,,
\end{aligned} \tag{2.130}$$

where $\Phi^i(r,u)$ is the Higgs field of the Cherkis & Durcan solution. The metric and scalar fields take the form

$$e^{-2U} = \tfrac{1}{2}(\mathcal{I}^0)^2 + 2(\mathcal{I}_0)^2 - \Phi^i\Phi^i\,, \tag{2.131}$$

$$Z^i = \frac{\mp\sqrt{2}\Phi^i}{\mathcal{I}^0 + 2i\mathcal{I}_0}\,. \tag{2.132}$$

The normalization of the metric and scalars at infinity leads to the same relations between the integration constants A^0, A_0, μ and the physical constants Z_∞, g as in the single-center case, namely

$$\tfrac{1}{\sqrt{2}}A^0 + \sqrt{2}iA_0 = \frac{Z_\infty^*}{|Z_\infty|}\frac{1}{\sqrt{1-|Z_\infty|^2}}\,, \qquad \mu = \frac{|Z_\infty|}{\sqrt{1-|Z_\infty|^2}}g\,. \tag{2.133}$$

The integrability conditions of Eq. (2.21) are, in this case,

$$\mathcal{I}_0\partial_{\underline{m}}\partial_{\underline{m}}\mathcal{I}^0 - \mathcal{I}^0\partial_{\underline{m}}\partial_{\underline{m}}\mathcal{I}_0 = 0\,, \tag{2.134}$$

and lead to the following relations between the integration constants:

$$A^0(q_{r,0}+q_{u,0}) - A_0(p_r^0+p_u^0) = 0\,, \tag{2.135}$$

$$J - \tfrac{1}{\sqrt{2}}d(A^0 q_{u,0} - A_0 p_u^0) = 0\,, \tag{2.136}$$

where we have defined the constant

$$J \equiv p_r^0 q_{u,0} - q_{r,0}p_u^0\,. \tag{2.137}$$

The first equation is equivalent to Eq. (2.59) for the total charges and the second equation determines the relative distance d in terms of J and $A^0q_{u,0} - A_0p_u^0$ provided that $J\neq 0$. When that is the case, the solution is not static and has an angular momentum J directed along the line that joins the monopoles $J^m = Jd^m/d$. The corresponding 1-form

[21] The location of the BPS 't Hooft-Polyakov anti-monopole is not completely clear: it is sometimes argued that the center of the monopole is the point at which the Higgs vanishes and the full gauge symmetry is restored. As we have discussed, that point is not $r=0$. We could try to place the poles of the harmonic functions at that point, but, given that its location is not known analytically and the expansion of $\Phi^a\Phi^a$ around it is difficult to compute, we will not try to do that here.

ω can be constructed by the standard procedure of the Abelian case. However, since this complicates the analysis of the regularity of the solutions, we will stick to the static case and require $J=0$.

In order to have regular solutions, the charges at each center must be chosen as in the corresponding single-center case: since there is an Abelian monopole at $u=0$, we must switch on either p^0_u or $q_{u,0}$ to have a regular horizon there. We can treat them both as non-vanishing with no loss of generality. Then, there are two possibilities:

I. $p^0_r = q_{r,0} = 0$**:** Only for $s=0$ ('t Hooft-Polyakov anti-monopole at $r=0$) has the solution a chance of being regular at $r=0$. Solving Eq. (2.135) the solution can be written in the form

$$e^{-2U} = \frac{1}{1-|Z_\infty|^2}H^2 - \Phi^i\Phi^i\,, \tag{2.138}$$

$$Z^i = \frac{2\beta}{p^0+2iq_0}\frac{\Phi^i}{H}\,, \tag{2.139}$$

where H is the harmonic function

$$H \equiv 1+\frac{\beta}{u}\,, \qquad \beta^2 = (1-|Z_\infty|^2)W_{\mathrm{RN}}(\mathcal{Q}_u)/2\,, \qquad W_{\mathrm{RN}}(\mathcal{Q}_u) \equiv \tfrac{1}{2}(p^0_u)^2 + 2(q_{u,0})^2\,. \tag{2.140}$$

The free parameters of this solution are the charges p^0_u, $q_{u,0}$ and the single modulus $|Z_\infty|$.

Studying the $u\to 0$ limit we find a black hole with entropy

$$S_u/\pi = \tfrac{1}{2}W_{\mathrm{RN}}(\mathcal{Q}_u) - \frac{1}{g^2}\,, \tag{2.141}$$

as in the corresponding single-center case.

In the $r\to 0$ limit e^{-2U} is constant. The positivity of the constant is guaranteed if S_u is positive. The total entropy of the solution is just the entropy of the black hole at $u=0$ and the Dirac monopole does contribute to it.

The mass of the solution, expressed in terms of the independent parameters of the solution, p^0_u, $q_{u,0}$ and $|Z_\infty|$ takes the form

$$M = M_r + M_u\,, \tag{2.142}$$

$$M_r = -M_{\mathrm{monopole}}\,, \tag{2.143}$$

$$M_u = \sqrt{\frac{1}{2}\frac{W_{RN}(\mathcal{Q}_u)}{1-|Z_\infty|^2}} + M_{\mathrm{monopole}}\,, \tag{2.144}$$

where M_{monopole} is given by Eq. (2.66). The contributions of the monopole and the 't Hooft-Polyakov monopole to the mass cancel each other.

II. p^0_r **or** $q_{r,0}\neq 0$ We can treat both charges as non-vanishing with no loss of generality. Solving Eqs. (2.135) and (2.137), we can write the solution as in Eqs. (2.138) and

(2.139) where, now,

$$\begin{aligned} H &\equiv 1+\frac{\beta_r}{r}+\frac{\beta_u}{u}\,, \qquad \beta_{r,u}^2=(1-|Z_\infty|^2)W_{\rm RN}(\mathcal{Q}_{r,u})/2\,, \\ W_{\rm RN}(\mathcal{Q}_{r,u}) &\equiv \tfrac{1}{2}(p^0_{r,u})^2+2(q_{r,u,0})^2\,. \end{aligned} \tag{2.145}$$

The free parameters of this solution are the charges p^0_u, $q_{u,0}$ and $|Z_\infty|$ and either p^0_r or $q_{r,0}$, since they must be proportional to those of the other center. The areas of each of the horizons are as in the single-center case. In particular, the BPS 't Hooft-Polyakov monopole ($s=0$) does not contribute to the entropy of the $r=0$ center. The mass is given by

$$M = M_r+M_u\,, \tag{2.146}$$

$$M_r = \sqrt{\tfrac{1}{2}\frac{W_{RN}(\mathcal{Q}_r)}{1-|Z_\infty|^2}}-M_{\rm monopole}\,, \tag{2.147}$$

$$M_u = \sqrt{\tfrac{1}{2}\frac{W_{RN}(\mathcal{Q}_u)}{1-|Z_\infty|^2}}+M_{\rm monopole}\,, \tag{2.148}$$

and the contributions of the monopole and anti-monopole cancel each other. In the $s\to\infty$ limit it can be easily seen that the solution is completely regular everywhere (e^{-2U} only vanishes at $r=0$ and $u=0$) if the Abelian charges as chosen so that the horizons are regular. This guarantees that all the terms in e^{-2U} are positive. For finite s this is more difficult to proof analytically, but, since the Higgs field has a better behavior than in the $s\to\infty$ case, it is reasonable to expect that it will also be true. We have checked numerically that this is so in several examples.

2.3.3 Embedding in the SU(2)-gauged ST[2, 4] model

The metric and scalar fields of the solution are now given by

$$e^{-2U} = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0[(\mathcal{I}^2)^2-2\Phi^a\Phi^a]}\,, \tag{2.149}$$

$$Z^1 \equiv \tau = i\frac{e^{-2U}}{2[(\mathcal{I}^2)^2-2\Phi^a\Phi^a]}\,, \qquad Z^2=\frac{\mathcal{I}^2}{\mathcal{I}^1}\tau\,, \qquad Z^a=\frac{\sqrt{2}\Phi^a}{\mathcal{I}^1}\tau\,, \tag{2.150}$$

where Φ^a is the Higgs field of the Cherkis & Durcan solution (deformed with the Protogenov hair parameter s) and where the harmonic functions $\mathcal{I}^1$, $\mathcal{I}^2$ and $\mathcal{I}_0$ are allowed to have poles at $r=0$ and $u=0$:

$$\begin{aligned} \mathcal{I}^1 &= A^1+\frac{p^1_r/\sqrt{2}}{r}+\frac{p^1_u/\sqrt{2}}{u}\,, \qquad \mathcal{I}^2 = A^2+\frac{p^2_r/\sqrt{2}}{r}+\frac{p^2_u/\sqrt{2}}{u}\,, \\ \mathcal{I}_0 &= A_0+\frac{q_{r,0}/\sqrt{2}}{r}+\frac{q_{u,0}/\sqrt{2}}{u}\,. \end{aligned} \tag{2.151}$$

As in the $\overline{\mathbb{CP}}^3$ case, the Abelian charges at each center must be chosen with the same criteria as in the corresponding single-center case. This means, in particular, that the Abelian charges at $u=0$, $p_u^1, q_{u,0}$ must be non-vanishing. p_u^2 may need to be activated, depending on the branch we are considering. At $r=0$, for $s\neq 0$ we get exactly the same possibilities, but, for $s=0$ there are two possibilities:

1. p_r^1, $q_{r,0}, p_r^2$ non-vanishing. We find a black hole at $r=0$ in the $+$ branch.
2. $p_r^1 = q_{r,0} = p_r^2 = 0$. e^{-2U} is a complicated d-dependent constant in the $r=0$ limit and we get a global monopole.

Here we find an important difference with the single-center case, due to the fact that $\Phi^a\Phi^a$ is a finite constant in the $r\to 0$ limit instead of going to zero as r^2: there is no solution with $p_r^1 q_{r,0}\neq 0$ and $p_r^2 = 0$. In order to have such a global monopole solution with $p^1 q_0 \neq 0$ and $p^2=0$ in equilibrium with the monopole at $u=0$ one may try to place those charges at the point at which $\Phi^a\Phi^a = 0$, but the resulting solution may not be well defined there because the limit of the metric function depends on the direction from which we approach that point.

The entropy of the solution is the sum of the entropies of both centers (vanishing for global monopoles). As in the $\overline{\mathbb{CP}}^3$ case, the monopole at each center does contribute to the center entropy (except for global monopoles). The contributions of the monopole and anti-monopole to the mass cancel each other:

$$M = \tfrac{1}{4}\frac{\chi_\infty}{|\Im\mathfrak{m}\tau_\infty|}|p_u^1 + p_r^1| + \frac{1}{2\chi_\infty}|q_{u,0}+q_{r,0}| \pm \tfrac{1}{2}\frac{|\Im\mathfrak{m}\tau_\infty \Im\mathfrak{m}\, Z_\infty^2|}{\chi_\infty}|p_u^2 + p_r^2|\,. \tag{2.152}$$

2.4 Conclusions

In this chapter we have discussed the construction of supersymmetric multi-object solutions in $\mathcal{N}=2$, $d=4$ EYM theories, specifically in the so-called $\overline{\mathbb{CP}}^{n\geq 3}$ and $\mathrm{ST}[2,n]$ models. These models were chosen due to their workability, the fact that they allow for a SU(2) gauging and (in the second case) for their stringy origin. Starting with a deformation of the solutions to the SU(2) Bogomol'nyi equation found by Cherkis and Durcan that adds to the 't Hooft-Polyakov monopole *Protogenov hair*, we have been able to construct *bona fide* two-center solutions. These solutions describe a Dirac monopole embedded in SU(2) in the presence of either a global monopole (the supergravity solution corresponding to the 't Hooft-Polyakov monopole) or a non-Abelian black hole (a supergravity solution with an 't Hooft-Polyakov-Protogenov monopole). In order to make the comparison with the single-object case easier, we included a detailed discussion of the embeddings of the spherically symmetric solutions to the SU(2) Bogomol'nyi equations into the two models, and expressed the whole solution in terms of charges and moduli of the physical fields.

The constructed solutions are all static. It would be very interesting to study dyonic solutions and to see how this interplays with the Denef constraint; the stumbling block in this respect is not so much the Bogomol'nyi equation as the equation (2.20); for the moment the only general solution we know of is to take $\mathcal{I}_\Lambda \sim \mathcal{I}^\Lambda$ in the gauged directions, but this automatically solves the Denef constraint. The only case for which we can find non-trivial dyonic solutions is for the multi-Wu-Yang solutions, or if you like the $s\to\infty$ limit of the deformed Cherkis and Durcan's solution; we refrain from discussing these solutions here as, due to gauge invariance, even taking into account the singular gauge transformation, the restriction coming from the Denef constraint is basically the one corresponding to the Abelian theory.

A natural question that follows from the results presented here and in Refs. [122, 123, 154] is whether we could use a charge k SU(2) monopole to construct globally regular solutions; the answer is yes: observe that the construction of globally regular solutions in section (2.2) hinges exclusively but crucially on the fact that the used monopole solution is regular and is such that $\Phi^a\Phi^a \leq \lim_{|\vec{x}|\to\infty}\Phi^a\Phi^a$. A charge-$k$ monopole may be rather difficult to construct but the regularity is guaranteed and also the last needed ingredient is known to be satisfied: indeed, using the Bogomol'nyi equation (A.38) one can show that

$$\partial_{\underline{m}}\partial_{\underline{m}}\ \Phi^a\Phi^a \;=\; F^a{}_{\underline{mm}}F^a{}_{\underline{mm}} \;\geq\; 0\ . \tag{2.153}$$

This equation together with the Hopf maximum principle and the regularity, implies that the function $\Phi^a\Phi^a$ is bounded from above by its value on the sphere at infinity, which is exactly what one needs.

As was said in the introduction, the creation and study of non-Abelian solutions to $d=4$ supergravity theories is in its infancy and this holds doubly so for the higher dimensional theories. One possible reason is that the structure of supersymmetric solutions to higher supergravities (see *e.g.* Refs. [23, 57]) is more entangled than the one given in the recipe in section 5.1. For example, naively one would expect that Kronheimer's link of monopoles on $\mathbb{R}^3$ to instantons on GH-spaces, would carry over to the supersymmetric solutions as in $d=4$ the base space is $\mathbb{R}^3$ and that in $d=5$ must be hyper-Kähler; *i.e.* one would expect the instanton equation to show up in the recipe for cooking up 5-dimensional supersymmetric solutions. Perhaps it does, but it definitely is not obvious where and how it is making its appearance in such a clear-cut manner as in $d=4$.

The 4- and 5-dimensional EYMH theories are, however, related by dimensional reduction/oxidation, whence the solutions to the cubic models presented in here could be oxidized to 5-dimensions and can be studied with the hope of unraveling the structure of 5-dimensional supersymmetric solutions.

3

Resolution of $SU(2)$ monopole singularities by oxidation

This chapter is based on

Pablo Bueno, Patrick Meessen, Tomas Ortín and Pedro F. Ramírez
"Resolution of $SU(2)$ monopole singularities by oxidation",
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It has been known for a long time that selfdual Yang–Mills (YM) instantons in 4-dimensional Euclidean space $\mathbb{E}^4$ and magnetic monopoles satisfying the Bogomol'nyi equation in $\mathbb{E}^3$ [42][1] are related by dimensional reduction. In its simplest setting, this relation can be described as follows: if $\hat{A}_{\hat{\mu}}$ ($\hat{\mu} = 0, 1, 2, 3$)[2] is the gauge potential of a selfdual YM instanton solution in $\mathbb{E}^4$ and is furthermore independent of one of the 4 Cartesian coordinates, z say, then the z-component $\hat{A}_z$ and the other three components $\hat{A}_m$ ($m = 1, 2, 3$) can be identified with the Higgs field $\Phi \equiv -\hat{A}_z$ and the gauge potential $A_m \equiv \hat{A}_m$ of a solution of the Yang–Mills–Higgs (YMH) system in the Prasad-Sommerfield limit satisfying the Bogomol'nyi equation:

$$\mathcal{D}_m \Phi = \tfrac{1}{2}\epsilon_{mnp} F_{np} \,. \tag{3.1}$$

The sign in the Bogomol'nyi equation depends on the orientation of the coordinates; we have taken the one corresponding to z to be x^0 and $\epsilon_{0123} = \epsilon_{123} = +1$.

The coordinate z has to be compactified for the instanton action to be finite:[3] $z \sim z + 4\pi$. Thus, in practice, we are performing the dimensional reduction in $S^1 \times \mathbb{E}^3$ and the z-independent solutions can be considered to be the Fourier zero modes of instanton solutions periodic in the direction z (the so-called *calorons*).

The paradigm of selfdual YM instanton in $\mathbb{E}^4$ is the BPST instanton [19], usually presented in Cartesian coordinates using the 't Hooft symbols. It belongs to a family of selfdual YM solutions depending on an arbitrary function K, harmonic on $\mathbb{E}^4$ (see *e.g.* Ref. [130] and references therein). With K asymptotically constant and with a single point-like pole at the origin $K = 1 + 4/(\lambda^2\rho^2)$, where $|\vec{x}_{(4)}|^2 \equiv \rho^2$, the solution

[1] This is the equation satisfied by the 't Hooft–Polyakov monopole [119,179] in the Prasad-Sommerfield limit [181]. We will henceforth refer to these monopoles as BPS monopoles. Since the time direction does not play any role here, we will also refer to the spatial parts of 4-dimensional Lorentzian solutions as "3-dimensional" solutions.

[2] We dress 4-dimensional objects with a hat; hatless objects are 3-dimensional.

[3] This choice of period is unconventional but convenient for what follows.

describes a single BPST instanton located at the origin. Replacing K by a harmonic function on $S^1 \times \mathbb{E}^3$ with a single pole at the origin and asymptotically constant in $\mathbb{E}^3$, $K = 1 + (\sinh r/2)/[\lambda^2 r^2(\cosh r/2 - \cos z/2)]$, where $\rho^2 = z^2 + r^2 = z^2 + |\vec{x}_{(3)}|^2$, we get a caloron [110] whose Fourier zero mode gives, upon dimensional reduction, the spatial part of a Wu-Yang SU(2) magnetic monopole [216], which is singular at the origin.

Since the BPST instanton and caloron are regular everywhere, the singularity of the Wu–Yang solution can be understood as the result of having ignored the massive Fourier modes in the dimensional reduction, but the mere oxidation of the 3-dimensional monopole does not automatically restore them: the 4-dimensional singular instanton corresponding to the Fourier zero mode of the BPST caloron is singular.

The above *redox* relation was generalized by Kronheimer in Ref. [141] to a relation between selfdual Yang–Mills instanton solutions in hyper–Kähler (HK) spaces [141] and BPS monopoles in $\mathbb{E}^3$. We are going to see that Kronheimer's scheme provides an alternative reduction of the BPST instanton which relates it to the *colored* BPS monopole solution of Protogenov [182]. Colored monopoles are a rather misterious type of monopole solutions that exist for many gauge groups [159] and are characterized by asymptotically vanishing Higgs field and magnetic charge which, nevertheless, can contribute to the Bekenstein–Hawking entropy of certain (supersymmetric) non-Abelian black holes [46, 154, 159].

Let us start by reviewing Kronheimer's result: consider a 4-dimensional HK space admitting a free U(1) action which shifts the adapted periodic coordinate $z \sim z + 4\pi$ by an arbitrary constant. Its metric can always be put in the form [96]

$$d\hat{s}^2 = H^{-1}(dz+\omega)^2 + H dx^m dx^m \qquad (m = 1,2,3)\,, \tag{3.2}$$

where the z-independent function H and 1-form ω are related by[4]

$$dH = \star d\omega\,. \tag{3.3}$$

The integrability condition of this equation implies that H is a harmonic function in $\mathbb{E}^3$ which is furthermore required to be strictly positive in order for the metric to be regular. Now, for any gauge group G, let us consider a gauge field $\hat{A}$ whose field strength $\hat{F}$ is selfdual $\hat{\star}\hat{F} = +\hat{F}$ in the above HK metric with respect to the frame and orientation

$$\hat{e}^0 = H^{-1/2}(dz+\omega)\,, \qquad \hat{e}^a = H^{1/2}\delta^a{}_m dx^m\,, \qquad \epsilon_{0123} = +1\,. \tag{3.4}$$

Then, the 3-dimensional gauge and Higgs fields A and Φ defined by

$$\begin{aligned} \Phi &\equiv -H\hat{A}_z\,, \\ A_m &\equiv \hat{A}_m - \omega_m \hat{A}_z\,, \end{aligned} \tag{3.5}$$

satisfy the Bogomol'nyi equation in $\mathbb{E}^3$ Eq. (4.50). It is worth stressing that, had we started with an anti-selfdual YM field we would have obtained the Bogomol'nyi equation with opposite sign, which is acceptable, but also Eq. (3.3) with opposite sign, which would be a contradiction: in this setup we can only reduce YM fields which are selfdual w.r.t. the above frame and orientation.

[4] Unhatted objects are always defined in 3-dimensional Euclidean space $\mathbb{E}^3$.

When $H = 1$, the HK space is just $S^1 \times \mathbb{E}^3$ and one recovers the result explained at the beginning. A more interesting choice is $H = 1/r$ with $r^2 = x^m x^m$. Writing the $\mathbb{E}^3$ metric $dx^m dx^m$ as $dr^2 + r^2 d\Omega^2_{(2)}$ and then redefining $r = \rho^2/4$ the HK metric Eq. (3.2) becomes the metric of $\mathbb{E}^4$ in spherical coordinates

$$ds^2 = d\rho^2 + \rho^2 d\Omega^2_{(3)}\,, \tag{3.6}$$

where $d\Omega^2_{(3)}$ is the round metric of the 3-sphere of unit radius in Eq. (B.14). This HK space is, therefore, $\mathbb{E}^4_{-\{0\}}$ and the shifts of z act freely on it because the origin $\rho = 0$ does not belong to it.

Obviously, the standard BPST instanton is a selfdual solution in this space and, provided that the gauge field is independent of z, we can reduce it directly (avoiding the caloron step) using Kronheimer's scheme to find a monopole in $\mathbb{E}^3_{-\{0\}}$. This is what we are going to do in the next section but, before, we want to review the relation between the Euclidean action of the instanton and the monopole charge.

The gauge field strength components in the frame Eq. (4.49) are

$$\left\{\begin{array}{rcl} \hat{F}_{ab} & = & H^{-1}F_{ab} - H^{-2}\Phi(d\omega)_{ab}\,, \\ \\ \hat{F}_{0a} & = & H^{-1}\mathcal{D}_a\Phi - H^{-2}\Phi\partial_a H\,, \end{array}\right. \tag{3.7}$$

Substituting them into the YM action and using repeatedly Eq. (3.3), the Bogomol'nyi equation (4.50) and Stokes' theorem we get

$$\tfrac{1}{4}\int d^4x\sqrt{|\hat{g}|}\hat{F}^2 = 4\pi\int_{V^3}\tfrac{1}{2}H^{-2}d\star dH\,\Phi^2 + 4\pi\int_{\partial V^3}\left[H^{-1}\Phi^A F^A + \tfrac{1}{2}\star dH^{-1}\Phi^2\right]\,, \tag{3.8}$$

where V^3 is $\mathbb{E}^3$ with the singular points of H removed: this means that the first term on the r.h.s. always vanishes. The end result therefore reads

$$\tfrac{1}{4}\int d^4x\sqrt{|\hat{g}|}\hat{F}^2 = 4\pi\int_{\partial V^3}\left[H^{-1}\Phi^A F^A + \tfrac{1}{2}\star dH^{-1}\Phi^2\right]\,, \tag{3.9}$$

and one must take into account that the boundary of V^3 includes the singularities of H as well as infinity.

For $H = 1$, $V^3 = \mathbb{E}^3$ and the r.h.s. is directly related to the monopole magnetic charge

$$p = \tfrac{1}{4\pi}\int_{S^2_\infty}\frac{\Phi^A F^A}{\sqrt{\Phi^B\Phi^B}}\,, \tag{3.10}$$

provided the Higgs field is asymptotically constant, as in the BPS 't Hooft–Polyakov monopole.

For $H = 1/r$, which is the case of interest here, $V^3 = \mathbb{E}^3_{-\{0\}}$, $\partial V^3 = \{0\} \cup S^2_\infty$, and the integral will diverge precisely for monopoles with well-defined magnetic charge at infinity and asymptotically constant Higgs fields. Thus, we can only expect convergence for colored magnetic monopoles [159]. If the selfdual YM field has a finite action, then it must lead to a colored monopole in $\mathbb{E}^3$ by Kronheimer's dimensional reduction. In the next section we are going to see that this is indeed the case for the BPST instanton.

3.1 Singular reduction of the BPST instanton

In order to reduce the BPST instanton *à la Kronheimer* in the HK space with $H = 1/r$, it is convenient to write it in spherical coordinates and, actually, it is easier to rederive it directly using the following ansatz for the components of the SU(2) gauge potential

$$\hat{A}^A_{\substack{L\\R}} = b_{\substack{L\\R}}(\rho) v^A_{\substack{L\\R}}\,, \qquad A = 1, 2, 3\,, \tag{3.11}$$

where the $v^A_{\substack{L\\R}}$ are the components of the SU(2) Maurer–Cartan (MC) 1-forms defined in Eqs. (B.12), satisfying Eq. (B.13), and $b_{\substack{L\\R}}(\rho)$ is a function of ρ to be determined by imposing the selfduality of the gauge field strength. To this end it is most convenient to use the frames

$$\hat{e}^{\,0}_{\substack{L\\R}} = d\rho\,, \qquad \hat{e}^{\,a}_{\substack{L\\R}} = \tfrac{1}{2}\rho\delta^a{}_A v^A_{\substack{L\\R}}\,, \tag{3.12}$$

for the metric Eq. (3.6). Using the MC 1-forms it is straightforward to compute the gauge field strength $\hat{F}^A_{\substack{L\\R}}$:

$$\hat{F}_{\substack{L\\R}}{}^A = \frac{2\dot{b}}{\rho}\delta^A{}_a\,\hat{e}_{\substack{L\\R}}{}^0 \wedge \hat{e}_{\substack{L\\R}}{}^a + \frac{2b(b\mp 1)}{\rho^2}\epsilon^A{}_{ab}\,\hat{e}_{\substack{L\\R}}{}^a \wedge \hat{e}_{\substack{L\\R}}{}^b\,. \tag{3.13}$$

Requiring $\hat{F}^A_{\substack{L\\R}}$ to be (anti-)selfdual ($\hat{F}^{A(\pm)}{}_{0a} = \pm\frac{1}{2}\epsilon_{abc}\hat{F}^{A(\pm)}{}_{bc}$) in these two frames we arrive at a differential equation for $b^{\pm}_{\substack{L\\R}}(\rho)$ leading to two self- and two anti-selfdual solutions describing a single BPST instanton or anti-instanton, of size[5] determined by the parameter λ, at the origin:

$$\hat{\star}\hat{F} = +\hat{F}\left\{\begin{array}{rcl}\hat{A}^{A(+)}_L &=& \dfrac{1}{1+\lambda^2\rho^2/4}\,v^A_L\,,\\[2ex] \hat{A}^{A(+)}_R &=& -\dfrac{\lambda^2\rho^2/4}{1+\lambda^2\rho^2/4}\,v^A_R\,,\end{array}\right. \qquad \hat{\star}\hat{F} = -\hat{F}\left\{\begin{array}{rcl}\hat{A}^{A(-)}_L &=& +\dfrac{\lambda^2\rho^2/4}{1+\lambda^2\rho^2/4}\,v^A_L\,,\\[2ex] \hat{A}^{A(-)}_R &=& -\dfrac{1}{1+\lambda^2\rho^2/4}\,v^A_R\,.\end{array}\right. \tag{3.14}$$

The gauge fields $\hat{A}^{A(\pm)}_L$ are gauge-equivalent to the $\hat{A}^{A(\pm)}_R$ owing to

$$U\hat{A}^{A(\pm)}_L U^{-1} + dUU^{-1} = \hat{A}^{A(\pm)}_R\,, \tag{3.15}$$

and the property Eq. (B.11). Then, we could just work with $\hat{A}^{A(+)}_R$ and $\hat{A}^{A(-)}_L$, which are regular (they vanish at $\rho = 0$ while the other two are multivalued there). However, if we want to use Kronheimer's results we are forced to work with the singular ones, $\hat{A}^{A(+)}_L$ and

[5] In the instanton literature it is customary to denote the size of the (anti-)instanton by ρ, see *e.g.* Refs. [205], but here we'll denote it by ρ_0. It is then easy to see that $\lambda = 2/\rho_0$.

$\hat{A}_R^{A(-)}$, because as one can see the transformation between the frame $\hat{e}^{\hat{a}}_{\substack{L\\R}}$ in Eqs. (3.12) and Kronheimer's frame $\hat{e}^{\hat{a}}$ in Eqs. (4.49) preserves the orientation for $\hat{e}^{\hat{a}}_L$ but reverses it for $\hat{e}^{\hat{a}}_R$. In other words: the regular gauge fields $\hat{A}_R^{A(+)}$ and $\hat{A}_L^{A(-)}$ are anti-selfdual in Kronheimer's frame and can therfore not be consistently reduced.

Let us, then, consider $\hat{A}_L^{A(+)}$ and $\hat{A}_R^{A(-)}$. By construction, these gauge fields are invariant under the free U(1) actions in Eqs. (B.5) and (B.4), respectively.

In other words: $\hat{A}_L^{A(+)}$ is φ-independent and $\hat{A}_R^{A(-)}$ is ψ-independent and can be dimensionally reduced along those directions because the only invariant point under these actions (the origin $\rho = 0$) does not belong to our HK space. We can expect 3-dimensional monopoles which are singular there.

Using directly Eqs. (4.51), from $\hat{A}_L^{A(+)}$ we get the Yang–Mills and Higgs fields of a BPS monopole solution

$$\Phi_L^{A(+)} = \frac{1}{r(1+\lambda^2 r)}\delta^A{}_m \frac{y_L^m}{r}\,, \qquad A_L^{A(+)} = \frac{1}{(1+\lambda^2 r)}\epsilon^A{}_{mn} d\frac{y_L^m}{r}\frac{y_L^n}{r} \tag{3.16}$$

where we have defined the Cartesian coordinates $y^m/r \equiv -\delta^m{}_A v^A_{L\,\varphi}$:[6]

$$y_L^1 \equiv r\sin\theta\,\cos\psi\,, \quad y_L^2 \equiv r\sin\theta\,\sin\psi\,, \quad y_L^3 \equiv r\cos\theta\,. \tag{3.17}$$

The reduction of $\hat{A}_R^{A(-)}$ gives exactly the same 3-dimensional fields upon the replacement of the Cartesian coordinates y_L^m by $y_R^m \equiv +r\delta^m{}_A v^A_{R\,\psi}$:[7]

$$y_R^1 \equiv r\sin\theta\,\cos\varphi\,, \quad y_R^2 \equiv -r\sin\theta\,\sin\varphi\,, \quad y_R^3 \equiv -r\cos\theta\,. \tag{3.18}$$

As predicted by the arguments based on the Euclidean action, the 3-dimensional BPS monopole obtained by this procedure is the colored monopole found by Protogenov in Ref. [182]. The Higgs field vanishes at infinity and the magnetic charge, as defined in Eq. (3.10) vanishes identically. The solution approaches the Wu–Yang monopole [216] for $r \to 0$ (which corresponds to $\lambda^2 = 0$) and, therefore, one can argue that the solution describes a magnetic monopole at the origin whose charge is completely screened at infinity. This interpretation is supported by the computation of the Bekenstein–Hawking entropy $S_{\rm BH}$ of non-Abelian black holes with this kind of gauge fields: there is a contribution to $S_{\rm BH}$ corresponding to a magnetic charge [46, 154].

3.2 Oxidation of the singular Protogenov monopoles

Reversing the procedure we just carried out, we see that the singularity of the SU(2) colored BPS monopole disappears completely when it is oxidized to 4 Euclidean dimensions. Since there are other singular SU(2) BPS monopoles [182], it is natural to ask whether their singularities can also be cured by oxidizing them within this scheme.

The spherically symmetric solutions of the SU(2) Bogomol'nyi equations have the following *hedgehog* form [182]:

[6] We use the identity $v^A_L(\varphi = 0) - \cos\theta\, v^A_{L\,\varphi} d\psi = \epsilon^A{}_{mn} d\frac{y_L^m}{r}\frac{y_L^n}{r}$

[7] Now we use the identity $v^A_R(\psi = 0) - \cos\theta\, v^A_{R\,\psi} d\varphi = -\epsilon^A{}_{mn} d\frac{y_R^m}{r}\frac{y_R^n}{r}$

$$\begin{aligned} A^A &= -r^2 h(r)\epsilon^A{}_{mn}\frac{y^n}{r}d\left(\frac{y^m}{r}\right)\,, && (3.19)\\ \Phi^A &= -rf(r)\delta^A{}_m\frac{y^m}{r}\,, && (3.20)\end{aligned}$$

where the functions $f(r)$ and $h(r)$ must satisfy the differential equations

$$\begin{aligned} r\dot{h} + 2h + f(1+r^2h) &= 0\,, && (3.21)\\ r(\dot{h}-\dot{f}) - r^2h(h-f) &= 0\,, && (3.22)\end{aligned}$$

if the above Yang-Mills and Higgs fields are to satisfy the Bogomol'nyi equation (4.50). Apart from the family of colored solutions in Eq. (3.16), there is another 2-parameter (μ and s) family of solutions given by

$$rf = -\frac{1}{r}\left[1-\mu r\coth\left(\mu r+s\right)\right]\,, \qquad rh = \frac{1}{r}\left[\frac{\mu r}{\sinh\left(\mu r+s\right)}-1\right]\,. \tag{3.23}$$

The BPS limit of the 't Hooft–Polyakov monopole [119,179] is the $s=0$ member of this family, and the only regular one. Before oxidizing them, we can compute the action of the corresponding instanton using Eq. (3.9). The action turns out to diverge for all values of s. However, even if all hope of getting a regular instanton by oxidizing these solutions is lost, it is still worth finding the general expression of the singular instantons, since it may give us inspiration for making instanton ansätze directly in 4 dimensions. Using Kronheimer's relations, Eq. (4.51), we find

$$\hat{A}^A = -r^2 f(r)v_L^A + r^2\left[f(r)-h(r)\right]u^A\,, \tag{3.24}$$

where we have defined the 1-forms

$$\begin{aligned} u^1 &= \cos\psi\sin\theta\cos\theta d\psi + \sin\psi d\theta\,,\\ u^2 &= \sin\psi\sin\theta\cos\theta d\psi - \cos\psi d\theta\,,\\ u^3 &= -\sin^2\theta d\psi\,.\end{aligned} \tag{3.25}$$

These 1-forms depend only on two coordinates (ψ and θ) and they can be seen as projections of the left-invariant MC 1-forms v_L^A

$$u^A = v_L^B\left[\delta^A{}_B - \frac{y_B y^A}{r^2}\right]\,. \tag{3.26}$$

They satisfy differential equations identical to the ones satisfied by the left-invariant MC 1-forms v_L^A up to the 1/2 factor, *i.e.*

$$du^A = -\epsilon^A{}_{BC}u^B\wedge u^C\,, \tag{3.27}$$

which makes them well suited for a generalization of the ansatz Eq. (3.11):

$$\hat{A}^A = b(\rho)v_L^A + c(\rho)u^A\,. \tag{3.28}$$

Imposing selfduality of the corresponding field strength with the redefinition

$$b(\rho(r)) = -r^2 f(r)\,, \qquad c(\rho(r)) = -r^2\left[h(r) - f(r)\right]\,, \tag{3.29}$$

leads to Protogenov's equations (3.21) and (3.22); the oxidation of the BPS monopoles gives all the selfdual instantons of that form.

3.3 Conclusions

In this paper we have shown how a misterious kind of SU(2) BPS magnetic monopoles known as *colored monopoles*, which are singular at the origin and have vanishing asymptotic charge and Higgs field, can be understood as the result of the singular dimensional reduction of the BPST instanton, which is itself globally regular. The parameter appearing in the monopole family of solutions turns out to be related to the one that measures the instantons' size.

The mechanism is analogous to the well-known mechanism curing gravitational singularities by oxidation as for example the KK-monopole [196] or in certain 4-dimensional dilatonic black holes [97], but with the twist that here the fields are non-Abelian. The mechanism that cures the singularity of the colored monopole does not, however, work for the rest of the spherically-symmetric BPS monopoles of the theory: they always have infinite action, but depending on the application this may or may not be a problem.

We have argued, based on the relation between the instanton action and the monopole magnetic charge, that this relation between regular instantons and singular, colored magnetic monopoles should be general. It has recently been shown in Ref. [159] that colored magnetic monopoles are present in the Yang–Mills–Higgs theory for all SU(N) groups and the results of that paper can be used to construct regular selfdual SU(N) instantons, as we will see in the following chapters. Possibly, the transmutation monopoles discovered in Ref. [159], which have different (non-vanishing) charges at infinity and at the origin, can be related to regular solutions by a similar mechanism.

The case studied here is just the simplest and most special of those comprised in Kronheimer's work Ref. [141], since it just involves $\mathbb{E}^4_{-\{0\}}$. One may wonder if the rest can be of any relevance in physics. It turns out that the relation between $\mathcal{N} = 1, d = 5$ and $\mathcal{N} = 2, d = 4$ super-Einstein–Yang–Mills (SEYM) theories must include the relation between selfdual instantons in HK spaces and BPS monopoles in $\mathbb{E}^3$ discovered by Kronheimer: the timelike supersymmetric solutions of $\mathcal{N} = 1, d = 5$ [23] (as it happens in the Abelian case [92]) involve a 4-dimensional Euclidean base space of HK type and the YM field strengths have a piece which is selfdual in that space. On the other hand the YM fields of the timelike supersymmetric solutions of $\mathcal{N} = 2, d = 4$ SEYM [156] are required to satisfy the Bogomol'nyi equation in $\mathbb{E}^3$ in combination with an effective *Higgs* field. These two classes of theories and their solutions are related by dimensional reduction. Explicit solutions of the latter describing non-Abelian black holes have been obtained in [46,122,123,154,159]. Some of the solutions are powered by the colored BPS monopoles that we have shown to be related to the BPST instanton. It is then natural to expect that the oxidation of the complete supergravity solutions will provide us with explicit solutions of the $\mathcal{N} = 1, d = 5$ SEYM theory[8] involving the BPST instanton. These solutions, whose form is quite intriguing, may be globally regular. The oxidation *à la Kronheimer* of solutions involving other monopoles will give potentially singular solutions, but, just as it happens with singular monopoles in $d = 4$, gravity may cover the singularities with event horizons. All these new possibilities opened by the result presented in this paper are very interesting and well worth investigating.

[8]So far, no explicit solutions of these theories have been constructed.

4

Non-Abelian, supersymmetric black holes and strings in 5 dimensions

This chapter is based on

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The search for classical solutions of General Relativity and theories of gravity in general has proven to be one of the most fruitful approaches to study this universal and mysterious interaction. This is partially due to the non-perturbative information they provide, which we do not know how to obtain otherwise. It is fair to say that some of the solutions discovered (such as the Schwarzschild and Kerr black-hole solutions, the cosmological ones or the $\mathrm{AdS}_5\times\mathrm{S}^5$ solution of type IIB supergravity) have opened entire fields of research.

Some of the most interesting solutions are supported by fundamental matter fields and a large part of the search for gravity solutions has been carried out in theories in which gravity is coupled to different forms of matter, usually scalar fields, Abelian vector and p-form fields coupled in gauge-invariant ways among themselves and to scalars, as suggested by superstring and supergravity theories, for instance. The solutions of gravity coupled to non-Abelian vector fields have been much less studied because of the complexity of the equations. Most of the genuinely non-Abelian solutions found so far, such as the Bartnik-McKinnon particle [11] and its black hole-type generalizations [206], in the SU(2) Einstein-Yang-Mills (EYM) theory, are only known numerically, which makes them more difficult to study and generalize.

Supersymmetry can simplify dramatically the construction of classical solutions, providing in some cases recipes to construct systematically whole families of solutions that have the property of being "supersymmetric" or "having unbroken supersymmetry", or being "BPS" (a much less precise term) because these solutions satisfy much easier to solve first-order differential equations.[1] These techniques can be applied to non-supersymmetric theories if we can "embed" them in a larger supersymmetric theory from which they can be obtained by a consistent truncation that, in particular, gets rid of the fermionic fields.

In order to apply these techniques to the case of theories of gravity coupled to fundamental matter fields we must embed the theories first in supergravity theories. $d = 4$ EYM

[1] For a general review on the construction of supersymmetric solutions of supergravity theories, including some of those that we are going to study here, see Ref. [170].

theories can be embedded almost trivially in $\mathcal{N}=1, d=4$ gauged supergravity coupled to vector supermultiplets, but there are no supersymmetric black-hole or more general particle-like solutions in $\mathcal{N}=1, d=4$ supergravity: all the supersymmetric solutions of these theories belong to the null class[2] and describe, generically, massless solutions such as gravitational waves and also black strings (whose tension does not count as a mass). This could well explain why there are no simple analytic solutions of the EYM theory.

Embedding of $d=4$ EYM theories in extended ($\mathcal{N}>1$) $d=4$ supergravity theories turns out to be impossible, since the latter always include additional scalar fields charged under the non-Abelian fields which cannot be consistently truncated away. On the other hand, these scalar fields (or part of them) can also be interpreted as Higgs fields and we can think of those supergravities (which we will call Super-Einstein-Yang-Mills (SEYM) theories) as the minimal supersymmetric generalizations of the Einstein-Yang-Mills-Higgs (EYMH) theory. Actually, some solutions of the SEYM theories are also solutions of the EYMH theory, but this is not generically true and we cannot say that the EYMH theory is embedded in some SEYM theory.

At any rate, analytic supersymmetric solutions of SEYM or more general gauged supergravity theories should be much easier to find than solutions of the EYM theory and, at the same time, much more realistic, since we know there are scalar fields charged under non-Abelian vector fields in Nature.

This expectation turns out to be true. In 1991 Harvey and Liu [112] and in 1997 Chamseddine and Volkov [60] found globally regular gravitating monopole ("global monopole") solutions to gauged $\mathcal{N}=4$, $d=4$ supergravity, a theory that can be related to the Heterotic string. In 1994, a 4-dimensional black-hole solution with non-Abelian hair was obtained by adding stringy (Heterotic) α' corrections to an $a=1$ dilaton black hole [132]. This solution was singular in the Einstein frame.[3] More recently, the timelike supersymmetric solutions of gauged $\mathcal{N}=2, d=4$ and $\mathcal{N}=1, d=5$ were characterized, respectively, in Refs. [123, 156] and [20, 23],[4] so the form of all the fields in those solutions is given in terms of a few functions that satisfy first-order equations.

In the 4-dimensional case, these first-order equations are straightforward generalizations of the well-known Bogomol'nyi monopole equations [42] whose more general static and spherically symmetric solutions for the gauge group SU(2) were obtained by Protogenov in Ref. [182]. Then, the characterization of timelike supersymmetric solutions was immediately used to construct, apart from global monopole solutions, the first analytical, regular, static, non-Abelian black-hole solutions which cannot be considered as pure Abelian embeddings [123], showing how the attractor mechanism works in the non-Abelian setting [122, 123]. Colored black holes[5] and two-center non-Abelian solutions were constructed, respectively, in [154] and [46] by using, respectively, "colored monopole" and two-center solutions of the Bogomol'nyi equations.

In the $\mathcal{N}=1, d=5$ SEYM case, the characterization obtained in Refs. [20, 23] has not yet been exploited. Doing so to construct non-Abelian black-hole and black-string

[2] The Killing spinor of the supersymmetric solutions in the null (resp. timelike) class gives rise to a null (resp. timelike) Killing vector bilinear.

[3] We will see, though, that it is closely related to the 4-dimensional black-hole solutions studied in [46] and to the 5-dimensional ones presented here.

[4] In the $\mathcal{N}=1, d=5$ case, the null supersymmetric solutions were characterized as well.

[5] Colored black holes have non-Abelian hair but vanishing asymptotic charges. The charges must be screened at infinity because they contribute to the near-horizon geometry and to the entropy.

solutions is our main goal in this paper. It is a well-known fact, one that also holds in the Abelian (ungauged) case that the vector field strengths of the timelike supersymmetric solutions of these theories are the sum of two pieces, one of them self-dual in the hyperKähler base space, *i.e.* an instanton in the base space. In the non-Abelian case we are interested in, this fact can be exploited in an obvious way to add non-Abelian hair to black hole solutions.

As we are going to see, it will be convenient to refine the general characterization obtained in those references to obtain a simpler recipe to construct supersymmetric solutions with one additional isometry. These solutions are still general enough and can also be related to the timelike supersymmetric solutions of $\mathcal{N}=2, d=4$ SEYM. In the timelike-to-timelike reduction, we recover the relation between self-dual instantons in hyperKähler spaces with one isometry and BPS monopoles in $\mathbb{E}^3$ found by Kronheimer in Ref. [141]. As we have shown in Ref. [47] this redox relation brings us from singular colored monopoles to globally regular BPST instantons and vice-versa and it will allow us to obtain regular black holes with a BPST instanton field.

The recipes we have obtained can be applied to any model of $\mathcal{N}=1, d=5$ supergravity coupled to vector multiplets in which a non-Abelian subgroup of the perturbative duality group can be gauged. The explicit solutions we will construct will belong to a particular model, the ST$[2,5]$ model which is the smallest of the ST$[2,n]$ family of models admitting a SU(2) gauging. These models are consistent truncations of $\mathcal{N}=1, d=10$ supergravity coupled to a number of vector multiplets on T^5 and, for low values of n, they can be embedded in Heterotic string theory. The SU(2) gauging can be associated to the enhancement of symmetry at the self-dual radius U(1)×U(1) →U(1)×SU(2), although, in order to study the details of the embedding of our model in Heterotic string theory (which will be our next goal) more work will be necessary.

This paper is organized as follows: in Section 4.1 we review the gauging of a non-Abelian group of isometries of an $\mathcal{N}=1, d=5$ supergravity theory coupled to vector multiplets. The result of this procedure is what we call an $\mathcal{N}=1, d=5$ Super-Einstein-Yang-Mills (SEYM) theory. In Section 4.2 we review and extend the results of Ref. [23] on the characterization of the supersymmetric solutions of $\mathcal{N}=1, d=5$ SEYM theories, giving the recipe to construct those admitting additional isometries and showing how they are related to the analogous supersymmetric solutions of $\mathcal{N}=2, d=4$ SEYM theories characterized in Ref. [122,156]. We will then use these results in Sec. 8.1 to construct black holes and black strings (in the timelike and null cases, respectively) of the SU(2)-gauged ST$[2,5]$ model of $\mathcal{N}=1, d=5$ supergravity and to study their relations, via dimensional reduction, to the non-Abelian timelike supersymmetric solutions (black holes and global monopoles) of the SU(2)-gauged ST$[2,5]$ model of $\mathcal{N}=2, d=4$ supergravity (see Ref. [46]). Our conclusions are given in Section 7.2. Appendix C.4 reviews the reduction of ungauged $\mathcal{N}=1, d=5$ supergravity to a cubic model of $\mathcal{N}=2, d=4$ supergravity, with the relation between the 5- and 4-dimensional fields for any kind of solution (supersymmetric or not). This relation remains true for gauged supergravity theories under standard dimensional reduction (which does not change the gauge group). Finally, Appendix A.5 review the spherically-symmetric solutions of the Bogomol'nyi equation in $\mathbb{E}^3$ for SU(2).

4.1 $\mathcal{N} = 1, d = 5$ SEYM theories

In this section we give a brief description of general $\mathcal{N} = 1, d = 5$ Super-Einstein-Yang-Mills (SEYM) theories. These are theories of $\mathcal{N} = 1, d = 5$ supergravity coupled to n_v vector supermultiplets (no hypermultiplets) in which a necessarily non-Abelian group of isometries of the Real Special manifold has been gauged. These theories can be considered the simplest supersymmetrization of non-Abelian Einstein-Yang-Mills theories in $d = 5$. Our conventions are those in Refs. [21, 23] which are those of Ref. [35] with minor modifications.

The supergravity multiplet is constituted by the graviton $e^a{}_\mu$, the gravitino ψ^i_μ and the graviphoton A_μ. All the spinors are symplectic Majorana spinors and carry a fundamental $SU(2)$ R-symmetry index. The n_v vector multiplets, labeled by $x = 1, ..., n_v$ consist of a real vector field $A^x{}_\mu$, a real scalar ϕ^x and a gaugino $\lambda^{i\,x}$.

The full theory is formally invariant under a $SO(n_v + 1)$ group[6] that mixes the matter vector fields $A^x{}_\mu$ with the graviphoton $A_\mu \equiv A^0{}_\mu$ and it is convenient to combine them into an $SO(n_v + 1)$ vector $(A^I{}_\mu) = (A^0{}_\mu, A^x{}_\mu)$. It is also convenient to define a $SO(n_v + 1)$ vector of functions of the scalars $h^I(\phi)$. These $n_v + 1$ functions of n_v scalar must satisfy a constraint. $\mathcal{N} = 1, d = 5$ supersymmetry determines that this constraint is of the form

$$C_{IJK}h^I(\phi)h^J(\phi)h^K(\phi) = 1, \tag{4.1}$$

where the constant symmetric tensor C_{IJK} completely characterizes the theory and the *Special Real geometry* of the scalar manifold. In particular, the kinetic matrix of the vector fields $a_{IJ}(\phi)$ and the metric of the scalar manifold $g_{xy}(\phi)$ can be derived from it as follows: first, we define

$$h_I \equiv C_{IJK}h^J h^K, \quad \Rightarrow \quad h^I h_I = 1, \tag{4.2}$$

and

$$h^I_x \equiv -\sqrt{3}h^I{}_{,x} \equiv -\sqrt{3}\frac{\partial h^I}{\partial \phi^x}, \qquad h_{Ix} \equiv +\sqrt{3}h_{I,x}, \quad \Rightarrow \quad h_I h^I_x = h^I h_{Ix} = 0. \tag{4.3}$$

Then, a_{IJ} is defined implicitly by the relations

$$h_I = a_{IJ}h^I, \qquad h_{Ix} = a_{IJ}h^J{}_x. \tag{4.4}$$

It can be checked that

$$a_{IJ} = -2C_{IJK}h^K + 3h_I h_J. \tag{4.5}$$

The metric of the scalar manifold $g_{xy}(\phi)$, which we will use to raise and lower x, y indices is (proportional to) the pullback of a_{IJ}

$$g_{xy} \equiv a_{IJ}h^I{}_x h^J{}_y = -2C_{IJK}h^I_x h^J_y h^K. \tag{4.6}$$

[6] The theory will only be invariant under a subgroup of $SO(n_v + 1)$.

The functions h^I and their derivatives h^I_x satisfy the following completeness relation:

$$a_{IJ} = h_I h_J + g_{xy} h^x_I h^y_J . \tag{4.7}$$

By assumption, the real Real Special structure is invariant under reparametrizations generated by vectors $k_I{}^x(\phi)$[7]

$$\delta\phi^x = c^I k_I{}^x , \tag{4.8}$$

satisfying the Lie algebra[8]

$$[k_I, k_J] = -f_{IJ}{}^K k_K . \tag{4.9}$$

The invariance of the metric g_{xy} implies that the vectors $k_I{}^x(\phi)$ are Killing vectors. The invariance of the constraint Eq. (4.1) implies the invariance of the C_{IJK} tensor

$$-3 f_{I(J}{}^M C_{KL)M} = 0. \tag{4.10}$$

Multiplying this identity by $h^J h^K h^L$ we get another important relation:

$$f_{IJ}{}^K h^J h_K = 0. \tag{4.11}$$

The functions $h^I(\phi)$, in their turn, must be invariant up to $SO(n_v + 1)$ rotations, that is

$$k_I{}^x \partial_x h^J - f_{IK}{}^J h^K = 0, \quad \Rightarrow \quad k_I{}^x = -\sqrt{3} f_{IJ}{}^K h^x_K h^J , \quad \Rightarrow \quad h^I k_I{}^x = 0, \tag{4.12}$$

where we have used the completeness relation Eq. (4.7) and Eq. (4.11).

If the real special manifold is a symmetric space, then the tensor C_{IJK} satisfies the identity

$$C^{IJK} C_{J(LM} C_{NP)K} = \tfrac{1}{27} \delta^I{}_{(L} C_{MNP)} , \tag{4.13}$$

where $C^{IJK} = C_{IJK}$. In these spaces we can solve immediately h^I in terms of the h_I

$$h^I = 27 C^{IJK} h_J h_K , \quad \Rightarrow \quad C^{IJK} h_I h_J h_K = \frac{1}{27} . \tag{4.14}$$

To gauge this global symmetry group we promote the constant parameters c^I to arbitrary spacetime functions identifying them with the gauge parameters of the vector fields $\Lambda^I(x)$ $c^I \to -g\Lambda^I(x)$. The gauge transformations scalars ϕ^x, the functions h^I and the $A^I{}_\mu$ take the form

[7] Some of these vectors may be identically zero. This is price paid for labeling the gauge vectors and the Killing vectors with the same indices.

[8] Some of the structure constants may vanish identically, but it is assumed that some of them do not because, otherwise, we would be dealing with an ungauged supergravity.

$$\begin{aligned}\delta_\Lambda \phi^x &= -g\Lambda^I k_I{}^x, && (4.15)\\ \delta_\Lambda h^I &= -g f_{JK}{}^I \Lambda^J h^K, && (4.16)\\ \delta_\Lambda A^I{}_\mu &= \partial_\mu \Lambda^I + g f_{JK}{}^I A^J{}_\mu \Lambda^K \equiv \mathfrak{D}_\mu \Lambda^I, && (4.17)\end{aligned}$$

where $\mathfrak{D}_\mu$ is the gauge-covariant derivative. $\mathfrak{D}_\mu h^I$ has the same expression as $\mathfrak{D}_\mu \Lambda^I$ and have the same gauge transformations as h^I and Λ^I. We also have

$$\begin{aligned}\mathfrak{D}_\mu h_I &= \partial_\mu h_I + g f_{IJ}{}^K A^J{}_\mu h_K, && (4.18)\\ \mathfrak{D}_\mu C_{IJK} &= 0. && (4.19)\end{aligned}$$

On the other hand, the gauge-covariant derivative of the scalars is given by

$$\mathfrak{D}_\mu \phi^x = \partial_\mu \phi^x + g A^I{}_\mu k_I{}^x, \tag{4.20}$$

and transforms as

$$\delta_\Lambda \mathfrak{D}_\mu \phi^x = -g\Lambda^I \partial_y k_I{}^x \mathfrak{D}_\mu \phi^x. \tag{4.21}$$

The gauginos $\lambda^{i\,x}$ transform in exactly the same way as $\mathfrak{D}\phi^x$ and their gauge-covariant derivatives are identical to the second covariant derivative of ϕ^x:

$$\mathfrak{D}_\mu \mathfrak{D}_\nu \phi^x = \partial_\mu \mathfrak{D}_\nu \phi^x - \Gamma^\rho_{\mu\nu} \mathfrak{D}_\rho \phi^x + \Gamma_{yz}{}^x \mathfrak{D}_\mu \phi^y \mathfrak{D}_\nu \phi^z + g A^I{}_\mu \partial_y k_I{}^x \mathfrak{D}_\nu \phi^y. \tag{4.22}$$

The gauge-covariant vector field strength has the standard form

$$F^I{}_{\mu\nu} = 2\partial_{[\mu} A^I{}_{\nu]} + g f_{JK}{}^I A^J{}_\mu A^K{}_\nu. \tag{4.23}$$

The bosonic action of $\mathcal{N}=1, d=5$ SEYM is given in terms of a_{IJ}, g_{xy}, C_{IJK} and the structure constants $f_{IJ}{}^K$ by

$$\begin{aligned} S \;=\; \int d^5x\sqrt{g}\, \Big\{ & R + \tfrac{1}{2} g_{xy} \mathfrak{D}_\mu \phi^x \mathfrak{D}^\mu \phi^y - \tfrac{1}{4} a_{IJ} F^{I\,\mu\nu} F^J{}_{\mu\nu} + \tfrac{1}{12\sqrt{3}} C_{IJK} \frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} \big[F^I{}_{\mu\nu} F^J{}_{\rho\sigma} A^K{}_\alpha \\ & -\tfrac{1}{2} g f_{LM}{}^I F^J{}_{\mu\nu} A^K{}_\rho A^L{}_\sigma A^M{}_\alpha + \tfrac{1}{10} g^2 f_{LM}{}^I f_{NP}{}^J A^K{}_\mu A^L{}_\nu A^M{}_\rho A^N{}_\sigma A^P{}_\alpha \big] \Big\}. \end{aligned} \tag{4.24}$$

Observe that this action does not contain a scalar potential $V(\phi)$ because

$$V(\phi) = \tfrac{3}{2} g^2 h^I h^J k_I{}^x k_J{}^y g_{xy}\,, \tag{4.25}$$

(the expression that follows from the general formula in Ref. [35]) vanishes identically for the kind of gaugings considered here, owing to the property Eq. (4.12). This fact is associated to the vanishing of the corresponding fermion shift in the gauginos' supersymmetry transformations.

The equations of motion for the bosonic fields are

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &\equiv \frac{1}{2\sqrt{g}} e_{a(\mu} \frac{\delta S}{\delta e_a{}^{\nu)}} \\
&= G_{\mu\nu} - \tfrac{1}{2} a_{IJ} \left(F^I{}_\mu{}^\rho F^J{}_{\nu\rho} - \tfrac{1}{4} g_{\mu\nu} F^{I\,\rho\sigma} F^J{}_{\rho\sigma} \right) \\
&\quad + \tfrac{1}{2} g_{xy} \left(\mathfrak{D}_\mu \phi^x \mathfrak{D}_\nu \phi^y - \tfrac{1}{2} g_{\mu\nu} \mathfrak{D}_\rho \phi^x \mathfrak{D}^\rho \phi^y \right) \qquad (4.26)
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}_I{}^\mu &\equiv \frac{1}{\sqrt{g}} \frac{\delta S}{\delta A^I{}_\mu} \\
&= \mathfrak{D}_\nu \left(a_{IJ} F^{J\,\nu\mu} \right) + \tfrac{1}{4\sqrt{3}} \frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} C_{IJK} F^J{}_{\nu\rho} F^k{}_{\sigma\alpha} + g k_{I\,x} \mathfrak{D}^\mu \phi^x \qquad (4.27)
\end{aligned}$$

$$\begin{aligned}
\mathcal{E}^x &\equiv -\frac{g^{xy}}{\sqrt{g}} \frac{\delta S}{\delta \phi^y} \\
&= \mathfrak{D}_\mu \mathfrak{D}^\mu \phi^x + \tfrac{1}{4} g^{xy} \partial_y a_{IJ} F^{I\,\rho\sigma} F^J{}_{\rho\sigma}. \qquad (4.28)
\end{aligned}$$

The supersymmetry transformation rules for the bosonic fields are

$$\begin{aligned}
\delta_\epsilon e^a{}_\mu &= \tfrac{i}{2} \bar{\epsilon}_i \gamma^a \psi^i_\mu, \\
\delta_\epsilon A^I{}_\mu &= -\tfrac{i\sqrt{3}}{2} h^I \bar{\epsilon}_i \psi^i_\mu + \tfrac{i}{2} h^I_x \bar{\epsilon}_i \gamma_\mu \lambda^{i\,x}, \qquad (4.29) \\
\delta_\epsilon \phi^x &= \tfrac{i}{2} \bar{\epsilon}_i \lambda^{i\,x}.
\end{aligned}$$

and the corresponding transformation rules for the fermionic fields evaluated on vanishing fermions are

$$\delta_\epsilon \psi^i_\mu = \nabla_\mu \epsilon^i - \tfrac{1}{8\sqrt{3}} h_I F^{I\,\alpha\beta} \left(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha} \gamma_\beta \right) \epsilon^i, \qquad (4.30)$$

$$\delta_\epsilon \lambda^{i\,x} = \tfrac{1}{2} \left(\not{\mathfrak{D}} \phi^x - \tfrac{1}{2} h^x_I \not{F}^I \right) \epsilon^i, \qquad (4.31)$$

where $\nabla_\mu \epsilon^i$ is just the Lorentz-covariant derivative on the spinors, given in our conventions by

$$\nabla_\mu \epsilon^i = \left(\partial_\mu - \tfrac{1}{4} \not{\omega}_\mu \right) \epsilon^i. \qquad (4.32)$$

The equations of motion and the supersymmetry transformation rules are the straightforward covariantization of those of the ungauged theory, except for the addition of a source to the Maxwell equations corresponding to the charge carried by the scalar fields.

4.2 The supersymmetric solutions of $\mathcal{N}=1, d=5$ SEYM theories

In this section we are going to review first the results of Ref. [23] particularized to the case in which there are no hypermultiplets nor Fayet-Iliopoulos terms. We will simply focus on the final characterization of the supersymmetric solutions. Then, we will analyze the form of the solutions that admit an additional isometry and can, therefore, be dimensionally reduced to $d=4$, following Refs. [21, 92].

Let us start by reminding the reader that a solution of one of the $\mathcal{N}=1, d=5$ SEYM theories is said supersymmetric if the so-called *Killing spinor equations*

$$\delta_\epsilon \psi^i_\mu = 0\,, \qquad \delta_\epsilon \lambda^{i\,x} = 0\,, \tag{4.33}$$

written in the background of the solution can be solved for at least one spinor $\epsilon^i(x)$, which is then called *Killing spinor*. The supersymmetric solutions of these theories can be classified according to the causal nature of the Killing vector that one can construct as a bilinear of the Killing spinor $V^a = i\bar{\epsilon}_i\gamma^a\epsilon^i$ as *timelike* ($V^aV_a > 0$) or *null* ($V^aV_a = 0$). These two cases must be discussed separately.

4.2.1 Timelike supersymmetric solutions

The fields of the timelike supersymmetric solutions of $\mathcal{N}=1, d=5$ SEYM theories are completely determined by

1. A choice of 4-dimensional (obviously Euclidean) hyperKähler metric

$$d\hat{s}^2 = h_{\underline{mn}}(x)dx^m dx^n\,. \tag{4.34}$$

   Fields and operators defined in this space are customarily hatted.

2. Vector fields defined in the hyperKähler space, $\hat{A}^I$, such that their 2-form field strengths, $\hat{F}^I(\hat{A})$ are self-dual

$$\hat{\star}\hat{F}^I = +\hat{F}^I\,, \tag{4.35}$$

   with respect to the hyperKähler metric. This implies that $\hat{A}^I$ defines an instanton solution of the Yang-Mills equations in the hyperKähler space.

3. A set of functions in the hyperKähler space $\hat{f}_I$ satisfying the equation[9]

$$\hat{\mathfrak{D}}^2\hat{f}_I - \tfrac{1}{6}C_{IJK}\hat{F}^J\cdot\hat{F}^K = 0\,. \tag{4.36}$$

Given $h_{\underline{mn}}, \hat{A}^I, \hat{f}_I$, the physical fields can be reconstructed as follows:

[9] The coefficient of the second term is wrong by a factor of 2 in Refs. [21, 23] although all subsequent formulae are correct.

1. The functions $\hat{f}_I$ are proportional to the $h_I(\phi)$ defined in Eq. (4.2). The proportionality coefficient is called $1/\hat{f}$:

$$h_I/\hat{f} = \hat{f}_I\,. \tag{4.37}$$

The functions $h_I(\phi)$ satisfy a model-dependent constraint (analogous to the constraint satisfied by the functions $h^I(\phi)$, Eq. (4.1)). This constraint can be obtained by solving Eq. (4.2) for the h^I and substituting the result into Eq. (4.1). Therefore, the constraint has the form $F(h.) = 1$ where F is a function homogeneous of degree $3/2$ in the h_I and, substituting the above equation, one gets

$$\hat{f}^{-3/2} = F(\hat{f}.)\,. \tag{4.38}$$

Using this result in Eq. (4.37) one gets all the h_I as in terms of the $\hat{f}_I$

$$h_I = \hat{f}_I F^{-2/3}(\hat{f}.)\,, \tag{4.39}$$

and, using the expression of the h^I in terms of the h_I, one also gets the h^I in terms of the functions $\hat{f}_I$.

If the real special scalar manifold is symmetric, then we can use Eq. (4.14) to get

$$\hat{f}^{-3} = 27C^{IJK}\hat{f}_I\hat{f}_J\hat{f}_K\,. \tag{4.40}$$

2. The scalar fields ϕ^x can be obtained by inverting the functions $h_I(\phi)$ or $h^I(\phi)$. A parametrization which is always available is

$$\phi^x = h_x/h_0 = \hat{f}_x/\hat{f}_0\,. \tag{4.41}$$

3. Next, we define the 1-form $\hat{\omega}$ through the equation

$$\left(\hat{f}d\hat{\omega}\right)^+ = \frac{\sqrt{3}}{2}h_I\hat{F}^{I+}\,. \tag{4.42}$$

4. Having solved the above equation for $\hat{\omega}$ we have determined completely the metric of the timelike supersymmetric solutions, which is given by

$$ds^2 = \hat{f}^{\,2}(dt+\hat{\omega})^2 - \hat{f}^{\,-1}h_{\underline{mn}}dx^m dx^n\,, \tag{4.43}$$

5. Also, the complete 5-dimensional vector fields are given by

$$A^I = -\sqrt{3}h^I e^0 + \hat{A}^I\,, \quad \text{where} \quad e^0 \equiv \hat{f}(dt+\hat{\omega})\,, \tag{4.44}$$

so that the spatial components are

$$A^I{}_{\underline{m}} = \hat{A}^I{}_{\underline{m}} - \sqrt{3}h^I\hat{f}\hat{\omega}_{\underline{m}}\,. \tag{4.45}$$

The field strength can be written in the form

$$F^I = -\sqrt{3}\hat{\mathfrak{D}}(h^I e^0) + \hat{F}^I \,, \tag{4.46}$$

where $\hat{\mathfrak{D}}$ is the covariant derivative in the hyperKähler space with connection $\hat{A}^I$.

Timelike supersymmetric solutions with one isometry

We are particularly interested in the supersymmetric solutions that have an additional isometry. Following Refs. [91, 92] we assume that the additional isometry is a triholomorphic isometry of the hyperKähler metric (*i.e.* an isometry respecting the hyperKähler structure), in which case, as shown in Ref. [98] it must be a Gibbons-Hawking multi-instanton metric [96]. Assuming z is the coordinate associated to the additional isometry, these metrics can always be written in the form

$$h_{\underline{mn}}dx^m dx^n = H^{-1}(dz+\chi)^2 + H dx^r dx^r \,, \quad r=1,2,3\,, \tag{4.47}$$

where the z-independent function H and 1-form $\chi = \chi_{\underline{r}}dx^r$ are related by

$$d\chi = \star_3 dH \,, \tag{4.48}$$

$\star_3$ being the Hodge operator in $\mathbb{E}^3$. Assuming now that the rest of the bosonic fields of the timelike supersymmetric solutions are z-independent one can simplify Eqs. (4.35),(4.36) and (4.42).

Let us start with Eq. (4.35) and let us assume that the selfduality of $\hat{F}^I$ has been defined with respect to the frame and orientation

$$\hat{e}^{\,z} = H^{-1/2}(dz+\chi)\,, \qquad \hat{e}^{\,r} = H^{1/2}\delta^r{}_{\underline{r}}dx^r\,, \qquad \varepsilon_{z123} = +1\,. \tag{4.49}$$

Then, following Kronheimer [141],[10] Eq. (4.35) can be rewritten as Bogomol'nyi equations for a Yang-Mills-Higgs (YMH) system in the BPS limit in $\mathbb{E}^3$ [42]

$$\breve{\mathfrak{D}}_r \Phi^I = \tfrac{1}{2}\varepsilon_{rst}\breve{F}^I{}_{st}\,, \tag{4.50}$$

where the 3-dimensional Higgs field and the vector fields are given by[11]

$$\begin{aligned} 2\sqrt{6}\Phi^I &\equiv H\hat{A}^I{}_{\underline{z}}\,, \\ 2\sqrt{6}\breve{A}_{\underline{r}} &\equiv -\hat{A}^I{}_{\underline{r}} + \chi_{\underline{r}}\hat{A}^I{}_{\underline{z}}\,. \end{aligned} \tag{4.51}$$

Thus, we can always construct a selfdual YM instanton in a Gibbons-Hawking space from a (monopole) solution of the Bogomol'nyi equation of a YMH system in $\mathbb{E}^3$ $(\Phi^I, \breve{A}^I{}_{\underline{r}})$ [141]. Many solutions of these equations are known, specially in the spherically symmetric case[12]. In Ref. [47] this relation has been explored precisely for the SU(2) monopoles and instantons we are interested in, and we will make use of those results later.

[10] See also Ref. [47].

[11] We have rescaled the 3-dimensional fields by a factor of $-1/(2\sqrt{6})$ to conform to the normalization of the fields in $\mathcal{N}=2, d=4$ supergravity. See Appendix C.4.

[12] see Ref. [182] for the SU(2) case and Ref. [159] and references therein for more general gauge groups.

We can now use this result into Eq. (4.36), rewriting the 4-dimensional gauge vector in terms of the 3-dimensional gauge vector and Higgs field defined above and using the harmonicity of H and the Bogomol'nyi equation to get rid of $\breve{F}^I$ and $\breve{\mathfrak{D}}^2\Phi^I$ (which vanishes identically). The result is the equation in $\mathbb{E}^3$

$$\breve{\mathfrak{D}}^2\hat{f}_I - g^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J\Phi^K\hat{f}_M - 8C_{IJK}\breve{\mathfrak{D}}^2\left(\Phi^J\Phi^K/H\right) = 0\,. \tag{4.52}$$

Defining

$$\hat{f}_I \equiv L_I + 8C_{IJK}\Phi^J\Phi^K/H\,, \tag{4.53}$$

and using the condition Eq. (4.10) we find a linear equation for the functions L_I:

$$\breve{\mathfrak{D}}^2 L_I - g^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J\Phi^K L_M = 0\,. \tag{4.54}$$

Finally, let us consider Eq. (4.42). Defining $\hat{\omega}$ as

$$\hat{\omega} = \omega_5(dz+\chi) + \omega\,, \quad \text{where} \quad \omega = \omega_{\underline{r}}dx^r\,, \tag{4.55}$$

Eq. (4.42) gives an equation for ω_5 whose general solution is

$$\omega_5 = M + 16\sqrt{2}H^{-2}C_{IJK}\Phi^I\Phi^J\Phi^K + 3\sqrt{2}H^{-1}L_I\Phi^I\,, \quad \text{where} \quad d\star_3 dM = 0\,, \tag{4.56}$$

and the following equation for ω:

$$\star_3 d\omega = HdM - MdH + 3\sqrt{2}\left(\Phi^I\breve{\mathfrak{D}}L_I - L_I\breve{\mathfrak{D}}\Phi^I\right)\,, \tag{4.57}$$

whose integrability condition $d^2\omega = 0$ is satisfied wherever the above equations for H, M, Φ^I, L_I are satisfied.

Summarizing: we have identified a set of z-independent functions M, H, Φ^I, L_I and 1-forms ω, A^I, χ in $\mathbb{E}^3$ in terms of which we can write all the building blocks of the 5-dimensional timelike supersymmetric solutions admitting an isometry as follows:

$$h_I/\hat{f} = L_I + 8C_{IJK}\Phi^J\Phi^K/H\,, \tag{4.58}$$

$$\hat{\omega} = \omega_5(dz+\chi) + \omega\,, \tag{4.59}$$

$$\omega_5 = M + 16\sqrt{2}H^{-2}C_{IJK}\Phi^I\Phi^J\Phi^K + 3\sqrt{2}H^{-1}L_I\Phi^I\,, \tag{4.60}$$

$$\hat{A}^I = 2\sqrt{6}\left[H^{-1}\Phi^I(dz+\chi) - \breve{A}^I\right]\,, \tag{4.61}$$

$$\hat{F}^I = 2\sqrt{6}H^{-1}\left[\breve{\mathfrak{D}}\Phi^I\wedge(dz+\chi) - \star_3 H\breve{\mathfrak{D}}\Phi^I\right]\,, \tag{4.62}$$

provided that they satisfy the following set of equations:

$$\begin{aligned} d\star_3 dM &= 0\,, & (4.63)\\ \star_3 dH - d\chi &= 0\,, & (4.64)\\ \star_3\breve{\mathfrak{D}}\Phi^I - \breve{F}^I &= 0\,, & (4.65)\\ \breve{\mathfrak{D}}^2 L_I - g^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J\Phi^K L_M &= 0\,, & (4.66)\\ \star_3 d\omega - \Big\{HdM - MdH + 3\sqrt{2}(\Phi^I\breve{\mathfrak{D}}L_I - L_I\breve{\mathfrak{D}}\Phi^I)\Big\} &= 0\,. & (4.67)\end{aligned}$$

For symmetric real special manifolds we can use Eq. (4.40) to write the metric function $\hat{f}$ explicitly in terms of the tensor C_{IJK} and the functions M, H, Φ^I, L_I:

$$\begin{aligned}\hat{f}^{-3} = {} & 3^3 C^{IJK}L_IL_JL_K + 3^4\cdot 2^3 C^{IJK}C_{KLM}L_IL_J\Phi^L\Phi^M/H \\ & +3\cdot 2^6 L_I\Phi^I C_{JKL}\Phi^J\Phi^K\Phi^L/H^2 + 2^9\left(C_{IJK}\Phi^I\Phi^J\Phi^K\right)^2/H^3\,.\end{aligned} \tag{4.68}$$

Let us compare the above formulae with those of the ungauged case (in Ref. [21] in our conventions). It is easy to see that all the functions M, H, Φ^I, L_I become standard harmonic functions in $\mathbb{E}^3$. Furthermore, the functions Φ^I are related to the functions K^I used in that reference by

$$\Phi^I = +\tfrac{1}{2\sqrt{2}}K^I\,. \tag{4.69}$$

Dimensional reduction of the timelike supersymmetric solutions with one isometry

The supersymmetric solutions that admit an additional isometry can be dimensionally reduced to supersymmetric solutions of $\mathcal{N}=2, d=4$ supergravity using the formulae in Appendix C.4[13]. Performing explicitly this reduction will allow us to simplify the tasks of oxidation and reduction of supersymmetric solutions.

First of all, the metric of the 4-dimensional solutions obtained through the dimensional reduction takes the conventional conformastationary form of the timelike supersymmetric solutions of the $\mathcal{N}=2, d=4$ theory

$$ds^2 = e^{2U}(dt+\omega)^2 - e^{-2U}dx^rdx^r\,, \tag{4.70}$$

where the 1-form $\omega = \omega_{\underline{r}}dx^r$ is precisely the 1-form given in Eq. (4.57) and the metric function e^{-2U} is given by

$$e^{-2U} = 2\sqrt{\frac{(\hat{f}^{\,-1}H)^3 - (\omega_5 H^2)^2}{4H^2}}\,. \tag{4.71}$$

[13]These formulae are valid for any field configuration, supersymmetric or not.

We can compare the equations satisfied by the building blocks of the timelike supersymmetric solutions of gauged $\mathcal{N}=1, d=5$ supergravity (C.19)-(6.17) with the equations satisfied by the building blocks of the timelike supersymmetric solutions of gauged $\mathcal{N}=2, d=4$ supergravity Ref. [122,156], which we rewrite here for convenience adapting slightly the notation to avoid confusion with the different accents used to distinguish the different gauge fields:

$$-\tfrac{1}{\sqrt{2}}\star_3 \breve{\mathfrak{D}}\mathcal{I}^\Lambda - \breve{F}^\Lambda = 0\,, \tag{4.72}$$

$$\breve{\mathfrak{D}}^2\mathcal{I}_\Lambda - \tfrac{1}{2}g^2 f_{\Lambda\Sigma}{}^{\Omega} f_{\Delta\Omega}{}^{\Gamma}\mathcal{I}^\Sigma\mathcal{I}^\Delta\mathcal{I}_\Gamma = 0\,, \tag{4.73}$$

$$\star_3 d\omega - 2\left[\mathcal{I}_\Lambda\breve{\mathfrak{D}}\mathcal{I}^\Lambda - \mathcal{I}^\Lambda\breve{\mathfrak{D}}\mathcal{I}_\Lambda\right] = 0\,, \tag{4.74}$$

where $\breve{\mathfrak{D}}$ is the gauge covariant derivative associated to the modified gauge connection in $\mathbb{E}^3$

$$\breve{A}^\Lambda{}_{\underline{m}} \equiv A^\Lambda{}_{\underline{m}} - \omega_{\underline{m}}A^\Lambda{}_t\,. \tag{4.75}$$

The notation that we are using has implicit the identification of the gauge potentials $\breve{A}$ coming from 5 and 4 dimensions, except for $\Lambda=0$. Using the formulae in Appendix C.4 with the modifications explained in the last paragraph we can identify[14]

$$\chi_{\underline{m}} = -2\sqrt{2}\breve{A}^0{}_{\underline{m}}\,, \tag{4.76}$$

which leads to the identifications

$$\Phi^I = -\tfrac{1}{\sqrt{2}}\mathcal{I}^{I+1}\,, \qquad L_I = \tfrac{2}{3}\mathcal{I}_{I+1}\,, \qquad H = 2\mathcal{I}^0\,, \qquad M = -\mathcal{I}_0\,. \tag{4.77}$$

These are the only formulae we need to relate timelike supersymmetric solutions in $\mathcal{N}=1, d=5$ supergravity with one additional isometry to timelike supersymmetric solutions in cubic model of $\mathcal{N}=2, d=4$ supergravity with $\mathcal{I}^0 \neq 0$[15].

For symmetric real special scalar manifolds we can use the explicit form of $\hat{f}$ in Eq. (5.8) together with the expression for ω_5 in Eq. (6.18) to get

$$\begin{aligned} e^{-2U} = {} & 2\Big\{\tfrac{3^3}{4}HC^{IJK}L_IL_JL_K - 2^{7/2}MC_{IJK}\Phi^I\Phi^J\Phi^K + 2\cdot 3^4 C^{IJK}C_{KLM}L_IL_J\Phi^L\Phi^M \\ & -\tfrac{3^2}{2}\left(L_I\Phi^I\right)^2 - \tfrac{3}{\sqrt{2}}HML_I\Phi^I - \tfrac{1}{4}M^2H^2\Big\}^{1/2}\,. \end{aligned} \tag{4.78}$$

Then, using the identifications Eqs. (4.77) together with the second of Eqs. (C.35) we get

[14]The 0th components are never gauged if the dimensional reduction is simple (not generalized).
[15]Those with $\mathcal{I}^0 = 0$ are related to null supersymmetric 5-dimensional solutions.

$$
\begin{aligned}
e^{-2U} &= 2\left\{(d^{ijk}\mathcal{I}_j\mathcal{I}_l - \tfrac{2}{3}\mathcal{I}_0\mathcal{I}^i)(d_{ilm}\mathcal{I}^l\mathcal{I}^m + \tfrac{2}{3}\mathcal{I}^0\mathcal{I}_i) + \tfrac{4}{9}\mathcal{I}^0\mathcal{I}_0\mathcal{I}^i\mathcal{I}_i\right. \\
&\qquad \left. -(\mathcal{I}^0\mathcal{I}_0 + \mathcal{I}^i\mathcal{I}_i)^2\right\}^{1/2} .
\end{aligned}
\tag{4.79}
$$

4.2.2 Null supersymmetric solutions

The general form of the null supersymmetric solutions of $\mathcal{N}=1, d=5$ SEYM is quite involved [23], but it simplifies dramatically when one assumes the existence of an additional isometry so that all the fields are independent of the two null coordinates u and v. These are the solutions which will become timelike supersymmetric solutions of $\mathcal{N}=2, d=4$ SEYM upon dimensional reduction and, therefore, we are going to describe only these.

u-independent null supersymmetric solutions

The metric of the general null supersymmetric solutions of $\mathcal{N}=1, d=5$ SEYM can always brought into the form [23][16]

$$
ds^2 = 2\ell du(dv + Kdu + \sqrt{2}\omega) - \ell^{-2}dx^r dx^r\,, \tag{4.80}
$$

where the functions ℓ, K and the 1-form $\omega = \omega_{\underline{r}}dx^r$ are v-independent. We are going to assume also u-independence of all the fields throughout.

After the partial gauge fixing $A^I{}_{\underline{v}} = 0$, the gauge fields are decomposed as[17]

$$
A^I = A^I{}_{\underline{u}}du - 2\sqrt{6}\breve{A}^I\,, \qquad \breve{A}^I = \breve{A}^I{}_{\underline{r}}dx^r\,, \tag{4.81}
$$

and the vector field strengths take the form[18]

$$
F^I = (\sqrt{2/3}\ell^2 h^I \star_3 d\omega - \psi^I)\wedge du + \sqrt{3}\star_3 \breve{\mathfrak{D}}(h^I/\ell)\,, \tag{4.82}
$$

where the ψ^I are some 1-forms in $\mathbb{E}^3$ satisfying

$$
h_I\psi^I = 0\,, \tag{4.83}
$$

to be determined and $\breve{\mathfrak{D}}$ is the gauge-covariant derivative on $\mathbb{E}^3$ with respect to the connection $\breve{A}^I$.

Finally, the scalar fields will be determined by the equations obeyed by the scalar functions h^I, which follow from the equations of motion.[19]

Let us start by analyzing the Bianchi identities of the vector field strength. They lead to the following two sets of equations:

[16] We have changed the notation and normalization with respect to [23] to avoid possible confusions between the objects that appear in the null and timelike cases.

[17] As the notation suggests, the gauge fields $\breve{A}^I$ are the same as the $\mathcal{N}=2, d=4$ fields denoted with the same symbols, according to the general formulae of Appendix C.4. The same is true of the 1-form ω.

[18] All the operators in the r.h.s. are defined in $\mathbb{E}^3$.

[19] The field configurations that we have just described are automatically supersymmetric, but not necessarily solutions of all the equations of motion and Bianchi identities [23].

$$-\tfrac{1}{2\sqrt{2}}\star_3\breve{\mathfrak{D}}(h^I/\ell)-\breve{F}^I \;=\; 0\,, \tag{4.84}$$

$$\breve{\mathfrak{D}}A^I{}_{\underline{u}}-\sqrt{2/3}\ell^2h^I\star_3 d\omega+\psi^I \;=\; 0\,. \tag{4.85}$$

Eq. (4.84) is the Bogomol'nyi equation on $\mathbb{E}^3$ and, thus, we define the Higgs field

$$\Sigma^I\equiv-\tfrac{1}{2\sqrt{2}}h^I/\ell\,. \tag{4.86}$$

Multiplying Eq. (4.85) by h_I and using Eq. (4.83) together with $h_Ih^I=1$ we get the equation that defines ω

$$d\omega=\sqrt{3/2}\ell^{-2}\star_3\left\{h_I\breve{\mathfrak{D}}A^I{}_{\underline{u}}\right\}\,. \tag{4.87}$$

Defining the functions

$$K_I\equiv C_{IJK}\Sigma^JA^K{}_{\underline{u}}\,, \tag{4.88}$$

the above equation takes a much more familiar form

$$d\omega=4\sqrt{6}\star_3\left\{\Sigma^I\breve{\mathfrak{D}}K_I-K_I\breve{\mathfrak{D}}\Sigma^I\right\}\,, \tag{4.89}$$

whose integrability condition is

$$\Sigma^I\breve{\mathfrak{D}}^2K_I=0\,. \tag{4.90}$$

Given the functions Σ^I, K_I and the gauge fields $\breve{A}^I$ we can solve this equation for ω. It should be possible to find the functions $A^I{}_{\underline{u}}$ in terms of Σ^I, K_I[20] and, plugging these result in Eq. (4.85), compute directly the 1-forms ψ^I.

From the Maxwell equations one obtains the equations that determine the functions K_I:

$$\breve{\mathfrak{D}}^2K_I-g^2f_{IJ}{}^Lf_{KL}{}^M\Sigma^J\Sigma^KK_M=0\,, \tag{4.91}$$

from which the integrability condition Eq. (4.90) follows automatically.

Finally, defining

$$N\equiv K-\sqrt{2}A^I{}_{\underline{u}}K_I\,, \tag{4.92}$$

the last non-trivial equation of motion, from the Einstein equations, takes the simple form

$$\nabla^2N=0\,. \tag{4.93}$$

[20] This will certainly be the case for the particular model we are going to study, but we have not found (even for just the symmetric case) a general way of solving Eq. (4.88) for $A^I{}_{\underline{u}}$.

Summarizing: we have identified a set of u-independent functions Σ^I, K_I, N and 1-forms $\omega, \breve{A}^I$ on $\mathbb{E}^3$ in terms of which we can write all the building blocks of the 5-dimensional u-independent null supersymmetric solutions, assuming we can solve Eq. (4.88) for $A^I{}_{\underline{u}}$, as follows:

$$h^I/\ell = -2\sqrt{2}\Sigma^I\,, \tag{4.94}$$

$$K = N + \sqrt{2}A^I{}_{\underline{u}}K_I\,, \tag{4.95}$$

$$A^I = A^I{}_{\underline{u}}du + 2\sqrt{6}\breve{A}^I\,, \tag{4.96}$$

$$F^I = \breve{\mathfrak{D}}A^I{}_{\underline{u}} \wedge du + \sqrt{3}\star_3 \breve{\mathfrak{D}}(h^I/\ell)\,, \tag{4.97}$$

provided the following equations are satisfied[21]:

$$\star_3\breve{\mathfrak{D}}\Sigma^I - \breve{F}^I = 0\,, \tag{4.98}$$

$$\breve{\mathfrak{D}}^2K_I - g^2 f_{IJ}{}^L f_{KL}{}^M\Sigma^J\Sigma^K K_M = 0\,, \tag{4.99}$$

$$d\omega - 4\sqrt{6}\star_3\left\{\Sigma^I\breve{\mathfrak{D}}K_I - K_I\breve{\mathfrak{D}}\Sigma^I\right\} = 0\,, \tag{4.100}$$

$$\nabla^2 N = 0\,. \tag{4.101}$$

Using Eq. (4.1), we find a general expression for ℓ:

$$\ell^{-3} = -2^{9/2}C_{IJK}\Sigma^I\Sigma^J\Sigma^K\,. \tag{4.102}$$

Dimensional reduction of the u-independent null supersymmetric solutions

Using the general formulae in Appendix C.4, the u-independent solutions that we have considered can be dimensionally reduced to timelike supersymmetric solutions of $\mathcal{N} = 2, d = 4$ SEYM along the spacelike coordinate z defined by

$$u = \tfrac{1}{\sqrt{2}}(t+z)\,, \qquad v = \tfrac{1}{\sqrt{2}}(t-z)\,, \tag{4.103}$$

with metrics of the form Eq. (4.70) where the 1-form $\omega = \omega_{\underline{r}}dx^r$ is precisely the 1-form given in Eq. (4.80) and the metric function e^{-2U} is given by

$$e^{-2U} = \sqrt{\ell^{-3}(1-K)} = \sqrt{-2^{9/2}C_{IJK}\Sigma^I\Sigma^J\Sigma^K(1-N-\sqrt{2}A^I{}_{\underline{u}}K_I)}\,. \tag{4.104}$$

[21] The gauge coupling constant is the 4-dimensional one.

In order to express entirely the metric function in terms of the functions K_I, Σ^I, N we need to solve Eq. (4.88) for $A^I{}_{\underline{u}}$ as a function of K_I, Σ^I, which we do not know how to do in general. We can still compare the equations satisfied by these functions (4.98)-(4.101) with those satisfied by $\mathcal{I}^\Lambda, \mathcal{I}_\Lambda$ in $\mathcal{N}=2, d=4$ SEYM (4.72)-(4.74) knowing that the vector fields $\breve{A}^I$ and the 1-form ω are the same objects. We find that

$$\Sigma^I = -\tfrac{1}{\sqrt{2}}\mathcal{I}^{I+1}\,, \qquad K_I = -\tfrac{1}{2\sqrt{3}}\mathcal{I}_{I+1}\,, \tag{4.105}$$

while N must be proportional to either $\mathcal{I}^0$ or $\mathcal{I}_0$. Since a wave moving in the internal z direction should give rise to a 4-dimensional electric charge, it must be

$$N \sim \mathcal{I}_0\,, \tag{4.106}$$

but the precise coefficient cannot be determined from this comparison alone. We have to find a more explicit expression for e^{-2U}.

4.3 5-dimensional supersymmetric non-Abelian solutions of the SU(2)-gauged ST[2, 5] model

In this section we are going to consider a particular model of $\mathcal{N}=1, d=5$ supergravity that admits an SU(2) gauging. This model is related to the SU(2)-gauged ST[2, 5] model of $\mathcal{N}=2, d=4$ supergravity some of whose solutions we have studied in Ref. [46]. We will use the relations derived in the previous section to find relations between the non-Abelian supersymmetric solutions of both theories.

We start by describing the 4- and 5-dimensional models and their SU(2) gauging.

4.3.1 The models

The ST[2, 5] model is a cubic model of $\mathcal{N}=2, d=4$ supergravity coupled to 5 vector multiplets *i.e.* a model with a prepotential of the form

$$\mathcal{F} = -\tfrac{1}{3!}\frac{d_{ijk}\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0}\,, \quad i = 1, 2\cdots, 5 \tag{4.107}$$

where the fully symmetric tensor d_{ijk} has as only non-vanishing components

$$d_{1\alpha\beta} = \eta_{\alpha\beta}\,, \quad \text{where} \quad (\eta_{\alpha\beta}) = \text{diag}(+-\cdots-)\,, \quad \text{and} \quad \alpha,\beta = 2,\cdots,5\,. \tag{4.108}$$

The 5 complex scalars parametrize the coset space

$$\frac{\text{SL}(2,\mathbb{R})}{\text{SO}(2)} \times \frac{\text{SO}(2,4)}{\text{SO}(2)\times\text{SO}(4)}\,, \tag{4.109}$$

and the group SO(3) acts in the adjoint on the coordinates $\alpha = 3, 4, 5$. These are the directions we are going to gauge and we will denote them with capital $A, B, \ldots$. This is the only information we need in order to construct supersymmetric solutions, but more details on the construction of this theory can be found in Ref. [46]. We will need the form of the metric function in terms of the functions $\mathcal{I}^M$:

$$e^{-2U} = 2\sqrt{(\mathcal{I}^\alpha\mathcal{I}^\beta\eta_{\alpha\beta} + 2\mathcal{I}^0\mathcal{I}_1)(\mathcal{I}_\alpha\mathcal{I}_\beta\eta^{\alpha\beta} - 2\mathcal{I}^1\mathcal{I}_0) - (\mathcal{I}^0\mathcal{I}_0 - \mathcal{I}^1\mathcal{I}_1 + \mathcal{I}^\alpha\mathcal{I}_\alpha)^2}\,., \tag{4.110}$$

The models of the ST$[2,n]$ family are related to the effective theory of the Heterotic string and compactified on T^6 by a consistent truncation: the 10-dimensional effective theory is $\mathcal{N}=1, d=10$ supergravity coupled to 16 10-dimensional vector multiplets with gauge group U(1). Upon dimensional reduction on a generic T^6 one gets $\mathcal{N}=4, d=4$ supergravity coupled to $16+6=22$ vector multiplets, whose duality group is

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \times \frac{\mathrm{SO}(6,22)}{\mathrm{SO}(6)\times\mathrm{SO}(22)}\,. \tag{4.111}$$

Observe that SO(6) acts on the 6 vectors in the supergravity multiplet and SO(22) on the 22 matter vector fields. The coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is parametrized by the only scalar in the supergravity multiplet. A consistent truncation to $\mathcal{N}=2, d=4$ eliminates 4 vectors from the $\mathcal{N}=4$ supergravity multiplet and one of the remaining two vectors becomes a matter vector field from the $\mathcal{N}=2$ point of view and comes in the same multiplet as the complex scalar that parametrizes the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The result is a ST$[2,23]$ model from which one can consistently eliminate vector multiplets to arrive to the ST$[2,5]$ model we are dealing with.

This is the story at a generic point in the moduli space of the Heterotic strings on T^6. At certain points, though, there is a enhancement of gauge symmetry usually associated to an increase in the number of massless vector fields that we must take into account in the effective theory. Our SU(2)-gauged model of $\mathcal{N}=2, d=4$ supergravity can be interpreted as the effective theory describing the simplest of these situations in which the enhancement of gauge symmetry arises in the sector of the 16 original 10-dimensional vector fields.

The ST$[2,5]$ model is related to a model of $\mathcal{N}=1, d=5$ supergravity coupled to 4 vector multiplets determined by the tensor $C_{i-1,j-1,k-1} = \frac{1}{6}d_{ijk}$ so its only non-vanishing components are

$$C_{0xy} = \tfrac{1}{6}\eta_{xy}\,,\text{where}\quad (\eta_{xy}) = \mathrm{diag}(+-\cdots-)\,,\quad \text{and}\quad x,y=1,\cdots,4\,. \tag{4.112}$$

The 4 real scalars in the vector multiplets parametrize the coset space

$$\frac{\mathrm{SO}(1,4)}{\mathrm{SO}(4)}\,. \tag{4.113}$$

Now the group SO(3) acts in the adjoint on the coordinates $x=2,3,4$ and, if we gauge it, the theory goes to the gauged 4-dimensional model we just discussed. It should be obvious after the 4-dimensional discussion that that this model can be interpreted as a truncation of the effective theory of the Heterotic string compactified on T^5.

Again, we do not need many more details of the theory in order to construct supersymmetric solutions. For timelike supersymmetric solutions admitting an additional isometry we will need the metric function, which follows directly from the generic expression Eq. (5.8)

$$\begin{aligned}\hat{f}^{\,-1} &= H^{-1}\left\{\tfrac{1}{4}\left(6HL_0+8\eta_{xy}\Phi^x\Phi^y\right)\left[9H^2\eta^{xy}L_xL_y+48H\Phi^0L_x\Phi^x\right.\right.\\ &\quad \left.\left.+64(\Phi^0)^2\eta_{xy}\Phi^x\Phi^y\right]\right\}^{1/3}\end{aligned} \tag{4.114}$$

This metric function and the 4-dimensional one e^{-2U} are related by Eq. (4.71) using Eq. (6.18) and the relations between the functions $\mathcal{I}^M$ and H, M, L_I, Φ^I in Eqs. (4.77), which we rewrite for this specific pair of models for convenience:

$$\begin{aligned} H &= 2\mathcal{I}^0\,, & \Phi^0 &= -\tfrac{1}{\sqrt{2}}\mathcal{I}^1\,, & \Phi^1 &= -\tfrac{1}{\sqrt{2}}\mathcal{I}^2\,, & \Phi^A &= -\tfrac{1}{\sqrt{2}}\mathcal{I}^A\,,\\ M &= -\mathcal{I}_0\,, & L_0 &= \tfrac{2}{3}\mathcal{I}_1\,, & L_1 &= \tfrac{2}{3}\mathcal{I}_2\,, & L_A &= \tfrac{2}{3}\mathcal{I}_A\,,\end{aligned} \tag{4.115}$$

For u-independent null supersymmetric solutions we first need to solve Eq. (4.88) for $A^I{}_{\underline{u}}$. For this model, we find

$$A^0{}_{\underline{u}} = 6\frac{\Sigma^x K_x - \Sigma^0 K_0}{(\eta\Sigma\Sigma)}\,, \qquad A^x{}_{\underline{u}} = 6\frac{\eta^{xy}K_y(\eta\Sigma\Sigma) - \Sigma^x(\Sigma^y K_y - \Sigma^0 K_0)}{\Sigma^0(\eta\Sigma\Sigma)}\,, \tag{4.116}$$

where $(\eta\Sigma\Sigma) \equiv \eta_{xy}\Sigma^x\Sigma^y$, so that

$$e^{-2U} = 2\sqrt{(\mathcal{I}^\alpha\mathcal{I}^\beta\eta_{\alpha\beta})[\mathcal{I}_\alpha\mathcal{I}_\beta\eta^{\alpha\beta}+\mathcal{I}^1(1-N)] - (-\mathcal{I}^1\mathcal{I}_1+\mathcal{I}^\alpha\mathcal{I}_\alpha)^2}\,.\,, \tag{4.117}$$

and we arrive at the following identifications

$$\begin{aligned} 0 &= \mathcal{I}^0\,, & \Sigma^0 &= -\tfrac{1}{\sqrt{2}}\mathcal{I}^1\,, & \Sigma^1 &= -\tfrac{1}{\sqrt{2}}\mathcal{I}^2\,, & \Sigma^A &= -\tfrac{1}{\sqrt{2}}\mathcal{I}^A\,,\\ N &= 1+2\mathcal{I}_0\,, & K_0 &= -\tfrac{1}{2\sqrt{3}}\mathcal{I}_1\,, & K_1 &= -\tfrac{1}{2\sqrt{3}}\mathcal{I}_2\,, & K_A &= -\tfrac{1}{2\sqrt{3}}\mathcal{I}_A\,.\end{aligned} \tag{4.118}$$

4.3.2 The solutions

We are ready to put to work the machinery developed in the previous sections. We are going to consider the simplest cases first.

A simple $5d$ black hole with non-Abelian hair

In order to add non-Abelian fields to our solutions it is exceedingly useful to consider metrics with one additional isometry, because, then, we can make use of our knowledge of the spherically symmetric solutions of the Bogomol'nyi equations of the SU(2) YMH system found by Protogenov in Ref. [182]. However, this isometry cannot be translational if we want to find spherically-symmetric black holes because, then, the full 5-dimensional solution will have a translational isometry. Thus, we will start with the choice $H = 1/r$ $(r^2 = y^r y^r)$[22] which, as we have shown in Ref. [47], relates the *colored monopole* solution[23] to the the BPST instanton, which is spherically symmetric in $\mathbb{E}^4$.

We are, thus, going to consider a configuration with the following non-vanishing functions:

$$H = \frac{1}{r}\,, \quad L_0 = A_0 + \frac{q_0}{4r}\,, \quad L_1 = A_1 + \frac{q_1}{4r}\,, \quad \Phi^A = -f(r)\delta^A{}_r y^r\,, \tag{4.119}$$

where q_0, q_1 are electric charges in some convenient normalization, A_0, A_1 are constants to be determined through the normalization of the metric and the scalar fields at infinity and $f(r)$ is the function (not to be mistaken by $\hat{f}$) that characterizes the Higgs field in the spherically-symmetric monopole solutions of Ref. [182][24]).

The next step consists in finding the 1-forms $\chi, \breve{A}^I, \omega$ and functions L_I that satisfy Eqs. (6.14)-(6.17) for the above non-vanishing functions. ω is closed and can be set to zero, the functions L_I can also be set to zero while[25]

$$\chi = d\varphi + \cos\theta d\psi\,, \qquad \breve{A}^A = h(r)\varepsilon^A{}_{\underline{rs}} y^r dy^s\,, \tag{4.120}$$

where $h(r)$ is the function that characterizes the gauge field of the monopole solution (see Appendix A.5)). The spacetime metric is, then,

$$ds^2 = \hat{f}^{\,2}dt^2 - \hat{f}^{\,-1}\left[r(d\varphi + \cos\theta d\psi)^2 + \frac{1}{r}(dr^2 + r^2 d\Omega^2_{(2)})\right]\,, \tag{4.121}$$

where

$$d\Omega^2_{(2)} = d\theta^2 + \sin^2\theta d\psi^2\,, \tag{4.122}$$

and, upon the change of coordinates $r = \rho^2/4$, it becomes

$$ds^2 = \hat{f}^{\,2}dt^2 - \hat{f}^{\,-1}dx^m dx^m\,, \quad \text{where} \quad dx^m dx^m = d\rho^2 + \rho^2 d\Omega^2_{(3)}\,. \tag{4.123}$$

For this configuration, the metric function Eq. (5.16) is given by

$$\hat{f}^{\,-1} = 3\sqrt[3]{\tfrac{1}{2}\left(L_0 - \tfrac{4}{3}r^3 f^2\right)(L_1)^2}\,, \tag{4.124}$$

and it immediately follows that in order for the solution to be asymptotically regular, the monopole must be the colored one for which $r^3 f_\lambda^2 \sim 1/r$, because for all the rest $r^3 f^2 \sim r$ (see Appendix A.5). With this choice,[26] as shown in Ref. [47][27], the gauge field $\hat{A}^A = \hat{A}^A{}_{\underline{m}} dx^m$ that follows from the use of Eq. (4.61) is that of a BPST instanton in $\mathbb{E}^4$:

[22] We need to distinguish between the Cartesian coordinates in $\mathbb{E}^3$, which we will denote by y^r and the Cartesian coordinates in $\mathbb{E}^4$, which we will denote by x^m. The former are not a simple subset of the latter.

[23] This monopole is characterized by a vanishing magnetic charge.

[24] See Appendix A.5 in which we have written all of Protogenov's solutions.

[25] The choice of angular coordinates is conditioned by the relation between the monopole and instanton as explained in Ref. [47]. We will identify the compact coordinate z with the angular coordinate φ.

[26] We are going to study the consequences of the other choices in Section 4.3.2.

[27] More specifically, the gauge field one gets is $\hat{A}_L^{A(+)}$.

$$\hat{A}^A = \tfrac{1}{\tilde{g}} \frac{1}{1+\lambda^2\rho^2/4}\, v_L^A \,, \tag{4.125}$$

where v_L^A are the SU(2) left-invariant Maurer-Cartan 1-forms[28]. Since the scalar functions h^A vanish for this configuration, the full 5-dimensional vector fields are, according to Eq. (4.44), given by

$$\begin{aligned}
A^0 &= \tfrac{3^{5/2}}{2}(L_1)^2 \hat{f}^{\,3} dt \,,\\
A^1 &= 3^{5/2} L_1 \left(L_0 - \tfrac{4}{3} r^3 f_\lambda^2\right) \hat{f}^{\,3} dt \,,\\
A^A &= \tfrac{1}{\tilde{g}} \frac{1}{1+\lambda^2\rho^2/4}\, v_L^A \,.
\end{aligned} \tag{4.127}$$

Finally, the only non-vanishing scalar is given by by

$$\phi \equiv h_1/h_0 = \frac{L_1}{L_0 - \frac{4}{3} r^3 f_\lambda^2} \,. \tag{4.128}$$

The integration constants are readily identified in terms of the asymptotic value of the scalar as

$$A_0 = \tfrac{2^{1/3}}{3}\phi_\infty^{-2/3} \,, \qquad A_1 = \tfrac{2^{1/3}}{3}\phi_\infty^{1/3} \,, \tag{4.129}$$

while the mass and the area of the event horizon are given by

$$M = 2^{-1/3} 3^{1/2} \left[\phi_\infty^{2/3} q_0 + 2\phi_\infty^{-1/3} q_1\right] \,, \tag{4.130}$$

$$\frac{A}{2\pi^2} = \sqrt{\tfrac{3^3}{2}\left(q_0 - \frac{2^7}{\tilde{g}^2}\right)(q_1)^2} \,. \tag{4.131}$$

This solution can be understood as the result of the addition of a BPST instanton to a standard 2-charge Abelian solution. This addition does not produce any observable effects at spatial infinity, like, for instance, a change in the mass, but does produce a change in the near-horizon geometry and in the entropy.

The metric function of the 4-dimensional solution e^{-2U} that one obtains by dimensional reduction is related to the metric function of the 5-dimensional solution by

$$e^{-4U} = \frac{1}{r}\hat{f}^{-3} \,, \tag{4.132}$$

[28] In our conventions, these are given by

$$\left\{\begin{aligned}
v_L^1 &= \sin\psi\, d\theta - \sin\theta\cos\psi\, d\varphi \,,\\
v_L^2 &= -\cos\psi\, d\theta - \sin\theta\sin\psi\, d\varphi \,,\\
v_L^3 &= -(d\psi + \cos\theta\, d\varphi) \,,
\end{aligned}\right. \qquad \text{and} \qquad dv_L^A + \tfrac{1}{2}\epsilon_{ABC}\, v_L^B \wedge v_L^C = 0 \,. \tag{4.126}$$

which implies that the 4- and 5-dimensional solutions cannot be asymptotically flat at the same time. In particular, with the choice made above (corresponding to a colored monopole in $d = 4$) $e^{-2u} \sim r^{-1/2}$ at spatial infinity, a behavior that does not correspond to any known vacuum. With the monopoles we discarded, however, we get an asymptotically-flat solution. The near-horizon behavior is simultaneously good in $d = 4$ and $d = 5$.

A rotating $5d$ black hole with non-Abelian hair

In the context of timelike supersymmetric solutions of $\mathcal{N} = 1, d = 5$ supergravity rotation can be added by switching on the harmonic function M [118]. More specifically, we add to the static solution we just constructed the harmonic function

$$M = \frac{J/2}{4r}\,, \tag{4.133}$$

which only appears in Eq. (6.18). The metric of the new solution is

$$ds^2 = \hat{f}^{\,2}\left[dt + \frac{J/2}{4r}(d\varphi + \cos\theta d\psi)\right]^2 - \hat{f}^{\,-1}\left[r(d\varphi + \cos\theta d\psi)^2 + \frac{1}{r}(dr^2 + r^2 d\Omega^2_{(2)})\right]\,, \tag{4.134}$$

where the metric function $\hat{f}$ is still given by Eq. (4.124). The scalar field ϕ and the non-Abelian vector field A^A take the same value as in the static solution while the two Abelian vector fields are modified by the change

$$dt \longrightarrow dt + \frac{J/2}{4r}(d\varphi + \cos\theta d\psi)\,, \tag{4.135}$$

which describes the presence of a magnetic dipole moment associated to the rotation.

Asymptotically, the only novelty is the off-diagonal term $\sim J/\rho^2 dt(d\varphi + \cos\theta d\psi)$ which corresponds to identical values of the two Casimirs of the angular momentum, both proportional to J, so this solution is a non-Abelian generalization of the Breckenridge–Myers-Peet–Vafa (BMPV) spinning black hole [43, 93]. The mass has the same expression in terms of the charges as in the static case.

In the near-horizon limit, if the behavior of the metric function $\hat{f}$ is

$$\hat{f}^{-1} \sim R^2/r\,, \tag{4.136}$$

for some constant R, the metric can be rewritten in the form

$$ds^2 \sim R^2 d\Pi^2_{(2)} - R^2 d\Omega^2_{(2)} - R^2\left[\cos\alpha(d\varphi + \cos\theta d\psi) - \sin\alpha \frac{r}{R^2} d\phi\right]^2\,, \tag{4.137}$$

where ϕ is the rescaled time coordinate, defined as follows

$$\phi \equiv t/X\,, \qquad X/R \equiv \sqrt{1 - [J/(2R)^3]^2} \equiv \cos\alpha\,, \qquad (2R)^3 \equiv \sqrt{\tfrac{3^3}{2}\left(q_0 - \frac{2^7}{\tilde{g}^2}\right)(q_1)^2}\,, \tag{4.138}$$

and $d\Pi^2_{(2)}, d\Omega^2_{(2)}$ are the metrics of the 2-dimensional Anti-de Sitter and sphere of unit radius

$$d\Pi^2_{(2)} \equiv \left(\frac{r}{R^2}\right)^2 d\phi^2 - \frac{dr^2}{r^2}\,. \tag{4.139}$$

The constant-time sections of the event horizon are squashed 3-spheres with metric

$$-\,ds^2 = R^2\left\{\cos^2\alpha(d\varphi+\cos\theta d\psi)^2 + d\Omega^2_{(2)}\right\}\,, \tag{4.140}$$

and area

$$\frac{A}{2\pi^2} = \sqrt{\tfrac{3^3}{2}\left(q_0 - \frac{2^7}{\tilde{g}^2}\right)(q_1)^2 - J^2}\,. \tag{4.141}$$

A more general solution

In Section 4.3.2 we used the colored monopole solution in order to obtain an asymptotically flat black-hole solution in the simplest way. However, we can also use the monopoles in the 2-parameter family, for which, asymptotically, $r^3 f^2 \sim r$ if we switch on additional harmonic functions and choose the values of the integration constants appropriately so that the metric functions $\hat{f}(r), \omega_5, \omega$ give an asymptotically-flat solution.

Throughout the following discussion, it is convenient to have the explicit form of these functions for $H = 1/r$, $\Phi^A = -f(r)\delta^A{}_r y^r$ and $L_A = 0$ at hand:

$$\begin{aligned}
\hat{f}^{\,-3} &= 27\left[\tfrac{1}{2}L_0 + \tfrac{2}{3}r[(\Phi^1)^2 - r^2 f^2]\right]\left[(L_1)^2 + \tfrac{16}{3}r\Phi^0 L_1\Phi^1 + \tfrac{64}{9}(r\Phi^0)^2[(\Phi^1)^2 - r^2 f^2]\right]\,,\\
\omega_5 &= M + 8\sqrt{2}\,r^2\Phi^0[(\Phi^1)^2 - r^2f^2] + 3\sqrt{2}\,rL_i\Phi^i\,,\\
\star_3 d\omega &= \tfrac{1}{r}dM - Md\tfrac{1}{r} + 3\sqrt{2}\left(\Phi^i dL_i - L_i d\Phi^i\right)\,,
\end{aligned} \tag{4.142}$$

where $i = 0, 1$. Apart from the functions H and Φ^A, we are going to consider the following non-vanishing harmonic functions

$$\{\Phi^0, \Phi^1, L_0, L_1, M\}\,, \tag{4.143}$$

with

$$\Phi^{0,1} = A^{0,1} + \frac{p^{0,1}}{4r}\,, \qquad L_{0,1} = A_{0,1} + \frac{q_{0,1}}{4r}\,, \qquad M = a + \frac{b}{4r}\,. \tag{4.144}$$

$\hat{f}^{-3}$ is a product of two factors. Our strategy will be to make the constant piece of Φ^1, A^1, cancel the constant piece in $rf(r)$, μ/g so that $[(\Phi^1)^2 - r^2f^2]$ is asymptotically $\mathcal{O}(1/r)$[29]:

[29]We choose the positive sign for simplicity.

$$A^1 = \mu/g\,. \tag{4.145}$$

This ensures that the second term in $\hat{f}^{-3}$ diverges asymptotically at most as $\mathcal{O}(r)$ while the first is asymptotically constant. This constant can be made to vanish by choosing the constant piece of L_0, A_0, to be

$$A_0 = -\tfrac{8}{3}\frac{\mu}{g}\left(\frac{1}{g}+\frac{p^1}{4}\right)\,, \tag{4.146}$$

and now the first term is asymptotically $\mathcal{O}(1/r)$ and $\hat{f}^{-3}$ is asymptotically constant.

Next, we require that all the $\mathcal{O}(r^2)$, $\mathcal{O}(r)$ and $\mathcal{O}(1)$ terms in ω_5 vanish[30]. This gives two new relations[31] between the constants A_i, A^i and a. The vanishing of ω gives another relation between the same constants. Thus, requiring asymptotic flatness fixes the values of all these constants in terms of the Abelian charges p^i, q_i and μ and g. Finally the normalization of the metric at infinity also fixes the value of μ and the solution has no free moduli!

The values of the integration constants A_0, A^1 has been given above and the values of the rest are[32]

$$\begin{aligned}
A_1 &= -\tfrac{88}{3}A^0\left(\frac{1}{g}+\frac{p^1}{4}\right)\,,\\
A^0 &= \left\{\frac{\left(16p^0+4gp^0p^1+gq^1\right)\left(4+gp^1\right)^{-1}}{40\left(3q_0+(p^1)^2-\frac{16}{g^2}\right)\left(q_0+2(p^1)^2-\frac{32}{g^2}\right)}\right\}^{1/3}\,,\\
\mu &= A^0\left[\frac{32-2g^2(p^1)^2-g^2q_0}{16p^0+4gp^0p^1+gq_1}\right]\,,\\
a &= \sqrt{2}A^0\left[\frac{48}{g^2}+\frac{22p^1}{g}+\frac{5(p^1)^2}{2}-\frac{3q_0}{4}\right]-\sqrt{2}\left[\frac{22\mu p^0}{g^2}+\frac{11\mu p^0p^1}{2g}+\frac{3\mu q_1}{4g}\right]\,,\\
b &= J/2-6\sqrt{2}\left[\frac{p^0(p^1)^2}{2}+\frac{p^0q_0+p^1q^1}{8}-8\frac{p^0}{g^2}\right]\,,
\end{aligned} \tag{4.147}$$

where J is the angular momentum.

The mass of this solution is given by

[30] Observe that this does not imply the complete vanishing of ω_5: there are $\mathcal{O}(1/r)$ terms that give angular momentum (which could be cancelled by the integration constant b in M) and also $\mathcal{O}(e^{-4\mu r})$ terms that cannot be cancelled. Therefore, the metric is not static even if the angular momentum is set to zero.

[31] The above values of A_0 and A^1 make the $\mathcal{O}(r^2)$ term vanish.

[32] We have not reexpressed the 4-dimensional gauge coupling constant g in terms of the 5-dimensional, $\tilde{g}$ to have simpler expressions.

$$M = \frac{\pi A^0}{2G}\left[3q_0 + (p^1)^2 - \frac{16}{g^2}\right]\left[3\frac{\mu}{g}q_1 + 8\left(\frac{1}{g}+\frac{p^1}{4}\right)\left(10A^0\left(\frac{24}{g}+5p^1\right) - 9\frac{\mu}{g}p^0\right)\right]. \tag{4.148}$$

and the area of the horizon is

$$\frac{A}{2\pi^2} = \sqrt{\tfrac{1}{2}\left[3q_0 + (p^1)^2 - \frac{16}{g^2}\right]\left[3q_1 + 2p^1p^0 - \frac{8p^0}{g}\right]\left[3q_1 + 2p^0p^1 + \frac{8p^0}{g}\right] - J^2}\,. \tag{4.149}$$

Null supersymmetric non-Abelian $5d$ solutions from $4d$ black holes and global monopoles

Using the general results of the preceding sections it is very easy to construct null supersymmetric solutions by uplifting 4-dimensional timelike supersymmetric solutions with $\mathcal{I}^0$. In particular, we can uplift the black-hole and global-monopole solutions of the ST[2, 5] model recently constructed in Ref. [46]. In this paper we will focus on the single center solutions only.

The 4-dimensional solutions depend on the following non-vanishing $\mathcal{I}^M$

$$\begin{aligned}
\mathcal{I}^1 &= A^1 + \frac{p^1/\sqrt{2}}{r}\,, \quad \mathcal{I}^2 = A^2 + \frac{p^2/\sqrt{2}}{r}\,, \quad \mathcal{I}^A = \sqrt{2}\,\delta^A{}_p x^p f(r)\,,\\
\mathcal{I}_0 &= A_0 + \frac{q_0/\sqrt{2}}{r}\,,
\end{aligned} \tag{4.150}$$

where $f(r)$ is the function $f_{\mu,s}$ or f_λ in Appendix A.5 corresponding to one of the spherically-symmetric BPS SU(2) monopoles, p^1, p^2, q_0 are magnetic and electric charges and A^1, A^2, A_0 integration constants to be determined in terms of the asymptotic values of the scalars and the metric.

The 5-dimensional metric is that of an intersection of a string lying along the z direction and a pp-wave propagating along the same direction:

$$ds^2 = 2\ell du(dv + Kdu) - \ell^{-2}d\vec{x}^2_{(3)}\,, \tag{4.151}$$

where

$$\ell^{-3} = 4\mathcal{I}^1[(\mathcal{I}^2)^2 - 2r^2f^2]\,, \qquad K = 1 + 2\mathcal{I}_0\,. \tag{4.152}$$

The scalar fields, defined by $\phi^x \equiv h^x/h^0$, are given by

$$\phi^1 = \mathcal{I}^2/\mathcal{I}^1\,, \qquad \phi^A = -\delta^A{}_p x^p f(r)/\mathcal{I}^1\,, \tag{4.153}$$

and the vector fields are given by

$$A^{0,1} = -2\sqrt{6}p^{1,2}A\,, \qquad A^A = 2\sqrt{6}h(r)\epsilon^A{}_{rs}x^r dx^s\,, \tag{4.154}$$

where A is the vector field of a Dirac magnetic monopole of unit charge, satisfying $dA = \star_3 d\frac{1}{r}$ and $h(r)$ is the function $h_{\mu,s}$ or h_λ in Appendix A.5 corresponding to one of the spherically-symmetric BPS SU(2) monopoles.

The 4-dimensional electric charge q_0 corresponds to the momentum of the 5-dimensional gravitational wave in the z direction and none of the scalar and vector fields depend on it. For the sake of simplicity we are going to set it to zero ($q_0 = 0$ and $\mathcal{I}_0 = -1/2$ so $K = 0$) and we are going to analyze the string solutions with the above scalar and vector fields and with metric

$$ds^2 = \ell(dt^2 - dz^2) - \ell^{-2} d\vec{x}^2_{(3)}\,, \tag{4.155}$$

with the metric function ℓ given as above.

The metric will be regular in the $r \to 0$ limit if $\ell \sim r$ or $\ell \sim$ constant. These two behaviors are, respectively, those of extremal black strings in the near-horizon limit and those of global monopoles. Let us consider each case separately.

Global string-monopoles These are the string-like solutions that, upon dimensional reduction along z, give the spherically-symmetric global monopoles constructed in Ref. [46]. They can be constructed with $f(r) = f_{\mu,s=0}(r)$ (the BPST 't Hooft-Polyakov monopole) and with $p^1 = p^2 = 0$, so that

$$\ell^{-3} = 4A^1[(A^2)^2 - 2r^2 f^2_{\mu,\,s=0}]\,, \quad \phi^1 = A^2/A^1\,, \quad \phi^A = -\sqrt{2}\delta^A{}_r x^r f_{\mu,s=0}(r)/A^1\,, \tag{4.156}$$

and the only non-trivial vector field is A^A.

The integration constants $A^{1,2}, \mu$ are given by

$$A^1 = \frac{1}{\chi_\infty^{1/3}}\,, \quad A^2 = \frac{\phi^1_\infty}{\chi_\infty^{1/3}}\,, \quad \mu = \frac{g|\phi_\infty|}{\sqrt{2}\chi_\infty^{1/3}}\,, \quad \chi_\infty \equiv 4[(\phi^1_\infty)^2 - |\phi_\infty|^2]\,, \tag{4.157}$$

where $|\phi_\infty|^2$ is the asymptotic value of the gauge-invariant combination $\phi^A\phi^A$, and the string's tension (simply defined as minus the coefficient of $1/r$ in the large-r expansion of g_{tt}) is given by [71]

$$T_{\text{monopole}} = \frac{32|\phi_\infty|}{\sqrt{3}\chi_\infty^{2/3}}\frac{1}{|\tilde{g}|}\,. \tag{4.158}$$

These are globally regular solutions with no horizons, like their 4-dimensional analogues.

Black strings They must necessarily have non-vanishing magnetic charges $p^{1,2}$ in order to have a regular horizon. This horizon will be a 2-dimensional surface characterized by being normal to 2 linearly independent null vectors. The mass and entropy of the black string will depend on the choice of monopole.

Let us first consider the BPST 't Hooft-Polyakov monopole (or equivalently, let us add magnetic charges $p^{1,2}$ to the above global monopole). In this case, the relation between the integration constants $A^{1,2}, \mu$ and the asymptotic values of the scalars will be the same as before. The string's tension and the area of the horizon contain contributions from the magnetic charges p^1, p^2:

$$\begin{aligned} T &= \tfrac{1}{3\sqrt{2}}\chi_\infty^{1/3}\left[p^1 + 8\frac{\phi^1_\infty}{\chi_\infty}p^2\right] + T_{\text{monopole}}\,, && (4.159)\\ \frac{A}{4\pi} &= 2\left[p^1(p^2)^2\right]^{2/3}\,. && (4.160)\end{aligned}$$

When we consider the more general 't Hooft-Polyakov-Protogenov monopole we find that the area of the horizon receives a contribution from the non-Abelian charge,

$$\frac{A}{4\pi} = 2\left\{p^1\left[(p^2)^2 - \frac{2}{g^2}\right]\right\}^{2/3}\,. \tag{4.161}$$

4.4 Conclusions

In this paper we have studied the general procedure to construct timelike and null supersymmetric solutions of $\mathcal{N}=1, d=5$ SEYM theories that can be dimensionally reduced to timelike solutions of $\mathcal{N}=2, d=4$ SEYM theories. These solutions, therefore, can also be constructed by oxidation of the 4-dimensional solutions and we have striven to clarify this procedure and find the relations between the 4- and 5-dimensional fields and the 4- and 5-dimensional equations they satisfy. The relation between instantons in 4-dimensional hyperKähler spaces and monopoles satisfying the Bogomol'nyi equation in $\mathbb{E}^3$ found by Kronheimer plays a crucial role in this relation and, in combination with the results obtained in Ref. [47], it allows us to construct spherically-symmetric 5-dimensional solutions that contain YM instantons. The standard oxidation of monopoles gives rise to 5-dimensional solutions that have an additional translational isometry and cannot be spherically symmetric.

We have exploited the general results to construct the first 5-dimensional black-hole and black-string solutions with non-Abelian YM fields. The simplest black-hole solutions contain the field of a BPST instanton in the so-called *base space* and their behavior is similar to that of the colored black holes found in 4-dimensional SEYM theories [154,159]: the non-Abelian YM field cannot be "seen" at spatial infinity, it does not contribute to the mass, but it can be seen in the near-horizon limit and it contributes to the entropy. One can compare the entropies of the simplest non-Abelian black hole with that of another black hole with the same Abelian charges and moduli (and, henceforth, with the same mass). The entropy of the former is always smaller, so it is entropically favorable to lose the non-Abelian field. It is not clear by which mechanism this can happen.

We have also found more complicated black-hole solutions which contain the field of the instantons that one obtains by reducing Protogenov monopoles in the so-called *base space*. Those instantons are not regular in flat space and, in general, the spacetime metrics they give rise to are not asymptotically flat. We have shown that a judicious choice of the integration constants (and, hence, of the moduli) in terms of the charges produces a metric that is not only asymptotically flat with positive mass but also has a regular horizon. Thus, at special points in the moduli space of the scalar manifold, additional non-Abelian black-hole solutions are possible. In these solutions, the YM fields do contribute to the mass and to the entropy.

Finally, we have also found black-string solutions by conventional oxidation of non-Abelian black-hole solutions from 4 dimensions. One of them is a globally-regular string-monopole solution and the rest are more conventional solutions.

It is clear that the new solutions that we have constructed need further study. Their string-theoretic interpretation could be very interesting. The model we have chosen to construct explicit solutions is a truncation of the effective theory of the heterotic string compactified to 5 dimensions and can, alternatively, be seen as associated to the compactification of the type IIB theory in K3 times a circle. This should simplify a bit the task and, perhaps, open the way to a microscopic interpretation of entropies that depend on parameters that do not appear at infinity. Work in this direction is in progress.

5

A non-Abelian Black Ring

This chapter is based on

Tomas Ortín and Pedro F. Ramírez
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The discovery of black rings by Emparan and Reall in Ref. [80] showed how two important properties of 4-dimensional asymptotically-flat black holes, uniqueness/no-hair and spherical topology of the event horizon (which, for the 5-dimensional black ring, is $S^2 \times S^1$), could be violated in higher dimensions.[1] For a range of values of the conserved charges (mass, angular momenta) that may characterize an uncharged black ring, a different black-ring and a black-hole solutions are also possible. For charged black rings (the first of which was constructed in Ref. [76]) the non-uniqueness becomes infinite; for the same conserved electric charges one can construct black rings with regular horizons with magnetic dipole momenta taking continuous values in some interval [79]. Despite being innocuous to the conserved charges, these dipole momenta do contribute to the BH entropy. The construction of supersymmetric black-ring solutions in minimal [77] or matter-coupled $\mathcal{N} = 1, d = 5$ supergravity [30, 78, 90, 91, 169] using the general classification of supersymmetric solutions of these theories started in Ref. [92] opened up the possibility of constructing very general families of black-ring solutions with various kinds of electric charges and moduli in which these issues could be studied.

The violation of the no-hair conjecture by non-Abelian fields in 4-dimensions is also a well-known but less stressed fact, perhaps because the first solutions in which this was observed [39, 142, 206], black-hole generalizations of the "Bartnik-McKinnon particle" [11] with asymptotically vanishing gauge charges, were purely numerical, which makes more difficult their study and understanding.[2] The first black-holes with non-Abelian hair (not related to the embedding of an Abelian field into a non-Abelian one through a singular gauge transformation) given in an analytical form were found using supersymmetry techniques in the context of $\mathcal{N} = 2, d = 4$ Super-Einstein-Yang-Mills (SEYM) theory[3]

[1]See, for instance, the reviews [32, 81, 82] and references therein.

[2]For a review on hairy and non-Abelian black-hole solutions see Ref. [208] or the more recent Ref. [207].

[3]This theory is the simplest $\mathcal{N} = 2$ supersymmetric generalization of the Einstein-Yang-Mills theory. This supersymmetrization requires the addition of scalar fields to the pure Einstein-Yang-Mills theory in order to complete $\mathcal{N} = 2, d = 4$ vector supermultiplets and, often, the addition of full vector supermultiplets to fulfill the requirements of Special Geometry. There may be more than one way of performing this supersymmetrization. Thus, there are more than one $\mathcal{N} = 2, d = 4$ SEYM theory with gauge group SU(2), for instance. These theories are also known as non-Abelian gauged $\mathcal{N} = 2, d = 4$ supergravity coupled to vector supermultiplets.

in Refs. [123] and [154] using the general classification of the timelike supersymmetric solutions of these theories made in Ref. [122]. The black-hole solutions constructed in Ref. [154] include the field of an SU(2) *coloured* monopole found by Protogenov in [182] which also has asymptotically vanishing gauge charge. The monopole charge does contribute to the entropy, though. These black holes, which can be seen as the result of adding the coloured monopole to a standard black hole with Abelian charges, modifying the entropy but none of the asymptotic charges, were called *coloured black holes* and they seem to be ubiquitous [159].

The results of Ref. [122] have been used more recently to construct new single-center and two-center non-Abelian solutions of $\mathcal{N} = 2, d = 4$ SEYM models that can be obtained by dimensional reduction of $\mathcal{N} = 1, d = 5$ SEYM models[4] in Ref. [46].

One of the main goals of that exercise was to open the possibility for the construction of the first non-Abelian black-hole solutions in $d = 5$ by oxidation to $d = 5$ of those solutions, because the direct construction using the general classification of timelike supersymmetric solutions of Refs. [20,23] turns out to be too complicated. This can only be done for certain models of the lower dimensional theory. The oxidation itself turned out to be a non-trivial exercise if one wanted to construct solutions without spatial translation isometries (which would be black strings instead of black holes), but, as was shown in Ref. [47], one can use non-trivial cycles to perform the reduction and still preserve supersymmetry, basically using Kronheimer's mechanism [141]. Both kinds of black solutions (strings and holes) were recently constructed in Ref. [158].

The $d = 5$ non-Abelian black holes constructed there are, again, coloured black holes, with asymptotically vanishing gauge fields. They can be understood as the result of adding a BPST instanton to a black hole with Abelian charges, leaving the mass and electric charges unmodified. Just as in the 4-dimensional case, the non-Abelian field does contribute to the entropy. The BPST instanton field turns out to be related by dimensional redox to the coloured monopole at the heart of the 4-dimensional coloured black holes.

It is natural to try to see if black-rings also admit the addition non-Abelian instanton fields and the effect this addition may have on the mass and entropy. In this paper we are going to construct and study a regular supersymmetric black-ring solution of $\mathcal{N} = 1, d = 5$ SEYM with a *distorted* BPST instanton. We start by reviewing in Section 5.1 the recipe that we are going to use to construct timelike supersymmetric solutions, which was obtained in Ref. [158]. In Section 5.2 we will carry out the construction of the solution after which we will study its regularity and we will compute its essential properties. In Section 5.3 we will study the limit in which the black ring becomes a non-Abelian rotating black hole. Our conclusions are in Section 7.2.

5.1 The recipe to construct solutions

In Ref. [158] we have found a procedure to construct systematically timelike supersymmetric solutions admitting an additional spacelike isometry (with adapted coordinate z) of any $\mathcal{N} = 1, d = 5$ Super-Einstein-Yang-Mills (SEYM) characterized by the tensor C_{IJK}

[4] Again, these are the simplest, but not unique $\mathcal{N} = 1$ (minimal) supersymmetrizations of the $d = 5$ Einstein-Yang-Mills theory and the supersymmetrization requires the addition of, at least, scalars. They also go by the name of non-Abelian-gauged $\mathcal{N} = 1, d = 5$ coupled to vector supermultiplets.

and the structure constants ${f_{IJ}}^K$:[5]

1. Find a set of t- and z-independent functions M, H, Φ^I, L_I and 1-forms ω, A^I, χ in $\mathbb{E}^3$ satisfying the equations (defined in $\mathbb{E}^3$ as well)

$$d \star_3 dM = 0\,, \tag{5.1}$$

$$\star_3 dH - d\chi = 0\,, \tag{5.2}$$

$$\star_3 \breve{\mathfrak{D}}\Phi^I - \breve{F}^I = 0\,, \tag{5.3}$$

$$\breve{\mathfrak{D}}^2 L_I - g^2 {f_{IJ}}^L {f_{KL}}^M \Phi^J \Phi^K L_M = 0\,, \tag{5.4}$$

$$\star_3 d\omega - \left\{HdM - MdH + 3\sqrt{2}(\Phi^I \breve{\mathfrak{D}} L_I - L_I \breve{\mathfrak{D}}\Phi^I)\right\} = 0\,. \tag{5.5}$$

The first two equations state that H and M are harmonic functions on $\mathbb{E}^3$. Once H is given, the second equation (which is the Abelian Bogomol'nyi equation on $\mathbb{E}^3$ [42]) can be solved for χ. Eq. (6.15) is the general Bogomol'nyi equation on $\mathbb{E}^3$. In the ungauged (Abelian) directions, it implies that the Φ^I are harmonic functions on $\mathbb{E}^3$ and, once they are chosen, the corresponding vectors $\breve{A}^I$ can be determined. In the non-Abelian directions, the equation becomes non-linear and one has to find simultaneously solutions for the functions Φ^I and gauge fields $\breve{A}^I$ through adequate ansatzs or other methods. Eq. (6.16) is automatically solved if we choose $L_I \propto \Phi^I$ (or zero). Finally, Eq. (6.17) can always be solved if the other equations are solved (because they solve its integrability condition), except, perhaps, at the singularities of the functions where, strictly speaking, the other equations are not solved. In most cases, the integrability condition can be solved by a choice of integration constants in the functions H, M, L_I, Φ^I. Then, of course, one has to integrate explicitly Eq. (6.17) to obtain ω.

2. Using them, reconstruct the solution's 5-dimensional spacetime fields as follows:

   (a) The scalars can be found from this equation for the quotients $h_I(\phi)/\hat{f}$

$$h_I/\hat{f} = L_I + 8C_{IJK}\Phi^J\Phi^K/H\,, \tag{5.6}$$

   because there is always a parametrization of the scalar manifold such that

$$\phi^x \equiv h_x/h_0\,. \tag{5.7}$$

   With the above equation for the quotients $h_I(\phi)/\hat{f}$ one can also determine the function $\hat{f}$. For the special case of symmetric scalar manifolds, it is given by[6]

[5] Our conventions are those of Refs. [21,23] and are based on Ref. [35]. The supersymmetric solutions of the most general $\mathcal{N} = 1, d = 5$ supergravity theory including vector supermultiplets and hypermultiplets and generic gaugings were characterized in Ref. [23]. The inclusion of tensor supermultiplets was considered in Ref. [20].

[6] In this expression, $C^{IJK} \equiv C_{IJK}$.

$$
\begin{aligned}
\hat{f}^{-3} &= 3^3 C^{IJK} L_I L_J L_K + 3^4 \cdot 2^3 C^{IJK} C_{KLM} L_I L_J \Phi^L \Phi^M / H \\
&\quad + 3 \cdot 2^6 L_I \Phi^I C_{JKL} \Phi^J \Phi^K \Phi^L / H^2 + 2^9 \left(C_{IJK} \Phi^I \Phi^J \Phi^K\right)^2 / H^3 \,.
\end{aligned} \tag{5.8}
$$

(b) The metric has the form

$$
ds^2 = \hat{f}^{\,2}(dt + \hat{\omega})^2 - \hat{f}^{\,-1} d\hat{s}^2 \,, \tag{5.9}
$$

where $\hat{f}$ has been determined above, the 1-form $\hat{\omega}$ is given by[7]

$$
\hat{\omega} = \omega_5 (dz + \chi) + \omega \,, \tag{5.10}
$$

$$
\omega_5 = M + 16\sqrt{2} H^{-2} C_{IJK} \Phi^I \Phi^J \Phi^K + 3\sqrt{2} H^{-1} L_I \Phi^I \,, \tag{5.11}
$$

and where the 4-dimensional Euclidean metric $d\hat{s}^2$ is given by[8]

$$
d\hat{s}^2 = H^{-1}(dz + \chi)^2 + H dx^r dx^r \,, \quad r = 1, 2, 3 \,. \tag{5.12}
$$

(c) The vector fields and their corresponding field strengths are given by

$$
\begin{aligned}
A^I &= -\sqrt{3} h^I \hat{f} (dt + \hat{\omega}) + \hat{A}^I \,, \\
F^I &= -\sqrt{3} \hat{\mathfrak{D}} [h^I \hat{f} (dt + \hat{\omega})] + \hat{F}^I \,,
\end{aligned} \tag{5.13}
$$

where the vector fields $\hat{A}^I$, defined on the 4-dimensional Euclidean space $d\hat{s}^2$, and their field strengths are given by

$$
\begin{aligned}
\hat{A}^I &= 2\sqrt{6} \left[H^{-1} \Phi^I (dz + \chi) - \breve{A}^I \right] \,, \\
\hat{F}^I &= 2\sqrt{6} H^{-1} \left[\breve{\mathfrak{D}} \Phi^I \wedge (dz + \chi) - \star_3 H \breve{\mathfrak{D}} \Phi^I \right] \,,
\end{aligned} \tag{5.14}
$$

where $\hat{\mathfrak{D}}$ (resp. $\breve{\mathfrak{D}}$) is the exterior gauge-covariant derivative with respect to the connection $\hat{A}^I$ (resp. $\breve{A}^I$).

In Ref. [158] we used this recipe to construct black-hole solutions with non-Abelian gauge and scalar fields for the SU(2)-gauged ST[2, 5] model.[9] This model has 4 vector multiplets and, hence, 4 scalar fields that parametrize the symmetric space SO(1, 3)/SO(3). It is defined by a tensor C_{IJK} with the following non-vanishing components

[7] The unhatted ω is the one occurring in Eq. (6.17).

[8] With H and χ related by Eq. (6.14), this is a hyperKähler metric admitting a triholomorphic Killing vector, also known as Gibbons-Hawking metric [96, 98]. We will also denote the compact coordinate z by φ. It will be assumed to take values in $[0, 4\pi)$.

[9] Actually, this is the name of the model of $\mathcal{N} = 2, d = 4$ supergravity one obtains by dimensional reduction.

$$C_{0xy} = \tfrac{1}{6}\eta_{xy}\,,\text{where} \quad (\eta_{xy}) = \text{diag}(+-\cdots-)\,, \quad \text{and} \quad x,y = 1,\cdots,4\,. \tag{5.15}$$

The directions to be gauged are the last three, which we will denote by indices $\alpha, \beta, \ldots = 2, 3, 4$. the ungauged directions will be denoted by indices $i, j, \ldots = 0, 1$.

Being a symmetric space, we can use Eq. (5.8) to write the metric function $\hat{f}$ as a function of the building blocks H, L_I, Φ^I:

$$\begin{aligned} \hat{f}^{\,-1} &= H^{-1}\left\{\tfrac{1}{4}\left(6HL_0 + 8\eta_{xy}\Phi^x\Phi^y\right)\left[9H^2\eta^{xy}L_xL_y + 48H\Phi^0L_x\Phi^x\right.\right. \\ &\quad \left.\left. +64(\Phi^0)^2\eta_{xy}\Phi^x\Phi^y\right]\right\}^{1/3}\,. \end{aligned} \tag{5.16}$$

Now, in order to find solutions of this model, we just need to find building blocks that satisfy Eqs. (C.19)-(6.17). In the next section we will just do this to find a solution that describes a black ring.

5.2 Non-Abelian Black Rings

5.2.1 Construction of the Solution

Inspired by Refs. [90, 91], we choose a point $\vec{x}_0 \equiv \left(0, 0, -R^2/4\right)$ in $\mathbb{E}^3$ and a harmonic function N with a pole at that point,

$$N \equiv \frac{1}{|\vec{x} - \vec{x}_0|} \equiv \frac{1}{r_n}\,, \tag{5.17}$$

in terms of which we can write the non-vanishing building blocks in the ungauged directions as

$$H = \frac{1}{r}\,, \qquad M = \tfrac{3}{4}\lambda_i q^i\left(1 - |\vec{x}_0|N\right)\,, \qquad \Phi^i = -\frac{q^i}{4\sqrt{2}}N\,, \qquad L_i = \lambda_i + \frac{Q_i - C_{ijk}q^jq^k}{4}N\,. \tag{5.18}$$

These functions contain the integration constants q^i, Q_i and λ_i. The first two can be interpreted as charges. The latter, whose value will be restricted by requirements such as the normalization of the metric at infinity, are moduli. Eq. (C.19) is satisfied automatically. Eq. (6.14) is satisfied with

$$\chi = \cos\theta d\psi\,, \tag{5.19}$$

where $r, \theta \in (0, \pi)$ and $\psi \in [0, 2\pi)$ are spherical coordinates centered at $r = |\vec{x}| = 0$ with the definitions and orientation

$$\left\{\begin{array}{lcl} x^1 &=& r\sin\theta\sin\psi\,, \\ x^2 &=& r\sin\theta\cos\psi\,, \\ x^3 &=& -r\cos\theta\,, \end{array}\right. \qquad \epsilon^{123} = \epsilon^{r\theta\psi} = +1\,. \tag{5.20}$$

Eqs. (6.15) are satisfied with

$$\breve{A}^i = -\frac{q^i}{4\sqrt{2}} \cos\theta_n d\psi_n \,, \tag{5.21}$$

where $r_n, \theta_n \in (0, \pi)$ and $\psi_n \in [0, 2\pi)$ are spherical coordinates centered at $r_n = |\vec{x}_n| = 0$ with the definitions

$$\left\{ \begin{array}{lcl} x_n^1 & \equiv & x^1 - x_0^1 = r_n \sin\theta_n \sin\psi_n \,, \\ x_n^2 & \equiv & x^2 - x_0^2 = r_n \sin\theta_n \cos\psi_n \,, \\ x_n^3 & \equiv & x^3 - x_0^3 = -r_n \cos\theta_n \,, \end{array} \right. \tag{5.22}$$

and the same orientation as the spherical coordinates centered at $r = 0$.

Eqs. (6.16) in the Abelian directions are trivially satisfied because all $f_{ij}{}^k = 0$ and, finally, the integrability condition of Eq. (6.17) is identically satisfied for the chosen integration constants and ω can be found by integration. We will compute ω for the complete solution later.

The above functions are enough to construct an Abelian black ring. Now, we excite the gauged directions of this solution by adding to it a solution of the SU(2) Bogomol'nyi equations on $\mathbb{E}^3$ (6.15)

$$\Phi^\alpha = \frac{1}{g r_n \left(1 + \lambda^2 r_n\right)} \delta^\alpha_{s+1} \frac{x_n^s}{r_n} \,, \qquad \breve{A}^\alpha = \frac{1}{g r_n \left(1 + \lambda^2 r_n\right)} \epsilon^\alpha{}_{rs} \frac{x_n^s}{r_n} dx_n^r \,. \tag{5.23}$$

This solution, originally found by Protogenov in Ref. [182], describes a magnetic colored monopole placed at $r_n = 0$. It is singular at $r_n = 0$ as a field configuration in $\mathbb{E}^3$, but this behaviour can change when we analyze the whole picture. In fact, we showed in Ref. [47] that the monopole field gives rise to a BPST instanton in $\mathbb{E}^4$ through (6.11), and we used this result in Ref. [158] to construct a regular black hole of the same supergravity theory we consider in this work.

In the present case we obtain a different instanton field configuration from (6.11), which we call *distorted* BPST, because the pole of the harmonic function H is placed in a different point ($r = 0$) than that of the coloured monopole ($r_n = 0$). This distorted BPST is singular at $r_n = 0$, which might turn the black ring solution ill-defined. Happily this is not the case. The complete vector field contains the instanton plus an additional term, see (6.4), where the latter cancels precisely this divergence at that critical point

$$\lim_{r_n \to \infty} \left(-\sqrt{3} h^I \hat{f} \omega_5 + 2\sqrt{6} H^{-1} \Phi^I\right) (dz + \chi) = 0 \,. \tag{5.24}$$

Observe that in the ungauged case the Φ^αs would have been harmonic functions $-q^\alpha N/(4\sqrt{2})$ and the combinations $C_{ijk} q^j q^k$ should have been replaced by $C_{iJK} q^J q^K$. Here the asymptotic behaviour of the non-Abelian gauge field indicates that the "non-Abelian q^αs" do not contribute in the same way the q^is do. However, they have a similar near-horizon behaviour.

The above functions define completely the solution. In what follows we are going to analyze its metric to show that it describes a regular black ring and to compute its main properties.

5.2.2 Analysis of the Solution

In this analysis it is convenient to use two set of coordinates: those centered at $r = 0$, (r, θ, ψ, defined in Eq. (5.20)) supplemented by the time coordinate t and the angular coordinate φ, and those centered at $r_n = 0$ (r_n, θ_n, ψ_n, defined in Eq. (5.22)) supplemented by the time coordinate t_n and the angular coordinate φ_n. The relations

$$\begin{aligned} r_n &= \left(r^2 + |\vec{x}_0|^2 - 2|\vec{x}_0| r \cos\theta\right)^{1/2}, \\ r &= \left(r_n^2 + |\vec{x}_0|^2 + 2|\vec{x}_0| r_n \cos\theta_n\right)^{1/2}, \\ |\vec{x}_0| &= r\cos\theta - r_n \cos\theta_n, \end{aligned} \tag{5.25}$$

will be useful in the computations.

The metric function $\hat{f}$ can be obtained by substituting the functions H, L_I, Φ^I in Eq. (5.8). At this moment we just want to impose the standard asymptotic normalization

$$\lim_{r\to\infty} \hat{f} = 1, \quad \Rightarrow \quad 3^3 C^{ijk}\lambda_i\lambda_j\lambda_k = \frac{3^3}{2}\lambda_0\lambda_1^2 = 1. \tag{5.26}$$

Now let us compute the only missing ingredient in the metric (6.3): the 1-form $\hat{\omega}$. Let us consider Eq. (6.17), which, upon substitution of the chosen functions H, M, L_I, Φ^I, can be written as

$$\begin{aligned} \star_3 d\omega &= -\tfrac{3}{4}\lambda_i q^i \Bigg\{ -\frac{1}{r^2}\left[1 - \frac{|\vec{x}_0| + r}{r_n} + \frac{r|\vec{x}_0|\left(r + |\vec{x}_0|\right)}{r_n^3}(1-\cos\theta)\right] dr \\ &+ \left[\frac{|\vec{x}_0|\sin\theta}{r_n^3}\left(r - |\vec{x}_0|\right)\right] d\theta \Bigg\}, \end{aligned} \tag{5.27}$$

and a solution can be readily found assuming ω has only one non-vanishing component, ω_ψ:[10]

$$\omega = -\tfrac{3}{4}\lambda_i q^i \left(\cos\theta - 1\right)\left[1 - \left(r + \frac{R^2}{4}\right)\frac{1}{r_n}\right] d\psi. \tag{5.28}$$

Observe that, since $L_\alpha = 0$ the non-Abelian terms do not affect ω. However, they do affect the whole 5-dimensional $\hat{\omega}$ given in Eq. (6.13) via ω_5 in Eq. (6.18):

$$\hat{\omega} = (F - G)\, d\varphi + (F - G\cos\theta)\, d\psi, \tag{5.29}$$

$$F = \frac{3\lambda_i q^i}{4}\left[1 - \left(r + \frac{R^2}{4}\right)\frac{1}{r_n}\right], \tag{5.30}$$

$$G = \frac{q^i}{16}\left[3\left(Q_i - C_{ijk}q^j q^k\right) + 2C_{ijk}q^j q^k \frac{r}{r_n}\right]\frac{r}{r_n^2} - \frac{2q^0}{g^2}\frac{r^2}{r_n^3\left(1 + \lambda^2 r_n\right)^2}. \tag{5.31}$$

[10]The expression coincides with that of [170] despite we have chosen $\vec{x}_0$ to be on the negative x^3 axis. This is because the coordinate θ has also a relative sign with respect to the used in that reference.

The last term in G has a non-Abelian origin. In the $r \to \infty$ limit in which the metric tends to Minkowski's (so we have an asymptotically flat solution), though, it is subdominant and we do not expect it to contribute to the the angular momentum of the solution.

So far we have been working in coordinates in which the hyperKähler metric Eq. (6.10) is of the form

$$d\hat{s}^2 = r\,(d\varphi + \cos\theta d\psi)^2 + \frac{1}{r}\left[dr^2 + r^2\left(d\theta^2 + \sin\theta^2 d\psi^2\right)\right]\,, \tag{5.32}$$

but, in order to compute mass and angular momentum, it is convenient to use a different coordinate system (also centered at $\vec{x} = 0$) $t, \Theta, \phi_1, \phi_2$, related to the former by

$$r = \frac{\rho^2}{4}\,, \qquad \theta = 2\Theta\,, \qquad \psi = \phi_1 - \phi_2\,, \qquad \varphi = \phi_1 + \phi_2\,, \tag{5.33}$$

in which the complete 5-dimensional metric is of the form

$$ds^2 = \hat{f}^2\,(dt + \hat{\omega})^2 - \hat{f}^{-1}\left[d\rho^2 + \rho^2\left(d\Theta^2 + \cos^2\Theta d\phi_1^2 + \sin^2\Theta d\phi_2^2\right)\right]\,, \tag{5.34}$$

with

$$\hat{\omega} = \left(2F - 2G\cos^2\Theta\right)d\phi_1 - 2G\sin^2\Theta d\phi_2\,. \tag{5.35}$$

The independent components of the angular momentum are now obtained from the metric behaviour in the $\rho \to \infty$ limit[11]

$$J_{\phi_1} \;=\; \lim_{\rho\to\infty}\frac{\pi|g_{t\phi_1}|\rho^2}{4G_N\cos^2\Theta} = \tfrac{1}{2\sqrt{3}}q^i\left(3Q_i - C_{ijk}q^jq^k\right)\,, \tag{5.36}$$

$$J_{\phi_2} \;=\; \lim_{\rho\to\infty}\frac{\pi|g_{t\phi_2}|\rho^2}{4G_N\sin^2\Theta} = \tfrac{1}{2\sqrt{3}}q^i\left(3Q_i - C_{ijk}q^jq^k + 6\lambda_iR^2\right)\,, \tag{5.37}$$

and, from the absence of contribution proportional to g, we see that they coincide with those of the Abelian black ring, as we expected.

Observe that these formulae allow us to identify

$$q^i\lambda_iR^2 = \tfrac{1}{\sqrt{3}}(J_{\phi_2} - J_{\phi_1})\,. \tag{5.38}$$

Before we move to study the possible presence of an event horizon, let us point out that the solution does not contain any Dirac-Misner strings.[12] Indeed, the $g_{t\phi_1}$ (resp. $g_{t\phi_2}$) metric component vanishes when the coordinate ϕ_1 (resp. ϕ_2) is not well defined, which happens at $\Theta = \pi/2$ ($\Theta = 0$).

The solution may have an event horizon at $\vec{x} = \vec{x}_0$, where the norm of the timelike Killing vector of the metric vanishes. In order to study the near horizon limit we need to

[11] We use units in which $G_N = \sqrt{3}\pi/4$.

[12] They could have been removed but only at the price of introducing closed timelike curves [161].

use a different coordinate system because several components of the metric blow up there in the coordinates we have been using so far. Recall the expression for the metric in the original frame centered at $\vec{x} = 0$

$$
\begin{aligned}
ds^2 &= \hat{f}^2 (dt + \omega)^2 + 2\hat{f}^2 \omega_5 (dt + \omega)(d\varphi + \cos\theta d\psi) \\
&\quad - \hat{f}^2 \left(\hat{f}^{-3} r - \omega_5^2\right) (d\varphi + \cos\theta d\psi)^2 - \hat{f}^{-1} r^{-1} dx^r dx^r \,.
\end{aligned}
\tag{5.39}
$$

We first go to the auxiliary frame centered at the horizon with spherical coordinates and take the $r_n \to 0$ limit. The functions that appear in the metric behave in this limit as follows

$$\hat{f} = \frac{16}{R^2 v^2} r_n^2 + \mathcal{O}(r_n^3) \,, \tag{5.40}$$

$$\omega_{\psi_n} = -\frac{3}{R^2} \lambda_i q^i \sin^2\theta_n r_n + \mathcal{O}(r_n^2) \tag{5.41}$$

$$\hat{f}^{-1} r^{-1} = \frac{v^2}{4} r_n^{-2} + k_1 r_n^{-1} + \mathcal{O}(r_n) \,, \tag{5.42}$$

$$\hat{f}^2 \omega_5 = -\frac{2}{v} r_n + k_2 r_n^2 + \mathcal{O}(r_n^3) \,, \tag{5.43}$$

$$\hat{f}^2 (\hat{f}^{-3} r - \omega_5^2) = \frac{l^2}{4} + k_3 r_n + \mathcal{O}(r_n^2) \,, \tag{5.44}$$

where we have defined the constants

$$v = \left(C_{ijk} q^i q^j q^k - 16 \frac{q^0}{g^2}\right)^{1/3} \,, \tag{5.45}$$

$$
\begin{aligned}
l &= \frac{1}{2v^2} \Big[9 \cdot 6^2 C^{ijk} C_{klm} \left(Q_i - C_{ihn} q^h q^n\right) \left(Q_j - C_{jpq} q^p q^q\right) q^l q^m \\
&\quad - 9 \left(q^i Q_i - C_{ijk} q^i q^j q^j\right)^2 - 12 q^i \lambda_i R^2 v^3 - 9 \left(Q_1 - \frac{q^0 q^1}{3}\right)^2 \left(\frac{32}{g^2}\right) \Big]^{1/2} \,.
\end{aligned}
\tag{5.46}
$$

These expression for the constants v and l resemble those of the Abelian case [91], with an additional non-Abelian term. The precise form of the constants k_1, k_2 and k_3 in terms of the charges are messy. They do not occur in the calculation of any physical quantity, but they play a role in the near horizon analysis,[13] since they are responsible for the disappearance of $\mathcal{O}(r_n^{-1})$ in the metric after we perform the following coordinate transformation,

$$dt_n = d\tau_n + \left(\frac{b_2}{r_n^2} + \frac{b_1}{r_n}\right) dr_n\,, \qquad d\varphi_n = -d\psi_n + 2d\xi_n + \frac{c_1}{r_n} dr_n\,, \tag{5.50}$$

where the constants b_1, b_2 and c_1 can be chosen such that all divergences in the metric in the $r_n \to 0$ limit disappear:

$$c_1 = \mp\frac{v}{l}\,, \qquad b_2 = \pm\frac{lv^2}{8}\,, \qquad b_1 = \pm\frac{4l^2 k_1 + l^2 v^3 k_2 + 4v^2 k_3}{16l}\,. \tag{5.51}$$

With this choice we find in the $r_n \to 0$ limit that the horizon has the following metric

$$ds_h^2 = -l^2 d\xi_n^2 - \frac{v^2}{4}\left(d\theta_n^2 + \sin^2\theta_n d\psi_n^2\right)\,. \tag{5.52}$$

with the topology $S^1 \times S^2$, so the solution is a black ring with non-Abelian hair, *i.e.* a non-Abelian black ring. Using this metric we can compute the area of the horizon[14],

$$\frac{A_h}{2\pi^2} = \frac{1}{2\pi^2}\int d^3x\sqrt{|g_h|} = lv^2\,, \tag{5.53}$$

so the entropy of the non-Abelian black ring can be written in terms of the charges and angular momenta using the expressions for the constants v and l Eqs. (5.45) and (5.46) together with Eq. (5.38) as follows:

---

[13] We give their form here for the sake of completeness,

$$\begin{aligned}
k_1 &= \frac{16\lambda^2 R^2 \frac{q^0}{g^2} + 3\left(q^i Q_i - C_{ijk}q^i q^j q^k\right)}{3R^2 v}\,, && (5.47)\\
k_2 &= \frac{4k_1}{v}\,, && (5.48)\\
k_3 &= \frac{1}{2g^2R^2v^4}\Bigg\{\frac{3R^2k_1}{v^3}\left[(q^0q^1/3)^2\left(96 - 3g^2(q^1)^2\right)\right.\\
&\quad +6(q^0q^1/3)\left(-32Q_1 + g^2q^1(-2(q^1)^2/6q^0 + 2q^0Q_0 + q^1Q_1)\right)\\
&\quad +3\left(4(q^1)^2/6g^2q^0q^1Q_1 + 32Q_1^2 + g^2\left(-q^1Q_1(4q^0Q_0 + q^1Q_1) + (C_{ijk}q^iq^jq^k - q^iQ_i)^2\right)\right)\\
&\quad \left.+4\lambda_iq^i\left(-16q^0 + g^2C_{ijk}q^iq^jq^k\right)R^2\right] - 3\left[9g^2((q^1)^2/6 - Q_0)(q^0q^1/3 - Q_1)^2\right.\\
&\quad +\left(-24(q^0q^1/3)^2\lambda^2 + 6(q^1)^2/6g^2\lambda_1q^0q^1 - 6g^2\lambda_1q^0Q_0q^1 + 96\lambda_1Q_1 - 6g^2\lambda_0q^0q^1Q_1\right.\\
&\quad -3g^2\lambda_1(q^1)^2Q_1 - 24\lambda^2Q_1^2 + 3q^0q^1/3\left(-32\lambda_1 + 2g^2\lambda_0q^0q^1 + g^2\lambda_1(q^1)^2 + 16\lambda^2Q_1\right)\\
&\quad \left.\left.+32\lambda_iq^iq^0 - 8C_{ijk}q^iq^jq^kg^2\lambda_iq^i + 6g^2\lambda_iq^iQ_jq^j\right)R^2 + 16\lambda^2\lambda_iq^iq^0R^4\right]\Bigg\} && (5.49)
\end{aligned}$$

[14] Notice that $\xi_n \in [0, 2\pi)$, as can be deduced from expression (5.50) together with $\int d\Omega_{(3)} = 2\pi^2$.

$$
\begin{aligned}
S &= \pi \Bigg[3 \cdot 6^2 C^{ijk} C_{klm} \left(Q_i - C_{ihn} q^h q^n \right) \left(Q_j - C_{jpq} q^p q^q \right) q^l q^m - 3 \left(q^i Q_i - C_{ijk} q^i q^j q^k \right)^2 \\
&\quad - \frac{4}{\sqrt{3}} (J_{\phi_2} - J_{\phi_1}) \left(C_{ijk} q^i q^j q^k - 16 \frac{q^0}{g^2} \right) - 3 \left(Q_1 - \frac{q^0 q^1}{3} \right)^2 \left(\frac{32}{g^2} \right) \Bigg]^{1/2} . \quad (5.54)
\end{aligned}
$$

Finally, we would like to compute the mass of the solution. We do so by comparing the asymptotic behavior of the metric component g_{tt} with that of the Schwarzschild solution, $g_{tt} \sim 1 - \frac{8MG}{3\pi\rho^2} + \cdots$. We get

$$
M = \frac{3^{5/2}\lambda_1}{2} \left(\lambda_1 Q_0 + 2\lambda_0 Q_1 \right) . \quad (5.55)
$$

The constants λ_i can be expressed in terms of physical constants. If we define the physical scalars of the theory as $\phi^x \equiv h_x/h_0$ we find that the only scalar with a non-vanishing asymptotic value is the Abelian one and this value is $\phi^1_\infty = \lambda_1/\lambda_0$. On the other hand, the asymptotic normalization of the metric Eq. (5.26) implied $\lambda_0\lambda_1^2 = 2/3^3$. Then,

$$
\lambda_0 = 2^{1/3} 3^{-1} \left(\phi^1_\infty \right)^{-2/3} , \qquad \lambda_1 = 2^{1/3} 3^{-1} \left(\phi^1_\infty \right)^{1/3} . \quad (5.56)
$$

and M takes the form

$$
M = 2^{-1/3} 3^{1/2} \left[\left(\phi^1_\infty \right)^{2/3} Q_0 + 2 \left(\phi^1_\infty \right)^{-1/3} Q_1 \right] , \quad (5.57)
$$

and depends only on the moduli and on the electric charges Q_0, Q_1 while the q^i, which correspond to magnetic dipole momenta do not contribute to it [79]. The non-Abelian field do not contribute, either.

This expression looks identical to that of the non-Abelian black hole solution constructed in Ref. [158], but the charges Q_0 and Q_1 are not the same than the charges q_0 and q_1 that appear in the black-hole mass formula given in that reference. They are, actually, related by $Q_i^{\rm BR} = q_i^{\rm BH} + C_{ijk} q^j_{\rm BR} q^k_{\rm BR}$. This is just reflecting the fact that the conserved electrical charges in the black ring receive contributions from the magnetic dipole momenta via the Chern-Simons term in the action. This effect is commonly described as "charges dissolved in fluxes" [30].

This non-Abelian black-ring mass formula, is, however, identical to that of the Abelian black ring that one would obtain by removing the non-Abelian fields from this solution. In other words: the presence of non-Abelian fields is not observable at spatial infinity. They do contribute to the entropy, though, as in the black-hole case, their entropy being smaller than that of their Abelian siblings.

5.3 Non-Abelian Rotating Black Holes

In the $R \to 0$ limit, several things happen:

1. All the harmonic functions are now centered at $r = 0$ (except for M which becomes constant):

$$H = N = \frac{1}{r}\,, \qquad M = \tfrac{3}{4}\lambda_i q^i\,, \qquad \Phi^i = -\frac{q^i}{4\sqrt{2}}N\,, \qquad L_i = \lambda_i + \frac{Q_i - C_{ijk}q^j q^k}{4}H\,. \tag{5.58}$$

2. The non-Abelian gauge field is also centered at $r = 0$:

$$\Phi^\alpha = \frac{1}{gr\,(1+\lambda^2 r)}\delta^\alpha_{s+1}\frac{x^s}{r}\,, \qquad \breve{A}^\alpha = \frac{1}{gr\,(1+\lambda^2 r)}\epsilon^\alpha{}_{rs}\frac{x^s}{r}dx^r\,, \tag{5.59}$$

and the distorted BPST instanton is not distorted anymore.

3. The metric function $\hat{f}$ is now given by

$$\hat{f}^{-3} = \left[\tfrac{3}{2}\left(\lambda_0 + \frac{Q_0}{4r}\right) - \frac{2}{g^2}\frac{1}{r(1+\lambda^2 r)^2}\right]\left[9\left(\lambda_1 + \frac{Q_1}{4r}\right)^2 - \frac{2(q^0)^2}{g^2}\frac{1}{r^2(1+\lambda^2 r)^2}\right]. \tag{5.60}$$

The mass of this object is identical to that of the black ring Eqs. (5.55) and (5.57). it has no non-Abelian contributions. The near-horizon limit, though, includes non-Abelian terms

$$\hat{f}^{-1} \sim \frac{Y}{r}\,, \quad \text{with} \quad Y^3 = \left(\tfrac{3}{8}Q_0 - \frac{2}{g^2}\right)\left(\tfrac{9}{16}Q_1^2 - \frac{2}{g^2}(q^0)^2\right) \tag{5.61}$$

4. ω vanishes identically and $\hat{\omega}$ is determined only by ω_5, which takes the form

$$\begin{aligned} \hat{\omega} &= \omega_5(d\varphi + \cos\theta d\psi)\,, \\ \omega_5 &= \frac{q^i}{16}\left(3Q_i - C_{ijk}q^j q^k\right)\frac{1}{r} - \frac{2q^0}{g^2}\frac{1}{r\,(1+\lambda^2 r)^2}\,. \end{aligned} \tag{5.62}$$

As a result, the two angular momenta become identical

$$J_{\phi_1} = J_{\phi_2} = \tfrac{1}{2\sqrt{3}}q^i\left(3Q_i - C_{ijk}q^j q^k\right) \equiv J\,. \tag{5.63}$$

Observe that the non-Abelian term in ω_5, which does not contribute to the angular momentum, does contribute to the $r \to 0$ limit just as the Abelian terms:

$$\omega_5 \sim Z/r\,, \quad \text{where} \quad Z = \tfrac{\sqrt{3}}{8}J - \frac{2q^0}{g^2}\,. \tag{5.64}$$

Let us study the near-horizon limit $\to 0$. Using Eqs. (5.61) and (5.64), we find that the metric Eq. (6.3) behaves in this limit as

$$ds^2 \sim \frac{r^2}{Y^2}dt^2 - \frac{Y}{r^2}dr^2 - Y d\Omega^2_{(2)} + \frac{2Z}{Y^2}rdt(d\varphi + \cos\theta d\psi) + \left(\frac{Z^2}{Y^2} - Y\right)(d\varphi + \cos\theta d\psi)^2\,, \tag{5.65}$$

which can be rewritten in the form

$$ds^2 \sim Y d\Pi^2_{(2)} - Y d\Omega^2_{(2)} - Y \left[\sin\alpha\rho dt - \cos\alpha(d\varphi + \cos\theta d\psi)\right]^2 \, , \tag{5.66}$$

where $r = (Y^3 - Z^2)^{1/2}\rho$, $d\Pi^2_{(2)} = \rho^2 dt^2 - \frac{d\rho^2}{\rho^2}$ is the metric of the AdS_2 of unit radius and $\sin^2\alpha = Z^2/Y^3$. This space is the near-horizon limit of the BMPV black hole [43], but, due to the non-Abelian contribution to Z (which can be understood as a sort of "near-horizon angular momentum"), now α does not vanish for vanishing asymptotic angular momentum J and we can have a stationary black hole with $J = 0$ whose near-horizon limit is not $\mathrm{AdS}_2\times\mathrm{S}^3$. The converse is also possible: we can make $\alpha = Z = 0$ for $J = \frac{16}{\sqrt{3}} q^0/g^2$ and have a rotating black hole whose near-horizon limit is $\mathrm{AdS}_2\times\mathrm{S}^3$.

The area of the horizon is

$$\frac{A}{2\pi^2} = 8\sqrt{Y^3 - Z^2}\, . \tag{5.67}$$

5.4 Conclusions

The existence of black-hole and black-ring solutions with identical asymptotic behaviour but with non-Abelian hair that contributes to the entropy [46, 154, 158, 159] challenges our understanding of black-hole hair and the microscopic interpretation of the black-hole/black-ring entropy, just as the Abelian hair discovered in Ref. [79] did. More research is necessary to gain a better understanding of these solutions. In particular, the stability of these supersymmetric non-Abelian solutions (which are entropically disfavored) needs to be addressed and their possible non-supersymmetric and non-extremal generalizations have to be constructed and studied. Work in these directions is in progress.

6

Non-Abelian bubbles in microstate geometries

This chapter is based on

Pedro F. Ramírez
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The construction and study of smooth microstate geometries in supergravity theories has become a fruitful area of research since the pioneering work, more than a decade ago, of Bena and Warner [31] and independently of Berglund, Gimon and Levi [34], where a strategy to obtain ample families of microstate geometries was given, generalizing earlier results [101–103, 144–146, 152]. This kind of solutions can be roughly described as a black hole configuration in which the horizon and its interior have been replaced by some complicated, although smooth horizonless geometry while keeping the rest of the field configuration looking like the unmodified solution. Any solution with such remarkable properties is interesting *per se*, although it is in the context of the *fuzzball proposal* [150] in which these configurations acquire their greatest significance.

The proposal originated as a possible solution to the information paradox and conjectures that the entropy of a black hole has its microscopic origin in the degeneracy of a quantum bound state, the fuzzball. In this picture, the classical black hole would provide an effective description of the system, that would consist in a quantum *ensamble of geometries*. These microstate geometries, when considered individually, would correspond to string theory configurations with unitary scattering and hopefully a subset of these states might be captured as smooth horizonless supergravity solutions. Since the proposal suggests a modification at the horizon scale, such geometries should have the same asymptotics as the black hole.

This conjecture opened a whole program in the quest to construct smooth microstate geometries in theories of supergravity. Much progress has been made in this direction and vast classes of such solutions have already been described in the literature, see [9,32,33,64, 191] and references therein. The direct identification of these configurations as representing typical microstates of a particular black hole is generally unclear due to the absence of a description in terms of a dual CFT. However very recently this identification has been performed for a particular type of configurations known as *superstrata*, constituting a major achievement of the fuzzball program [24]. Nevertheless, even though general microstate geometries lack of this identification, they are still very useful in providing valuable information about the physics of black holes in string theory, see for instance [26–28, 153, 186].

Typically these are described as topologically non-trivial spacetimes in five and six dimensions, in the context of supergravity coupled to Abelian matter multiplets or pure supergravity. In the present work we perform the inclusion of non-Abelian degrees of freedom for the first time[1]. The reason why this class of microstate geometries has remained unexplored so far seems to be clear: the construction of explicit analytic non-Abelian solutions in five- and six-dimensional supergravity theories has become accessible only in the last few months [56,158,171]. The solutions that we present here constitute a non-Abelian extension of the BPS three-charge smooth geometries described in [32]. We work in $\mathcal{N}=1$, $d=5$ Super-Einstein-Yang-Mills (SEYM) theories. One can think of these theories as an extension of the five-dimensional STU model of supergravity, that describes a supergravity multiplet coupled to two Abelian vector multiplets. SEYM theories are then obtained by consistently coupling the STU model to a set of additional vector multiplets that transform under the local action of a non-Abelian group[2]. Although this nomenclature might seem unfamiliar in the literature of microstate geometries, in fact the underlying theory where this solutions are constructed is quite frequently the STU model: five-dimensional three-charge configurations are naturally described in this framework.

It is worth mentioning how $\mathcal{N}=1$, $d=5$ SEYM theories are embedded in string theory. The 10-dimensional effective theory of the Heterotic string describes $\mathcal{N}=1$ supergravity coupled to 16 Abelian vector multiplets. When the Heterotic string theory is compactified on T^5, there are special points in the moduli space for which there is an enhancement of the gauge symmetry. Then, besides the *Kaluza-Klein* vectors, the effective supergravity description contains additional massless vector fields taking values in the algebra of some non-Abelian group. A consistent truncation can reduce the supermultiplets content (as well as their number) and result in the $\mathcal{N}=1$, $d=5$ SEYM theories that we consider here. The explicit realization of this particular compactification and truncation is discussed in the following chapters.

The procedure by which non-Abelian microstate geometries are found has a similar structure than that of the Abelian case, but requires the introduction of some modifications. Just like in the case of supersymmetric solutions of STU supergravity, the construction of BPS configurations satisfying the equations of motion of SEYM theory relies on the specification of a reduced set of *seed functions* defined in $\mathbb{R}^3$. In the case of the familiar STU model, these are simply harmonic functions that satisfy certain differential equations whose integrability condition is the Laplace equation. The SEYM procedure conserves these harmonic functions and introduces a new set of seed functions satisfying the *covariant* version of these differential equations.

We find that the bubbling equations, which determine the size of the bubbles leading to physically sensible geometries, contain a new contribution that appears standing next to the magnetic fluxes threading the bubbles, see (6.45). This new term can be given a physical interpretation in terms of the topological charge, or instanton number, associated to the endpoints of the bubble of a non-Abelian instanton that builds up the vector fields. As a consequence it should be possible to have stable bubbles without some magnetic fluxes placed on them or, inversely, a bubble can collapse even though the fluxes are non-zero.

Another interesting peculiarity introduced by the non-Abelian fields is that the solution depends on a set of continuous parameters that can be modified with no apparent

[1]Notice that globally regular non-Abelian gravitating configurations on contractible spaces have been known since the late 80*s*, see [12,60,61,112]. These are usually referred as *global monopoles*.

[2]One can consider as well the introduction of additional Abelian vector multiplets.

restriction whose influence is only local, i.e. their modification does not change any of the asymptotic charges. This is a shocking feature that allows the construction of huge amounts of microstate geometries with the same topology for a unique black hole, and its proper interpretation requires further study.

Having said that, let us start talking about the details of non-Abelian microstate geometries. We give a general description of the solutions that can be found using our generating technique in Section 6.1. In Section 6.2 we describe how this method can be utilized for the construction of smooth horizonless solutions. We conclude in Section 6.3 with some comments about the results and discuss future directions. In Appendix C.1 we give a brief summary of $\mathcal{N}=1$, $d=5$ SEYM theories, describing its matter content and its action. Appendix C.3 contains the solution generating technique written in a *step-by-step* language.

6.1 Supersymmetric solutions of $\mathcal{N}=1$, $d=5$ Super-Einstein-Yang-Mills

A technique to construct supersymmetric timelike solutions with a spacelike isometry in these theories was recently developed in [158], where it was used to describe the first non-Abelian analytic black holes in five dimensions[3]. This method has also been used in [171] to find non-Abelian generalizations of the Emparan-Reall black ring solution, [80], and the BMPV rotating black hole, [43]. In the simplest settings, the configurations can be roughly interpreted as three-charge Abelian solutions on top of which we place a non-Abelian instanton that, interestingly, does not produce any change on the mass of the solution while it reduces its entropy.

The solutions of $\mathcal{N}=1$, $d=5$ SEYM[4] are specified by the form of the metric ds^2, the vector fields A^I and the scalars ϕ^x. The indices labeling the vectors take values in $\{I,J,\ldots=0,\ldots,5\}$, with the Abelian sector contained in the first values $\{i,j,\ldots=0,1,2\}$ and the non-Abelian sector in the last three $\{\alpha,\beta,\ldots=3,4,5\}$. We make a continuous use of this division in two sectors through the text. The scalars are conveniently codified in terms of a set of functions h_I labeled with the same indices than the vectors, such that $\phi^x \equiv h_x/h_0$. We also define the functions of the scalars with upper indices as

$$h^I \equiv 27C^{IJK}h_I h_J\,, \qquad\qquad h^I h_I = 1\,, \tag{6.1}$$

where $C^{IJK}=C_{IJK}$ is a constant symmetric tensor that characterizes the supergravity theory. We work on the SU(2)-gauged ST$[2,6]$ model, that contains $n_v=5$ vector multiplets and, as we mentioned in the introduction, can be understood as a non-Abelian extension of the STU model. This model is characterized by a constant symmetric tensor with the following non-vanishing components

$$C_{0xy}=\tfrac{1}{6}\eta_{xy}\,,\text{where}\quad (\eta_{xy})=\text{diag}(+-\cdots-)\,,\quad\text{and}\quad x,y=1,\cdots,5\,. \tag{6.2}$$

In [23] it was shown that timelike supersymmetric solutions of this theory are of the

[3]A method for the systematic construction of null solutions and some explicit examples describing black strings and regular string-monopoles are also given in that reference.
[4]See Apendix C.1 for a brief description of the theory.

form

$$ds^2 = f^{2}(dt+\omega)^2 - f^{-1}d\hat{s}^2\,, \tag{6.3}$$

$$A^I = -\sqrt{3}h^I f(dt+\omega) + \hat{A}^I\,, \tag{6.4}$$

where $d\hat{s}^2$ is a four-dimensional hyperKähler metric and the rest of elements that appear in this decomposition are defined on this four-dimensional space. These elements satisfy the system of *BPS equations*:

$$\hat{F}^I = \star_4 \hat{F}^I\,, \tag{6.5}$$

$$\hat{\mathfrak{D}}^2\left(h_I/f\right) = \tfrac{1}{6}C_{IJK}\hat{F}^J\cdot\hat{F}^K\,, \tag{6.6}$$

$$d\omega + \star_4 d\omega = \tfrac{\sqrt{3}}{2}(h_I/f)\hat{F}^I\,. \tag{6.7}$$

Here $\star_4$ is the Hodge dual in the four-dimensional metric $d\hat{s}^2$ and $\hat{F}^I$ is the field strength of the vector $\hat{A}^I$

$$\hat{F}^I{}_{\mu\nu} = 2\partial_{[\mu}\hat{A}^I{}_{\nu]} + \hat{g}f_{JK}{}^I\hat{A}^J{}_\mu\hat{A}^K{}_\nu\,, \tag{6.8}$$

where $f_{IJ}{}^K$ are only non-vanishing when the indices take values in the non-Abelian sector, in which case they are the structure constants of $SU(2)$, $f_{\alpha\beta}{}^\gamma = \varepsilon_{\alpha\beta\gamma}$.

Some words about notation are necessary. Notice that we use *hats* to distinguish objects that are defined in four spatial dimensions. For example, A^I is used to represent the five-dimensional physical vectors and $\hat{A}^I$ is a vector in the four-dimensional hyperKähler space. In a few lines we will introduce another collection of objects that are labeled with *inverse hats* and that are defined in three-dimensional Euclidean space. In particular we define the vectors $\breve{A}^I$. We use all these vectors to define covariant derivatives in five, four and three dimensions for objects with upper and lower vector indices. For example the four-dimensional covariant derivatives are defined by

$$\hat{\mathfrak{D}}h^I = dh^I + \hat{g}f_{JK}{}^I\hat{A}^J h^K\,, \qquad \hat{\mathfrak{D}}h_I = dh_I + \hat{g}f_{IJ}{}^K\hat{A}^J h_K\,. \tag{6.9}$$

The system of BPS equations can be drastically simplified under the assumption that the solutions admit a global spacelike isometry along a compact direction [158]. Then the mathematical objects that build up the physical fields can be further decomposed in terms of elements defined in three dimensional flat space in the following manner

$$d\hat{s}^2 = H^{-1}(d\varphi+\chi)^2 + H dx^r dx^r\,, \quad r=1,2,3\,, \tag{6.10}$$

$$\hat{A}^I = -2\sqrt{6}\left[-H^{-1}\Phi^I(d\varphi+\chi) + \breve{A}^I\right]\,, \tag{6.11}$$

$$h_I/f = L_I + 8C_{IJK}\Phi^J\Phi^K H^{-1}\,, \tag{6.12}$$

$$\omega = \omega_5(d\varphi+\chi) + \breve{\omega}\,, \tag{6.13}$$

where φ is a coordinate adapted to the direction of the isometry. Substituting back these expressions in the BPS system of equations, we obtain the conditions that $H, \chi, \Phi^I, \breve{A}^I, L_I, \omega_5$ and $\breve{\omega}$ need to satisfy

$$\star_3 dH = d\chi\,, \tag{6.14}$$

$$\star_3 \breve{\mathfrak{D}}\Phi^I = \breve{F}^I\,, \tag{6.15}$$

$$\breve{\mathfrak{D}}^2 L_I = \breve{g}^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J \Phi^K L_M\,, \tag{6.16}$$

$$\star_3 d\breve{\omega} = HdM - MdH + 3\sqrt{2}(\Phi^I \breve{\mathfrak{D}} L_I - L_I \breve{\mathfrak{D}}\Phi^I)\,, \tag{6.17}$$

$$\omega_5 = M + 16\sqrt{2}H^{-2}C_{IJK}\Phi^I\Phi^J\Phi^K + 3\sqrt{2}H^{-1}L_I\Phi^I\,, \tag{6.18}$$

where M is just a harmonic function in $\mathbb{E}^3$, i.e. $\nabla^2 M = 0$.

Equations (6.14), (6.15) and (6.16) in the Abelian sector imply that H, Φ^i and L_i are just harmonic functions on $\mathbb{E}^3$. Once these are specified it is straightforward to find the 1-forms χ and $\breve{A}^i$.

In the non-Abelian sector (6.15) is the Bogomol'nyi equation [42], which is non-linear and hard to solve in general. Fortunately this system, that describes a non-Abelian monopole in Yang-Mills-Higgs theory, has been studied by many authors and the space of solutions available in the bibliography is rich enough for the purposes of our work.

Equation (6.16) in the non-Abelian sector is easily solved if we choose $L_\alpha \propto \Phi^\alpha$ or just $L_\alpha = 0$. However none of these choices is completely satisfying if one pursues the construction of general smooth horizonless geometries. If one takes $L_\alpha \propto \Phi^\alpha$ then there are some potential restrictions on the space of possible Φ^i that can result in smooth geometries. We will need to find a more general solution.

Finally, (6.17) can always be solved if its integrability condition is satisfied. This condition gives a set of algebraic equations, which in this context are known as *bubbling equations*, that impose restrictions on the distance between the different centers of the solution (the points were the seed functions are singular). Then, of course, one has to integrate explicitly equation (6.17) to obtain $\breve{\omega}$.

In summary, we have described a procedure to construct supersymmetric timelike solutions in terms of a set of *seed functions* defined on three-dimensional flat space: H, Φ^I, L_I and M.

6.2 Smooth bubbling geometries in SEYM supergravity

Smooth microstate geometries are defined as horizonless, regular field configurations without any brane sources but with the asymptotic charges of a black hole. At a technical level this statement implies several conditions that we shall address in the following subsections, being perhaps the most important of those the requirement of working with manifolds with non-trivial topology[5]. This fact can be roughly understood from the fact

[5]By this we mean that they describe non-contractible spaces.

that the existence of non-trivial cycles allows for the presence of measurable asymptotic charges without the introduction of localized brane sources. See for instance [32] for a detailed discussion about this topic.

The systematic procedure for finding solutions described in the previous section can naturally accommodate ambipolar Gibbons-Hawking spaces, which have just the right properties for these purposes. Let us start with a brief description of these manifolds.

6.2.1 Ambipolar Gibbons-Hawking spaces

Much of the very interesting physics exhibited by these solutions is related to the use of ambipolar Gibbons-Hawking spaces, which are a particular example of ambipolar hyperKähler manifolds [166]. These have the form of a $U(1)$ fibration over a $\mathbb{R}^3$ base, with the fiber collapsing to a point at a finite collection $X = \{\vec{x}_a | a = 1, \dots, n\}$ of points in $\mathbb{R}^3$ which we will call *centers*. Any path in the base manifold connecting two centers, γ_{ab}, defines a non-contractible 2-cycle through the inclusion of the $U(1)$ fiber, $\Delta_{\gamma_{ab}}$. A different path γ'_{ab} between the same centers describes an homologically equivalent 2-cycle $\Delta_{\gamma'_{ab}} \simeq \Delta_{\gamma_{ab}}$. We will denote any of the equivalent 2-cycles simply as Δ_{ab}.

These spaces have the metric

$$d\hat{s}^2 = H^{-1}(d\varphi + \chi)^2 + H\left[dr^2 + r^2\left(d\theta^2 + sin^2\theta d\psi^2\right)\right] , \qquad \star_3 dH = d\chi \, , \tag{6.19}$$

with the angular coordinates taking values in $\theta \in [0, \pi)$, $\psi \in [0, 2\pi)$, $\varphi \in [0, 4\pi)$. H is a harmonic function on $\mathbb{E}^3$ of the form

$$H = \sum_a \frac{q_a}{r_a} \, , \qquad \text{with} \quad r_a \equiv |\vec{x} - \vec{x}_a| \, , \quad \vec{x}_a \in X \, , \tag{6.20}$$

while the 1-form χ plays the role of local connection of the fiber bundle and can be written as

$$\chi = \sum_a q_a cos\theta_a d\psi_a \, , \tag{6.21}$$

where θ_a and ψ_a are coordinates on a spherical frame centered in $\vec{x}_a$.

Although H is singular when evaluated at the centers it is straightforward to check that if all q_a, aka Gibbons-Hawking charges, are integers then the metric remains regular at these points[6]. Indeed under the redefinition of the radial coordinate $\rho_a = 2\sqrt{r_a}$ we find that locally

$$d\hat{s}^2|_{\rho_a \to 0} \sim d\rho_a^2 + \rho_a^2 d\Omega^2_{(3)/q_a} \, , \tag{6.22}$$

being $d\Omega^2_{(3)/q_a}$ the standard metric on $S^3/\mathbb{Z}_{|q_a|}$. Asymptotically the manifold is also of this form, $d\hat{s}^2|_{\rho \to \infty} \sim d\rho^2 + \rho^2 d\Omega^2_{(3)/Q}$, with the orbifold given in this case by $S^3/\mathbb{Z}_{|Q|}$, being $Q \equiv \sum_a q_a$.

Physically, smooth bubbling geometries are claimed to represent microstate configurations of some particular black hole, being both solutions indistinguishable asymptotically. Therefore we are interested in having the ambipolar Gibbons-Hawking space asymptotic to $\mathbb{R}^4$, which we can achieve imposing $Q = 1$. This condition requires that

[6] When $|q_a| \neq 1$ there is an orbifold singularity at $\vec{x} = \vec{x}_a$, but we will not worry about it since these singularities are innocuous in the context of string theory.

some of the Gibbons-Hawking charges be negative, and therefore the function H interpolates between $-\infty$ and $+\infty$. Each negatively charged center is surrounded by a connected open region with $H < 0$, whose boundary is a surface where H vanishes.

Then the signature of the metric interpolates between $(++++)$ and $(----)$, being clearly ill-defined at the surfaces where $H = 0$. It is this characteristic what renders this space be ambipolar. This harmful properties, however, can be made compatible with having a smooth five-dimensional supergravity solution due to the presence of both, the conformal factor f^{-1} multiplying $d\hat{s}^2$ and the additional terms in the full metric, see equation (6.3). We will elaborate on this in subsequent sections.

6.2.2 Seed functions for horizonless spacetimes

In the language of the solution generating technique outlined in Section 6.1, we have given the first small step in the way to obtain a supersymmetric solution, that can be synthesized as

$$H = \sum_a \frac{q_a}{r_a}\,, \qquad \text{with} \quad q_a \in \mathbb{Z}\,, \quad \sum_a q_a = 1\,. \tag{6.23}$$

The remaining seed functions in the Abelian sector Φ^i, L_i and M are also harmonic,

$$\Phi^i = k_0^i + \sum_a \frac{k_a^i}{r_a}\,, \qquad L_i = l_0^i + \sum_a \frac{l_a^i}{r_a}\,, \qquad M = m_0 + \sum_a \frac{m_a}{r_a}\,, \tag{6.24}$$

and from equation (6.15) we readily obtain

$$\breve{A}^i = \sum_a k_a^i cos\theta_a d\psi_a\,. \tag{6.25}$$

Notice that we imposed that the location of the singularities coincides with a Gibbons-Hawking center. With this requirement we will be able to avoid that the building blocks h_I/f as defined in (6.12) become singular whenever any of the seed functions individually diverge. This is the mathematical version of what at the beginning of the section we called absence of brane sources, and it is the mechanism responsible of obtaining horizonless geometries[7]. Also, the fact that the harmonic seed functions are singular at the Gibbons-Hawking centers is directly responsible for much of the very interesting physics captured by these solutions. Consequently, we would like the non-Abelian seed functions to display a similar qualitative behavior, i.e. $(\Phi^\alpha, L_\alpha)|_{r_a\to 0} \sim r_a^{-1} + \mathcal{O}(r_a^0)$.

Protogenov's $SU(2)$ colored monopole [182] is a solution to the Bogomol'nyi equation with this property, with only one single center. Colored monopoles are rather intriguing objects. They describe a point with unit local magnetic charge surrounded by a magnetic cloud that completely screens the charge as seen from infinity[8]. Despite its singular nature when interpreted in the context of Yang-Mills-Higgs theory, single center colored monopole solutions have been fruitfully used in the literature to obtain regular non-Abelian black

[7] Clearly this naming is pointing at the physical origin of these potential singularities once the solutions are interpreted in the context of string theory.

[8] The magnetic charge is defined as $p = \frac{\breve{g}}{4\pi}\int_{S^2}\frac{\Phi^\alpha \breve{F}^\alpha}{\sqrt{\Phi^\alpha\Phi^\alpha}}$.

holes in four- [46,122,154,159] and five-dimensional [158,171] theories of gauged supergravity. Their presence has an interesting impact on black hole thermodynamics, modifying the entropy without altering the mass.

Therefore, a family of well-suited non-Abelian seed functions Φ^α is given by a multicenter generalization of colored monopoles, which we construct now. From now on we will assume the gauged group is $SU(2)$ for the sake of simplicity, so the index α can take three possible values. Nevertheless, following the ideas of Meessen and Ortín [159], it should be possible to embed these monopoles in a more general group $SU(N)$ and use them in the construction of smooth bubbling geometries in $SU(N)$-gauged supergravity.

Plugging in the Bogomoln'yi equation (6.15) the ansatz of the *hedgehog* form

$$\Phi^\alpha = -\frac{1}{\breve{g}P}\frac{\partial P}{\partial x^s}\delta^\alpha_s\,, \qquad \breve{A}^\alpha_\mu = -\frac{1}{\breve{g}P}\frac{\partial P}{\partial x^s}\varepsilon^\alpha{}_{\mu s}\,, \tag{6.26}$$

we find that this configuration describes a monopole solution if P is a harmonic function,

$$P = \lambda_0 + \sum_a \frac{\lambda_a}{r_a}\,, \qquad \lambda_0 \neq 0\,. \tag{6.27}$$

Substituting back in (6.26), we can write the solution as

$$\Phi^\alpha = \sum_a \frac{\lambda_a}{\breve{g}r_a^2 P}\delta^\alpha_s\frac{(x^s - x^s_a)}{r_a}\,, \qquad \breve{A}^\alpha_\mu = \sum_a \frac{\lambda_a}{\breve{g}r_a^2 P}\varepsilon^\alpha{}_{\mu s}\frac{(x^s - x^s_a)}{r_a}\,. \tag{6.28}$$

The Higgs field of the monopole is singular at the centers and vanishes at infinity

$$\lim_{r_a\to 0}\Phi^\alpha = \frac{k^\alpha_a}{r_a} + \mathcal{O}(r_a^0)\,, \qquad \lim_{r\to\infty}\Phi^\alpha \sim \mathcal{O}(r^{-2})\,, \qquad k^\alpha_a \equiv \delta^\alpha_s\frac{(x^s - x^s_a)}{\breve{g}r_a}\,. \tag{6.29}$$

This solution corresponds to a *multicenter colored monopole* configuration.

The last seed functions we need to find are L_α, which are solutions of equation (6.16), that we repeat here for convenience

$$\breve{\mathfrak{D}}^2 L_\alpha - \breve{g}^2 f_{\alpha\beta}{}^\lambda f_{\gamma\lambda}{}^\rho \Phi^\beta\Phi^\gamma L_\rho = 0\,. \tag{6.30}$$

We can solve this differential system by making use of the ansatz

$$L_\alpha = -\frac{1}{\breve{g}P}\frac{\partial Q}{\partial x^s}\delta^\alpha_s\,, \tag{6.31}$$

the equation reduces to the condition of Q being harmonic. We choose Q to be of the form

$$Q = \sum_a \frac{\sigma_a\lambda_a}{r_a}\,. \tag{6.32}$$

The functions L_α behave similarly to Φ^α near the centers and at infinity

$$\lim_{r_a\to 0} L_\alpha = \frac{l^\alpha_a}{r_a} + \mathcal{O}(r_a^0)\,, \qquad \lim_{r\to\infty} L_\alpha \sim \mathcal{O}(r^{-2})\,, \qquad l^\alpha_a \equiv \sigma_a\delta^\alpha_s\frac{(x^s - x^s_a)}{\breve{g}r_a}\,, \tag{6.33}$$

only differentiated by the presence of the parameters σ_a in the near-center limit. The appearance of these factors will be fundamental for obtaining horizonless geometries.

After having fixed the general form of all the seed functions, we can start analyzing the regularity of the metric. In order to construct horizonless solutions we need to avoid having brane sources at the centers. In other words, we want the building blocks h_I/f that constitute the metric function, given by (6.12), to remain finite at these points. Keeping the charges q_a and k_a^i arbitrary, it is possible to remove the brane sources by taking

$$l_a^I = -8C_{IJK}\frac{k_a^J k_a^K}{q_a}\,. \tag{6.34}$$

Notice that this expression is valid in both the Abelian and the non-Abelian sector. In the former it fixes the value of the parameters l_a^i, while in the latter it fixes the parameters σ_a. Regularity of the metric at the centers also requires ω_5 to be finite there, something that we achieve by choosing

$$m_a = 8\sqrt{2}C_{IJK}\frac{k_a^I k_a^J k_a^K}{q_a^2}\,. \tag{6.35}$$

The constant terms in the harmonic seed functions (6.24) define the solution at infinity. In order to have an asymptotically flat metric ($f_\infty \sim 1$, $\omega_{5,\infty} \sim 0$) we need to satisfy the constrains

$$k_0^i = 0\,, \qquad 27C^{ijk}l_0^i l_0^j l_0^k = 1\,, \qquad m_0 = -3\sqrt{2}\sum_{i,a} l_0^i k_a^i\,. \tag{6.36}$$

6.2.3 Closed timelike curves and bubbling equations

By using an ambipolar Gibbons-Hawking metric we are taking a clear risk: the spacetime metric might contain closed timelike curves (CTC's) or even be ill-defined at the critical surfaces where $H = 0$. We now study the conditions under which CTC's are absent, so the microstate geometries are physically sensible.

Let us expand the expression of the spacetime metric (6.3) and write it in the following manner

$$ds^2 = f^2dt^2 + 2f^2dt\omega - \frac{\mathcal{I}}{f^{-2}H^2}\left(d\varphi + \chi - \frac{\omega_5 H^2}{\mathcal{I}}\breve{\omega}\right)^2 - f^{-1}H\left(d\vec{x}\cdot d\vec{x} - \frac{\breve{\omega}^2}{\mathcal{I}}\right)\,, \tag{6.37}$$

where $\mathcal{I}$ is defined as

$$\mathcal{I} \equiv f^{-3}H - \omega_5^2 H^2\,. \tag{6.38}$$

There is one general restriction that needs to be satisfied in order to avoid the presence of CTC's

$$\mathcal{I} \geq 0\,. \tag{6.39}$$

Apparently there is one additional condition, $f^{-1}H \geq 0$, but this is implied by the inequality in (6.39). Let us express this condition in more detail by evaluating $\mathcal{I}$ in terms of the seed functions

$$\begin{aligned}\mathcal{I} \;\;=\;\; & -M^2H^2 - 18\left(\Phi^I L_I\right)^2 - 32\sqrt{2}MC_{IJK}\Phi^I\Phi^J\Phi^K - 6\sqrt{2}MHL_I\Phi^I \\ & +27HC^{IJK}L_IL_JL_K + 3^4 2^3 C^{IJK}C_{KLM}L_IL_J\Phi^L\Phi^M \geq 0\,.\end{aligned} \tag{6.40}$$

The first point to notice is that the form of this expression coincides with that of ungauged supergravity originally derived in [31], where it was identified as the quartic invariant of $E_{7(7)}$. The analysis of the positivity of this quantity is hard to do in general, although we can assert that this bound can be satisfied for large families of configurations. The reason behind this statement is that this has been shown to be the case for ungauged supergravities, and many techniques to construct solutions satisfying this bound have been developed. In any case, it is fair to say that this restriction definitely makes the process of constructing explicit solutions more complicated.

There is one additional factor that can result in the appearance of CTC's, and this is the formation of Dirac-Misner strings. Those arise when the integrability condition of the last differential equation that still remains to be solved, (6.17), is not satisfied. This condition is obtained acting with the operator $d\star_3$ in that expression, which gives

$$\left\{H\nabla^2 M - M\nabla^2 H + 3\sqrt{2}(\Phi^i\nabla^2 L_i - L_i\nabla^2\Phi^i + \Phi^\alpha\breve{\mathfrak{D}}^2 L_\alpha - L_\alpha\breve{\mathfrak{D}}^2\Phi^\alpha)\right\} = 0\,. \tag{6.41}$$

This condition is identically satisfied as a consequence of equations (6.14)-(6.16) everywhere except at the centers, where technically those equations cease to apply. The bubbling equations are algebraic constrains that guarantee that the integrability condition is satisfied everywhere, setting the requirements that avoid the presence of Dirac-Misner strings.

To make further progress it is convenient to define the symplectic vector of seed functions

$$S^M = \left(H, 3\sqrt{2}\Phi^I, M, L_I\right)\,, \qquad\qquad S_M = \left(M, L_I, -H, -3\sqrt{2}\Phi^I\right)\,, \tag{6.42}$$

and a symplectic vector of charges at each center

$$Q^M_a = \left(q_a, 3\sqrt{2}k^I_a, m_a, l^I_a\right)\,, \qquad\qquad Q_{M,a} = \left(m_a, l^I_a, -q_a, -3\sqrt{2}k^I_a\right)\,. \tag{6.43}$$

Now we can write the integrability condition as

$$S^M\breve{\mathfrak{D}}^2 S_M = 0\,. \tag{6.44}$$

Interestingly the non-Abelian sector vanishes in the last expression due to the symplectic product and the expression is reduced to $S^m Q_{m,a}\delta(\vec{x}-\vec{x}_a) = 0$ with the understanding that S^m and Q^m_a are the components of the symplectic vectors in the Abelian sector. Then, one could naively expect that the bubbling equations coincide with those in the case of ungauged supergravity theories. However, this does not happen because the charges l^i_a are affected by the presence of the non-Abelian fields according to (6.34). After a few lines of algebraic computation, the resulting bubbling equations are conveniently written as

$$\sum_{b\neq a}\frac{q_a q_b}{r_{ab}}\left[C_{ijk}\Pi^i_{ab}\Pi^j_{ab}\Pi^k_{ab}-\frac{1}{2\breve{g}^2}\Pi^0_{ab}\mathbb{T}_{ab}\right]=\frac{3}{8}l^i_0\left(\sum_b q_a k^i_b-k^i_a\right)\,, \tag{6.45}$$

where Π^i_{ab} is the i^{th}- flux threading the 2-cycle Δ_{ab} and $\mathbb{T}_{ab}$ contains information about the topological charge associated to the centers a and b, see (6.63)

$$\Pi^i_{ab}\equiv\left(\frac{k^i_b}{q_b}-\frac{k^i_a}{q_a}\right)\,,\qquad\qquad \mathbb{T}_{ab}\equiv\breve{g}^2\left(\frac{k^\alpha_a k^\alpha_a}{q_a^2}+\frac{k^\alpha_b k^\alpha_b}{q_b^2}\right)\,. \tag{6.46}$$

We are now ready to integrate (6.17). It is convenient to decompose the 1-form $\breve{\omega}$ into two parts, $\breve{\omega}^A$ and $\breve{\omega}^B$, satisfying

$$\star_3 d\breve{\omega}^A \;=\; HdM-MdH+3\sqrt{2}(\Phi^i dL_i-L_i d\Phi^i)\,, \tag{6.47}$$

$$\star_3 d\breve{\omega}^B \;=\; 3\sqrt{2}(\Phi^\alpha\breve{\mathfrak{D}}L_\alpha-L_\alpha\breve{\mathfrak{D}}\Phi^\alpha)\,, \tag{6.48}$$

The first equation can be solved independently for each pair of centers (a,b), with $\breve{\omega}^A=\sum_a\sum_{b>a}\breve{\omega}^A_{ab}$. For each pair we use adapted coordinates such that $\vec{x}_a=(0,0,0)$ and $\vec{x}_b=(0,0,-r_{ab})$, with spherical angles given by

$$x^1_{ab}=r_a sin\theta_{ab}sin\psi_{ab}\qquad x^2_{ab}=r_a sin\theta_{ab}cos\psi_{ab}\qquad x^3_{ab}=-r_a cos\theta_{ab}\,. \tag{6.49}$$

Upon substitution of the seed functions H,M,L_i,Φ^i, (6.47) can be written as

$$\begin{aligned}\star_3 d\breve{\omega}^A_{ab} \;=\; &\frac{Q_{m,a}Q^m_b}{r_{ab}}\left\{-\frac{1}{r_a^2}\left[1-\frac{r_{ab}+r_a}{r_b}+\frac{r_a r_{ab}\left(r_a+r_{ab}\right)}{r_b^3}\left(1-\cos\theta_{ab}\right)\right]dr_a\right.\\ &\left.+\left[\frac{r_{ab}\sin\theta_{ab}}{r_b^3}\left(r_a-r_{ab}\right)\right]d\theta_{ab}\right\}\,,\end{aligned} \tag{6.50}$$

being r_b the radial distance as measured from $\vec{x}_b$. A solution can be readily found provided $\breve{\omega}^A_{ab}$ has only one non-vanishing component, $\breve{\omega}^A_{ab,\psi_{ab}}$

$$\breve{\omega}^A_{ab}=\frac{8\sqrt{2}q_a q_b}{r_{ab}}\left[C_{ijk}\Pi^i_{ab}\Pi^j_{ab}\Pi^k_{ab}-\frac{1}{2\breve{g}^2}\Pi^0_{ab}\mathbb{T}_{ab}\right](\cos\theta_{ab}-1)\left(1-\frac{r_a+r_{ab}}{r_b}\right)d\psi_{ab}\,. \tag{6.51}$$

Now we turn our attention to (6.48). Notice that this expression contains three-point interactions due to the presence of the connection $\breve{A}^\alpha$ in the covariant derivative, so at first sight its structure is more involved than that of its Abelian counterpart. However, despite this complexity, the general solution for an arbitrary number of centers can be found. It is most remarkable that the interactions among all of them can be written in a very compact form! We obtain

$$\breve{\omega}^B=\frac{3\sqrt{2}\varepsilon_{rst}}{\breve{g}^2P^2}\frac{\partial Q}{\partial x^s}\frac{\partial P}{\partial x^t}dx^r\,. \tag{6.52}$$

While deriving (6.51) and (6.52) we have assumed that the integrability condition is satisfied by making use of the bubbling equations (6.45). As a consistency check we can perform an inspection to confirm the absence Dirac-Misner strings in $\breve{\omega}^A$ and $\breve{\omega}^B$. For the former, it is straightforward to verify that the only component of the one form, $\breve{\omega}^A_{ab,\psi_{ab}}$, vanishes when the coordinate ψ_{ab} is not well defined. In particular this happens along the x^3_{ab} axis both in the positive direction, where $\left(1 - \frac{r_a + r_{ab}}{r_b}\right)|_{x^{3,+}_{ab}} = 0$, and in the negative direction, with $(\cos\theta_{ab} - 1)\,|_{x^{3,-}_{ab}} = 0$. In the case of the latter it suffices to check that $\breve{\omega}^B$ is regular at the centers as a consequence of the antisymmetric character of the 1-form components.

6.2.4 Fluxes and topological charge

We now turn our attention to the vector fields. We shall recall their expressions

$$
\begin{aligned}
A^I &= -\sqrt{3}h^I f(dt+\omega) + \hat{A}^I\,, && (6.53)\\
\hat{A}^I &= -2\sqrt{6}\left[-\Phi^I H^{-1}(d\varphi+\chi) + \breve{A}^I\right]\,, && (6.54)
\end{aligned}
$$

where $\breve{A}^I$ is determined in terms of Φ^I by the Bogomol'nyi equation (6.15) and whose explicit form is (6.25) in the Abelian sector and (6.26) in the non-Abelian. From these expressions we see that these fields can be understood in terms of three layers: the physical vectors A^I, a four-dimensional instanton $\hat{A}^I$ with selfdual field strength and a three-dimensional static magnetic monopole $\breve{A}^I$. Each of them is used to build up those preceding it, in a configuration that resembles the structure of the Russian *matryoshka* dolls.

In the Abelian sector $\breve{A}^i$ describes a configuration with several Dirac monopoles, which is singular due to the presence of Dirac strings attached to each center. These strings are eliminated in $\hat{A}^i$ by the new term in (6.54), although this term introduces new strings in the compact direction φ,

$$
\lim_{r_a\to 0}\hat{A}^i \sim -2\sqrt{6}\left[-\frac{k^i_a}{q_a}(d\varphi + q_a cos\theta_a d\psi_a) + k^i_a cos\theta_a d\psi_a\right] \sim 2\sqrt{6}\frac{k^i_a}{q_a}d\varphi\,. \qquad (6.55)
$$

The component in the local coordinate ψ_a is compensated by the new term, but now $\hat{A}^i_\varphi$ is finite at the centers, where the coordinate φ is not well defined. Besides $\hat{A}^i$ is not regular either at the critical surfaces characterized by $H = 0$. Yet again, this singularity is cured at the next stage and the physical vectors A^i are globally regular up to gauge transformations. In this case the first term in (6.53) compensates the divergence at the critical surface,

$$
\lim_{H\to 0}\left(-\sqrt{3}h^i f\omega_5(d\varphi+\chi)\right) = -2\sqrt{6}H^{-1}\Phi^i(d\varphi+\chi) + \mathcal{O}(H^0)\,, \qquad (6.56)
$$

without introducing any anomaly elsewhere, which is guaranteed because ω has been designed to be free of Dirac-Misner strings.

To every non-trivial 2-cycle at the ambipolar space it is naturally associated a magnetic flux for each vector, defined as the integral of the field strength F^i along the 2-cycle. To compute this quantity we make use of our standard decomposition for A^i, which is valid everywhere except at the centers. Nevertheless since the field strength is globally regular the flux can be equally computed by taking the integral along the 2-cycle with the poles excised. In this region the integrand is an exact form and we can make use of Stokes' theorem. We get

$$\Pi^i_{ab} \equiv \frac{1}{(2\sqrt{6})4\pi} \int_{\Delta_{ab}} F^i = \left(\frac{k^i_b}{q_b} - \frac{k^i_a}{q_a} \right) . \tag{6.57}$$

We now consider the non-Abelian sector. Our recipe for constructing solutions of $\mathcal{N} = 1$, $d = 5$ SEYM theory naturally incorporates Kronheimer's scheme [141], that relates any static monopole $\breve{A}^\alpha$ to an instanton over a Gibbons-Hawking base, $\hat{A}^\alpha$, through equation (6.54). For example, in [47] this mechanism has been utilized to *oxidize* the single center colored monopole, that has turned out to be the counterpart of the BPST instanton [19]. On the other hand, Etesi and Hausel showed in [84] that families of regular Yang-Mills instantons over an Asymptotically Locally Euclidean space (ALE) are related to multicenter colored monopoles in Kronheimer's scheme[9]. However, although our instanton is related to the same monopole, it is necessarily different than the Etesi-Hausel solution because they are defined on different bases: our Gibbons-Hawking space is ambipolar, not ALE. In particular this means that our instanton is singular at the critical surfaces. This is cured for the five-dimensional physical vector in the same manner than it is for the Abelian vectors.

Even though the instanton $\hat{A}^\alpha$ is ill-defined at the critical surfaces, we would like to study if we can associate to it a topological charge, also known as instanton number[10]. Here we need to remark that this topological charge is associated to the vector $\hat{A}^\alpha$ defined on the ambipolar Gibbons-Hawking space. Therefore this quantity may not be a true invariant of the physical spacetime. Nevertheless its computation is interesting by itself and, as we are about to see, this quantity is finite even though the connection blows up. We define the topological charge as

$$\mathbb{T} = \frac{g^2}{32\pi^2} \int_{\mathcal{M}_4 \backslash S} d^4\Sigma \hat{F}^2 \,, \tag{6.58}$$

where $d^4\Sigma$ is the volume form of the manifold, $\hat{F}^2$ is the scalar obtained by taking the trace of the field strength contracted with itself, $\hat{F}^2 \equiv \hat{F}^\alpha_{\mu\nu} \hat{F}^{\alpha\,\mu\nu}$, and $\mathcal{M}_4 \backslash S$ is the ambipolar space without the critical surfaces. These have to be necessarily removed because the canonical volume form associated to the metric vanishes there and the above integral cannot be defined over them. To perform the calculation it is convenient to work in the following flat frame of the cotangent bundle

$$e^0 = s|H|^{-1/2}(d\varphi + \chi) \,, \qquad e^a = |H|^{1/2} dx^s \delta^a_s \,, \qquad \epsilon^{0123} = \epsilon_{0123} = 1 \,. \tag{6.59}$$

where s is $+1$ when H is positive and -1 when H is negative. The volume form is expressed in terms of the vielbeins as $e^0 \wedge e^1 \wedge e^2 \wedge e^3 = H d\varphi \wedge d^3x$, where d^3x is a shorthand for $dx^1 \wedge dx^2 \wedge dx^3$. The gauge field strength is obtained from (6.54) and its components in this coframe are

[9] In fact, to the best of our knowledge, multicenter colored monopoles have only appeared in the literature so far in [84], where they are used as valuable intermediates for computing the topological charge of their instanton counterparts.

[10] It would be very interesting to study rigorously the construction of $SU(2)$ fiber bundles over ambipolar Gibbons-Hawking bases, but this goes beyond the scope of the present work.

$$\hat{F}^{\alpha}_{0a} = -2\sqrt{6}s\breve{\mathfrak{D}}_a\left(\Phi^{\alpha}H^{-1}\right)\,, \qquad \hat{F}^{\alpha}_{ab} = -2\sqrt{6}s\left[H^{-1}\breve{F}^{\alpha}_{ab} - H^{-2}\Phi^{\alpha}(d\chi)_{ab}\right]\,. \tag{6.60}$$

Substituting back into (6.58), using (6.14), (6.15) and integrating by parts we get

$$\mathbb{T} = \frac{\breve{g}^2}{32\pi^2}\int_{\mathcal{M}_4\backslash(S\cup X)} d\varphi\wedge d^3x\left[2\nabla^2\left(\frac{\Phi^{\alpha}\Phi^{\alpha}}{H}\right) - 4H^{-1}\Phi^{\alpha}\breve{\mathfrak{D}}^2\Phi^{\alpha} + 2H^{-2}\Phi^{\alpha}\Phi^{\alpha}\nabla^2 H\right]\,. \tag{6.61}$$

Notice that in this step the centers have also been removed from the integration space because the decomposition (6.54) is not well-defined there. This does not change the value of the integral because $\hat{F}^2$ is regular at these points. The second and third terms in the above expression vanish identically in the region. We can integrate on φ and apply Stokes theorem to get

$$\mathbb{T} = \frac{\breve{g}^2}{4\pi}\int_{V^3} d^3x\nabla^2\left(\frac{\Phi^{\alpha}\Phi^{\alpha}}{H}\right) = \frac{\breve{g}^2}{4\pi}\int_{\partial V^3} d^2\Sigma\, n_a\partial_a\left(\frac{\Phi^{\alpha}\Phi^{\alpha}}{H}\right)\,. \tag{6.62}$$

Here V^3 is $\mathbb{R}^3$ with the centers and the critical surfaces excised, $d^2\Sigma$ is the volume form induced on ∂V^3 and n_a are the components of a unit vector normal to ∂V^3. Thus the problem is reduced to a computation at the boundary of V^3, which is composed of the critical surfaces, the centers and infinity. Formally at the critical surfaces we receive an infinite contribution to the topological charge, but notice that each connected critical surface is the boundary of two disconnected regions of V_3 and therefore it appears twice in the computation. Since the normal unitary vector $\vec{n}$ has opposite direction in each case, both infinite contributions cancel out because $\lim_{\vec{x}\to\partial V^3}\partial_a\left(\frac{\Phi^{\alpha}\Phi^{\alpha}}{H}\right)|n_a|$ takes the same value when $\vec{x}$ is evaluated at both sides of the critical surface.

After having got rid of the critical surfaces, the computation of (6.62) is straightforward. The contributions at each center and at infinity are

$$\mathbb{T}_a = \breve{g}^2\frac{k^{\alpha}_a k^{\alpha}_a}{q_a}\,, \qquad \mathbb{T}_{\infty} = 0\,, \tag{6.63}$$

Assuming that we placed non-Abelian seed functions at every center, the total topological charge is

$$\mathbb{T} = \sum_a \frac{1}{q_a}\,. \tag{6.64}$$

6.2.5 Critical surfaces

As we have already discussed at previous stages, the critical surfaces defined by having $H = 0$ are worth special attention. Not only is the ambipolar Gibbons-Hawking metric ill-defined there, but also many of the other auxiliary building blocks that make up the solution contain inverse powers of H. Nevertheless, the spacetime metric and all physical fields remain completely regular at the critical surfaces. It is interesting to illustrate in some detail how this happens.

Let us consider the metric as written in (6.37). In the purely spatial part there are no singularities in these surfaces because the product $f^{-1}H$ defines a finite positive quantity,

$$\lim_{H\to 0} f^{-1}H = 8\left(C_{IJK}\Phi^I\Phi^J\Phi^K\right)^{2/3} + \mathcal{O}(H)\,, \tag{6.65}$$

and $\mathcal{I}$ is also regular, as easily seen from its expression in terms of the seed functions (6.40). Of course, this is only possible because $\lim_{H\to 0} f \sim 0$ and this, in particular, means that the critical surfaces are determined by the vanishing of the norm of the Killing vector that generates time translations, $V = \partial_t$, $V^\mu V_\mu = f^2$.

One might get worried by this statement, since timelike supersymmetric solutions in supergravity quite frequently have event horizons at the regions where the timelike Killing vector becomes null. Happily this does not happen here. First, because as we just saw the spatial part remains regular, and second, because of the presence of the additional finite term in the metric that keeps the determinant non-vanishing at these regions,

$$\lim_{H\to 0} f^2\omega_5 dt(d\varphi+\chi) = \frac{1}{2\sqrt{2}}\left(C_{IJK}\Phi^I\Phi^J\Phi^K\right)^{-1/3} dt(d\varphi+\chi) + \mathcal{O}(H)\,. \tag{6.66}$$

Then any massive particle sitting at the surface is unavoidably dragged along some spatial direction. Critical surfaces have the same properties as the boundary of an ergosphere, except from the fact that they do not actually surround an ergosphere since the Killing vector V remains timelike at both of their sides. As a consequence of this they have been named *evanescent ergosurfaces* [99].

In the previous subsection we already showed that the physical vectors are well-behaved at the evanescent ergospheres. The physical scalars, constructed by $\phi^x \equiv h_I/h_0$, are also regular here

$$\lim_{H\to 0} \phi^x = \frac{C_{xIJ}\Phi^I\Phi^J}{C_{0LM}\Phi^L\Phi^M} + \mathcal{O}(H)\,. \tag{6.67}$$

6.3 Final comments

The set of continuous parameters λ_a that appear in the definition of the colored monopole, (6.26), have no impact on the physics of the solution neither at the centers nor at infinity, but they do affect the physical fields at intermediate regions. This means that the geometry of a particular solution can be continuously distorted in some manner as long as the modification does not introduce CTC's. Therefore we can build a *classically* infinite number of microstate geometries with the same topology for the same black hole or black ring.

It is useful to explain in some detail why these parameters are special in this sense. First, one has to notice that asymptotically the non-Abelian seed functions Φ^α are subleading with respect to the Abelian seed functions Φ^i (6.24). Second, the functions Φ^α have the same limit at leading order at all the centers, whose value is independent of these parameters. These characteristics imply that the mass, angular momenta and electric charges of the solution are invisible to the parameters λ_a. The size of the bubbles are also unaffected by them, see (6.45).

The *colored* non-Abelian black hole solutions discovered so far are constructed from a single-center colored monopole. They incorporate one parameter, say λ_1, interpreted as the *size* of the instanton field of the solution, that modifies the geometry outside the horizon but does not alter any of the *observables* of the solution, like the mass, entropy, electric charges or instanton number. In this context this parameter is interpreted as non-Abelian hair. On the other hand microstate geometries have one parameter for each center. Although we do not have a complete interpretation of the multicenter instanton field contained in these solutions, preliminary analysis based on the expansion of the instanton field $\hat{A}^{\alpha}$ near the centers suggest that each parameter codifies the information of the size of an instanton placed at the corresponding center whose individual topological charge is $1/q_a$.

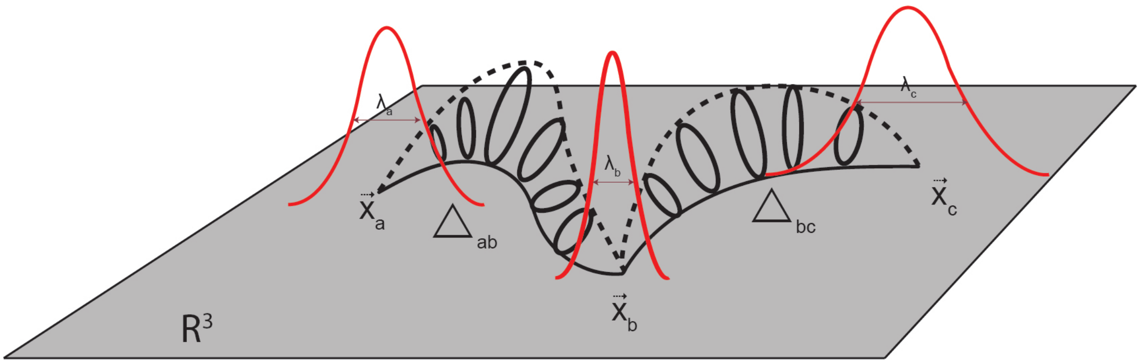

Figure 6.1: Representation of the multicenter instanton on the Gibbons-Hawking space.

On the other hand, the gauge coupling constant $\breve{g}$ controls the relative weight of the non-Abelian versus the Abelian fields. The closer this parameter is to zero the more influent the non-Abelian ingredients are. This is in particular reflected in the bubbling equations (6.45), from what we see that the size of the bubble can be dominated by one or the other contributions for different values of the coupling constant.

Clearly these solutions require further study. The explicit construction of concrete solutions with specific charges would be of course very interesting.

7

A gravitating Yang-Mills instanton

This chapter is based on

Pablo A. Cano, Tomas Ortín and Pedro F. Ramírez
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Fueled by the research on theories of elementary particles and fundamental fields (Yang-Mills, Kaluza-Klein, Supergravity, Superstrings...), over the last 30 years, the search for and study of solutions of theories of gravity coupled to fundamental matter fields (scalars and vectors in $d = 4$ and higher-rank differential forms in higher dimensions) has been enormously successful and it has revolutionized our knowledge of gravity itself. Each new classical solution to the Einstein equations (vacua, black holes, cosmic strings, domain walls, black rings, black branes, multi-center solutions...) sheds new light on different aspects of gravity and, often, on the underlying fundamental theories. For instance, although the string effective field theories (supergravities, typically) only describe the massless modes of string theory, it is possible to learn much through them about the massive non-perturbative states of the fundamental theory because they appear as classical solutions of the effective theories.[1] Beyond this, there is a definite program in the quest to construct horizonless *microstate geometries* as classical solutions of Supergravity theories [32, 33]. When interpreted within the context of the fuzzball conjecture [150], these geometries have been proposed to correspond to the classical description of black hole microstates. Therefore, in the best case scenario, it might be possible to find a large collection ($\sim e^S$) of microstate geometries with the same asymptotic charges as a particular black hole, and, furthermore, to identify explicitly their role in the ensemble of black-hole microstates. See Refs. [24, 29] for recent progress in that direction.

Apart from the fact that they describe gravity, one of the most interesting features of string theories is that their spectra include non-Abelian Yang-Mills (YM) gauge fields. This aspect is crucial for their use in BSM phenomenology but has often been neglected in the search for classical solutions of their effective field theories, specially in lower dimensions, which have been mostly focused on theories with Abelian vector fields and with, at most, an Abelian gauging. Thus, the space of extremal (supersymmetric and non-supersymmetric, spherically-symmetric and multi-center) black-hole solutions of 4- and 5-dimensional ungauged supergravities has been exhaustively explored and progress has been made in the Abelian gauged case, motivated by the AdS/CFT correspondence, but the non-Abelian case has drawn much less attention in the string community and, corre-

[1] See, *e.g.*, Refs. [75, 170, 197].

spondingly, there are just a few solutions of the string effective action (and of supergravity theories in general) with non-Abelian fields in the literature.

One of the main reasons for that is the intrinsic difficulty of solving the highly non-linear equations of motion. This difficulty, however, has not prevented the General Relativity community from attacking the problem in simpler theories such as the Einstein-Yang-Mills (EYM) or Einstein-Yang-Mills-Higgs (EYMH) theories, although it has prevented them from finding analytical solutions: most of the genuinely non-Abelian solutions[2] are known only numerically.[3] Another reason is that non-Abelian YM solutions are much more difficult to understand than the Abelian ones (specially when they are known only numerically): in the Abelian case we can characterize the electromagnetic field of a black hole, say, by its electric and magnetic charge, dipoles and higher multipoles. In the non-Abelian case the fields are usually characterized by topological invariants or constructions such as t' Hooft's magnetic monopole charge.

In general, the systems studied by the GR community (the EYM or EYMH theories in particular) are not part of any theory with extended local supersymmetry (a $\mathcal{N} > 1$ supergravity with more than 4 supercharges)[4] and, therefore, the use of supersymmetric solution-generating techniques is not possible. One can, however, consider the minimal $\mathcal{N} > 1$ supergravity theories that include non-Abelian YM fields, which are amenable to those methods. Some time ago we started the search for supersymmetric solution-generating methods in $\mathcal{N} = 2, d = 4$ [122] and $\mathcal{N} = 1, d = 5$ [20,23,47,158] Super-Einstein-Yang-Mills (SEYM) theories. The results obtained have allowed to construct, for the first time (at least in fully analytical form), several interesting supersymmetric solutions with genuine non-Abelian hair: *global monopoles* and extremal static black holes in 4 [46,154, 159] and 5 dimensions [159], rotating black holes and black rings in 5 dimensions [171], non-Abelian 2-center solutions in 4 dimensions [46] and the first non-Abelian microstate geometries [183].

Many of the black-hole solutions found by these methods can be embedded in string theory and, in that framework, one can try to address the microscopic interpretation of their entropy, which seems to have relevant contributions from the non-Abelian fields, even though, typically, they decay so fast at infinity that they do not seem to contribute to the mass. Following the pioneer's route [148,199] requires an understanding of the stringy objects (D-branes etc.) that contribute to the 4- and 5-dimensional solutions' charges. Furthermore, the interpretation of the non-Abelian microstate geometries would benefit from the knowledge of their stringy origin. In this paper, as a previous step towards the microscopic interpretation of the 5-dimensional non-Abelian black holes' entropy which we will undertake in the following chapter, we identify the elementary component of the simplest, static, spherically symmetric, non-Abelian 5-dimensional black hole that carries all the non-Abelian hair. The solution that describes this component turns out to be asymptotically flat, globally regular, and horizonless and the non-Abelian field is that of a BPST instanton [19] living in constant-time hypersurfaces. Only a few solutions supported

[2]That is, solutions whose non-Abelian fields cannot be rotated into Abelian ones using (singular or non-singular) gauge transformations. When they can be rotated into a purely Abelian one, it is often referred to as an "Abelian embedding".

[3]The most complete review on non-Abelian solutions containing the most relevant developments until 2001 is Ref. [208] complemented with the update Ref. [88]. Ref. [213] reviews the anti-De Sitter case. A more recent but less exhaustive review is Ref. [207], although it omits most of the non-Abelian solutions found recently in the supergravity/superstring context.

[4]The supersymmetric solutions of $\mathcal{N} = 1$ supergravity are massless (waves) or not asymptotically flat.

by elementary fields with these characteristics are known analytically: the *global monopoles* found in gauged $\mathcal{N}=4, d=4$ supergravity [60, 61, 112] and also in $\mathcal{N}=2, d=4$ SEYM theories [46, 122] whose non-Abelian field is that of a BPS 't Hooft-Polyakov monopole.

The simplest string embedding of this solution is in the Heterotic Superstring and the 10-dimensional solution whose dimensional reduction over T^5 gives this 5-dimensional *global instanton* turns out to be the gauge 5-brane found in Ref. [198]. This is, therefore, the non-Abelian ingredient present in the non-Abelian 5-dimensional black holes and rings constructed in Refs. [159, 171].

In what follows, we are going to derive the global instanton solution as a component of the 5-dimensional non-Abelian black holes, we show that it is the Heterotic String gauge 5-brane compactified on T^5 and we study the dependence of the distribution of energy on the instanton's scale parameter, showing that, no matter how small it is, there is never more energy concentrated in a 3-sphere of radius R than that of a Schwarzschild-Tangerlini black hole of radius R.

7.1 The global instanton solution

We are going to work in the context of the ST[2, 6] model of $\mathcal{N}=1, d=5$ supergravity (which is a model with 5 vector supermultiplets) with an SU(2) gauging in the $I=3,4,5$ sector. This theory is briefly described in Appendix C.2 and the solution-generating technique that allows us to construct timelike supersymmetric solutions of this theory with one isometry is explained in Appendix 4.2.

Our goal is to construct the minimal non-singular solution that includes in the SU(2) sector the following solution of the Bogomol'nyi equations

$$\Phi^A = \frac{1}{g_4 r(1+\lambda^2 r)}\frac{x^A}{r}\,, \qquad \breve{A}^A{}_B = \varepsilon^A{}_{BC}\frac{1}{g_4 r(1+\lambda^2 r)}\frac{x^C}{r}\,. \qquad r^2 \equiv x^s x^s\,, \tag{7.1}$$

This solution describes a *coloured monopole* [154, 159], one of the singular solutions found by Protogenov [182]. Observe that this solution is written in terms of the 4(= 1 + 3)-dimensional Yang-Mills coupling constant g_4. As shown in [47], the 4-dimensional Euclidean SU(2) gauge field $\hat{A}^A$ that one obtains via Eq. (4.61) for $H = 1/r$ is the BPST instanton [19], which justifies our choice. Using the 4-dimensional radial coordinate $\rho^2 = 4r$, the 5-dimensional Yang-Mills coupling constant $g_4 = -2\sqrt{6}g$, and renaming $4\lambda^{-2} = \kappa^2$ (the instanton scale parameter) it takes the form[5]

$$\hat{A}^A = \frac{\kappa^2}{g(\rho^2+\kappa^2)}v_R^A\,, \tag{7.2}$$

where the v_R^A are the three SU(2) left-invariant Maurer-Cartan 1-forms.

Let us now consider the ungauged sector. As it is well known, 5-dimensional asymptotically-flat, static, regular black holes need to be sourced by at least three charges,

[5] Our conventions for the SU(2) gauge fields are slightly different from the ones used in Refs. [158, 171]: in this paper the generators satisfy the algebra $[T_A, T_B] = +\epsilon_{ABC}T_C$ (which is equivalent to changing the sign of all the generators), and the gauge field strength is defined by $F = dA + gA\wedge A$. The left- and right-invariant Maurer-Cartan 1-forms $v_{L,R}$ have the same definitions, but the overall signs of the components are different, as a consequence of the change of sign in the generators T_A.

associated to three different kind of branes. A popular example is the D1D5W black hole considered by Strominger and Vafa in Ref. [199]. The corresponding solution of the (supergravity) effective action is expressed in terms of three independent harmonic functions. In the basis that we are using, these functions are $L_{0,1,2}$, where the last two will be used in the the combinations $L_\pm = L_1 \pm L_2$ in order to make contact with the literature.

Thus, we take[6]

$$L_{0,\pm} = B_{0,\pm} + q_{0,\pm}/\rho^2\,, \tag{7.3}$$

and we will assume that all the constants are positive.

This choice gives a static solution ($\hat{\omega} = 0$, see the appendices for more information) with the following active fields function

$$\begin{aligned}
ds^2 &= \hat{f}^2 dt^2 - \hat{f}^{-1}(d\rho^2 + \rho^2 d\Omega^2_{(3)})\,,\\
A^0 &= -\tfrac{1}{\sqrt{3}}\frac{1}{\tilde{L}_0}dt\,, \qquad A^1 \pm A^2 = -\tfrac{2}{\sqrt{3}}\frac{1}{L_\pm}dt\,, \qquad A^A = \frac{\kappa^2}{g(\rho^2+\kappa^2)}v^A_R\,,\\
e^{2\phi} &= 2\frac{\tilde{L}_0}{L_+}\,, \qquad k = (3\hat{f}L_-)^{3/4}\,,
\end{aligned} \tag{7.4}$$

where the metric function $\hat{f}$ is given by

$$\hat{f}^{-1} = \left\{\tfrac{27}{2}\tilde{L}_0 L_+ L_-\right\}^{1/3}\,, \tag{7.5}$$

and we have defined the combination

$$\tilde{L}_0 \equiv L_0 - \tfrac{1}{3}\rho^2\Phi^2\,, \quad \text{and} \quad \Phi^2 \equiv \Phi^A\Phi^A = \frac{2\kappa^4}{3g^2\rho^4(\rho^2+\kappa^2)^2}\,. \tag{7.6}$$

The normalization of the metric at spatial infinity demands $\frac{27}{2}B_0B_+B_- = 1$ and we can express the three integration constants B in terms of the values of the 2 scalars at infinity:

$$B_0 = \tfrac{1}{3}e^{\phi_\infty}k_\infty^{-2/3}\,, \qquad B_+ = \tfrac{2}{3}e^{-\phi_\infty}k_\infty^{-2/3}\,, \qquad B_- = \tfrac{1}{3}k_\infty^{4/3}\,, \tag{7.7}$$

and the metric takes the form

$$\hat{f}^{-1} = \left\{(\tilde{L}_0/B_0)\,(L_+/B_+)\,(L_-/B_-)\right\}^{1/3}\,, \tag{7.8}$$

where

[6]The simplest 5-dimensional non-Abelian black hole constructed in Ref. [158] has $L_2 = 0$, or $L_+ = L_-$ and, therefore, it has three Abelian charges as well, but two of them are equal, which obscures the interpretation of the solution from the string theory point of view.

$$
\begin{aligned}
\tilde{L}_0/B_0 &= 1+\frac{2e^{-\phi_\infty}k_\infty^{2/3}}{3g^2}\frac{\rho^2+2\kappa^2}{(\rho^2+\kappa^2)^2}+3e^{-\phi_\infty}k_\infty^{2/3}\left(q_0-\frac{2}{9g^2}\right)\frac{1}{\rho^2}\,,\\
L_+/B_+ &= 1+3e^{\phi_\infty}k_\infty^{2/3}q_+/(2\rho^2)\,,\\
L_-/B_- &= 1+3k_\infty^{-4/3}q_-/\rho^2\,.
\end{aligned}
\tag{7.9}
$$

If $\tilde{q}_0 \equiv q_0 - \frac{2}{9g^2} > 0$ and $q_\pm \neq 0$ there is a regular event horizon with entropy

$$
S = \frac{\pi^2}{2G_N^{(5)}}\sqrt{(3\tilde{q}_0)\,(3q_+/2)\,(3q_-)}\,. \tag{7.10}
$$

The mass, however, depends on q_0, not on $\tilde{q}_0$

$$
M = \frac{\pi}{4G_N^{(5)}}\left[e^{-\phi_\infty}k_\infty^{2/3}(3q_0)+e^{\phi_\infty}k_\infty^{2/3}(3q_+/2)+k_\infty^{-4/3}(3q_-)\right]\,, \tag{7.11}
$$

so that the Yang-Mills fields only appear to be relevant in the near-horizon region, a behavior also observed in 4-dimensional *colored* black holes Refs. [154, 159]. Explaining this behavior and finding a stringy microscopic interpretation for the entropy of these black holes will be the subject of next chapter.

One of the main ingredients needed to reach that goal is the list of elementary components (branes, waves, KK monopoles...) of the black-hole solution. In the Abelian case, these are typically associated to the harmonic functions in which the brane charges occur as coefficients of the $1/\rho^2$ terms (in 5 dimensions) and these are the charges that appear in the entropy formula. In the present case $\tilde{L}_0/B_0$ has a term which is finite in the $\rho \to 0$ limit and another term, proportional to $\tilde{q}_0$, which goes like $1/\rho^2$ in that limit, as an ordinary Abelian contribution would. The presence of the finite term suggests the presence of a solitonic brane which does not contribute to the entropy.

In order to identify this brane we set $\tilde{q}_0 = q_\pm = 0$ in the above solution (but $q_0 = \frac{2}{9g^2} \neq 0$) and we obtain[7]

$$
\begin{aligned}
ds^2 &= \hat{f}^2dt^2-\hat{f}^{-1}(d\rho^2+\rho^2d\Omega^2_{(3)})\,,\\
\hat{f}^{-3} &= 1+\frac{2e^{-\phi_\infty}k_\infty^{2/3}}{3g^2}\frac{\rho^2+2\kappa^2}{(\rho^2+\kappa^2)^2}\,,\\
A^0 &= -\frac{1}{\sqrt{3}B_0}\hat{f}^3dt\,, \qquad A^A = \frac{\kappa^2}{g(\rho^2+\kappa^2)}v^A_R\,,\\
e^{2\phi} &= e^{2\phi_\infty}\hat{f}^{-3}\,, \qquad k = k_\infty\hat{f}^{3/4}\,,
\end{aligned}
\tag{7.12}
$$

[7]Notice that the cancellation of the term that diverges in the $\rho \to 0$ limit can only be achieved in the branch in which $L_0 > 0$. In particular, if either $L_+ < 0$ or $L_- < 0$ we are forced to work in the $L_0 > 0$ branch and that contribution cannot be made to vanish,

This solution depends on one function, $\hat{f}$ which has the same profile as the one appearing in the gauge 5-brane [198].[8] The similarity can be made more manifest by using the relation between the 5-dimensional Yang-Mills coupling constant g, the Regge slope α', the string coupling constant $g_s = e^{\phi_\infty}$ and the radius of compactification from 6 to 5 dimensions $k_\infty = R_z/\ell_s$, where $\ell_s = \sqrt{\alpha'}$ is the string length parameter:

$$g = k_\infty^{1/3} e^{-\phi_\infty/2} / \sqrt{12\alpha'}\,, \tag{7.13}$$

which brings $e^{2\phi}$ to the form[9]

$$e^{2\phi} = e^{2\phi_\infty} \hat{f}^{-3} = e^{2\phi_\infty} \left\{ 1 + 8\alpha' \frac{\rho^2 + 2\kappa^2}{(\rho^2+\kappa^2)^2} \right\} . \tag{7.15}$$

It is not difficult to show that, indeed, this solution is nothing but the double dimensional reduction of the gauge 5-brane compactified on T^5 [54].

From the purely 5-dimensional point of view, apart from the instanton field, the solution has a vector field A^0 which is dual to the Kalb-Ramond 2-form and is sourced by the instanton number density only, as in the gauge 5-brane [74]. Observe that this means that the parameter q_0 is the sum of the instanton-number contributions (associated to a gauge 5-brane, as we are going to argue) which amount to just $\frac{2}{9g^2}$ and electric sources of a different origin which amount to $\tilde{q}_0 = q_0 - \frac{2}{9g^2}$ which we have set to zero in the above solution. The complete identification of the higher-dimensional stringy components of the general solution will be the subject of the forthcoming chapter, see also [54]. Here we just want to study the above solution, which in its 5-dimensional form is, apart from supersymmetric, clearly globally regular (at least for finite values of κ), asymptotically flat and horizonless and they are the higher-dimensional analogue of the *global monopole* solutions found in gauged $\mathcal{N} = 4, d = 4$ supergravity [60, 61, 112] and also in $\mathcal{N} = 2, d = 4$ SEYM theories [46, 122].

The mass of the global instanton is obtained by replacing q_0 by $\frac{2}{9g^2}$ and setting $q_\pm = 0$ in Eq. (8.10):

$$M = \frac{\pi}{6g^2 G_N^{(5)}} e^{-\phi_\infty} k_\infty^{2/3} = 8 \frac{R_9 \cdots R_5}{g_s^2 \ell_s^6}\,, \tag{7.16}$$

where R_i is the compactification radius of the x^i coordinate and where we have used

$$G_N^{(5)} = \frac{G_N^{(10)}}{(2\pi)^5 R_9 \cdots R_5}\,, \quad \text{and} \quad G_N^{(10)} = 8\pi^6 g_s^2 \ell_s^8\,. \tag{7.17}$$

[8] More precisely, the function $H = e^{2\phi_\infty} \hat{f}^{-3}$.

[9] In our conventions, which coincide essentially with those of Ref. [198], the 10-dimensional Heterotic String effective action is written in the string frame as

$$S_{\text{Het}} = \frac{g_s^2}{16\pi G_N^{(10)}} \int dx^{10} \sqrt{|g|}\, e^{-2\phi} \left[R - 4(\partial\phi)^2 + \tfrac{1}{12} H^2 - \alpha' F^A F^A \right] . \tag{7.14}$$

The 10-dimensional string-frame metric solution is normalized such that it becomes $(+1, -1, \cdots, -1)$ at spatial infinity. The same is true for the 5-dimensional metric, which can be seen as the modified-Einstein-frame metric in the language of Ref. [148]. The relation between these two metrics involves rescalings by powers of $e^{\phi-\phi_\infty}$ and k/k_∞.

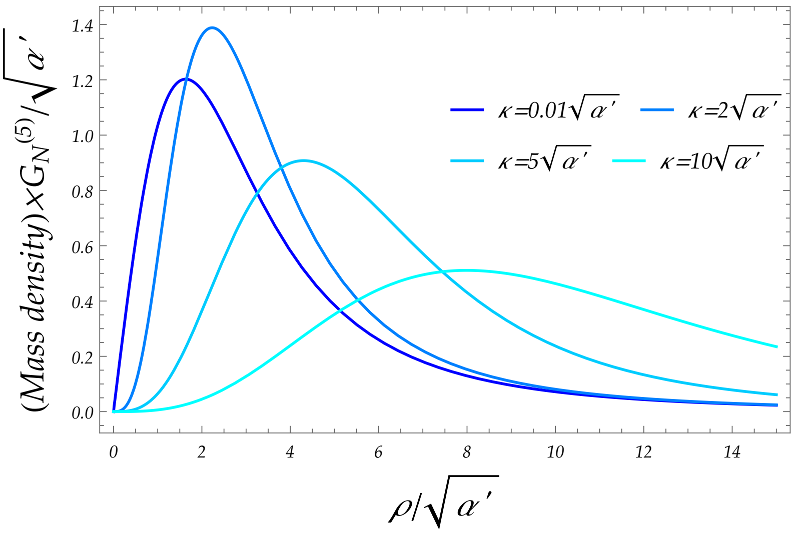

Figure 7.1: Radial mass density function of the global instanton solution for different values of the instanton scale, κ^2.

This value is eight times that of a single neutral (solitonic) 5-brane [50, 51].

The metric depends on the instanton scale κ^2, and it becomes singular when $\kappa = 0$. It is tempting to regard that singular metric as the result of concentrating all the mass, which is independent of κ, in a single point. Thus, one may wonder how the radial distribution of the energy depends on κ and whether there is a value of κ and ρ such that the energy enclosed in a 3-sphere of that radius is larger than the mass of a Schwarzschild black hole of that Schwarzschild radius ($R_S^2 = 3\pi M/(8G_N^{(5)})$).

The radial mass density, given by $\sqrt{|g|}T^{00}$ (T^{00} being the tangent-space basis component of the energy-momentum tensor) is represented in Fig. 7.1 for different values of the instanton scale and its integral over a sphere of radius R (the *mass function*) is represented in Fig. 7.2. The values of the integrals at infinity are not exactly equal because, after all, there is no well-defined concept of energy density in General Relativity and we are just using a reasonable approximation. In Fig. 7.3 we have represented the quotient between the mass function and the Schwarzschild mass as a function of R and we see that it never goes above 5/9 for any finite, non-vanishing value of the instanton scale.

7.2 Conclusions

Globally regular solutions supported by elementary fields are quite remarkable. In the case of the 4-dimensional global monopoles [46, 60, 61, 112, 122] we have argued that they represent elementary, non-perturbative states of the theory because they do not modify the entropy of a given Abelian black hole solution when they are added to it. They do contribute to the mass, though. Adding the global instanton to 5-dimensional black holes should have the same result: unmodified entropy and increased mass. However, the reverse seems to happen: the entropy is modified while the mass is not. The construction of the global instanton solution seems to suggest that this is a false appearance caused by an inappropriate definition of the charges involved. The exact role in 4-dimensional non-Abelian black-hole solutions (in which it must appear disguised as a *coloured monopole*) has

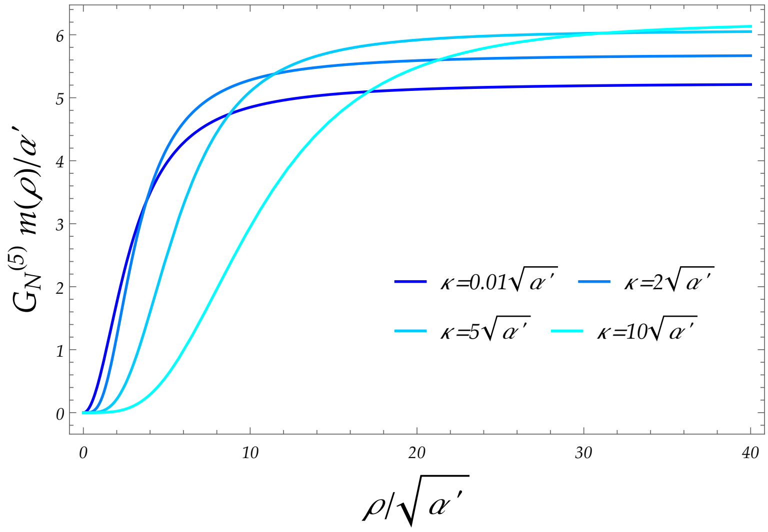

Figure 7.2: Radial mass function of the global instanton solution for different values of κ^2 obtained by integration of the mass density function in Fig. 7.1 with respect to ρ.

to be investigated. It is also unclear if a global instanton can be added to a Schwarzschild-Tangerlini (or any other non-extremal black hole) and what the effect would be.

We have tried to deform this solution by adding angular momentum, which in these theories is always possible, although the simplest ways to do it (adding a non-trivial harmonic function M to generate a non-vanishing ω_5) would also introduce a singularity at the origin. While we have succeeded in producing an ω_5 regular at $\rho = 0$ and dropping at infinity as ρ^{-2}, the metric function $\hat{f}^{-1}$ becomes singular at $\rho = 0$. It is possible to cancel those singularities by introducing additional Abelian harmonic functions with fine-tuned coefficients but the resulting $\hat{f}^{-1}$ either has zeroes, or leads to negative mass or both.

The non-Abelian solutions found so far in the supergravity/superstring context are the simplest to construct. One can expect, however, a space of solutions far richer than that of the Abelian ones. Work in this direction is under way.

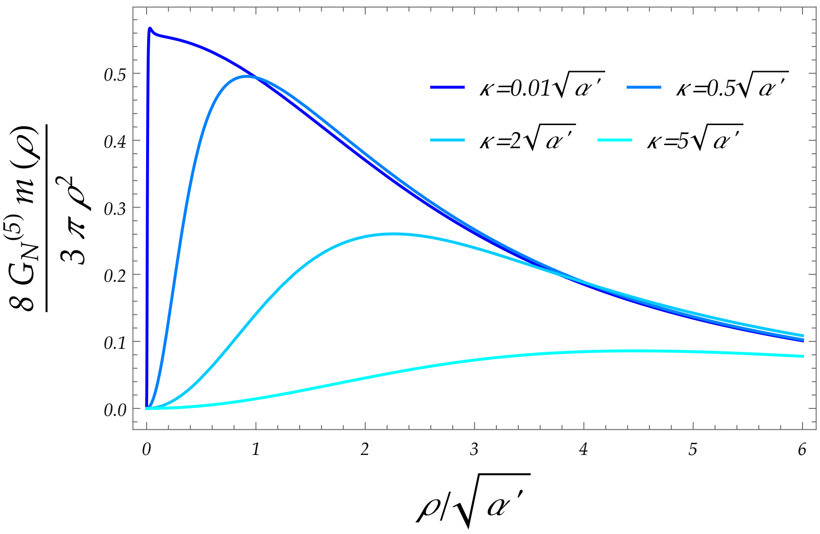

Figure 7.3: Quotient between the radial mass function of the global instanton solution and the mass of the 5-dimensional Schwarzschild black hole for with that Schwarzschild radius for different values of κ^2. This figure is obtained by integration of the quotient of the mass density function in Fig. 7.1 and $M_S = 3\pi\rho^2/8$ respect to ρ.

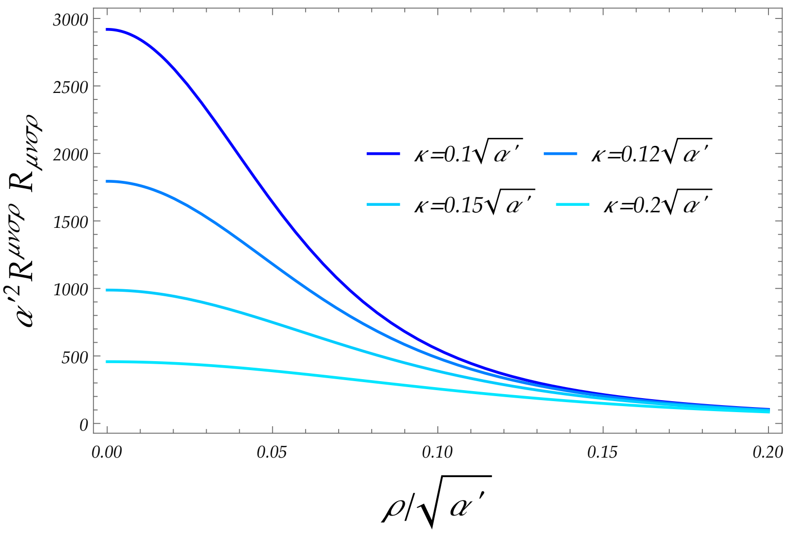

Figure 7.4: Value of the Kretschmann invariant for the global instanton solution for different values of κ^2.

8

Non-abelian black holes in string theory

This chapter is based on

Pablo A. Cano, Patrick Meessen, Tomas Ortín and Pedro F. Ramírez
"Non-abelian black holes in string theory",
FPAU-O17-05-IFT-UAM-CSIC-17-025. [arXiv:1704.01134 [hep-th]] [54].

One of the common features of black holes or black rings with genuinely non-Abelian fields[1] in Einstein-Yang-Mills (EYM) theory, where they are only known numerically [39, 206], or in $\mathcal{N}=2, d=4,5$ Super-EYM (SEYM) theories [154,158,159,171], where they are known analytically, is that their non-Abelian fields fall off at spatial infinity so fast that they cannot be characterized by a conserved charge. For this reason they are sometimes called "colored" black holes, as opposed to "charged" black holes. As a consequence, the parameters that characterize the black holes must be understood as pure non-Abelian hair.

In the SEYM it has also been observed that the non-Abelian fields seem to contribute in a non-trivial way to the BH entropy because their near-horizon behavior is similar to that of their Abelian counterparts [154, 158, 159, 171]. Thus, apparently, the entropy of these non-Abelian black holes and rings depends on non-Abelian hair! If the BH entropy admits a microscopic interpretation, this conclusion is clearly unacceptable.

In this paper we are going to solve this puzzle for a family of particularly simple non-Abelian 5-dimensional black holes that can be embedded in String Theory [158] and which can be seen as the well-known 3-charge D1D5W black-hole solutions discussed in Ref. [49][2] with the addition of a BPST instanton [19], which is genuinely non-Abelian in the sense discussed above.[3]

In this case at least, the solution to the non-Abelian hair puzzle lies in the correct interpretation of the different charges that characterize the black hole. As we have shown in Ref. [55], the charges that count the underlying String-Theory objects are combinations of the naive ones. The correctly identified charges can be switched off one by

[1]That is: non-Abelian fields that cannot be related to an Abelian embedding via a (possibly singular) gauge transformation [195]. Gauge transformations, whether regular or singular, have no effect whatsoever on the spacetime metric and, therefore, if the non-Abelian fields can be related to an Abelian embedding, the metric is effectively that of a solution with an Abelian field. This was the only kind of regular solutions thought to exist in the Einstein-Yang-Mills theory, basically because the non-Abelian fields were expected to behave at infinity like the Abelian ones [40, 83, 89]. See also See Refs. [88, 208] and references therein.

[2]More information on these black holes and the String Theory computation of their BH entropy can be found in Ref. [70] and references therein.

[3]Technically, this family of black holes is a solution of the SU(2)-gauged ST[2, 6] model of $\mathcal{N}=1, d=5$ supergravity. This model and the solution-generating technique used to obtain the black-hole family is described in full detail in an Appendix of Ref. [55].

one and, switching off those that count the objects that give rise to the Abelian charges (that is, setting to zero the number of D1s, D5s and the momentum) one is left with the object that produces the net non-Abelian field. In 5 dimensions, this object is a globally regular, horizonless gravitating instanton [55] which, when uplifted to 10-dimensional Heterotic Supergravity (the effective field theory of the Heterotic Superstring), is nothing but Strominger's gauge 5-brane [198].[4] In terms of these charges, as we will see, there is a non-Abelian contribution to the mass and the non-Abelian contribution to the entropy disappears, solving the puzzle.

This is a very important clue that we are going to apply to these solutions. In Section 8.1 we are going to introduce them and rewrite them in terms of the charges that describe the underlying String-Theory objects. In Section 8.2 we are going to uplift them to 10-dimensional Heterotic Supergravity, a theory that has non-Abelian vector fields in 10 dimensions, and, in Section 8.3 we will reinterpret the solution in terms of intersections of fundamental strings, solitonic 5-branes and gauge 5-branes, plus momentum along the strings, and we will dualize it into a solution of Type-I Supergravity (the effective field theory of Type-I Superstring Theory) [69, 124, 178] with D-strings with momentum, D5-branes and "gauge D5-branes", the duals of the gauge 5-branes, also referred to as D5-branes dissolved into the D9 branes. Then, in Section 8.4 we discuss how this brane configuration leads to the same entropy as the Abelian one, pointing to directions for future work.

8.1 5-dimensional non-Abelian black holes

We consider the SU(2)-gauged ST[2, 6] model of $\mathcal{N} = 1, d = 5$ supergravity, which can be obtained from $d = 10$ Heterotic Supergravity by compactification on T^5 followed by a truncation. This is most conveniently done in two stages: first, compactification on T^4 followed by a truncation to $\mathcal{N} = (2, 0), d = 6$ supergravity coupled to a tensor multiplet and a triplet of SU(2) vectors and, second, further compactification on S^1. The first stage is almost trivial: all the 6-dimensional fields are identical (up to rescalings) to the first 6 components of the 10-dimensional ones. The second stage is described in detail in Ref. [56].

This model is determined by the symmetric tensor $C_{0xy} = \frac{1}{6}\eta_{xy}$, with $x, y = 1, 2, A$, $A, B, \ldots = 3, 4, 5$ and $\eta_{xy} = (+, -, -, -, -)$.[5] The $A, B, \ldots$ are adjoint SU(2) indices. The bosonic content of this model consists of the metric $g_{\mu\nu}$, 3 Abelian vectors, A^0, A^1 and A^2 a triplet of SU(2) vectors A^A, and 5 scalars which we choose as ϕ, k and ℓ^A where ϕ can be directly identified with the 10-dimensional heterotic dilaton and k is the Kaluza-Klein scalar of the last compactification from $d = 6$ to $d = 5$.

A particularly simple family of non-Abelian black-hole solutions of $\mathcal{N} = 1, d = 5$ supergravity can be constructed by adding a BPST instanton to the standard 3-charge solution [47, 55, 158]. The family of solutions is determined by 3 harmonic functions $L_{0,\pm}$ which depend on three constants $B_{0,\pm}$ satisfying $\frac{27}{2}B_0B_+B_- = 1$ and three independent charges $q_{0,\pm}$

$$L_{0,\pm} = B_{0,\pm} + q_{0,\pm}/\rho^2\,, \tag{8.1}$$

[4]For recent work on Abelian black-hole solutions of Heterotic Supergravity (with R^2terms, the Hull-Strominger system) see Ref. [108] and references therein.

[5]A more detailed description of this model can be found in Appendix A of Ref. [55], for instance.

and a non-Abelian contribution that depends on the 5-dimensional gauge coupling constant g and on the instanton scale κ

$$\Phi^2 \equiv \frac{2\kappa^4}{3g^2\rho^4(\rho^2+\kappa^2)^2}\,. \tag{8.2}$$

The non-Abelian contribution appears combined with the harmonic function L_0 as follows:

$$\tilde{L}_0 \equiv L_0 - \tfrac{1}{3}\rho^2\Phi^2\,, \tag{8.3}$$

and, since it goes like $1/\rho^4$ at spatial infinity while L_0 goes like B_0+q_0/ρ^2, it is not expected to contribute to the mass. However, both the Abelian and non-Abelian contributions diverge like $1/\rho^2$ near the horizon at $\rho=0$, and, naively, one expects both of them to contribute to the entropy. This can be manifest by rewriting $\tilde{L}_0$ as

$$\tilde{L}_0 = B_0 + (q_0 - \frac{2}{9g^2})\frac{1}{\rho^2} + \frac{2}{9g^2}\frac{\rho^2+2\kappa^2}{(\rho^2+\kappa^2)^2}\,, \tag{8.4}$$

where we have combined Abelian and non-Abelian $1/\rho^2$ terms in $\tilde{L}_0$, leaving a purely non-Abelian contribution which is finite at $\rho=0$. As in Ref. [55], we will call $\tilde{q}_0 \equiv q_0 - \frac{2}{9g^2}$ the coefficient of the $1/\rho^2$ term.

The constants $B_{0,\pm}$ are related to the moduli *i.e.* the values of the 2 scalars at infinity.[6] as follows

$$B_0 = \tfrac{1}{3}e^{\phi_\infty}k_\infty^{-2/3}\,, \qquad B_- = \tfrac{2}{3}e^{-\phi_\infty}k_\infty^{-2/3}\,, \qquad B_+ = \tfrac{1}{3}k_\infty^{4/3}\,. \tag{8.5}$$

Is is convenient to use the functions $\tilde{\mathcal{Z}}_0 \equiv \tilde{L}_0/B_0$ and $\mathcal{Z}_\pm \equiv L_\pm/B_\pm$ and the charges $\tilde{\mathcal{Q}}_0 \equiv \tilde{q}_0/B_0 = (q_0 - \frac{2}{9g^2})/B_0$ and $\mathcal{Q}_\pm \equiv q_\pm/B_\pm$.

It is also convenient to transform the BPST instanton field from the gauge used in Refs. [158, 171] to one in which the 10-dimensional solution will be easier to recognize:[7],[8]

$$A^A_R = \frac{1}{g}\frac{1}{(1+\lambda^2\rho^2/4)}v^A_R \quad\longrightarrow\quad A^A_L = -\frac{1}{g}\frac{\rho^2}{(\kappa^2+\rho^2)}v^A_L, \tag{8.6}$$

where $\kappa^2 = 4/\lambda^2$. In the first gauge, the instanton field is not defined on the horizon, while in the second one, it is not defined at infinity. The vector field strength is, evidently, the same, but the Chern-Simons term is not and this difference will also affect the 10-dimensional 2-form. The functions Φ^A must be transformed as well but they only appear in the gauge-invariant combination Φ^2 and we will not need to compute them explicitly in the new gauge.

[6]We will relate the charges to the numbers of branes in $d=10$ after embedding the solution in Heterotic Supergravity.

[7]The reason why this gauge was not used in Refs. [158, 171] is that, in it, the gauge field cannot be consistently reduced following Kronheimer.

[8]Our conventions for the SU(2) gauge fields are slightly different from the ones used in Refs. [158, 171] Here the the generators satisfy the algebra $[T_A, T_B] = +\epsilon_{ABC}T_C$, the left-invariant Maurer-Cartan 1-forms are defined by $v_L \equiv -U^{-1}dU$ and the right-invariant ones by $v_R \equiv -dUU^{-1}$. the gauge field strength is defined by $F = dA + gA\wedge A$.

After all these transformations, the active fields of the solutions are[9]

$$
\begin{aligned}
ds^2 &= f^2 dt^2 - f^{-1}(d\rho^2 + \rho^2 d\Omega^2_{(3)})\,, && \\
A^0 &= -\sqrt{3}e^{-\phi_\infty} k_\infty^{2/3}\frac{dt}{\tilde{\mathcal{Z}}_0}\,, & A^1 + A^2 &= -2\sqrt{3}k_\infty^{-4/3}\frac{dt}{\mathcal{Z}_+}\,, \\
A^A &= -\frac{1}{g}\frac{\rho^2}{(\kappa^2+\rho^2)}v_L^A\,, & A^1 - A^2 &= -\sqrt{3}e^{\phi_\infty}k_\infty^{2/3}\frac{dt}{\mathcal{Z}_-}\,, \\
e^{2\phi} &= e^{2\phi_\infty}\frac{\tilde{\mathcal{Z}}_0}{\mathcal{Z}_-}\,, & k &= k_\infty (f\mathcal{Z}_+)^{3/4}\,,
\end{aligned}
\tag{8.7}
$$

where the metric function f is given by

$$
f^{-3} = \tilde{\mathcal{Z}}_0\,\mathcal{Z}_+\,\mathcal{Z}_-\,, \tag{8.8}
$$

and the $\mathcal{Z}$ functions take the form

$$
\begin{aligned}
\tilde{\mathcal{Z}}_0 &= 1 + \frac{\tilde{\mathcal{Q}}_0}{\rho^2} + \frac{2e^{-\phi_\infty}k_\infty^{2/3}}{3g^2}\frac{\rho^2+2\kappa^2}{(\rho^2+\kappa^2)^2}\,, \\
\mathcal{Z}_\pm &= 1 + \frac{\mathcal{Q}_\pm}{\rho^2}\,.
\end{aligned}
\tag{8.9}
$$

The mass and entropy of this family of black-hole solutions take the form

$$
M = \frac{\pi}{4G_N^{(5)}}\left[\tilde{\mathcal{Q}}_0 + \frac{2e^{-\phi_\infty}k_\infty^{2/3}}{3g^2} + \mathcal{Q}_+ + \mathcal{Q}_-\right]\,, \tag{8.10}
$$

$$
S = \frac{\pi^2}{2G_N^{(5)}}\sqrt{\tilde{\mathcal{Q}}_0\mathcal{Q}_+\mathcal{Q}_-}\,. \tag{8.11}
$$

Using the charge $\tilde{\mathcal{Q}}_0$ instead of $\mathcal{Q}_0 \equiv q_0/B_0$, and assuming that $\tilde{\mathcal{Q}}_0$ is not related to the non-Abelian fields, the mass contains a net $\mathcal{O}(1/g^2)$ contribution from the instanton while the entropy does not, against the naive expectations exposed above. We are going to argue that, indeed, $\tilde{\mathcal{Q}}_0$ is a charge completely unrelated to the non-Abelian vector fields, showing that it counts the number of neutral 5-branes (also known as solitonic or NSNS 5-branes) while $\mathcal{Q}_-$ and $\mathcal{Q}_+$ count, respectively, the number of fundamental strings and the momentum along them. Setting these three charges to zero we are left with the only non-Abelian component of this solution which is the globally regular and horizonless gravitating Yang-Mills instanton that we have found in Ref. [55], showing that it is is nothing but the dimensional reduction of Strominger's gauge 5-brane [198].

In Ref. [55] we have argued that the gravitating Yang-mills instanton (or the gauge 5-branes) should not contribute to the entropy while, obviously, it must contribute to

[9]Since we are going to use hats to denote 10-dimensional fields, we have removed the hats that we use in our notation for the metric function f.

the total mass of black-hole solutions, just as the global monopole does in 4 dimensions [46, 122]. the above mass and entropy formulae reflect this fact.

8.2 Embedding in $d = 10$ Heterotic Supergravity

In order to embed our solutions in 10-dimensional Heterotic Supergravity we are going to show how the reduction and truncation of the latter leads to the SU(2)-gauged ST[2, 6] model of $\mathcal{N} = 1, d = 5$ supergravity we are working with.

The action of Heterotic Supergravity in the string frame, including only a SU(2) triplet of vector fields, is

$$\hat{S} = \frac{g_s^2}{16\pi G_N^{(10)}} \int d^{10}x \sqrt{|\hat{g}|}\, e^{-2\hat{\phi}} \left[\hat{R} - 4(\partial\hat{\phi})^2 + \tfrac{1}{2\cdot 3!}\hat{H}^2 - \alpha' \hat{F}^A \hat{F}^A\right] , \tag{8.12}$$

where the field strengths are defined as

$$\hat{F}^A = d\hat{A}^A + \tfrac{1}{2}\epsilon^{ABC}\hat{A}^B \wedge \hat{A}^C , \tag{8.13}$$

$$\hat{H} = d\hat{B} + 2\alpha'\omega_{\text{CS}} , \tag{8.14}$$

and ω_{CS} is the Chern-Simons 3-form

$$\omega_{\text{CS}} \equiv \hat{F}^A \wedge \hat{A}^A - \tfrac{1}{3!}\epsilon^{ABC}\hat{A}^A \wedge \hat{A}^B \wedge \hat{A}^C , \qquad d\omega_{\text{CS}} = \hat{F}^A \wedge \hat{F}^A . \tag{8.15}$$

In the above expressions, α', the Regge slope, is related to the string length ℓ_s by $\alpha' = \ell_s^2$, and g_s, the string coupling constant, is the value of the exponential of the dilaton at infinity: $g_s = e^{\phi_\infty}$ in asymptotically-flat configurations. The somewhat unconventional factor of g_s^2 in front of the action ensures that, after a rescaling from the string frame to the modified Einstein frame defined in Ref. [148] with powers of $e^{\phi-\phi_\infty}$, the action has the standard normalization factor $(16\pi G_N^{(10)})^{-1}$. The 10-dimensional Newton constant is given by

$$G_N^{(10)} = 8\pi^6 g_s^2 \ell_s^8 . \tag{8.16}$$

If we compactify this theory on T^4, it is not difficult to see that truncating all the components of the fields with indices in the internal coordinates y^i, $i = 1, \cdots, 4$, is a consistent truncation. The resulting 6-dimensional action and field strengths have exactly the same form as the 10-dimensional ones, although the action carries an extra factor $(2\pi\ell_s)^4$ which is the volume of the T^4:

$$\hat{S} = \frac{(2\pi\ell_s)^4 g_s^2}{16\pi G_N^{(10)}} \int d^6x \sqrt{|g|}\, e^{-2\hat{\phi}} \left[\hat{R} - 4(\partial\hat{\phi})^2 + \tfrac{1}{2\cdot 3!}\hat{H}^2 - \alpha' \hat{F}^A \hat{F}^A\right] . \tag{8.17}$$

The 6-dimensional modified Einstein metric $\hat{g}_{E\,\hat{\mu}\hat{\nu}}$ is related to the the 6-dimensional string metric $\hat{g}_{\hat{\mu}\hat{\nu}}$ by

$$\hat{g}_{\hat{\mu}\hat{\nu}} = g_s^{-1} e^{\hat{\phi}} \hat{g}_{E\,\hat{\mu}\hat{\nu}}\,, \tag{8.18}$$

and, in this frame, the action takes the form

$$\hat{S} = \frac{(2\pi\ell_s)^4}{16\pi G_N^{(10)}} \int d^6x \sqrt{|g|}\, \left[\hat{R}_E + (\partial\hat{\phi})^2 + \tfrac{1}{2\cdot 3!} g_s^2 e^{-2\hat{\phi}} \hat{H}^2 - \alpha' g_s e^{-\hat{\phi}} \hat{F}^A \hat{F}^A\right]\,, \tag{8.19}$$

which coincides exactly with the action of the theory of gauged $\mathcal{N} = (2,0), d = 6$ supergravity that we called $\mathcal{N} = 2A$ in Ref. [56] upon the redefinitions

$$\hat{\phi} = -\tilde{\varphi}/\sqrt{2}\,, \qquad g_s\hat{H}/2 = \tilde{H}\,, \qquad \sqrt{g_s\alpha'}\hat{F}^A = \tilde{F}^A\,, \tag{8.20}$$

which lead to the introduction of the 6-dimensional Yang-Mills coupling constant $g_6 = (g_s\alpha')^{-1/2}$.

Further compactification of this theory on a circle leads to the SU(2)-gauged ST[2, 6] model of $\mathcal{N} = 1, d = 5$ supergravity we are working with, with Newton and Yang-Mills constants given by

$$G_N^{(5)} = \frac{G_N^{(10)}}{(2\pi)^5 \ell_s^4 R_z} = \frac{\pi g_s^2 \ell_s^4}{4R_z}\,, \quad \text{and} \quad g = \frac{g_6 k_\infty^{1/3}}{\sqrt{12}} = \frac{k_\infty^{1/3}}{\sqrt{12 g_s \ell_s^2}}\,. \tag{8.21}$$

This reduction was carried out in detail in Ref. [56] and we can use its results, but we have to take into account that we have to rescale the 5-dimensional metric with the Kaluza-Klein scalar k divided by its asymptotic value, k_∞ in order to preserve the normalization of asymptotically-flat metrics. This introduces an additional factor of $k_\infty^{1/3}$ in the relations between higher-dimensional fields and 5-dimensional vector fields and an additional factor of $k_\infty^{2/3}$ in the relations between higher-dimensional fields and 5-dimensional 2-form fields.

Combining the k_∞-corrected rules given in Ref. [56] to uplift 5-dimensional configurations to $d = 6$ and the relations given above between 6- and 10-dimensional fields in the string frame, we arrive to the following rules that allow us to uplift any solution of the SU(2)-gauged ST[2, 6] model of $\mathcal{N} = 1, d = 5$ supergravity to a solution of 10-dimensional Heterotic Supergravity preserving the normalization of the fields at spatial infinity:

$$\begin{aligned}
d\hat{s}^2 &= e^{\phi-\phi_\infty}\left[(k/k_\infty)^{-2/3} ds^2 - k^2\mathcal{A}^2\right] - dy^i dy^i\,,\\
\hat{\phi} &= \phi\,,\\
\hat{A}^A &= \frac{k_\infty^{1/3}}{\sqrt{12 g_s\alpha'}} A^A - \frac{k^2\ell^A}{\sqrt{\alpha' g_s}}\mathcal{A}\,,\\
\hat{H} &= -\frac{k_\infty^{2/3}}{g_s\sqrt{3}} e^{2\phi} k^{-4/3} \star_{(5)} F^0 + \frac{k_\infty^{1/3}}{g_s\sqrt{3}}\mathcal{A}\wedge\mathcal{F}\,,
\end{aligned} \tag{8.22}$$

where we have introduced the auxiliary fields

$$
\begin{aligned}
&\mathcal{A} \equiv dz + \frac{k_\infty^{1/3}}{\sqrt{12}} A^+ \,, \qquad A^+ \equiv A^1 + A^2 \,, \\
&\mathcal{F} \equiv F^+ + \ell^2 F^- - 2\ell^A F^A \,.
\end{aligned}
\tag{8.23}
$$

Notice that the map gives us the 3-form field strength $\hat{H}$, but not the 2-form potential $\hat{B}$ because the process involves a dualization. Therefore $\hat{B}$ must be obtained from 8.14 once the field strengths $\hat{H}$ and $\hat{F}^A$ have been computed.

8.3 String Theory interpretation

Using the uplifting formulae of the previous section, and defining the coordinate $u = k_\infty z$ (whose period is $2\pi R_z$) we get the following solution of $d = 10$ Heterotic Supergravity

$$
\begin{aligned}
d\hat{s}^2 &= \frac{2}{\mathcal{Z}_-} du \left(dv - \tfrac{1}{2}\mathcal{Z}_+ du\right) - \tilde{\mathcal{Z}}_0 (d\rho^2 + \rho^2 d\Omega^2_{(3)}) - dy^i dy^i \,, \qquad i = 1,2,3,4 \,, \\
\hat{B} &= -\frac{1}{\mathcal{Z}_-} dv \wedge du + \tfrac{1}{4}\tilde{\mathcal{Q}}_0 \cos\theta d\psi \wedge d\phi \,, \\
\hat{A}^A &= -\frac{\rho^2}{(\kappa^2 + \rho^2)} v_L^A \,, \\
e^{-2\hat{\phi}} &= e^{-2\hat{\phi}_\infty} \frac{\mathcal{Z}_-}{\tilde{\mathcal{Z}}_0} \,,
\end{aligned}
\tag{8.24}
$$

where $\tilde{\mathcal{Z}}_0$ and $\mathcal{Z}_\pm$ are given in Eqs. (8.9). In terms of the stringy constants, $\tilde{\mathcal{Z}}_0$ is given by

$$
\tilde{\mathcal{Z}}_0 = 1 + \frac{\tilde{\mathcal{Q}}_0}{\rho^2} + 8\alpha' \frac{\rho^2 + 2\kappa^2}{(\rho^2 + \kappa^2)^2} \,,
\tag{8.25}
$$

showing that the charge $\tilde{\mathcal{Q}}_0$ which is the coefficient of the $1/\rho^2$ term is probably associated to *neutral* (or solitonic or NSNS) 5-branes while the last term should be associated to *gauge* 5-branes. We are first going to discuss this point in more detail.

We start by noticing that, in absence of the Yang-Mills instanton, this supergravity solution is the one found in Refs. [68, 204] which describes solitonic 5-branes wrapped on T^5, and fundamental strings wrapped around one cycle of the T^5 with momentum along the same direction.

Let us consider the coupling of N_{S5} solitonic 5-branes lying in the directions $\frac{1}{2}(u + v), y^1, \cdots, y^4$, to the Heterotic Supergravity action given in Eq. (8.12). Since the effective action of the solitonic 5-branes is written in terms of the NSNS 6-form $\tilde{B}$, we must first rewrite the action in terms of that field. It is convenient to use the language of differential forms, so the action Eq. (8.12) takes the form

$$\hat{S} = \frac{g_s^2}{16\pi G_N^{(10)}} \int e^{-2\hat{\phi}} \left[\star\hat{R} - 4d\hat{\phi}\wedge\star d\hat{\phi} + \tfrac{1}{2}\hat{H}\wedge\star\hat{H} + 2\alpha'\hat{F}^A\wedge\star\hat{F}^A\right], \tag{8.26}$$

and, after dualization $\star e^{-2\hat{\phi}}\hat{H} = \hat{\tilde{H}} \equiv d\hat{\tilde{B}}$

$$\begin{aligned}\hat{S} \;=\; & \frac{g_s^2}{16\pi G_N^{(10)}} \int \Big\{ e^{-2\hat{\phi}} \left[\star\hat{R} - 4d\hat{\phi}\wedge\star d\hat{\phi} + 2\alpha'\hat{F}^A\wedge\star\hat{F}^A\right] \\ & + \tfrac{1}{2}e^{2\hat{\phi}}\hat{\tilde{H}}\wedge\star\hat{\tilde{H}} + 2\alpha'\hat{\tilde{B}}\wedge\hat{F}^A\wedge\hat{F}^A\Big\}\,.\end{aligned} \tag{8.27}$$

The 6-form will couple to the Wess-Zumino term in the effective action of N_{S5} coincident solitonic 5-branes via its pullback over the worldvolume

$$N_{S5}T_{S5}\,g_s^2\int \phi_*\hat{\tilde{B}}\,, \qquad \text{where} \qquad T_{S5} = \frac{1}{(2\pi\ell_s)^5\ell_s g_s^2}\,, \tag{8.28}$$

and the 6-form equation of motion is

$$\frac{g_s^2}{16\pi G_N^{(10)}}\left\{d(\star e^{2\hat{\phi}}\hat{\tilde{H}}) - 2\alpha'\hat{F}^A\wedge\hat{F}^A\right\} = g_s^2 N_{S5}T_{S5}\,\star_{(4)}\,\delta^{(4)}(\rho)\,, \tag{8.29}$$

where $\star_{(4)}\delta^{(4)}(\rho)$ is a 4-form in the 5-branes' transverse space whose integral gives 1.

Integrating both sides of this equation over the transverse space[10] we get

$$\tilde{\mathcal{Q}}_0 = \mathcal{Q}_0 - 8\alpha' = \ell_s^2 N_{S5}\,, \tag{8.31}$$

which confirms that $\tilde{\mathcal{Q}}_0 = \mathcal{Q}_0 - 8\alpha' n$, where n would the instanton number in more general configurations counts solitonic (neutral) 5-branes. The number of gauge 5-branes N_{G5} coincides with the instanton number n. Thus, we conclude that the parameter $\mathcal{Q}_0$ of the solution is

$$\mathcal{Q}_0 = \ell_s^2(N_{S5} + 8N_{G5})\,. \tag{8.32}$$

The function $\mathcal{Z}_-$ is clearly associated to 10-dimensional fundamental strings wrapped around the coordinate $\frac{1}{2}(u-v)$. If we couple N_{F1} fundamental strings lying in the direction $\frac{1}{2}(u-v)$ we have

$$T_{F1}N_{F1} = \frac{g_s^2}{16\pi G_N^{(10)}}\int_{V^8} d(\star e^{-2\hat{\phi}}\hat{H})\,, \qquad \text{where} \qquad T_{F1} = \frac{1}{2\pi\alpha'}\,, \tag{8.33}$$

[10] We replace $\star e^{2\hat{\phi}}\hat{\tilde{H}}$ by $\hat{H}$ for simplicity and use Stokes' theorem in the first term. For the second term we have

$$\frac{1}{16\pi^2}\int_{\mathbb{R}^4}\hat{F}^A\wedge\hat{F}^A = 1\,, \tag{8.30}$$

the instanton number.

where V^8 is the space transverse to worldsheet parametrized by u and v, whose boundary is the product $T_4 \times S^3_\infty$. Using Stokes' theorem and the value of volume of T^4 $(2\pi\ell_S)^4$, we get

$$\mathcal{Q}_- = \ell_s^2 g_s^2 N_{F1}\,. \tag{8.34}$$

Finally, the function $\mathcal{Z}_+$ is associated to a gravitational wave moving in the compact direction $\frac{1}{2}(u-v)$ at the speed of light. The simplest way to compute its momentum is to T-dualize the solution along that direction. This operation interchanges winding number (N_{F1}) and momentum (N_W) and, at the level of the solution, it interchanges the functions $\mathcal{Z}_-$ and $\mathcal{Z}_+$ or, equivalently, the constants $\mathcal{Q}_-$ and $\mathcal{Q}_+$. Thus,

$$\mathcal{Q}_+ = \ell_s^2 g_s'^2 N'_{F1} = \ell_s^2 \left(g_s \ell_s / R_z\right)^2 N_W = \frac{g_s^2 \ell^4}{R_z^2} N_W\,, \tag{8.35}$$

where we have taken into account the transformation of the string coupling constant under T-duality.

We conclude that the fields that give rise to the 5-dimensional non-Abelian black hole in Eq. (8.7),(8.8) and (8.9) correspond to those sourced by N_{F1} fundamental strings wrapped around the 6th dimension with N_w units of momentum moving in the same direction and N_{S5} solitonic (neutral) and $N_{G5} = 1$ gauge 5-branes wrapped around the 6th direction and a T^4. In terms of these numbers, the black hole's mass and the entropy in Eqs. (8.10) and (8.11) take the form

$$\begin{aligned} M &= \frac{R_z}{g_s^2 \ell_s^2}(N_{S5} + 8N_{G5}) + \frac{R_z}{\ell_s^2} N_{F1} + \frac{1}{R_z} N_W\,, &(8.36)\\ S &= 2\pi\sqrt{N_{F1} N_W N_{S5}}\,. &(8.37) \end{aligned}$$

Unfortunately, the dynamics of String Theory in the background of non-perturbative objects such as solitonic and gauge 5-branes is not as well understood as its dynamics in the background of D-branes. Therefore, it is convenient to perform a string-weak coupling Heterotic-Type-I duality transformation [69, 124, 178] which acts on the fields as follows:[11,12]

$$\hat{g}_{\hat{\mu}\hat{\nu}} = e^{-(\hat{\varphi}-\hat{\varphi}_\infty)}\hat{\jmath}_{\hat{\mu}\hat{\nu}}\,, \qquad \hat{\phi} = -\hat{\varphi}\,, \qquad \hat{C}^{(2)}_{\hat{\mu}\hat{\nu}} = e^{-\hat{\varphi}_\infty}\hat{B}_{\hat{\mu}\hat{\nu}} \qquad \hat{A}^A_{\hat{\mu}} = g_I^{1/2}\hat{\mathcal{A}}^A_{\hat{\mu}}\,, \tag{8.40}$$

where $g_I \equiv e^{\hat{\varphi}_\infty}$ is the Type-I string coupling constant. These transformations lead to the Type-I supergravity action

$$g_I^{-4}\hat{S}_I = \frac{g_I^2}{16\pi G^{(10)}_{N,I}}\int\left\{e^{-2\hat{\varphi}}\left[\star\hat{R} - 4d\hat{\varphi}\wedge\star d\hat{\varphi}\right] + \tfrac{1}{2}\hat{G}^{(3)}\wedge\star\hat{G}^{(3)} + 2\alpha' e^{-\hat{\varphi}}\hat{\mathcal{F}}^A\wedge\star\hat{\mathcal{F}}^A\right\}, \tag{8.41}$$

and our solution takes the form

$$\begin{aligned}
d\hat{s}^2_I &= \frac{2}{\sqrt{\tilde{\mathcal{Z}}_0\mathcal{Z}_-}}du\left(dv - \tfrac{1}{2}\mathcal{Z}_+du\right) - \sqrt{\tilde{\mathcal{Z}}_0\mathcal{Z}_-}(d\rho^2 + \rho^2 d\Omega^2_{(3)}) - \sqrt{\frac{\mathcal{Z}_-}{\tilde{\mathcal{Z}}_0}}dy^i dy^i\,,\\
\hat{C}^{(2)} &= -\frac{e^{-\hat{\varphi}_\infty}}{\mathcal{Z}_-}dv\wedge du + \frac{e^{-\hat{\varphi}_\infty}}{4}\tilde{\mathcal{Q}}_0\cos\theta d\psi\wedge d\phi\,,\\
\hat{\mathcal{A}}^A &= -e^{-\hat{\varphi}_\infty/2}\frac{\rho^2}{(\kappa^2+\rho^2)}v^A_L\,,\\
e^{-2\hat{\varphi}} &= e^{-2\hat{\varphi}_\infty}\frac{\tilde{\mathcal{Z}}_0}{\mathcal{Z}_-}\,.
\end{aligned} \tag{8.42}$$

In agreement with the fact that under Heterotic/Type-I duality fundamental strings and solitonic 5-branes transform into D1- and D5-branes, respectively, gravitational waves

[11] These are the transformations that preserve the normalization of the string metric at spatial infinity and lead to the correct normalization of the action of the Type-I theory. In particular, the rescaling of the gauge fields is required in order to reproduce correctly the term that appears in the expansion of the Born-Infeld action of the O9-D9-brane system (in the Abelian case). The effective worldvolume action of the D9-brane (Born-Infeld plus Wess-Zumino (WZ) terms) is

$$\hat{S}_{D9} = T_{D9}g_I\int d\xi^{10}e^{-\hat{\varphi}}\sqrt{\det(\hat{\jmath}_{ij} + 2\pi\alpha'\hat{\mathcal{F}}_{ij})} + WZ\,, \tag{8.38}$$

where g_I is the Type I string coupling constant. In the physical gauge, ignoring the cosmological constant-type term because it will be cancelled by the O9-planes, and using $T_{D9} = [(2\pi\ell_s)^9\ell_s g_I]^{-1}$ we get

$$\hat{S}_{D9} \sim \frac{g_I^2}{16\pi G^{(10)}_{N,I}}\int d^{10}x\sqrt{|\hat{\jmath}|}\left[\alpha' e^{-\hat{\varphi}}\hat{\mathcal{F}}^2\right] + WZ\,, \tag{8.39}$$

where, now, $16\pi G^{(10)}_{N,I} = (2\pi\ell_s)^7\ell_s g_I^2$. If we rewrite the Type-I supergravity action in terms of the RR 6-form $\hat{C}^{(6)}$, just as in the Heterotic case, we get a term $\hat{C}^{(6)}\wedge\hat{\mathcal{F}}^A\wedge\hat{\mathcal{F}}^A$. This term originates in the WZ term of the D9 effective action as well.

[12] The same procedure (a strong-weak coupling duality transformation within Type-IIB supergravity) was followed in Ref. [49] to derive the D5D1W solution without non-Abelian fields from the solution in [204, 204] which can be embedded directly in the Type-IIB NSNS sector. The presence of non-Abelian vector fields suggests the route we have taken.

remain gravitational waves with the same momentum, this solution describes the fields produced by a D5-brane intersecting a D1-brane in the z direction with a wave propagating along that direction. The Yang-Mills instanton is a nor-perturbative configuration of the non-Abelian Born-Infeld field that occurs in the worldvolume of the parallel D9-branes that give rise to the Type-I theory from the Type-IIB and sources D5-branes. Thus $N_{D1} = N_{F1}$, $N_{D5} = N_{S5}$, $N_{GD5} = N_{G5}$ and, in Type-I variables, the mass and entropy formulae take the form

$$M = \frac{R_z}{g_I \ell_s^2}(N_{D5} + 8N_{GD5}) + \frac{R_z}{g_I \ell_s^2} N_{D1} + \frac{1}{R_z} N_W \,, \tag{8.43}$$

$$S = 2\pi\sqrt{N_{D1} N_{D5} N_W} \,. \tag{8.44}$$

In absence of the instanton ($N_{GD5} = 0$) this solution is identical to the one originally considered in Ref. [49], which is itself very closely related to Strominger and Vafa's original model [199].[13] The same conditions (namely, that all the Ns are large and $N_W >> N_{D1,D5}$) ensure that this solution describes at leading order in α' (low curvature) and in g_s (perturbative string theory) a good background for Type-IIB string theory.

8.4 Discussion

In the previous sections we have shown that the 5-dimensional supergravity black holes with 3 quantized Abelian charges N_{D1}, N_{D5}, N_W and a non-Abelian instanton can be seen, up to dualities, as the fields associated to a 10-dimensional Type-IIB configuration with

1. An orientifold $O9_+$-plane and 16 D9-branes and their mirror images, that give rise to the Type-I superstring theory with gauge group SO(32) (see, *e.g.* [5] and references therein).

2. N_{D5} D5-branes wrapped around the 5th-9th directions and N_{D1} D-strings wrapped around the 5th direction with N_W units of momentum along the 5th direction. Open strings can end on these D-strings and D5-branes.

3. $N_{GD5} = 1$ "gauge D5-brane", sourced by an instanton field located in the 1st-4th dimensions, which are not compact. This brane, which is the dual of the heterotic gauge 5-brane is often referred to as a D5-brane "dissolved" into the spacetime-filling D9-branes and differs essentially from standard D5-branes because no strings can end on them.

Since the entropy of the D1D5W black holes can be understood as associated to the massless states associated to strings with one endpoint on a D1 and the other on a D5 (1-5 states) and this fact, as discussed in in Ref. [49] is unchanged by the presence of the D9-branes and $O9_+$-plane that defines the Type-I theory[14] the microscopic interpretation of the entropy of these non-Abelian black holes must be the same as in the Abelian case

[13] See also Refs. [70, 148, 174].

[14] The counting of states is, however, different since, as mentioned in Ref. [49] one has to take into account the SU(2) degrees of freedom associated to the D5-brane of the Type-I string found in [215].

and should give the same result at leading order. Observe that, as an intermediate step in the uplift of the solution to 10 dimensions one obtains a non-Abelian string solution in 6 dimensions with an $\mathrm{AdS}_3\times\mathrm{S}_3$ near-horizon geometry where the AdS_3 radius only depends on 3 quantized Abelian charges N_{D1}, N_{D5}, N_W.

It is important to stress that the correct identification of the charges and their meaning in terms of branes plays a crucial rôle to reach this conclusion as well as in solving the apparent non-Abelian hair problem explained in the Introduction. A more detailed study is, however, necessary to find corrections to the entropy.

In the last few years we have constructed non-Abelian stating and rotating black-hole solutions in 4 and 5 dimensions [46, 122, 123, 154, 158, 159], as well as black-ring solutions [171] and microstate geometries [183] in 5 dimensions. All those constructed with "coloured monopoles" in 4 dimensions and many of the 5-dimensional solutions exhibit non-Abelian hair which seems to contribute to the entropy or the angular momentum on the horizon but cannot be seen at infinity. Many of them can be uplifted to 10-dimensional Heterotic Supergravity and then dualized into Type-I Supergravity solutions and it is likely that the correct interpretation of the charges of those solutions is enough to understand the non-Abelian hair problem. Work in this direction is in progress.

A
SU(2) Yang-Mills solutions

A.1 The SU(2) Lorentzian meron

A Lorentzian meron is a classical solution to the pure SU(2) (Lorentzian) Yang-Mills theory such that the 1-form gauge field A defining it, is proportional to a pure-gauge configuration, which in our conventions would be $\frac{1}{g}dUU^{-1}$ where $U(x) \in \mathrm{SU}(2)$. In Ref. [53] $U(x)$ was chosen to be of the *hedgehog form*

$$U \equiv 2\frac{x^m}{r}\delta^a_m T_a\,, \qquad U^\dagger = U^{-1} = -U\,, \quad \Rightarrow U^2 = -\mathbb{1}_{2\times 2}\,. \tag{A.1}$$

and it was shown that A solves the Yang-Mills equations if the proportionality coefficient is 1/2, that is

$$A = \frac{1}{2g}dUU^{-1} = -\frac{1}{gr^2}\varepsilon^a{}_{mn}x^m dx^n T_a\,. \tag{A.2}$$

As we will see, this gauge field is nothing but the gauge field of the Wu-Yang SU(2) monopole given in Eq. (A.15).

Since the field strength of a pure gauge configuration vanishes, we find that $F(A)$ can be written in these two specially simple ways which we will use in Appendix A.3:

$$F(A) = \tfrac{1}{2}dA = g[A,A] = \star_{(3)}d\frac{1}{2gr}U\,, \tag{A.3}$$

Now we can write the non-Abelian field strength $F(A)$ in terms of $F(B)$, where $F(B)$ is the field strengths of the Dirac monopole of unit charge Eq. (A.6) that we will review in the next section

$$F(A) = F(B)\,U\,, \qquad F(B) = \star_{(3)}d\frac{1}{2gr}\,, \tag{A.4}$$

and the energy-momentum tensor of A in terms of that of B

$$T_{\mu\nu}(A) = -\tfrac{1}{2}\mathrm{Tr}[F_{\mu\rho}(A)F_\nu{}^\rho(A) - \tfrac{1}{4}\eta_{\mu\nu}F^2(A)] = F_{\mu\rho}(B)F_\nu{}^\rho(B) - \tfrac{1}{4}\eta_{\mu\nu}F^2(B) = T_{\mu\nu}(B)\,. \tag{A.5}$$

A.2 The Wu-Yang SU(2) monopole

The Wu-Yang SU(2) monopole [216] is a solution of the SU(2) Yang-Mills theory that can be obtained from the embedding of the Dirac monopole in SU(2) via a singular gauge transformation (see, *e.g.* Ref. [190] and references therein). To fix our conventions, it is convenient to start by reviewing the Wu-Yang construction of the Dirac monopole [217].

A.2.1 The Dirac monopole

The U(1) field of the Dirac monopole, that we will denote by B is defined to satisfy the Dirac monopole equation[1], which can be written in several forms:

$$F(B) \equiv dB = \star_{(3)} d\frac{1}{2gr} = -\frac{1}{2g} d\Omega^2 \,, \qquad 2\partial_{[m}B_{n]} = -\frac{1}{2g}\varepsilon_{mnp}\frac{x^p}{r^3} \,, \tag{A.6}$$

where $d\Omega^2$ is the volume 2-form of the round 2-sphere of unit radius

$$d\Omega^2 = -\tfrac{1}{2}\varepsilon_{mnp}\frac{x^m}{r}d\frac{x^n}{r} \wedge d\frac{x^p}{r} = \sin\theta d\theta \wedge d\varphi \,. \tag{A.7}$$

The value of the magnetic charge has been set to g^{-1} and it is the minimal charge allowed if the unit of electric charge is g.

The above equation does not admit a global regular solution.

$$B^{(\pm)} = -\frac{1}{2g}(\cos\theta \mp 1)d\varphi \,, \tag{A.8}$$

are local solutions regular everywhere except on the negative (resp. positive) z axis (the Dirac strings). A globally regular solution can be constructed by using $B^{\pm}$ in the upper (lower) hemisphere and using the gauge transformation

$$B^{(+)} - B^{(-)} = -d\left(\frac{1}{g}\varphi\right) \,, \tag{A.9}$$

to relate them in the overlap region. If the gauge group is U(1) where the radius of the circle is the inverse coupling constant $1/g$, the gauge transformation parameter can have a periodicity $2\pi n/g$ with $n \in \mathbb{N}$. This is the well-known Abelian Wu-Yang monopole construction [217]. In our case, since the period of φ is 2π, we get $2\pi/g$, which is the smallest value allowed $p = 1/g$. The solution that describes the monopole of charge n times the minimum is n times this one $p = n/g$.

It is useful to have the expression of $B^{(\pm)}$ in Cartesian coordinates:

$$B^{(\pm)} = \frac{1}{2g}\frac{[(0,0,\mp 1) \times (x^1,x^2,x^3)] \cdot d\vec{x}}{r^2(r \pm x^3)} \,, \tag{A.10}$$

in which the singularity at $r = \mp x^3$ becomes evident. In this form, one can easily change the position of the monopole from the origin to some other point x_0^m and the position of the Dirac string from the half line that starts from the origin in the direction $-(0,0,\mp 1)$ to

[1]This equation is just the Abelian version of the Bogomol'nyi equation.

the half line that starts at the monopole's position x_0^m hand has the direction s^m relative to that point:

$$B^{(s)} = \frac{1}{2g}\left(1 - \frac{s^m}{s}\frac{u^m}{u}\right)^{-1} \varepsilon_{mnp}\frac{s^m}{s}\frac{u^n}{u}d\frac{u^p}{u}\,, \tag{A.11}$$

with

$$u^m \equiv x^m - x_0^m\,, \quad u^2 \equiv u^m u^m\,, \quad s^2 \equiv s^m s^m\,. \tag{A.12}$$

A.2.2 From the Dirac monopole to the Wu-Yang SU(2) monopole

Let us consider the Abelian $B^{(+)}$ solution in Eq. (A.8) and let us embed it in SU(2) as the 3rd component of the gauge field

$$A^{(+)} \equiv 2B^{(+)}T_3\,, \qquad F(A^{(+)}) = 2F(B)T_3\,. \tag{A.13}$$

The SU(2) gauge transformation (which is evidently singular along the negative z axis and makes the whole Dirac string singularity, but the endpoint at the coordinate origin, disappear)

$$U^{(+)} \equiv \frac{1}{\sqrt{2(1+\frac{z}{r})}}\left[1 + \frac{z}{r} + 2\left(\frac{x}{r}T_2 - \frac{y}{r}T_1\right)\right]\,, \tag{A.14}$$

relates the gauged field $A^{(+)}$ to

$$A = \frac{1}{g}\varepsilon^a{}_{mn}dx^m\frac{x^n}{r^2}T_a\,, \qquad A^{(+)} = U^{(+)}A(U^{(+)})^{-1} + \frac{1}{g}dU^{(+)}(U^{(+)})^{-1}\,, \tag{A.15}$$

which is the gauge field of the Wu-Yang SU(2) monopole. As we have mentioned in the previous appendix, this is also the gauge field of the Lorentzian meron Eq. (A.2). The gauge transformation also relates T_3 to $\mathcal{U}$ in Eq. (A.1) and the Abelian vector

$$U^{(+)}\mathcal{U}(U^{(+)})^{-1} = 2T_3\,. \tag{A.16}$$

The fact that the Lorentzian meron is the Wu-Yang monopole, which is related by a gauge transformation to the Dirac monopole makes the relation Eq. (A.5) trivial.

This construction can be generalized to more general positions of the Dirac string: if we consider embedding of the Dirac monopole solution $B^{(s)}$ in Eq. (A.11) into SU(2)

$$A^{(s)} \equiv -2B^{(s)}\frac{s^m}{s}\delta_m{}^a T_a\,, \tag{A.17}$$

it is easy to see that the gauge transformation

$$U^{(s)} \equiv \frac{1}{\sqrt{2\left(1 - \frac{s^m}{s}\frac{u^m}{u}\right)}}\left[1 - \frac{s^m}{s}\frac{u^m}{u} - 2\varepsilon_{mn}{}^a\frac{s^m}{s}\frac{u^n}{u}T_a\right]\,, \tag{A.18}$$

relates it to the same Wu-Yang monopole field Eq. (A.15)

$$A^{(s)} = U^{(s)} A (U^{(s)})^{-1} + \frac{1}{g} dU^{(s)} (U^{(s)})^{-1} \,. \tag{A.19}$$

A.3 The SU(2) Skyrme model

In this appendix we are going to show that the Lorentzian meron (Wu-Yang monopole) is also associated to a solution of the equations of motion of the SU(2) Skyrme model [192] written in the form [52]

$$S_{\text{Skyrme}} = -\tfrac{1}{2} \int d^4x \left\{ \tfrac{1}{2} R_\mu R^\mu + \frac{\lambda}{16} S_{\mu\nu} S^{\mu\nu} \right\} , \tag{A.20}$$

where

$$R_\mu \equiv V^{-1} \partial_\mu V \,, \qquad S_{\mu\nu} \equiv [R_\mu, R_\nu] \,, \qquad V(x) \in \text{SU}(2) \,. \tag{A.21}$$

The equations of motion are

$$\partial_\mu R^\mu + \frac{\lambda}{4} \partial_\mu [R_\nu, F^{\mu\nu}] = 0 \,. \tag{A.22}$$

If we take $V = U^{-1}$ (U given by Eq. (A.1)), then we can write $R = 2gA$ where A is Lorentzian meron's gauge field Eq. (A.2) and

$$\begin{aligned} \partial_\mu R^{i\,\mu} &= -2g \partial_m A^i{}_m = 0 \,, \\ \partial_\mu [R_\nu, F^{\mu\nu}]^i &\sim \partial_m \left(\frac{A^i{}_m}{r^2} \right) = 0 \,. \end{aligned} \tag{A.23}$$

A.4 Higher-charge Lorentzian merons and Wu-Yang monopoles

The construction of a Lorentzian meron can be generalized by using a generalization of the unit outward-pointing vector x^m/r denoted by ξ^m and defined by [8]

$$(\xi^m) \equiv \frac{1}{r} \left(\frac{\mathfrak{Im}(x^2 + ix^1)^n}{\rho^{n-1}}, \frac{\mathfrak{Re}(x^2 + ix^1)^n}{\rho^{n-1}}, x^3 \right) , \qquad \rho^2 \equiv (x^1)^2 + (x^2)^2 \,, \tag{A.24}$$

or, in spherical coordinates,

$$(\xi^m) \equiv (\sin\theta \sin n\varphi, \sin\theta \cos n\varphi, \cos\theta) \,, \tag{A.25}$$

and which reduces to x^m/r for $n = 1$. The essential properties of ξ^m are

$$d\xi^m \wedge d\xi^n = -n \varepsilon_{mnp} \xi^p d\Omega^2 \,, \tag{A.26}$$

$$-\tfrac{1}{2} \varepsilon_{mnp} \xi^m d\xi^n \wedge d\xi^p = n d\Omega^2 = \star_{(3)} d\frac{n}{r} \,, \tag{A.27}$$

The generalization of the meron solution is constructed in terms of the generalization SU(2) matrix in Eq. (A.1)

$$U_{(n)} \equiv 2\xi^m \delta_m^a T_a\,, \qquad U_{(n)}^\dagger = U_{(n)}^{-1} = -U_{(n)}\,, \tag{A.28}$$

and takes the form

$$A \equiv \frac{1}{2g} dU_{(n)} U_{(n)}^{-1}\,. \tag{A.29}$$

The field strength is given by

$$F(A_{(n)}) = \tfrac{1}{2} dA = g[A, A] = \star_{(3)} d\frac{n}{2gr}\, U_{(n)}\,, \tag{A.30}$$

and can be related to that of a Dirac monopole of charge $p = n/g$

$$F(B_{(n)}) = \star_{(3)} d\frac{n}{2gr}\,, \quad F(A_{(n)}) = F(B_{(n)})\, U_{(n)}\,, \tag{A.31}$$

which is given by the expressions studied at the beginning. The energy-momentum tensor of A is also equal to that of the Abelian monopole of charge n/g B. These fields can also be related to the embedding of the charge n/g Dirac monopole into SU(2) with a generalization of the gauge transformation Eq. (A.18)

$$U_{(n)}^{(s)} \equiv \frac{1}{\sqrt{2\left(1 - \frac{s^m}{s}\xi^m\right)}} \left[1 - \frac{s^m}{s}\xi^m - 2\varepsilon_{mn}{}^a \frac{s^m}{s}\xi^n T_a\right]\,, \tag{A.32}$$

relates it to the meron gauge field:

$$U_{(n)}^{(s)} U_{(n)} (U_{(n)}^{(s)})^{-1} = -2\frac{s^m}{s}\delta_m{}^a T_a\,, \qquad U_{(n)}^{(s)} A_{(n)} (U_{(n)}^{(s)})^{-1} + \frac{1}{g} dU_{(n)}^{(s)} (U_{(n)}^{(s)})^{-1} = nB_{(n)}^{(s)} 2\frac{s^m}{s}\delta_m{}^a T_a\,. \tag{A.33}$$

To check that this gauge field solves the Yang-Mills equations of motion we first stress that, with the above connection, $U_{(n)}$ is a covariantly-constant adjoint field. Then, auxiliary the adjoint Higgs field

$$\Phi_{(n)} \equiv \left(-\frac{\mu}{2g} + \frac{n}{2gr}\right) U_{(n)}\,, \tag{A.34}$$

satisfies

$$D\Phi_{(n)} = d\frac{n}{2gr}\, U_{(n)}\,, \tag{A.35}$$

and the pair $A_{(n)}, \Phi_{(n)}$ satisfies the Bogomol'nyi equations (A.38) and, as a consequence the equations of motion of the Yang-Mills-Higgs system. The last equation implies that $\Phi_{(n)}$ and $D\Phi_{(n)}$ commute so the Higgs current vanishes and $A_{(n)}$ also solves the sourceless Yang-Mills equations.

A.5 Spherically-symmetric solutions of the SU(2) Bogomol'nyi equations in $\mathbb{E}^3$

The equations of motion of the SU(2) Yang-Mills-Higgs (YMH) theory in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit in which the the Higgs potential vanishes read

$$\mathfrak{D}_\mu F^{A\,\mu\nu} = -g\varepsilon_{BC}{}^{A}\Phi^{B}\mathfrak{D}^{\nu}\Phi^{C}\,, \tag{A.36}$$

$$\mathfrak{D}^2\Phi^{A} = 0\,. \tag{A.37}$$

Static configurations satisfying the first-order *Bogomol'nyi equations* [42]

$$F^{A}{}_{\underline{rs}} = \varepsilon_{rst}\mathfrak{D}_{\underline{t}}\Phi A\,, \tag{A.38}$$

can be seen to satisfy all the above second-order YMH equations of motion.

BPS magnetic monopole solutions such as the (BPS) 't Hooft-Polyakov monopole found by Prasad and Sommerfield in Ref. [181] satisfy the Bogomol'nyi equations and, therefore, it is of some interest to identify all their solutions. In the spherically-symmetric case this problem was solved by Protogenov in Ref. [182] and his solution can be described as follows: the Higgs and gauge field can always be brought to this form (*hedgehog ansatz*)

$$\Phi^{A} = -\delta^{A}{}_{s}f(r)y^{s}\,, \qquad A^{A}{}_{\underline{r}} = -\varepsilon^{A}{}_{rs}y^{s}h(r)\,, \tag{A.39}$$

in which they are characterized by just two functions, $f(r), h(r)$ of the radial coordinate $r = \sqrt{y^s y^s}$. There is only a 2-parameter family for which these functions, denoted by $(f_{\mu,s}, h_{\mu,s})$, are given by

$$rf_{\mu,s} = \frac{1}{gr}\left[1-\mu r\coth(\mu r+s)\right]\,, \qquad rh_{\mu,s} = \frac{1}{gr}\left[\frac{\mu r}{\sinh(\mu r+s)}-1\right]\,, \tag{A.40}$$

and a 1-parameter family for which these functions, denoted by (f_λ, h_λ), are given by

$$rf_\lambda = \frac{1}{gr}\left[\frac{1}{1+\lambda^2 r}\right]\,, \qquad rh_\lambda = -rf_\lambda\,. \tag{A.41}$$

The BPS 't Hooft-Polyakov monopole [181] is the only globally regular solution and corresponds to $f_{\mu,s=0}$. The $f_{\mu,s=\infty}$ solution is given by

$$-rf_{\mu,\infty} = \frac{\mu}{g}-\frac{1}{gr}, \qquad rh_{\mu,\infty} = -\frac{1}{gr}\,, \tag{A.42}$$

and, for $\mu = 0$, it is the Wu-Yang monopole [216]. The latter solution is also recovered in the 1-parameter family for $f_{\lambda=0}$.

The asymptotic behavior of $rf(r)$ (which is the combination that occurs in the metrics we study) for the different solutions is

$$rf_{\mu,s} \sim -\frac{\mu}{g}+\frac{1}{gr}+\mathcal{O}(e^{-4\mu r})\,, \qquad -rf_\lambda \sim \frac{1}{g\lambda^2 r^2}+\mathcal{O}(r^{-3})\,, \tag{A.43}$$

and the behavior near the origin (where the black-hole horizons may be in the metrics under study) are

$$r f_{\mu,0} \sim -\frac{\mu^2}{2g} r + \mathcal{O}(r^3)\,, \qquad r f_{\mu,s} \sim \frac{1}{gr} - \frac{\mu}{g} \coth s + \mathcal{O}(r)\,, \qquad r f_{\lambda} \sim \frac{1}{gr} - \frac{\lambda^2}{g} r + \mathcal{O}(r^3)\,. \tag{A.44}$$

If we define the magnetic monopole charge by

$$p \equiv \frac{1}{4\pi} \int_{S^2_\infty} \mathrm{Tr}(\hat{\Phi} F)\,, \qquad \hat{\Phi} \equiv \frac{\Phi}{\sqrt{|\mathrm{Tr}(\Phi^2)|}}\,, \tag{A.45}$$

then, we always find $p = 1/g$ except in the 1-parameter family for finite λ, for which we find $p = 0$. As we have argued in Ref. [46], the $\lambda \neq 0$ *colored* monopoles can be seen as a magnetic monopole placed at the origin whose charge is completely screened at infinity.

B
The metrics of the round S^3 and S^2

In this appendix we will review the well-known construction of the SO(4)-invariant metric on S^3 using its identification with the SU(2) group manifold, the construction of SO(3)-invariant metric on S^2 using its identification with the SU(2)/U(1) coset space and the relation between both of them.

All matrices $U \in \mathrm{SU}(2)$ ($U^\dagger = U^{-1}$, $\det U = +1$) can be parametrized by two complex numbers z_0, z_1

$$U \equiv \begin{pmatrix} z_0 & z_1 \\ -\bar{z}_1 & \bar{z}_0 \end{pmatrix}, \qquad |z_0|^2 + |z_1|^2 = 1\,. \tag{B.1}$$

Therefore, the SU(2) manifold can be identified with S^3. Both are traditionally parametrized by the Euler angles $\{\theta, \varphi, \psi\}$:

$$z_0 = \cos(\theta/2)\; e^{i(\varphi+\psi)/2}\,, \qquad z_1 = \sin(\theta/2)\; e^{i(\varphi-\psi)/2}\,. \tag{B.2}$$

The main property of this parametrization is that any SU(2) rotation can be written as the product of three rotations with these angles:

$$U(\varphi,\theta,\psi) = U(\varphi,0,0)U(0,\theta,0)U(0,0,\psi)\,. \tag{B.3}$$

The Euler angles are usually assumed to take values in the intervals $\theta \in [0,\pi]$, $\varphi \in [0,2\pi)$, and $\psi \in [0,4\pi)$. Other choices are possible: for instance, $\theta \in [0,\pi]$, $\varphi \in [0,4\pi)$, and $\psi \in [0,2\pi)$ also covers once S^3. Only the coordinate chosen to take values in $[0,4\pi)$ should be considered periodic. There is a free U(1) action on S^3 associated to constant shifts of the periodic coordinate. For the standard choice, this action is

$$U(\varphi,\theta,\psi) \to U(\varphi,\theta,\psi)U(0,0,2\alpha)\,, \qquad \alpha \in [0,2\pi)\,. \tag{B.4}$$

Being a right action, it is adequate to define the right coset space SU(2)/U(1). If we choose instead φ to be the periodic coordinate, the U(1) action is

$$U(\varphi,\theta,\psi) \to U(2\alpha,0,0)U(\varphi,\theta,\psi)\,, \qquad \alpha \in [0,2\pi)\,. \tag{B.5}$$

Being a left action, it is adequate to define the left coset space $\mathrm{U}(1)\backslash\mathrm{SU}(2)$, which is a more unusual option.

A convenient basis of the $\mathfrak{su}(2)$ Lie algebra is provided by the anti-Hermitian matrices[1]

$$T_A = \tfrac{i}{2}\sigma^A\,, \qquad [T_A, T_A] = -\epsilon_{ABC}T_C\,. \tag{B.7}$$

In this basis

$$U(\varphi,0,0) = e^{\varphi T_3}\,, \qquad U(0,\theta,0) = e^{\theta T_2}\,, \qquad U(0,0,\psi) = e^{\psi T_3}\,. \tag{B.8}$$

The left- (resp. right-)invariant Maurer–Cartan (MC) 1-form V_L (resp. V_R) are defined by

$$V_L \equiv -U^{-1}dU\,, \qquad V_R \equiv -dUU^{-1}\,, \tag{B.9}$$

and as a consequence of their definition they satisfy the MC equations

$$dV_{\substack{L\\R}} \mp V_{\substack{L\\R}} \wedge V_{\substack{L\\R}} = 0\,. \tag{B.10}$$

Observe that the left- and right-invariant MC 1-forms are related by the following *gauge* transformations:

$$V_R = UV_LU^{-1}\,. \tag{B.11}$$

The components of the MC 1-forms in the above basis $V_{\substack{L\\R}} \equiv v^A_{\substack{L\\R}}T_A$ are given by

$$\left\{\begin{array}{rcl} v^1_L &=& \sin\psi\, d\theta - \sin\theta\cos\psi\, d\varphi\,,\\ v^2_L &=& -\cos\psi\, d\theta - \sin\theta\sin\psi\, d\varphi\,,\\ v^3_L &=& -(d\psi + \cos\theta\, d\varphi)\,, \end{array}\right. \qquad \left\{\begin{array}{rcl} v^1_R &=& -\sin\varphi\, d\theta + \sin\theta\cos\varphi\, d\psi\,,\\ v^2_R &=& -\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\psi\,,\\ v^3_R &=& -(d\varphi + \cos\theta\, d\psi)\,, \end{array}\right. \tag{B.12}$$

and the MC equations in components take the form

$$dv^A_{\substack{L\\R}} \pm \tfrac{1}{2}\epsilon_{ABC}\, v^B_{\substack{L\\R}} \wedge v^C_{\substack{L\\R}} = 0\,. \tag{B.13}$$

As their name indicates, the left- (resp. right-)invariant MC 1-forms are invariant under the left (resp. right) U(1) action in Eq. (B.5) (resp. Eq. (B.4)).

Both the left- or the right-invariant MC 1-forms can be used as Dreibeins to construct a bi-invariant (that is $\mathrm{SU}(2)\times\mathrm{SU}(2) \sim \mathrm{SO}(4)$ -invariant) metric on SU(2) ($\sim \mathrm{S}^3$) with tangent space metric δ_{AB}. The result is exactly the same in both cases: normalizing the metric so as to get the volume of the 3-sphere of unit radius, we find

$$d\Omega^2_{(3)} = \tfrac{1}{4}v^A_Lv^A_L = \tfrac{1}{4}v^A_Rv^A_R = \tfrac{1}{4}\left[d\theta^2 + d\varphi^2 + d\psi^2 + 2\cos\theta\, d\varphi d\psi\right]\,. \tag{B.14}$$

[1]The σ^A are the Pauli matrices, which we take to satisfy

$$\sigma^A\sigma^B = \delta^{AB} + i\epsilon^{ABC}\sigma^C\,. \tag{B.6}$$

It is customary to rewrite this metric so that the invariance under the chosen U(1) action is manifest. For the standard choice in which $\psi \in [0, 4\pi)$ is the periodic coordinate and there is invariance under the right action in Eq. (B.4)

$$d\Omega^2_{(3)} = \tfrac{1}{4}\left[d\Omega^2_{(2)}(\theta,\varphi) + v^3_L v^3_L\right]\,, \tag{B.15}$$

where $d\Omega^2_{(2)}(\theta,\varphi)$ is the standard metric of the round 2-sphere of unit radius

$$d\Omega^2_{(2)}(\theta,\varphi) = d\theta^2 + \sin^2\theta d\varphi^2 = v^1_L v^1_L + v^2_L v^2_L\,. \tag{B.16}$$

For the other choice, we just have to interchange φ and ψ and L by R in the above expressions.

C

$\mathcal{N}=1$, $d=5$ Super-Einstein-Yang-Mills

C.1 The theory

In this appendix we give a very brief, workable description of SEYM theories and their known analytic solutions adapted to the purpose of this letter. $\mathcal{N}=1, d=5$ gauged supergravities can be interpreted as the minimal supersymmetric realization of Einstein-Yang-Mills-Higgs theories[1]. They describe the coupling between a supergravity multiplet and n_v vector multiplets, a subset of which transform under the local action of a non-Abelian group. The supergravity multiplet is constituted by the graviton $e^a{}_\mu$, the gravitino ψ^i_μ and the graviphoton A^0_μ, while each vector multiplet, labeled by $x=1, ..., n_v$, contains a real vector field $A^x{}_\mu$, a real scalar ϕ^x and a gaugino $\lambda^{i\,x}$. The vector fields can be collectively denoted as $A^I{}_\mu$, with $\{I, J, \ldots = 0, 1, \cdots, n_v\}$. The set over which these indices take values is conveniently split in two sectors denoted as $\{i, j, \cdots = 0, \cdots, i_{max}\}$ and $\{\alpha, \beta, \cdots = i_{max}+1, \cdots, n_v\}$, referred as the Abelian and the non-Abelian sectors respectively.

The n_v scalars ϕ^x parametrize a σ-model equipped with a Riemannian metric g_{xy} and can be understood as coordinates on a scalar manifold. On general grounds the σ-model metric is invariant under coordinate transformations in the scalar manifold of the form

$$\delta_\Lambda \phi^x = -\hat{g} c^I k_I{}^x \,, \tag{C.1}$$

where $\hat{g}$ is interpreted as the gauge coupling constant (see below) and $k_I{}^x(\phi)$ is a set of Killing vectors of the scalar metric[2]. The requirement that the σ-model is compatible with the supersymmetric structure that controls the coupling between scalars and vectors gives rise to the mathematical construct known as Real Special Geometry, see [35, 170], that completely characterizes the supergravity theory. Then, a Killing vector of the scalar metric generates an isometry of the full supergravity theory if it respects the real special structure of the theory, see Appendix H in [170].

The parameters that generate these isometries in the non-Abelian sector are spacetime functions, i.e. $c^\alpha = c^\alpha(x)$, while the corresponding Killing vectors satisfy the algebra

$$[k_\alpha, k_\beta] = -f_{\alpha\beta}{}^\gamma k_\gamma \,, \tag{C.2}$$

[1]Those were first considered in [106], see [21, 23, 35, 158] for more detailed expositions in our same conventions.

[2]Here the index I is for labeling each one of these vectors. We use it in order to keep notation simple, and it should be understood that the Killing vectors will be non-zero only for a subset of the possible values of the index.

where $f_{\alpha\beta}{}^{\gamma}$ are the structure constants of some non-Abelian group (we will often use the notation $f_{IJ}{}^{K}$, understanding that the structure constants just vanish whenever any index take values in the Abelian sector).

The vectors in the non-Abelian sector, i.e. $A^{\alpha}{}_{\mu}$, play the role of gauge fields under the action of (C.1). That is, they transform in an appropriate way such that the covariant derivative of the scalars defined as

$$\mathfrak{D}_{\mu}\phi^{x} = \partial_{\mu}\phi^{x} + \hat{g}A^{\alpha}{}_{\mu}k_{\alpha}{}^{x}\,, \tag{C.3}$$

transforms, indeed, covariantly. The field strengths are defined in the standard manner in both the Abelian and non-Abelian sectors,

$$F^{I}{}_{\mu\nu} = 2\partial_{[\mu}A^{I}{}_{\nu]} + \hat{g}f_{JK}{}^{I}A^{J}{}_{\mu}A^{K}{}_{\nu}\,. \tag{C.4}$$

We will set all the fermionic fields to zero, which is always a consistent truncation in these theories. The bosonic action of $\mathcal{N}=1, d=5$ SEYM is given by

$$\begin{aligned} S \;=\; & \int d^{5}x\sqrt{g}\,\Big\{R + \tfrac{1}{2}g_{xy}\mathfrak{D}_{\mu}\phi^{x}\mathfrak{D}^{\mu}\phi^{y} - \tfrac{1}{4}a_{IJ}F^{I\,\mu\nu}F^{J}{}_{\mu\nu} + \tfrac{1}{12\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\lambda}}{\sqrt{g}}\Big[F^{I}{}_{\mu\nu}F^{J}{}_{\rho\sigma}A^{K}{}_{\lambda} \\ & -\tfrac{1}{2}\hat{g}f_{LM}{}^{I}F^{J}{}_{\mu\nu}A^{K}{}_{\rho}A^{L}{}_{\sigma}A^{M}{}_{\lambda} + \tfrac{1}{10}\hat{g}^{2}f_{LM}{}^{I}f_{NP}{}^{J}A^{K}{}_{\mu}A^{L}{}_{\nu}A^{M}{}_{\rho}A^{N}{}_{\sigma}A^{P}{}_{\lambda}\Big]\Big\}. \end{aligned} \tag{C.5}$$

The Real Special Geometry, and therefore the full supergravity theory, is completely determined by the constant symmetric tensor C_{IJK}. In particular the σ-model metric $g_{xy}(\phi)$ and the kinetic matrix $a_{IJ}(\phi)$ are directly derived from this tensor, see for example [158] for the explicit expressions.

We make use of the SU(2)-gauged ST[2,6] model, that contains $n_v = 5$ vector multiplets and the constant symmetric tensor C_{IJK} that characterizes it has the following non-vanishing components

$$C_{0xy} = \tfrac{1}{6}\eta_{xy}\,, \text{where} \quad (\eta_{xy}) = \mathrm{diag}(+-\cdots-)\,, \quad \text{and} \quad x,y = 1,\cdots,5\,. \tag{C.6}$$

C.2 A special parametrization

The theory we are considering is a truncation of the effective field theory of the Heterotic Superstring compactified on T^5 that preserves an SU(2) triplet of vector fields. The compactification and truncation reduce the theory to a particular model of gauged $\mathcal{N}=1, d=5$ supergravity to which one can apply the solution-generating techniques based on the characterization of supersymmetric solutions described in Appendix 4.2. The dimensional reduction of this model on a circle gives the so-called ST[2,6] model of $\mathcal{N}=2, d=4$ supergravity coupled to 6 vector multiplets and we will, therefore, refer to it by that name in the 5-dimensional context as well. Here we are going to give a minimal description of the bosonic sector of these theories and of the particular model we are considering. More information can be found in Refs. [35, 85, 170].[3]

[3]Our conventions are those in Refs. [21, 23, 170] which are those of Ref. [35] with minor modifications.

The ST[2,6] model model $\mathcal{N}=1, d=5$ supergravity contains 5 vector supermultiplets labeled by $x,y=1,\cdots,5$, each containing a vector field $A^x{}_\mu$ and a scalar ϕ^x. Together with the graviphoton $A^0{}_\mu$, all the vectors are written $A^I{}_\mu$, $I,J,\ldots=0,1,\cdots,5$. The only remaining bosonic field is the spacetime metric $g_{\mu\nu}$. The C_{IJK} tensor has the non-vanishing components

$$C_{0xy}=\tfrac{1}{6}\eta_{xy}\,,\qquad \text{where}\quad (\eta_{xy})=\mathrm{diag}(+-\cdots-)\,, \tag{C.7}$$

and the Real Special manifold parametrized by the physical scalars can be identified with the Riemannian symmetric space

$$\mathrm{SO}(1,1)\times\frac{\mathrm{SO}(1,4)}{\mathrm{SO}(4)}\,. \tag{C.8}$$

A convenient parametrization of the scalar manifold is

$$h^0=e^{-\phi}k^{2/3}\,,\quad h^{1,2}=k^{-4/3}\left[1\pm(\ell^2+\tfrac{1}{2}e^{\phi}k^2)\right]\,,\quad h^{3,4,5}=-2k^{-4/3}\ell^{3,4,5}\,, \tag{C.9}$$

where ϕ coincides with the 10-dimensional Heterotic Superstring dilaton field, k is the Kaluza-Klein scalar of the dimensional reduction from $d=6$ to $d=5$ and the ℓ^A are the fifth components of the 6-dimensional vector fields. The rest of the components that make up the 10-dimensional vector fields have been truncated [56].

The group SO(3) acts in the adjoint on the coordinates $x=3,4,5$ which we are going to denote by $A,B,\ldots$ and this is the sector that is gauged without the use of Fayet-Iliopoulos terms. This means that R-symmetry is not gauged and there is no scalar potential.[4] The structure constants are $f_{AB}{}^C=+\varepsilon_{AB}{}^C$.[5] We will denote with $a,b,\ldots=1,2$ the ungauged directions. Observe that this sector of the theory corresponds to the so-called STU model: in absence of the h^As we can make the linear redefinitions

$$h^{1\prime}\equiv\tfrac{1}{\sqrt{2}}(h^1+h^2)\,,\qquad h^{2\prime}\equiv\tfrac{1}{\sqrt{2}}(h^1-h^2)\,,\ \Rightarrow\ C_{abc}h^ah^bh^c=h^0h^{1\prime}h^{2\prime}\,. \tag{C.10}$$

Thus, our model can be also understood as the STU model with an additional SU(2) triplet of vector multiplets.

With the above parametrization of the scalar manifold, the action for this model can be brought to the form

$$\begin{aligned} S \;=\; \int d^5x\sqrt{g}\,\Big\{ & R+\partial_\mu\phi\partial^\mu\phi+\tfrac{4}{3}\partial_\mu\log k\partial^\mu\log k+2e^{-\phi}k^{-2}\mathfrak{D}_\mu\ell^A\mathfrak{D}^\mu\ell^A \\ & -\tfrac{1}{12}e^{2\phi}k^{-4/3}F^0\cdot F^0+\tfrac{1}{12}\left(\eta_{xy}e^{-\phi}k^{2/3}-9h_xh_y\right)F^x\cdot F^y \\ & +\tfrac{1}{24\sqrt{3}}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}A^0{}_\mu\eta_{xy}F^x{}_{\nu\rho}F^y{}_{\sigma\alpha}\Big\}\,, \end{aligned} \tag{C.11}$$

[4] Models of this kind are called model of $\mathcal{N}=1, d=5$ Super-Einstein-Yang-Mills (SEYM), which are the simplest $\mathcal{N}=1$ supersymmetrization of the 5-dimensional Einstein-Yang-Mills (EYM) theories.

[5] These indices will always be raised and lowered with δ_{AB}, just for esthetical reasons.

where

$$
\begin{aligned}
\mathfrak{D}_\mu \ell^A &= \partial_\mu \ell^A + g\varepsilon^A{}_{BC} A^B{}_\mu \ell^C\,, && \text{(C.12)}\\
F^{0,a}{}_{\mu\nu} &= 2\partial_{[\mu} A^{0,a}{}_{\nu]}\,, && \text{(C.13)}\\
F^A{}_{\mu\nu} &= 2\partial_{[\mu} A^A{}_{\nu]} + g\varepsilon^A{}_{BC} A^B{}_\mu A^C{}_\nu\,. && \text{(C.14)}
\end{aligned}
$$

Notice that $A^0{}_\mu$ is sourced by $\varepsilon^{\mu\nu\rho\sigma\alpha}\eta_{xy}F^x{}_{\nu\rho}F^y{}_{\sigma\alpha}$ which is related to the instanton number on the constant-time hypersurfaces. In differential-form language, its equation of motion is

$$
d(e^{2\phi}k^{-4/3}\star F^0) = \tfrac{1}{2\sqrt{3}}\eta_{xy}F^x \wedge F^y = 0\,, \tag{C.15}
$$

which is similar to that of the Kalb-Ramond 2-form B. This is because A^0 is the 5-dimensional dual of the dimensionally reduced Heterotic Kalb-Ramond form B. The duality relation is

$$
F^0 = e^{-2\phi}k^{4/3}\star H\,, \quad \text{with} \quad H \equiv dB + \tfrac{1}{2\sqrt{3}}\omega_{\rm CS}\,, \tag{C.16}
$$

where $\omega_{\rm CS}$ is the Chern-Simons 3-form of all the vector fields but A^0 itself

$$
\omega_{\rm CS} = \tfrac{1}{2}F^+ \wedge A^- + \tfrac{1}{2}F^- \wedge A + F^A \wedge A^A - \tfrac{1}{3!}g\epsilon_{ABC}A^A \wedge A^B \wedge A^C\,, \tag{C.17}
$$

satisfying

$$
d\omega_{\rm CS} = \eta_{xy}F^x \wedge F^y\,. \tag{C.18}
$$

C.3 Procedure for constructing solutions

1. Timelike supersymmetric solutions of $\mathcal{N}=1$, $d=5$ SEYM with a spacelike isometry are constructed from a set of $(2n_v+4)$ seed functions defined on $\mathbb{E}^3$. These are denoted[6] as M, H, Φ^I, L_I and satisfy the following equations

[6]Notice that the seed functions Φ^I should not be confused with the physical scalars ϕ^x appearing in the action (C.5).

$$d \star_3 dM = 0\,, \tag{C.19}$$

$$\star_3 dH - d\chi = 0\,, \tag{C.20}$$

$$\star_3 \breve{\mathfrak{D}}\Phi^I - \breve{F}^I = 0\,, \tag{C.21}$$

$$\breve{\mathfrak{D}}^2 L_I - \breve{g}^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J \Phi^K L_M = 0\,, \tag{C.22}$$

$$\star_3 d\breve{\omega} - \Big\{ H dM - M dH + 3\sqrt{2}(\Phi^I \breve{\mathfrak{D}} L_I - L_I \breve{\mathfrak{D}}\Phi^I) \Big\} = 0\,, \tag{C.23}$$

for some 1-forms $\chi, \breve{\omega}$ and $\breve{A}^I$ (with field strength $\breve{F}^I$) defined also in $\mathbb{E}^3$. Here the covariant derivative $\breve{\mathfrak{D}}$ is defined in three-dimensional Euclidean space with respect to the gauge field $\breve{A}^I$ for objects transforming in the (dual) adjoint representation. More explicitly,

$$\breve{\mathfrak{D}}\Phi^I = d\Phi^I + \breve{g} f_{JK}{}^I \breve{A}^J \Phi^K\,, \qquad \breve{\mathfrak{D}} L_I = dL_I + \breve{g} f_{IJ}{}^K \breve{A}^J L_K\,. \tag{C.24}$$

Two subtleties about these expressions are worth mentioning. First, notice that the structure constants are only non-trivial in the non-Abelian sector so the covariant derivative reduce to the standard exterior derivative in the Abelian sector. Second, the gauge coupling constant in this expression is rescaled with respect to the *physical* gauge constant appearing in the action of the theory[7], $\hat{g} = -\breve{g}/2\sqrt{6}$.

2. Using the seed functions, the five-dimensional fields of the solution are obtained as follows:

   (a) We define the intermediate building blocks

   $$h_I/f = L_I + 8 C_{IJK} \Phi^J \Phi^K / H\,, \tag{C.25}$$

   that can be used to compute the physical scalars

   $$\phi^x \equiv h_x/h_0\,, \tag{C.26}$$

   and the metric function

   $$\begin{aligned} f^{-3} \;=\; & 3^3 C^{IJK} L_I L_J L_K + 3^4 \cdot 2^3 C^{IJK} C_{KLM} L_I L_J \Phi^L \Phi^M / H \\ & + 3 \cdot 2^6 L_I \Phi^I C_{JKL} \Phi^J \Phi^K \Phi^L / H^2 + 2^9 \left(C_{IJK} \Phi^I \Phi^J \Phi^K \right)^2 / H^3\,. \end{aligned} \tag{C.27}$$

   This is derived from the Real Special Geometry constrain $27 C^{IJK} h_I h_J h_K = 1$, which is valid for symmetric scalar manifolds[8]. In these spaces we can also define

   $$h^I = 27 C^{IJK} h_J h_K\,. \tag{C.28}$$

[7] This fact is an indirect consequence of the rescaling factor appearing in equation (C.34).

[8] This is always the case in the supergravity models that we consider here. In this expression, $C^{IJK} \equiv C_{IJK}$.

(b) The spacetime metric is of the conformastationary form

$$ds^2 = f^{\,2}(dt+\omega)^2 - f^{\,-1}d\hat{s}^2\,, \tag{C.29}$$

where the 1-form ω is obtained as

$$\omega \;=\; \omega_5(d\varphi+\chi)+\breve{\omega}\,, \tag{C.30}$$

$$\omega_5 \;=\; M+16\sqrt{2}H^{-2}C_{IJK}\Phi^I\Phi^J\Phi^K+3\sqrt{2}H^{-1}L_I\Phi^I\,, \tag{C.31}$$

being the inverse-hatted $\breve{\omega}$ the one in (C.23), and $d\hat{s}^2$ is a four-dimensional Gibbons-Hawking metric [96, 98]

$$d\hat{s}^2 = H^{-1}(d\varphi+\chi)^2 + Hdx^r dx^r\,,\quad r=1,2,3\,. \tag{C.32}$$

(c) The physical vector fields and their field strengths are

$$\begin{aligned} A^I &= -\sqrt{3}h^I f(dt+\omega)+\hat{A}^I\,,\\ F^I &= -\sqrt{3}\hat{\mathfrak{D}}[h^I f(dt+\omega)]+\hat{F}^I\,, \end{aligned} \tag{C.33}$$

where the auxiliary vectors $\hat{A}^I$ are four-dimensional gauge fields defined on the Gibbons-Hawking space as

$$\begin{aligned} \hat{A}^I &= -2\sqrt{6}\left[-H^{-1}\Phi^I(d\varphi+\chi)+\breve{A}^I\right]\,,\\ \hat{F}^I &= -2\sqrt{6}\left[-\breve{\mathfrak{D}}\left[\Phi^I H^{-1}(d\varphi+\chi)\right]+\star_3\breve{\mathfrak{D}}\Phi^I\right]\,, \end{aligned} \tag{C.34}$$

By this construction, which is due to Kronheimer [141], the field strength $\hat{F}^I$ is self-dual in the Gibbons-Hawking space, describing an instanton configuration intimately related to a lower dimensional static monopole.
Notice that $\hat{\mathfrak{D}}$ is the covariant derivative with associated connection $\hat{A}^I$ in the Gibbons-Hawking space, while $\breve{\mathfrak{D}}$ is the covariant derivative with associated connection $\breve{A}^I$ in $\mathbb{E}^3$.

C.4 Dimensional reduction of $\mathcal{N}=1, d=5$ SEYM theories

$\mathcal{N}=1, d=5$ supergravity coupled to vector multiplets gives $\mathcal{N}=2, d=4$ supergravity coupled to vector multiplets upon dimensional reduction over a spacelike circle[9]. If some non-Abelian subgroup of the isometry group of the scalar manifold of the 5-dimensional theory has been gauged, and we perform a simple (as opposed to a generalized) dimensional reduction, the 4-dimensional theory will have exactly the same non-Abelian subgroup of the (now bigger) isometry group gauged. Thus $\mathcal{N}=1, d=5$ and $\mathcal{N}=2, d=4$ SEYM theories are related by dimensional reduction over a spacelike circle.

It should be clear that, under the above conditions, the relation between the 5- and 4-dimensional fields in the gauged theories is exactly the same as in the ungauged one and

[9] See, for instance, Refs. [85] and references therein

is, therefore, well known. In the conventions we follow here[10] the relation between the bosonic fields of an $\mathcal{N}=1, d=5$ supergravity model defined by C_{IJK} (tilded) and the bosonic fields of a cubic model of $\mathcal{N}=2, d=4$ supergravity defined by the symmetric tensor d_{ijk} (untilded) are [11]

$$
\begin{aligned}
g_{\mu\nu} &= |\tilde{g}_{\underline{zz}}|^{\frac{1}{2}}\left(\tilde{g}_{\mu\nu} - \tilde{g}_{\mu\underline{z}}\tilde{g}_{\nu\underline{z}}/\tilde{g}_{\underline{zz}}\right), & d_{ijk} &= 6C_{i-1\,j-1\,k-1}, \\
A^{0}{}_{\mu} &= \tfrac{1}{2\sqrt{2}}\tilde{g}_{\mu\underline{z}}/\tilde{g}_{\underline{zz}}\,, & A^{i}{}_{\mu} &= -\tfrac{1}{2\sqrt{6}}\left(\tilde{A}^{i-1}{}_{\mu} - \tilde{A}^{i-1}{}_{\underline{z}}\tilde{g}_{\mu\underline{z}}/\tilde{g}_{\underline{zz}}\right), \\
Z^{i} &= \tfrac{1}{\sqrt{3}}\tilde{A}^{i-1}{}_{\underline{z}} + i|\tilde{g}_{\underline{zz}}|^{\frac{1}{2}}\tilde{h}^{i-1}\,, & &
\end{aligned}
\tag{C.35}
$$

and the inverse relations are

$$
\begin{aligned}
\tilde{g}_{\underline{zz}} &= -k^{2}\,, & \tilde{A}^{I}{}_{\underline{z}} &= \sqrt{3}\,\mathfrak{Re}Z^{I+1}\,, \\
\tilde{g}_{\mu\underline{z}} &= -2\sqrt{2}k^{2}A^{0}{}_{\mu}\,, & \tilde{A}^{I}{}_{\mu} &= -2\sqrt{6}\left(A^{I+1}{}_{\mu} - \mathfrak{Re}Z^{I+1}A^{0}{}_{\mu}\right), \\
\tilde{g}_{\mu\nu} &= k^{-1}g_{\mu\nu} - 8k^{2}A^{0}{}_{\mu}A^{0}{}_{\nu}\,, & \tilde{h}^{I} &= k^{-1}\mathfrak{Im}Z^{I+1}\,.
\end{aligned}
\tag{C.36}
$$

In these relations it has been taken into account that, if n_v denotes the number of vector multiplets in $d=5$, then, the 4-dimensional theory has n_v+1 vector multiplets so that $I,J,K=0,\cdots,n_v$, $i,j,k=0,\cdots,n_v+1$. The additional 4-dimensional vector multiplet is the $i=0$ one and, therefore, the 5-dimensional vector labeled by I corresponds to the 4-dimensional vector labeled by $i=I+1$.

While this is the whole story for the fields, it is important to realize that the factor that related the 4- and 5-dimensional gauge fields changes the standard form of the covariant derivatives and gauge field strengths and it must be absorbed into a redefinition of the gauge coupling constant. Thus, we also have

$$
\tilde{g} = -2\sqrt{6}g\,. \tag{C.37}
$$

Observe that this result has been obtained using the orientation $\varepsilon^{0123z}=+1$, which is not the one we are using in the main text ($\varepsilon^{0z123}=+1$). However, in practice, the result can be adapted to that orientation by reversing the sign of each z tensor index. This operation only changes the sign of $A^{0}{}_{\mu}$ and $\mathfrak{Re}Z^{i}$.

[10] That is, the conventions used in Refs. [20, 21, 23] for the $\mathcal{N}=1, d=5$ theories and in the conventions used in Refs. [46, 47, 121–123, 154–156, 159] for the $\mathcal{N}=2, d=4$ theories.

[11] See, for instance, Ref. [170] which follows the conventions used here.

D
Resumen

Esta tesis está dedicada al estudio de la interacción de campos de Yang-Mills no abelianos con el campo gravitatorio así como de su realización en el contexto de teoría de cuerdas a través de teorías efectivas de supergravedad. Esta clase de interacciones se ha venido considerando en diversos marcos teóricos desde hace varias décadas y sin embargo el grado de entendimiento alcanzado durante todos estos años ha sido limitado, especialmente si lo comparamos con el que se tiene de la interacción de campos abelianos con la gravedad. Buena parte de esta falta de entendimiento se debe sin duda a la enorme complejidad de estos sistemas; las ecuaciones diferenciales que rigen el comportamiento tanto de campos gravitatorios como de campos de Yang-Mills no abelianos son de carácter no lineal y su resolución es una tarea harto complicada.

La complejidad de estos sistemas, sin embargo, puede reducirse a través de la consideración de configuraciones de campos supersimétricas. Esta clase de configuraciones dan lugar a soluciones con propiedades muy especiales, como es el caso de los agujeros negros extremos. Aún así, a través de ellas no sólo es posible comprender propiedades sobre el acoplamiento entre ciertos campos y a la gravedad, sino que también es posible atisbar la naturaleza cuántica de ciertos sistemas gravitacionales. El agujero negro abeliano de "tres cargas", una solución supersimétrica de supergravedad $\mathcal{N} = 1$ en 5 dimensiones, constituye el paradigma de ello. Esta solución se puede entender como un agujero negro clásico en supergravedad o como un sistema cuántico en teoría de cuerdas. Uno de los mayores logros de esta teoría consiste precisamente en que la entropía de este agujero negro puede calcularse en estos dos esquemas, obteniéndose el mismo resultado.

El principal resultado de esta tesis es la construcción de agujeros negros no abelianos de "tres cargas" en teorías de supergravedad y su interpretación en teoría de cuerdas, lo que a su vez permite una identificación microscópica de su entropía. Esto a su vez requiere (o implica) la resolución del problema de la presencia de "pelo" en estos agujeros negros no abelianos. Es decir, mientras que los agujeros negros abelianos quedan completamente determinados por sus cargas conservadas (no tienen "pelo"), es sabido que los agujeros negros no abelianos necesitan de algunos parámetros adicionales para especificar la solución.

Otro destacado resultado de esta tesis es el desarrollo de una técnica de generación que permite la descripción de amplias familias de soluciones en teorías de supergravedad. La interpretación de éstas en función de objetos fundamentales de teoría de cuerdas no ha hecho más que comenzar.

E
Conclusiones

En el capítulo 2 describimos las primeras soluciones analíticas multicentro de campos no abelianos acoplados a la gravedad, junto a otras soluciones con un sólo centro de diversas características. Este capítulo se desarrolla en el contexto de una teoría de supergravedad cuatro dimensional en la que se ha gaugeado un subgrupo $SU(2)$ del grupo de isometrías de la variedad escalar. Este trabajo sirve como un punto de partida en la familiarización con teorías de supergravedad con campos de Yang-Mills no abelianos, así como con sus soluciones.

El capítulo 3 hace uso de la relación entre monopolos estáticos en espacio plano e instantones en variedades hyperKähler con una isometría descubierta por Kronheimer para ilustrar la conexión existente entre los singulares *monopolos colorados* y los populares instantones BPST. Esta relación resulta clave para el desarrollo de una técnica de construcción de soluciones en teorías gaugeadas de supergravedad $\mathcal{N} = 1$ en cinco dimensiones, iniciando el camino hacia una posible interpretación en teoría de cuerdas.

En el capítulo 4 describimos dicha técnica de obtención de soluciones y la ponemos en práctica para describir los primeros agujeros negros no abelianos en cinco dimensiones. Además de esto, describimos algunas soluciones de tipo nulo como cuerdas negras. A continuación, en el capítulo 5, utilizamos nuestra técnica de generación de soluciones para describir un anillo negro (cuyo horizonte tiene topología $S^2 \times S^1$) así como agujeros negros rotatorios.

En el capítulo 6 describimos cómo obtener solitones regulares multicentro, también conocidos como geometrías de microestados, en estas teorías. Las soluciones descritas tienen las mismas cargas asintóticas que ciertos agujeros negros, aunque carecen de horizonte de eventos. Este tipo de configuraciones se hace posible gracias al descubrimiento de una solución diónica no abeliana multicentro, en la cual las componentes eléctricas no son triviales e interactúan con las magnéticas dando lugar a geometrías no estáticas. Otra propiedad de estos diones no abelianos es que, al contrario de lo que sucede en los de tipo abeliano, las posiciones de los centros no están sometidas a restricción alguna.

Los capítulos 7 y 8 suponen la culminación de esta tesis. En ellos se explica cómo es posible obtener las teorías de supergravedad cinco dimensionales $\mathcal{N} = 1$ con campos de Yang-Mills no abelianos a través de la compactificación toroidal de la teoría de supergravedad heterótica, acompañada de una truncación consistente que reduce el número de supersimetrías. A continuación, se identifican los elementos fundamentales de teoría de cuerdas que originan estas soluciones de supergravedad para dos clases especiales de soluciones: un instanton global regular y un agujero negro no abeliano de “tres cargas”. Esta identificación hace posible el cálculo de la entropía de estas soluciones desde un punto de vista microscópico.

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$\mathcal{N}=2$ Einstein-Yang-Mills' static two-center solutions

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Abstract: We construct *bona fide* one- and two-center supersymmetric solutions to $\mathcal{N}=2$, $d=4$ supergravity coupled to SU(2) non-Abelian vector multiplets. The solutions describe black holes and global monopoles alone or in equilibrium with each other and exhibit non-Abelian hairs of different kinds.

Keywords: Black Holes in String Theory, Solitons Monopoles and Instantons, Black Holes, Supergravity Models

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1 Introduction

Contrary to what one might think, multi-black hole solutions need not be related to supersymmetry or, like in the case of Kastor and Traschen's solution in ref. [1–4], fake-supersymmetry. Proof of this is given by various solutions e.g. the ones presented in refs. [5] and [6]. The benefit of using supersymmetry, however, is that after a few decades' worth of investigations there are workable recipes for creating supersymmetric solutions, which greatly facilitate the construction and study of multi-black hole solutions.

The construction is particularly straightforward in ungauged $\mathcal{N}=2$, $d=4$ supergravity coupled to vector multiplets where there are clear-cut rules for a supersymmetric multi-object solution to give rise to a well-defined multi-black hole solution [7–14]: i) positive mass of the constituents, ii) the near-horizon limit has to have definite entropy, iii) the 2^{nd} law of thermodynamics must hold in the coalescence of constituents, and iv) the Denef constraints [12] must be satisfied. Depending on the charges the latter may constrain the distance between the constituents but it always implies the absence of NUT charge.

The oft forgotten case of ungauged $\mathcal{N}=2$, $d=4$ supergravity coupled to non-Abelian vector multiplets, which we will refer to as $\mathcal{N}=2$ Einstein-Yang-Mills, is similar to the Abelian case in that there is a well-defined recipe for constructing supersymmetric solutions [15, 16]. However, the construction of supersymmetric solutions is greatly hindered not only by the fact that not every Abelian theory can be non-Abelianized, but doubly more so by the fact that the supersymmetric recipe requires the use of solutions of the (non-Abelian) Bogomol'nyi equation on $\mathbb{R}^3$ [18]. Our lack of knowledge of the space of all solutions to this equation is a serious limitation to the application of the supersymmetric recipe: there exists a vast literature on single monopole solutions, i.e. regular single-center solutions to the Bogomol'nyi equation (see e.g. refs. [19, 20]). Depending on the chosen $\mathcal{N}=2$, $d=4$ model, these can be used to construct globally regular supergravity solutions known as *global monopoles*. A lot less is known about the singular solutions to the Bogomol'nyi equation which are the ones which give rise to black holes with different degrees of non-Abelian hair [15–17]. Finally, even less is known about multi-center solutions to the Bogomol'nyi equation. These are the ones we need in order to apply the supersymmetric recipe to the construction of multi-center supergravity solutions, with centers that correspond to global monopoles or black holes.

Luckily enough, some explicit solutions are known.[1] In this paper we are going to use the solutions of the SU(2) Bogomol'nyi equation found by Cherkis and Durcan [24, 25] and Blair and Cherkis [26, 27] (which we will generalize by adding *Protogenov hair* [17]). These solutions describe an 't Hooft-Polyakov (-Protogenov) monopole in the presence of an arbitrary number of Dirac monopoles embedded in SU(2), all having charge opposite to the one carried by the former. These solutions can (in principle) give rise to supergravity solutions describing black holes in the presence of a global monopole. The construction of these solutions is, at the same time, our main goal and our main result.

[1] Finite-energy, multi-center solutions of the Yang-Mills or Yang-Mills-Higgs system which do not satisfy the Bogomol'nyi equation like those in refs. [21–23] are also known.

Before we start constructing multi-black hole solutions, however, it is worth reviewing briefly some of the previous work on solutions of YM theories coupled to gravity.[2] Most of the previous work on this topic was focused on pure Einstein-Yang-Mills (EYM) theories, (the minimal non-Abelian extension of the Einstein-Maxwell theory), ignoring the possible existence of unbroken supersymmetry which is, however, one of our main concerns here.

Soon after the discovery of the 't Hooft-Polyakov monopole [30, 31] several groups found solutions to the pure EYM theory [32–36] whose SU(2) gauge field is that of the Wu-Yang SU(2) monopole [37]. The metric of all these solutions is that of the (dS or AdS) non-extremal Reissner-Nordström black hole and the singularity in the gauge field (generically expected for static YM solutions [38]) is covered by an event horizon.

This coincidence of the metrics is due to the relation between the Wu-Yang SU(2) monopole and the non-Abelian embedding of the Dirac monopole eq. (B.10): they are related by a singular gauge transformation and therefore give rise to exactly the same energy-momentum tensor as it is gauge invariant whether the gauge transformation is singular or not. For this reason, these solutions have been regarded as not truly non-Abelian, even though there are potentially measurable differences, see e.g. refs. [39–41].

Finding less trivial ("genuinely or essentially non-Abelian") solutions proved much more difficult and a *non-Abelian baldness theorem* stating that the only black-hole solutions of the EYM SU(2) theory with a regular horizon and non-vanishing magnetic charge had to be non-Abelian embeddings of the Reissner-Nordström solution was proven in [42]. This theorem was subsequently generalized to prove the absence of regular monopole or dyon solutions to the EYM theory in refs. [43, 44].

An "essentially non-Abelian" solution, globally regular [45] to EYM theory had already been found: the Bartnik-McKinnon particle [46]. The Bartnik-McKinnon particle and its black hole-type generalizations [47], are in fact families of unstable solutions indexed by a discrete parameter and evade the non-Abelian baldness theorem by being bald, i.e. they have no asymptotic charge. It is worth pointing out that even though these solutions are only known numerically, they have been proven to exist [48, 49].

The classification of the possible EYM solutions for the gauge group SU(2) [50] suggests that one has to add more fields to the theory in order to get "essentially non-Abelian" black-hole or gravitating monopole solutions with non-vanishing charges. Investigations of solutions to the EYM theory coupled to a Higgs field in the adjoint representation [51–53] in the BPS-limit, a theory that is closer to the one we are going to study than EYM, shows that a globally well-defined 't Hooft-Polyakov monopole exists and furthermore the existence of other Bartnik-McKinnon-like solutions.

As far as 4-dimensional supergravity is concerned we have the (supersymmetric) Harvey-Liu [54] and the Chamseddine-Volkov [55, 56] regular gravitating monopole solutions to gauged $\mathcal{N}=4$, $d=4$ supergravity; in $\mathcal{N}=2$, $d=4$ theories there are analytical solutions describing global monopole solutions and non-Abelian black hole solutions with and without asymptotic magnetic charge. Needless to say, all the solutions mentioned in this little historical *exposé* describe the fields corresponding to a single object. To our

[2]For more comprehensive reviews see e.g. refs. [28, 29].

knowledge, there are no known, essentially non-Abelian multi-object analytic[3] solutions and this article intends to fill this gap by constructing static solutions describing the interplay between an 't Hooft-Polyakov monopole and a Dirac monopole of opposite charge in two generic classes of gauged $\mathcal{N}=2$, $d=4$ models.

It is convenient to stress that in the theories we have called $\mathcal{N}=2$, $d=4$ SEYM the gauge group does not contain any part of the R-symmetry group. Indeed, in general (ungauged) $\mathcal{N}=2$, $d=4$ theories, the global symmetry group G can be written as

$$\mathrm{G} = \mathrm{G}_{\mathrm{V}} \times \mathrm{G}_{\mathrm{hyper}} \times \mathrm{SU}(2)_{\mathrm{R}} \times \mathrm{U}(1)_{\mathrm{R}}\,, \tag{1.1}$$

where G_{V} and $\mathrm{G}_{\mathrm{hyper}}$ stand for the isometry groups of the special and quaternionic Kähler manifolds respectively. When a (necessarily non-Abelian) subgroup of G_{V} is gauged (as in $\mathcal{N}=2$, $d=4$ SEYM theories) the scalar potential is positive semidefinite, which allows for asymptotically De Sitter and asymptotically flat solutions (such as the ones we construct in this paper). This is in contradistinction to theories in which a subgroup of $\mathrm{SU}(2)_{\mathrm{R}}$ (or the complete $\mathrm{SU}(2)_{\mathrm{R}}$) is gauged via Fayet-Iliopoulos terms[4] in which case the scalar potential becomes negative definite, the solutions thus being asymptotically anti-De Sitter. Lately, an intense effort has been devoted to the construction of black-hole solutions of theories with Abelian gaugings (that is, theories in which a subgroup $\mathrm{U}(1) \in \mathrm{SU}(2)_{\mathrm{R}}$ has been gauged); see, for instance, refs. [59–64] and references therein. The case in which the full $\mathrm{SU}(2)_{\mathrm{R}}$ has been gauged remains as unexplored as challenging, even though the general form of the timelike supersymmetric solutions of this theory has been given in ref. [65].

This paper is organized as follows: in section 2 we review the theories we are going to work with ($\mathcal{N}=2$, $d=4$ Super-Einstein-Yang-Mills theories) and the recipe for constructing timelike supersymmetric solutions (black holes, in particular). In section 3 we apply that recipe to construct single, static supersymmetric black-hole and monopole solutions of two particular examples of SU(2)-gauged $\mathcal{N}=2$, $d=4$ SEYM: the $\overline{\mathbb{CP}}^3$ model (quadratic) 3.2 and the ST[2, 4] model (cubic) 3.3.1. We use as seeds for these solutions the single-center solutions of the Bogomol'nyi equations reviewed in section 3.1. In section 4 we construct multi-black-hole solutions for the same models using the multi-center solutions of the Bogomol'nyi equations reviewed in section 4.1. Our conclusions are contained in section 5. In the appendices we review a particularly interesting single-center solution of the SU(2) Bogomol'nyi equations which appears in different guises: as a "Lorentzian meron" (appendix A), as the Wu-Yang monopole (appendix B) or as a solution of the Skyrme model (appendix C). A higher-charge generalization of this solution is reviewed in appendix D.

[3]Numerical, multi-center solutions have been found previously, though. See, e.g. refs. [57, 58]. Some of those solutions can be embedded in $\mathcal{N}=1, d=4$ supergravity. However, representing massive objects, they can never be supersymmetric in that theory. The embedding in higher-$\mathcal{N}$ supergravities is much more difficult (if possible at all). We thank J. Kunz for pointing these works to us.

[4]The overall $\mathrm{U}(1)_R$ group cannot be gauged in this way. The Abelian gaugings discussed in the literature deal with a subgroup $\mathrm{U}(1) \in \mathrm{SU}(2)_{\mathrm{R}}$.

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2 $\mathcal{N}=2$, $d=4$ SEYM and its supersymmetric black-hole solutions (SBHSs)

In this section we are going to introduce the class of theories that we have called $\mathcal{N}=2$, $d=4$ SEYM theories and we are going to review the recipe to construct all their timelike supersymmetric solutions, presented in ref. [15]. We shall be extremely brief. The interested reader can find more details in refs. [16, 66, 67]; our conventions are those of refs. [15, 16, 67].

2.1 The theory

$\mathcal{N}=2$, $d=4$ SEYM theories can be seen as the simplest $\mathcal{N}=2$ supersymmetrization of the Einstein-Yang-Mills (EYM) theories. They are nothing but theories of $\mathcal{N}=2$, $d=4$ supergravity coupled to n vector multiplets in which a (necessarily non-Abelian)[5] subgroup of the isometry group of the (Special Kähler) scalar manifold has been gauged using some of the vector fields of the theory as gauge fields.[6]

We will only be concerned with the bosonic sector of the theory, which consists on the metric $g_{\mu\nu}$, the vector fields $A^{\Lambda}{}_{\mu}$ ($\Lambda=0,1,\cdots,n$) and the complex scalars Z^i ($i=1,\cdots,n$). The action of the bosonic sector reads

$$S[g_{\mu\nu},A^{\Lambda}{}_{\mu},Z^i]=\int d^4x\sqrt{|g|}\Big[R+2\mathcal{G}_{ij^*}\mathfrak{D}_{\mu}Z^i\mathfrak{D}^{\mu}Z^{*\,j^*}+2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu}$$
$$-2\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu}-V(Z,Z^*)\Big]\,. \tag{2.1}$$

In this expression, $\mathcal{G}_{ij^*}$ is the Kähler metric, $\mathfrak{D}_{\mu}Z^i$ is the gauge-covariant derivative

$$\mathfrak{D}_{\mu}Z^i=\partial_{\mu}Z^i+gA^{\Lambda}{}_{\mu}k_{\Lambda}{}^i\,, \tag{2.2}$$

$F^{\Lambda}{}_{\mu\nu}$ is the vector field strength

$$F^{\Lambda}{}_{\mu\nu}=2\partial_{[\mu}A^{\Lambda}{}_{\nu]}-gf_{\Sigma\Gamma}{}^{\Lambda}A^{\Sigma}{}_{\mu}A^{\Gamma}{}_{\nu}\,, \tag{2.3}$$

$\mathcal{N}_{\Lambda\Sigma}$ is the period matrix and, finally, $V(Z,Z^*)$ is the scalar potential

$$V(Z,Z^*)=-\tfrac{1}{4}g^2\Im\mathfrak{m}\mathcal{N}^{\Lambda\Sigma}\mathcal{P}_{\Lambda}\mathcal{P}_{\Sigma}\,. \tag{2.4}$$

Since the imaginary part of the period matrix is negative definite, the scalar potential is positive semidefinite, which leads to asymptotically-flat or -De Sitter solutions.

In the above equations, $k_{\Lambda}{}^i(Z)$ are the holomorphic Killing vectors of the isometries that have been gauged[7] and $\mathcal{P}_{\Lambda}(Z,Z^*)$ the corresponding momentum maps, which are related to the Killing vectors and to the Kähler potential $\mathcal{K}$ by

$$i\mathcal{P}_{\Lambda}=k_{\Lambda}{}^i\partial_i\mathcal{K}-\lambda_{\Lambda}\,, \tag{2.5}$$
$$k_{\Lambda\,i^*}=i\partial_{i^*}\mathcal{P}_{\Lambda}\,, \tag{2.6}$$

[5]The theory becomes identical to the ungauged one when the gauge group is Abelian.

[6]A global symmetry group can be gauged if it acts on the vector fields in the adjoint representation. Furthermore, it is required to be a symmetry of the prepotential; see e.g. ref. [15] for more details.

[7]The employed notation associates a Killing vector to each value of the index Λ in order to avoid the introduction of yet another class of indices and the embedding tensor (see e.g. the reviews [68–70]); it is understood that not all the k_{Λ} need to be non-vanishing.

for some holomorphic functions $\lambda_\Lambda(Z)$. Furthermore, the holomorphic Killing vectors and the generators T_Λ of the gauge group satisfy the Lie algebras

$$[k_\Lambda, k_\Sigma] = -f_{\Lambda\Sigma}{}^\Gamma k_\Gamma\,, \qquad [T_\Lambda, T_\Sigma] = +f_{\Lambda\Sigma}{}^\Gamma T_\Gamma\,. \tag{2.7}$$

For the gauge group SU(2), which is the only one we are going to consider, we use lowercase indices[8] $a, b, c = 1, 2, 3$ and the structure constants are $f_{ab}{}^c = -\varepsilon_{abc}$, so

$$[k_a, k_b] = +\varepsilon_{abc} k_c\,, \qquad [T_a, T_b] = -\varepsilon_{abc} T_c\,. \tag{2.8}$$

We will use the fundamental representation, in which the generators are proportional to the standard Pauli matrices[9] σ^a

$$T_a \equiv +\frac{i}{2}\sigma^a\,, \quad \Rightarrow \quad \mathrm{Tr}(T_a T_b) = -\frac{1}{2}\delta_{ab}\,. \tag{2.10}$$

The equations of motion of the theory can be written in the following form:

$$\begin{aligned} G_{\mu\nu} + 2\mathcal{G}_{ij^*}[\mathfrak{D}_{(\mu} Z^i \mathfrak{D}_{\nu)} Z^{*j^*} - \tfrac{1}{2} g_{\mu\nu} \mathfrak{D}_\rho Z^i \mathfrak{D}^\rho Z^{*j^*}] & \\ +4\mathcal{M}_{MN} \mathcal{F}^M{}_\mu{}^\rho \mathcal{F}^N{}_{\nu\rho} + \tfrac{1}{2} g_{\mu\nu} V(Z, Z^*) &= 0, \end{aligned} \tag{2.11}$$

$$\mathfrak{D}^2 Z^i + \partial^i G_{\Lambda\,\mu\nu} \star F^{\Lambda\,\mu\nu} + \tfrac{1}{2}\partial^i V(Z, Z^*) = 0, \tag{2.12}$$

$$\mathfrak{D}_\nu \star G_\Lambda{}^\nu{}_\mu + \tfrac{1}{4} g \left(k_{\Lambda\,i^*} \mathfrak{D}_\mu Z^{*i^*} + k^*_{\Lambda\,i} \mathfrak{D}_\mu Z^i \right) = 0\,, \tag{2.13}$$

where $G_{\Lambda\,\mu\nu}$ is the dual vector field strength

$$G_\Lambda \equiv \Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma} F^\Sigma + \Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma} \star F^\Sigma\,, \tag{2.14}$$

$\mathcal{F}^M{}_{\mu\nu}$ is the symplectic vector of vector field strengths

$$(\mathcal{F}^M) \equiv \begin{pmatrix} F^\Lambda \\ G_\Lambda \end{pmatrix}\,, \tag{2.15}$$

$\mathcal{M}_{MN}$ is the symmetric $2(n+1) \times 2(n+1)$ matrix defined by

$$(\mathcal{M}_{MN}) \equiv \begin{pmatrix} \Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma} + R_{\Lambda\Gamma} \Im\mathfrak{m}\mathcal{N}^{-1|\Gamma\Omega} R_{\Omega\Sigma} & -R_{\Lambda\Gamma} \Im\mathfrak{m}\mathcal{N}^{-1|\Gamma\Sigma} \\ -\Im\mathfrak{m}\mathcal{N}^{-1|\Lambda\Omega} R_{\Omega\Sigma} & \Im\mathfrak{m}\mathcal{N}^{-1|\Lambda\Sigma} \end{pmatrix}\,, \tag{2.16}$$

and

$$\mathfrak{D}_\nu \star G_\Lambda{}^{\nu\mu} = \partial_\nu \star G_\Lambda{}^{\nu\mu} + g f_{\Lambda\Sigma}{}^\Gamma A^\Sigma{}_\nu \star G_\Lambda{}^{\nu\mu}\,. \tag{2.17}$$

Most of the literature and earlier work on non-Abelian black-hole and monopole solutions has been carried out in the context of the Einstein-Yang-Mills (EYM) and Einstein-Yang-Mills-Higgs (EYMH) theories. Before closing this introduction, it is worth discussing

[8] These will be a certain subset of those represented by $\Lambda, \Sigma, \ldots$.

[9] These are

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^a \sigma^b = \delta^{ab} + i\varepsilon^{abc}\sigma^c\,. \tag{2.9}$$

the relation between those and the theories we are considering here. The main differences of the latter w.r.t. the former are the complexification of the Higgs field and the presence of a non-trivial period matrix. A further difference is the possibility of having more general scalar manifolds, which is reflected in the expressions of the gauge-covariant derivatives of the scalar fields. Solutions to the $\mathcal{N}=2$, $d=4$ SEYM theory have a chance of being also solutions of the EYMH theory if they have covariantly-constant scalars with identical phases (e.g. all of them purely imaginary). Then, if the scalar potential vanishes on the solutions, they also have a chance of being solutions to the EYM system as well; as we are going to see, some of the solutions found in refs. [15, 16] are also solutions of the EYM theory and have the same metric as the EYM solutions of refs. [32–36, 41].

2.2 The recipe to construct SBHSs of $\mathcal{N}=2$, $d=4$ SEYM

To construct timelike supersymmetric solutions of the $\mathcal{N}=2$, $d=4$ SEYM theory, it suffices to follow this recipe [15, 16] to find the elementary building blocks of the solutions, which are the $2(n+1)$ time-independent functions $(\mathcal{I}^M) = \begin{pmatrix} \mathcal{I}^\Lambda \\ \mathcal{I}_\Lambda \end{pmatrix}$:

1. Take a solution of the Bogomol'nyi equations

$$\tilde{F}^\Lambda{}_{\underline{mn}} = -\frac{1}{\sqrt{2}}\varepsilon_{mnp}\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^\Lambda, \tag{2.18}$$

for a gauge field $\tilde{A}^\Lambda{}_{\underline{m}}$ ($\underline{m}=1,2,3$ labels the 3 spatial coordinates) and a real "Higgs" field $\mathcal{I}^\Lambda$. $\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^\Lambda$ is the covariant derivative in the adjoint representation with gauge field $\tilde{A}^\Lambda{}_{\underline{m}}$. Observe that this equation has to be solved in the gauged (non-Abelian) and ungauged (Abelian) directions. The integrability condition in the Abelian directions is the familiar requirement that the $\mathcal{I}^\Lambda$ be harmonic functions on $\mathbb{R}^3$.

2. Find the functions $\mathcal{I}_\Lambda$ by solving these equations:

$$\tilde{\mathfrak{D}}_{\underline{m}}\tilde{\mathfrak{D}}_{\underline{m}}\mathcal{I}_\Lambda = \frac{1}{2}g^2\left[f_{\Lambda(\Sigma}{}^{\Gamma}f_{\Delta)\Gamma}{}^{\Omega}\,\mathcal{I}^\Sigma\mathcal{I}^\Delta\right]\mathcal{I}_\Omega\,. \tag{2.19}$$

In the non-Abelian directions these equations can, in many cases, be solved by taking $\mathcal{I}_\Lambda \propto \mathcal{I}^\Lambda$, but currently we only know how to generate non-trivial solutions to them in the cases where the gauge doublet $(\tilde{A}^\Lambda, \mathcal{I}^\Lambda)$ describes a non-Abelian Wu-Yang monopole; observe that $\mathcal{I}_\Lambda = 0$ is always a solution, but the physical fields may be singular in some models.

In the Abelian directions, the $\mathcal{I}_\Lambda$ are just independent harmonic functions on $\mathbb{R}^3$.

3. Given the functions $\mathcal{I}^M$, we must find the 1-form on $\mathbb{R}^3$ $\omega_{\underline{m}}$ by solving the following equation:

$$\partial_{[\underline{m}}\omega_{\underline{n}]} = \varepsilon_{mnp}\mathcal{I}_M\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^M = \varepsilon_{mnp}\left(\mathcal{I}_\Lambda\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}^\Lambda - \mathcal{I}^\Lambda\tilde{\mathfrak{D}}_{\underline{p}}\mathcal{I}_\Lambda\right). \tag{2.20}$$

The integrability conditions of this equation impose constraints on the integration constants of the functions $\mathcal{I}^M$ in exactly the same manner as in the ungauged case [12, 71].

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In the case of static solutions, i.e. when $\omega = 0$, the above equation becomes a constraint on the integration constants of the functions $\mathcal{I}^M$ that will have to be solved. Observe, however, that this constraint is independent of the specific $\mathcal{N} = 2$, $d = 4$ model and only depends on the choice of gauge group; possible restrictions on the solution to said constraint can come from the desired behaviour of the physical fields in the full solution.

4. To reconstruct the physical fields from the functions $\mathcal{I}^M$ we need to solve the stabilization equations, a.k.a. *Freudenthal duality equations*, which give the components of the Freudenthal dual[10] $\tilde{\mathcal{I}}^M(\mathcal{I})$ in terms of the functions $\mathcal{I}^M$ [73]; these relations completely characterize the model of $\mathcal{N} = 2$, $d = 4$ supergravity.

   Equivalently, the $\tilde{\mathcal{I}}$ can be derived from a homogeneous function of degree 2 $W(\mathcal{I})$ called the *Hesse potential* as [71, 74, 75]
$$\tilde{\mathcal{I}}_M = \tfrac{1}{2}\frac{\partial W}{\partial \mathcal{I}^M} \quad \longrightarrow \quad W(\mathcal{I}) = \tilde{\mathcal{I}}_M \mathcal{I}^M\,. \tag{2.21}$$

5. The metric takes the form
$$ds^2 = e^{2U}(dt+\omega)^2 - e^{-2U}dx^m dx^m\,, \tag{2.22}$$
   where $\omega = \omega_{\underline{m}} dx^m$ is the above spatial 1-form and the metric function e^{-2U} is given by
$$e^{-2U} = \tilde{\mathcal{I}}_M(\mathcal{I})\mathcal{I}^M = W(\mathcal{I})\,. \tag{2.23}$$

6. The scalar fields are given by
$$Z^i = \frac{\tilde{\mathcal{I}}^i + i\mathcal{I}^i}{\tilde{\mathcal{I}}^0 + i\mathcal{I}^0}\,. \tag{2.24}$$

7. The components of the vector fields are given by
$$A^{\Lambda}{}_t = -\frac{1}{\sqrt{2}} e^{2U}\tilde{\mathcal{I}}^{\Lambda}\,, \tag{2.25}$$
$$A^{\Lambda}{}_{\underline{m}} = \tilde{A}^{\Lambda}{}_{\underline{m}} + \omega_{\underline{m}}\, A^{\Lambda}{}_t\,. \tag{2.26}$$

After having gone through the steps of the recipe, one ends up with a supersymmetric solution to a chosen $\mathcal{N} = 2$, $d = 4$ EYM theory and what remains to be done is to analyze the constraints coming from imposing appropriate regularity conditions such as the absence of naked singularities.

3 Static, single-SBHSs of SU(2) $\mathcal{N} = 2$, $d = 4$ SEYM and pure EYM

Following the recipe given in section 2.2, we are going to construct static, single-center SBHSs of SU(2) $\mathcal{N} = 2$, $d = 4$ SEYM. Some of the solutions will simultaneously solve the equations of motion of the EYM and EYMH theories.

The first step consists in finding a solution $\tilde{A}^{\Lambda}{}_{\underline{m}}, \mathcal{I}^{\Lambda}$ of the SU(2) Bogomol'nyi equations in $\mathbb{R}^3$ eqs. (2.18).

[10] In refs. [15, 16, 72] the components of the Freudenthal dual are denoted by $\mathcal{R}^M$.

3.1 Single-center solutions of the SU(2) Bogomol'nyi equations in $\mathbb{R}^3$

Before we search for solutions of the Bogomol'nyi equations it is worth reviewing the origin and meaning of those equations in the context of the SU(2) Yang-Mills-Higgs theory (in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit in which the Higgs potential vanishes).

3.1.1 The SU(2) Yang-Mills-Higgs system

With the normalization in eq. (2.10) and writing $F \equiv F^a T_a, \Phi \equiv \Phi^a T_a$, the action of the YMH theory in our conventions reads

$$S_{\mathrm{YMH}} = -2 \int d^4x \, \mathrm{Tr} \left\{ \frac{1}{2} \mathfrak{D}_\mu \Phi \mathfrak{D}^\mu \Phi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\} , \tag{3.1}$$

and the corresponding equations of motion are

$$\mathfrak{D}_\mu F^{\mu\nu} = g[\Phi, \mathfrak{D}^\nu \Phi] , \tag{3.2}$$

$$\mathfrak{D}^2 \Phi = 0 . \tag{3.3}$$

For static configurations $F_{t\underline{m}} = \mathfrak{D}_t \Phi = 0$, the action can be written, up to a total derivative, in the manifestly positive form

$$S_{\mathrm{YMH}} = -2 \int d^4x \, \mathrm{Tr} \left\{ -\frac{1}{4} \left(F_{\underline{mn}} \mp \varepsilon_{mnp} \mathfrak{D}_{\underline{p}} \Phi \right) \left(F_{\underline{mn}} \mp \varepsilon_{mnp} \mathfrak{D}_{\underline{p}} \Phi \right) \right\} , \tag{3.4}$$

which leads to the conclusion that static field configurations satisfying the first-order *Bogomol'nyi equations* [18]

$$F_{\underline{mn}} = \pm \varepsilon_{mnp} \mathfrak{D}_{\underline{p}} \Phi , \tag{3.5}$$

extremize the action eq. (3.1) and are solutions of the full Yang-Mills-Higgs equations. Indeed, if we act with $\mathfrak{D}_{\underline{m}}$ on both sides of the equation and use the Ricci identity and the Bogomol'nyi equation we get the Yang-Mills equation:

$$\mathfrak{D}_{\underline{m}} F_{\underline{mn}} = \mp \varepsilon_{nmp} \mathfrak{D}_{\underline{m}} \mathfrak{D}_{\underline{p}} \Phi = \mp \frac{1}{2} g \varepsilon_{nmp} [F_{\underline{mp}}, \Phi] = -g[\mathfrak{D}_{\underline{n}} \Phi, \Phi] . \tag{3.6}$$

If, instead, we act with $\varepsilon_{pmn} \mathfrak{D}_{\underline{p}}$ and use the Bianchi identity, we get the Higgs equation:

$$0 = \varepsilon_{pmn} \mathfrak{D}_{\underline{p}} F_{\underline{mn}} = \pm \mathfrak{D}_{\underline{p}} \mathfrak{D}_{\underline{p}} \Phi . \tag{3.7}$$

Observe that the source of the Yang-Mills field, the *Higgs current* $g[\Phi, \mathfrak{D}\Phi]$, not only vanishes when the Higgs field is covariantly constant $\mathfrak{D}\Phi = 0$ but also when Φ and $\mathfrak{D}\Phi$ are parallel in $\mathfrak{su}(2)$.

Eqs. (3.5) are identical to the ones that arise in $\mathcal{N} = 2$, $d = 4$ SEYM theory, (2.18) upon the identification of the vector fields and

$$\frac{1}{\sqrt{2}} \mathcal{I}^a = \mp \Phi^a . \tag{3.8}$$

3.1.2 The hedgehog ansatz

In order to construct static, single-center black-hole-type solutions, it is natural to look for spherically symmetric solutions of eqs. (3.5). Substituting the *hedgehog ansatz*

$$\mp\Phi^a = \delta^a{}_m f(r) x^m\,, \qquad A^a{}_{\underline{m}} = -\varepsilon^a{}_{mn} x^n h(r) \tag{3.9}$$

in the Bogomol'nyi eqs. (3.5) we get an equivalent system of differential equations for $f(r)$ and $h(r)$:

$$\begin{aligned} r\partial_r h + 2h - f(1+gr^2 h) &= 0\,,\\ r\partial_r (h+f) - gr^2 h(h+f) &= 0\,. \end{aligned} \tag{3.10}$$

After Prasad and Sommerfield [76] found the solution describing the 't Hooft-Polyakov monopole in the BPS limit, Protogenov [77] classified all spherically symmetric solutions to the SU(2) Bogomol'nyi equations: the ones that can be used to generate BH-like spacetimes are a 2-parameter family $(f_{\mu,s}, h_{\mu,s})$ plus a 1-parameter family (f_λ, h_λ) given by

$$\begin{aligned} rf_{\mu,s} &= \frac{1}{gr}\left[1-\mu r\coth(\mu r+s)\right], & rh_{\mu,s} &= \frac{1}{gr}\left[\frac{\mu r}{\sinh(\mu r+s)}-1\right],\\ rf_\lambda &= \frac{1}{gr}\left[\frac{1}{1+\lambda^2 r}\right], & rh_\lambda &= -rf_\lambda\,. \end{aligned} \tag{3.11}$$

The parameter s is known in the black-hole context as the *Protogenov hair* parameter [17]. The BPS 't Hooft-Polyakov monopole [76] is the only globally regular solution of this family (which explains why it is the only one usually considered in the monopole literature[11]) and corresponds to $s=0$. In the $s\to\infty$ limit we get

$$-rf_{\mu,\infty} = \frac{\mu}{g} - \frac{1}{gr}, \qquad rh_{\mu,\infty} = -\frac{1}{gr}\,, \tag{3.12}$$

which, for $\mu = 0$, coincides with the Wu-Yang monopole [37] given in eq. (B.10), and is a solution of the pure Yang-Mills theory. This is possible because the Higgs current $g[\Phi,\mathfrak{D}\Phi]$ vanishes even though Φ is neither zero nor covariantly constant.[12] With a non-trivial Higgs field, though, we can assign a well-defined monopole charge to it: for any μ and s

$$\frac{1}{4\pi}\int_{S^2_\infty} \mathrm{Tr}(\hat{\Phi}F) = \frac{1}{g}\,, \qquad \hat{\Phi} \equiv \frac{\Phi}{\sqrt{|\mathrm{Tr}(\Phi^2)|}}\,. \tag{3.13}$$

The same field configuration can be seen as a Lorentzian meron (see appendix A) and as a solution to the Skyrme model (see appendix C), and, crucially, it is related to the SU(2)-embedded Dirac monopole by a singular gauge transformation (see appendix B). Since the metric is oblivious to gauge transformations, singular or not, the Wu-Yang monopole gives rise to solutions whose metric is identical to that of the Abelian case.[13] The same applies to the higher-charge generalizations of the Lorentzian meron/Wu-Yang monopole reviewed in appendix D.

[11] After coupling the system to gravity, the singularities of the other solutions may become "harmless" if they can be covered by regular event horizons.

[12] Actually, the only field configuration in this ansatz with a vanishing Higgs current is this one.

[13] Of course there are measurable differences between these two situations, see e.g. refs. [39–41].

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If fact, this mechanism can be used to generate Wu-Yang monopoles of higher charge from the well-known Dirac monopole solutions of charge higher than 1 embedded in SU(2), as reviewed in appendix D. The metric cannot see the difference between the non-Abelian and the Abelian fields given in eq. (3.12).

The 1-parameter family is singular for all values of the parameter λ, which also appears in black-hole solutions as hair. The magnetic charge measured at spatial infinity vanishes according to the above definition. However, it can be argued that these solutions do describe a magnetic monopole placed at the origin whose charge is screened: the entropy of black hole associated to this field has the same form as that of the black hole associated to the Wu-Yang monopole. Observe that, for $\lambda = 0$, the solution is identical to the Wu-Yang monopole with $\mu = 0$, eqs. (3.12).

3.1.3 The Protogenov trick

As it turns out, many regular monopole solutions can be deformed by adding a parameter s to the argument μr, generating a family of solutions that contains the original one ($s = 0$) and, typically, a new and simpler solution in the $s \to \infty$ limit. We will refer to this procedure as *the Protogenov trick* and it can be justified as follows: let us consider, for instance, the 't Hooft-Polyakov monopole. Since the Bogomol'nyi equation is polynomial, having elementary functions such as hyperbolic functions in the solution means that they must cancel amongst themselves and that only their derivatives contribute to the polynomial part of the solution. This means that one should be able to deform the dependency of the elementary functions introducing a shift s of the radial coordinate and still solve the Bogomol'nyi equations.

Of course, the cancellations necessary for having a regular solution will not work out anymore (assuming they did work for $s = 0$) and one will end up with a family of singular solutions. We will use this trick later.

3.2 Embedding in the SU(2)-gauged $\overline{\mathbb{CP}}^3$ model

3.2.1 The $\overline{\mathbb{CP}}^3$ model

The $\overline{\mathbb{CP}}^n$ models have n vector supermultiplets and are defined by the quadratic prepotentials

$$\mathcal{F} = -\frac{i}{4}\eta_{\Lambda\Sigma}\mathcal{X}^\Lambda\mathcal{X}^\Sigma\,, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+-\cdots-)\,. \tag{3.14}$$

The n physical scalar fields can be defined as

$$Z^i \equiv \mathcal{X}^i/\mathcal{X}^0\,, \tag{3.15}$$

and they parametrize the symmetric space $\mathrm{U}(1,n)/(\mathrm{U}(1)\times\mathrm{U}(n))$. It is convenient to define $Z^0 \equiv 1$, $Z^\Lambda \equiv \mathcal{X}^\Lambda/\mathcal{X}^0$ and $Z_\Lambda \equiv \eta_{\Lambda\Sigma}Z^\Sigma$. In the $\mathcal{X}^0 = 1$ gauge, the Kähler potential and the Kähler metric are given by

$$\mathcal{K} = -\log\left(Z^{*\Lambda}Z_\Lambda\right), \quad \mathcal{G}_{ij^*} = -e^{\mathcal{K}}\left(\eta_{ij^*} - e^{\mathcal{K}}Z^*_i Z_{j^*}\right)\,, \quad \Rightarrow \quad 0 \leq \sum_i |Z^i|^2 < 1\,. \tag{3.16}$$

The above metric is the standard (Bergman) metric for the $\mathrm{U}(1,n)/(\mathrm{U}(1)\times\mathrm{U}(n))$ symmetric spaces [78]. The covariantly holomorphic symplectic section $\mathcal{V}$ and the period matrix $\mathcal{N}_{\Lambda\Sigma}$ are given by

$$\mathcal{V} = e^{\mathcal{K}/2}\begin{pmatrix} Z^\Lambda \\ -\frac{i}{2}Z_\Lambda \end{pmatrix}, \qquad \mathcal{N}_{\Lambda\Sigma} = \frac{i}{2}\left[\eta_{\Lambda\Sigma} - 2\frac{Z_\Lambda Z_\Sigma}{Z^\Gamma Z_\Gamma}\right]. \tag{3.17}$$

The isometry subgroup $\mathrm{SU}(1,n)$ acts linearly, in the fundamental representation, on the coordinates $\mathcal{X}^\Lambda$

$$\mathcal{X}'^\Lambda = \Lambda^\Lambda{}_\Sigma \mathcal{X}^\Sigma, \quad \text{with} \quad \Lambda^\dagger \eta \Lambda = \eta, \quad \text{and} \quad \det \Lambda = 1. \tag{3.18}$$

This linear action induces a non-linear action on the special coordinates:

$$Z'^\Lambda = \frac{\Lambda^\Lambda{}_\Sigma Z^\Sigma}{\Lambda^0{}_\Sigma Z^\Sigma}. \tag{3.19}$$

The Kähler potential is invariant under these transformations up to Kähler transformations $\mathcal{K}' = \mathcal{K} + f + f^*$ with

$$f(Z) = \log\left(\Lambda^0{}_\Sigma Z^\Sigma\right). \tag{3.20}$$

The $n(n+2)$ infinitesimal generators T_m of $\mathfrak{su}(1,n)$ in the fundamental representation are defined by

$$\Lambda^\Lambda{}_\Sigma \sim \delta^\Lambda{}_\Sigma + \alpha^m\, T_m{}^\Lambda{}_\Sigma, \quad \text{with} \quad \eta T_m^\dagger \eta = -T_m, \quad \text{and} \quad T_m{}^\Lambda{}_\Lambda = 0. \tag{3.21}$$

Substituting this definition into eq. (3.19) we find an expression for the holomorphic Killing vectors.[14]

$$Z'^\Lambda = Z^\Lambda + \alpha^m k_m{}^\Lambda(Z), \qquad k_m{}^\Lambda(Z) = T_m{}^\Lambda{}_\Sigma\, Z^\Sigma - T_m{}^0{}_\Omega\, Z^\Omega Z^\Lambda, \tag{3.22}$$

and, from this expression, we also find explicit expressions for the holomorphic functions $\lambda_m(Z)$ and the momentum maps

$$\lambda_m = T_m{}^0{}_\Sigma Z^\Sigma, \qquad \mathcal{P}_m = ie^{\mathcal{K}} T_m{}^\Lambda{}_\Sigma Z^\Sigma Z^*_\Lambda = ie^{\mathcal{K}}\eta_{\Lambda\Omega} T_m{}^\Lambda{}_\Sigma Z^\Sigma Z^{*\,\Omega}. \tag{3.23}$$

Although the theory is invariant under the whole $\mathrm{SU}(1,n)$ group, the prepotential is invariant only under the subgroup of $\mathrm{SU}(1,n)$ with real matrices, $\mathrm{SO}(1,n)$, which is the largest group that we could eventually gauge. However, the requirements that the vectors must transform in the adjoint representation restricts the possibilities to either $\mathrm{SO}(1,2)$ or $\mathrm{SO}(3)$ (if $n \geq 2$ or $n \geq 3$, respectively); we are going to consider the latter case embedded into the minimal model admitting this gauge group, namely $\overline{\mathbb{CP}}^3$.

In this model, the adjoint indices $a, b, c, \ldots$ and the fundamental indices $i, j, k, \ldots$ take the same values $1, 2, 3$ and we will only use the latter. The infinitesimal transformations of the scalars are

$$\delta_\alpha Z^i = \alpha^j T_j{}^i{}_k Z^k, \quad \text{where} T_j{}^i{}_k = f_{jk}{}^i = -\epsilon_{jki}, \tag{3.24}$$

[14] The $k_m{}^0(Z)$ component vanishes identically, as it must, but it is convenient to keep it.

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and the momentum maps, holomorphic Killing vectors etc. take the values

$$\mathcal{P}_i = -ie^{\mathcal{K}}\epsilon_{ijk}Z^j Z^{*\,k}\,, \qquad k_i{}^j = \epsilon_{ijk}Z^k\,, \qquad \lambda_i = 0\,. \tag{3.25}$$

This means that the gauge-covariant derivative of the scalars is just that of a complex adjoint SO(3) scalar

$$\mathfrak{D}_\mu Z^i = \partial_\mu Z^i - g\epsilon_{ijk}A^j{}_\mu Z^k, \tag{3.26}$$

and that the scalar potential takes the form

$$V(Z,Z^*) \;=\; -\frac{1}{2}g^2 e^{\mathcal{K}}\epsilon_{ijk}\epsilon_{imn}Z^j Z^{*k^*}Z^m Z^{*n^*} \;=\; \frac{1}{2}g^2\,\left|\vec{Z}\times\vec{Z}^*\right|^2\,. \tag{3.27}$$

3.2.2 The solutions

To construct the solutions of this model[15] we just have to follow the recipe spelled out in section 2.2. We will only consider static solutions (so $\omega = 0$ and $\tilde{A}^\Lambda{}_{\underline{m}} = A^\Lambda{}_{\underline{m}}$). First of all, we need a solution of the Bogomol'nyi eqs. (2.18). These equations split into an Abelian part (the 0th component) and the non-Abelian part (the $i = 1,2,3$ components):

$$F^0{}_{\underline{mn}} = -\frac{1}{\sqrt{2}}\epsilon_{mnp}\partial_{\underline{p}}\mathcal{I}^0\,, \tag{3.28}$$

$$F^i{}_{\underline{mn}} = -\frac{1}{\sqrt{2}}\epsilon_{mnp}\mathfrak{D}_{\underline{p}}\mathcal{I}^i\,. \tag{3.29}$$

The Abelian equation is solved by

$$\mathcal{I}^0 = A^0 + \frac{p^0/\sqrt{2}}{r}\,, \tag{3.30}$$

where A^0 is an integration constant and p^0 is the normalized Abelian magnetic charge. The non-Abelian set of equations can be solved making the identification eq. (3.8) and using Protogenov's solutions eqs. (3.11).

The second step in the recipe (finding solutions $\mathcal{I}_\Lambda$ to eqs. (2.19)) will be solved, for the sake of simplicity, by choosing another harmonic function in the Abelian direction and vanishing functions in the rest:

$$\mathcal{I}_0 = A_0 + \frac{q_0/\sqrt{2}}{r}\,, \qquad \mathcal{I}_i = 0\,. \tag{3.31}$$

The third point in the recipe, combined with the staticity of the solutions implies the following constraint on the integration constants:

$$A^0 q_0 - A_0 p^0 = 0\,. \tag{3.32}$$

Another constraint will arise from the normalization of the metric at infinity. The solution is completely determined and, now, we only have to write the physical fields and make, if

[15] All these solutions have already been presented in refs. [15–17]. We review them here for pedagogical reasons and also for the sake of making easier the comparison with the solutions of other models.

necessary, sensible choices of the values of the physical parameters to make the solutions regular.

In order to write the physical fields we need the solutions of the Freudenthal duality equations of this model. These are given by (see, e.g. ref. [79])

$$(\tilde{\mathcal{I}}^M) = \begin{pmatrix} \tilde{\mathcal{I}}^\Lambda \\ \tilde{\mathcal{I}}_\Lambda \end{pmatrix} = \begin{pmatrix} -2\eta^{\Lambda\Sigma}\mathcal{I}_\Sigma \\ \frac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Sigma \end{pmatrix}, \quad \Rightarrow \quad e^{-2U} = \frac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Lambda\mathcal{I}^\Sigma + 2\eta^{\Lambda\Sigma}\mathcal{I}_\Lambda\mathcal{I}_\Sigma\,, \tag{3.33}$$

and the metric function and the physical scalars are given by

$$e^{-2U} = \frac{1}{2}(\mathcal{I}^0)^2 + 2(\mathcal{I}_0)^2 - (rf)^2, \tag{3.34}$$

$$Z^i = \frac{\sqrt{2}rf}{\mathcal{I}^0 + 2i\mathcal{I}_0}\delta^i{}_m\frac{x^m}{r}. \tag{3.35}$$

At least one of the two functions $\mathcal{I}^0, \mathcal{I}_0$ must be different from zero for the metric function to be positive. Then, there are two possible cases, depending on the vanishing of the Abelian charges p^0, q_0:

I. $p^0 = q_0 = 0$ The only regular solution is the one with $s = 0$ (the 't Hooft-Polyakov monopole). In this solution, the integration constants satisfy the normalization condition

$$\frac{1}{2}(A^0)^2 + 2(A_0)^2 = 1 + (\mu/g)^2\,. \tag{3.36}$$

They are also related to the asymptotic values of the scalars. These cannot be constant, in general, because the scalars transform under local SU(2) transformations, but they are covariantly constant and their asymptotic values are determined by a single gauge-invariant complex parameter that we call Z_∞:[16]

$$Z^i \sim Z_\infty\delta^i{}_m\frac{x^m}{r}\,, \quad Z_\infty \equiv \frac{\mu/g}{1+(\mu/g)^2}\left(\frac{1}{\sqrt{2}}A^0 - \sqrt{2}iA_0\right), \quad 0 \leq |Z_\infty|^2 < 1\,. \tag{3.37}$$

These expressions lead to the following identification of the integration constant μ in terms of the physical parameters:

$$\mu^2 = \frac{|Z_\infty|^2}{1-|Z_\infty|^2}g^2\,, \tag{3.38}$$

and to the following expression for the mass of the solution

$$M_{\text{monopole}} = \sqrt{\frac{|Z_\infty|^2}{1-|Z_\infty|^2}}\frac{1}{g}\,. \tag{3.39}$$

This asymptotically flat solution has no singularities nor horizons (one finds a Minkowski spacetime in the $r \to 0$ limit, hence zero entropy and temperature). Globally-regular solutions of this kind [54–56] are sometimes called *global monopoles.*

[16] Observe that the scalar potential of this theory, eq. (3.27), vanishes at infinity for those solutions, which is why they are asymptotically flat.

Observe that a solution of the ungauged theory with

$$H^0 = A^0\,, \qquad H_0 = A_0\,, \qquad H^1 = A^1 + \frac{\sqrt{2}}{gr}\,, \tag{3.40}$$

in which the non-Abelian monopole is replaced by an Abelian monopole with the same charge, would have the same asymptotic behavior but it would mean having a naked singularity at some value of $r > 0$.

II. $p^0 q_0 \neq 0$ [17] Solving eq. (3.32) the metric can be written in the form

$$e^{-2U} = \frac{1}{1-|Z_\infty|^2} H^2 - (rf)^2, \tag{3.41}$$

$$Z^i = \frac{2\beta}{p^0 + 2iq_0} \frac{rf}{H} \delta^i{}_m \frac{x^m}{r}\,, \tag{3.42}$$

where H is the harmonic function

$$H \equiv 1 + \frac{\beta}{r}\,, \quad \beta^2 = (1-|Z_\infty|^2) W_{\rm RN}(\mathcal{Q})/2\,, \quad W_{\rm RN}(\mathcal{Q}) \equiv \frac{1}{2}(p^0)^2 + 2(q_0)^2\,, \tag{3.43}$$

and the integration constant μ is still given by eq. (3.38). We have expressed all the constants (except for Protogenov's hair parameter s and λ) in terms of physical constants. Observe that the isolated solution f_* has $\mu = 0$ and corresponds to $Z_\infty = 0$. These identifications allow us to compute the mass and entropy of all the possible solutions in terms of the physical parameters. We get a completely general mass formula and two formulae for the entropy, one for the $s \neq 0$ solutions and another one for the $s = 0$ and the isolated solutions (which corresponds to $Z_\infty = 0$):

$$M = \sqrt{\frac{1}{2}\frac{W_{RN}(\mathcal{Q})}{1-|Z_\infty|^2}} + M_{\rm monopole}, \tag{3.44}$$

$$S/\pi = \frac{1}{2} W_{\rm RN}(\mathcal{Q}) - \frac{1}{g^2}\,, \qquad \text{for} \quad s \neq 0 \ \text{ and } \ Z_\infty = 0, \tag{3.45}$$

$$S/\pi = \frac{1}{2} W_{\rm RN}(\mathcal{Q}), \qquad \text{for} \quad s = 0\,, \tag{3.46}$$

where $M_{\rm monopole}$ is given by eq. (3.39).

The entropy is moduli-independent as in the ungauged case and both the entropy and the mass are independent of the hair parameters s and λ.

Observe that the charge of the BPS 't Hooft-Polyakov monopole $s = 0$ does not contribute to the entropy which suggests that it must be associated to a pure state in the quantum theory. The non-Abelian field of the isolated solution does not contribute to the mass at infinity ($M_{\rm monopole} = 0$ for $Z_\infty = 0$) but there is a magnetic-charge contribution to the entropy, which suggests that there really is a magnetic charge but it is screened at infinity. Further support for this interpretation comes

[17] It is easier to work with both charges non-vanishing. The results will still be valid when we set one of them to zero.

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from the near-horizon limit of the scalars, which is the covariantly-constant function of the charges

$$Z^i_{\rm h} = \frac{1/g}{\frac{1}{2}p^0 + iq_0}\,\delta^i{}_m \frac{x^m}{r}. \tag{3.47}$$

even for the isolated case, when no magnetic charge is observed at infinity.

In the case of the 1-parameter (λ) family, neither the mass nor the entropy depend on λ.

Some of the solutions in this family can also be seen as solutions of the pure EYM theory. They are identical to those obtained in refs. [32–36, 41]. As discussed at the end of section 2.1, we need to tune the parameters of the solutions so as to get covariantly constant scalars which do not contribute to the energy-momentum tensor. This is only possible for the $s \to \infty$ solutions (Wu-Yang monopoles) for which rf is a harmonic function. In that case

$$Z^i = Z\,\delta^i{}_m \frac{x^m}{r}\,, \qquad Z = \frac{1/g}{\frac{1}{2}p^0 + iq_0} = Z_\infty\,. \tag{3.48}$$

The metric is identical to that of a Reissner-Nordström black hole. These solutions were called *black hedgehogs* in ref. [15] and *black merons* in ref. [41] because the gauge field of the Wu-Yang monopole can also be understood as Lorentzian meron solution.

A closely related solution with non-covariantly constant scalars was obtained in a different context in ref. [80].

3.3 Embedding in SU(2)-gauged ST[2, n] models

3.3.1 The ST[2, n] models

The $ST[2,n]$ models are cubic models with $n_V = n+1$ vector supermultiplets and as many complex scalars and, as all other cubic models, they can be embedded in type II String Theory compactified Calabi-Yau 3-folds and then uplifted to M-theory. They can also be obtained from corresponding models of $N=1$, $d=5$ supergravity compactified on S^1.

A generic cubic model is defined by the prepotential

$$\mathcal{F} \;=\; -\frac{1}{3!}\,d_{ijk}\,\frac{\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0}\,, \tag{3.49}$$

where d is completely symmetric in its indices; the $ST[2,n]$ models are characterized by d-tensors with non-vanishing components $d_{1\alpha\beta} = \eta_{\alpha\beta}$ where $(\eta_{\alpha\beta}) = \mathrm{diag}(+-\cdots-)$ and where the indices α, β take n values between 2 and $n+1$.

The scalar $Z^1 = \mathcal{X}^1/\mathcal{X}^0$ plays a special role and parametrizes a $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ coset space. For this and other reasons, it is called axidilaton and we will denote it by τ. The other n scalars parametrize a $\mathrm{SO}(2,n)/(\mathrm{SO}(2)\times\mathrm{SO}(n))$ coset space and will be denoted by $Z^\alpha = \mathcal{X}^\alpha/\mathcal{X}^0$ ($\alpha = 2,\cdots,n$). The Kähler metric and 1-form connection are the products of those of the two spaces.

Using this notation and using the gauge $\mathcal{X}^0 = 1$, the canonical symplectic section Ω, the Kähler potential $\mathcal{K}$ and the components of Kähler 1-form $\mathcal{Q}_i$ and of the Kähler metric

$\mathcal{G}_{ij^*}$ are given by

$$\Omega = \begin{pmatrix} 1 \\ \tau \\ Z^\alpha \\ \frac{1}{2}\tau\eta_{\alpha\beta}Z^\alpha Z^\beta \\ -\frac{1}{2}\eta_{\alpha\beta}Z^\alpha Z^\beta \\ -\tau\eta_{\alpha\beta}Z^\beta \end{pmatrix}, \qquad e^{-\mathcal{K}} = 4\Im\mathfrak{m}\,\tau\;\eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\;\Im\mathfrak{m}\,Z^\beta,$$
$$\mathcal{Q}_\tau = \frac{1}{4\Im\mathfrak{m}\,\tau}, \qquad \mathcal{Q}_\alpha = \frac{\eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\beta}{2\eta_{\gamma\delta}\Im\mathfrak{m}\,Z^\gamma\;\Im\mathfrak{m}\,Z^\delta}, \tag{3.50}$$
$$\mathcal{G}_{\tau\tau^*} = \frac{1}{4(\Im\mathfrak{m}\,\tau)^2}, \qquad \mathcal{G}_{\alpha\beta^*} = \frac{\eta_{\alpha\gamma}\Im\mathfrak{m}\,Z^\gamma\;\eta_{\beta\delta}\Im\mathfrak{m}\,Z^\delta}{[\eta_{\epsilon\varphi}\Im\mathfrak{m}\,Z^\epsilon\;\Im\mathfrak{m}\,Z^\varphi]^2} - \frac{\eta_{\alpha\beta}}{2\eta_{\epsilon\varphi}\Im\mathfrak{m}\,Z^\epsilon\;\Im\mathfrak{m}\,Z^\varphi}.$$

The reality of the Kähler potential constrains the values of the scalars. The model has two branches characterized by

$$\Im\mathfrak{m}\,\tau > 0\,, \qquad \eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\;\Im\mathfrak{m}\,Z^\beta > 0\,, \tag{3.51}$$

and

$$\Im\mathfrak{m}\,\tau\; < 0\,, \qquad \eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\;\Im\mathfrak{m}\,Z^\beta < 0\,, \tag{3.52}$$

that will be distinguished where required by $+$ and $-$ indices, respectively.

Only the subgroup $\mathrm{SO}(1,n) \subset \mathrm{SO}(2,n)$ acts linearly (in the fundamental representation) on the special coordinates Z^α and the group SO(3) acts in the adjoint (for instance) on the coordinates $\alpha = 3,4,5$ if $n \geq 4$. We take $n = 4$ for simplicity and denote the $\alpha = 3,4,5$ indices by $a, b, \cdots = 1,2,3$. For the sake of simplicity we will write Z^a instead of Z^{a+2} for Z^3, Z^4, Z^5 etc. The generators and structure constants of $\mathfrak{so}(3)$ and their action on the scalars are the same as in the $\overline{\mathbb{CP}}^3$ model with obvious changes of notation:

$$(T_a)^b{}_c = f_{ac}{}^b = -\varepsilon_{acb}\,, \qquad \delta_\alpha Z^a = \alpha^b (T_b)^a{}_c Z^c = -\epsilon_{abc}\alpha^b Z^c = \alpha^b k_b{}^a(Z)\,, \tag{3.53}$$

(τ and Z^2 are inert) so the holomorphic Killing vectors and the momentum maps are

$$k_a{}^b(Z) = \epsilon_{abc}Z^c\,, \qquad \mathcal{P}_a = -\frac{i}{2}\frac{\epsilon_{abc}Z^b Z^{*\,c^*}}{\eta_{\alpha\beta}\Im\mathfrak{m}\,Z^\alpha\;\Im\mathfrak{m}\,Z^\beta}\,. \tag{3.54}$$

The scalar potential has a structure similar to that of the $\overline{\mathbb{CP}}^3$ model, but more complicated. We will not give it here since it is not needed anyway.

3.3.2 The solutions

To find solutions to this non-Abelian model we just need to follow the recipe. First, we find the functions $\mathcal{I}^\Lambda$ and the spatial components of the vector fields $A^\Lambda{}_{\underline{m}}$ by solving the Bogomol'nyi equations

$$F^\Lambda{}_{\underline{mn}} = -\frac{1}{\sqrt{2}}\epsilon_{mnp}\partial_{\underline{p}}\mathcal{I}^\Lambda\,, \qquad I = 0,1,2, \tag{3.55}$$

$$F^{a+2}{}_{\underline{mn}} = -\frac{1}{\sqrt{2}}\epsilon_{mnp}\mathfrak{D}_{\underline{p}}\mathcal{I}^{a+2}\,, \qquad a = 1,2,3, \tag{3.56}$$

(we will suppress the $+2$ in the non-Abelian indices in most places). The Abelian equations are solved by harmonic functions and the non-Abelian ones by making the identification eq. (3.8) with the Higgs field and using Protogenov's solutions eqs. (3.11), as we did in the $\overline{\mathbb{CP}}^3$ model.

Next, we have to find the functions $\mathcal{I}_\Lambda$ by solving eqs. (2.19). In the Abelian directions $\Lambda = 0, 1, 2$ we can simply choose harmonic functions and in the non-Abelian ones we take $\mathcal{I}_a = 0$. This choice gives non-singular solutions, as we are going to see. We will also set some of the harmonic functions to zero for simplicity.

The Hesse potential defined in eq. (2.21) can be found from Shmakova's solution of the stabilization (or Freudenthal duality) equations for cubic models [81]; it can be written as

$$\mathsf{W}(\mathcal{I}) = 2\sqrt{J_4(\mathcal{I})}\,, \tag{3.57}$$

with the quartic invariant $J_4(\mathcal{I})$ given by

$$J_4(\mathcal{I}) \equiv (\mathcal{I}^\alpha \mathcal{I}^\beta \eta_{\alpha\beta} + 2\mathcal{I}^0\mathcal{I}_1)(\mathcal{I}_\alpha \mathcal{I}_\beta \eta^{\alpha\beta} - 2\mathcal{I}^1\mathcal{I}_0) - (\mathcal{I}^0\mathcal{I}_0 - \mathcal{I}^1\mathcal{I}_1 + \mathcal{I}^\alpha \mathcal{I}_\alpha)^2\,. \tag{3.58}$$

This potential does not vanish for the choice $\mathcal{I}_a = 0$, as we advanced and it will remain non-singular if we set $\mathcal{I}^0 = \mathcal{I}_1 = \mathcal{I}_2 = 0$. In other words: the only non-trivial components of $\mathcal{I}^M$ are $\mathcal{I}^1, \mathcal{I}^2, \mathcal{I}^{a+2}, \mathcal{I}_0$. With this choice the metric function is given by

$$e^{-2U} = \mathsf{W}(\mathcal{I}) = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0\,\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta} = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0[(\mathcal{I}^2)^2 - \mathcal{I}^a\mathcal{I}^a]}\,. \tag{3.59}$$

As instructed by the recipe in section 2.2, we can calculate the $\tilde{\mathcal{I}}$ from eq. (2.21), which for our choice of non-trivial components of $\mathcal{I}^M$ means that $\tilde{\mathcal{I}}^i = 0$ $(i = 1, \cdots, 5)$; this implies that all the scalars are purely imaginary and given by

$$Z^i = i\frac{\mathcal{I}^i}{\tilde{\mathcal{I}}^0}\,, \qquad \text{where} \qquad \tilde{\mathcal{I}}^0 = \frac{2\mathcal{I}^1\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta}{\mathsf{W}(\mathcal{I})}\,. \tag{3.60}$$

It is convenient to write all of them in terms of $\tau = Z^1$

$$Z^\alpha = \frac{\mathcal{I}^\alpha}{\mathcal{I}^1}\tau\,, \qquad \tau = i\frac{e^{-2U}}{2\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta}\,. \tag{3.61}$$

In the two ($+$ and $-$) branches of the model corresponding, respectively, to the upper and lower signs $\pm\Im\mathfrak{m}\,\tau_{(\pm)} > 0$ and, since $e^{-2U} > 0$, we must choose the functions $\mathcal{I}^\alpha_{(\pm)}$ so that

$$\pm\,\eta_{\alpha\beta}\mathcal{I}^\alpha_{(\pm)}\mathcal{I}^\beta_{(\pm)} = \pm\left[(\mathcal{I}^2_{(\pm)})^2 - \mathcal{I}^a_{(\pm)}\mathcal{I}^a_{(\pm)}\right] > 0\,. \tag{3.62}$$

In order for $\mathsf{W}(\mathcal{I})$ to be real the $\mathcal{I}_{(\pm)\,0}$ and $\mathcal{I}^1_{(\pm)}$ must be chosen so as to satisfy

$$\pm\,\mathcal{I}^1_{(\pm)}\mathcal{I}_{(\pm)\,0} < 0\,. \tag{3.63}$$

(We will suppress the $\pm$ subindices in what follows, to simplify the notation, except where this may lead to confusion.)

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Observe that with our choice of non-vanishing components of $\mathcal{I}^M$ the r.h.s. of eq. (2.20) vanishes automatically, whence the staticity condition $\omega = 0$ does not impose any constraint.

According to the preceding discussions, the non-vanishing components of $\mathcal{I}^M$ will be assumed to take the form

$$\begin{aligned} &\mathcal{I}^1 = A^1 + \frac{p^1/\sqrt{2}}{r}, \qquad \mathcal{I}^2 = A^2 + \frac{p^2/\sqrt{2}}{r}, \qquad \mathcal{I}^a = \sqrt{2}\,\delta^a{}_m x^m f(r), \\ &\mathcal{I}_0 = A_0 + \frac{q_0/\sqrt{2}}{r}, \end{aligned} \tag{3.64}$$

where $f(r)$ is $f_{\mu,s}$ or f_λ in eqs. (3.11), p^1, p^2, q_0 are magnetic and electric charges and A^1, A^2, A_0 are integration constants to be determined in terms of the asymptotic values of the scalars and the metric. These constants must have the same sign as the corresponding charges

$$\mathrm{sign}(A^{1,2}) = \mathrm{sign}(p^{1,2}), \qquad \mathrm{sign}(A_0) = \mathrm{sign}(q_0), \tag{3.65}$$

as the functions $\mathcal{I}^1$, $\mathcal{I}^2$ and $\mathcal{I}_0$ are required to have no zeroes on the interval $r \in (0, +\infty)$ in order to avoid naked singularities there. Then, the above constraint on the signs of $\mathcal{I}^1$ and $\mathcal{I}_0$ translates into the following constraints on the signs of the charges in the two branches:

$$\mathrm{sign}(p^1)\mathrm{sign}(q_0) = \mp 1. \tag{3.66}$$

Defining as in the $\overline{\mathbb{CP}}^3$ case the asymptotic value Z_∞ of the adjoint scalars by

$$Z^a_\infty \equiv Z_\infty\, \delta^a{}_m \frac{x^m}{r}, \tag{3.67}$$

and imposing the normalization of the metric at infinity it is not hard to express the integration constants μ, A^1, A^2, A_0 in terms of the moduli (the asymptotic values of the scalars $\Im\mathfrak{m}\tau_\infty, \Im\mathfrak{m}\, Z^2_\infty$ and $\Im\mathfrak{m}\, Z_\infty$) and the coupling constant g

$$\begin{aligned} A^1 &= \frac{\mathrm{sign}(p^1)|\Im\mathfrak{m}\tau_\infty|}{\sqrt{2}\chi_\infty}, \\ A^2 &= \frac{\mathrm{sign}(p^2)|\Im\mathfrak{m}\, Z^2_\infty|}{\sqrt{2}\chi_\infty}, \\ \mu &= \frac{g|\Im\mathfrak{m}\, Z_\infty|}{2\chi_\infty}, \\ A_0 &= \tfrac{1}{2\sqrt{2}}\mathrm{sign}(q_0)\chi_\infty, \end{aligned} \tag{3.68}$$

where we have defined the combination (real in both branches of the theory)

$$\chi_\infty \equiv \sqrt{\Im\mathfrak{m}\tau_\infty\,[(\Im\mathfrak{m}\, Z^2_\infty)^2 - (\Im\mathfrak{m}\, Z_\infty)^2]}. \tag{3.69}$$

The mass of the solutions in terms of the moduli and the charges is

$$M = \tfrac{1}{4}\frac{\chi_\infty}{|\Im\mathfrak{m}\tau_\infty|}|p^1| + \frac{1}{2\chi_\infty}|q_0| \pm \tfrac{1}{2}\frac{|\Im\mathfrak{m}\tau_\infty \Im\mathfrak{m}\, Z^2_\infty|}{\chi_\infty}|p^2| \pm \frac{|\Im\mathfrak{m}\tau_\infty \Im\mathfrak{m}\, Z_\infty|}{\chi_\infty}\frac{1}{g}. \tag{3.70}$$

In the above expressions we have used two consistency conditions:

$$\mathrm{sign}(\Im\mathfrak{m}\, Z_\infty) = \mp\mathrm{sign}(p^1), \qquad \mathrm{sign}(\Im\mathfrak{m}\, Z^2_\infty) = \pm\mathrm{sign}(p^1)\mathrm{sign}(p^2). \tag{3.71}$$

These expressions for the integration constants and the mass are valid both for the 2- and 1-parameter families, the latter being recovered by setting $\Im\mathfrak{m}\, Z_\infty = 0$ everywhere. The contribution of the monopole charge $1/g$ to the mass disappears because it is screened.

Observe that the positivity of the mass is not guaranteed in the $-$ branch for arbitrary values of the charges and moduli: it has to be imposed by hand.

Let us now study the behavior of the solution in the near-horizon limit $r \to 0$. For $f_{\mu,s\neq 0}$ and f_λ the metric function behaves as

$$e^{-2U} \sim \sqrt{-2p^1 q_0 \left[(p^2)^2 - (2/g)^2\right]}\, \frac{1}{r^2}\,, \tag{3.72}$$

which corresponds to a regular horizon in both branches. The solutions will describe regular black holes if the charges and moduli are such that $M > 0$. Observe that in the $-$ branch it is possible to chose those such that $M = 0$ with a non-vanishing entropy.

In the $f_{\mu,s=0}$ case with $p^2 \neq 0$ the solution is only well defined in the $+$ branch because there is no $1/r$ contribution from the monopole in the $r \to 0$ limit and it is impossible to satisfy the inequality $-\eta_{\alpha\beta}\mathcal{I}^\alpha\mathcal{I}^\beta > 0$ in that limit. In this case (the $+$ branch with $p^2 \neq 0$) we have

$$e^{-2U} \sim \sqrt{-2p^1 q_0 (p^2)^2}\, \frac{1}{r^2}\,, \tag{3.73}$$

which corresponds to a regular horizon.

In the $f_{\mu,s=0}$ case with $p^2 = 0$ there are two possibilities:

1. We can set $p^1 = q_0 = 0$. Then, in the $r \to 0$ limit, e^{-2U} is the moduli-dependent constant $2\sqrt{-2A^1 A_0 (A^2)^2}$. There is neither horizon nor singularity and the solution, which is a global monopole, belongs to the $+$ branch (this also guarantees that the mass is positive).

2. We can keep both $p^1 \neq 0$ and $q_0 \neq 0$, setting $A^2 = 0$ and profit from the fact that, in this limit $\Phi^a\Phi^a$ goes to zero as r^2. The solution is only well defined in the $-$ branch. The metric function takes the constant value

$$e^{-2U} \sim \sqrt{+p^1 q_0 \frac{\mu^4}{g^2}}\,, \tag{3.74}$$

We have, as far as the metric is concerned, a global monopole solution (as long as $M > 0$), but since we need two Abelian charges switched on, namely p^1 and q_0, the scalar fields and the gauge fields are singular at $r = 0$. As before, it is possible to tune the moduli and charges so that $M = 0$.

The near-horizon limits of the scalars are, in the $f_{\mu,s\neq 0}$ and f_λ cases

$$\begin{aligned} \Im\mathfrak{m}\tau_{\rm h} &= \frac{\sqrt{-2p^1 q_0 \left[(p^2)^2 - (2/g)^2\right]}}{2\left[(p^2)^2 - (2/g)^2\right]}\,, \\ \Im\mathfrak{m}\, Z^2_{\rm h} &= \frac{p^2}{p^1}\Im\mathfrak{m}\tau_{\rm h}\,, \\ \Im\mathfrak{m}\, Z^a_{\rm h} &= \frac{2\Im\mathfrak{m}\tau_{\rm h}}{gp^1}\delta^a{}_m \frac{x^m}{r}\,, \end{aligned} \tag{3.75}$$

and, in the $f_{\mu,s=0}$ case with $p^2 \neq 0$, we get the same results up to the contribution of the monopole which disappears (formally, $1/g = 0$).

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3.4 Embedding in pure SU(2) EYM

The scalars can only be trivialized for the Wu-Yang monopole $s = \infty$. In that case, it is easy to construct a double-extremal black hole with constant scalars and the metric is, as usual, Reissner-Nordström's.

4 Multi-center SBHSs

To construct multi-center SBHSs we can use the same recipe as in the single-center case but we need multi-center solutions of the Bogomol'nyi equations. We start by discussing these.

4.1 Multi-center solutions of the SU(2) Bogomol'nyi equations on $\mathbb{R}^3$

In the Abelian case, the multicenter solutions of the Bogomol'nyi equations are associated to harmonic functions with isolated point-like singularities. They are the seed solutions of the multi-black-hole solutions of the Einstein-Maxwell theory [7–11, 13] and $\mathcal{N} = 2$, $d = 4$ supergravities [12, 14, 71, 82]. In the non-Abelian case, the hedgehog ansatz is clearly inappropriate and more sophisticated methods need to be used. Only a few explicit solutions are known, even though solutions describing several BPS objects in equilibrium are, on general grounds, expected to exist. For instance, there is no explicit solution describing two BPS 't Hooft-Polyakov monopoles in equilibrium (see however ref. [83]).

Perhaps not surprisingly, the only general families of explicit solutions available involve an arbitrary number of Wu-Yang or Dirac monopoles embedded in SU(2). The simplest of these only involve Wu-Yang monopoles and formally, it can be obtained from solutions describing Dirac monopoles embedded in SU(2) via singular gauge transformations [84], generalizing the constructions reviewed in appendices B (minimal charge) and D (higher charge). As we have explained at length in the preceding sections, the metric is completely oblivious to these gauge transformations and takes the same form as in the Abelian cases. We will not study such solutions in this section.

In refs. [24, 25], using the Nahm equations [85], Cherkis and Durcan found new solutions describing one or two, charge 1, Wu-Yang monopoles embedded in SU(2) in the background of a single BPS 't Hooft-Polyakov monopole.[18] We are going to use the first of them to construct multi-center solutions of the $\overline{\mathbb{CP}}^3$ and $ST[2,4]$ models of $\mathcal{N} = 2$, $d = 4$ SEYM. Let us review the Cherkis-Durcan solution first: take the BPS 't Hooft-Polyakov monopole to be located at $x^n = x_0^n$ and the Wu-Yang monopole at $x^m = x_1^m$. We define the coordinates relative to each of those centers and the relative position by

$$r^m \equiv x^m - x_0^m\,, \qquad u^m \equiv x^m - x_1^m\,, \qquad d^m \equiv u^m - r^m = x_0^m - x_1^m\,, \tag{4.1}$$

[18] In ref. [26, 27] Blair and Cherkis generated a solution describing an arbitrary number of charge 1 Wu-Yang monopoles in the presence of an 't Hooft-Polyakov monopole; one can easily generalize this solution to one describing an arbitrary number of charge $n(> 0)$ Wu-Yang monopoles in the background of an 't Hooft-Polyakov monopole, by coalescing n charge 1 Wu-Yang monopoles. Needless to say, the Protogenov trick works as expected. For the sake of simplicity of exposition, we will not consider this more general solution in this article.

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and their norms by respectively, r, u and d. The Higgs field and gauge potential of this solution (adapted to our conventions) are given by [24, 25]

$$\pm\Phi^a = \frac{1}{g}\delta^a{}_m \left\{ \left[\frac{1}{r} - \left(\mu + \frac{1}{u}\right)\frac{K}{L}\right]\frac{r^m}{r} + \frac{2r}{uL}\left(\delta^{mn} - \frac{r^m r^n}{r^2}\right) d^n \right\} , \tag{4.2}$$

$$\begin{aligned} A^a &= -\frac{1}{g}\left[\frac{1}{r} - \frac{\mu \mathrm{D} + 2d + 2u}{\mathrm{L}}\right]\frac{\varepsilon^a{}_{mn} r^m dx^n}{r} + 2\frac{\mathrm{K}}{\mathrm{L}}\frac{\varepsilon_{npq} d^n u^p dx^q}{u\mathrm{D}}\,\delta^a{}_m \frac{r^m}{r} \\ &\quad - \frac{2r}{u\mathrm{L}}\,\delta^a{}_m \left(\delta^{mn} - \frac{r^m r^n}{r^2}\right)\varepsilon_{npq} u^p dx^q , \end{aligned} \tag{4.3}$$

where the functions K, L, D of u and r are defined by

$$K \equiv \left[(u+d)^2 + r^2\right]\cosh \mu r + 2r(u+d)\sinh \mu r , \tag{4.4}$$

$$L \equiv \left[(u+d)^2 + r^2\right]\sinh \mu r + 2r(u+d)\cosh \mu r , \tag{4.5}$$

$$\mathrm{D} = 2\,(ud + u^m d^m) = (d+u)^2 - r^2 . \tag{4.6}$$

The function D is clearly zero along the direction[19] $u^m/u = -d^m/d$ signaling the possible presence of a Dirac string in eq. (4.3); that this is however not the case is demonstrated in ref. [26, 27].

In the models that we are going to study, the Higgs field enters the metric in the combination $\Phi^a\Phi^a$, which takes the value

$$\Phi^a\Phi^a = \frac{1}{g^2}\left\{ \left[\frac{1}{r} - \left(\mu + \frac{1}{u}\right)\frac{K}{L}\right]^2 + \frac{4|\vec{r}\times\vec{d}|^2}{u^2L^2} \right\} . \tag{4.7}$$

To better understand this solution one will consider several limits:

1. The limit in which we take the BPS 't Hooft-Polyakov anti-monopole infinitely far away, keeping the Dirac monopole at x_1^m: in this limit $d \to \infty$, $r^m \sim -d^m$ while u remains finite. The Higgs and gauge fields take the form

$$\pm\Phi^a \sim -\frac{1}{g}\delta^a{}_m\left(\mu + \frac{1}{u}\right)\frac{d^m}{d} , \tag{4.8}$$

$$A^a \sim -\frac{1}{g}\left(1 + \frac{d^m}{d}\frac{u^m}{u}\right)^{-1}\varepsilon_{mnp}\frac{d^m}{d}\frac{u^m}{u}d\frac{u^p}{u} . \tag{4.9}$$

   The gauge field should be compared with the embedding of a Dirac monopole with a string in the direction $-d^m$ into the direction $\delta^a{}_m d^m T^a$ of the gauge group, eqs. (B.6) and (B.12) with $s^m = -d^m$.

2. The limit in which we take the Dirac monopole infinitely away, keeping the BPS 't Hooft-Polyakov anti-monopole at x_0^m: in this limit $d \to \infty$, $u^m \sim d^m$ while r remains finite. The Higgs and gauge fields become those of a single BPS 't Hooft-Polyakov anti-monopole at x_0^m.

[19] This is the half of the line that joins $r = 0$ to $u = 0$ that stretches from the Dirac monopole $u = 0$ to infinity in the direction opposite to the 't Hooft-Polyakov monopole at $r = 0$.

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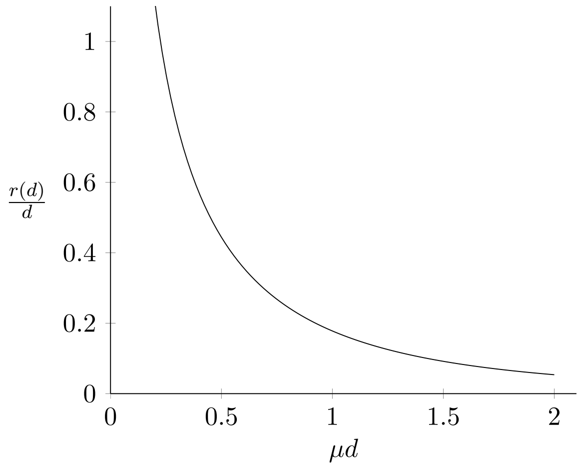

Figure 1. The zeros of the Higgs density as measured by r as a function of the dimensionless separation μd.

3. In the limit in which we are infinitely far away from both monopoles ($r \to \infty$, $u \to \infty$), which remain at a finite relative distance, the Higgs and gauge fields take the form

$$\pm\Phi^a = -\left[\frac{\mu}{g} + \mathcal{O}(|x|^{-2})\right]\delta^a{}_m \frac{x^m}{|x|}\,, \tag{4.10}$$

$$A^a = -\frac{1}{g}\varepsilon^a{}_{mn}\frac{x^m dx^n}{|x|^2} + \frac{1}{2g}\delta^a{}_m \frac{x^m}{|x|}\left(\frac{\varepsilon_{npq}d^n x^p dx^q}{|x|^2}\right)\,. \tag{4.11}$$

The first term in the gauge potential is identical to that of a Wu-Yang anti-monopole (compare with eq. (A.2)). This is also the asymptotic behavior of the BPS 't Hooft-Polyakov monopole. The Higgs field is asymptotically covariantly constant and, in particular

$$\Phi^a\Phi^a \sim \frac{\mu^2}{g^2} + \mathcal{O}(\frac{1}{|x|^2})\,. \tag{4.12}$$

4. The limit in which we approach the center of the BPS 't Hooft-Polyakov anti-monopole $r^m \to 0$, $u^m \to d^m$

$$\Phi^a\Phi^a \sim \frac{1}{4g^2d^2(1+\mu d)^2} + \mathcal{O}(r)\,. \tag{4.13}$$

This limit is finite and only vanishes when the Dirac monopole is taken to infinity $d \to \infty$.

For finite values of d, eq. (4.7) says that $\Phi^a\Phi^a$ can only vanish along the line that stretches from $r = 0$ to $u = 0$ so $\vec{r} \times \vec{d} = 0$. Substituting $r^m = \alpha d^m$ in $\Phi^a\Phi^a$ we get a function of α and of the parameter μd. (See figure 1.) Plotting the functions of α for different values of μd we find that they have a single zero, which is also a local minimum. At this minimum the second derivative does not vanish, and therefore, there, $\Phi^a\Phi^a \sim \mathcal{O}(r^2)$, as in the single-monopole case. However, the value of this second derivative depends on the direction.

5. The limit in which we approach the singularity of the Dirac monopole $u^m \to 0$, $r^m \to -d^m$

$$\Phi^a\Phi^a \to \frac{1}{g^2}\left\{\frac{1}{u^2}+\left(\frac{1}{d}-\mu\right)\frac{1}{u}\right\}+\mathcal{O}(1)\,. \tag{4.14}$$

4.1.1 Growing Protogenov hair

As we have argued in section 3.1.3 we can add a Protogenov hair parameter s to the Cherkis & Durcan solution by simply replacing the argument μr of the hyperbolic sines and cosines in the functions K and L by the shifted on $\mu r + s$. We do not need to write explicitly the solution, but we do need to reconsider the different limits studied for the $s = 0$ case:

1. In the limit in which we take the BPS 't Hooft-Polyakov-Protogenov anti-monopole infinitely away, keeping the Dirac monopole at x_1^m the Higgs and gauge fields become, to leading order, those of the Dirac monopole with the Dirac string in the direction $-d^m$, as in the $s = 0$ case (see eqs. (4.8) and (4.3)).

2. In the limit in which we take the Dirac monopole infinitely away, keeping the BPS 't Hooft-Polyakov-Protogenov anti-monopole at x_0^m the Higgs and gauge fields become those of a single BPS 't Hooft-Polyakov-Protogenov anti-monopole at $x^m = x_0^m$ (the first two equations (3.11)).

3. In the limit in which we are infinitely far away from both monopoles ($r \to \infty$, $u \to \infty$), which remain at a finite relative distance, the Higgs and gauge fields take the same form as in the $s = 0$ case, eqs. (4.10)–(4.12).

4. The limit in which we approach the singularity of the BPS 't Hooft-Polyakov-Protogenov anti-monopole $r^m \to 0$, $u^m \to d^m$ (for $s \neq 0$)

$$\pm\Phi^a \sim \frac{1}{g}\delta^a{}_m\left[\frac{1}{r}-\left(\mu+\frac{1}{d}\right)\coth s+\mathcal{O}(r)\right]\frac{r^m}{r}\,, \tag{4.15}$$

$$\Rightarrow \Phi^a\Phi^a \sim \frac{1}{g^2r^2}+\mathcal{O}\left(\frac{1}{r}\right)\,, \tag{4.16}$$

   which is similar to the behaviour near the Dirac monopole as in eq. (4.14) (with u replaced by r).

5. The limit in which we approach the singularity of the Dirac monopole $u^m \to 0$, $r^m \to -d^m$ we have the same behavior as in the $s = 0$ case eq. (4.14).

The solutions with Protogenov hair have another limit, namely the one in which $s \to \infty$; this case will be studied separately.

4.1.2 The $s \to \infty$ limit solution

In this limit we get a solution that describes the same Dirac monopole together with a ($\mu \neq 0$) Wu-Yang anti-monopole:[20]

$$\pm\Phi^a = \frac{1}{g}\delta^a{}_m\left[-\mu + \frac{1}{r} - \frac{1}{u}\right]\frac{r^m}{r}, \tag{4.17}$$

$$A^a = \frac{1}{g}\frac{\varepsilon^a{}_{mn}r^m dx^n}{r^2} + \frac{1}{g}\frac{\varepsilon_{npq}d^n u^p du^q}{u(ud+u^r d^r)}\delta^a{}_m\frac{r^m}{r}. \tag{4.18}$$

This solution is a particular example of a more general family describing an arbitrary number of Dirac monopoles in the background of a Wu-Yang anti-monopole. These solutions can be obtained from a solution describing only Dirac monopoles embedded in SU(2) via a singular gauge transformation that only removes the Dirac string of one of them, which becomes the Wu-Yang anti-monopole. The general family of solutions can be written in the form:

$$\Phi = \Phi_{\rm WY} + HU, \qquad A = A_{\rm WY} + CU, \tag{4.19}$$

where U is the SU(2) (and $\mathfrak{su}(2)$) matrix defined in eq. (A.1) and where $\Phi_{\rm WY}$ and $A_{\rm WY}$ are the Higgs and Yang-Mills fields of a Wu-Yang monopole, given, respectively, by

$$\mp\Phi_{\rm WY} = \frac{1}{2g}\left[-\mu + \frac{1}{r}\right]U, \tag{4.20}$$

and by eq. (A.2) and where H is a function and C a 1-form on $\mathbb{R}^3$. If we substitute into the Bogomol'nyi equations (3.5) and use, on the one hand, that they are satisfied by the pair $A_{\rm WY}, \Phi_{\rm WY}$, and, on the other hand, that U is covariantly constant with the connection $A_{\rm WY}$ we arrive at the Dirac monopole equation

$$dC = \star_{(3)} dH. \tag{4.21}$$

The integrability condition of this equation is $d\star_{(3)} dH = 0$ so H is any harmonic function. We can choose it to have isolated poles at the points $x^m = x^m_i$ $i = 1, \cdots, N$

$$H = \sum_i \frac{p_i}{2u_i}, \qquad u^m_i \equiv x^m - x^m_i, \tag{4.22}$$

in which case C is the 1-form potential of N Dirac monopoles with charges p_i which can be constructed by summing over the potentials of each individual monopole:

$$C = \sum C_i, \qquad dC_i = \star_{(3)} d\frac{p_i}{2u_i}. \tag{4.23}$$

The expression for each of the C_i is of the form eq. (B.6) where we can, in principle, choose the direction s^m_i of each Dirac string independently:

$$C_i = \frac{p_i}{2}\left(1 - \frac{s^m_i}{s_i}\frac{u^m_i}{u_i}\right)^{-1}\varepsilon_{mnp}\frac{s^m_i}{s_i}\frac{u^m_i}{u_i}d\frac{u^p_i}{u_i}, \qquad \text{(no sum over } i\text{)}. \tag{4.24}$$

[20] One can see fairly easily that in the limiting solution one can, as far as the Bogomol'nyi equations are concerned, allow for μ to be negative; for finite values of s this is impossible.

This solution of the Yang-Mills-Higgs system shares two important properties with the original Wu-Yang monopole and which are related to the fact that they are related to Abelian embeddings by singular gauge transformations:

1. Both Φ and $D\Phi$ are proportional to U:
$$\Phi = \left(-\frac{\mu}{2g} + \frac{1}{2gr} + H\right) U\,, \qquad D\Phi = d\left(-\frac{\mu}{2g} + \frac{1}{2gr} + H\right) U\,, \tag{4.25}$$
and, therefore, commute with each other, so the Higgs current vanishes and the gauge field is, by itself, a solution of the pure Yang-Mills theory.

2. The gauge field strength is also proportional to U, the coefficient being the field strength of an Abelian gauge field:
$$F(A) = d(B+C)U\,, \tag{4.26}$$
which implies that the energy-momentum tensors are related as in the single-center case.

These solutions can be generalized even further, by allowing the the charge of the "original" Wu-Yang monopole at $r = 0$ to be n/g (that is: using the generalization of the Wu-Yang monopole due to Bais [86] which is studied in appendix D). If we now substitute into the Bogomol'nyi equations (3.5) the ansatz
$$\Phi = \Phi_{(n)} + HU_{(n)}\,, \qquad A = A_{(n)} + CU_{(n)}\,, \tag{4.27}$$
where $U_{(n)}, A_{(n)}$ and $\Phi_{(n)}$ are given, respectively, in eqs. (D.5), (D.6) and (D.11), H is a function and C a 1-form on $\mathbb{R}^3$, and use that they are satisfied by the pair $A_{(n)}, \Phi_{(n)}$ and that $U_{(n)}$ is covariantly constant with the connection $A_{(n)}$, we arrive again at the Dirac monopole equation (4.21).

Since all these solutions are related to Abelian embeddings, they contribute to the black-hole solutions as the Abelian solutions. We will not consider them in what follows.

4.2 Embedding in the SU(2)-gauged $\overline{\mathbb{CP}}^3$ model

We can use the Cherkis & Durcan solution of the SU(2) Bogomol'nyi equations reviewed in the previous section as a seed solution for a multicenter solution of $\mathcal{N} = 2$, $d = 4$ SEYM, adding the same harmonic functions as in the single-center case $(\mathcal{I}^0, \mathcal{I}_0)$ or a generalization with poles at the locations of the monopoles $r = 0$,[21] and $u = 0$. More explicitly, we take
$$\begin{aligned}
\mathcal{I}^0 &= A^0 + \frac{p^0_r/\sqrt{2}}{r} + \frac{p^0_u/\sqrt{2}}{u}\,,\\
\mathcal{I}_0 &= A_0 + \frac{q_{r,0}/\sqrt{2}}{r} + \frac{q_{u,0}/\sqrt{2}}{u}\,,\\
\mathcal{I}^i &= \mp\sqrt{2}\Phi^i(r,u)\,,\\
\mathcal{I}_i &= 0\,,
\end{aligned} \tag{4.28}$$

[21] The location of the BPS 't Hooft-Polyakov anti-monopole is not completely clear: it is sometimes argued that the center of the monopole is the point at which the Higgs vanishes and the full gauge symmetry is restored. As we have discussed, that point is not $r = 0$. We could try to place the poles of the harmonic functions at that point, but, given that its location is not known analytically and the expansion of $\Phi^a\Phi^a$ around it is difficult to compute, we will not try to do that here.

where $\Phi^i(r,u)$ is the Higgs field of the Cherkis & Durcan solution. The metric and scalar fields take the form

$$e^{-2U} = \frac{1}{2}(\mathcal{I}^0)^2 + 2(\mathcal{I}_0)^2 - \Phi^i\Phi^i\,, \tag{4.29}$$

$$Z^i = \frac{\mp\sqrt{2}\Phi^i}{\mathcal{I}^0 + 2i\mathcal{I}_0}\,. \tag{4.30}$$

The normalization of the metric and scalars at infinity leads to the same relations between the integration constants A^0, A_0, μ and the physical constants Z_∞, g as in the single-center case, namely

$$\frac{1}{\sqrt{2}}A^0 + \sqrt{2}iA_0 = \frac{Z^*_\infty}{|Z_\infty|}\frac{1}{\sqrt{1-|Z_\infty|^2}}\,, \qquad \mu = \frac{|Z_\infty|}{\sqrt{1-|Z_\infty|^2}}g\,. \tag{4.31}$$

The integrability conditions of eq. (2.20) are, in this case,

$$\mathcal{I}_0\partial_{\underline{m}}\partial_{\underline{m}}\mathcal{I}^0 - \mathcal{I}^0\partial_{\underline{m}}\partial_{\underline{m}}\mathcal{I}_0 = 0\,, \tag{4.32}$$

and lead to the following relations between the integration constants:

$$A^0(q_{r,0}+q_{u,0}) - A_0(p^0_r + p^0_u) = 0\,, \tag{4.33}$$

$$J - \frac{1}{\sqrt{2}}d(A^0 q_{u,0} - A_0 p^0_u) = 0\,, \tag{4.34}$$

where we have defined the constant

$$J \equiv p^0_r q_{u,0} - q_{r,0}p^0_u\,. \tag{4.35}$$

The first equation is equivalent to eq. (3.32) for the total charges and the second equation determines the relative distance d in terms of J and $A^0 q_{u,0} - A_0 p^0_u$ provided that $J \neq 0$. When that is the case, the solution is not static and has an angular momentum J directed along the line that joins the monopoles $J^m = Jd^m/d$. The corresponding 1-form ω can be constructed by the standard procedure of the Abelian case. However, since this complicates the analysis of the regularity of the solutions, we will stick to the static case and require $J=0$.

In order to have regular solutions, the charges at each center must be chosen as in the corresponding single-center case: since there is an Abelian monopole at $u=0$, we must switch on either p^0_u or $q_{u,0}$ to have a regular horizon there. We can treat them both as non-vanishing with no loss of generality. Then, there are two possibilities:

I. $p^0_r = q_{r,0} = 0$**:** Only for $s=0$ ('t Hooft-Polyakov anti-monopole at $r=0$) has the solution a chance of being regular at $r=0$. Solving eq. (4.33) the solution can be written in the form

$$e^{-2U} = \frac{1}{1-|Z_\infty|^2}H^2 - \Phi^i\Phi^i\,, \tag{4.36}$$

$$Z^i = \frac{2\beta}{p^0 + 2iq_0}\frac{\Phi^i}{H}\,, \tag{4.37}$$

where H is the harmonic function

$$H \equiv 1 + \frac{\beta}{u}, \quad \beta^2 = (1-|Z_\infty|^2) W_{\rm RN}(\mathcal{Q}_u)/2\,, \quad W_{\rm RN}(\mathcal{Q}_u) \equiv \frac{1}{2}(p_u^0)^2 + 2(q_{u,0})^2\,. \tag{4.38}$$

The free parameters of this solution are the charges p_u^0, $q_{u,0}$ and the single modulus $|Z_\infty|$.

Studying the $u \to 0$ limit we find a black hole with entropy

$$S_u/\pi = \frac{1}{2} W_{\rm RN}(\mathcal{Q}_u) - \frac{1}{g^2}\,, \tag{4.39}$$

as in the corresponding single-center case.

In the $r \to 0$ limit e^{-2U} is constant. The positivity of the constant is guaranteed if S_u is positive. The total entropy of the solution is just the entropy of the black hole at $u = 0$ and the Dirac monopole does contribute to it.

The mass of the solution, expressed in terms of the independent parameters of the solution, p_u^0, $q_{u,0}$ and $|Z_\infty|$ takes the form

$$M = M_r + M_u\,, \tag{4.40}$$

$$M_r = -M_{\rm monopole}\,, \tag{4.41}$$

$$M_u = \sqrt{\frac{1}{2}\frac{W_{RN}(\mathcal{Q}_u)}{1-|Z_\infty|^2}} + M_{\rm monopole}\,, \tag{4.42}$$

where $M_{\rm monopole}$ is given by eq. (3.39). The contributions of the monopole and the 't Hooft-Polyakov monopole to the mass cancel each other.

II. p_r^0 or $q_{r,0} \neq 0$ We can treat both charges as non-vanishing with no loss of generality. Solving eqs. (4.33) and (4.35), we can write the solution as in eqs. (4.36) and (4.37) where, now,

$$\begin{aligned} H \equiv 1 + \frac{\beta_r}{r} + \frac{\beta_u}{u}\,, \qquad \beta_{r,u}^2 = (1-|Z_\infty|^2) W_{\rm RN}(\mathcal{Q}_{r,u})/2\,, \\ W_{\rm RN}(\mathcal{Q}_{r,u}) \equiv \tfrac{1}{2}(p_{r,u}^0)^2 + 2(q_{r,u,0})^2\,. \end{aligned} \tag{4.43}$$

The free parameters of this solution are the charges p_u^0, $q_{u,0}$ and $|Z_\infty|$ and either p_r^0 or $q_{r,0}$, since they must be proportional to those of the other center. The areas of each of the horizons are as in the single-center case. In particular, the BPS 't Hooft-Polyakov monopole ($s = 0$) does not contribute to the entropy of the $r = 0$ center. The mass is given by

$$M = M_r + M_u\,, \tag{4.44}$$

$$M_r = \sqrt{\frac{1}{2}\frac{W_{RN}(\mathcal{Q}_r)}{1-|Z_\infty|^2}} - M_{\rm monopole}\,, \tag{4.45}$$

$$M_u = \sqrt{\frac{1}{2}\frac{W_{RN}(\mathcal{Q}_u)}{1-|Z_\infty|^2}} + M_{\rm monopole}\,, \tag{4.46}$$

and the contributions of the monopole and anti-monopole cancel each other. In the $s \to \infty$ limit it can be easily seen that the solution is completely regular everywhere (e^{-2U} only vanishes at $r = 0$ and $u = 0$) if the Abelian charges are chosen so that the horizons are regular. This guarantees that all the terms in e^{-2U} are positive. For finite s this is more difficult to proof analytically, but, since the Higgs field has a better behavior than in the $s \to \infty$ case, it is reasonable to expect that it will also be true. We have checked numerically that this is so in several examples.

4.3 Embedding in the SU(2)-gauged ST[2, 4] model

The metric and scalar fields of the solution are now given by

$$e^{-2U} = 2\sqrt{-2\mathcal{I}^1\mathcal{I}_0[(\mathcal{I}^2)^2 - 2\Phi^a\Phi^a]}\,, \tag{4.47}$$

$$Z^1 \equiv \tau = i\frac{e^{-2U}}{2[(\mathcal{I}^2)^2 - 2\Phi^a\Phi^a]}\,, \qquad Z^2 = \frac{\mathcal{I}^2}{\mathcal{I}^1}\tau\,, \qquad Z^a = \frac{\sqrt{2}\Phi^a}{\mathcal{I}^1}\tau\,, \tag{4.48}$$

where Φ^a is the Higgs field of the Cherkis & Durcan solution (deformed with the Protogenov hair parameter s) and where the harmonic functions $\mathcal{I}^1$, $\mathcal{I}^2$ and $\mathcal{I}_0$ are allowed to have poles at $r = 0$ and $u = 0$:

$$\begin{aligned} \mathcal{I}^1 &= A^1 + \frac{p^1_r/\sqrt{2}}{r} + \frac{p^1_u/\sqrt{2}}{u}\,, \qquad \mathcal{I}^2 = A^2 + \frac{p^2_r/\sqrt{2}}{r} + \frac{p^2_u/\sqrt{2}}{u}\,, \\ \mathcal{I}_0 &= A_0 + \frac{q_{r,0}/\sqrt{2}}{r} + \frac{q_{u,0}/\sqrt{2}}{u}\,. \end{aligned} \tag{4.49}$$

As in the $\overline{\mathbb{CP}}^3$ case, the Abelian charges at each center must be chosen with the same criteria as in the corresponding single-center case. This means, in particular, that the Abelian charges at $u = 0$, $p^1_u, q_{u,0}$ must be non-vanishing. p^2_u may need to be activated, depending on the branch we are considering. At $r = 0$, for $s \neq 0$ we get exactly the same possibilities, but, for $s = 0$ there are two possibilities:

1. p^1_r, $q_{r,0}, p^2_r$ non-vanishing. We find a black hole at $r = 0$ in the + branch.
2. $p^1_r = q_{r,0} = p^2_r = 0$. e^{-2U} is a complicated d-dependent constant in the $r = 0$ limit and we get a global monopole.

Here we find an important difference with the single-center case, due to the fact that $\Phi^a\Phi^a$ is a finite constant in the $r \to 0$ limit instead of going to zero as r^2: there is no solution with $p^1_r q_{r,0} \neq 0$ and $p^2_r = 0$. In order to have such a global monopole solution with $p^1 q_0 \neq 0$ and $p^2 = 0$ in equilibrium with the monopole at $u = 0$ one may try to place those charges at the point at which $\Phi^a\Phi^a = 0$, but the resulting solution may not be well defined there because the limit of the metric function depends on the direction from which we approach that point.

The entropy of the solution is the sum of the entropies of both centers (vanishing for global monopoles). As in the $\overline{\mathbb{CP}}^3$ case, the monopole at each center does contribute to the center entropy (except for global monopoles). The contributions of the monopole and anti-monopole to the mass cancel each other:

$$M = \frac{1}{4}\frac{\chi_\infty}{|\Im\mathfrak{m}\tau_\infty|}|p^1_u + p^1_r| + \frac{1}{2\chi_\infty}|q_{u,0} + q_{r,0}| \pm \frac{1}{2}\frac{|\Im\mathfrak{m}\tau_\infty \Im\mathfrak{m}\, Z^2_\infty|}{\chi_\infty}|p^2_u + p^2_r|\,. \tag{4.50}$$

5 Conclusions

In this article we have discussed the construction of supersymmetric multi-object solutions in $\mathcal{N} = 2$, $d = 4$ EYM theories, specifically in the so-called $\overline{\mathbb{CP}}^{n\geq 3}$ and $\mathrm{ST}[2,n]$ models. These models were chosen due to their workability, the fact that they allow for a SU(2) gauging and (in the second case) for their stringy origin. Starting with a deformation of the solutions to the SU(2) Bogomol'nyi equation found by Cherkis and Durcan that adds to the 't Hooft-Polyakov monopole *Protogenov hair*, we have been able to construct *bona fide* two-center solutions. These solutions describe a Dirac monopole embedded in SU(2) in the presence of either a global monopole (the supergravity solution corresponding to the 't Hooft-Polyakov monopole) or a non-Abelian black hole (a supergravity solution with an 't Hooft-Polyakov-Protogenov monopole). In order to make the comparison with the single-object case easier, we included a detailed discussion of the embeddings of the spherically symmetric solutions to the SU(2) Bogomol'nyi equations into the two models, and expressed the whole solution in terms of charges and moduli of the physical fields.

The constructed solutions are all static. It would be very interesting to study dyonic solutions and to see how this interplays with the Denef constraint; the stumbling block in this respect is not so much the Bogomol'nyi equation as the equation (2.19); for the moment the only general solution we know of is to take $\mathcal{I}_\Lambda \sim \mathcal{I}^\Lambda$ in the gauged directions, but this automatically solves the Denef constraint. The only case for which we can find non-trivial dyonic solutions is for the multi-Wu-Yang solutions, or if you like the $s \to \infty$ limit of the deformed Cherkis and Durcan's solution; we refrain from discussing these solutions here as, due to gauge invariance, even taking into account the singular gauge transformation, the restriction coming from the Denef constraint is basically the one corresponding to the Abelian theory.

A natural question that follows from the results presented here and in refs. [15–17] is whether we could use a charge-k SU(2) monopole to construct globally regular solutions; the answer is yes: observe that the construction of globally regular solutions in section 3 hinges exclusively but crucially on the fact that the used monopole solution is regular and is such that $\Phi^a\Phi^a \leq \lim_{|\vec{x}|\to\infty} \Phi^a\Phi^a$. A charge-$k$ monopole may be rather difficult to construct but the regularity is guaranteed and also the last needed ingredient is known to be satisfied: indeed, using the Bogomol'nyi equation (3.5) one can show that

$$\partial_{\underline{m}}\partial_{\underline{m}}\ \Phi^a\Phi^a \ =\ F^a{}_{\underline{mn}}F^a{}_{\underline{mn}} \ \geq\ 0\ . \tag{5.1}$$

This equation together with the Hopf maximum principle and the regularity, implies that the function $\Phi^a\Phi^a$ is bounded from above by its value on the sphere at infinity, which is exactly what one needs.

As was said in the introduction, the creation and study of non-Abelian solutions to $d = 4$ supergravity theories is in its infancy and this holds doubly so for the higher dimensional theories. One possible reason is that the structure of supersymmetric solutions to higher-dimensional supergravities (see e.g. refs. [87–89]) is more entangled than the one given in the recipe in section 2.2. For example, naively one would expect that Kronheimer's link of monopoles on $\mathbb{R}^3$ to instantons on GH-spaces, would carry over to the supersymmetric solutions as in $d = 4$ the base space is $\mathbb{R}^3$ and that in $d = 5$ must be hyper-Kähler; i.e. one would expect the instanton equation to show up in the recipe for cooking up 5-dimensional supersymmetric solutions. Perhaps it does, but it definitely is not obvious where and how it is making its appearance in such a clear-cut manner as in $d = 4$.

The 4- and 5-dimensional EYMH theories are, however, related by dimensional reduction/oxidation, whence the solutions to the cubic models presented in this article can be oxidized to 5-dimensions and can be studied with the hope of unraveling the structure of 5-dimensional supersymmetric solutions. Work along these lines is in progress.

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A The SU(2) Lorentzian meron

A Lorentzian meron is a classical solution to the pure SU(2) (Lorentzian) Yang-Mills theory such that the 1-form gauge field A defining it, is proportional to a pure-gauge configuration, which in our conventions would be $\frac{1}{g}dUU^{-1}$ where $U(x) \in \mathrm{SU}(2)$. In ref. [41] $U(x)$ was chosen to be of the *hedgehog form*

$$U \equiv 2\frac{x^m}{r}\delta^a_m T_a\,, \qquad U^\dagger = U^{-1} = -U\,, \quad \Rightarrow U^2 = -\mathbb{1}_{2\times 2}\,. \tag{A.1}$$

and it was shown that A solves the Yang-Mills equations if the proportionality coefficient is 1/2, that is

$$A = \frac{1}{2g}dUU^{-1} = -\frac{1}{gr^2}\varepsilon^a{}_{mn}x^m dx^n T_a\,. \tag{A.2}$$

As we will see, this gauge field is nothing but the gauge field of the Wu-Yang SU(2) monopole given in eq. (B.10).

Since the field strength of a pure gauge configuration vanishes, we find that $F(A)$ can be written in these two specially simple ways which we will use in appendix C:

$$F(A) = \frac{1}{2}dA = g[A, A] = \star_{(3)} d\frac{1}{2gr}U\,, \tag{A.3}$$

Now we can write the non-Abelian field strength $F(A)$ in terms of $F(B)$, where $F(B)$ is the field strengths of the Dirac monopole of unit charge eq. (B.1) that we will review in the next section

$$F(A) = F(B)\,U\,, \qquad F(B) = \star_{(3)} d\frac{1}{2gr}\,, \tag{A.4}$$

and the energy-momentum tensor of A in terms of that of B

$$T_{\mu\nu}(A) = -\frac{1}{2}\mathrm{Tr}[F_{\mu\rho}(A)F_{\nu}{}^{\rho}(A) - \frac{1}{4}\eta_{\mu\nu}F^2(A)] = F_{\mu\rho}(B)F_{\nu}{}^{\rho}(B) - \frac{1}{4}\eta_{\mu\nu}F^2(B) = T_{\mu\nu}(B)\,. \tag{A.5}$$

B The Wu-Yang SU(2) monopole

The Wu-Yang SU(2) monopole [37] is a solution of the SU(2) Yang-Mills theory that can be obtained from the embedding of the Dirac monopole in SU(2) via a singular gauge transformation (see, e.g. ref. [90] and references therein). To fix our conventions, it is convenient to start by reviewing the Wu-Yang construction of the Dirac monopole [91].

B.1 The Dirac monopole

The U(1) field of the Dirac monopole, that we will denote by B is defined to satisfy the Dirac monopole equation,[22] which can be written in several forms:

$$F(B) \equiv dB = \star_{(3)} d\frac{1}{2gr} = -\frac{1}{2g}d\Omega^2\,, \qquad 2\partial_{[m}B_{n]} = -\frac{1}{2g}\varepsilon_{mnp}\frac{x^p}{r^3}\,, \tag{B.1}$$

where $d\Omega^2$ is the volume 2-form of the round 2-sphere of unit radius

$$d\Omega^2 = -\frac{1}{2}\varepsilon_{mnp}\frac{x^m}{r}d\frac{x^n}{r}\wedge d\frac{x^p}{r} = \sin\theta d\theta\wedge d\varphi\,. \tag{B.2}$$

The value of the magnetic charge has been set to g^{-1} and it is the minimal charge allowed if the unit of electric charge is g.

The above equation does not admit a global regular solution.

$$B^{(\pm)} = -\frac{1}{2g}(\cos\theta \mp 1)d\varphi\,, \tag{B.3}$$

are local solutions regular everywhere except on the negative (resp. positive) z axis (the Dirac strings). A globally regular solution can be constructed by using $B^{\pm}$ in the upper (lower) hemisphere and using the gauge transformation

$$B^{(+)} - B^{(-)} = -d\left(\frac{1}{g}\varphi\right)\,, \tag{B.4}$$

to relate them in the overlap region. If the gauge group is U(1) where the radius of the circle is the inverse coupling constant $1/g$, the gauge transformation parameter can have a periodicity $2\pi n/g$ with $n \in \mathbb{N}$. This is the well-known Abelian Wu-Yang monopole

[22] This equation is just the Abelian version of the Bogomol'nyi equation.

construction [91]. In our case, since the period of φ is 2π, we get $2\pi/g$, which is the smallest value allowed $p = 1/g$. The solution that describes the monopole of charge n times the minimum is n times this one $p = n/g$.

It is useful to have the expression of $B^{(\pm)}$ in Cartesian coordinates:

$$B^{(\pm)} = \frac{1}{2g} \frac{[(0,0,\mp 1) \times (x^1, x^2, x^3)] \cdot d\vec{x}}{r^2(r \pm x^3)}\,, \tag{B.5}$$

in which the singularity at $r = \mp x^3$ becomes evident. In this form, one can easily change the position of the monopole from the origin to some other point x_0^m and the position of the Dirac string from the half line that starts from the origin in the direction $-(0,0,\mp 1)$ to the half line that starts at the monopole's position x_0^m and has the direction s^m relative to that point:

$$B^{(s)} = \frac{1}{2g} \left(1 - \frac{s^m}{s}\frac{u^m}{u}\right)^{-1} \varepsilon_{mnp} \frac{s^m}{s}\frac{u^n}{u} d\frac{u^p}{u}\,, \tag{B.6}$$

with

$$u^m \equiv x^m - x_0^m\,, \qquad u^2 \equiv u^m u^m\,, \qquad s^2 \equiv s^m s^m\,. \tag{B.7}$$

B.2 From the Dirac monopole to the Wu-Yang SU(2) monopole

Let us consider the Abelian $B^{(+)}$ solution in eq. (B.3) and let us embed it in SU(2) as the 3rd component of the gauge field

$$A^{(+)} \equiv 2B^{(+)} T_3\,, \qquad F(A^{(+)}) = 2F(B) T_3\,. \tag{B.8}$$

The SU(2) gauge transformation (which is evidently singular along the negative z axis and makes the whole Dirac string singularity, but the endpoint at the coordinate origin, disappear)

$$U^{(+)} \equiv \frac{1}{\sqrt{2(1+\frac{z}{r})}} \left[1 + \frac{z}{r} + 2\left(\frac{x}{r}T_2 - \frac{y}{r}T_1\right)\right]\,, \tag{B.9}$$

relates the gauged field $A^{(+)}$ to

$$A = \frac{1}{g}\varepsilon^a{}_{mn} dx^m \frac{x^n}{r^2} T_a\,, \qquad A^{(+)} = U^{(+)} A (U^{(+)})^{-1} + \frac{1}{g} dU^{(+)} (U^{(+)})^{-1}\,, \tag{B.10}$$

which is the gauge field of the Wu-Yang SU(2) monopole. As we have mentioned in the previous appendix, this is also the gauge field of the Lorentzian meron eq. (A.2). The gauge transformation also relates T_3 to $\mathcal{U}$ in eq. (A.1) and the Abelian vector

$$U^{(+)} U (U^{(+)})^{-1} = 2T_3\,. \tag{B.11}$$

The fact that the Lorentzian meron is the Wu-Yang monopole, which is related by a gauge transformation to the Dirac monopole makes the relation eq. (A.5) trivial.

This construction can be generalized to more general positions of the Dirac string: if we consider embedding of the Dirac monopole solution $B^{(s)}$ in eq. (B.6) into SU(2)

$$A^{(s)} \equiv -2B^{(s)} \frac{s^m}{s} \delta_m{}^a T_a\,, \tag{B.12}$$

it is easy to see that the gauge transformation

$$U^{(s)} \equiv \frac{1}{\sqrt{2\left(1-\frac{s^m}{s}\frac{u^m}{u}\right)}}\left[1-\frac{s^m}{s}\frac{u^m}{u}-2\varepsilon_{mn}{}^{a}\frac{s^m}{s}\frac{u^n}{u}T_a\right], \tag{B.13}$$

relates it to the same Wu-Yang monopole field eq. (B.10)

$$A^{(s)} = U^{(s)}A(U^{(s)})^{-1}+\frac{1}{g}dU^{(s)}(U^{(s)})^{-1}. \tag{B.14}$$

C The SU(2) Skyrme model

In this appendix we are going to show that the Lorentzian meron (Wu-Yang monopole) is also associated to a solution of the equations of motion of the SU(2) Skyrme model [92] written in the form [93]

$$S_{\text{Skyrme}} = -\frac{1}{2}\int d^4x\left\{\frac{1}{2}R_\mu R^\mu+\frac{\lambda}{16}S_{\mu\nu}S^{\mu\nu}\right\}, \tag{C.1}$$

where

$$R_\mu \equiv V^{-1}\partial_\mu V\,, \qquad S_{\mu\nu}\equiv[R_\mu,R_\nu]\,, \qquad V(x)\in \text{SU}(2)\,. \tag{C.2}$$

The equations of motion are

$$\partial_\mu R^\mu+\frac{\lambda}{4}\partial_\mu[R_\nu,F^{\mu\nu}]=0\,. \tag{C.3}$$

If we take $V=U^{-1}$ (U given by eq. (A.1)), then we can write $R=2gA$ where A is Lorentzian meron's gauge field eq. (A.2) and

$$\begin{aligned}\partial_\mu R^{i\,\mu} &= -2g\partial_m A^i{}_m = 0\,,\\ \partial_\mu[R_\nu,F^{\mu\nu}]^i &\sim \partial_m\left(\frac{A^i{}_m}{r^2}\right)=0\,.\end{aligned} \tag{C.4}$$

D Higher-charge Lorentzian merons and Wu-Yang monopoles

The construction of a Lorentzian meron can be generalized by using a generalization of the unit outward-pointing vector x^m/r denoted by ξ^m and defined by [86]

$$(\xi^m)\equiv\frac{1}{r}\left(\frac{\mathfrak{Im}(x^2+ix^1)^n}{\rho^{n-1}},\frac{\mathfrak{Re}(x^2+ix^1)^n}{\rho^{n-1}},x^3\right), \qquad \rho^2\equiv(x^1)^2+(x^2)^2\,, \tag{D.1}$$

or, in spherical coordinates,

$$(\xi^m)\equiv(\sin\theta\sin n\varphi,\sin\theta\cos n\varphi,\cos\theta)\,, \tag{D.2}$$

and which reduces to x^m/r for $n=1$. The essential properties of ξ^m are

$$d\xi^m\wedge d\xi^n = -n\varepsilon_{mnp}\xi^p d\Omega^2\,, \tag{D.3}$$

$$-\frac{1}{2}\varepsilon_{mnp}\xi^m d\xi^n\wedge d\xi^p = nd\Omega^2 = \star_{(3)}d\frac{n}{r}\,, \tag{D.4}$$

The generalization of the meron solution is constructed in terms of the generalization SU(2) matrix in eq. (A.1)

$$U_{(n)} \equiv 2\xi^m \delta_m^a T_a\,, \qquad U^\dagger_{(n)} = U^{-1}_{(n)} = -U_{(n)}\,, \tag{D.5}$$

and takes the form

$$A \equiv \frac{1}{2g} dU_{(n)} U^{-1}_{(n)}\,. \tag{D.6}$$

The field strength is given by

$$F(A_{(n)}) = \frac{1}{2} dA = g[A, A] = \star_{(3)} d \frac{n}{2gr} U_{(n)}\,, \tag{D.7}$$

and can be related to that of a Dirac monopole of charge $p = n/g$

$$F(B_{(n)}) = \star_{(3)} d \frac{n}{2gr}\,, \quad F(A_{(n)}) = F(B_{(n)})\, U_{(n)}\,, \tag{D.8}$$

which is given by the expressions studied at the beginning. The energy-momentum tensor of A is also equal to that of the Abelian monopole of charge n/g B. These fields can also be related to the embedding of the charge n/g Dirac monopole into SU(2) with a generalization of the gauge transformation eq. (B.13)

$$U^{(s)}_{(n)} \equiv \frac{1}{\sqrt{2\left(1 - \frac{s^m}{s}\xi^m\right)}} \left[1 - \frac{s^m}{s}\xi^m - 2\varepsilon_{mn}{}^a \frac{s^m}{s}\xi^n T_a\right]\,, \tag{D.9}$$

relates it to the meron gauge field:

$$U^{(s)}_{(n)} U_{(n)} (U^{(s)}_{(n)})^{-1} = -2\frac{s^m}{s}\delta_m{}^a T_a\,, \quad U^{(s)}_{(n)} A_{(n)} (U^{(s)}_{(n)})^{-1} + \frac{1}{g} dU^{(s)}_{(n)} (U^{(s)}_{(n)})^{-1} = nB^{(s)}_{(n)} 2\frac{s^m}{s}\delta_m{}^a T_a\,. \tag{D.10}$$

To check that this gauge field solves the Yang-Mills equations of motion we first stress that, with the above connection, $U_{(n)}$ is a covariantly-constant adjoint field. Then, auxiliary the adjoint Higgs field

$$\Phi_{(n)} \equiv \left(-\frac{\mu}{2g} + \frac{n}{2gr}\right) U_{(n)}\,, \tag{D.11}$$

satisfies

$$D\Phi_{(n)} = d\frac{n}{2gr} U_{(n)}\,, \tag{D.12}$$

and the pair $A_{(n)}, \Phi_{(n)}$ satisfies the Bogomol'nyi equations (3.5) and, as a consequence the equations of motion of the Yang-Mills-Higgs system. The last equation implies that $\Phi_{(n)}$ and $D\Phi_{(n)}$ commute so the Higgs current vanishes and $A_{(n)}$ also solves the sourceless Yang-Mills equations.

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Resolution of SU(2) monopole singularities by oxidation

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ABSTRACT

We show how *colored* SU(2) BPS monopoles (that is: SU(2) monopoles satisfying the Bogomol'nyi equation whose Higgs field and magnetic charge vanish at infinity and which are singular at the origin) can be obtained from the BPST instanton by a singular dimensional reduction, explaining the origin of the singularity and implying that the singularity can be cured by the oxidation of the solution. We study the oxidation of other monopole solutions in this scheme.

1. Introduction: monopoles and instantons

It has been known for a long time that selfdual Yang–Mills (YM) instantons in 4-dimensional Euclidean space $\mathbb{E}^4$ and magnetic monopoles satisfying the Bogomol'nyi equation in $\mathbb{E}^3$ [1] [1] are related by dimensional reduction. In its simplest setting, this relation can be described as follows: if $\hat{A}_{\hat{\mu}}$ ($\hat{\mu} = 0, 1, 2, 3$) [2] is the gauge potential of a selfdual YM instanton solution in $\mathbb{E}^4$ and is furthermore independent of one of the 4 Cartesian coordinates, z say, then the z-component $\hat{A}_z$ and the other three components $\hat{A}_m$ ($m = 1, 2, 3$) can be identified with the Higgs field $\Phi \equiv -\hat{A}_z$ and the gauge potential $A_m \equiv \hat{A}_m$ of a solution of the Yang–Mills–Higgs (YMH) system in the Prasad–Sommerfield limit satisfying the Bogomol'nyi equation:

$$\mathcal{D}_m \Phi = \tfrac{1}{2}\epsilon_{mnp} F_{np} \,. \tag{1.1}$$

The sign in the Bogomol'nyi equation depends on the orientation of the coordinates; we have taken the one corresponding to z to be x^0 and $\epsilon_{0123} = \epsilon_{123} = +1$.

The coordinate z has to be compactified for the instanton action to be finite [3]: $z \sim z + 4\pi$. Thus, in practice, we are performing the dimensional reduction in $S^1 \times \mathbb{E}^3$ and the z-independent solutions can be considered to be the Fourier zero modes of instanton solutions periodic in the direction z (the so-called *calorons*).

The paradigm of selfdual YM instanton in $\mathbb{E}^4$ is the BPST instanton [5], usually presented in Cartesian coordinates using the 't Hooft symbols. It belongs to a family of selfdual YM solutions depending on an arbitrary function K, harmonic on $\mathbb{E}^4$ (see *e.g.* Ref. [6] and the references therein). With K asymptotically constant and with a single point-like pole at the origin $K = 1 + 4/(\lambda^2\rho^2)$, where $|\vec{x}_{(4)}|^2 \equiv \rho^2$, the solution describes a single BPST instanton located at the origin. Replacing K by a harmonic function on $S^1 \times \mathbb{E}^3$ with a single pole at the origin and asymptotically constant in $\mathbb{E}^3$, $K = 1 + (\sinh r/2)/[\lambda^2 r^2(\cosh r/2 - \cos z/2)]$, where $r^2 = |\vec{x}_{(3)}|^2$ and z is the fourth, compact, Euclidean coordinate, we get a caloron [7] whose Fourier zero mode gives, upon dimensional reduction, the spatial part of a Wu–Yang SU(2) magnetic monopole [8], which is singular at the origin.

Since the BPST instanton and caloron are regular everywhere, the singularity of the Wu–Yang solution can be understood as the result of having ignored the massive Fourier modes in the dimensional reduction, but the mere oxidation of the 3-dimensional monopole does not automatically restore them: the 4-dimensional instanton corresponding to the Fourier zero mode of the BPST caloron is singular.

The above *redox* relation was generalized by Kronheimer in Ref. [9] to a relation between selfdual Yang–Mills instanton solutions in hyper-Kähler (HK) spaces [9] and BPS monopoles in $\mathbb{E}^3$. We are going to see that Kronheimer's scheme provides an alternative reduction of the BPST instanton which relates it to the *colored* BPS monopole solution of Protogenov [10]. Colored monopoles are a rather misterious type of monopole solutions that exist for many

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[1] This is the equation satisfied by the 't Hooft–Polyakov monopole [2,3] in the Prasad–Sommerfield limit [4]. We will henceforth refer to these monopoles as BPS monopoles. Since the time direction does not play any role here, we will also refer to the spatial parts of 4-dimensional Lorentzian solutions as "3-dimensional" solutions.

[2] We dress 4-dimensional objects with a hat; hatless objects are 3-dimensional.

[3] This choice of period is unconventional but convenient for what follows.

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gauge groups [11] and are characterized by asymptotically vanishing Higgs field and magnetic charge which, nevertheless, can contribute to the Bekenstein–Hawking entropy of certain (supersymmetric) non-Abelian black holes [12,13,11].

Let us start by reviewing Kronheimer's result: consider a 4-dimensional HK space admitting a free U(1) action which shifts the adapted periodic coordinate $z \sim z + 4\pi$ by an arbitrary constant. Its metric can always be put in the form [14]

$$d\hat{s}^2 = H^{-1}(dz+\omega)^2 + H dx^m dx^m \qquad (m=1,2,3)\,, \tag{1.2}$$

where the z-independent function H and 1-form ω are related by[4]

$$dH = \star d\omega\,. \tag{1.3}$$

The integrability condition of this equation implies that H is a harmonic function in $\mathbb{E}^3$ which is furthermore required to be strictly positive in order for the metric to be regular. Now, for any gauge group G, let us consider a gauge field $\hat{A}$ whose field strength $\hat{F}$ is selfdual $\hat{\star}\hat{F} = +\hat{F}$ in the above HK metric with respect to the frame and orientation

$$\hat{e}^{\,0} = H^{-1/2}(dz+\omega)\,, \qquad \hat{e}^{\,a} = H^{1/2}\delta^a{}_m dx^m\,,$$
$$\epsilon_{0123} = +1\,. \tag{1.4}$$

Then, the 3-dimensional gauge and Higgs fields A and Φ defined by

$$\Phi \equiv -H\hat{A}_z\,,$$
$$A_m \equiv \hat{A}_m - \omega_m \hat{A}_z\,, \tag{1.5}$$

satisfy the Bogomol'nyi equation in $\mathbb{E}^3$ Eq. (1.1). It is worth stressing that, had we started with an anti-selfdual YM field we would have obtained the Bogomol'nyi equation with opposite sign, which is acceptable, but also Eq. (1.3) with opposite sign, which would be a contradiction: in this setup we can only reduce YM fields which are selfdual w.r.t. the above frame and orientation.

When $H=1$, the HK space is just $S^1 \times \mathbb{E}^3$ and one recovers the result explained at the beginning. A more interesting choice is $H = 1/r$ with $r^2 = x^m x^m$. Writing the $\mathbb{E}^3$ metric $dx^m dx^m$ as $dr^2 + r^2 d\Omega^2_{(2)}$ and then redefining $r = \rho^2/4$ the HK metric Eq. (1.2) becomes the metric of $\mathbb{E}^4$ in spherical coordinates

$$ds^2 = d\rho^2 + \rho^2 d\Omega^2_{(3)}\,, \tag{1.6}$$

where $d\Omega^2_{(3)}$ is the round metric of the 3-sphere of unit radius in Eq. (A.14). This HK space is, therefore, $\mathbb{E}^4_{-\{0\}}$ and the shifts of z act freely on it because the origin $\rho = 0$ does not belong to it.

Obviously, the standard BPST instanton is a selfdual solution in this space and, provided that the gauge field is independent of z, we can reduce it directly (avoiding the caloron step) using Kronheimer's scheme to find a monopole in $\mathbb{E}^3_{-\{0\}}$. This is what we are going to do in the next section but, before, we want to review the relation between the Euclidean action of the instanton and the monopole charge.

The gauge field strength components in the frame Eq. (1.4) are

$$\begin{cases} \hat{F}_{ab} \;=\; H^{-1}F_{ab} - H^{-2}\Phi(d\omega)_{ab}\,, \\ \hat{F}_{0a} \;=\; H^{-1}\mathcal{D}_a\Phi - H^{-2}\Phi\partial_a H\,. \end{cases} \tag{1.7}$$

Substituting them into the YM action and using repeatedly Eq. (1.3), the Bogomol'nyi equation (1.1) and Stokes' theorem we get

$$\tfrac14\int d^4x\sqrt{|\hat{g}|}\hat{F}^2 = 4\pi\int_{V^3}\tfrac12 H^{-2}d\star dH\,\Phi^2 + 4\pi\int_{\partial V^3}\left[H^{-1}\Phi^A F^A + \tfrac12\star dH^{-1}\Phi^2\right], \tag{1.8}$$

where V^3 is $\mathbb{E}^3$ with the singular points of H removed: this means that the first term on the r.h.s. always vanishes. The end result therefore reads

$$\tfrac14\int d^4x\sqrt{|\hat{g}|}\hat{F}^2 = 4\pi\int_{\partial V^3}\left[H^{-1}\Phi^A F^A + \tfrac12\star dH^{-1}\Phi^2\right], \tag{1.9}$$

and one must take into account that the boundary of V^3 includes the singularities of H as well as infinity.

For $H=1$, $V^3 = \mathbb{E}^3$ and the r.h.s. is directly related to the monopole magnetic charge

$$p = \tfrac{1}{4\pi}\int_{S^2_\infty}\frac{\Phi^A F^A}{\sqrt{\Phi^B\Phi^B}}\,, \tag{1.10}$$

provided the Higgs field is asymptotically constant, as in the BPS 't Hooft–Polyakov monopole.

For $H = 1/r$, which is the case of interest here, $V^3 = \mathbb{E}^3_{-\{0\}}$, $\partial V^3 = \{0\} \cup S^2_\infty$, and the integral will diverge precisely for monopoles with well-defined magnetic charge at infinity and asymptotically constant Higgs fields. Thus, we can only expect convergence for colored magnetic monopoles [11]. If the selfdual YM field has a finite action, then it must lead to a colored monopole in $\mathbb{E}^3$ by Kronheimer's dimensional reduction. In the next section we are going to see that this is indeed the case for the BPST instanton.

2. Singular reduction of the BPST instanton

In order to reduce the BPST instanton *à la Kronheimer* in the HK space with $H = 1/r$, it is convenient to write it in spherical coordinates and, actually, it is easier to rederive it directly using the following ansatz for the components of the SU(2) gauge potential

$$\hat{A}^A_{\substack{L\\R}} = b_{\substack{L\\R}}(\rho)v^A_{\substack{L\\R}}\,, \qquad A=1,2,3\,, \tag{2.1}$$

where the $v^A_{\substack{L\\R}}$ are the components of the SU(2) Maurer–Cartan (MC) 1-forms defined in Eqs. (A.12), satisfying Eq. (A.13), and $b_{\substack{L\\R}}(\rho)$ is a function of ρ to be determined by imposing the selfduality of the gauge field strength. To this end it is most convenient to use the frames

$$\hat{e}^{\,0}_{\substack{L\\R}} = d\rho\,, \qquad \hat{e}^{\,a}_{\substack{L\\R}} = \tfrac12\rho\,\delta^a{}_A v^A_{\substack{L\\R}}\,, \tag{2.2}$$

for the metric Eq. (1.6). Using the MC 1-forms it is straightforward to compute the gauge field strength $\hat{F}^A_{\substack{L\\R}}$:

$$\hat{F}_{\substack{L\\R}}{}^A = \frac{2\dot{b}}{\rho}\,\delta^A{}_a\,\hat{e}_{\substack{L\\R}}{}^0\wedge\hat{e}_{\substack{L\\R}}{}^a + \frac{2b(b\mp1)}{\rho^2}\epsilon^A{}_{ab}\,\hat{e}_{\substack{L\\R}}{}^a\wedge\hat{e}_{\substack{L\\R}}{}^b\,. \tag{2.3}$$

Requiring $\hat{F}^A_{\substack{L\\R}}$ to be (anti-)selfdual $(\hat{F}^{A(\pm)}{}_{0a} = \pm\tfrac12\epsilon_{abc}\hat{F}^{A(\pm)}{}_{bc})$ in these two frames we arrive at a differential equation for $b^{\pm}_{\substack{L\\R}}(\rho)$ leading to two self- and two anti-selfdual solutions describing a

[4] Unhatted objects are always defined in 3-dimensional Euclidean space $\mathbb{E}^3$.

single BPST instanton or anti-instanton, of size[5] determined by the parameter λ, at the origin:

$$\hat{\star}\hat{F} = +\hat{F} \begin{cases} \hat{A}_L^{A(+)} &= \dfrac{1}{1+\lambda^2\rho^2/4}\, v_L^A\,, \\ \hat{A}_R^{A(+)} &= -\dfrac{\lambda^2\rho^2/4}{1+\lambda^2\rho^2/4}\, v_R^A\,, \end{cases}$$

$$\hat{\star}\hat{F} = -\hat{F} \begin{cases} \hat{A}_L^{A(-)} &= +\dfrac{\lambda^2\rho^2/4}{1+\lambda^2\rho^2/4}\, v_L^A\,, \\ \hat{A}_R^{A(-)} &= -\dfrac{1}{1+\lambda^2\rho^2/4}\, v_R^A\,. \end{cases} \tag{2.4}$$

The gauge fields $\hat{A}_L^{A(\pm)}$ are gauge-equivalent to the $\hat{A}_R^{A(\pm)}$ owing to

$$U\hat{A}_L^{A(\pm)}U^{-1} + dUU^{-1} = \hat{A}_R^{A(\pm)}\,, \tag{2.5}$$

and the property Eq. (A.11). Then, we could just work with $\hat{A}_R^{A(+)}$ and $\hat{A}_L^{A(-)}$, which are regular (they vanish at $\rho = 0$ while the other two are multivalued there). However, if we want to use Kronheimer's results we are forced to work with the singular ones, $\hat{A}_L^{A(+)}$ and $\hat{A}_R^{A(-)}$, because as one can see the transformation between the frame $\hat{e}^{\hat{a}}_{\substack{L\\R}}$ in Eqs. (2.2) and Kronheimer's frame $\hat{e}^{\hat{a}}$ in Eqs. (1.4) preserves the orientation for $\hat{e}^{\hat{a}}_L$ but reverses it for $\hat{e}^{\hat{a}}_R$. In other words: the regular gauge fields $\hat{A}_R^{A(+)}$ and $\hat{A}_L^{A(-)}$ are anti-selfdual in Kronheimer's frame and can therefore not be consistently reduced.

Let us, then, consider $\hat{A}_L^{A(+)}$ and $\hat{A}_R^{A(-)}$. By construction, these gauge fields are invariant under the free U(1) actions in Eqs. (A.5) and (A.4), respectively.

In other words: $\hat{A}_L^{A(+)}$ is φ-independent and $\hat{A}_R^{A(-)}$ is ψ-independent and can be dimensionally reduced along those directions because the only invariant point under these actions (the origin $\rho = 0$) does not belong to our HK space. We can expect 3-dimensional monopoles which are singular there.

Using directly Eqs. (1.5), from $\hat{A}_L^{A(+)}$ we get the Yang–Mills and Higgs fields of a BPS monopole solution

$$\Phi_L^{A(+)} = \frac{1}{r(1+\lambda^2 r)}\delta^A{}_m \frac{y_L^m}{r}\,,$$
$$A_L^{A(+)} = \frac{1}{(1+\lambda^2 r)}\epsilon^A{}_{mn} d\frac{y_L^m}{r}\frac{y_L^n}{r} \tag{2.6}$$

where we have defined the Cartesian coordinates $y^m/r \equiv -\delta^m{}_A v^A_{L\,\varphi}$ [6]:

$$y_L^1 \equiv r\sin\theta\,\cos\psi\,,\quad y_L^2 \equiv r\sin\theta\,\sin\psi\,,\quad y_L^3 \equiv r\cos\theta\,. \tag{2.7}$$

The reduction of $\hat{A}_R^{A(-)}$ gives exactly the same 3-dimensional fields upon the replacement of the Cartesian coordinates y_L^m by $y_R^m \equiv +r\delta^m{}_A v^A_{R\,\psi}$ [7]:

$$y_R^1 \equiv r\sin\theta\,\cos\varphi\,,\quad y_R^2 \equiv -r\sin\theta\,\sin\varphi\,,\quad y_R^3 \equiv -r\cos\theta\,. \tag{2.8}$$

As predicted by the arguments based on the Euclidean action, the 3-dimensional BPS monopole obtained by this procedure is the colored monopole found by Protogenov in Ref. [10]. The Higgs field vanishes at infinity and the magnetic charge, as defined in Eq. (1.10) vanishes identically. The solution approaches the Wu–Yang monopole [8] for $r \to 0$ (which corresponds to $\lambda^2 = 0$) and, therefore, one can argue that the solution describes a magnetic monopole at the origin whose charge is completely screened at infinity. This interpretation is supported by the computation of the Bekenstein–Hawking entropy $S_{\rm BH}$ of non-Abelian black holes with this kind of gauge fields: there is a contribution to $S_{\rm BH}$ corresponding to a magnetic charge [12,13].

3. Oxidation of the singular Protogenov monopoles

Reversing the procedure we just carried out, we see that the singularity of the SU(2) colored BPS monopole disappears completely when it is oxidized to 4 Euclidean dimensions. Since there are other singular SU(2) BPS monopoles [10], it is natural to ask whether their singularities can also be cured by oxidizing them within this scheme.

The spherically symmetric solutions of the SU(2) Bogomol'nyi equations have the following *hedgehog* form [10]:

$$A^A = -r^2 h(r)\epsilon^A{}_{mn}\frac{y^n}{r} d\left(\frac{y^m}{r}\right)\,, \tag{3.1}$$

$$\Phi^A = -rf(r)\delta^A{}_m \frac{y^m}{r}\,, \tag{3.2}$$

where the functions $f(r)$ and $h(r)$ must satisfy the differential equations

$$r\dot{h} + 2h + f(1 + r^2 h) = 0\,, \tag{3.3}$$
$$r(\dot{h} - \dot{f}) - r^2 h(h - f) = 0\,, \tag{3.4}$$

if the above Yang–Mills and Higgs fields are to satisfy the Bogomol'nyi equation (1.1). Apart from the family of colored solutions in Eq. (2.6), there is another 2-parameter (μ and s) family of solutions given by

$$rf = -\frac{1}{r}\left[1 - \mu r\coth(\mu r + s)\right]\,,$$
$$rh = \frac{1}{r}\left[\frac{\mu r}{\sinh(\mu r + s)} - 1\right]\,. \tag{3.5}$$

The BPS limit of the 't Hooft–Polyakov monopole [2,3] is the $s = 0$ member of this family, and the only regular one. Before oxidizing them, we can compute the action of the corresponding instanton using Eq. (1.9). The action turns out to diverge for all values of s. However, even if all hope of getting a regular instanton by oxidizing these solutions is lost, it is still worth finding the general expression of the singular instantons, since it may give us inspiration for making instanton ansätze directly in 4 dimensions. Using Kronheimer's relations, Eq. (1.5), we find

$$\hat{A}^A = -r^2 f(r) v_L^A + r^2\left[f(r) - h(r)\right] u^A\,, \tag{3.6}$$

where we have defined the 1-forms

$$u^1 = \cos\psi\,\sin\theta\cos\theta d\psi + \sin\psi d\theta\,,$$
$$u^2 = \sin\psi\,\sin\theta\cos\theta d\psi - \cos\psi d\theta\,,$$
$$u^3 = -\sin^2\theta d\psi\,. \tag{3.7}$$

These 1-forms depend only on two coordinates (ψ and θ) and they can be seen as projections of the left-invariant MC 1-forms v_L^A

$$u^A = v_L^B\left[\delta^A{}_B - \frac{y_B y^A}{r^2}\right]\,. \tag{3.8}$$

[5] In the instanton literature it is customary to denote the size of the (anti-)instanton by ρ, see *e.g.* Refs. [15], but here we'll denote it by ρ_0. It is then easy to see that $\lambda = 2/\rho_0$.

[6] We use the identity $v_L^A(\varphi = 0) - \cos\theta\, v^A_{L\,\varphi} d\psi = \epsilon^A{}_{mn} d\frac{y_L^m}{r}\frac{y_L^n}{r}$.

[7] Now we use the identity $v_R^A(\psi = 0) - \cos\theta\, v^A_{R\,\psi} d\varphi = -\epsilon^A{}_{mn} d\frac{y_R^m}{r}\frac{y_R^n}{r}$.

They satisfy differential equations identical to the ones satisfied by the left-invariant MC 1-forms v_L^A up to the 1/2 factor, *i.e.*

$$du^A = -\epsilon^A{}_{BC} u^B \wedge u^C \,, \tag{3.9}$$

which makes them well suited for a generalization of the ansatz Eq. (2.1):

$$\hat{A}^A = b(\rho) v_L^A + c(\rho) u^A \,. \tag{3.10}$$

Imposing selfduality of the corresponding field strength with the redefinition

$$b(\rho(r)) = -r^2 f(r) \,, \qquad c(\rho(r)) = -r^2 [h(r) - f(r)] \,, \tag{3.11}$$

leads to Protogenov's equations (3.3) and (3.4); the oxidation of the BPS monopoles gives all the selfdual instantons of that form.

4. Conclusions

In this paper we have shown how a misterious kind of SU(2) BPS magnetic monopoles known as *colored monopoles*, which are singular at the origin and have vanishing asymptotic charge and Higgs field, can be understood as the result of the singular dimensional reduction of the BPST instanton, which is itself globally regular. The parameter appearing in the monopole family of solutions turns out to be related to the one that measures the instantons' size.

The mechanism is analogous to the well-known mechanism curing gravitational singularities by oxidation as for example the KK-monopole [16] or in certain 4-dimensional dilatonic black holes [17], but with the twist that here the fields are non-Abelian. The mechanism that cures the singularity of the colored monopole does not, however, work for the rest of the spherically-symmetric BPS monopoles of the theory: they always have infinite action, but depending on the application this may or may not be a problem.

We have argued, based on the relation between the instanton action and the monopole magnetic charge, that this relation between regular instantons and singular, colored magnetic monopoles should be general. It has recently been shown in Ref. [11] that colored magnetic monopoles are present in the Yang–Mills–Higgs theory for all SU(N) groups and the results of that paper can be used to construct regular selfdual SU(N) instantons [18]. Possibly, the transmutation monopoles discovered in Ref. [11], which have different (non-vanishing) charges at infinity and at the origin, can be related to regular solutions by a similar mechanism.

The case studied here is just the simplest and most special of those comprised in Kronheimer's work Ref. [9], since it just involves $\mathbb{E}^4_{-\{0\}}$. One may wonder if the rest can be of any relevance in physics. It turns out that the relation between $\mathcal{N}=1$, $d=5$ and $\mathcal{N}=2$, $d=4$ super-Einstein–Yang–Mills (SEYM) theories must include the relation between selfdual instantons in HK spaces and BPS monopoles in $\mathbb{E}^3$ discovered by Kronheimer: the timelike supersymmetric solutions of $\mathcal{N}=1$, $d=5$ [19] (as it happens in the Abelian case [20]) involve a 4-dimensional Euclidean base space of HK type and the YM field strengths have a piece which is selfdual in that space. On the other hand the YM fields of the timelike supersymmetric solutions of $\mathcal{N}=2$, $d=4$ SEYM [21] are required to satisfy the Bogomol'nyi equation in $\mathbb{E}^3$ in combination with an effective *Higgs* field. These two classes of theories and their solutions are related by dimensional reduction. Explicit solutions of the latter describing non-Abelian black holes have been obtained in [22,23,12,13,11]. Some of the solutions are powered by the colored BPS monopoles that we have shown to be related to the BPST instanton. It is then natural to expect that the oxidation of the complete supergravity solutions will provide us with explicit solutions of the $\mathcal{N}=1$, $d=5$ SEYM theory[8] involving the BPST instanton. These solutions, whose form is quite intriguing, may be globally regular. The oxidation *à la Kronheimer* of solutions involving other monopoles will give potentially singular solutions, but, just as it happens with singular monopoles in $d=4$, gravity may cover the singularities with event horizons. All these new possibilities opened by the result presented in this paper are very interesting and well worth investigating. Work in this direction is already under way [24].

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Appendix A. The metrics of the round S^3 and S^2

In this appendix we will review the well-known construction of the SO(4)-invariant metric on S^3 using its identification with the SU(2) group manifold, the construction of SO(3)-invariant metric on S^2 using its identification with the SU(2)/U(1) coset space and the relation between both of them.

All matrices $U \in SU(2)$ ($U^\dagger = U^{-1}$, $\det U = +1$) can be parametrized by two complex numbers z_0, z_1

$$U \equiv \begin{pmatrix} z_0 & z_1 \\ -\bar{z}_1 & \bar{z}_0 \end{pmatrix}, \qquad |z_0|^2 + |z_1|^2 = 1 \,. \tag{A.1}$$

Therefore, the SU(2) manifold can be identified with S^3. Both are traditionally parametrized by the Euler angles $\{\theta, \varphi, \psi\}$:

$$z_0 = \cos(\theta/2)\, e^{i(\varphi+\psi)/2} \,, \qquad z_1 = \sin(\theta/2)\, e^{i(\varphi-\psi)/2} \,. \tag{A.2}$$

The main property of this parametrization is that any SU(2) rotation can be written as the product of three rotations with these angles:

$$U(\varphi,\theta,\psi) = U(\varphi,0,0)U(0,\theta,0)U(0,0,\psi) \,. \tag{A.3}$$

The Euler angles are usually assumed to take values in the intervals $\theta \in [0,\pi]$, $\varphi \in [0,2\pi)$, and $\psi \in [0,4\pi)$. Other choices are possible: for instance, $\theta \in [0,\pi]$, $\varphi \in [0,4\pi)$, and $\psi \in [0,2\pi)$ also covers once S^3. Only the coordinate chosen to take values in $[0,4\pi)$ should be considered periodic. There is a free U(1) action on S^3 associated to constant shifts of the periodic coordinate. For the standard choice, this action is

$$U(\varphi,\theta,\psi) \to U(\varphi,\theta,\psi)U(0,0,2\alpha) \,, \qquad \alpha \in [0,2\pi) \,. \tag{A.4}$$

Being a right action, it is adequate to define the right coset space SU(2)/U(1). If we choose instead φ to be the periodic coordinate,

[8] So far, no explicit solutions of these theories have been constructed.

the U(1) action is

$$U(\varphi,\theta,\psi) \to U(2\alpha,0,0)U(\varphi,\theta,\psi)\,, \qquad \alpha\in[0,2\pi)\,. \tag{A.5}$$

Being a left action, it is adequate to define the left coset space U(1)\SU(2), which is a more unusual option.

A convenient basis of the $\mathfrak{su}(2)$ Lie algebra is provided by the anti-Hermitian matrices[9]

$$T_A = \tfrac{i}{2}\sigma^A\,, \qquad [T_A, T_B] = -\epsilon_{ABC}T_C\,. \tag{A.7}$$

In this basis

$$U(\varphi,0,0) = e^{\varphi T_3}\,, \qquad U(0,\theta,0) = e^{\theta T_2}\,,$$
$$U(0,0,\psi) = e^{\psi T_3}\,. \tag{A.8}$$

The left- (resp. right-)invariant Maurer–Cartan (MC) 1-form V_L (resp. V_R) are defined by

$$V_L \equiv -U^{-1}dU\,, \qquad V_R \equiv -dUU^{-1}\,, \tag{A.9}$$

and as a consequence of their definition they satisfy the MC equations

$$dV_{\substack{L\\R}} \mp V_{\substack{L\\R}} \wedge V_{\substack{L\\R}} = 0\,. \tag{A.10}$$

Observe that the left- and right-invariant MC 1-forms are related by the following *gauge* transformations:

$$V_R = UV_LU^{-1}\,. \tag{A.11}$$

The components of the MC 1-forms in the above basis $V_{\substack{L\\R}} \equiv v^A_{\substack{L\\R}}T_A$ are given by

$$\begin{cases} v^1_L &= \sin\psi\, d\theta - \sin\theta\cos\psi\, d\varphi\,,\\ v^2_L &= -\cos\psi\, d\theta - \sin\theta\sin\psi\, d\varphi\,,\\ v^3_L &= -(d\psi + \cos\theta\, d\varphi)\,,\end{cases}$$
$$\begin{cases} v^1_R &= -\sin\varphi\, d\theta + \sin\theta\cos\varphi\, d\psi\,,\\ v^2_R &= -\cos\varphi\, d\theta - \sin\theta\sin\varphi\, d\psi\,,\\ v^3_R &= -(d\varphi + \cos\theta\, d\psi)\,,\end{cases} \tag{A.12}$$

and the MC equations in components take the form

$$dv^A_{\substack{L\\R}} \pm \tfrac{1}{2}\epsilon_{ABC}\, v^B_{\substack{L\\R}} \wedge v^C_{\substack{L\\R}} = 0\,. \tag{A.13}$$

As their name indicates, the left- (resp. right-)invariant MC 1-forms are invariant under the left (resp. right) U(1) action in Eq. (A.5) (resp. Eq. (A.4)).

Both the left- or the right-invariant MC 1-forms can be used as Dreibeins to construct a bi-invariant (that is SU(2) × SU(2) ∼ SO(4)-invariant) metric on SU(2) ($\sim S^3$) with tangent space metric δ_{AB}. The result is exactly the same in both cases: normalizing the metric so as to get the volume of the 3-sphere of unit radius, we find

$$\begin{aligned} d\Omega^2_{(3)} &= \tfrac{1}{4}v^A_L v^A_L = \tfrac{1}{4}v^A_R v^A_R\\ &= \tfrac{1}{4}\Big[d\theta^2 + d\varphi^2 + d\psi^2 + 2\cos\theta\, d\varphi d\psi\Big]\,. \end{aligned} \tag{A.14}$$

It is customary to rewrite this metric so that the invariance under the chosen U(1) action is manifest. For the standard choice in which $\psi\in[0,4\pi)$ is the periodic coordinate and there is invariance under the right action in Eq. (A.4)

$$d\Omega^2_{(3)} = \tfrac{1}{4}\Big[d\Omega^2_{(2)}(\theta,\varphi) + v^3_L v^3_L\Big]\,, \tag{A.15}$$

where $d\Omega^2_{(2)}(\theta,\varphi)$ is the standard metric of the round 2-sphere of unit radius

$$d\Omega^2_{(2)}(\theta,\varphi) = d\theta^2 + \sin^2\theta d\varphi^2 = v^1_L v^1_L + v^2_L v^2_L\,. \tag{A.16}$$

For the other choice, we just have to interchange φ and ψ and L by R in the above expressions.

[9] The σ^A are the Pauli matrices, which we take to satisfy

$$\sigma^A\sigma^B = \delta^{AB} + i\epsilon^{ABC}\sigma^C\,. \tag{A.6}$$

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Non-Abelian, supersymmetric black holes and strings in 5 dimensions

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Abstract: We construct and study the first supersymmetric black-hole and black-string solutions of non-Abelian-gauged $\mathcal{N} = 1, d = 5$ supergravity ($\mathcal{N} = 1, d = 5$ Super-Einstein-Yang-Mills theory) with non-trivial SU(2) gauge fields: BPST instantons for black holes and BPS monopoles of different kinds ('t Hooft-Polyakov, Wu-Yang and Protogenov) for black strings and also for certain black holes that are well defined solutions only for very specific values of all the moduli. Instantons, as well as colored monopoles do not contribute to the masses and tensions but do contribute to the entropies.

The construction is based on the characterization of the supersymmetric solutions of gauged $\mathcal{N} = 1, d = 5$ supergravity coupled to vector multiplets achieved in ref. [1] which we elaborate upon by finding the rules to construct supersymmetric solutions with one additional isometry, both for the timelike and null classes. These rules automatically connect the timelike and null non-Abelian supersymmetric solutions of $\mathcal{N} = 1, d = 5$ SEYM theory with the timelike ones of $\mathcal{N} = 2, d = 4$ SEYM theory [2, 3] by dimensional reduction and oxidation. In the timelike-to-timelike case the singular Kronheimer reduction recently studied in ref. [4] plays a crucial role.

Keywords: Black Holes, Supergravity Models, Black Holes in String Theory

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Contents

1 Introduction

The search for classical solutions of General Relativity and theories of gravity in general has proven to be one of the most fruitful approaches to study this universal and mysterious interaction. This is partially due to the non-perturbative information they provide, which we do not know how to obtain otherwise. It is fair to say that some of the solutions discovered (such as the Schwarzschild and Kerr black-hole solutions, the cosmological ones or the $AdS_5 \times S^5$ solution of type IIB supergravity) have opened entire fields of research.

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Some of the most interesting solutions are supported by fundamental matter fields and a large part of the search for gravity solutions has been carried out in theories in which gravity is coupled to different forms of matter, usually scalar fields, Abelian vector and p-form fields coupled in gauge-invariant ways among themselves and to scalars, as suggested by superstring and supergravity theories, for instance. The solutions of gravity coupled to non-Abelian vector fields have been much less studied because of the complexity of the equations. Most of the genuinely non-Abelian solutions found so far, such as the Bartnik-McKinnon particle [5] and its black hole-type generalizations [6], in the SU(2) Einstein-Yang-Mills (EYM) theory, are only known numerically, which makes them more difficult to study and generalize.

Supersymmetry can simplify dramatically the construction of classical solutions, providing in some cases recipes to construct systematically whole families of solutions that have the property of being "supersymmetric" or "having unbroken supersymmetry", or being "BPS" (a much less precise term) because these solutions satisfy much easier to solve first-order differential equations.[1] These techniques can be applied to non-supersymmetric theories if we can "embed" them in a larger supersymmetric theory from which they can be obtained by a consistent truncation that, in particular, gets rid of the fermionic fields.

In order to apply these techniques to the case of theories of gravity coupled to fundamental matter fields we must embed the theories first in supergravity theories. $d = 4$ EYM theories can be embedded almost trivially in $\mathcal{N} = 1, d = 4$ gauged supergravity coupled to vector supermultiplets, but there are no supersymmetric black-hole or more general particle-like solutions in $\mathcal{N} = 1, d = 4$ supergravity: all the supersymmetric solutions of these theories belong to the null class[2] and describe, generically, massless solutions such as gravitational waves and also black strings (whose tension does not count as a mass). This could well explain why there are no simple analytic solutions of the EYM theory.

Embedding of $d = 4$ EYM theories in extended ($\mathcal{N} > 1$) $d = 4$ supergravity theories turns out to be impossible, since the latter always include additional scalar fields charged under the non-Abelian fields which cannot be consistently truncated away. On the other hand, these scalar fields (or part of them) can also be interpreted as Higgs fields and we can think of those supergravities (which we will call Super-Einstein-Yang-Mills (SEYM) theories) as the minimal supersymmetric generalizations of the Einstein-Yang-Mills-Higgs (EYMH) theory. Actually, some solutions of the SEYM theories are also solutions of the EYMH theory, but this is not generically true and we cannot say that the EYMH theory is embedded in some SEYM theory.

At any rate, analytic supersymmetric solutions of SEYM or more general gauged supergravity theories should be much easier to find than solutions of the EYM theory and, at the same time, much more realistic, since we know there are scalar fields charged under non-Abelian vector fields in Nature.

This expectation turns out to be true. In 1991 Harvey and Liu [8] and in 1997 Chamseddine and Volkov [9, 10] found globally regular gravitating monopole ("global monopole")

[1]For a general review on the construction of supersymmetric solutions of supergravity theories, including some of those that we are going to study here, see ref. [7].

[2]The Killing spinor of the supersymmetric solutions in the null (resp. timelike) class gives rise to a null (resp. timelike) Killing vector bilinear.

solutions to gauged $\mathcal{N}=4$, $d=4$ supergravity, a theory that can be related to the Heterotic string. In 1994, a 4-dimensional black-hole solution with non-Abelian hair was obtained by adding stringy (Heterotic) α' corrections to an $a=1$ dilaton black hole [11]. This solution was singular in the Einstein frame.[3] More recently, the timelike supersymmetric solutions of gauged $\mathcal{N}=2, d=4$ and $\mathcal{N}=1, d=5$ were characterized, respectively, in refs. [2, 13] and [1, 14],[4] so the form of all the fields in those solutions is given in terms of a few functions that satisfy first-order equations.

In the 4-dimensional case, these first-order equations are straightforward generalizations of the well-known Bogomol'nyi monopole equations [15] whose more general static and spherically symmetric solutions for the gauge group SU(2) were obtained by Protogenov in ref. [16]. Then, the characterization of timelike supersymmetric solutions was immediately used to construct, apart from global monopole solutions, the first analytical, regular, static, non-Abelian black-hole solutions which cannot be considered as pure Abelian embeddings [2], showing how the attractor mechanism works in the non-Abelian setting [2, 3]. Colored black holes[5] and two-center non-Abelian solutions were constructed, respectively, in [17] and [12] by using, respectively, "colored monopole" and two-center solutions of the Bogomol'nyi equations.

In the $\mathcal{N}=1, d=5$ SEYM case, the characterization obtained in refs. [1, 14] has not yet been exploited. Doing so to construct non-Abelian black-hole and black-string solutions is our main goal in this paper. It is a well-known fact, one that also holds in the Abelian (ungauged) case, that the vector field strengths of the timelike supersymmetric solutions of these theories are the sum of two pieces, one of them self-dual in the hyperKähler base space, i.e. an instanton in the base space. In the non-Abelian case we are interested in, this fact can be exploited in an obvious way to add non-Abelian hair to black hole solutions.

As we are going to see, it will be convenient to refine the general characterization obtained in those references to obtain a simpler recipe to construct supersymmetric solutions with one additional isometry. These solutions are still general enough and can also be related to the timelike supersymmetric solutions of $\mathcal{N}=2, d=4$ SEYM. In the timelike-to-timelike reduction, we recover the relation between self-dual instantons in hyperKähler spaces with one isometry and BPS monopoles in $\mathbb{E}^3$ found by Kronheimer in ref. [18]. As we have shown in ref. [4] this redox relation brings us from singular colored monopoles to globally regular BPST instantons and vice-versa and it will allow us to obtain regular black holes with a BPST instanton field.

The recipes we have obtained can be applied to any model of $\mathcal{N}=1, d=5$ supergravity coupled to vector multiplets in which a non-Abelian subgroup of the perturbative duality group can be gauged. The explicit solutions we will construct will belong to a particular model, the ST$[2,5]$ model which is the smallest of the ST$[2,n]$ family of models admitting a SU(2) gauging. These models are consistent truncations of $\mathcal{N}=1, d=10$

[3] We will see, though, that it is closely related to the 4-dimensional black-hole solutions studied in [12] and to the 5-dimensional ones presented here.

[4] In the $\mathcal{N}=1, d=5$ case, the null supersymmetric solutions were characterized as well.

[5] Colored black holes have non-Abelian hair but vanishing asymptotic charges. The charges must be screened at infinity because they contribute to the near-horizon geometry and to the entropy.

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supergravity coupled to a number of vector multiplets on T^5 and, for low values of n, they can be embedded in Heterotic string theory. The SU(2) gauging can be associated to the enhancement of symmetry at the self-dual radius U(1)×U(1) →U(1)×SU(2), although, in order to study the details of the embedding of our model in Heterotic string theory (which will be our next goal) more work will be necessary.

This paper is organized as follows: in section 2 we review the gauging of a non-Abelian group of isometries of an $\mathcal{N}=1, d=5$ supergravity theory coupled to vector multiplets. The result of this procedure is what we call an $\mathcal{N}=1, d=5$ Super-Einstein-Yang-Mills (SEYM) theory. In section 3 we review and extend the results of ref. [1] on the characterization of the supersymmetric solutions of $\mathcal{N}=1, d=5$ SEYM theories, giving the recipe to construct those admitting additional isometries and showing how they are related to the analogous supersymmetric solutions of $\mathcal{N}=2, d=4$ SEYM theories characterized in ref. [3, 13]. We will then use these results in section 4 to construct black holes and black strings (in the timelike and null cases, respectively) of the SU(2)-gauged ST[2,5] model of $\mathcal{N}=1, d=5$ supergravity and to study their relations, via dimensional reduction, to the non-Abelian timelike supersymmetric solutions (black holes and global monopoles) of the SU(2)-gauged ST[2,5] model of $\mathcal{N}=2, d=4$ supergravity (see ref. [12]). Our conclusions are given in section 5. Appendix A reviews the reduction of ungauged $\mathcal{N}=1, d=5$ supergravity to a cubic model of $\mathcal{N}=2, d=4$ supergravity, with the relation between the 5- and 4-dimensional fields for any kind of solution (supersymmetric or not). This relation remains true for gauged supergravity theories under standard dimensional reduction (which does not change the gauge group). Finally, appendix B review the spherically-symmetric solutions of the Bogomol'nyi equation in $\mathbb{E}^3$ for SU(2).

2 $\mathcal{N}=1, d=5$ SEYM theories

In this section we give a brief description of general $\mathcal{N}=1, d=5$ Super-Einstein-Yang-Mills (SEYM) theories. These are theories of $\mathcal{N}=1, d=5$ supergravity coupled to n_v vector supermultiplets (no hypermultiplets) in which a necessarily non-Abelian group of isometries of the Real Special manifold has been gauged. These theories can be considered the simplest supersymmetrization of non-Abelian Einstein-Yang-Mills theories in $d=5$. Our conventions are those in refs. [1, 19] which are those of ref. [20] with minor modifications.

The supergravity multiplet is constituted by the graviton $e^a{}_\mu$, the gravitino ψ^i_μ and the graviphoton A_μ. All the spinors are symplectic Majorana spinors and carry a fundamental SU(2) R-symmetry index. The n_v vector multiplets, labeled by $x=1,\ldots,n_v$ consist of a real vector field $A^x{}_\mu$, a real scalar ϕ^x and a gaugino $\lambda^{i\,x}$.

The full theory is formally invariant under a SO(n_v+1) group[6] that mixes the matter vector fields $A^x{}_\mu$ with the graviphoton $A_\mu \equiv A^0{}_\mu$ and it is convenient to combine them into an SO(n_v+1) vector $(A^I{}_\mu) = (A^0{}_\mu, A^x{}_\mu)$. It is also convenient to define a SO(n_v+1) vector of functions of the scalars $h^I(\phi)$. These n_v+1 functions of n_v scalar must satisfy a

[6]The theory will only be invariant under a subgroup of SO(n_v+1).

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constraint. $\mathcal{N}=1, d=5$ supersymmetry determines that this constraint is of the form

$$C_{IJK}h^I(\phi)h^J(\phi)h^K(\phi) = 1, \tag{2.1}$$

where the constant symmetric tensor C_{IJK} completely characterizes the theory and the *Special Real geometry* of the scalar manifold. In particular, the kinetic matrix of the vector fields $a_{IJ}(\phi)$ and the metric of the scalar manifold $g_{xy}(\phi)$ can be derived from it as follows: first, we define

$$h_I \equiv C_{IJK}h^J h^K, \quad \Rightarrow \quad h^I h_I = 1, \tag{2.2}$$

and

$$h^I_x \equiv -\sqrt{3}h^I{}_{,x} \equiv -\sqrt{3}\frac{\partial h^I}{\partial \phi^x}, \qquad h_{Ix} \equiv +\sqrt{3}h_{I,x}, \quad \Rightarrow \quad h_I h^I_x = h^I h_{Ix} = 0. \tag{2.3}$$

Then, a_{IJ} is defined implicitly by the relations

$$h_I = a_{IJ}h^I, \qquad h_{Ix} = a_{IJ}h^J{}_x. \tag{2.4}$$

It can be checked that

$$a_{IJ} = -2C_{IJK}h^K + 3h_I h_J. \tag{2.5}$$

The metric of the scalar manifold $g_{xy}(\phi)$, which we will use to raise and lower x, y indices is (proportional to) the pullback of a_{IJ}

$$g_{xy} \equiv a_{IJ}h^I{}_x h^J{}_y = -2C_{IJK}h^I_x h^J_y h^K. \tag{2.6}$$

The functions h^I and their derivatives h^I_x satisfy the following completeness relation:

$$a_{IJ} = h_I h_J + g_{xy}h^x_I h^y_J. \tag{2.7}$$

By assumption, the real Real Special structure is invariant under reparametrizations generated by vectors $k_I{}^x(\phi)$[7]

$$\delta\phi^x = c^I k_I{}^x, \tag{2.8}$$

satisfying the Lie algebra[8]

$$[k_I, k_J] = -f_{IJ}{}^K k_K. \tag{2.9}$$

The invariance of the metric g_{xy} implies that the vectors $k_I{}^x(\phi)$ are Killing vectors. The invariance of the constraint eq. (2.1) implies the invariance of the C_{IJK} tensor

$$-3f_{I(J}{}^M C_{KL)M} = 0. \tag{2.10}$$

Multiplying this identity by $h^J h^K h^L$ we get another important relation:

$$f_{IJ}{}^K h^J h_K = 0. \tag{2.11}$$

[7]Some of these vectors may be identically zero. This is the price to be paid for labeling the gauge vectors and the Killing vectors with the same indices.

[8]Some of the structure constants may vanish identically, but it is assumed that some of them do not because, otherwise, we would be dealing with an ungauged supergravity.

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The functions $h^I(\phi)$, in their turn, must be invariant up to $\mathrm{SO}(n_v+1)$ rotations, that is

$$k_I{}^x\partial_x h^J - f_{IK}{}^J h^K = 0, \quad \Rightarrow \quad k_I{}^x = -\sqrt{3} f_{IJ}{}^K h^x_K h^J, \quad \Rightarrow \quad h^I k_I{}^x = 0\,. \tag{2.12}$$

If the real special manifold is a symmetric space, then the tensor C_{IJK} satisfies the identity

$$C^{IJK}C_{J(LM}C_{NP)K} = \frac{1}{27}\delta^I{}_{(L}C_{MNP)}\,, \tag{2.13}$$

where $C^{IJK} = C_{IJK}$. In these spaces we can solve immediately h^I in terms of the h_I

$$h^I = 27C^{IJK}h_J h_K\,, \quad \Rightarrow \quad C^{IJK}h_I h_J h_K = \frac{1}{27}\,. \tag{2.14}$$

To gauge this global symmetry group we promote the constant parameters c^I to arbitrary spacetime functions identifying them with the gauge parameters of the vector fields $\Lambda^I(x)$ $c^I \to -g\Lambda^I(x)$. The gauge transformations scalars ϕ^x, the functions h^I and the $A^I{}_\mu$ take the form

$$\begin{aligned}
\delta_\Lambda \phi^x &= -g\Lambda^I k_I{}^x, &(2.15)\\
\delta_\Lambda h^I &= -g f_{JK}{}^I \Lambda^J h^K, &(2.16)\\
\delta_\Lambda A^I{}_\mu &= \partial_\mu \Lambda^I + g f_{JK}{}^I A^J{}_\mu \Lambda^K \equiv \mathfrak{D}_\mu \Lambda^I, &(2.17)
\end{aligned}$$

where $\mathfrak{D}_\mu$ is the gauge-covariant derivative. $\mathfrak{D}_\mu h^I$ has the same expression as $\mathfrak{D}_\mu\Lambda^I$ and have the same gauge transformations as h^I and Λ^I. We also have

$$\begin{aligned}
\mathfrak{D}_\mu h_I &= \partial_\mu h_I + g f_{IJ}{}^K A^J{}_\mu h_K, &(2.18)\\
\mathfrak{D}_\mu C_{IJK} &= 0. &(2.19)
\end{aligned}$$

On the other hand, the gauge-covariant derivative of the scalars is given by

$$\mathfrak{D}_\mu \phi^x = \partial_\mu \phi^x + g A^I{}_\mu k_I{}^x, \tag{2.20}$$

and transforms as

$$\delta_\Lambda \mathfrak{D}_\mu \phi^x = -g\Lambda^I \partial_y k_I{}^x \mathfrak{D}_\mu \phi^x. \tag{2.21}$$

The gauginos $\lambda^{i\,x}$ transform in exactly the same way as $\mathfrak{D}\phi^x$ and their gauge-covariant derivatives are identical to the second covariant derivative of ϕ^x:

$$\mathfrak{D}_\mu \mathfrak{D}_\nu \phi^x = \partial_\mu \mathfrak{D}_\nu \phi^x - \Gamma^\rho_{\mu\nu}\mathfrak{D}_\rho\phi^x + \Gamma_{yz}{}^x \mathfrak{D}_\mu\phi^y \mathfrak{D}_\nu \phi^z + g A^I{}_\mu \partial_y k_I{}^x \mathfrak{D}_\nu \phi^y. \tag{2.22}$$

The gauge-covariant vector field strength has the standard form

$$F^I{}_{\mu\nu} = 2\partial_{[\mu}A^I{}_{\nu]} + g f_{JK}{}^I A^J{}_\mu A^K{}_\nu. \tag{2.23}$$

The bosonic action of $\mathcal{N}=1, d=5$ SEYM is given in terms of a_{IJ}, g_{xy}, C_{IJK} and the structure constants $f_{IJ}{}^K$ by

$$S = \int d^5x\sqrt{g}\,\left\{R + \frac{1}{2}g_{xy}\mathfrak{D}_\mu\phi^x\mathfrak{D}^\mu\phi^y - \frac{1}{4}a_{IJ}F^{I\,\mu\nu}F^J{}_{\mu\nu} + \frac{1}{12\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}\left[F^I{}_{\mu\nu}F^J{}_{\rho\sigma}A^K{}_\alpha\right.\right.$$

$$-\frac{1}{2}g f_{LM}{}^{I} F^{J}{}_{\mu\nu} A^{K}{}_{\rho} A^{L}{}_{\sigma} A^{M}{}_{\alpha} + \frac{1}{10} g^2 f_{LM}{}^{I} f_{NP}{}^{J} A^{K}{}_{\mu} A^{L}{}_{\nu} A^{M}{}_{\rho} A^{N}{}_{\sigma} A^{P}{}_{\alpha} \Big] \Big\} . \tag{2.24}$$

Observe that this action does not contain a scalar potential $V(\phi)$ because

$$V(\phi) = \frac{3}{2} g^2 h^I h^J k_I{}^x k_J{}^y g_{xy} , \tag{2.25}$$

(the expression that follows from the general formula in ref. [20]) vanishes identically for the kind of gaugings considered here, owing to the property eq. (2.12). This fact is associated to the vanishing of the corresponding fermion shift in the gauginos' supersymmetry transformations.

The equations of motion for the bosonic fields are

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &\equiv \frac{1}{2\sqrt{g}} e_{a(\mu} \frac{\delta S}{\delta e_a{}^{\nu)}} \\
&= G_{\mu\nu} - \frac{1}{2} a_{IJ} \left(F^{I}{}_{\mu}{}^{\rho} F^{J}{}_{\nu\rho} - \frac{1}{4} g_{\mu\nu} F^{I\,\rho\sigma} F^{J}{}_{\rho\sigma} \right) \\
&\quad + \frac{1}{2} g_{xy} \left(\mathfrak{D}_\mu \phi^x \mathfrak{D}_\nu \phi^y - \frac{1}{2} g_{\mu\nu} \mathfrak{D}_\rho \phi^x \mathfrak{D}^\rho \phi^y \right) && (2.26) \\
\mathcal{E}_I{}^{\mu} &\equiv \frac{1}{\sqrt{g}} \frac{\delta S}{\delta A^I{}_\mu} \\
&= \mathfrak{D}_\nu \left(a_{IJ} F^{J\,\nu\mu} \right) + \frac{1}{4\sqrt{3}} \frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}} C_{IJK} F^{J}{}_{\nu\rho} F^{k}{}_{\sigma\alpha} + g k_{I\,x} \mathfrak{D}^\mu \phi^x && (2.27) \\
\mathcal{E}^x &\equiv -\frac{g^{xy}}{\sqrt{g}} \frac{\delta S}{\delta \phi^y} \\
&= \mathfrak{D}_\mu \mathfrak{D}^\mu \phi^x + \frac{1}{4} g^{xy} \partial_y a_{IJ} F^{I\,\rho\sigma} F^{J}{}_{\rho\sigma} . && (2.28)
\end{aligned}$$

The supersymmetry transformation rules for the bosonic fields are

$$\begin{aligned}
\delta_\epsilon e^a{}_\mu &= \frac{i}{2} \bar{\epsilon}_i \gamma^a \psi^i_\mu , \\
\delta_\epsilon A^I{}_\mu &= -\frac{i\sqrt{3}}{2} h^I \bar{\epsilon}_i \psi^i_\mu + \frac{i}{2} h^I_x \bar{\epsilon}_i \gamma_\mu \lambda^{i\,x} , \\
\delta_\epsilon \phi^x &= \frac{i}{2} \bar{\epsilon}_i \lambda^{i\,x} .
\end{aligned} \tag{2.29}$$

and the corresponding transformation rules for the fermionic fields evaluated on vanishing fermions are

$$\delta_\epsilon \psi^i_\mu = \nabla_\mu \epsilon^i - \frac{1}{8\sqrt{3}} h_I F^{I\,\alpha\beta} \left(\gamma_{\mu\alpha\beta} - 4 g_{\mu\alpha} \gamma_\beta \right) \epsilon^i , \tag{2.30}$$

$$\delta_\epsilon \lambda^{i\,x} = \frac{1}{2} \left(\not{\mathfrak{D}} \phi^x - \frac{1}{2} h^x_I \not{F}^I \right) \epsilon^i , \tag{2.31}$$

where $\nabla_\mu \epsilon^i$ is just the Lorentz-covariant derivative on the spinors, given in our conventions by

$$\nabla_\mu \epsilon^i = \left(\partial_\mu - \frac{1}{4} \not{\omega}_\mu \right) \epsilon^i . \tag{2.32}$$

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The equations of motion and the supersymmetry transformation rules are the straightforward covariantization of those of the ungauged theory, except for the addition of a source to the Maxwell equations corresponding to the charge carried by the scalar fields.

3 The supersymmetric solutions of $\mathcal{N}=1, d=5$ SEYM theories

In this section we are going to review first the results of ref. [1] particularized to the case in which there are no hypermultiplets nor Fayet-Iliopoulos terms. We will simply focus on the final characterization of the supersymmetric solutions. Then, we will analyze the form of the solutions that admit an additional isometry and can, therefore, be dimensionally reduced to $d=4$, following refs. [19, 21].

Let us start by reminding the reader that a solution of one of the $\mathcal{N}=1, d=5$ SEYM theories is said supersymmetric if the so-called *Killing spinor equations*

$$\delta_\epsilon \psi^i_\mu = 0\,, \qquad \delta_\epsilon \lambda^{i\,x} = 0\,, \tag{3.1}$$

written in the background of the solution can be solved for at least one spinor $\epsilon^i(x)$, which is then called *Killing spinor*. The supersymmetric solutions of these theories can be classified according to the causal nature of the Killing vector that one can construct as a bilinear of the Killing spinor $V^a = i\bar{\epsilon}_i \gamma^a \epsilon^i$ as *timelike* ($V^a V_a > 0$) or *null* ($V^a V_a = 0$). These two cases must be discussed separately.

3.1 Timelike supersymmetric solutions

The fields of the timelike supersymmetric solutions of $\mathcal{N}=1, d=5$ SEYM theories are completely determined by

1. A choice of 4-dimensional (obviously Euclidean) hyperKähler metric

$$d\hat{s}^2 = h_{\underline{mn}}(x) dx^m dx^n\,. \tag{3.2}$$

   Fields and operators defined in this space are customarily hatted.

2. Vector fields defined in the hyperKähler space, $\hat{A}^I$, such that their 2-form field strengths, $\hat{F}^I(\hat{A})$ are self-dual

$$\hat{\star}\hat{F}^I = +\hat{F}^I\,, \tag{3.3}$$

   with respect to the hyperKähler metric. This implies that $\hat{A}^I$ defines an instanton solution of the Yang-Mills equations in the hyperKähler space.

3. A set of functions in the hyperKähler space $\hat{f}_I$ satisfying the equation[9]

$$\hat{\mathfrak{D}}^2 \hat{f}_I - \frac{1}{6} C_{IJK} \hat{F}^J \cdot \hat{F}^K = 0\,. \tag{3.4}$$

Given $h_{\underline{mn}}$, $\hat{A}^I$, $\hat{f}_I$, the physical fields can be reconstructed as follows:

[9] The coefficient of the second term is wrong by a factor of 2 in refs. [1, 19] although all subsequent formulae are correct.

1. The functions $\hat{f}_I$ are proportional to the $h_I(\phi)$ defined in eq. (2.2). The proportionality coefficient is called $1/\hat{f}$:

$$h_I/\hat{f} = \hat{f}_I\,. \tag{3.5}$$

The functions $h_I(\phi)$ satisfy a model-dependent constraint (analogous to the constraint satisfied by the functions $h^I(\phi)$, eq. (2.1)). This constraint can be obtained by solving eq. (2.2) for the h^I and substituting the result into eq. (2.1). Therefore, the constraint has the form $F(h.) = 1$ where F is a function homogeneous of degree $3/2$ in the h_I and, substituting the above equation, one gets

$$\hat{f}^{-3/2} = F(\hat{f}.)\,. \tag{3.6}$$

Using this result in eq. (3.5) one gets all the h_I as in terms of the $\hat{f}_I$

$$h_I = \hat{f}_I F^{-2/3}(\hat{f}.)\,, \tag{3.7}$$

and, using the expression of the h^I in terms of the h_I, one also gets the h^I in terms of the functions $\hat{f}_I$.

If the real special scalar manifold is symmetric, then we can use eq. (2.14) to get

$$\hat{f}^{-3} = 27C^{IJK}\hat{f}_I\hat{f}_J\hat{f}_K\,. \tag{3.8}$$

2. The scalar fields ϕ^x can be obtained by inverting the functions $h_I(\phi)$ or $h^I(\phi)$. A parametrization which is always available is

$$\phi^x = h_x/h_0 = \hat{f}_x/\hat{f}_0\,. \tag{3.9}$$

3. Next, we define the 1-form $\hat{\omega}$ through the equation

$$\left(\hat{f}d\hat{\omega}\right)^+ = \frac{\sqrt{3}}{2}h_I\hat{F}^{I+}\,. \tag{3.10}$$

4. Having solved the above equation for $\hat{\omega}$ we have determined completely the metric of the timelike supersymmetric solutions, which is given by

$$ds^2 = \hat{f}^{\,2}(dt+\hat{\omega})^2 - \hat{f}^{\,-1}h_{\underline{mn}}dx^m dx^n\,, \tag{3.11}$$

5. Also, the complete 5-dimensional vector fields are given by

$$A^I = -\sqrt{3}h^I e^0 + \hat{A}^I\,, \quad \text{where} \quad e^0 \equiv \hat{f}(dt+\hat{\omega})\,, \tag{3.12}$$

so that the spatial components are

$$A^I{}_{\underline{m}} = \hat{A}^I{}_{\underline{m}} - \sqrt{3}h^I\hat{f}\hat{\omega}_{\underline{m}}\,. \tag{3.13}$$

The field strength can be written in the form

$$F^I = -\sqrt{3}\hat{\mathfrak{D}}(h^I e^0) + \hat{F}^I\,, \tag{3.14}$$

where $\hat{\mathfrak{D}}$ is the covariant derivative in the hyperKähler space with connection $\hat{A}^I$.

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3.1.1 Timelike supersymmetric solutions with one isometry

We are particularly interested in the supersymmetric solutions that have an additional isometry. Following refs. [21, 22] we assume that the additional isometry is a triholomorphic isometry of the hyperKähler metric (i.e. an isometry respecting the hyperKähler structure), in which case, as shown in ref. [23] it must be a Gibbons-Hawking multi-instanton metric [24]. Assuming z is the coordinate associated to the additional isometry, these metrics can always be written in the form

$$h_{\underline{mn}}dx^m dx^n = H^{-1}(dz+\chi)^2 + H dx^r dx^r\,, \quad r=1,2,3\,, \tag{3.15}$$

where the z-independent function H and 1-form $\chi = \chi_{\underline{r}} dx^r$ are related by

$$d\chi = \star_3 dH\,, \tag{3.16}$$

$\star_3$ being the Hodge operator in $\mathbb{E}^3$. Assuming now that the rest of the bosonic fields of the timelike supersymmetric solutions are z-independent one can simplify eqs. (3.3), (3.4) and (3.10).

Let us start with eq. (3.3) and let us assume that the selfduality of $\hat{F}^I$ has been defined with respect to the frame and orientation

$$\hat{e}^{\,z} = H^{-1/2}(dz+\chi)\,, \qquad \hat{e}^{\,r} = H^{1/2}\delta^r{}_{\underline{r}} dx^r\,, \qquad \varepsilon_{z123} = +1\,. \tag{3.17}$$

Then, following Kronheimer [18],[10] eq. (3.3) can be rewritten as Bogomol'nyi equations for a Yang-Mills-Higgs (YMH) system in the BPS limit in $\mathbb{E}^3$ [15]

$$\breve{\mathfrak{D}}_r \Phi^I = \frac{1}{2}\varepsilon_{rst}\breve{F}^I{}_{st}\,, \tag{3.18}$$

where the 3-dimensional Higgs field and the vector fields are given by[11]

$$\begin{aligned} 2\sqrt{6}\Phi^I &\equiv H\hat{A}^I{}_{\underline{z}}\,, \\ 2\sqrt{6}\breve{A}^I_{\underline{r}} &\equiv -\hat{A}^I{}_{\underline{r}} + \chi_{\underline{r}}\hat{A}^I{}_{\underline{z}}\,. \end{aligned} \tag{3.19}$$

Thus, we can always construct a selfdual YM instanton in a Gibbons-Hawking space from a (monopole) solution of the Bogomol'nyi equation of a YMH system in $\mathbb{E}^3$ $(\Phi^I, \breve{A}^I{}_{\underline{r}})$ [18]. Many solutions of these equations are known, specially in the spherically symmetric case.[12] In ref. [4] this relation has been explored precisely for the SU(2) monopoles and instantons we are interested in, and we will make use of those results later.

We can now use this result into eq. (3.4), rewriting the 4-dimensional gauge vector in terms of the 3-dimensional gauge vector and Higgs field defined above and using the harmonicity of H and the Bogomol'nyi equation to get rid of $\breve{F}^I$ and $\breve{\mathfrak{D}}^2\Phi^I$ (which vanishes identically). The result is the equation in $\mathbb{E}^3$

$$\breve{\mathfrak{D}}^2 \hat{f}_I - g^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J\Phi^K \hat{f}_M - 8C_{IJK}\breve{\mathfrak{D}}^2\left(\Phi^J\Phi^K/H\right) = 0\,. \tag{3.20}$$

[10] See also ref. [4].

[11] We have rescaled the 3-dimensional fields by a factor of $-1/(2\sqrt{6})$ to conform to the normalization of the fields in $\mathcal{N}=2, d=4$ supergravity. See appendix A.

[12] See ref. [16] for the SU(2) case and ref. [25] and references therein for more general gauge groups.

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Defining

$$\hat{f}_I \equiv L_I + 8C_{IJK}\Phi^J\Phi^K/H\,, \tag{3.21}$$

and using the condition eq. (2.10) we find a linear equation for the functions L_I:

$$\breve{\mathfrak{D}}^2 L_I - g^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J\Phi^K L_M = 0\,. \tag{3.22}$$

Finally, let us consider eq. (3.10). Defining $\hat{\omega}$ as

$$\hat{\omega} = \omega_5(dz+\chi) + \omega\,, \quad \text{where} \quad \omega = \omega_{\underline{r}} dx^r\,, \tag{3.23}$$

eq. (3.10) gives an equation for ω_5 whose general solution is

$$\omega_5 = M + 16\sqrt{2}H^{-2}C_{IJK}\Phi^I\Phi^J\Phi^K + 3\sqrt{2}H^{-1}L_I\Phi^I\,, \quad \text{where} \quad d\star_3 dM = 0\,, \tag{3.24}$$

and the following equation for ω:

$$\star_3\, d\omega = HdM - MdH + 3\sqrt{2}\left(\Phi^I\breve{\mathfrak{D}}L_I - L_I\breve{\mathfrak{D}}\Phi^I\right)\,, \tag{3.25}$$

whose integrability condition $d^2\omega = 0$ is satisfied wherever the above equations for H, M, Φ^I, L_I are satisfied.

Summarizing: we have identified a set of z-independent functions M, H, Φ^I, L_I and 1-forms ω, A^I, χ in $\mathbb{E}^3$ in terms of which we can write all the building blocks of the 5-dimensional timelike supersymmetric solutions admitting an isometry as follows:

$$h_I/\hat{f} = L_I + 8C_{IJK}\Phi^J\Phi^K/H\,, \tag{3.26}$$

$$\hat{\omega} = \omega_5(dz+\chi) + \omega\,, \tag{3.27}$$

$$\omega_5 = M + 16\sqrt{2}H^{-2}C_{IJK}\Phi^I\Phi^J\Phi^K + 3\sqrt{2}H^{-1}L_I\Phi^I\,, \tag{3.28}$$

$$\hat{A}^I = 2\sqrt{6}\left[H^{-1}\Phi^I(dz+\chi) - \breve{A}^I\right]\,, \tag{3.29}$$

$$\hat{F}^I = 2\sqrt{6}H^{-1}\left[\breve{\mathfrak{D}}\Phi^I \wedge (dz+\chi) - \star_3 H\breve{\mathfrak{D}}\Phi^I\right]\,, \tag{3.30}$$

provided that they satisfy the following set of equations:

$$d\star_3 dM = 0\,, \tag{3.31}$$

$$\star_3 dH - d\chi = 0\,, \tag{3.32}$$

$$\star_3\breve{\mathfrak{D}}\Phi^I - \breve{F}^I = 0\,, \tag{3.33}$$

$$\breve{\mathfrak{D}}^2 L_I - g^2 f_{IJ}{}^L f_{KL}{}^M\Phi^J\Phi^K L_M = 0\,, \tag{3.34}$$

$$\star_3 d\omega - \left\{HdM - MdH + 3\sqrt{2}(\Phi^I\breve{\mathfrak{D}}L_I - L_I\breve{\mathfrak{D}}\Phi^I)\right\} = 0\,. \tag{3.35}$$

For symmetric real special manifolds we can use eq. (3.8) to write the metric function $\hat{f}$ explicitly in terms of the tensor C_{IJK} and the functions M, H, Φ^I, L_I:

$$\begin{aligned}\hat{f}^{-3} = {} & 3^3 C^{IJK}L_IL_JL_K + 3^4\cdot 2^3 C^{IJK}C_{KLM}L_IL_J\Phi^L\Phi^M/H \\ & + 3\cdot 2^6 L_I\Phi^I C_{JKL}\Phi^J\Phi^K\Phi^L/H^2 + 2^9\left(C_{IJK}\Phi^I\Phi^J\Phi^K\right)^2/H^3\,.\end{aligned} \tag{3.36}$$

Let us compare the above formulae with those of the ungauged case (in ref. [19] in our conventions). It is easy to see that all the functions M, H, Φ^I, L_I become standard harmonic functions in $\mathbb{E}^3$. Furthermore, the functions Φ^I are related to the functions K^I used in that reference by

$$\Phi^I = +\frac{1}{2\sqrt{2}} K^I \,. \tag{3.37}$$

3.1.2 Dimensional reduction of the timelike supersymmetric solutions with one isometry

The supersymmetric solutions that admit an additional isometry can be dimensionally reduced to supersymmetric solutions of $\mathcal{N} = 2, d = 4$ supergravity using the formulae in appendix A.[13] Performing explicitly this reduction will allow us to simplify the tasks of oxidation and reduction of supersymmetric solutions.

First of all, the metric of the 4-dimensional solutions obtained through the dimensional reduction takes the conventional conformastationary form of the timelike supersymmetric solutions of the $\mathcal{N} = 2, d = 4$ theory

$$ds^2 = e^{2U}(dt + \omega)^2 - e^{-2U} dx^r dx^r \,, \tag{3.38}$$

where the 1-form $\omega = \omega_{\underline{r}} dx^r$ is precisely the 1-form given in eq. (3.25) and the metric function e^{-2U} is given by

$$e^{-2U} = 2\sqrt{\frac{(\hat{f}^{\,-1}H)^3 - (\omega_5 H^2)^2}{4H^2}} \,. \tag{3.39}$$

We can compare the equations satisfied by the building blocks of the timelike supersymmetric solutions of gauged $\mathcal{N} = 1, d = 5$ supergravity (3.31)–(3.35) with the equations satisfied by the building blocks of the timelike supersymmetric solutions of gauged $\mathcal{N} = 2, d = 4$ supergravity ref. [3, 13], which we rewrite here for convenience adapting slightly the notation to avoid confusion with the different accents used to distinguish the different gauge fields:

$$-\frac{1}{\sqrt{2}} \star_3 \breve{\mathfrak{D}}\mathcal{I}^\Lambda - \breve{F}^\Lambda = 0 \,, \tag{3.40}$$

$$\breve{\mathfrak{D}}^2 \mathcal{I}_\Lambda - \frac{1}{2} g^2 f_{\Lambda\Sigma}{}^{\Omega} f_{\Delta\Omega}{}^{\Gamma} \mathcal{I}^\Sigma \mathcal{I}^\Delta \mathcal{I}_\Gamma = 0 \,, \tag{3.41}$$

$$\star_3 d\omega - 2\left[\mathcal{I}_\Lambda \breve{\mathfrak{D}}\mathcal{I}^\Lambda - \mathcal{I}^\Lambda \breve{\mathfrak{D}}\mathcal{I}_\Lambda\right] = 0 \,, \tag{3.42}$$

where $\breve{\mathfrak{D}}$ is the gauge covariant derivative associated to the modified gauge connection in $\mathbb{E}^3$

$$\breve{A}^\Lambda{}_{\underline{m}} \equiv A^\Lambda{}_{\underline{m}} - \omega_{\underline{m}} A^\Lambda{}_t \,. \tag{3.43}$$

The notation that we are using has implicit the identification of the gauge potentials $\breve{A}$ coming from 5 and 4 dimensions, except for $\Lambda = 0$. Using the formulae in appendix A

[13] These formulae are valid for any field configuration, supersymmetric or not.

with the modifications explained in the last paragraph we can identify[14]

$$\chi_{\underline{m}} = -2\sqrt{2}\breve{A}^{0}{}_{\underline{m}}\,, \tag{3.44}$$

which leads to the identifications

$$\Phi^{I} = -\frac{1}{\sqrt{2}}\mathcal{I}^{I+1}\,, \qquad L_{I} = \frac{2}{3}\mathcal{I}_{I+1}\,, \qquad H = 2\mathcal{I}^{0}\,, \qquad M = -\mathcal{I}_{0}\,. \tag{3.45}$$

These are the only formulae we need to relate timelike supersymmetric solutions in $\mathcal{N} = 1, d = 5$ supergravity with one additional isometry to timelike supersymmetric solutions in cubic model of $\mathcal{N} = 2, d = 4$ supergravity with $\mathcal{I}^{0} \neq 0$.[15]

For symmetric real special scalar manifolds we can use the explicit form of $\hat{f}$ in eq. (3.36) together with the expression for ω_5 in eq. (3.28) to get

$$\begin{aligned} e^{-2U} = 2\Bigg\{ &\frac{3^3}{4}HC^{IJK}L_{I}L_{J}L_{K} - 2^{7/2}MC_{IJK}\Phi^{I}\Phi^{J}\Phi^{K} + 2\cdot 3^4 C^{IJK}C_{KLM}L_{I}L_{J}\Phi^{L}\Phi^{M} \\ &-\frac{3^2}{2}\left(L_{I}\Phi^{I}\right)^2 - \frac{3}{\sqrt{2}}HML_{I}\Phi^{I} - \frac{1}{4}M^2H^2\Bigg\}^{1/2}\,. \end{aligned} \tag{3.46}$$

Then, using the identifications eqs. (3.45) together with the second of eqs. (A.1) we get

$$\begin{aligned} e^{-2U} = 2\Bigg\{&\left(d^{ijk}\mathcal{I}_{j}\mathcal{I}_{l} - \frac{2}{3}\mathcal{I}_{0}\mathcal{I}^{i}\right)\left(d_{ilm}\mathcal{I}^{l}\mathcal{I}^{m} + \frac{2}{3}\mathcal{I}^{0}\mathcal{I}_{i}\right) + \frac{4}{9}\mathcal{I}^{0}\mathcal{I}_{0}\mathcal{I}^{i}\mathcal{I}_{i} \\ &- (\mathcal{I}^{0}\mathcal{I}_{0} + \mathcal{I}^{i}\mathcal{I}_{i})^2\Bigg\}^{1/2}\,. \end{aligned} \tag{3.47}$$

3.2 Null supersymmetric solutions

The general form of the null supersymmetric solutions of $\mathcal{N} = 1, d = 5$ SEYM is quite involved [1], but it simplifies dramatically when one assumes the existence of an additional isometry so that all the fields are independent of the two null coordinates u and v. These are the solutions which will become timelike supersymmetric solutions of $\mathcal{N} = 2, d = 4$ SEYM upon dimensional reduction and, therefore, we are going to describe only these.

3.2.1 u-independent null supersymmetric solutions

The metric of the general null supersymmetric solutions of $\mathcal{N} = 1, d = 5$ SEYM can always brought into the form [1][16]

$$ds^2 = 2\ell du(dv + Kdu + \sqrt{2}\omega) - \ell^{-2}dx^{r}dx^{r}\,, \tag{3.48}$$

where the functions ℓ, K and the 1-form $\omega = \omega_{\underline{r}}dx^{r}$ are v-independent. We are going to assume also u-independence of all the fields throughout.

[14] The 0th components are never gauged if the dimensional reduction is simple (not generalized).

[15] Those with $\mathcal{I}^{0} = 0$ are related to null supersymmetric 5-dimensional solutions.

[16] We have changed the notation and normalization with respect to [1] to avoid possible confusions between the objects that appear in the null and timelike cases.

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After the partial gauge fixing $A^I{}_{\underline{v}} = 0$, the gauge fields are decomposed as[17]

$$A^I = A^I{}_{\underline{u}} du - 2\sqrt{6}\breve{A}^I \,, \qquad \breve{A}^I = \breve{A}^I{}_{\underline{r}} dx^r \,, \tag{3.49}$$

and the vector field strengths take the form[18]

$$F^I = (\sqrt{2/3}\ell^2 h^I \star_3 d\omega - \psi^I) \wedge du + \sqrt{3} \star_3 \breve{\mathfrak{D}}(h^I/\ell) \,, \tag{3.50}$$

where the ψ^I are some 1-forms in $\mathbb{E}^3$ satisfying

$$h_I \psi^I = 0 \,, \tag{3.51}$$

to be determined and $\breve{\mathfrak{D}}$ is the gauge-covariant derivative on $\mathbb{E}^3$ with respect to the connection $\breve{A}^I$.

Finally, the scalar fields will be determined by the equations obeyed by the scalar functions h^I, which follow from the equations of motion.[19]

Let us start by analyzing the Bianchi identities of the vector field strength. They lead to the following two sets of equations:

$$-\frac{1}{2\sqrt{2}} \star_3 \breve{\mathfrak{D}}(h^I/\ell) - \breve{F}^I = 0 \,, \tag{3.52}$$

$$\breve{\mathfrak{D}} A^I{}_{\underline{u}} - \sqrt{2/3}\ell^2 h^I \star_3 d\omega + \psi^I = 0 \,. \tag{3.53}$$

Eq. (3.52) is the Bogomol'nyi equation on $\mathbb{E}^3$ and, thus, we define the Higgs field

$$\Sigma^I \equiv -\frac{1}{2\sqrt{2}} h^I/\ell \,. \tag{3.54}$$

Multiplying eq. (3.53) by h_I and using eq. (3.51) together with $h_I h^I = 1$ we get the equation that defines ω

$$d\omega = \sqrt{3/2}\ell^{-2} \star_3 \left\{ h_I \breve{\mathfrak{D}} A^I{}_{\underline{u}} \right\} \,. \tag{3.55}$$

Defining the functions

$$K_I \equiv C_{IJK} \Sigma^J A^K{}_{\underline{u}} \,, \tag{3.56}$$

the above equation takes a much more familiar form

$$d\omega = 4\sqrt{6} \star_3 \left\{ \Sigma^I \breve{\mathfrak{D}} K_I - K_I \breve{\mathfrak{D}} \Sigma^I \right\} \,, \tag{3.57}$$

whose integrability condition is

$$\Sigma^I \breve{\mathfrak{D}}^2 K_I = 0 \,. \tag{3.58}$$

Given the functions Σ^I, K_I and the gauge fields $\breve{A}^I$ we can solve this equation for ω. It should be possible to find the functions $A^I{}_{\underline{u}}$ in terms of Σ^I, K_I[20] and, plugging these result in eq. (3.53), compute directly the 1-forms ψ^I.

[17] As the notation suggests, the gauge fields $\breve{A}^I$ are the same as the $\mathcal{N} = 2, d = 4$ fields denoted with the same symbols, according to the general formulae of appendix A. The same is true of the 1-form ω.

[18] All the operators in the r.h.s. are defined in $\mathbb{E}^3$.

[19] The field configurations that we have just described are automatically supersymmetric, but not necessarily solutions of all the equations of motion and Bianchi identities [1].

[20] This will certainly be the case for the particular model we are going to study, but we have not found (even for just the symmetric case) a general way of solving eq. (3.56) for $A^I{}_{\underline{u}}$.

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From the Maxwell equations one obtains the equations that determine the functions K_I:

$$\breve{\mathfrak{D}}^2 K_I - g^2 f_{IJ}{}^L f_{KL}{}^M \Sigma^J \Sigma^K K_M = 0\,, \tag{3.59}$$

from which the integrability condition eq. (3.58) follows automatically.

Finally, defining

$$N \equiv K - \sqrt{2} A^I{}_{\underline{u}} K_I\,, \tag{3.60}$$

the last non-trivial equation of motion, from the Einstein equations, takes the simple form

$$\nabla^2 N = 0\,. \tag{3.61}$$

Summarizing: we have identified a set of u-independent functions Σ^I, K_I, N and 1-forms $\omega, \breve{A}^I$ on $\mathbb{E}^3$ in terms of which we can write all the building blocks of the 5-dimensional u-independent null supersymmetric solutions, assuming we can solve eq. (3.56) for $A^I{}_{\underline{u}}$, as follows:

$$\begin{aligned} h^I/\ell &= -2\sqrt{2}\Sigma^I\,, & (3.62)\\ K &= N + \sqrt{2} A^I{}_{\underline{u}} K_I\,, & (3.63)\\ A^I &= A^I{}_{\underline{u}} du + 2\sqrt{6}\breve{A}^I\,, & (3.64)\\ F^I &= \breve{\mathfrak{D}} A^I{}_{\underline{u}} \wedge du + \sqrt{3} \star_3 \breve{\mathfrak{D}}(h^I/\ell)\,, & (3.65) \end{aligned}$$

provided the following equations are satisfied:[21]

$$\begin{aligned} \star_3 \breve{\mathfrak{D}}\Sigma^I - \breve{F}^I &= 0\,, & (3.66)\\ \breve{\mathfrak{D}}^2 K_I - g^2 f_{IJ}{}^L f_{KL}{}^M \Sigma^J \Sigma^K K_M &= 0\,, & (3.67)\\ d\omega - 4\sqrt{6} \star_3 \left\{\Sigma^I \breve{\mathfrak{D}} K_I - K_I \breve{\mathfrak{D}}\Sigma^I\right\} &= 0\,, & (3.68)\\ \nabla^2 N &= 0\,. & (3.69) \end{aligned}$$

Using eq. (2.1), we find a general expression for ℓ:

$$\ell^{-3} = -2^{9/2} C_{IJK} \Sigma^I \Sigma^J \Sigma^K\,. \tag{3.70}$$

3.2.2 Dimensional reduction of the u-independent null supersymmetric solutions

Using the general formulae in appendix A, the u-independent solutions that we have considered can be dimensionally reduced to timelike supersymmetric solutions of $\mathcal{N} = 2, d = 4$ SEYM along the spacelike coordinate z defined by

$$u = \frac{1}{\sqrt{2}}(t + z)\,, \qquad v = \frac{1}{\sqrt{2}}(t - z)\,, \tag{3.71}$$

with metrics of the form eq. (3.38) where the 1-form $\omega = \omega_{\underline{r}} dx^r$ is precisely the 1-form given in eq. (3.48) and the metric function e^{-2U} is given by

$$e^{-2U} = \sqrt{\ell^{-3}(1 - K)} = \sqrt{-2^{9/2} C_{IJK} \Sigma^I \Sigma^J \Sigma^K (1 - N - \sqrt{2} A^I{}_{\underline{u}} K_I)}\,. \tag{3.72}$$

[21] The gauge coupling constant is the 4-dimensional one.

In order to express entirely the metric function in terms of the functions K_I, Σ^I, N we need to solve eq. (3.56) for $A^I{}_{\underline{u}}$ as a function of K_I, Σ^I, which we do not know how to do in general. We can still compare the equations satisfied by these functions (3.66)–(3.69) with those satisfied by $\mathcal{I}^\Lambda, \mathcal{I}_\Lambda$ in $\mathcal{N}=2, d=4$ SEYM (3.40)–(3.42) knowing that the vector fields $\breve{A}^I$ and the 1-form ω are the same objects. We find that

$$\Sigma^I = -\frac{1}{\sqrt{2}}\mathcal{I}^{I+1}\,, \qquad K_I = -\frac{1}{2\sqrt{3}}\mathcal{I}_{I+1}\,, \tag{3.73}$$

while N must be proportional to either $\mathcal{I}^0$ or $\mathcal{I}_0$. Since a wave moving in the internal z direction should give rise to a 4-dimensional electric charge, it must be

$$N \sim \mathcal{I}_0\,, \tag{3.74}$$

but the precise coefficient cannot be determined from this comparison alone. We have to find a more explicit expression for e^{-2U}.

4 5-dimensional supersymmetric non-Abelian solutions of the SU(2)-gauged ST[2, 5] model

In this section we are going to consider a particular model of $\mathcal{N}=1, d=5$ supergravity that admits an SU(2) gauging. This model is related to the SU(2)-gauged ST[2, 5] model of $\mathcal{N}=2, d=4$ supergravity some of whose solutions we have studied in ref. [12]. We will use the relations derived in the previous section to find relations between the non-Abelian supersymmetric solutions of both theories.

We start by describing the 4- and 5-dimensional models and their SU(2) gauging.

4.1 The models

The ST[2, 5] model is a cubic model of $\mathcal{N}=2, d=4$ supergravity coupled to 5 vector multiplets i.e. a model with a prepotential of the form

$$\mathcal{F} = -\frac{1}{3!}\frac{d_{ijk}\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0}\,, \quad i = 1, 2\cdots, 5 \tag{4.1}$$

where the fully symmetric tensor d_{ijk} has as only non-vanishing components

$$d_{1\alpha\beta} = \eta_{\alpha\beta}\,, \quad \text{where} \quad (\eta_{\alpha\beta}) = \text{diag}(+-\cdots-)\,, \quad \text{and} \quad \alpha,\beta = 2,\cdots,5\,. \tag{4.2}$$

The 5 complex scalars parametrize the coset space

$$\frac{\text{SL}(2,\mathbb{R})}{\text{SO}(2)} \times \frac{\text{SO}(2,4)}{\text{SO}(2)\times\text{SO}(4)}\,, \tag{4.3}$$

and the group SO(3) acts in the adjoint on the coordinates $\alpha = 3, 4, 5$. These are the directions we are going to gauge and we will denote them with capital $A, B, \ldots$. This is the only information we need in order to construct supersymmetric solutions, but more details on the construction of this theory can be found in ref. [12]. We will need the form of the metric function in terms of the functions $\mathcal{I}^M$:

$$e^{-2U} = 2\sqrt{(\mathcal{I}^\alpha\mathcal{I}^\beta\eta_{\alpha\beta} + 2\mathcal{I}^0\mathcal{I}_1)(\mathcal{I}_\alpha\mathcal{I}_\beta\eta^{\alpha\beta} - 2\mathcal{I}^1\mathcal{I}_0) - (\mathcal{I}^0\mathcal{I}_0 - \mathcal{I}^1\mathcal{I}_1 + \mathcal{I}^\alpha\mathcal{I}_\alpha)^2}\,., \tag{4.4}$$

The models of the ST$[2,n]$ family are related to the effective theory of the Heterotic string and compactified on T^6 by a consistent truncation: the 10-dimensional effective theory is $\mathcal{N}=1, d=10$ supergravity coupled to 16 10-dimensional vector multiplets with gauge group U(1). Upon dimensional reduction on a generic T^6 one gets $\mathcal{N}=4, d=4$ supergravity coupled to $16+6=22$ vector multiplets, whose duality group is

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \times \frac{\mathrm{SO}(6,22)}{\mathrm{SO}(6)\times\mathrm{SO}(22)}\,. \tag{4.5}$$

Observe that SO(6) acts on the 6 vectors in the supergravity multiplet and SO(22) on the 22 matter vector fields. The coset $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ is parametrized by the only scalar in the supergravity multiplet. A consistent truncation to $\mathcal{N}=2, d=4$ eliminates 4 vectors from the $\mathcal{N}=4$ supergravity multiplet and one of the remaining two vectors becomes a matter vector field from the $\mathcal{N}=2$ point of view and comes in the same multiplet as the complex scalar that parametrizes the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$. The result is a ST$[2,23]$ model from which one can consistently eliminate vector multiplets to arrive to the ST$[2,5]$ model we are dealing with.

This is the story at a generic point in the moduli space of the Heterotic strings on T^6. At certain points, though, there is an enhancement of gauge symmetry usually associated to an increase in the number of massless vector fields that we must take into account in the effective theory. Our SU(2)-gauged model of $\mathcal{N}=2, d=4$ supergravity can be interpreted as the effective theory describing the simplest of these situations in which the enhancement of gauge symmetry arises in the sector of the 16 original 10-dimensional vector fields.

The ST$[2,5]$ model is related to a model of $\mathcal{N}=1, d=5$ supergravity coupled to 4 vector multiplets determined by the tensor $C_{i-1,j-1,k-1} = \frac{1}{6}d_{ijk}$ so its only non-vanishing components are

$$C_{0xy} = \frac{1}{6}\eta_{xy}\,, \text{where} \quad (\eta_{xy}) = \mathrm{diag}(+-\cdots-)\,, \quad \text{and} \quad x,y=1,\cdots,4\,. \tag{4.6}$$

The 4 real scalars in the vector multiplets parametrize the coset space

$$\frac{\mathrm{SO}(1,3)}{\mathrm{SO}(3)}\,. \tag{4.7}$$

Now the group SO(3) acts in the adjoint on the coordinates $x=2,3,4$ and, if we gauge it, the theory goes to the gauged 4-dimensional model we just discussed. It should be obvious after the 4-dimensional discussion that this model can be interpreted as a truncation of the effective theory of the Heterotic string compactified on T^5.

Again, we do not need many more details of the theory in order to construct supersymmetric solutions. For timelike supersymmetric solutions admitting an additional isometry we will need the metric function, which follows directly from the generic expression eq. (3.36)

$$\hat{f}^{\,-1} = H^{-1}\left\{\frac{1}{4}\left(6HL_0 + 8\eta_{xy}\Phi^x\Phi^y\right)\left[9H^2\eta^{xy}L_xL_y + 48H\Phi^0L_x\Phi^x \right.\right.$$
$$\left.\left. +64(\Phi^0)^2\eta_{xy}\Phi^x\Phi^y\right]\right\}^{1/3} \tag{4.8}$$

This metric function and the 4-dimensional one e^{-2U} are related by eq. (3.39) using eq. (3.28) and the relations between the functions $\mathcal{I}^M$ and H, M, L_I, Φ^I in eqs. (3.45), which we rewrite for this specific pair of models for convenience:

$$\begin{aligned} H &= 2\mathcal{I}^0\,, \quad \Phi^0 = -\frac{1}{\sqrt{2}}\mathcal{I}^1\,, \quad \Phi^1 = -\frac{1}{\sqrt{2}}\mathcal{I}^2\,, \quad \Phi^A = -\frac{1}{\sqrt{2}}\mathcal{I}^A\,, \\ M &= -\mathcal{I}_0\,, \quad L_0 = \frac{2}{3}\mathcal{I}_1\,, \qquad L_1 = \frac{2}{3}\mathcal{I}_2\,, \qquad L_A = \frac{2}{3}\mathcal{I}_A\,, \end{aligned} \tag{4.9}$$

For u-independent null supersymmetric solutions we first need to solve eq. (3.56) for $A^I{}_{\underline{u}}$. For this model, we find

$$A^0{}_{\underline{u}} = 6\frac{\Sigma^xK_x - \Sigma^0K_0}{(\eta\Sigma\Sigma)}\,, \qquad A^x{}_{\underline{u}} = 6\frac{\eta^{xy}K_y(\eta\Sigma\Sigma) - \Sigma^x(\Sigma^yK_y - \Sigma^0K_0)}{\Sigma^0(\eta\Sigma\Sigma)}\,, \tag{4.10}$$

where $(\eta\Sigma\Sigma) \equiv \eta_{xy}\Sigma^x\Sigma^y$, so that

$$e^{-2U} = 2\sqrt{(\mathcal{I}^\alpha\mathcal{I}^\beta\eta_{\alpha\beta})[\mathcal{I}_\alpha\mathcal{I}_\beta\eta^{\alpha\beta} + \mathcal{I}^1(1-N)] - (-\mathcal{I}^1\mathcal{I}_1 + \mathcal{I}^\alpha\mathcal{I}_\alpha)^2}\,.\,, \tag{4.11}$$

and we arrive at the following identifications

$$\begin{aligned} 0 &= \mathcal{I}^0\,, \qquad \Sigma^0 = -\frac{1}{\sqrt{2}}\mathcal{I}^1\,, \quad \Sigma^1 = -\frac{1}{\sqrt{2}}\mathcal{I}^2\,, \quad \Sigma^A = -\frac{1}{\sqrt{2}}\mathcal{I}^A\,, \\ N &= 1 + 2\mathcal{I}_0\,, \quad K_0 = -\frac{1}{2\sqrt{3}}\mathcal{I}_1\,, \quad K_1 = -\frac{1}{2\sqrt{3}}\mathcal{I}_2\,, \quad K_A = -\frac{1}{2\sqrt{3}}\mathcal{I}_A\,. \end{aligned} \tag{4.12}$$

4.2 The solutions

We are ready to put to work the machinery developed in the previous sections. We are going to consider the simplest cases first.

4.2.1 A simple 5d black hole with non-Abelian hair

In order to add non-Abelian fields to our solutions it is exceedingly useful to consider metrics with one additional isometry, because, then, we can make use of our knowledge of the spherically symmetric solutions of the Bogomol'nyi equations of the SU(2) YMH system found by Protogenov in ref. [16]. However, this isometry cannot be translational if we want to find spherically-symmetric black holes because, then, the full 5-dimensional solution will have a translational isometry. Thus, we will start with the choice $H = 1/r$ ($r^2 = y^ry^r$)[22] which, as we have shown in ref. [4], relates the *colored monopole* solution[23] to the the BPST instanton, which is spherically symmetric in $\mathbb{E}^4$.

[22]We need to distinguish between the Cartesian coordinates in $\mathbb{E}^3$, which we will denote by y^r and the Cartesian coordinates in $\mathbb{E}^4$, which we will denote by x^m. The former are not a simple subset of the latter.
[23]This monopole is characterized by a vanishing magnetic charge.

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We are, thus, going to consider a configuration with the following non-vanishing functions:

$$H=\frac{1}{r}\,,\quad L_0=A_0+\frac{q_0}{4r}\,,\quad L_1=A_1+\frac{q_1}{4r}\,,\quad \Phi^A=-f(r)\delta^A{}_r y^r\,, \tag{4.13}$$

where q_0, q_1 are electric charges in some convenient normalization, A_0, A_1 are constants to be determined through the normalization of the metric and the scalar fields at infinity and $f(r)$ is the function (not to be mistaken by $\hat{f}$) that characterizes the Higgs field in the spherically-symmetric monopole solutions of ref. [16][24]).

The next step consists in finding the 1-forms $\chi, \breve{A}^I, \omega$ and functions L_I that satisfy eqs. (3.32)–(3.35) for the above non-vanishing functions. ω is closed and can be set to zero, the functions L_I can also be set to zero while[25]

$$\chi=\cos\theta d\psi\,,\qquad \breve{A}^A=h(r)\varepsilon^A{}_{\underline{rs}}y^r dy^s\,, \tag{4.14}$$

where $h(r)$ is the function that characterizes the gauge field of the monopole solution (see appendix B)). The spacetime metric is, then,

$$ds^2=\hat{f}^{\,2}dt^2-\hat{f}^{\,-1}\left[r(d\varphi+\cos\theta d\psi)^2+\frac{1}{r}(dr^2+r^2 d\Omega^2_{(2)})\right]\,, \tag{4.15}$$

where

$$d\Omega^2_{(2)}=d\theta^2+\sin^2\theta d\psi^2\,, \tag{4.16}$$

and, upon the change of coordinates $r=\rho^2/4$, it becomes

$$ds^2=\hat{f}^{\,2}dt^2-\hat{f}^{\,-1}dx^m dx^m\,,\quad \text{where}\quad dx^m dx^m=d\rho^2+\rho^2 d\Omega^2_{(3)}\,. \tag{4.17}$$

For this configuration, the metric function eq. (4.8) is given by

$$\hat{f}^{\,-1}=3\sqrt[3]{\frac{1}{2}\left(L_0-\frac{4}{3}r^3 f^2\right)(L_1)^2}\,, \tag{4.18}$$

and it immediately follows that in order for the solution to be asymptotically regular, the monopole must be the colored one for which $r^3 f_\lambda^2\sim 1/r$, because for all the rest $r^3 f^2\sim r$ (see appendix B). With this choice,[26] as shown in ref. [4],[27] the gauge field $\hat{A}^A=\hat{A}^A{}_{\underline{m}}dx^m$ that follows from the use of eq. (3.29) is that of a BPST instanton in $\mathbb{E}^4$:

$$\hat{A}^A=\frac{1}{\tilde{g}}\frac{1}{1+\lambda^2\rho^2/4}\,v_L^A\,, \tag{4.19}$$

where v_L^A are the SU(2) left-invariant Maurer-Cartan 1-forms.[28] Since the scalar functions h^A vanish for this configuration, the full 5-dimensional vector fields are, according to

[24] See appendix B in which we have written all of Protogenov's solutions.

[25] The choice of angular coordinates is conditioned by the relation between the monopole and instanton as explained in ref. [4]. We will identify the compact coordinate z with the angular coordinate φ.

[26] We are going to study the consequences of the other choices in section 4.2.3.

[27] More specifically, the gauge field one gets is $\hat{A}_L^{A(+)}$.

[28] In our conventions, these are given by

$$\begin{cases} v_L^1=\sin\psi\, d\theta-\sin\theta\cos\psi\, d\varphi\,,\\ v_L^2=-\cos\psi\, d\theta-\sin\theta\sin\psi\, d\varphi\,,\\ v_L^3=-(d\psi+\cos\theta\, d\varphi)\,,\end{cases}\qquad \text{and}\qquad dv_L^A+\frac{1}{2}\epsilon_{ABC}\,v_L^B\wedge v_L^C=0\,. \tag{4.20}$$

eq. (3.12), given by

$$
\begin{aligned}
A^0 &= \frac{3^{5/2}}{2}(L_1)^2 \hat{f}^{\,3} dt\,, \\
A^1 &= 3^{5/2} L_1 \left(L_0 - \frac{4}{3} r^3 f_\lambda^2\right) \hat{f}^{\,3} dt\,, \\
A^A &= \frac{1}{\tilde{g}} \frac{1}{1+\lambda^2\rho^2/4}\, v_L^A\,.
\end{aligned}
\tag{4.21}
$$

Finally, the only non-vanishing scalar is given by by

$$
\phi \equiv h_1/h_0 = \frac{L_1}{L_0 - \frac{4}{3} r^3 f_\lambda^2}\,. \tag{4.22}
$$

The integration constants are readily identified in terms of the asymptotic value of the scalar as

$$
A_0 = \frac{2^{1/3}}{3}\phi_\infty^{-2/3}\,, \qquad A_1 = \frac{2^{1/3}}{3}\phi_\infty^{1/3}\,, \tag{4.23}
$$

while the mass and the area of the event horizon are given by

$$
M = 2^{-1/3} 3^{1/2} \left[\phi_\infty^{2/3} q_0 + 2\phi_\infty^{-1/3} q_1\right]\,, \tag{4.24}
$$

$$
\frac{A}{2\pi^2} = \sqrt{\frac{3^3}{2}\left(q_0 - \frac{2^7}{\tilde{g}^2}\right)(q_1)^2}\,. \tag{4.25}
$$

This solution can be understood as the result of the addition of a BPST instanton to a standard 2-charge Abelian solution. This addition does not produce any observable effects at spatial infinity, like, for instance, a change in the mass, but does produce a change in the near-horizon geometry and in the entropy.

The metric function of the 4-dimensional solution e^{-2U} that one obtains by dimensional reduction is related to the metric function of the 5-dimensional solution by

$$
e^{-4U} = \frac{1}{r}\hat{f}^{-3}\,, \tag{4.26}
$$

which implies that the 4- and 5-dimensional solutions cannot be asymptotically flat at the same time. In particular, with the choice made above (corresponding to a colored monopole in $d=4$) $e^{-2u} \sim r^{-1/2}$ at spatial infinity, a behavior that does not correspond to any known vacuum. With the monopoles we discarded, however, we get an asymptotically-flat solution. The near-horizon behavior is simultaneously good in $d=4$ and $d=5$.

4.2.2 A rotating 5d black hole with non-Abelian hair

In the context of timelike supersymmetric solutions of $\mathcal{N}=1, d=5$ supergravity rotation can be added by switching on the harmonic function M [26]. More specifically, we add to the static solution we just constructed the harmonic function

$$
M = \frac{J/2}{4r}\,, \tag{4.27}
$$

which only appears in eq. (3.28). The metric of the new solution is

$$ds^2 = \hat{f}^{\,2}\left[dt + \frac{J/2}{4r}(d\varphi + \cos\theta d\psi)\right]^2 - \hat{f}^{\,-1}\left[r(d\varphi+\cos\theta d\psi)^2 + \frac{1}{r}(dr^2 + r^2 d\Omega^2_{(2)})\right], \quad (4.28)$$

where the metric function $\hat{f}$ is still given by eq. (4.18). The scalar field ϕ and the non-Abelian vector field A^A take the same value as in the static solution while the two Abelian vector fields are modified by the change

$$dt \longrightarrow dt + \frac{J/2}{4r}(d\varphi + \cos\theta d\psi)\,, \quad (4.29)$$

which describes the presence of a magnetic dipole moment associated to the rotation.

Asymptotically, the only novelty is the off-diagonal term $\sim J/\rho^2 dt(d\varphi + \cos\theta d\psi)$ which corresponds to identical values of the two Casimirs of the angular momentum, both proportional to J, so this solution is a non-Abelian generalization of the Breckenridge-Myers-Peet-Vafa (BMPV) spinning black hole [27, 28]. The mass has the same expression in terms of the charges as in the static case.

In the near-horizon limit, if the behavior of the metric function $\hat{f}$ is

$$\hat{f}^{-1} \sim R^2/r\,, \quad (4.30)$$

for some constant R, the metric can be rewritten in the form

$$ds^2 \sim R^2 d\Pi^2_{(2)} - R^2 d\Omega^2_{(2)} - R^2\left[\cos\alpha(d\varphi + \cos\theta d\psi) - \sin\alpha\frac{r}{R^2}d\phi\right]^2, \quad (4.31)$$

where ϕ is the rescaled time coordinate, defined as follows

$$\phi \equiv t/X\,, \quad X/R \equiv \sqrt{1-[J/(2R)^3]^2} \equiv \cos\alpha\,, \quad (2R)^3 \equiv \sqrt{\frac{3^3}{2}\left(q_0 - \frac{2^7}{\tilde{g}^2}\right)(q_1)^2}\,, \quad (4.32)$$

and $d\Pi^2_{(2)}, d\Omega^2_{(2)}$ are the metrics of the 2-dimensional Anti-de Sitter and sphere of unit radius

$$d\Pi^2_{(2)} \equiv \left(\frac{r}{R^2}\right)^2 d\phi^2 - \frac{dr^2}{r^2}\,. \quad (4.33)$$

The constant-time sections of the event horizon are squashed 3-spheres with metric

$$-\,ds^2 = R^2\left\{\cos^2\alpha(d\varphi + \cos\theta d\psi)^2 + d\Omega^2_{(2)}\right\}, \quad (4.34)$$

and area

$$\frac{A}{2\pi^2} = \sqrt{\frac{3^3}{2}\left(q_0 - \frac{2^7}{\tilde{g}^2}\right)(q_1)^2 - J^2}\,. \quad (4.35)$$

4.2.3 A more general solution

In section 4.2.1 we used the colored monopole solution in order to obtain an asymptotically flat black-hole solution in the simplest way. However, we can also use the monopoles in the 2-parameter family, for which, asymptotically, $r^3 f^2 \sim r$ if we switch on additional harmonic functions and choose the values of the integration constants appropriately so that the metric functions $\hat{f}(r), \omega_5, \omega$ give an asymptotically-flat solution.

Throughout the following discussion, it is convenient to have the explicit form of these functions for $H = 1/r$, $\Phi^A = -f(r)\delta^A{}_r y^r$ and $L_A = 0$ at hand:

$$\begin{aligned}
\hat{f}^{\,-3} &= 27\left[\frac{1}{2}L_0 + \frac{2}{3}r[(\Phi^1)^2 - r^2 f^2]\right]\left[(L_1)^2 + \frac{16}{3}r\Phi^0 L_1\Phi^1 + \frac{64}{9}(r\Phi^0)^2[(\Phi^1)^2 - r^2 f^2]\right], \\
\omega_5 &= M + 8\sqrt{2}\,r^2\Phi^0[(\Phi^1)^2 - r^2 f^2] + 3\sqrt{2}\,rL_i\Phi^i\,, \\
\star_3 d\omega &= \frac{1}{r}dM - Md\frac{1}{r} + 3\sqrt{2}\left(\Phi^i dL_i - L_i d\Phi^i\right)\,,
\end{aligned} \tag{4.36}$$

where $i = 0, 1$. Apart from the functions H and Φ^A, we are going to consider the following non-vanishing harmonic functions

$$\{\Phi^0, \Phi^1, L_0, L_1, M\}\,, \tag{4.37}$$

with

$$\Phi^{0,1} = A^{0,1} + \frac{p^{0,1}}{4r}\,, \qquad L_{0,1} = A_{0,1} + \frac{q_{0,1}}{4r}\,, \qquad M = a + \frac{b}{4r}\,. \tag{4.38}$$

$\hat{f}^{-3}$ is a product of two factors. Our strategy will be to make the constant piece of Φ^1, A^1, cancel the constant piece in $rf(r)$, μ/g so that $[(\Phi^1)^2 - r^2 f^2]$ is asymptotically $\mathcal{O}(1/r)$:[29]

$$A^1 = \mu/g\,. \tag{4.39}$$

This ensures that the second term in $\hat{f}^{-3}$ diverges asymptotically at most as $\mathcal{O}(r)$ while the first is asymptotically constant. This constant can be made to vanish by choosing the constant piece of L_0, A_0, to be

$$A_0 = -\frac{8}{3}\frac{\mu}{g}\left(\frac{1}{g} + \frac{p^1}{4}\right)\,, \tag{4.40}$$

and now the first term is asymptotically $\mathcal{O}(1/r)$ and $\hat{f}^{-3}$ is asymptotically constant.

Next, we require that all the $\mathcal{O}(r^2)$, $\mathcal{O}(r)$ and $\mathcal{O}(1)$ terms in ω_5 vanish.[30] This gives two new relations[31] between the constants A_i, A^i and a. The vanishing of ω gives another relation between the same constants. Thus, requiring asymptotic flatness fixes the values of all these constants in terms of the Abelian charges p^i, q_i and μ and g. Finally the normalization of the metric at infinity also fixes the value of μ and the solution has no free moduli!

[29]We choose the positive sign for simplicity.

[30]Observe that this does not imply the complete vanishing of ω_5: there are $\mathcal{O}(1/r)$ terms that give angular momentum (which could be cancelled by the integration constant b in M) and also $\mathcal{O}(e^{-4\mu r})$ terms that cannot be cancelled. Therefore, the metric is not static even if the angular momentum is set to zero.

[31]The above values of A_0 and A^1 make the $\mathcal{O}(r^2)$ term vanish.

The values of the integration constants A_0, A^1 has been given above and the values of the rest are[32]

$$
\begin{aligned}
A_1 &= -\frac{88}{3}A^0\left(\frac{1}{g}+\frac{p^1}{4}\right),\\
A^0 &= \left\{\frac{\left(16p^0+4gp^0p^1+gq^1\right)\left(4+gp^1\right)^{-1}}{40\left(3q_0+(p^1)^2-\frac{16}{g^2}\right)\left(q_0+2(p^1)^2-\frac{32}{g^2}\right)}\right\}^{1/3},\\
\mu &= A^0\left[\frac{32-2g^2(p^1)^2-g^2q_0}{16p^0+4gp^0p^1+gq_1}\right],\\
a &= \sqrt{2}A^0\left[\frac{48}{g^2}+\frac{22p^1}{g}+\frac{5(p^1)^2}{2}-\frac{3q_0}{4}\right]-\sqrt{2}\left[\frac{22\mu p^0}{g^2}+\frac{11\mu p^0p^1}{2g}+\frac{3\mu q_1}{4g}\right],\\
b &= J/2-6\sqrt{2}\left[\frac{p^0(p^1)^2}{2}+\frac{p^0q_0+p^1q^1}{8}-8\frac{p^0}{g^2}\right],
\end{aligned}
\tag{4.41}
$$

where J is the angular momentum.

The mass of this solution is given by

$$
M=\frac{\pi A^0}{2G}\left[3q_0+(p^1)^2-\frac{16}{g^2}\right]\left[3\frac{\mu}{g}q_1+8\left(\frac{1}{g}+\frac{p^1}{4}\right)\left(10A^0\left(\frac{24}{g}+5p^1\right)-9\frac{\mu}{g}p^0\right)\right]. \tag{4.42}
$$

and the area of the horizon is

$$
\frac{A}{2\pi^2}=\sqrt{\frac{1}{2}\left[3q_0+(p^1)^2-\frac{16}{g^2}\right]\left[3q_1+2p^1p^0-\frac{8p^0}{g}\right]\left[3q_1+2p^0p^1+\frac{8p^0}{g}\right]-J^2}\,. \tag{4.43}
$$

4.2.4 Null supersymmetric non-Abelian 5d solutions from 4d black holes and global monopoles

Using the general results of the preceding sections it is very easy to construct null supersymmetric solutions by uplifting 4-dimensional timelike supersymmetric solutions with $\mathcal{I}^0$. In particular, we can uplift the black-hole and global-monopole solutions of the ST$[2,5]$ model recently constructed in ref. [12]. In this paper we will focus on the single center solutions only.

The 4-dimensional solutions depend on the following non-vanishing $\mathcal{I}^M$

$$
\begin{aligned}
&\mathcal{I}^1 = A^1+\frac{p^1/\sqrt{2}}{r}\,,\quad \mathcal{I}^2 = A^2+\frac{p^2/\sqrt{2}}{r}\,,\quad \mathcal{I}^A=\sqrt{2}\,\delta^A{}_p x^p f(r)\,,\\
&\mathcal{I}_0 = A_0+\frac{q_0/\sqrt{2}}{r}\,,
\end{aligned}
\tag{4.44}
$$

where $f(r)$ is the function $f_{\mu,s}$ or f_λ in appendix B corresponding to one of the spherically-symmetric BPS SU(2) monopoles, p^1, p^2, q_0 are magnetic and electric charges and A^1, A^2, A_0 integration constants to be determined in terms of the asymptotic values of the scalars and the metric.

[32] We have not reexpressed the 4-dimensional gauge coupling constant g in terms of the 5-dimensional, $\tilde{g}$ to have simpler expressions.

The 5-dimensional metric is that of an intersection of a string lying along the z direction and a pp-wave propagating along the same direction:

$$ds^2 = 2\ell du(dv + K du) - \ell^{-2} d\vec{x}^2_{(3)}\,, \tag{4.45}$$

where

$$\ell^{-3} = 4\mathcal{I}^1[(\mathcal{I}^2)^2 - 2r^2 f^2]\,, \qquad K = 1 + 2\mathcal{I}_0\,. \tag{4.46}$$

The scalar fields, defined by $\phi^x \equiv h^x/h^0$, are given by

$$\phi^1 = \mathcal{I}^2/\mathcal{I}^1\,, \qquad \phi^A = -\delta^A{}_p x^p f(r)/\mathcal{I}^1\,, \tag{4.47}$$

and the vector fields are given by

$$A^{0,1} = -2\sqrt{6}p^{1,2}A\,, \qquad A^A = 2\sqrt{6}h(r)\epsilon^A{}_{rs}x^r dx^s\,, \tag{4.48}$$

where A is the vector field of a Dirac magnetic monopole of unit charge, satisfying $dA = \star_3 d\frac{1}{r}$ and $h(r)$ is the function $h_{\mu,s}$ or h_λ in appendix B corresponding to one of the spherically-symmetric BPS SU(2) monopoles.

The 4-dimensional electric charge q_0 corresponds to the momentum of the 5-dimensional gravitational wave in the z direction and none of the scalar and vector fields depend on it. For the sake of simplicity we are going to set it to zero ($q_0 = 0$ and $\mathcal{I}_0 = -1/2$ so $K = 0$) and we are going to analyze the string solutions with the above scalar and vector fields and with metric

$$ds^2 = \ell(dt^2 - dz^2) - \ell^{-2} d\vec{x}^2_{(3)}\,, \tag{4.49}$$

with the metric function ℓ given as above.

The metric will be regular in the $r \to 0$ limit if $\ell \sim r$ or $\ell \sim$ constant. These two behaviors are, respectively, those of extremal black strings in the near-horizon limit and those of global monopoles. Let us consider each case separately.

Global string-monopoles. These are the string-like solutions that, upon dimensional reduction along z, give the spherically-symmetric global monopoles constructed in ref. [12]. They can be constructed with $f(r) = f_{\mu,s=0}(r)$ (the BPST 't Hooft-Polyakov monopole) and with $p^1 = p^2 = 0$, so that

$$\ell^{-3} = 4A^1[(A^2)^2 - 2r^2 f^2_{\mu,\,s=0}]\,, \quad \phi^1 = A^2/A^1\,, \quad \phi^A = -\sqrt{2}\delta^A{}_r x^r f_{\mu,s=0}(r)/A^1\,, \tag{4.50}$$

and the only non-trivial vector field is A^A.

The integration constants $A^{1,2}, \mu$ are given by

$$A^1 = \frac{1}{\chi_\infty^{1/3}}\,, \qquad A^2 = \frac{\phi^1_\infty}{\chi_\infty^{1/3}}\,, \qquad \mu = \frac{g|\phi_\infty|}{\sqrt{2}\chi_\infty^{1/3}}\,, \qquad \chi_\infty \equiv 4[(\phi^1_\infty)^2 - |\phi_\infty|^2]\,, \tag{4.51}$$

where $|\phi_\infty|^2$ is the asymptotic value of the gauge-invariant combination $\phi^A\phi^A$, and the string's tension (simply defined as minus the coefficient of $1/r$ in the large-r expansion of g_{tt}) is given by [29, 30]

$$T_{\text{monopole}} = \frac{32|\phi_\infty|}{\sqrt{3}\chi_\infty^{2/3}} \frac{1}{|\tilde{g}|}\,. \tag{4.52}$$

These are globally regular solutions with no horizons, like their 4-dimensional analogues.

Black strings. They must necessarily have non-vanishing magnetic charges $p^{1,2}$ in order to have a regular horizon. This horizon will be a 2-dimensional surface characterized by being normal to 2 linearly independent null vectors. The mass and entropy of the black string will depend on the choice of monopole.

Let us first consider the BPST 't Hooft-Polyakov monopole (or equivalently, let us add magnetic charges $p^{1,2}$ to the above global monopole). In this case, the relation between the integration constants $A^{1,2}, \mu$ and the asymptotic values of the scalars will be the same as before. The string's tension and the area of the horizon contain contributions from the magnetic charges p^1, p^2:

$$T = \frac{1}{3\sqrt{2}}\chi_\infty^{1/3}\left[p^1 + 8\frac{\phi^1_\infty}{\chi_\infty}p^2\right] + T_{\text{monopole}}\,, \tag{4.53}$$

$$\frac{A}{4\pi} = 2\left[p^1(p^2)^2\right]^{2/3}\,. \tag{4.54}$$

When we consider the more general 't Hooft-Polyakov-Protogenov monopole we find that the area of the horizon receives a contribution from the non-Abelian charge,

$$\frac{A}{4\pi} = 2\left\{p^1\left[(p^2)^2 - \frac{2}{g^2}\right]\right\}^{2/3}\,. \tag{4.55}$$

5 Conclusions

In this paper we have studied the general procedure to construct timelike and null supersymmetric solutions of $\mathcal{N}=1, d=5$ SEYM theories that can be dimensionally reduced to timelike solutions of $\mathcal{N}=2, d=4$ SEYM theories. These solutions, therefore, can also be constructed by oxidation of the 4-dimensional solutions and we have striven to clarify this procedure and find the relations between the 4- and 5-dimensional fields and the 4- and 5-dimensional equations they satisfy. The relation between instantons in 4-dimensional hyperKähler spaces and monopoles satisfying the Bogomol'nyi equation in $\mathbb{E}^3$ found by Kronheimer plays a crucial role in this relation and, in combination with the results obtained in ref. [4], it allows us to construct spherically-symmetric 5-dimensional solutions that contain YM instantons. The standard oxidation of monopoles gives rise to 5-dimensional solutions that have an additional translational isometry and cannot be spherically symmetric.

We have exploited the general results to construct the first 5-dimensional black-hole and black-string solutions with non-Abelian YM fields. The simplest black-hole solutions contain the field of a BPST instanton in the so-called *base space* and their behavior is similar to that of the colored black holes found in 4-dimensional SEYM theories [17, 25]: the non-Abelian YM field cannot be "seen" at spatial infinity, it does not contribute to the mass, but it can be seen in the near-horizon limit and it contributes to the entropy. One can compare the entropies of the simplest non-Abelian black hole with that of another black hole with the same Abelian charges and moduli (and, henceforth, with the same

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mass). The entropy of the former is always smaller, so it is entropically favorable to lose the non-Abelian field. It is not clear by which mechanism this can happen.

We have also found more complicated black-hole solutions which contain the field of the instantons that one obtains by oxidizing Protogenov monopoles in the so-called *base space*. Those instantons are not regular in flat space and, in general, the spacetime metrics they give rise to are not asymptotically flat. We have shown that a judicious choice of the integration constants (and, hence, of the moduli) in terms of the charges produces a metric that is not only asymptotically flat with positive mass but also has a regular horizon. Thus, at special points in the moduli space of the scalar manifold, additional non-Abelian black-hole solutions are possible. In these solutions, the YM fields do contribute to the mass and to the entropy.

Finally, we have also found black-string solutions by conventional oxidation of non-Abelian black-hole solutions from 4 dimensions. One of them is a globally-regular string-monopole solution and the rest are more conventional solutions.

It is clear that the new solutions that we have constructed need further study. Their string-theoretic interpretation could be very interesting. The model we have chosen to construct explicit solutions is a truncation of the effective theory of the heterotic string compactified to 5 dimensions and can, alternatively, be seen as associated to the compactification of the type IIB theory in K3 times a circle. This should simplify a bit the task and, perhaps, open the way to a microscopic interpretation of entropies that depend on parameters that do not appear at infinity. Work in this direction is in progress.

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A Dimensional reduction of $\mathcal{N}=1, d=5$ SEYM theories

$\mathcal{N}=1, d=5$ supergravity coupled to vector multiplets gives $\mathcal{N}=2, d=4$ supergravity coupled to vector multiplets upon dimensional reduction over a spacelike circle.[33] If some non-Abelian subgroup of the isometry group of the scalar manifold of the 5-dimensional theory has been gauged, and we perform a simple (as opposed to a generalized) dimensional reduction, the 4-dimensional theory will have exactly the same non-Abelian subgroup of the (now bigger) isometry group gauged. Thus $\mathcal{N}=1, d=5$ and $\mathcal{N}=2, d=4$ SEYM theories are related by dimensional reduction over a spacelike circle.

[33] See, for instance, refs. [31] and references therein.

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It should be clear that, under the above conditions, the relation between the 5- and 4-dimensional fields in the gauged theories is exactly the same as in the ungauged one and is, therefore, well known. In the conventions we follow here[34] the relation between the bosonic fields of an $\mathcal{N}=1, d=5$ supergravity model defined by C_{IJK} (tilded) and the bosonic fields of a cubic model of $\mathcal{N}=2, d=4$ supergravity defined by the symmetric tensor d_{ijk} (untilded) are[35]

$$\begin{aligned}
g_{\mu\nu} &= |\tilde{g}_{\underline{zz}}|^{\frac{1}{2}}\left(\tilde{g}_{\mu\nu}-\tilde{g}_{\mu\underline{z}}\tilde{g}_{\nu\underline{z}}/\tilde{g}_{\underline{zz}}\right), & d_{ijk} &= 6C_{i-1\,j-1\,k-1},\\
A^0{}_\mu &= \frac{1}{2\sqrt{2}}\tilde{g}_{\mu\underline{z}}/\tilde{g}_{\underline{zz}}\,, & A^i{}_\mu &= -\frac{1}{2\sqrt{6}}\left(\tilde{A}^{i-1}{}_\mu-\tilde{A}^{i-1}{}_{\underline{z}}\tilde{g}_{\mu\underline{z}}/\tilde{g}_{\underline{zz}}\right),\\
Z^i &= \frac{1}{\sqrt{3}}\tilde{A}^{i-1}{}_{\underline{z}}+i|\tilde{g}_{\underline{zz}}|^{\frac{1}{2}}\tilde{h}^{i-1}\,,
\end{aligned} \tag{A.1}$$

and the inverse relations are

$$\begin{aligned}
\tilde{g}_{\underline{zz}} &= -k^2\,, & \tilde{A}^I{}_{\underline{z}} &= \sqrt{3}\,\mathfrak{Re}Z^{I+1}\,,\\
\tilde{g}_{\mu\underline{z}} &= -2\sqrt{2}k^2A^0{}_\mu\,, & \tilde{A}^I{}_\mu &= -2\sqrt{6}\left(A^{I+1}{}_\mu-\mathfrak{Re}Z^{I+1}A^0{}_\mu\right),\\
\tilde{g}_{\mu\nu} &= k^{-1}g_{\mu\nu}-8k^2A^0{}_\mu A^0{}_\nu\,, & \tilde{h}^I &= k^{-1}\mathfrak{Im}Z^{I+1}\,.
\end{aligned} \tag{A.2}$$

In these relations it has been taken into account that, if n_v denotes the number of vector multiplets in $d=5$, then, the 4-dimensional theory has n_v+1 vector multiplets so that $I,J,K=0,\cdots,n_v$, $i,j,k=0,\cdots,n_v+1$. The additional 4-dimensional vector multiplet is the $i=0$ one and, therefore, the 5-dimensional vector labeled by I corresponds to the 4-dimensional vector labeled by $i=I+1$.

While this is the whole story for the fields, it is important to realize that the factor that related the 4- and 5-dimensional gauge fields changes the standard form of the covariant derivatives and gauge field strengths and it must be absorbed into a redefinition of the gauge coupling constant. Thus, we also have

$$\tilde{g}=-2\sqrt{6}g\,. \tag{A.3}$$

Observe that this result has been obtained using the orientation $\varepsilon^{0123z}=+1$, which is not the one we are using in the main text ($\varepsilon^{0z123}=+1$). However, in practice, the result can be adapted to that orientation by reversing the sign of each z tensor index. This operation only changes the sign of $A^0{}_\mu$ and $\mathfrak{Re}Z^i$.

B Spherically-symmetric solutions of the SU(2) Bogomol'nyi equations in $\mathbb{E}^3$

The equations of motion of the SU(2) Yang-Mills-Higgs (YMH) theory in the Bogomol'nyi-Prasad-Sommerfield (BPS) limit in which the the Higgs potential vanishes read

$$\mathfrak{D}_\mu F^{A\,\mu\nu}=-g\varepsilon_{BC}{}^A\Phi^B\mathfrak{D}^\nu\Phi^C\,, \tag{B.1}$$

[34]That is, the conventions used in refs. [1, 14, 19] for the $\mathcal{N}=1, d=5$ theories and in the conventions used in refs. [2–4, 12, 13, 17, 25, 32, 33] for the $\mathcal{N}=2, d=4$ theories.

[35]See, for instance, ref. [7] which follows the conventions used here.

$$\mathfrak{D}^2\Phi^A = 0\,. \tag{B.2}$$

Static configurations satisfying the first-order *Bogomol'nyi equations* [15]

$$F^A{}_{\underline{rs}} = \varepsilon_{rst}\mathfrak{D}_{\underline{t}}\Phi A\,, \tag{B.3}$$

can be seen to satisfy all the above second-order YMH equations of motion.

BPS magnetic monopole solutions such as the (BPS) 't Hooft-Polyakov monopole found by Prasad and Sommerfield in ref. [34] satisfy the Bogomol'nyi equations and, therefore, it is of some interest to identify all their solutions. In the spherically-symmetric case this problem was solved by Protogenov in ref. [16] and his solution can be described as follows: the Higgs and gauge field can always be brought to this form (*hedgehog ansatz*)

$$\Phi^A = -\delta^A{}_s f(r) y^s\,, \qquad A^A{}_{\underline{r}} = -\varepsilon^A{}_{rs} y^s h(r)\,, \tag{B.4}$$

in which they are characterized by just two functions, $f(r), h(r)$ of the radial coordinate $r = \sqrt{y^s y^s}$. There is only a 2-parameter family for which these functions, denoted by $(f_{\mu,s}, h_{\mu,s})$, are given by

$$rf_{\mu,s} = \frac{1}{gr}\left[1 - \mu r \coth(\mu r + s)\right]\,, \qquad rh_{\mu,s} = \frac{1}{gr}\left[\frac{\mu r}{\sinh(\mu r + s)} - 1\right]\,, \tag{B.5}$$

and a 1-parameter family for which these functions, denoted by (f_λ, h_λ), are given by

$$rf_\lambda = \frac{1}{gr}\left[\frac{1}{1+\lambda^2 r}\right]\,, \qquad rh_\lambda = -rf_\lambda\,. \tag{B.6}$$

The BPS 't Hooft-Polyakov monopole [34] is the only globally regular solution and corresponds to $f_{\mu,s=0}$. The $f_{\mu,s=\infty}$ solution is given by

$$-rf_{\mu,\infty} = \frac{\mu}{g} - \frac{1}{gr}, \qquad rh_{\mu,\infty} = -\frac{1}{gr}\,, \tag{B.7}$$

and, for $\mu = 0$, it is the Wu-Yang monopole [35]. The latter solution is also recovered in the 1-parameter family for $f_{\lambda=0}$.

The asymptotic behavior of $rf(r)$ (which is the combination that occurs in the metrics we study) for the different solutions is

$$rf_{\mu,s} \sim -\frac{\mu}{g} + \frac{1}{gr} + \mathcal{O}(e^{-4\mu r})\,, \qquad -rf_\lambda \sim \frac{1}{g\lambda^2 r^2} + \mathcal{O}(r^{-3})\,, \tag{B.8}$$

and the behavior near the origin (where the black-hole horizons may be in the metrics under study) are

$$rf_{\mu,0} \sim -\frac{\mu^2}{2g} r + \mathcal{O}(r^3)\,, \quad rf_{\mu,s} \sim \frac{1}{gr} - \frac{\mu}{g}\coth s + \mathcal{O}(r)\,, \quad rf_\lambda \sim \frac{1}{gr} - \frac{\lambda^2}{g} r + \mathcal{O}(r^3)\,. \tag{B.9}$$

If we define the magnetic monopole charge by

$$p \equiv \frac{1}{4\pi}\int_{S^2_\infty} \mathrm{Tr}(\hat{\Phi} F)\,, \qquad \hat{\Phi} \equiv \frac{\Phi}{\sqrt{|\mathrm{Tr}(\Phi^2)|}}\,, \tag{B.10}$$

then, we always find $p = 1/g$ except in the 1-parameter family for finite λ, for which we find $p = 0$. As we have argued in ref. [12], the $\lambda \neq 0$ *colored* monopoles can be seen as a magnetic monopole placed at the origin whose charge is completely screened at infinity.

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A non-Abelian black ring

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ABSTRACT

We construct a supersymmetric black ring solution of SU(2) $\mathcal{N}=1$, $d=5$ Super-Einstein–Yang–Mills (SEYM) theory by adding a *distorted* BPST instanton to an Abelian black ring solution of the same theory. The change cannot be observed from spatial infinity: neither the mass, nor the angular momenta or the values of the scalars at infinity differ from those of the Abelian ring. The entropy is, however, sensitive to the presence of the non-Abelian instanton, and it is smaller than that of the Abelian ring, in analogy to what happens in the supersymmetric colored black holes recently constructed in the same theory and in $\mathcal{N}=2$, $d=4$ SEYM. By taking the limit in which the two angular momenta become equal we derive a non-Abelian generalization of the BMPV rotating black-hole solution.

0. Introduction

The discovery of black rings by Emparan and Reall in Ref. [1] showed how two important properties of 4-dimensional asymptotically-flat black holes, uniqueness/no-hair and spherical topology of the event horizon (which, for the 5-dimensional black ring, is $S^2\times S^1$), could be violated in higher dimensions.[1] For a range of values of the conserved charges (mass, angular momenta) that may characterize an uncharged black ring, a different black-ring and a black-hole solutions are also possible. For charged black rings (the first of which was constructed in Ref. [5]) the non-uniqueness becomes infinite; for the same conserved electric charges one can construct black rings with regular horizons with magnetic dipole momenta taking continuous values in some interval [6]. Despite being innocuous to the conserved charges, these dipole momenta do contribute to the BH entropy. The construction of supersymmetric black-ring solutions in minimal [7] or matter-coupled $\mathcal{N}=1$, $d=5$ supergravity [8–12] using the general classification of supersymmetric solutions of these theories started in Ref. [13] opened up the possibility of constructing very general families of black-ring solutions with various kinds of electric charges and moduli in which these issues could be studied.

The violation of the no-hair conjecture by non-Abelian fields in 4-dimensions is also a well-known but less stressed fact, perhaps because the first solutions in which this was observed [14–16], black-hole generalizations of the "Bartnik–McKinnon particle" [17] with asymptotically vanishing gauge charges, were purely numerical, which makes more difficult their study and understanding.[2] The first black-holes with non-Abelian hair (not related to the embedding of an Abelian field into a non-Abelian one through a singular gauge transformation) given in an analytical form were found using supersymmetry techniques in the context of $\mathcal{N}=2$, $d=4$ Super-Einstein–Yang–Mills (SEYM) theory[3] in Refs. [20] and [21] using the general classification of the timelike supersymmetric solutions of these theories made in Ref. [22]. The black-hole solutions constructed in Ref. [21] include the field of an SU(2) *colored* monopole found by Protogenov in [23] which also has asymptotically vanishing gauge charge. The monopole charge does contribute to the entropy, though. These black holes, which can be seen as the result of adding the colored monopole to a standard black hole with Abelian charges, modifying the entropy but none of the asymptotic charges, were called *colored black holes* and they seem to be ubiquitous [24].

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[1] See, for instance, the reviews [2–4] and references therein.

[2] For a review on hairy and non-Abelian black-hole solutions see Ref. [18] or the more recent Ref. [19].

[3] This theory is the simplest $\mathcal{N}=2$ supersymmetric generalization of the Einstein–Yang–Mills theory. This supersymmetrization requires the addition of scalar fields to the pure Einstein–Yang–Mills theory in order to complete $\mathcal{N}=2$, $d=4$ vector supermultiplets and, often, the addition of full vector supermultiplets to fulfill the requirements of Special Geometry. There may be more than one way of performing this supersymmetrization. Thus, there are more than one $\mathcal{N}=2$, $d=4$ SEYM theory with gauge group SU(2), for instance. These theories are also known as non-Abelian gauged $\mathcal{N}=2$, $d=4$ supergravity coupled to vector supermultiplets.

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The results of Ref. [22] have been used more recently to construct new single-center and two-center non-Abelian solutions of $\mathcal{N}=2$, $d=4$ SEYM models that can be obtained by dimensional reduction of $\mathcal{N}=1$, $d=5$ SEYM models[4] in Ref. [25].

One of the main goals of that exercise was to open the possibility for the construction of the first non-Abelian black-hole solutions in $d=5$ by oxidation to $d=5$ of those solutions, because the direct construction using the general classification of timelike supersymmetric solutions of Refs. [26,27] turns out to be too complicated. This can only be done for certain models of the lower dimensional theory. The oxidation itself turned out to be a non-trivial exercise if one wanted to construct solutions without spatial translation isometries (which would be black strings instead of black holes), but, as was shown in Ref. [28], one can use non-trivial cycles to perform the reduction and still preserve supersymmetry, basically using Kronheimer's mechanism [29]. Both kinds of black solutions (strings and holes) were recently constructed in Ref. [30].

The $d=5$ non-Abelian black holes constructed there are, again, colored black holes, with asymptotically vanishing gauge fields. They can be understood as the result of adding a BPST instanton to a black hole with Abelian charges, leaving the mass and electric charges unmodified. Just as in the 4-dimensional case, the non-Abelian field does contribute to the entropy. The BPST instanton field turns out to be related by dimensional redox to the colored monopole at the heart of the 4-dimensional colored black holes.

It is natural to try to see if black-rings also admit the addition of non-Abelian instanton fields and the effect this addition may have on the mass and entropy. In this paper we are going to construct and study a regular supersymmetric black-ring solution of $\mathcal{N}=1$, $d=5$ SEYM with a *distorted* BPST instanton. We start by reviewing in Section 1 the recipe that we are going to use to construct timelike supersymmetric solutions, which was obtained in Ref. [30]. In Section 2 we will carry out the construction of the solution after which we will study its regularity and we will compute its essential properties. In Section 3 we will study the limit in which the black ring becomes a non-Abelian rotating black hole. Our conclusions are in Section 4.

1. The recipe to construct solutions

$\mathcal{N}=1$, $d=5$ Super-Einstein–Yang–Mills (SEYM) theories describe a supergravity multiplet (constituted by the graviton $e^a{}_\mu$, the gravitino ψ^i_μ and the graviphoton $A^0{}_\mu$) coupled to n_v vector multiplets labeled by $x=1,\cdots,n_v$ (each containing a real vector field $A^x{}_\mu$, a real scalar ϕ^x and a gaugino λ^{ix}). The graviphoton and the matter vector fields are collectively denoted by $A^I{}_\mu$, $I,J,\ldots=0,1,\cdots,n_v$. The ungauged theory (the couplings between scalars and vector fields dictated by the σ-model metric $g_{xy}(\phi)$, the kinetic matrix $a_{IJ}(\phi)$ and the Chern–Simons couplings) is completely determined by a constant symmetric tensor C_{IJK}.[5]

In the gauged theory, a subset of the vector fields plays the role of gauge field of some non-Abelian group whose structure constants will be denoted by $f_{IJ}{}^K$ in the understanding that they will just vanish for the values of the indices that do not correspond to the gauge fields. The transformations of the scalars under the gauge group are generated by Killing vectors of the σ-model metric $k_I{}^x(\phi)$ satisfying the Lie algebra of the gauge group. Again, it is assumed that they can be identically zero for the values of $I,J,\cdots$ corresponding to the ungauged directions.

Thus, the bosonic action of $\mathcal{N}=1$, $d=5$ SEYM is given by

$$
\begin{aligned}
S = \int d^5x\sqrt{g}\,\Big\{ & R+\tfrac{1}{2}g_{xy}\mathfrak{D}_\mu\phi^x\mathfrak{D}^\mu\phi^y-\tfrac{1}{4}a_{IJ}F^{I\,\mu\nu}F^J{}_{\mu\nu}\\
&+\tfrac{1}{12\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\alpha}}{\sqrt{g}}\Big[F^I{}_{\mu\nu}F^J{}_{\rho\sigma}A^K{}_\alpha\\
&-\tfrac{1}{2}\hat{g}f_{LM}{}^IF^J{}_{\mu\nu}A^K{}_\rho A^L{}_\sigma A^M{}_\alpha\\
&+\tfrac{1}{10}\hat{g}^2f_{LM}{}^If_{NP}{}^JA^K{}_\mu A^L{}_\nu A^M{}_\rho A^N{}_\sigma A^P{}_\alpha\Big]\Big\},
\end{aligned}
\tag{1.1}
$$

where $\hat{g}$ is the gauge coupling constant, $F^I{}_{\mu\nu}=2\partial_{[\mu}A^I{}_{\nu]}+\hat{g}f_{JK}{}^IA^J{}_\mu A^K{}_\nu$ are the non-Abelian vector field strengths and $\mathfrak{D}_\mu\phi^x=\partial_\mu\phi^x+\hat{g}A^I{}_\mu k_I{}^x$ are the gauge-covariant derivatives of the scalars.

In Ref. [30] we have found a procedure to construct systematically timelike supersymmetric solutions admitting an additional spacelike isometry (with adapted coordinate z) of any $\mathcal{N}=1$, $d=5$ Super-Einstein–Yang–Mills (SEYM):

1. Find a set of t- and z-independent functions M, H, Φ^I, L_I and 1-forms ω, A^I, χ in $\mathbb{E}^3$ satisfying the equations (defined in $\mathbb{E}^3$ as well)[6]

$$d\star_3 dM=0\,, \tag{1.2}$$

$$\star_3 dH-d\chi=0\,, \tag{1.3}$$

$$\star_3\breve{\mathfrak{D}}\Phi^I-\breve{F}^I=0\,, \tag{1.4}$$

$$\breve{\mathfrak{D}}^2L_I-g^2f_{IJ}{}^Lf_{KL}{}^M\Phi^J\Phi^KL_M=0\,, \tag{1.5}$$

$$\star_3 d\omega-\Big\{HdM-MdH+3\sqrt{2}(\Phi^I\breve{\mathfrak{D}}L_I-L_I\breve{\mathfrak{D}}\Phi^I)\Big\}=0\,. \tag{1.6}$$

The first two equations state that H and M are harmonic functions on $\mathbb{E}^3$. Once H is given, the second equation (which is the Abelian Bogomol'nyi equation on $\mathbb{E}^3$ [33]) can be solved for χ. Eq. (1.4) is the general Bogomol'nyi equation on $\mathbb{E}^3$. In the ungauged (Abelian) directions, it implies that the Φ^I are harmonic functions on $\mathbb{E}^3$ and, once they are chosen, the corresponding vectors $\breve{A}^I$ can be determined. In the non-Abelian directions, the equation becomes non-linear and one has to find simultaneously solutions for the functions Φ^I and gauge fields $\breve{A}^I$ through adequate ansatzs or other methods. Eq. (1.5) is automatically solved if we choose $L_I\propto\Phi^I$ (or zero). Finally, Eq. (1.6) can always be solved if the other equations are solved (because they solve its integrability condition), except, perhaps, at the singularities of the functions where, strictly speaking, the other equations are not solved. In most cases, the integrability condition can be solved by a choice of integration constants in the functions H, M, L_I, Φ^I. Then, of course, one has to integrate explicitly Eq. (1.6) to obtain ω.

2. Using them, reconstruct the solution's 5-dimensional spacetime fields as follows:
   (a) The scalars can be found from this equation for the quotients $h_I(\phi)/\hat{f}$

$$h_I/\hat{f}=L_I+8C_{IJK}\Phi^J\Phi^K/H\,, \tag{1.7}$$

[4] Again, these are the simplest, but not unique $\mathcal{N}=1$ (minimal) supersymmetrizations of the $d=5$ Einstein–Yang–Mills theory and the supersymmetrization requires the addition of, at least, scalars. They also go by the name of non-Abelian-gauged $\mathcal{N}=1$, $d=5$ coupled to vector supermultiplets.

[5] Our conventions are those of Refs. [31,26] and are based on Ref. [32]. In those references it is explained how to obtain $g_{xy}(\phi)$ and $a_{IJ}(\phi)$ from C_{IJK}. The Chern–Simons coupling are directly determined by C_{IJK}.

The supersymmetric solutions of the most general $\mathcal{N}=1$, $d=5$ supergravity theory including vector supermultiplets and hypermultiplets and generic gaugings were characterized in Ref. [26]. The inclusion of tensor supermultiplets was considered in Ref. [27].

[6] The gauge coupling constant appearing in these expressions has been rescaled with respect to that occuring in the action, $g=-2\sqrt{6}\hat{g}$.

because there is always a parametrization of the scalar manifold such that

$$\phi^x \equiv h_x/h_0\,. \tag{1.8}$$

With the above equation for the quotients $h_I(\phi)/\hat{f}$ one can also determine the function $\hat{f}$. For the special case of symmetric scalar manifolds, it is given by[7]

$$\begin{aligned}\hat{f}^{-3} \;=\; & 3^3 C^{IJK} L_I L_J L_K + 3^4\cdot 2^3 C^{IJK} C_{KLM} L_I L_J \Phi^L \Phi^M/H \\ & +3\cdot 2^6 L_I \Phi^I C_{JKL}\Phi^J\Phi^K\Phi^L/H^2 \\ & +2^9\left(C_{IJK}\Phi^I\Phi^J\Phi^K\right)^2/H^3\,.\end{aligned} \tag{1.9}$$

(b) The metric has the form

$$ds^2 = \hat{f}^{\,2}(dt+\hat{\omega})^2 - \hat{f}^{\,-1} d\hat{s}^2\,, \tag{1.10}$$

where $\hat{f}$ has been determined above, the 1-form $\hat{\omega}$ is given by[8]

$$\hat{\omega} = \omega_5(dz+\chi)+\omega\,, \tag{1.11}$$

$$\begin{aligned}\omega_5 = {} & M + 16\sqrt{2}H^{-2}C_{IJK}\Phi^I\Phi^J\Phi^K \\ & + 3\sqrt{2}H^{-1}L_I\Phi^I\,,\end{aligned} \tag{1.12}$$

and where the 4-dimensional Euclidean metric $d\hat{s}^2$ is given by[9]

$$d\hat{s}^2 = H^{-1}(dz+\chi)^2 + H dx^r dx^r\,,\quad r=1,2,3\,. \tag{1.13}$$

(c) The vector fields and their corresponding field strengths are given by

$$\begin{aligned} A^I \;&=\; -\sqrt{3}h^I\hat{f}(dt+\hat{\omega})+\hat{A}^I\,,\\ F^I \;&=\; -\sqrt{3}\hat{\mathfrak{D}}[h^I\hat{f}(dt+\hat{\omega})]+\hat{F}^I\,,\end{aligned} \tag{1.14}$$

where the vector fields $\hat{A}^I$, defined on the 4-dimensional Euclidean space $d\hat{s}^2$, and their field strengths are given by

$$\begin{aligned} \hat{A}^I \;&=\; 2\sqrt{6}\left[H^{-1}\Phi^I(dz+\chi)-\breve{A}^I\right],\\ \hat{F}^I \;&=\; 2\sqrt{6}H^{-1}\left[\breve{\mathfrak{D}}\Phi^I\wedge(dz+\chi)-\star_3 H\breve{\mathfrak{D}}\Phi^I\right],\end{aligned} \tag{1.15}$$

where $\hat{\mathfrak{D}}$ (resp. $\breve{\mathfrak{D}}$) is the exterior gauge-covariant derivative with respect to the connection $\hat{A}^I$ (resp. $\breve{A}^I$).

In Ref. [30] we used this recipe to construct black-hole solutions with non-Abelian gauge and scalar fields for the SU(2)-gauged ST[2, 5] model.[10] This model has 4 vector multiplets and, hence, 4 scalar fields that parametrize the symmetric space SO(1, 3)/SO(3). It is defined by a tensor C_{IJK} with the following non-vanishing components

$$C_{0xy} = \tfrac{1}{6}\eta_{xy}\,,\text{where}\quad (\eta_{xy}) = \mathrm{diag}(+-\cdots-)\,,\quad\text{and}$$
$$x,y = 1,\cdots,4\,. \tag{1.16}$$

The directions to be gauged are the last three, which we will denote by indices $\alpha,\beta,\ldots = 2,3,4$. The ungauged directions will be denoted by indices $i,j,\ldots = 0,1$.

[7] In this expression, $C^{IJK} \equiv C_{IJK}$.

[8] The unhatted ω is the one occurring in Eq. (1.6).

[9] With H and χ related by Eq. (1.3), this is a hyperKähler metric admitting a triholomorphic Killing vector, also known as Gibbons–Hawking metric [34,35]. We will also denote the compact coordinate z by φ. It will be assumed to take values in $[0, 4\pi)$.

[10] Actually, this is the name of the model of $\mathcal{N}=2$, $d=4$ supergravity one obtains by dimensional reduction.

Being a symmetric space, we can use Eq. (1.9) to write the metric function $\hat{f}$ as a function of the building blocks H, L_I, Φ^I:

$$\begin{aligned}\hat{f}^{\,-1} \;=\; & H^{-1}\Big\{\tfrac{1}{4}\left(6HL_0 + 8\eta_{xy}\Phi^x\Phi^y\right) \\ & \times\left[9H^2\eta^{xy}L_xL_y + 48H\Phi^0 L_x\Phi^x \right.\\ & \left. + 64(\Phi^0)^2\eta_{xy}\Phi^x\Phi^y\right]\Big\}^{1/3}\,.\end{aligned} \tag{1.17}$$

Now, in order to find solutions of this model, we just need to find building blocks that satisfy Eqs. (1.2)–(1.6). In the next section we will just do this to find a solution that describes a black ring.

2. Non-Abelian black rings

2.1. Construction of the solution

Inspired by Refs. [11,8], we choose a point $\vec{x}_0 \equiv \left(0,0,-R^2/4\right)$ in $\mathbb{E}^3$ and a harmonic function N with a pole at that point,

$$N \equiv \frac{1}{|\vec{x}-\vec{x}_0|} \equiv \frac{1}{r_n}\,, \tag{2.1}$$

in terms of which we can write the non-vanishing building blocks in the ungauged directions as

$$\begin{aligned} & H = \frac{1}{r}\,, \qquad M = \tfrac{3}{4}\lambda_i q^i\left(1-|\vec{x}_0|N\right)\,,\\ & \Phi^i = -\frac{q^i}{4\sqrt{2}}N\,, \qquad L_i = \lambda_i + \frac{Q_i - C_{ijk}q^jq^k}{4}N\,.\end{aligned} \tag{2.2}$$

These functions contain the integration constants q^i, Q_i and λ_i. The first two can be interpreted as charges. The latter, whose value will be restricted by requirements such as the normalization of the metric at infinity, are moduli. Eq. (1.2) is satisfied automatically. Eq. (1.3) is satisfied with

$$\chi = \cos\theta d\psi\,, \tag{2.3}$$

where $r, \theta \in (0,\pi)$ and $\psi \in [0,2\pi)$ are spherical coordinates centered at $r = |\vec{x}| = 0$ with the definitions and orientation

$$\left\{\begin{aligned} x^1 \;&=\; r\sin\theta\sin\psi\,,\\ x^2 \;&=\; r\sin\theta\cos\psi\,,\\ x^3 \;&=\; -r\cos\theta\,,\end{aligned}\right. \qquad \epsilon^{123} = \epsilon^{r\theta\psi} = +1\,. \tag{2.4}$$

Eqs. (1.4) are satisfied with

$$\breve{A}^i = -\frac{q^i}{4\sqrt{2}}\cos\theta_n d\psi_n\,, \tag{2.5}$$

where $r_n, \theta_n \in (0,\pi)$ and $\psi_n \in [0,2\pi)$ are spherical coordinates centered at $r_n = |\vec{x}_n| = 0$ with the definitions

$$\left\{\begin{aligned} x_n^1 \;&\equiv\; x^1 - x_0^1 = r_n\sin\theta_n\sin\psi_n\,,\\ x_n^2 \;&\equiv\; x^2 - x_0^2 = r_n\sin\theta_n\cos\psi_n\,,\\ x_n^3 \;&\equiv\; x^3 - x_0^3 = -r_n\cos\theta_n\,,\end{aligned}\right. \tag{2.6}$$

and the same orientation as the spherical coordinates centered at $r = 0$.

Eqs. (1.5) in the Abelian directions are trivially satisfied because all ${f_{ij}}^k = 0$ and, finally, the integrability condition of Eq. (1.6) is identically satisfied for the chosen integration constants and ω can be found by integration. We will compute ω for the complete solution later.

The above functions are enough to construct an Abelian black ring. Now, we excite the gauged directions of this solution by adding to it a solution of the SU(2) Bogomol'nyi equations on $\mathbb{E}^3$ (1.4)

$$
\begin{aligned}
\Phi^\alpha &= \frac{1}{g r_n\left(1+\lambda^2 r_n\right)}\delta^\alpha_{s+1}\frac{x^s_n}{r_n}\,,\\
\breve{A}^\alpha &= \frac{1}{g r_n\left(1+\lambda^2 r_n\right)}\epsilon^\alpha{}_{rs}\frac{x^s_n}{r_n}dx^r_n\,.
\end{aligned}
\tag{2.7}
$$

This solution, originally found by Protogenov in Ref. [23], describes a magnetic colored monopole placed at $r_n=0$. It is singular at $r_n=0$ as a field configuration in $\mathbb{E}^3$, but this behavior can change when we analyze the whole picture. In fact, we showed in Ref. [28] that the monopole field gives rise to a BPST instanton in $\mathbb{E}^4$ through (1.15), and we used this result in Ref. [30] to construct a regular black hole of the same supergravity theory we consider in this work.

In the present case we obtain a different instanton field configuration from (1.15), which we call *distorted* BPST, because the pole of the harmonic function H is placed in a different point ($r=0$) than that of the colored monopole ($r_n=0$). This distorted BPST is singular at $r_n=0$, which might turn the black ring solution ill-defined. Happily this is not the case. The complete vector field contains the instanton plus an additional term, see (1.14), where the latter cancels precisely this divergence at that critical point

$$
\lim_{r_n\to\infty}\left(-\sqrt{3}h^I\hat{f}\omega_5+2\sqrt{6}H^{-1}\Phi^I\right)(dz+\chi)=0\,. \tag{2.8}
$$

Observe that in the ungauged case the Φ^αs would have been harmonic functions $-q^\alpha N/(4\sqrt{2})$ and the combinations $C_{ijk}q^jq^k$ should have been replaced by $C_{iJK}q^Jq^K$. Here the asymptotic behavior of the non-Abelian gauge field indicates that the "non-Abelian q^αs" do not contribute in the same way the q^is do. However, they have a similar near-horizon behavior.

The above functions define completely the solution. In what follows we are going to analyze its metric to show that it describes a regular black ring and to compute its main properties.

2.2. Analysis of the solution

In this analysis it is convenient to use two set of coordinates: those centered at $r=0$ (r,θ,ψ, defined in Eq. (2.4)) supplemented by the time coordinate t and the angular coordinate φ, and those centered at $r_n=0$ (r_n,θ_n,ψ_n, defined in Eq. (2.6)) supplemented by the time coordinate t_n and the angular coordinate φ_n. The relations

$$
\begin{aligned}
r_n &= \left(r^2+|\vec{x}_0|^2-2|\vec{x}_0|r\cos\theta\right)^{1/2}\,,\\
r &= \left(r_n^2+|\vec{x}_0|^2+2|\vec{x}_0|r_n\cos\theta_n\right)^{1/2}\,,\\
|\vec{x}_0| &= r\cos\theta-r_n\cos\theta_n\,,
\end{aligned}
\tag{2.9}
$$

will be useful in the computations.

The metric function $\hat{f}$ can be obtained by substituting the functions H, L_I, Φ^I in Eq. (1.9). At this moment we just want to impose the standard asymptotic normalization

$$
\lim_{r\to\infty}\hat{f}=1\,,\quad\Rightarrow\quad 3^3C^{ijk}\lambda_i\lambda_j\lambda_k=\frac{3^3}{2}\lambda_0\lambda_1^2=1\,. \tag{2.10}
$$

Now let us compute the only missing ingredient in the metric (1.10): the 1-form $\hat{\omega}$. Let us consider Eq. (1.6), which, upon substitution of the chosen functions H, M, L_I, Φ^I, can be written as

$$
\begin{aligned}
\star_3 d\omega = -\tfrac{3}{4}\lambda_i q^i\Bigg\{ & -\frac{1}{r^2}\Bigg[1-\frac{|\vec{x}_0|+r}{r_n}\\
&+\tfrac{r|\vec{x}_0|(r+|\vec{x}_0|)}{r_n^3}(1-\cos\theta)\Bigg]dr\\
&+\left[\frac{|\vec{x}_0|\sin\theta}{r_n^3}\left(r-|\vec{x}_0|\right)\right]d\theta\Bigg\}\,,
\end{aligned}
\tag{2.11}
$$

and a solution can be readily found assuming ω has only one non-vanishing component, ω_ψ [11]:

$$
\omega=-\tfrac{3}{4}\lambda_i q^i(\cos\theta-1)\left[1-\left(r+\frac{R^2}{4}\right)\frac{1}{r_n}\right]d\psi\,. \tag{2.12}
$$

Observe that, since $L_\alpha=0$ the non-Abelian terms do not affect ω. However, they do affect the whole 5-dimensional $\hat{\omega}$ given in Eq. (1.11) via ω_5 in Eq. (1.12):

$$
\hat{\omega}=(F-G)\,d\varphi+(F-G\cos\theta)\,d\psi\,, \tag{2.13}
$$

$$
F=\frac{3\lambda_i q^i}{4}\left[1-\left(r+\frac{R^2}{4}\right)\frac{1}{r_n}\right], \tag{2.14}
$$

$$
\begin{aligned}
G=&\frac{q^i}{16}\left[3\left(Q_i-C_{ijk}q^jq^k\right)+2C_{ijk}q^jq^k\frac{r}{r_n}\right]\frac{r}{r_n^2}\\
&-\frac{2q^0}{g^2}\frac{r^2}{r_n^3\left(1+\lambda^2 r_n\right)^2}\,.
\end{aligned}
\tag{2.15}
$$

The last term in G has a non-Abelian origin. In the $r\to\infty$ limit in which the metric tends to Minkowski's (so we have an asymptotically flat solution), though, it is subdominant and we do not expect it to contribute to the angular momentum of the solution.

So far we have been working in coordinates in which the hyperKähler metric Eq. (1.13) is of the form

$$
\begin{aligned}
d\hat{s}^2 = & r\left(d\varphi+\cos\theta d\psi\right)^2\\
&+\frac{1}{r}\left[dr^2+r^2\left(d\theta^2+\sin\theta^2 d\psi^2\right)\right],
\end{aligned}
\tag{2.16}
$$

but, in order to compute mass and angular momentum, it is convenient to use a different coordinate system (also centered at $\vec{x}=0$) t,Θ,ϕ_1,ϕ_2, related to the former by

$$
r=\frac{\rho^2}{4}\,,\qquad \theta=2\Theta\,,\qquad \psi=\phi_1-\phi_2\,,\qquad \varphi=\phi_1+\phi_2\,, \tag{2.17}
$$

in which the complete 5-dimensional metric is of the form

$$
\begin{aligned}
ds^2 = & \hat{f}^2\left(dt+\hat{\omega}\right)^2-\hat{f}^{-1}\Big[d\rho^2\\
&+\rho^2\left(d\Theta^2+\cos^2\Theta d\phi_1^2+\sin^2\Theta d\phi_2^2\right)\Big]\,,
\end{aligned}
\tag{2.18}
$$

with

$$
\hat{\omega}=\left(2F-2G\cos^2\Theta\right)d\phi_1-2G\sin^2\Theta d\phi_2\,. \tag{2.19}
$$

The independent components of the angular momentum are now obtained from the metric behavior in the $\rho\to\infty$ limit [12]

$$
J_{\phi_1}=\lim_{\rho\to\infty}\frac{\pi|g_{t\phi_1}|\rho^2}{4G_N\cos^2\Theta}=\tfrac{1}{2\sqrt{3}}q^i\left(3Q_i-C_{ijk}q^jq^k\right)\,, \tag{2.20}
$$

$$
J_{\phi_2}=\lim_{\rho\to\infty}\frac{\pi|g_{t\phi_2}|\rho^2}{4G_N\sin^2\Theta}=\tfrac{1}{2\sqrt{3}}q^i\left(3Q_i-C_{ijk}q^jq^k+6\lambda_i R^2\right)\,, \tag{2.21}
$$

and, from the absence of contribution proportional to g, we see that they coincide with those of the Abelian black ring, as we expected.

[11] The expression coincides with that of [36] despite we have chosen $\vec{x}_0$ to be on the negative x^3 axis. This is because the coordinate θ has also a relative sign with respect to the used in that reference.

[12] We use units in which $G_N=\sqrt{3}\pi/4$.

Observe that these formulae allow us to identify

$$q^i\lambda_i R^2 = \frac{1}{\sqrt{3}}(J_{\phi_2} - J_{\phi_1})\,. \tag{2.22}$$

Before we move to study the possible presence of an event horizon, let us point out that the solution does not contain any Dirac–Misner strings.[13] Indeed, the $g_{t\phi_1}$ (resp. $g_{t\phi_2}$) metric component vanishes when the coordinate ϕ_1 (resp. ϕ_2) is not well defined, which happens at $\Theta = \pi/2$ ($\Theta = 0$).

The solution may have an event horizon at $\vec{x} = \vec{x}_0$, where the norm of the timelike Killing vector of the metric vanishes. In order to study the near horizon limit we need to use a different coordinate system because several components of the metric blow up there in the coordinates we have been using so far. Recall the expression for the metric in the original frame centered at $\vec{x} = 0$

$$\begin{aligned} ds^2 \;=\; & \hat{f}^2(dt+\omega)^2 + 2\hat{f}^2\omega_5(dt+\omega)(d\varphi + \cos\theta d\psi) \\ & - \hat{f}^2\left(\hat{f}^{-3}r - \omega_5^2\right)(d\varphi + \cos\theta d\psi)^2 - \hat{f}^{-1}r^{-1}dx^r dx^r\,. \end{aligned} \tag{2.23}$$

We first go to the auxiliary frame centered at the horizon with spherical coordinates and take the $r_n \to 0$ limit. The functions that appear in the metric behave in this limit as follows

$$\hat{f} = \frac{16}{R^2v^2}r_n^2 + \mathcal{O}(r_n^3)\,, \tag{2.24}$$

$$\omega_{\psi_n} = -\frac{3}{R^2}\lambda_i q^i \sin^2\theta_n r_n + \mathcal{O}(r_n^2) \tag{2.25}$$

$$\hat{f}^{-1}r^{-1} = \frac{v^2}{4}r_n^{-2} + k_1 r_n^{-1} + \mathcal{O}(r_n)\,, \tag{2.26}$$

$$\hat{f}^2\omega_5 = -\frac{2}{v}r_n + k_2 r_n^2 + \mathcal{O}(r_n^3)\,, \tag{2.27}$$

$$\hat{f}^2(\hat{f}^{-3}r - \omega_5^2) = \frac{l^2}{4} + k_3 r_n + \mathcal{O}(r_n^2)\,, \tag{2.28}$$

where we have defined the constants

$$v = \left(C_{ijk}q^iq^jq^k - 16\frac{q^0}{g^2}\right)^{1/3}\,, \tag{2.29}$$

$$\begin{aligned} l = \frac{1}{2v^2}\Bigg[& 9\cdot 6^2 C^{ijk}C_{klm}\left(Q_i - C_{ihn}q^hq^n\right)\left(Q_j - C_{jpq}q^pq^q\right)q^lq^m \\ & - 9\left(q^iQ_i - C_{ijk}q^iq^jq^j\right)^2 - 12q^i\lambda_i R^2v^3 \\ & - 9\left(Q_1 - \frac{q^0q^1}{3}\right)^2\left(\frac{32}{g^2}\right)\Bigg]^{1/2}. \end{aligned} \tag{2.30}$$

These expression for the constants v and l resemble those of the Abelian case [11], with an additional non-Abelian term. The precise form of the constants k_1, k_2 and k_3 in terms of the charges are messy. They do not occur in the calculation of any physical quantity, but they play a role in the near horizon analysis,[14] since they are responsible for the disappearance of $\mathcal{O}(r_n^{-1})$ in the metric after we perform the following coordinate transformation,

$$dt_n = d\tau_n + \left(\frac{b_2}{r_n^2} + \frac{b_1}{r_n}\right)dr_n\,, \qquad d\varphi_n = -d\psi_n + 2d\xi_n + \frac{c_1}{r_n}dr_n\,, \tag{2.34}$$

where the constants b_1, b_2 and c_1 can be chosen such that all divergences in the metric in the $r_n \to 0$ limit disappear:

$$c_1 = \mp\frac{v}{l}\,, \qquad b_2 = \pm\frac{lv^2}{8}\,, \qquad b_1 = \pm\frac{4l^2k_1 + l^2v^3k_2 + 4v^2k_3}{16l}\,. \tag{2.35}$$

With this choice we find in the $r_n \to 0$ limit that the horizon has the following metric

$$ds_h^2 = -l^2 d\xi_n^2 - \frac{v^2}{4}\left(d\theta_n^2 + \sin^2\theta_n d\psi_n^2\right)\,, \tag{2.36}$$

with the topology $S^1 \times S^2$, so the solution is a black ring with non-Abelian hair, *i.e.* a non-Abelian black ring. Using this metric we can compute the area of the horizon,[15]

$$\frac{A_h}{2\pi^2} = \frac{1}{2\pi^2}\int d^3x\sqrt{|g_h|} = lv^2\,, \tag{2.37}$$

so the entropy of the non-Abelian black ring can be written in terms of the charges and angular momenta using the expressions for the constants v and l Eqs. (2.29) and (2.30) together with Eq. (2.22) as follows:

$$\begin{aligned} S = \pi\Bigg[& 3\cdot 6^2 C^{ijk}C_{klm}\left(Q_i - C_{ihn}q^hq^n\right)\left(Q_j - C_{jpq}q^pq^q\right)q^lq^m \\ & - 3\left(q^iQ_i - C_{ijk}q^iq^jq^k\right)^2 \\ & - \frac{4}{\sqrt{3}}(J_{\phi_2} - J_{\phi_1})\left(C_{ijk}q^iq^jq^k - 16\frac{q^0}{g^2}\right) \\ & - 3\left(Q_1 - \frac{q^0q^1}{3}\right)^2\left(\frac{32}{g^2}\right)\Bigg]^{1/2}. \end{aligned} \tag{2.38}$$

[13] They could have been removed but only at the price of introducing closed timelike curves [37].

[14] We give their form here for the sake of completeness,

$$k_1 = \frac{16\lambda^2R^2\frac{q^0}{g^2} + 3\left(q^iQ_i - C_{ijk}q^iq^jq^k\right)}{3R^2v}\,, \tag{2.31}$$

$$k_2 = \frac{4k_1}{v}\,, \tag{2.32}$$

$$\begin{aligned} k_3 = \frac{1}{2g^2R^2v^4}\Bigg\{ & \frac{3R^2k_1}{v^3}\Big[(q^0q^1/3)^2\left(96 - 3g^2(q^1)^2\right) \\ & + 6(q^0q^1/3)\left(-32Q_1 + g^2q^1(-2(q^1)^2/6q^0 + 2q^0Q_0 + q^1Q_1)\right) \\ & + 3\Big(4(q^1)^2/6g^2q^0q^1Q_1 + 32Q_1^2 \\ & + g^2\left(-q^1Q_1(4q^0Q_0 + q^1Q_1) + (C_{ijk}q^iq^jq^k - q^iQ_i)^2\right)\Big) \\ & + 4\lambda_iq^i\left(-16q^0 + g^2C_{ijk}q^iq^jq^k\right)R^2\Big] \\ & - 3\Big[9g^2((q^1)^2/6 - Q_0)(q^0q^1/3 - Q_1)^2 \\ & + \Big(-24(q^0q^1/3)^2\lambda^2 + 6(q^1)^2/6g^2\lambda_1q^0q^1 \\ & - 6g^2\lambda_1q^0Q_0q^1 + 96\lambda_1Q_1 - 6g^2\lambda_0q^0q^1Q_1 \\ & - 3g^2\lambda_1(q^1)^2Q_1 - 24\lambda^2Q_1^2 \\ & + 3q^0q^1/3\left(-32\lambda_1 + 2g^2\lambda_0q^0q^1 + g^2\lambda_1(q^1)^2 + 16\lambda^2Q_1\right) \\ & + 32\lambda_iq^iq^0 - 8C_{ijk}q^iq^jq^kg^2\lambda_iq^i + 6g^2\lambda_iq^iQ_jq^j\Big)R^2 + 16\lambda^2\lambda_iq^iq^0R^4\Big]\Bigg\}. \end{aligned} \tag{2.33}$$

[15] Notice that $\xi_n \in [0, 2\pi)$, as can be deduced from expression (2.34) together with $\int d\Omega_{(3)} = 2\pi^2$.

Finally, we would like to compute the mass of the solution. We do so by comparing the asymptotic behavior of the metric component g_{tt} with that of the Schwarzschild solution, $g_{tt} \sim 1 - \frac{8MG}{3\pi\rho^2} + \cdots$. We get

$$M = \frac{3^{5/2}\lambda_1}{2}\left(\lambda_1 Q_0 + 2\lambda_0 Q_1\right) . \tag{2.39}$$

The constants λ_i can be expressed in terms of physical constants. If we define the physical scalars of the theory as $\phi^x \equiv h_x/h_0$ we find that the only scalar with a non-vanishing asymptotic value is the Abelian one and this value is $\phi^1_\infty = \lambda_1/\lambda_0$. On the other hand, the asymptotic normalization of the metric Eq. (2.10) implied $\lambda_0\lambda_1^2 = 2/3^3$. Then,

$$\lambda_0 = 2^{1/3}3^{-1}\left(\phi^1_\infty\right)^{-2/3} , \qquad \lambda_1 = 2^{1/3}3^{-1}\left(\phi^1_\infty\right)^{1/3} , \tag{2.40}$$

and M takes the form

$$M = 2^{-1/3}3^{1/2}\left[\left(\phi^1_\infty\right)^{2/3} Q_0 + 2\left(\phi^1_\infty\right)^{-1/3} Q_1\right] , \tag{2.41}$$

and depends only on the moduli and on the electric charges Q_0, Q_1 while the q^i, which correspond to magnetic dipole momenta do not contribute to it [6]. The non-Abelian field do not contribute, either.

This expression looks identical to that of the non-Abelian black hole solution constructed in Ref. [30], but the charges Q_0 and Q_1 are not the same than the charges q_0 and q_1 that appear in the black-hole mass formula given in that reference. They are, actually, related by $Q_i^{\rm BR} = q_i^{\rm BH} + C_{ijk}q^j_{\rm BR}q^k_{\rm BR}$. This is just reflecting the fact that the conserved electrical charges in the black ring receive contributions from the magnetic dipole momenta via the Chern–Simons term in the action. This effect is commonly described as "charges dissolved in fluxes" [9].

This non-Abelian black-ring mass formula, is, however, identical to that of the Abelian black ring that one would obtain by removing the non-Abelian fields from this solution. In other words: the presence of non-Abelian fields is not observable at spatial infinity. They do contribute to the entropy, though, as in the black-hole case, their entropy being smaller than that of their Abelian siblings.

3. Non-Abelian rotating black holes

In the $R \to 0$ limit, several things happen:

1. All the harmonic functions are now centered at $r = 0$ (except for M which becomes constant):

$$H = N = \frac{1}{r} , \quad M = \tfrac{3}{4}\lambda_i q^i , \quad \Phi^i = -\frac{q^i}{4\sqrt{2}}N ,$$
$$L_i = \lambda_i + \frac{Q_i - C_{ijk}q^jq^k}{4}H . \tag{3.1}$$

2. The non-Abelian gauge field is also centered at $r = 0$:

$$\Phi^\alpha = \frac{1}{gr\left(1+\lambda^2 r\right)}\delta^\alpha_{s+1}\frac{x^s}{r} , \qquad \breve{A}^\alpha = \frac{1}{gr\left(1+\lambda^2 r\right)}\epsilon^\alpha{}_{rs}\frac{x^s}{r}dx^r , \tag{3.2}$$

and the distorted BPST instanton is not distorted anymore.

3. The metric function $\hat{f}$ is now given by

$$\hat{f}^{-3} = \left[\tfrac{3}{2}\left(\lambda_0 + \frac{Q_0}{4r}\right) - \frac{2}{g^2}\frac{1}{r(1+\lambda^2 r)^2}\right] \times \left[9\left(\lambda_1 + \frac{Q_1}{4r}\right)^2 - \frac{2(q^0)^2}{g^2}\frac{1}{r^2(1+\lambda^2 r)^2}\right] . \tag{3.3}$$

The mass of this object is identical to that of the black ring Eqs. (2.39) and (2.41). It has no non-Abelian contributions. The near-horizon limit, though, includes non-Abelian terms

$$\hat{f}^{-1} \sim \frac{Y}{r} , \quad \text{with} \quad Y^3 = \left(\tfrac{3}{8}Q_0 - \frac{2}{g^2}\right)\left(\tfrac{9}{16}Q_1^2 - \frac{2}{g^2}(q^0)^2\right) \tag{3.4}$$

4. ω vanishes identically and $\hat{\omega}$ is determined only by ω_5, which takes the form

$$\hat{\omega} = \omega_5(d\varphi + \cos\theta d\psi) ,$$
$$\omega_5 = \frac{q^i}{16}\left(3Q_i - C_{ijk}q^jq^k\right)\frac{1}{r} - \frac{2q^0}{g^2}\frac{1}{r\left(1+\lambda^2 r\right)^2} . \tag{3.5}$$

As a result, the two angular momenta become identical

$$J_{\phi_1} = J_{\phi_2} = \tfrac{1}{2\sqrt{3}}q^i\left(3Q_i - C_{ijk}q^jq^k\right) \equiv J . \tag{3.6}$$

Observe that the non-Abelian term in ω_5, which does not contribute to the angular momentum, does contribute to the $r \to 0$ limit just as the Abelian terms:

$$\omega_5 \sim Z/r , \quad \text{where} \quad Z = \tfrac{\sqrt{3}}{8}J - \frac{2q^0}{g^2} . \tag{3.7}$$

Let us study the near-horizon limit $\to 0$. Using Eqs. (3.4) and (3.7), we find that the metric Eq. (1.10) behaves in this limit as

$$ds^2 \sim \frac{r^2}{Y^2}dt^2 - \frac{Y}{r^2}dr^2 - Yd\Omega^2_{(2)} + \frac{2Z}{Y^2}rdt(d\varphi + \cos\theta d\psi) + \left(\frac{Z^2}{Y^2} - Y\right)(d\varphi + \cos\theta d\psi)^2 , \tag{3.8}$$

which can be rewritten in the form

$$ds^2 \sim Yd\Pi^2_{(2)} - Yd\Omega^2_{(2)} - Y\left[\sin\alpha\rho dt - \cos\alpha(d\varphi + \cos\theta d\psi)\right]^2 , \tag{3.9}$$

where $r = (Y^3 - Z^2)^{1/2}\rho$, $d\Pi^2_{(2)} = \rho^2 dt^2 - \frac{d\rho^2}{\rho^2}$ is the metric of the AdS_2 of unit radius and $\sin^2\alpha = Z^2/Y^3$. This space is the near-horizon limit of the BMPV black hole [38], but, due to the non-Abelian contribution to Z (which can be understood as a sort of "near-horizon angular momentum"), now α does not vanish for vanishing asymptotic angular momentum J and we can have a stationary black hole with $J = 0$ whose near-horizon limit is not $AdS_2 \times S^3$. The converse is also possible: we can make $\alpha = Z = 0$ for $J = \frac{16}{\sqrt{3}}q^0/g^2$ and have a rotating black hole whose near-horizon limit is $AdS_2 \times S^3$.

The area of the horizon is

$$\frac{A}{2\pi^2} = 8\sqrt{Y^3 - Z^2} . \tag{3.10}$$

4. Conclusions

The existence of black-hole and black-ring solutions with identical asymptotic behavior but with non-Abelian hair that contributes to the entropy [21,25,24,30] challenges our understanding of black-hole hair and the microscopic interpretation of the black-hole/black-ring entropy, just as the Abelian hair discovered in Ref. [6] did. More research is necessary to gain a better understanding of these solutions. In particular, the stability of these supersymmetric non-Abelian solutions (which are entropically disfavored) needs to be addressed and their possible non-supersymmetric and non-extremal generalizations have to be constructed and studied. Work in these directions is in progress.

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Non-Abelian bubbles in microstate geometries

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Abstract: We find the first smooth bubbling microstate geometries with non-Abelian fields. The solutions constitute an extension of the BPS three-charge smooth microstates. These consist in general families of regular supersymmetric solutions with non-trivial topology, i.e. *bubbles*, of $\mathcal{N} = 1$, $d = 5$ Super-Einstein-Yang-Mills theory, having the asymptotic charges of a black hole or black ring but with no horizon. The non-Abelian fields make their presence at the very heart of the microstate structure: the physical size of the bubbles is affected by the non-Abelian topological charge they carry, which combines with the Abelian flux threading the bubbles to hold them up. Interestingly the non-Abelian fields carry a set of adjustable continuous parameters that do not alter the asymptotics of the solutions but modify the local geometry. This feature can be used to obtain a *classically infinite* number of microstate solutions with the asymptotics of a single black hole or black ring.

Keywords: Black Holes in String Theory, Gauge Symmetry

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Contents

1 Introduction

The construction and study of smooth microstate geometries in supergravity theories has become a fruitful area of research since the pioneering work, more than a decade ago, of Bena and Warner [1] and independently of Berglund, Gimon and Levi [2], where a strategy to obtain ample families of microstate geometries was given, generalizing earlier results [3–9]. This kind of solutions can be roughly described as a black hole configuration in which the horizon and its interior have been replaced by some complicated, although smooth horizonless geometry while keeping the rest of the field configuration looking like the unmodified solution. Any solution with such remarkable properties is interesting *per se*, although it is in the context of the *fuzzball proposal* [10] in which these configurations acquire their greatest significance.

The proposal originated as a possible solution to the information paradox and conjectures that the entropy of a black hole has its microscopic origin in the degeneracy of a quantum bound state, the fuzzball. In this picture, the classical black hole would provide an effective description of the system, that would consist in a quantum *ensamble of geometries*. These microstate geometries, when considered individually, would correspond to string theory configurations with unitary scattering and hopefully a subset of these states might be captured as smooth horizonless supergravity solutions. Since the proposal suggests a modification at the horizon scale, such geometries should have the same asymptotics as the black hole.

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This conjecture opened a whole program in the quest to construct smooth microstate geometries in theories of supergravity. Much progress has been made in this direction and vast classes of such solutions have already been described in the literature, see [11–15] and references therein. The direct identification of these configurations as representing typical microstates of a particular black hole is generally unclear due to the absence of a description in terms of a dual CFT. However very recently this identification has been performed for a particular type of configurations known as *superstrata*, constituting a major achievement of the fuzzball program [16]. Nevertheless, even though general microstate geometries lack of this identification, they are still very useful in providing valuable information about the physics of black holes in string theory, see for instance [17–21].

Typically these are described as topologically non-trivial spacetimes in five and six dimensions, in the context of supergravity coupled to Abelian matter multiplets or pure supergravity. In the present work we perform the inclusion of non-Abelian degrees of freedom for the first time.[1] The reason why this class of microstate geometries has remained unexplored so far seems to be clear: the construction of explicit analytic non-Abelian solutions in five- and six-dimensional supergravity theories has become accessible only in the last few months [26–28]. The solutions that we present here constitute a non-Abelian extension of the BPS three-charge smooth geometries described in [11]. We work in $\mathcal{N}=1$, $d=5$ Super-Einstein-Yang-Mills (SEYM) theories. One can think of these theories as an extension of the five-dimensional STU model of supergravity, that describes a supergravity multiplet coupled to two Abelian vector multiplets. SEYM theories are then obtained by consistently coupling the STU model to a set of additional vector multiplets that transform under the local action of a non-Abelian group.[2] Although this nomenclature might seem unfamiliar in the literature of microstate geometries, in fact the underlying theory where this solutions are constructed is quite frequently the STU model: five-dimensional three-charge configurations are naturally described in this framework.

It is worth mentioning how $\mathcal{N}=1$, $d=5$ SEYM theories are embedded in string theory. The 10-dimensional effective theory of the Heterotic string describes $\mathcal{N}=1$ supergravity coupled to 16 Abelian vector multiplets. When the Heterotic string theory is compactified on T^5, there are special points in the moduli space for which there is an enhancement of the gauge symmetry. Then, besides the *Kaluza-Klein* vectors, the effective supergravity description contains additional massless vector fields taking values in the algebra of some non-Abelian group. A consistent truncation can reduce the supermultiplets content (as well as their number) and result in the $\mathcal{N}=1$, $d=5$ SEYM theories that we consider here. The explicit realization of this particular compactification and truncation might be interesting [29].

The procedure by which non-Abelian microstate geometries are found has a similar structure than that of the Abelian case, but requires the introduction of some modifications. Just like in the case of supersymmetric solutions of STU supergravity, the construction of BPS configurations satisfying the equations of motion of SEYM theory relies on the specification of a reduced set of *seed functions* defined in $\mathbb{R}^3$. In the case of the familiar

[1] Notice that globally regular non-Abelian gravitating configurations on contractible spaces have been known since the late 80s, see [22–25]. These are usually referred as *global monopoles*.

[2] One can consider as well the introduction of additional Abelian vector multiplets.

STU model, these are simply harmonic functions that satisfy certain differential equations whose integrability condition is the Laplace equation. The SEYM procedure conserves these harmonic functions and introduces a new set of seed functions satisfying the *covariant* version of these differential equations.

We find that the bubbling equations, which determine the size of the bubbles leading to physically sensible geometries, contain a new contribution that appears standing next to the magnetic fluxes threading the bubbles, see (3.27). This new term can be given a physical interpretation in terms of the topological charge, or instanton number, associated to the endpoints of the bubble of a non-Abelian instanton that builds up the vector fields. As a consequence it should be possible to have stable bubbles without some magnetic fluxes placed on them or, inversely, a bubble can collapse even though the fluxes are non-zero.

Another interesting peculiarity introduced by the non-Abelian fields is that the solution depends on a set of continuous parameters that can be modified with no apparent restriction whose influence is only local, i.e. their modification does not change any of the asymptotic charges. This is a shocking feature that allows the construction of huge amounts of microstate geometries with the same topology for a unique black hole, and its proper interpretation requires further study.

Having said that, let us start talking about the details of non-Abelian microstate geometries. We give a general description of the solutions that can be found using our generating technique in section 2. In section 3 we describe how this method can be utilized for the construction of smooth horizonless solutions. We conclude in section 4 with some comments about the results and discuss future directions. In appendix A we give a brief summary of $\mathcal{N}=1$, $d=5$ SEYM theories, describing its matter content and its action. Appendix B contains the solution generating technique written in a *step-by-step* language.

2 Supersymmetric solutions of $\mathcal{N}=1$, $d=5$ super-Einstein-Yang-Mills

A technique to construct supersymmetric timelike solutions with a spacelike isometry in these theories was recently developed in [26], where it was used to describe the first non-Abelian analytic black holes in five dimensions.[3] This method has also been used in [27] to find non-Abelian generalizations of the Emparan-Reall black ring solution, [30], and the BMPV rotating black hole, [31]. In the simplest settings, the configurations can be roughly interpreted as three-charge Abelian solutions on top of which we place a non-Abelian instanton that, interestingly, does not produce any change on the mass of the solution while it reduces its entropy.

The solutions of $\mathcal{N}=1$, $d=5$ SEYM[4] are specified by the form of the metric ds^2, the vector fields A^I and the scalars ϕ^x. The indices labeling the vectors take values in $\{I,J,\ldots=0,\ldots,5\}$, with the Abelian sector contained in the first values $\{i,j,\ldots=0,1,2\}$ and the non-Abelian sector in the last three $\{\alpha,\beta,\ldots=3,4,5\}$. We make a continuous use of this division in two sectors through the text. The scalars are conveniently codified

[3] A method for the systematic construction of null solutions and some explicit examples describing black strings and regular string-monopoles are also given in that reference.

[4] See Apendix A for a brief description of the theory.

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in terms of a set of functions h_I labeled with the same indices than the vectors, such that $\phi^x \equiv h_x/h_0$. We also define the functions of the scalars with upper indices as

$$h^I \equiv 27C^{IJK}h_Ih_J\,, \qquad h^Ih_I = 1\,, \tag{2.1}$$

where $C^{IJK} = C_{IJK}$ is a constant symmetric tensor that characterizes the supergravity theory. We work on the SU(2)-gauged ST[2,6] model, that contains $n_v = 5$ vector multiplets and, as we mentioned in the introduction, can be understood as a non-Abelian extension of the STU model. This model is characterized by a constant symmetric tensor with the following non-vanishing components

$$C_{0xy} = \frac{1}{6}\eta_{xy}\,, \text{where} \quad (\eta_{xy}) = \mathrm{diag}(+-\cdots-)\,, \quad \text{and} \quad x,y = 1,\cdots,5\,. \tag{2.2}$$

In [32] it was shown that timelike supersymmetric solutions of this theory are of the form

$$ds^2 = f^2(dt+\omega)^2 - f^{-1}d\hat{s}^2\,, \tag{2.3}$$
$$A^I = -\sqrt{3}h^If(dt+\omega) + \hat{A}^I\,, \tag{2.4}$$

where $d\hat{s}^2$ is a four-dimensional hyperKähler metric and the rest of elements that appear in this decomposition are defined on this four-dimensional space. These elements satisfy the system of *BPS equations*:

$$\hat{F}^I = \star_4\hat{F}^I\,, \tag{2.5}$$
$$\hat{\mathfrak{D}}^2\,(h_I/f) = \frac{1}{6}C_{IJK}\hat{F}^J\cdot\hat{F}^K\,, \tag{2.6}$$
$$d\omega + \star_4 d\omega = \frac{\sqrt{3}}{2}(h_I/f)\hat{F}^I\,. \tag{2.7}$$

Here $\star_4$ is the Hodge dual in the four-dimensional metric $d\hat{s}^2$ and $\hat{F}^I$ is the field strength of the vector $\hat{A}^I$

$$\hat{F}^I{}_{\mu\nu} = 2\partial_{[\mu}\hat{A}^I{}_{\nu]} + \hat{g}f_{JK}{}^I\hat{A}^J{}_\mu\hat{A}^K{}_\nu\,, \tag{2.8}$$

where $f_{IJ}{}^K$ are only non-vanishing when the indices take values in the non-Abelian sector, in which case they are the structure constants of SU(2), $f_{\alpha\beta}{}^\gamma = \varepsilon_{\alpha\beta\gamma}$.

Some words about notation are necessary. Notice that we use *hats* to distinguish objects that are defined in four spatial dimensions. For example, A^I is used to represent the five-dimensional physical vectors and $\hat{A}^I$ is a vector in the four-dimensional hyperKähler space. In a few lines we will introduce another collection of objects that are labeled with *inverse hats* and that are defined in three-dimensional Euclidean space. In particular we define the vectors $\breve{A}^I$. We use all these vectors to define covariant derivatives in five, four and three dimensions for objects with upper and lower vector indices. For example the four-dimensional covariant derivatives are defined by

$$\hat{\mathfrak{D}}h^I = dh^I + \hat{g}f_{JK}{}^I\hat{A}^Jh^K\,, \qquad \hat{\mathfrak{D}}h_I = dh_I + \hat{g}f_{IJ}{}^K\hat{A}^Jh_K\,. \tag{2.9}$$

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The system of BPS equations can be drastically simplified under the assumption that the solutions admit a global spacelike isometry along a compact direction [26]. Then the mathematical objects that build up the physical fields can be further decomposed in terms of elements defined in three dimensional flat space in the following manner

$$d\hat{s}^2 = H^{-1}(d\varphi + \chi)^2 + H dx^r dx^r\,, \quad r = 1,2,3\,, \tag{2.10}$$

$$\hat{A}^I = -2\sqrt{6}\left[-H^{-1}\Phi^I(d\varphi+\chi) + \breve{A}^I\right]\,, \tag{2.11}$$

$$h_I/f = L_I + 8C_{IJK}\Phi^J\Phi^K H^{-1}\,, \tag{2.12}$$

$$\omega = \omega_5(d\varphi+\chi) + \breve{\omega}\,, \tag{2.13}$$

where φ is a coordinate adapted to the direction of the isometry. Substituting back these expressions in the BPS system of equations, we obtain the conditions that $H, \chi, \Phi^I, \breve{A}^I, L_I, \omega_5$ and $\breve{\omega}$ need to satisfy

$$\star_3 dH = d\chi\,, \tag{2.14}$$

$$\star_3\breve{\mathfrak{D}}\Phi^I = \breve{F}^I\,, \tag{2.15}$$

$$\breve{\mathfrak{D}}^2 L_I = \breve{g}^2 f_{IJ}{}^L f_{KL}{}^M \Phi^J\Phi^K L_M\,, \tag{2.16}$$

$$\star_3 d\breve{\omega} = HdM - MdH + 3\sqrt{2}(\Phi^I\breve{\mathfrak{D}}L_I - L_I\breve{\mathfrak{D}}\Phi^I)\,, \tag{2.17}$$

$$\omega_5 = M + 16\sqrt{2}H^{-2}C_{IJK}\Phi^I\Phi^J\Phi^K + 3\sqrt{2}H^{-1}L_I\Phi^I\,, \tag{2.18}$$

where M is just a harmonic function in $\mathbb{E}^3$, i.e. $\nabla^2 M = 0$.

Equations (2.14), (2.15) and (2.16) in the Abelian sector imply that H, Φ^i and L_i are just harmonic functions on $\mathbb{E}^3$. Once these are specified it is straightforward to find the 1-forms χ and $\breve{A}^i$.

In the non-Abelian sector (2.15) is the Bogomol'nyi equation [33], which is non-linear and hard to solve in general. Fortunately this system, that describes a non-Abelian monopole in Yang-Mills-Higgs theory, has been studied by many authors and the space of solutions available in the bibliography is rich enough for the purposes of our work.

Equation (2.16) in the non-Abelian sector is easily solved if we choose $L_\alpha \propto \Phi^\alpha$ or just $L_\alpha = 0$. However none of these choices is completely satisfying if one pursues the construction of general smooth horizonless geometries. If one takes $L_\alpha \propto \Phi^\alpha$ then there are some potential restrictions on the space of possible Φ^i that can result in smooth geometries. We will need to find a more general solution.

Finally, (2.17) can always be solved if its integrability condition is satisfied. This condition gives a set of algebraic equations, which in this context are known as *bubbling equations*, that impose restrictions on the distance between the different centers of the solution (the points were the seed functions are singular). Then, of course, one has to integrate explicitly equation (2.17) to obtain $\breve{\omega}$.

In summary, we have described a procedure to construct supersymmetric timelike solutions in terms of a set of *seed functions* defined on three-dimensional flat space: H, Φ^I, L_I and M.

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3 Smooth bubbling geometries in SEYM supergravity

Smooth microstate geometries are defined as horizonless, regular field configurations without any brane sources but with the asymptotic charges of a black hole. At a technical level this statement implies several conditions that we shall address in the following subsections, being perhaps the most important of those the requirement of working with manifolds with non-trivial topology.[5] This fact can be roughly understood from the fact that the existence of non-trivial cycles allows for the presence of measurable asymptotic charges without the introduction of localized brane sources. See for instance [11] for a detailed discussion about this topic.

The systematic procedure for finding solutions described in the previous section can naturally accommodate ambipolar Gibbons-Hawking spaces, which have just the right properties for these purposes. Let us start with a brief description of these manifolds.

3.1 Ambipolar Gibbons-Hawking spaces

Much of the very interesting physics exhibited by these solutions is related to the use of ambipolar Gibbons-Hawking spaces, which are a particular example of ambipolar hyperKähler manifolds [34]. These have the form of a U(1) fibration over a $\mathbb{R}^3$ base, with the fiber collapsing to a point at a finite collection $X = \{\vec{x}_a | a = 1, \ldots, n\}$ of points in $\mathbb{R}^3$ which we will call *centers*. Any path in the base manifold connecting two centers, γ_{ab}, defines a non-contractible 2-cycle through the inclusion of the U(1) fiber, $\Delta_{\gamma_{ab}}$. A different path γ'_{ab} between the same centers describes an homologically equivalent 2-cycle $\Delta_{\gamma'_{ab}} \simeq \Delta_{\gamma_{ab}}$. We will denote any of the equivalent 2-cycles simply as Δ_{ab}.

These spaces have the metric

$$d\hat{s}^2 = H^{-1}(d\varphi + \chi)^2 + H\left[dr^2 + r^2\left(d\theta^2 + sin^2\theta d\psi^2\right)\right] , \qquad \star_3 dH = d\chi \, , \tag{3.1}$$

with the angular coordinates taking values in $\theta \in [0, \pi)$, $\psi \in [0, 2\pi)$, $\varphi \in [0, 4\pi)$. H is a harmonic function on $\mathbb{E}^3$ of the form

$$H = \sum_a \frac{q_a}{r_a} \, , \qquad \text{with} \quad r_a \equiv |\vec{x} - \vec{x}_a| \, , \quad \vec{x}_a \in X \, , \tag{3.2}$$

while the 1-form χ plays the role of local connection of the fiber bundle and can be written as

$$\chi = \sum_a q_a cos\theta_a d\psi_a \, , \tag{3.3}$$

where θ_a and ψ_a are coordinates on a spherical frame centered in $\vec{x}_a$.

Although H is singular when evaluated at the centers it is straightforward to check that if all q_a, aka Gibbons-Hawking charges, are integers then the metric remains regular at these points.[6] Indeed under the redefinition of the radial coordinate $\rho_a = 2\sqrt{r_a}$ we find that locally

$$d\hat{s}^2|_{\rho_a \to 0} \sim d\rho_a^2 + \rho_a^2 d\Omega^2_{(3)/q_a} \, , \tag{3.4}$$

[5] By this we mean that they describe non-contractible spaces.

[6] When $|q_a| \neq 1$ there is an orbifold singularity at $\vec{x} = \vec{x}_a$, but we will not worry about it since these singularities are innocuous in the context of string theory.

being $d\Omega^2_{(3)/q_a}$ the standard metric on $S^3/\mathbb{Z}_{|q_a|}$. Asymptotically the manifold is also of this form, $d\hat{s}^2|_{\rho\to\infty} \sim d\rho^2 + \rho^2 d\Omega^2_{(3)/Q}$, with the orbifold given in this case by $S^3/\mathbb{Z}_{|Q|}$, being $Q \equiv \sum_a q_a$.

Physically, smooth bubbling geometries are claimed to represent microstate configurations of some particular black hole, being both solutions indistinguishable asymptotically. Therefore we are interested in having the ambipolar Gibbons-Hawking space asymptotic to $\mathbb{R}^4$, which we can achieve imposing $Q = 1$. This condition requires that some of the Gibbons-Hawking charges be negative, and therefore the function H interpolates between $-\infty$ and $+\infty$. Each negatively charged center is surrounded by a connected open region with $H < 0$, whose boundary is a surface where H vanishes.

Then the signature of the metric interpolates between $(++++)$ and $(----)$, being clearly ill-defined at the surfaces where $H = 0$. It is this characteristic what renders this space be ambipolar. These harmful properties, however, can be made compatible with having a smooth five-dimensional supergravity solution due to the presence of both, the conformal factor f^{-1} multiplying $d\hat{s}^2$ and the additional terms in the full metric, see equation (2.3). We will elaborate on this in subsequent sections.

3.2 Seed functions for horizonless spacetimes

In the language of the solution generating technique outlined in section 2, we have given the first small step in the way to obtain a supersymmetric solution, that can be synthesized as

$$H = \sum_a \frac{q_a}{r_a}\,, \qquad \text{with} \quad q_a \in \mathbb{Z}\,, \quad \sum_a q_a = 1\,. \tag{3.5}$$

The remaining seed functions in the Abelian sector Φ^i, L_i and M are also harmonic,

$$\Phi^i = k^i_0 + \sum_a \frac{k^i_a}{r_a}\,, \qquad L_i = l^i_0 + \sum_a \frac{l^i_a}{r_a}\,, \qquad M = m_0 + \sum_a \frac{m_a}{r_a}\,, \tag{3.6}$$

and from equation (2.15) we readily obtain

$$\breve{A}^i = \sum_a k^i_a cos\theta_a d\psi_a\,. \tag{3.7}$$

Notice that we imposed that the location of the singularities coincides with a Gibbons-Hawking center. With this requirement we will be able to avoid that the building blocks h_I/f as defined in (2.12) become singular whenever any of the seed functions individually diverge. This is the mathematical version of what at the beginning of the section we called absence of brane sources, and it is the mechanism responsible of obtaining horizonless geometries.[7] Also, the fact that the harmonic seed functions are singular at the Gibbons-Hawking centers is directly responsible for much of the very interesting physics captured by these solutions. Consequently, we would like the non-Abelian seed functions to display a similar qualitative behavior, i.e. $(\Phi^\alpha, L_\alpha)|_{r_a\to 0} \sim r_a^{-1} + \mathcal{O}(r_a^0)$.

[7] Clearly this naming is pointing at the physical origin of these potential singularities once the solutions are interpreted in the context of string theory.

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Protogenov's SU(2) colored monopole [35] is a solution to the Bogomol'nyi equation with this property, with only one single center. Colored monopoles are rather intriguing objects. They describe a point with unit local magnetic charge surrounded by a magnetic cloud that completely screens the charge as seen from infinity.[8] Despite its singular nature when interpreted in the context of Yang-Mills-Higgs theory, single center colored monopole solutions have been fruitfully used in the literature to obtain regular non-Abelian black holes in four- [36–39] and five-dimensional [26, 27] theories of gauged supergravity. Their presence has an interesting impact on black hole thermodynamics, modifying the entropy without altering the mass.

Therefore, a family of well-suited non-Abelian seed functions Φ^α is given by a multi-center generalization of colored monopoles, which we construct now. From now on we will assume the gauged group is SU(2) for the sake of simplicity, so the index α can take three possible values. Nevertheless, following the ideas of Meessen and Ortín [36], it should be possible to embed these monopoles in a more general group SU(N) and use them in the construction of smooth bubbling geometries in SU(N)-gauged supergravity.

Plugging in the Bogomoln'yi equation (2.15) the ansatz of the *hedgehog* form

$$\Phi^\alpha = -\frac{1}{\breve{g}P}\frac{\partial P}{\partial x^s}\delta^\alpha_s\,, \qquad \breve{A}^\alpha_\mu = -\frac{1}{\breve{g}P}\frac{\partial P}{\partial x^s}\varepsilon^\alpha{}_{\mu s}\,, \tag{3.8}$$

we find that this configuration describes a monopole solution if P is a harmonic function,

$$P = \lambda_0 + \sum_a \frac{\lambda_a}{r_a}\,, \qquad \lambda_0 \neq 0\,. \tag{3.9}$$

Substituting back in (3.8), we can write the solution as

$$\Phi^\alpha = \sum_a \frac{\lambda_a}{\breve{g}r_a^2 P}\delta^\alpha_s\frac{(x^s - x^s_a)}{r_a}\,, \qquad \breve{A}^\alpha_\mu = \sum_a \frac{\lambda_a}{\breve{g}r_a^2 P}\varepsilon^\alpha{}_{\mu s}\frac{(x^s - x^s_a)}{r_a}\,. \tag{3.10}$$

The Higgs field of the monopole is singular at the centers and vanishes at infinity

$$\lim_{r_a\to 0}\Phi^\alpha = \frac{k^\alpha_a}{r_a} + \mathcal{O}(r_a^0)\,, \qquad \lim_{r\to\infty}\Phi^\alpha \sim \mathcal{O}(r^{-2})\,, \qquad k^\alpha_a \equiv \delta^\alpha_s\frac{(x^s - x^s_a)}{\breve{g}r_a}\,. \tag{3.11}$$

This solution corresponds to a *multicenter colored monopole* configuration.

The last seed functions we need to find are L_α, which are solutions of equation (2.16), that we repeat here for convenience

$$\breve{\mathfrak{D}}^2 L_\alpha - \breve{g}^2 f_{\alpha\beta}{}^\lambda f_{\gamma\lambda}{}^\rho\Phi^\beta\Phi^\gamma L_\rho = 0\,. \tag{3.12}$$

We can solve this differential system by making use of the ansatz

$$L_\alpha = -\frac{1}{\breve{g}P}\frac{\partial Q}{\partial x^s}\delta^\alpha_s\,, \tag{3.13}$$

[8] The magnetic charge is defined as $p = \frac{\breve{g}}{4\pi}\int_{S^2}\frac{\Phi^\alpha\breve{F}^\alpha}{\sqrt{\Phi^\alpha\Phi^\alpha}}$.

the equation reduces to the condition of Q being harmonic. We choose Q to be of the form

$$Q = \sum_a \frac{\sigma_a \lambda_a}{r_a}\,. \tag{3.14}$$

The functions L_α behave similarly to Φ^α near the centers and at infinity

$$\lim_{r_a\to 0} L_\alpha = \frac{l_a^\alpha}{r_a} + \mathcal{O}(r_a^0)\,, \qquad \lim_{r\to\infty} L_\alpha \sim \mathcal{O}(r^{-2})\,, \qquad l_a^\alpha \equiv \sigma_a \delta_s^\alpha \frac{(x^s - x_a^s)}{\breve{g} r_a}\,, \tag{3.15}$$

only differentiated by the presence of the parameters σ_a in the near-center limit. The appearance of these factors will be fundamental for obtaining horizonless geometries.

After having fixed the general form of all the seed functions, we can start analyzing the regularity of the metric. In order to construct horizonless solutions we need to avoid having brane sources at the centers. In other words, we want the building blocks h_I/f that constitute the metric function, given by (2.12), to remain finite at these points. Keeping the charges q_a and k_a^i arbitrary, it is possible to remove the brane sources by taking

$$l_a^I = -8 C_{IJK} \frac{k_a^J k_a^K}{q_a}\,. \tag{3.16}$$

Notice that this expression is valid in both the Abelian and the non-Abelian sector. In the former it fixes the value of the parameters l_a^i, while in the latter it fixes the parameters σ_a. Regularity of the metric at the centers also requires ω_5 to be finite there, something that we achieve by choosing

$$m_a = 8\sqrt{2} C_{IJK} \frac{k_a^I k_a^J k_a^K}{q_a^2}\,. \tag{3.17}$$

The constant terms in the harmonic seed functions (3.6) define the solution at infinity. In order to have an asymptotically flat metric ($f_\infty \sim 1$, $\omega_{5,\infty} \sim 0$) we need to satisfy the constrains

$$k_0^i = 0\,, \qquad 27 C^{ijk} l_0^i l_0^j l_0^k = 1\,, \qquad m_0 = -3\sqrt{2} \sum_{i,a} l_0^i k_a^i\,. \tag{3.18}$$

3.3 Closed timelike curves and bubbling equations

By using an ambipolar Gibbons-Hawking metric we are taking a clear risk: the spacetime metric might contain closed timelike curves (CTC's) or even be ill-defined at the critical surfaces where $H = 0$. We now study the conditions under which CTC's are absent, so the microstate geometries are physically sensible.

Let us expand the expression of the spacetime metric (2.3) and write it in the following manner

$$ds^2 = f^2 dt^2 + 2f^2 dt\omega - \frac{\mathcal{I}}{f^{-2}H^2}\left(d\varphi + \chi - \frac{\omega_5 H^2}{\mathcal{I}}\breve{\omega}\right)^2 - f^{-1}H\left(d\vec{x}\cdot d\vec{x} - \frac{\breve{\omega}^2}{\mathcal{I}}\right)\,, \tag{3.19}$$

where $\mathcal{I}$ is defined as

$$\mathcal{I} \equiv f^{-3}H - \omega_5^2 H^2\,. \tag{3.20}$$

There is one general restriction that needs to be satisfied in order to avoid the presence of CTC's

$$\mathcal{I} \geq 0\,. \tag{3.21}$$

Apparently there is one additional condition, $f^{-1}H \geq 0$, but this is implied by the inequality in (3.21). Let us express this condition in more detail by evaluating $\mathcal{I}$ in terms of the seed functions

$$\begin{aligned}\mathcal{I} = &-M^2H^2 - 18\left(\Phi^I L_I\right)^2 - 32\sqrt{2}MC_{IJK}\Phi^I\Phi^J\Phi^K - 6\sqrt{2}MHL_I\Phi^I \\ &+ 27HC^{IJK}L_IL_JL_K + 3^4 2^3 C^{IJK}C_{KLM}L_IL_J\Phi^L\Phi^M \geq 0\,.\end{aligned} \tag{3.22}$$

The first point to notice is that the form of this expression coincides with that of ungauged supergravity originally derived in [1], where it was identified as the quartic invariant of $E_{7(7)}$. The analysis of the positivity of this quantity is hard to do in general, although we can assert that this bound can be satisfied for large families of configurations. The reason behind this statement is that this has been shown to be the case for ungauged supergravities, and many techniques to construct solutions satisfying this bound have been developed. In any case, it is fair to say that this restriction definitely makes the process of constructing explicit solutions more complicated.

There is one additional factor that can result in the appearance of CTC's, and this is the formation of Dirac-Misner strings. Those arise when the integrability condition of the last differential equation that still remains to be solved, (2.17), is not satisfied. This condition is obtained acting with the operator $d\star_3$ in that expression, which gives

$$\left\{H\nabla^2 M - M\nabla^2 H + 3\sqrt{2}(\Phi^i\nabla^2 L_i - L_i\nabla^2\Phi^i + \Phi^\alpha\breve{\mathfrak{D}}^2 L_\alpha - L_\alpha\breve{\mathfrak{D}}^2\Phi^\alpha)\right\} = 0\,. \tag{3.23}$$

This condition is identically satisfied as a consequence of equations (2.14)–(2.16) everywhere except at the centers, where technically those equations cease to apply. The bubbling equations are algebraic constrains that guarantee that the integrability condition is satisfied everywhere, setting the requirements that avoid the presence of Dirac-Misner strings.

To make further progress it is convenient to define the symplectic vector of seed functions

$$S^M = \left(H, 3\sqrt{2}\Phi^I, M, L_I\right)\,, \qquad\qquad S_M = \left(M, L_I, -H, -3\sqrt{2}\Phi^I\right)\,, \tag{3.24}$$

and a symplectic vector of charges at each center

$$Q_a^M = \left(q_a, 3\sqrt{2}k_a^I, m_a, l_a^I\right)\,, \qquad\qquad Q_{M,a} = \left(m_a, l_a^I, -q_a, -3\sqrt{2}k_a^I\right)\,. \tag{3.25}$$

Now we can write the integrability condition as

$$S^M\breve{\mathfrak{D}}^2 S_M = 0\,. \tag{3.26}$$

Interestingly the non-Abelian sector vanishes in the last expression due to the symplectic product and the expression is reduced to $S^m Q_{m,a}\delta(\vec{x}-\vec{x}_a) = 0$ with the understanding that S^m and Q_a^m are the components of the symplectic vectors in the Abelian sector. Then,

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one could naively expect that the bubbling equations coincide with those in the case of ungauged supergravity theories. However, this does not happen because the charges l^i_a are affected by the presence of the non-Abelian fields according to (3.16). After a few lines of algebraic computation, the resulting bubbling equations are conveniently written as

$$\sum_{b\neq a} \frac{q_a q_b}{r_{ab}} \left[C_{ijk}\Pi^i_{ab}\Pi^j_{ab}\Pi^k_{ab} - \frac{1}{2\breve{g}^2}\Pi^0_{ab}\mathbb{T}_{ab}\right] = \frac{3}{8} l^i_0 \left(\sum_b q_a k^i_b - k^i_a \right) , \tag{3.27}$$

where Π^i_{ab} is the i^{th}- flux threading the 2-cycle Δ_{ab} and $\mathbb{T}_{ab}$ contains information about the topological charge associated to the centers a and b, see (3.45)

$$\Pi^i_{ab} \equiv \left(\frac{k^i_b}{q_b} - \frac{k^i_a}{q_a} \right) , \qquad \mathbb{T}_{ab} \equiv \breve{g}^2 \left(\frac{k^\alpha_a k^\alpha_a}{q^2_a} + \frac{k^\alpha_b k^\alpha_b}{q^2_b} \right) . \tag{3.28}$$

We are now ready to integrate (2.17). It is convenient to decompose the 1-form $\breve{\omega}$ into two parts, $\breve{\omega}^A$ and $\breve{\omega}^B$, satisfying

$$\star_3 d\breve{\omega}^A = HdM - MdH + 3\sqrt{2}(\Phi^i dL_i - L_i d\Phi^i) \,, \tag{3.29}$$
$$\star_3 d\breve{\omega}^B = 3\sqrt{2}(\Phi^\alpha \breve{\mathfrak{D}} L_\alpha - L_\alpha \breve{\mathfrak{D}}\Phi^\alpha) \,, \tag{3.30}$$

The first equation can be solved independently for each pair of centers (a, b), with $\breve{\omega}^A = \sum_a \sum_{b>a} \breve{\omega}^A_{ab}$. For each pair we use adapted coordinates such that $\vec{x}_a = (0, 0, 0)$ and $\vec{x}_b = (0, 0, -r_{ab})$, with spherical angles given by

$$x^1_{ab} = r_a sin\theta_{ab} sin\psi_{ab} \qquad x^2_{ab} = r_a sin\theta_{ab} cos\psi_{ab} \qquad x^3_{ab} = -r_a cos\theta_{ab} \,. \tag{3.31}$$

Upon substitution of the seed functions H, M, L_i, Φ^i, (3.29) can be written as

$$\begin{aligned} \star_3 d\breve{\omega}^A_{ab} = \frac{Q_{m,a}Q^m_b}{r_{ab}} \Bigg\{ &-\frac{1}{r^2_a} \left[1 - \frac{r_{ab}+r_a}{r_b} + \frac{r_a r_{ab}\,(r_a + r_{ab})}{r^3_b} (1-\cos\theta_{ab}) \right] dr_a \\ &+ \left[\frac{r_{ab}\sin\theta_{ab}}{r^3_b} (r_a - r_{ab}) \right] d\theta_{ab} \Bigg\} \,, \end{aligned} \tag{3.32}$$

being r_b the radial distance as measured from $\vec{x}_b$. A solution can be readily found provided $\breve{\omega}^A_{ab}$ has only one non-vanishing component, $\breve{\omega}^A_{ab,\psi_{ab}}$

$$\breve{\omega}^A_{ab} = \frac{8\sqrt{2} q_a q_b}{r_{ab}} \left[C_{ijk}\Pi^i_{ab}\Pi^j_{ab}\Pi^k_{ab} - \frac{1}{2\breve{g}^2}\Pi^0_{ab}\mathbb{T}_{ab} \right] (\cos\theta_{ab} - 1) \left(1 - \frac{r_a + r_{ab}}{r_b} \right) d\psi_{ab} \,. \tag{3.33}$$

Now we turn our attention to (3.30). Notice that this expression contains three-point interactions due to the presence of the connection $\breve{A}^\alpha$ in the covariant derivative, so at first sight its structure is more involved than that of its Abelian counterpart. However, despite this complexity, the general solution for an arbitrary number of centers can be found. It is most remarkable that the interactions among all of them can be written in a very compact form! We obtain

$$\breve{\omega}^B = \frac{3\sqrt{2}\varepsilon_{rst}}{\breve{g}^2 P^2} \frac{\partial Q}{\partial x^s} \frac{\partial P}{\partial x^t} dx^r \,. \tag{3.34}$$

While deriving (3.33) and (3.34) we have assumed that the integrability condition is satisfied by making use of the bubbling equations (3.27). As a consistency check we can perform an inspection to confirm the absence Dirac-Misner strings in $\breve{\omega}^A$ and $\breve{\omega}^B$. For the former, it is straightforward to verify that the only component of the one form, $\breve{\omega}^A_{ab,\psi_{ab}}$, vanishes when the coordinate ψ_{ab} is not well defined. In particular this happens along the x^3_{ab} axis both in the positive direction, where $\left(1-\frac{r_a+r_{ab}}{r_b}\right)|_{x^{3,+}_{ab}}=0$, and in the negative direction, with $(\cos\theta_{ab}-1)\,|_{x^{3,-}_{ab}}=0$. In the case of the latter it suffices to check that $\breve{\omega}^B$ is regular at the centers as a consequence of the antisymmetric character of the 1-form components.

3.4 Fluxes and topological charge

We now turn our attention to the vector fields. We shall recall their expressions

$$A^I = -\sqrt{3}h^I f(dt+\omega)+\hat{A}^I\,, \tag{3.35}$$
$$\hat{A}^I = -2\sqrt{6}\left[-\Phi^I H^{-1}(d\varphi+\chi)+\breve{A}^I\right]\,, \tag{3.36}$$

where $\breve{A}^I$ is determined in terms of Φ^I by the Bogomol'nyi equation (2.15) and whose explicit form is (3.7) in the Abelian sector and (3.8) in the non-Abelian. From these expressions we see that these fields can be understood in terms of three layers: the physical vectors A^I, a four-dimensional instanton $\hat{A}^I$ with selfdual field strength and a three-dimensional static magnetic monopole $\breve{A}^I$. Each of them is used to build up those preceding it, in a configuration that resembles the structure of the Russian *matryoshka* dolls.

In the Abelian sector $\breve{A}^i$ describes a configuration with several Dirac monopoles, which is singular due to the presence of Dirac strings attached to each center. These strings are eliminated in $\hat{A}^i$ by the new term in (3.36), although this term introduces new strings in the compact direction φ,

$$\lim_{r_a\to 0}\hat{A}^i \sim -2\sqrt{6}\left[-\frac{k^i_a}{q_a}(d\varphi+q_a cos\theta_a d\psi_a)+k^i_a cos\theta_a d\psi_a\right]\sim 2\sqrt{6}\frac{k^i_a}{q_a}d\varphi\,. \tag{3.37}$$

The component in the local coordinate ψ_a is compensated by the new term, but now $\hat{A}^i_\varphi$ is finite at the centers, where the coordinate φ is not well defined. Besides $\hat{A}^i$ is not regular either at the critical surfaces characterized by $H=0$. Yet again, this singularity is cured at the next stage and the physical vectors A^i are globally regular up to gauge transformations. In this case the first term in (3.35) compensates the divergence at the critical surface,

$$\lim_{H\to 0}\left(-\sqrt{3}h^i f\omega_5(d\varphi+\chi)\right) = -2\sqrt{6}H^{-1}\Phi^i(d\varphi+\chi)+\mathcal{O}(H^0)\,, \tag{3.38}$$

without introducing any anomaly elsewhere, which is guaranteed because ω has been designed to be free of Dirac-Misner strings.

To every non-trivial 2-cycle at the ambipolar space it is naturally associated a magnetic flux for each vector, defined as the integral of the field strength F^i along the 2-cycle. To compute this quantity we make use of our standard decomposition for A^i, which is valid everywhere except at the centers. Nevertheless since the field strength is globally regular the flux can be equally computed by taking the integral along the 2-cycle with the poles

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excised. In this region the integrand is an exact form and we can make use of Stokes' theorem. We get

$$\Pi^i_{ab} \equiv \frac{1}{(2\sqrt{6})4\pi}\int_{\Delta_{ab}} F^i = \left(\frac{k^i_b}{q_b} - \frac{k^i_a}{q_a}\right) . \tag{3.39}$$

We now consider the non-Abelian sector. Our recipe for constructing solutions of $\mathcal{N}=1$, $d=5$ SEYM theory naturally incorporates Kronheimer's scheme [40], that relates any static monopole $\breve{A}^\alpha$ to an instanton over a Gibbons-Hawking base, $\hat{A}^\alpha$, through equation (3.36). For example, in [41] this mechanism has been utilized to *oxidize* the single center colored monopole, that has turned out to be the counterpart of the BPST instanton [42]. On the other hand, Etesi and Hausel showed in [43] that families of regular Yang-Mills instantons over an Asymptotically Locally Euclidean space (ALE) are related to multicenter colored monopoles in Kronheimer's scheme.[9] However, although our instanton is related to the same monopole, it is necessarily different than the Etesi-Hausel solution because they are defined on different bases: our Gibbons-Hawking space is ambipolar, not ALE. In particular this means that our instanton is singular at the critical surfaces. This is cured for the five-dimensional physical vector in the same manner than it is for the Abelian vectors.

Even though the instanton $\hat{A}^\alpha$ is ill-defined at the critical surfaces, we would like to study if we can associate to it a topological charge, also known as instanton number.[10] Here we need to remark that this topological charge is associated to the vector $\hat{A}^\alpha$ defined on the ambipolar Gibbons-Hawking space. Therefore this quantity may not be a true invariant of the physical spacetime. Nevertheless its computation is interesting by itself and, as we are about to see, this quantity is finite even though the connection blows up. We define the topological charge as

$$\mathbb{T} = \frac{g^2}{32\pi^2}\int_{\mathcal{M}_4\backslash S} d^4\Sigma \hat{F}^2 , \tag{3.40}$$

where $d^4\Sigma$ is the volume form of the manifold, $\hat{F}^2$ is the scalar obtained by taking the trace of the field strength contracted with itself, $\hat{F}^2 \equiv \hat{F}^\alpha_{\mu\nu}\hat{F}^{\alpha\,\mu\nu}$, and $\mathcal{M}_4\backslash S$ is the ambipolar space without the critical surfaces. These have to be necessarily removed because the canonical volume form associated to the metric vanishes there and the above integral cannot be defined over them. To perform the calculation it is convenient to work in the following flat frame of the cotangent bundle

$$e^0 = s|H|^{-1/2}(d\varphi + \chi) , \qquad e^a = |H|^{1/2}dx^s\delta^a_s , \qquad \epsilon^{0123} = \epsilon_{0123} = 1 . \tag{3.41}$$

where s is $+1$ when H is positive and -1 when H is negative. The volume form is expressed in terms of the vielbeins as $e^0\wedge e^1\wedge e^2\wedge e^3 = H d\varphi\wedge d^3x$, where d^3x is a shorthand for $dx^1\wedge dx^2\wedge dx^3$. The gauge field strength is obtained from (3.36) and its components in this coframe are

$$\hat{F}^\alpha_{0a} = -2\sqrt{6}s\breve{\mathfrak{D}}_a\left(\Phi^\alpha H^{-1}\right) , \qquad \hat{F}^\alpha_{ab} = -2\sqrt{6}s\left[H^{-1}\breve{F}^\alpha_{ab} - H^{-2}\Phi^\alpha(d\chi)_{ab}\right] . \tag{3.42}$$

[9] In fact, to the best of our knowledge, multicenter colored monopoles have only appeared in the literature so far in [43], where they are used as valuable intermediates for computing the topological charge of their instanton counterparts.

[10] It would be very interesting to study rigorously the construction of SU(2) fiber bundles over ambipolar Gibbons-Hawking bases, but this goes beyond the scope of the present work.

Substituting back into (3.40), using (2.14), (2.15) and integrating by parts we get

$$\mathbb{T} = \frac{\breve{g}^2}{32\pi^2}\int_{\mathcal{M}_4\setminus(S\cup X)} d\varphi\wedge d^3x\left[2\nabla^2\left(\frac{\Phi^\alpha\Phi^\alpha}{H}\right) - 4H^{-1}\Phi^\alpha\breve{\mathfrak{D}}^2\Phi^\alpha + 2H^{-2}\Phi^\alpha\Phi^\alpha\nabla^2 H\right]. \tag{3.43}$$

Notice that in this step the centers have also been removed from the integration space because the decomposition (3.36) is not well-defined there. This does not change the value of the integral because $\hat{F}^2$ is regular at these points. The second and third terms in the above expression vanish identically in the region. We can integrate on φ and apply Stokes theorem to get

$$\mathbb{T} = \frac{\breve{g}^2}{4\pi}\int_{V^3} d^3x\nabla^2\left(\frac{\Phi^\alpha\Phi^\alpha}{H}\right) = \frac{\breve{g}^2}{4\pi}\int_{\partial V^3} d^2\Sigma\, n_a\partial_a\left(\frac{\Phi^\alpha\Phi^\alpha}{H}\right). \tag{3.44}$$

Here V^3 is $\mathbb{R}^3$ with the centers and the critical surfaces excised, $d^2\Sigma$ is the volume form induced on ∂V^3 and n_a are the components of a unit vector normal to ∂V^3. Thus the problem is reduced to a computation at the boundary of V^3, which is composed of the critical surfaces, the centers and infinity. Formally at the critical surfaces we receive an infinite contribution to the topological charge, but notice that each connected critical surface is the boundary of two disconnected regions of V_3 and therefore it appears twice in the computation. Since the normal unitary vector $\vec{n}$ has opposite direction in each case, both infinite contributions cancel out because $\lim_{\vec{x}\to\partial V^3}\partial_a\left(\frac{\Phi^\alpha\Phi^\alpha}{H}\right)|n_a|$ takes the same value when $\vec{x}$ is evaluated at both sides of the critical surface.

After having got rid of the critical surfaces, the computation of (3.44) is straightforward. The contributions at each center and at infinity are

$$\mathbb{T}_a = \breve{g}^2\frac{k_a^\alpha k_a^\alpha}{q_a}, \qquad \mathbb{T}_\infty = 0, \tag{3.45}$$

Assuming that we placed non-Abelian seed functions at every center, the total topological charge is

$$\mathbb{T} = \sum_a \frac{1}{q_a}. \tag{3.46}$$

3.5 Critical surfaces

As we have already discussed at previous stages, the critical surfaces defined by having $H=0$ are worth special attention. Not only is the ambipolar Gibbons-Hawking metric ill-defined there, but also many of the other auxiliary building blocks that make up the solution contain inverse powers of H. Nevertheless, the spacetime metric and all physical fields remain completely regular at the critical surfaces. It is interesting to illustrate in some detail how this happens.

Let us consider the metric as written in (3.19). In the purely spatial part there are no singularities in these surfaces because the product $f^{-1}H$ defines a finite positive quantity,

$$\lim_{H\to 0} f^{-1}H = 8\left(C_{IJK}\Phi^I\Phi^J\Phi^K\right)^{2/3} + \mathcal{O}(H), \tag{3.47}$$

and $\mathcal{I}$ is also regular, as easily seen from its expression in terms of the seed functions (3.22). Of course, this is only possible because $\lim_{H\to 0} f \sim 0$ and this, in particular, means that the critical surfaces are determined by the vanishing of the norm of the Killing vector that generates time translations, $V = \partial_t$, $V^\mu V_\mu = f^2$.

One might get worried by this statement, since timelike supersymmetric solutions in supergravity quite frequently have event horizons at the regions where the timelike Killing vector becomes null. Happily this does not happen here. First, because as we just saw the spatial part remains regular, and second, because of the presence of the additional finite term in the metric that keeps the determinant non-vanishing at these regions,

$$\lim_{H\to 0} f^2\omega_5 dt(d\varphi+\chi) = \frac{1}{2\sqrt{2}}\left(C_{IJK}\Phi^I\Phi^J\Phi^K\right)^{-1/3} dt(d\varphi+\chi) + \mathcal{O}(H)\,. \tag{3.48}$$

Then any massive particle sitting at the surface is unavoidably dragged along some spatial direction. Critical surfaces have the same properties as the boundary of an ergosphere, except from the fact that they do not actually surround an ergosphere since the Killing vector V remains timelike at both of their sides. As a consequence of this they have been named *evanescent ergosurfaces* [44].

In the previous subsection we already showed that the physical vectors are well-behaved at the evanescent ergospheres. The physical scalars, constructed by $\phi^x \equiv h_I/h_0$, are also regular here

$$\lim_{H\to 0}\phi^x = \frac{C_{xIJ}\Phi^I\Phi^J}{C_{0LM}\Phi^L\Phi^M} + \mathcal{O}(H)\,. \tag{3.49}$$

4 Final comments

The set of continuous parameters λ_a that appear in the definition of the colored monopole, (3.8), have no impact on the physics of the solution neither at the centers nor at infinity, but they do affect the physical fields at intermediate regions. This means that the geometry of a particular solution can be continuously distorted in some manner as long as the modification does not introduce CTC's. Therefore we can build a *classically* infinite number of microstate geometries with the same topology for the same black hole or black ring.

It is useful to explain in some detail why these parameters are special in this sense. First, one has to notice that asymptotically the non-Abelian seed functions Φ^α are subleading with respect to the Abelian seed functions Φ^i (3.6). Second, the functions Φ^α have the same limit at leading order at all the centers, whose value is independent of these parameters. These characteristics imply that the mass, angular momenta and electric charges of the solution are invisible to the parameters λ_a. The size of the bubbles are also unaffected by them, see (3.27).

The *colored* non-Abelian black hole solutions discovered so far are constructed from a single-center colored monopole. They incorporate one parameter, say λ_1, interpreted as the *size* of the instanton field of the solution, that modifies the geometry outside the horizon but does not alter any of the *observables* of the solution, like the mass, entropy, electric charges or instanton number. In this context this parameter is interpreted as non-Abelian hair.

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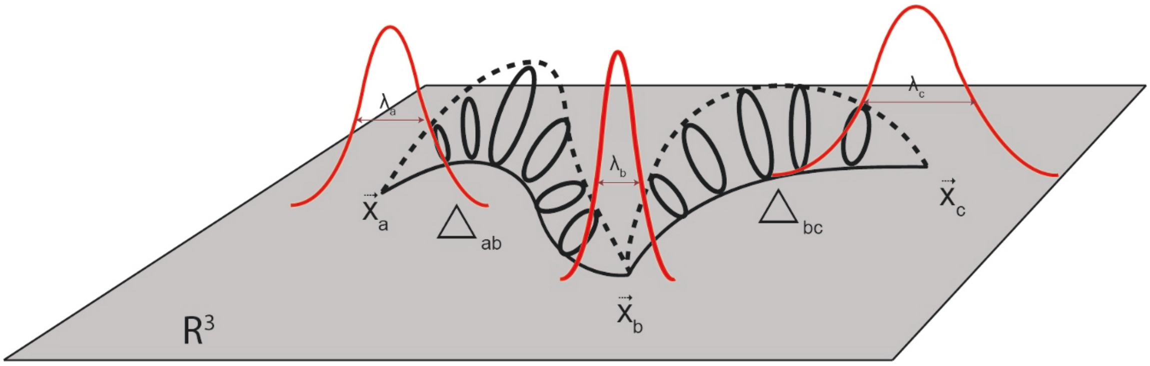

Figure 1. Representation of the multicenter instanton on the Gibbons-Hawking space.

On the other hand microstate geometries have one parameter for each center. Although we do not have a complete interpretation of the multicenter instanton field contained in these solutions, preliminary analysis based on the expansion of the instanton field $\hat{A}^{\alpha}$ near the centers suggest that each parameter codifies the information of the size of an instanton placed at the corresponding center whose individual topological charge is $1/q_a$.

On the other hand, the gauge coupling constant $\breve{g}$ controls the relative weight of the non-Abelian versus the Abelian fields. The closer this parameter is to zero the more influent the non-Abelian ingredients are. This is in particular reflected in the bubbling equations (3.27), from what we see that the size of the bubble can be dominated by one or the other contributions for different values of the coupling constant.

Clearly these solutions require further study. The explicit construction of concrete solutions with specific charges would be of course very interesting. Work in this direction is in progress [45].

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A The theory

In this appendix we give a very brief, workable description of SEYM theories and their known analytic solutions adapted to the purpose of this letter. $\mathcal{N} = 1, d = 5$ gauged supergravities can be interpreted as the minimal supersymmetric realization of Einstein-Yang-

Mills-Higgs theories.[11] They describe the coupling between a supergravity multiplet and n_v vector multiplets, a subset of which transform under the local action of a non-Abelian group. The supergravity multiplet is constituted by the graviton $e^a{}_\mu$, the gravitino ψ^i_μ and the graviphoton A^0_μ, while each vector multiplet, labeled by $x = 1, \ldots, n_v$, contains a real vector field $A^x{}_\mu$, a real scalar ϕ^x and a gaugino λ^{ix}. The vector fields can be collectively denoted as $A^I{}_\mu$, with $\{I, J, \ldots = 0, 1, \cdots, n_v\}$. The set over which these indices take values is conveniently split in two sectors denoted as $\{i, j, \cdots = 0, \cdots, i_{\max}\}$ and $\{\alpha, \beta, \cdots = i_{\max} + 1, \cdots, n_v\}$, referred as the Abelian and the non-Abelian sectors respectively.

The n_v scalars ϕ^x parametrize a σ-model equipped with a Riemannian metric g_{xy} and can be understood as coordinates on a scalar manifold. On general grounds the σ-model metric is invariant under coordinate transformations in the scalar manifold of the form

$$\delta_\Lambda \phi^x = -\hat{g} c^I k_I{}^x \,, \tag{A.1}$$

where $\hat{g}$ is interpreted as the gauge coupling constant (see below) and $k_I{}^x(\phi)$ is a set of Killing vectors of the scalar metric.[12] The requirement that the σ-model is compatible with the supersymmetric structure that controls the coupling between scalars and vectors gives rise to the mathematical construct known as Real Special Geometry, see [48, 49], that completely characterizes the supergravity theory. Then, a Killing vector of the scalar metric generates an isometry of the full supergravity theory if it respects the real special structure of the theory, see appendix H in [49].

The parameters that generate these isometries in the non-Abelian sector are spacetime functions, i.e. $c^\alpha = c^\alpha(x)$, while the corresponding Killing vectors satisfy the algebra

$$[k_\alpha, k_\beta] = -f_{\alpha\beta}{}^\gamma k_\gamma \,, \tag{A.2}$$

where $f_{\alpha\beta}{}^\gamma$ are the structure constants of some non-Abelian group (we will often use the notation $f_{IJ}{}^K$, understanding that the structure constants just vanish whenever any index take values in the Abelian sector).

The vectors in the non-Abelian sector, i.e. $A^\alpha{}_\mu$, play the role of gauge fields under the action of (A.1). That is, they transform in an appropriate way such that the covariant derivative of the scalars defined as

$$\mathfrak{D}_\mu \phi^x = \partial_\mu \phi^x + \hat{g} A^\alpha{}_\mu k_\alpha{}^x \,, \tag{A.3}$$

transforms, indeed, covariantly. The field strengths are defined in the standard manner in both the Abelian and non-Abelian sectors,

$$F^I{}_{\mu\nu} = 2\partial_{[\mu} A^I{}_{\nu]} + \hat{g} f_{JK}{}^I A^J{}_\mu A^K{}_\nu \,. \tag{A.4}$$

[11] Those were first considered in [46], see [26, 32, 47, 48] for more detailed expositions in our same conventions.

[12] Here the index I is for labeling each one of these vectors. We use it in order to keep notation simple, and it should be understood that the Killing vectors will be non-zero only for a subset of the possible values of the index.

We will set all the fermionic fields to zero, which is always a consistent truncation in these theories. The bosonic action of $\mathcal{N}=1, d=5$ SEYM is given by

$$S=\int d^5x\sqrt{g}\,\Big\{R+\frac{1}{2}g_{xy}\mathfrak{D}_\mu\phi^x\mathfrak{D}^\mu\phi^y-\frac{1}{4}a_{IJ}F^{I\,\mu\nu}F^J{}_{\mu\nu}+\frac{1}{12\sqrt{3}}C_{IJK}\frac{\varepsilon^{\mu\nu\rho\sigma\lambda}}{\sqrt{g}}\Big[F^I{}_{\mu\nu}F^J{}_{\rho\sigma}A^K{}_\lambda \\ -\frac{1}{2}\hat{g}f_{LM}{}^IF^J{}_{\mu\nu}A^K{}_\rho A^L{}_\sigma A^M{}_\lambda+\frac{1}{10}\hat{g}^2f_{LM}{}^If_{NP}{}^JA^K{}_\mu A^L{}_\nu A^M{}_\rho A^N{}_\sigma A^P{}_\lambda\Big]\Big\}. \quad (A.5)$$

The Real Special Geometry, and therefore the full supergravity theory, is completely determined by the constant symmetric tensor C_{IJK}. In particular the σ-model metric $g_{xy}(\phi)$ and the kinetic matrix $a_{IJ}(\phi)$ are directly derived from this tensor, see for example [26] for the explicit expressions.

We make use of the SU(2)-gauged ST[2, 6] model, that contains $n_v=5$ vector multiplets and the constant symmetric tensor C_{IJK} that characterizes it has the following non-vanishing components

$$C_{0xy}=\frac{1}{6}\eta_{xy}\,,\text{where}\quad(\eta_{xy})=\text{diag}(+-\cdots-)\,,\quad\text{and}\quad x,y=1,\cdots,5\,. \quad (A.6)$$

B Procedure for constructing solutions

1. Timelike supersymmetric solutions of $\mathcal{N}=1$, $d=5$ SEYM with a spacelike isometry are constructed from a set of $(2n_v+4)$ seed functions defined on $\mathbb{E}^3$. These are denoted[13] as M, H, Φ^I, L_I and satisfy the following equations

$$d\star_3 dM=0\,, \quad (B.1)$$
$$\star_3 dH-d\chi=0\,, \quad (B.2)$$
$$\star_3\breve{\mathfrak{D}}\Phi^I-\breve{F}^I=0\,, \quad (B.3)$$
$$\breve{\mathfrak{D}}^2L_I-\breve{g}^2f_{IJ}{}^Lf_{KL}{}^M\Phi^J\Phi^KL_M=0\,, \quad (B.4)$$
$$\star_3 d\breve{\omega}-\Big\{HdM-MdH+3\sqrt{2}(\Phi^I\breve{\mathfrak{D}}L_I-L_I\breve{\mathfrak{D}}\Phi^I)\Big\}=0\,, \quad (B.5)$$

for some 1-forms $\chi,\breve{\omega}$ and $\breve{A}^I$ (with field strength $\breve{F}^I$) defined also in $\mathbb{E}^3$. Here the covariant derivative $\breve{\mathfrak{D}}$ is defined in three-dimensional Euclidean space with respect to the gauge field $\breve{A}^I$ for objects transforming in the (dual) adjoint representation. More explicitly,

$$\breve{\mathfrak{D}}\Phi^I=d\Phi^I+\breve{g}f_{JK}{}^I\breve{A}^J\Phi^K\,,\qquad\qquad \breve{\mathfrak{D}}L_I=dL_I+\breve{g}f_{IJ}{}^K\breve{A}^JL_K\,. \quad (B.6)$$

Two subtleties about these expressions are worth mentioning. First, notice that the structure constants are only non-trivial in the non-Abelian sector so the covariant derivative reduce to the standard exterior derivative in the Abelian sector. Second, the gauge coupling constant in this expression is rescaled with respect to the *physical* gauge constant appearing in the action of the theory,[14] $\hat{g}=-\breve{g}/2\sqrt{6}$.

[13]Notice that the seed functions Φ^I should not be confused with the physical scalars ϕ^x appearing in the action (A.5).

[14]This fact is an indirect consequence of the rescaling factor appearing in equation (B.16).

2. Using the seed functions, the five-dimensional fields of the solution are obtained as follows:

   (a) We define the intermediate building blocks

$$h_I/f = L_I + 8C_{IJK}\Phi^J\Phi^K/H\,, \tag{B.7}$$

   that can be used to compute the physical scalars

$$\phi^x \equiv h_x/h_0\,, \tag{B.8}$$

   and the metric function

$$\begin{aligned} f^{-3} &= 3^3 C^{IJK}L_IL_JL_K + 3^4\cdot 2^3 C^{IJK}C_{KLM}L_IL_J\Phi^L\Phi^M/H \\ &\quad + 3\cdot 2^6 L_I\Phi^I C_{JKL}\Phi^J\Phi^K\Phi^L/H^2 + 2^9\left(C_{IJK}\Phi^I\Phi^J\Phi^K\right)^2/H^3\,. \end{aligned} \tag{B.9}$$

   This is derived from the Real Special Geometry constrain $27C^{IJK}h_Ih_Jh_K = 1$, which is valid for symmetric scalar manifolds.[15] In these spaces we can also define

$$h^I = 27C^{IJK}h_Jh_K\,. \tag{B.10}$$

   (b) The spacetime metric is of the conformastationary form

$$ds^2 = f^{\,2}(dt+\omega)^2 - f^{\,-1}d\hat{s}^2\,, \tag{B.11}$$

   where the 1-form ω is obtained as

$$\omega = \omega_5(d\varphi + \chi) + \breve{\omega}\,, \tag{B.12}$$

$$\omega_5 = M + 16\sqrt{2}H^{-2}C_{IJK}\Phi^I\Phi^J\Phi^K + 3\sqrt{2}H^{-1}L_I\Phi^I\,, \tag{B.13}$$

   being the inverse-hatted $\breve{\omega}$ the one in (B.5), and $d\hat{s}^2$ is a four-dimensional Gibbons-Hawking metric [50, 51]

$$d\hat{s}^2 = H^{-1}(d\varphi+\chi)^2 + H dx^r dx^r\,,\quad r = 1,2,3\,. \tag{B.14}$$

   (c) The physical vector fields and their field strengths are

$$\begin{aligned} A^I &= -\sqrt{3}h^I f(dt+\omega) + \hat{A}^I\,, \\ F^I &= -\sqrt{3}\hat{\mathfrak{D}}[h^I f(dt+\omega)] + \hat{F}^I\,, \end{aligned} \tag{B.15}$$

   where the auxiliary vectors $\hat{A}^I$ are four-dimensional gauge fields defined on the Gibbons-Hawking space as

$$\begin{aligned} \hat{A}^I &= -2\sqrt{6}\left[-H^{-1}\Phi^I(d\varphi+\chi) + \breve{A}^I\right]\,, \\ \hat{F}^I &= -2\sqrt{6}\left[-\breve{\mathfrak{D}}\left[\Phi^I H^{-1}(d\varphi+\chi)\right] + \star_3\breve{\mathfrak{D}}\Phi^I\right]\,, \end{aligned} \tag{B.16}$$

[15] This is always the case in the supergravity models that we consider here. In this expression, $C^{IJK} \equiv C_{IJK}$.

By this construction, which is due to Kronheimer [40], the field strength $\hat{F}^I$ is self-dual in the Gibbons-Hawking space, describing an instanton configuration intimately related to a lower dimensional static monopole.

Notice that $\hat{\mathfrak{D}}$ is the covariant derivative with associated connection $\hat{A}^I$ in the Gibbons-Hawking space, while $\breve{\mathfrak{D}}$ is the covariant derivative with associated connection $\breve{A}^I$ in $\mathbb{E}^3$.

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