Algebra, Geometry and Topology of the Riordan Group

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a mi abuelo, a Jimena
y a toda mi familia

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Introducción/Introduction

Español

El protagonista de este trabajo es el grupo de Riordan, que denotaremos por $\mathcal{R}(\mathbb{K})$. El grupo de Riordan fue nombrado así en honor de J. Riordan, 1903-1988 y nació (aparece por primera vez con este nombre) en el trabajo de L. Shapiro, S. Getu, W. J. Woan y L. C. Woodson en [105] (aunque la definición ha variado ligeramente: el grupo que se definió en ese artículo es de hecho un subgrupo del grupo que se considera actualmente) y fue utilizado posteriormente (por ejemplo en los trabajos de R. Sprugnoli [107] y [108]) como herramienta adecuada para problemas de naturaleza combinatoria como la demostración de identidades. Los directores de este trabajo también participaron en este desarrollo, cuando se encontraron con el grupo de Riordan desde una aproximación bastante diferente (véase [66]).

Los elementos del grupo de Riordan suelen describirse como matrices triangulares infinitas invertibles, esto es, de la forma:

$$(a_{ij})_{0\leq i,j<\infty} = \begin{bmatrix} a_{00} & & & \\ a_{10} & a_{11} & & \\ a_{20} & a_{21} & a_{22} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(inicialmente con entradas en $\mathbb{C}$, aunque pueden considerarse con entradas en cualquier cuerpo $\mathbb{K}$) asociadas a un par de series formales de potencias $d(x), h(x) \in \mathbb{K}[[x]]$ donde:

$$d(x) = d_0 + d_1x + d_2x^2 + \dots \qquad h(x) = h_1x + h_2x^2 + h_3x^3 + \dots \qquad \mathit{con}\ f_0, g_1 \neq 0$$

con la propiedad de que la función generatriz de las entradas de la columna i-ésima es:

$$d(x)\cdot(h(x))^i,\ \text{esto es: } d(x)\cdot(h(x))^i = a_{ii}x^i + a_{i+1,i}x^{i+1} + a_{i+2,i}x^{i+2} + \dots$$

En este caso dicha matriz se denota por $R(d(x), h(x))$. Gracias a esta estructura, hay una interpretación en términos de series formales de potencias de la multiplicación de estas matrices por un vector columna (el llamado Primer Teorema Fundamental de las Matrices de Riordan o simplemente 1FTRM) y de la multiplicación de dos de estas matrices entre sí (véase la subsección 0.3.2 para más detalles).

Por lo anteriormente expuesto, el grupo de Riordan podría clasificarse en el área de Combinatoria. Pero por otra parte, como cualquier objeto matemático de interés, el grupo de

Riordan tiene aspectos que pueden llamar la atención de expertos de otros ramas del desarrollo matemático. Al exponer al grupo de Riordan a distintas lupas, y buscando diferentes características se aumenta el conocimiento sobre dicho objeto y, por lo tanto, las posibilidades de su uso en problemas de distinta índole. Por eso en esta tesis vamos a tratar fundamentalmente tres tipos de aspectos:

1. **Estudio de la estructura de grupo de** $\mathcal{R}(\mathbb{K})$, que se realiza en los capítulos 1, 2 y 3, y al que hemos dedicado más trabajo. Contiene una fundamentación importante para el resto del trabajo relacionada con la estructura como límite inverso del grupo de Riordan. La correspondiente sucesión inversa involucra unos grupos de matrices de dimensión finita que llamaremos grupos parciales de Riordan. Además, en esta parte se estudia la serie derivada del grupo (con varias aplicaciones), las clases de conjugación desde diferentes puntos de vista y los elementos de orden finito, haciendo especial hincapié en las involuciones y encontrando el grupo generado por las involuciones. También se estudia la relación entre problemas algebraicos en el grupo de Riordan y ecuaciones y sistemas de ecuaciones funcionales en series formales de potencias.

2. **Estudio de una estructura de grupo de Lie infinito-dimensional de** $\mathcal{R}(\mathbb{K})$ **para** $\mathbb{K} = \mathbb{R}, \mathbb{C}$, que se realiza en el capítulo 4, como ejemplo de grupo de Lie sobre un espacio de Frechet, que además es un pro-grupo de Lie (límite inverso de grupos de Lie clásicos). Se estudian también parametrizaciones globales de los grupos parciales y del grupo de Riordan infinito.

3. **Aplicaciones a problemas de Topología Combinatoria de Complejos Simpliciales**, que se realiza en el capítulo 5. Se aplican las técnicas algebraicas desarrolladas en el resto del trabajo al estudio del problema del f-vector. Se pretende también mostrar la presencia de objetos con patrones del tipo Riordan en este campo.

Un procedimiento común en los resultados conseguidos es considerar el grupo de Riordan $\mathcal{R}(\mathbb{K})$ como límite inverso de una sucesión inversa:

$$\{(\mathcal{R}_n(\mathbb{K}), P_n)\}_{n=0}^{\infty}$$

donde los grupos $\mathcal{R}_n(\mathbb{K})$ son unos grupos de matrices triangulares inferiores de tamaño $(n+1) \times (n+1)$ que llamaremos los grupos parciales, estudiar primero qué ocurre en las proyecciones finitas y la posibilidad de *elevación* de propiedades al grupo de matrices infinitas (esta construcción de límite inverso será presentada con detalle en la sección 1.4).

También hay que tener en cuenta que la identificación de las matrices de Riordan y la interpretación del producto de las mismas en términos de series formales de potencias establece una forma de traducir problemas algebraicos en el grupo (conjugación, serie derivada, caracterización de elementos de orden finito, etc.) a sistemas de ecuaciones funcionales (en series formales de potencias) y viceversa.

Existen diferentes maneras de construir matrices de Riordan. Tenemos por ejemplo la construcción horizontal (que involucra a la A-*sequence*) y la construcción vertical (que involucra a la *g-sequence* y es uno de los motivos del uso de la notación $T(f \mid g)$ por parte de los directores de este trabajo introducida en [66] y utilizada en sus trabajos siguientes, como por ejemplo

[62, 64, 67]) y otras, como las introducidas en [21, 64]. También existen ciertas construcciones para encontrar nuevas matrices de Riordan a partir de otras conocidas (véase [16, 65, 116, 118]).

Hay también una fuerte relación entre matrices de Riordan y sucesiones clásicas de polinomios (véase [26, 44, 68, 71, 114]).

Es importante conocer también que el grupo de Riordan contiene un subgrupo isomorfo (el subgrupo asociado o de Lagrange) a $\mathcal{F}_1(\mathbb{K})$, el grupo de series formales de la forma:

$$g(x) = g_1x + g_2x^2 + g_3x^3 + \dots \qquad \textit{con } g_1 \neq 0$$

respecto a la composición, que ha sido bastante estudiado. Principalmente ha recibido mucha atención su subgrupo de series formales de potencias con $g_1 = 1$ llamado el *composition group of formal power series* (véase por ejemplo [2] para más información). Cuando $\mathbb{K}$ es un cuerpo cíclico finito, este último grupo se conoce como el grupo de Nottingham (véase [18]). A lo largo de este trabajo, en ocasiones algunos de los resultados se particularizan en resultados de interés para $\mathcal{F}_1(\mathbb{K})$.

En resumen, podemos decir que la literatura del grupo de Riordan se generó en gran parte en la última década del siglo XX (véase [76, 77, 80, 105, 107, 108]) y que sigue actualmente en desarrollo (véase [22–25, 29, 48, 62, 64, 66, 78, 79, 104, 117]).

Hemos incluido en este trabajo un capítulo inicial para revisar algunos conceptos básicos que intervienen en los demás y ofrecer bibliografía complementaria.

En el Capítulo 1, la primera cuestión que trataremos será la estructura del grupo de Riordan como límite inverso (teorema 1.4.2) de la sucesión inversa antes mencionada:

$$\{(\mathcal{R}_n(\mathbb{K}), P_n)\}_{n=0}^{\infty}$$

donde $P_n : \mathcal{R}_{n+1}(\mathbb{K}) \to \mathcal{R}_n(\mathbb{K})$ es la aplicación que elimina la última fila y columna de la matriz sobre la que se aplica. El concepto de límite inverso se utiliza en prácticamente todas las ramas de las Matemáticas, algunas veces también llamado límite proyectivo. Es un método para aproximar ciertos objetos por otros que se comportan o se conocen mejor. Los sistemas inversos, de los cuales se derivan los límites inversos pueden, en general, definirse en cualquier categoría. Un texto introductorio para estos temas es [72]. Como ya hemos dicho, para nosotros este enfoque será de gran utilidad ya que es el adecuado para hacer, entre otras cosas, demostraciones por inducción. Estudiaremos, por lo tanto, como extender una matriz $M_n \in \mathcal{R}_n(\mathbb{K})$ a una matriz $M_{n+1} \in \mathcal{R}_{n+1}(\mathbb{K})$ de forma que $P_n(M_{n+1}) = M_n$ (proposición 1.3.1). Relacionaremos estos grupos parciales con la ultramétrica que se propone en [66] y con una nueva.

También mostraremos otra representación del grupo de Riordan como grupo de matrices bi-infinitas, esto es, matrices de la forma:

$$(a_{ij})_{-\infty<i,j<\infty} = \begin{bmatrix} \ddots & & & & & & \\ \cdots & a_{-2,-2} & & & & & \\ \cdots & a_{-2,-1} & a_{-1,-1} & & & & \\ \cdots & a_{0,-2} & a_{0,-1} & a_{00} & & & \\ \cdots & a_{1,-2} & a_{1,-1} & a_{10} & a_{11} & & \\ \cdots & a_{2,-2} & a_{2,-1} & a_{2,0} & a_{21} & a_{22} & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

que también puede estudiarse como límite inverso de diferentes sucesiones (que involucran matrices finitas o no) utilizando las construcciones verticales y horizontales anteriormente citadas. Debido a trabajos previos [62, 64, 65, 109], se hacía necesaria una mejor comprensión de las diferentes simetrías en las matrices de Riordan bi-infinitas. Gracias a la estructura de límite inverso, podremos describir mucho más facilmente dichas simetrías *reflejando* las matrices finitas involucradas en el límite inverso y *elevando* al grupo bi-infinito esta simetría (dando lugar a las matrices duales y complementarias). En esta dirección estudiaremos y daremos respuesta a cuestiones propuestas en [65] sobre matrices que sean autoduales y autocomplementarias (recogida en los teoremas 1.12.3 y 1.12.4). Algunos de los conceptos anteriores han sido independientemente utilizados en [21] para construir matrices de Riordan. Como aplicación final, a partir de la estructura de límite inverso en la representación infinita, podremos recuperar fácilmente usando inducción en términos de los límites inversos (sección 1.14) algunos resultados clásicos sobre las ecuaciones funcionales de Schröder y de Schröder con pesos (para $\mathbb{K}$ de característica 0) en series formales de potencias (recogidos en el teorema 1.14.6 y en el teorema 1.14.7, véase también la bibliografía [2, 59, 99, 100, 112]). Razonamientos de este tipo utilizando el límite inverso puede que permitan en un futuro el tratamiento de otras ecuaciones funcionales en series formales de potencias de interés. Las ecuaciones de Schröder y de Schröder con pesos se pueden formular como un problema de autovectores de matrices de Riordan, lo que será relevante en capítulos posteriores de este trabajo.

En el capítulo 2, de nuevo gracias a la estructura de límite inverso que nos ayuda a realizar con comodidad pruebas por inducción, estudiaremos algunas propiedades de grupo de $\mathcal{R}(\mathbb{K})$, con $\mathbb{K}$ de característica 0.

Nos centraremos en primer lugar en el estudio de la serie derivada del grupo de Riordan (véase [91]) dando una caracterización del n-ésimo grupo derivado en el teorema 2.3.2 dando así respuesta, con mucha generalidad, a una pregunta abierta sobre el grupo de Riordan propuesta por L. Shapiro en [103]. Para hacer esto, necesitaremos caracterizar antes el grupo derivado n-ésimo de $\mathcal{F}_1(\mathbb{K})$ (teorema 2.1.2, nótese que el primer grupo derivado $\mathcal{F}_1(\mathbb{K})$ es precisamente el anteriormente citado *substitution group of formal power series*, véase [2]). Hasta donde nosotros sabemos, la caracterización de los grupos derivados sólo era conocida para $\mathbb{K}$ cuerpo cíclico finito (véase [18]). Todos estos resultados los obtendremos utilizando la estrategia que mencionamos antes: estudiar primero lo que ocurre en los grupos parciales y *elevar* los resultados a las matrices infinitas.

Después de esto, se realiza un cierto estudio sobre conjugación en $\mathcal{R}(\mathbb{K})$ (que no está completo), para $\mathcal{F}_1(\mathbb{K})$ (véase [58, 59, 84] para bibliografía sobre conjugación en $\mathcal{F}_1(\mathbb{K})$) y en el primer subgrupo derivado de ambos, para $\mathbb{K}$ de característica 0. La estrategia seguida es trasladar el problema a un sistema de ecuaciones funcionales en las que intervienen las ecuaciones funcionales de Schröder y de Schröder con pesos y después, de nuevo, *elevar* el resultado de los grupos parciales al grupo infinito. Se estudia en primer lugar el problema de conjugación en los subgrupos derivados (véase proposition 2.4.1) y $(\mathcal{R}(\mathbb{K}))'$(donde se da también un conjunto de representantes canónicos de cada clase de conjugación, teorema 2.4.5). Despues se estudia completamente el problema de conjugación para $\mathcal{F}_1(\mathbb{K})$ (recuperándose teoremas conocidos: teorema 2.5.1 y teorema 2.5.2) y se estudian las clases de conjugación de elementos $R(d(x), h(x))$ tales que el multiplicador de $h(x)$ es distinto de raiz de la unidad o bien $R(d(x), h(x))$ es de orden finito (teorema 2.5.1).

El capítulo 3 es el último capítulo sobre cuestiones puramente algebraicas. Su centro de estudio son los elementos de orden finito, problema que había recibido cierta atención en la bibliografía del grupo de Riordan (véase [23–25, 69, 103]).

Comenzaremos por el estudio de los elementos de orden 2: las involuciones. Hay una literatura extensa que trata involuciones en grupos, mucha de la cual se apoya, si es posible, en una representación lineal de los grupos (como matrices finitas o incluso infinitas).

El interés en el estudio del grupo de Riordan fue iniciado por algunas preguntas propuestas por L. Shapiro en [103]. Algunos años más tarde, aparecieron algunos artículos de G. S Cheon y sus alumnos y en algunas ocasiones también co-autorizados por el propio L. Shapiro contestando algunas de esas preguntas (véase por ejemplo [22, 23, 25]). Como antes mencionamos, las matrices de Riordan están intrínsecamente relacionadas con sucesiones de polinomios. El artículo [69] es de interés en este punto, ya que relaciona sucesiones de Sheffer auto-inversas e involuciones en el grupo de Riordan.

En primer lugar se expresa el problema de las involuciones como un sistema de ecuaciones funcionales en serie formales de potencias, en el que aparece la ecuación funcional de Babbage. En el teorema 3.2.2 se da una nueva caracterización de las entradas de las involuciones en el grupo de Riordan, que puede utilizarse también para estudiar las involuciones en el grupo $\mathcal{F}_1(\mathbb{K})$ mediante la representación natural de $\mathcal{F}_1(\mathbb{K})$ como el subgrupo de Lagrange de $\mathcal{R}(\mathbb{K})$. De nuevo esto se ha conseguido probando el resultado primero para los grupos parciales y permite construir gran cantidad de involuciones de una manera sencilla. Además, se prueba con este resultado una conjetura propuesta por He en [43].

Después, se realiza un estudio similar para elementos de cualquier orden finito k y se consigue también una caracterización de las entradas de dichas matrices (teorema 3.4.3).

Por último se estudia un problema general que ha sido de interés para muchos grupos, entre ellos los grupos de matrices (véase el trabajo de W. H. Gustafson, P. R. Halmos y H. Radjavi [40], de P. R. Halmos y S. Kakutani [41], de D. Z. Djokovic [28], o el de M. J. Wonenburger [115]): determinar el grupo generado por las involuciones $\mathcal{I}(\mathbb{K})$ en $\mathcal{R}(\mathbb{K})$ y el número máximo de involuciones necesarias para expresar como producto cualquiera de los elementos de $\mathcal{I}(\mathbb{K})$ (teorema 3.6.5). Incluso, de nuevo, podemos recuperar un resultado reciente de A. O'Farrell sobre este mismo problema para $\mathcal{F}_1(\mathbb{K})$ (véase [88]).

El capítulo 4 pasa a abordar el estudio de la estructura de grupo de Lie infinito-dimensional de $\mathcal{R}(\mathbb{K})$ para $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

Merece epecial mención el artículo de R. Bacher [3] porque algunos de los resultados obtenidos en este capítulo fueron primero descritos en este artículo para $\mathbb{K} = \mathbb{C}$. Por ejemplo y entre otras cosas, apareció por vez primera la representación del álgebra de Lie como matrices infinitas, la descripción del corchete de Lie y la descomposición en el producto semidirecto Toeplitz-Lagrange del álgebra de Lie. Tiene también un capítulo muy interesante sobre el cálculo explícito de la exponencial. Implícitamente, Bacher parece utilizar la parametrización del grupo de Riordan dado por los coeficientes de las series formales de potencias $d(x), h(x)$ que describen cada elemento $R(d(x), h(x))$. Bacher no utiliza hasta bastante avanzado el trabajo la estructura diferencial. Pero, en su trabajo, la forma de justificar la existencia de esta estructura diferencial es incorrecta (cita textual del *abstract* de [3]): *This group has a faithful representation into infinite lower triangular matrices and carries thus a natural structure as a Lie group.* Esto no parece cierto en general: supongamos que G es el grupo entero de matrices invertibles triangulares inferiores infinitas. G recibe, como límite inverso de:

$$\{(G_n(\mathbb{K}), P_n\}$$

donde G_n son los subgrupos de GL_n de matrices triangualres inferiores y las aplicaciones P_n extienden a las que se describieron antes, una estructura de grupo topológico y de pro-grupo de Lie y no es un grupo de Lie (si $\mathbb{K} = \mathbb{R}$ ni si quiera es localmente conexo y si $\mathbb{K} = \mathbb{C}$ no es localmente contractible). Por supuesto, la estructura diferenciable propuesta por Bacher es correcta y es la misma que describiremos aquí (aunque la parametrización no será igual) así que este trabajo puede considerarse también que describe un marco teórico para [3].

Queremos hacer también notar que el grupo de Riordan contiene una copia isomorfa al antes mencionado *substitution group of formal power series*, que ya ha sido tratado desde el punto de vista de la teoría de grupos de Lie infinito dimensionales. La correspondiente álgebra de Lie ha sido identificada como $W_1(1)$: la parte nilpotente de la conocida álgebra de Witt $W(1)$ de campos de vectores formales en la recta real. En cualquier caso aquí se ha seguido una aproximación diferente.

Dicho lo anterior, nuestro trabajo trata de describir en primer lugar un contexto adecuado para dotar al grupo de Riordan de una estructura de grupo de Lie infinito dimensional. Existen dos marcos que resultan adecuados principalmente:

- La aproximación de J. Milnor en [82]. Consideramos $\mathbb{K}^{\mathbb{N}}$ con la topología producto y utilizando la identificación entre $\mathbb{K}^{\mathbb{N}}$ y $\mathbb{K}[[x]]$ pasando de sucesiones a funciones generatrices podemos inducir dicha topología. De esta forma se convierte a $\mathbb{K}[[x]]$ en un espacio de Frechet, esto es, un espacio vectorial topológico localmente convexo y completamente metrizable. Este es el punto de partida para describir una estructura natural en el grupo de Riordan de grupo de Lie sobre un espacio de Frechet.

- Como $\mathcal{R}(\mathbb{K})$ es el límite inverso de una sucesión inversa en la que aparecen los grupos parciales $\mathcal{R}_n(\mathbb{K})$ (que son grupos de matrices finito-dimensionales y por lo tanto grupos de Lie en el sentido clásico) recibe naturalmente una estructura de pro-grupo de Lie sobre $\mathbb{K}$, que es además en sí mismo, un pro-grupo de Lie. Véase [45, 46] para un desarrollo exhaustivo del estudio de la teoría de pro-grupos de Lie.

Hemos seguido preferiblemente la primera de las opciones, aunque relacionamos lo obtenido con la otra, ya que $\mathcal{R}(\mathbb{K})$ es un ejemplo ilustrador de esta segunda. De hecho uno de nuestros objetivos es precisamente mostrar que gracias a la estructura de $\mathcal{R}(\mathbb{K})$ como pro-grupo de Lie podemos *elevar* algunas propiedades de los grupos parciales (que son grupos de Lie clásicos), como por ejemplo la descripción del álgebra de Lie y el cálculo de la exponencial.

Después de describir este marco, estudiaremos la estructura de grupos de Lie que tienen naturalmente los grupos parciales $\mathcal{R}_n(\mathbb{K})$ como grupos de matrices (véase por ejemplo [5] para una introducción general a los grupos de Lie clásicos de matrices finito-dimensionales). En la proposición 4.3.1 describimos la estructura diferenciable de los mismos que son, en efecto, grupos de Lie. En la proposición 4.5.4 y su correspondiente corolario encontramos el álgebra de Lie en términos de matrices triangulares inferiores, describimos el corchete de Lie y la aplicación exponencial (que es la exponencial usual de matrices).

Una vez comprendida la estructura de Lie de los grupos parciales, estudiaremos la estructura de Lie sobre el espacio de Frechet $\mathbb{K}^{\mathbb{N}}$ del grupo $\mathcal{R}(\mathbb{K})$ (proposición 4.7.2) y la relacionamos con la estructura que hereda como pro-grupo de Lie (sección 4.8). Con el teorema 4.10.1 se describe una representación en términos de matrices triangulares inferiores infinitas del álgebra de Lie de $\mathcal{R}(\mathbb{K})$ y en el corolario 4.11.1 se describe la exponencial. También se indica brevemente en la sección sección 4.12 como se extiende naturalmente esta construcción para la representación con matrices bi-infinitas.

Sorprendentemente, las matrices infinitas que representan elementos del algebra de Lie de $\mathcal{R}(\mathbb{K})$, cumplen también un patrón parecido a las matrices de Riordan: para una tal matriz L, existen dos series $\alpha(x), \beta(x) \in \mathbb{K}[[x]]$ tales que la función generatriz de la columna i-ésima de L es $x^i \cdot (\alpha(x) + i \cdot \beta(x))$, esto es, las columnas están en progresión aritmetico-geométrica. Gracias a esto, también hay una interpretación en términos de series formales de potencias de la multiplicación de estas matrices por un vector columna infinito análoga al 1FTRM (proposición 4.13.2). Esta interpretación, permite entender las ecuaciones diferenciales de matrices (similares a los sistemas lineales clásicos de primer orden, véase proposición 4.11.2) como un cierto tipo de problemas de valor inicial que gracias a este marco sabremos resolver en series formales de potencias utilizando exponenciales de matrices (véase corolario 4.14.1), de las que mostramos varios ejemplos.

Por último, para cerrar el capítulo, dedicaremos nuestra atención a considerar una extensión del (esto es, un grupo que contiene al) grupo de Riordan, motivado por la construcción del fibrado tangente, veremos como se traslada al álgebra de Lie la descomposición del grupo de Riordan como producto semidirecto de los subgrupos de Toeplitz y Lagrange y apuntaremos brevemente algunas cuestiones sobre las algebras de Lie de subgrupos estabilizadores de $\mathcal{R}(\mathbb{K})$ (respecto de la acción de las matrices sobre vectores columna infinitos, o equivalentemente sobre series formales de potencias, dada en el 1FTRM, véase el teorema 4.18.4).

Por último, el capítulo 5 de este trabajo trata sobre aplicaciones de las técnicas desarrolladas en los anteriores a ciertos problemas de topología combinatoria, principalmente relacionados con el problema del f-vector. El f-vector de un complejo simplicial K de dimensión d (véase [39, 119]) es una sucesión:

$$(f_0, f_1, f_2, \ldots)$$

donde la entrada f_i indica el número de caras de dimensión i de K (y por lo tanto $f_k = 0$ para todo $k > d$). El problema del f-vector consiste en caracterizar las posibles sucesiones que pueden ser el f-vector de un complejo simplicial que cumpla una cierta condición topológica (ser una esfera, por ejemplo). Es un problema abierto en el que actualmente hay mucho trabajo activo (véase [13, 15, 27]). Se conocen pocos resultados fuertes en este sentido. Uno de ellos es el g-teorema (véase los trabajos de L. J. Billera y C. W. Lee [8, 9] y de R. P. Stanley [110]) y otro es la caracterización de los posibles f-vectores de un complejo K conocida la sucesión de números de Betti (también llamada simplemente la sucesión de Betti) del mismo y que aparece en el artículo de A. Björner y G. Kalai [14].

Existen otras sucesiones de enteros (como los g-, h- y γ-vectores) que aparecen en la bibliografía sobre el problema del f-vector y que contienen esencialmente la misma información expuesta de manera más conveniente para el problema con el que están relacionadas. Mostraremos que no son sino la imágen del f-vector mediante una matriz de Riordan (véanse los resultados de la sección 5.3). Esto además, motiva a preguntarse si algunos resultados clásicos (que tratan unimodalidad, positividad, log-concavidad, etc. de alguna de las sucesiones anteriormente citadas) podrían probarse exclusivamente en términos de matrices de Riordan.

Por otra parte, se muestra que las ecuaciones clásicas de Dehn-Sommerville (véase [39]) pueden formularse como un problema de autovectores de unas ciertas involuciones en los grupos finitos $\mathcal{R}_n(\mathbb{R})$ (proposición 5.4.1) y se recuperan y enfocan de diferente manera algunos resultados clásicos sobre sus soluciones (véase la sección dedicada a las soluciones de las ecuaciones de Dehn-Sommerville en el libro [39]).

Además, mostramos que los f-vectores de los símplices y los *cross-polytopes* si son colocados por filas en una matriz dan lugar a una matriz de Riordan. Como los símplices y las *cross-polytopes* no son sino los complejos simpliciales obtenidos partiendo de los complejos:

- $L_1 = \Delta_0^1$, que consiste en un solo punto
- $L_2 = \Delta_0^2$, que consiste en dos puntos aislados

y realizando iterativamente *joins* iterados:

$$L_1, L_1 * L_1, L_1 * L_1 * L_1, \dots \qquad y \qquad L_2, L_2 * L_2, L_2 * L_2 * L_2, \dots$$

respectivamente, estudiamos también otros complejos simpliciales obtenidos mediante *joins* iterados. Mostraremos como este proceso iterativo da lugar a un cierto patrón Riordan al colocar los f-vectores por filas, pero veremos que hay sólo un modo de conseguir que los f-vectores formen realmente una matriz de Riordan. A estas familias que resultan las hemos llamado m, q-conos (ver proposición 5.5.6). El hecho de que esta matriz de f-vectores sea de Riordan es para nosotros de gran interés porque permiten calcular muy rápidamente algunas relaciones en los f-vectores utilizando el 1FTRM (véase por ejemplo el ejemplo 5.5.8). También las sucesiones de Betti de estas familias de complejos simpliciales son matrices diagonales de Riordan (ver proposición 5.8.2).

También mostraremos algunos ejemplos de como la acción de los llamados métodos de subdivision (como por ejemplo la subdivision baricéntrica) sobre el f-vector pueden describirse como una multiplicaión de una matriz infinita por un f-vector (véase la proposición 5.6.2). Esto permite por ejemplo (sección 5.7) recuperar algún resultado ya conocido y mostrar alguno nuevo sobre la imposibilidad de existencia de relaciones lineales en el f-vector de familias de complejos simpliciales cumpliendo una cierta propiedad topológica (véase la proposición 5.7.5).

Por último en la sección 5.9 estudiaremos “supercomplejos”: complejos simpliciales en los que cada símplice se sustituye a su vez por otro complejo simplicial perteneciente a una “familia de ladrillos". No cualquier familia de complejos simpliciales puede ser una "familia de ladrillos", pero entre las que sí pueden, se encuentran los q, q-conos. Por lo tanto, dado cualquier complejo simplicial K, podemos estudiar el complejo obtenido al sustituir los símplices por q, q-conos, que es a su vez otro complejo simplicial llamado el q-engrosamiento de K. El f-vector de este q- engrosamiento es el f-vector del complejo original multiplicado por una matriz de Riordan.

Incluimos al final de la tesis un último apartado con preguntas abiertas que serán las que intentaremos contestar en un futuro próximo.

El trabajo realizado en esta tesis ha dado lugar a la publicación de los siguientes artículos:

- [63] junto con los directores de este trabajo, D. Merlini y R. Sprugnoli.
- [70] junto con los directores de este trabajo.

a los siguientes *preprints* enviados ya para su evaluación y posible publicación:

- *Finite and infinite dimensional Lie group structures on Riordan groups* con los directores de este trabajo, G. S. Cheon y M. Song.
- *A formula to construct all involutions in Riordan matrix groups* con los directores de este trabajo.

a los siguientes *preprints* aún no enviados:

- *The derived series of the Riordan group* con los directores de este trabajo.
- *The group generated by the involutions in the Riordan group* con los directores de este trabajo.

y por último al trabajo en progreso:

- *The Riordan group and the f-vector problem* con los directores de este trabajo.

English

The main object of study in this work is the Riordan group, that will be denoted by $\mathcal{R}(\mathbb{K})$. The Riordan group was named in honor of J. Riordan, 1903-1988 and appeared for the first time (with this name) in the work by L. Shapiro, S. Getu, W. J. Woan and L. C. Woodson in [105] (although the definition has changed slightly: the group defined in this article is in fact a subgroup of the group considered nowadays) and was succesufully used after this as a tool for problems of combinatorial flavour, as the proof of combinatorial identities (for example in the articles by R. Sprugnoli [107] and [108]). The advisors of this work also were part of this development, when they met the Riordan group coming from a very different approach (see [66]).

The elements in the Riordan group are usually described as invertible infinite lower triangular matrices, that is, matrices of the form:

$$(a_{ij})_{0\leq i,j<\infty} = \begin{bmatrix} a_{00} & & & \\ a_{10} & a_{11} & & \\ a_{20} & a_{21} & a_{22} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(initially with entries in $\mathbb{C}$, although the can be considered with entries in any field $\mathbb{K}$) associated to a couple of formal power series $d(x), h(x) \in \mathbb{K}[[x]]$ where:

$$d(x) = d_0 + d_1x + d_2x^2 + \dots \qquad h(x) = h_1x + h_2x^2 + h_3x^3 + \dots \qquad con\ f_0, g_1 \neq 0$$

with the property of the generating function of the entries in the i-th column being $d(x)\cdot(h(x))^i$, that is:

$$d(x) \cdot (h(x))^i = a_{ii}x^i + a_{i+1,i}x^{i+1} + a_{i+2,i}x^{i+2} + \dots$$

In this case, this matrix is denoted by $R(d(x), h(x))$. Thanks to this structure, there is an interpretation in terms of formal power series of the multiplication of these matrices by a column vector (the so called First Fundamental Theorem of Riordan matrices, or simply 1FTRM) and of the multiplication of two of those matrices (see subsection 0.3.2 for more details).

In view of what we have explained above, Riordan group could be clasified in the branch of Combinatorics. But on the other hand, as any mathematical object of interest, the Riordan group has aspects that could be of the interest of experts in other branches of Mathematics. Looking at different aspects of the Riordan group and from different points of view, we improve the understanding of this object and so the possibilities of application to several kinds of problems. Becouse of that, in this thesis we will study three type of aspects:

1. **The group structure of** $\mathcal{R}(\mathbb{K})$, that will be fulfilled in chapters 1, 2 and 3. Most of this work is devoted to this part. It contains the basics for the rest of this work concerning the inverse limit structure of the Riordan group. The corresponding inverse sequence involves finite dimensional matrix groups that will be called partial Riordan groups. Moreover, here the derived series of the Riordan group will be studied (including some applications), conjugacy classes from different viewpoints and the elements of finite order, specially involutions. Also the group generated by involutions will be studied. Also the relation between algebraic problems in the Riordan group and equations and functional equations in formal power series will be considered.

2. **The infinite-dimensional Lie group structure of** $\mathcal{R}(\mathbb{K})$ **for** $\mathbb{K} = \mathbb{R}, \mathbb{C}$, which is done in chapter 4. The Riordan group is an example of Lie group over a Frechet spaces which is also a pro-Lie group (inverse limit of an inverse sequence involving classical Lie groups). Global parametrizations of the partial groups and of the infinite Riordan group are studied.

3. **Applications to some problems in Topological Combinatorics of Simplicial Complexes**, that will be done in chapter 5. The tools developed in the rest of this work will be applied to the f-vector problem. We also want to show the presence of Riordan type patterns in this field.

A common process of working in the results obtained in this work is considering the Riordan group $\mathcal{R}(\mathbb{K})$ as the inverse limit of the inverse sequence:

$$\{(\mathcal{R}_n(\mathbb{K}), P_n)\}_{n=0}^{\infty}$$

where the groups $\mathcal{R}_n(\mathbb{K})$ are lower triangular matrix groups of size $(n+1) \times (n+1)$ that we will call partial groups This allow us to study what happens in the partial groups and the possibility of *liftingón* properties to the infinite matrix group (this inverse limit construction will be presented in detail in section 1.4).

We should also take into account that the interpretation of the product of Riordan matrices in terms of formal power series stablishes a way to translate algebraic problems in the group (conjugacy, derived series, characterization of elements of finite order, etc.) into systems of functional equations (in formal power series) and viceversa.

There exist several ways to construct Riordan matrices. For example we have the horizontal construction (involving the A-sequence) and the vertical construction (involving the g-sequence, being one of the reasons of the notation $T(f \mid g)$ introduced by the advisors of this work in [66] and used in most of the rest of their work, like for example [62, 64, 67]) and others, as those introduced in [21, 64]. There also exist some construction to find new Riordan matrices from older ones (see [16, 65, 116, 118]).

There is also a strong relation between Riordan matrices and classical polynomial sequences (see [26, 44, 68, 71, 114]).

It is also important that the Riordan group contains a subgroup isomorphic to $\mathcal{F}_1(\mathbb{K})$, the group of formal power series of the type:

$$g(x) = g_1 x + g_2 x^2 + g_3 x^3 + \dots \qquad \qquad con\ g_1 \neq 0$$

with respect to the composition that has already been widely studied. Its subgroup with $g_1 = 1$ has received even more attention. It is usually referred as the substitution group of formal power series (see for example [2] for more information) which is known as the Nottingham group when $\mathbb{K}$ is a finite cyclic field (see [18]). Through this work, sometimes some or the result particularize into results in $\mathcal{F}_1(\mathbb{K})$.

To sum up, we could say that the literature of the Riordan group was generated mainly in the last decade of XX century (see [76, 77, 80, 105, 107, 108]) and that is currently growing (see [22–25, 29, 48, 62, 64, 66, 78, 79, 104, 117]).

We have included in this work an initial chapter to review some of the contcepts involved in the rest of the chapters and to provide some aditional bibliography.

In chapter 1, the first point considered (theorem 1.4.2) will be the structure of the Riordan group as an inverse limite of the inverse sequence introduced above:

$$\{(\mathcal{R}_n(\mathbb{K}), P_n)\}_{n=0}^{\infty}$$

where $P_n : \mathcal{R}_{n+1}(\mathbb{K}) \to \mathcal{R}_n(\mathbb{K})$ is the map that deletes last row and column of the matrix. The concept of inverse limit is used in almost all branches of Mathematics, sometimes referred as projective limit. It is a method for approaching certain objetcs by others that are well behaved or better understood. Inverse systems, from which inverse limits are derived, may be defined, in general, in any category. An introductory text to cover those topics is [72]. As we have already said, for us this point of view will be very useful to do, among other things, proofs by induction. We will study then how to extend a matrix $M_n \in \mathcal{R}_n(\mathbb{K})$ into a matrix $M_{n+1} \in \mathcal{R}_{n+1}(\mathbb{K})$ in such a way that $P_n(M_{n+1}) = M_n$ (proposition 1.3.1). We will relate those partial groups to the ultrametric proposed in [66] and to a new one.

We will also show another representation of the Riordan group: as a group of bi-infinite matrices, that is, matrices of the form:

$$(a_{ij})_{-\infty<i,j<\infty} = \begin{bmatrix} \ddots & & & & & & \\ \cdots & a_{-2,-2} & & & & & \\ \cdots & a_{-2,-1} & a_{-1,-1} & & & & \\ \cdots & a_{0,-2} & a_{0,-1} & a_{00} & & & \\ \cdots & a_{1,-2} & a_{1,-1} & a_{10} & a_{00} & & \\ \cdots & a_{2,-2} & a_{2,-1} & a_{2,0} & a_{21} & a_{22} & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

that can also be studied as the inverse limit of different sequences (involving finite or infinite matrices) using the horizontal and vertical construction cited above. Due to the previous works [62, 64, 65, 109], it seemed to be necessary a better understanding of the different symetries in bi-infinite Riordan matrices. Thanks to the structure of inverse limit, we will be able to describe easily those symmetries *reflecting* the finite matrices involved in the inverse limit and *lifting* this reflection into the bi-infinite group (giving rise to complementary and dual matrices). In this direction we will also study and give an answer to some questions proposed in [65] about matrices being self-dual and self-complementary (theorems 1.12.3 and 1.12.4). The previous concepts have already been independently used in [21] to construct Riordan matrices. As a final application, from the inverse limit structure of the inverse representation of the Riordan group we will be able to recover, using induction in terms of the inverse limits, some classical results about the Schröder and weighted Schröder functional equations for K of characteristic 0 (section 1.14) in formal power series (see theorem 1.14.6 and theorem 1.14.7, see also the bibliography [2, 59, 99, 100, 112]), and in the future, we think that this will allow the treatment of other functional equations in formal power series. Schröder and weighted Schróder equations can be stated as an eigenvector problem for Riordan matrices. This fact will be of importance for future chapters of this work.

In chapter 2, again using the inverse limit structure that helps us to do proof by induction, we will study some group properties of $\mathcal{R}(\mathbb{K})$, with $\mathbb{K}$ of characteristic 0.

Firstly we will focus on the derived series of the Riordan group (see [91] for general information) giving a characterization of the n-th derivative subgroup in theorem 2.3.2, giving then an answer (with much more generality) to an open question posted by L. Shapiro in [103]. To do this, we will need to characterize before the n-th derivative groups of $\mathcal{F}_1(\mathbb{K})$ (theorem 2.1.2. Note that the first derivative subgroup of $\mathcal{F}_1(\mathbb{K})$ is precisely what it is usually called in the bibliophary the *substitution group of formal power series* (see [2]). As far as we know, a characterization of the derived series of $\mathcal{F}_1(\mathbb{K})$ was only known for finite cyclic fields (see [18]).

All those results will be obtained following the strategy suggested before: studying first the partial groups and then *lifting* the results to the infinite matrices.

After this, a certain (not complete) study of conjugacy is made for $\mathcal{R}(\mathbb{K})$, for $\mathcal{F}_1(\mathbb{K})$ (see [58,59,84] for some bibliography about conjugation in $\mathcal{F}_1(\mathbb{K})$) and in the first derived subgroup of both, for $\mathbb{K}$ of characteristic 0. The strategy followed is translating the problem into a system of functional equations where Schröder and weighted Schröder equations appear and then, again, *lifting* the result from the partial groups to the infinite group. Firstly the problem of conjugation is studied for the first derivated subgroups $(\mathcal{F}_1(\mathbb{K}))'$ (see proposition 2.4.1) and $(\mathcal{R}(\mathbb{K}))'$ (where also a set of canonical representatives is given for each conjugacy class, theorem 2.4.5). After this, the problem of conjugacy is completely studied for $\mathcal{F}_1(\mathbb{K})$ (recovering known theorems: theorem 2.5.1 and theorem 2.5.2). And finally conjugacy classes in $\mathcal{R}(\mathbb{K})$ is considered for elements $R(d(x), h(x))$ so that the multiplier of $h(x)$ is different from a root of unity or $R(d(x), h(x))$ is of finite order (theorem 2.5.1).

Chapter 3 is the last chapter devoted to purely algebraic aspects. The object of study is this time the elements of finite order, a problem that has received some attention in the bibliography of the Riordan group (see [23–25, 69, 103]).

This study starts with the study of elements of finite order 2: the involutions. There is an extensive literature treating the study of involutions in groups, some of it relays in the linear representation, if possible, of those groups (with finite or even infinite matrices).

The interest in the study of Riordan involutions started due to some open questions proposed by L. Shapiro in [103]. Some years later, some articles appeared by G. S Cheon and his students and sometimes co-authoriced by L. Shapiro answering some of these questions (see for example [22, 23, 25]). As we have already mentioned, Riordan matrices are intrinsically related to polynomial sequences. The article [69] is of interest in this sense, since it relates self-inverse Sheffer sequences and involutions in the Riordan group.

At first, the problem of finding involutions is expressed as a system of funcional equations in formal power series in which Babbage's functional equation appears. In theorem 3.2.2 a new characterization of the entries of involutions in the Riordan group is given, that can be applied to find a characterization of the coefficients of involutions in $\mathcal{F}_1(\mathbb{K})$ using the natural identification of $\mathcal{F}_1(\mathbb{K})$ with the Lagrange subgroup. Again, this has been obtained proving the result at first for the partial groups, and allow us to construct several involutions in a simple way. Moreover, a conjecture proposed by He in [43] is proved with this result.

After this, a similar study for elements of any finite order k is made, obtaining also a characterization of the entries of those matrices (theorem 3.4.3).

Finally, a general problem that has been of interest for many groups (between them also matrix groups, see for example the work by W. H. Gustafson, P. R. Halmos and H. Radjavi [40], of P. R. Halmos and S. Kakutani [41], of D. Z. Djokovic [28], or of M. J. Wonenburger [115]) is studied: determining the group generated by the involutions in $\mathcal{R}(\mathbb{K})$, denoted by $\mathcal{I}(\mathbb{K})$, and the maximal number of involutions required to express any element in this group as a product of involutions (theorem 3.6.5). Again, particularizing our work we obtain a recent result by A. O'Farrell about this same problem but for $\mathcal{F}_1(\mathbb{K})$ (see [88]).

In chapter 4 we start the study of the infinite-dimensional Lie group structure of $\mathcal{R}(\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

It deserves special attention the article by R. Bacher [3], becouse some of the obtained results in this chapter, were first described there. For example and among other things, there appeared the representation of the Lie algebra in terms of infinite matrices, the description of the Lie bracket and the decomposition associated to the Toeplitz-Lagrange subgroup decomposition in the Lie algebra. It also has a very interesting chapter about the explicit computation of the exponential. Implicitely Bacher seems to use the parametrization of the Riordan group given by the coefficient of the formal power series $d(x), h(x)$ that describe each element $R(d(x), h(x))$. Bacher does not use until the work is already more than started the differential structure. But in his work the way of justifying this differential structure is incorrect (taken exactly from the english version of the abstract of [3]): *This group has a faithful representation into infinite lower triangular matrices and carries thus a natural structure as a Lie group.* This does not seem to be true in general: suppose that G is the group of invertible infinite lower triangular matrices. G is the inverse limit of the inverse sequence

$$\{(G_n(\mathbb{K}), P_n\}$$

where G_n are the subgroups of lower triangular matrices in GL_n and the bonding maps P_n act as described before. Thus G receives a topological group structure and a pro-Lie group structure and G is not a Lie group itself (if $\mathbb{K} = \mathbb{R}$ it is not even locally connected and if $\mathbb{K} = \mathbb{C}$ it is not locally contractible). Of course the differential structure proposed by Bacher is correct, and is the same used here (the parametrization is not) so this chapter may be considered as a theoretical framework for [3].

Note also that, as explained before, the Riordan group contains a subgroup isomorphic to the *substitution group of formal power series*, that has already been studied from the point of view of the theory of infinite-dimensional Lie group. Its Lie algebra has already been identified as $W_1(1)$: the nilpotent part of the well known Witt algebra $W(1)$ of vector fields in the real line. Anyway our approach is quite different from this.

In view of this, our work tries firstly to describe an adequated framework to endow the Riordan group with an infinite dimensional Lie group structure. There are two suitable frameworks to do this:

- The approach by J. Milnor in [82]. If we consider $\mathbb{K}^{\mathbb{N}}$ with the product topology and then we use the natural identification between $\mathbb{K}^{\mathbb{N}}$ and $\mathbb{K}[[x]]$ changing from sequences to generating functions, we can induce a topology in $\mathbb{K}[[x]]$. This topology makes $\mathbb{K}[[x]]$ a Frechet space, that is, a topological vector space locally convex and totally metrizable. This is the starting point to describe a natural structure of Frechet Lie group in the Riordan group.

- Since $\mathcal{R}(\mathbb{K})$ is the inverse limit of the inverse sequence of the partial groups $\mathcal{R}_n(\mathbb{K})$ (which are finite dimensional matrix groups and so Lie groups in the classical sense) it receives a pro-Lie groups structure over $\mathbb{K}$ and is itself a Lie group.See [45, 46] for an exhaustive development of pro-Lie groups theory.

We have followed mainly the first of the two options although we have related it with the other, since $\mathcal{R}(\mathbb{K})$ is an illustrative example of this second one. In fact, one of the aims of this work is precisely showing that thanks to the pro-Lie group structure we can *lift* some properties from the partial groups (which are classical Lie groups), since, for example, the description of the Lie algebra and the computation of the exponential map.

After describing this thoretical framework, we will study the natural Lie group structure of the partial groups $\mathcal{R}_n(\mathbb{K})$ as matrix groups (see [5] for a general introduction to classical finite-dimensional matrix Lie groups). In proposition 4.3.1 we describe the differential structure of them and then we show that they are actually Lie groups. In proposition 4.5.4 and its corollary we find the Lie algebra in terms of infinite lower triangular matrices, we describe the Lie bracket and finally the exponential map (which is the usual exponential of matrices).

Once the Lie group structure of the partial groups is understood, we will study the Lie group structure over the Frechet space $\mathbb{K}^{\mathbb{N}}$ of the group $\mathcal{R}(\mathbb{K})$ (proposition 4.7.2) and we relate it with the structure that it inherits as a pro-Lie group (section 4.8). With theorem 4.10.1 the representation in terms of lower triangular matrices of the Lie algebra of $\mathcal{R}(\mathbb{K})$ is obtained, and in corollary 4.11.1 a description of the exponential is included too. It is also briefly explained in section 4.12 how we can naturally extend this construction for the representation of $\mathcal{R}(\mathbb{K})$ in terms of bi-infinite matrices.

Surprisingly, the infinite matrices that represent elements in the Lie algebra of $\mathcal{R}(\mathbb{K})$, also satisfy a pattern similar to this in Riordan matrices: for any of those matrices L, there are two formal power series $\alpha(x), \beta(x) \in \mathbb{K}[[x]]$ so that the generating function of the i-th column in L is $x^i \cdot (\alpha(x) + i \cdot \beta(x))$, that is, the columnsin L are in arithmetic-geometric progression. Thanks to this, there is also an interpretation in terms of formal power series of the multiplication of those matrices by an infinite column vector analogous to 1FTRM (proposition 4.13.2). This interpretation, allow us to understand matrix differential equations (similar to those classi first order linear systems, see proposition 4.11.2) as a certain type of initial value problems that, in this framework, we can solve in formal power series using matrix exponentials (see corollary 4.14.1). We will provide some examples of this.

Finally to close this chapter we will focus our attention in considering an extension of (that is, a group that contains) the Riordan group, motivated by the tangent bundle. We will see how the decomposition of the Riordan group as a semidirect product of the Toeplitz and Lagrange subgroups is traslated to the Lie algebra and we will discuss briefly some questions about the Lie algebras of stabilizer subgroups in $\mathcal{R}(\mathbb{K})$ (with respect to the action of the matrices over infinite column vectors described in 1FTRM, see theorem 4.18.4).

Finally chapter 5 in this work deals with applications of the tools developed in previous chapters to a certain type of problems in topological combinatorics, principally related to the f-vector problem. The f-vector of a simplicial complex K of dimension d (see [39, 119]) is a sequence:

$$(f_0, f_1, f_2, \dots)$$

where the entry f_i represents the number of faces of dimension i in K (and all the entries f_k equal 0 for $k > d$). The f-vector problem consist in characterizing the possible sequences that can be the f-vector of a simplicial complex satisfying a given topolocial condition (being an sphere, for example) Is an open problem in which there is currently a lot of active work (see [13, 15, 27]). There are few strong results in this sense. One of them is the g-theorem (see the work by L. J. Billera and C. W. Lee [8, 9] and by R. P. Stanley [110]) and the other one is the characterization of the conditions on the f-vector provided the sequence of Betti numbers (Betti sequence for short) of the complex (see the article by A. Björner and G. Kalai [14]).

There exist other classical sequences of integers (like the g-, h- and γ-vectors), that appear in the bibliography of the f-vector problem and that contain essentially the same information than the f-vector in a more convenient way for the problem they are treating. We will show that those sequences are no other things than the image of the f-vector through a Riordan matrix (see the results in section 5.3). This also leads to wondering if some of the classical results (dealing with unimodality, positivity, log-concavity, etc.) of any of the previous sequences cannot be proved in terms of Riodan matrices.

On the other hand, we will show that the classical Dehn-Sommerville equations (see [39]) can be formulated as an eigenvector problem for some involutions in the partial groups $\mathcal{R}_n(\mathbb{R})$ (propositión 5.4.1) and then we recover and study from a different viewpoint some classical results about their solutions (see the section devoted to solutions of the Dehn-Sommerville equations in [39]).

Moreover, we will show that the f-vectors of the simplices and of the cross-polytopes, if placed as rows of a matrix, give rise to a Riordan matrix. Simplices and cross polytopes are no other thing but simplicial complex obtained starting from the complexes:

- $L_1 = \Delta_0^1$ consisting in a simple point
- $L_2 = \Delta_0^2$ consisting in two single points

and doing iteratively the joins:

$$L_1, L_1 * L_1, L_1 * L_1 * L_1, \dots \qquad y \qquad L_2, L_2 * L_2, L_2 * L_2 * L_2, \dots$$

respectively. So we will also study other simplicial complexes obtained iterating joins. We will show how this kind of processes lead to a Riordan pattern when we place the f-vectors as rows in a matrix. But there is essentially only one way to do this process obtaining actually a Riordan matrix. The familiy of simplicial complexes obtained doing this will be called m, q-cones (see proposition 5.5.6). The fact of this matrix of f-vectors being Riordan is for us of great interest, since it allows us to compute quickly relations in the f-vectors using the 1FTRM (see example 5.5.8). Also the Betti sequences of these families of simplicial complexes are diagonal Riordan matrices (see proposition 5.8.2).

We will also show some examples of how the action of the so called subdivision methods (like for example barycentric subdivision) over the f-vector may be described as the multiplication of an infinite matrix by an f-vector (see propostion 5.6.2). This allow us in section 5.7, for example, to recover a known result and to prove a new one on the non-existence of linear relations in the f-vector of families of simplicial complexes satisfying a certain topological property (see proposition 5.7.5).

Finally in section 5.9 we will study "supercomplexes", that is, simplicial complexes in which each simplex is substituted by another simplicial complex belonging to a "family of building-blocks". We will see that not any family of simplicial complex can be a family of building blocks, but between those that can, we can find q, q-cones. So, given any simplicial complex K, we can study the simplicial complex obtained replacing the simplices by q, q-cones obtaining another simplicial complex called the q-widening of K. The f-vector of this q-widening of K is the f-vector of the original simplicial complex multiplied by a Riordan matrix.

We have decided to include a last chapter with open questions that we will try to answer in the future.

This work has given rise to the publication of the following articles:

- [63] with the advisors of this work, D. Merlini and R. Sprugnoli.
- [70] with the advisors of this work.

to the following pre-print, already submitted for review:

- *Finite and infinite dimensional Lie group structure on Riordan groups* with the advisors of this work, G. S. Cheon and M. Song.
- *A formula to construct all involutions in Riordan matrix groups* with the advisors of this work.

the following preprints which have not been sent yet:

- *The derived series of the Riordan group* with the advisors of this work.
- *The group generated by the involutions in the Riordan group* with the advisors of this work.

and finally to the following work in progress:

- *The Riordan group and the f-vector problem* with the advisors of this work.

Chapter 0

Basics

In this chapter we will introduce briefly some of the basic concepts involved in this work, to facilitate its reading and to fix some of the notation.

We will also provide some bibliography for further reading about the topics covered.

- In section 0.1 we will introduce the concept of inverse limit of a sequence, that will play an important role throughout all this work.
- In section 0.2 we will review some basic concepts about formal power series, which are closely related to Riordan matrices as we will show later. We will also introduce two groups of power series: $\mathcal{F}_0(\mathbb{K})$ and $\mathcal{F}_1(\mathbb{K})$ of great interest for the study of Riordan matrices.
- In section 0.3 we will provide a brief introduction to Riordan matrices in their classical (infinite) representation. We will remark most of the aspects concerning the notation used in this work for Riordan matrices.
- In section 0.4 we will include more recommended bibliography covering other topics involved in this work: Lie groups (classical and over Frechet spaces) and simplicial complexes.

0.1 Inverse Limits

The concept of inverse limit has been and is still being widely used in practically all branches of mathematics, sometimes under the name of projective limit. Usually it is a way to approximate objects by better behaved or widely known ones.

The concept of inverse system, from which the inverse limit is derived, can be defined in any category, where the related concept of pro-category appears.

An introductory text for Category Theory is [72]. On the other hand, in [30] the authors developed, from the categorical point of view, the Algebraic Topology, and there it can be found a study of inverse limit in the categories of both, groups with homomorphisms and topological spaces with continuous maps.

Concerning this work, we will be mainly interested in a simple case of inverse limits: the inverse limit, not of an arbitrary inverse system, but of an inverse sequence and in the categories of groups and Lie groups. For this first glimpse, we will restrict ourselves to the category of groups:

Definition 0.1.1 (Inverse Limit (I)) *Given a sequence of groups* $\{G_0, G_1, G_2, \ldots\}$ *and a sequence of homomorphisms* $(\Psi_0, \Psi_1, \Psi_2, \ldots)$ *called* ***bonding maps*** *where for each* $n \geq 0$*:*

$$\Psi_n : G_{n+1} \longrightarrow G_n$$

we say that the sequence of pairs $\{(G_n, p_n)\}_{0 \leq n}$ *is an* ***inverse sequence****. The* ***inverse limit*** *of this sequence is the group made from the set:*

$$\lim_{\leftarrow}\{G_n, \Psi_n\} = \left\{ (g_0, g_1, g_2, \ldots) \in \prod_{n=0}^{\infty} G_n : \Psi_n(g_{n+1}) = g_n \right\}$$

with the componentwise operation:

$$(g_0, g_1, g_2, \ldots) * (h_0, h_1, h_2, \ldots) = (g_0 *_0 h_0, g_1 *_1 *h_1, g_2 *_2 h_2, \ldots)$$

where $*_i$ *is the operation in* G_i*. There is a natural map called* ***projection****:*

$$\Pi_n : \lim_{\leftarrow}\{G_k, \Psi_k\} \longrightarrow G_n$$

that maps $(g_0, g_1, g_2, \ldots)$ *to* g_n*.*

Another equivalent way to introduce inverse limits is to present them as universal objects:

Definition 0.1.2 (Inverse Limit (II)) *A group* X *together with a collection of homomorphism* $\pi_i : X \longrightarrow G_i$ *is the* ***inverse limit*** *of an inverse sequence of groups* $\{(G_n, p_n)\}$ *if:*

- *For any* $i \geq 0$*, the following diagram commute:*

$$\begin{tikzcd}
& X \arrow[dl, "\pi_{n+1}"'] \arrow[dr, "\pi_n"] & \\
G_{n+1} \arrow[rr, "\Psi_n"] & & G_n
\end{tikzcd}$$

- *For any other group* Y *together with a collection of homomorphisms* $\widetilde{\pi}_i$ *satisfying the property above, there exists a unique homomorphism* $\widetilde{\pi}$ *that makes the following diagram commutative:*

$$\begin{tikzcd}
& Y \arrow[d, "\widetilde{\pi}"] \arrow[ddl, "\widetilde{\pi}_{n+1}"'] \arrow[ddr, "\widetilde{\pi}_{n+1}"] & \\
& X \arrow[dl, "\pi_{n+1}"'] \arrow[dr, "\pi_n"] & \\
G_{n+1} \arrow[rr, "\Psi_n"] & & G_n
\end{tikzcd}$$

As it was said before, in the category of groups, both definitions coincide (we will not check this here). But we have to keep the following in mind:

Remark 0.1.3 *Unlike what happens in the category of groups, not for any sequence in any category there exists an inverse limit for any inverse sequence. This is the case of the category of Lie groups, which is also of interest for this work and where we will have to make some effort to ensure and understand the coherence of the definition of inverse limits of certain sequences.*

Therefore, this second definition is more convenient for categories where the product is not so well behaved. For instance replacing "group" by "Lie group" and "homomorhism" by "Lie group homomorphism" in this second definition we can define also the inverse limits of an inverse sequence of Lie groups.

But obviously from the universal object definition, if the inverse limit exists in a given category, it is unique up to isomorphism. Moreover, the isomoprhism is also unique.

0.2 Formal Power Series

In this chapter we will provide some basic definitions and notation for power series that will be used throughout the rest of this work. For this section, let $\mathbb{K}$ denote a field of any characteristic, unless otherwise specified.

A basic reference covering most ot the topics in this section, could be the paper by I. Niven [87]. For an introduction to formal power series from the point of view of combinatorics (were power series are usually generating functions of sequences) a good reference could be chapter 2 of the book by J. Riordan [95]. With respect to the substitution group of formal power series (a subgroup of the group that we have denoted in this work as $\mathcal{F}_1(\mathbb{K})$) and even covering certain aspects of the multiplication group of formal power series (denoted in this work by $\mathcal{F}_0(\mathbb{K})$) we recommend the reader the wide survey by I. K. Babenko [2]. Concerning aspects related to convergent power series (Taylor series of analytic functions are formal power series) any book of complex analysis can provide many information, such as the one by M. Rao and H. Stetkaer [93].

0.2.1 Basic Definitions

Definition 0.2.1 ***Formal power series*** *in the variable x over a field $\mathbb{K}$ are formal algebraic objects of the type:*

$$f(x) = f_0 + f_1 x + f_2 x^2 + \dots \quad (1)$$

The set of all formal power series over the field $\mathbb{K}$ will be denoted by $\mathbb{K}[[x]]$.

We will use the notation $[x^k]f(x)$ to refer to the k-th coefficient of the power series, that is, f_k in the power series described in (1).

Polynomials are also formal power series (which coefficients are 0 from certain k on). The set of polynomials in the variable x over the field $\mathbb{K}$ will be denoted by $\mathbb{K}[x]$. The set of polynomials of degree less or equal than n in the variable x over the field $\mathbb{K}$ will be denoted by $\mathbb{K}_n[x]$.

Although the notation suggests it the other way round, in general the **evaluation** of a formal power series in a point $0 \neq p \in \mathbb{K}$, that is the infinite sum:

$$f_0 + f_1 \cdot p^1 + f_2 \cdot p^2 + \dots$$

doesn't make any sense (the evaluation of $f(x)$ in 0 is f_0). It might do in some important cases like $\mathbb{K} = \mathbb{R}, \mathbb{C}$ where the field has a norm. In those cases, for a given $p \in \mathbb{K}$ we can define the evaluation to be:

$$f(p) = \lim_{n \to \infty} \sum_{k=0}^{n} f_k (p)^k$$

when this limit exists. In this case, we will say that $f(x)$ is a **convergent** power series in p. The set of formal power series which are convergent in any neigbourhood of 0 will be denoted by $\mathbb{K}_{hol}[[x]]$.

Formal power series are very useful in combinatorics due to the following bridge between sequences and formal power series:

Definition 0.2.2 *For a given sequence $(d_0, d_1, d_2, \dots)$ of elements in a field (or ring) $\mathbb{K}$, we say that:*

$$d(x) = d_0 + d_1 x + d_2 x^2 + \dots \in \mathbb{K}[[x]]$$

*is the **generating function**. By convention, if we have a finite sequence, that is $(d_0, \dots, d_n) \in \mathbb{K}^{n+1}$, we will say that its generating function is the polynomial:*

$$d(x) = d_0 + \dots + d_n x^n$$

Conversely, from a given formal power series or a polynomial, we can recover the correspondent **sequence of coefficients**.

Analogously to what is done in analysis, we define:

Definition 0.2.3 *The **Taylor Polynomial** of degree n of a formal power series:*

$$f(x) = f_0 + f_1 x + f_2 x^2 + \dots \in \mathbb{K}[[x]]$$

is the polynomial:

$$Taylor_n(f(x)) = f_0 + f_1 x + \dots + f_n x^n \in \mathbb{K}[x]$$

There is a natural way to define the addition and product elements in $\mathbb{K}[[x]]$. With respect to those operations, $\mathbb{K}[[x]]$ is a ring. Also there is a natural multiplication of the elements in $\mathbb{K}[[x]]$ by scalars. With respect to the addition and product by scalars, $\mathbb{K}[[x]]$ is a vector space. Also a formal derivative and a formal integral can be defined for formal power series.

In some sense, a concept that generalizes the one of formal power series is the following:

Definition 0.2.4 *The set of formal Laurent series over the field $\mathbb{K}$ denoted by $\mathbb{K}((x))$ is the set of formal objects of the type:*

$$\sum_{k=n}^{\infty} f_k x^k \text{ with } n \in \mathbb{Z}$$

Remark 0.2.5 *Another way to understand formal Laurent series is supposing that* $\mathbb{K}((x))$ *is the set of quotients of power series:*

$$\mathbb{K}((x)) = \left\{ \frac{f(x)}{g(x)} : f(x), g(x) \in \mathbb{K}[[x]] \text{ with } g(x) \neq 0 \right\}$$

since if $g(x) \in \mathbb{K}((x)) \setminus \mathbb{K}[[x]]$ *there always exists a formal power series* $\widetilde{g}(x) \in \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]$ *such that:*

$$\frac{f(x)}{g(x)} = \frac{1}{x^n} \frac{f(x)}{\widetilde{g}(x)}$$

Then we can define the sum and product of formal Laurent series out of the sum and product of power series in the natural way:

- $\frac{f(x)}{g(x)} + \frac{u(x)}{v(x)} = \frac{f(x)v(x)+g(x)u(x)}{g(x)v(x)}$
- $\frac{f(x)}{g(x)} \cdot \frac{u(x)}{v(x)} = \frac{f(x)\cdot u(x)}{g(x)\cdot v(x)}$

It is easy to check that, with respect to these operations, $\mathbb{K}((x))$ *is a field.*

Also formal power series in two (or more) variables can be defined:

Definition 0.2.6 *A formal power series in two variables* x, t *over a field* $\mathbb{K}$ *are formal algebraic objects of the type:*

$$\phi(x,t) = \sum_{i,j=0}^{\infty} a_{ij} x^i t^j = a_{00} + a_{10}x + a_{01}t + a_{20}x^2 + a_{02}t^2 + a_{11}xt + \dots \tag{2}$$

The set of all these formal power series will be denoted by $\mathbb{K}[[x,t]]$.

Those series can be multiplied according to certain rules, substitution of one of the variable x by a formal power series in one variable is possible, etc.

0.2.2 The group $\mathcal{F}_0(\mathbb{K})$

Two groups of formal power series will be of great impact on the rest of this work. The first one is the following:

Proposition 0.2.7 *The set:*

$$\mathcal{F}_0(\mathbb{K}) = \mathbb{K}[[x]] \setminus x\mathbb{K}[[x]] = \{f_0 + f_1x + f_2x^2 + \dots \ : f_0 \neq 0\}$$

with respect to the multiplication of formal power series is a group.

When there is no possibility of misunderstanding we will write simply $\mathcal{F}_0$ instead of $\mathcal{F}_0(\mathbb{K})$.

Concerning the description of the inverses in this group, we have the following formula:

Proposition 0.2.8 (Reciprocation Formula) *Let $f(x) \in \mathbb{K}[[x]]$, $g(x) \in \mathcal{F}_0$. Then if:*

$$u(x) = u_0 + u_1 x + u_2 x^2 + \ldots = \frac{f(x)}{g(x)}$$

we have that $u_0 = \frac{f_0}{g_0}$, and for $n \geq 1$:

$$u_n = -\frac{g_1}{g_0} u_{n-1} - \frac{g_2}{g_0} u_{n-2} - \ldots + \frac{g_n}{g_0} u_0 + \frac{f_n}{g_0}$$

A nice proof of this result may be found in the paper by the advisors of this work [66] as a consequence of the Generalized Banach Fixed Point Theorem.

Remark 0.2.9 *Note that for any elements $f(x), g(x) \in \mathcal{F}_0$:*

$$Taylor_n(f(x) \cdot g(x)) = Taylor_n(Taylor_n(f(x) \cdot Taylor_n(g(x))$$

$$Taylor_n\left(\frac{1}{f(x)}\right) = Taylor_n\left(\frac{1}{Taylor_n(f(x)}\right)$$

We will omit details, but this would allow us to show that $\mathcal{F}_0$ is the inverse limit of the inverse sequence $\{(\mathbb{K}_n[x] \setminus x\mathbb{K}_{n-1}[x], \Psi_n)\}$ where the operation is $p(x) *_n q(x) = Taylor_n(p(x) \cdot q(x))$ and Ψ_n is given by:

$$a_0 + a_1 x + \ldots + a_n x^n + a_{n+1} x^{n+1} \longmapsto a_0 + a_1 x + \ldots + a_n x^n$$

This is an example of how inverse sequence and inverse limits (of groups in this case) allow us to approximate complicated structures by simpler ones.

Regarding convergent power series:

Remark 0.2.10 *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we can define $\mathcal{F}_{0,hol}(\mathbb{K}) = \mathcal{F}_0(\mathbb{K}) \cap \mathbb{K}_{hol}[[x]]$ and we have:*

$$\mathcal{F}_{0,hol} < \mathcal{F}_0$$

0.2.3 The group F_1

The second group of formal power series of great importance for this work is $\mathcal{F}_1(\mathbb{K})$.

Proposition 0.2.11 *The set:*

$$\mathcal{F}_1(\mathbb{K}) = x\mathbb{K}[[x]] \setminus x^2\mathbb{K}[[x]] = \{f_0 + f_1x + f_2x^2 + \dots \ : f_0 = 0, f_1 \neq 0\}$$

*with respect to the **composition** of formal power series is also a group. The composition of two power series:*

$$f(x) = f_0 + f_1x + f_2x^2 + \dots, \ g(x) = g_1x + g_2x^2 + \dots$$

in $\mathcal{F}_1(\mathbb{K})$ is given by:

$$f(g(x)) = (f \circ g)(x) = f_0 + f_1 \cdot g(x) + f_2 \cdot (g(x))^2 + \dots$$

Independently to the development of the study of the Riordan group, many years before, the groups $\mathcal{F}_1(\mathbb{K})$ were studied by many authors and from many points of view (and using many different notations). We highly recommend the survey by I. Babenko [2] for the study of one of its subgroups. In the case $\mathbb{K} = \mathcal{F}_p$ (finite fields of positive characteristic) they are closely related to **Nottingham groups** and have been widely studied (see [18]).

A description of the inverse of elements in this group is known:

Proposition 0.2.12 (Lagrange Inversion Formula) *Let $v(x) \in \mathcal{F}_1$. Then if $v(x) = \frac{x}{g(x)}$, we have that:*

$$[x^{n+1}]v^{-1} = \frac{1}{n+1}[x^n]g^{n+1} \tag{3}$$

The proof of this result can be found in page 38 of [111] but also a nice proof of it using the Banach Fixed Point Theorem may be found in the paper [62] by one of the advisors of this work.

We will borrow the following notation from [83]:

- Let $f(x) = f_1x + f_2x^2 + \dots$ be an element in $\mathcal{F}_1$. Then f_1 is called the **multiplier** of $f(x)$. It is easy to see (section 2.8 in [2]) that:

  $$\mathcal{F}_1 = \mathbb{K}^* \ltimes \left(x + x^2\mathbb{K}[[x]]\right)$$

  so we can see that the multiplier plays an important role in the algebraic structure of $\mathcal{F}_1$.

- If the multiplier of $g(x) \in \mathcal{F}_C$ is 1, we say that 0 is a fixed point with multiplicity k, or simply that the **multiplicity** of $g(x)$ is k if:

$$g(x) - x \in x^k \mathbb{K}[[x]] \setminus x^{k+1}\mathbb{K}[[x]]$$

As we can see in [2], the group F_1 has a natural structure of Lie group but we are not going to give a detailed description of this structure here.

Similarly to what happened for $\mathcal{F}_0(\mathbb{K})$:

Remark 0.2.13 *Note that for any elements $f(x), g(x) \in \mathcal{F}_1$, let $\widetilde{f}(x), \widetilde{g}(x)$ be their Taylor polynomials of degree $n \geq 1$ we have:*

$$Taylor_n(f(g(x))) = Taylor_n(\widetilde{f}(\widetilde{g}(x)))$$

$$Taylor_n\left(f^{-1}(x)\right) = Taylor_n\left(\widetilde{f}^{-1}(x)\right)$$

Again we will omit details, but this would allow us to show that $\mathcal{F}_1$ is the inverse limit of the inverse sequence $\{(x\mathbb{K}_{n-1}[x] \setminus x^2\mathbb{K}_{n-2}[x], \Psi_n)\}$ where the operation is $p(x) *_n q(x) = Taylor_n(p(q(x)))$ and Ψ_n is given by:

$$a_0 + a_1 x + \ldots + a_n x^n + a_{n+1}x^{n+1} \longmapsto a_0 + a_1 x + \ldots + a_n x^n$$

This is an example of how inverse sequence and inverse limits (of groups in this case) allow us to approximate complicated structures by simpler ones.

And with regards to convergent power series:

Remark 0.2.14 *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we can define $\mathcal{F}_{1,hol}(\mathbb{K}) = \mathcal{F}_1(\mathbb{K}) \cap \mathbb{K}_{hol}[[x]]$ and:*

$$\mathcal{F}_{1,hol} < \mathcal{F}_1$$

0.2.4 Power Series in one variable over Rings

Formal power series in a variable x can also be defined over a unitary ring R. In this case we will use the analogue notation $R[[x]]$. This has been considered in combinatorics for example when dealing with generating functions of integer sequences.

We can again define the multiplication of an element in R by a power series, and the sum, multiplication and composition of two given power series as done before.

The problem is that as in a ring not any element has a multiplicative inverse, the existence of multiplicative and compositional inverses is not guaranteed.

To have something similar to the group $\mathcal{F}_0$, we will need to find sets as big as possible of elements in $R[[x]]$ with multiplicative and compositional inverses respectively:

Proposition 0.2.15 *Let R be a unitary ring. $(1 + xR[[x]])$ is a group with respect to the multiplication of power series.*

Analogously, we can try to define a group of the type $\mathcal{F}_1$ finding a set as big as possible of elements in $\mathcal{R}[[x]]$ with compositional inverses:

Proposition 0.2.16 (Proposition 2.1 in [2]**)** *Let R be a unitary ring with unity element 1_R, then $(1_R x + x^2 R[[x]])$ is a group with respect to the composition of power series.*

0.3 The Riordan Group (Infinite Representation)

This section is an introduction to the basic concepts, notation and tools related to the classical Riordan group that will be necessary for the rest of this work.

The Riordan group is usually introduced as a group of matrices with entries in an arbitrary field $\mathbb{K}$ (again, a generalization for entries over a ring is possible). It can be viewed either:

- in its infinite representation (the classical one): as a subgroup of $ILT_\infty(\mathbb{K})$, that is, the group of invertible matrices (those which have non-zero entries in the main diagonal) of the type:
$$(a_{ij})_{0\le i,j<\infty} = \begin{bmatrix} a_{00} & & & \\ a_{10} & a_{11} & & \\ a_{20} & a_{21} & a_{22} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$
with entries in $\mathbb{K}$, with respect to the multiplication (which is well defined).

- in its bi-infinite representation (that will appear in chapter 1): as a subgroup of $ILT_{\infty\infty}(\mathbb{K})$, that is, the group of invertible matrices (those which have non-zero entries in the main diagonal) of the type:
$$(a_{ij})_{\infty<i,j<\infty} = \begin{bmatrix} \ddots & & & & \\ \cdots & a_{-1,-1} & & & \\ \cdots & a_{0,-1} & a_{00} & & \\ \cdots & a_{1,-1} & a_{10} & a_{11} & \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

The Riordan group appears in many branches of Mathematics, since it is an useful tool for doing computations with power series (and power series appear in many branches of Mathematics and from many different points of view). Although neither the name "Riordan group" may not be so "old" (it first appeared in the paper of L. Shapiro and his colaborators [105], in 1991) nor is the sistematic study of its properties (see for example [17, 23–25, 51, 63, 66, 76]), some of the ideas involved were know in the 19th century in relation to some problems in Analysis (where formal power series play an important role when studying functions which are analytic in a neighbourhood of a point).

Anyway it was the interest of the Riordan group in combinatorics (generating functions of sequences are formal power series) the reason why L.W. Shapiro, S. Getu, W.-J. Woan and L.C. Woodson started the "modern" study of this group in [105] from a different point of view. The group was named after John Riordan (April 22, 1903 to August 26, 1988) with ocassion of his death, since he already setted the base for the study of this group in his previous work (see for example his book [94]).

This combinatorial point of view is the one that we will follow in this section to introduce the main concepts. We will introduce several changes in the notation and we may focus on slightly different aspects from the classical approaches.

As we said before, many ideas included in this section were known before the foundation of the term "Riordan matrix". As far as we know, the principal ones are the following:

- One of the important subgroups of the Riordan group, the associated subgroup or Lagrange subgroup, and the actions of its elements on $\mathbb{K}[[x]]$ were already studied before (see the work by L. Leau in the 19-th century [60], the work by A. A. Benett [7] and the work by E. Javotinski [49], [50]).

- The associated or Lagrange subgroup is naturally anti-isomorphic (and then isomorphic) to $\mathcal{F}_1(\mathbb{K})$ which, as we have already showed, has been widely studied by I. K. Babenko and his colaborators (see for example the survey [2]), and by other authors (see for example the work by D. L. Johnson [53] and by S. A. Jennings [52]). In the case $\mathbb{K}$ being a finite field, the group $\mathcal{F}_1(\mathbb{K})$ is closely related to **Nottingham subgroup** and was also widely studied (see for example the corresponding chapter writen in the book [18] for references).

- Another of the important subgroups, the Toeplitz subgroup, is isomorphic to $\mathcal{F}_0(\mathbb{K})$ with respect to the product. Also the action of its elements on $\mathbb{K}[[x]]$ has been already studied.

- Riordan matrices (with respect to their action on $\mathbb{K}[[x]]$) can be considered to be a "symbolic version" of weighted composition operators which has been also widely studied in analysis (see for example the book by J. Shapiro [101]).

0.3.1 Basic Definitions

Definition 0.3.1 *An infinite lower-triangular matrix $(d_{ij})_{0\leq i,j<\infty}$ with entries in a field $\mathbb{K}$ is a **Riordan matrix**, if there exist two power series:*

$$d(x) \in \mathcal{F}_0(\mathbb{K}),\ h(x) \in \mathcal{F}_1(\mathbb{K}) \tag{4}$$

in which case we will denote it by $R(d(x), h(x))$, such that the generating function of the j-column $\begin{bmatrix} d_{0,j} \\ d_{1,j} \\ \vdots \end{bmatrix}$ is $d(x)\cdot(h(x))^j$, that is:

$$d(x)\cdot(h(x))^j = \sum_{k=0}^{\infty} d_{kj}x^k$$

Sometimes it will be convenient to keep in mind the existence of a more general object: the **improper Riordan matrix** (introduced in [107]) which are the matrices that satisfy all the above conditions replacing (1.4) by:

$$d(x) \in \mathbb{K}[[x]] \qquad h(x) \in x\mathbb{K}[[x]]$$

This matrix will be denoted by $\widetilde{R}(d(x), h(x))$. If we want to remark that a given Riordan matrix is not improper, we will talk about **proper** Riordan matrices.

The difference is that improper Riordan matrices may have zeros in the main diagonal, while Riordan matrices cannot. From the discussion made in the chapter about basic concepts about matrices, we know that a lower triangular infinite matrix with zeros in its diagonal is not invertible.

Note the following:

Remark 0.3.2 *The fact that the generating functions of the columns in $D = (d_{ij})_{0 \le i,j < \infty} \in ILT_\infty$ are in geometric progression of ratio $h(x)$, can be equivalently stated in terms of the exist a sequence $(h_1, h_2, h_3, \ldots)$ of elements in $\mathbb{K}$ satisfying:*

$$\forall 1 \le j \le i < \infty \qquad d_{ij} = h1 d_{i-1,j-1} + h_2 d_{i-2,j-1} + \ldots + h_i d_{j-1,j-1} \tag{5}$$

Obviously, the generating function of this sequence is $h(x)$.

Riordan matrices appear frequently in combinatorics. The most famous example is:

Example 0.3.3 (Pascal's Triangle) *The Pascal's Triangle, which for this work will be the matrix:*

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

that is, the matrix $(d_{ij})_{0 \le i,j < \infty}$ such that $d_{ij} = \binom{i}{j}$ (with the convention $\binom{n}{k} = 0$ if $k > n$), is the Riordan matrix:

$$R\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$$

Since:

- *The generating function of the 0-column is $\frac{1}{1-x}$.*
- *The generating function of the 1-column is $\frac{1}{(1-x)^2}$.*
- *The generating function of the 2-column is $\frac{1}{(1-x)^3}$*

and so on.

We will see another example in section 0.3.3 once we have introduced the A-sequence.

0.3.2 The natural action of a infinite Riordan matrix on $\mathbb{K}[[x]]$. The Riordan group $\mathcal{R}(\mathbb{K})$

One of the most useful properties that Riordan matrices have is the following result, sometimes called the **First Fundamental Theorem of the Riordan matrices** (**1FTRM**), concerning the action of Riordan matrices over the set of power series:

Proposition 0.3.4 (1FT) *Let $R(d(x), h(x))$ be a Riordan matrix (the result also holds for improper Riordan matrices), let $f(x) = f_0 + f_1x + f_2x^2 + \ldots \in \mathbb{K}[[x]]$. Then if:*

$$\begin{bmatrix} w_0 \\ w_1 \\ \vdots \end{bmatrix} = R(d(x), h(x)) \begin{bmatrix} f_0 \\ f_1 \\ \vdots \end{bmatrix}$$

we have that the generating function of the sequence $(w_0, w_1, w_2, \ldots)$ satisfies:

$$w_0 + w_1x + w_2x^2 + \ldots = d(x) \cdot f(h(x))$$

Proof: We have that (via the identification with the sequences and their generating functions):

$$R(d(x), h(x)) \begin{bmatrix} f_0 \\ f_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} \uparrow & \uparrow & \uparrow & \\ d(x) & d(x) \cdot h(x) & d(x) \cdot (h(x))^2 & \ldots \\ \downarrow & \downarrow & \downarrow & \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ \vdots \end{bmatrix} =$$

$$= f_0 \cdot \begin{bmatrix} \uparrow \\ d(x) \\ \downarrow \end{bmatrix} + f_1 \cdot \begin{bmatrix} \uparrow \\ d(x) \cdot h(x) \\ \downarrow \end{bmatrix} + f_2 \cdot \begin{bmatrix} \uparrow \\ d(x) \cdot (h(x))^2 \\ \downarrow \end{bmatrix} + \ldots$$

Although this is an infinite sum of vectors, in the position k there is only k vectors with a non-zero entry. So this sum makes sense and equals to $d(x) \cdot f(h(x))$.

□

From now on, for this action over the set of power series, we will use the notation:

$$R(d(x), h(x)) \otimes f(x)$$

As an application, we will prove a well known formula for the binomial coefficients by using the 1FT:

Example 0.3.5 *Let $P = R\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ (the Pascal's triangle). According to the 1FT:*

$$P \otimes \left(\frac{1}{1+x}\right) = 1$$

But this is equivalent to say that:

$$\begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}$$

which reduces to the well known formula:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0 \qquad \text{if } n \geq 1$$

The 1FT also allow us to endow the set of all Riordan matrices with a group structure, as we can see below:

Theorem 0.3.6 *The set of Riordan matrices with entries in $\mathbb{K}$ together with the product of matrices is a group, called the **Riordan group**, and denoted by $\mathcal{R}(\mathbb{K})$. In fact we have that:*

(1) $R(d(x), h(x)) \cdot R(u(x), v(x)) = R(d(x) \cdot u(h(x)), v(h(x)))$.

(2) $R(1, x)$ *is the neutral element.*

(3) $R(d(x), h(x))^{-1} = R\left(\frac{1}{d(h^{-1}(x))}, h^{-1}(x)\right)$.

If there is no possibility of misunderstanding, we will denote the Riordan group simply by $\mathcal{R}$.

Proof: It follows from theorem 0.3.4. We omit the details, that can be found together with a different approach in [66].

□

Improper Riordan matrices need to be excluded since they have no multiplicative inverse. But for them (1) still holds.

This result stablishes an useful bridge between matrix multiplication and the correspondent operations between power series.

0.3.3 A-sequence

One characteristic property of the Riordan matrices is the existence of an A-sequence (first introduced in [98], see also [76]). The following result is sometimes called the **Second Fundamental Theorem of Riordan Matrices (2FT)**. :

Proposition 0.3.7 (2FT) *Let $D = (d_{ij})_{0\leq i,j<\infty} \in ILT_\infty$ be an infinite lower triangular matrix. D is Riordan matrix $R(d(x), h(x))$ for certain power series $d(x), h(x)$ if and only if there exists a sequence $(a_0, a_1, a_2, \ldots)$ of elements in $\mathbb{K}$ with $a_0 \neq 0$, called the **A-sequence**, satisfying:*

$$\forall i,j \geq 1,\ d_{ij} = a_0 d_{i-1,j-1} + a_1 d_{i-1,j} + \ldots + a_{i-j} d_{i-1,i-1} \tag{6}$$

In this case, if we denote by $A(x)$ to the generating function of the A-sequence we have that:

$$h(x) = \left(\frac{x}{A(x)}\right)^{-1}$$

We will show a direct proof for this result found by the advisors of this work in section 0.3.5 as a consequence of the algebraic structure of the Riordan group.

This pattern followed by the entries of the Riordan group have, for instance, the two following consequences:

Corollary 0.3.8 *Let $R(d(x), h(x)) = (d_{ij})_{0\leq i,j<\infty} \in \mathcal{R}$, with A-sequence $(a_0, a_1, a_2, \ldots)$.*

- *The main diagonal of a Riordan matrix, that is, the diagonal $d_{00}, d_{11}, d_{22}, \ldots$ is a geometric progression of rate a_0 (or equivalently, of rate h_0).*
- *The following diagonal, that is, the diagonal $d_{10}, d_{21}, d_{32}, \ldots$ is an arithmetic-geometric progression:*

$$d_{10}, a_0(d_{10} + a_1 d_{00}), a_0^2(d_{10} + 2a_1 d_{00}), \ldots$$

We will omit the proof, which is a direct application of the formula (6). Understanding the structure of the rest of the diagonals in a Riordan matrix is still an open problem (see Open Question 1). As far as we know, R. Sprugnoli has obtained some results in this sense that we are looking forward to seeing published.

As an example, we will find the A-sequence for the Pascal's Triangle, and we will also recover a well known identity for the binomial coefficients:

Example 0.3.9 *The pattern to construct the Pascal Triangle:*

$$R\left(\frac{1}{1-x},\frac{x}{1-x}\right)=\begin{bmatrix}1 & & & & & \\ 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots\end{bmatrix}$$

was (with the convention $\binom{n}{k}=0$ *if* $k>n$*):*

$$\binom{n}{k}=1\cdot\binom{n-1}{k-1}+1\cdot\binom{n-1}{k}\ \textit{for}\ 0<k\leq n$$

so the A-sequence in this case is $(1,1,0,0,0,0,\ldots)$.

Moreover, since many sequences $d_{n,k}$ in combinatorics satisfy a recurrence relation of the type (6) now is easy to believe that we will find loads of examples of Riordan matrices in combinatorics apart from Pascal's Triangle (see for example the survey [102]).

0.3.4 Alternative notation for Riordan matrices. The g-sequence.

In this section we will introduce two alternative notations for Riordan matrices.

The first one is the adequate one to work with the A-sequence and has not been used yet in the bibliography:

Definition 0.3.10 *We define* $H(u(x),A(x))$ *to be the Riordan matrix which first column and A-sequence have generating functions* $u(x)$ *and* $A(x)$ *respectively. So the equivalence between both notations is:*

$$H(u(x),A(x))=R\left(u(x),\left(\frac{x}{A(x)}\right)^{-1}\right)$$

The second one, was proposed by the advisors of this work in [66] due to their different approach to this subject:

Definition 0.3.11 *For two power series* $f(x),g(x)\in\mathcal{F}_0$ *we define:*

$$T(f(x)\mid g(x))=R\left(\frac{f(x)}{g(x)},\frac{x}{g(x)}\right)$$

In those terms also the 1FT can be stated, and a formula for the group operation can be obtained. We will not give the details.

One more thing remains to be announced in this subsection: the notation proposed in definition 0.3.11 implies a power series $g(x) = g_0 + g_1x + g_2x^2 + \ldots$ The sequence $(g_0, g_1, g_2, \ldots)$ is known as the **g-sequence**, and it is as useful as the A-sequence thanks to the following result anologous to remark 0.3.2 and proposition 0.3.7 (see [102]):

Proposition 0.3.12 *An element $D = (d_{ij})_{0\leq i,j\leq\infty} \in ILT_\infty$ is a Riordan matrix $R(d(x), h(x))$ if and only if there exists a sequence $(g_0, g_1, g_2, \ldots)$ of elements in $\mathbb{K}$ called the g-sequence of D satisfying:*

$$(7) \qquad 0 \leq \forall k \leq n, \qquad d_{nk} = g_0 d_{n+1,k+1} + g_1 d_{n,k+1} + \ldots + g_n d_{k+1,k+1}$$

In this case, if we denote by $g(x)$ to the generating function of the g-sequence we have that:

$$h(x) = \frac{x}{g(x)} \text{ or equivalently } \left(\frac{x}{g(x)}\right) = \left(\frac{x}{A(x)}\right)^{-1}$$

where $A(x)$ is the generating function of the A-sequence of $R(d(x), h(x))$.

0.3.5 The subgroups $\mathcal{T}(\mathbb{K}), \mathcal{A}(\mathbb{K})$

We will meet the following subgroups of the Riordan group through the rest of this work. Those subgroups have been largely studied in the bibliography: see the survey [102] or the paper [104].

Definition 0.3.13 *The **Toeplitz subgroup** (also denoted in the bibliography as the **Appel Subgroup**) is the set of Riordan matrices:*

$$\mathcal{T}(\mathbb{K}) = \{R(d(x), x)\} < \mathcal{R}(\mathbb{K})$$

It is easy to check the closure of this set by the operation and that it is a subgroup. This subgroup is normal in $\mathcal{R}(\mathbb{K})$ since:

$$\forall R(d(x), h(x)) \in \mathcal{R}(\mathbb{K}),\ R(F(x), x) \in \mathcal{T}(\mathbb{K}),\ R(d(x), h(x))^{-1} R(F(x), x) R(d(x), h(x)) \in \mathcal{T}(\mathbb{K})$$

as it is easily checked by the group law in $\mathcal{R}(\mathbb{K})$.

$\mathcal{T}(\mathbb{K})$ is naturally isomorphic to the group $\mathcal{F}_0(\mathbb{K})$, with the product of power series.

Definition 0.3.14 *The **associated subgroup** (also denoted in the bibliography as the **Lagrange subgroup**, for instance in* [64]*) is the set of Riordan matrices:*

$$\mathcal{A}(\mathbb{K}) = \{R(1, h(x))\} < \mathcal{R}(\mathbb{K})$$

It is very easy to see the closure of this set by the operation, and that it is a subgroup.

In view of the group law in $\mathcal{R}$, the natural identification:

$$h(x) \longmapsto R(1, h(x))$$

is an anti-isomoprhism between $\mathcal{F}_1(\mathbb{K})$ and $\mathcal{A}(\mathbb{K})$. Then, we can construct an isomorphism between $\mathcal{F}_1$ and $\mathcal{A}(\mathbb{K})$ by:

$$h(x) \longmapsto R(1, h(x))^{-1}$$

We have the following important result:

Proposition 0.3.15 (see [66]**)** $\mathcal{R}(\mathbb{K}) = \mathcal{T}(\mathbb{K}) \rtimes \mathcal{A}(\mathbb{K})$

0.3.6 Riordan matrices with entries in a unitary ring

Most of the discussion in this section has been taken from [2]. Since formal power series are not only defined in fields but in unitary rings, we can extend the definition of Riordan and generalized Riordan matrices (definition 0.3.1) for matrices with entries on a unitary ring. But this time, in order to have an inverse, we need to restrict ourselves to other set of matrices:

Remark 0.3.16 *For any commutative ring with unity R, for any two elements:*

$$d(x) \in (1 + xR[[x]]),\ h(x) \in (x + x^2R[[x]])$$

we can define as before a Riordan type matrix $R(d(x), h(x))$, and it will have an inverse and an A-sequence.

This was the original definition of Riordan matrix in [105] in order to admit to define Riordan matrices over $\mathbb{Z}$. So we have:

Definition 0.3.17 *Let R be a ring. We wil define $\mathcal{C}(\mathcal{R})$ to be the set of Riordan type matrices $R(d(x), h(x))$ where:*

$$\begin{cases} d(x) \in (1 + xR[[x]]) \\ h(x) \in (x + x^2R[[x]]) \end{cases}$$

$\mathcal{C}(\mathcal{R})$ is a group according to the previous remark.

This group via the identification made before between $\mathcal{F}_1$ and $\mathcal{A}$ contains a copy of $\mathcal{J}(R)$ (see the definition in [2]).

As discussed before, the Riordan group arises from combinatorics. Due to this combinatorial flavour the group many papers have been devoted to studying aspects of elements in the subgroup $\mathcal{J}(\mathbb{Z}) < \mathcal{R}(\mathbb{R})$ (see for example [107] or [78]).

0.3.7 Other relevant subgroups of the Riordan group

For basic information of subgroups of the Riordan Group see the survey [102] or the paper [104].

$\mathcal{R}$ is a group which elements are associated with some formal power series. In the field of Analysis, the power series considered are always convergent in a neigbourhood of a point. So the following subgroups are interesting in this field:

Definition 0.3.18 *For* $\mathbb{K} = \mathbb{R}, \mathbb{C}$*:*

- $\mathcal{T}_{hol} = \{R(f(x), x) \in \mathcal{T} : f(x) \in \mathcal{F}_{P,hol}\}$
- $\mathcal{A}_{hol} = \{R(1, h(x)) \in \mathcal{A} : h(x) \in \mathcal{F}_{C,hol}\}$
- $\mathcal{R}_{hol} = \{R(d(x), h(x)) \in \mathcal{R} : d(x) \in \mathcal{F}_{P,hol}, h(x) \in \mathcal{F}_{P,hol}\}$. *In other words:*

$$\mathcal{R}_{hol} = \mathcal{T}_{hol} \rtimes \mathcal{A}_{hol}$$

Remark 0.3.19 *The fact that* $\mathcal{T}_{hol}, \mathcal{A}_{hol}, \mathcal{R}_{hol}$ *are subgroups of* $\mathcal{R}$ *comes from the fact that* $\mathcal{F}_{0,hol}, \mathcal{F}_{1,hol}$ *are subgroups of* $\mathcal{F}_0, \mathcal{F}_1$ *respectively.*

Definition 0.3.20 *The* ***Checkerboard subgroup*** *is the set of Riordan matrices of the type* $R(d(x), h(x))$ *where:*

$$\begin{cases} d(x) = d_0 + d_2x^2 + d_4x^4 + \dots & (d(x) \textit{ is } \textbf{\textit{even}}) \\ h(x) = h_1x + h_3x^3 + x_5x^5 + \dots & (h(x) \textit{ is } \textbf{\textit{odd}}) \end{cases}$$

Remark 0.3.21 *The reason of the name "checkerboard" is that* $R(d(x), h(x)) \in \mathcal{R}$ *belongs to this subgroup if and only if it is of the form:*

$$\begin{bmatrix} * & & & & & & \\ 0 & * & & & & & \\ * & 0 & * & & & & \\ 0 & * & 0 & * & & & \\ * & 0 & * & 0 & * & & \\ 0 & * & 0 & * & 0 & * & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Definition 0.3.22 *We will denote by* $\mathcal{D}$ *to the* ***diagonal subgroup****, the set of all the Riordan matrices that are diagonal matrices. This subgroup is obviously commutative.*

Remark 0.3.23 *If* $(d_{ij})_{0 \le i,j < \infty} \in \mathcal{R}$ *the diagonal of a Riordan matrix is always a geometric progression since if its A sequence is* $a_0, a_1, a_2, \dots$*, then the diagonal must be* $d_{00}, a_0d_{00}, a_0^2d_{00}, \dots$ *according to the (2FT).*

So $\mathcal{D}$ *is isomorphic to the group:*

$$\{(d_0, a_0) \in \mathbb{K}^2 : d_0, a_0 \neq 0\}$$

with the binary operation $(d_0, a_0) * (\widetilde{d}_0, \widetilde{a}_0) = (d \cdot \widetilde{d}, a \cdot \widetilde{a})$.

Definition 0.3.24 *The* ***renewal subgroup*** *(also denoted in the bibliography by the name* ***Bell subgroup****) is the set of Riordan matrices of the form $R(h(x), h(x))$.*

Remark 0.3.25 *The renewal subgroup is naturally isomorphic to $\mathcal{A}$ through the map:*

$$\begin{bmatrix} 1 & & & & \\ 0 & d_{11} & & & \\ 0 & d_{21} & d_{22} & & \\ 0 & d_{31} & d_{32} & d_{33} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \longmapsto \begin{bmatrix} d_{11} & & & & \\ d_{21} & d_{22} & & & \\ d_{31} & d_{32} & d_{33} & & \\ d_{41} & d_{42} & d_{43} & d_{44} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

To see this, again the key is block multiplication in those matrices.

0.4 Final Comments

Other objects will appear in this work. Mainly Lie groups over Frechet spaces and simplicial complexes.

Since we don't want to enlarge this introductory chapter we will only provide some bibliography to cover those topics.

0.4.1 Lie groups over Frechet spaces

A readable introduction to the general theory of Lie groups can be found in [37, 90].

It is well known that the groups of matrices are Lie groups. A good reference describing the Lie group structure of matrix groups is [5].

We are not so interested in classical Lie groups but in infinite dimensional Lie groups, since in chapter 4 our purpouse will be to study the Lie group structure of $\mathcal{R}(\mathbb{K})$. Up to our knowledge, there are two main options to define infinite dimensional Lie groups:

- Defining the concept of manifold modelled over an infinite dimensional vector space (for example a Frechet space) and then defining a Lie group as a manifold modelled over one of those spaces with a group structure for which multiplication and inversion are smooth. A good reference for this is the work by J. Milnor [82], and also [57] may be of interest.

- Considering inverse limits of inverse sequences of classical (finite dimensional) Lie groups and trying to endow those objects with a reasonable structure, similar to the one of classical Lie groups. This is the path followed in [45, 46].

The Riordan group is a good example for both points of view: the structure of manifold modelled over a Frechet space is quite simple, and the Riordan group (as we will see in chapters 1 and 4) has a natural structure as inverse limit of an inverse sequence of classical Lie groups.

0.4.2 Simplicial Complexes

Simplicial complexes are objects that have both: a combinatoric structure and a topological structure. So to talk about them, a basic background of topology. Good references for this may be [42, 85].

Only finite simplicial complexes will be considered here. Simplicial complexes are built from certain "building blocks" of different dimensions called simplices. Two definition are possible:

- Geometric definition: where simplices are considered to be certain convex subsets in $\mathbb{R}^N$ and a simplicial complex is considered to be a topological space obtained gluing together some (a finite number) simplices by following certain rules.
- Abstract definition: where simplices are considered to be subsets of a fixed (finite) set V and a simplicial complex is a set of simplices closed under inclusion.

Both definition are equivalent and there is a canonical way to obtain "geometric simplicial complexes" from "abstract simplicial complexes" and viceversa.

For more information about the basic theory of simplicial complexes we recommend [42, 85].

Since a simplicial complex is constructed out of some simplices, for each simplicial complex K, we define its **f-vector** to be the sequence:

$$f(K) = (f_0(K), f_1(K), f_2(K), \dots, f_d(K))$$

where $f_i(K)$ is the number of simplicial complex of dimension i in K.

Characterizing the set of f-vectors of a family of simplicial complex with a given property in common is a classical problem in topological combinatorics called the f-vector problem. The f-vector problem has been widely studied for example for the set of simplicial complexes with a given Betti sequence (see [14]) or for the set of simplicial complexes PL-homeomorphic to the boundary of a polytope (g-theorem). We recommend the reader [14, 39, 119].

Chapter 1

Some Inverse Limit Approaches to the Riordan Group

Most of the topics covered in this chapter are contained in [63]. In this chapter unless otherwise specified (this will only happen in section 1.14) $\mathbb{K}$ will denote a field of any characteristic.

The key tool used in this chapter is the inverse limit (see section 0.1). We will introduce finite Riordan matrices, which are lower triangular submatrices of a Riordan matrix, and by using them together with the inverse limit tool, we will obtain the Riordan group in its infinite and bi-infinite representations.

Some of those concepts of the finite framework have been independently used in [21] to obtain certain constructions for Riordan arrays.

As it was revealed in some of the previous work in Riordan matrices: by the advisors of this work in collaboration with D. Merlini and R. Sprugnoli [64, 65], by R. Sprugnoli in [107] and A. Luzón in [62], there is a necessity of a deeper understanding of different reflections in bi-infinite Riordan matrices. Obviously, the transposition of a Riordan matrix (either finite or infinite) is not a Riordan matrix. But from the point of view of finite Riordan matrices, an internal transformation reflecting accross "$y = x$" is naturally found:

$$\begin{bmatrix} a_{00} & & \\ a_{10} & a_{11} & \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{22} & & \\ a_{21} & a_{11} & \\ a_{20} & a_{10} & a_{00} \end{bmatrix}$$

$$\begin{bmatrix} a_{00} & & & \\ a_{10} & a_{11} & & \\ a_{20} & a_{21} & a_{22} & \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{33} & & & \\ a_{32} & a_{22} & & \\ a_{31} & a_{21} & a_{11} & \\ a_{30} & a_{20} & a_{10} & a_{00} \end{bmatrix}$$

producing again another finite Riordan matrix.

This transformation makes no sense and has no analogue in usual infinite Riordan matrices, but it does in bi-infinite Riordan matrices. Using inverse sequences of finite Riordan matrices and the inverse limit concept, we are able to define essentially two different reflections in bi-infinite Riordan matrices. This allows us to reformulate and give answers to some questions left open in [65].

The horizontal and vertical constructions that were explored in [64] enable us to understand these reflections. In fact, the fundamental point is the analogy between the sequence g for a vertical construction and the A-sequence for an horizontal construction of a Riordan matrix.

In section 1.1 we obtain the groups of finite Riordan matrices $\mathcal{R}_n(\mathbb{K})$ of size $(n+1)\times(n+1)$ for $n = 0, 1, 2, \ldots$ by means of natural projections (that are also studied in section 1.2) from the Riordan group $\mathcal{R}$. We also give an internal characterization of such finite matrices. Later (section 1.4) we recover the Riordan group in its infinite representation as an inverse limit of these groups of finite matrices with appropriate bonding maps.

In section 1.7 we will recall from the bibliography the concept of complementary and dual Riordan matrices. After this, in section 1.8 we describe both matrices in terms of inverse limits and reflections. After this, we will recover the bi-infinite representation of the Riordan group as the inverse limit of two different sequences: in section 1.9 we achieve it from the usual infinite one and in section 1.10 we will do it as inverse limit of sequences of groups of finite Riordan matrices (by two different natural ways depending overall on the parity of the size of finite matrices in the inverse sequences).

Reflecting term by term the finite matrices involved in both sequences above, in section 1.11 we get two different, but related, reflections on bi-infinite Riordan matrices and the two corresponding concepts of symmetric matrices where the dual and the complementary appears. This allows us to translate symmetries in bi-infinite matrices in terms of the problems of self-complementarity and self-duality in Riordan matrices left open in [65]. We will solve those problems in section 1.12.

The idea of understanding (infinite or bi-infinite) matrices in the Riordan group as elements in the inverse limit of certain inverse sequences will be the suitable framework to do proofs by induction, and will be key in the rest of this work (see sections 1.13 and 1.14 for examples some examples of the application of this technique).

1.1 Partial Riordan matrices and groups

In a similar fashion to the groups $ILT_\infty(\mathbb{K})$ introduced in section 0.3, for every $n \in \mathbb{N}$ consider the group of invertible lower triangular matrices $ILT_{n+1}(\mathbb{K})$. Let $\mathcal{R}(\mathbb{K})$ be the Riordan group. Since every Riordan matrix is lower triangular, we can define a natural homomorphism:

$$\Pi_n : \mathcal{R}(\mathbb{K}) \to ILT_{n+1}(\mathbb{K})$$

$$(d_{i,j})_{i,j\in\mathbb{N}} = \left[\begin{array}{ccc|c} d_{00} & & & \\ \vdots & \ddots & & \\ d_{n0} & \ldots & d_{nn} & \\ \hline \vdots & & \vdots & \ddots \end{array}\right] \mapsto \begin{bmatrix} d_{00} & & \\ \vdots & \ddots & \\ d_{n0} & \ldots & d_{nn} \end{bmatrix} = (d_{i,j})_{0\le i,j\le n}$$

For obvious reasons, many times we will refer to this homomorphism as the **n-th projection** of a certain infinite Riordan matrix.

With the aim of describing the Riordan group as an inverse limit of an inverse sequence of groups of finite matrices, we consider first:

Definition 1.1.1 *The* ***partial Riordan group*** *(or* ***finite****, to distinguish it from the infinite one) $\mathcal{R}_n(\mathbb{K})$ is the subgroup of $ILT_{n+1}(\mathbb{K})$ defined by $\mathcal{R}(\mathbb{K})_n = \Pi_n(\mathcal{R}(\mathbb{K}))$ (recall that the image under a group homomorphism is a subgroup of the target group).*

The elements of the groups $\mathcal{R}_n(\mathbb{K})$ (for any $n \in \mathbb{N}$) are called ***finite Riordan matrices****, and we will use the notation:*

$$R_n(d(x), h(x)) = \Pi_n(R(d(x), h(x)))$$

Remark 1.1.2 *Beware of the index: the size of any element in $\mathcal{R}_n(\mathbb{K})$ is $(n+1) \times (n+1)$, that is, $\mathcal{R}_n(\mathbb{K}) < ILT_{n+1}(\mathbb{K})$.*

This is because, as we will see later, we will identify the columns in the elements in $\mathcal{R}_n(\mathbb{K})$ (of size $n+1$) with Taylor polynomials $a_0 + a_1 x + \ldots + a_n x^n \in \mathbb{K}_n[x]$.

If there is no possibility of misunderstanding, we will write simply $\mathcal{R}_n$, ILT_{n+1} instead of $\mathcal{R}_n(\mathbb{K})$, $ILT_{n+1}(\mathbb{K})$.

Example 1.1.3 *For $n = 0, 1, 2, 3$ we have:*

- *$\mathcal{R}_0 = \mathbb{K}^*$ with the usual product in $\mathbb{K}$ (being $\mathbb{K}^* = \mathbb{K} \setminus \{0\}$).*
- *$\mathcal{R}_1$ is the group of 2×2 lower triangular invertible matrices.*
- *$\mathcal{R}_2$ is the group of 3×3 lower triangular invertible matrices where the main diagonal is formed by three consecutive terms of a geometric progression.*

$$\begin{bmatrix} d_{00} & & \\ d_{10} & d_{00}a_0 & \\ d_{20} & d_{21} & d_{00}a_0^2 \end{bmatrix}$$

- *$\mathcal{R}_3$ is the group of 4×4 lower triangular invertible matrices where the main diagonal is formed by four consecutive terms of a geometric progression and the first sub-diagonal is formed by three consecutive terms of an arithmetic-geometric progression with the same ratio as the geometric one in the main diagonal:*

$$\begin{bmatrix} d_{00} & & & \\ d_{10} & d_{00}a_0 & & \\ d_{20} & a_0(d_{10} + a_1 d_{00}) & d_{00}a_0^2 & \\ d_{30} & d_{31} & a_0^2(d_{10} + 2a_{10}d_{00}) & d_{00}a_0^3 \end{bmatrix}$$

In view of the characterizations of infinite Riordan matrices given by 2FT and proposition 0.3.12 we immediately obtain:

Proposition 1.1.4 (Partial A, g -sequences) *Let $D_n = (d_{ij})_{0\leq i,j<n} \in ILT_{n+1}$. $D_n \in \mathcal{R}_n$ if and only if there exists a sequence $(a_0, a_1, \ldots, a_{n-1})$ of elements in $\mathbb{K}$ with $a_0 \neq 0$, called the **A-sequence** (or the **partial A-sequence** if we need to make a distinction), satisfying:*

$$\forall i, j \geq 1,\ d_{ij} = a_0 d_{i-1,j-1} + a_1 d_{i-1,j} + \ldots + a_{i-j} d_{i-1,i-1} \tag{1.1}$$

*or equivalently it there exists a sequence $g_0, \ldots, g_{n-1}$ of elements in $\mathbb{K}$ with $g_0 \neq 0$, called the **g-sequence** (or the **partial g-sequence** if we need to make a distinction), satisfying:*

$$\forall i, j < n,\ d_{ij} = g_0 d_{i+1,j+1} + g_1 d_{i,j+1} + \ldots + g_i d_{j+1,j+1} \tag{1.2}$$

Both sequences are unique.

1.2 Riordan matrices with the same n-th projection

It is obvious that we can have a finite Riordan matrix as the projection of different infinite Riordan matrices. In fact, there are infinitely many different infinite Riordan matrices with the same n-projection.

Example 1.2.1 *Take $d(x) = 1$, $h(x) = \frac{1}{1-x}$ and $u(x) = 1 + x^4$, $v(x) = \frac{1}{1-x} + x^4$. Obviously:*

$$R(d(x), h(x)) = \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 1 & 2 & 1 & & \\ 0 & 1 & 3 & 3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \neq \begin{bmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 1 & 2 & 1 & & \\ 1 & 2 & 3 & 3 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = R(u(x), v(x))$$

but:

$$R_3(d(x), h(x)) = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 1 & 2 & 1 \end{bmatrix} = R_3(u(x), v(x))$$

Infering from this example, we will clarify the general situation on equality of n-projections with the following result:

Proposition 1.2.2 *Two Riordan matrices $D = R(d(x), h(x))$, $T = R(u(x), v(x))$ have the same projections $D_n = \Pi_n(D)$, $T_n\Pi_n(T)$ if and only if $Taylor_n(d(x)) = Taylor_n(u(x))$ and $Taylor_n(h(x)) = Taylor_n(v(x))$.*

On the other hand, given a finite Riordan matrix $D_n \in \mathcal{R}_n$ with $n \geq 1$, there are two unique polynomials $\tilde{d}(x) \in \mathcal{F}_0(\mathbb{K})$ and $\tilde{h}(x) \in \mathcal{F}_1(\mathbb{K})$ with $\deg(\tilde{d}(x)), \deg(\tilde{h}(x)) \leq n$ such that:

$$\Pi_n(R(\tilde{d}(x), \tilde{h}(x))) = D_n$$

*We call $R(\tilde{d}(x), \tilde{h}(x))$ **the canonical Riordan representative** of the finite Riordan matrix D_n.*

Proof: Let $R(d(x), h(x)) = (b_{ij})_{0\leq i,j<\infty}$, $R(u(x), v(x)) = (c_{ij})_{0\leq i,j<\infty}$.

- The entries:
$$b_{00}, \ldots, b_{n0} \quad c_{00}, \ldots, c_{n0}$$
determine univocally $Taylor_n(d(x))$ and $Taylor_n(u(x))$
- The entries:
$$b_{11}, \ldots, b_{n1} \quad c_{11}, \ldots, c_{n1}$$
determine univocally $Taylor_n(d(x) \cdot h(x))$ and $Taylor_n(u(x) \cdot v(x))$.
- To complete the proof, note that:
$$\begin{cases} Taylor_n(d(x)) = Taylor_n(u(x)) \\ Taylor_n(d(x) \cdot h(x)) = Taylor_n(u(x) \cdot v(x)) \end{cases} \Leftrightarrow \begin{cases} Taylor_n(d(x)) = Taylor_n(u(x)) \\ Taylor_n(h(x)) = Taylor_n(v(x)) \end{cases}$$

A proof in terms of the A-sequence is also possible: if two matrices have the same projection, they obviously have the same A-sequence and the proof of the result follows from the first point in this proof and from Lagrange inversion formula (3).

$\square$

The previous result leads to the following consequence which is a very important structural property of infinite and bi-infinite Riordan matrices:

Proposition 1.2.3 (The equality of banded matrices) *Let two finite Riordan matrices $R(d(x), h(x)), R(u(x), v(x))$ satisfying for certain $n \geq 1$:*

$$R_n(d(x), h(x)) = R_n(u(x), v(x)) \qquad \begin{bmatrix} \bullet & & & & & & \\ \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & \\ \bullet & \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

or even the weaker condition:

$$\begin{cases} R_{n-1}(d(x), h(x)) = R_{n-1}(u(x), v(x)) \\ R_{n-1}\left(d(x) \cdot \frac{h(x)}{x}, h(x)\right) = R_{n-1}\left(u(x) \cdot \frac{v(x)}{x}, v(x)\right) \end{cases} \qquad \begin{bmatrix} \bullet & & & & & & \\ \bullet & \bullet & & & & & \\ \bullet & \bullet & \bullet & & & & \\ \bullet & \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

then:

(i) The terms $0, \ldots, n-1$ of the A-sequence and g-sequence of both matrices coincide.

(ii) $\forall m \in \mathbb{Z}$, $R_{n-1}\left(d(x) \cdot \left(\frac{h(x)}{x}\right)^m, h(x)\right) = R_{n-1}\left(u(x) \cdot \left(\frac{v(x)}{x}\right)^m, v(x)\right)$.

(iii) So in particular, for the entire bi-infinite matrices:

$$R(d(x), h(x)) = (c_{ij})_{i,j \in \mathbb{Z}}, \qquad R(u(x), v(x)) = (d_{ij})_{i,j \in \mathbb{Z}}$$

we have the equality of the entire band, that is, that for $k = 0, \ldots, n-1$

$$d_{i,j} = c_{i,j} \quad \textit{for} \quad i - j = k \qquad \begin{bmatrix} \ddots & & & & & & \\ \ddots & \bullet & & & & & \\ \cdots & \bullet & \bullet & & & & \\ \cdots & \bullet & \bullet & \bullet & & & \\ \cdots & \bullet & \bullet & \bullet & \bullet & & \\ \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix}$$

Proof: We have that:

(i) follows directly from proposition 1.1.4.

(ii) is immediately equivalent to (iii).

(iii) If the terms $0, \dots, n-1$ of the A-sequence are fixed and coincide in both matrices, obviously from the formula (1.7.9) of the A-sequence all the entries c_{ij} d_{ij} for $0 \leq i-j \leq n-1$ are determined and are equal for $j \geq 0$. To prove the result for $j < 0$, we apply the same argument with the terms $0, \dots, n-1$ of the g-sequence according to the formula (7).

□

1.3 Extending an involution in $\mathcal{R}_n(\mathbb{K})$ to obtain an involution in $\mathcal{R}_{n+1}(\mathbb{K})$

As we described the Riordan group as an inverse limit of the finite Riordan groups, it is natural to ask for all extensions in $\mathcal{R}_{n+1}$ of any given $D_n \in \mathcal{R}_n$.

Proposition 1.3.1 *We will study the equation $P_n(D_{n+1}) = D_n$ with $D_{n+1} \in \mathcal{R}_{n+1}$ and a fixed $D_n \in \mathcal{R}_n$. If $D_n = (d_{i,j})_{i,j=0,1,\cdots,n} \in \mathcal{R}_n$ with $n \geq 1$, then $P_n(D_{n+1}) = D_n$ if and only if*

$$D_{n+1} = \begin{bmatrix} d_{0,0} & & & & & & & \\ d_{1,0} & d_{1,1} & & & & & & \\ d_{2,0} & d_{2,1} & d_{2,2} & & & & & \\ d_{3,0} & d_{3,1} & d_{3,2} & d_{3,3} & & & & \\ d_{4,0} & d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & & \\ d_{n,0} & d_{n,1} & d_{n,2} & d_{n,3} & d_{n,4} & \cdots & d_{n,n} & \\ \alpha & \beta & d_{n+1,2} & d_{n+1,3} & d_{n+1,4} & \cdots & d_{n+1,n} & d_{n+1,n+1} \end{bmatrix} \quad \text{with} \quad \alpha, \beta \in \mathbb{K}$$

and $d_{n+1,j}$, $j \geq 2$ are univocally determined by the matrix D_n.

Proof: This is a direct corollary of the equality of banded matrices.

□

Remark 1.3.2 *Suppose that in the construction above $D_n = R_n(d(x), h(x))$. Any D_{n+1} with $\Pi_n(D_{n+1}) = D_n$ is of the type $R_n(\widetilde{d}(x), \widetilde{h}(x))$ with $Taylor_n(\widetilde{d}(x)) = Taylor_n(d(x))$ and $Taylor_n(\widetilde{h}(x)) = Taylor_n(h(x))$.*

Determining in the matrix D_{n+1} that the entry $(n+1, 1)$ is β is equivalent to determine $[x^{n+1}]\widetilde{h}(x)$ or equivalently the last term in the A-sequence of D_{n+1} which is of the type $(a_0, \dots, a_n)$.

Similarly, determininng that the entry $(n+1, 1)$ is α is equivalent to determine $[x^{n+1}]\widetilde{d}(x)$.

In this work, many times we are going to proceed by induction. So, we will need to extend a given finite Riordan matrix in $\mathcal{R}_n$ to others in $\mathcal{R}_{n+1}$.

1.4 $\mathcal{R}(\mathbb{K})$ as an inverse limit

Apart from the sequence of groups $\{\mathcal{R}_0, \mathcal{R}_1, \ldots\}$, in order to obtain an inverse sequence the bonding maps are also required:

Definition 1.4.1 *We define the maps:*

$$P_n : \mathcal{R}_{n+1} \to \mathcal{R}_n$$

$$(d_{ij})_{0\le i,j\le n+1} = \left[\begin{array}{ccc|c} d_{00} & & & \\ \vdots & \ddots & & \\ d_{n0} & \ldots & d_{nn} & \\ \hline d_{n+1,0} & \ldots & d_{n+1,n} & d_{n+1,n+1} \end{array}\right] \longmapsto \begin{bmatrix} d_{00} & & \\ \vdots & \ddots & \\ d_{n0} & \ldots & d_{nn} \end{bmatrix} = (d_{ij})_{0\le i,j\le n}$$

$P_n(D_{n+1})$ is obtained from D_{n+1} by deleting its last row and its last column.

It is easy to see that P_n is a group homomorphism for every n because the matrices are lower triangular. Moreover, the diagram below is commutative:

$$\begin{tikzcd} \mathcal{R} \arrow[d, "\Pi_n"'] \arrow[dr, "\Pi_{n+1}"] & \\ \mathcal{R}_n & \mathcal{R}_{n+1} \arrow[l, "P_n"] \end{tikzcd}$$

Consequently we get:

Theorem 1.4.2 *The Riordan group $\mathcal{R}$ is isomorphic to $\varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$.*

Proof: According to definition 0.1.1 of the inverse limit, we only need to see that the natural map:

$$D = R(d(x), h(x)) = \begin{bmatrix} d_{00} & & & \\ d_{10} & d_{11} & & \\ d_{20} & d_{21} & d_{22} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \longmapsto$$

$$\longmapsto (\Pi_0(D),\ \Pi_1(D),\ \Pi_1(D),\ \ldots) = \left([d_{00}], \quad \begin{bmatrix} d_{00} & \\ d_{10} & d_{11} \end{bmatrix} \begin{bmatrix} d_{00} & & \\ d_{10} & d_{11} & \\ d_{20} & d_{21} & d_{22} \end{bmatrix}, \quad \ldots \right)$$

is an isomorphism. It is obvious that it is an injective homomorphism. In order to see that this map is onto we just see that, by using proposition 1.2.2, an element:

$$(D_0, D_1, D_2, \ldots) \in \varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$$

is the image of $R(d(x), h(x))$ where $[x^n]d(x)$ and $[x^n]h(x)$ are determined (in a compatible way, since $P_m(D_{m+1}) = D_m$) by any of the D_N with $N \geq n$.

□

The above proposition means that a Riordan matrix can be uniquely described by a sequence of finite matrices $(D_n)_{n\in\mathbb{N}}$ with $D_n \in \mathcal{R}_n$ (corresponding to the projections) and such that $P_n(D_{n+1}) = D_n$ for every $n \in \mathbb{N}$. Furthermore, the product in the Riordan group corresponds to the component-wise products in the sequences.

1.5 Finite matrices and metrics in $\mathcal{R}(\mathbb{K})$

In [66] an ultrametric for $\mathcal{R}$ was introduced. Intuitively, this ultrametric measures the distance between two Riordan matrices looking at the first different row. This ultrametric can be trivially formulated in terms of the maps Π_n:

Proposition 1.5.1 *Let $A, B \in \mathcal{R}$, then:*

$$d^*(A,B) = \frac{1}{2^n} \text{ where } n = \min\{k : \Pi_k(A) \neq \Pi_k(B)\}$$

Moreover:

Remark 1.5.2 *Denote only for a moment by φ to the isomorphism between $\mathcal{R}$ and $\varprojlim(\mathcal{R}_n, P_n)$. Through φ the distance d^* induces a well known metric for sequences:*

$$d^{**}((A_0, A_1, A_2, \ldots), \quad (B_0, B_1, B_2, \ldots)) = \frac{1}{2^n} \text{ where } n = min\{k : A_k \neq B_k\}$$

From the idea of the equality of banded submatrices, it is possible to define another metric for $\mathcal{R}$:

Definition 1.5.3 *For $A, B \in \mathcal{R}$, we will define:*

$$d^{**}(A,B) = \frac{1}{2^n} \text{ where } n = min\{k : \text{ the } k\text{-th diagonals of } A, B \text{ are distinct}\}$$

where the k-th diagonal is the one passing through $d_{k0}, d_{k+1,1}, \ldots$

We can see that there is a relation between both matrices:

Remark 1.5.4 *It is easy to see that d^* and d^{**} are Lipschitz equivalent: we have that for all $A, B \in \mathcal{R}$:*

$$\frac{1}{2} d^{**}(A,B) \leq d^*(A,B) \leq d^{**}(A,B)$$

Check this inequality for example with the matrices:

$$A = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 1 & 1 & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We have not explored this metric d^{**} yet, but we consider that it may have aspects to be studied as done with d^{*} in [66]. For example:

Remark 1.5.5 *As it was already announced in* [66]*, the ultrametric d^{*} induces a sequence of normal subgroups:*

$$G_k = \{T \in \mathcal{R} : d^{*}(T, R(1,x)) \leq \frac{1}{2^k}\}$$

with the property that:

$$\bigcup_{k=0}^{\infty} G_k = \{R(1,x)\}$$

Naturally we could do an analogue definition for d^{**}. This, together with the study of the importance of those sequences of normal subgroups will be left as open question 2.

1.6 Reflection of Finite Riordan Matrices

Once this inverse limit context has been set, we will jump into the question about simmetries and reflections.

The usual reflection studied for matrices is usually transposition. Matrix transposition can be viewed as a reflection on the matrix across "$y = -x$". But the transpose of a Riordan matrix is not a Riordan matrix (it is not even lower triangular).

However, if we reflect a Riordan matrix across "$y = x$" we obtain another lower triangular matrix:

$$\begin{bmatrix} a_{00} & & \\ a_{10} & a_{11} & \\ a_{20} & a_{21} & a_{22} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{22} & & \\ a_{21} & a_{11} & \\ a_{20} & a_{10} & a_{00} \end{bmatrix}$$

$$\begin{bmatrix} a_{00} & & & \\ a_{10} & a_{11} & & \\ a_{20} & a_{21} & a_{22} & \\ a_{30} & a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{33} & & & \\ a_{32} & a_{22} & & \\ a_{31} & a_{21} & a_{11} & \\ a_{30} & a_{20} & a_{10} & a_{00} \end{bmatrix}$$

which in fact is again a Riordan matrix:

Proposition 1.6.1 (The reflected Riordan matrix) *Let $D = (d_{ij})_{0\leq i,j\leq n}$ be a finite Riordan matrix and consider the matrix $D^R = (c_{ij})_{0\leq i,j\leq n}$ with $c_{i,j} = d_{n-j,n-i}$. Then D^R is a finite Riordan matrix that we call the* ***reflected matrix*** *of D. Moreover, the A-sequence of D is the g-sequence of D^R and viceversa.*

Proof: Let $a_0, \ldots, a_{n-1}$ be the partial A-sequence of D_n. Then, according to (6) we have that for all $0 < j \leq i \leq n$:

$$c_{ij} = d_{n-j,n-i} = a_0 d_{n-j-1,n-i-1} + \ldots + a_{i-j} d_{n-j-1,n-j-1} =$$

$$= a_0 c_{i+1,j+1} + \ldots + a_{i-j} c_{j+1,j+1}$$

So $a_0, \ldots, a_{n-1}$ acts as a g-sequence. The fact that this reflected matrix is a Riordan matrix follows from the fact that an element in ILT_{n+1} is a finite Riordan matrix if and only if it has a g-sequence (proposition 1.1.4).

□

Remark 1.6.2 *Due to the importance of the A, g-sequences for those kind of manipulations the notation $T(f \mid g)$ introduced by the advisors of this work and discussed in subsection 0.3.4 is sometimes very convenient to work with reflections, since it is easier to recover information about those sequences in this notation.*

Remark 1.6.3 *The constants used to construct D_n^R by rows are the same as those used to construct D by columns. Moreover, if $D_n = R_n(d(x), h(x))$ the first column of D_n^R can be calculated by the expression*

$$c_{i0} = d_{n,n-i} = [x^i] d(x) \cdot (h(x))^n. \tag{1.3}$$

Let us include an example of reflection:

Example 1.6.4 *Let P be the Pascal Triangle (introduced in example 0.3.3). As usual, denote $P_5 = \Pi_5(P)$. Then:*

$$P_5^R = \begin{pmatrix} 1 & & & & & \\ 5 & 1 & & & & \\ 10 & 4 & 1 & & & \\ 10 & 6 & 3 & 1 & & \\ 5 & 4 & 3 & 2 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$P_5^R = \Pi_5(T((1+x)^6 \mid 1+x)) = \Pi_5\left(\mathcal{R}\left((1+x)^5, \frac{x}{1+x}\right)\right).$$

Some properties that can be deduced from the above proposition are:

Corollary 1.6.5 *Let $D_n, C_n \in \mathcal{R}_n$:*

(i) $(D^R)^R = D$.

(ii) $(D_n C_n)^R = C_n^R D_n^R$.

(iii) $(D_n^R)^{-1} = (D_n^{-1})^R$.

(iv) If D_n a Toeplitz matrix, then $D_n = D_n^R$.

(v) $D_n D_n^R$ is a Toeplitz matrix, i.e., $D_n D_n^R = R_n(d(x), x)$. In particular:

$$D^{-1} = D_n^R \cdot R_n\left(\frac{1}{d(x)}, x\right)$$

Proof: (i), (ii) and (iv) are a direct consequence of the definition of reflected matrix in proposition 1.6.1. (iii) follows from (ii) since:

$$R(1,x) = (D_n \cdot D_n^{-1})^R = D_n^R \cdot (D_n^{-1})^R$$

In order to prove (v), just see that as we said before, the A-sequence of D_n is the g-sequence of D_n^R and according to the relation between the A-sequence and the g-sequence discussed in proposition 0.3.12 we have that if $D_n = R_n(u(x), v(x))$, then D_n^R is of the type $R_n(\widetilde{u}(x), v^{-1}(x))$ for some $\widetilde{u}(x) \in \mathcal{F}_0$ (more will be said about this $\widetilde{u}(x)$).

□

1.7 The Riordan group in its bi-infinite representation. Complementary and Dual Riordan matrices.

As already mentioned in section 0.3, it is possible to consider the Riordan group as a subgroup of the set of invertible lower triangular bi-infinite matrices $ILT_{\infty\infty}(\mathbb{K})$. This representation is called the **bi-infinite representation of the Riordan group**. The elements in $ILT_{\infty\infty}(\mathbb{K})$ are of the type:

$$(d_{ij})_{-\infty<i,j<\infty} = \begin{bmatrix} \ddots & & & & \\ \cdots & d_{-1,-1} & & & \\ \cdots & d_{0,-1} & d_{00} & & \\ \cdots & d_{1,-1} & d_{10} & d_{11} & \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad \text{with } d_{ii} \neq 0,\ \forall i \in \mathbb{Z}$$

Note that:

Remark 1.7.1 *In order to define the multiplication, it is important for these bi-infinite matrices the assumption that the entries are labelled, that is, there is a row and a column labelled as 0-column and 0-row used as reference. Then, since they are lower triangular, we can define the product as usual:*

$$\cdot : ILT_{\infty\infty} \times ILT_{\infty\infty} \longrightarrow ILT_{\infty\infty}$$

$$(b_{ij})_{-\infty\leq i,j<\infty} \cdot (d_{ij})_{-\infty\leq i,j<\infty} = (p_{ij})_{-\infty\leq i,j<\infty} \text{ where } p_{ij} = \sum_{k=-\infty}^{\infty} b_{ik}d_{kj}$$

This product is well defined since only a finite set of elements in this sum are non-zero.

The first papers to present this bi-infinite representation are [64] and [65]. But the idea of using bi-infinite matrices appeared previously in the bibliography (see for example the paper by E. Jabotinsky [49]).

Definition 1.7.2 *A bi-infinite lower-triangular matrix $(d_{ij})_{-\infty<i,j<\infty}$ with entries in a field $\mathbb{K}$ is a* ***bi-infinite Riordan matrix****, if there exist two power series:*

$$d(x) \in \mathcal{F}_0(\mathbb{K}),\ h(x) \in \mathcal{F}_1(\mathbb{K}) \tag{1.4}$$

in which case we will denote it by $R_{\infty\infty}(d(x), h(x))$, such that the generating function of the j-column $\begin{bmatrix} d_{0,j} \\ d_{1,j} \\ \vdots \end{bmatrix}$ is $d(x)\cdot(h(x))^j$, that is:

$$d(x)\cdot(h(x))^j = \sum_{k=0}^{\infty} d_{kj}x^k$$

The Riordan bi-infinite matrix $R_{\infty\infty}(d(x), h(x))$ does not contain more algebraic information than the infinite one $R(d(x), h(x))$. But due to the combinatorial aspects, bi-infinite matrices deserve to be studied.

Again, the set of all bi-infinite Riordan matrices with respect to the product is a group, called the **bi-infinite Riordan group**. Also **improper invertible bi-infinite Riordan matrices** can be defined. This time, the (1FT) is stated as:

Proposition 1.7.3 (1FT for bi-infinite matrices) *Let $R_{\infty\infty}(d(x), h(x))$ be a bi-infinite Riordan matrix, let:*

$$f(x) = \frac{f_{-k}}{x^k} + \ldots + \frac{f_{-1}}{x} + f_0 + f_1x + \ldots \in \mathbb{K}((x))$$

Then if:

$$\begin{bmatrix} \vdots \\ 0 \\ w_{-k} \\ w_{-k+1} \\ \vdots \end{bmatrix} = R_{\infty\infty}(d(x), h(x)) \begin{bmatrix} \vdots \\ 0 \\ f_{-k} \\ f_{-k+1} \\ \vdots \end{bmatrix}$$

we have that the generating function of the sequence $(w_{-k}, w_{-k+1}, \ldots)$ *satisfies:*

$$\frac{w_{-k}}{x^k} + \ldots + \frac{w_{-1}}{x} + w_0 + w_1 x + w_2 x^2 + \ldots = \frac{d(x)}{(h(x))^k} \cdot f(h(x))$$

According to this, we again have an interpretation of the group operation and of the inverse. It is easy to see that, again, an element in $ILT_{\infty\infty}$ is a Riordan matrix if and only if there exists an A-sequence, and the entries satisfy (6).

In view of this, we have that:

Proposition 1.7.4 *The map* $\mathcal{R} \to \mathcal{R}_{\infty\infty}$ *given by:*

$$R(d(x), h(x)) \longmapsto R_{\infty\infty}(d(x), h(x))$$

is a group isomorphism.

We will skip the proof. The ismorphism between $\mathcal{R}$ and $\mathcal{R}_{\infty\infty}$ also allow us to push forward the ultrametrics discussed in section 1.5 to $\mathcal{R}_{\infty\infty}$. We will omit details.

Since the elements in $\mathcal{R}_{\infty\infty}$ are objects with combinatorial meaning, it is reasonable to study some simmetries in the entries on the matrices. The bases for this study may be found in [65]. In this work, for a given bi-infinite matrix D, the $[m]$-complementary of D is presented. But later in section 4, the attention is restricted to the cases $m = 0, -1$.

Definition 1.7.5 *Let:*

$$D_{\infty\infty} = (d_{ij})_{-\infty<i,j<\infty} = \begin{bmatrix} \ddots & & & & \\ \ldots & d_{00} & & & \\ \ldots & d_{10} & d_{11} & & \\ \ldots & d_{20} & d_{21} & d_{22} & \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in \mathcal{R}_{\infty\infty}$$

- *The **dual** of* $D_{\infty\infty}$ *(not named like this in* [65]*) is the matrix denoted by* $D^{\diamondsuit}_{\infty\infty} = (c_{ij})_{-\infty<i,j<\infty}$ *and given by* $c_{kn} = d_{-m,-k}$

$$\begin{bmatrix} \ddots & & & & \\ \ldots & d_{00} & & & \\ \ldots & d_{0,-1} & d_{-1,-1} & & \\ \ldots & d_{0,-2} & d_{-1,-2} & d_{-2,-2} & \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- *The **complementary** of $D_{\infty\infty}$ (not named like this in* [65]*) is the matrix denoted by $D^{\perp}_{\infty\infty} = (c_{ij})_{-\infty<i,j<\infty}$ and given by $c_{kn} = d_{-1-m,-1-k}$*

$$\begin{bmatrix} \ddots & & & & \\ \cdots & d_{-1,-1} & & & \\ \cdots & d_{-1,-2} & d_{-2,-2} & & \\ \cdots & d_{-1,-3} & d_{-2,-3} & d_{-3,-3} & \\ & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We will now recall some results needed for the rest of this chapter. As we will recall in a moment, we have that $D^{\diamondsuit}_{\infty\infty}, D^{\perp}_{\infty\infty}$ belong to $\mathcal{R}_{\infty\infty}$. Due to this reflection in the bi-infinite Riordan matrices, for any infinite Riordan matrix we associate two infinite Riordan matrices:

Proposition 1.7.6 (section 4 in [65]**)** *Let:*

$$D = R(d(x), h(x)) \quad D_{\infty\infty} = R_{\infty\infty}(d(x), h(x))$$

and:

$$D^{\diamondsuit}_{\infty\infty} = (c_{ij})_{-\infty<i,j<\infty}, \qquad D^{\perp}_{\infty\infty} = (d_{ij})_{-\infty<i,j<\infty}$$

We define:

- *The **dual** of D is the matrix $D^{\diamondsuit} = (c_{ij})_{0\leq i,j<\infty}$. We have that:*

$$D^{\diamondsuit} = R\left(d(h^{-1}(x)) \cdot (h^{-1}(x))' \cdot \frac{x}{h^{-1}(x)}, h^{-1}(x)\right) = R\left(\frac{d(h^{-1}(x))}{h'(h^{-1}(x))} \cdot \frac{x}{h^{-1}(x)}, h^{-1}(x)\right)$$

- *The **complementary** of D is the matrix $D^{\perp} = (d_{ij})_{0\leq i,j<\infty}$. We have that:*

$$D^{\perp} = R\left(d(h^{-1}(x)) \cdot (h^{-1}(x))', h^{-1}(x)\right) = R\left(\frac{d(h^{-1}(x))}{h'(h^{-1}(x))}, h^{-1}(x)\right)$$

Remark 1.7.7 (Section 4 in [65]**)** *There is an obvious relation between the dual and the complementary matrices: $D^{\perp}$ is the matrix that we obtain if we delete the first row and column from $D^{\diamondsuit}$.*

We also have the following:

Proposition 1.7.8 (Involutory condition, corollary 3.5 in [65]**)** *For any infinite or bi-infinite Riordan matrix D:*

$$(D^{\diamondsuit})^{\diamondsuit} = D \qquad (D^{\perp})^{\perp} = D$$

Finally, note that analogously to what we had for infinite Riordan matrices:

Proposition 1.7.9 ([65]) *An element in $D_{\infty\infty} = (d_{ij})_{-\infty<i,j<\infty} ILT_{\infty\infty}$ is a bi-infinite Riordan matrix if and only if one of the following two equivalent conditions hold:*

(1) There is a sequence $(a_0, a_1, a_2, \ldots)$ called the ***A-sequence*** *such that, for every n, k:*

$$d_{nk} = a_{00} d_{n-1,k-1} + \ldots + a_{n-k} d_{n-1,n-1}$$

Moreover, If $D_{\infty\infty} = R_{\infty\infty}(d(x), h(x))$, then the generating function of the A-sequence is $A(x)$ where:

$$h(x) = \left(\frac{x}{A(x)}\right)^{-1}$$

(2) There is a sequence $(g_0, g_1, g_2, \ldots)$ called the ***g-sequence*** *such that:*

$$\forall n, k, \qquad d_{nk} = g_0 d_{n+1,k+1} + g_1 d_{n,k+1} + \ldots + g_{n-k} d_{k+1,k+1}$$

If $D_{\infty\infty} = R(d(x), h(x))$, then the generating function of the g-sequence, $g(x)$, satisfies:

$$h(x) = \frac{x}{g(x)}$$

In this case, the g-sequence is just the A-sequence of the complementary and the dual of $D_{\infty\infty}$.

1.8 Reflections and complementary and dual Riordan matrices

In Definition 1.4.1 we constructed the maps $P_n(D_{n+1})$ that delete the last row and the last column of D_{n+1}. Anologously, we will consider:

Definition 1.8.1 *We define:*

$$Q_n : \mathcal{R}_{n+1} \to \mathcal{R}_n$$

by:

$$Q_n((d_{i,j})_{0 \le i,j \le n+1}) = (\tilde{d}_{i,j})_{i,j=0,\cdots,n} \quad \textit{with} \qquad \tilde{d}_{i,j} = d_{i+1,j+1}$$

that is:

$$\left[\begin{array}{c|ccc} d_{00} & & & \\ \hline d_{10} & d_{11} & & \\ \vdots & \vdots & \ddots & \\ d_{n+1,0} & d_{n+1,1} & \ldots & d_{n+1,n+1} \end{array}\right] \longmapsto \begin{bmatrix} d_{11} & & \\ \vdots & \ddots & \\ d_{n+1,1} & \ldots & d_{n+1,n+1} \end{bmatrix}$$

The fact that $Q_n(D_{n+1}) \in \mathcal{R}_n$, when $D_{n+1} \in \mathcal{R}_{n+1}$, is immediate: the columns of D_{n+1} and of $Q_n(D_{n+1})$ form two geometric progressions of the same ratio. The difference between both matrices is the first term of this progression. We can see that an equivalent way to describe those maps is the following:

$$Q_n(R_{n+1}(d(x), h(x))) = R_n\left(d(x) \cdot \frac{h(x)}{x}, h(x)\right) \tag{1.5}$$

It is easy to see that Q_n is a group homomorphism. Therefore, we get the following result:

Proposition 1.8.2 *$(D_n)_{n\in\mathbb{N}} \in \varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$ if and only if $D_n^R = Q_n(D_{n+1}^R)$ for all $n \in \mathbb{N}$.*

Proof: Recall that $(D_n)_{n\in\mathbb{N}} \in \varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$ if and only if for all $n \in \mathbb{N}$, $P_n(D_{n+1}) = D_n$, that is, D_n is obtained by deleting the last column and the last row in D_{n+1}. This is equivalent to delete the first row and the first column in D_{n+1}^R.

$\square$

Thanks to this result we will be able to relate reflections in finite matrices to the dual and complementary Riordan matrices:

Theorem 1.8.3 *Let $D = R(d(x), h(x))$ be any Riordan matrix. Then:*

(i) The sequence:

$$\left(\left(R_n\left(d(x) \cdot \left(\frac{x}{h(x)}\right)^{n+1}, \quad h(x)\right)\right)^R\right)_{n\in\mathbb{N}} \tag{1.6}$$

is a Riordan matrix. In fact, the above sequence, as an element in $\varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$, is the complementary Riordan matrix of D, that is $D^\perp$.

(ii) The sequence:

$$\left(\left(R_n\left(d(x) \cdot \left(\frac{x}{h(x)}\right)^{n}, h(x)\right)\right)^R\right)_{n\in\mathbb{N}}$$

is a Riordan matrix. In fact, the above sequence, as an element in in $\varprojlim\{(\mathcal{R}_n)_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$, is the dual Riordan matrix of D, that is $D^\diamondsuit$.

Proof: To prove (i), it suffices to show that it coincides with the complementary Riordan matrix, since the complementary Riordan matrix is a Riordan matrix (anyway, checking that the sequence in (1.6) belongs to the inverse limit $\varprojlim(\mathcal{R}_n, P_n)$ is easy by using the previous result). See that the entry $(n-j, n-i)$ in:

$$R_n\left(d(x)\cdot\left(\frac{x}{h(x)}\right)^{n+1}, \quad h(x)\right)$$

is:

$$[x^{n-j}]\left(d(x)\left(\frac{x}{h(x)}\right)^{n+1}\cdot(h(x))^{n-i}\right)=[x^{-1-j}]\left(d(x)\frac{1}{(h(x))^{i+1}}\right)$$

which is, by definition of reflexion the entry (i,j) in:

$$\left(R_n\left(d(x)\cdot\left(\frac{x}{h(x)}\right)^{n+1}, \quad h(x)\right)\right)^R$$

In order to prove that this matrix is actually the complementary, just see that this entry (i,j) is the entry $(-1-j,-1-i)$ of $R_{\infty\infty}(d(x),h(x))$. The proof of (ii) is totally analogous. □

Remark 1.8.4 *We have not introduced them here, but in fact any* $[m]$*-complementary in the sense of* [65] *can be described in a similar way.*

1.9 $\mathcal{R}_{\infty\infty}$ as an inverse limit I: from infinite to bi-infinite Riordan matrices

As we showed previously, the Riordan group $\mathcal{R}$ is, in some sense, the asymptotic behaviour of the inverse sequence $\{(\mathcal{R}_n)_{n\in\mathbb{N}},(P_n)_{n\in\mathbb{N}}\}$. Similarly, we can study the asymptotic behaviour of the sequence of homomorphisms $(Q_n)_{n\in\mathbb{N}}$. In this sense, by using the definition of Q_n, we easily have:

Proposition 1.9.1 *There is an unique isomorphism* $\Phi:\mathcal{R}\to\mathcal{R}$ *such that the following diagram commutes for all* $n\geq 0$*:*

$$\begin{tikzcd} \mathcal{R} \arrow[r, "\Phi"] \arrow[d, "\Pi_{n+1}"] & \mathcal{R} \arrow[d, "\Pi_n"] \\ \mathcal{R}_{n+1} \arrow[r, "Q_n"] & \mathcal{R}_n \end{tikzcd}$$

The action of this isomorphism on any infinite Riordan matrix can also be understood as deleting the first row and column, or equivalently, it can be defined as:

$$\Phi(R(d(x),h(x))=R\left(d(x)\cdot\frac{h(x)}{x},h(x)\right)$$

This isomorphism was first used in [65]. Now we are going to use this result to get the bi-infinite representation of the Riordan group found in [64] by a different approach and using again the concept of inverse limit of an inverse sequence.

Proposition 1.9.2 *The Riordan group $\mathcal{R}$ is isomorphic to the $\varprojlim \{G_n, \Psi_n\}_{n\in\mathbb{N}}$, where $G_n = \mathcal{R}$ and $\Psi_n = \Phi$ for every $n \in \mathbb{N}$.*

Proof: Let $\mathcal{G} = \varprojlim(G_n, \Psi_n)_{n\in\mathbb{N}}$. Consider the map $\tau : \mathcal{G} \longrightarrow G_0$ (which obviously preserves the operation), let us prove that this map is an isomorphism.

- Note first that $\alpha \in \varprojlim \mathcal{G}$ if and only if there is a $R(d(x), h(x)) \in \mathcal{R}$ such that

  $$\alpha = \left(R(d(x), h(x)), R\left(d(x)\cdot\left(\frac{h(x)}{x}\right), h(x)\right), \ldots, R\left(d(x)\left(\frac{h(x)}{x}\right)^n, h(x)\right)\ldots\right)$$

  So it is now obvious that τ is onto.

- In order to prove the injectivity, take $\alpha \in \ker\tau$, that is $\tau(\alpha) = R(1, x)$. Consequently:

  $$\alpha = (R(1, x), R(1, x), \ldots, R(1, x), \ldots)$$

  which is the neutral element in $\mathcal{G}$.

$\square$

The above construction allows us to get the following representation of the elements in the Riordan group. Let $\alpha \in \varprojlim \mathcal{G}$ and $R(d(x), h(x)) \in \mathcal{R} = G_0$ be its 0-coordinate. We have that:

$$\begin{array}{cccccccccc} \ldots & \xrightarrow{\Phi} & \mathcal{R} & \xrightarrow{\Phi} & \mathcal{R} & \xrightarrow{\Phi} & \ldots & \xrightarrow{\Phi} & \mathcal{R} & \xrightarrow{\Phi} \mathcal{R} \\ \alpha = \Big(\cdots, & & R\left(d\cdot\left(\frac{h}{x}\right)^{n+1}, h\right), & & R\left(d\cdot\left(\frac{h}{x}\right)^{n}, h\right), & & \cdots & & R\left(d\cdot\frac{h}{x}, h\right), & R(d, h)\Big) \end{array}$$

Then:

- The 0-approximation of α is $\Big(R(d(x), h(x))\Big)$.
- The 1-approximation is $\left(R(d(x), h(x), R\left(d(x)\cdot\frac{h(x)}{x}, h(x)\right)\right)$ with:

  $$\Phi\left(R\left(d(x)\cdot\frac{h(x)}{x}, h(x)\right)\right) = (R(d(x), h(x)$$

  that is $R\left(d(x)\cdot\frac{h(x)}{x}, h(x)\right)$ is obtained from the 0-approximation by adding adequately a column (to the left) and a row (on the top).

In a similar way, we can associate a unique matrix, the matrix $R\left(d(x)\cdot\left(\frac{h(x)}{x}\right)^n, h(x)\right)$, with the n-approximation which is:

$$\left(R(d(x),h(x)), R\left(d(x)\cdot\left(\frac{h(x)}{x}\right)^{n-1}, h(x)\right), \ldots, R\left(d(x)\cdot\left(\frac{h(x)}{x}\right)^{n}, h(x)\right)\right)$$

Then now, it is very natural to identify α with the bi-infinite matrix $R_{\infty\infty}(d(x),h(x)) = (d_{n,k})_{n,k\in\mathbb{Z}}$ where

$$d_{n,k} = [x^n]d(x)\cdot(h(x))^k \quad n,k\in\mathbb{Z},\ n\geq k.$$

Moreover, the product of two elements of $\varprojlim \mathcal{G}$ turns into the usual row-by-column product of the corresponding bi-infinite representations.

Remark 1.9.3 *It is clear that Φ induces an isomorphism, denoted again by Φ, in $\mathcal{R}_{\infty\infty}$ given by:*

$$\Phi(R_{\infty\infty}(d(x),h(x))) = R_{\infty\infty}\left(d(x)\cdot\frac{h(x)}{x}, h(x)\right)$$

or equivalently:

$$\Phi((d_{n,k})_{n,k\in\mathbb{Z}}) = (d_{n+1,k+1})_{n,k\in\mathbb{Z}}$$

Note that the action of Φ changes the location of the reference axis in the bi-ininite matrix, that is, the entry considered as (0,0).

1.10 $\mathcal{R}_{\infty\infty}$ as an inverse limit II: from finite to bi-infinite Riordan matrices

At this time we are going to get the bi-infinite representation of the Riordan group in a new way, by using only finite Riordan matrices.

Definition 1.10.1 *For $n\geq 0$, we define the projection $\gamma_n^{even}:\mathcal{R}_{\infty\infty}\to\mathcal{R}_{2n}$ by:*

$$\gamma_n^{even}(R_{\infty\infty}(d(x),h(x))) = R_{2n}\left(d(x)\cdot\left(\frac{h(x)}{x}\right)^n, h(x)\right)$$

Thus, for $D_{\infty\infty}\in\mathcal{R}_{\infty\infty}$ we can define the sequence:

$$(\gamma_0^{even}(D_{\infty\infty}), \gamma_1^{even}(D_{\infty\infty}), \gamma_2^{even}(D_{\infty\infty}), \ldots)$$

which is a sequence of matrices of the type:

$$\left([d_{00}],\quad \begin{bmatrix} d_{-1,-1} & & \\ d_{0,-1} & d_{00} & \\ d_{1,-1} & d_{1,0} & d_{11}\end{bmatrix},\quad \begin{bmatrix} d_{-2,-2} & & & & \\ d_{-1,-2} & d_{-1,-1} & & & \\ d_{0,-2} & d_{0,-1} & d_{00} & & \\ d_{1,-2} & d_{1,-1} & d_{1,0} & d_{11} & \\ d_{2,-2} & d_{2,-1} & d_{2,0} & d_{2,1} & d_{22}\end{bmatrix},\quad \ldots\right) \tag{1.7}$$

Remark 1.10.2 *Note that if $n \geq 0$, since the diagram:*

$$\begin{tikzcd} \mathcal{R}_{n+1} \arrow[r, "P_n"] \arrow[d, "Q_n"] & \mathcal{R}_n \arrow[d, "Q_{n-1}"] \\ \mathcal{R}_n \arrow[r, "P_{n-1}"] & \mathcal{R}_{n-1} \end{tikzcd}$$

is commutative, we have that:

$$\gamma_n^{even} = Q_{2n} \circ P_{2n+1} \circ \gamma_{n+1}^{even} = P_{2n} \circ Q_{2n+1} \circ \gamma_{n+1}^{even}$$

So we will define $\mu_{n-1}^{even} : \mathcal{R}_{2n} \to \mathcal{R}_{2n-2}$ as:

$$\mu_{n-1}^{even} = Q_{2n-1} \circ P_{2n-2} = P_{2n-1} \circ Q_{2n-2}$$

to make the following diagram commutative:

$$\begin{tikzcd} & \mathcal{R}_{\infty\infty} \arrow[dl, "\gamma_n^{even}"'] \arrow[dr, "\gamma_{n-1}^{even}"] & \\ \mathcal{R}_{2n} \arrow[rr, "\mu_n^{even}"] & & \mathcal{R}_{2n-2} \end{tikzcd}$$

Analogously:

Definition 1.10.3 *For $n \geq 0$, we define the projection $\gamma_n^{odd} : \mathcal{R}_{\infty\infty} \to R_{2n+1}$ by:*

$$\gamma_n^{odd}(R_{\infty\infty}(d(x), h(x))) = R_{2n+1}\left(d(x) \cdot \left(\frac{h(x)}{x}\right)^n, h(x)\right)$$

Thus, for $D_{\infty\infty} \in \mathcal{R}_{\infty\infty}$ we can define the sequence:

$$(\gamma_0^{odd}(D_{\infty\infty}), \gamma_1^{odd}(D_{\infty\infty}), \gamma_2^{odd}(D_{\infty\infty}), \ldots)$$

which is a sequence of matrices of the type:

(1.8)

$$\left(\begin{bmatrix} d_{00} & \\ d_{10} & d_{11} \end{bmatrix}, \quad \begin{bmatrix} d_{-1,-1} & & & \\ d_{0,-1} & d_{00} & & \\ d_{1,-1} & d_{1,0} & d_{11} & \\ d_{2,-1} & d_{20} & d_{21} & d_{22} \end{bmatrix}, \quad \begin{bmatrix} d_{-2,-2} & & & & & \\ d_{-1,-2} & d_{-1,-1} & & & & \\ d_{0,-2} & d_{0,-1} & d_{00} & & & \\ d_{1,-2} & d_{1,-1} & d_{1,0} & d_{11} & & \\ d_{2,-2} & d_{2,-1} & d_{2,0} & d_{2,1} & d_{22} & \\ d_{3,-2} & d_{3,-1} & d_{30} & d_{31} & d_{32} & d_{33} \end{bmatrix}, \quad \ldots \right)$$

Thus, as we had before, for $n \geq 0$:

$$\gamma_n^{odd} = Q_{2n+1} \circ P_{2n+2} \circ \gamma_{n+1}^{odd} = P_{2n+1} \circ Q_{2n+2} \circ \gamma_{n+1}^{odd}$$

and we can define $\mu_{n-1}^{odd} : \mathcal{R}_{2n+1} \to \mathcal{R}_{2n-1}$ in order to make the following diagram commutative:

$$\begin{tikzcd} & \mathcal{R}_{\infty\infty} \arrow[dl, "\gamma_n^{odd}"'] \arrow[dr, "\gamma_{n-1}^{odd}"] & \\ \mathcal{R}_{2n+1} \arrow[rr, "\mu_{n-1}^{odd}"] & & \mathcal{R}_{2n-1} \end{tikzcd}$$

So we obtain:

Proposition 1.10.4

(i) The bi-infinite representation of the Riordan group $\mathcal{R}$ is isomorphic to the inverse limit $\varprojlim(\mathcal{R}_{2n}, \mu_n^{even})$ where recall that $\mu_n^{even} = Q_{2n} \circ P_{2n+1} = P_{2n} \circ Q_{2n+1}$.

(ii) The bi-infinite representation of the Riordan group $\mathcal{R}$ is isomorphic to the inverse limit $\varprojlim(\mathcal{R}_{2n+1}, \mu_n^{odd})_{n\geq 0}$ where recall that $\mu_n^{odd} = Q_{2n+1} \circ P_{2n+2} = P_{2n+1} \circ Q_{2n+2}$.

Proof: We will prove (i), and (ii) is totally analogous. Let X be the inverse limit, which is the set of sequences:

$$(D_0, D_2, D_4, \ldots) \text{ where } \forall n \geq 0, \mu_n^{even}(D_{2n+1}) = D_{2n}$$

We will look for an isomorphism $\widetilde{\pi}$ by making the following diagram commutative:

$$\begin{tikzcd}
& \mathcal{R}_{\infty\infty} \arrow[d, "\widetilde{\pi}"] \arrow[ddl, "\gamma_n^{even}"'] \arrow[ddr, "\gamma_{n-1}^{even}"] & \\
& X \arrow[dl, "\tau_n"] \arrow[dr, "\tau_{n-1}"'] & \\
\mathcal{R}_{2n} \arrow[rr, "\mu_{n-1}^{even}"] & & \mathcal{R}_{2n-2}
\end{tikzcd}$$

where $\tau_n : \mathcal{R}_{\infty\infty} \to \mathcal{R}_{2n}$ is the natural projection into the corresponding coordinate. Take $\widetilde{\pi}$ given by:

$$\widetilde{\pi}(D) = (\gamma_0^{even}(D), \gamma_1^{even}(D), \ldots)$$

which is obviously an injective homomorphism. We only need to see that this map is onto. Fixed a sequence $\alpha = (D_0, D_2, D_4 \ldots) \in X$, we can find a Riordan matrix $D = R(d(x), h(x))$ such that $\widetilde{\pi}(D) = \alpha$ doing the following:

- $d(x) = d_0 + d_1 x + d_2 x^2 + \ldots$ is given by the following formula:

  $$d_k = (D_{2k})_{k0} \text{ where } D_{2k} = [(D_{2k})_{ij}]_{-k\leq i,j\leq k}$$

  The condition that for all $n \geq 0$, $\alpha_n(D_{n+1}) = D_n$ ensures that $(D_{2k})_{k0} = (D_{2m})_{k0}$ for any $m \geq k$.

- As we already know, an element in $\mathcal{R}_{2n}$ has a partial A-sequence of terms numbered $0, \ldots, 2n-1$. We will set that the A-sequence of D is $a_0, a_1, a_2, \ldots$ where the terms a_{2k}, a_{2k+1} for any $k \geq 0$ are the terms $2k, 2k+1$ in D_{2k+2}.

  The condition $\alpha_n(D_{n+1}) = D_n$ for all $n \geq 0$ ensures that the terms $2k, 2k+1$ of the A-sequence of D_{2k+2} are also the terms $2k, 2k+2$ of the A-sequence of any other D_{2m+2} for $m \geq k$.

The unique matrix D determined by this $d(x)$ and this A-sequence satisfies $\gamma_n^{even}(D) = D_{2n}$ for all $n \geq 0$.

$\square$

1.11 Symmetries in bi-infinite Riordan matrices

In section 1.8 the concepts of complementary and dual Riordan matrices were introduced. It is natural to look for some symmetries related to these "reflections". We would like to clarify the solutions of the following two problems:

$$(Problem\ 1) \qquad D = D^{\perp} \quad D \in \mathcal{R}$$

$$(Problem\ 2) \qquad D = D^{\diamondsuit} \quad D \in \mathcal{R}$$

that is, to characterize Riordan matrices which coincide with their complementary matrices ($Problem$ 1) and their dual arrays ($Problem$ 2). These problems will be solved in Theorem 1.12.3 and Theorem 1.12.4, respectively.

What we are going to do in this section is to reinterpret these problems in terms of two different reflections of bi-infinite Riordan matrices involving even an odd finite matrices and reflections.

Theorem 1.11.1 *Let $D = (d_{ij})_{-\infty<i,j<\infty} = R_{\infty\infty}(d(x), h(x)) \in \mathcal{R}_{\infty\infty}$.*

(i) If we identify D with:

$$(D_{2n})_{n\geq 0} \in \varprojlim\,(\mathcal{R}_{2n}, \mu_n^{even})_{n\geq 0} \ \ \textit{where}\ D_{2n} = \gamma_n^{even}(D)$$

then:

$$(D_{2n}^R)_{n\geq 0} \in \varprojlim\,(\mathcal{R}_{2n}, \mu_n^{even})_{n\geq 0}$$

and it can be identified with a bi-infinite Riordan matrix which is precisely $D^{\diamondsuit}$.

$$\begin{bmatrix} \ddots & & & & & & \\ \cdots & a_{22} & & & & & \\ \cdots & a_{21} & a_{11} & & & & \\ \cdots & a_{20} & a_{10} & a_{00} & & & \\ \cdots & a_{2,-1} & a_{1,-1} & a_{0,-1} & a_{-1,-1} & & \\ \cdots & a_{2,-2} & a_{1,-2} & a_{0,-2} & a_{-1,-2} & a_{-2,-2} & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(ii) If we identify D with:

$$(D_{2n+1})_{n\geq 0} \in \varprojlim \left(\mathcal{R}_{2n+1}, \mu_n^{odd}\right)_{n\geq 0} \ \ \textit{where}\ D_{2n+1} = \gamma_n^{odd}(D)$$

then:

$$(D^R_{2n+1})_{n\geq 0} \in \varprojlim \left(\mathcal{R}_{2n+1}, \mu_n^{odd}\right)_{n\geq 0}$$

and it can be identified with a bi-infinite Riordan matrix which is precisely $R^{\perp}_{\infty\infty}\left(d(x)\cdot\left(\frac{h(x)}{x}\right)^2, h(x)\right)$.

$$\begin{bmatrix} \ddots & & & & & \\ \cdots & a_{22} & & & & \\ \cdots & a_{21} & a_{11} & & & \\ \cdots & a_{20} & a_{10} & a_{00} & & \\ \cdots & a_{2,-1} & a_{1,-1} & a_{0,-1} & a_{-1,-1} & \\ & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Proof: Let $(c_{ij})_{-\infty<i,j<\infty}$ the matrix identified with $(D^R_{2n})_{n\geq 0}$. According to proposition 1.6.1, $c_{km} = d_{-m,-k}$, so (i) follows directly from the definition of the complementary matrix. The proof of (ii) is totally analogous.

□

Recall that, for any $D \in \mathcal{R}_{\infty\infty}$:

$$\Phi(D^{\diamondsuit}) = D^{\perp}$$

1.12 Solution of (Problem 1) and (Problem 2)

We first need the following result to solve (*Problem* 1) and (*Problem* 2):

Proposition 1.12.1 *Let $D_m = (d_{i,j}) \in \mathcal{R}_m$ be such that $D_m = D^R_m$ with $m \geq 1$.*

(a) If m is odd then D_m is a Toeplitz matrix.

(b) If m is even and $d_{0,0} = d_{1,1}$, then D_m is a Toeplitz matrix.

Proof: In order to prove (a) we will proceed by induction. Let $m = 2n+1$. For any matrix in $\mathcal{R}_m$, being Toeplitz means that the g-sequence $(g_0, \ldots, g_{m-1})$ is (1,0,0,...,0). We can see that:

- If $n = 0$ then:

$$\begin{pmatrix} d_{00} & \\ d_{10} & d_{11} \end{pmatrix} = \begin{pmatrix} d_{11} & \\ d_{10} & d_{00} \end{pmatrix}, \;\Rightarrow\; d_{00} = d_{11}, \;\Rightarrow\; g_0 = 1.$$

- If $n = 1$, take:

$$D_3 = \begin{pmatrix} d_{0,0} & & & \\ d_{1,0} & d_{1,1} & & \\ d_{2,0} & d_{2,1} & d_{2,2} & \\ d_{3,0} & d_{3,1} & d_{3,2} & d_{3,3} \end{pmatrix} = \begin{pmatrix} d_{3,3} & & & \\ d_{3,2} & d_{2,2} & & \\ d_{3,1} & d_{2,1} & d_{1,1} & \\ d_{3,0} & d_{2,0} & d_{1,0} & d_{0,0} \end{pmatrix} = D_3^R.$$

  - In particular we must have:

$$\mu_0^{odd}(D_3) = \begin{pmatrix} d_{11} & \\ d_{21} & d_{22} \end{pmatrix} = \begin{pmatrix} d_{22} & \\ d_{21} & d_{11} \end{pmatrix} = (\mu_0^{odd}(D_3))^R$$

    so $d_{11} = d_{22}$ and consequently $g_0 = 1$.
  - Since:

$$d_{10} = d_{32}, \quad d_{10} = d_{21} + g_1 d_{22}, \quad d_{21} = d_{32} + g_1 d_{22}$$

    we get:

$$d_{10} = d_{21} = d_{32}, \quad g_1 = 0$$

    Proceeding in the same way from the equality $d_{20} = d_{31}$, we get $g_2 = 0$.

- Suppose now that this is true for n and consider $D_{2n+3} \in \mathcal{R}_{2n+3}$ with $D_{2n+3} = D_{2n+3}^R$.

  - We know that $\mu_n^{odd}(D_{2n+3}) \in \mathcal{R}_{2n+1}$ satisfies $\mu_n^{odd}(D_{2n+3}) = \mu_n^{odd}(D_{2n+3})^R$. Hence $g_0 = 1$ and $g_i = 0$ for all $i = 1, \cdots, 2n$.
  - Since $D_{2n+3} = D_{2n+3}^R$, then $d_{2n+1,0} = d_{2n+3,2}$ and since $D_{2n+3} \in \mathcal{R}_{2n+3}$, consequently:

$$d_{2n+1,0} = d_{2n+2,1} + g_{2n+1} d_{00}, \quad d_{2n+2,1} = d_{2n+3,2} + g_{2n+1} d_{00}$$

    This implies that $g_{2n+1} = 0$. Moreover, since:

$$d_{2n+2,0} = d_{2n+3,1} \quad d_{2n+2,0} = d_{2n+3,1} + g_{2n+2} d_{0,0}$$

    we obtain $g_{2n+2} = 0$ and D is a Toeplitz matrix.

Finally, putting all above together we have $g_0 = 1$, $g_n = 0$ for all $n \geq 1$ and then D_m is a Toeplitz matrix. In order to prove (b), we note that the condition $d_{00} = d_{11}$, implies that $g_0 = 1$ and we proceed in a similar way as in (a).

□

An immediate corollary is:

Corollary 1.12.2 *Let $R(d(x), h(x))$ be a Riordan matrix. This matrix is Toeplitz if and only if $(R_n(d(x), h(x)))^R = R_n(d(x), h(x))$ for all $n \in \mathbb{N}$.*

Before going on, recall that in the notation used in this work:

$$R(d(x),h(x)) = R^{\perp}(d(x),h(x)) \Leftrightarrow R_{\infty\infty}(d(x),h(x)) = R^{\perp}_{\infty\infty}(d(x),h(x))$$

$$R(d(x),h(x)) = R^{\diamondsuit}(d(x),h(x)) \Leftrightarrow R_{\infty\infty}(d(x),h(x)) = R^{\diamondsuit}_{\infty\infty}(d(x),h(x))$$

Now, the solution of the ($Problem$ 1) is:

Theorem 1.12.3 *Let $D \in \mathcal{R}$. $D^{\perp} = D$ if and only if D is a Toeplitz matrix.*

Proof: If $D \in \mathcal{T}$, obviously $D = D^{\perp}$. On the other hand, let $D \in \mathcal{R}_{\infty\infty}$. According to theorem 1.11.1:

- D can be identified with $(D_{2n+1})_{n\geq 0} \in \varprojlim(R_{2n+1}, \mu_n^{odd})_{n\geq 0}$ where $D_{2n+1} = \gamma_n^{odd}(D)$.
- $D^{\perp}$ can be identified with $(D^R_{2n+1})_{n\geq 0} \in \varprojlim(R_{2n+1}, \mu_n^{odd})_{n\geq 0}$ where $D_{2n+1} = \gamma_n^{odd}(D)$.

Now note that $D = D^{\perp}$ if and only if $D_{2n+1} = D^R_{2n+1}$. According to proposition 1.12.1 this implies that for all $n \geq 0$, D_{2n+1} is Toeplitz, which implies that D is Toeplitz.

□

We will now discuss ($Problem$ 2) whose answer is very different from that of ($Problem$ 1). The result is the following:

Theorem 1.12.4 *For $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, the solutions of ($Problem$ 2) are the Riordan matrices $R(d(x),h(x))$ such that:*

- *$h(x) = h^{-1}(x)$, that is either $h(x)$ is an **involution** in $\mathcal{F}_1$ (element of order 2) or $h(x) = x$ (element of order 1).*
- *$d(x) = \lambda \cdot \left(\sqrt{x\frac{h'(x)}{h(x)}}\right) \cdot e^{\phi(x,h(x))}$ with $\lambda \in \mathbb{K}^*$ and $\phi(x,t) = \sum_{0=i,j}^{\infty} a_{ij}x^i t^j$ is a **symmetric** bivariate power series in $\mathbb{K}[[x,t]]$ that is:*

$$\phi(x,t) = \phi(t,x) \textit{ or equivalently } \forall 0 \leq ij < \infty,\ a_{ij} = a_{ji}$$

with $\phi(0,0) = 0$.

Proof: According to the formula given after the definition in section 1.8, $R^{\diamondsuit}(d(x),g(x)) = R(d(x),h(x))$ if and only if:

$$\begin{cases} d(x) = \frac{d(h^{-1}(x))}{h'(h^{-1}(x))} \cdot \frac{x}{h^{-1}(x)} \\ h(x) = h^{-1}(x) \end{cases}$$

So we only need to prove that the second point is equivalent to the first equation. By using the first equation, and composing the first one with $h(x)$ (or equivalently with $h^{-1}(x)$) we obtain easily that the first equation is equivalent to:

$$\frac{d(h(x))}{d(x)} = \frac{h(x)}{x \cdot h'(x)}$$

Now we can take logarithms (the formal power series version of the logarithm, which makes sense since both terms in the equation belong to $\mathcal{F}_0$) in both sides of the equation:

$$\log\left(\frac{d(h(x))}{d(x)}\right) = F(x) \quad \text{where } F(x) = \log\left(\frac{h(x)}{x \cdot h'(x)}\right)$$

that we will write as:

$$\log\left(\frac{\frac{d(h(x))}{d(0)}}{\frac{d(x)}{d(0)}}\right) = F(x) \quad \text{where } F(x) = \log\left(\frac{h(x)}{x \cdot h'(x)}\right)$$

to ensure that the following expression makes sense:

$$\log\left(\frac{d(h(x))}{d(0)}\right) - \log\left(\frac{d(x)}{d(0)}\right) = F(x)$$

Now denote $y(x) = \log\left(\frac{d(x)}{d(0)}\right)$ and then, the previous equation is now:

$$y(h(x)) - y(x) = F(x)$$

Now, if order to solve this equation:

- Since $F(x) = -F(h(x))$, then $\frac{1}{2}F(x)$ is a particular solution.
- In order to get the general solution, note that the corresponding homogeneous equation has as general solution $y_H(x) = \phi(x, \omega(x))$ where $\phi(x, t)$ is a symmetric bivarite formal power series in $\mathbb{K}[[x, t]]$. It is clear that such a $\phi(x, \omega(x))$ is a solution. On the other hand, if $y_H(x)$ is a solution then:

$$\phi(x, t) = \frac{y_H(x) + y_H(t)}{2}$$

  satisfies all the required conditions.
- So the general solution is:

$$y(x) = \frac{1}{2}F(x) + \phi(x, \omega(x)) \text{ with } \phi \text{ symmetric}$$

Since $y(x) = \log\left(\frac{d(x)}{d(0)}\right)$, the result follows from taking exponentials in this expression.

$\square$

A consequence of the above result is a way to construct self-dual bi-infinite Riordan matrices:

Example 1.12.5 *Take $h(x) = \frac{x}{2x-1}$ and $\phi(x,z) = 0$. Then:*

$$D = R_{\infty\infty}\left(\sqrt{\frac{1}{1-2x}}, \frac{x}{2x-1}\right)$$

is self-dual. Below we write $\gamma_3^{even}(D) = R_6\left(\left(\frac{x}{h(x)}\right)^3 \cdot \sqrt{\frac{1}{1-2x}}, \ \frac{x}{2x-1}\right)$ which is obviously symmetric under the correspondent reflection in $\mathcal{R}^6$:

$$\left(\begin{array}{ccc|c|ccc}
1 & & & & & & \\
-5 & -1 & & & & & \\
15/2 & 3 & 1 & & & & \\
\cline{4-7}
-5/2 & -3/2 & -1 & -1 & & & \\
\cline{1-4}
-5/8 & -1/2 & -1/2 & -1 & 1 & & \\
-3/8 & -3/8 & -1/2 & -3/2 & 3 & -1 & \\
-5/16 & -3/8 & -5/8 & -5/2 & 15/2 & -5 & 1
\end{array}\right).$$

Example 1.12.6 *In general, if we take $h(x) = \frac{x}{\alpha x-1}$ (see* [67]*, where these kind of matrices are studied in the notation $T(f \mid g)$ introduced in subsection 0.3.4 which is much more comfortable in this case) and $\phi(x,z) = 0$ and we proceed as in the previous example we get that for any $\lambda \in \mathbb{K}^*$:*

$$\mathcal{R}\left(\frac{-\lambda}{\sqrt{1-\alpha x}}, \frac{x}{\alpha x - 1}\right)$$

is a family of self-dual matrices. The case $\alpha = -1$ and $\alpha = 4$ was first detected as self-dual in [65] *page 82.*

1.13 Relation to Functional Equations in Power Series

Most of the results in this work, obtained by combinatoric ways of reasoning from Riordan matrices, have an interpretation in terms of existence and uniqueness of solutions of functional equations or system of two functional equations. We have decided not to devote an entire chapter to solve functional equations, but we will point out some of those reasonings as often as possible. For instance, theorem 1.12.3 is a good example of this double interest, since its consequence in terms of functional equations is the following:

Corollary 1.13.1 *The solutions system of functional equations:*

$$\begin{cases} d(x) = d(h(x)) \cdot h'(x) \\ h(h(x)) = x \end{cases} \tag{1.9}$$

such that $d(x) \in \mathcal{F}_0$, $h(x) \in \mathcal{F}_1$ are:

$$d(x) \text{ is arbitrary}, \qquad h(x) = x$$

Proof: Obviously this is a solution for (1.9). Now see that:

- (1.9) is equivalent to $R(d(x), h(x))$ being self-complementary. As we have seen before, $R(d(x), h(x))$ being self-complementary is equivalent to:

$$\begin{cases} d(x) = \frac{d(h^{-1}(x))}{h'(h^{-1}(x))} \\ h(x) = h^{-1}(x) \end{cases}$$

  The second equation of this system is obviously equivalent to the second equation in (1.9). Now composing in both sides of the first equation by $h^{-1}(x)$ (or equivalently by $h(x)$) we obtain the equivalence.

- A matrix $R(d(x), h(x))$ is self-complementary if and only if belongs to $\mathcal{T}$ (theorem 1.12.3), that is, $d(x)$ is an arbitrary element in $\mathcal{F}_0$ and $h(x) = x$.

□

The inverse limit is the adequate framework to do proofs by induction when working with the Riordan group. Proposition 1.12.1 and the way of obtaining its consequence for the infinite case (theorem 1.12.3) is an example of this, but these kind of proofs will appear all the time throughout the rest of this work, and will turn out to be very fruitful. Another way to organize this induction will appear in the next section.

1.14 Application: Schröder and weighted Schröder equations

Before ending this chapter we will show an application to this inverse limit structure proving by induction some known results about the so called Schröder equation (linearisation problem) in $\mathcal{F}_1$ and for the weighted Schröder equation. Those equations are very related to structural algebraic properties of $\mathcal{A}$ and $\mathcal{R}$.

Firstly we will assert one of the ideas that we will be used frequently in this work:

Remark 1.14.1 *In* [1, 58] *a formal definition of functional equation in one variable appears. In this moment, simplifying, we can think that a functional equation is an equality between two power series that depend on one unknown power series $y(x) \in \mathbb{K}[[x]]$:*

$$\Phi_1(y(x)) = \Phi_2(y(x))$$

Obviously this equality between power series holds if and only if $\forall n \geq 0$:

$$Taylor_n(\Phi_1(y(x))) = Taylor_n(\Phi_2(y(x))) \tag{1.10}$$

So for functional equations satisfying $Taylor_n(\phi_i(y(x))) = Taylor_n(\phi_i(Taylor_n(y(x))))$ we can study the solutions:

$$y(x) \cong (Taylor_0(y(x)), Taylor_1(y(x)), Taylor_2(y(x)), \ldots) \in \varprojlim(\mathbb{K}[[x]], Taylor_n)$$

by induction determining for each n $Taylor_n(y(x))$ from (1.10)*.*

Now we will use this idea for the following two examples:

Definition 1.14.2 *Let $h(x) \in \mathcal{F}_1(\mathbb{K})$, $\lambda \in \mathbb{K}$. The following equation in the indeterminate $g(x) \in \mathcal{F}_1[[x]]$ is called the **Schröder equation**:*

$$g(h(x)) = \lambda g(x) \tag{1.11}$$

*If λ is the multiplier of $h(x)$ then this is known as the **linearisation** problem and $h(x)$ is said to be the **linearisation** of $h(x)$.*

Definition 1.14.3 *Let $d(x) \in \mathcal{F}_0(\mathbb{K})$, $h(x) \in \mathcal{F}_1(\mathbb{K})$, $\lambda \in \mathbb{K}$. The following equation in the indeterminate $f(x) \in \mathbb{K}[[x]]$ is called the **weighted Schröder equation**:*

$$d(x) \cdot f(h(x)) = \lambda f(x) \tag{1.12}$$

*If $\lambda = d(0)$ we will call this equation the **main case** of equation 1.12.*

The study of the Schröder equation is a classical problem in analysis tipically in the case $\mathbb{K} = \mathbb{C}$ and in $\mathcal{F}_{1,hol}(\mathbb{C})$. It was first studied in the 1871 paper [100] and has also been explored in our formal power series context (for $\mathbb{K} = \mathbb{R}, \mathbb{C}$): see for example the survey by I. K. Babenko [2], or any of the works [59, 99, 112] for more information.

The weighted Schröder equation is a natural generalization of the previous equation, triggered by the study of weighted composition operators (see for example [47], where this name is used) which, as mentioned before, are closely related to Riordan matrices. Those equations have also been studied under the name of **linear homogeneous functional equations** (chapter II in [58]). When this last equation is studied from other points of view, the condition $h(x) \in \mathcal{F}_1$ is sometimes removed, but it is convenient to mantain it in our setting.

Note that, both equations can be related to a problem of eigenvectors and to problems of conjugation in $\mathcal{A}, \mathcal{R}$:

Remark 1.14.4 *Equation* (1.11) *can be written in terms of Riordan matrices as:*

$$R(1, h(x)) \begin{bmatrix} 0 \\ g_1 \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ g_1 \\ \vdots \end{bmatrix} \tag{1.13}$$

Equivalently, equation (1.12) *can be written as:*

$$R(d(x), h(x)) \begin{bmatrix} f_0 \\ f_1 \\ \vdots \end{bmatrix} = \lambda \begin{bmatrix} f_0 \\ f_1 \\ \vdots \end{bmatrix} \tag{1.14}$$

The list of eigenvalues of a Riordan matrix is easy to find, since it is lower triangular, so now we know for which possible values of λ we can expect to have a solution.

Remark 1.14.5 *Equation* (1.12) *can be writen in terms of Riordan matrices as:*

$$R(1, h(x)) = R(1, g(x))R(1, \lambda x)R(1, g^{-1}(x)) \tag{1.15}$$

Equivalently, the system made out from the two functional equations (1.11), (1.12) *can be writen as:*

$$R(d(x), h(x)) = R(f(x), g(x))R(d(0), h'(0)x)R(f(x), g^{-1}(x)) \tag{1.16}$$

Now we will recover two theorems which are already well known (see section 6 in [2] and chapter II in [58]) but that we can prove in a new and simple way by using inverse limit structure in $\mathcal{R}$ with the advantage that our proof easily works for any field of characteristic 0:

Theorem 1.14.6 (Linearisation problem, see section 6 in [2]**)** *In relation to the linearisation problem* (1.11) *(λ is the multiplier of $h(x)$) in power series over a field $\mathbb{K}$ of charactersitic 0:*

(1) If the multiplier of $h(x)$ is not a root of unity, equation (1.11) *has a unique solution in $\mathcal{F}_1(\mathbb{K})$ up to multiplication by a constant.*

(2) If the multiplier of $h(x)$ is a root of unity of order q, and $h(x)$ is an element of finite order q in $\mathcal{F}_1(\mathbb{K})$, (1.11) *has infinitely many solutions in $\mathcal{F}_1(\mathbb{K})$. Concretely, for every choice of $g_1, g_{1+q}, g_{1+2q}, \ldots$ there is a unique solution $g(x) = g_1x + g_2x + \ldots \in \mathcal{F}_1(\mathbb{K})$.*

(3) If the multiplier of $h(x)$ is a root of unity of order q and $h(x)$ is not of order q, there is no solution in in $\mathcal{F}_1(\mathbb{K})$.

Proof using the inverse limit structure in $\mathcal{R}$: Let's look at (1.11) in its formulation (1.13). In view of remark 1.14.1 each solution $y(x)$ of (1.13) can be identified with an element:

$$(Taylor_0(y(x)), Taylor_1(y(x)), Taylor_2(y(x)), \ldots) \in \varprojlim(\mathbb{K}_n[x], Taylor_n)$$

or equivalently, taking only the coefficients of each polynomial with a sequence o row vectors:

$$\left(\begin{bmatrix}0\end{bmatrix}, \quad \begin{bmatrix}0\\g_1\end{bmatrix}, \quad \begin{bmatrix}0\\g_1\\g_2\end{bmatrix}, \quad \ldots\right) \tag{1.17}$$

such that, for each n:

$$R_n(1, h(x)) \begin{bmatrix}0\\g_1\\\vdots\\g_n\end{bmatrix} = \lambda \begin{bmatrix}0\\g_1\\\vdots\\g_n\end{bmatrix} \tag{1.18}$$

or equivalently:

$$R_n(1, h(x)) = R_n(1, g_1x + \ldots + g_nx^n) R_n(1, \lambda x) R_n(1, Taylor_n((g_1x + \ldots + g_nx^n)^{-1})) \tag{1.19}$$

So we can use induction over n to determine the set of possible solutions to (1.13) in each case:

(1) If the multiplier of $h(x)$ is not a root of unity we have that:

- Obviously $R_0(1, h(x))\begin{bmatrix}0\end{bmatrix} = \lambda\begin{bmatrix}0\end{bmatrix}$.
- For every choice of g_1, we have that:

$$R_1(1, h(x))\begin{bmatrix}0\\ g_1\end{bmatrix} = \begin{bmatrix}1 & \\ 0 & \lambda\end{bmatrix}\begin{bmatrix}0\\ g_1\end{bmatrix} = \lambda\begin{bmatrix}0\\ g_1\end{bmatrix}$$

- Suppose $\begin{bmatrix}0\\ g_1\\ \vdots\\ g_n\end{bmatrix}$ is a solution of (1.18). Then the extension $\begin{bmatrix}0\\ g_1\\ \vdots\\ g_n\\ g_{n+1}\end{bmatrix}$ is a solution of:

$$R_{n+1}(1, h(x))\begin{bmatrix}0\\ g_1\\ \vdots\\ g_n\\ g_{n+1}\end{bmatrix} = \left[\begin{array}{ccc|c} & R_n(d(x), h(x)) & & \\ \hline m_{n+1,0} & \dots & m_{n+1,n} & m_{n+1,n+1}\end{array}\right]\left[\begin{array}{c}0\\ g_1\\ \vdots\\ g_n\\ \hline g_{n+1}\end{array}\right] =$$

$$= \lambda\begin{bmatrix}0\\ g_1\\ \vdots\\ g_n\\ g_{n+1}\end{bmatrix}$$

  if and only if (remember that $m_{n+1,n+1} = \lambda^{n+1}$):

$$g_{n+1} = \frac{1}{\lambda - \lambda^{n+1}} = m_{n+1,0}\cdot 0 + m_{n+1,1}g_1 + \ldots + m_{n+1,n}g_n$$

  Note that since λ is, by hypothesis, not a root of unity, $\lambda - \lambda^{n+1} \neq 0$.
- Since for every choice of g_1 we have a unique solution of the form (1.17), and for every solution of (1.13) any multiple by a constant is again a solution, we obtain the desired result.

(2) If the multiplier λ of $h(x)$ is a root of unity of order q and $h(x)$ is of order q we have:

- Obviously $R_0(1, h(x))\begin{bmatrix}0\end{bmatrix} = \lambda\begin{bmatrix}0\end{bmatrix}$.
- For every choice of g_1, we have that:

$$R_1(1, h(x))\begin{bmatrix}0\\ g_1\end{bmatrix} = \begin{bmatrix}1 & \\ 0 & \lambda\end{bmatrix}\begin{bmatrix}0\\ g_1\end{bmatrix} = \lambda\begin{bmatrix}0\\ g_1\end{bmatrix}$$

- Suppose we have a solution $\begin{bmatrix} 0 \\ g_1 \\ \vdots \\ g_n \end{bmatrix}$ of (1.18).
  - If $n \not\equiv 0 \mod q$ there is a unique extension satisfying:

$$R_{n+1}(1, h(x)) \begin{bmatrix} 0 \\ g_1 \\ \vdots \\ g_n \\ g_{n+1} \end{bmatrix} = \left[\begin{array}{cccc|c} & & R_n(d(x), h(x)) & & \\ \hline m_{n+1,0} & & \dots & m_{n+1,n} & m_{n+1,n+1} \end{array}\right] \left[\begin{array}{c} 0 \\ g_1 \\ \vdots \\ g_n \\ \hline g_{n+1} \end{array}\right] =$$

$$= \lambda \begin{bmatrix} 0 \\ g_1 \\ \vdots \\ g_n \\ g_{n+1} \end{bmatrix}$$

which is again given by:

$$g_{n+1} = \frac{1}{\lambda - \lambda^{n+1}} = m_{n+1,0} \cdot 0 + m_{n+1,1} g_1 + \ldots + m_{n+1,n} g_n$$

where since $n \not\equiv 0 \mod q$, $\lambda - \lambda^{n+1} \neq 0$
  - If $n \equiv 0 \mod q$ obviously other strategy is needed. We need formulations of type (1.19) instead of (1.19). Then in the equation:

$$R_{n+1}(1, h_1 x + \ldots + h_{n+1} x^{n+1}) =$$

$$= R(1, g_1 x + \ldots + g_{n+1} x^{n+1}) R(1, \lambda) R(1, Taylor_{n+1}(g_1 x + \ldots + g_{n+1} x^{n+1}))$$

we have that regardless g_{n+1} the matrix on the right hand side of the equation is of order q. On the left side of the equation we can see that there is a unique h_{n+1} making $R_{n+1}(1, h_1 x + \ldots + h_{n+1} x^{n+1})$ an element of order q (we will omit this proof now, since this will be proved in chapter 3, where the hypothesis of $\mathbb{K}$ being of characteristic 0 is used). The only possible conclusion is that equation (1.19) holds regardless the choice of g_{n+1}.

(3) Putting all the pieces above together, we obtain the desired result.

(4) The existence of such a linerisation of $h(x)$ is equivalent to say that:

$$h(x) = g(\lambda g^{-1}(x))$$

but the expression in the right side of this equation is a power series of finite order in $\mathcal{F}_1(\mathbb{K})$ and the left side expression is not by hypothesis.

□

Theorem 1.14.7 (Main Case of W. Schröder Eq., chapter II in [58]**)** *In relation to the main case of the weighted Schröder equation* (1.12) *($\lambda = d(0)$) in power series over a field $\mathbb{K}$ of characteristic 0:*

(1) If the multiplier of $h(x)$ is not a root of unity, there is a unique solution up to multiplication by a constant in $\mathcal{F}_0(\mathbb{K})$.

(2) If the multiplier of $h(x)$ is a root of unity of order q and $R(d(x), h(x))$ is either an element of order q in $\mathcal{R}(\mathbb{K})$ or an element of order q in $\mathcal{R}(\mathbb{K})$ multiplied by a constant, then there are infinitely many solutions in $\mathcal{F}_0(\mathbb{K})$.

Proof using the inverse limit structure in $\mathcal{R}$: The proof is analogous to the previous one. We are looking for a sequence of column vectors:

$$\left(\begin{bmatrix} f_0 \end{bmatrix}, \quad \begin{bmatrix} f_0 \\ f_1 \end{bmatrix}, \quad \begin{bmatrix} f_0 \\ f_1 \\ f_2 \end{bmatrix}, \quad \dots \right) \tag{1.20}$$

each of them satisfying the correspondent equations.

(1) In this case we will see by induction that there is a unique solution of (1.14) up to multiplication by a constant. Again we have that:

- Obviously $R_0(d(x), h(x))(f_0) = \lambda f(0)$ for any λ.
- Now suppose that we have a fixed column vector $\begin{bmatrix} f_0 \\ \vdots \\ f_n \end{bmatrix}$ satisfying:

$$R_n(d(x), h(x)) \begin{bmatrix} f_0 \\ \vdots \\ f_n \end{bmatrix} = \lambda \begin{bmatrix} f_0 \\ \vdots \\ f_n \end{bmatrix}$$

we can see that there is a unique f_{n+1} making the following equation hold:

$$R_{n+1}(d(x), h(x)) \begin{bmatrix} f_0 \\ \vdots \\ f_n \\ f_{n+1} \end{bmatrix} =$$

$$= \left[\begin{array}{ccc|c} & R_n(d(x), h(x)) & & \\ \hline m_{n+1,0} & \cdots & m_{n+1,n} & m_{n+1,n+1} \end{array} \right] \begin{bmatrix} f_0 \\ \vdots \\ f_n \\ f_{n+1} \end{bmatrix} = \lambda \begin{bmatrix} f_0 \\ \vdots \\ f_n \\ f_{n+1} \end{bmatrix}$$

which is:

$$f_{n+1} = \frac{m_{n+1,0}f_0 + \ldots + m_{n+1,n}f_n}{\lambda - m_{n+1,n+1}} = \frac{m_{n+1,0}f_0 + \ldots + m_{n+1,n}f_n}{d(0)(1-(h'(0)^{n+1})}$$

(2) We only need to look at the case of $R(d(x), h(x))$ being an element of finite order q and the other case follows trivially. As in the proof or theorem 1.14.6, the reasoning of (1) works as long as $n+1 \not\equiv 0 \mod q$. Otherwise, we can use theorem 1.14.6 to ensure the existence of $g(x)$ satisfying $h(x) = g^{-1}(h'(0)g(x)))$. Thus, for those cases of n, instead of looking at:

$$R_{n+1}(d(x), h(x)) \begin{bmatrix} f_0 \\ \vdots \\ f_n \\ f_{n+1} \end{bmatrix} = \lambda \begin{bmatrix} f_0 \\ \vdots \\ f_n \\ f_{n+1} \end{bmatrix}$$

we can look at the equivalent equation (see remark 1.14.5):

$$R_{n+1}(d(x), h(x)) = R(f(x), g(x))R(d(0), h'(0)x)R(f(x), g^{-1}(x))$$

Again, as in the proof of 1.14.6, we postpone for the future (chapter 3, where the hypothesis of 0 characteristic is needed) to proof that there is a unique $[x^n]d(x)$ making $R(Taylor_n(d(x)) + [x^n]d(x), h(x))$ an element of finite order q. So, since regardless the choice of f_{n+1} we obtain an element of finite order on the left hand of the equation, we conclude that this equation holds for any f_{n+1}.

□

Those theorems will have many applications in this work. We have the following remarks to do:

Remark 1.14.8 *We will omit here the details since they are not related to our discussion, but we have only studied the linearisation problem and the main case of the weighted Schröder equations because other cases of the Schröder and weighted Schröder equations respectively can be reduced to those.*

Remark 1.14.9 *In section 6 in* [2] *and in chapter II of* [58] *more precise descriptions of the solutions of the Schröder and the weighted Schröder equations can be found.*

Remark 1.14.10 *Later on this work, another case of the main case of the Weighted Schröder equation will appear for $\lambda = 1$ (see section 2.2) but up to our knowledge, nothing is known for the case λ being a root of unity but $R(d(x), h(x))$ not an element of finite order, neither a multiple of an element of finite order.*

Chapter 2

Some aspects of the algebraic structure of the Riordan Group

In this chapter, we will study some aspects of the algebraic structure of $\mathcal{R}(\mathbb{K})$ for different fields $\mathbb{K}$ (most of the times of characteristic 0), mainly the derived series, some results about conjugacy and some consequences. Some aspects of the elements of finite order will be also studied here, but a more detailed study of involutions (elements of order 2) has been put aside for next chapter and closes the study of the purely algebraic aspects of the Riordan group made in this work.

Since the Riordan group contains a copy (modulo isomorphism) of $\mathcal{F}_1(\mathbb{K})$, a good starting point can be found in [2], where a very exhaustive study of $\mathcal{F}_1(\mathbb{K})$ is made. The derived series of $\mathcal{F}_1(\mathbb{K})$ for $\mathbb{K}$ being a field of characteristic 0 is partially explored: the elements of each n-th derived subgroup of $\mathcal{F}_1(\mathbb{K})$ are proved to be contained in the subgroups $x + x^{2n}\mathbb{K}[[x]]$ of $\mathcal{F}_1$. As a consequence of our study we will improve the results surveyed in [2] proving the equality between both subgroups. Also a complete catalogue is made in this work about the conjugacy classes in $\mathcal{F}_1$. Concerning the derived series of $\mathcal{F}_1$, more is already known about the derived series in the case $\mathbb{K}$ being a finite field, and also this equality is known (see for example [18]).

Regarding the known results about the algebraic structure of the Riordan group, in the previous work a great effort has been made. A lot is known about the subgroups of $\mathcal{R}(\mathbb{K})$: see for example the survey [102], and [104]. Also in the survey [102], the question of finding the derived series of $\mathcal{R}(\mathbb{K})$ is posted (question that it is answered in this chapter) together with other questions about conjugacy and elements of finite order, that were later analyzed for $\mathbb{K} = \mathbb{C}$ in [24] (those questions will also be studied here too, proving in a different way some of the known results and extending them). In [51], some aspects of the algebraic structure of $\mathcal{R}(\mathbb{K})$ has also been studied. Some centralizers and stabilizers are computed, some group isomorphisms are also given and some properties connecting similar Riordan matrices and pseudo-involutions in the Riordan group are proved.

In section 2.1 we will compute the derived series of $\mathcal{A}$ (and so of $\mathcal{F}_1$ completing the study presented in [2]). After an auxiliary result is proved in 2.2 treating another case of the weighted Schröder equation, we will be ready to study the derived series of $\mathcal{R}$ in section 2.3.

After this, we will jump into the study of conjugacy. Firstly we will study in section 2.4 the problem of conjugacy in $\mathcal{A}'(\mathbb{K})$ (or equivalently $\mathcal{F}_1(\mathbb{K})$) and $\mathcal{R}'(\mathbb{K})$ (at this point we will have studied the derived series of $\mathcal{R}(\mathbb{K})$) for various reasons: due to the importance of those groups (recall that in the starting point of the study of Riordan matrices [105] the Riordan group was $\mathcal{R}'(\mathbb{K})$) and second because we will be able to find unified results for any field $\mathbb{K}$ of characteristic 0. Then in section 2.5 we will state some facts about the general problem of conjugacy in $\mathcal{A}(\mathbb{K})$ (or equivalently in $\mathcal{F}_1(\mathbb{K})$). After this in section 2.6, as an example of the previous study, we will study the conjugacy class of the extended and the non-extended Pascal Triangles.

Finally, there are two sections where we will remark two applications of the study of conjugacy in $\mathcal{R}(\mathbb{K})$: computing centralizers and computing powers of matrices and in section 2.9 a description of the abelianized of $\mathcal{R}_n$ and $\mathcal{R}$ will be made.

2.1 Derived Series of $\mathcal{F}_1$

As explained in the introduction, our first point of interest will be the derived series of $\mathcal{F}_1(\mathbb{K})$ for $\mathbb{K}$ being a field of characteristic 0. Recall that:

Definition 2.1.1 *Let $(G, *)$ be a group.*

- *Let $g, h \in G$. We define the **commutator** of g, h as:*

$$[g, h] := g^{-1} * h^{-1} * g * h$$

- *The set of all the commutators of elements in G is denoted by $[G, G]$. The subgroup of G generated by the commutators in G (the elements in $[G, G]$) is called the **first derivative subgroup** of G, which is denoted by G'.*
- *Inductively, for $n \geq 2$, we can define the n-th derivative subgroup of G, denoted by $G^{(n)}$, as the subgroup of $G^{(n-1)}$ generated by the commutators in $G^{(n-1)}$ (the elements in $[G^{(n-1)}, G^{(n-1)}]$).*

It follows from the definition that $[G, G] \subset G'$. If for a group G, equality holds, this group is said to satisfy the **Ore property** (see [91]).

The main result of this section is the following:

Theorem 2.1.2 *Let $\mathbb{K}$ be a field of characteristic 0. Then:*

(i) $\mathcal{F}_1' = \{v(x) \in \mathcal{F}_1 : v'(0) = 1\}$ *and all of its elements are commutators.*

(ii) For $n \geq 2$:

$$\mathcal{F}_1^{(n)} = \{v(x) \in \mathcal{F}_1 : v(x) = x + \mathcal{O}(x^{2^n})\} \tag{2.1}$$

and all of its elements are commutators of elements in $\mathcal{F}_1^{(n-1)}$.

Recall that when $\mathbb{K}$ is a finite field, $\mathcal{F}(\mathbb{K})$ is said to be a Nottingham group (see for example [18]). An analogue of the previous theorem was known for Nottingham groups, and proved by using different techniques. But the case $\mathbb{K}$ of characteristic 0 and $\mathbb{K}$ being an infinite field of positive characteristic were still up in the air, although partial results were known (see below). The first case will be solved in this section, but the case $\mathbb{K}$ being an infinite field of positive characteristic is still unknown up to our knowledge (open question 9).

In relation to Riordan matrices, thanks to the natural anti-isomorphism $h(x) \longmapsto R(1, h(x))$ we have that:

$$h(x) \in \mathcal{F}_1^{(n)} \iff R(1, h(x)) \in \mathcal{A}^{(n)}$$

since:

$$h(x) = v^{-1}(g^{-1}(v(g(x)))) \iff$$
$$\iff R(1, h(x)) = R(1, g(x)) \cdot R(1, v(x)) \cdot R(1, g(x))^{-1} \cdot R(1, v(x))^{-1}$$

so this theorem has an immediate and simple translation to the derived series of $\mathcal{A}$.

Example 2.1.3 *For instance, this theorem asserts that the Riordan matrix:*

$$R(1, x + x^4) = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 0 & 1 & & & & \\ 0 & 0 & 0 & 1 & & & \\ 0 & 1 & 0 & 0 & 1 & & \\ 0 & 0 & 2 & 0 & 0 & 1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

belongs to $\mathcal{A}''$.

We will prove Theorem 2.1.2 by induction over n (by using the inverse limit setting). The following lemma is the base case:

Lemma 2.1.4 *Let $\mathbb{K}$ be a field of characteristic 0 (the proof still holds for other field $\mathbb{K}$ as long as $\exists r \in \mathbb{K}$, not a root of unity). Then:*

- *$h(x) = h_1x + h_2x^2 + \ldots$ is in $\mathcal{F}_1'$ if and only if $h_1 = 1$. Equivalently, $R(1, h(x))$ is in $\mathcal{A}'$ if and only if it has ones in the diagonal.*
- *$\mathcal{F}_1$ or equivalently $\mathcal{A}$ satisfy the Ore property ($[\mathcal{F}_1, \mathcal{F}_1] = \mathcal{F}_1'$ and $[\mathcal{A}, \mathcal{A}] = \mathcal{A}'$ respectively).*

Proof: We will prove the Riordan matrix statement. The subset of $\mathcal{A}$ with ones in its diagonal, is clearly a subgroup of $\mathcal{A}$, so it suffices to prove that any element in this subgroup is, in fact a commutator of element sin $\mathcal{A}$. Take any $R(1, h(x))$ such that $h_1 = 1$, and take any $\lambda \in \mathbb{K}$ not a root of unity. According to remark 1.14.5 and to theorem 1.14.6:

$$R(1, h(\lambda x)) = R(1, \lambda x)R(1, h(x)) = R^{-1}(1, v(x))R(1, \lambda x)R(1, v(x))$$

and then:

$$R(1, \lambda x) = R^{-1}(1, h(x))R^{-1}(1, v(x))R(1, \lambda x)R(1, v(x))$$

□

Now, in order to prove the induction hypothesis in the induction of the proof of theorem 2.1.2, we will take two steps: showing that any element in $\mathcal{A}^{(n)}$ must be of the type announced, and showing that any element of this type is in $\mathcal{A}^{(n)}$.

The first of those steps is taken in the following lemma. As announced before, this result was already proved by Babenko and Bogatyi (see for example [2]). Anyway we have decided to prove it here by using our own techniques.

Lemma 2.1.5 (proved in section 2.3 of [2]**)** *Let $\mathbb{K}$ be any field. Let:*

$$v(x), w(x) \in \left(x + x^k \mathbb{K}[[x]]\right)$$

Then $R_{2k-1}(1, v(x))$ and $R_{2k-1}(1, w(x))$ cummute. In particular, we are interested in:

$$I_{2k-1} = [R_{2k-1}(1, v(x)), R_{2k-1}(1, w(x))]$$

where I_{2k-1} is the $(2k) \times (2k)$ identity matrix.

Proof: Let the partial A-sequences of $R_{2k-1}(1, v(x)), R_{2k-1}(1, w(x))$ be respectively:

$$(1, 0, \ldots, 0, \alpha_{k-1}, \ldots),\ (1, 0, \ldots, 0, \beta_{k-1}, \ldots)$$

Consider that the k diagonal of a Riordan matrix $(a_{ij})_{i,j\in\mathbb{N}}$ is the sequence:

$$a_{k0}, a_{k+1,1}, a_{k+2,2}, \ldots$$

Then, for the elements $R_{2k-1}(1, v(x)), R_{2k-1}(1, w(x))$, according to the vertical construction for Riordan matrices using the A-sequence:

- The 0 diagonals are: $1, 1 \ldots$
- The diagonals $1, \ldots, k-2$ are: $0, 0, \ldots$
- The diagonals $k-1$ are arithmetic progressions, they are respectively:

$$0, \alpha_{k-1}, 2\alpha_{k-1}, 3\alpha_{k-1}, \ldots \text{ and } 0, \beta_{k-1}, 2\beta_{k-1}, 3\beta_{k-1}, \ldots$$

In order to check the commutativity of $R_{2k-1}(1, v(x)) = (a_{ij})_{0\le i,j\le 2k-1}$, $R_{2k-1}(1, w(x)) = (b_{ij})_{0\le i,j\le 2k-1}$ is equivalent to check the equality for all $0 \le i, j \le 2k-1$ between:

(a) c_{ij}, being the position (i, j) of the product $R_{2k-1}(1, v(x)) \cdot R_{2k-1}(1, w(x))$, that is (exceptionally (0), will denote either a row or a column of zeros):

$$c_{ij} = \left[0, a_{i,1} \ldots, a_{i,i-k+1}, (0), 1, (0)\ \right] \begin{bmatrix} (0) \\ 1 \\ (0) \\ b_{k+j-1,j} \\ \vdots \\ b_{m,j} \end{bmatrix}$$

(b) d_{ij}, being the position (i,j) of the product $R_{2k-1}(1,w(x)) \cdot R_{2k-1}(1,v(x))$ (again exceptionally (0) denotes either a row or a column of zeros):

$$d_{ij} = \left[0, b_{i,1} \ldots, b_{i,i-k+1}, (0), 1, (0)\right] \begin{bmatrix} (0) \\ 1 \\ (0) \\ a_{k+j-1,j} \\ \vdots \\ a_{m,j} \end{bmatrix}$$

We can do the following case by case comprobation:

(1) For every $1 \leq j < i \leq 2k-2$ we have that:

$$i-k+1 < k+j-1$$

and then:

$$\begin{cases} c_{ij} = a_{ij} + b_{ij} \\ d_{ij} = a_{ij} + b_{ij} \end{cases}$$

(2) The only position left is $i = 2k$, $j = 1$, and in this case:

$$\begin{cases} c_{ij} = a_{ij} + b_{ij} + a_{2k-1,k} b_{k,1} = a_{ij} + b_{ij} + k \cdot \alpha_{k-1} \cdot \beta_{k-1} \\ d_{ij} = a_{ij} + b_{ij} + b_{2k-1,k} a_{k1} = a_{ij} + b_{ij} + k \cdot \alpha_{k-1} \cdot \beta_{k-1} \end{cases}$$

□

Now we are ready to complete the proof:

Proof of Theorem 2.1.2: For each n, the sets of elements satisfying equation (2.1) in Theorem 2.1.2 are obviously a group. So if for a given n, the set of elements satisfying (2.1) is the set of commutators of elements in $\mathcal{R}^{(n-1)}$, then this set is $\mathcal{R}^{(n)}$.

We will prove the following statement: let $X = R(1, F(x)) \in \mathcal{A}$ such that $X_{2k-1} = \pi_{2k-1}(X) = I_{2k-1}$ (the $(2k) \times (2k)$ identity matrix), where:

$$F(x) = \sum_{i=1}^{\infty} F_i x^i$$

then for every $A \in \mathcal{A}$, such that $\pi_{k-1}(A) = I_{k-1}$,$\pi_k(A) \neq I_k$ and for every choice of $\beta_k \in \mathbb{K}$ there exists a unique $B = R(1, x + \beta_k x^k + \ldots) \in \mathcal{A}$ such that $\pi_{k-1}(B) = I_{k-1}$ satisfying:

$$X = A^{-1} B^{-1} A B$$

Let $(1, 0, \ldots, 0, \beta_{k-1}, \ldots)$ the A-secuence of $B = (b_{ij})_{0 \leq i,j < \infty}$. We will prove by induction over n that for every choice of β_{k-1} there exists a unique partial A-sequence:

$$(1, 0, \ldots, 0, \beta_{k-1}, \ldots, \beta_{n-k})$$

such that:

(i) $\pi_{k-1}(B) = I_{k-1}$

(ii) $\pi_n(B)\pi_n(A)\pi_n(X) = \pi_n(A)\pi_n(B)$

Recall that for the elements of $\mathcal{A}$, a partial A-sequence $(a_0, \dots, a_m)$ determines the diagonals $0, \dots, m$.

- Case $n = 2k-1$ If we want B to satisfy (i), then the partial A-sequence of $\pi_k(B)$ must be:
$$(1, 0, \dots, 0, \beta_{k-1})$$
So for every choice of β_{k-1} the induction hypothesis holds trivially according to the previous lemma.

- Case $n > 2k-1$ We want the following equation to hold (exceptionally $\begin{pmatrix}0\end{pmatrix}$ denotes a column of zeros of the adequate size):
$$\left[\begin{array}{ccc|c} & \pi_{n-1}(B) & & \\ \hline b_{n0} & \dots & b_{n,n-1} & 1 \end{array}\right] \left[\begin{array}{ccc|c} & \pi_{n-1}(A) & & \\ \hline a_{n0} & \dots & a_{n,n-1} & 1 \end{array}\right] \begin{bmatrix} 0 \\ 1 \\ (0) \\ F_{2k-1} \\ \vdots \\ F_n \end{bmatrix} =$$
$$= \left[\begin{array}{ccc|c} & \pi_{n-1}(A) & & \\ \hline a_{n0} & \dots & a_{n,n-1} & 1 \end{array}\right] \begin{bmatrix} 0 \\ 1 \\ (0) \\ b_{k,1} \\ \vdots \\ b_{n1} \end{bmatrix}$$
where by induction hypothesis the equations for the rows $0, \dots, n-1$ hold.

Then, we only need to check the last entry of this equation, which by block multiplication is:
$$\left(\begin{bmatrix} b_{n0} & \dots & b_{n,n-1} \end{bmatrix} \pi_{n-1}(A) + \begin{bmatrix} a_{n0} & \dots & a_{n,n-1} \end{bmatrix}\right) \begin{bmatrix} 0 \\ 1 \\ (0) \\ F_{2k-1} \\ F_{n-1} \end{bmatrix} + F_n = \tag{2.2}$$
$$= \begin{bmatrix} a_{n0} & \dots & a_{n,n-1} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ (0) \\ b_{k,1} \\ \vdots \\ b_{n-1,1} \end{bmatrix} + b_{n1}$$

(a) Taking into account that $\pi_{k-1}(A) = \pi_{k-1}(B) = I_{k-1}$ and that if we call:

$$[d_0, \dots, d_{n-1}] = \begin{bmatrix} b_{n0} & \dots & b_{n,n-1} \end{bmatrix} A_{n-1}$$

we have that:

$$d_m = \sum_{r=0}^{n} b_{nr} a_{rm} = b_{nm} + \left[\sum_{r=m+k-1}^{n-k-1} b_{nr} a_{rm} \right] \tag{2.3}$$

– For $m = 1$ is:

$$d_1 = b_{n1} + a_{k1} b_{nk} + \left[\sum_{r=k+1}^{n-k-1} b_{nr} a_{r1} \right]$$

where the b_{nr} inside the brackets are already determined, because they lie in digonals lower or equal to $n-k-1$ since:

$$n - r \leq n - k - 1 \Leftrightarrow r \geq k + 1$$

and then, they are determined by the partial A-sequence $(1, \dots, \beta_{n-k-1})$

– For $2 \leq m \leq n-1$ the b_{nr} inside the brackets in (2.3) are already determined for $m \geq 2$, becouse they lie in digonals lower or equal to $n-k-1$ since:

$$n - r \leq n - k - 1 \Leftrightarrow r \geq k + 1 \Leftarrow m + k - 1 \geq k + 1 \Leftrightarrow m \geq 2$$

and then, they are determined by the partial A-sequence $(1, \dots, \beta_{n-k-1})$

(b) The left hand side of this equation is:

$$(a_{n1} + d_1) + \sum_{m=2k-1}^{n-1} (a_{nm} + d_m) F_m = \tag{2.4}$$

$$= \left[a_{n1} + \sum_{m=2k-1}^{n-1} a_{nm} F_m \right] + \left[d_1 + \sum_{m=2k-1}^{n-1} d_m F_m \right]$$

where everything in the first bracket and in the summatory of the second bracket is known or determined

So this expression is of the form:

$$d_1 + [\dots] = b_{n1} + a_{k1} b_{nk} + [\dots]$$

where all the b_{ij} inside the $[\dots]$ are already determined.

(c) The right hand side of this equation is:

$$a_{n1} + a_{n,n-k+1} b_{n-k+1,1} + \left(\sum_{m=k}^{n-k} a_{nm} b_{m1} \right) + b_{n1} \tag{2.5}$$

that is, it has the form:

$$b_{n1} + a_{n,n-k+1} b_{n-k+1,1} + [\dots]$$

where all the b_{m1} in $[\dots]$ are already determined because they lie in diagonals lower or equal to $n-k-1$ since $m - 1 \leq k - 1$.

(d) So the equation we need to solve is of the type:

$$b_{n1} + a_{k1}b_{nk} + [\dots] = b_{n1} + a_{n,n-k+1}b_{n-k+1,1} + [\dots]$$

that is:

$$a_{k1}b_{nk} - a_{n,n-k+1}b_{n-k+1,1} = [\dots] \tag{2.6}$$

If $(\beta_0, \dots, \beta_{n-1})$ is the partial A-sequence of B_n, where the terms $\beta_0, \dots, \beta_{n-k-1}$ are already determined, and $(\alpha_0, \alpha_1 \dots)$ is the A-sequence of A, then:

$$a_{k1}b_{nk} - a_{n,n-k+1}b_{n-k+1,1} = a_{k1}([\dots] + k \cdot \beta_{n-k}) - a_{n,n-k+1}([\dots] + \beta_{n-k}) =$$

$$= (a_{k1} \cdot k - a_{n,n-k+1})\beta_{n-k} + [\dots] = (2k - n - 1) \cdot \alpha_k \cdot \beta_{n-k} + [\dots]$$

where the terms $[\dots]$ only depend on the a_{ij}, on the b_{ij} lying in the diagonals $0, \dots, n-k-1$ and on the $\beta_0, \dots, \beta_{n-k-1}$.

(e) So equation (2.6) is finally of the type:

$$(2k - n - 1) \cdot \alpha_k \cdot \beta_{n-k} = [\dots] \tag{2.7}$$

where everything inside the brackets is known by hypothesis and the indeterminate is β_{n-k}. This equation must have a unique solution.

□

2.2 Again the weighted Schröder Equation: main case with multiplier equal 1

Recall that a, as defined in section 1.14, a weighted Schröder equation is a functional equation (in formal power series) in the indeterminate $y(x)$ of the type:

$$u(x) \cdot y(v(x)) = \lambda y(x)$$

where $u(x) = d_0 + d_1 x + d_2 x^2 + \dots$, $v(x) = v_1 x + v_2 x^2 + \dots$, and $0 \neq \lambda = d_0$. Recall also that those equations are suitable to be treated in terms of Riordan arrays, since:

$$u(x) \cdot y(v(x)) = \lambda y(x) \Longleftrightarrow R(u(x), v(x)) \otimes y(x) = y(x)$$

and dividing in both sides of the equation by λ, we can reduce our study to equations of the type:

$$u(x) \cdot y(v(x)) = y(x) \tag{2.8}$$

with $d(x) = 1 + d_1 x + \dots$.

The main result in this section is the following:

Proposition 2.2.1 *Let $\mathbb{K}$ be a field of characteristic 0. Let $R(d(x), h(x)) = (d_{ij})_{0\leq i,j<\infty}$. If $h(x) \in \left(x + x^k\mathbb{K}[[x]] \setminus x^{k+1\mathbb{K}[[x]]}\right)$. Then there exists a power series:*

$$u(x) = 1 + \sum_{m=1}^{\infty} u_m x^m$$

solution of (2.8) *if and only if $d(x) \in \left(1 + x^k\mathbb{K}[[x]]\right)$. In this case, this $u(x)$ is unique.*

Moreover, the solution $u(x)$ lie in $1 + x^r\mathbb{K}[[x]]$ if and only if:

$$d(x) \in \left(1 + x^{r+k-1}\mathbb{K}[[x]]\right)$$

Proof: Seeing that necessarily $u(x) \in \left(1 + x^k\mathbb{K}[[x]]\right)$ is easy, since:

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ \vdots & \ddots & \ddots & & \\ 0 & \dots & 0 & 1 & \\ d_{i0} & 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_i \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_i \end{bmatrix} \Rightarrow d_{i0} = 0$$

In order to prove the converse, we will prove by induction over n that there exists a unique sequence $(u_1, \dots, u_{n-k+1})$ such that:

$$R_n(d(x), h(x)) \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_n \end{bmatrix} \tag{2.9}$$

regarless the sequence $(u_{n-k+2}, \dots, u_n)$.

- Case $n = k-1$ The result is obviously true, since $R_n(d(x), h(x)) = I_n$ (the $(n+1)\times(n+1)$ identity matrix).
- Case $n > k-1$ By induction hypothesis there exist a unique sequence $(u_1, \dots, u_{n-k})$ such that:

$$R_{n-1}(d(x), h(x)) \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix} = \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix}$$

for all the sequences $(u_{n-k+1}, \ldots, u_{n-1})$.

So we only need to check that there exists a unique u_{n-k+1} such that the last equation in 2.9, that is:

$$d_{n0} + \left(\sum_{m=1}^{n-k} d_{nm} u_m\right) + d_{n,n-k+1} \cdot u_{n-k+1} + u_n = u_n$$

and since the A-sequence of $R_n(d(x), h(x))$ is of the type $(1, 0, \ldots, 0, \alpha_{k-1}, \ldots)$, this is equivalent to say that there exists a unique u_{n-k+1} such that:

$$(n-k+1)\alpha_{k-1} u_{n-k+1} = -d_{n0} - \left(\sum_{m=1}^{n-k} d_{nm} u_m\right) \tag{2.10}$$

holds for all the sequences $(u_{n-k+2}, \ldots, u_n)$, which is obviously true.

The condition saying that $u(x)$ lie in $1 + x^r\mathbb{K}[[x]]$ if and only if $d(x)$ lie in $1 + x^{r+k-1}\mathbb{K}[[x]]$ come from induction on n in the equation (2.10).

□

2.3 Derived Series of $\mathcal{R}$

Once we have understood the derived series in $\mathcal{F}_1(\mathbb{K})$ and $\mathcal{A}(\mathbb{K})$ for $\mathbb{K}$ of characteristic zero, we will go for the derived series of $\mathcal{R}(\mathbb{K})$.

First note that:

Remark 2.3.1 *Analogously to the proof of 2.1.4, for $\mathbb{K}$ being a field of characteristic 0, $\mathcal{R}'(\mathbb{K})$ is precisely the set of Riordan matrices with ones in the diagonal. Obviously the set of Riordan matrices with ones in the diagonal is a group and we can see that each of its elements $R(d(x), h(x))$ with $d(0) = 1$, $h'(0) = 1$ is a commutator of elements in $\mathcal{R}$, since according to remark 1.14.5, for any $\lambda \in \mathbb{K}$ not a root of unity:*

$$R(d(x), h(x)) \cdot R(1, \lambda x) = R^{-1}(u(x), v(x))R(1, \lambda x)R(u(x), v(x))$$

for certain $R(u(x), v(x)) \in \mathcal{R}(\mathbb{K})$, and thus:

$$R(d(x), h(x)) = R^{-1}(1, \lambda x)R^{-1}(u(x), v(x))R(1, \lambda x)R(u(x), v(x))$$

In this case, it is easy to compute the coefficients of this $R(u(x), (x))$. If we suppose that:

$$R(d(x), h(x)) = R(1, rx)R(u(x), v(x))R^{-1}(1, rx)R^{-1}(u(x), v(x)) =$$

$$= R(1, rx)R(u(x), v(x))R\left(1, \frac{x}{r}\right) R\left(\frac{1}{u(v^{-1}(x))}, v^{-1}(x)\right) =$$

$$= R\left(\frac{u(rx)}{u(v^{-1}(\frac{v(rx)}{r}))}, v^{-1}\left(\frac{v(rx)}{r}\right)\right)$$

So we have:

- $h(x) = v^{-1}\left(\frac{v(rx)}{r}\right)$. *Composing in both sides of the equation by* $v(x)$ *we get:*

$$v(h(x)) = \frac{v(rx)}{r} \tag{2.11}$$

or equivalently:

$$R(1, h(x)) \otimes v(x) = R\left(\frac{1}{r}, rx\right) \otimes v(x)$$

from where if $v(x) = v_1 x + v_2 x^2 + \ldots$, *and for each* $0 \neq v_1 \in \mathbb{K}$ *we obtain that:*

$$v_n = \frac{1}{r^{n-1} - 1} \sum_{k=1}^{n-1} v_k \cdot ([x^k](h(x))^k)$$

- *On the other hand, by using* (2.11) *in the other equation we get:*

$$d(x) = \frac{u(rx)}{u(h(x))}$$

which is equivalent to:

$$d(x) \cdot u(h(x)) = u(rx)$$

or equivalently:

$$R(d(x), h(x)) \otimes u(x) = R(1, rx) \otimes u(x)$$

and then, if $u(x) = u_0 + u_1 x + u_2 x^2 + \ldots$*:*

$$u_n = \frac{1}{r^n - 1} \sum_{k=0}^{n-1} d_{nk} u_k$$

In the past Riordan arrays were defined with the restriction on having ones in the diagonal (see [105]) to also obtain a group when we choose the entries to lie in the ring $\mathbb{Z}$. Here the notation $\mathcal{C}(\mathbb{K})$, $\mathcal{C}(R)$ has been introduced for the same reason in section 0.3.6. In particular, finding $\mathcal{R}''$ (which was the first commutator of the original Riordan group) answers an open question from L. Shapiro in [103].

The main result of this section is the following:

Theorem 2.3.2 *Let* $\mathbb{K}$ *be a field of characteristic 0. Then, for* $n \geq 1$*:*

$$\mathcal{R}^{(n)} = \{R(d(x), h(x)) \in \mathcal{R} : h(x) \in \mathcal{F}^{(n)}, d(x) = 1 + \mathcal{O}(x^{2^n - n})\} \tag{2.12}$$

and all of its elements are commutators of elements in $\mathcal{R}^{(n-1)}$.

Example 2.3.3 *This theorem asserts that the following Riordan matrix lie in $\mathcal{R}''$:*

$$R(1+x^2, x+x^4) = R(1+x^2, x)\cdot R(1, x+x^4) =$$

$$= \begin{bmatrix} 1 \\ 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} 1 \\ 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 & 0 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

We will give a sketch of the proof of Theorem 2.3.2 and one of the points will be proved separatedly in lemma 2.3.4.

Sketch of the proof of Theorem 2.3.2: We will prove it by induction over n. The base case has been studied in remark 2.3.1. Now we will go on with the induction:

- Assume the result is true for $\mathcal{R}^{(n)}$
- $R(d(x), h(x))$ is a commutators of elements in $\mathcal{R}^{(n)}$ if there exists $R(u(x), v(x)), R(f(x), g(x)) \in \mathcal{R}^{(n)}$ with:
$$R(d(x), h(x)) = [R(u(x), v(x)), R(f(x), g(x))]$$
that is:
$$\begin{cases} d(x) = \frac{1}{u(x)}\cdot\frac{1}{f(v^{-1}(x))}\cdot u(g^{-1}(v^{-1}(x)))\cdot f(v(g^{-1}(v^{-1}(x)))) \\ h(x) = v^{-1}(g^{-1}(v(g(x))) \end{cases}$$
which is equivalent to:
$$\begin{cases} u(x) = \left[\frac{f(v(g^{-1}(v^{-1}(x))))}{d(x)\cdot f(v^{-1}(x))}\right]\cdot u(g^{-1}(v^{-1}(x)))) \\ h(x)\in\mathcal{F}^{(n+1)} \end{cases} \tag{2.13}$$
- The first of these two equations:
$$u(x) = \left[\frac{f(v(g^{-1}(v^{-1}(x))))}{d(x)\cdot f(v^{-1}(x))}\right]\cdot u(g^{-1}(v^{-1}(x)))) \tag{2.14}$$
will be studied in 2.3.4, and will be shown to have an adequated solution.
- It is a simple comprobation to check that the set of commutators of $\mathcal{R}^{(n)}$ is a group (and then equal to $\mathcal{R}^{(n+1)}$) since the set of matrices satisfying (2.13) is closed under products and inversions.

$\square$

Lemma 2.3.4 *Equation* (2.14) *has a solution if and only if* $d(x) \in \left(1 + x^{2^{n+1}-(n+1)}\mathbb{K}[[x]]\right)$

Proof: We have two different cases:

- For $n = 0$ we are looking at the equation:

$$u(x) = \left[\frac{f(v(g^{-1}(v^{-1}(x))))}{d(x)\cdot f(v^{-1}(x))}\right]\cdot u(g^{-1}(v^{-1}(x))))$$

  where $v(x), g(x) \in \mathcal{F},\ f(x), u(x) \in \Big(\mathbb{K}[[x]] \setminus x\mathbb{K}[[x]]\Big)$.

  According to proposition 2.2.1, there exists such an $u(x)$ if and only if:

$$\frac{f(v(g^{-1}(v^{-1}(x))))}{d(x)\cdot f(v^{-1}(x))} \in \Big(1 + x\mathbb{K}[x]]\Big)$$

  and this happens if and only if $\frac{1}{d(x)}$ and equivalently $d(x)$ lie in $\Big(1 + x\mathbb{K}[x]]\Big)$

- For each $n \geq 1$, we are looking at the equation:

$$u(x) = \left[\frac{f(v(g^{-1}(v^{-1}(x))))}{d(x)\cdot f(v^{-1}(x))}\right]\cdot u(g^{-1}(v^{-1}(x))))$$

  where $v(x), g(x) \in \Big(x + x^{2^n}\mathbb{K}[[x]]\Big),\ f(x), u(x) \in \Big(1 + x^{2^n - n}\mathbb{K}[[x]]\Big)$.

  According to proposition 2.2.1, there exists such an $u(x)$ if and only if:

$$\frac{f(v(g^{-1}(v^{-1}(x))))}{d(x)\cdot f(v^{-1}(x))} \in \Big(1 + x^{2^{n+1}-(n+1)}\mathbb{K}[x]]\Big)$$

  and this happens if and only if $\frac{1}{d(x)}$ and equivalently $d(x)$ lie in $\Big(1 + x^{2^{n+1}-(n+1)}\mathbb{K}[x]]\Big)$

$\square$

2.4 The Conjugacy Problem in $\mathcal{A}'$ (or $\mathcal{F}_1'$), $\mathcal{R}'$

Recall that, for any group $(G, \cdot)$ (multiplicative notation), we say that $g, h \in G$ are **conjugated elements**, and we write $g \sim h$ if there exists $p \in G$ such that:

$$g = p^{-1}\cdot h\cdot p$$

This is an equivalence relation. In each equivalence class, any element can be said to be a **representative** of this conjugacy class.

The understanding of the conjugacy clases of a given group is very important for clarifying its algebraic structure. So obviously we will be interested in studying conjugacy in $\mathcal{A}$, $\mathcal{R}$

The reason why we are now studying conjugacy in $\mathcal{A}', \mathcal{R}'$ instead of in $\mathcal{A}, \mathcal{R}$ is double:

- Firstly, it has its own interest as originally the Riordan group was actually $\mathcal{R}'$.

- Secondly, the study of conjugacy in $\mathcal{A}'(\mathbb{K}), \mathcal{R}'(\mathbb{K})$ shows less dependence on the choice of the field $\mathbb{K}$ than the study of conjugacy in $\mathcal{A}(\mathbb{K})$, $\mathcal{R}(\mathbb{K})$, as we will see in the following section. So it is, in some sense, a way to present a unified approach to the study of conjugacy that can be concreted for particular choices of $\mathbb{K}$ in the next section for $\mathcal{A}(\mathbb{K})$, $\mathcal{R}(\mathbb{K})$.

In relation to conjugacy in $\mathcal{A}'$, or equivalently in $\mathcal{F}'_1$ we have:

Proposition 2.4.1 *Let $\mathbb{K}$ be a field of characteristic 0 and $f(x) \in \mathcal{F}'(\mathbb{K})$. If:*

$$f(x) = x + \ldots + f_n x^n + \ldots$$

then there exists $v(x)$ of the type $v(x) = x + \alpha x^n + \beta x^{2n-1}$ such that:

$$f(x) \sim v(x)$$

Before going on with the proof, note that this result is easily translated in terms of $\mathcal{A}$ where it is more visual. Let's denote with a bullet ($\bullet$) a nonzero entry in the matrix, leave in blank the entries that are zeroes, and write a cross product sign ($\times$) for an entry that may be zero or not. Then we can represent a matrix $R(1, h(x))$ in $\mathcal{A}'$, where $h(x)$ has multiplicity n as:

$$\begin{bmatrix}
1 &&&&&&&& \\
& 1 &&&&&&& \\
&& \ddots &&&&&& \\
& \bullet && 1 &&&&& \\
& \times & \bullet && 1 &&&& \\
& \times & \times & \bullet && 1 &&& \\
& \times & \times & \times & \bullet && 1 && \\
& \times & \times & \times & \times & \bullet && 1 & \\
& \vdots & \vdots & \vdots & \vdots & \ddots & \ddots && \ddots
\end{bmatrix}$$

where the "bullet" in the 1-column is in the entry $(n, 1)$. Then its canonical representative would be of the type:

$$\begin{bmatrix}
1 &&&&&&&& \\
& 1 &&&&&&& \\
&& \ddots &&&&&& \\
& \bullet && 1 &&&&& \\
&& \bullet && 1 &&&& \\
& \bullet && \bullet && 1 &&& \\
&& \bullet && \bullet && 1 && \\
&&& \bullet && \bullet && 1 & \\
&&&& \ddots && \ddots && \ddots
\end{bmatrix}$$

This matrix has two nonzero diagonals, the first one passing through $(n,1)$ and the second one through $(2n-1,1)$. It seems to be a banded matrix, but is not: for instance the entry:

$$(3n-1,2) = [x^{3n-1}](h(x))^2$$

is nonzero in general. Recall that no Riordan matrix is a banded matrix except from the identity matrix.

Remark 2.4.2 *Recall that in section 1.5 (we were talking about metrics in the Riordan group) we already presented the set of the elements in $\mathcal{A}'$ with multiplicity n, and was denoted by $\mathcal{G}_k$. It was shown to be a normal subgroup of $\mathcal{R}$, so it is not surprising now that all the conjugated elements to an element of multiplicity n have also multiplicity n.*

Proof or Proposition 2.4.1: We will prove this result in terms of matrices in $\mathcal{A}$. We want to show the existence of a Riordan matrix $R(1,g(x))$ satisfying:

$$R(1,f(x)) = R(1,g^{-1}(x))R(1,x+\alpha x^n+\beta x^{2n-1})R(1,g(x))$$

or equivalently:

$$R(1,g(x))\otimes f(x) = R(1,x+\alpha x^n+\beta x^{2n-1})\otimes g(x) \tag{2.15}$$

We will denote $R(1,g(x)) = (d_{i,j})_{0\leq i,j<\infty}$.
As usual, we will prove this result by induction over k in the partial grous $\mathcal{R}_k$. This time we will have to distinguish some cases:

- $k=n$. We have that the corresponding Π_k projection of both sides in equation (2.15) leads to the equation:

$$\left[\begin{array}{cccccc|c} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & g_2 & 1 & & & & \\ \vdots & \vdots & & \ddots & & & \\ 0 & g_{k-1} & \dots & \dots & 1 & & \\ \hline 0 & g_k & \dots & \dots & \dots & & 1 \end{array}\right]\begin{bmatrix}0\\1\\0\\\vdots\\0\\f_k\end{bmatrix} = \left[\begin{array}{ccccc|c} & & R_{k-1}(1,x) & & & \\ \hline 0 & \alpha & 0 & \dots & 0 & 1\end{array}\right]\begin{bmatrix}0\\1\\g_2\\\vdots\\g_k\end{bmatrix}$$

  This is a system of $k+1$ equation. For $0\leq i\leq k-1$ the first equations are $g_i=g_i$. Only the last equation needs to be studied, which is:

$$g_k+f_k = \alpha + g_k$$

  that is:

$$\alpha = f_k$$

- $n < k < 2n-1$. We have the corresponding Π_k projection of both sides in equation (2.15) leads to the equation:

$$\left[\begin{array}{cccccc|c} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & g_2 & 1 & & & & \\ \vdots & \vdots & & \ddots & & & \\ 0 & g_{k-1} & \dots & \dots & 1 & & \\ \hline 0 & g_k & \dots & \dots & \dots & & 1 \end{array}\right]\begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ f_n \\ \vdots \\ f_k \end{bmatrix} = \left[\begin{array}{ccc|c} & R_{k-1}(1, x+\alpha x^n) & & \\ \hline \dots & (k-n+1)\alpha & \dots & 1 \end{array}\right]\begin{bmatrix} 0 \\ 1 \\ g_2 \\ \vdots \\ g_k \end{bmatrix}$$

All the elements in the last row of the matrix in the right are zeros except, of course, the entry (k,k) and the entry $(k, k-n+1)$. In order to understand this, firstly we have to understand the structure of the matrix $R(1, x+\alpha x^n)$

$$\begin{bmatrix} 1 & & & & & & & \\ & 1 & & & & & & \\ & & 1 & & & & & \\ 0 & & & 1 & & & & \\ & \alpha & & & 1 & & & \\ & & 2\alpha & & & 1 & & \\ & & & 3\alpha & & & 1 & \\ & & & & \ddots & & & \ddots \end{bmatrix}$$

which according to the formula of the A-sequence, equals the identity matrix except from the elements in the diagonal $(n-1,0), (n,1), (n+1,2), \dots$ which are $0, \alpha, 2\alpha, \dots$

Thus, we have a system of $k+1$ equation. Equations $0, \dots, k-1$ are already fixed by induction hypothesis. And the k-th equation is:

$$g_k + f_n d_{kn} + \dots + d_{kk} f_k = (k-n+1)\alpha g_{k-n+1} + g_k \tag{2.16}$$

Note that:

$$d_{ki} = [x^k](g(x))^i = \sum_{j_1, \dots, j_i = k} g_{j_1} \dots g_{j_i} = i g_{k-i+1} + [\dots]$$

where everything inside the brackets depends on entries g_j with $j < k-i+1$. So equation (2.16) is of the type:

$$g_k + f_n n g_{k-n+1} + [\dots] = (k-n+1)\alpha g_{k-n+1} + g_k$$

or equivalently (recall that $\alpha = f_n$):

$$(k-2n+1)\alpha g_{k-n+1} = [\dots] \tag{2.17}$$

where the things inside the brackets depend on the variables g_i with $i < k-n+1$ (fixed by induction hypothesis), and where the coefficient of g_{k-n+1}, $(k-2n+1)\alpha$ is obviously nonzero. So this equation has a solution.

- For $k = 2n - 1$, this time we have:

$$\left[\begin{array}{cccccc|c} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & g_2 & 1 & & & & \\ \vdots & \vdots & & \ddots & & & \\ 0 & g_{k-1} & \dots & \dots & 1 & & \\ \hline 0 & g_k & \dots & \dots & \dots & & 1 \end{array}\right] \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ f_n \\ \vdots \\ f_k \end{bmatrix} = \left[\begin{array}{ccccc|c} & & R_{k-1}(1, x + \alpha x^n) & & & \\ \hline 0 & \beta & \dots & (k-n+1)\alpha & \dots & 1 \end{array}\right] \begin{bmatrix} 0 \\ 1 \\ g_2 \\ \vdots \\ g_k \end{bmatrix}$$

  where the entries in the dots in the last row of the matrix on the left are zero. Again, we have a system of $k+1$ equations. Equations $0, \dots, k-1$ are already fixed by induction hypothesis. And the k-th equation is now of the type (compare with equation (2.17)):

$$0 = \beta + [\dots]$$

  (this time the coefficient of $g_{k-n+1} = g_n$ is 0) so there is a unique β satisfying this equation. We fix $g_n = 0$, for example.

- For $k > 2n - 1$ we do the same as in case $k < 2n - 1$ and again we obtain an equation of the same type of (2.17), where again the coefficient of the variable g_{k-n+1} is nonzero.

□

From the proof above, we have also obtained:

Remark 2.4.3 *In terms of matrices in $\mathcal{A}(\mathbb{K})$ with $\mathbb{K}$ being a field of characteristic 0, the above result means that the conjugacy class of an element $R(d(x), h(x)) \in \mathcal{A}'$ with $h(x)$ of multiplicity n is determined by the finite matrix $R_{2n-1}(1, h(x))$.*

Concerning conjugacy in $\mathcal{R}'$ (as well as in $\mathcal{R}$), as in the case of the derived series, it is possible to reduce the problem to the corresponding problem (of conjugacy in this case) in $\mathcal{A}$ and to solve a weighted Schrder equation, as we can see in the following:

Remark 2.4.4 *$R(d(x), h(x)) = (R(f(x), g(x)))^{-1} R(u(x), v(x)) R(f(x), g(x))$ if and only if $R(f(x), g(x)) R(d(x), h(x)) = R(u(x), v(x)) R(f(x), g(x))$ which is equivalent to:*

(1) $R(f(x), g(x)) \otimes d(x) = R(u(x), v(x)) \otimes f(x)$

(2) $R(f(x), g(x)) \otimes h(x) = R(u(x), v(x)) \otimes g(x)$

Using the 1FTRM (1) is equivalent to:

$$f(x) \cdot d(g(x)) = u(x) \cdot f(v(x))$$

and then to:

$$\frac{u(x)}{d(g(x))} f(v(x)) = f(x)$$

which is a weighted Schröder equation in the indeterminate $f(x)$. (2) is equivalent to $h(x) \sim u(x)$.

We have not even tried to find a canonical representative of each conjugacy class in $\mathcal{R}'(\mathbb{K})$. but it is easy to find, at least a simple one:

Theorem 2.4.5 *Let $\mathbb{K}$ be a field of characteristic 0. Let $R(d(x), h(x)) \in \mathcal{R}'(\mathbb{K})$. If the multiplicity of $v(x)$ is k then:*

$$R(d(x), h(x)) \sim R(u(x), v(x)) \Leftrightarrow \begin{cases} h(x) = g^{-1}(v(g(x))) \\ Taylor_{k-1}(u(x)) = Taylor_{k-1}(d(g(x))) \end{cases}$$

This means that we can find a canonical representative $R(u(x), v(x))$ where $u(x)$ is a polynomial of degree $k-1$ and $v(x)$ is of the type $x + \alpha x^k + \beta x^{2k-1}$ of the conjugacy class of $R(d(x), h(x))$.

Proof: According to the previous remark, we want to solve the functional equation:

$$\frac{u(x)}{d(g(x))} f(v(x)) = f(x)$$

According to proposition 2.2.1 this has a solution (and this solution is unique) if and only if:

$$\frac{f(x)}{u(h(x))} \in 1 + x^k \mathbb{K}[[x]]$$

that is, if and only if:

$$Taylor_{k-1}(f(x)) = Taylor_{k-1}(u(h(x)))$$

This same argument can easily show that two of those representatives will never be conjugated.

$\square$

Using the same way of picturing as before, the representative used in the theorem are of the type:

$$\begin{bmatrix}
1 & & & & & & & & \\
\bullet & 1 & & & & & & & \\
\bullet & & 1 & & & & & & \\
\bullet & & & 1 & & & & & \\
 & \bullet & & & 1 & & & & \\
 & & \bullet & & & 1 & & & \\
 & & & \bullet & & & 1 & & \\
 & \bullet & & & \bullet & & & 1 & \\
 & & \ddots & & & \ddots & & & \ddots
\end{bmatrix}$$

To conclude this section we have to say that, up to our knowledge, there are no results concerning conjugacy in $\mathcal{A}(\mathbb{K})$, $\mathcal{R}(\mathbb{K})$ if $\mathbb{K}$ is an infinite field of positive characteristic. So, as far as we know, this question remains open (open question 13).

On the other hand, recall that we have already introduced the notation $\mathcal{C}(\mathbb{K}) = \mathcal{R}'(\mathbb{K})$ precisely to consider the groups $\mathcal{C}(R)$ where R is a commutative ring with unity. For this group, the proof above does not work, so the question of studying conjugacy in $\mathcal{C}(R)$ remains open (open question 14).

2.5 Some words on the general Conjugacy Problem in $\mathcal{A}$, $\mathcal{R}$

In this thesis, we have already stated some results about conjugacy, we will present them together in the following:

Theorem 2.5.1 *Let $\mathbb{K}$ be a field of characteristic 0:*

(i) Let $h(x) \in \mathcal{F}_1(\mathbb{K})$ and let λ be its multiplier. If either λ is not a root of unity or either $h(x)$ is an element of finite order then $h(x) \sim \lambda x$.

(ii) Let $D = R(d(x), h(x)) \in \mathcal{R}(\mathbb{K}))$ and let λ be the multiplier of $h(x)$. If either λ is not a root of unity or either D is an element of finite order or an element of finite order multiplied by a constant $R(d_{00}, x)$ then:

$$R(d(x), h(x)) \sim R(d(0), h'(0))$$

Proof: (i) follows from theorem 1.14.6 and from the first statement in remark 1.14.5. (ii) follows from theorem 1.14.7 and from the second statement in 1.14.5.

□

Results about conjugacy of the elements of finite order in $\mathcal{F}_1(\mathbb{C})$ were known a long time ago (see for example the work by J. F. Ritt [96]), in both, the formal an analytic categories. Also the case with the multiplier not a root of unity is also well known as a consequence of Köning's theorem (equivalent statement to theorem 1.14.6).

The general study of the conjugacy classes in $\mathcal{F}_1(\mathbb{C})$ has been done by B. Muckenhoupt in [84], and by S. Scheinberg in [99], and it also appears in the book by M. Kuczma, B. Choczewski and R. Ger in [59]. For a brief discussion of the analytic case (conjugacy in the group $\mathcal{F}_{1,hol}(\mathbb{C})$), see also [2]. The next result contains all the cases left concerning conjugacy in $\mathcal{F}_1(\mathbb{C})$:

Theorem 2.5.2 *Let $f(x) \in \mathcal{F}_1(\mathbb{C})$.*

(1) (theorem 1, [84]) If the multiplier of $f(x)$ is 1 and the multiplicity is n, there exists exactly one α such that:

$$f(x) \sim x + x^n + \alpha x^{2n-1}$$

(2) (theorem 5, [84]) If the multiiplier of $f(x)$ is primitive q-th root of unity a_1 and $f^{[q]}(x) \neq x$, and we have another $g(x) \in \mathcal{F}_1(\mathbb{C})$ with multiplier b_1:

$$f(x) \sim g(x) \text{ if and only if } a_1 = b_1 \text{ and } f^{[q]}(x) \sim g^{[q]}$$

(3) (proposition 10 in [99]) In the previous case, since we have:

$$f^{[q]}(x) \sim x + x^n + cx^{2n+1}$$

then:

$$f(x) \sim a_1 x + b_1 x^n + c_1 x^{2n-1} \text{ where: } \begin{cases} b_1 = \frac{1}{qa_1} \\ c_1 = \frac{ca_1}{q} - \frac{(q-1)n}{2q^2 a_1^3} \end{cases}$$

We are not going to prove this result here, since it is well known, but it can be proved by using our methods similarly to the proof of proposition 2.4.1. Before going on, just remark the following:

Remark 2.5.3 *There is an interpretation for those n, α in (1). Those numbers are related to the Julia equation and the iterative logarithm. We refer the reader to section 8.5 in* [59].

Unlike the results for conjugacy in $\mathcal{F}_1'(\mathbb{K})$, $\mathcal{A}'(\mathbb{K})$, conjugacy in $\mathcal{F}_1(\mathbb{K})$, $\mathcal{A}(\mathbb{K})$ depends strongly on $\mathbb{K}$ even if we restrict on fields of characteristic 0. It remains open to us the question of understanding conjugacy classes in $\mathcal{F}_1(\mathbb{K})$, $\mathcal{A}(\mathbb{K})$ for a general field of characteristic 0 (see open question 12).

We will include in the following remark what we know about conjugacy for other fields of interest as $\mathbb{K} = \mathbb{R}, \mathbb{Q}$, although again we will not give a proof, since it is exactly the same of proposition 2.4.1.

Remark 2.5.4 *Let $\mathbb{K}$ be a field of characteristic 0. Let $h(x) \in F(\mathbb{K})$ with multiplier 1 and multiplicity n. If we want to find a conjugacy representative of $h(x)$ in $\mathcal{F}(\mathbb{K})$ we will always be able to find a representative $v(x) = x + \alpha x^n + \beta x^{2n-1}$ (because $h(x)$ and $v(x)$ are conjugated as elements in $\mathcal{F}'(\mathbb{K})$).*

If $\mathbb{K} = \mathbb{C}$ we have already seen that we can choose $\alpha = 1$. It is easy to see that for $\mathbb{K} = \mathbb{R}$, we only can ensure that we can find such a representative with α being ± 1. For $\mathbb{K} = \mathbb{Q}$ in general the representant will have an arbitrary α.

We don't know much more about conjugacy in $\mathcal{R}(\mathbb{K})$ even for $\mathbb{K} = \mathbb{C}$. As explained in remark 2.4.4 the problem of conjugacy in $\mathcal{R}(\mathbb{K})$ can be reduced to studying conjugacy in $\mathcal{F}_1(\mathbb{K})$ (or in $\mathcal{A}(\mathbb{K})$) and studying a weighted Schröder equation. So since we have not studied all the cases of the weighted Schröder equation even for $\mathbb{K} = \mathbb{C}$ we cannot expect to complete such an study, that remains open for the future (open question 12).

2.6 Example: the Conjugacy Class of the Pascal Triangle

As an application, we will describe explicitly the conjugacy class of the Pascal Triangle. This was suggested by G. S. Cheon and it could be a good way to clarify the results above, specially being the Pacal Triangle our favourite example of Riordan matrix.

With "Pascal Triangle" this time we will refer to both: the Extended Pascal Triangle $\widetilde{P} = R\left(1, \frac{x}{1-x}\right)$ and the Classical Pascal Triangle $P = R\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$.

Both elements belong to the corresponding commutator: $\widetilde{P} \in \mathcal{A}'(\mathbb{K})$, $P \in \mathcal{R}'(\mathbb{K})$, where we can consider that $\mathbb{K} = \mathbb{R}, \mathbb{C}$.

According to the previous study, another matrix T is conjugated (in any of the groups listed above) to $\widetilde{P}$ or P respectively if and only if $\Pi_3(T)$ is, to $\Pi_3(\widetilde{P})$, $\Pi_3(P)$ in each case.

So we have:

Proposition 2.6.1 *The elements conjugated to $\widetilde{P}$ are those matrices $R(1, v(x))$ with $v(x)$ such that:*

- *If we are looking at conjugacy in $\mathcal{A}'(\mathbb{K})$ for any $\mathbb{K}$ of characteristic 0:*

$$v(x) = x + x^2 + x^3 + \mathcal{O}(x^4)$$

  or in other words, those elements in $\mathcal{A}'(\mathbb{K})$ with A-sequence of the type $(1, 1, 0, A_3, A_4, \ldots)$.

- *If we are looking at conjugacy in $\mathcal{A}(\mathbb{K})$ for $\mathbb{K} = \mathbb{C}, \mathbb{R}, \mathbb{Q}$:*

$$v(x) = x + \alpha x^2 + \alpha^2 x^3 + v_4 x^4 + \ldots \text{ with } \alpha \neq 0$$

  in other words, the elements in $\mathcal{A}(\mathbb{K})$ with A-sequence of the type $(1, \alpha, 0, A_3, A_4, \ldots)$ with $\alpha \neq 0$.

Proof: We have already stated that we only need to solve the problem in $\mathcal{A}_3(\mathbb{K})$. Now we are going to check this 4×4 case in detail. We are looking for the matrices of the type:

$$\begin{bmatrix} 1 & & & \\ 0 & a_0 & & \\ 0 & a_0a_1 & a_0^2 & \\ 0 & a_0a_1^2 + a_0^2a_2 & 2a_0^2a_1 & a_0^3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & a_0 & & \\ 0 & a_0a_1 & a_0^2 & \\ 0 & a_0a_1^2 + a_0^2a_2 & 2a_0^2a_1 & a_0^3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & & & \\ 0 & \frac{1}{a_0} & & \\ 0 & -\frac{a_1}{a_0^2} & \frac{1}{a_0^2} & \\ 0 & \frac{a_1^2}{a_0^3} - \frac{a_2}{a_0^2} & -2\frac{a_1}{a_0^3} & \frac{1}{a_0^3} \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 1 & 1 & \\ 0 & 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & a_0 & & \\ 0 & a_0a_1 & a_0^2 & \\ 0 & a_0a_1^2 + a_0^2a_2 & 2a_0^2a_1 & a_0^3 \end{bmatrix} =$$

$$= \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & \frac{1}{a_0} & 1 & \\ 0 & \frac{1}{a_0^2} & \frac{2}{a_0} & 1 \end{bmatrix}$$

The result follows from a case by case comprobation in the matrix above.

□

Proposition 2.6.2 *The set of matrices $R(u(x), v(x))$ in $\mathcal{R}(\mathbb{K})$ for $\mathbb{K} = \mathbb{Q}, \mathbb{R}, \mathbb{C}$ conugated to $R\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ are those of the type:*

$$\begin{bmatrix} 1 & & & & \\ \frac{1}{v_2} & 1 & & & \\ d_{20} & d_{21} & 1 & & \\ d_{30} & (d_{20} + 2d_{21}^2 - \frac{d_{21}}{v_2}) & (d_{21} - \frac{1}{v_2}) & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $v(x) = v_1 x + v_2 x^2 + \ldots$, and all the entries of the type d_{ij} together with the entries not included above are free.

Proof: Concerning the first column, we only need to look at $\mathcal{R}_2(\mathbb{K})$. We are looking for the matrices of the type (we can suppose that the entry $(0,0)$ of the matrix we are using to conjugate is 1):

$$\begin{bmatrix} 1 & & \\ d_{10} & d_{11} & \\ d_{20} & d_{21} & d_{11}^2 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ d_{10} & d_{11} & \\ d_{20} & d_{21} & d_{11}^2 \end{bmatrix}^{-1}$$

We only need the 0-column of this product, which is:

$$\begin{bmatrix} 1 & & \\ d_{10} & d_{11} & \\ d_{20} & d_{21} & d_{11}^2 \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -\frac{d_{10}}{d_{11}} \\ -\frac{d_{20}}{d_{11}^2} + \frac{d_{10}d_{21}}{d_{11}^3} \end{bmatrix} = \begin{bmatrix} 1 \\ d_{11} \\ [d_{21} + d_{11}^2] - (2d_{11})d_{10} \end{bmatrix}$$

It is easy to see that, since $2d_{11} \neq 0$, varying d_{10} we can get any value in the last entry of this column. The same would have happened if we had tried to solve the problem in $\mathcal{R}_k(\mathbb{K})$: fixing adequately $d_{k-1,0}$ we can obtain any entry in the last column of this product, as explained in section 2.2.

□

2.7 Application of Conjugacy I: centralizers

The study of the problem of finding the centralizer of elements in $\mathcal{R}$ is closely related to the study of the problem of conjugacy. This problem has been also partially studied by C. Jean-Louis and A. Nkwanta, who found the centralizer of $R(1, -x)$ (theorem 5 in [51]).

Recall that:

Definition 2.7.1 *For any group $(G, \cdot)$ the* ***centralizer*** *of an element $g \in G$ is the set:*

$$Z_G(h) = \{h \in G \ : \ hg = gh\}$$

Analogously, we can define the centralizer of a subset S of G as:

$$Z_G(S) = \{h \in G \ : \ \forall g \in S, \ hg = gh\}$$

The centralizer of G is called the ***center*** *of the group, is the set of elements that commute with all the elements in G.*

According to the following lemma (we omit the proof):

Lemma 2.7.2 *Let $(G, \cdot)$ be a group. Let an element $g \in G$ such that $g = v^{-1}pv$ for some $v, p \in G$. Then h is in the centralizer of g if and only if $g = v^{-1}\widetilde{g}v$ for some $\widetilde{g}$ in the centralizer of p.*

Having found some canonical representative for the conjugacy classes in $\mathcal{A}$ and $\mathcal{R}$, we have made a step towards a way to find the centralizer of any element in $\mathcal{A}$ or $\mathcal{R}$, since it is easy to compute the centralizer of some of those canonical representatives:

Proposition 2.7.3 *For any field $\mathbb{K}$ of characteristic 0, let $T = R(\lambda, \mu x) \in \mathcal{R}(\mathbb{K})$.*

(i) If μ is not a root of unity, then $Z_{\mathcal{R}}(T)$ is the set of diagonal matrices $R(a, bx) \in \mathcal{R}$.

(ii) If $\mu = 1$, then $Z_{\mathcal{R}}(T)$ is $\mathcal{R}$.

(iii) If $\mu = -1$, (root of unity of order 2), then $Z_{\mathcal{R}}(T)$ is the Checkerboard Subgroup.

Proof: We will prove each case separatedly:

(i) Obviously, diagonal matrices commute. On the other hand, we can see that if μ is not a root of unity, from:

$$\begin{bmatrix} 1 & \\ d_{10} & d_{11} \end{bmatrix} \begin{bmatrix} \lambda & \\ & \lambda\mu \end{bmatrix} = \begin{bmatrix} \lambda & \\ & \lambda\mu \end{bmatrix} \begin{bmatrix} 1 & \\ d_{10} & d_{11} \end{bmatrix}$$

looking at the position $(1, 0)$, we obtain that:

$$d_{10}\lambda = \lambda\mu d_{10} \Rightarrow d_{10} = \mu d_{10}$$

and since $\mu \neq 0, 1$, this can only hold if $d_{10} = 0$. Assuming the result is true for $n-1 \geq 1$, then from:

$$\left[\begin{array}{ccccc|c} 1 & & & & & \\ 0 & d_{11} & & & & \\ \vdots & \ddots & \ddots & & & \\ 0 & \dots & 0 & d_{n-1,n-1} & & \\ \hline d_{n0} & d_{n1} & \dots & 0 & & d_{nn} \end{array}\right] \begin{bmatrix} \lambda & & & & \\ & \lambda\mu & & & \\ & & \ddots & & \\ & & & \lambda\mu^{n-1} & \\ & & & & \lambda\mu^n \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda & & & & \\ & \lambda\mu & & & \\ & & \ddots & & \\ & & & \lambda\mu^{n-1} & \\ & & & & \lambda\mu^n \end{bmatrix} \left[\begin{array}{ccccc|c} 1 & & & & & \\ 0 & d_{11} & & & & \\ \vdots & \ddots & \ddots & & & \\ 0 & \dots & 0 & d_{n-1,n-1} & & \\ \hline d_{n0} & d_{n1} & \dots & 0 & & d_{nn} \end{array}\right]$$

we obtain that $d_{n0} = d_{n1} = 0$.

(ii) This part is obvious using the 1FTRM.

(iii) In the case $\mu = -1$, repeating the argument above we obtain that d_{10} must be 0. Then assuming the result is true for certain $n-1 \geq 1$ we have that from the equations in the last row:

$$\left[\begin{array}{ccc|c} 1 & & & \\ \vdots & \ddots & & \\ d_{n-1,0} & \dots & d_{n-1,n-1} & \\ \hline d_{n0} & d_{n1} & \dots & d_{nn} \end{array}\right] \begin{bmatrix} \lambda & & & \\ & \ddots & & \\ & & \lambda\mu^{n-1} & \\ & & & \lambda\mu^n \end{bmatrix} =$$

$$= \begin{bmatrix} \lambda & & & \\ & \ddots & & \\ & & \lambda\mu^{n-1} & \\ & & & \lambda\mu^n \end{bmatrix} \left[\begin{array}{ccc|c} 1 & & & \\ \vdots & \ddots & & \\ d_{n-1,0} & \dots & d_{n-1,n-1} & \\ \hline d_{n0} & d_{n1} & \dots & d_{nn} \end{array}\right]$$

we obtain the desired result, since they are:

$$\begin{cases} (1-\mu^n)d_{n0} = 0 \\ (1-\mu^{n-1})d_{n1} = 0 \end{cases}$$

and the coefficients of d_{n0}, d_{n1} vanish when $\mu^n = 1, \mu^{n-1} = 0$ respectively.

□

Note that:

Remark 2.7.4 *We are not going to check it out but, in general, if μ is a root of unity of order $q \geq 2$, then $Z_{\mathcal{R}}(T)$ is the* ***generalized Checkerboard subgroup****:*

$$GCh_{(q)} = \{R(d(x), h(x)) : [x^n]d(x) = [x^{n+1}]h(x) = 0, \text{ unless } n \equiv 0 \mod q\}$$

Remark 2.7.5 *For any field $\mathbb{K}$ of characteristic 0, the center of $\mathcal{R}(\mathbb{K})$ is the set of diagonal matrices which diagonal is a constant pregression:*

$$\{R(\lambda, x) \in \mathcal{R}\}$$

It is easy to check that for any Riordan matrix $R(d(x), h(x))$, and for all $0 \neq \lambda \in \mathbb{K}$ we have that ((ii) in the previos proposition):

$$R(d(x), h(x))R(\lambda, x) = R(\lambda, x)R(d(x), h(x))$$

On the other hand, the matrices in the center must commute with every element in $\mathcal{R}(\mathbb{K})$ including the diagonal ones. So by (i) in the previous proposition we also know that the elements in the center must be diagonal matrices. Finally, looking only at the corresponding part in $\mathcal{F}_1(\mathbb{K})$, we can see that if:

$$g(\lambda x) = \lambda g(x)$$

for $g(x) = x + x^2$, then λ must be 1.

Remark 2.7.6 *We can particularize (we will skip the proof) the above results to the group $\mathcal{A}$ or equivalently to $\mathcal{F}_1$ we obtain, the following results, most of them known:*
For any field $\mathbb{K}$, let $T = R(1, \mu x) \in \mathcal{A}(\mathbb{K})$.

(i) If μ is not a root of unity, then the centralizer of T is the set of diagonal matrices $R(1, cx) \in \mathcal{A}$.

(ii) If $\mu = 1$, the centralizer of T is $\mathcal{A}$.

(iii) If $\mu = -1$, (root of unity of order 2), then the centralizer of T is the intersection of the Checkerboard subgroup with $\mathcal{A}$.

(iv) In general, if μ is a root of unity of order $q \geq 2$, the centralizer of T is the intersection of the ***generalized Checkerboard subgroup*** *with $\mathcal{A}$.*

We are neither going to compute the centralizer of any element in $\mathcal{R}(\mathbb{K})$, nor in $\mathcal{A}(\mathbb{K})$ (or equivalently in $\mathcal{F}_1(\mathbb{K})$).

The problem of finding the centralizer of elements in $\mathcal{F}_1(\mathbb{C})$ is treated in the book [59] section 8.7A.

On other occasions in this work, the problem of finding the centralizer of an element $R(d(x), h(x)) \in \mathcal{R}(\mathbb{K})$ can be reduced to finding the centralizer of $h(x) \in \mathcal{F}_1(\mathbb{K})$ and then solving a weighted Schröder equation. So in order to complete the study of centralizers of elements in $\mathcal{R}(\mathbb{K})$ (see open question 16) we would need to be able to solve all the cases of the weighted Schröder equation.

2.8 Application of Conjugacy II: powers of Riordan matrices

When studying the group $\mathcal{F}_1(\mathbb{C})$, specially if we want to study one parameter groups in order to study the Lie group structure of $\mathcal{F}_1(\mathbb{C})$, it is reasonable to study the iterative powers and iteratives roots of elements in $\mathcal{F}_1(\mathbb{C})$.

The idea is widely developed in the books [58, 59]. It is obvious what we understand by the n-th iterative power of an element $g(x) \in \mathcal{F}_1(\mathbb{C})$ when $n \in \mathbb{Z}$: it is $g^{[n]}(x)$. On the other hand we say that $v(x)$ is an n-th iterative root of $g(x)$ for $n \in \mathbb{Z}$ if:

$$v^{[n]}(x) = g(x)$$

Using this idea, we can say in general that $v(x)$ is the $\frac{a}{b}$-th iterative power of $g(x)$ if:

$$v(x)^{[a]} = g^{[b]}(x)$$

for $a \in \mathbb{Z}$, $0 \neq a \in \mathbb{N}$.

Using the natural identification $h(x) \leftrightarrow R(1, h(x))$ between elements in $F_1(\mathbb{K})$ and elements in $\mathcal{A}(\mathbb{K})$, we can see that, for $r \in \mathbb{Q}$, the r-th iterative power of $h(x)$ corresponds to r-th power or the corresponding matrix: $(R(1, h(x)))^r$.

Thanks to our study of conjugacy, it is sometimes easy to compute the powers or those matrices:

Remark 2.8.1 *For all $a \in \mathcal{R}$, for all T in either $\mathcal{R}(\mathbb{R})$ or $\mathcal{R}(\mathbb{C})$ we have that if:*

$$T = P^{-1}SP$$

then:

$$T^a = P^{-1}S^aP$$

Which allow us, for instance to compute the powers and root of any matrix conjugated to diagonal matrices in $\mathcal{R}(\mathbb{R})$ or in $\mathcal{R}(\mathbb{C})$ respectively.

Analogously we could think about powers of matrices in $\mathcal{R}(\mathbb{C})$ and in $\mathcal{R}(\mathbb{R})$ which in fact are very relevant to find the one parameter subgroups in relation to the Lie group structure of the Riordan group. But we will leave this for future work (open question 15).

2.9 The abelianized of $\mathcal{R}_n$

In any group G, the **abelianization** is the quotient $G/[G, G]$. If G is abelian, this quotient is G, so in some sense the abelianization is a measure of the "commutativity" of G: the bigger is the abelianization the "more commutative" is G. We have that:

Proposition 2.9.1 *Let $n \geq 1$ and let $\mathbb{K}$ be a field of 0 characteristic. Then the abelianized group $\mathcal{R}_n/\mathcal{R}'_n$ is a group isomorphic to:*

(1) The product $\mathbb{K}^\star \times \mathbb{K}^\star$.

(2) The group $\left\{\dfrac{\lambda}{1 - rx},\ \lambda, r \in \mathbb{K}^\star\right\}$ with the Hadamard product of series.

Finally $\mathcal{R}_n$ is isomorphic to the semidirect product

$$\mathcal{R}'_n \rtimes \mathcal{R}_n/\mathcal{R}'_n$$

Proof: The groups in (1) and (2) are obviously isomorphic. So we only need to check that $\mathcal{R}_n/\mathcal{R}'_n$. We only need to apply the first isomorphism theorem for the diagram:

$$\begin{tikzcd} \mathcal{R}_n \arrow[r, "\Phi"] \arrow[d] & \mathbb{K}^* \times \mathbb{K}^* \\ \mathcal{R}_n/ker(\Phi) \arrow[ur] & \end{tikzcd}$$

for Φ given by $\Phi(R(d(x), h(x))) = (d(0), h'(0))$.

The set of diagonal matrices $\mathcal{D}_n$ and the first derivative subgroup $\mathcal{R}'_n$ only intersect in the neutral element $R_n(1, x)$ and we have that $\mathcal{R}_n = \mathcal{D}_n\mathcal{R}'_n$. So:

$$\mathcal{R}_n = \mathcal{R}'_n \rtimes \mathcal{R}_n/\mathcal{R}'_n$$

$\square$

A complete characterization of the quotients $\mathcal{R}_n^{(k+1)}/\mathcal{R}_n^{(k)}$ would be desirable. This will be left as an open question (open question 11).

Chapter 3

Involutions and elements of finite order in the Riordan Group

For a start, we will remember that:

Definition 3.0.1 *In any group $(G, *)$ with identity element e, we say that $g \in G$ is an **involution** if and only if g is an **element of order 2**, that is, $g * g = e$, or in other words, g is the inverse of g.*

Involutions are a particular case of **elements of finite order** q in G, that is, elements such that $g^q = e$ (multiplicative notation) and such that there is no other $1 \leq k < q$ such that $g^k = e$.

Unless otherwise specified, in this chapter $\mathbb{K}$ will denote one of the fields $\mathbb{Q}, \mathbb{R}$ or $\mathbb{C}$.

This chapter deals mainly with some questions related to involutions in the Riordan group, although also some aspects of elements of general finite order will be visited.

In section 3.1, some basic aspects about involutions in $\mathcal{R}(\mathbb{K})$ will be included. In section 3.2 a characterization of the entries of the involutions in $\mathcal{R}(\mathbb{K})$ will be made. It will be followed by section 3.3, where some explicit examples of this characterization may be found, together with a particularization of this result to $\mathcal{A}(\mathbb{K})$ (or equivalently to $\mathcal{F}_1(\mathbb{K})$).

After this, in section 3.4 a similar characterization will be proved for elements of finite order q in $\mathcal{R}(\mathbb{C})$.

After the revision of some aspects concerning the A-sequence in sections 3.6 and 3.7 we will find an algebraic study of the group generated by the involutions $\mathcal{I} < \mathcal{R}(\mathbb{K})$ including the minimal number of involutions needed to express an element of $\mathcal{I}$ as a product of involutions. Some motivation of those results will also be given. In section 3.8 we will include some consequences of the results in 3.6 and 3.7.

3.1 Basics about involutions

Particularizing definition 3.0.1 to the Riordan group, we have that an involution in $\mathcal{R}(\mathbb{K})$ is a matrix $R(d(x), h(x))$ such that :

$$R(d(x), h(x)) \cdot R(d(x), h(x)) = R(d(x) \cdot d(h(x)), h(h(x))) = R(1, x)$$

or equivalently, using the group law, such that:

$$R(d(x), h(x)) = (R(d(x), h(x)))^{-1} = R\left(\frac{1}{d(h^{-1}(x)))}, h^{-1}(x)\right) \tag{3.1}$$

This last expression establishes the bridge between the problem of finding involutions and the problem of solving a system of functional equations, which has shown to be useful in the previous chapters.

First of all, note that:

Remark 3.1.1 *Using the identification between Riordan matrices and elements in the inverse limit* $\varprojlim\{(\mathcal{R}_n), (P_n)\}_{n\geq 0}$, *it is easy to see that* $D \in \mathcal{R}(\mathbb{K})$ *is an involution, if and only if* $D_n = \Pi_n(D)$ *is an involution for every* $n \geq 0$.

It is clear that in any involution in $\mathcal{R}_n(\mathbb{K})$ the elements in the main diagonal should be ± 1 (in lower triangular matrices the elements in the position (i, i) of the product are the product of the elements in the position (i, i) of each factor).

There are two obvious Riordan matrices whose squares are the identity $R_n(1, x)$, the identity itself and $-R_n(1, x) = R_n(-1, x)$. In fact:

Remark 3.1.2 *If* $D_n = R_n(d(x), h(x))$ *with* $D_n^2 = R_n(1, x)$ *and* $h'(0) = 1$, *then* $D_n = \pm R(1, x)$ *depending on* $d(0) = \pm 1$.

This is because the minimal polynomial of D_n *must divide the polynomial* $x^2 - 1$. *Since* $D_n \in \mathcal{R}_n(\mathbb{K})$ *and* $h'(0) = 1$ *then all the elements in the main diagonal are* 1 *if* $d(0) = 1$ *and all elements are* -1 *if* $d(0) = -1$. *So the characteristic polynomial of* D_n *is* $(x \pm 1)^{n+1}$. *Consequently, the minimal polynomial of* D_n *is either* $x - 1$ *or* $x + 1$, *hence* $D_n = R_n(1, x)$ *or* $D_n = -R(1, x)$.

Considering now any infinite Riordan matrix as an element in $\varprojlim\{(\mathcal{R}_n), (P_n)\}_{n\geq 0}$ *we have the above result for infinite Riordan matrices.*

$R(1, x)$ is not considered to be an involution, since it has order 1, not order 2. $R(-1, x)$ is actually an involution, that is called (either in the finite or infinite case) the **trivial involution**. The rest of the involutions in $\mathcal{R}(\mathbb{K})$ (which diagonal is a geometric progression of rate -1, and so the 0-term of its A-sequence is -1) will be called **non-trivial** involutions.

Even the previous remark has its interpretation in terms of formal power series, proving a result in an elegant way:

Remark 3.1.3 *A consequence of the above reasoning is that the unique solution of the Babbage equation:*

$$\begin{cases} \omega(\omega(x)) = x \\ \omega'(0) = 1 \end{cases}$$

is $\omega(x) = x$.

To talk about Riordan involutions, it is sometime much more comfortable to use the notation $T(f \mid g)$ (see section 0.3.4) since, for instance:

Remark 3.1.4

$$T^{-1}(1 \mid g) = T(1 \mid A) \qquad \Leftrightarrow \qquad T^{-1}(1 \mid A) = T(1 \mid g)$$

So, if $T(f \mid g)$ *is an involution* $A = g$.

3.2 Finite and infinite non-trivial involutions.

Our first step to construct all Riordan involutions, finite or infinite, is the following:

Lemma 3.2.1 (Band of Involutions) *Let* $D_n = R_n(d(x), g(x))$ *be an involution. Then* $R_{n-1}\left(d(x)\cdot\left(\frac{h(x)}{x}\right)^j, h(x)\right)$ *is an involution for every* $j \in \mathbb{Z}$.

Let us ilustrate this result, before its proof. If the indicated 3×3 matrix is an involution:

$$\begin{bmatrix} \times & & & & & \\ \bullet & \bullet & & & & \\ \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

then so are the following 2×2 matrices, and so are the rest of the matrices of this type that lie along the diagonal up and below in the correspondent bi-infinite matrix:

$$\begin{bmatrix} \ddots & & & & & & \\ \cdots & \times & & & & & \\ \cdots & \bullet & \bullet & & & & \\ \cdots & \bullet & \bullet & \bullet & & & \\ \cdots & \bullet & \bullet & \bullet & \bullet & & \\ \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \begin{bmatrix} \ddots & & & & & & \\ \cdots & \times & & & & & \\ \cdots & \bullet & \bullet & & & & \\ \cdots & \bullet & \bullet & \bullet & & & \\ \cdots & \bullet & \bullet & \bullet & \bullet & & \\ \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad \begin{bmatrix} \ddots & & & & & & \\ \cdots & \times & & & & & \\ \cdots & \bullet & \bullet & & & & \\ \cdots & \bullet & \bullet & \bullet & & & \\ \cdots & \bullet & \bullet & \bullet & \bullet & & \\ \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\begin{bmatrix} \ddots & & & & & & \\ \cdots & \times & & & & & \\ \cdots & \bullet & \bullet & & & & \\ \cdots & \bullet & \bullet & \bullet & & & \\ \cdots & \bullet & \bullet & \bullet & \bullet & & \\ \cdots & \bullet & \bullet & \bullet & \bullet & \bullet & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Proof: Let $R_n(d(x), h(x)) \in \mathcal{R}_n(\mathbb{K})$. Suppose that $(R_n(d(x), h(x))^2 = R(1, x)$. Then we know that:

$$Taylor_n(d(x)\cdot d(h(x)) = 1, \qquad Taylor_n(h(h(x))) = x \tag{3.2}$$

We only need to check that for any j:

$$Taylor_{n-1}\left(d(x)\cdot\left(\frac{h(x)}{x}\right)^j\cdot d(h(x))\cdot\left(\frac{h(h(x))}{h(x)}\right)^j\right) = 1 \qquad Taylor_{n-1}(h(h(x))) = x \tag{3.3}$$

The second equation in (3.2) obviously implies the second equation in (3.3). The first one in (3.3) is also true since, by using both equations in (3.2):

$$Taylor_{n-1}\left(d(x)\cdot\left(\frac{h(x)}{x}\right)^j\cdot d(h(x))\cdot\left(\frac{h(h(x))}{h(x)}\right)^j\right)=$$

$$=Tylr_{n-1}\left[\left(Tylr_{n-1}\left(d(x)\cdot d(h(x))\right)\right)\cdot\left(Tylr_{n-1}\left(\left(\frac{h(x)}{x}\right)\right)\right)^j\cdot\left(Tylr_{n-1}\left(\left(\frac{h(h(x))}{h(x)}\right)\right)\right)^j\right]=$$

$$=Taylor_{n-1}\left[(1)\cdot\left(Tylr_{n-1}\left(\left(\frac{h(x)}{x}\right)\right)\right)^j\cdot\left(Tylr_{n-1}\left(\left(\frac{x}{h(x)}\right)\right)\right)^j\right]=1$$

□

This result is in the core of the proof of the main result in this section. There is an alternative and more general proof (it will be included in section 3.4, since it also works for elements of order $q\geq 2$) of this main result bypassing some points in this proof. We consider that both proofs are interesting: the first one is more direct and combinatorial, and checks directly the needed property doing computations with the entries of the matrix, the second one is more abstract, is much more of a linear algebraic flavour, and also uses the results about conjugacy in chapter 2.

Theorem 3.2.2 *Suppose $n\geq 2$. Let $D_{n-1}=(d_{i,j})_{0\leq i,j<n-1}\in\mathcal{R}_{n-1}(\mathbb{K})$ be an involution and take $\hat{D}_n=(d_{i,j})\in\mathcal{R}_n$ such that $P_{n-1}(\hat{D}_n)=D_{n-1}$.*

(a) If n is even, $\hat{D}_n$ is an involution if and only if ($d_{n,1}$ is arbitrary) and:

$$d_{n,0}=-\frac{1}{2d_{0,0}}\sum_{k=1}^{n-1}d_{k,0}d_{n,k}\tag{3.4}$$

(b) If n is odd, $\hat{D}_n$ is an involution if and only if ($d_{n,0}$ is arbitrary) and:

$$d_{n,1}=-\frac{1}{2d_{1,1}}\sum_{k=2}^{n-1}d_{k,1}d_{n,k}\tag{3.5}$$

Moreover, if the A-sequence of D_{n-1} is $(a_0,\cdots,a_{n-2})$ then the needed a_{n-1} to construct $\hat{D}_n$ is given by the formula:

$$a_{n-1}=\frac{1}{d_{n-1,n-1}}\left(d_{n,1}-\sum_{j=0}^{n-2}a_jd_{n-1,j}\right)\tag{3.6}$$

Proof: It is clear that the unique elements in $\mathcal{R}_0(\mathbb{K})$ that multiplied by themselves are the unity are (1) and (-1) ($\mathcal{R}_0(\mathbb{K})$ is isomorphic to $(\mathbb{K}*.\cdot)$ as a group. In $\mathcal{R}_1$ are those of the type:

$$\begin{bmatrix} d_{0,0} & \\ d_{1,0} & -d_{0,0} \end{bmatrix}$$

where $d_{0,0} = \pm 1$ and $d_{1,0}$ is arbitrary.

The bigger cases, will be studied by following the iterative method that will be described below.

We will divide the proof in two parts. In part 1, we will prove a claim related to how the different steps of this iterative method fix. In part 2 we will describe this iterative method to extend involutions in $\mathcal{R}_n(\mathbb{K})$ to involutions in $\mathcal{R}_{n+1}(\mathbb{K})$.

PART 1: **Claim:** *Let $m \geq 2$. Suppose $B_{m-1}, C_{m-1} \in \mathcal{R}_{m-1}(\mathbb{K})$ are two involutions with $a_0 = -1$ and such that $Q_{m-2}(B_{m-1}) = P_{m-2}(C_{m-1})$, that is, the two indicated submatrices are equal:*

$$B_{m-1} = \begin{bmatrix} \times & & & \\ \times & \bullet & & \\ \times & \bullet & \bullet & \\ \times & \bullet & \bullet & \bullet \end{bmatrix}, \qquad C_{m-1} = \begin{bmatrix} \bullet & & & \\ \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \\ \star & \star & \star & \star \end{bmatrix}$$

Let:

$$\mathcal{E}(B_{m-1}, C_{m-1}) = \{D_m \in \mathcal{R}_m(\mathbb{K}) : \ P_{m-1}(D_m) = B_{m-1} \textit{ and } \ Q_{m-1}(D_m) = C_{m-1}\}$$

that is, in relation to the previous diagram, the set of matrices of the type:

$$\begin{bmatrix} \times & & & & \\ \times & \bullet & & & \\ \times & \bullet & \bullet & & \\ \times & \bullet & \bullet & \bullet & \\ d_{m0} & \star & \star & \star & \star \end{bmatrix}$$

where $d_{m0} \in \mathbb{K}$. Then,

(i) If m is even, there is a unique involution in $\mathcal{E}(B_{m-1}, C_{m.1})$

(ii) If m is odd, any matrix in $\mathcal{E}(B_{m-1}, C_{m-1})$ is an involution.

In order to prove this claim, suppose $B_{m-1} = (b_{i,j})_{i,j=0\cdots m-1}$ and $C_{m-1} = (c_{i,j})_{i,j=0\cdots m-1}$. Let $D_m = (d_{i,j})_{i,j=0\cdots m} \in \mathcal{E}(B_{m-1}, C_{m-1})$. Then:

$$d_{i,j} = b_{i,j} = c_{i-1,j-1} \text{ for } i,j = 1, \cdots, m-1$$

$$d_{i,0} = b_{i,0} \ i = 0, \cdots, m-1 \text{ and } d_{m,j} = c_{m-1,j-1} \ i = 1, \cdots, m$$

Consequently, the unique entry in D_m which is not determined by B_{m-1} and C_{m-1} is $d_{m,0}$.

By choosing suitable blocks and using matrix multiplication by blocks we can compute $D_m^2 = (\bar{d}_{i,j})_{i,j=0,\cdots,m}$ where, except for the place $(m,0)$, $\bar{d}_{i,j} = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases}$ and in the place $(m,0)$ we have $\bar{d}_{m,0} = \sum_{k=0}^{m} d_{m,k}d_{k,0}$. Hence D_m is an involution if and only if:

$$\bar{d}_{m,0} = \sum_{k=0}^{m} d_{m,k}d_{k,0} = 0 \tag{3.7}$$

(i) Suppose m is even. Since D_m is a Riordan matrix with $a_0 = -1$, then $d_{m,m} = d_{0,0}$ and the unique value of $d_{m,0}$ making (3.7) hold is

$$d_{m,0} = -\frac{1}{2d_{0,0}} \sum_{k=1}^{m-1} d_{m,k}d_{k,0}$$

(ii) Suppose m is odd and write $m = 2n+1$ with $n \geq 1$. Since D_m is a Riordan matrix with $a_0 = -1$, then $d_{m,m} = -d_{0,0}$ consequently (3.7) is equivalent to:

$$\sum_{k=1}^{2n} d_{k,0}d_{2n+1,k} = 0$$

Note that the above equality depends only on the matrices B_{m-1} and C_{m-1} and that:

$$\sum_{k=1}^{2n} d_{k,0}d_{2n+1,k} = d_{1,0}d_{2n+1,1} + \sum_{k=2}^{2n} d_{k,0}d_{2n+1,k} = (*)$$

Since C_{m-1} is an involution we get:

$$d_{2n+1,1} = \frac{1}{2d_{0,0}} \sum_{k=2}^{2n} d_{k,1}d_{2n+1,k}$$

and then:

$$(*) = \frac{d_{1,0}}{2d_{0,0}} \sum_{k=2}^{2n} d_{k,1}d_{2n+1,k} + \sum_{k=2}^{2n} d_{k,0}d_{2n+1,k} = \frac{1}{2d_{0,0}} \sum_{k=2}^{2n} (d_{1,0}d_{k,1} + 2d_{0,0}d_{k,0})d_{2n+1,k} = \frac{1}{2d_{0,0}}(**)$$

So we need to check that $(**)$ equals 0.

- Dividing the expression above into even and odd rows we get:

$$(**) = \sum_{i=1}^{n} (d_{1,0}d_{2i,1} + 2d_{0,0}d_{2i,0})d_{2n+1,2i} + \sum_{i=1}^{n-1} (d_{1,0}d_{2i+1,1} + 2d_{0,0}d_{2i+1,0})d_{2n+1,2i+1}$$

- As we know that $P_2 \circ \cdots \circ P_{m-2}(B_{m-1})$ is an involution, then:

$$d_{1,0}d_{2,1} + 2d_{0,0}d_{2,0} = 0$$

thus:

$$(**) = \sum_{i=2}^{n}(d_{1,0}d_{2i,1} + 2d_{0,0}d_{2i,0})d_{2n+1,2i} + \sum_{i=1}^{n-1}(d_{1,0}d_{2i+1,1} + 2d_{0,0}d_{2i+1,0})d_{2n+1,2i+1}$$

- As $P_{2i} \circ \cdots \circ P_{m-2}(B_{m-1})$ is an involution, then:

$$2d_{0,0}d_{2i,0} + \sum_{k=1}^{2i-1} d_{k,0}d_{2i,k} = 0$$

or equivalently

$$2d_{0,0}d_{2i,0} + d_{1,0}d_{2i,1} = -\sum_{k=2}^{2i-1} d_{k,0}d_{2i,k} \tag{3.8}$$

- And since $P_{2i+1} \circ \cdots \circ P_{m-2}(B)$ is an involution then:

$$\sum_{k=1}^{2i} d_{k,0}d_{2i+1,k} = 0$$

or equivalently

$$d_{1,0}d_{2i+1,1} = -\sum_{k=2}^{2i} d_{k,0}d_{2i+1,k} \tag{3.9}$$

- Hence, by using (3.8) and (3.9) in $(**)$ we get:

$$(**) = \sum_{i=2}^{n}\left(-\sum_{k=2}^{2i-1} d_{k,0}d_{2i,k}\right) d_{2n+1,2i} + \sum_{i=1}^{n-1}\left(-\sum_{k=2}^{2i} d_{k,0}d_{2i+1,k} + 2d_{0,0}d_{2i+1,0}\right) d_{2n+1,2i+1}$$

gathering again together the even and odd rows and removing common factors we get:

$$(**) = \sum_{i=1}^{n-1} 2d_{0,0}d_{2i+1,0}d_{2n+1,2i+1} - \sum_{j=2}^{2n-1} d_{j,0} \sum_{k=j+1}^{2n} d_{k,j}d_{2n+1,k}$$

- On the other hand, as $Q_{m-2i} \circ \cdots \circ Q_{m-2}(C_{m-1})$ is an involution then:

$$\sum_{k=2i+1}^{2n} d_{k,2i}d_{2n+1,k} = 0$$

- We only have to take care of odd columns $j = 2i + 1$ then:

$$(**) = \sum_{i=1}^{n-1} 2d_{0,0}d_{2i+1,0}d_{2n+1,2i+1} - \sum_{i=1}^{n-1} d_{2i+1,0} \sum_{k=2i+2}^{2n} d_{k,2i+1}d_{2n+1,k}$$

- As $Q_{m-(2i+1)} \circ \cdots \circ Q_{m-2}(C)$ is an involution then:

$$-2d_{0,0}d_{2n+1,2i+1} + \sum_{k=2i+2}^{2n} d_{k,2i+1}d_{2n+1,k} = 0$$

- So finally we get:

$$(**) = -\sum_{i=1}^{n-1} d_{2i+1,0} \left(-2d_{0,0}d_{2n+1,2i+1} + \sum_{k=2i+2}^{2n} d_{k,2i+1}d_{2n+1,k} \right) = 0$$

PART 2: Let $D_{n-1} = R_{n-1}(d(x), h(x)) = (d_{i,j})_{0 \le i,j \le n-1} \in \mathcal{R}_{n-1}$ be an involution. Suppose that $\hat{D}_n \in \mathcal{R}_n(\mathbb{K})$ with $P_{n-1}(\hat{D}_n) = D_{n-1}$.

- We have that:

$$Q_{n-2}(Q_{n-1}(\hat{D}_n)) = R_{n-2}\left(d(x) \cdot (h(x))^2, h(x)\right) = (d_{i,j})_{2 \le i,j \le n}$$

  is an involution.
- In order to $\hat{D}_n$ to be an involution, $Q_{n-1}(\hat{D}_n)$ must be an involution. Note that:

$$Q_{n-1}(\hat{D}) \in \mathcal{E}(Q_{n-2}(D_{n-1}), Q_{n-2}(Q_{n-1}(\hat{D}_n)))$$

- From part 1 we get all the results related to $d_{n,1}$ in order to $Q_{n-1}(\hat{D})$ to be an involution. In this case, note that

$$\hat{D}_n \in \mathcal{E}(D_{n-1}, Q_{n-1}(\hat{D}_n))$$

- From part 1 again we obtain all the results in the theorem related to $d_{n,0}$. Finally, by using the Riordan structure of $\hat{D}_n$ we get the formula for a_{n-1} and the proof is finished.

□

It is clear that there are Riordan matrices such that for some $n \in \mathbb{N}$, their n-th projections are involutions but their $(n+1)$-th projections are not. Those matrices are not involutions. But, in the result below, we will point out that the property to be an involution can be lifted from finite Riordan groups to the infinite one by using the corresponding maps Π_n concerning the inverse limit structure in $\mathcal{R}(\mathbb{K})$.

Corollary 3.2.3 *Any Riordan involution $D_n \in \mathcal{R}_n$ can be extended to a Riordan involution $D_{n+1} \in \mathcal{R}_{n+1}$, i.e. $P_n(D_{n+1}) = D_n$.*

Equivalently, for any finite Riordan involution $D_n \in \mathcal{R}_n$ there is an infinite Riordan involution $D \in \mathcal{R}$ such that $\Pi_n(D) = D_n$.

An immediate consequence of Theorem 3.2.2 is the following:

Corollary 3.2.4 *Let $\alpha = \sum_{i\in\mathbb{N}} \alpha_i x^i$ be an arbitrary formal power series then:*

(i) There is an unique nontrivial involution that we will denote by $I^{+,\alpha} = (d_{i,j})_{i,j\in\mathbb{N}}$ such that:

$$d_{0,0} = 1, \qquad d_{2i+1,0} = \alpha_{2i} \qquad \textit{and} \qquad d_{2i+2,1} = \alpha_{2i+1} \qquad \textit{for} \qquad i = 0, 1, \cdots$$

(ii) There is an unique nontrivial involution that we will denote by $I^{-,\alpha} = (d_{i,j})_{i,j\in\mathbb{N}}$ such that:

$$d_{0,0} = -1, \qquad d_{2i+1,0} = \alpha_{2i} \qquad \textit{and} \qquad d_{2i+2,1} = \alpha_{2i+1} \qquad \textit{for} \qquad i = 0, 1, \cdots$$

Moreover, any nontrivial Riordan involution can be constructed by this way.

Corollary 3.2.5 *We can construct nontrivial involutions $D = (d_{i,j})_{i,j\in\mathbb{N}}$ with A-sequence $A = \sum_{i\in\mathbb{N}} a_i x^i$ such that*

$$d_{2i+1,0} \quad \textit{and} \quad a_{2i+1} \quad \textit{are arbitrary.}$$

3.3 Examples and Related Aspects

Example 3.3.1 (Non trivial involutions $I^{+,\alpha}$ and $I^{-,\alpha}$) *Let* $\alpha(x) = \alpha_0 + \alpha_1 x + \ldots \in \mathbb{K}[[x]]$*:*

$$I^{+,\alpha} = \begin{bmatrix} 1 &&&&&& \\ \boxed{\alpha_0} & -1 &&&&& \\ d_{2,0} & \boxed{\alpha_1} & 1 &&&& \\ \boxed{\alpha_2} & d_{3,1} & d_{3,2} & -1 &&& \\ d_{4,0} & \boxed{\alpha_3} & d_{4,2} & d_{4,3} & 1 && \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad I^{-,\alpha} = \begin{bmatrix} -1 &&&&&& \\ \boxed{\alpha_0} & 1 &&&&& \\ c_{2,0} & \boxed{\alpha_1} & -1 &&&& \\ \boxed{\alpha_2} & c_{3,1} & c_{3,2} & 1 &&& \\ c_{4,0} & \boxed{\alpha_3} & c_{4,2} & c_{4,3} & -1 && \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Remark 3.3.2 (Non-trivial involutions and the sign) *Non trivial diagonal involutions are:*

$$I^{+,0} = R(1,-x) \qquad \textit{and} \qquad I^{-,0} = R(-1,-x)$$

and thus:

$$I^{-,0} = -I^{+,0}$$

But, in general

$$-I^{+,\alpha} = I^{-,-\alpha}$$

Note that we have the relation:

$$I^{\pm,\alpha} = I^{+,0} I^{\mp,\alpha} I^{-,0}$$

Example 3.3.3 (Some involtutions for different values of $\alpha(x)$)

$$I^{+,\frac{1}{1-x}} = \begin{bmatrix} 1 &&&&&&& \\ \boxed{1} & -1 &&&&&& \\ -\frac{1}{2} & \boxed{1} & 1 &&&&& \\ \boxed{1} & -\frac{3}{2} & -3 & -1 &&&& \\ -\frac{9}{8} & \boxed{1} & \frac{15}{2} & 5 & 1 &&& \\ \boxed{1} & \frac{17}{8} & -15 & -\frac{35}{2} & -7 & -1 && \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

A-sequence: $(-1,-2,\boxed{0},1,1,-2,\ldots)$

$$I^{+,\frac{-2x}{1-4x^2}} = \begin{bmatrix} 1 &&&&&&& \\ \boxed{0} & -1 &&&&&& \\ 0 & \boxed{-2} & 1 &&&&& \\ \boxed{0} & -4 & 4 & -1 &&&& \\ 0 & \boxed{-8} & 12 & -6 & 1 &&& \\ \boxed{0} & -16 & 32 & -24 & 8 & -1 && \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

A-sequence: $(-1,2,\boxed{0},0,0,\ldots)$

$$I^{+,\frac{1}{(1-x)^2}} = \begin{bmatrix} 1 & & & & & & \\ \boxed{1} & -1 & & & & & \\ -1 & \boxed{2} & 1 & & & & \\ \boxed{3} & -5 & -5 & -1 & & & \\ -4 & \boxed{4} & 20 & 8 & 1 & & \\ \boxed{5} & 32 & -56 & -44 & -11 & -1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad \textit{A-sequence: } (-1,-3,\boxed{0},1,8,-24,\ldots)$$

Let $C(x)$ be the generating function of the Catalan numbers:

$$I^{+,C(x)} = \begin{bmatrix} 1 & & & & & & \\ \boxed{1} & -1 & & & & & \\ -\frac{1}{2} & \boxed{1} & 1 & & & & \\ \boxed{2} & -\frac{3}{2} & -3 & -1 & & & \\ -\frac{45}{8} & \boxed{5} & \frac{15}{2} & 5 & 1 & & \\ \boxed{14} & -\frac{131}{8} & -24 & -\frac{35}{2} & -7 & -1 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad \textit{A-sequence: } (-1,-2,\boxed{0},-4,8,\ldots)$$

As it can be easily checked in the above examples, all of those involutions has an A-sequence of the type $(-1, a_1, 0, a_2, \ldots)$. This property will be formally proved (proposition 3.5.1).

A more interesting example appears when we take α an arbitrary odd power series.

Proposition 3.3.4 (Involutions in $\mathcal{F}_1(\mathbb{K})$) *An involution $I^{+,\alpha} \in \mathcal{R}(\mathbb{K})$ belongs to $\mathcal{A}(\mathbb{K})$ if and only if $\alpha(x)$ is an odd power series.*

Equivalently, this result could be stated in the following way: for any sequence $(\omega_2, \omega_4, \omega_6, \ldots)$ there is a unique non-trivial involution $\omega(x) \in \mathcal{F}_1(x)$ (non-trivial this time means that $\omega'(0) = -1$) of the type:

$$\omega(x) = -x + \omega_2 x^2 + \omega_3 x^3 + \omega_4 x^4 + \omega_5 x^5 + \ldots$$

Moreover, for any n odd ω_n is given by:

$$\omega_n = \frac{1}{2} \sum_{k=2}^{n-1} ([x^n]\omega(x))^k \cdot \omega_k$$

Proof: It is a direct consequence of the formulas of theorem 3.2.2 taking all the parameters d_{k0} equal 0.

□

In the realm of formal power series, the functional equation $\omega(\omega(x)) = x$ is known as the **Babbage equation**. As a corollary of our result, we have proved a conjecture posted by T. X. He (page 349 in [43]).

Corollary 3.3.5 *The Taylor polynomial of order 10 of any nontrivial solution, $\omega(x) \in K[[x]]$, of Babbage's equation is*

$$(3.10)\quad -x+\beta_0x^2-\beta_0^2x^3+\beta_1x^4+(2\beta_0^4-3\beta_0\beta_1)x^5+\beta_2x^6+(-13\beta_0^6+18\beta_0^3\beta_1-4\beta_0\beta_2-2\beta_1^2)x^7+$$

$$+\beta_3x^8+(145\beta_0^8-221\beta_0^5\beta_1+35\beta_0^3\beta_2+50\beta_0^2\beta_1^2-5\beta_0\beta_3-5\beta_1\beta_2)x^9+\beta_4x^{10}$$

where $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{K}$. Moreover, for any values $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{K}$ the expression (3.10) is the Taylor polynomial of order 10 of a nontrivial solutions of Babbage's equation.

Another interesting example arises when we take α an arbitrary even power series and we construct $I^{-,\alpha}$. In this case, Theorem 3.2.2 gives us an iterative process to compute its A-sequence which appears in the first column in that involution:

Proposition 3.3.6 *Let $\alpha(x)$ be an even formal power series and $I^{-,\alpha} = (d_{i,j})_{i,j\in\mathbb{N}}$. Then the generating function of the A-sequence of $I^{-,\alpha}$ is:*

$$A(x) = \sum_{n=0}^{\infty} d_{n,0}x^n$$

Proof: In order to see this, note that if $\alpha(x)$ is even $d_{2i+1,1} = \alpha_{2i+1} = 0$ and by using (3.5) we get $d_{n,1} = 0$ for all $n \geq 2$. Let us proceed by induction. For $n = 2$ from (3.6) in Theorem 3.2.2

$$a_1 = \frac{1}{d_{11}}(d_{2,1} - a_0d_{1,0}) = d_{1,0}$$

Suppose true to n. What happens in $n+1$? Taking into account that $d_{n+1,1} = 0$, by induction hypothesis $a_k = d_{k,0}$ for all $k \leq n-1$ and using again (3.6), we obtain

$$a_n = \frac{1}{d_{n,n}}\left(d_{n+1,1} - \sum_{j=0}^{n-1} a_jd_{n,j}\right) = \frac{1}{d_{n,n}}\left(-\sum_{j=0}^{n-1} d_{n,j}d_{j,0}\right)$$

As $I^{-,\alpha}$ is an involution, the product of its n-row by its 0-colunm is 0, then

$$\sum_{j=0}^{n} d_{n,j}d_{j,0} = 0 \quad\Leftrightarrow\quad -\sum_{j=0}^{n-1} d_{n,j}d_{j,0} = d_{n,n}d_{n,0}.$$

so

$$a_n = \frac{1}{d_{n,n}} \left(-\sum_{j=0}^{n-1} d_{n,j} d_{j,0} \right) = \frac{1}{d_{n,n}} (d_{n,n} d_{n,0}) = d_{n,0}$$

□

Some special nice examples of involutions are the self-dual ones, due to their symmetries (see sections 1.11, 1.12).

Proposition 3.3.7 *$R(d(x), h(x)) \in \mathcal{R}$ is a self-dual involution if and only if:*

$$h(h(x)) = x \qquad \textit{and} \qquad d(x) = \pm \left(\sqrt{\frac{x h'(x)}{h(x)}} \right)$$

Proof: According to (the proof of) theorem 1.12.4 and the introduction of section 3.1 the conditions that must be satisfied simultaneously by $d(x)$ and by $h(x)$ if $R(d(x), h(x))$ is a self dual involutions are, respectively:

$$\begin{cases} d \cdot d(h(x)) = 1 \\ h(h(x)) = x \end{cases} \qquad \text{and} \qquad \begin{cases} d(x) = \frac{d(h^{-1}(x))}{h'(h^{-1}(x))} \cdot \frac{x}{h^{-1}(x)} \\ h(h(x)) = x \end{cases}$$

In theorem 1.12.4 we already showed that in order to satisfy both equations in the second system:

$$d(x) = \lambda \cdot \left(\sqrt{x \frac{h'(x)}{h(x)}} \right) \cdot e^{\Phi(x, h(x))}$$

with $\lambda \in \mathbb{K}^*$ and $\Phi(x, t)$ a symmetric bivariate power series in $\mathbb{K}[[x, t]]$. If we impose to this $d(x)$ to satisfy also the first equation in the first system we have, at the first sight, that $\lambda = \pm 1$ and thus this equation is:

$$\left[\left(\sqrt{x \frac{h'(x)}{h(x)}} \right) \cdot e^{\Phi(x, h(x))} \right] \cdot \left[\left(\sqrt{h(x) \frac{h'(h(x))}{h(h(x))}} \right) \cdot e^{\Phi(h(x), h(h(x)))} \right] = 1$$

and since $h(h(x)) = x$ and $\Phi(x, t)$ is symmetric, this is equivalent to:

$$\left(\sqrt{h'(h(x)) \cdot h'(x)} \right) \cdot e^{2\Phi(x, h(x)))} = 1$$

By taking derivatives from $h(h(x)) = x$ we obtain that $h'(h(x)) \cdot h'(x) = 1$ and so this equation is in fact:

$$e^{2\Phi(x, h(x)))} = 1$$

where, modulo the sign that we have considered with the parameter λ implies that:

$$e^{\Phi(x, h(x)))} = 1$$

□

Example 3.3.8 (Constructing a self-dual involution.) *Below the first step for constructing a self-dual involution is shown:*

$$d_{0,0}=-1,\quad \begin{bmatrix} 1 & & \\ \boxed{\gamma_0} & -1 & \\ d_{1,-1} & \boxed{\gamma_0} & 1 \end{bmatrix},\quad \begin{bmatrix} -1 & & & & \\ d_{-1,-2} & 1 & & & \\ d_{0,-2} & \boxed{\gamma_0} & -1 & & \\ \boxed{\gamma_1} & d_{1,-1} & \boxed{\gamma_0} & 1 & \\ d_{2,-2} & \boxed{\gamma_1} & d_{2,0} & d_{2,1} & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & & & & & & \\ d_{-2,-3} & -1 & & & & & \\ d_{-1,-3} & d_{-1,-2} & 1 & & & & \\ d_{0,-3} & d_{0,-2} & \boxed{\gamma_0} & -1 & & & \\ d_{1,-3} & \boxed{\gamma_1} & d_{1,-1} & \boxed{\gamma_0} & 1 & & \\ \boxed{\gamma_2} & d_{2,-2} & \boxed{\gamma_1} & d_{2,0} & d_{2,1} & -1 & \\ d_{3,-3} & \boxed{\gamma_2} & d_{1,3} & d_{3,0} & d_{3,1} & d_{3,2} & 1 \end{bmatrix}$$

with $d_{-j,-i}=d_{i,j}$ and those parameters $\gamma_i\in\mathbb{K}$ being arbitrary.

The study of **pseudo-involutions** is also of our interest. We say that $D\in\mathcal{R}$ is a pseudo-involution if and only if $D\cdot R(1,-x)$ is an involution. The interest in the study of pseudo-involutions is related to the fact that, in case we are considering entries in $\mathbb{Z}$, the more convenient group to be consider is $\mathcal{C}(\mathbb{Z})$, and so we cannot expect to find regular involutions there.

Remark 3.3.9 *Note that once obtained an involution by using the formula, to get the corresponding pseudo-involution we have only to change signs suitably.*

For instance, we are going to compute:

Proposition 3.3.10 (Pseudo-involutions in the Toeplitz subgroup) *Let $\alpha(x)$ be a formal power series such that $\alpha_{2i+1}=-\alpha_{2i}$. Then $I^{+,\alpha\alpha}I^{+,0}$ and $I^{+,\alpha}I^{+,0}$ are pseudo-involutions in the Toeplitz subgroup. Moreover, any pseudo-involution in the Toeplitz subgroup can be obtained by this way.*

Proof: For $n=2$ we get

$$I^{+,\alpha}I^{+,0}=\begin{pmatrix} 1 & & \\ \alpha_0 & -1 & \\ d_{2,0} & -\alpha_0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}=\begin{pmatrix} 1 & & \\ \alpha_0 & 1 & \\ d_{2,0} & \alpha_0 & 1 \end{pmatrix}$$

By induction, we suppose that is true for every $k\le n$, this means that $a_0=-1$ and $a_k=0$ for $1\le k\le n-1$, then, in particular $d_{k+1,1}=-d_{k,0}$ for $1\le k\le n-1$. Once again by Riordan involution formula:

$$a_n=(-1)^n(d_{n+1,1}+d_{n,0})=0$$

As an example, note that the product:

$$\begin{bmatrix} 1 & & & & & \\ \alpha_0 & -1 & & & & \\ \frac{1}{2}\alpha_0^2 & -\alpha_0 & 1 & & & \\ \alpha_2 & -\frac{1}{2}\alpha_0^2 & \alpha_0 & -1 & & \\ \alpha_0\alpha_2 - \frac{1}{8}\alpha_0^4 & -\alpha_2 & \frac{1}{2}\alpha_0^2 & -\alpha_0 & 1 & \\ \alpha_4 & \frac{1}{8}\alpha_0^4 - \alpha_0\alpha_2 & \alpha_2 & -\frac{1}{2}\alpha_0^2 & \alpha_0 & -1 \end{bmatrix} \cdot I^{+,0}$$

belongs to the Toeplitz subgroup.

In [25], G.-S. Cheon, H. Kim and L.W. Shapiro treat, from a different point of view, this kind of involutions.

3.4 Elements of finite order in $\mathcal{R}(\mathbb{K})$

Totally analogous to our previous study of involutions, we can produce similar results for finite elements of order q. Similarly to what happened with involutions, we will talk about **non-trivial** elements of finite order q for those matrices such that the rate of the geometric progression in the main diagonal is a root of unity of order q.

Remark 3.4.1 *As happened in the case of involutions,* ***trivial*** *elements of finite order q are elements of finite order q' with q' dividing q multiplied by a constant.*

This time, we are only interested in elements in $\mathcal{R}(\mathbb{C})$, $\mathcal{F}_1(\mathbb{C})$, since among the fields considered in the previous sections, is the one which has roots of unity of order q.

Remark 3.4.2 *Fon non-trivial involutions we can restrict our study to those matrices $R(d(x), h(x))$ with $d(0) = 1$, since the rest of them are one of those multiplied by a constant. We will do this assumption throughout the rest of this section.*

So our main result is the following:

Theorem 3.4.3 (Analogous to Theorem 3.2.2) *Suppose $n \geq 2$. Let $D_{n-1} = (d_{ij})_{0\leq i,j\leq n-1} \in \mathcal{R}_{n-1}(\mathbb{C})$ be an involution and take $\hat{D}_n = (d_{ij})_{0\leq i,j\leq n} \in \mathcal{R}_n(\mathbb{C})$ such that $P_{n-1}(\hat{D}_n) = D_{n-1}$.*

- *If $n \equiv 0 \mod q$ for any $d_{n,1}$, there is a unique $d_{n,0}$ such that $\hat{D}_n$ is an element of finite order q.*
- *If $n \equiv 1 \mod q$ for any $d_{n,0}$, there is a unique $d_{n,1}$ such that $\hat{D}_n$ is an element of finite order q.*
- *In other case any X_{n+1} extending X_n is an element of finite order q.*

Proof: As usual we may assume that $D_{n-1} = R_{n-1}(d(x), h(x))$ where $d(0) = 1$. Let λ be the multiplier of $h(x)$.

- Obviously there is at least one matrix in $\mathcal{R}_n(\mathbb{C})$ being an involution and extending D_{n-1}. According to our results of conjugacy (see theorems 1.14.6, 1.14.7 and their proofs):

$$D_{n-1} = X_{n-1}^{-1} R_{n-1}(1, \lambda x) X_{n-1}, \text{ for some } X_{n-1} \in \mathcal{R}_{n-1}(\mathbb{C})$$

  Let $\hat{X}_n$ be any matrix extending X_{n-1}, that is, such that $P_{n-1}(\hat{X}_n) = X_{n-1}$. Then obviously:

$$\hat{X}_n^{-1} R_n(1, \lambda x) \hat{X}_n$$

  is a matrix of finite order q that extends D_{n-1}.

- Similarly to the case of involutions, $d_{n,0}, d_{n1}$ must satisfy an equation for $\hat{D}_n$ being an element of finite order q. By using block multiplication we can see that the last row of the following product:

$$\hat{D}_n^q = \left[\begin{array}{ccc|c} & D_{n-1} & & \\ \hline d_{n,0} & \dots & d_{n,n-1} & d_{nn} \end{array}\right]^q$$

  is the row:

$$[d_{n0}, \dots, d_{n,n-1}] D_{n-1}^{q-1} + d_{nn}[d_{n0}, \dots, d_{n,n-1}] D_{n-1}^{q-2} + \dots + d_{nn}^{q-1}[d_{n0}, \dots, d_{n,n-1}] \tag{3.11}$$

  and taking into account that $d_{nn} = \lambda^n$ (which is a q-root of unity) we have that the 0-entry and the 1-entry in this row are, respectively of the type:

$$(1 + (\lambda^n) + \dots + (\lambda^n)^{q-1}) d_{n0} + [\dots]$$

  and:

$$\frac{1}{\lambda}(1 + \lambda^n \lambda^{q-2} + \dots + (\lambda^n)^{q-1}) d_{n1} + [\dots]$$

  or equivalently (we refer to this last equation):

$$\frac{1}{\lambda}(1 + \lambda^{n-1} + \dots + (\lambda^{n-1})^{q-1}) d_{n1} + [\dots]$$

  where nothing inside the brackets depend on d_{n0}, d_{n1}. Since we want the expression in (3.11) to equal 0, those expressions give two equations of the type:

$$(1 + (\lambda^n) + \dots + (\lambda^n)^{q-1}) d_{n0} + [\dots]$$

$$\frac{1}{\lambda}(1 + \lambda^{n-1} + \dots + (\lambda^{n-1})^{q-1}) d_{n1} = [\dots]$$

- For $n \not\equiv 0, 1$ the two equations above have a nonzero coefficient of the corresponding indeterminate (d_{n0} or d_{n1}). For $n \equiv 0, 1$ the corresponding equation has a zero coefficient of the indeterminate. Since we know that this system has a solution, this means that regarless of the corresponding indeterminate, $\hat{D}_n$ is an element of order q.

$\square$

And again analogously to the case of involutions:

Corollary 3.4.4 *Let two sequences sequence* $(a_1, \ldots, a_{q-1}, a_{q+1}, \ldots a_{2q-1}, a_{2q+1}, \ldots)$ *and* $(b_2, \ldots, b_q, b_{q+2}, \ldots b_{2q}, b_{2q+2}, \ldots)$. *We can construct a unique Riordan element of order* q $D = (d_{ij})_{0 \leq i,j < \infty}$ *such that* $d_{00} = 1$, $d_{11} = \lambda$ *an element of order* q *in* $\mathbb{C}$ *and:*

$$\forall n \in \mathbb{N}, n \not\equiv 0 \mod q, \;\; d_{n0} = a_n, \qquad \forall n \in \mathbb{N}, n \not\equiv 1 \mod q, \;\; d_{n1} = b_n$$

Corollary 3.4.5 *Let a sequence* $(b_2, \ldots, b_q, b_{q+2}, \ldots b_{2q}, b_{2q+2}, \ldots)$ *and let* b_1 *an element of order* q *in* $\mathcal{C}$. *We can construct a unique element of finite order* q *in* $\mathcal{F}_1(\mathbb{C})$ *of the type:*

$$b_1 x + b_2 x^2 + b_3 x^3 + \ldots$$

3.5 More about the A-sequence

We will start by recalling that $\mathcal{R}(\mathbb{K}) = \mathcal{T}(\mathbb{K}) \rtimes \mathcal{A}(\mathbb{K})$. A consequence of this algebraic fact, and of independent interest of our current discussion, is the following proof of proposition 0.3.7 that we are including now as a part of this work:
Proof of the 2FT: Let $(G, *)$ be a group, and $(H, *)$ a normal subgroup of it. Let $g, h \in H$. The equation:

$$h * g = g * x$$

in the indeterminate $x \in G$ has a unique solution $x \in H$. Now by taking $G = \mathcal{R}(\mathbb{K})$ and $H = \mathcal{T}(\mathbb{K})$, we have that there exists a unique solution:

$$R\left(\frac{h(x)}{x}, x\right) \cdot R\left(d(x), h(x)\right) = R(d(x), h(x)) \cdot X$$

which is equivalent to:

$$R\left(d(x) \cdot \frac{h(x)}{x}, h(x)\right) = R(d(x), h(x)) \cdot X$$

that is:

$$\begin{bmatrix} d_{00} & & & \\ d_{10} & d_{11} & & \\ d_{20} & d_{21} & d_{22} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ a_2 & a_1 & a_0 & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} d_{11} & & & \\ d_{21} & d_{22} & & \\ d_{31} & d_{32} & d_{33} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.12}$$

which is exactly the identity we wanted (remember that X must belong to $\mathcal{T}(\mathbb{K})$, so is of the type $X = R(A(x), x)$). It is easy to see that:

$$A(x) = \frac{x}{h^{-1}(x)}$$

□

As we have already mentioned, an important property of the A-sequence of the involutions is the following:

Proposition 3.5.1 *If $R(d(x), g(x))$ is a non-trivial Riordan involution then its A-sequence is of the type $(-1, a_1, 0, a_3, \ldots)$.*

Proof: Given a Riordan matrix $D = (d_{ij})_{0 \le i,j < \infty}$ with A-sequence $(a_0, a_1, \ldots)$, the terms (a_0, a_1, a_2) are already determined by $\Pi_3(D) = (d_{ij})_{0 \le i,j \le 3} \in \mathcal{R}_3$. According to theorem 3.2.2, the elements in $\mathcal{R}_3$ that are involutions are those of the type:

$$\pm \begin{bmatrix} 1 & & & \\ d_{10} & -1 & & \\ -\frac{1}{2}d_{10}d_{21} & d_{21} & 1 & \\ d_{30} & -\frac{1}{2}(d_{10}+d_{21})d_{21} & -d_{10} - 2d_{21} & -1 \end{bmatrix}$$

a_0 is always the ration of the geometric progression in the diagonal, which in this case equals 1. a_1 can be obtained from d_{21}, d_{11}, d_{10} and a_0 according to the formula that must be satisfied for Riordan matrices:

$$d_{21} = a_0 d_{10} + a_1 d_{11}$$

so in this case we obtain that $a_1 = -d_{10} - d_{21}$. Analogously we can obtain a_2 from the relation:

$$d_{31} = a_0 d_{21} + a_1 d_{21} + a_2 d_{22}$$

we can see that in the previous matrix, $a_2 = 0$.

□

There are several different ways to prove the above proposition. We have chosen just one. In fact, all Riordan matrices with this condition form a subgroup:

Proposition 3.5.2 *The set of Riordan matrices with the 2-term in the A-sequence equal 0 is a subgroup of $\mathcal{R}(\mathbb{K})$ (for any field $\mathbb{K}$, this time).*

Proof: Let T denote this subset of $\mathcal{R}$. To see that T is a group, we only need to see that:

- If two matrices belong to T, then so does their product. Since:

$$R(f(x), g(x)) = R(d(x), h(x)) = R(h(x) \cdot d(g(x)), h(g(x)))$$

  and the A-sequence of $R(F(x), G(x))$ is totally determined by $G(x)$, we only need to check that if $R(1, g(x))$, $R(1, h(x))$ belong to T, that is:

$$R(1,g(x)) = \begin{bmatrix} 1 & & & & \\ 0 & a_0 & & & \\ 0 & a_0a_1 & a_0^2 & & \\ 0 & a_0a_1^2 & 2a_0^2a_1 & a_0^3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad R(1,h(x)) = \begin{bmatrix} 1 & & & & \\ 0 & b_0 & & & \\ 0 & b_0b_1 & b_0^2 & & \\ 0 & b_0b_1^2 & 2b_0^2b_1 & b_0^3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

  then so does $R(1, h(g(x))) = R(1, g(x)) \cdot R(1, h(x))$, which is:

$$\begin{bmatrix} 1 & & & & \\ 0 & a_0b_0 & & & \\ 0 & a_0a_1b_0 + a_0^2b_0b_1 & a_0^2b_0^2 & & \\ 0 & a_0a_1^2b_0 + 2a_0^2a_1b_0b_1 + a_0^3b_0b_1^2 & 2a_0^2a_1b_0^2 + 2a_0^3b_0^2b_1 & a_0^3b_0^3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

  It is easy to check that this matrix belongs to T.

- If a matrix belongs to T, then so does its inverse. Following a similar reasoning, we only need to prove that for a given element in $\mathcal{A} \cap T$:

$$R(1,g(x)) = \begin{bmatrix} 1 & & & & \\ 0 & a_0 & & & \\ 0 & a_0a_1 & a_0^2 & & \\ 0 & a_0a_1^2 & 2a_0^2a_1 & a_0^3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

  its inverse, which is:

$$R(1,g^{-1}(x)) = \begin{bmatrix} 1 & & & & \\ 0 & \frac{1}{a_0} & & & \\ 0 & -\frac{a_1}{a_0^2} & \frac{1}{a_0^2} & & \\ 0 & \frac{-a_1^2+2a_1^2}{a_0^3} & -\frac{2a_1}{a_0^3} & \frac{1}{a_0^3} & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

  and obviously belongs to T.

□

Remark 3.5.3 *It is not difficult to prove that the above subgroup is not normal.*

Remark 3.5.4 *The group described above can be described as the set of Riordan matrices $R(d(x), h(x))$ such that:*

$$h(x) = h_1 x + h_2 x^2 + \frac{h_2^2}{h_1} x^3 + h_4 x^4 + \dots$$

In view of the result above, we can search for groups similarly described:

Remark 3.5.5 *The set of Riordan matrices with A-sequence of the type $(a_0, 0, a_2, \dots)$ is also a subgroup of $\mathcal{R}(\mathbb{K})$. In general, for $m \geq 3$ the set of matrices with the m-term of the A-sequence equal to 0 is not a group. Another groups of this flavour can be found, also related to the metrics that we have already introduced for the Riordan group, but we are not going to develop this here.*

3.6 The group generated by the involutions I: Statement of the Problem

Recall that an element $M \in \mathcal{R}(\mathbb{K})$ is an involution if and only if $M = M^{-1}$. There is a generalization in some sense of the concept of involution which is the concept of **reversible element**. In this case M is not equal but conjugated to M^{-1}.

It is well known that:

Remark 3.6.1 *For every group, if an element M of any group is the product of two involutions then is conjugated to M^{-1}. In order to see this just note that:*

$$M = I_1 I_2 \Rightarrow I_1 M_1 = I_2 \Rightarrow I_1 M I_1 = I_2 I_1 \Rightarrow M^{-1} = I_2 I_1 \Rightarrow M^{-1} = I_1 M I_1$$

Pseudo-involutions are a particular case of reversible elements.

For groups of matrices, there is a lot of work in the bibliography (see the work by W. H. Gustafson, P. R. Halmos and H. Radjavi [40], by W. H. Gustafson and S. Kakutani [41], by D. Z. Djokovic [28], or by M. J. Wonenburger [115]) studying the set of matrices which can be writen as a product of involutions. For example:

Theorem 3.6.2 (Gustafson-Halmos-Radjavi, [40]**)** *Every square matrix over a field, with determinant ± 1, is the product of no more than four involutions.*

Even there is some work for infinite matrices (not in the Riordan matrices framework) done by R. Slowik in [106]. Concerning the same problem for the group $\mathcal{F}_1(\mathbb{C})$ a lot has been also writen (see the work by E. Kasner in 1916 [55] and the recent ones by A. G. O'Farrell [88, 89]).

Now, our main result in this section deals with the analogue problem in $\mathcal{R}(\mathbb{K})$: to study the group and the structure of it generated by the involutions in $\mathcal{R}(\mathbb{K})$.

We will use the following notation:

Definition 3.6.3 *Let $SI(\mathbb{K})$ be the set of all Riordan involutions and by $\mathcal{I}(\mathbb{K})$ the group generated by Riordan involutions.*

It is known, and easy to prove, that:

Remark 3.6.4 *The group generated by involutions in any group is a normal subgroup of such a group.*

Other subgroup that we are going to use in the description of the subgroup generated by involutions is $\mathcal{K} = \{R(1,x), -R(1-x), I^{+,0}, I^{-,0}\}$. It is easy to prove that any set of the type $\{R(1,x), -R(1-x), A, B\}$ for $A, B \in SI(\mathbb{K})$ is in fact a subgroup of $\mathcal{R}(\mathbb{K})$ isomorphic to the Klein group of order 4.

The main result of this part of this chapter is:

Theorem 3.6.5 *The subgroup generated by the involutions in $\mathcal{R}(\mathbb{K})$, denoted by $\mathcal{I}(\mathbb{K})$ is the set of Riordan matrices with ± 1 in the entry $(0,0)$ and A-sequence of the type $(\pm 1, a_1, 0, a_3, a_4, \ldots)$.*

Every element in $\mathcal{I}(\mathbb{K})$ is the product of no more that four involutions.

The analogue result holds for $\mathcal{R}_n(\mathbb{K})$.

3.7 The group generated by the involutions II: Proof of the Main Result

Since we already know that the elements in the commutator of $\mathcal{R}(\mathbb{K})$ are those with ones in the main diagonal, and we also know that a product of involutions in $\mathcal{R}(\mathbb{K})$ will have A-sequence of the type $(\pm 1, a_1, 0, a_3, \ldots)$, we will call **special commutators** to those elements in $\mathcal{R}'(\mathbb{K})$ with A-sequence of the type $(1, a_1, 0, a_3 \ldots)$. We will denote the set of special commutators in $\mathcal{R}(\mathbb{K})$ by $\mathcal{SC}(\mathbb{K})$, and this set is obviously a group

Note that, actually:

Remark 3.7.1 *In order to prove theorem 3.6.5, we only need to show that for every special commutator D, there exists three involutions $H, I, J \in SI(\mathbb{K})$ such that:*

$$D \cdot R(1, -x) = H \cdot I \cdot J \tag{3.13}$$

This statement is lemma 3.7.4 below. In order to see that lemma 3.7.4 implies theorem 3.6.5, just see that:

- *Every special commutator D is a product of at most 4 involutions.*

- *Let T be matrix with A-sequence of the type $(-1, a_1, 0, a_3, \ldots)$. Then either $T \cdot R(1, -x)$ or $-T \cdot R(1, -x)$ is a special commutator. So there exists $H, I, J \in SI(\mathbb{K})$ such that:*

$$(T \cdot R(1,-x))R(1,-x) = H \cdot I \cdot J \quad \text{or} \quad (-T \cdot R(1,-x))R(1,-x) = \cdot I \cdot J$$

In any case, we can write T as a product of 3 involutions:

$$T = H \cdot I, \cdot J \quad \text{or} \quad T = (-H) \cdot I \cdot J$$

On the other hand, no matrix of other type can be writen as a product of involutions for obvious reasons.

Not only have we reduced our problem to prove lemma 3.7.4 but also, in the following remark, we will split the proof of lemma 3.7.4 in two parts:

Remark 3.7.2 *In the notation of the previous remark, if we consider that $D \cdot R(1, -x) = R(d(x), h(x))$ and:*

$$H = R(f(x), g(x)), \quad I = R(u(x), v(x)), \quad J = R(\alpha(x), \omega(x))$$

finding I, J, K satisfying (3.17) *is equivalent to find $f(x), u(x), \alpha(x) \in \mathcal{F}_0(\mathbb{K})$ and $g(x), v(x), \omega(x) \in \mathcal{F}_1(\mathbb{K})$ such that:*

$$\begin{cases} d(x) = f(x) \cdot u(g(x)) \cdot \alpha(v(g(x)))) \\ h(x) = \omega(v(g(x))) \end{cases}$$

For any given $h(x)$ showing the existence of those $\omega(v(g(x)))$ is lemma 3.7.3. So the proof of lemma 3.7.4 is just to find those $f(x)$, $u(x)$, $\alpha(x)$ for any possible combination of $d(x)$, $g(x)$, $v(x)$, $\omega(x)$.

Apart from the utility of the following lemma according to the previous remark, it is also interesing itself and has its own consequences as we will see in section 3.8.

Lemma 3.7.3 *Let $h(x)$ such that the A-sequence of $R(1, h(x))$ is of the type $(-1, a_1, 0, a_3, a_4, \ldots)$ or equivalently that:*

$$h(x) = -x + h_2 x^2 - h_2^2 x^2 + h_4 x^4 + h_5 x^5 + \ldots$$

Then there exists three involutions $v(x), g(x), \omega(x) \in \mathcal{F}_1(\mathbb{K})$ such that:

$$h(x) = \omega(v(g(x)))$$

Proof: This result can be easily re-stated in terms of matrices in $\mathcal{A}(\mathbb{K})$,that is, we want to show the existence of three involutions $A = (a_{ij})_{0 \leq i,j}$, $B = (a_{ij})_{0 \leq i,j}$ and $C = (a_{ij})_{0 \leq i,j}$ in $\mathcal{A}(\mathbb{K})$ satisfying:

$$R(1, h(x)) = A \cdot B \cdot C$$

or equivalently:

$$A \cdot R(1, h(x)) = B \cdot C$$

or even better:

$$A \otimes h(x) = B \otimes \omega(x) \tag{3.14}$$

This last statement will be more convenient for us. As usual, we will prove the corresponding analogue for the partial groups $\mathcal{A}_n(\mathbb{K})$ by induction.

- $\boxed{\text{Base case: } n=3}$ We have already characterized involutions in $\mathcal{A}(\mathbb{K})$ (see proposition 3.3.4). So the equation we want to solve is:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & a_{21} & 1 & 0 \\ 0 & -a_{21}^2 & -2a_{21} & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ h_2 \\ -h_2^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & b_{21} & 1 & 0 \\ 0 & -b_{21}^2 & -2b_{21} & -1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ c_{21} \\ -c_{21}^2 \end{bmatrix}$$

The above equality is equivalent to the system

$$\begin{cases} a_{21} - h_2 = b_{21} - c_{21} \\ (a_{21} - h_2)^2 = (b_{21} - c_{21})^2 \end{cases}$$

and it is clear that this system reduces to the following linear equation:

$$a_{21} - b_{21} + c_{21} = h_2$$

which, obviously, has solutions.

- $\boxed{\text{Induction step: } n+1 \geq 4}$ As induction hypothesis, suppose now that the analogue of (3.14) in $\mathcal{A}_n(\mathbb{K})$ has solution:

$$A_n = (a_{ij})_{0\leq i,j\leq n}, \qquad B_n = (b_{ij})_{0\leq i,j\leq n}, \qquad C_n = (c_{ij})_{0\leq i,j\leq n}$$

In order to solve the corresponding analogue of equation (3.14) in $\mathcal{A}_{n+1}(\mathbb{K})$:

$$\left[\begin{array}{ccc|c} & A_n & & \\ \hline 0 & a_{n+1,1} & \ldots & a_{n+1,n+1} \end{array}\right] \begin{bmatrix} 0 \\ -1 \\ \vdots \\ h_{n+1} \end{bmatrix} = \left[\begin{array}{ccc|c} & B_n & & \\ \hline 0 & b_{n+1,1} & \ldots & b_{n+1,n+1} \end{array}\right] \begin{bmatrix} 0 \\ -1 \\ \vdots \\ c_{n+1,1} \end{bmatrix}$$

only a new equation appears (corresponding to the last entry of the column vector) since the rest can be solved by induction hypothesis:

$$\sum_{k=1}^{n+1} a_{n+1,k} h_k = \sum_{k=1}^{n+1} b_{n+1,k} c_{k,1} \tag{3.15}$$

We have two different cases:

  - In the case n odd (3.15) can be written as:

$$a_{n+1,1} - b_{n+1,1} + c_{n+1,1} = h_{n+1} + \sum_{k=2}^{n} (a_{n+1,k} h_k - b_{n+1,k} c_{k,1}) \tag{3.16}$$

    According to the characterization of involutions made in proposition 3.3.4, for every choice of $a_{n+1,1}$, $b_{n+1,1}$, $c_{n+1,1}$ we obtain that the Riordan matrices:

$$A_{n+1} = (a_{ij})_{0\leq i,j\leq n+1}, \qquad B_{n+1} = (b_{ij})_{0\leq i,j\leq n+1}, \qquad C_{n+1} = (c_{ij})_{0\leq i,j\leq n+1}$$

provided that the matrices obtained from the induction hypothesis:

$$A_n = (a_{ij})_{0\le i,j\le n}, \qquad B_n = (b_{ij})_{0\le i,j\le n}, \qquad C_n = (c_{ij})_{0\le i,j\le n}$$

also are.

– In the case n even, we obtain an equation similar to (3.16). But in order to construct the involutions the coefficients $a_{n+1,1},\ b_{n+1,1},\ c_{n+1,1}$ are not arbitrary, they depend, in particular, on $a_{n,1}, b_{n,1}$ and $c_{n,1}$.
  So equation (3.16) can be reduced to an expression of the form:

$$\left(2h_2 - \left(\frac{n}{2}+1\right) a_{2,1}\right) a_{n,1} + \left(\left(\frac{n}{2}+1\right) b_{2,1} - 2c_{2,1}\right) b_{n,1} + \left(\left(\frac{n}{2}+1\right) c_{2,1} - nb_{2,1}\right) c_{n,1} = \mathbf{K}$$

where everything inside this $\mathbf{K}$ are elements that do not depend on the elements a_{k1}, b_{k1}, c_{k1} for $k < n$.

Since (for $n \ge 4$) in the study of the even case $n+2$, we involve parameters of the previous odd case $n+1$, we have to study compatibility between the equations:

$$\begin{cases} a_{n+1,1} - b_{n+1,1} + c_{n+1,1} = h_{n+1} + \sum_{k=2}^{n} (a_{n+1,k} h_k - b_{n+1,k} c_{k1}) \\[2ex] \left(2h_2 - \left(\frac{n+1}{2}+1\right) a_{21}\right) a_{n+1,1} + \left(\left(\frac{n+1}{2}+1\right) b_{21} - 2c_{21}\right) b_{n+1,1} + \\ \qquad\qquad + \left(\left(\frac{n+1}{2}+1\right) c_{21} - (n+1) b_{21}\right) c_{n+1,1} = \mathbf{K} \end{cases}$$

together all equations in the system have solution if

$$a_{21} - b_{21} + c_{21} = h_2 \quad \text{and} \quad a_{21} \ne b_{21}$$

Putting all the piece together we have that as long as we make a choice of the parameters a_{21}, b_{21}, c_{21}:

$$\begin{cases} a_{21} - b_{21} + c_{21} = h_2 \\ a_{21} \ne b_{21} \end{cases}$$

we can solve the problem for $\mathcal{A}_3(\mathbb{K})$. After this, we can do extensions by induction progressively extending two rows at each time.

□

Now finally we can prove the result that implies theorem 3.6.5:

Lemma 3.7.4 *For every special commutator $D \in \mathcal{SC}(\mathbb{K})$, there exists three involutions $H, I, J \in SI(\mathbb{K})$ such that:*

$$D \cdot R(1, -x) = H \cdot I \cdot J \tag{3.17}$$

Proof: As we already explained in remark 3.7.2, after having proved lemma 3.7.3, we only need to find, for any:

$$d(x) = d_0 + d_1 x + d_2 x^2 + \ldots \in \mathcal{F}_0(x)$$

and for any involutions:

$$g(x) = -x + g_2 x + g_3 x^3 + \ldots \qquad v(x) = -x + v_2 x^2 + v_3 x^3 + \ldots \qquad \omega(x) = -x + \omega_2 x + \omega_3 x^3 + \ldots$$

in $\mathcal{F}_1(x)$, some $f(x)$, $u(x)$, $\alpha(x)$ in such a way that:

(i) $d(x) = f(x) \cdot u(g(x)) \cdot \alpha(v(g(x))))$

(ii) $H = R(f(x), g(x))$, $I = R(u(x), v(x))$, $J = R(\alpha(x), \omega(x))$

After intensive inspection we conclude that we can choose $\alpha(x) = 1$, so (i) reduces to:

$$d(x) = f(x) \cdot u(g(x)) \tag{3.18}$$

But this equation is equivalent to:

$$d(x) = R(f(x), g(x)) \otimes u(x) \tag{3.19}$$

We will construct two power series:

$$f(x) = f_0 + f_1 x + f_2 x^2 + \ldots \qquad u(x) = u_0 + u_1 x + u_2 x^2 + \ldots$$

satisfying (3.18) and (ii) by induction. Denote by:

$$R(f(x), g(x)) = (a_{ij})_{0 \le i,j < \infty} \qquad R(u(x), v(x)) = (b_{ij})_{0 \le i,j < \infty}$$

where obviously $a_{i0} = f_i$, $b_{i0} = u_i$ for all $i \in \mathbb{N}$.

- $\boxed{\text{Base Step: } n = 2}$ The analogue version of (3.19) in terms of partial Riordan matrices is:

$$\begin{bmatrix} 1 & & \\ f_1 & -1 & \\ \frac{f_1(f_1-g_2)}{2} & g_2 - f_1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ u_1 \\ \frac{u_1(u_1-v_2)}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ d_1 \\ d_2 \end{bmatrix}$$

  where some entries in the matrices and column vectors above are determined by the characterization of involutions made in theorem 3.2.2 in order to satisfy (ii). This equation is equivalent to:

$$\begin{cases} f_1 - u_1 & = d_1 \\ (g_2 - v_2)u_1 & = 2d_2 + g_2 d_1 - d_1^2 \end{cases}$$

  that has solution if $g_2 \neq v_2$ which is provided in the proof of the previous lemma.

- $\boxed{\text{Induction Step: } n+1 \geq 3}$ We want to solve:

$$\left[\begin{array}{ccc|c} & R_n(f(x),g(x)) & & \\ \hline a_{n+1,0} & \cdots & a_{n+1,n} & (-1)^{n+1} \end{array}\right] \begin{bmatrix} 1 \\ u_1 \\ \vdots \\ u_{n+1} \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ d_n \\ d_{n+1} \end{bmatrix}$$

which is the analogue equation of (3.19). This is a system of equations and assuming that the case n is solved by induction hypothesis, we only need to fix the equation corresponding to the $(n+1)$-row:

$$a_{n+1,0} + a_{n+1,1}u_1 + \sum_{k=2}^{n-1} a_{n+1,k}u_k + a_{n+1,n}u_n + (-1)^{n+1}u_{n+1} = d_{n+1}$$

or equivalently:

$$a_{n+1,0} + a_{n+1,1}u_1 + a_{n+1,n}u_n + (-1)^{n+1}u_{n+1} = \mathbf{K}^{(1)}_{n+1} \tag{3.20}$$

where $\mathbf{K}^{(1)}_{n+1} = d_{n+1} - \sum_{k=2}^{n-1} a_{n+1,n}u_n$ and so it does not depend on f_n, f_{n+1}, u_n, u_{n+1}. Again we have to distinguish between two cases for this last equation:

  - If $n+1$ is odd, since we want $R(f(x),g(x))$ and $R(u(x),v(x))$ to be involutions, according to the characterization of involutions made in theorem 3.2.2, $a_{n+1,1}$, $b_{n+1,1}$ are fixed but do not depend on $a_{n+1,0} = f_{n+1}$ and on $b_{n+1,0} = u_{n+1}$ and so (3.20) is of the form:

$$f_{n+1} - u_{n+1} = \mathbf{K}^{(2)}_{n+1} \tag{3.21}$$

    where $\mathbf{K}^{(2)}_{n+1} = \mathbf{K}^{(1)}_{n+1} - a_{n+1,1}u_1 - a_{n+1,n}u_n$ and does not depend on f_{n+1}, u_{n+1}.
  - If $n+1$ is even, since we want $R(f(x),g(x))$ and $R(u(x),v(x))$ to be involutions, according to the characterization of involutions made in theorem 3.2.2, $a_{n+1,0} = f_{n+1}$, $b_{n+1,0} = u_{n+1}$ are fixed, and so (3.20) is of the form:

$$-\frac{1}{2}\sum_{k=1}^{n} a_{k0}a_{n+1,k} + a_{n+1,1}u_1 + a_{n+1,n}u_n - \frac{1}{2}\sum_{k=1}^{n} b_{k0}b_{n+1,k} = \mathbf{K}^{(1)}_{n+1} \tag{3.22}$$

Since (for $n \geq 3$) in the study of the even case $n+2$ we involve parameters of the odd case $n+1$, we must study compatibility between odd an even equations. So first we will simplify (3.22).

Firstly, note that in (3.22) ($n+1$ is even) $a_{n+1,1}$, $b_{n+1,1}$ depend on f_n and u_n. So we can think that (3.22) is of the type:

$$\left(-\frac{1}{2}a_{n+1,n}\right)a_{n0} + \left(u_1 - \frac{1}{2}a_{10}\right)a_{n+1,1} + \left(a_{n+1,n} - \frac{1}{2}b_{n+1,n}\right)b_{n0} - \left(\frac{1}{2}b_{10}\right)b_{n+1,1} = \mathbf{K}^{(3)}_{n+1} \tag{3.23}$$

where $\mathbf{K}^{(3)}_{n+1} = \mathbf{K}^{(1)}_{n+1} + \frac{1}{2}\sum_{k=2}^{n-1} a_{k0}a_{n+1,k} + \frac{1}{2}\sum_{k=2}^{n-1} b_{k0}b_{n+1,k}$ and does not depend on $f_n, u_n, f_{n+1}, u_{n+1}$.

As we mentioned before, $a_{n+1,1}, b_{n+1,1}$ depend on a_{n0}, b_{n0} since by definition the Riordan matrices $R(f(x), h(x))$ and $R(u(x), v(x))$ satisfy:

$$a_{n+1,1} = [x^{n+1}](f(x)\cdot g(x)) = f_0 g_{n+1} + f_1 g_n + \ldots + f_n g_1$$
$$b_{n+1,1} = [x^{n+1}](u(x)\cdot v(x)) = u_0 v_{n+1} + u_1 v_n + \ldots + u_n v_1$$

so (3.23) is in fact of the form:

$$\left(-\frac{1}{2}a_{n+1,n} + \frac{1}{2}a_{10} - u_1\right) a_{n0} + \left(a_{n+1,n} - \frac{1}{2}b_{n+1,n} + \frac{1}{2}b_{10}\right) b_{n0} = \mathbf{K}^{(4)}_{n+1} \tag{3.24}$$

where:

$$\mathbf{K}^{(4)}_{n+1} = \mathbf{K}^{(3)}_{n+1} - \left(u_1 - \frac{1}{2}a_{10}\right)(f_0 g_{n+1} + f_1 g_n + \ldots + f_{n-1} g_2) + \left(\frac{1}{2}b_{10}\right)(u_0 v_{n+1} + u_1 v_n + \ldots + u_n v_1)$$

and since for the Riordan matrices $R(f(x), g(x))$ and $R(u(x), v(x))$ by definition we have that ($f_0 = u_0 = 1$ and $g_1 = v_1 = -1$):

$$a_{n+1,n} = -f_1 + nf_0 g_2, \qquad b_{n+1,n} = -u_1 + nu_0 v_2$$

and so (3.24) is equivalent to:

$$\left(-\frac{n}{2}g_2 + (f_1 - u_1)\right) a_{n0} + \left(ng_2 - \frac{n}{2}v_2 - (f_1 - u_1)\right) b_{n0} = \mathbf{K}^{(4)}_{n+1}$$

Remember from the base step $n = 2$, $f_1 - u_1 = d_1$ and so this equation is:

$$\left(-\frac{n}{2}g_2 + d_1\right) a_{n0} + \left(ng_2 - \frac{n}{2}v_2 - d_1\right) b_{n0} = \mathbf{K}^{(4)}_{n+1}$$

Now putting all the pieces together, we have found a solution in $\mathcal{R}_2(\mathbb{K})$ that we can extend progressively by induction (extending two rows at each step) to a solution of equation (3.19) if and only if the following system has a solution for every n:

$$\begin{cases} f_{n+1} - u_{n+1} = \mathbf{K}^{(2)}_{n+1} \\ \left(-\frac{n+1}{2}g_2 + d_1\right) f_{n+1} + \left((n+1)g_2 - \frac{n+1}{2}v_2 - d_1\right) b_{n+1,0} = \mathbf{K}^{(4)}_{n+2} \end{cases}$$

In order to discuss the existence of a solution for this system, we only need to look at its matrix of coefficients:

$$\begin{bmatrix} 1 & -1 \\ \left(-\frac{n+1}{2}g_2 + d_1\right) & \left((n+1)g_2 - \frac{n+1}{2}v_2 - d_1\right) \end{bmatrix}$$

This matrix has the same determinant than:

$$\begin{bmatrix} 1 & -1 \\ 0 & (g_2 - v_2) \end{bmatrix}$$

and since from the proof of the previous result we know that $g_2 \neq v_2$ we conclude that this determinant is not equal to 0.

□

3.8 Some Consequences

Interpreting elements in $\mathcal{F}_1(\mathbb{K})$ as elements in $\mathcal{A}(\mathbb{K})$ via the usual identification, we have proved by the way (as a direct corollary of theorem 3.6.5) that:

Corollary 3.8.1 *Consider $\mathbb{K} = \mathbb{R}, \mathbb{C}$. In $\mathcal{F}_1(\mathbb{K})$ we have that:*

(1) The product of any odd number of involutions is the product of three. A power series is the product of three involution if and only if it is of the type:

$$-x + ax^2 - a^2x^+ \dots \tag{3.25}$$

(2) The product of any even number of involutions is the product of four. A power series is of this type if and only if:

$$x + ax^2 + a^2x^3 + \dots \tag{3.26}$$

Note that the conditions (3.25), (3.26) are equivalent to impose that the corresponding matrices have A-sequences of the type $(1, a_1, 0, a_3, \dots)$, $(-1, a_1, 0, a_3, \dots)$ in each case.

This was already known by O'Farrell (see [88]), but his paper did not came to our knowledge after we had already proved theorem 3.6.5. It would have been possible to obtain a shorter proof of theorem 3.6.5 by using this result, but we have decided to maintain our technique of induction in the partial Riordan groups to prove it.

On the other hand, a first immediate consequence of theorem 3.6.5 (we will skip the proof) for the algebraic structure of $\mathcal{R}(\mathbb{K})$ is the following:

Corollary 3.8.2

$$\mathcal{I}(\mathbb{K}) \cong \mathcal{SC}(\mathbb{K}) \rtimes \mathcal{K}(\mathbb{K})$$

Remark 3.8.3 *Finally we must say that, as we announced, $\mathcal{I}$ is a normal subgroup of $\mathcal{R}(\mathbb{K})$, so it would be possible to wonder how the quotient $\mathcal{R}/\mathcal{I}$ is. This quotient can be easily found, but the description is quite long, so we have decided to skip it from this thesis.*

Chapter 4

Infinite Dimensional Lie Group Structure for the Riordan group

In this chapter, $\mathbb{K}$ will be $\mathbb{R}$ or $\mathbb{C}$.

As we have already presented in chapter 1, the Riordan group can be described as the inverse limit of an inverse sequence involving the groups $\mathcal{R}_n(\mathbb{K}) < GL_{n+1}(\mathbb{K})$. As we will show in a moment, those groups are naturally Lie groups and so, in fact, we are obtaining automatically a pro-Lie group for the Riordan group $\mathcal{R}(\mathbb{K})$.

Not any pro-Lie group happens to be a Lie group (see [45] for an exhaustive topological treatment of pro-Lie groups) but $\mathcal{R}(\mathbb{K})$ does (it is a Frechet Lie group modelled over $\mathbb{K}^{\mathbb{N}}$).

As happened before, this pro-Lie group or inverse limit structure will allow us to lift some features from the Lie group structure of the partial Riordan group to the Lie group structure of $\mathcal{R}$ (the characterization of the tangent algebra, the exponential, stabilizers,...).

The Lie group structure over $\mathcal{R}(\mathbb{K})$, among other things:

- Is interesting itself as a toy example of the pro-Lie group theory.
- Presents another interesting class of infinite matrices that represent the tangent algebra of $\mathcal{R}(\mathbb{K})$, and that also has a natural action over $\mathbb{K}[[x]]$ that remains to be explored.
- Will allow us, among other things, to solve some initial value problems, recover some known subgroups as stabilizers and compute their tangent algebra.

The organization of this chapter is the following:

- Some bibliography was already proposed in section 0.4. Anyway some basic definitions will be recalled in sections 4.1, 4.2.
- In sections 4.3, 4.4 and 4.5 the Lie group structure of the finite matrix groups $\mathcal{R}(\mathbb{K})$ will be studied. In the first one, we will present their manifold structure, in the second one, we will make some comments about their Lie group structure and finally in the last one we will present their Lie algebra.
- In section 4.6 we will show that the bonding maps $P_n : \mathcal{R}_{n+1} \to \mathcal{R}_n$, that were already presented in chapter 1 are Lie groups homomorphisms in this new setting. This will allow us to describe $\mathcal{R}(\mathbb{K})$ as a pro-Lie group below.

- In sections 4.7 and 4.8 we will study the Frechet Lie group structure of $\mathcal{R}(\mathbb{K})$. In the first one we will study some basic facts about this structure and in the second one we will study the compatibility between this structure and the natural structure of pro-Lie group of $\mathcal{R}(\mathbb{K})$.
- In sections 4.9, 4.10 4.11 we will explain in detail the structure of the Lie algebra and the exponential for $\mathcal{R}(\mathbb{K})$.
- In section 4.12 we will shortly comment the analogous Lie group structure with which $\mathcal{R}_{\infty\infty}$ is endowed.
- In section 4.13 an interpretation in terms of power series for the multiplication of an element in the Lie algebra of $\mathcal{R}(\mathcal{K})$ by a column vector is given.
- The rest of the sections in this chapter are devoted to the applications described above of the Lie group structure of $\mathcal{R}(\mathbb{K})$.

4.1 Some Basic Definitions: classical Lie groups

In this chapter we will recall some basic concepts and notation about classical Lie Groups. Many books could be used as a reference for that, for example [37, 90]. Specific references for matrix Lie groups are also available, for example [5].

Definition 4.1.1 *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. An analytic $\mathbb{K}$-Lie group G is a $\mathbb{K}$-analytic manifold jointly with a product operation converting G into a group and such that*

(i) The multiplication
$$\begin{array}{ccc} G \times G & \to & G \\ (a,b) & \mapsto & ab \end{array}$$
is a $\mathbb{K}$-analytic map.

(ii) The inversion
$$\begin{array}{ccc} G & \to & G \\ a & \mapsto & a^{-1} \end{array}$$
is a $\mathbb{K}$-analytic map.

Example 4.1.2 *For all $n \geq 0$ $GL_n(\mathbb{K})$ is a Lie group. It is considered to be a differentiable submanifold of $\mathbb{K}^{n^2}$ via the natural identification:*

$$\widetilde{\varphi}_n : \mathcal{M}_n(\mathbb{K}) \longrightarrow \mathbb{K}^{n^2}$$

$$\begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \equiv (a_{11}, \dots, a_{n1}, \dots, a_{1n}, \dots, a_{nn})$$

that maps $GL_n(\mathbb{K})$ into an open subset of $\mathbb{K}^{n^2}$.

Let G be a Lie group. Let $H \leq G$ be a subgroup which is also a closed submanifold of G. Then H is said to be **Lie subgroup** of G. Lie subgroups of $GL_n(\mathbb{K})$ are usually called **matrix groups**.

Lie group homomorphisms are defined as smooth maps preserving the operation and this leads to the definition of Lie group isomorphism.

Also of our interst is the following:

Definition 4.1.3 *Let G be a Lie group and X a differentiable manifold. An homomorphism:*

$$\alpha : G \longrightarrow Diff(X)$$

where Diff(X) is the ***group of diffeomorphisms of*** X ***into itself****, is called an* ***action*** *on X if the following map, bi-univocally determined by the previous one, is differentiable:*

$$G \times X \longrightarrow X$$

$$(g, x) \longmapsto (\alpha(g))(x)$$

where $G \times X$ is endowed with the differentiable structure of product of differentiable manifolds.

For any Lie group G one can easily find three actions of G on itself:

Example 4.1.4 *The following three maps are actions of G on itself:*

$$L_g(x) = gx \qquad R_g(x) = xg \qquad Adj_g = gxg^{-1}$$

Example 4.1.5 *Since elements in GL_n can be naturally identified with diffeomorphisms of $\mathbb{K}^n$ into intself, there is a natural action:*

$$GL_n(\mathbb{K}) \longrightarrow Diff(\mathbb{K}^n)$$

given by the map:

$$GL_n(\mathbb{K}) \times \mathbb{K}^n \longrightarrow \mathbb{K}^n$$

$$(M, (x_1, \ldots, x_n)) \longmapsto M \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

One of the basic methods of the theory of Lie groups consists of reducing questions concerning Lie groups to certain problems of linear algebra. This is done by assigning to every Lie group G its tangent algebra $\mathcal{L}(G)$ which is a Lie algebra, a vector space with a Lie bracket (definition can be found in the bibliography, for example [37] chapter 2).

The map assigning to each Lie group a Lie algebra is called the **Lie functor** (between the category of Lie groups and the category of Lie algebras) and will be denoted by $\mathcal{L}$. As a vector space, the Lie algebra associated to a Lie group G is the tangent space of G at its identity element e, denoted by T_eG.

Proposition 4.1.6 *Let $\mathcal{L}(GL_n(\mathbb{K}))$ be the Lie algebra of $GL_n(\mathbb{K})$. Consider the set of matrices $\mathcal{M}_n(\mathbb{K})$ with respect to the Lie bracket:*

$$[A, B] = AB - BA$$

Then $(\mathcal{M}_n(\mathbb{K}), [\cdot, \cdot])$ is a Lie algebra and is a ***full and faithful representation*** *of $\mathcal{L}(GL_n(\mathbb{K}))$, that is, there exists an isomorphism of Lie algebras between $\mathcal{L}(GL_n)$ and $(\mathcal{M}_n(\mathbb{K}), [\cdot, \cdot])$.*

Analogously, for any matrix group S, $\mathcal{L}(S)$ has a full and faithful representation as a subalgebra of $(M_n(\mathbb{K}), [\cdot, \cdot])$.

The Lie algebra "almost" determines the Lie group. For example simply connected Lie groups are determined up to isomorphism by their Lie algebra (p. 43 in [37]).

An interesting result for us will be the following:

Proposition 4.1.7 *Let $\alpha : G \longrightarrow Diff(X)$ be an action of the Lie group G over the differentiable manifold X. We define the* ***stabilizer*** *of an element in X with respect to this action to be the set:*

$$G_x = \{g \in G : \alpha(g)(x) = x\}$$

(i) (Theorem 2.1 in chapter 1 in [37]) For any $x \in X$, G_x is a Lie subgroup of G.

(ii) (Theorem 1.1 in chapter 2 in [37]) $\mathcal{L}(G_x) = \{v \in \mathcal{L}(G) : (D(\alpha)(v))(x) = 0\}$

Recall also that:

Definition 4.1.8 *A* ***one-parameter subgroup*** *in the Lie group G is Lie group homomorphism:*

$$\varphi : (\mathbb{R}, +) \longrightarrow G$$

However, sometimes we will also use the term "one-parameter subgroup" for the image of this map.

Remark 4.1.9 *It is easy to see (proposition 3.1 in chapter 2 in [37]) that a path on a Lie group G with identity element e:*

$$\gamma : \mathbb{R} \longrightarrow G$$

is a one-parameter subgroup if and only if its velocity is constant and $\gamma(0) = e$.

There is a natural bijection between one parameter subgroups in G and tangent vectors in $v \in T_e(G)$. The one-parameter subgroup with velocity $v \in T_e(G)$ will be denoted by γ_v.

Thanks to remark 4.1.9 we can define the following:

Definition 4.1.10 *Let G be a Lie group and $\mathcal{L}(G)$ be its Lie algebra. The* ***exponential map*** *is the application:*

$$\exp\colon \mathcal{L}(G) \longrightarrow G$$

$$\exp(v) = \gamma_v(1)$$

It is easy to verify that:

$$\gamma_v(t) = exp(tv)$$

If G is a matrix group, the exponential map is given by the usual exponential of matrices.

These exponential matrices allow us to solve certain differential equation for matrices as we will see later.

4.2 Some Basic Definitions: Lie groups modelled over Frechet spaces

For this section we will follow the work by J. Milnor [82] although a readible development of the subject is also available in [37].

A classical Lie group is a smooth manifold (that is, every point has a neighbourhood diffeomorphic to an open subset in $\mathbb{R}^n$) with a compatible group structure. Now with the new definition, Lie groups will not need to be manifolds in this classical sense but Frechet manifolds (that is, every point has a neighbourhood diffeomorphic to an open subset of a Frechet space) and having a compatible group structure.

Recall that:

Definition 4.2.1 *A real or complex* ***topological vector space*** *X is a vector space with a topology which is Hausdorff and such that the operations of sum and scalar multiplication (with respect to the correspondent product topology) are continous maps.*

A ***Frechet space*** *is a topological vector space X such that:*

- *X is* ***locally convex*** *, that is, every neighborhood of zero U, contains another neighborhood of zero U' which is convex.*
- *The topology on X can be induced by a translation invariant metric, i.e., a metric d such that:*

$$\forall x, y, a \in X, \quad d(x, y) = d(x + a, y + a)$$

- *X is a complete metric space.*

In the original notes by Milnor [82] a more general class of spaces is considered for doing differential calculus (locally convex vector spaces) but the framework of Frechet spaces will be enough for us. The reason of the interest of the study of differential calculus and differential geometry over Frechet spaces is other apart from just generalizing, but we are not going to discuss this here.

Roughly speaking, it is possible to do differential calculus between frechet spaces, since the **Gateaux derivative** can be well defined, and so we can define C^k or analytic functions. Some basic properties of the differential calculus still hold. For example the chain rule works. But this is not the case of the Inverse function theorem.

The Frechet space we are interested in is the following:

Example 4.2.2 $\mathbb{K}^{\mathbb{N}}$ *with the product topology is a Frechet space. A complete metric, which is also translation invariant and that induces this topology in* $\mathbb{K}^{\mathbb{N}}$ *is:*

$$d : \mathbb{K}^{\mathbb{N}} \times \mathbb{K}^{\mathbb{N}} \longrightarrow \mathbb{R}$$

$$d((u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}) = \sum_{n=0}^{\infty} \frac{1}{2^n} \frac{|u_n - v_n|}{1 + |u_n - v_n|}$$

Now that we have a notion of derivative in Frechet spaces, we can think about doing (infinite dimensional) geometry on spaces modelled over them. The definition of manifold can be naturally extended to the definition of **manifold over a Frechet space** and the definition of Lie group to the definition of **Lie group modelled over a Frechet space**. We omit details that can be found in the bibliography. One-parameter sugroups, the tangent algebra and the exponential map can be also defined analogously to the finite dimensional case.

Another approach to the theory of infinite dimensional Lie groups is possible and it is also interesting for our purposes. Inverse limits of sequences of finite dimensional Lie groups, the so called **pro-Lie groups**, are good candidates to generalize the notion of Lie group. This is the approach made by K. H. Hofmann and S. A. Morris in their book [45].

Inverse sequences work well in the category of topological groups: given an inverse sequence of topological groups joined by appropiate bonding maps we can always define the inverse limit of this sequence. But this is not true in the category of Lie groups: not any pro-Lie group has an acceptable structure of Lie group, not even an acceptable structure of infinite dimensional manifold.

Finally we want to point out the fact that there is a Lie group modelled over a Frechet space which Lie group structure has been studied. We are talking about $\mathcal{F}_1(\mathbb{K})$, for $\mathbb{K} = \mathbb{R}, \mathbb{C}$. The study of its Lie group structure is interesting due to its applications to some problems related to the existence of fractional iterates and one parameter subgroups. This topic is discussed in sections 5.2, 5.3 and 5.4 of [2]. See also the work by S. A. Jennings [52] for an expression to compute the exponential.

4.3 $\mathcal{R}_n(\mathbb{K})$ as a manifold.

As we have already mentioned, $\mathcal{R}_n(\mathbb{K})$ is a subgroup of the classical Lie group $GL_{n+1}(\mathbb{K})$. Therefore if we consider any topology in $GL_{n+1}(\mathbb{K})$ automatically it induces a topology in $\mathcal{R}_n(\mathbb{K})$.

The description for the topology in $GL_{n+1}(\mathbb{K})$ is obtained considering $GL_{n+1}(\mathbb{K}) \subset M_{n+1}(\mathbb{K})$ as an open set in $\mathbb{K}^{(n+1)^2}$ by using the chart:

$$\widetilde{\varphi}_{n+1} : \mathcal{M}_{n+1}(\mathbb{K}) \longrightarrow \mathbb{K}^{(n+1)^2} \tag{4.1}$$

given by:

$$\begin{bmatrix} x_{11} & \dots & x_{1,n+1} \\ \vdots & & \vdots \\ x_{n+1,1} & \dots & x_{n+1,n+1} \end{bmatrix} \longmapsto (x_{11}, \dots, x_{1,n+1}, \dots, x_{n+1,1}, \dots, x_{n+1,n+1})$$

So according to the characterization of the elements in $GL_{n+1}(\mathbb{K})$ that are Riordan matrices given in chapter 0, we have:

Proposition 4.3.1 *Let $n \in \mathbb{N}$.*

(i) $\mathcal{R}_n(\mathbb{R})$ has a natural structure of differentiable manifold diffeomorphic to an open subspace $\mathcal{U}_n$ of $\mathbb{R}^{2n+1}$ compatible with the differentiable structure that it has as submanifold of $GL_{n+1}(\mathbb{K})$.

(ii) $\mathcal{R}_n(\mathbb{C})$ has a natural structure of complex manifold holomorphic to an open subspace $\mathcal{U}_n$ of $\mathbb{C}^{2n+1}$ compatible with the differentiable structure that it has as submanifold of $GL_{n+1}(\mathbb{K})$.

Proof: Let us denote by $\mathbf{u} = (x_0, a_0, x_1, a_1, \dots, x_{n-1}, a_{n-1}, x_n) \in \mathbb{K}^{2n+1}$,

- Consider the following open subspace of $\mathbb{K}^{2n+1}$:

$$\mathcal{U}_n = \{\mathbf{u} \in \mathbb{K}^{2n+1} \mid x_0 a_0 \neq 0\}$$

- Define:

$$\varphi_n : \mathcal{U}_n \to \mathcal{R}_n(\mathbb{K})$$
$$\varphi_n(\mathbf{u}) = (d_{i,j}(\mathbf{u}))_{i,j=0,1,\dots,n}$$

  to be the map that assigns the corresponding partial Riordan matrix with first column $x_1, \dots, x_n$ and partial A-sequence $(a_0, \dots, a_{n-1})$ to each element $\mathbf{u}$.

  Iteratively we can easily prove that $d_{i,j}(\mathbf{u})$ is a polynomial on the components of $\mathbf{u}$. Therefore, by using the chart described before (see equation (4.1)) we can see that φ_n is analytic or holomorphic in each case.

- Moreover, using the natural characterization of partial Riordan matrices in terms of the A-sequence it is immediate that φ_n is injective and surjective. In fact, we can compute the global chart in the manifold:

$$\varphi_n^{-1} : \mathcal{R}_n(\mathbb{K}) \to \mathcal{U}_n$$
$$R_n(d(x), h(x)) \longmapsto \mathbf{u}$$

  where $x_0, \dots, x_n$ is the first column of $R_n(d(x), h(x))$ and $(a_0, \dots, a_{n-1})$ is its partial A-sequence.

- We can also compute this inverse. Note that if:

$$\varphi_n^{-1}((d_{i,j})_{i,j=0,1,\ldots,n}) = (x_0, a_0, \ldots, x_{n-1}, a_{n-1}, x_n)$$

then $x_i = d_{i,0}$ for every $i = 0, 1, \ldots, n$. Furthermore:

$$a_0 = \frac{d_{1,1}}{d_{0,0}},\ a_1 = \frac{1}{d_{1,1}}(d_{2,1} - a_0 d_{1,0}) = \frac{1}{d_{1,1}}\left(d_{2,1} - \frac{d_{1,0}d_{1,1}}{d_{0,0}}\right)$$

and in general:

$$a_k = \frac{1}{d_{k,k}}\left(d_{k+1,1} - \sum_{j=0}^{k-1} a_j d_{k,j}\right) \text{ for } k = 1, \ldots, n-1.$$

Thus, iteratively, we obtain that a_k is a rational function on the entries d_{ij} where the denominators are always non-null because they are product of elements in the main diagonal.

- So we have proved that φ_n is an analytic or holomorphic (in each case) global parametrization of $\mathcal{R}_n(\mathbb{K})$.

□

Remark 4.3.2 *From the result above we can consider $\mathcal{R}_n(\mathbb{K})$ as a closed topological subspace of $GL_{n+1}(\mathbb{K})$, therefore is a matrix group.*

Since φ_n above is a homeomorphism and by using basic facts in homotopy theory we get the following topological properties:

Proposition 4.3.3 *Consider $\mathcal{R}_n(\mathbb{K})$ as a topological subspace of $\mathbb{K}^{(n+1)^2}$ (via the identification $\widetilde{\varphi}_n$). Then:*

(i) For $\mathbb{K} = \mathbb{R}$:

- *$\mathcal{R}_0(\mathbb{R})$ has two connected components each of them is contractible.*
- *If $n \geq 1$, $\mathcal{R}_n(\mathbb{R})$ has four connected components each of them is contractible.*

(ii) For $\mathbb{K} = \mathbb{C}$:

- *$\mathcal{R}_0(\mathbb{C})$ is homeomorphic to the cylinder $S^1 \times \mathbb{R}$ where S^1 is the unit sphere in $\mathbb{R}^2$. Consequently $\mathcal{R}_0(\mathbb{C})$ is path-connected and:*
  - *With respect to the homotopy groups, the fundamental group $\pi_1(\mathcal{R}_0(\mathbb{C}))$ is isomorphic to the group of integers $\mathbb{Z}$. The homotopic groups for higher dimensions are null.*

  - *For homology, $H_0(\mathcal{R}_0(\mathbb{C})) \equiv H_1(\mathcal{R}_0(\mathbb{C})) \equiv \mathbb{Z}$. All the higher dimensional homology groups are null.*
- *If $n \geq 1$ $\mathcal{R}_n(\mathbb{C})$ is homeomorphic to the topological product $\mathbb{T} \times \mathbb{R}^{4n}$ where $\mathbb{T} = S^1 \times S^1$ is the torus. Consequently, $\mathcal{R}_n(\mathbb{C})$ is path-connected and:*
  - *With respect to the homotopy groups, $\pi_1(\mathcal{R}_n(\mathbb{C}) \equiv \mathbb{Z} \oplus \mathbb{Z}$ and all higher homotopy groups are null.*
  - *$H_0(\mathcal{R}_n(\mathbb{C})) \equiv H_2(\mathcal{R}_n(\mathbb{C})) \equiv \mathbb{Z}$, $H_1(\mathcal{R}_n(\mathbb{C})) = \mathbb{Z} \oplus \mathbb{Z}$ and all higher dimensional homology groups are null.*

We will skip the proof.

4.4 The Lie group structure of $\mathcal{R}_n(\mathbb{K})$.

Once we have described the differentiable structure on $\mathcal{R}_n(\mathbb{K})$ (just the $\mathbb{K}$-differentiable structure generated by the global parametrization $(\mathcal{U}_n, \varphi_n)$) the second step we are going to take is to describe the Lie group structure of $\mathcal{R}_n(\mathbb{K})$.

Since we have already noticed that $\mathcal{R}_n(\mathbb{K})$ is a closed subgroup of $GL_{n+1}(\mathbb{K})$, then as we announced in the previous section:

Proposition 4.4.1 *For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, the manifold $\mathcal{R}_n(\mathbb{K})$ with the product of matrices as operation is an analytic $\mathbb{K}$-Lie group for every $n \in \mathbb{N}$.*

$\mathcal{R}_n(\mathbb{K})$ has a natural structure of Lie group over $\mathbb{K}$ of dimension $2n+1$. Note that $\mathcal{R}_n(\mathbb{C})$ can be also considered as a real Lie group of dimension $4n+2$.

Remark 4.4.2 *For further developments and in order to understand better the Lie group structure of $\mathcal{R}_n(\mathbb{K})$ we prefer to check that the conditions in the definition of a Lie group are satisfied:*

(i) Consider the product:

$$\begin{array}{ccc} \mathcal{R}_n(\mathbb{K}) \times \mathcal{R}_n(\mathbb{K}) & \to & \mathcal{R}_n(\mathbb{K}) \\ (D, E) & \mapsto & F = DE \end{array}$$

where $\mathcal{R}_n(\mathbb{K}) \times \mathcal{R}_n(\mathbb{K})$ is endowed with the structure of product of analytic manifolds. Suppose:

$$D = \varphi_n(x_0, a_0, \ldots, x_{n-1}, a_{n-1}, x_n) = (d_{ij})_{0 \leq i,j \leq n}$$

$$E = \varphi_n(y_0, b_0, \ldots, y_{n-1}, b_{n-1}, y_n)$$

$$F = \varphi_n(z_0, c_0, \ldots, z_{n-1}, c_{n-1}, z_n)$$

- *Observe that:*
$$z_k = \sum_{j=0}^{k} y_j d_{k,j}(x_0, a_0, \ldots, x_{n-1}, a_{n-1}, x_n)$$
which is a polynomial in the variables:
$$(x_0, a_0, \ldots, x_{n-1}, a_{n-1}, x_n, y_0, b_0, \ldots, y_{n-1}, b_{n-1}, y_n)$$
In fact, the polynomials z_k do not depend on the variables $b_0, \ldots, b_{n-1}$.
- *Furthermore, the variables c_k for $k = 0, 1, \ldots, n-1$ depend only on the variables $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1}$. Indeed we obtain that c_k for $k = 0, 1, \ldots, n-1$ are differentiable functions on the variables $a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}$, that is, the product is analytic.*

(ii) For the inversion, consider:
$$D = \varphi_n(x_0, a_0, \ldots, x_{n-1}, a_{n-1}, x_n) = (d_{i,j})_{0 \le i,j \le n}, \qquad D^{-1} = \varphi_n(y_0, b_0, y_1, \ldots, b_{n-1}, y_n)$$
As usual, denote $\mathbf{u} = (x_0, a_0, \ldots, x_{n-1}, a_{n-1}, x_n)$. Then we get $x_0 y_0 = 1$ and
$$x_k y_0 + \sum_{\ell=1}^{k} y_\ell d_{k,\ell}(\mathbf{u}) = 0, \text{ for } k = 1, \ldots, n.$$

- *Iteratively we obtain that y_k is a rational function on the coordinates $\mathbf{u}$, where the denominator is of the form $x_0 a_0^k$. Consequently, it is analytic for any $k = 0, 1, \ldots, n$.*
- *From the Lagrange inversion formula we obtain that $b_0, \ldots, b_n$ are analytic functions on $a_0, \ldots, a_n$.*

4.5 The Lie Algebra of $\mathcal{R}_n(\mathbb{K})$

The next step is to describe the Lie algebra of $\mathcal{R}_n(\mathbb{K})$ that we denote by $\mathcal{L}(\mathcal{R}_n(\mathbb{K}))$.

Recall that in this case the tangent algebra $\mathcal{L}(\mathcal{R}_n(\mathbb{K}))$ is the tangent space to $\mathcal{R}_n(\mathbb{K})$ at its neutral element, which is the identity matrix $R_n(1, x)$, together with the corresponding Lie bracket.

As explained before, there is a full and faithful representation of the Lie algebra of $GL_{n+1}(\mathbb{K})$ and of any matrix group as a subalgebra of $(M_{n+1}(\mathbb{K}), [\cdot, \cdot])$, where this bracket is given by:
$$[A, B] = AB - BA$$

Before going on, note that:

Remark 4.5.1 *Using the global parametrization we get:*
$$R_n(1, x) = \varphi_n(\mathbf{e_n})$$
where $\mathbf{e_n} = (1, 1, 0, \ldots, 0)$, that is, $\mathbf{e_n} = (x_0, a_0, x_1, \ldots, a_{n-1}, x_n)$ with $x_0 = 1 = a_0$ and $x_j = 0 = a_k$ for $1 \le j \le n$ and $1 \le j \le n-1$.

We will start by computing the Lie algebra of the $\mathcal{R}_1(\mathbb{K})$ and of $\mathcal{R}_2(\mathbb{K})$. To motivate the result we will obtain, let us develop the lower dimensional cases.

Example 4.5.2 *In the case $n = 1$ the global parametrization is given by:*

$$\varphi_1(x_0, a_0, x_1) = \begin{bmatrix} x_0 & 0 \\ x_1 & a_0x_0 \end{bmatrix}$$

We know that $\left\{ \frac{\partial\varphi_1}{\partial x_0}(x_0, a_0, x_1), \frac{\partial\varphi_1}{\partial a_0}(x_0, a_0, x_1), \frac{\partial\varphi_1}{\partial x_1}(x_0, a_0, x_1) \right\}$ *is a base of the tangent space to $\mathcal{R}_1(\mathbb{K})$ at the point $\varphi_1(x_0, a_0, x_1)$. Since (using the usual identification):*

$$\frac{\partial\varphi_1}{\partial x_0}(x_0, a_0, x_1) = \begin{bmatrix} 1 & 0 \\ 0 & a_0 \end{bmatrix}, \qquad \frac{\partial\varphi_1}{\partial a_0}(x_0, a_0, x_1) = \begin{bmatrix} 0 & 0 \\ 0 & x_0 \end{bmatrix}, \qquad \frac{\partial\varphi_1}{\partial x_1}(x_0, a_0, x_1) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

evaluating at $\mathbf{e} = (1, 1, 0)$:

$$\frac{\partial\varphi_1}{\partial x_0}(\mathbf{e}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \frac{\partial\varphi_1}{\partial a_0}(\mathbf{e}) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \frac{\partial\varphi_1}{\partial x_1}(\mathbf{e}) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and forming all linear combinations with coefficients in $\mathbb{K}$ we have:

$$\mathcal{L}(\mathcal{R}_1(\mathbb{K})) = \left\{ \begin{bmatrix} \chi_0 & 0 \\ \chi_1 & \chi_0 + \alpha_0 \end{bmatrix} \;\middle|\; \chi_0, \alpha_0, \chi_1 \in \mathbb{K} \right\}.$$

Example 4.5.3 *In the case $n = 2$ the global parametrization is given by:*

$$\varphi_2(\mathbf{u}) = \begin{bmatrix} x_0 & 0 & 0 \\ x_1 & a_0x_0 & \\ x_2 & a_0x_1 + a_0a_1x_0 & a_0^2x_0 \end{bmatrix}$$

where $\mathbf{u} = (x_0, a_0, x_1, a_1, x_2)$ with $x_0a_0 \neq 0$.

$$\frac{\partial\varphi_2}{\partial x_0}(\mathbf{u}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a_0 & 0 \\ 0 & a_0a_1 & a_0^2 \end{bmatrix}, \qquad \frac{\partial\varphi_2}{\partial a_0}(\mathbf{u}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & x_0 & 0 \\ 0 & x_1 + a_1x_0 & 2a_0x_0 \end{bmatrix}, \qquad \frac{\partial\varphi_2}{\partial x_1}(\mathbf{u}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & a_0 & 0 \end{bmatrix},$$

$$\frac{\partial\varphi_2}{\partial a_1}(\mathbf{u}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a_0x_0 & 0 \end{bmatrix}, \qquad \frac{\partial\varphi_2}{\partial x_2}(\mathbf{u}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

evaluating at $\mathbf{e} = (1, 1, 0, 0, 0)$:

$$\frac{\partial\varphi_2}{\partial x_0}(\mathbf{e}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \frac{\partial\varphi_2}{\partial a_0}(\mathbf{e}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \frac{\partial\varphi_2}{\partial x_1}(\mathbf{e}) = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

$$\frac{\partial\varphi_2}{\partial a_1}(\mathbf{e}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \frac{\partial\varphi_2}{\partial x_2}(\mathbf{e}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and forming all linear combinations with coefficients in $\mathbb{K}$ *we have*

$$\mathcal{L}(\mathcal{R}_2(\mathbb{K})) = \left\{ \begin{bmatrix} \chi_0 & 0 & 0 \\ \chi_1 & \chi_0 + \alpha_0 & 0 \\ \chi_2 & \chi_1 + \alpha_1 & \chi_0 + 2\alpha_0 \end{bmatrix} \;\middle|\; \chi_0, \alpha_0, \chi_1, \alpha_1, \chi_2 \in \mathbb{K} \right\}.$$

Therefore, to give a general result we are going to use the following notation analogous to the previous examples:

- χ_k the coordinate respect to $\dfrac{\partial \varphi_n}{\partial x_k}(\mathbf{e_n})$ for $0 \leq k \leq n$.
- α_j the coordinate respect to $\dfrac{\partial \varphi_n}{\partial a_j}(\mathbf{e_n})$ for $0 \leq j \leq n-1$

Once this has been settled, we can state the main result of this section:

Proposition 4.5.4 *For every n, we have:*

$$\frac{\partial \varphi_n}{\partial x_k}(\mathbf{e_n}) = (\delta_{i-j,k})_{i,j=0,1\ldots,n} \text{ for } k = 0, 1, \ldots, n,$$

$$\frac{\partial \varphi_n}{\partial a_\ell}(\mathbf{e_n}) = (j\delta_{i-j,\ell})_{i,j=0,1\ldots,n} \text{ for } \ell = 0, 1, \ldots, n-1$$

where $\delta_{r,s}$ is the Kronecker delta. Consequently, the Lie algebra $\mathcal{L}(\mathcal{R}_n(\mathbb{K}))$ is given by the set:

$$\begin{bmatrix} \chi_0 & & & & & & \\ \chi_1 & \chi_0 + \alpha_0 & & & & & \\ \chi_2 & \chi_1 + \alpha_1 & \chi_0 + 2\alpha_0 & & & & \\ \chi_3 & \chi_2 + \alpha_2 & \chi_1 + 2\alpha_1 & \chi_0 + 3\alpha_0 & & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & & \\ \chi_{n-1} & \chi_{n-2} + \alpha_{n-2} & \chi_{n-3} + 2\alpha_{n-3} & \chi_{n-4} + 3\alpha_{n-4} & \cdots & \chi_0 + (n-1)\alpha_0 & \\ \chi_n & \chi_{n-1} + \alpha_{n-1} & \chi_{n-2} + 2\alpha_{n-2} & \chi_{n-3} + 3\alpha_{n-3} & \cdots & \chi_1 + (n-1)\alpha_1 & \chi_0 + n\alpha_0 \end{bmatrix}$$

where $\chi_k, \alpha_j \in \mathbb{K}$. The correspondent Lie bracket is defined by:

$$[M, N] = MN - NM$$

Proof: The result will be proved by induction. We have to prove that:

$$\frac{\partial \varphi_n}{\partial x_k}(\mathbf{e_n}) = (\delta_{i-j,k})_{i,j=0,1,\ldots,n} \qquad k = 0, 1, \ldots, n$$

and that:

$$\frac{\partial \varphi_n}{\partial a_\ell}(\mathbf{e_n}) = (j\delta_{i-j,\ell})_{i,j=0,1,\ldots,n} \qquad \ell = 0, 1, \ldots, n-1.$$

- We have already proved the basic cases. Suppose the result is true for n.
- Again the elements of $\mathcal{U}_n \subset \mathbb{K}^{2n+1}$ will be denoted by:

$$\mathbf{u_n} = (x_0, a_0, \ldots, x_{n-1}, a_{n-1}, x_n)$$

  Note that we can interpret $\mathbf{u_{n+1}} = (\mathbf{u_n}, a_n, x_{n+1})$. For example, $\mathbf{e_{n+1}} = (\mathbf{e_n}, 0, 0)$.
- Take now:

$$\varphi_{n+1}(\mathbf{u_{n+1}}) = (d_{i,j}(\mathbf{u_{n+1}}))_{i,j=0,1,\ldots,n+1} =$$

$$= \left[\begin{array}{ccc|c} & \varphi_n(\mathbf{u_n}) & & \\ \hline d_{n+1,0}(\mathbf{u_{n+1}}) & d_{n+1,1}(\mathbf{u_{n+1}}) & \cdots & d_{n+1,n+1}(\mathbf{u_{n+1}}) \end{array}\right]$$

  Note that we have:

$$P_n(\varphi_{n+1}(\mathbf{u_{n+1}})) = \varphi_n(\mathbf{u_n})$$

  where P_n is the corresponding bonding map introduced in chapter 1.
- Fix k with $0 \leq k \leq n$. Then:

$$\frac{\partial \varphi_{n+1}}{\partial x_k}(\mathbf{u_{n+1}}) = \left[\begin{array}{ccc|c} & \frac{\partial \varphi_n}{\partial x_k}(\mathbf{u_n}) & & \\ \hline \frac{\partial d_{n+1,0}}{\partial x_k}(\mathbf{u_{n+1}}) & \frac{\partial d_{n+1,1}}{\partial x_k}(\mathbf{u_{n+1}}) & \cdots & \frac{\partial d_{n+1,n+1}}{\partial x_k}(\mathbf{u_{n+1}}) \end{array}\right] \tag{4.2}$$

- Using the characterization of the Riordan matrices in terms of the A-sequence, we have:

$$d_{n+1,j}(\mathbf{u_{n+1}}) = \sum_{s=0}^{n+1-j} a_s d_{n,j-1+s}(\mathbf{u_n}) \qquad \text{for} \qquad j > 0 \tag{4.3}$$

  and $d_{n+1,0}(\mathbf{u_{n+1}}) = x_{n+1}$.
- Consequently: $\dfrac{\partial d_{n+1,0}}{\partial x_k}(\mathbf{u_{n+1}}) = 0$ and

$$\frac{\partial d_{n+1,j}}{\partial x_k}(\mathbf{u_{n+1}}) = \sum_{s=0}^{n+1-j} a_s \frac{\partial d_{n,j-1+s}}{\partial x_k}(\mathbf{u_n}) \qquad \text{for} \qquad j > 0.$$

  and so evaluating at $\mathbf{e_{n+1}}$ we obtain:

$$\frac{\partial d_{n+1,j}}{\partial x_k}(\mathbf{e_{n+1}}) = \frac{\partial d_{n,j-1}}{\partial x_k}(\mathbf{e_n}) = \delta_{n+1-j,k} \qquad \text{for} \qquad j > 0$$

  because $a_0 = 1$ and $a_s = 0$ for $s = 1, \ldots, n$ and using the induction hypotheses. Evaluating now (4.2) at $\mathbf{e_{n+1}}$ and using the induction hypotheses again we have the result.

- We have to compute: $\dfrac{\partial \varphi_{n+1}}{\partial a_\ell}(\mathbf{e_{n+1}})$

  - Let us compute $\dfrac{\partial \varphi_{n+1}}{\partial a_n}(\mathbf{e_{n+1}})$. Since only $d_{n+1,1}((\mathbf{u_{n+1}}))$ depends on a_n, we get:

$$\frac{\partial d_{n+1,1}}{\partial a_n}(\mathbf{e_{n+1}}) = d_{n,n}(\mathbf{e_n}) = 1$$

    Consequently $\dfrac{\partial \varphi_{n+1}}{\partial a_n}(\mathbf{e_{n+1}})$ fits the desired formula.

  - Suppose now $0 \leq \ell \leq n-1$, we have to compute $\dfrac{\partial \varphi_{n+1}}{\partial a_\ell}(\mathbf{e_{n+1}})$.

    * From (4.3) it is easy to see that if $\ell > n+1-j$ then $\dfrac{\partial d_{n+1,j}}{\partial a_\ell}(\mathbf{e_{n+1}}) = 0$.
    * In the case that $\ell \leq n+1-j$ we obtain:

$$\frac{\partial d_{n+1,j}}{\partial a_\ell}(\mathbf{u_{n+1}}) = d_{n,j-1+\ell}(\mathbf{u_n}) + \sum_{s=0}^{n+1-j} a_s \frac{\partial d_{n,j-1+s}}{\partial a_\ell}(\mathbf{u_n}).$$

      Evaluating at $\mathbf{e_{n+1}}$ and using the induction hypothesis, we get:

$$\frac{\partial d_{n+1,j}}{\partial a_\ell}(\mathbf{e_{n+1}}) = \delta_{n+1-j,\ell} + \frac{\partial d_{n,j-1}}{\partial a_\ell}(\mathbf{e_n}) =$$

$$= \delta_{n+1-j,\ell} + (j-1)\delta_{n+1-j,\ell} = j\delta_{n+1-j,\ell}.$$

- Finally if we denote by χ_k and α_ℓ the coordinates with respect to $\dfrac{\partial \varphi_n}{\partial x_k}(\mathbf{e_n})$ and $\dfrac{\partial \varphi_n}{\partial a_\ell}(\mathbf{e_n})$ respectively, we get the formula in the proposition.

$\square$

Now that we have described the Lie algebra $\mathcal{L}(\mathcal{R}_n(\mathbb{K}))$ we can describe the exponential map (that, since R_n is a matrix group, coincides with the exponential):

Corollary 4.5.5 *For any $M \in \mathcal{L}(\mathcal{R}_n(\mathbb{K}))$ and $t \in \mathbb{K}$ we have that $e^{tM} \in \mathcal{R}_n(\mathbb{K})$ where e^{tM} is the usual matrix exponential. In fact e^{tM} is always contained in the connected component of $R_n(1,x)$.*

It would be of great interest to study one-parameter groups in $\mathcal{R}_n$. Unfortunately, we have decided to leave it for future work (see open question 19).

4.6 Bonding maps

We have already mentioned that we are going to consider the Riordan group as an inverse limit of finite dimensional pro-Lie groups. So we have to study the corresponding bonding maps:

Proposition 4.6.1 *$P_n : \mathcal{R}_{n+1}(\mathbb{K}) \to \mathcal{R}_n(\mathbb{K})$ is a Lie group homomorphism for every $n \in \mathbb{N}$. Moreover:*

$$DP_n(R_{n+1}(1,x))\left(\frac{\partial \varphi_{n+1}}{\partial x_k}(\mathbf{e_{n+1}})\right) = \begin{cases} \frac{\partial \varphi_n}{\partial x_k}(\mathbf{e_n}) & \text{if } 0 \leq k \leq n \\ 0 & \text{if } k = n+1 \end{cases}$$

$$DP_n(R_{n+1}(1,x))\left(\frac{\partial \varphi_{n+1}}{\partial a_j}(\mathbf{e_{n+1}})\right) = \begin{cases} \frac{\partial \varphi_n}{\partial a_j}(\mathbf{e_n}), & \text{if } 0 \leq j \leq n-1 \\ 0 & \text{if } j = n \end{cases}$$

Proof: The expression of P_n in terms of global parametrization in each of the groups is given by:

$$\varphi_n^{-1} \circ P_n \circ \varphi_{n+1}(x_0, a_0, x_1, \ldots, a_n, x_{n+1}) = (x_0, a_0, x_1, \ldots, a_{n-1}, x_n).$$

Since the matrix of $DP_n(R_{n+1}(1,x))$ in this parametrization is the Jacobian matrix $J(\varphi_n^{-1} \circ P_n \circ \varphi_{n+1})(\mathbf{e_{n+1}})$ we get the results above.

□

Remark 4.6.2 *The above proposition shows us that the action of:*

$$DP_n(R_{n+1}(1,x)) : \mathcal{L}(\mathcal{R}_{n+1}(\mathbb{K})) \to \mathcal{L}(\mathcal{R}_n(\mathbb{K}))$$

is to delete the last row and the last column in every matrix in $\mathcal{L}(\mathcal{R}_{n+1}(\mathbb{K}))$.

4.7 $\mathcal{R}(\mathbb{K})$ as a Lie group I: Frechet Lie Group Structure

Consider $\mathbb{K}^{\mathbb{N}}$ with the product topology for the usual topology in $\mathbb{K}$, which we have already stated that is a Frechet space. This is the starting point to describe a natural infinite dimensional Lie group structure for the Riordan group.

Firstly, let's introduce the following:

Definition 4.7.1 (Global Parametrization for $\mathcal{R}(\mathbb{K})$) *Think about $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ with the usual Euclidean topology and take the product topology in $\mathbb{K}^{\mathbb{N}}$. Consider the open set:*

$$\mathcal{U}_\infty = \left\{ \boldsymbol{u} = (u_k)_{k \in \mathbb{N}} \in \mathbb{K}^{\mathbb{N}} \mid u_0 \neq 0,\ u_1 \neq 0 \right\} \subset \mathbb{K}^{\mathbb{N}}$$

and then define:

$$\begin{array}{cccc} \varphi_\infty & : & \mathcal{U}_\infty & \longrightarrow & \mathcal{R}(\mathbb{K}) \\ & & \boldsymbol{u} & \longmapsto & \varphi_\infty(\boldsymbol{u}). \end{array}$$

that maps a sequence:

$$\boldsymbol{u} = (u_k)_{k\in\mathbb{N}} = (x_0, a_0, x_1, a_1, x_2, a, \ldots)$$

with $u_0 \neq 0$, $u_1 \neq 0$ *or equivalently* $x_0 \neq 0$, $a_0 \neq 0$ *into a Riordan matrix:*

$$\varphi_\infty(\boldsymbol{u}) = D = (d_{i,j})_{i,j\in\mathbb{N}}$$

given by:

$$x_k = d_{k,0} \qquad d_{i,j} = \sum_{k=0}^{i-j} a_k d_{i-1,j-1+k} \text{ for } j \geq 1$$

Obviously φ_∞ is a bijective function. So, we consider the unique topology on $\mathcal{R}(\mathbb{K})$ converting φ_∞ into an homeomorphism. Then we have:

Proposition 4.7.2 $\mathcal{R}(\mathbb{K})$ *with the smooth structure induced by the global parametrization* $(\mathcal{U}_\infty, \varphi_\infty)$ *described above is a smooth manifold modelled on the locally convex vector space* $\mathbb{K}^{\mathbb{N}}$.

Moreover, $\mathcal{R}(\mathbb{K})$ *with the smooth structure defined above is an infinite dimensional Lie group in the sense described before.*

Proof: Note that the topological space $\mathcal{R}(\mathbb{K})$ and the map $\varphi_\infty : \mathcal{U}_\infty \to \mathcal{R}(\mathbb{K})$ fit all conditions in the definition of Lie group:

- To prove the smoothness of the product we consider, as in finite dimensional cases, the natural smooth structure on the product $\mathcal{R}(\mathbb{K})\times\mathcal{R}(\mathbb{K})$ is given by the global parametrization:

$$\begin{aligned} \varphi_\infty \times \varphi_\infty \ &: \ \mathcal{U}_\infty \times \mathcal{U}_\infty \to \mathcal{R}(\mathbb{K}) \times \mathcal{R}(\mathbb{K}) \\ (\mathbf{u}, \mathbf{v}) &\longmapsto (\varphi_\infty(\mathbf{u}), \varphi_\infty(\mathbf{v})) \end{aligned}$$

  Now the smoothness of the product follows as in Proposition 4.4.1 because the factor matrices are lower triangular and the dependence of the entries in the product is just the same as in finite cases.

- For the inverse, let $D \in \mathcal{R}(\mathbb{K})$. Again by Lagrange Inversion Formula, the dependence of the entries of D^{-1} depend analitically on the coefficients of the A-sequence of D. The differentiability of the entries of first column in D^{-1} respect to the first column and the A-sequence of D follows exactly as in the last part of the proof of Proposition 4.4.1.

□

Since $\varphi_\infty : \mathcal{U}_\infty \to \mathcal{R}(\mathbb{K})$ is an homeomorphism, we have:

Corollary 4.7.3

(a) *$\mathcal{R}(\mathbb{R})$ is homeomorphic to $\mathbb{R}^* \times \mathbb{R}^* \times \mathbb{R}^{\mathbb{N}}$ with the product topology. Consequently $\mathcal{R}(\mathbb{R})$ has four connected components each of them is contractible.*

(b) *$\mathcal{R}(\mathbb{C})$ is homeomorphic to $\mathbb{T} \times \mathbb{R}^{\mathbb{N}}$ where $\mathbb{T} = S^1 \times S^1$ is the torus. Consequently $\mathcal{R}(\mathbb{C})$ is path-connected and has the same homotopy type as the torus.*

So, they share the same algebraic invariants as in Proposition 4.3.3.

4.8 $\mathcal{R}(\mathbb{K})$ as a Lie group II: Lie group structure as a pro-Lie group

Beside this, and as it was pointed out in chapter 1, the Riordan group can be described as the inverse limit of an inverse sequence of finite dimensional matrix groups, obtaining so a pro-Lie group structure on $\mathcal{R}(\mathbb{K})$ (again, we recommend [45] for an exhaustive topological treatment of pro-Lie groups).

In this section we will show how the structure of pro-Lie group can be used to get information on basic facts of the infinite dimensional Lie group structure of $\mathcal{R}(\mathbb{K})$ like, for example, the description of the Lie algebra and the exponential map.

Denote by:

$$\Upsilon(\mathbb{K}) = \varprojlim\{(\mathcal{R}_n(\mathbb{K}))_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$$

If we consider $\Upsilon(\mathbb{K})$ as the inverse limit of an inverse sequence of groups then $\Upsilon(\mathbb{K})$ and $\mathcal{R}(\mathbb{K})$ are isomorphic as groups. Denote by $\zeta : \mathcal{R} \longrightarrow \Upsilon$ this isomorphism.

But since the groups $\mathcal{R}_n$ are also Lie groups and the bonding maps P_n are homomorphisms of Lie groups, $\Upsilon(\mathbb{K})$ is a pro-Lie group. We will show that $\Upsilon(\mathbb{K})$ is also the inverse limit of the inverse sequence $\varprojlim\{(\mathcal{R}_n(\mathbb{K}))_{n\in\mathbb{N}}, (P_n)_{n\in\mathbb{N}}\}$ in the category of Lie groups. The key is that unlike most of the cases considered in [45], the pro-Lie group $\Upsilon(\mathbb{K})$ does have a natural structure of Frechet manifold:

Proposition 4.8.1 *The bijective map $\overline{\varphi}_\infty : \mathcal{U}_\infty \to \Upsilon$ given by $\overline{\varphi}_\infty = \zeta \circ \varphi_\infty$ is a global parametrization which induces a Frechet Lie group structure in Υ (with respect to the componentwise operations).*

(1) *If $\Upsilon(\mathbb{K})$ is endowed with this differentiable structure then $\zeta : \mathcal{R}(\mathbb{K}) \to \Upsilon$ is a Frechet Lie group isomorphism.*

(2) *The topology that* Υ *inherits as an inverse limit and the topology induced on* Υ *by the Frechet Lie group structure above are the same.*

Proof: The first part of the proposition is an immediate consequence of the commutativity of the following diagram:

$$\begin{tikzcd} \mathcal{R}(\mathbb{K}) \arrow[r, "\xi"] & \Upsilon \\ \mathcal{U}_\infty \arrow[u, "\varphi_\infty"] \arrow[r, "Id_{\mathcal{U}_\infty}"] & \mathcal{U}_\infty \arrow[u, "\overline{\varphi}_\infty"'] \end{tikzcd}$$

where $Id_{\mathcal{U}_\infty}$ is the identity map on $\mathcal{U}_\infty$.

To prove the second part, it is easy to see ([31] page 99 Example 2.5.3) that if we consider the inverse sequence $\{(\mathcal{U}_n)_{n\in\mathbb{N}}, (\mu_n)_{n\in\mathbb{N}}\}$ where:

$$\mu_n : \mathcal{U}_{n+1} \to \mathcal{U}_n$$

$$\mu_n(x_0, a_0, x_1, \ldots, a_{n-1}, x_n, a_n, x_{n+1}) = (x_0, a_0, \ldots, x_{n-1}, a_{n-1}, x_n)$$

then the corresponding topological inverse limit is just $\mathcal{U}_\infty = \mathbb{K}^* \times \mathbb{K}^* \times \mathbb{K}^{\mathbb{N}}$ with the product topology. Since the diagram:

$$\begin{tikzcd} \mathcal{R}_n(\mathbb{K}) & \mathcal{R}_{n+1}(\mathbb{K}) \arrow[l, "\psi_n"'] \\ \mathcal{U}_n \arrow[u, "\varphi_n"] & \mathcal{U}_{n+1} \arrow[l, "\mu_n"'] \arrow[u, "\varphi_{n+1}"'] \end{tikzcd}$$

is commutative for every $n \in \mathbb{N}$ and φ_k is a homeomorphism for every $k \in \mathbb{N}$, then the natural induced map $\overline{\varphi}$ between the corresponding inverse limits is an homeomorphism.

□

It is not casual the choice of the differentiable structure of $\mathcal{R}$ (as a Frechet manifold structure over $\mathbb{K}^N$). The partial Riordan groups are modelled with a global chart over spaces $\mathbb{K}^n$. As we have seen in the previous proof, $\mathbb{K}^N$ is the inverse limit of a certain inverse sequence involving those spaces and we also want the differentiable structure to be given by a global chart.

The result above shows certain compatibility between this differentiable structure that we have proposed for Υ and with the topology that it inherits as inverse limit of topological spaces. Still to determined is wether this differential structure is compatible with the differential structure of the groups $\mathcal{R}_n(\mathbb{K})$.

Proposition 4.8.2 *For every* $n \in \mathbb{N}$, $\pi_n : \mathcal{R}(\mathbb{K}) \to \mathcal{R}_n(\mathbb{K})$ *is a Lie group homomorphism.*

Proof: This is because the expression of π_n for the parametrizations $(\mathcal{U}_\infty, \varphi_\infty)$, $(\mathcal{U}_n, \varphi_n)$ is given by:

$$\varphi_n^{-1} \circ \pi_n \circ \varphi_\infty(x_0, a_0, x_1, \ldots, a_n, x_{n+1}, \ldots) = (x_0, a_0, x_1, \ldots, a_{n-1}, x_n)$$

and then corresponds to the restriction of a Linear continuous map between Frechet spaces. □

So $\mathcal{R}$ has happened to be a pro-Lie group which is a Lie group and in this sense it can be viewed as a toy example of the theory described in [45].

4.9 Curves and One-parameter subgroups in $\mathcal{R}$

Before going on, we are going to give a natural criterion to detect C^∞ curves into the Riordan group $\mathcal{R}(\mathbb{K})$ which will be useful for recognizing the exponential map.

The proof is based on some results about calculus of smooth mappings as established by A. Kriegl and P.W. Michor in [56, 57].

Proposition 4.9.1 *Let $I \subset \mathbb{K}$ be a connected open subset and:*

$$\gamma : I \longrightarrow \mathcal{R}(\mathbb{K})$$

be a map. Suppose that $(u_k(t))_{k\geq 0}$ are the coordinates of γ in the global parametrization $(\mathcal{U}_\infty, \varphi_\infty)$. Then γ is a C^∞ curve if and only if $u_k : I \longrightarrow \mathbb{K}$ is C^∞ for any $k \geq 0$.

Proof: As we saw before, $\mathcal{R}(\mathbb{K})$ is, as manifold, diffeomorphic to the open subspace $\mathcal{U}_\infty$ of the Frechet space $\mathbb{K}^\mathbb{N}$. The differentiability of γ is then, by definition, the differentiability of $\varphi_\infty^{-1} \circ \gamma : I \longrightarrow \mathcal{U}_\infty$. By composing with the inclusion of $\mathcal{U}_\infty$ into $\mathbb{K}^\mathbb{N}$, it is also equivalent to the differentiability of $\varphi_\infty^{-1} \circ \beta : I \longrightarrow \mathbb{K}^\mathbb{N}$.

This is equivalent to the fact that $\ell \circ (\varphi_\infty^{-1} \circ \beta)$ is C^∞ for any linear continuous map $\ell : \mathbb{K}^\mathbb{N} \longrightarrow \mathbb{K}$ in the dual of $\mathbb{K}^\mathbb{N}$. Finally remember that the topological dual of the space $\mathbb{K}^\mathbb{N}$ of all sequences is the space $\mathbb{K}_0^\mathbb{N}$ of sequences with only a finite number of nonzero terms. This implies the statement in the proposition. □

Another formulation of this proposition is the following, which fits perfectly with the ideas of using the inverse limit structure in the Riordan group:

Theorem 4.9.2 *Let $I \subset \mathbb{K}$ be a connected open subset and:*

$$\gamma : I \longrightarrow \mathcal{R}(\mathbb{K})$$

be a map. Then γ is a C^∞ curve if and only if:

$$\forall n \in \mathbb{N}, \;\; \pi_n \circ \gamma : I \longrightarrow \mathcal{R}_n(\mathbb{K})$$

is a C^∞ curve.

To study the Lie group structure we are interested in one-parameter subgroups. Recall that a one-parameter subgroup in $\mathcal{R}$ is a Lie group homomorphism from $(\mathbb{R}, +)$ to $\mathcal{R}$.

Proposition 4.9.3 *A path $\gamma : I \to \mathcal{R}$ is a one-parameter group if and only if for every n $\pi_n \circ \gamma$ is a one-parameter group.*

Proof: A path $\gamma : I \to \mathcal{R}$ is a group homomorphism if and only if for every n $\pi_n \circ \gamma$ is a group homomorphism since $\mathcal{R}$ is isomorphic to the inverse limit of the inverse sequence of $(\mathcal{R}_n, P_n)_{n\in\mathbb{N}}$.

A path $\gamma : I \to \mathcal{R}$ is differentiable if and only if for every n $\pi_n \circ \gamma$ is differentiable according to the previous result.

□

4.10 The Lie Algebra of $\mathcal{R}(\mathbb{K})$

As it is usual in Lie group theory, $T_{R(1,x)}\mathcal{R}(\mathbb{K})$ can be identified with $\mathbb{K}^{\mathbb{N}}$. Even more, since $\{\epsilon_i = (\delta_{ij})_{j\geq 1}\}_{i\in\mathbb{N}}$ is a Schauder base for $\mathbb{K}^{\mathbb{N}}$ then we can topologize $T_{R(1,x)}\mathcal{R}(\mathbb{K})$ in such a way that $\{D\varphi_\infty(\mathbf{e})(\epsilon_i)\}_{i\in\mathbb{N}}$ is a Schauder base for $T_{R(1,x)}\mathcal{R}(\mathbb{K})$.

Now, we are going to get a representation of $T_{R(1,x)}\mathcal{R}(\mathbb{K})$ by means of infinite lower triangular matrices and after that we are going to define the Lie bracket in $T_{R(1,x)}\mathcal{R}(\mathbb{K})$ to get the Lie algebra of $\mathcal{R}(\mathbb{K})$.

Theorem 4.10.1 *A full and faithful representation of the Lie algebra $\mathcal{L}(\mathcal{R}(\mathbb{K}))$ of the Lie group $\mathcal{R}(\mathbb{K})$ is given by the set of matrices of the form $L(\chi(x), \alpha(x))$, where for two power series:*

$$\chi(x) = \chi_0 + \chi_1 x + \chi_2 x^2 + \ldots, \qquad \alpha(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \ldots$$

$L(\chi, \alpha)$ is the lower triangular matrix:

$$L = \begin{pmatrix} \chi_0 & & & & & & \\ \chi_1 & \chi_0 + \alpha_0 & & & & & \\ \chi_2 & \chi_1 + \alpha_1 & \chi_0 + 2\alpha_0 & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ \chi_{n-1} & \chi_{n-2} + \alpha_{n-2} & \chi_{n-3} + 2\alpha_{n-3} & \cdots & \chi_0 + (n-1)\alpha_0 & & \\ \chi_n & \chi_{n-1} + \alpha_{n-1} & \chi_{n-2} + 2\alpha_{n-2} & \cdots & \chi_1 + (n-1)\alpha_1 & \chi_0 + n\alpha_0 & \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with the usual sum of matrices and the usual product by scalars in $\mathbb{K}$. The Lie bracket is

$$[L_1, L_2] = L_1 L_2 - L_2 L_1$$

where the product and the difference are the usual ones for infinite lower triangular matrices.

Proof: Consider the inverse sequence of Lie algebras:

$$\mathcal{L}_\infty = \varprojlim (\mathcal{L}(\mathcal{R}_{n+1}(\mathbb{K})), DP_n(I_{n+1}))_{n \in \mathbb{N}}$$

where recall that the Lie brackets are given by $[A_n, B_n]_n = A_n B_n - B_n A_n$ for any $A_n, B_n \in \mathcal{L}(\mathcal{R}_n(\mathbb{K})))$. Through all this proof we will identify the elements in $\mathcal{L}_\infty$ with infinite matrices when necessary.

We will denote by $\widetilde{\pi}_n$ to the corresponding projection $\mathcal{L}_\infty \longrightarrow \mathcal{L}(\mathcal{R}_n(\mathbb{K}))$. If we identify the elements in $\mathcal{L}_\infty$ by infinite matrices, then $\widetilde{\pi}$ makes the analogous action to $\pi_n : \mathcal{R} \to \mathcal{R}_n$, that is:

$$\left[\begin{array}{cccccc|c} \chi_0 & & & & & & \\ \chi_1 & \chi_0 + \alpha_0 & & & & & \\ \chi_2 & \chi_1 + \alpha_1 & \chi_0 + 2\alpha_0 & & & & \\ \vdots & \vdots & \vdots & \ddots & & & \\ \chi_{n-1} & \chi_{n-2} + \alpha_{n-2} & \chi_{n-3} + 2\alpha_{n-3} & \cdots & \chi_0 + (n-1)\alpha_0 & & \\ \chi_n & \chi_{n-1} + \alpha_{n-1} & \chi_{n-2} + 2\alpha_{n-2} & \cdots & \chi_1 + (n-1)\alpha_1 & \chi_0 + n\alpha_0 & \\ \hline \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \end{array}\right] \longmapsto$$

$$\longmapsto \begin{bmatrix} \chi_0 & & & & & \\ \chi_1 & \chi_0 + \alpha_0 & & & & \\ \chi_2 & \chi_1 + \alpha_1 & \chi_0 + 2\alpha_0 & & & \\ \vdots & \vdots & \vdots & \ddots & & \\ \chi_{n-1} & \chi_{n-2} + \alpha_{n-2} & \chi_{n-3} + 2\alpha_{n-3} & \cdots & \chi_0 + (n-1)\alpha_0 & \\ \chi_n & \chi_{n-1} + \alpha_{n-1} & \chi_{n-2} + 2\alpha_{n-2} & \cdots & \chi_1 + (n-1)\alpha_1 & \chi_0 + n\alpha_0 \end{bmatrix}$$

(1) Firstly we will focus on the vector space of the Lie algebra. We want to show that there is a map:

$$\mathfrak{L} : T_{R(1,x)}(\mathcal{R}(\mathbb{K})) \to \mathcal{L}_\infty$$

which is a linear homeomorphism ($\mathcal{L}_\infty$ is endowed with the corresponding topological and linear structure as inverse limit in both categories). The map $\mathfrak{L}$ is defined as follows:

- The diagram:

$$\begin{tikzcd} \mathcal{L}_\infty \arrow[d, "\widetilde{\pi}_n"'] \arrow[drr, "\widetilde{\pi}_{n+1}"] & & \\ \mathcal{L}(\mathcal{R}_n(\mathbb{K})) & & \mathcal{L}(\mathcal{R}_{n+1}(\mathbb{K})) \arrow[ll, "(DP_n)_{R_{n+1}(1,x)}"] \end{tikzcd}$$

  is commutative for every n. For every tangent vector at $R(1,x)$, $v \in T_{R(1,x)}(\mathcal{R}(\mathbb{K}))$ (since in our framework the chain rule works) we have:

$$(DP_n)_{R(1,x)}(D\widetilde{\pi}_{n+1})_{R(1,x)}(v)) = (D\widetilde{\pi}_n)_{R(1,x)}(v).$$

- Consequently, the sequence $((D\widetilde{\pi}_n)_{R_n(1,x)}(v))_{n\in\mathbb{N}} \in \mathcal{L}_\infty$. So we can define:

$$\mathfrak{L}(v) = ((D\widetilde{\pi}_n)_{R_n(1,x)}(v))_{n\in\mathbb{N}}$$

- This map is obviously linear and continuous.

Now we will define a map:

$$\mathfrak{T} : \mathcal{L}_\infty \to T_{R(1,x)}(\mathcal{R}(\mathbb{K}))$$

which will be shown to be the inverse of $\mathfrak{L}$ and also linear and continous:

- Take the infinite matrix $L \in \mathcal{L}_\infty$. L which can be identified with a sequence:

$$(\widetilde{\pi}_n(L))_{n\in\mathbb{N}} \text{ with } (D\pi_n)_{R_{n+1}(1,x)}(\widetilde{\pi}_{n+1}(L)) = \widetilde{\pi}_n(L)$$

- For every $n \in \mathbb{N}$, consider the one-parameter group:

$$\gamma_{\widetilde{\pi}_n(L)} : \mathbb{R} \to \mathcal{R}_n(\mathbb{K})$$

$$\gamma_{\widetilde{\pi}_n(L)}(t) = e^{t\widetilde{\pi}_n(L)}$$

  It is clear that $\pi_n \circ \gamma_{\widetilde{\pi}_{n+1}(L)} = \gamma_{\widetilde{\pi}_n(L)}$, so we can define a continous group homomorphism:

$$\gamma_L : \mathbb{R} \to \mathcal{R}(\mathbb{K})$$

$$t \longmapsto (\gamma_{\widetilde{\pi}_n}(L)(t))_{k\in\mathbb{N}}$$

- Since the expression of γ_L in the parametrization $(\mathcal{U}_\infty, \varphi_\infty)$ is:

$$\gamma_L(t) = (x_0(t), a_0(t), x_1(t), a_1(t), \dots)$$

  and:

$$\gamma_{\widetilde{\pi}_n(L)}(t) = (x_0(t), a_0(t), x_1(t), a_1(t), \dots, a_{n-1}(t), x_n(t))$$

  in the parametrization $(\mathcal{U}_n, \varphi_n)$ of $\mathcal{R}_n(\mathbb{K})$ we have that $x_k(t)$ and $a_k(t)$ are C^∞ for any $k \in \mathbb{N}$ and hence, by proposition 4.9.2, γ_L is C^∞. Thus we can define:

$$\mathfrak{T}(L) = \gamma_L'(0)$$

- It is easy to see that $\mathfrak{T} \circ \mathfrak{L}$ is the identity in $T_{R(1,x)}(\mathcal{R}(\mathbb{K}))$. Consequently, $\mathfrak{T}$ is also a linear map. Moreover, since:

$$D\pi_n(I)(\gamma'_L(0)) = \gamma'_{\widetilde{\pi}_n(L)}(0) = \widetilde{\pi}_n(L)$$

  then $\mathfrak{L} \circ \mathfrak{T}$ is the identity map in $\mathcal{L}_\infty$.
- Applying now the Open Mapping Theorem for Frechet spaces we get that $\mathfrak{L}$ is a linear homeomorphism.

(2) It remains only to proof that $[v, w] \in T_I(\mathcal{R}(\mathbb{K}))$ is represented by

$$\mathfrak{L}(v) \cdot \mathfrak{L}(w) - \mathfrak{L}(w) \cdot \mathfrak{L}(v)$$

where the product and the difference are the usual ones for infinite lower triangular matrices.

- Note that:

$$(DP_n)_{R_{n+1}(1,x)}([A, B]_{n+1}) = [(DP_n)_{R_{n+1}(1,x)}(A), (DP_n)_{R_{n+1}(1,x)}(B)]_n$$

- Suppose $v, w \in \mathcal{L}(\mathcal{R}(\mathbb{K})) \equiv T_I(\mathcal{R}(\mathbb{K})) \equiv \mathcal{L}_\infty$ and consider the Lie bracket $[v, w]$. Since $\pi_n : \mathcal{R}(\mathbb{K}) \to \mathcal{R}_n(\mathbb{K})$ is a Lie group homomorphism then:

$$(D\pi_n)_{R(1,x)} : \mathcal{L}(\mathcal{R}(\mathbb{K})) \to \mathcal{L}(\mathcal{R}_n(\mathbb{K}))$$

  is a Lie algebra homomorphism.
- Consider the infinite lower triangular matrices:

$$\mathfrak{L}(v), \mathfrak{L}(w), \mathfrak{L}([v, w]) \in \mathcal{L}_\infty$$

  We have:

$$(D\pi_n)_{R(1,x)}([v, w]) = [(D\pi_n)_{R(1,x)}(v), (D\pi_n)_{R(1,x)}(w)]_n = [\widetilde{\pi}_n(\mathfrak{L}(v)), \widetilde{\pi}_n(\mathfrak{L}(w))]_n$$

  where $[\ ,\ ]_n$ represents the Lie bracket in $\mathcal{L}(\mathcal{R}_n(\mathbb{K}))$. Since P_n is also a Lie group homomorphism we have:

$$(DP_n)_{R_{n+1}(1,x)}([\widetilde{\pi}_{n+1}(\mathfrak{L}(v)), \widetilde{\pi}_{n+1}(\mathfrak{L}(w))]_{n+1}) = [\widetilde{\pi}_n(\mathfrak{L}(v)), \widetilde{\pi}_n(\mathfrak{L}(w))]_n$$

  but:

$$[\widetilde{\pi}_n(\mathfrak{L}(v)), \widetilde{\pi}_n(\mathfrak{L}(w))]_n = \widetilde{\pi}_n(\mathfrak{L}(v)) \cdot \widetilde{\pi}_n(\mathfrak{L}(w)) - \widetilde{\pi}_n(\mathfrak{L}(w)) \cdot \widetilde{\pi}_n(\mathfrak{L}(v)).$$

- Consequently:

$$\mathfrak{L}([v, w]) = \mathfrak{L}(v) \cdot \mathfrak{L}(w) - \mathfrak{L}(w) \cdot \mathfrak{L}(v).$$

□

4.11 The exponential map

Recall that the exponental is given by:

$$exp : \mathcal{L}(\mathcal{R}) \longrightarrow \mathcal{R}$$

$$exp(v) = \gamma_L(1)$$

where γ_L is the unique one-parameter subgroup of $\mathcal{R}$ whose tangent vector at the identity is v. So note that the one-parameter group γ_L used in the proof above, that is:

$$\gamma_L : \mathbb{R} \longmapsto \mathcal{R}(\mathbb{K})$$

$$t \longmapsto (e^{t\widetilde{\pi}_n(L)})_{k\in\mathbb{N}}$$

defines the exponential map in $\mathcal{R}$:

Corollary 4.11.1 *The* **exponential map** *in $\mathcal{R}(\mathbb{K})$ for any $L \in \mathcal{L}(\mathcal{R})$ is given by:*

$$exptL = \gamma_L(t) = e^{tL}$$

where $\gamma_L(t)$ is the one parameter group described above and e^{tL} is the usual exponential of matrices (in this infinite matrix context):

$$\gamma_L(t) = \sum_{n\geq 0} \frac{(tL)^n}{n!}$$

In particular, $e^L = \gamma_L(1)$.

It is interesting to note that, in this framework, the behaviour of the exponential resembles that of the real or complex exponential, in the sense that it transforms arithmetic progressions (in the rows of the matrices in $\mathcal{L}(\mathcal{R})$) on geometric progressions (in the rows of the matrices in $\mathcal{R}$). Arithmetic progressions are intrinsically related to the structure of the columns of the matrices in the Lie algebra and geometric progressions describe just the structure of the columns in a Riordan matrix.

Analogously to the finite dimensional case:

Proposition 4.11.2 (Matrix Differential Equations) *Let $L \in \mathcal{L}(\mathcal{R})$. The solution of the equation:*

$$\begin{cases} \gamma'(t) = L \cdot \gamma(t) \\ \gamma(0) = R \end{cases} \tag{4.4}$$

where $R \in \mathcal{R}$ and $\gamma_t \in \mathcal{R}$ for every t has a unique solution:

$$\gamma(t) = e^{tL} R$$

Proof: The groups $\mathcal{R}_n(\mathbb{K})$ are matrix groups so according to the classical Lie theory (see for example [5]) for each n, the equation:

$$\begin{cases} \gamma_n'(t) = \pi_n(L) \cdot \gamma_n(t) \\ \gamma(0) = \pi_n(R) \end{cases}$$

has as unique solution:

$$\gamma_n(t) = e^{t\pi_n(L)} \pi_n(R)$$

The result follows from the inverse limit structure. We omit the details.

□

Apart from the already mentioned paper by S. A. Jennings [52], some results about computation of the exponential of Riordan matrices can be found in the paper by R. Bacher [4].

4.12 Lie group structure for $\mathcal{R}_{\infty\infty}$

Later on we will want to compute the Lie algebra of some stabilizer subgroups for which the bi-infinite representation of the Riordan matrices will be more convenient.

Remark 4.12.1 *We have already mentioned that there is a natural group isomorphism:*

$$B : \mathcal{R} \longrightarrow \mathcal{R}_{\infty\infty}$$

$$B(R(d(x), h(x))) = R_{\infty\infty}(d(x), h(x))$$

This isomorphism induces automatically a structure of Frechet manifold on $\mathcal{R}_{\infty\infty}$ since we can define the parametrization $\varphi_{\infty\infty}$ by making the following diagram commutative:

$$\begin{tikzcd} \mathcal{R} \arrow[r, "B"] & \mathcal{R}_{\infty\infty} \\ \mathcal{U}_\infty \arrow[u, "\varphi_\infty"] & \mathcal{U}_\infty \arrow[l, "Id_{\mathcal{U}_\infty}"] \arrow[u, "\varphi_{\infty\infty}"'] \end{tikzcd}$$

just taking $\varphi_{\infty\infty} = B \circ \varphi_\infty \circ Id_{U_\infty}$, where $Id_{\mathcal{U}_\infty}$ is the identity map in $\mathcal{U}_\infty$. Moreover, this manifold structure is compatible with the product and the inverse in $\mathcal{R}_{\infty\infty}$ since B is a group isomoprhism so B also induces a Lie group structure for $\mathcal{R}_{\infty\infty}$ compatible with the natural product and inverse in $\mathcal{R}_{\infty\infty}$.

The elements $L(\chi,\alpha) \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$ have also a natural bi-infinite representation:

$$L_{\infty\infty}(\chi,\alpha) = \begin{pmatrix} \ddots & & & & & & \\ \cdots & \chi_0 - 2\alpha_0 & & & & & \\ \cdots & \chi_1 - 2\alpha_1 & \chi_0 - \alpha_0 & & & & \\ \cdots & \chi_2 - 2\alpha_2 & \chi_1 - \alpha_1 & \chi_0 & & & \\ \cdots & \chi_3 - 2\alpha_3 & \chi_2 - \alpha_2 & \chi_1 & \chi_0 + \alpha_0 & & \\ \cdots & \chi_4 - 2\alpha_4 & \chi_3 - \alpha_3 & \chi_2 & \chi_1 + \alpha_1 & \chi_0 + 2\alpha_0 & \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

and it is easy to see that the set of elements $L_{\infty\infty}(\chi,\alpha)$ with respect to the Lie bracket:

$$[T,S] = TS - ST$$

is again a full and faithful representation of the Lie algebra $\mathcal{L}(\mathcal{R})$.

4.13 Multplication of $L(\chi,\alpha)$ by a column vector on $\mathbb{K}^{\mathbb{N}}$

We are going to use the natural identification $\mathbb{K}^{\mathbb{N}} \equiv \mathbb{K}[[x]]$ by means of considering the ordinary generating function of any sequence. The topology used in $\mathbb{K}^{\mathbb{N}}$ is always the product topology for the usual topology in $\mathbb{K}$.

Recall that for any matrix in the Lie algebra $L(\chi,\alpha) \in \mathcal{L}(\mathcal{R})$ the generating function of the j-column in L is given by $x^j(\chi(x) + j\alpha(x))$, $j \in \mathbb{N}$. This fact can be turned into the following.

Proposition 4.13.1 *Any $L(\chi,\alpha) \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$ induces a linear continuous map $\mathbb{K}^{\mathbb{N}} \to \mathbb{K}^{\mathbb{N}}$, given by:*

$$L(\chi,\alpha)\begin{bmatrix} h_0 \\ h_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \\ \vdots \end{bmatrix}$$

If $h(x)$ is the generating function of $(h_0, h_1, \ldots)$, then the generating function of $(y_0, y_1, \ldots)$ is:

$$\chi(x)h(x) + x\alpha(x)h'(x)$$

So via the identification, $L(\chi,\alpha)$ also induces a linear continous map $\mathbb{K}[[x]] \longrightarrow \mathbb{K}[[x]]$. As done before, we will use the notation:

$$L(\chi,\alpha) \otimes h(x) = \chi(x)h(x) + x\alpha(x)h'(x)$$

Proof: It is clear that any infinite lower triangular matrix induces a linear continuous map in $\mathbb{K}^{\mathbb{N}}$ by using the product of matrices.

Let:

$$\chi = \chi_0 + \chi_1 x + \dots \qquad \alpha(x) = \alpha_0 + \alpha_1 x + \dots$$

Note that, analogously to the proof or the First Fundamentl Theorem of Riordan matrices:

$$L(\chi,\alpha)\begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \end{bmatrix} = h_0 \begin{bmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \\ \vdots \end{bmatrix} + h_1 \begin{bmatrix} 0 \\ \chi_0+\alpha_0 \\ \chi_1+\alpha_1 \\ \vdots \end{bmatrix} + h_2 \begin{bmatrix} 0 \\ 0 \\ \chi+2\alpha_0 \\ \vdots \end{bmatrix} + \dots =$$

$$= \left(h_0 \begin{bmatrix} \chi_0 \\ \chi_1 \\ \chi_2 \\ \vdots \end{bmatrix} + h_1 \begin{bmatrix} 0 \\ \chi_0 \\ \chi_1 \\ \vdots \end{bmatrix} + h_2 \begin{bmatrix} 0 \\ 0 \\ \chi_0 \\ \vdots \end{bmatrix} + \dots \right) + \left(1\cdot h_1 \begin{bmatrix} 0 \\ \alpha_0 \\ \alpha_1 \\ \vdots \end{bmatrix} + 2\cdot h_2 \begin{bmatrix} 0 \\ 0 \\ \alpha_0 \\ \vdots \end{bmatrix} + \dots \right) =$$

it is easy to see that the generating function of this column vector is what we need.

□

Analogously, we can think of the elements in the Lie algebra acting on $\mathbb{K}((x))$ in the following way:

Corollary 4.13.2 *Any $L_{\infty\infty}(\chi,\alpha) \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$ induces a natural map given by:*

$$L_{\infty\infty}(\chi,\alpha)\begin{bmatrix} \vdots \\ 0 \\ h_{-k} \\ h_{-k+1} \\ \vdots \end{bmatrix} = \begin{bmatrix} \vdots \\ 0 \\ y_{-k} \\ y_{-k+1} \\ \vdots \end{bmatrix}$$

where if $h(x) = h_{-k}x^{-k} + h_{-k+1}x_{-k+1}, \dots$, then:

$$y_{-k}x^{-k} + y_{-k+1}x^{-k+1} \dots = (\chi(x) - k\cdot\alpha(x))h(x) + x\alpha(x)h'(x)$$

So via the identification, $L(\chi,\alpha)$ also induces a linear map $\mathbb{K}((x)) \longrightarrow \mathbb{K}((x))$. As done before, we will use the notation:

$$L_{\infty\infty}(\chi,\alpha)\otimes h(x) = \frac{1}{x^k}L(\chi - k\alpha,\alpha)\otimes(x^k h(x)) \tag{4.5}$$

Proof: The proof is totally analogous to the previous one.

□

4.14 Initial Value Problems

Consider the continuous linear map $L(\chi, \alpha)$ as a C^∞ vector field in the Frechet space $\mathbb{K}^{\mathbb{N}}$ under the canonical identification $T_h\mathbb{K}^{\mathbb{N}} = \mathbb{K}^{\mathbb{N}}$ in the tangent space at any $h \in \mathbb{K}^{\mathbb{N}}$.

We have already stated proposition 4.11.2 (which has a natural analogue for bi-infinite matrices, we omit details). Thanks to the interpretation that we have given to the multiplication of elements in the tangent algebra by row vectors, another way to interpret the above proposition is:

Corollary 4.14.1 *Let $\chi(x), \alpha(x) \in \mathbb{K}[[x]]$. The unique solution of the initial value problem:*

$$\begin{cases} \frac{\partial u}{\partial t} = \chi(x)u(x,t) + x\alpha(x)\frac{\partial u}{\partial x} \\ u(x,0) = F(x) \end{cases} \tag{4.6}$$

in $\mathbb{K}[[x,t]]$ is given by:

$$u(x,t) = e^{tL(\chi(x),\alpha(x))} \otimes F(x)$$

The same holds for the bi-infinite case doing the corresponding modifications.

Remark 4.14.2 *There are well known methods for the solution of problems of the type* (4.6)*. This equation is a first order quasilinear differential equation (see* [92] *or* [54]*, where this problem is solved in terms of characteristic curves).*

On the othe hand, this new method may be computationally useful. We leave this as an open question (see open question 22).

We will now solve two of those initial value problems, just to see these ideas working. In those examples, we will choose two matrices in the tangent algebra whose exponential is particularly easy to compute:

Example 4.14.3 *Let $c_1, c_2 \in \mathbb{K}$ and take $c_1\dfrac{\partial\varphi_\infty}{\partial x_0} + c_2\dfrac{\partial\varphi_\infty}{\partial a_0} \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$. The corresponding matrix representation is the infinite diagonal matrix:*

$$L(c_1, c_2) = \begin{bmatrix} c_1 & & & \\ 0 & c_1 + c_2 & & \\ 0 & 0 & c_1 + 2c_2 & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

or a given initial condition $F(x) \in \mathbb{K}[[x]]$, the associated initial value problem is:

$$\begin{cases} \frac{\partial u}{\partial t} = c_1 u(x,t) + xc_2\frac{\partial u}{\partial x} \\ u(x,0) = F(x) \end{cases} \tag{4.7}$$

Then:

$$u(x,t) = e^{tL(c_1,c_2)} \otimes F(x)$$

is the unique solution of the initial value problem. In this case, we can compute easily:

$$e^{tL(c_1,c_2)} = R(e^{c_1 t}, e^{c_2 t}x)$$

Hence, the solution is:

$$u(x,t) = e^{c_1 t} \cdot F(xe^{c_2 t}).$$

Example 4.14.4 *Consider* $\dfrac{\partial \varphi_\infty}{\partial x_1} + \dfrac{\partial \varphi_\infty}{\partial a_1} \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$. *The corresponding matrix representation is* ***the creation matrix***, *that is*

$$L(x,x) = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ 0 & 2 & 0 & & \\ 0 & 0 & 3 & 0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Given any $F(x) \in \mathbb{K}[[x]]$ *as initial condition, the associated initial value problem is:*

$$\begin{cases} \frac{\partial u}{\partial t} = xu(x,t) + x^2 \frac{\partial u}{\partial x} \\ u(x,0) = F(x) \end{cases}$$

Then:

$$u(x,t) = e^{tL(x,x)} \otimes F(x)$$

is the unique solution. It is a well-known example (it can be computed directly) that:

$$e^{tL(x,x)} = R\left(\frac{1}{1-xt}, \frac{x}{1-xt}\right)$$

Then:

$$u(x,t) = \frac{1}{1-xt} \cdot F\left(\frac{x}{1-xt}\right).$$

4.15 Conjugation in $\mathcal{L}(\mathcal{R})$

Consider the left and right translations in $\mathcal{R}(\mathbb{K})$ given respectively by

$$\begin{array}{rccc} L_A : & \mathcal{R}(\mathbb{K}) & \longrightarrow & \mathcal{R}(\mathbb{K}) \\ & X & \longmapsto & L_A(X) = AX \end{array} \qquad \begin{array}{rccc} R_A : & \mathcal{R}(\mathbb{K}) & \longrightarrow & \mathcal{R}(\mathbb{K}) \\ & X & \longmapsto & R_A(X) = XA \end{array}$$

Remark 4.15.1 *Since the product is a* C^∞*-function then both* L_A *and* R_A *are diffeomorphisms.*

(i) *Moreover, it is a well-known fact that if G is a finite group of matrices (in particular this holds for $G = \mathcal{R}_n$) and $A, B \in G$ then, the tangent map:*

$$(D(L_A))_B : T_B G \to T_{AB} G$$

is represented again by multiplying by the matrix $M \in G$, i.e.

$$Y \in T_B G \longmapsto AY \in T_{AB} G$$

(ii) *Analogously it happens happen for R_M, we have that $(D(R_A))_B(Y) = YA \in T_{BA}G$.*

(iii) *Consequently, the conjugation by an element $A_n \in \mathcal{R}_n(\mathbb{K})$ defined by*

$$\begin{array}{rcccc} conj_{A_n} & : & \mathcal{R}_n(\mathbb{K}) & \longrightarrow & \mathcal{R}_n(\mathbb{K}) \\ & & X & \longmapsto & A_n X A_n^{-1} \end{array}$$

is a C^∞ isomorphism in the groups $\mathcal{R}_n(\mathbb{K})$. This implies that $(D(conj_{A_n}))_{R_n(1,x)}$ is a linear isomorphism defined by:

$$(D conj_{A_n})_{R_n(1,x)} \; : \; \mathcal{L}(\mathcal{R}_n(\mathbb{K})) \longrightarrow \mathcal{L}(\mathcal{R}_n(\mathbb{K}))$$

$$(D conj_{A_n})_{R_n(1,x)}(L_n) = A_n L_n A_n^{-1}.$$

(iv) *By this way we get:*

$$T_{A_n}\mathcal{R}_n(\mathbb{K}) = \{v \in \mathcal{M}_{n+1}(\mathbb{K}) \mid v = A_n L_n, \; L_n \in \mathcal{L}(\mathcal{R}_n(\mathbb{K}))\} =$$

$$= \{w \in \mathcal{M}_{n+1}(\mathbb{K}) \mid w = L_n A_n, \; L_n \in \mathcal{L}(\mathcal{R}_n(\mathbb{K}))\}$$

By using analogous arguments as in the proof of Theorem 4.10.1 and some direct computations we get:

Proposition 4.15.2 *Let $A = R(d(x), h(x))$ be a Riordan matrix.*

(1) *The tangent space of $\mathcal{R}(\mathbb{K})$ at $R(d(x), h(x))$ is given by:*

$$T_{R(d(x),h(X)}\mathcal{R}(\mathbb{K}) = \{R(d(x), h(x)L(\chi, \alpha), \; \chi, \alpha \in \mathbb{K}[[x]]\} =$$

$$= \{L(\chi, \alpha)R(d(x), h(x)), \; \chi, \alpha \in \mathbb{K}[[x]]\}$$

(2) *The conjugation by A defined by*

$$\begin{array}{rcccc} conj_{A} & : & \mathcal{R}(\mathbb{K}) & \longrightarrow & \mathcal{R}(\mathbb{K}) \\ & & X & \longmapsto & A \cdot X \cdot A^{-1} \end{array}$$

is a C^∞-diffeomorphism and the differential map is given by:

$$(D conj_A)_{R(1,x)} \; : \; \mathcal{L}(\mathcal{R}(\mathbb{K})) \longrightarrow \mathcal{L}(\mathcal{R}(\mathbb{K}))$$

$$L(\chi, \alpha) \longmapsto A \cdot L(\chi, \alpha) \cdot A^{-1}$$

(3) *For any $t \in \mathbb{R}$ (actually $t \in \mathbb{K}$) we have that:*

$$exp\Big(t\big(D(conj_A)\big)_{R(1,x)}\big(L(\chi,\alpha)\big)\Big) = conj_A(e^{tL(\chi,\alpha)})$$

in particular the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{L}(\mathcal{R}) \arrow[r, "D(conj_A)_{R(1,x)}"] \arrow[d, "exp"] & \mathcal{L}(\mathcal{R}) \arrow[d, "exp"] \\
\mathcal{R} \arrow[r, "conj_A"] & \mathcal{R}
\end{tikzcd}$$

(4) *Given $L(\chi,\alpha) \in \mathcal{L}(\mathcal{R}(\mathbb{K}))$, a direct computation shows that:*

$$\Big((Dconj_{R(d(x),h(x))})_{R(1,x)}\Big)(L(\chi,\alpha)) = L\left(\chi(h(x)) - \frac{h(x)}{h'(x)}\frac{d'(x)}{d(x)}\alpha(h(x)), \frac{h(x)}{xh'(x)}\alpha(h(x))\right)$$

(1) in this proposition will be very interesting for the next section. (2), (3) and (4) stablish a bridge between conjugation in $\mathcal{L}(\mathcal{R})$ and conjugation in $\mathcal{R}$.

We have the following:

Remark 4.15.3 *Let $L \in \mathcal{L}(\mathcal{R})$. If $L' = ALA^{-1}$ for some $A \in \mathcal{R}$ and L' is a diagonal matrix, then $e^{L'} = Ae^{L}A^{-1}$ where e^{L} is also a diagonal matrix. This remark relates conjugacy in $\mathcal{L}(\mathcal{R})$ to conjugacy in $\mathcal{R}$ (see open question 20).*

Determining wether $L' = ALA^{-1}$ is equivalent to solve a system of functional-differential equations. According to (4) in proposition 4.15.2 we have that:

$$L(\tilde{\chi},\tilde{\alpha}) = R(d(x),h(x))R(\chi,\alpha)R^{-1}(d(x),h(x))$$

if and only if:

$$\begin{cases} \tilde{\chi}(x) = \chi(h(x)) - \frac{h(x)}{h'(x)}\frac{d'(x)}{d(x)}\alpha(h(x)) \\ \tilde{\alpha}(x) = \frac{h(x)}{xh'(x)}\alpha(h(x)) \end{cases}$$

which is equivalent to:

$$\begin{cases} d'(x) = \left[\frac{\chi(h(x)) - \tilde{\chi}(x)}{x\tilde{\alpha}(x)}\right] d(x) \\ h'(x)\tilde{\alpha}(x) = \frac{h(x)}{x}\alpha(h(x)) \end{cases}$$

The questions above are not only interesting for conjugation in $\mathcal{R}$. Look for instance at the following example:

Example 4.15.4 *Let $A = R(d(x),h(x))$. If we replace $L(c_1,c_2)$ by $(D(conj_A))_{R(1,x)}(L(c_1,c_2))$ in example 4.14.3 the corresponding initial value problem is:*

$$\begin{cases} \frac{\partial u}{\partial t} = \left(c_1 h(x) - c_2\frac{h(x)}{h'(x)}\frac{d'(x)}{d(x)}\right)u(x,t) + c_2\frac{h(x)}{h'(x)}\frac{\partial u}{\partial x} \\ u(x,0) = F(x) \end{cases}$$

But since:

$$exp\Big(t(Dconj_A)_{R(1,x)}(L(c_1,c_2))\Big) = conj_A(e^{tL(c_1,c_2)}) = conj_A(R(e^{c_1t}, e^{c_2t}x))$$

it is easy to compute the solution:

$$u(x,t) = \frac{e^{c_1t}d(x)}{d(h^{-1}(e^{c_2t}h(x)))} \cdot F(h^{-1}(e^{c_2t}h(x)))$$

4.16 A natural group extension of the Riordan group: the tangent bundle.

The **tangent bundle** of a differentiable manifold M is another manifold TM which assembles all the tangent vectors in M: as a set it is the disjoint union of the tangent spaces TM_x of M, or equivalently it can be thought as a set of pairs (x, v) with $x \in M$, $v \in TM_x$.

So for $M = \mathcal{R}(\mathbb{K})$ we can identify (as a set) the tangent bundle with:

$$T\mathcal{R}(\mathbb{K}) = \mathcal{R}(\mathbb{K}) \times \mathcal{L}(\mathcal{R}(\mathbb{K})) = \{(R(d,h), L(\chi,\alpha)) : R(d,h) \in \mathcal{R}(\mathbb{K}),\ L(\chi,\alpha) \in \mathcal{L}(\mathcal{R}(\mathbb{K}))\}.$$

and as a smooth Frechet manifold the tangent bundle is diffeomorphic to the product of manifolds $\mathcal{R}(\mathbb{K}) \times \mathcal{L}(\mathcal{R}(\mathbb{K}))$ considering $\mathcal{L}(\mathcal{R}(\mathbb{K}))$ as a linear isomorphic copy of $\mathbb{K}^{\mathbb{N}}$.

To end this section, following Milnor [82] pages 1033-1036, we have that:

Remark 4.16.1 *The tangent bundle $T\mathcal{R}(\mathbb{K})$ has a natural structure of Lie group in such a way that $T\mathcal{R}(\mathbb{K})$ is isomorphic to a semidirect product $\mathcal{L}(\mathcal{R}(\mathbb{K})) \rtimes \mathcal{R}(\mathbb{K})$, where the operation in $\mathcal{L}(\mathcal{R}(\mathbb{K}))$ is the sum and the operation in $\mathcal{R}(\mathbb{K})$ is the matrix product. So the tangent bundle $T\mathcal{R}(\mathbb{K})$ is naturally endowed with the operation:*

$$(R(d,h), L(\chi,\alpha)) \star (R(\tilde{d},\tilde{h}), L(\tilde{\chi},\tilde{\alpha})) =$$

$$= (R(d,h)R(\tilde{d},\tilde{h}), R^{-1}(\tilde{d},\tilde{h})L(\chi,\alpha)R(\tilde{d},\tilde{h}) + L(\tilde{\chi},\tilde{\alpha}))$$

Besides, the additive group $\mathcal{L}(\mathcal{R}(\mathbb{K}))$ is embedded as a normal subgroup in $T\mathcal{R}(\mathbb{K})$ and with $\mathcal{R}(\mathbb{K})$ identified with the subset of elements of the form $(R(d(x), h(x), 0)$ in the tangent bundle. The additive group $\mathcal{L}(\mathcal{R}(\mathbb{K}))$ is also naturally embedded as a normal subgroup of $T\mathcal{R}(\mathbb{K})$ in such a way that $T\mathcal{R}(\mathbb{K})$ is isomorphic to a semidirect product $\mathcal{L}(\mathcal{R}(\mathbb{K})) \rtimes \mathcal{R}(\mathbb{K})$, where $\mathcal{R}(\mathbb{K})$ acts on $\mathcal{L}(\mathcal{R}(\mathbb{K}))$ by conjugation.

We will skip those details since we are not going to use this in this work.

Among other things we have included this subsection to give a motivation and a geometric framework for open question 21, which states the possibility of studying the subgroup of $ILT_\infty(\mathbb{K})$ generated by the elements in $\mathcal{R}$ and the invertible elements in $\mathcal{L}(\mathcal{R})$ that may have combinatorial interest.

4.17 The Toeplitz-Lagrange semidirect product in $\mathcal{R}(\mathbb{K})$ and the corresponding decomposition in $\mathcal{L}(\mathcal{R}(\mathbb{K}))$

We have already stated (in chapter 0) that the Riordan group is a semidirect product of the Toeplitz and the associated subgroups (also called Lagrange subgroup).

Moreover, the Toeplitz subgroup is normal in $\mathcal{R}$. Consequently the associated subgroup acts by conjugation on the Toeplitz subgroup in the corresponding semidirect product structure.

We have the following:

Proposition 4.17.1

(1) For $0 \leq n \leq \infty$, the subgroups $\mathcal{T}_n$ and $\mathcal{A}_n$ are closed Lie subgroups of the matrix Lie group $\mathcal{R}_n$.

(2) The tangent algebra of the Toeplitz subgroup is the following subalgebra of $\mathcal{L}(\mathcal{R})$:

$$\mathcal{L}(\mathcal{T}) = \{L(\chi, 0) \in \mathcal{L}(\mathcal{R})\}$$

(3) The tangent algebra of the associated subgroup is the following subalgebra of $\mathcal{L}(\mathcal{R})$:

$$\mathcal{L}(\mathcal{A}) = \{L(0, \alpha) \in \mathcal{L}(\mathcal{R})\}$$

(4) As a vector space $\mathcal{L}(R) = \mathcal{L}(\mathcal{T}) \oplus \mathcal{L}(\mathcal{A})$.

Proof: We have:

- In (i), the proof for a finite n is immediate. To prove $n = \infty$ (that follows the same idea), consider the global parametrization of the Riordan group:

$$\begin{array}{rccl} \varphi_\infty & : & \mathcal{U}_\infty & \longrightarrow & \mathcal{R}(\mathbb{K}) \\ & & \mathbf{u} & \longmapsto & \varphi_\infty(\mathbf{u}) \end{array}$$

  where
$$\mathcal{U}_\infty = \left\{\mathbf{u} = (u_k)_{k\in\mathbb{N}} = (x_0, a_0, x_1, a_1, \ldots) \in \mathbb{K}^{\mathbb{N}} \mid u_0 \neq 0,\ u_1 \neq 0\right\}.$$

  Define:
$$u' = (x_0, 1, x_1, 0, x_2, \ldots), \qquad u'' = (1, a_0, 0, a_1, 0, \ldots)$$

  and:
$$\mathcal{U}'_\infty = \{\mathbf{u}' : \mathbf{u} \in \mathcal{U}_\infty\} \qquad \mathcal{U}''_\infty = \{\mathbf{u}'' : \mathbf{u} \in \mathcal{U}_\infty\}$$

  $\mathcal{U}'_\infty$ and $\mathcal{U}''_\infty$ are closed in $\mathcal{U}_\infty$ because each of them is a product of closed subset in each factor in the topological product space $\mathcal{U}_\infty$. Since φ_∞ is a homeomorphism, then $\varphi_\infty(\mathcal{U}'_\infty)$ and $\varphi_\infty(\mathcal{U}''_\infty)$ are closed in $\mathcal{R}(\mathbb{K})$ and they are, respectively, the Toeplitz and the Lagrange subgroups.

- For (ii), (iii), obviously,

$$\{L(\chi,0)\in\mathcal{L}(\mathcal{R})\}\qquad\{L(0,\alpha)\in\mathcal{L}(\mathcal{R})\}$$

are linear subspaces of the vectorial space associated to $\mathcal{L}(\mathcal{R})$. Consider now the global parametrizations for the Toeplitz and Associated subgroups given, respectively, by:

$$(\varphi_\infty\circ i_1)((x_n)_{n\geq 0})=\begin{pmatrix} x_0 & & & & \\ x_1 & x_0 & & & \\ x_2 & x_1 & x_0 & & \\ x_3 & x_2 & x_1 & x_0 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

$$(\varphi_\infty\circ i_2)((a_n)_{n\geq 0})=\begin{pmatrix} 1 & & & & \\ 0 & a_0 & & & \\ 0 & a_1a_0 & {a_0}^2 & & \\ 0 & a_1^2a_0+a_2{a_0}^2 & 2a_1a_0^2 & {a_0}^3 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where:

$$i_1(x_0,x_1,\ldots)=(x_0,0,x_1,0,\ldots),\qquad i_2(a_0,a_1,\ldots)=(0,a_0,0,a_1,\ldots)$$

and for $(x_n)_{n\geq 0},(a_n)_{n\geq 0}\in\mathbb{K}^*\times\mathbb{K}^{\mathbb{N}}$. Take now:

$$\mathbf{u}=(x_0,a_0,x_1,a_1,\ldots)$$

We have:

$$\varphi_\infty(\mathbf{u})=(\varphi_\infty\circ i_1)((x_n))(\varphi_\infty\circ i_2)((a_n))$$

Using now the Chain rule and the basic properties for the derivation of the product of matrices, we get:

$$\frac{\partial\varphi_\infty}{\partial x_n}\mid_{(1,1,0,0,\ldots)}=\frac{\partial(\varphi_\infty\circ i_1)}{\partial x_n}\mid_{(1,0,0,0,\ldots)}$$
$$\frac{\partial\varphi_\infty}{\partial a_n}\mid_{(1,1,0,0,\ldots)}=\frac{\partial(\varphi_\infty\circ i_2)}{\partial a_n}\mid_{(1,0,0,0,\ldots)}$$

for $n\geq 0$. Finally, from Theorem 4.10.1, we obtain that the Lie algebra of each subgroup is the one proposed above.

- (4) is immediate.

□

So we could say that the decomposition of $\mathcal{R}$ as a semidirect product of the Toeplitz and Associated subgroups is coherent with the inverse limits structures of both $\mathcal{R}(\mathbb{K})$ and $\mathcal{L}(\mathcal{R}(\mathbb{K}))$.

4.18 Stabilizers in $\mathcal{R}(\mathbb{K})$ and the corresponding Lie algebras

In this section we will obtain another application of the pro-Lie group structure of $\mathcal{R}$.

As we have already mentioned, there is a natural linear action of $\mathcal{R}_n(\mathbb{K})$ on $\mathbb{K}^{n+1}$. Consider now the elements in $\mathbb{K}^{n+1}$ as polynomials of degree less than or equal to n. We have already introduced stabilizers too in proposition 4.1.7. In this context:

Remark 4.18.1 *Suppose a fixed polynomial $h(x) = \sum_{k=0}^{n} h_k x^k$. The stabilizer of $h(x)$ under the natural linear action is:*

$$(\mathcal{R}_n(\mathbb{K}))_{h(x)} = \{D \in \mathcal{R}_n(\mathbb{K}) : D \otimes h(x) = h(x)\}$$

Recall from proposition 4.1.7 that the Lie algebra of the stabilizer subgroup $\mathcal{R}_n(\mathbb{K})_{h(x)}$ is given by

$$\mathcal{L}((\mathcal{R}_n(\mathbb{K})_{h(x)})) = \{L \in \mathcal{L}(\mathcal{R}_n(\mathbb{K})) : h(x) \in \mathrm{Ker} L\}$$

Some results on stabilizers in the Riordan group can be already found in [51].

We will verify the above remark with an example of subgroup whose tangent algebra has already been computed in this work:

Example 4.18.2 *Consider the associated subgroup $\mathcal{A}_n(\mathbb{K})$. The elements of this subgroup are the Riordan matrices of the form $R_n(1, h(x))$. The associated subgroup is the stabilizer of the power series* 1*. We have already proved that the Lie algebra of the associated subgroup is formed just by the elements:*

$$\{L_n(0, \alpha) : \mathcal{L}(\mathcal{R}_n(\mathbb{K}))\}$$

Note also that Proposition 4.13.2 confirms this result because $L_n(\chi, \alpha) \otimes (1) = \chi(x)$ and then $1 \in \mathrm{Ker} L_n(\chi, \alpha)$ if and only if $\chi = 0$.

Fortunately we will be able to extend this result for the infinite group $\mathcal{R}(\mathbb{K})$. Obviously, we have the following:

Remark 4.18.3 *Let $\gamma \in \mathbb{K}[[x]]$. Represent by $(\mathcal{R}(\mathbb{K}))_\gamma$ the stabilizer of γ in $\mathcal{R}(\mathbb{K})$. Then, $R(d(x), h(x)) \in (\mathcal{R}(\mathbb{K}))_\gamma$ if and only if $\Pi_n(R(d(x), h(x))) \in (R_n(\mathbb{K}))_{Taylor_n(\gamma)}$ for any $n \geq 0$.*

As a consequence we get our main result in this section that explains the behaviour of the stabilizers in $\mathcal{R}(\mathbb{K})$:

Theorem 4.18.4 *Let $\gamma \in \mathbb{K}[[x]]$ be non-null. Consider the stabilizer of γ:*

$$(\mathcal{R}(\mathbb{K}))_\gamma = \{D \in \mathcal{R}(\mathbb{K}) : D \otimes (\gamma) = \gamma\}$$

Then:

$$T_\gamma : \mathcal{R} \longrightarrow \mathcal{R}$$

$$T_\gamma(R(d(x),h(x))) = R\left(d(x)\frac{\gamma(x)}{\gamma(h(x))}, h(x)\right)$$

defines, by restriction, an isomorphism from the associated subgroup $\mathcal{A}(\mathbb{K})$ onto $(\mathcal{R}(\mathbb{K}))_\gamma$. Consequently $\mathcal{R}(\mathbb{K})_\gamma$ is a closed Lie subgroup of the Riordan group $\mathcal{R}(\mathbb{K})$.

Moreover the differential map defines, by restriction, a Lie algebra isomorphism:

$$DR_\gamma(R(1,x)) : \mathcal{L}(\mathcal{A}) \to \mathcal{L}((\mathcal{R}(\mathbb{K}))_\gamma)$$

Finally,

$$\mathcal{L}((\mathcal{R}(\mathbb{K}))_\gamma) = \{L(\chi,\alpha) \in \mathcal{L}(\mathcal{R}(\mathbb{K})) \ : \ \gamma \in \mathrm{Ker}L(\chi,\alpha)\}$$

or equivalently:

$$\chi \cdot \gamma + x\alpha\gamma' = 0$$

Proof:

- Suppose an element $R(1,h(x))$ in the associated subgroup, then, obviously:

  $$T_\gamma(R(1,h(x))) \otimes \gamma(x) = R\left(\frac{\gamma(x)}{\gamma(h(x))}, h(x)\right) \otimes \gamma(x) = \gamma(x)$$

  On the other hand if:

  $$R(d(x),h(x)) \otimes \gamma(x) = \gamma(x)$$

  it is easy to see that:

  $$R(d(x),h(x)) = T_\gamma(R(1,h(x)))$$

- The map T_γ is a Lie group isomorphism, since:

  $$T_\gamma = conj_{R(1,\gamma(x))}$$

- Consider $L(\chi,\alpha) \in \mathcal{L}(\mathcal{R}(\mathbb{K})_\gamma)$. This implies that $e^{tL(\chi,\alpha)} \in \mathcal{R}(\mathbb{K})_\gamma$ for any $t \in \mathbb{R}$. So, $e^{tL(\chi,\alpha)}(\gamma) = \gamma$ and, by proposition 4.18.3, this is equivalent to $\Pi_n(e^{tL(\chi,\alpha)}) \in \mathcal{R}_n(\mathbb{K})_{Taylor_n(\gamma)}$ for every $n \in \mathbb{N}$, $t \in \mathbb{R}$. But, recall here the proof of Theorem 4.10.1, that:

  $$\Pi_n(e^{tL(\chi,\alpha)}) = e^{tD\Pi_n(I)(L(\chi,\alpha))}$$

  where $I = R(1,x)$ is the neutral element in $\mathcal{R}(\mathbb{K})$. For the finite dimensional Lie group $\mathcal{R}_n(\mathbb{K})$, the above equality implies that:

  $$D\Pi_n(I)(L(\chi,\alpha)) \in \mathcal{L}(\mathcal{R}_n(\mathbb{K})_{Taylor_n(\gamma)})$$

  for any $n \in \mathbb{N}$. Consequently $Taylor_n(\gamma) \in \mathrm{Ker}D\Pi_n(I)(L(\chi,\alpha))$ for any $n \in \mathbb{N}$.

- Proceeding now as in the proof of Theorem 4.10.1, $L(\chi, \alpha)$ is represented by the sequence $(D\Pi_n(I)(L(\chi, \alpha)))_{n\in\mathbb{N}}$ in the inverse limit description of $\mathcal{L}(\mathcal{R}(\mathbb{K}))$ pointed out there. Hence $L(\chi, \alpha) \otimes \gamma = 0$ and so:

$$\chi(x)\gamma(x) + x\alpha(x)\gamma'(x) = 0$$

- Suppose now that $L(\chi, \alpha)(\gamma) = 0$. Passing again through the finite dimensional Lie groups $\mathcal{R}_n(\mathbb{K})$ and by using similar arguments we have that $e^{tL(\chi,\alpha)} \in \mathcal{R}(\mathbb{K})_\gamma$ for any $t \in \mathbb{R}$. So the curve $\delta : \mathbb{R} \longrightarrow \mathcal{R}(\mathbb{K})_\gamma$ given by $\delta(t) = e^{tL(\chi,\alpha)}$, is well defined and differentiable for the infinite dimensional Lie group structure in $\mathcal{R}(\mathbb{K})_\gamma$ whose existence was proved at the beginning of this proof. Since $\delta'(0) = L(\chi, \alpha)$, it means, by definition, that $(\chi, \alpha) \in \mathcal{L}((\mathcal{R}(\mathbb{K}))_\gamma)$.

□

As we have already mentioned in this chapter, stabilizer of elements in $\mathbb{K}((x))$ are also useful. Note that:

Remark 4.18.5 *With essentially the same proof the above proposition remains true if we replace finite matrices by bi-infinite matrices and $\mathbb{K}[[x]]$ by $\mathbb{K}((x))$.*

Moreover, there is again some interesting symmetry when considering the bi-infinite case:

Corollary 4.18.6 *For any $L(\chi, \alpha) \in \mathcal{L}(\mathcal{R}_{\infty\infty}(\mathbb{K}))$ and any $\gamma \in \mathbb{K}((x))$ we have that:*

$$L(\chi, \alpha) \in \mathcal{L}(\mathcal{R}_{\infty\infty}(\mathbb{K})_\gamma) \Leftrightarrow L(-\chi, \alpha) \in \mathcal{L}(\mathcal{R}_{\infty\infty}(\mathbb{K})_{\frac{1}{\gamma}})$$

Now, for the rest of this section, we will recover some of the known subgroups of the Riordan group as stabilizer of certain elements and so we will also compute their Lie algebras.

Example 4.18.7 *The* ***Bell subgroup*** *$\mathcal{B}$ is formed by all Riordan matrices of the form:*

$$R\left(\frac{h(x)}{x}, h(x)\right)$$

There is an obvious relation between the Bell and the associated subgroups. In fact, it is easy to see that the Bell subgroup is the stabilizer of $\frac{1}{x}$.

Thus applying the formula in theorem 4.18.4, the matrices $L(\chi, \alpha)$ lying in $\mathcal{L}(\mathcal{B})$ are those of the type:

$$L(-\alpha, \alpha)$$

Example 4.18.8 *The* ***Stochastic subgroup*** $\mathcal{S}$ *of the Riordan group is, by definition, the stabilizer of the geometric series* $\frac{1}{1-x}$.

Then $\mathcal{L}(\mathcal{S})$ *is the set of matrices in* $L(\chi, \alpha)$ *satisfying:*

$$L(\chi, \alpha) \otimes \left(\frac{1}{1-x} \right) = \chi(x) \cdot \frac{1}{1-x} + x\alpha(x) \cdot \frac{1}{(1-x)^2} = 0$$

that is, the set of matrices of the type:

$$L\left(-\frac{x\alpha(x)}{(1-x)}, \alpha(x) \right)$$

Chapter 5

Riordan Matrices and Simplicial Complexes

In this final chapter, we will show the presence of Riordan matrices in some classical combinatorial problems related to simplicial complexes.

We will start recalling the main concepts involved in the rest of this chapter: for simplicial complex (section 5.1) and for the f-vector problem (section 5.2).

In section 5.3 we will show how the relation between f-vector, g-vector, h-vector and γ-vector may be understood as a Riordan change of basis.

In section 5.4 we will see how Dehn-Sommerville equations (a classical result when studying the f-vector problem, see the books [39, 119]) in the formulation made in [39] (in terms of the f-vector, not in terms of the h-vector) can be stated as a problem of finding eigenvectors of a Riordan matrix which is an involution.

We will also show in section 5.5 that fixing two simplicial complexes K and L and considering the simplicial complexes $K, K * L, K * L * L, \dots$ (recall that the join of simplicial complexes is associative) we obtain a certain Riordan pattern in the f-vector if we write the corresponding f-vectors as rows in a matrix. We will also introduce the m, q-cones which are the unique choice of L and K that makes this matrix actually a Riordan matrix. This last fact may be of interest since it allows us to test some properties about linear relations in the f-vector via the 1FTRM.

After this, we will introduce subdivision methods (section 5.6). Then in section 5.7 we will see new proofs of some classical results in a new context (the non-existence of other linear relations on the entries in the f-vector being an homotopical invariant, the invariance of the Euler characteristic under barycentric subdivision, and the non-existence of other possible linear relations for certain PL-topologically closed families apart from the Dehn-Sommerville equations) and even a new one (proposition 5.7.5, (ii)), all of it relying only on linear algebra and Riordan matrices.

In section 5.8 we will study a matrix concerning the Betti-sequence of m, q-cones that happens to be a Riordan matrix again.

Finally, we will study the complexes obtained by replacing the simplices by q, q-cones as building blocks (section 5.9).

5.1 Simplicial Complexes

We will assume that the reader has a basic topological background (basic references for this can be [42, 85]).

Simplicial complexes are formed from a family of more simple objects: the simplices.

Definition 5.1.1 *Let $\{a_0, \dots, a_1\}$ a set of point affinely independent in $\mathbb{R}^N$. We define the* n***-simplex*** σ *generated by $a_0, \dots, a_n$ as the set of convex combinations of the points $a_0, \dots, a_n$, that is:*

$$\sigma = \left\{ \sum_{i=0}^{n} t_i a_i : \sum_{i=0}^{n} t_i = 1 \right\}$$

Note that if σ is generated by $n + 1$ point we will say that is an n-simplex, or a ***simplex of dimension*** n*. By agreement, $\emptyset$ is considered to be the simplex of dimension -1.*

The points $\{a_0, \dots, a_1\}$ are called **vertices**. Any subset of the set of vertices is again affinely independent and we can use them to generate another (smaller) simplex. Any simplex generated by a subset of the original set of vertices is said to be a **face** of the simplex σ.

Once this is settled, there are two possible ways to define simplicial complexes:

Definition 5.1.2 (geometric definition, see [85]**)** *A* ***(geometric) simplicial complex****, denoted by $|K|$, is the union of a nonempty collection K of simplices satisfying the following conditions:*

1. *$\emptyset \in K$*
2. *(**Closure under inclusions**) For any simplex in K, the faces are also in K.*
3. *The intersection of two simplices in K, if not empty, is also a simplex in K.*

For the rest of this work, we will only considered ***finite simplicial complexes****, that is, simplicial complexes which are a finite collection of simplices.*

Definition 5.1.3 (abstract or combinatorial definition) *Let V be a set, that will always be finite for us, and whose elements are called* ***vertices****. An* ***(abstract) simplicial complex*** *K over the vertex set V is a non-empty family of subsets of V satisfying the following conditions:*

(a) $\emptyset \in K$

(b) ***(Closure under inclusions)*** $A \in K,\ B \subset A \ \Rightarrow\ B \in K$.

*Again we will always consider that K must be a finite family of sets (****finite simplicial complex****).*

In the first one of those definitions, simplicial complexes may be thought as object with a double structure: geometric objects (since they are topological spaces embedded in $\mathbb{R}^N$) with some extra combinatorial structure: vertices, edges,...and d-dimensional faces in general.

Remark 5.1.4 *Identifying each simplex with its set of vertices we can obtain an abstract simplicial complex out of a geometric simplicial complex and there is a canonical way called* ***canonical geometric realization*** *to obtain a geometric simplicial complex out of an abstract simplicial complex K with vertex set V associating to each element in K a simplicial complex in $\mathbb{R}^{\#V}$. We omit details that can be found in the bibliography proposed above.*

For both definitions we define ***vertices*** *and* ***faces*** *and subcomplexes in the natural way. Again we will omit the details.*

The maximal faces, that is, the simplices which are not the face of any other simplex in K are called **facets**. The simplicial complexes whose facets are of the same dimension are said to be **pure**. The family of pure simplicial complexes is very interesting since it contains the familly of simplicial complexes with a structure of topological manifold.

For both definitions (and of course in a compatible way) we have a notion of **simplicial map** and of **isomorphism**. Roughly speaking, a simplicial map is a function that maps vertices to vertices and simplices of dimension d to simplices of dimension d. This allow us, for example, to consider different geometric realizations for an abstract simplicial complexes, modulo isomorphism.

For the geometric definition of simplicial complex an isomorphism is not only an homeomorphism, but an homeomorphism that preserves the combinatorial structure (vertices, edges and faces in general) of the simplicial complex. One way to preserve the geometric structure but not the combinatorial one is the use of subdivisions.

Definition 5.1.5 *Let K be a simplicial complex. We say that another simplicial complex K' is a* ***subdivision*** *of K, if:*

- $\forall \sigma \in L$, $\exists \tau \in K$ *so that* $\sigma \subset \tau$.
- $\forall \tau \in K$, τ *is a finite union of simplicies in* L.

If $|K'|$ is a subdivision (modulo isomorphism) of $|K|$, then both are homeomorphic as topological spaces. Two simplicial complexes K and L are said to be **PL-homeomorphic** (for Pieceswise-Linear) if there exists a third simplicial complex S which is a subdivision of K and of L.

Initially, it was thought that for two simplicial complex being homeomorphic and being PL-homeomorphic were equivalent conditions (Hauptvermutung conjecture), but shi was disproved by J. Milnor in [81] one can find “two complexes which are homeomorphic but combinatorially distinct”.

As we have showed, homemorphisms are not a natural language for simplicial complex since they do not take into account the combinatorial structure so they are replaced by isomorphisms or PL-homeomorphisms. The same happens with homotopy equivalences that may be replaced by simple homotopy equivalences. A good reference to cover this topic may be the book by J. A. Barmak [6].

Finally, recall that there is an homotopical invariant defined specially for simplicial complexes: **simplicial homology groups**. We are not going to describe them here. The definition can be found in [42, 85]. We have put all we need to know in the following remark:

Remark 5.1.6 *We have that:*

- *Homology groups are finitely generated. They can be defined over any ring but typically they are defined over $\mathbb{Z}$ or over a field. If they are defined over a field, there is no torsion.*
- *We can define an homology group for each $n \geq 0$ which is denoted by $H_n(K, \mathbb{Z})$ or simply $H_n(K)$.*
- *$H_0(K)$ is free abelian (see [85], section 7) and its rank equals to the number of arcwise connected components of $|K|$ (in a simplicial complex, the arcwise connected components and the connected components coincide). If K is connected, then $H_1(K)$ is the abelianization of the fundamental group of K (see [73] or [42]).*
- *Simplicial homology groups are homotopy invariants.*
- *Sometimes is more convenient to use the **reduced homology groups**. The n-th reduced homology group is denoted by $\widetilde{H}_n(K)$ and the only difference is that for $n = 0$ we have that:*
$$rank(H_0(K)) = rank(\widetilde{H}_0(K) + 1)$$
- *For a simplicial complex of dimension d, it follows from the definition that $\forall n > 0,\ H_n(K) = 0$.*
- *If K is a contractible simplicial complex of dimension d, then for all $0 \leq n \leq d$, $\widetilde{H}_n(K) = 0$.*
- *If a simplicial complex K is homeomorphic to the n-th dimensional sphere S^d we have that:*
$$\widetilde{H}_n(K, \mathbb{Z}) = \begin{cases} 0 & for\ 0 \leq n \leq d-1 \\ \mathbb{Z} & for\ n = d \end{cases}$$

Once the homology groups have been presented, the Betti numbers can be introduced. They contain simplified information from the homology groups and they come from one of the first attempts to assign invariants to the simplicial complexes.

Simplicial homology groups are finitely generated. Let $H_p(K)$ one of those homology groups and let T_p its **torsion subgroup** (the subgroup generated by all the elements of finite order). It must exist ([85], section 4) a free abelian subgroup $G_p < H_p(K)$ of finite rank $b_p(K)$, so that $H_p(K) = T_p \oplus G_p$ and this b_p is univocally determined by the group $H_p(K)$. Those numbers $b_p(K)$ are the so called ***p*-th Betti number** of K.

Intuitively, $b_0(K)$ is the number of connected components of a simplicial complex K and $b_p(K)$ is the "number of $p+1$ dimensional holes" (a hole is $d+1$ dimensional if it is contained in a d dimensional simplex but not in a $d-1$ dimensional simplex) of K.

Also an analogue of Betti numbers can be defined for reduced homology, the so called **reduced Betti numbers**:

$$\widetilde{b}_0(K), \widetilde{b}_1(K), \widetilde{b}_2(K), \ldots$$

They are equal to Betti numbers except for:

$$\widetilde{b}_0(K) = b_0(K) - 1$$

For a simplicial complex K of dimension d, the sequence $(\beta_0(K), \ldots, \beta_d(K))$ is called the **Betti sequence**. The sequence $(\widetilde{\beta}_0(K), \ldots, \widetilde{\beta}_d(K))$ is called the **reduced Betti sequence**.

Remark 5.1.7 *As homology groups are homotopically invariant, so are the following numbers:*

$$\chi(K) = b_0 - b_1 + b_2 - \ldots \pm b_d, \ \widetilde{\chi}(K) = \widetilde{b}_0 - \widetilde{b}_1 + \widetilde{b}_2 - \ldots \pm \widetilde{b}_d$$

*called respectively the **Euler** and the **reduced Euler characteristics**.*

5.2 The f-vector Problem

One of the objects that we will be very interested in along this last chapter are **face-vectors** or **f-vectors** since we will study the connection between those objects and Riordan matrices.

Definition 5.2.1 *Let K a simplicial complex of dimension d. We define its **f-vector** to be the sequence of integers:*

$$f(K) = (f_0(K), \ldots, f_d(K))$$

such that for all $0 \leq i \leq d$, $f_i(K)$ counts the number of faces of dimension i of K.

*We will sometimes use **extended f-vectors** , which are sequences:*

$$ef(K) = (f_{-1}(K), f_0(K), \ldots, f_d(K))$$

where for all $0 \leq i \leq d$, $f_i(K)$ has the same interpretation as in the previous case and by agreement we always consider that $f_{-1}(K) = 1$, thinking that we are counting in this case the unique face with 0 points (dimension -1) that any simplicial complex has: $\emptyset$.

Moreover, it will be sometimes convenient to consider (extended or not) f-vectors to be sequences of infinite length with the convention $f_k(K) = 0$ if $d > k$.

As usual, it will be sometimes necessary to associate to these vectors some power series (polynomials in this case). The **f-polynomial** and the **extended f-polynomial** of a given simplicial complex K are respectively the polynomials:

$$f_K(x) = f_0 + f_1 x + \ldots + f_d x^d, \quad ef_K(x) = f_{-1} + f_0 x + \ldots + f_d x^{d+1} = 1 + x f_K(x)$$

One of the main problems in topological combinatorics is the so called **f-vector problem**: the problem of describing the possible set of f-vectors $(f(S))$ of a given family of simplicial complexes (S). Of course, this may not be interesting for any possible family $\mathcal{S}$ but for families that have any characteristic property, tipically a topological one. For example an open and hard question is computing $f(\mathcal{S})$ for $\mathcal{S}$, being the set of simplicial complexes homeomorphic to a fixed simplicial complex K.

There exist some other vectors of integers containing the same information as the f-vector. They have happened to be very useful in the study of the f-vector problem. We will list some of them now.

Definition 5.2.2 *Let S a simplicial complex of dimension d. Let $ef(K) = (f_{-1}, \dots, f_d)$ be its extended f-vector. We define the **h-vector** of K to be the sequence $h(K) = (h_0, \dots, h_{d+1})$ satisfying:*

$$h_k = \sum_{i=0}^{k} (-1)^{k-i} \binom{d+1-i}{d+1-k} f_{i-1} \tag{5.1}$$

Again, associated to this sequence, we have a polynomial $h_S(x)$, which will be called the **h-polynomial**.

In most of the cases that will be studied here, the h-vector will happen to be a symmetric vector (see Dehn-Sommerville equations below).

Definition 5.2.3 *Let K be a simplicial complex of dimension d whose h-vector is $h(K) = (h_0, \dots, h_{d+1})$. If $h(K)$ is symmetric, it is usually defined the **g-vector** by:*

$$g_0 = h_0, \; g_i = h_i - h_{i-1}$$

Definition 5.2.4 *Let K be a simplicial complex, let $h(K) = (h_0, \dots, h_{d+1})$ be its h-vector. If $(h(K)$ is symmetric, then we can define the **γ-vector** to be the unique sequence of integers $\gamma(K) = (\gamma_0, \dots, \gamma_{\lceil \frac{d}{2} \rceil})$ satisfying:*

$$h(x) = \sum_{i=0}^{\lceil \frac{d}{2} \rceil} \gamma_i t^i (1+t)^{d+1-2i}$$

In this direction (of the f-vector problem) there are many important results and conjectures, some of which will be listed in the following. The main references for the known results are: the book by G.M Ziegler [119] and the article by A- Björner and G. Kalai [14].

1. Kruskal-Katona theorem, that characterizes $f(\mathcal{S})$ for $\mathcal{S}$ being the set of all finite simplicial complexes.

Theorem 5.2.5 (Kruskal-Katona Theorem 8.32 in [119], corrections [120]) *We have that:*

(i) *For any $n, k \geq 0$, there is a unique k-**cascade** (see section 8.5 in* [119]*, the term k-cascade appears in* [33]*) of n of the form:*

$$n = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \ldots + \binom{a_1}{1} \text{ with } a_k > a_{k-1} > \ldots > a_1 \geq 0$$

From this unique decomposition the following operator is defined:

$$\partial_{k-1}(n) = \binom{a_k}{k-1} + \binom{a_{k-1}}{k-2} + \ldots + \binom{a_1}{0}$$

(ii) *A sequence of integers:*

$$f = (f_0, f_1, \ldots, f_d)$$

*is the f-vector of a simplicial complex of dimension d if and only if it satisfies the **Kruskal-Katona condition**, that is, $\forall k \geq 1$:*

$$\partial_k(f_k) \leq f_{k-1}$$

2. The Björner-Kalai theorem, that characterizes $f(\mathcal{S}_\beta)$ for $\mathcal{S}_\beta$ being the set of simplicial complexes with a fixed Betti sequence $\beta = (b_0, b_1, \ldots)$ and that we are not going to include here.

3. The Dehn-Sommerville equations, that must be satisfied by pure and shellable simplicial complexes (in fact by larger classes of simplicial complexes such as Eulerian simplicial complexes, see for example [34]).

Theorem 5.2.6 (Dehn-Sommerville equations, 8.21 in [119]**)** *Let K be the boundary of a simplicial polytope. Then the following statements are equivalent and all of them hold. They are the so called **Dehn-Sommerville equations**:*

(a) *Let f(K)=$(f_{-1}, \ldots, f_d)$, then $\forall 0 \leq k \leq d+1$:*

$$f_{k-1} = \sum_{i=k}^{d+1} (-1)^{d+1-i} \binom{i}{k} f_{i-1} \tag{5.2}$$

(b) *Let $h(K) = (h_0, \ldots, h_{d+1})$, then $\forall k = 0, \ldots, d+1$:*

$$h_k = h_{d+1-k} \tag{5.3}$$

4. The g-theorem, that determines $f(\mathcal{S})$ with $\mathcal{S}$ being the set of simplicial complex PL-homeomorphic to the boundary of a simplicial polytope (we are not going to include it here, see [119]).

5. Conjecture concerning **flag homology spheres**, a class of simplicial complexes that contains spheres.

   - **Gal's conjecture** (2.1.7 in [34]) If S is a $(2k-1)$-dimensional flag homology sphere, then all the coefficients in the γ-polynomial are nonnegative.
   - **Charney-Davis conjecture** (conjecture D in [19]) If S is a $(2k-1)$-dimensional flag homology sphere and $h_S(x)$ is its h-polynomial, then:

     $$(-1)^k h_S(-1) \geq 0$$

     or equivalently:

     $$\gamma_k(S) \geq 0$$

     Of course, also upper bounds for the γ-polynomial would be desirable.

     As pointed out by Gal and Januszkiewicz in [36], this conjecture also has implications for even flag homology spheres.
   - **Nevo-Petersen conjecture** (problem 6.4 in [86]) If S is a flag homology sphere, then γ is the f-polynomial of a flag simplicial complex.

5.3 f-, h-, g- and γ-vectors

Riordan change of basis for the f-vector: h-, g- and γ-vectors

One of the first appearance of the Riordan group when working with the f-vector is the one described in this section: the h-vector, g-vector and γ-vector are nothing more than the product of a Riodan matrix by the f-vector.

Proposition 5.3.1 *Let S a simplicial complex of dimension d. Let $ef(K) = (f_{-1}, \ldots, f_d)$ be its extended f-vector. Then:*

$$\begin{bmatrix} h_0 \\ h_1 \\ \vdots \\ h_{d+1} \end{bmatrix} = \begin{bmatrix} (-1)^0\binom{d+1}{d+1} & & & \\ (-1)^1\binom{d+1}{d} & (-1)^0\binom{d}{d} & & \\ \vdots & & \ddots & \\ (-1)^{d+1}\binom{d+1}{0} & \ldots & \ldots & (-1)^0\binom{0}{0} \end{bmatrix} \begin{bmatrix} f_{-1} \\ f_0 \\ \vdots \\ f_d \end{bmatrix} = \tag{5.4}$$

$$= R_{d+1}\left((1-x)^{d+1}, \frac{x}{1-x}\right) \begin{bmatrix} f_{-1} \\ f_0 \\ \vdots \\ f_d \end{bmatrix}$$

Proof: It follows directly from 5.1.

□

A similar trick for changing between f-vector and h-vector is referred in the bibliography as "Stanley's Trick" (see for exmple [119]).

As we already mentioned in section 5.2 in most of the classic cases studied, the h-vector is symmetric. In this case, all the information of the simplicial complex is contained in the first half of the sequence, that is, in the numbers $(h_0, \dots, h_{\lceil \frac{d}{2} \rceil})$. For this reason, the g-vector and the γ-vector are usually only considered if the h-vector is symmetric and only taking into account the entries $(h_0, \dots, h_{\lceil \frac{d}{2} \rceil})$.

We have:

Proposition 5.3.2 *Let K be a simplicial complex of dimension d which h-vector is $h_0, \dots, h_{d+1}$. Consider the vector:*

$$\begin{bmatrix} g_0 \\ \vdots \\ g_{d+1} \\ g_{d+2} \end{bmatrix} := R(1-x, x) \begin{bmatrix} h_0 \\ \vdots \\ h_{d+1} \\ 0 \end{bmatrix} \tag{5.5}$$

If the h-vector of K is a symmetric sequence, then $(g_0, \dots, g_{\lceil \frac{d}{2} \rceil})$ is what is usually named the g-vector in the bibliography and the sequence $g_0, \dots, g_{d+2}$ is anti-symmetric.

Proof: It follows directly from definition 5.2.3.

□

Finally, we can see that:

Proposition 5.3.3 *Let K be a simplicial complex of dimension d which h-vector is $h_0, \dots, h_{d+1}$. Consider the vector:*

$$\begin{bmatrix} \gamma_0 \\ \vdots \\ \gamma_{d+1} \end{bmatrix} := \left(R\left((1+x)^{d+1}, \frac{x}{(1+x)^2} \right) \right)^{-1} \begin{bmatrix} h_0 \\ \vdots \\ h_{d+1} \end{bmatrix} \tag{5.6}$$

If the h-vector of K is a symmetric sequence, then $(\gamma_0, \dots, \gamma_{\lceil \frac{d}{2} \rceil})$ is what is usually named the γ-vector in the bibliography and the rest of the entries, $\gamma_{\lceil \frac{d}{2} \rceil + 1}, \dots, \gamma_{d+1}$, equal to 0.

Proof: It follows from definition 5.2.4.

□

Remark 5.3.4 *A. Björner proposed the **unimodality conjecture** in* [11] *that states that the f-vector of certain simplicial complexes is always an unimodal sequence (it grows and then decreases). He later disproved it in* [12] *proving that it is "almost" unimodal and exhibited a case where it was not. The h-vector of the simplicial complexes considered are always symmetric and strictly positive.*

Björner also conjectured in [13] *that the unique matrix M_d satisfying:*

$$\begin{bmatrix} f_0 \\ \vdots \\ f_d \end{bmatrix} = M_d \begin{bmatrix} g_0 \\ \dots \\ g_d \end{bmatrix}$$

*is **totally non-negative**, i.e. all its minors are non-negative. This was later proved by M. Björklund and A. Engström in* [10] *and in a different way by S. R Gal in* [35].

Both results are then related to the study of the sequences obtained by multiplying certain partial Riordan matrices by sequences satisfying certain properties.

The results above yield a lot of open questions:

- *Which is the significance of those extended g- and γ-vectors if the h-vector is not symmetric? When is it positive, log-concave, unimodal,...? (open question 23).*
- *On the other hand, when is, in general, the image of a column vector multiplied by the left by a Riordan matrix symmetric or anti-symmetric? (open question 24). Some work has already been done in this sense in* [20].
- *How do Riordan matrices behave with respect to the Kruskal-Katona conditions for f-vectors (see theorem 5.2.5 and open question 25).*

5.4 Dehn-Sommervile Equations

The f-vectors of large classes of simplicial complexes satisfy the Dehn-Sommerville equations (see section 5.2 for a brief discussion of this).

Let K be a simplicial complex of dimension d, whose extended f-vector is $(f_{-1}, \dots, f_d)$ and whose h-vector is $(h_0, \dots, h_{d+1})$. As stated in theorem 5.2.6, the Dehn-Sommerville equations are nothing more than imposing the h-vector to be symmetric. The formulation in terms of the f-vector is more interesting for us:

Proposition 5.4.1 *The Dehn-Sommerville equations can be stated as an eigenvector problem:*

$$\begin{bmatrix} f_d \\ f_{d-1} \\ \vdots \\ f_{-1} \end{bmatrix} = \begin{bmatrix} \binom{d+1}{d+1} & & & \\ \binom{d+1}{d} & -\binom{d}{d} & & \\ \vdots & \vdots & \ddots & \\ \binom{d+1}{0} & -\binom{d}{0} & \dots & (-1)^{d+1}\binom{0}{0} \end{bmatrix} \begin{bmatrix} f_d \\ f_{d-1} \\ \vdots \\ f_{-1} \end{bmatrix} \tag{5.7}$$

The matrix above is the partial Riordan matrix $R_{d+1}\left((1+x)^{d+1}, -\frac{x}{1+x}\right)$, *which is an involultion.*

Proof: It follows directly from the formulation (5.3) of the Dehn-Sommerville equations (in terms of the f-vector).

□

Finding the sequences satisfying the Dehn-Sommerville equations in their formulation (5.3) or equivalently (5.7) was a problem already studied by M. Grunbaun in [39], section 9.5, but by using Riordan matrices is easy to find them:

Remark 5.4.2 *Since:*

$$R_{d+1}\left((1+x)^{d+1}, -\frac{x}{1+x}\right) \cdot R_{d+1}\left((1+x)^{\frac{d+1}{2}}, \frac{x}{\sqrt{1+x}}\right) =$$

$$= R_{d+1}\left((1+x)^{\frac{d+1}{2}}, \frac{x}{\sqrt{1+x}}\right) \cdot R_{d+1}(1, -x)$$

a sequence of real numbers $(f_{-1}, \ldots, f_d)$ *satisfies* (5.7) *if and only if it is a linear combination of the columns* $0, 2, \ldots$ *of the Riordan matrix* $R_{d+1}\left((1+x)^{\frac{d+1}{2}}, \frac{x}{\sqrt{1+x}}\right)$ *(which are obviously independent)or equivalently its generating polynomial is:*

$$Taylor_{d+1}\left((1+x)^{\frac{d+1}{2}} \cdot \Phi\left(x\frac{x}{\sqrt{1+x}}\right)\right)$$

for some even power series $\phi(x) \in \mathbb{K}[[x]]$.

Remark 5.4.3 *Studying when such a sequence satisfies Kruskal-Katona conditions is related to open question 26.*

In this case, for small cases at least, it would be easy to find a set of inequalities that describe the coefficients that we have to choose in the linear combination of the columns in the matrix of the previous remark in order to obtain a sequence satisfying the Kruskal-Katona conditions.

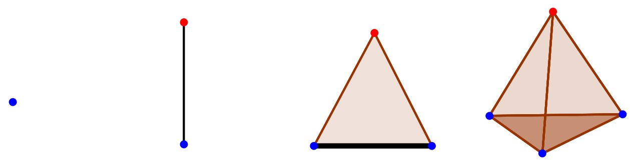
But the remark above gives another interpretation of this problem in terms of functional equations and is related to open question 27: what can we say about formal power series whose entries satisfy Kruskal-Katona conditions? Can we find any class of them?

5.5 Iterated Join of Simplicial Complexes as a Riordan pattern

The f-vectors of two well known families of simplicial complexes f-vectors exhibit a Riordan pattern:

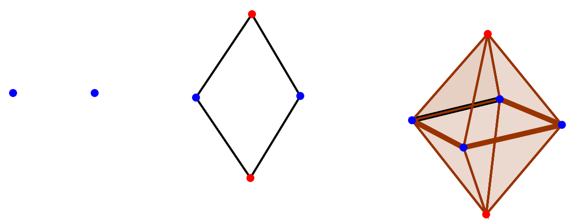
Proposition 5.5.1 (f-vectors of simplices and cross-polytopes)

(1) Let the matrix $(f_{ij})_{0\leq i,j} = R\left(\frac{1}{(1-x)^2}, \frac{x}{1-x}\right)$. *The* k*-row* $(f_{k0}, \ldots, f_{kk})$ *is the f-vector of the* k*-dimensional simplex. Moreover, from the matrix* $(a_{ij})_{0\leq i,j} = R\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ *if we consider the* $(k+1)$*-row* $(a_{k0}, \ldots, a_{k,k+1})$ *we obtain the extended f-vector of the* k*-dimensional simplex.*



$(1,0,0,0,\ldots)$ $(2,1,0,0,\ldots)$ $(3,3,1,0,\ldots)$ $(4,6,4,1,\ldots)$

(2) Let the matrix $(f_{ij})_{0\leq i,j} = R\left(\frac{2}{(1-x)^2}, \frac{2x}{1-x}\right)$. *The* k*-row* $(f_{k0}, \ldots, f_{kk})$ *is the f-vector of the* k*-dimensional cross polytope (see the definition in* [119] *or the definition in terms of joins below). Moreover, from the matrix* $(a_{ij})_{0\leq i,j} = R\left(\frac{1}{(1-x)}, \frac{2x}{1-x}\right)$ *if we consider the* $(k+1)$*-row* $(a_{k0}, \ldots, a_{k,k+1})$ *we obtain the extended f-vector of the* k*-dimensional cross polytope.*



$(2,0,0,0,\ldots)$ $(4,4,0,0,\ldots)$ $(6,12,8,0,\ldots)$

Proof: Let the simplex of dimension k with vertex set $V = \{v_0, \ldots, v_k\}$. Observed as an abstract simplicial complex is the family of subsets of V. The number of subsets of V of size $i+1$ is $\binom{k+1}{i+1}$. This completes the proof of (1), since $R\left(\frac{1}{1-x}, \frac{x}{1-x}\right)$ is then Pascal Triangle (which is our main example of Riordan matrix). The statement for the extended f-vector is a direct consequence of this.

With respect to (2), let $\Delta_2^{[0]}, \Delta_2^{[1]}, \Delta_2^{[2]}, \ldots$ be the cross polytopes of dimensions $0, 1, 2, \ldots$ We will proceed by induction. Let $(f_{ij})_{0 \leq i,j} = R\left(\frac{2}{(1-x)^2}, \frac{2x}{1-x}\right)$:

- $\Delta_2^{[2]}$ is the simplicial complex consisting in two single points so obviously its f-vector is the first row of
$$R\left(\frac{2}{(1-x)^2}, \frac{2x}{1-x}\right)$$
- Now suppose that $\Delta_2^{[n]}$ has f-vector $[f_{n0}, f_{n1}, \ldots, f_{nn}, 0, 0, \ldots]$ for $n \geq 0$. To obtain $\Delta_2^{[n+1]}$ we do the suspension of $\Delta_2^{[n]}$. When we do the suspension of a simplicial complex K, 2 new vertices appear (apart from those already in K) and 2 new faces of dimension $d+1$ appear for each face of dimension d in K (apart from those already in K). So, if:
$$f(\Delta_2^{[d+1]} = (F_0, F_1, \ldots, F_n, 0, 0, \ldots)$$
then $F_0 = f_{n,0} + 2$ and:
$$\forall d \geq 1, \qquad F_d = 2 \cdot f_{n,d-1} + 1 \cdot f_{n,d}$$
The result follows from the fact that the first column in $R\left(\frac{2}{(1-x)^2}, \frac{2x}{1-x}\right)$ is the arithmetic progression $2, 4, 6, 8, \ldots$ and that the A-sequence $R\left(\frac{2}{(1-x)^2}, \frac{2x}{1-x}\right)$ is precisely $(2, 1, 0, 0, \ldots)$ The statement for the extended f-vector in (2) follows directly from this.

$\square$

An immediate consequence of the above result, we have the following well known fact with a new proof by using the 1FTRM:

Corollary 5.5.2

(1) The Euler characteristic of the d-dimensional simplicial complex $\Delta^{[d]}$ *is* 1.

(2) The Euler characteristic of the d-dimensional cross polytope $\Delta_2^{[d]}$ *is* 2 *if* d *is even, 0 otherwise.*

Proof: We have already described the matrix which rows are the f-vectors of each family. Thus, to obtain the Euler characteristic of every complex in each family we only need to compute the product of the corresponding matrix by the column vector $\begin{bmatrix} 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix}$. And, as the generating function of this vector is $\frac{1}{1+x}$, we can use the 1FTRM to see that:

(1) $R\left(\frac{1}{(1-x)^2}, \frac{x}{1-x}\right) \otimes \frac{1}{1+x} = \frac{1}{1-x}$ which is the generating function of the column vector $\begin{bmatrix} 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix}$

(2) $R\left(\frac{2}{(1-x)^2}, \frac{2x}{1-x}\right) \otimes \frac{1}{1+x} = \frac{1}{1-x^2}$ which is the generating function of the column vector $\begin{bmatrix} 2 \\ 0 \\ 2 \\ \vdots \end{bmatrix}$

$\square$

The reason under the presence of a Riordan matrix for the f-vectors of those families of simplicial complex presented above is the iterative proccess that can be used to construct them.

Definition 5.5.3 *With more generality, given two simplicial complexes K and L with two sets of vertices V, W with empty intersection (we can obtain this re-labelling the vertices of one of those complexes) we define the* ***join*** *of K and L denoted by $K * L$ to be the simplicial complex given by the following rules:*

- *The vertex set of $K * L$ is $V \cup W$*
- *The simplex σ of dimension d is in $K * L$ if one of the following holds:*
  - *(a) $\sigma \in K$*
  - *(b) $\sigma \in L$*
  - *(c) $\sigma = \tau_K \cup \tau_L$ with $\tau_K \in K$, $\tau_L \in L$, in which case we denote it by $\sigma = \tau_K * \tau_L$.*

A geometric interpretation of the join is possible but we will not discuss it here (see again the book by J. Matousek [74]).

All the simplices can be obtained one from another by taking **cones** iteratively (the cone of K is the join of K with the simplicial complex consisting in a single point), starting from the 0-dimensional simplex (the simplicial complex consisting of a single point). Similarly, all the cross polytopes can be obtained one from another by taking **suspensions** iteratively (the suspension of K is the join of K with the simplicial complex consisting of two single points) and starting from the 0-dimensional sphere (which is the simplicial complex consisting of 2 single points).

Thus, note that:

Remark 5.5.4 *Let L be a simplicial complex. If we start from a given simplicial complex K and we define the simplicial complexes $K_L^{[0]}, K_L^{[1]}, K_L^{[2]}, \dots$ by the iterative proccess of joining described by:*

$$K_L^{[0]} = K, \qquad \forall i \geq 0,\ K_L^{[i+1]} = K_L^i * L$$

then the f-vector placed as rows follows a Riordan pattern in some sense. Since for any two simplicial complexes K, L where:

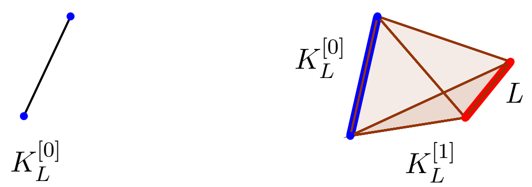
$$ef(K) = (f_{-1}, f_0, \dots), \qquad ef(L) = (f'_{-1}, f'_0, \dots)$$

the number of n dimensional faces of $K * L$ is:

$$F_n = f'_n \cdot f_{-1} + f'_{n-1} f_0 + \dots + f'_0 f_{n-1} + f'_{-1} f_n$$

the f-vectors $f(K_L^{[0]}), f(K_L^{[1]}), f(K_L^{[2]}), \dots$ have an A-sequence-like pattern.

For example if we consider K, L to be the simplicial complexes consisting of two vertices and the segment between them, then we define $f(K_L^{[0]}), f(K_L^{[1]}), f(K_L^{[2]}), \dots$ as above:



and finally we place the extended f-vectors $ef(K_L^{[0]}), ef(K_L^{[1]}), ef(K_L^{[2]}), \dots$ as the rows of a matrix what we obtain is the following (obviously not lower triangular) matrix:

$$\begin{bmatrix} 1 & 2 & 1 & & & & \\ 1 & 4 & 6 & 4 & 1 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

From the previous remark we easily deduce that although the matrices of f-vectors (placed as rows) of a family of simplicial complexes obtained by iterating the join with a fixed simplicial complex L, have certain pattern, they are not in general Riordan matrices. Determining the combinatorial meaning of those matrices and the combinatorial meaning of a Riordan matrix, if any, extending this one will be left as open question 28. If we want the corresponding matrix to be a Riordan matrix we need K_0 and L to be 0-dimensional. So we will define:

Definition 5.5.5 *The d-dimensional m, q-cone, which will be denoted by $\Delta_{m,q}^{[d]}$, is the simplicial complex obtained recursively as:*

- *$\Delta_{m,q}^{[0]}$ is the 0-dimensional simplicial complex consisting of m points.*

- *For $k \geq 0$, $\Delta_{m,q}^{[k+1]}$ is the join between $\Delta_{m,q}^{[k]}$ and the 0-dimensional simplex consisting of q points.*

So with this definition the family of $1,1$-cones is the family of the simplices. And the family of the $2,2$-cones is the family of the cross-polytopes. And now we have:

Proposition 5.5.6

$$(d_{ij})_{0\leq i,j} = R\left(\frac{m+(q-m)x}{(1-x)^2}, \frac{qx}{1-x}\right) = \begin{bmatrix} \leftarrow & f(\Delta_{m,q}^{[0]}) & \rightarrow \\ \leftarrow & f(\Delta_{m,q}^{[1]}) & \rightarrow \\ \leftarrow & f(\Delta_{m,q}^{[2]}) & \rightarrow \\ & \vdots & \end{bmatrix} \tag{5.8}$$

Moreover we have that:

(i) For all n, k:

$$d_{nk} = \left[q\binom{n}{k+1} + m\binom{n}{k}\right] q^k$$

(ii) The extended f-polynomial of $\Delta_{m,q}^{[n]}$ is $(mx+1)(qx+1)^n$.

(iii) The f-polynomial of $\Delta_{m,q}^{n}$ is the ***Jackson 0-derivative*** *of the extended f-polynomial and so is:*

$$\frac{(m+1)(qx+1)^n}{x}$$

Proof: If we prove (5.8), then we obtain immediately the statements (i),(ii)(iii) as a consequence. So we will only prove (i).

From the previous definition it is easy to see that the number of vertices of the m,q-cones are $m, m+q, m+2q, \dots$ respectively, which are the entries $d_{00}, d_{10}, d_{20}, \dots$. On the other hand, as we have already used before, since the m,q-cones are obtained by iterating a join with the space consisting of q points, for every k, d, $0 \leq d \leq k$, the number of d-dimensional faces of $\Delta_{m,q}^{[k+1]}$ is:

$$\#(d-\text{dimensional faces of } \Delta_{m,q}^{k})\#((d-1)-\text{dimensional faces of } \Delta_{m,q}^{k})$$

so:

$$d_{k+1,d} = q \cdot d_{k+1,d-1} + d_{k+1,d}$$

this proves that $(d_{ij})_{0\leq i,j}$ is a Riordan matrix (it has an A-sequence) and checking that it is in fact $R\left(\frac{m+(q-m)x}{(1-x)^2}, \frac{qx}{1-x}\right)$ follows from a direct computation.

□

Moreover, as we can change from f-vector to h-vector by doing a Riordan change of basis, not only the matrix from the f-vectors but the matrix from the h-vectors of the m, q-cones are a Riordan matrix and surprisingly it is known to us in this chapter:

Proposition 5.5.7 *For $m, q \geq 2$, the h-polynomial of $\Delta_{m,q}^{[n]}$ is the extended f-polynomial of $\Delta_{m-1,q-1}^{[n]}$. In other words, the matrix which rows are the h-vector of $\Delta_{m,q}^{[0]}, \Delta_{m,q}^{[1]}, \Delta_{m,q}^{[2]}, \dots$ is:*

$$R\left(\frac{(m-1)+(q-m)x}{(1-x)^2}, \frac{(q-1)x}{1-x}\right)$$

If $m = 1$ or $q = 1$, still the h-vector of $\Delta_{m,q}^{[n]}$ is:

$$\Big((m-1)x+1\Big)\Big((q-1)x+1\Big)^n$$

Proof: In proposition 5.5.6 is stated that the extended f-polynomial of $\Delta_{m,q}^{[n]}$ is $(mx+1)(qx+1)^n$. To obtain the h-polynomial we only need to multiply by the adequated Riordan matrix (proposition 5.3.1):

$$R_{n+1}\left((1-x)^{n+1}, \frac{x}{1-x}\right) \otimes \Big((mx+1)(qx+1)^n\Big) =$$

$$= Taylor_{n+1}\left[(1-x)^{n+1} \cdot \left(\left(\frac{mx}{1-x}+1\right)\left(\frac{qx}{1-x}+1\right)^n\right)\right] =$$

$$= \Big((m-1)x+1\Big)\Big((q-1)x+1\Big)^n$$

□

In [119] the h-vector is not introduced as a sequence containing the same information as the f-vector but as a sequence defining only for shellable simplicial complex and counting certain aspects of the shelling (the equivalence is proved later, showing also the independence of the chosen shelling). We leave open the question of trying to find some combinatorial meaning for the result above: is the h-vector counting something in the m, q-cones? (see open question 29).

To close this section and to show the utility of this description of the f-vectors as rows in a Riordan matrix in this setting of formal power series we will include two examples:

Example 5.5.8 *We can compute the Euler characteristic by using the 1FTRM in the following way:*

$$R\left(\frac{m+(q-m)x}{(1-x)^2}, \frac{qx}{1-x}\right)\begin{bmatrix} 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix} \cong R\left(\frac{m+(q-m)x}{(1-x)^2}, \frac{qx}{1-x}\right) \otimes \frac{1}{1+x} =$$

$$= \frac{m+(q-m)x}{1+(q-1)x} \cong \begin{bmatrix} m \\ (m+(q-m))(1-q) \\ (m+(q-m))(1-q)^2 \\ \vdots \end{bmatrix}$$

Example 5.5.9 *We can also compute the total number of faces (of any dimension) of the m,q-cones again by using the 1FTRM:*

$$R\left(\frac{m+(q-m)x}{(1-x)^2}, \frac{qx}{1-x}\right) \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{bmatrix} \cong R\left(\frac{m+(q-m)x}{(1-x)^2}, \frac{qx}{1-x}\right) \otimes \frac{1}{1-x} =$$

$$= \frac{m+(q-m)x}{1-(q+1)x} \cong \begin{bmatrix} m \\ (m+(q-m))(q+1) \\ (m+(q-m))(q+1)^2 \\ \vdots \end{bmatrix}$$

5.6 Subdivision methods and matrices in ILT_∞

By subdivision method we will mean a way of doing subdivisions making the same operation in all the simplices of the same dimension. This concept is rigurously defined in [15, 27].

Two subdivision methods will be used in this chapter:

Definition 5.6.1 *Let K be a (geometric) simplicial complex (an analogue definition is made for abstract simplicial complexes) with vertex set V and which collection of simplices is $K = \{\sigma_1, \ldots, \sigma_n\}$*

- ***Barycentric subdivision**: For every simplex:*

  $$\sigma_i = \{v_0^i, \ldots, v_{n_i}^i\} \in K$$

  let a_i be its barycenter, that is, the point:

  $$a_i = \sum_{j=0}^{n_i} \frac{1}{n+1} v_j^i$$

  The (first) barycentric subdivision is the simplicial complex $bs(K)$ which vertex set is:

  $$V' = V \cup \{a_i : 1 \leq i \leq n\}$$

  and such that, for every chain of simplices in K of the type $\sigma_{i_0} \subset \ldots \subset \sigma_{i_d}$, where σ_{i_j} is of dimension j, the simplex spanned by:

  $$\{a_{i_0}, \ldots, a_{i_d}\}$$

is in $bs(K)$.

*We will use the notation $bs^{(m)}(K)$ for the m-**th barycentric subdivision**, that is, $bs(bs(\ldots(K)\ldots))$*

- ***Simple stellar subdivision**: Let $\sigma_1, \ldots, \sigma_k$ be the facets of maximal dimension in K, let $a_1, \ldots, a_k$ be the correspondent barycenters. The (first) simple stellar subdivision is the simplicial complex $sss(K)$ which vertex set is:*

  $$V' = V \cup \{a_1, \ldots, a_k\}$$

  and which simplices are those elements not in $\{\sigma_1, \ldots, \sigma_k\}$, and for each facet $\sigma_i = \{v_{j_0}, \ldots, v_{j_d}\}$ of dimension d all the simplices (which are also facets) of the type:

  $$\{a_i, v_{m_0}, \ldots, v_{m_{d-1}}\}$$

  where a_i is the barycenter of σ_i and $v_{m_0}, \ldots, v_{m_{d-1}}$ span a face of diension $d-1$ of σ_i.

  We will use the notation $sss^{(m)}(K)$ for the m-th simple stellar subdivision, that is, $sss(sss(\ldots(K)\ldots))$.

The term "simple stellar subdivision" is not standard in the bibliography but we will use this name since it is related to the so called **stellar subdivisions** or **starring subdivisions** (see [85]) and this is why we have chosen this name.

It is very easy to check from the definition (in particular for those two) that if K is a simplicial complex of dimension d and K' is the simplicial complex obtained by a subdivision method then there exists a $(d+1)\times(d+1)$ lower triangular matrix M such that: $[\leftarrow f(K') \rightarrow] = [\leftarrow f(K) \rightarrow] M$

For example we have:

Proposition 5.6.2 *Let K be a simplicial complex of dimension d. We have that:*

(a) (see [15]) Let K' be the barycentric subdivision of K, then:

$$[\leftarrow f(K') \rightarrow] = [\leftarrow f(K) \rightarrow] \begin{bmatrix} \genfrac\{\}{0pt}{}{1}{1} & & & \\ \genfrac\{\}{0pt}{}{2}{1} & 2!\genfrac\{\}{0pt}{}{2}{2} & & \\ \genfrac\{\}{0pt}{}{3}{1} & 2!\genfrac\{\}{0pt}{}{3}{2} & 3!\genfrac\{\}{0pt}{}{3}{3} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

(b) Let K' be the simple stellar subdivision of K, then:

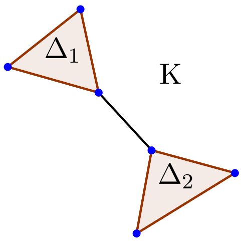
$$f(K') = f(K) + f_d(K)\left(\binom{d+1}{0}, \binom{d+1}{1}, \ldots, \binom{d+1}{d-1}, \binom{d+1}{d} - 1\right)$$

or equivalently:

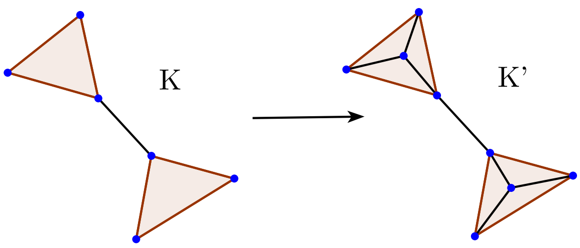
$$[\leftarrow f(K') \rightarrow] = [\leftarrow f(K) \rightarrow] \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ \vdots & \vdots & \ddots & & \\ 0 & & 0 & 1 & \\ \binom{d+1}{0} & \binom{d+1}{1} & & \binom{d+1}{d-1} & \binom{d+1}{d} \\ \vdots & \vdots & \vdots & \ddots & \end{bmatrix}$$

Proof: The first part is already proved in the bibliography proposed above. To prove (b) see that:

- Let $\Delta_1, \ldots, \Delta_r$ be the faces of dimension d in K (facets since K is of dimension d).



- We will replace Δ_i by the cone of the border (the simplicial complex consisting of the faces of Δ of dimension strictly less than d)



- This will be the new simplicial complex K' and it is by construction a subdivision of K and thus PL-homeomorphic to K.

If $f(K) = (f_0, \ldots, f_d)$, then:

$$f(K') = (f_0, \ldots, f_d) + f_d \cdot \left(\binom{d+1}{0}, \binom{d+1}{1}, \ldots, \binom{d+1}{d-1}, \binom{d+1}{d} - 1\right)$$

□

Characterising all possible matrices associated with subdivision methods in order to obtain results as the ones in the following sections could be very interesting and we will leave it for future work (open question 30).

5.7 Application: linear algebra and linear arithmetic relations for the f-vector

Definition 5.7.1 *Let $\mathcal{S}$ a family of simplicial complexes. A **linear arithmetic relation** $\mathcal{R}$ for the family S is the linear combination:*

$$\lambda_0 f_0 + \ldots + \lambda_n f_n + \ldots$$

that takes the same value for any element in $f(\mathcal{S})$ (this infinite sum can be taken, since any simplicial complex in this work must be finite dimensional).

The term “arithmetic relation” was already used by C. T. C. Wall in [113].

Determining the set of linear relation that must be satisfied by the f-vectors of a family of simplicial complexes $\mathcal{S}$ is a weaker version of the f-vector problem that has also been partially studied. As we announced before, most of the families of interest are defined by a topological property (for example $\mathcal{S}$ being the family of simplicial complexes homeomorhpic to a sphere). So in most of the cases the family $\mathcal{S}$ is closed under homeomorphisms or PL-homeomorphism, that is:

$$K \in \mathcal{S}, L \text{ homeomorphic or PL-homeomorphic to } K \Rightarrow L \in S$$

or under homotopy or simple homotopy or simple homotopy equivalence (analogous definition).

Definition 5.7.2 *Consider the linear combination $\mathcal{R}$ on the entries for the f-vector given by:*

$$\lambda_0 f_0 + \ldots + \lambda_n f_n + \ldots$$

For any simplicial complex, consider $\mathcal{S}_K$ to be the family of simplicial complexes consisting of any simplicial complex PL-homeomorphic to K (we could do an analogous definition for other relations, such as simple homotopy equivalence). If $\mathcal{R}$ satisfies the property:

$$\forall K, \quad \mathcal{R} \text{ is a linear arithmetic relation for } S_K$$

*we say that $\mathcal{S}$ is a **linear arithmetic PL-topological invariant**.*

This framework allows us to prove certain classical problems like the one below in a different way (this result was first proved by W. Mayer in [75] and more recently proved in different ways in [32, 61, 97]):

Proposition 5.7.3 (W. Mayer, [75]**)** *There is no other linear arithmetic PL-topological invariant, apart from the Euler characteristic and its multiples, which is also a simple-homotopy invariant.*

Proof: Take the family $\mathcal{S}$ formed by all the simplices (which are of the same simple-homotopy type). We have already proved (proposition 5.5.1) that if we put the f-vectors of the simplices ase rows we obtain the matrix $R\left(\frac{x}{(1-x)^2}, \frac{x}{1-x}\right)$. So for any linear arithmetic invariant for the f-vector of the family $\mathcal{S}$:

$$\lambda_0 f_0 + \ldots + \lambda_k f_k + \ldots$$

we have that:

$$R\left(\frac{1}{(1-x)^2}, \frac{x}{1-x}\right) \otimes u(x) = \frac{c}{1-x}, \qquad \text{where } c \in \mathbb{R},\ u(x) = \lambda_0 + \lambda_1 x + \lambda_2 x^2 + \ldots$$

Thus, we can see that:

$$u(x) = \left(R\left(\frac{1}{(1-x)^2}, \frac{x}{1-x}\right)\right)^{-1} \otimes \frac{c}{1-x} = R\left(\frac{1}{(1+x)^2}, \frac{x}{1+x}\right) \otimes \frac{c}{1-x} = \frac{c}{1+x}$$

□

By using exactly the same argument we can prove the following result for one of the families of simplicial complexes appearing in this chapter:

Proposition 5.7.4 *Consider now $\mathcal{S}$ to be the family consisting of the m, q-cones of any dimension (for m, q fixed). There is a unique (up to multiples) linear arithmetical relation satisfied for all the f-vectors of elements in $\mathcal{S}$, which is:*

$$\frac{1}{q} \cdot f_0 - \frac{1}{q^2} \cdot f_1 + \frac{1}{q^3} \cdot f_2 - \frac{1}{q^4} \cdot f_3 + \ldots \qquad \textit{(and their multiples)}$$

We will omit the proof.

Other proofs relying on Riordan matrices of results of this type are possible. For instance we have the following result. The first part improves the previous result and the second one explains in some sense why Dehn-Sommerville equations are so special.

Proposition 5.7.5 *Let $\mathcal{S}$ be PL-topologically closed family of simplicial complexes:*

(i) If $\mathcal{S}$ has at least one element of each dimension up to d with $1 \leq d \leq \infty$. Then, the unique linear arithmetical invariant that can hold for any element in $\mathcal{S}$ is the Euler characteristic and their multiples (if it does).

(ii) Note that the Dehn-Sommerville equations are not linear arithmetic relation in the sense we have defined here but they can be considered linear arithmetic relations when all the dimensions of the complexes considered are of the same parity. If $\mathcal{S}$ has at least one element of each dimension k even up to $1 \leq d \leq \infty$ (we can suppose then that d is even) the unique linear relation that can hold for any element in $\mathcal{S}$ are those from the Dehn-Sommerville equations (and their multiples) that can be stated as (compare with (5.7)*):*

$$[1, f_0, \ldots, f_d]\Big[R_{d+1}\left(-\frac{1}{1+x}, -\frac{x}{1+x}\right) - R_{d+1}(1, x)\Big] = [0, 0, \ldots, 0] \tag{5.9}$$

or equivalently, if we want to consider them as linear arithmetical relations, they are the linear combination obtained by multiplying $[f_0, f_1, \ldots, f_d]$ by the column vector:
(5.10)

$$\begin{bmatrix} \uparrow \\ Tylr_d\left(\frac{1}{1+x}\right) \\ \downarrow \end{bmatrix}, \begin{bmatrix} \uparrow \\ Tylr_d\left(\frac{1}{(1+x)^2} - 1\right) \\ \downarrow \end{bmatrix}, \ldots, \begin{bmatrix} \uparrow \\ Tylr_d\left((-1)^{d+1}\frac{x^{d-1}}{(1+x)^{d+1}} - x^{d-1}\right) \\ \downarrow \end{bmatrix}$$

If we are considering simplicial complexes of dimension less or equal than d, linear arithmetical relations are considered as linear combinations of the type:

$$\lambda_0 \cdot f_0 + \ldots + \lambda_d \cdot f_d \text{ instead of } \lambda_0 \cdot f_0 + \ldots + \lambda_n \cdot f_n + \ldots$$

and it is in this framework when we talk about "uniqueness" in this result.

Proof: If $\mathcal{S}$ is PL-topologically closed, then if $\mathbb{K} \in \mathcal{S}$ and L is PL-homeomorphic to K, $L \in S$. So for any K of dimension d we will define K' to be its simple stellar subdivision. According to proposition 5.6.2, if $f(K) = (f_0, \ldots, f_d)$, then:

$$f(K') = (f_0, \ldots, f_d) + f_d \cdot \left(\binom{d+1}{0}, \binom{d+1}{1}, \ldots, \binom{d+1}{d-1}, \binom{d+1}{d} - 1\right)$$

So if an arithmetic linear relation with coefficients $(\lambda_0, \ldots, \lambda_d)$ holds for K, K' then:

$$\Big[f_0, \ldots, f_d\Big]\begin{bmatrix}\lambda_0 \\ \vdots \\ \lambda_d\end{bmatrix} = \Big[(f_0, \ldots, f_d) + f_d \cdot \left(\binom{d+1}{0}, \ldots, \binom{d+1}{d-1}, \binom{d+1}{d} - 1\right)\Big]\begin{bmatrix}\lambda_0 \\ \vdots \\ \lambda_d\end{bmatrix}$$

or equivalently:

$$\left(\binom{d+1}{0},\dots,\binom{d+1}{d-1},\binom{d+1}{d}-1\right)\begin{bmatrix}\lambda_0\\ \vdots\\ \lambda_d\end{bmatrix}=0 \tag{5.11}$$

So:

(i) If we have a PL-topologically closed family S with an element of each dimension up to d (d could be ∞) and there is an arithmetic linear relation $\mathcal{R}$ for $\mathcal{S}$ with coefficients $(\lambda_0,\dots,\lambda_d)$, we have that by applying the reasoning above for different values of d, we obtain equations of type (5.11) that we can put in a system:

$$\left[R_{d+1}\left(\frac{1}{1-x},\frac{x}{1-x}\right)-R_{d+1}(1+x,x)\right]\begin{bmatrix}\lambda_0\\ \vdots\\ \lambda_d\\ 0\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\\ 0\end{bmatrix}$$

or equivalently:

$$R_{d+1}\left(\frac{1}{1-x^2},\frac{x}{1-x}\right)\begin{bmatrix}\lambda_0\\ \vdots\\ \lambda_d\\ 0\end{bmatrix}=\begin{bmatrix}\lambda_0\\ \vdots\\ \lambda_d\\ 0\end{bmatrix}$$

and it is easy to see that the only eigenvector of $R_{d+1}\left(\frac{1}{1-x^2},\frac{x}{1-x}\right)$ is $[1,-1,1,\dots,-1,0]$ and their multiples.

(ii) If we have a PL-topologically closed family S with an element of each even (the same for odd) dimension up to d (thus we may assume that d is even) by doing the same reasoning we have a system of $\frac{d}{2}$ linearly independent equations in $d+1$ variables:

$$\begin{cases}\left(\binom{1}{0},0,\dots,0\right)\begin{bmatrix}\lambda_0\\ \vdots\\ \lambda_d\end{bmatrix}=\lambda_0\\ \quad\vdots\\ \left(\binom{d+1}{0},\dots,\binom{d+1}{d-1},\binom{d+1}{d}\right)\begin{bmatrix}\lambda_0\\ \vdots\\ \lambda_d\end{bmatrix}=\lambda_d\end{cases} \tag{5.12}$$

thus, the solution is a linear subspace of rank $\frac{d}{2}+1$ of $\mathbb{R}^{d+1}$.

It was already proved in [39], page 146, that the linear space spanned by those columns has dimension $\frac{d}{2}+1$. All of those columns must be a solution (since Dehn-Sommerville equations hold for the family of simplicial polytopes of even dimension, which is obviously closed under the subdivision proposed above). So they are a basis for the linear subspace of solutions.

□

Remark 5.7.6 *It is possible to show explicitely that each of the columns (skipping the first row) of* (5.10) *satisfies* (5.12)*.*

Consider the following system of equations:

$$\left[R_{d+1}(1,-x)\cdot R_{d+1}\left(\frac{1}{1-x},\frac{x}{1-x}\right)-R_{d+1}(1+x,x)\right]\begin{bmatrix}\lambda_0\\ \vdots\\ \lambda_d\\ 0\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 0\\ 0\end{bmatrix}$$

or equivalently:

$$R_{d+1}\left(\frac{1}{(1+x)^2},-\frac{x}{1+x}\right)\begin{bmatrix}\lambda_0\\ \vdots\\ \lambda_d\\ 0\end{bmatrix}=\begin{bmatrix}\lambda_0\\ \vdots\\ \lambda_d\\ 0\end{bmatrix} \tag{5.13}$$

All the equations in (5.12) *are also equations in* (5.13)*, so the linear space of solutions of* (5.13) *must be contained in the linear space of solutions of* (5.12)*.*

Now, it is easy to check that all those columns are in fact eigenvectors of eigenvalue 1 of $R_{d+1}\left(\frac{1}{(1+x)^2},-\frac{x}{1+x}\right)$*:*

- $R_{d+1}\left(\frac{1}{(1+x)^2},-\frac{x}{1+x}\right)\otimes\left(\frac{1}{1+x}-x^{d+1}\right)=\left(\frac{1}{1+x}-x^{d+1}\right)$
- *For* $k\geq 1$*:*

$$R_{d+1}\left(\frac{1}{(1+x)^2},-\frac{x}{1+x}\right)\otimes Tylr_d\left((-1)^{k+1}\cdot\frac{x^{k-1}}{(1+x)^{k+1}}-x^{k-1}\right)=$$

$$=Tylr_d\left((-1)^{k+1}\cdot\frac{x^{k-1}}{(1+x)^{k+1}}-x^{k-1}\right)$$

A consequence of the second part of the previous result is that no other linear arithmetical relations hold for shellable simplicial complexes apart from the Dehn-Sommerville ones, which was already stated in [39], section 9.2.

Another subdivision method will appear in this section, together with the unique infinite lower triangular matrix neither being a Riordan matrix nor an element in the tangent algebra $\mathcal{L}(\mathcal{R})$ that will appear in this work.

Barycentric subdivision has already been introduced in this chapter together with a previous result describing the f-vector of the simplicial complex K' obtained by performing a barycentric subdivision on another simplicial complex K, in terms of the f-vector of K.

This framework of matrices in ILT_∞ allow us to prove the following result:

Proposition 5.7.7 *The invariance of the Euler characteristic by succesive barycentric subdivisions is not a topology matter in the sense that we do not need any topological argument to prove.*

It is a consequence of the fact that $\begin{bmatrix} 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix}$ *is an eigenvector of eigenvalue 1 of the matrix:*

$$B = \begin{bmatrix} \left\{ {1 \atop 1} \right\} & & & \\ \left\{ {2 \atop 1} \right\} & 2!\left\{ {2 \atop 2} \right\} & & \\ \left\{ {3 \atop 1} \right\} & 2!\left\{ {3 \atop 2} \right\} & 3!\left\{ {3 \atop 3} \right\} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Proof: In this proof, all the f-vectors are placed as rows. First of all note that according to proposition 5.6.2 if K' is the baricentric subdivision of K, then:

$$f(K') = f(K) \cdot B$$

So if $\begin{bmatrix} 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix}$ is an eigenvector of eigenvalue 1 of B we have that:

$$f(K') \begin{bmatrix} 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix} = f(K) \begin{bmatrix} 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix}$$

To see that it is this way:

- Note that $B = DS$ where:

$$D = \begin{bmatrix} 1! & & & \\ 0 & 2! & & \\ 0 & 0 & 3! & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \qquad S = \begin{bmatrix} \left\{ {1 \atop 1} \right\} & & & \\ \left\{ {2 \atop 1} \right\} & \left\{ {2 \atop 2} \right\} & & \\ \left\{ {3 \atop 1} \right\} & \left\{ {3 \atop 2} \right\} & \left\{ {3 \atop 3} \right\} & \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

- From [38], pages 259-264 we have that:

$$S^{-1} = \left((-1)^{i-j} \begin{bmatrix} i+1 \\ j+1 \end{bmatrix} \right)_{i,j<\infty}$$

  where $\begin{bmatrix} m \\ n \end{bmatrix}$ denotes the corresponding Stirling number of the first kind.

- Since for all $i \geq 1$ $\begin{bmatrix} i \\ 0 \end{bmatrix}$ and $\sum_{j=0}^{i} \begin{bmatrix} i \\ 0 \end{bmatrix} = i!$ we have that:

$$S^{-1} \begin{bmatrix} 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix} = D \begin{bmatrix} 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix}$$

  which completes the proof.

□

Obviously, the matrix described above is not a Riordan matrix but still it has its own structure that will be interesting to clarify: computing more eigenvectors, giving an analogue of the 1FTRM,...(see open question 31).

5.8 Betti numbers of the m, q-cones

Betti numbers of the m, q-cones: a topologically generated Riordan matrix

The proccess of making the joint between a simplicial complex with another one made of q points to obtain the m, q-cones, not only induces a Riordan matrix pattern in the f-vector but in the sequences of Betti numbers.

In general, we have that:

Lemma 5.8.1 *Let K be a simplicial complex of dimension d homotopically equivalent to a wedge of spheres and with reduced Betti sequence $(\widetilde{\beta}_0, \dots, \widetilde{\beta}_d)$. Let L be the 0-dimensional simplicial complex consisting in q points with $q > 1$. The joint $K * L$ is homotopically equivalent to a wedge of spheres wich reduced Betti sequence is* $(0, (q-1)\widetilde{\beta}_0, \dots, (q-1)\widetilde{\beta}_d)$

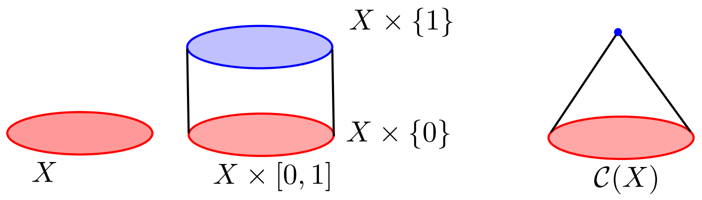
Proof:

(i) Firstly, given a topological space X, we will define the q-cone of X, denoted by $\mathcal{C}_q(X)$ as the space obtained gluing through the base q copies of the cone $C(X)$, that is:

  - For all $1 \leq i \leq q$ define the cone:

$$C_i(X) = \mathcal{C}(X) = (X \times I)/\sim$$

    where $\sim$ is the equivalence relation given by $(x_1, 1) \cong (x_2, 1)$, $\forall x_1, x_2 \in X$.



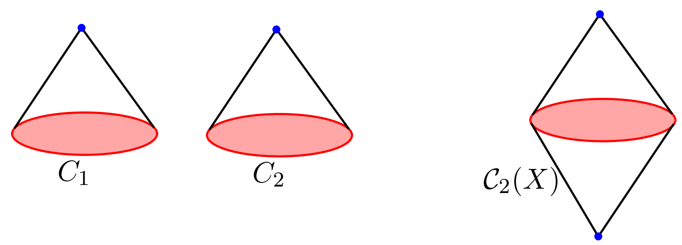
- Then define $\mathcal{C}_q(X)$ as:

$$(C_1(X) \cup \ldots \cup C_q(X))/\sim'$$

where $\sim'$ is the equivalence relation given for all $0 \leq i, j \leq q$ by:

$$[(x_i, 0)] \sim' [(x_j, 0)]$$

where $[(x_k, 0)] \in C_k(X)$ (so we must use this notation of equivalence class). For example, for $q = 2$:



(ii) Let K be a simplicial complex homotopically equivalent to the a wedge of spheres X. Then $\mathcal{C}_q(K)$ is homotopically equivalent to $\mathcal{C}_q(X)$

- As well as we can speak about the cone and suspension of a map, we can speak about the ***q*-cone** of a map. Given a map between two topological spaces $f : X \to Y$ we can define:

$$\mathcal{C}_q(f) : \mathcal{C}_q(X) \to \mathcal{C}_q(Y)$$

defining a map:

$$\widetilde{\mathcal{C}_q(f)} : C_1(X) \cup \ldots \cup C_q(X) \longrightarrow C_1(Y) \cup \ldots \cup C_q(Y)$$

$$[(x,t)] \in C_i(X) \longmapsto [(f(x), t)] \in C_i(Y)$$

that behaves well with respect to the equivalence relation $\sim'$ and then projecting into the quotient which is well defined in relation to $\sim'$.

- Now assume that there is an homotopy equivalence $f : K \to X$, that is, there is another map $g : X \to K$ such that $f \circ g$ and $g \circ f$ are homotopic to the identity maps in K and in X respectively, with respect to the homotopies H_1, H_2 respectively.

- We will show that $\mathcal{C}_q(f)$ is an homotopy equivalence (with respect to the also homotopy equivalence $\mathcal{C}_q(g)$). To see this, we construct the map:

$$\widetilde{H}_1 : \mathcal{C}(X) \times [0,1] \to \mathcal{C}(X)$$

defining the map:

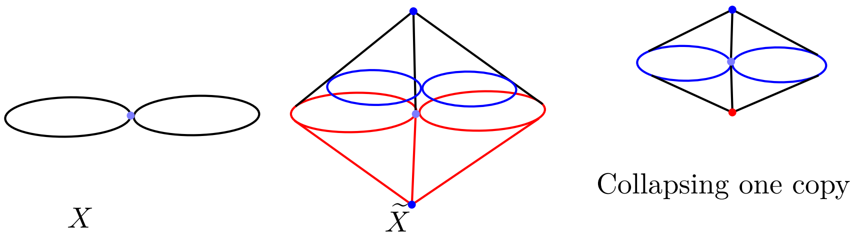
$$\widehat{H}_1 : (C_1(X) \cup \ldots \cup C_q(X)) \times [0,1] \to \mathcal{C}_1(X) \cup \ldots \cup C_q(X)$$

$$([(x,t)], s) \longmapsto (H_1([x,s]), t) \quad \text{for } [(x,t)] \in C_i(X)$$

and then projecting to the quotient. This map $\widetilde{H}_1$ is an homotopy between $\mathcal{C}_q(f) \circ \mathcal{C}_q(g)$ and the identity in $\mathcal{C}_q(X)$.

- The same argument applies to construct the other homotopy needed.

(iii) Since each of the copies $C_1, \ldots, C_q$ (that are glued through the base) is contractible, we can collapse one of them. We then obtained the wedge of $(q-1)$ suspensions of X.



□

So we again can considered a matrix having as rows the reduced Betti sequences of the m, q-cones and this is what we get:

Proposition 5.8.2 *If we define the matrix whose rows are the reduced Betti sequences of* $\Delta^0_{m,q}, \Delta^1_{m,q}, \Delta^2_{m,q}, \ldots$ *we obtain the diagonal matrix* $R(m-1, (q-1)x)$

Proof: $\Delta^0_{m,q}$ has as reduced Betti sequence $((m-1), 0, 0, \ldots)$, since it has m connected components. The rest of this result follows from the previous result.

□

5.9 New building blocks to build complexes: q-simplices. q-Euler characteristic

In this section we will only consider q,q-cones. We will denote the d-dimensional q,q-cone by $\Delta_q^{[d]}$.

Due to the similarities in construction between simplices and q,q-cones we may realize the following:

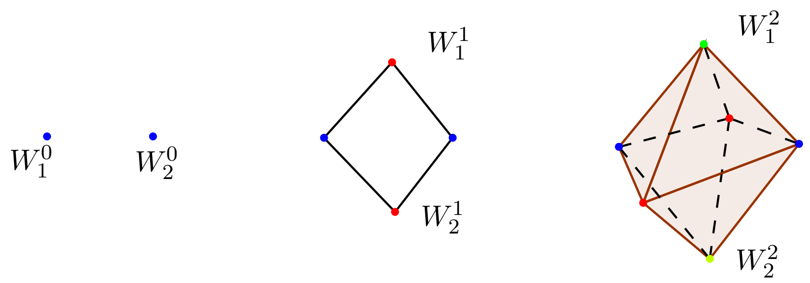
Remark 5.9.1 *Consider $\Delta_q^{[d]}$ and $\Delta^{[d]}$ that are constructed iteratively by making a join operation. There is a natural simplicial map:*

$$\varphi : \Delta_q^{[d]} \to \Delta^{[d]}$$

which is q to 1 restricted to the set of vertices, and is defined in the following way:

- *Label the points in $\Delta^{[d]}$ as $\{V_0, \ldots, V_d\}$*
- *Since $\Delta_q^{[d]}$ is obtained iteratively making a join, label the starting points $W_1^1, \ldots, W_q^1$. Label the new vertices after the first join $W_1^1, \ldots W_q^1$. And so on until labelling the vertices introduced in the last join as $W_1^d, \ldots, W_q^d$.*

  For example, if $q = 2$, $d = 3$:



- *Now the simplicial map is defined imposing that:*

$$\forall 1 \leq i \leq q, \forall 0 \leq k \leq d, \quad W_i^k \longmapsto V_k$$

  that is, in the previous example, by mapping all the vertices of the same colour to the same vertex of $\Delta^{[2]}$.

Under the previous equivalence see that:

Lemma 5.9.2 *In the notation of the previous remark, the pre-image by ϕ any proper face of dimension k in $\Delta^{[d]}$ is isomorphic to $\Delta_q^{[k]}$.*

Proof: By using the notation in the previous remark we only need to see that, in the notation of the previous remark, $\Delta^{[d]}$ is the simplicial complex whose simplices are or the type:

$$\{W_{i_0}^{j_0}, \ldots, W_{i_k}^{j_k}\} \qquad \textit{with } j_0, \ldots, j_k \textit{ distinct}$$

□

This structural similarity between q, q-cones and simplices, allow us to use them as "building blocks" in an analogous way as it is usually done with simplices:

Proposition 5.9.3 *Given a simplicial complex K, we can define another simplicial complex K_q, called the q-**widening** of K_q, replacing every k-dimensional simplicial complex in K by a copy of $\Delta_q^{[k]}$.*

Proof: It is a direct consequence of lemma 5.9.2. Perform the following reasoning in every facet of K:

- Let L be a facet of dimension d of K, and then also a copy of $\Delta^{[d]}$.
- L must be replaced by a copy of $\Delta_q^{[d]}$.
- We only need to check that this replacement is compatible with all the faces of L, that is, that every face of L of dimension k is replaced by a copy of $\Delta_q^{[k]}$. And this is a consequence of lemma 5.9.2.

□

Finding families of simplicial complexes of every dimension with this property of being suitable to be used as building blocks might be an interesting question that also have applications in Chemistry (see open question 32).

Remark 5.9.4 *Note that if a simplicial complex K_q is the q-widening of any other simplicial complex K, this simplicial complex has in some sense a double structure.*

Deciding whether a given simplicial complex is the q-widening of any other simplicial complex, is still an open question for us (see open question 33). This problem is similar to the one of determining if a given simplicial complex can be viewed as the subdivision of another one or not (hauptvermutung).

Given a simplicial complex K it is easy to obtain the f-vector of the q-widening K_q:

Proposition 5.9.5 *Let K be a simplicial complex with f-polynomial $p(x)$ and let K_q be its q-widening with f-polynomial $p_q(x)$. Then:*

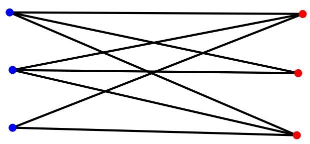
$$p_q(x) = R(q, qx) \otimes p(x)$$

Proof: Each simplicial complex of dimension d is spanned by $q+1$ vertices. For each of those vertex V_i we have q vertices $W_1^i, \ldots, W_q^i$ in the q-widening of the simplex.

It is easy to see that the number of faces of dimension d in $\Delta_q^{[d]}$ is q^{d+1} and it comes from the fact that there is a face of dimension d for any possible combination of vertices of the type:

$$\{W_{i_1}^0, \ldots, \ldots W_{i_q}^d\}$$

For example, see the following picture of $\Delta_3^{[1]}$



□

Thanks to the previous result, every linear arithmetic relation that holds for a simplicial complex K is automatically translated into another linear arithmetic relation that holds for its q-widening K_q. For instance:

Remark 5.9.6 *This yields an alternative proof for proposition 5.7.4. Since by transposition is easy to see that:*

$$[\leftarrow f(\Delta_q^{[d]}) \rightarrow] = [\leftarrow f(\Delta^{[d]}) \rightarrow] R(q, qx)$$

then necessarily:

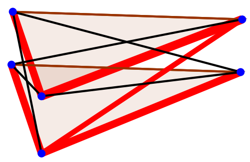
$$[\leftarrow f(\Delta_q^{[d]}) \rightarrow] \left((R(q, qx))^{-1} \begin{bmatrix} 1 \\ -1 \\ 1 \\ \vdots \end{bmatrix} \right)$$

It is still an open question for us to determine the action of this q-widening proccess on the Betti sequence (see open question 34) although we have already described the Betti sequence of $\Delta_q^{[d]}$. For 1-dimensional simplicial complexes (graphs) this problem is easy to solve but it does not seem to be for higher dimensions:

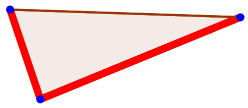
Proposition 5.9.7 *Let G be a graph with v vertices, e edges and such that the rank of the fundamental group of G is r. Then the rank of the q-widening of G is:*

$$r' = eq - qv + 1 = (eq + r - 1)(q - 1) + r$$

Proof: G must have a **minimum spanning tree** T with t edges.
Then the q-widening of G has a minimum spanning tree T' with $q \cdot t + (q - 1)$ edges. To see this, think as q copies of the spanning tree joined by "stairs" between each "floor":



$The\ q - widening\ with\ an\ spanning\ tree$



$K\ with\ an\ spanning\ tree$

We omit the details showing that this is actually an spanning tree.

□

Open Questions

In this last chapter, as an annex, we will include some questions that have been left open throughout this work.

Some of them may be easy to solve and some of them may not. Most of them have been left aside only for reasons of time, so they may be answered with little effort. On the other hand, some of those questions have a very concrete statement, and some others have a more vague formulation.

Those questions have been artificially grouped according to the chapter which they belong.

From Chapter 0

Open question 1 *The main diagonal of any Riordan matrix is a geometric progression. The diagonal under this one is a geometric-arithmetic progression.*

What can we say about the sequences in the rest of diagonals of Riordan matrices? As far as we know, R. Sprugnoli has been working in some results about this.

From Chapter 1

Open question 2 *In* [66] *the balls with respect to the ultrametric proposed are shown to form a nested sequence of group which is described in this article.*

A better study of the balls with respect to the new ultrametric proposed in section 1.5 would be desirable.

One of the problem is that the natural way to describe balls in this ultrametric is by using banded matrices, which are neither elements in the Riordan group nor a group.

Open question 3 *In chapter 1 a nice description of the involutions in $\mathcal{R}$ is given. A similar description for self-dual and self-complementary matrices would be desirable.*

Open question 4 *Studying the linearisation problem in the case of $\mathbb{K}$ being an infinite field of positive characteristic.*

Open question 5 *Studying the main case of the weighted Schröder equation in the cases not studied in section 1.14.*

Open question 6 *In chapter 9.5 A of the book by M. Kuczma , B. Chocewski and R. Ger* [59] *some problems are studied about systems of functional equations involving Schroder equations. What can we say with the methods introduced in chapter 1 about this kind of problems?*

Open question 7 *Which other functional equations, apart from the Schröder, weighted Schröder and Babbage equations, are suitable or interesting to be studied for formal power series by using those techniques?*

Open question 8 *It would be very interesting to study the existence of solutions of the functional equations studied in this work (Schröder, weighted Schröder and Babbage) for formal power series in* $\mathbb{Z}[[x]]$ *for combinatorial reasons. Specially for the Schröder and weighted Schröder equations due to the consequences it would have when studying the conjugacy.*

The behaviour is quite different. For example, let $h(x) = -x + x^2$*. Equation* (1.11) *does not have a solution for* $\lambda = g'(0)$, $g(x) \in \left(x + x^2\mathbb{Z}[[x]]\right)$*. Even in the partial case, we have that:*

$$\begin{bmatrix} 1 & & \\ 0 & -1 & \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ g_2 \end{bmatrix} = -1 \cdot \begin{bmatrix} 0 \\ 1 \\ g_2 \end{bmatrix}$$

yields $g_2 = -\frac{1}{2}$.

A general solution would be desirable for those equations in this different setting.

From Chapter 2

Open question 9 *For* $\mathbb{K}$ *a non-finite field of positive characteristic, a description of the derived series of* $\mathcal{A}(\mathbb{K})$ *(and then of* $\mathcal{F}_C(\mathbb{K})$*) and of* $\mathcal{R}(\mathbb{K})$ *is still unknown up to our knowledge.*

Open question 10 *For the groups* $\mathcal{C}(R)$ *and* $\mathcal{F}_1(R)$ *for* R *being a unitary ring a description of the derived series is unknown. It would be specially interesting for* $R = \mathbb{Z}$*, because of its applications in combinatorics.*

Open question 11 *Once we have described the derived series of the Riordan group, we could study the quotients:*

$$\mathcal{R}/\mathcal{R}', \mathcal{R}'/\mathcal{R}'', \ldots$$

We have already given a good description of the first one of those quotients and we have some ideas about the rest although we are not going to develop this here.

The same applies for $\mathcal{A}$ *and its derived series.*

Open question 12 *Finishing the study of conjugacy of elements in* $\mathcal{R}(\mathbb{K})$ *for* $\mathbb{K} = \mathbb{C}$ *and other fields of characteristic 0.*

Open question 13 *Studying conjugation in* $\mathcal{R}$, $\mathcal{A}$ *for infinite fields of positive characteristic.*

Open question 14 *Studying conjugation in $C(R)$, $\mathcal{F}_1(R)$ for R being an unitary ring.*

At least studying the conjugacy class in the group $\mathcal{C}(\mathbb{Z})$ (and even in the associated subgroup) of the Pascal Triangle would be interesting.

For example we can see that $R_3(1, x + x^2 + x^3)$ is the only element in the same conjugacy class of $R_3\left(1, \frac{x}{1-x}\right)$ in the associated subgroup of $\mathcal{C}_3(\mathbb{Z})$. If we want to solve:

$$v(h(x)) = h\left(\frac{x}{1-x}\right)$$

Take $R(1, h(x)) = (x_{ij})$, $R(1, v(x)) = (v_{ij})$. So this equation is equivalent to:

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & x_{21} & 1 & & \\ 0 & x_{31} & x_{32} & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ v_{21} \\ v_{31} \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & \binom{1}{0} & 1 & & \\ 0 & \binom{2}{0} & \binom{2}{1} & 1 & \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ x_{21} \\ x_{31} \\ \vdots \end{bmatrix}$$

See that:

$$\begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & x_{21} & 1 & \\ 0 & x_{31} & 2x_{21} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ v_{21} \\ v_{31} \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & \binom{1}{0} & 1 & \\ 0 & \binom{2}{0} & \binom{2}{1} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ x_{21} \\ x_{31} \end{bmatrix}$$

holds if and only if $v_{21} = v_{31} = 1$. In this case, it holds independently of the choice of x_{21}, x_{31}.

Open question 15 *Studying powers and roots of Riordan matrices and elements in $\mathcal{A}$, due to its relation to fractional iterates and other interesting problems.*

Open question 16 *Finding the centralizer of any element in $\mathcal{R}$, $\mathcal{A}$.*

From Chapter 3

Open question 17 *It would be desirable to find a better description of the coefficients of the power series satisfying the Babbage and the generalized Babbage equations.*

Open question 18 *For any solution $\omega(x)$ of the generalized Babbage equation it is possible to find a canonical $g(x)$ such that:*

$$\omega(x) = g(\lambda g^{-1}(x))$$

Can those $g(x)$ be chosen in a subgroup of $\mathcal{A}$?

From Chapter 4

Open question 19 *It would be desirable to characterise one-parameter subgroups in the Riordan group or even in the partial Riordan groups.*

Open question 20 *Looking at conjugacy in the Lie algebra in the sense of studying the existence of a matrix $P \in \mathcal{R}$ satisfying:*

$$L' = ALA^{-1}$$

for two given $L, L' \in \mathcal{L}(\mathcal{R})$ may be of great interest. It could help to finish our study of conjugacy in $\mathcal{R}$. See remark 4.15.3.

Open question 21 *It would be very interesting, due to its applications in combinatorics, to study supergroups of the Riordan group. For instance, which is the group generated by $\mathcal{R}$ and $\mathcal{L}(\mathcal{R})$? Do its element have an interpretation, analogous to the 1FTM, when multiplied by infinite column vectors?*

Open question 22 *Is it computationally interesting our way to compute the solutions of the initial value problems? In general, is it computationally interesting our way to compute solutions for functional equations?*

From Chapter 5

Open question 23 *Which is the significance of those extended g- and γ-vectors proposed in chapter 5 if the h-vector is not symmetric? When is it positive, log-concave, unimodal,...? Is it counting something? Does it have a combinatorial meaning?*

Open question 24 *On the other hand, when is, in general, the image of a column vector multiplied by the left by a Riordan matrix symmetric or anti-symmetric? or concave or log-concave?. Some work has already been done in this sense in* [20]*. It is very related to the Björner matrix and to Björner's partial unimodality theorem (theorem 8.39 in* [119]*).*

Open question 25 *How do Riordan matrices behave with respect to the Kruskal-Katona conditions for f-vectors (see theorem 5.2.5). Can we characterize when the image of an infinite row vector by a Riordan matrix satisfies those conditions? This is related to the McMullen correspondence (see* [119]*) and the Björner's partial unimodality theorem.*

Open question 26 *When does a linear combination of sequences satisfy the Kruskal Katona sequence? This question is posted in relation to the set of possible sequences satisfying Dehn-Sommerville equations and also in relation to the study of this set made in* [39]*.*

Open question 27 *What can be said about formal power series which coefficients satisfy Kruskal-Katona conditions? Is there any algebraic property on the formal power series related to Kruskal-Katona conditions?*

Open question 28 *The matrices obtained by iterating a join exhibit a Riordan pattern. What can we say about them? Can they be extended to Riordan matrices?*

Open question 29 *The matrix of h-vectors of the* m, q*-cones is the matrix of f-vector of the* $(m-1), (q-1)$*-cones. Why is this happening? Does the h-vector of the m,q-cones have any combinatorial meaning?*

Open question 30 *Is it possible to characterise the matrices associated to a subdivision method? This would be really interesting, since it would allow us to use linear algebraic tools for the f-vector problem.*

Open question 31 *The matrix appearing in proposition 5.6.2 is not Riordan but still has certain interesting pattern. Does this matrix have any property similar to those of Riordan matrices?*

Open question 32 *How should it be a family of simplicial complexes* $K_0, K_1, K_2, \ldots$ *where* K_i *is of dimension i if we want them to be used as building blocks? are* q, q*-cones such a family. Could we find another one?*

Some problems in Chemist are related to this (molecular nets), where we are considering some structures which vertices are, in fact, molecules themselves.

Open question 33 *When is a simplicial complex the q-widening of any other?*

Open question 34 *Which are the homology groups, or even the Betti sequence, of the q-widening of a given simplicial complex?*

Open question 35 *Claryfying the behaviour of the operators* ∂_k *(for a fixed* k *as functions in* n*, or varying the value* k*) could be of great importance. For example a simple question that may arise is: what could we say about the values n such that:*

$$\partial_k(n) = \partial_k(n+1)$$

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