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Algorithmic Resolution of Singularities and Nash multiplicity sequences

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Contents

Venimos de la larga noche
Los invisibles, hombres y mujeres libres
Por nuestros abuelos...
T. Mejías

Somos la alegría que regresa,
el día de la furia en primavera.
La vida fue un ensayo hasta ahora.
I. Serrano

Introduction: summary and conclusions

Singularities are object of study in many areas of Mathematics. They usually entail a difficulty for many results which are in general valid when singularities do not appear. From a geometric point of view, the singular points of a variety are those where the dimension of the tangent space is greater than the dimension of the variety itself. From an algebraic point of view, singular points correspond to multiple roots of polynomials. In Commutative Algebra, singular points correspond to non-regular local rings. An algebraic variety is called *singular* if it has singular points.

The problem of *Resolution of Singularities* inquires whether a singular variety can be approximated in some way by a non-singular one. More precisely, given an algebraic variety defined over a field k , by a resolution of singularities of X we mean a proper and birational morphism

$$X \xleftarrow{\pi} X', \tag{1}$$

where X' is a nonsingular variety. It is often asked also that π defines an isomorphism outside of the singular locus of X:

$$X \setminus \mathrm{Sing}(X) \cong X' \setminus \pi^{-1}(\mathrm{Sing}(X)).$$

The problem of Resolution of Singularities consists then on deciding whether such a morphism can be found for any singular variety X. A positive answer to this question would open the door to extending many results of algebraic geometry, which are only known for nonsingular varieties. But in addition it would enable the proof of some results, for instance, in motivic integration or positivity. As an example, some Lojasiewicz-type inequalities are proven via resolution of singularities.

It is known that a resolution of singularities can be found whenever X is defined over a field of characteristic zero. This is a theorem due to H. Hironaka [41]. When it comes to fields of positive characteristic, some partial results are known (due to S. Abhyankar, J. Lipman, V. Cossart-O. Piltant, A. Benito-O. Villamayor or H. Kawanoue-K. Matsuki among others), but the general case is still an open problem.

The answer that Hironaka gave to the problem in characteristic zero is that a resolution of the singularities can be found for any variety X, and that it can be defined as a sequence of blow ups at certain smooth closed centers:

$$X \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_r} X_r. \tag{2}$$

However, the proof of Hironaka is existential: it does not give a procedure to define such a sequence. After his result, other approaches have appeared, some of them being constructive, such as those from O. Villamayor, [77], [78], E. Bierstone-P. Milman [9]. See also S. Encinas-O. Villamayor [33], S. Encinas-H. Hauser [32], J. Włodarczyk [84] and J. Kollár [57].

Constructive Resolution of Singularities pursues the design of an algorithm which, for any X, determines univocally the construction of a birational map as in (1), given by a sequence of blow ups carefully chosen, as in (2). The algorithm must be able to choose, for each variety X, a smooth closed subset $Y \subset X$ which is the best center to blow up, according to some established criterion, oriented to concatenate blow ups which lead, eventually, to a resolution of the singularities of the present variety.

For the design of such an algorithm, we use *invariants* attached to the points of X. These invariants must distinguish between different kinds of singularities. Their study is already interesting for the design of the algorithm, but furthermore, they may also give some insight into the resolution phenomenon, in order to solve the problem for more general fields. Common invariants for this task are the Hilbert-Samuel function and the multiplicity.

The *multiplicity* of X at a point $\eta \in X$ is given by an upper semicontinuous function (see [22]):

$$\begin{aligned} \operatorname{mult}(X) : X &\longrightarrow \mathbb{N} \\ \eta &\longmapsto \operatorname{mult}(X)(\eta) := \operatorname{mult}(\mathcal{O}_{X,\eta}), \end{aligned}$$

where $\operatorname{mult}(\mathcal{O}_{X,\eta})$ stands for the multiplicity of the local ring $\mathcal{O}_{X,\eta}$ at the maximal ideal. In the particular case in which X is defined as the zeroset of a polynomial f in the affine space, the multiplicity of X at the origin is the order of f.

Since the multiplicity function is upper semicontinuous, it defines a stratification of X into locally closed sets

$$Z_m = \{\eta \in X : \operatorname{mult}(X)(\eta) = m\} \subset X.$$

This stratification is an example of how invariants distinguish between different singular points of X. For instance, the singular points of X are given by the closed set $\bigcup_{m\geq 2} Z_m$.

The problem of Resolution of Singularities is one motivation for the definition of invariants of singular points of varieties, and it is also connected to many other approaches to the study of singularities: from algebra, geometry or topology, for instance. However, the study of singularities is also interesting from the point of view, for example, of the classification of varieties.

Arc spaces are also oriented to the study of singularities of algebraic varieties. They have shown themselves useful in the study of geometrical and topological properties of varieties, as one can see in the works of Denef-Loeser, Ein, Ishii, Mustaţă, Reguera and Yasuda among many others.

The main motivation of this thesis is the study of arcs from the point of view of Constructive Resolution of Singularities. We have investigated possible connections between invariants of singularities that are given in terms of the arc spaces of varieties, and the information that one can use for defining algorithms of resolution. Let us introduce now the actors involved in this study, before stating the main results.

Arcs and singularities

Suppose that X is an algebraic variety over a field k. An *arc* (a K-arc) φ in X centered at a point $\xi \in X$ is a morphism

$$\varphi : \mathrm{Spec}(K[[t]]) \longrightarrow X$$

for some field $K \supset k$, mapping the closed point of $\mathrm{Spec}(K[[t]])$ to ξ. If $X = \mathrm{Spec}(B)$ is an affine variety over a field k, an arc can be regarded as a homomorphism of rings

$$\varphi^* : B \longrightarrow K[[t]].$$

We call $\varphi(\langle t \rangle) \in X$ the *center* of the arc φ. If $K = k$, then a K-arc in X centered at a closed point $\xi \in X$ describes a germ of a curve inside of X containing ξ.

The *arc space* of a variety X over a field k is a scheme (not of finite type) representing the functor from k-schemes to sets given by

$$Y \overset{\mathcal{A}}{\longmapsto} \mathrm{Hom}_k\left(Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[[t]]), X\right),$$

whose K-points, for a field $K \supset k$, are the K-arcs in X ([8]). The arc space of X can be constructed as the inverse limit of the schemes of m-jets of X for $m \in \mathbb{N}$.

There is a strong connection between arc (and jet) spaces and Hasse-Schmidt derivations. This is certainly useful to understand how one can give equations defining arc spaces. There is also a relation between arcs and valuations: all arcs in X define a valuation in a certain subvariety of X, and any valuation on the field of fractions of $\mathcal{O}_X$ gives an arc in X. This relation is also a motivation for the study of arc spaces and is a key fact, for instance, for the Nash problem ([69]).

Many authors have contributed to the understanding of arc and jet spaces by studying their structure, properties, connection with singularities, etc., see for instance [56], [38], [60], [26], [52], [45], [73], [49], [50], [59], [24].

Some invariants of varieties defined through their spaces of arcs and of jets have already been studied (see for instance [26], [30], [76], [25], [29], [51]), but here we focus our research on the definition of invariants which can be connected to Constructive Resolution of Singularities.

Our main object of study, framed in the context of arc spaces, is the *Nash multiplicity sequence*. Given a variety X defined over a field k and given an arc φ in X, the Nash multiplicity of φ is a non-increasing sequence

$$m_0 \geq m_1 \geq \ldots \geq m_l = m_{l+1} = \ldots \geq 1$$

of positive integers attached to the center of φ (which is a point in X). This sequence can be regarded as a refinement of the multiplicity of X at $\xi = \varphi(\langle t \rangle)$: it is, in some sense (see Remark 2.7.3), the multiplicity of X at ξ along the direction given by φ.

The Nash multiplicity sequence was first defined for arcs in germs of hypersurfaces by M. Lejeune-Jalabert in [58], and later generalized by M. Hickel in [40] for arbitrary codimension. It can be constructed as follows: Let us assume, for simplicity, that $X = \mathrm{Spec}(B)$ is affine, let ξ be a point in X, and let φ be an arc in X centered at ξ. Consider the graph of φ,

$$\Gamma_0^* = \varphi^* \otimes i : B \otimes K[t] \to K[[t]],$$

which is additionally an arc in $X_0 = X \times \mathbb{A}^1$ centered at the point $\xi_0 = (\xi, 0) \in X_0$. These elements determine completely a sequence of blow ups at points:

$$\begin{array}{ccccccccc}
\mathrm{Spec}(K[[t]]) & & \xrightarrow{\Gamma_1} & & & \xrightarrow{\Gamma_l} & & & \\
\downarrow \Gamma_0 & & & & & & & & \\
X_0 = X \times \mathbb{A}^1 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & \ldots & \xleftarrow{\pi_l} & X_l & & \ldots \\
\xi_0 = (\xi, 0) & & \xi_1 & & \ldots & & \xi_l & & \ldots
\end{array} \qquad (3)$$

Here, π_i is the blow up of X_{i-1} at ξ_{i-1}, where $\xi_i = \mathrm{Im}(\Gamma_i) \cap \pi_i^{-1}(\xi_{i-1})$ for $i = 1, \ldots, l, \ldots$, and Γ_i is the (unique) arc in X_i centered at ξ_i which is obtained by lifting Γ_0 via the proper morphism $\pi_i \circ \ldots \circ \pi_1$. The element m_i of the Nash multiplicity sequence corresponds to the multiplicity of X_i at ξ_i for each $i = 0, \ldots, l, \ldots$. Note that m_0 is nothing but the multiplicity of X at ξ.

We will refer to a sequence of blow ups as in (3) as the sequence of blow ups *directed by* φ. Note that, before the first blow up, X is multiplied by an affine line. Assuming that X is a singular variety, this implies that X_0 has non-isolated singularities. Note also that in (3) we are only blowing up closed points of X_0. Hence, a sequence of blow ups directed by an arc in X can never define a resolution of the singularities of X_0. Moreover, the maximum multiplicity cannot decrease along the sequence either, because the multiplicity function is upper semicontinuous (and it cannot increase either, see [22]). Still, if we choose ξ such that $\mathrm{mult}(X)(\xi) = m_0 > 1$, and φ is not contained in the stratum of multiplicity greater than or equal to m_0 of X:

$$\cup_{i \geq m_0} Z_i = \{\eta \in X : \mathrm{mult}(X)(\eta) \geq m_0\},$$

the Nash multiplicity sequence will eventually decrease (see [58]): indeed, if the generic point of φ is contained in the stratum of multiplicity m_i of X, the Nash multiplicity sequence will stabilize at the value m_i. The reason for this phenomenon is that at some step, say r, the center ξ_r of the transform Γ_r of the graph Γ_0 in X_r is no longer contained in the stratum of multiplicity m_0 of X_r:

$$(Z_r)_{m_0} = \{\eta \in X_r : \mathrm{mult}(X_r)(\eta) = m_0\}.$$

If we choose $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X) = \{\eta \in X : \mathrm{mult}(X)(\eta) = \max \mathrm{mult}(X)\}$, where $\max \mathrm{mult}(X)$ is the highest multiplicity in X, then $(Z_r)_{m_0}$ is in fact the subset $\underline{\mathrm{Max}}\,\mathrm{mult}(X_r)$ because of the previous discussion.

In [58], M. Lejeune-Jalabert defines the Nash multiplicity sequence of an arc in a germ of a hypersurface relating it to the understanding and the computation of Artin's β funcion. In the last part of Section 2.7, we explain roughly the idea behind the sequence from this point of view.

Invariants from arcs

For our work, we will be interested in considering those arcs whose center is a point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, but whose generic point $\varphi(\langle 0\rangle)$ is not contained in this subset. The latter condition guarantees that their Nash multiplicity sequence is not constant. It is reasonable to think of a notion of contact of an arc φ with $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, based on how many blow ups directed by φ it takes to separate φ from this subset.

Given a variety X, a point $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$ and an arc φ in X centered at ξ, we define the *persistance of* φ *in* $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, and denote it by $\rho_{X,\varphi}$, as the number of blow ups as in (3) which must be performed before the Nash multiplicity sequence decreases for the first time (see Definition 3.1.1). Whenever the generic point of φ is not contained in $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, the persistance of φ is a natural number:

$$\rho_{X,\varphi} = \min_{i\in\mathbb{N}} \{m_i < m_0\}.$$

Both, the Nash multiplicity sequence and the persistance of φ are invariants of (X, φ, ξ). If we consider the minimum of the $\rho_{X,\varphi}$ for all arcs φ in X centered at ξ, this is an invariant for (X, ξ). It turns out that these invariants are strongly related to constructive resolution, in a way that we will specify later. To study them, we use what we call local presentations for the multiplicity and Rees algebras.

We also construct here another invariant, which turns out to be a refinement of $\rho_{X,\varphi}$, and which we call the *order of contact* of φ with $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ and denote by

$$r_{X,\varphi} \in \mathbb{Q}_{\geq 1}.$$

This invariant is computed as the order of a certain Rees algebra (see Definition 3.2.16), and it turns out that

$$\rho_{X,\varphi} = [r_{X,\varphi}].$$

In principle, none of these invariants have any relation with any resolution of singularities of X.

The main tools used for the definition of $r_{X,\varphi}$ and the conclusions concerning it are Rees algebras and their resolution, which have been widely developed by O. Villamayor, S. Encinas, A. Bravo, A. Benito, R. Blanco, M. L. García-Escamilla and C. Abad in [34], [36], [80], [5], [10], [13], [16], [82], [1], [2].

The *order of an arc* $\varphi^* : \mathcal{O}_{X,\xi} \longrightarrow K[[t]]$ is the largest positive integer n such that $\varphi^*(\mathcal{M}_\xi) \subset (t^n)$, where $\mathcal{M}_\xi \subset \mathcal{O}_{X,\xi}$ is the maximal ideal. The quotient

$$\bar{r}_{X,\varphi} = \frac{r_{X,\varphi}}{\mathrm{ord}(\varphi)}, \tag{4}$$

which is also an invariant, is sometimes more interesting than the invariant $r_{X,\varphi}$, because it avoids the influence of the order of the arc. For instance, it assigns the same value to different parametrizations of the same germ of curve.

Constructive resolution and invariants

A *constructive resolution* is an algorithm that chooses, for any variety X, a closed subvariety $Y \subset X$ to be the center of a blow up that will, after iterating the process, lead to a resolution of the singularities of X (see [34]). The choice of Y is given by an upper semi-continuous function F defined on varieties

$$F(X) = F_X : X \longrightarrow (\Lambda, \geq) \tag{5}$$

whose highest value, max F_X, determines a closed smooth subset

$$\underline{\mathrm{Max}}\, F_X := \{\xi \in X : F(\xi) = \max F_X\} \subset X.$$

The subset $\underline{\mathrm{Max}}\, F_X$ will be the center of the first blow up $\pi_1 : X' \longrightarrow X$ in the construction of a resolution of singularities of X. After this blow up, a new function

$$F'_X := F_{X'} : X' \longrightarrow (\Lambda, \geq)$$

may be defined, satisfying

$$\begin{aligned} &F'_X(\pi_1^{-1}(\xi)) = F_X(\xi) && \text{if } \xi \in X \setminus Z \\ &F'_X(\xi') < F_X(\xi) && \text{if } \xi = \pi_1(\xi') \in Z. \end{aligned}$$

Then $\underline{\mathrm{Max}}\, F'_X$ will be the center of the second blow up $\pi_2 : X'' \longrightarrow X'$, and the process can be iterated: we define an upper semicontinuous function $F_X^{(i)} := F_{X^{(i)}}$ for each $i \geq 1$, and $\underline{\mathrm{Max}}\, F_X^{(i)} \subset X^{(i)}$ will be the center of the blow up $\pi_{i+1} : X^{(i+1)} \longrightarrow X^{(i)}$. In addition, each F^i may be constructed in a way such that it is constant if and only if X^i is smooth. In that case, a resolution of the singularities of X is achieved after finitely many iterations. We call these F^i the *resolution functions*. To construct resolution functions we use *invariants*. The mission of an invariant of singularities is to assign a value to each singular point, so that different singular points can be compared through this value. One way to define resolution functions is to assign to each point ξ a string of invariants $F^i(\xi)$. Following the methods in

[82], we will take the multiplicity as the first coordinate of this string. This means that we will keep our attention on the points of X where the multiplicity function reaches its highest value, since those are the points for which the first coordinate of the resolution function will be maximum. We write:

$$\begin{aligned}\underline{\operatorname{Max}}\ \operatorname{mult}(X) := \{\eta \in X : \operatorname{mult}(X)(\eta) \geq \max \operatorname{mult}(X)\} = \\ = \{\eta \in X : \operatorname{mult}(X)(\eta) = \max \operatorname{mult}(X)\}\,,\end{aligned}$$

which is a closed subset of X. However, this set is not necessarily smooth, so it cannot be chosen as the center of the first blow up. Hence, we need to add more invariants to construct the resolution function. As the second invariant, we will use $\operatorname{ord}_\xi^{(d)}(X)$, *Hironaka's order in dimension* $d = \dim(X)$ (see [15], and also [2]). The invariant $\operatorname{ord}_\xi^{(d)}(X)$ will be the first one in our resolution functions which can distinguish between points of maximum multiplicity of X.

Before stating the main results, let us give an impression of what this order in dimension d means, introducing at the same time one of the fundamental tools involved in this thesis: Rees algebras.

Local presentations and Rees algebras

When one is interested in studying the worst singularities of a given variety X, it is useful to have some equations describing them as a subset of some smooth scheme V. Assume that we are given an upper semicontinuous function F, and that we want to keep track of how the subset $\underline{\operatorname{Max}}\ F_X \subset X$ where F_X reaches its highest value behaves under a sequence of blow ups

$$X = X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_l} X_l$$

with "good" centers. For certain functions F, this can be done locally, and we call it a *local presentation*: for $\xi \in \underline{\operatorname{Max}}\ F_X$, a local presentation of X for F at ξ consists of a local (étale) immersion $X \hookrightarrow V$ for some smooth scheme V, a set of elements $\{f_1, \ldots, f_r\} \subset \mathcal{O}_V$ and weights $\{n_1, \ldots, n_r\} \subset \mathbb{N}$ such that, in a neighborhood of ξ:

- The closed subset

  $$\underline{\operatorname{Max}}\ F_X = \{\eta \in X : F_X(\eta) = \max F_X\} \subset X \subset V$$

  equals

  $$\{\eta \in V : \nu_\eta(f_i) \geq n_i, i = 1, \ldots, r\} \subset V,$$

  where $\nu_\eta(f_i)$ denotes the order of f_i in the local regular ring $\mathcal{O}_{V,\eta}$;

- There is a transformation rule for the f_i so that the previous condition on equality of sets is preserved by any sequence of blow ups:

  $$\begin{aligned}V = V_0 \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_l} V_l \\ X = X_0 \longleftarrow X_1 \longleftarrow \ldots \longleftarrow X_l,\end{aligned}$$

  where the center of π_i is a smooth closed subset $Y_{i-1} \subset \underline{\operatorname{Max}}\ F_{X_{i-1}}$ for $i = 1, \ldots, l$, as long as $\max F_X = \max F_{X_1} = \ldots = \max F_{X_{l-1}}$.

In the particular case in which F is the multiplicity function, this is possible (see [82]).

Rees algebras are a convenient tool to manipulate local presentations, and see how sequences of blow ups transform them. A *Rees algebra* over a regular Noetherian ring R, or over $V = \mathrm{Spec}(R)$, is a finitely generated R-algebra

$$\mathcal{G} = \oplus_{l \in \mathbb{N}} I_l W^l \subset R[W],$$

for some ideals $I_l \subset R$, satisfying $I_0 = R$ and $I_l I_j \subset I_{l+j}$. A Rees algebra over V defines a closed subset of V in a very natural way, which we refer to as the *singular locus of* $\mathcal{G}$:

$$\mathrm{Sing}(\mathcal{G}) = \{\eta \in V : \nu_\eta(f) \geq n \text{ for any } fW^n \in \mathcal{G}\}.$$

Assume that we are given a local presentation for the multiplicity of a variety X at a point $\xi \in \underline{\mathrm{Max}}\ \mathrm{mult}(X)$:

$$X \hookrightarrow V,\ \{f_1, \dots, f_r\} \subset \mathcal{O}_{V,\xi},\ \{n_1, \dots, n_r\} \subset \mathbb{N},$$

for some smooth V, with $\dim(V) = n > \dim(X)$. Then, we can attach the Rees algebra

$$\mathcal{G}_X = \mathcal{O}_{V,\xi}[f_1 W^{n_1}, \dots, f_r W^{n_r}] \tag{6}$$

to the multiplicity of X locally in an (étale) neighborhood of ξ. The singular locus of $\mathcal{G}_X$ will be exactly the subset $\underline{\mathrm{Max}}\ \mathrm{mult}(X) \subset X \subset V$. Rees algebras extend to sheaves of Rees algebras in the obvious way.

In the line of the transformations of local presentations by blowing up, there is a notion of transformation of Rees algebras. Given a Rees algebra $\mathcal{G}$ over V and a blow up $V' \longrightarrow V$ with center a regular closed subset $Y \subset \mathrm{Sing}(\mathcal{G})$, a new Rees algebra $\mathcal{G}'$ can be defined over V'. Moreover, there is a notion of *resolution of Rees algebras*, meaning a sequence of blow ups

$$\begin{array}{c} V_0 \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_l} V_l \\ \mathcal{G}_0 \longleftarrow \mathcal{G}_1 \longleftarrow \dots \longleftarrow \mathcal{G}_l, \end{array}$$

where π_i is a blow up at a regular closed center $Y_{i-1} \subset \mathrm{Sing}(\mathcal{G}_{i-1})$ for $i = 1, \dots, l$, and such that $\mathrm{Sing}(\mathcal{G}_l) = \emptyset$.

A resolution of a Rees algebra attached (as in 6) to the multiplicity of a variety X over k induces a sequence of blow ups in X that leads to a simplification of the multiplicity of X.

By means of Rees algebras attached to the multiplicity of a variety X at a point $\xi \in \underline{\mathrm{Max}}\ \mathrm{mult}(X)$, we can define an invariant:

$$\mathrm{ord}_\xi(\mathcal{G}) := \inf_{fW^n \in \mathcal{G}} \left\{ \frac{\nu_\xi(f)}{n} \right\}.$$

This is the most important invariant for the construction of a resolution of $\mathcal{G}$.

It can be proven that, if $\dim(X) = d$ and $\mathcal{G}_X$ is a Rees algebra over V attached to the multiplicity of X locally in an (étale) neighborhood of X, there exists a Rees algebra $\mathcal{G}_X^{(d)}$ over a smooth scheme $V^{(d)}$ of dimension d such that finding a resolution of the algebra $\mathcal{G}_X$ in dimension $n > d$ is somehow equivalent to finding a resolution of $\mathcal{G}_X^{(d)}$. Then, Hironaka's order in dimension d can be computed as the order of this Rees algebra:

$$\operatorname{ord}_\xi^{(d)}(X) := \operatorname{ord}_\xi^{(d)}(\mathcal{G}_X^{(d)}).$$

This invariant is intrinsic to the variety and does not depend on the choice of the Rees algebra (see [16], [2]).

Main results

Fixed a variety X over a field k of characteristic zero and a point $\xi \in \underline{\operatorname{Max}} \operatorname{mult}(X)$, one can consider the set

$$\Phi_{X,\xi} = \{\bar{r}_{X,\varphi}\}_\varphi , \tag{7}$$

where φ runs over all arcs in X centered at ξ which are not totally contained in $\underline{\operatorname{Max}} \operatorname{mult}(X)$ (see (4)). This set is a new invariant of X at ξ. It can be proven that $\Phi_{X,\xi}$ has a mimimum (which is again an invariant of X at ξ). Moreover:

Theorem 1. *Let X be a variety of dimension d over a field k of characteristic zero. Let ξ be a point in $\underline{\operatorname{Max}} \operatorname{mult}(X)$. Then,*

$$\operatorname{ord}_\xi^{(d)}(X) = \min(\Phi_{X,\xi}).$$

This means that one can read the invariant $\operatorname{ord}_\xi^{(d)}(X)$ in the space of arcs of X. Indeed, given $\varphi : \operatorname{Spec}(K[[t]]) \longrightarrow X$, centered at ξ, one can consider the family of arcs given as $\varphi_n = \varphi \circ i_n$ for $i > 1$, where $i_n^* : K[[t]] \longrightarrow K[[t^n]]$ maps t to t^n. It can be proven (see Corollary 4.3.4) that

$$\bar{r}_{X,\varphi} = \frac{1}{\operatorname{ord}(\varphi)} \cdot \lim_{n\to\infty} \frac{\rho_{X,\varphi_n}}{n},$$

and hence

$$\operatorname{ord}_\xi^{(d)}(X) = \inf_\varphi \left(\frac{1}{\operatorname{ord}(\varphi)} \cdot \lim_{n\to\infty} \frac{\rho_{X,\varphi_n}}{n} \right),$$

where φ runs over all arcs in X centered at ξ which are not contained in $\underline{\operatorname{Max}} \operatorname{mult}(X)$. This shows that the invariant $\operatorname{ord}_\xi^{(d)}(X)$ not only is independent of the choice of a particular Rees algebra attached to the multiplicity of X locally in a neighborhood of ξ as it was said already, but also does not need Rees algebras to be defined. It is certainly intrinsic to the variety since (as the formula shows) it can be expressed in terms of the space of arcs of X, and in some sense a natural invariant to be considered.

Additionally, the invariant $\bar{r}_{X,\varphi}$ suggests a classification of arcs in X centered at ξ according to their order of contact with $\underline{\operatorname{Max}} \operatorname{mult}(X)$. Among all arcs in X

centered at ξ, those with minimum $\bar{r}_{X,\varphi}$ give us the invariant $\operatorname{ord}_{\xi}^{(d)}(X)$, and also are separated from the subset $\underline{\operatorname{Max}}\operatorname{mult}(X)$ faster than the rest, via the sequence of blow ups as in (3).

A further investigation of the invariant $\Phi_{X,\xi}$ has given a criterion based on it to decide whether a point in the subset $\underline{\operatorname{Max}}\operatorname{mult}(X)$ is isolated in this set or not:

Theorem 2. *Let X be a variety over a field k of characteristic zero, and let ξ be a point in $\underline{\operatorname{Max}}\operatorname{mult}(X)$. Then ξ is an isolated point of $\underline{\operatorname{Max}}\operatorname{mult}(X)$ if and only if the set $\Phi_{X,\xi}$ is upper bounded.*

This has a consequence in terms of the Nash multiplicity sequence: under the hypotheses from the theorem, ξ is an isolated point of $\underline{\operatorname{Max}}\operatorname{mult}(X)$ if and only if one can find an upper bound, $D(X,\xi)$, such that for any arc φ in X centered at ξ and not totally contained in $\underline{\operatorname{Max}}\operatorname{mult}(X)$, the number of blow ups which are needed before the Nash multiplicity sequence decreases for the first time (normalized by $\operatorname{ord}(\varphi)$), is at most $D(X,\xi)$. In some sense this means that, if ξ is contained in a component of $\underline{\operatorname{Max}}\operatorname{mult}(X)$ of dimension at least 1, then the contact of the arcs centered at ξ with this set can be arbitrarily strong, while for isolated points of $\underline{\operatorname{Max}}\operatorname{mult}(X)$ this contact is somehow limited.

Moreover, under some conditions on X and ξ, we can compute the supremum of the set $\Phi_{X,\xi}$. These conditions involve another resolution invariant: the τ *invariant* (see [5]). If the n-dimensional Rees algebra $\mathcal{G}_X$ has maximum τ invariant at ξ, that is, $\tau_{\mathcal{G}_X,\xi} = n-1$, then it can be proven that finding a resolution of $\mathcal{G}_X$ is equivalent to finding a resolution of a 1-dimensional algebra $\mathcal{G}_X^{(1)}$. In this case, the invariant $\operatorname{ord}_{\xi}^{(d)}(X)$ gives no interesting information for the constructive resolution of X if $d>1$. The most interesting one is $\operatorname{ord}_{\xi}^{(1)}(X) := \operatorname{ord}_{\xi}(\mathcal{G}_X^{(1)})$. We have the following result:

Proposition 3. *If X has maximum τ invariant at ξ, then:*

$$\sup(\Phi_{X,\xi}) = \operatorname{ord}_{\xi}^{(1)}(X).$$

The results of this thesis are collected in:

- Bravo, A. and Encinas, S. and Pascual-Escudero, B., *Nash multiplicities and resolution invariants*, Collectanea Mathematica **68** (2017), 2, 175–217;
- Pascual-Escudero, B., *Nash multiplicities and isolated points of maximal multiplicity*, arXiv:1609.09008 [math.AG].

Contents

Along the first two chapters, we shall give the preliminary concepts and results needed for the development of our work, which will be exposed in the last three chapters.

The first chapter starts with a brief introduction to the problem of Resolution of Singularities. We introduce afterwards the notion of multiplicity, which will be fundamental along the whole work. The rest of the chapter is dedicated to developing the basics on Rees algebras, one of the main tools that we will use. We expose there all the concepts and results related to Rees algebras that will be necessary later, and finally show their connection with constructive resolution and the invariant $\mathrm{ord}_{\xi}^{(d)} X$.

The second chapter is devoted to arc spaces and their relation with singularities. We introduce the schemes of arcs and m-jets of an algebraic variety, and illustrate their construction by means of Hasse-Schmidt derivations. We will also show some properties and results regarding the structure of arc and m-jet spaces, specially those related to the singularities of varieties, as well as the relation of arcs and valuations. This chapter includes also the definition of the Nash multiplicity sequence.

In the third chapter, we define the invariants derived from the Nash multiplicity sequence which will be the center of our results. We also give there the construction of the *algebra of contact* of an arc φ with the set $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$. This algebra will be an essential element for the proof of Theorems 1 and 2.

Chapter 4 contains the results connecting the invariants defined in Chapter 3 with Constructive Resolution of Singularities, having Theorem 1 as the central piece. The content of this chapter will appear in [11].

Finally, Chapter 5 is dedicated to the relation between $\Phi_{X,\xi}$ and the isolation of points of $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$, anticipated by Theorem 2. We also explain there the conditions under which we can give $\sup(\Phi_{X,\xi})$. This part of the work can be found in [71].

Introducción: resumen y conclusiones

Las singularidades son objeto de estudio desde diversas ramas de las Matemáticas. Normalmente suponen un obstáculo para la aplicación de muchos resultados que son conocidos cuando no aparecen singularidades. Desde un punto de vista geométrico, los puntos singulares de una variedad son aquellos donde la dimensión del espacio tangente es mayor que la dimensión de la propia variedad. Desde el punto de vista algebraico, los puntos singulares corresponden a raíces múltiples de polinomios. En Álgebra Conmutativa, los puntos singulares se corresponden con anillos locales no regulares. Una variedad algebraica se dice *singular* si tiene puntos singulares.

El problema de *Resolución de Singularidades* plantea la pregunta de si una variedad singular puede aproximarse de algún modo por una no singular. Más concretamente, dada una variedad algebraica definida sobre algún cuerpo k, cuando hablamos de una resolución de singularidades de X nos referimos a un morfismo propio y biracional

$$X \xleftarrow{\pi} X', \tag{1}$$

donde X' es una variedad no singular. Con frecuencia se suele pedir también que π defina un isomorfismo fuera de los puntos singulares de X:

$$X \setminus \operatorname{Sing}(X) \cong X' \setminus \pi^{-1}(\operatorname{Sing}(X)).$$

El problema de Resolución de Singularidades consiste en decidir si se puede encontrar un morfismo así para cualquier variedad singular X. Tener una respuesta afirmativa abriría la posibilidad de extender muchos resultados de la geometría algebraica que sólo se saben ciertos para variedades no singulares. Además permitiría probar algunos resultados en campos como la integración motívica y la positividad. Por ejemplo, algunas identidades de tipo Lojasiewicz se prueban utilizando resolución de singularidades.

Actualmente, se sabe que la resolución de singularidades existe para X siempre que esta sea una variedad definida sobre un cuerpo de característica cero. Este resultado es un teorema de H. Hironaka [41]. Para cuerpos de característica positiva se conocen algunos resultados parciales (gracias a S. Abhyankar, J. Lipman, V. Cossart-O. Piltant, A. Benito-O. Villamayor y H. Kawanoue-K. Matsuki entre otros), pero el caso general es todavía un problema abierto.

La respuesta que dio Hironaka al problema en característica cero es que para cualquier variedad X se puede encontrar una resolución de singularidades, definida como una sucesión de explosiones en centros cerrados y lisos:

$$X \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_r} X_r. \tag{2}$$

Sin embargo, la prueba de Hironaka es existencial: no da ningún procedimiento que permita definir una sucesión de explosiones así. Posteriormente han ido apareciendo otros resultados, algunos de ellos constructivos, como los de O. Villamayor, [77], [78], E. Bierstone-P. Milman [9]. Ver también S. Encinas-O. Villamayor [33], S. Encinas-H. Hauser [32], J. Włodarczyk [84] and J. Kollár [57].

La *Resolución Constructiva de Singularidades* pretende diseñar un algoritmo que, para cualquier variedad X, determine de forma unívoca la construcción de un morfismo birracional como en (1), dado por una sucesión de explosiones, como en (2), escogidas con cuidado. El algoritmo debe ser capaz de escoger, para cada variedad X, un subconjunto cerrado $Y \subset X$ que sea el mejor centro para una explosión, de acuerdo con algún criterio establecido, orientado a concatenar explosiones que formen una resolución de singularidades de X.

Para el diseño de un algoritmo con esta propiedad, utilizamos *invariantes* asociados a los puntos de X. Estos invariantes deben ser capaces de distinguir entre diferentes tipos de singularidades. Su estudio es interesante para el diseño del algoritmo, pero también proporcionan información sobre el fenóneno de resolución, que puede ayudar a resolver el problema en contextos más generales. Algunos invariantes habitualmente usados para este fin son la función de Hilbert-Samuel y la multiplicidad.

La *multiplicidad* de X en un punto $\eta \in X$ viene dada por una función semicontinua superiormente (véase [22]):

$$\begin{aligned} \mathrm{mult}(X) : X &\longrightarrow \mathbb{N} \\ \eta &\longmapsto \mathrm{mult}(X)(\eta) := \mathrm{mult}(\mathcal{O}_{X,\eta}), \end{aligned}$$

donde $\mathrm{mult}(\mathcal{O}_{X,\eta})$ es la multiplicidad del anillo local $\mathcal{O}_{X,\eta}$ en el ideal maximal. En el caso particular en el que X se define como el conjunto de ceros de un polinomio f en el espacio afín, la multiplicidad de X en el origen es el orden del polinomio f.

Puesto que la multiplicidad es una función semicontinua superiormente, define una estratificación de X en conjuntos localmente cerrados

$$Z_m = \{\eta \in X : \mathrm{mult}(X)(\eta) = m\} \subset X.$$

Esta estratificación es un ejemplo de cómo los invariantes hacen distinción entre puntos singulares de X. Por ejemplo, los puntos singulares de X son los del cerrado $\bigcup_{m \geq 2} Z_m$.

El problema de Resolución de Singularidades es una motivación para la definición de invariantes de los puntos singulares de las variedades, y además tiene conexión con otros métodos de estudio de las singularidades, por ejemplo desde el punto de vista del álgebra, de la geometría o de la topología. Sin embargo, el estudio de invariantes de las singularidades también es interesante, por ejemplo, en la clasificación de variedades.

Los *espacios de arcos* también surgen como herramienta para el estudio de las singularidades de variedades algebraicas. Han resultado útiles para la comprensión de algunas propiedades geométricas y topológicas de las variedades, como muestran los trabajos de Denef-Loeser, Ein, Ishii, Mustaţă, Reguera y Yasuda entre otros.

La motivación principal de esta tesis es el estudio de los arcos desde el punto de vista de la resolución constructiva de singularidades. Hemos investigado posibles conexiones entre invariantes de singularidades que surgen en términos del espacio de arcos de una variedad y la información que se suele usar para definir algoritmos de resolución. A continuación presentaremos los elementos involucrados en este estudio para después enunciar los principales resultados obtenidos.

Arcos y singularidades

Supongamos que X es una variedad algebraica definida sobre un cuerpo k. Un *arco* (K-arco) φ en X centrado en un punto $\xi \in X$ es un morfismo

$$\varphi : \mathrm{Spec}(K[[t]]) \longrightarrow X,$$

para algún cuerpo $K \supset k$, que lleva el punto cerrado de $\mathrm{Spec}(K[[t]])$ a ξ. Si $X = \mathrm{Spec}(B)$ es una variedad algebraica afín sobre un cuerpo k, un arco se puede ver como un homomorfismo de anillos

$$\varphi^* : B \longrightarrow K[[t]].$$

Llamamos a $\varphi(\langle t \rangle) \in X$ el *centro* del arco φ. Si $K = k$, entonces un K-arco en X centrado en un punto cerrado $\xi \in X$ describe el germen de una curva en X que contiene a ξ.

El *espacio de arcos* de una variedad X sobre un cuerpo k es un esquema (no de tipo finito) que representa el funtor de k-esquemas a conjuntos dado por

$$Y \overset{\mathcal{A}}{\longmapsto} \mathrm{Hom}_k \left(Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(k[[t]]), X \right),$$

cuyos K-puntos para un cuerpo $K \supset k$ son los K-arcos en X ([8]). El espacio de arcos de X se puede contruir como el límite inverso de los esquemas de m-jets de X, para $m \in \mathbb{N}$.

Hay una conexión muy estrecha entre los espacios de arcos y de jets y las derivaciones de Hasse-Schmidt, que resulta especialmente útil para comprender cómo se pueden dar ecuaciones que describan los espacios de arcos. También hay una relación entre los arcos y las valoraciones: todos los arcos en X describen una valoración en alguna subvariedad de X, y cualquier valoración en el cuerpo de fracciones de $\mathcal{O}_X$ da un arco en X. Esta relación también es una motivación para estudiar los espacios de arcos, y de hecho es clave, por ejemplo, en el problema de Nash ([69]).

Muchos autores han contribuido a comprender los espacios de arcos y de jets estudiando su estructura, propiedades, su conexión con las singularidades, etc. (véase por ejemplo [56], [38], [60], [26], [52], [45], [73], [49], [50], [59], [24]).

Ya se han estudiado también algunos invariantes de una variedad definidos por medio de sus espacios de arcos y de jets (por ejemplo en [26], [30], [76], [25], [29], [51]), pero aquí nos centramos en la definición de invariantes que tengan una relación con la resolución constructiva de singularidades.

Nuestro principal objeto de estudio dentro del contexto de los espacios de arcos es la *sucesión de multiplicidades de Nash*. Dada una variedad X definida sobre un cuerpo k y dado un arco φ en X, la sucesión de multiplicidades de Nash de φ es una sucesión no creciente de enteros positivos

$$m_0 \geq m_1 \geq \ldots \geq m_l = m_{l+1} = ... \geq 1$$

asociada al centro de φ (que es un punto en X). Esta sucesión se puede entender como un refinamiento de la multiplicidad de X en $\xi = \varphi(\langle t\rangle)$: en cierto sentido (véase Remark 2.7.3) es la multiplicidad de X en ξ a lo largo de la dirección dada por φ.

La sucesión de multiplicidades de Nash fue definida en primer lugar por M. Lejeune-Jalabert en [58] para arcos en un germen de una hipersuperficie, y fue generalizada más tarde por M. Hickel en [40] para variedades de codimensión arbitraria. Se puede contruir de la siguiente forma. Asumimos primero, por simplicidad, que $X = \mathrm{Spec}(B)$ es afín. Sea $\xi \in X$ un punto, y φ un arco en X centrado en ξ. Consideramos el grafo de φ,

$$\Gamma_0^* = \varphi^* \otimes i : B \otimes K[t] \to K[[t]],$$

que es además un arco en $X_0 = X \times \mathbb{A}^1$ centrado en el punto $\xi_0 = (\xi, 0) \in X_0$. Estos elementos determinan completamente una sucesión de explosiones en puntos:

$$\begin{array}{ccccccccc} \mathrm{Spec}(K[[t]]) & & & & & & & & \\ \downarrow{\scriptstyle \Gamma_0} & \searrow^{\Gamma_1} & & & \searrow^{\Gamma_l} & & & & \\ X_0 = X \times \mathbb{A}^1 & \xleftarrow{\pi_1} & X_1 & \xleftarrow{\pi_2} & \ldots & \xleftarrow{\pi_l} & X_l & & \ldots \\ \xi_0 = (\xi, 0) & & \xi_1 & & \ldots & & \xi_l & & \ldots \end{array} \qquad (3)$$

Aquí, π_i es la explosión de X_{i-1} con centro ξ_{i-1}, donde $\xi_i = \mathrm{Im}(\Gamma_i) \cap \pi_i^{-1}(\xi_{i-1})$ para $i = 1, \ldots, l, \ldots$, y Γ_i es el (único) arco en X_i centrado en ξ_i que se obtiene levantando el arco Γ_0 por medio del morfismo propio $\pi_i \circ \ldots \circ \pi_1$. El elemento m_i de la sucesión de multiplicidades de Nash será la multiplicidad de X_i en ξ_i para cada $i = 0, \ldots, l, \ldots$. En realidad m_0 es exactamente la multiplicidad de X en ξ.

Nos referiremos a una sucesión de explosiones definida como en (3) para un arco φ como la sucesión de explosiones *dirigida por* φ. Nótese que antes de la primera explosión hemos multiplicado la variedad X por una recta afín. Por lo tanto, si X era una variedad singular, X_0 tendrá singularidades no aisladas. Nótese también que en (3) todas las explosiones tienen como centros puntos cerrados. Entonces una sucesión de explosiones dirigida por un arco en X nunca induce una resolución de singularidades para X_0. Además la máxima multiplicidad de X_0 no puede tampoco disminuir a lo largo de esta sucesión de explosiones, porque la función multiplicidad es semicontinua superiormente (y tampoco puede crecer, véase [22]). Aún así, si escogemos un punto ξ tal que $\mathrm{mult}(X)(\xi) = m_0 > 1$ y φ no es un arco en el cerrado de multiplicidad mayor o igual que m_0 de X,

$$\cup_{i \geq m_0} Z_i = \{\eta \in X : \mathrm{mult}(X)(\eta) \geq m_0\},$$

la sucesión de multiplicidades de Nash decrece en algún momento (véase [58]). De hecho, si el punto genérico de φ está contenido en el estrato de multiplicidad m_i de X, entonces la sucesión de multiplicidades de Nash estabiliza en el valor m_i. La razón es que en algún paso, digamos r, el centro ξ_r del arco Γ_r levantado a X_r de Γ_0 no estará contenido en el estrato de multiplicidad m_0 de X_r,

$$(Z_r)_{m_0} = \{\eta \in X_r : \mathrm{mult}(X_r)(\eta) = m_0\}.$$

Si escogemos $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X) = \{\eta \in X : \mathrm{mult}(X)(\eta) = \max \mathrm{mult}(X)\}$, donde $\max \mathrm{mult}(X)$ es la máxima multiplicidad de X, entonces $(Z_r)_{m_0}$ es el subconjunto $\underline{\mathrm{Max}}\,\mathrm{mult}(X_r)$, como consecuencia de la discusión anterior.

En [58], M. Lejeune-Jalabert define la sucesión de multiplicidades de Nash de un arco en un germen de una hipersuperficie relacionándola con la comprensión del cálculo de la función β de Artin. En la última parte de la Sección 2.7 se explicará brevemente la idea que hay detrás de esta definición.

Invariantes definidos por arcos

Para nuestro trabajo consideraremos arcos centrados en puntos del subconjunto $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, pero cuyo punto genérico $\varphi(\langle 0 \rangle)$ no está contenido allí. Esto último garantiza que su sucesión de multiplicidades de Nash no es constante. Es razonable pensar en una noción de contacto de un arco φ con $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ basada en cuántas explosiones dirigidas por φ son necesarias hasta que su punto cerrado se separa de este subconjunto.

Dados una variedad X, un punto $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$ y un arco φ en X centrado en ξ, definimos la *persistencia de φ (en $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$)*, y la denotamos por $\rho_{X,\varphi}$, como el número de explosiones en (3) que son necesarias para que la sucesión de multiplicidades de Nash decrezca por primera vez (ver Definición 3.1.1). Siempre que el punto genérico de φ no esté contenido en $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, la persistencia de φ

es un número natural:

$$\rho_{X,\varphi} = \min_{i\in\mathbb{N}} \{m_i < m_0\} .$$

Ambos, la sucesión de multiplicidades de Nash y la persistencia de φ son invariantes de (X, φ, ξ). Si consideramos el mínimo de los $\rho_{X,\varphi}$ para todos los arcos φ en X centrados en ξ, este es un invariante para (X, ξ). Resulta que estos invariantes están fuertemente conectados a la resolución constructiva, de un modo que se especificará más adelante. Para estudiarlos, utilizamos presentaciones locales para la multiplicidad y álgebras de Rees.

También construimos otro invariante, que resulta ser un refinamiento de $\rho_{X,\varphi}$, y que llamamos *orden de contacto de φ (con $\underline{\text{Max}}$ mult(X))* y denotamos por

$$r_{X,\varphi} \in \mathbb{Q}_{\geq 1}.$$

Este invariante se calcula como el orden de una cierta álgebra de Rees (ver Definición 3.2.16), y resulta satisfacer

$$\rho_{X,\varphi} = [r_{X,\varphi}] .$$

Obsérvese que, en principio, ninguno de estos invariantes tiene que ver con una resolución de singularidades de X.

Las herramientas principales utilizadas para la definición de $r_{X,\varphi}$ y para las conclusiones acerca de ellos son las álgebras de Rees y su resolución, los cuáles han sido desarrollados con detalle por O. Villamayor, S. Encinas, A. Bravo, A. Benito, R. Blanco, M. L. García-Escamilla y C. Abad en [34], [36], [80], [5], [10], [13], [16], [82], [1], [2].

El *orden de un arco* $\varphi^* : \mathcal{O}_{X,\xi} \longrightarrow K[[t]]$ es el entero positivo más grande n tal que $\varphi^*(\mathcal{M}_\xi) \subset (t^n)$, siendo $\mathcal{M}_\xi \subset \mathcal{O}_{X,\xi}$ el ideal maximal. El cociente

$$\bar{r}_{X,\varphi} = \frac{r_{X,\varphi}}{\text{ord}(\varphi)}, \tag{4}$$

que también es un invariante, es a veces más interesante que el propio $r_{X,\varphi}$, puesto que evita la influencia del orden del arco y asigna, por ejemplo, el mismo valor a distintas parametrizaciones del mismo germen de curva.

Resolución Constructiva e invariantes

Una *resolución constructiva* es un algoritmo que elige, para cualquier variedad X, una subvariedad cerrada $Y \subset X$ como centro de una explosión que llevará, tras iterar el proceso, a una resolución de las singularidades de X (véase [34]). La elección de Y viene dada por alguna función semicontinua superiormente F, definida en variedades

$$F(X) = F_X : X \longrightarrow (\Lambda, \geq) \tag{5}$$

cuyo máximo valor, max F_X, determina un cerrado liso

$$\underline{\text{Max}}\, F_X := \{\xi \in X : F(\xi) = \max F_X\} \subset X.$$

El subconjunto $\underline{\mathrm{Max}}\ F_X$ será el centro de la primera explosión $\pi_1 : X' \longrightarrow X$ en la construcción de una resolución de singularidades de X. Después de esta explosión, se puede definir una nueva función

$$F'_X := F_{X'} : X' \longrightarrow (\Lambda, \geq)$$

que satisfaga

$$\begin{aligned} F'_X(\pi_1^{-1}(\xi)) &= F_X(\xi) && \text{if } \xi \in X \setminus Z \\ F'_X(\xi') &< F_X(\xi) && \text{if } \xi = \pi_1(\xi') \in Z. \end{aligned}$$

Entonces $\underline{\mathrm{Max}}\ F'_X$ será el centro de la segunda explosión $\pi_2 : X'' \longrightarrow X'$, y el proceso se puede iterar: definiremos una función semicontinua superiormente $F_X^{(i)} := F_{X^{(i)}}$ para cada $i \geq 1$, y $\underline{\mathrm{Max}}\ F_X^{(i)} \subset X^{(i)}$ será el centro de la explosión $\pi_{i+1} : X(i+1) \longrightarrow X^{(i)}$. Además, cada F^i se puede construir de manera que sea constante si y sólo si X^i es lisa. En ese caso se alcanzará una resolución de singularidades de X tras un número finito de iteraciones. Llamamos a estas F^i *funciones de resolución*. Para construir funciones de resolución usamos *invariantes*. La misión de un invariante de singularidades es asignar un valor a cada punto singular, de modo que podamos comparar distintos puntos singulares por medio de estos valores. Una forma de definir funciones de resolución es asignar a cada punto ξ una cadena de invariantes $F^i(\xi)$. Siguiendo los métodos de [82], tomaremos como primera coordenada de esta cadena la multiplicidad. Eso significa que centramos nuestra atención en los puntos de X donde la función multiplicidad toma su valor más alto, porque serán también aquellos donde la primera coordenada de la función de resolución tomará su valor máximo. Escribimos:

$$\begin{aligned} \underline{\mathrm{Max}}\ \mathrm{mult}(X) := &\{\eta \in X : \mathrm{mult}(X)(\eta) \geq \max \mathrm{mult}(X)\} = \\ = &\{\eta \in X : \mathrm{mult}(X)(\eta) = \max \mathrm{mult}(X)\}, \end{aligned}$$

que es un subconjunto cerrado de X. Sin embargo este conjunto no es necesariamente liso, así que no podemos escogerlo como centro de la primera explosión. Por lo tanto necesitaremos añadir más invariantes para construir nuestra función de resolución. Como segundo invariante utilizaremos $\mathrm{ord}_\xi^{(d)}(X)$, el *orden de Hirnaka en dimensión* $d = \dim(X)$ (véase [15] y [2]). El invariante $\mathrm{ord}_\xi^{(d)}(X)$ será la primera coordenada de nuestras funciones de resolución que distinga entre puntos de máxima multiplicidad.

Antes de exponer nuestros resultados vamos a dar una idea de lo que significa este orden en dimensión d, aprovechando para presentar una de las herramientas fundamentales para esta tesis: las álgebras de Rees.

Presentaciones locales y álgebras de Rees

Cuando a uno le interesa estudiar las peores singularidades de una variedad X dada, es útil tener ecuaciones que describan esos puntos como un subconjunto de algún esquema liso V. Supongamos que tenemos una función semicontinua superiormente F, y que queremos observar cómo se comporta a lo largo de una sucesión de explosiones

$$X = X_0 \overset{\pi_1}{\longleftarrow} X_1 \overset{\pi_2}{\leftarrow} \dots \overset{\pi_l}{\longleftarrow} X_l$$

(a cuyos centros les pediremos unas ciertas propiedades) el subconjunto $\underline{\mathrm{Max}}\, F_X \subset X$ de puntos donde F_X alcanza su valor más alto. Para ciertas funciones F esto es posible localmente, y llamamos a estas ecuaciones una *presentación local*: dado $\xi \in \underline{\mathrm{Max}}\, F_X$, una presentación local de X para F (o asociada a F) en ξ consiste en una inmersión local (étale) $X \hookrightarrow V$ para algún esquema liso V, junto con un conjunto de elementos $\{f_1, \dots, f_r\} \subset \mathcal{O}_V$ y pesos $\{n_1, \dots, n_r\} \subset \mathbb{N}$ tales que, en un entorno de ξ:

- El subconjunto cerrado

$$\underline{\mathrm{Max}}\, F_X = \{\eta \in X : F_X(\eta) = \max F_X\} \subset X \subset V$$

  es igual que

$$\{\eta \in V : \nu_\eta(f_i) \geq n_i, i = 1, \dots, r\} \subset V,$$

  donde $\nu_\eta(f_i)$ denota el orden de f_i en el anillo local regular $\mathcal{O}_{V,\eta}$;

- Hay una regla de transformación para las f_i de modo que la condición anterior se preserva por sucesiones de explosiones:

$$\begin{array}{c} V = V_0 \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_l} V_l \\ X = X_0 \longleftarrow X_1 \longleftarrow \dots \longleftarrow X_l, \end{array}$$

  donde el centro de π_i es un subconjunto liso cerrado $Y_{i-1} \subset \underline{\mathrm{Max}}\, F_{X_{i-1}}$ para $i = 1, \dots, l$, siempre y cuando $\max F_X = \max F_{X_1} = \dots = \max F_{X_{l-1}}$.

En el caso particular en el que F es la multiplicidad, esto es posible (véase [82]).

Las álgebras de Rees son una herramienta apropiada para manipular las presentaciones locales y ver cómo se transforman por sucesiones de explosiones. Un *álgebra de Rees* sobre un anillo Noetheriano R, o sobre $V = \mathrm{Spec}(R)$, es una R-álgebra finitamente generada

$$\mathcal{G} = \oplus_{l \in \mathbb{N}} I_l W^l \subset R[W],$$

para algunos ideales $I_l \subset R$ con las propiedades $I_0 = R$ e $I_l I_j \subset I_{l+j}$. Un álgebra de Rees sobre V determina de forma natural un subconjunto cerrado de V que llamaremos el *lugar singular de* $\mathcal{G}$:

$$\mathrm{Sing}(\mathcal{G}) = \{\eta \in V : \nu_\eta(f) \geq n \text{ para todo } fW^n \in \mathcal{G}\}.$$

Supongamos que tenemos una presentación local para la multiplicidad de una variedad X en un punto $\xi \in \underline{\mathrm{Max}}\, \mathrm{mult}(X)$:

$$X \hookrightarrow V,\ \{f_1, \dots, f_r\} \subset \mathcal{O}_{V,\xi},\ \{n_1, \dots, n_r\} \subset \mathbb{N},$$

para algún medio ambiente liso V, con $\dim(V) = n > \dim(X)$. Entonces podemos asociarle un álgebra de Rees

$$\mathcal{G}_X = \mathcal{O}_{V,\xi}[f_1 W^{n_1}, \dots, f_r W^{n_r}] \tag{6}$$

a la multiplicidad de X localmente en un entorno (étale) de ξ. El lugar singular de $\mathcal{G}_X$ será exactamente el subconjunto $\underline{\text{Max}}\,\text{mult}(X) \subset X \subset V$. La noción de álgebra de Rees se extiende a haces de álgebras de Rees de la manera obvia.

En la línea de las tranformaciones de presentaciones locales por explosiones, hay una noción de transformación de álgebras de Rees. Dada un álgebra de Rees $\mathcal{G}$ sobre V y una explosión $V' \longrightarrow V$ que tenga como centro un subconjunto cerrado y liso $Y \subset \text{Sing}(\mathcal{G})$, se puede definir una nueva álgebra de Rees $\mathcal{G}'$ sobre V'. Además existe una noción de *resolución de álgebras de Rees* que consiste en una sucesión de explosiones

$$\begin{array}{c} V_0 \xleftarrow{\pi_1} V_1 \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_l} V_l \\ \mathcal{G}_0 \longleftarrow \mathcal{G}_1 \longleftarrow \ldots \longleftarrow \mathcal{G}_l, \end{array}$$

donde π_i es una explosión en un centro cerrado y liso $Y_{i-1} \subset \text{Sing}(\mathcal{G}_{i-1})$ para $i = 1, \ldots, l$, tal que $\text{Sing}(\mathcal{G}_l) = \emptyset$.

Una resolución de un álgebra de Rees asociada a la multiplicidad de una variedad X definida sobre un cuerpo k (como en (6)) induce una sucesión de explosiones en X que conlleva una bajada de su máxima multiplicidad.

Utilizando un álgebra de Rees asociada a la multiplicidad de una variedad X en un punto $\xi \in \underline{\text{Max}}\,\text{mult}(X)$, podemos definir un invariante:

$$\text{ord}_\xi(\mathcal{G}) := \inf_{fW^n \in \mathcal{G}} \left\{ \frac{\nu_\xi(f)}{n} \right\}.$$

Este es el invariante más importante para la construcción de una resolución de $\mathcal{G}$.

Se puede probar que, si $\dim(X) = d$ y $\mathcal{G}_X$ es un álgebra de Rees sobre V ($\dim(V) = n$) asociada a la multiplicidad de X localmente en un entorno (étale) de X, entonces existe un álgebra de Rees $\mathcal{G}_X^{(d)}$ sobre un esquema liso $V^{(d)}$ de dimensión d tal que encontrar una resolución del álgebra en dimensión $n > d$, $\mathcal{G}_X$, equivale de algún modo a encontrar una resolución del álgebra en dimensión d, $\mathcal{G}_X^{(d)}$. Entonces, el orden de Hironaka en dimensión d se puede calcular como el orden de esta última álgebra:

$$\text{ord}_\xi^{(d)}(X) := \text{ord}_\xi^{(d)}(\mathcal{G}_X^{(d)}).$$

Este invariante es intrínseco a la variedad y no depende de la elección del álgebra de Rees (véase [16], [2]).

Resultados principales

Fijada una variedad X sobre un cuerpo k de característica cero y un punto $\xi \in \underline{\text{Max}}\,\text{mult}(X)$, consideramos el conjunto

$$\Phi_{X,\xi} = \{\bar{r}_{X,\varphi}\}_\varphi, \tag{7}$$

donde φ recorre todos los arcos en X centrados en ξ que no están totalmente contenidos en $\underline{\text{Max}}\,\text{mult}(X)$ (véase (4)). Este conjunto es un nuevo invariante de X en

ξ. Se puede probar que $\Phi_{X,\xi}$ tiene un mínimo (que es de nuevo un invariante de X en ξ). Además:

Teorema 1. *Sea X una variedad de dimensión d sobre un cuerpo k de característica cero. Sea ξ un punto en $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. Entonces,*

$$\mathrm{ord}_{\xi}^{(d)}(X) = \min(\Phi_{X,\xi}).$$

Esto significa que es posible leer el invariante $\mathrm{ord}_{\xi}^{(d)}(X)$ en el espacio de arcos de X. Además, el invariante $\bar{r}_{X,\varphi}$ sugiere una clasificación de los arcos en X centrados en ξ de acuerdo con su orden de contacto con $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. De entre todos los arcos en X centrados en ξ, aquellos para los que el valor de $\bar{r}_{X,\varphi}$ es mínimo nos dan el invariante $\mathrm{ord}_{\xi}^{(d)}(X)$, y también se separan de $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ más rápido que los demás durante la sucesión de explosiones en (3).
Esto significa que el invariante $\mathrm{ord}_{\xi}^{(d)}(X)$ se puede leer en el espacio de arcos de X. De hecho, dado un arco $\varphi : \mathrm{Spec}(K[[t]]) \longrightarrow X$ centrado en ξ, podemos considerar la familia de arcos dada por $\varphi_n = \varphi \circ i_n$ para $i > 1$, donde $i_n^* : K[[t]] \longrightarrow K[[t^n]]$ lleva t a t^n. Se puede probar entonces (véase el Corolario 4.3.4) que

$$\bar{r}_{X,\varphi} = \frac{1}{\mathrm{ord}(\varphi)} \cdot \lim_{n\to\infty} \frac{\rho_{X,\varphi_n}}{n},$$

y por lo tanto

$$\mathrm{ord}_{\xi}^{(d)}(X) = \inf_{\varphi}\left(\frac{1}{\mathrm{ord}(\varphi)} \cdot \lim_{n\to\infty} \frac{\rho_{X,\varphi_n}}{n}\right),$$

donde φ recorre todos los arcos en X centrados en ξ y no contenidos totalmente en $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. Esto demuestra que el invariante $\mathrm{ord}_{\xi}^{(d)}(X)$ no sólo es independiente de la elección de un álgebra de Rees particular asociada a la multiplicidad de X localmente en un entorno de ξ como ya hemos dicho, sino que ni siquiera necesita a las álgebras de Rees para su definición. Es realmente intrínseco a la variedad ya que (como demuestra la fórmula) puede expresarse en términos del espacio de arcos de X y por lo tanto, de algún modo, es un invariante natural a considerar.

Una investigación más profunda del invariante $\Phi_{X,\xi}$ nos da un criterio para decidir cuándo un punto de $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ es aislado en dicho subconjunto:

Teorema 2. *Sea X una variedad definida sobre un cuerpo k de característica cero, y sea ξ un punto en $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. Entonces ξ es un punto aislado de $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ si y sólo si el conjunto $\Phi_{X,\xi}$ está acotado superiormente.*

Esto tiene una consecuencia en términos de la sucesión de multiplicidades de Nash: bajo las hipótesis de teorema, ξ es un punto aislado de $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ si y sólo si se puede encontrar una cota superior $D(X,\xi)$ tal que para cualquier arco φ en X centrado en ξ que no esté totalmente contenido en $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, el número de explosiones necesarias hasta que la sucesión de multiplicidades de Nash decrece por primera vez (normalizado por $\mathrm{ord}(\varphi)$), es como mucho $D(X,\xi)$. En cierto sentido, esto significa que ξ está contenido en una componente de $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ de dimensión al menos 1, entonces el contacto de los arcos centrados en ξ con este subconjunto puede ser arbitrariamente grande, mientras que para puntos aislados de $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ este contacto está limitado.

De hecho, bajo ciertas condiciones de X y ξ, podemos calcular el supremo del conjunto $\Phi_{X,\xi}$. Estas condiciones involucran otro invariante de resolución: el *invariante* τ (véase [5]). Si el álgebra de Rees en dimensión n, $\mathcal{G}_X$, tiene invariante τ máximo en ξ, es decir, si $\tau_{\mathcal{G}_X,\xi} = n-1$, entonces se puede probar que encontrar una resolución de $\mathcal{G}_X$ es equivalente a encontrar una resolución de un álgebra de Rees en dimensión 1, $\mathcal{G}_X^{(1)}$. En este caso, el invariante $\mathrm{ord}_\xi^{(d)}(X)$ no proporciona información interesante para la resolución constructiva de X si $d > 1$. El invariante más interesante será $\mathrm{ord}_\xi^{(1)}(X) := \mathrm{ord}_\xi(\mathcal{G}_X^{(1)})$, y se tiene el siguiente resultado:

Proposición 3. *Si X tiene invariante τ máximo en ξ, entonces*

$$\sup(\Phi_{X,\xi}) = \mathrm{ord}_\xi^{(1)}(X).$$

Los resultados de esta tesis se recogen en los siguientes trabajos:

- Bravo, A. and Encinas, S. and Pascual-Escudero, B., *Nash multiplicities and resolution invariants*, Collectanea Mathematica **68** (2017), 2, 175–217;

- Pascual-Escudero, B., *Nash multiplicities and isolated points of maximal multiplicity*, arXiv:1609.09008 [math.AG].

Contenidos

A lo largo de los dos primeros capítulos revisaremos los conceptos y resultados preliminares necesarios para el los resultados enunciados anteriormente, que serán desarrollados en los tres últimos capítulos.

El primer capítulo comienza con una breve introducción al problema de Resolución de Singularidades. Introducimos a continuación la noción de multiplicidad, que resultará fundamental a lo largo de todo el trabajo. El resto del capítulo está dedicado a las definiciones y resultados básicos de la teoría de las álgebras de Rees, una de las principales herramientas que usaremos más adelante, y por último se muestra su conexión con la resolución constructiva y el invariante $\mathrm{ord}_\xi^{(d)}X$.

El segundo capítulo está dedicado a los espacios de arcos y su relación con las singularidades. Introducimos los esquemas de arcos y de m-jets de una variedad algebraica e ilustramos su construcción mediante las derivaciones de Hasse-Schmidt. También mostramos allí algunas propiedades y resultados relacionados con la estructura de los espacios de arcos y de m-jets, especialmente aquellos relacionados con las singularidades de variedades, así como la relación entre los arcos y las valoraciones. Este capítulo incluye también la definición de la sucesión de multiplicidades de Nash.

En el tercer capítulo definimos los invariantes derivados de la sucesión de multiplicidades de Nash que serán el centro de nuestros resultados. También se hace allí la construcción del *algebra de contacto* de un arco φ con el conjunto $\underline{\text{Max}}\,\text{mult}(X)$. Este álgebra será un elemento esencial en la prueba de los Teoremas 1 y 2.

El capítulo 4 contiene los resultados que conectan los invariantes definidos en el capítulo 3 con la Resolución Constructiva de Singularidades, siendo el Teorema 1 su pieza central. El contenido de este capítulo aparece en [11].

Por último, el capítulo 5 trata la relación entre $\Phi_{X,\xi}$ y los puntos aislados de $\underline{\text{Max}}\,\text{mult}(X)$, que ya anticipamos en el Teorema 2. También se explican allí las condiciones bajo las cuales podemos calcular $\sup(\Phi_{X,\xi})$ explícitamente. Esta parte del trabajo se puede encontrar también en [71].

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Chapter 1

Rees Algebras and Resolution of Singularities

1.1 Resolution of singularities

Let X be a reduced scheme of finite type over a field k. A resolution of singularities of X consists of a proper birational morphism together with a new regular scheme of finite type:

$$X' \xrightarrow{\pi} X.$$

Sometimes, it is also asked that π defines an isomorphism outside of the singular points of X, that is

$$X' \setminus \pi^{-1}(\mathrm{Sing}(X)) \cong X \setminus \mathrm{Sing}(X)$$

and such that the exceptional locus $\pi^{-1}(\mathrm{Sing}(X))$ is a set of hypersurfaces with normal crossing support.

In [41], Hironaka proved that, if X is defined over a field of characteristic zero, then a resolution of the singularities of X can always be found and, moreover, it can be achieved through successive blow ups at regular closed centers. His proof is, however, existential, and does not give an answer to the question of how to find such a sequence of blow ups, leading to a resolution.

Outside of the zero characteristic hypothesis things get even more complicated, and the general problem is still open, although some partial positive results are known for schemes of dimension 2 ([3], [61], [62], [6], [55]) and 3 ([21],[19],[20]).

If we stay in the zero characteristic case, the problem of constructive resolution studies the design of an algorithm that, for each X, provides a choice of centers to define a sequence of blow ups that yield a resolution of its singularities. There is not a unique resolution of singularities of a given scheme, but it is possible to establish a criterion that makes a decision for each X, attending to some designed rules, and satisfying some compatibility conditions. Results in zero characteristic fields were given by [77],[78], [9].

To study the changes that blow ups perform in the varieties and to obtain the data that the algorithm will use for choosing the successive centers, it is necessary to codify the complexity of the singularities. One uses equations for this task, which describe, as a subset of a smooth ambient space, the varieties that appear along the sequence of blow ups in some specific sense that we will explain here. However, this codification is not simple if one expects being able to compare the equations before and after a blow up. Still, a set of equations with a "good behavior" S, describing in a smooth ambient space V the worst singularities of X in some way, can be carefully chosen. This means that after a blow up

$$\begin{array}{c} V \xleftarrow{\pi_1} V_1 \\ X \longleftarrow X_1 \end{array}$$

one can obtain, by performing a transformation of S, a set of equations S' describing the worst singularities of X_1 in the same sense. Furthermore, expressing S' in terms of S allows us to measure the improvement of the singularities by π_1. Such a set of equations is what we call a *local presentation*.

The idea of local presentations appears already in basic objects, pairs, and idealistic exponents (see [42], [77], [34]), and they are also the motivation for the use of Rees algebras in the problem of Resolution of Singularities.

This first chapter is introductory, focused on this relation of Rees algebras with the study of algorithmic Resolution of Singularities. We will first introduce one of the invariants used for constructive resolution, which will be particularly interesting for our results: the multiplicity. We will explain how to describe the singularities of a given variety X via local presentations attached to the points of worst multiplicity, and then give some basic definitions and results about Rees algebras, the tool which will help us to deal with these local presentations. We will show how we can use them in resolution of singularities, and finally we will introduce some other invariants used in constructive resolution that can be computed via Rees algebras.

1.2 Multiplicity

Let R be a local noetherian ring, and let $\mathcal{M}$ be its maximal ideal.

Definition 1.2.1. [44, Theorem 11.1.3] Let $J \subset R$ be an $\mathcal{M}$-primary ideal. Let M be a finitely generated R-module of dimension d. The *Hilbert-Samuel function* of R for J, M is defined as the map

$$\begin{aligned} \mathrm{H}_R(J,M) : \mathbb{N} &\longrightarrow \mathbb{N} \\ n &\longmapsto \lambda(M/J^nM), \end{aligned}$$

where $\lambda(M/J^nM)$ denotes the length of M/J^nM as an R-module. That is, the length of any maximal chain of submodules

$$M_l \subset \ldots \subset M_0 = M/J^nM,$$

which is always finite as long as M_0 is artinian [31, Theorem 2.13] (See [44] or [31] for more details.)

For a fixed R, and for J, M as above, there exists a polynomial approximating the Hilbert-Samuel function (see [31, Proposition 12.2] or [44, Theorem 11.1.3]): there exists

$$P_{R,J,M}(x) = a_d x^d + a_{d-1}x^{d-1} + \ldots + a_0 \in \mathbb{Q}[x]$$

satisfying $P_{R,J,M}(n) = \mathrm{H}_R(J,M)(n)$ for n large enough. The degree of this polynomial is $d = \dim(M) \leq \dim(R)$, and moreover, the *multiplicity* of M at J appears as

$$e_R(J,M) = a_d \cdot d! = \lim_{n\to\infty} \frac{\mathrm{H}_R(J,M)(n)}{n^d} \cdot d!,$$

which is an integer. We shall denote $e_R(J) := e_R(J,R)$.

Definition 1.2.2. We call $e(R) := e_R(\mathcal{M})$ the *multiplicity of the local ring R.*

We present here some useful properties of the multiplicity:

Proposition 1.2.3. *[44, Chapter 11]*

1. *Assume that $J, I \subset R$ are $\mathcal{M}$-primary ideals such that $\overline{J} = \overline{I}$ (where $\overline{I}$ denotes the integral closure of the ideal I in R) . Then, for any finitely generated R-module M,*

   $$e_R(I,M) = e_R(J,M).$$

   Moreover, if R is also formally equidimensional, then the converse is also true if we set $M = R$, that is:

   $$\overline{J} = \overline{I} \Leftrightarrow e_R(I) = e_R(J).$$

   In particular, for any $\mathcal{M}$-primary ideal $I \subset R$,

   $$e_R(I,M) \geq e(R),$$

   and the equality holds if and only if I is a reduction of $\mathcal{M}$.

2. *For any $\mathcal{M}$-primary ideal J, $J\hat{R}$ is an $\hat{\mathcal{M}}$-primary ideal, where $\hat{\mathcal{M}}$ is the maximal ideal at the completion $\hat{R}$, and*

   $$e_R(J,R) = e_{\hat{R}}(J\hat{R}).$$

   In particular,

   $$e(\hat{R}) = e(R).$$

3. *Suppose that R is a regular ring and let $f \in \mathcal{M}$. If we denote $A = R/f$, then*

$$e(A) = \nu_R(f)$$

is the order of f in the local regular ring R.

Remark 1.2.4. There is also an iterative definition of $e_R(J)$ which is useful for computations. Assume that R is Cohen Macaulay and of dimension d. Then, for any system of parameters $(x_1, \dots, x_d) \subset R$, if we denote $J = (x_1, \dots, x_d)$, then

$$e_R(J) = \begin{cases} e_{(R/(x_d))}(J \cdot (R/(x_d)) & \text{if } d > 1, \\ \lambda(R/(x_1)) & \text{if } d = 1. \end{cases}$$

(We refer to [39, Definition 1.2] or [44, Proposition 11.1.9] for the general formula.) In particular, if R is a noetherian local ring of dimension d with maximal ideal $\mathcal{M}$ which is also Cohen Macaulay, and if $I = (x_1, \dots, x_d)$ is an $\mathcal{M}$-primary ideal, then

$$e_R(I) = \lambda(R/I).$$

(See [44, Proposition 11.1.10].)

Example 1.2.5.

- If $A_1 = k[x, y]/(x^2 - y^3)$, then $e((A_1)_{(\overline{x},\overline{y})}) = 2$. One can compute this using Remark 1.2.4 or the last property in Proposition 1.2.3. Note that $e((A_1)_{(\overline{x},\overline{y})}) = e_{(A_1)_{(\overline{x},\overline{y})}}((\overline{y})) \leq e_{(A_1)_{(\overline{x},\overline{y})}}((\overline{x}))$.
- For $A_2 = k[x, y, z]/(x^2 - z^5, y^3 - z^4)$, we have $e((A_2)_{(\overline{x},\overline{y},\overline{z})}) = 6 = e_{(A_2)_{(\overline{x},\overline{y},\overline{z})}}((\overline{z}))$, by Remark 1.2.4.

Definition 1.2.6. Given a variety X over a field k, the multiplicity function for X is defined as

$$\begin{aligned} \operatorname{mult}(X) : X &\longrightarrow \mathbb{N} \\ \eta &\longmapsto \operatorname{mult}(X)(\eta) := e(\mathcal{O}_{X,\eta}). \end{aligned} \tag{1.1}$$

We will sometimes denote $\operatorname{mult}_\eta(X) : \operatorname{mult}(X)(\eta)$.

Example 1.2.7.

- The curve $X_1 = \mathbb{V}(x^2 - y^3) \subset \operatorname{Spec}(k[x, y])$ has multiplicity 2 at the origin and multiplicity 1 at any other point.
- Let $X_3 = \mathbb{V}((x^2 - y^3)^2 + z^2) \subset \operatorname{Spec}(k[x, y, z])$. The multiplicity of X_3 is 2 along the curve defined by $(x^2 - y^3, z)$ (this corresponds to computing $e((k[x, y, z]/((x^2 - y^3)^2 + z^2))_{(x^2-y^3,z)}))$, and 1 everywhere else in X_3.
- Consider now the surface $X_4 = \mathbb{V}(x^3 - y^3z^2) \subset \operatorname{Spec}(k[x, y, z])$. It reaches its maximum multiplicity, namely 3, along the curve $\mathbb{V}(x, y)$, has multiplicity 2 at every point of $\mathbb{V}(x, z)$ except from the origin, and multiplicity 1 at any other point.

- Let $X_5 = \mathbb{V}(x^2y^3 - z^3s^4) \subset \mathrm{Spec}(k[x,y,z,s])$. The multiplicity is maximal at the origin, $\mathrm{mult}_{(0,0,0,0)}(X_5) = 5$. Along the line defined by $\mathbb{V}(x,y,s)$, but outisde of the origin, X_5 has multiplicity 4. The multiplicity is 3 along $\mathbb{V}(y,zs) \setminus \mathbb{V}(x,y,s)$, and it is 2 along $\mathbb{V}(x,zs) \setminus \mathbb{V}(x,y,s)$. Outside of $\mathbb{V}(xy,zs)$, the multiplicity is 1.

Let us discuss here the notion of multiplicity at a point from a more geometric point of view, assisted by the following Corollary of a Theorem of Zariski:

Theorem 1.2.8. *[85, Chapter VIII, Theorem 24, Corollary 1] Let A be a local noetherian domain, let $\mathcal{M}$ be the maximal ideal, let k be the residue field, and K the quotient field. Let $B \supset A$ be a finite extension such that no element in A is a zero divisor in B. Let $L = K \otimes_A B$, let $\mathcal{Q}_1, \dots \mathcal{Q}_r$ be the maximal ideals of B, and k_i their respective residue fields. Assume that the localized rings $B_{\mathcal{Q}_i}$ have the same dimension as A for $i = 1, \dots, r$. Then*

$$e_A(\mathcal{M}) \cdot [L : K] = \sum_{1 \leq i \leq r} e_{B_{\mathcal{Q}_i}}(\mathcal{M}B_{\mathcal{Q}_i}) \cdot [k_i : k].$$

Suppose that R is a localization of the coordinate ring B of an algebraic variety of dimension d. That is, let B be a k-algebra of finite type for an infinite field k. Assume that for any system of parameters $x_1, \dots, x_d$ in R, $A = k[x_1, \dots, x_d] \subset B$ is finite as a module (that is, that any system of parameters gives a Noether normalization for B). Write $I = (x_1, \dots, x_d) \subset A$, and consider the finite projection

$$\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A),$$

induced by the K-morphism

$$\begin{aligned} p : A &\hookrightarrow B \\ x_i &\longmapsto x_i, \;\; i = 1, \dots, d. \end{aligned}$$

After localizing at $I \subset A$, we have a finite morphism

$$p_I : A_I \longrightarrow S = B \otimes_A A_I.$$

Let $\mathcal{M} \subset A_I$ be the maximal ideal, and let $\mathcal{M}_1, \dots, \mathcal{M}_r \subset S$ be the maximal ideals of the semilocal ring S. Theorem 1.2.8 yields

$$[L : K] = \sum e_{B_{\mathcal{M}_i}}(IB_{\mathcal{M}_i}) \cdot [k_i : k] \,,$$

where K is the quotient field of A, and $L = K \otimes_A S$. Then $R = B_{\mathcal{M}_i}$ for some $i = 1, \dots, r$, and

$$e(R) \leq [L : K] \,,$$

that is, the rank of the fibre over a general point gives an upper bound for the multiplicity of R. For any choice of $x_1, \dots, x_d \in R$, one has $e_R(IR) \leq e(R)$. But for a sufficiently general projection the equality holds (see the discussion in [44, p. 221]). This is an example of what we will understand as a transversal projection.

If R is a complete local ring, the same construction can be made: for any choice of a system of parameters $x_1, \dots, x_d$ in R, $A = k[[x_1, \dots, x_d]] \subset R$ is finite as a module (a Cohen subring exists for any choice of the parameters, see [44, Theorem 4.3.3]). Assume that $k = \mathbb{C}$. Geometrically, the multiplicity of X at η corresponds to the smallest rank over the generic fiber of all possible local morphisms $(X, \eta) \to (\mathbb{C}^d, 0)$ (see [63]).

Example 1.2.9. Let X_1 be as in Example 1.2.7, defined over $k = \mathbb{C}$. Consider the following projections:

- The projection

$$p_x : \operatorname{Spec}(\mathbb{C}[x,y]/(x^2 - y^3)) \longrightarrow \operatorname{Spec}(\mathbb{C}[x])$$

  has generic rank 3. That is, for any $x_0 \in \mathbb{A}^1_C$, $p_x^{-1}(x_0) = \left\{(x_0, x_0^{2/3})\right\}$. Hence, the multiplicity at the origin is at most 3.

- The projection

$$p_y : \operatorname{Spec}(\mathbb{C}[x,y]/(x^2 - y^3)) \longrightarrow \operatorname{Spec}(\mathbb{C}[y])$$

  has generic rank 2. For any $y_0 \in \mathbb{A}^1_C$, $p_y^{-1}(y_0) = \left\{(y_0^{3/2}, y_0)\right\}$, so the multiplicity at the origin is at most 2. The ideal generated by $\overline{y}$ in $\mathbb{C}[x,y]/(x^2 - y^3)$ is a reduction of the maximal ideal $(\overline{x}, \overline{y})$, so it gives exactly the multiplicity at $(0,0)$. This is a transversal projection.

Multiplicity, local presentations and Resolution of Singularities

For the achievement of a resolution, we will be interested in measuring the singularities of X (see Section 1.5), and keeping track of the evolution of this measure after blowing up. For this task, we start by considering functions which stratify the varieties into locally closed strata, matching up points which share the same complexity. To this end, we use functions defined on varieties

$$F(X) = F_X : X \longrightarrow (\Lambda, \geq) \tag{1.2}$$

where $(\Lambda, \geq)$ is a well ordered set. The sets $\{\eta \in X : F_X(\eta) \geq n\}$ will be hence closed for each $n \in \Lambda$. As we said, we want to keep track of the evolution of the strata when we perform blow ups.

One possible measurement of the complexity of the singularities is given by the multiplicity:

Proposition 1.2.10. *[68, Theorem 40.6] Let R be a local ring, and assume that its completion $\hat{R}$ is equidimensional and that is has no embedded primes. Then R is regular if and only if $e(R) = 1$.*

Theorem 1.2.11. *[22], [70] Let X be a scheme of finite type over a perfect field k. Then*

1. *The multiplicity function, as defined in (1.1), is an upper semicontinuous function.*
2. *If $\pi : X' \longrightarrow X$ is the blow up at a smooth equimultiple center Y (that is, all points in Y have the same multiplicity), then for any $\eta \in X'$ we have* $\mathrm{mult}(X)(\pi(\eta)) \geq \mathrm{mult}(X')(\eta)$.

See also [1], [2] and [82, Theorem 6.12] for alternative proofs.

Hence, a reasonable strategy would be focusing on the points of X where the multiplicity is greater than 1, and performing monoidal transformations transformations until we reach a variety X' satisfying $e(\mathcal{O}_{X',\eta}) = 1$ for all $\eta \in X'$. In fact, the feasibility of this strategy was already asked by Hironaka in [41]. A positive answer in given by Villamayor in [82].

To perform this program, it is necessary to describe the subset of worst singularities, in this case of points of maximum multiplicity, in a way that is consistent along the resolution process. This is a motivation for local presentations:

Definition 1.2.12. Let X be a d dimensional scheme of finite type over a perfect field k. Let F be an upper semicontinuous function defined on X as before. A *local presentation* of F for X at $\xi \in \underline{\mathrm{Max}}\ F_X$ is a local (étale) immersion $X \hookrightarrow V^{(n)} = \mathrm{Spec}(R)$ in a smooth scheme of dimension $n > d$, and a set of elements $\{f_1, \dots, f_r\}$ in R together with a set of integers $\{n_1, \dots, n_r\}$ such that, in an (étale) neighborhood of ξ,

$$\underline{\mathrm{Max}}\ F_X = \cap_{i=1}^{r} \left\{ \eta \in V^{(n)} : \nu_\eta(H_i) = n_i \right\},$$

where H_i is the hypersurface defined by f_i in $V^{(n)}$ and $\nu_\eta(H_i)$ denotes the order of H_i at η, for $i = 1, \dots, r$. Moreover, this condition must be preserved by blow ups at regular closed centers contained in $\underline{\mathrm{Max}}\ F_X$. That is, if

$$\begin{array}{ccc} V & \xleftarrow{\pi_1} & V_1 \\ X & \hookleftarrow & X_1 \end{array}$$

is a blow up at a regular closed subset $Y \subset \underline{\mathrm{Max}}\ F_X \subset X$, then

$$\underline{\mathrm{Max}}\ F_{X_1} = \cap_{i=1}^{r} \left\{ \eta \in V_1^{(n)} : \nu_\eta(H_i') = n_i \right\}, \tag{1.3}$$

as long as $\max\ F_{X_1} = \max\ F_X$ (here H_i' is the strict transform of H_i by π_1 for $i = 1, \dots, r$). If $\max\ F_{X_1} < \max\ F_X$, then the set on the right hand side of (1.3) must be empty.

Example 1.2.13. In [82], it is proven that a local presentation can always be found for $F_X = \mathrm{mult}(X)$. For simplicity, assume that $X = \mathrm{Spec}(B)$ and that $\xi \in$

<u>Max</u> mult(X). Then, it is possible to construct, maybe in an étale neigborhood of X, a finite morphism

$$\beta : X \longrightarrow V^{(d)} = \mathrm{Spec}(S)$$

of generic rank $m = \mathrm{mult}_\xi(X)$, where S is a smooth k-algebra. Then, one can consider a presentation for B over S, that is, a finite set of integral elements $\theta_1, \ldots, \theta_{n-d}$ over S, such that

$$B = S\left[\theta_1, \ldots, \theta_{n-d}\right].$$

It can be shown that, for $i = 1, \ldots, n-d$, the element f_i is the minimal polynomial of θ_i in the quotient field L of S. Moreover, f_i has the form

$$f_i(x_i) = x_i^{n_i} + a_{i,n_i-1}x_i^{n_i-1} + \ldots + a_{i,0},$$

where $a_{i,j} \in S$ for $j = 0, \ldots n_i - 1$. Hence $f_i(x_i) \in S[x_i]$ for $i = 1, \ldots, n-d$. Then $B = S[x_1, \ldots, x_{n-d}]/I$ for some $I \supset (f_1, \ldots, f_{n-d})$, and there are surjective maps

$$S[x_1, \ldots, x_{n-d}] \longrightarrow S[x_1, \ldots, x_{n-d}]/(f_1, \ldots, f_{n-d}) \longrightarrow B.$$

This induces an immersion

$$X = \mathrm{Spec}(B) \hookrightarrow V^{(n)} = \mathrm{Spec}(S[x_1, \ldots, x_{n-d}])$$

Then, it can be shown that the local embedding $X \hookrightarrow V^{(n)}$ together with the set $\{f_i, n_i\}_{i=1,\ldots,n-d}$ form a local presentation for the multiplicity of X at ξ (see [82, 7.1]).

1.3 Rees Algebras

We already explained why local presentations are useful to describe closed sets that are of interest in resolution on singularities. Le us now introduce Rees algebras as the main tool that we will use to manipulate local presentations. The main references here are [80] and [36]

Definition 1.3.1. Let R be a Noetherian ring. A *Rees algebra* $\mathcal{G}$ *over* R is a finitely generated graded R-algebra

$$\mathcal{G} = \bigoplus_{l \in \mathbb{N}} I_l W^l \subset R[W]$$

for some ideals $I_l \in R$, $l \in \mathbb{N}$ such that $I_0 = R$ and $I_l I_j \subset I_{l+j}$, $\forall l, j \in \mathbb{N}$. Here, W is just a variable in charge of the degree of the ideals I_l. That is, if $\mathcal{G}$ is a Rees algebra over R, there exist some $f_1, \ldots, f_r \in R$ and positive integers (weights) $n_1, \ldots, n_r \in \mathbb{N}$ such that

$$\mathcal{G} = R[f_1 W^{n_1}, \ldots, f_r W^{n_r}]. \tag{1.4}$$

Note that this definition is more general than the (usual) one considering only algebras of the form $R[IW]$ for some ideal $I \subset R$, which we call Rees rings.

Rees algebras can be defined over Noetherian schemes as follows:

Definition 1.3.2. Let V be a noetherian scheme over a field k. A *Rees algebra on* V is a sheaf of finitely generated graded $\mathcal{O}_V$-algebras

$$\mathcal{G} = \bigoplus_{l \geq 0} I_l W^l$$

where the $I_l \subset \mathcal{O}_V$ are sheaves of ideals satisfying $I_0 = \mathcal{O}_V$ and $I_l I_j \subset I_{l+j}$ for all $l, j \in \mathbb{N}$. That is, there exists a covering of open affine subsets $\{U_i\} \subset V$ such that

$$\mathcal{G}(U_i) = \bigoplus_{l \geq 0} I_l(U_i) W^l \subset \mathcal{O}_V(U_i)[W]$$

is a Rees algebra over $\mathcal{O}_{U_i}$.

Definition 1.3.3. Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two Rees algebras. We denote by $\mathcal{G}_1 \odot \mathcal{G}_2$ the smallest Rees algebra containing both. If $\mathcal{G}_1 = R[f_1 W^{n_1}, \dots, f_r W^{n_r}]$ and $\mathcal{G}_2 = R[g_1 W^{m_1}, \dots, g_l W^{m_l}]$, then

$$\mathcal{G}_1 \odot \mathcal{G}_2 = R[f_1 W^{n_1}, \dots, f_r W^{n_r}, g_1 W^{m_1}, \dots, g_l W^{m_l}].$$

If $\mathcal{G}_2' = R'[g_1 W^{m_1}, \dots, g_l W^{m_l}]$, where $R' \subset R$ is a subring, by abuse of notation we will sometimes denote by $\mathcal{G}_1 \odot \mathcal{G}_2'$ the Rees algebra $\mathcal{G}_1 \odot \mathcal{G}_2$, where $\mathcal{G}_2$ is the extension of $\mathcal{G}_2'$ to a Rees algebra over R.

In what follows, we will assume k to be a perfect field. We will specify characteristic zero when needed. We will also assume R to be a smooth k-algebra, or V to be a smooth scheme over k. We will often work locally: for many computations, we will assume that we fix a point and an open subset of V containing it, so that we can reduce it to the affine case, $V = \mathrm{Spec}(R)$.

One can attach to a Rees algebra a closed set as follows:

Definition 1.3.4. Let $\mathcal{G}$ be a Rees algebra over V. The *singular locus* of $\mathcal{G}$, $\mathrm{Sing}(\mathcal{G})$, is the closed set given by all the points $\xi \in V$ such that $\nu_\xi(I_l) \geq l$, $\forall l \in \mathbb{N}$, where $\nu_\xi(I)$ denotes the order of the ideal I in the regular local ring $\mathcal{O}_{V,\xi}$.

Proposition 1.3.5. *([36, Proposition 1.4]) For any Rees algebra over R,*

$$\mathcal{G} = R[f_1 W^{n_1}, \dots, f_r W^{n_r}],$$

the singular locus of $\mathcal{G}$ can be computed as

$$\mathrm{Sing}(\mathcal{G}) = \{\xi \in \mathrm{Spec}(R) : \nu_\xi(f_i) \geq n_i, \ \forall i = 1, \dots, r\}.$$

Note that the singular locus of the Rees algebra on V generated by $f_1 W^{n_1}, \dots, f_r W^{n_r}$ does not coincide with the usual definition of the singular locus of the subvariety of V defined by $f_1, \dots, f_r$.

Corollary 1.3.6. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two Rees algebras over V, then*

$$\mathrm{Sing}(\mathcal{G}_1 \odot \mathcal{G}_2) = \mathrm{Sing}(\mathcal{G}_1) \cap \mathrm{Sing}(\mathcal{G}_2).$$

Definition 1.3.7. We define the *order of an element* $fW^n \in \mathcal{G}$ *at* $\xi \in \mathrm{Sing}(\mathcal{G})$ as

$$\mathrm{ord}_\xi(fW^n) = \frac{\nu_\xi(f)}{n}.$$

We define the *order of the Rees algebra* $\mathcal{G}$ *at* $\xi \in \mathrm{Sing}(\mathcal{G})$ as the infimum of the orders of the elements of $\mathcal{G}$ at ξ, that is

$$\mathrm{ord}_\xi(\mathcal{G}) = \inf_{l \geq 0} \left\{ \frac{\mathrm{ord}_\xi(I_l)}{l} \right\}.$$

Theorem 1.3.8. *[36, Proposition 6.4.1] Let* $\mathcal{G} = R[f_1W^{n_1}, \dots, f_rW^{n_r}]$ *be a Rees algebra over* R *and let* $\xi \in \mathrm{Sing}(\mathcal{G})$*. Then*

$$\mathrm{ord}_\xi(\mathcal{G}) = \min_{i=1\dots r} \{\mathrm{ord}_\xi(f_iW^{n_i})\}.$$

Transformations and resolutions of Rees algebras

As we will soon show, Rees algebras are a convenient tool for handling local presentations. There is a notion of transformation of Rees algebras which eases the task of finding a local presentation for a variety after blowing up, in terms of a local presentation of the initial variety.

Definition 1.3.9. Let $\mathcal{G}$ be a Rees algebra over V. A closed set $Y \subset V$ is a *permissible center for* $\mathcal{G}$ if it is a regular subvariety contained in $\mathrm{Sing}(\mathcal{G})$.

Definition 1.3.10. Let $\mathcal{G}$ be a Rees algebra on V. A $\mathcal{G}$*-permissible (monoidal) transformation*

$$V \overset{\pi}{\leftarrow} V_1,$$

is the blow up of V at a permissible center $Y \subset V$ for $\mathcal{G}$. We denote then by $\mathcal{G}_1$ the (weighted) transform of $\mathcal{G}$ by π, which is defined as

$$\mathcal{G}_1 := \bigoplus_{l \in \mathbb{N}} I_{l,1}W^l,$$

where

$$I_{l,1} = I_l\mathcal{O}_{V_1} \cdot I(E)^{-l} \tag{1.5}$$

for $l \in \mathbb{N}$ and E the exceptional divisor of the blow up $V \longleftarrow V_1$.

Let $V = \mathrm{Spec}(R)$, and let $\mathcal{G} = R[f_1W^{n_1}, \dots, f_rW^{n_r}]$ be a Rees algebra over V. Then, $< f_iW^{n_i} > \mathcal{O}_{V_1}$ is an element in the total transform of $\mathcal{G}$, and the weighted transform of $\mathcal{G}$ by π is locally generated by $\{f_{1,1}W^{n_1}, \dots, f_{r,1}W^{n_1}\}$, where $f_{i,1}W^{n_i}$ is a weighted transform of $f_iW^{n_i}$ by π for $i = 1, \dots, r$ (see [36, 1.6]). That is, a generator of the principal ideal

$$I(E)^{-n_i}(f_i)\mathcal{O}_{V_1}.$$

Definition 1.3.11. Let $\mathcal{G}$ be a Rees algebra over V. A *resolution of* $\mathcal{G}$ is a finite sequence of transformations

$$\begin{array}{ccccccc} V = V_0 & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_l} & V_l \\ \mathcal{G} = \mathcal{G}_0 & \longleftarrow & \mathcal{G}_1 & \longleftarrow & \dots & \longleftarrow & \mathcal{G}_l \end{array} \tag{1.6}$$

at permissible centers $Y_i \subset \operatorname{Sing}(\mathcal{G}_i)$, $i = 0, \dots, l-1$, such that $\operatorname{Sing}(\mathcal{G}_l) = \emptyset$, and such that the exceptional divisor of the composition $V_0 \longleftarrow V_l$ is a union of hypersurfaces with normal crossings. Recall that a set of hypersurfaces $\{H_1, \dots, H_r\}$ in a smooth n-dimensional V has normal crossings at a point $\xi \in V$ if there is a regular system of parameters $x_1, \dots, x_n \in \mathcal{O}_{V,\xi}$ such that if $\xi \in H_{i_1} \cap \dots \cap H_{i_s}$, and $\xi \notin H_l$ for $l \in \{1, \dots, r\} \setminus \{i_1, \dots, i_s\}$, then $\mathcal{I}(H_{i_j})_\xi = \langle x_{i_j} \rangle$ for $i_j \in \{i_1, \dots, i_s\}$; we say that $H_1, \dots, H_r$ have normal crossings in V if they have normal crossings at each point of V.

A *pair* (V, E) is a couple given by a smooth scheme V and a set of hypersurfaces with normal crossings E. A *permissible transformation* for (V, E) is a blow up

$$(V, E) \overset{\pi}{\leftarrow} (V_1, E_1)$$

at a regular closed center $Y \subset V$ which has normal crossings with E. The transform will be a new pair $(V_1, E_1 = \{E, H_1\})$, where $H_1 = \pi^{-1}(Y)$. A *basic object* is a triple $(V, \mathcal{G}, E)$ where (V, E) is a pair and $\mathcal{G}$ is a Rees algebra over V. A transformation

$$(V, \mathcal{G}, E) \overset{\pi}{\leftarrow} (V_1, \mathcal{G}_1, E_1)$$

is *permissible for* $(V, \mathcal{G}, E)$ if it is a permissible transformation for $\mathcal{G}$ in the sense of Definition 1.3.10 and the center of π has normal crossings with E. A *resolution of a basic object* $(V, \mathcal{G}, E)$ is a sequence of permissible transformations

$$(V, \mathcal{G}, E) \overset{\pi_1}{\leftarrow} (V_1, \mathcal{G}_1, E_1) \overset{\pi_2}{\leftarrow} (V_2, \mathcal{G}_2, E_2) \overset{\pi_2}{\leftarrow} \dots \overset{\pi_l}{\leftarrow} (V_l, \mathcal{G}_l, E_l) \tag{1.7}$$

where $\operatorname{Sing}(\mathcal{G}_l) = \emptyset$.

We mention here a few examples that may help getting an overall impression of the use of Rees algebras in resolution of singularities.

Example 1.3.12. **Resolution of singularities of a hypersurface:** Consider a hypersurface $X \subset V$. Then $I(X)$ is locally principal. Set $\mathcal{G} = \mathcal{O}_V[I(X)W^b]$, where b is the maximum multiplicity of X. Then $\operatorname{Sing}(\mathcal{G}) = \{\eta \in V : \operatorname{mult}_\eta(X) = b\} = \underline{\operatorname{Max}} \operatorname{mult}(X)$. A resolution of the basic object $(V, \mathcal{G}, E = \{\emptyset\})$ as (1.7) gives a simplification of the points of multiplicity b of X, that is, the induced sequence $X \longleftarrow X_l$ will be such that X_l has maximum multiplicity strictly smaller than b, because

$$\operatorname{Sing}(\mathcal{G}_l) = \{\eta \in V_l : \operatorname{mult}_\eta(X_l) = b\} = \emptyset.$$

Hence, if one can resolve this Rees algebra, then one can resolve the singularities of X by iterating this process until X_r is such that its maximum multiplicity is 1.

Example 1.3.13. **Resolution of $\mathcal{G} = \mathcal{O}_V[I(X)W]$:** Let V be a smooth scheme over a field of characteristic zero. Let now $X \subset V$ be a closed reduced equidimensional subscheme, defined by $I(X) \subset \mathcal{O}_V$. Let $\mathcal{G} = \mathcal{O}_V[I(X)W]$. By Theorem 1.3.16, one can construct a resolution of the basic object $(V, \mathcal{G}, E = \{\emptyset\})$. Let us show now how a resolution of singularities of X can be obtained: For any $i \in \{1, \dots, l\}$, the transform $I(X)^{(i)}$ of $I(X)$ in $\mathcal{O}_{V_i}$, defined by $I(X)^{(i)} := I_{1,i}$ as in (1.5), is supported in the exceptional locus (which has normal crossings) as well as in the strict transform of X by $V \longleftarrow V_i$. The condition $\mathrm{Sing}(\mathcal{G}_l) = \emptyset$ implies that, for some $j \in \{1, \dots, l\}$, the strict transform X_{j-1} of X in V_{j-1} is a connected component of the center of the transform π_j, and hence is permissible. In particular, this implies that X_{j-1} is regular and has normal crossings with the exceptional divisor E_{j-1}. Therefore

$$\begin{array}{ccccccc} V = V_0 & \xleftarrow{\pi_1} & V_1 & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_j} & V_j \\ \cup \quad \cup & & \cup & & & & \cup \\ X = X_0 & \longleftarrow & X_1 & \longleftarrow & \dots & \longleftarrow & X_j \end{array} \tag{1.8}$$

is a resolution of singularities of X (see [35, proof of Theorem 1.5] for a precise proof of this result in the language of basic objects).

Example 1.3.14. **Log-resolution of ideals:** A *Log-resolution* of an ideal I on a smooth scheme V is a proper birational morphism of smooth schemes, say $V' \longrightarrow V$, so that the total transform of I, $I\mathcal{O}_{V'}$, is an invertible ideal in V' supported on smooth hypersurfaces having only normal crossings. A resolution of

$$(V, \mathcal{G} = R[IW], E = \{\emptyset\})$$

gives a Log-resolution of I. In [36], Encinas and Villamayor proved, by using Rees algebras, that for two ideals with the same integral closure, one obtains the same algorithmic Log-resolution.

Resolution functions and local presentations

In [41], H. Hironaka proves resolution of singularities of varieties over fields of characteristic zero by showing that the maximum value of the Hilbert Samuel function can be lowered after a sequence of blow ups at suitable regular centers. To this end, he used the following main idea: let X be an algebraic variety over a (perfect) field k, let $\max \mathrm{H}(X)$ be the maximum value of the Hilbert Samuel function on X, let $\underline{\mathrm{Max}}\, \mathrm{H}(X)$ be the maximum stratum of this function, and let $\xi \in \underline{\mathrm{Max}}\, \mathrm{H}(X)$. Then in some (étale) neighborhood of ξ there is an immersion of X in some smooth V and a Rees algebra $\mathcal{G}$ attached to $\underline{\mathrm{Max}}\, \mathrm{H}(X)$ in some sense that will be explained later (see Example 1.3.17 below; see also [42]). Then he shows that a resolution of $\mathcal{G}$ induces a sequence of blow ups over X that ultimately leads to a lowering of $\max \mathrm{H}(X)$. To conclude, he proves that such resolution exists when the characteristic is zero:

Theorem 1.3.15. *[41] Let k be a field of characteristic zero, and let R be a smooth k-algebra. Given a Rees algebra $\mathcal{G}$ over R, there exists a resolution of $\mathcal{G}$.*

The proof of the previous result is existencial. The following theorem says that, in fact, resolution of Rees algebras can be constructed; i.e., given a Rees algebra $\mathcal{G}$ on a smooth V defined over a field of characteristic zero, there is a procedure that indicates how to actually construct a sequence of blow ups that leads to a resolution. See [77], [78] and [9], and see also [34] for a later reformulation.

Theorem 1.3.16. *[34, Theorem 3.1] Let k be a field of characteristic zero, and let R be a smooth k-algebra. Given a Rees algebra $\mathcal{G}$ over R, it is possible to construct a resolution of $\mathcal{G}$.*

Example 1.3.17. Let X be a variety over a perfect field k. Let $\mathcal{H}(X)$ be the following version of the Hilbert-Samuel function on X:

$$\begin{array}{rccl} \mathcal{H}(X): & X & \longrightarrow & (\mathbb{N}^{\mathbb{N}}, \leq) \\ & \xi & \mapsto & \mathcal{H}(X)(\xi) = \left(\mathrm{H}_{\mathcal{O}_X}(\mathcal{M}_\xi(n))\right)_{n\in\mathbb{N}}, \end{array}$$

where $\mathbb{N}^{\mathbb{N}}$ is ordered lexicographically. This is an upper semicontinuous function[1]. Let max $\mathcal{H}(X)$ and $\underline{\mathrm{Max}}\,\mathcal{H}(X)$ denote the maximum value of $\mathcal{H}(X)$ in X and the closed subset of points where $\mathcal{H}(X)$ reaches this value respectively. Pick $\xi \in \underline{\mathrm{Max}}\,\mathcal{H}(X)$. Then (see [42]), it is possible to find, locally in an étale neighborhood of ξ, an immersion of X in a smooth scheme V and equations $f_1, \ldots, f_r$ such that $I(X) =< f_1, \ldots, f_r >$,

$$\underline{\mathrm{Max}}\,\mathcal{H}(X) = \cap_{i=1}^{r} \underline{\mathrm{Max}}\,\mathcal{H}(\{f_i = 0\}),$$

and such that this condition is preserved by blow ups at smooth centers contained in $\underline{\mathrm{Max}}\,\mathcal{H}(X)$, in terms of the strict transforms of X and of the f_i. Let us translate this into the language of Rees algebras: let $\mathcal{G} = \mathcal{O}_{V,\xi}[f_1W^{\mu_1}, \ldots, f_rW^{\mu_r}]$, where μ_i is the maximum order of f_i for $i = 1, \ldots, r$. Then

$$\mathrm{Sing}(\mathcal{G}) = \underline{\mathrm{Max}}\,\mathcal{H}(X),$$

and this condition is preserved after permissible blow ups. Resolving the Rees algebra $\mathcal{G}$ is equivalent to making max $\mathcal{H}(X)$ decrease after a finite sequence of blow ups. (See [41],[43], [42].)

Example 1.3.18. Let X be a variety over a perfect field k. For any $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$, Example 1.2.13 shows how to find a local presentation for the multiplicity of X in an (étale) neighborhood of ξ. The Rees algebra

$$\mathcal{G}_X = S[x_1, \ldots, x_r][f_1W^{n_1}, \ldots, f_rW^{n_r}] \subset \mathcal{O}_{V^{(n)}}[W]$$

for $\{f_1, \ldots, f_r; n_1, \ldots, n_r\}$ as there is such that

$$\mathrm{Sing}(\mathcal{G}_X) = \underline{\mathrm{Max}}\,\mathrm{mult}(X),$$

and that this is preserved by permissible transformations. A resolution of this Rees algebra induces a simplification of the multiplicity of X.

[1] Actually, the Hilbert-Samuel function has to be modified in order to be semicontinuous (see [7]). We use here this modification of the Hilbert-Samuel function.

In section 1.2, we introduced local presentations as a way of describing with equations the set C of the worst singularities of a variety. The previous examples show that Rees algebras appear as an appropriate language to represent such a set of equations and weights, and allow us to describe how certain transformations of a variety X affect C, via well defined transformations of the associated Rees algebra (see (1.5)). It is very important to understand to which extent a given algebra can represent a given closed set C. It is clear that all this construction would not be useful if different presentations lead to different simplifications of the singularities under the same transformations. This induces us to consider an identification of Rees algebras (of local presentations) which have a compatible behavior under permissible transformations. In addition, we expect this presentations to also behave well up to smooth morphisms and restriction to open subsets of X. Let us give some definitions here that will play a role in the construction of such an identification on the set of Rees algebras over a given scheme V.

Definition 1.3.19. A *local sequence on* a smooth scheme V over a field k is a sequence of morphisms

$$V = V_0 \xleftarrow{\phi_1} V_1 \xleftarrow{\phi_2} \dots \xleftarrow{\phi_r} V_r$$

where each ϕ_i is either a blow up at a regular center or a smooth morphism, such as an open immersion or a projection from a product by an affine line.

Definition 1.3.20. Let $\mathcal{G}$ be a Rees algebra over $\mathcal{O}_V$. A *$\mathcal{G}$-local sequence over* V is a local sequence on V as in Definition 1.3.19,

$$\begin{array}{c} V = V_0 \xleftarrow{\phi_1} V_1 \xleftarrow{\phi_2} \dots \xleftarrow{\phi_r} V_r \\ \mathcal{G} = \mathcal{G}_0 \xleftarrow{\psi_1} \mathcal{G}_1 \xleftarrow{\psi_2} \dots \xleftarrow{\psi_r} \mathcal{G}_r, \end{array} \tag{1.9}$$

such that whenever ϕ_i is a blow up, it is in particular a blow up at a permissible center $Y_{i-1} \subset \operatorname{Sing}(\mathcal{G}_{i-1}) \subset V_{i-1}$, and then $\mathcal{G}_i$ is the transform of $\mathcal{G}_{i-1}$ by the rule in Definition 1.3.10; if ϕ_i is a smooth morphism, then $\mathcal{G}_i$ is the pullback of $\mathcal{G}_{i-1}$ by ϕ_i (see [13, Definition 3.2]).

Definition 1.3.21. Let $\mathcal{G}$ be a Rees algebra over V, and consider a $\mathcal{G}$-local sequence over V as in (1.9). This sequence determines a collection of closed sets, namely $\{\operatorname{Sing}(\mathcal{G}), \operatorname{Sing}(\mathcal{G}_1), \dots, \operatorname{Sing}(\mathcal{G}_r)\}$. We will refer to this collection (or *branch*) of closed sets as the one defined by or attached to the sequence (1.9). If we consider all possible $\mathcal{G}$-local sequences over V, we obtain a *tree of closed sets* for $\mathcal{G}$, which we denote by $\mathcal{F}_V(\mathcal{G})$ (see [13, Section 3]).

Hironaka uses this kind of constructions to obtain resolution invariants.

Definition 1.3.22. Let F be an upper semicontinuous function defined on varieties. An *F_X-local sequence* is a local sequence on X (Definition 1.3.19) such that, whenever ϕ_i is a blow up, the center is contained in $\underline{\operatorname{Max}} F_{X_{i-1}}$.

Definition 1.3.23. (see [16, Definition 28.4]) An upper semicontinuous function F defined on varieties as (1.2) is said to be *representable via local embeddings* if, for each X and each $\xi \in X$, in an étale neighborhood of ξ, we can find a closed immersion $X \hookrightarrow V$ and a Rees algebra $\mathcal{G}$ over $\mathcal{O}_{V,\xi}$ such that

1. The Rees algebra $\mathcal{G}$ satisfies:

$$\operatorname{Sing}(\mathcal{G}) = \underline{\operatorname{Max}}\, F_X; \tag{1.10}$$

2. Any F_X-local sequence

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_r \tag{1.11}$$

such that

$$m = \max F_X = \max F_{X_1} = \ldots = \max F_{X_{r-1}} \geq \max F_{X_r} \tag{1.12}$$

induces a $\mathcal{G}$-local sequence of Rees algebras over V

$$V = V_0 \leftarrow V_1 \leftarrow \ldots \leftarrow V_r \tag{1.13}$$

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_r \tag{1.14}$$

$$\mathcal{G} = \mathcal{G}_0 \leftarrow \mathcal{G}_1 \leftarrow \ldots \leftarrow \mathcal{G}_r \tag{1.15}$$

such that for $i = 1, \ldots, r$,

$$\operatorname{Sing}(\mathcal{G}_i) = \{\eta \in X_i : F_{X_i}(\eta) = m\}, \tag{1.16}$$

with $\operatorname{Sing}(\mathcal{G}_r) = \emptyset$ if and only if $\max F_{X_r} < m$.

3. Any $\mathcal{G}$-local sequence over V induces an F_X-local sequence as (1.13) satisfying (1.16).

Theorem 1.3.24. *([42]) The Hilbert-Samuel function is representable for any variety X via local embeddings. Thus, for each point $\xi \in X$ we can find, in an étale neighborhood of ξ, an immersion of X into a smooth scheme V and an $\mathcal{O}_{V,\xi}$-Rees algebra $\mathcal{G}_X$ such that* $\operatorname{Sing}(\mathcal{G}_X) = \underline{\operatorname{Max}}\ \mathcal{H}(X)$ *and this identity is preserved by $\mathcal{G}_X$-local sequences over V as long as the maximum value of the Hilbert-Samuel function of X does not decrease.*

Theorem 1.3.25. *([82, Proposition 5.7 and Theorem 7.1]) The multiplicity function is representable via local embeddings for X. That is, for each point $\xi \in$* $\underline{\operatorname{Max}}\operatorname{mult}(X)$*, we may find (in an étale neighborhood of ξ) an immersion of X in a smooth V, and a Rees algebra $\mathcal{G}_X$ over V such that* $\operatorname{Sing}(X) = \underline{\operatorname{Max}}\operatorname{mult}(X)$*, and that this condition is preserved by $\mathcal{G}_X$-local sequences over V while the maximum multiplicity does not decrease.*

Therefore, just as for the Hilbert-Samuel function in Example 1.3.17, we can attach a Rees algebra $\mathcal{G}$ to $\operatorname{mult}(X)$ so that resolving $\mathcal{G}$ is equivalent to decreasing the maximum value of $\operatorname{mult}(X)$.

Example 1.3.26.

- Let $R = k[x,y]$. The Rees algebra $\mathcal{G}_1 = R[(x^2-y^3)W^2]$ represents the multiplicity of X_1 from Example 1.2.7 in $\operatorname{Spec}(k[x,y])$.
- Consider now $R = k[x,y,z]$. The Rees algebra $\mathcal{G}_2 = R[(x^2-z^5)W^2,(y^3-z^4)W^3]$ represents the multiplicity of $X_2 = \mathbb{V}(x^2-z^5, y^3-z^4) \subset \operatorname{Spec}(R)$.
- The Rees algebra $\mathcal{G}_3 = R[((x^2-y^3)^2+z^2)W^2]$ represents the multiplicity of $X_3 \subset \operatorname{Spec}(R)$ as in Example 1.2.7, where $R = k[x,y,z]$ again.
- The Rees algebra over $R = k[x,y,z]$ given by $\mathcal{G}_4 = R[(x^3-y^3z^2)W^3]$ represents the multiplicity of X_4 from Example 1.2.7 in $\operatorname{Spec}(R)$.
- The Rees algebra over $R = k[x,y,z]$ given by $\mathcal{G}_6 = R[(x^3-xyz^2-yz^3+z^5)W^3]$ represents the multiplicity of $X_6 = \mathbb{V}(x^3-xyz^2-yz^3+z^5) \subset \operatorname{Spec}(R)$ locally at the origin.
- The Rees algebra $\mathcal{G}_7 = R[(xy-z^4)W^2]$ for $R = k[x,y,z]$ represents the multiplicity of $X_{10} = \mathbb{V}(xy-z^4) \subset \operatorname{Spec}(R)$.

Weak equivalence

Given an upper semicontinuous function F as in (1.2) which is representable via local embeddings, the choice of a Rees algebra satisfying the properties of Definition 1.3.23 is not unique. Note that, when we construct a Rees algebra $\mathcal{G}$ attached to $\underline{\operatorname{Max}}\,\mathrm{F}(X)$ at a point $\xi \in X$, we are considering this closed set as a closed set of the ambient space V. However, there are many possible choices for the immersion $X \hookrightarrow V$, as well as for a Rees algebras over V. Therefore, given two possible choices of Rees algebras, $\mathcal{G}$ and $\mathcal{G}'$ over V, attached to a fixed point $\xi \in \underline{\operatorname{Max}}\,\mathrm{F}(X)$, it would be desirable to compare the algorithmic resolution of $\mathcal{G}$ to that of $\mathcal{G}'$, and vice versa. To deal with this problem, we need to use the following notion of weak equivalence of Rees algebras.

Definition 1.3.27. [13, Definition 3.5] We say that two $\mathcal{O}_V$-Rees algebras $\mathcal{G}$ and $\mathcal{H}$ are *weakly equivalent* if:

1. $\operatorname{Sing}(\mathcal{G}) = \operatorname{Sing}(\mathcal{H})$,
2. Any $\mathcal{G}$-local sequence over V

   $$\mathcal{G} = \mathcal{G}_0 \longleftarrow \mathcal{G}_1 \longleftarrow \ldots \longleftarrow \mathcal{G}_r$$

   induces an $\mathcal{H}$-local sequence over V

   $$\mathcal{H} = \mathcal{H}_0 \longleftarrow \mathcal{H}_1 \longleftarrow \ldots \longleftarrow \mathcal{H}_r$$

   and vice versa, and moreover the equality in 1. is preserved, that is
3. $\operatorname{Sing}(\mathcal{G}_j) = \operatorname{Sing}(\mathcal{H}_j)$ for $j = 0, \ldots, r$.

This is an equivalence relation, and preserves properties of the Rees algebras that will be necessary for the correct definition of some invariants used in resolution, as we will see along the rest of this chapter.

The following concepts and results give a flavor of what this equivalence relation means and provide tools to compare different algebras under it. First, it is interesting to take into account that Rees algebras with the same integral closure have the same resolution invariants:

Theorem 1.3.28. *([54, Proposition 5.2.1]) Let R be a normal excellent (noetherian) domain, and let $\mathcal{G}$ be a Rees algebra over R. The integral closure of $\mathcal{G} \subset R[W]$ as a ring is also a Rees algebra over R.*

Definition 1.3.29. Two Rees algebras are *integrally equivalent* if their integral closure in $\mathrm{Quot}(\mathcal{O}_V)[W]$ coincide. We say that a Rees algebra over V, $\mathcal{G} = \oplus_{l \geq 0} I_l W^l$ is *integrally closed* if it is integrally closed as an $\mathcal{O}_V$-ring in $\mathrm{Quot}(\mathcal{O}_V)[W]$. We denote by $\overline{\mathcal{G}}$ the integral closure of $\mathcal{G}$.

Remark 1.3.30. [36, Proposition 5.4] If $\mathcal{G}_1$ and $\mathcal{G}_2$ are two integrally equivalent Rees algebras over R, then they are weakly equivalent.

If we apply differential operators to a Rees algebra, the resulting algebra is also a Rees algebra and it is weakly equivalent to the original one:

Given a k-algebra A, a differential operator on A of order s is a k-linear map $D : A \longrightarrow A$ such that for any $a \in A$ the map

$$[a, D] = a \cdot D(\bullet) - D(a \cdot \bullet)$$

is a differential operator on A of order $s - 1$ if $s > 0$, and an A-linear map if $s = 0$.

Definition 1.3.31. A Rees algebra $\mathcal{G} = \oplus_{l \geq 0} I_l W^l$ over V is *differentially closed* (or a *Diff-algebra*) if there is an affine open covering of V, $\{U_i\}$ such that for every $D \in \mathrm{Diff}^r(U_i)$ and $h \in I_l(U_i)$, we have $D(h) \in I_{l-r}(U_i)$ whenever $l \geq r$, where $\mathrm{Diff}^r(U_i)$ is the locally free sheaf of k-linear differential operators of order r or less. In particular, $I_{l+1} \subset I_l$ for $l \geq 0$. We denote by $\mathrm{Diff}(\mathcal{G})$ the smallest differential Rees algebra containing $\mathcal{G}$ (its *differential closure*). (See [80, Theorem 3.4] for the existence and construction.)

Remark 1.3.32. ([80, proof of Theorem 3.4], [13, Remark 4.2]) If $\mathcal{G}$ is a Rees algebra over a smooth V, locally generated by a set $\{f_1 W^{n_1}, \ldots, f_r W^{n_r}\} \subset \mathcal{G}$, then $\mathrm{Diff}(\mathcal{G})$ is (locally) generated by the set

$$\left\{D(f_i) W^{n_i - \alpha} : D \in \mathrm{Diff}^{\alpha},\ 0 \leq \alpha < n_i,\ i = 1, \ldots, r\right\}.$$

If $\mathcal{G}$ is differentially closed, then every I_l contains the information of the singular locus of $\mathcal{G}$:

Proposition 1.3.33. *[80, Proposition 3.9] Let $\mathcal{G} = \oplus_{l \geq 0} I_l W^l$ be a differential Rees algebra. Then, for any $l \in \mathbb{Z}_{\geq 0}$, $\mathrm{Sing}(\mathcal{G}) = \mathbb{V}(I_l)$.*

Remark 1.3.34. [13, Section 4] A Rees algebra $\mathcal{G}$ and its differential closure $\mathrm{Diff}(\mathcal{G})$ are weakly equivalent. This is a consequence of Giraud's Lemma (see [37]).

Example 1.3.35.

- Let $R = k[x, y]$ as in 1.3.26, with $\mathrm{char}(k) = 0$. Consider $\mathcal{G}_1 = R[(x^2 - y^3)W^2]$. Then
$$\mathrm{Diff}(\mathcal{G}_1) = R[xW, y^3W^2, y^2W].$$
- Consider now $R = k[x, y, z]$ as in 1.3.26, the differential closure of the Rees algebras described there are, respectively,
$$\mathcal{G}_2 = R[(x^2 - z^5)W^2, (y^3 - z^4)W^3] \to \mathrm{Diff}(\mathcal{G}_2) = R[xW, yW, z^4W^3, z^3W^2, z^2W],$$
$$\mathcal{G}_3 = R[((x^2 - y^3)^2 + z^2)W^2] \to \mathrm{Diff}(\mathcal{G}_3) = R[zW, (x^3 - xy^3)W, (y^5 - x^2y^2)W],$$
$$\mathcal{G}_4 = R[(x^3 - y^3z^2)W^3] \to \mathrm{Diff}(\mathcal{G}_4) = R[xW, yz^2W, y^3W, y^2zW, y^2z^2W^2, y^3zW^2, y^3z^2W^3],$$
$$\mathcal{G}_6 = R[(x^3 - xyz^2 - yz^3 + z^5)W^3] \to \mathrm{Diff}(\mathcal{G}_6) = R[(x, yz, z^2)W, (yz^2, z^3)W^2, yz^3W^3],$$
$$\mathcal{G}_7 = R[(xy - z^4)W^2] \to \mathrm{Diff}(\mathcal{G}_7) = R[xW, yW, z^3W, z^4W^2].$$

Theorem 1.3.36. *[13, Theorem 3.11] Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two Rees algebras over V. Then $\mathcal{G}_1$ and $\mathcal{G}_2$ are weakly equivalent if and only if $\overline{\mathrm{Diff}(\mathcal{G}_1)} = \overline{\mathrm{Diff}(\mathcal{G}_2)}$.*

In [2, Chapter 5], results in this direction in the more general context of certain classes of regular excellent schemes are developed.

Finally, this equivalence relation satisfies the compatibility properties that we asked for their use in Resolution of Singularities:

Corollary 1.3.37. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two weakly equivalent Rees algebras over V. Then for all $\eta \in \mathrm{Sing}(\mathcal{G}_1) = \mathrm{Sing}(\mathcal{G}_2)$, we have $\mathrm{ord}_\eta\mathcal{G}_1 = \mathrm{ord}_\eta\mathcal{G}_2$.*

The importance of this result relies on the fact that, given a Rees algebra $\mathcal{G}$ and $\xi \in \mathrm{Sing}(\mathcal{G})$, then $\mathrm{ord}_\xi\mathcal{G}$ is the most important invariant for the construction of a resolution of $\mathcal{G}$ (over a field of characteristic zero). In fact:

Corollary 1.3.38. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two weakly equivalent Rees algebras. Then a constructive resolution of $\mathcal{G}_1$ induces a constructive resolution of $\mathcal{G}_2$ and vice versa (see [16, Remark 11.8]).*

Remark 1.3.39. Let X be a variety, and fix an immersion $X \hookrightarrow V$. Any two local presentations of X attached to the multiplicity or to the Hilbert-Samuel function are weakly equivalent by definition, and therefore Corollary 1.3.37 applies: fixed an immersion for X, the order of a Rees algebra attached to a local presentation at any point of its singular locus does not depend on the local presentation, and neither does the resolution. This gives an answer to the problem of compatibility of Rees algebras over V.

Moreover, for different choices of the immersion of X into a smooth V, resolution invariants can be proven to only depend on X. For a discussion on identification of local presentations with different immersions and results about invariants, we refer to [16].

1.4 Elimination

In the following examples one can observe that, in some cases, the relevant information regarding the simplification of the multiplicity of a variety $X^{(d)} \hookrightarrow V^{(n)}$ can be reflected in a lower dimensional smooth variety. In order to explain this idea, we will use the concept of *elimination*, introduced by Villamayor in [80]. We will explain some of the main ideas along this section.

We assume here that all varieties we consider are defined over a field of characteristic zero.

Example I: Hypersurface case

Example 1.4.1. Let S be a regular d-dimensional k-algebra of finite type, with $d > 0$. Let $V^{(n)} = \mathrm{Spec}(S[x])$, where $n = d + 1$. Consider the natural injective morphism

$$S \xrightarrow{\beta^*} S[x],$$

and the induced smooth projection

$$V^{(n)} \xrightarrow{\beta} V^{(d)} = \mathrm{Spec}(S). \tag{1.17}$$

Let X be a hypersurface in $V^{(n)}$, $X = \mathrm{Spec}(S[x]/f(x))$, where f is a polynomial in x of degree $b > 1$ with coefficients in S. Let $\xi^{(n)}$ be a point in the closed set of multiplicity b of X. We are going to assume that the maximal ideal $\mathcal{M}_{\xi^{(n)}}$ of $\xi^{(n)}$ in $S[x]$ is given by $< x, z_1, \dots, z_d >$ for a regular system of parameters $\{z_1, \dots, z_d\}$ in S. The image $\xi^{(d)}$ of $\xi^{(n)}$ by the projection (1.17) is defined by the maximal ideal $\mathcal{M}_{\xi^{(d)}} =< z_1, \dots, z_d >$. Then, the Rees algebra $\mathcal{G}_X^{(n)}$ over $S[x]$

$$\mathcal{G}_X^{(n)} = \mathrm{Diff}(S[x][fW^b]) \subset S[x][W]$$

represents the multiplicity function on $X \subset V^{(n)}$ locally at $\xi^{(n)}$.

Let us suppose that, in addition, f has the form of Tschirnhausen:

$$f(x) = x^b + B_{b-2}x^{b-2} + \ldots + B_i x^i + \ldots + B_0 \in S[x], \tag{1.18}$$

where $B_i \in S$ for $i = 0, \dots, b-2$.

The following lemma shows that for X as in Example 1.4.1, the meaningful part of $f \in S[x]$ (regarding the maximum multiplicity) is given by the coefficients B_i, which are already in S.

Lemma 1.4.2. *Let X be given by f as in (1.18). Then*

$$\mathcal{G}_X^{(n)} = S[x][xW] \odot \operatorname{Diff}(S[x][B_{b-2}W^2, \ldots, B_i W^{b-i}, \ldots, B_0 W^b]).$$

Proof. In order to compute the differential closure of $S[x][fW^b]$, let us start by computing the $(b-1)$-th derivative of fW^b with respect to x, from where it follows that $xW \in \mathcal{G}_X^{(n)}$. Therefore $f_2 W^b = fW^b - (xW)^b \in \mathcal{G}_X^{(n)}$ and, if we consider xW and $f_2 W^b$ among the generators of $\mathcal{G}_X^{(n)}$, there is no need to include fW^b. To continue, we compute the $(b-2)$-th derivative of $f_2 W^b$ with respect to x obtaining, up to a nonzero constant, $B_{b-2}W^2 \in \mathcal{G}_X^{(n)}$. Just like in the previous step, it is possible to verify that $f_3 W^b = f_2 W^b - (B_{b-2}W^2)(xW)^{b-2} \in \mathcal{G}_X^{(n)}$, and that $f_2 W^b$ can be generated by xW, $B_{b-2}W^2$ and $f_3 W^b$. By iterating this argument, one concludes that the set consisting of xW and $B_i W^{b-i}$ for $i = 0, \ldots b-2$ is contained in $\mathcal{G}_X^{(n)}$ and, in addition, the differential closure of the $S[x]$-Rees algebra generated by this set corresponds exactly to $\mathcal{G}_X^{(n)}$. □

Remark 1.4.3. For $X = \mathbb{V}(f) \subset \operatorname{Spec}(S[x])$ as above, X has multiplicity $b = \max \operatorname{mult}(X)$ at a point $\xi \in X$ if and only if for all $i = 0, \ldots, b-2$, the order of B_i at $\xi^{(d)} \in \operatorname{Spec}(S)$ is greater or equal than $b - i$.

Remark 1.4.4. Since the generators of $S[x][B_{b-2}W^2, \ldots, B_i W^{b-i}, \ldots, B_0 W^b]$ are elements in $S[W]$, they also generate a Rees algebra $\mathcal{H}^{(d)}$ over S. This algebra is already differentially closed with respect to x. Then, the algebra $\mathcal{G}_X^{(n)}$ from Lemma 1.4.2 can be written as

$$\mathcal{G}_X^{(n)} = \operatorname{Diff}(S[x][xW] \odot \mathcal{H}^{(d)}).$$

The S-Rees algebra $\mathcal{H}^{(d)}$ already tells us if $\xi \in X$ is a point of maximum multiplicity of X or not, by Remark 1.4.3. Moreover, we will see that finding a resolution of the $S[x]$-Rees algebra $\mathcal{G}_X^{(n)}$ is equivalent to finding a resolution of the S-Rees algebra $\mathcal{H}^{(d)}$. This reduces the problem (in dimension n) of decreasing the multiplicity of $X = \mathbb{V}(f) \subset V^{(n)}$ to solving a problem in a d-dimensional smooth scheme $V^{(d)}$.

Example 1.4.5. Instead of (1.18), suppose now that f is of the form

$$f(x) = x^b + D_{b-1}x^{b-1} + \ldots + D_i x^i + \ldots + D_0 \in S[x], \tag{1.19}$$

where $D_i \in S$, $D_{b-1} \neq 0$ and $\nu_\xi(D_i) \geq b - i$ for $i = 0, \ldots b-1$. After a suitable change, namely $\tilde{x} = x + \frac{D_{b-1}}{b}$, we obtain

$$f(x) = \tilde{f}(\tilde{x}) = \tilde{x}^b + B_{b-2}\tilde{x}^{b-2} + \ldots + B_0 \in S[\tilde{x}], \;\; B_i \in S, \;\; \nu_\xi(B_i) \geq b - i.$$

Remark 1.4.6. By means of the Weierstrass preparation Theorem ([85, Chapter VII]) when X is locally a hypersurface we obtain, maybe after considering an étale extension, that X is given locally by some f as in (1.18) (see [79, 1.1 and Proposition 1.8] and Example 1.4.5). Hence, for (local) hypersurfaces defined over fields of characteristic zero, we may assume to be under the hypotheses of Lemma 1.4.2, up to étale extension.

Example II: General case

Example 1.4.7. (See [82, 7.1]) Let X be a variety of dimension d over a perfect field k of maximum multiplicity b, and let $\xi \in X$ be a point in $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. We have, after possibly replacing X by an étale neighborhood of ξ, a smooth k-algebra S, of dimension d, and a finite and *transversal* projection

$$\beta_X : X \longrightarrow \mathrm{Spec}(S) = V^{(d)} \tag{1.20}$$

over a smooth variety of dimension d. By transversal we mean a finite projection of generic rank $b = \max\mathrm{mult}(X)$. It can be shown that β_X induces a homeomorphism between $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ and its image ([16, Appendix A], [82, 4.8]), and an injective finite morphism

$$S \longrightarrow B = S[\theta_1, \dots, \theta_{n-d}] \cong S[x_1, \dots, x_{n-d}]/I(X).$$

As a consequence, we have a local immersion of X in a smooth n-dimensional space

$$V^{(n)} = \mathrm{Spec}(S[x_1, \dots, x_{n-d}])$$

in a neighborhood of ξ.

As we already explained when we discussed local presentations attached to the multiplicity in Section 1.2 (see Examples 1.2.13 and 1.3.18), there exist $f_1, \dots, f_{n-d} \in I(X) \subset S[x_1, \dots, x_{n-d}]$ such that for some positive integers $b_1, \dots, b_{n-d}$ the Rees algebra

$$\mathcal{G}_X^{(n)} = \mathrm{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_1 W^{b_1}, \dots, f_{n-d} W^{b_{n-d}}]) \tag{1.21}$$

represents $\mathrm{mult}(X) : X \longrightarrow \mathbb{N}$ locally at ξ. In addition, for $j = 1, \dots, n-d$,

$$f_j \in S[x_j] \tag{1.22}$$

is the minimum polynomial of θ_j over the quotient field of S (it can be proven that its coefficients are actually in S, see [82, 7.1] for more details). For any $j = 1, \dots n-d$ the following diagram commutes:

$$\begin{array}{ccccccc} S[x_1, \dots, x_{n-d}] & \longrightarrow & S[x_1, \dots, x_{n-d}]/(f_1, \dots, f_{n-d}) & \longrightarrow & B & \longrightarrow & 0 \\ \uparrow & & \uparrow & & & & \\ S[x_j] & \longrightarrow & S[x_j]/(f_j) & & & & \\ \uparrow & \nearrow & & & & & \\ S & & & & & & \end{array} \tag{1.23}$$

Due to (1.22) and Remark 1.4.6, if $\text{char}(k) = 0$, we can perform changes of variables for all of the x_j as in 1.4.5 in order to obtain an expression as in (1.18) for each of the f_j. We will therefore assume that, when we consider a local presentation attached to the multiplicity for X as (1.21), the f_j have the form of Tschirnhausen:

$$f_j(x) = x^b + B_{b-2}^{(j)} x^{b-2} + \ldots + B_i^{(j)} x^i + \ldots + B_0^{(j)} \in S[x],$$

and similarly to Remark 1.4.4:

$$\mathcal{G}_X^{(n)} = \text{Diff}(S[x_1, \ldots, x_{n-d}][x_1 W, \ldots, x_{n-d} W]) \odot \mathcal{H}_1^{(d)} \odot \ldots \odot \mathcal{H}_{n-d}^{(d)},$$

where each $\mathcal{H}_j^{(d)}$ is the S-algebra generated by the coefficients of f_j as in Lemma 1.4.2. As we will see, a resolution of $\mathcal{H}_1^{(d)} \odot \ldots \odot \mathcal{H}_{n-d}^{(d)}$ leads to a simplification of the multiplicity of X.

Let us formalize the ideas from Examples 1.4.1 and 1.4.7 next.

Elimination algebras

Given an n-dimensional smooth scheme of finite type $V^{(n)}$ and a Rees algebra $\mathcal{G}^{(n)}$ over $V^{(n)}$, which we will refer to as a *pair* from now on, one may wonder if it would be possible to find a new pair $(V^{(n-e)}, \mathcal{G}^{(n-e)})$ of dimension $n-e < n$, as in Examples 1.4.1 and 1.4.7, so that a resolution of $\mathcal{G}^{(n-e)}$ induces a resolution of $\mathcal{G}^{(n)}$, since the first one could be easier to find. This can be done for suitable values of e, limited by the invariant τ of an algebra $\mathcal{G}$ at a point $\xi \in \text{Sing}(\mathcal{G})$:

Definition 1.4.8. By the *tangent space of* $V^{(n)}$ *at* ξ we mean the spectrum of the graduate ring of the local ring $\mathcal{O}_{V^{(n)},\xi}$ at ξ, $\mathfrak{gr}_{\mathcal{M}_\xi}(\mathcal{O}_{V^{(n)},\xi})$, that is,

$$\text{Spec}(k(\xi) \oplus \mathcal{M}_\xi/\mathcal{M}_\xi^2 \oplus \mathcal{M}_\xi^2/\mathcal{M}_\xi^3 \oplus \ldots).$$

Let $\mathcal{G}^{(n)} = \oplus_{l \geq 0} I_l W^l$ be a Rees algebra over $V^{(n)}$. The *tangent cone* $\mathcal{C}_{\mathcal{G}^{(n)},\xi}$ of $\mathcal{G}^{(n)}$ at ξ is the subset of the tangent space of $V^{(n)}$ at ξ defined by the homogeneous ideal $I_{\mathcal{G}^{(n)},\xi} \subset \mathfrak{gr}_{\mathcal{M}_\xi}(\mathcal{O}_{V^{(n)},\xi})$ generated by the class of I_l in $\mathcal{M}_\xi^l/\mathcal{M}_\xi^{l+1}$ for all $l \geq 1$ (see [5, Section 4] for details).

One may consider now the largest linear subspace $\mathcal{L}_{\mathcal{G}^{(n)},\xi} \subset \mathcal{C}_{\mathcal{G}^{(n)},\xi}$ which acts on the tangent cone of $\mathcal{G}^{(n)}$ at ξ by translations.

Definition 1.4.9. The invariant $\tau_{\mathcal{G}^{(n)},\xi}$ is the codimension of $\mathcal{L}_{\mathcal{G}^{(n)},\xi}$ in the tangent space of $V^{(n)}$ at ξ. One can prove that this also corresponds to the minimum number of variables which are needed to define the tangent cone of $\mathcal{G}^{(n)}$ at ξ, that is, the minimum number of variables needed to define the ideal $I_{\mathcal{G}^{(n)},\xi} \subset \mathfrak{gr}_{\mathcal{M}_\xi}(\mathcal{O}_{V^{(n)},\xi})$ (see [18, Appendix II, p. 100]).

It can be shown that, if X is defined over a field of characteristic zero and $\mathcal{G}^{(n)}$ is assumed to be differentially closed, then $\mathcal{L}_{\mathcal{G}^{(n)},\xi} = \mathcal{C}_{\mathcal{G}^{(n)},\xi}$, and $\tau_{\mathcal{G}^{(n)},\xi}$ is the codimension of the smallest regular subvariety containing $\mathrm{Sing}(\mathcal{G}^{(n)})$ in a neighborhood of ξ (see [5, Remark 4.5]).

The τ invariant does not vary under weak equivalence:

Theorem 1.4.10. *[5, Remark 4.5, Theorem 5.2] Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two Rees algebras over $V^{(n)}$ and assume that they are weakly equivalent. Then, for any $\xi \in \mathrm{Sing}(\mathcal{G}_1) = \mathrm{Sing}(\mathcal{G}_2)$, we have $\tau_{\mathcal{G}_1,\xi} = \tau_{\mathcal{G}_2,\xi}$.*

Consider now a Rees algebra $\mathcal{G}^{(n)}$ over $V^{(n)}$, and let $\xi \in \mathrm{Sing}(\mathcal{G}^{(n)})$, with $\tau_{\mathcal{G}^{(n)},\xi} \geq 1$. Assume that there exists a smooth projection

$$\beta : V^{(n)} \longrightarrow V^{(n-1)} \tag{1.24}$$

inducing a homeomorphism between $\mathrm{Sing}(\mathcal{G}^{(n)})$ and $\beta(\mathrm{Sing}(\mathcal{G}^{(n)}))$ in a neighborhood of ξ. The idea here is to explore whether one can define a Rees algebra $\mathcal{G}^{(n-1)}$ over $V^{(n-1)}$ via the projection β such that finding a resolution of $\mathcal{G}^{(n)}$ is somehow equivalent to finding a resolution of $\mathcal{G}^{(n-1)}$. It turns out that this is possible by asking for a few technical conditions on β.

Definition 1.4.11. Assume that $\mathcal{G}^{(n)}$ is such that $\tau_{\mathcal{G}^{(n)},\xi} \geq 1$. Then a projection β as in (1.24) is *transversal to $\mathcal{G}^{(n)}$ at ξ* if the intersection of $\ker(d\beta)$ and the linear space $\mathcal{L}_{\mathcal{G}^{(n)},\xi}$ is just the origin, where $d\beta$ is the map induced by β on the tangent spaces. Is can be shown that if β is transversal to $\mathcal{G}^{(n)}$ at ξ, then it is transversal to $\mathcal{G}^{(n)}$ in a neighborhood of ξ (see [14, Remmark 8.5]).

Definition 1.4.12. Assume that $\mathrm{ord}_\xi(\mathcal{G}^{(n)}) = 1$ and that ξ is not contained in a component of $\mathrm{Sing}(\mathcal{G}^{(n)})$ of codimension 1[2]. Then a smooth projection β as in (1.24) is *$\mathcal{G}^{(n)}$-admissible* if it is transversal to $\mathcal{G}^{(n)}$ in a neighborhood of ξ and differentially closed with respect to β, meaning with respect to the relative differential operators $\mathrm{Diff}_{V^{(n)}/V^{(n-1)}}$.

Such an admissible projection can be found whenever $\mathrm{ord}_\xi(\mathcal{G}^{(n)}) = 1$ or, equivalently, whenever $\mathrm{Sing}(\mathcal{G}^{(n)})$ is locally contained in a regular subvariety of $V^{(n)}$ of dimension $n-1$, and additionally $\mathcal{G}^{(n)}$ is differentially closed with respect to β and ξ is not contained in a component of codimension 1 of $\mathrm{Sing}(\mathcal{G}^{(n)})$. In fact, there are many projections which are suitable for this role.

Let $\mathcal{G}^{(n)}$ be a differential Rees algebra over $V^{(n)}$ (with respect to $\mathrm{Diff}_{V^{(n)}/V^{(n-e)}}$ at least), and let $\xi \in \mathrm{Sing}(\mathcal{G}^{(n)})$ be a closed point. For some $e \geq 1$, assume that a smooth projection

$$\beta : V^{(n)} \longrightarrow V^{(n-e)} \tag{1.25}$$

[2] In case ξ is contained in a component $Y \subset \mathrm{Sing}(\mathcal{G}^{(n)})$ of codimension 1, then it can be proven that Y is a smooth component, and elimination is not useful in that case (see [14, Lemma 13.2])

can be constructed by repeating e times the previous process. This can be done for all e satisfying $\tau_{\mathcal{G}^{(n)},\xi} \geq e \geq 1$. It can be proven that this projection will induce a homeomorphism between $\mathrm{Sing}(\mathcal{G}^{(n)})$ and its image $\beta(\mathrm{Sing}(\mathcal{G}^{(n)}))$ in a neighborhood of ξ (see [80, Definition 4.10, Theorem 4.11] and [16, 16.1]). A projection β defined this way will be $\mathcal{G}^{(n)}$-*admissible*.

Definition 1.4.13. [80] We define the *elimination algebra* of $\mathcal{G}^{(n)}$ with respect to β in (1.25) as

$$\mathcal{G}^{(n-e)} = \mathcal{G}^{(n)} \cap \mathcal{O}_{V^{(n-e)}}$$

up to integral closure.

For a complete description of the properties asked to the projections, and of elimination algebras, we refer to [14], [16, 16 and Appendix A], [82] and [80, Theorem 4.11 and Theorem 4.13].

To discuss the behavior of elimination algebras under blow ups, we need some important properties. Let us start with the following:

Theorem 1.4.14. *([80, Theorem 4.11], [14, Theorem 9.1],[16, 16.7]) Let $\mathcal{G}^{(n)}$ be a Rees algebra over $V^{(n)}$, and let $\beta : V^{(n)} \longrightarrow V^{(n-e)}$ be a $\mathcal{G}^{(n)}$-admissible projection as in (1.25). Then*

$$\beta(\mathrm{Sing}(\mathcal{G}^{(n)})) \subset \mathrm{Sing}(\mathcal{G}^{(n-e)}).$$

If $V^{(n)}$ is defined over a field k of characteristic zero or if $\mathcal{G}^{(n)}$ is differentially closed, then both closed sets are equal.

Remark 1.4.15. Another important property of a $\mathcal{G}^{(n)}$-admissible projection as in the Theorem is that, if $Y \subset \mathrm{Sing}(\mathcal{G}^{(n)})$ is a regular closed subset, then $\beta(Y)$ is a regular closed subset of $\mathrm{Sing}(\mathcal{G}^{(n-e)})$ (see [14, 8.4]). The converse is also true. This means that any permissible center $Y \subset V^{(n)}$ for $\mathcal{G}^{(n)}$ induces a permissible center $\beta(Y) \subset V^{(n-e)}$ for $\mathcal{G}^{(n-e)}$ (see [80, 6.7] and [79, Lemma 1.7]).

It can be proven that any $\mathcal{G}^{(n)}$-permissible transformation

$$\begin{array}{ccc} V^{(n)} & \xleftarrow{\pi} & V_1^{(n)} \\ \mathcal{G}^{(n)} & \longleftarrow & \mathcal{G}_1^{(n)} \end{array}$$

yields a commutative diagram

$$\begin{array}{ccc} V^{(n)} & \xleftarrow{\pi} & V_1^{(n)} \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta_1} \\ V^{(n-e)} & \xleftarrow{\pi^{(n-e)}} & V_1^{(n-e)} \end{array} \qquad (1.26)$$

where β_1 might be defined only in an open subset of $V_1^{(n)}$, but this set will necessarily contain $\mathrm{Sing}(\mathcal{G}_1^{(n)})$ so, to ease the notation, we will write $V_1^{(n)}$ meaning this open subset. Moreover, β_1 is $\mathcal{G}_1^{(n)}$-permissible (see [14, Theorem 9.1]). It is important to observe that, although we will normally have a differentially closed Rees algebra $\mathcal{G}^{(n)}$, the transform $\mathcal{G}_1^{(n)}$ by π needs not to be differentially closed. However, it can be checked that $\mathcal{G}_1^{(n)}$ will be differentially closed with respect to β_1 (meaning with respect to the differential operators $\mathrm{Diff}_{V_1^{(n)}/V_1^{(n-e)}}$).

In addition, the transform of $\mathcal{G}^{(n-e)}$ by $\pi^{(n-e)}$ is exactly the elimination of $\mathcal{G}_1^{(n)}$ by β_1 (see [80, 6.9] and [14, Theorem 9.1]). As a consequence,

$$\beta_1(\mathrm{Sing}(\mathcal{G}_1^{(n)})) \subset \mathrm{Sing}(\mathcal{G}_1^{(n-e)})$$

by Theorem 1.4.14, having an equality between both sets over fields of characteristic zero.

These facts lead to the following properties of elimination:

1.4.16. Properties

1. Any $\mathcal{G}^{(n)}$-local sequence over $V^{(n)}$ induces a $\mathcal{G}^{(n-e)}$-local sequence over $V^{(n-e)}$ and a commutative diagram

$$\begin{array}{ccccccc} \mathcal{G}^{(n)} = \mathcal{G}_0^{(n)} & & \mathcal{G}_1^{(n)} & & \dots & & \mathcal{G}_r^{(n)} \\ & & & & & & \\ V^{(n)} = V_0^{(n)} & \longleftarrow & V_1^{(n)} & \longleftarrow & \dots & \longleftarrow & V_r^{(n)} \\ \downarrow \beta & & \downarrow \beta_1 & & & & \downarrow \beta_r \\ V^{(n-e)} = V_0^{(n-e)} & \longleftarrow & V_1^{(n-e)} & \longleftarrow & \dots & \longleftarrow & V_r^{(n-e)} \\ & & & & & & \\ \mathcal{G}^{(n-e)} = \mathcal{G}_0^{(n-e)} & & \mathcal{G}_1^{(n-e)} & & \dots & & \mathcal{G}_r^{(n-e)} \end{array} \tag{1.27}$$

   where $\mathcal{G}_i^{(n-e)}$ is an elimination algebra of $\mathcal{G}_i^{(n)}$ for $i = 0, \dots, r$, and the β_i are smooth $\mathcal{G}^{(n)}$-admissible projections, and $\beta_i(\mathrm{Sing}(\mathcal{G}_i^{(n)})) \subset \mathrm{Sing}(\mathcal{G}_i^{(n-e)})$. Moreover, over fields of characteristic zero, each β_i induces a homeomorphism between $\mathrm{Sing}(\mathcal{G}_i^{(n)})$ and $\mathrm{Sing}(\mathcal{G}_i^{(n-e)})$.

2. Any $\mathcal{G}^{(n-e)}$-local sequence over $V^{(n-e)}$ induces a $\mathcal{G}^{(n)}$-local sequence over $V^{(n)}$ and a commutative diagram as above where $\mathcal{G}_i^{(n-e)}$ is an elimination algebra of $\mathcal{G}_i^{(n)}$ for $i = 0, \dots, r$, and with β_i smooth $\mathcal{G}^{(n)}$-admissible projections, and $\beta_i(\mathrm{Sing}(\mathcal{G}_i^{(n)})) \subset \mathrm{Sing}(\mathcal{G}_i^{(n-e)})$ for all $i = 1, \dots, r$ (having homeomorphisms between $\mathrm{Sing}(\mathcal{G}_i^{(n)})$ and $\mathrm{Sing}(\mathcal{G}_i^{(n-e)})$ if the characteristic of the base field is zero).

3. Under the characteristic zero hypothesis, properties 1-2, together with the fact that β induces a homeomorphism on $\mathrm{Sing}(\mathcal{G}^{(n)})$ characterize the elimination algebra $\mathcal{G}^{(n-e)}$ up to weak equivalence.

4. Also for fields of characteristic zero, any resolution of $\mathcal{G}^{(n)}$ induces a resolution of $\mathcal{G}^{(n-e)}$ and vice versa.

5. For any two elimination algebras $\mathcal{G}^{(n-e)}$ and $\breve{\mathcal{G}}^{(n-e)}$ of $\mathcal{G}^{(n)}$, given by admissible projections $V^{(n)} \xrightarrow{\beta} V^{(n-e)}$ and $V^{(n)} \xrightarrow{\breve{\beta}} \breve{V}^{(n-e)}$ respectively, we have the same order at the image of ξ (see [80] and [14, Theorem 10.1]). That is, for any diagram

$$\begin{array}{ccc} & V^{(n)} & \\ {}^{\beta}\swarrow & & \searrow^{\breve{\beta}} \\ V^{(n-e)} & & \breve{V}^{(n-e)} \end{array}$$

where β and $\breve{\beta}$ are $\mathcal{G}^{(n)}$-admissible projections, we have

$$\mathrm{ord}_{\beta(\xi)}\mathcal{G}^{(n-e)} = \mathrm{ord}_{\breve{\beta}(\xi)}\breve{\mathcal{G}}^{(n-e)}.$$

Let us define

$$\mathrm{ord}_{\xi}^{(n-e)}(\mathcal{G}^{(n)}) := \mathrm{ord}_{\beta(\xi)}\mathcal{G}^{(n-e)}$$

for any elimination algebra $\mathcal{G}^{(n-e)}$ of $\mathcal{G}^{(n)}$ of dimension $(n-e)$ via some admissible projection β, which is an invariant for $\mathcal{G}^{(n)}$ at ξ.

In particular, given $X \subset V^{(n)}$ and a Rees algebra $\mathcal{G}^{(n)}$ representing the multiplicity of X, as in Example 1.4.7, we wish to find a Rees algebra in dimension $d = dim(X)$ which is an elimination algebra of $\mathcal{G}^{(n)}$. The reason for this will be explained in Section 1.5. The following theorem guarantees that this is possible:

Theorem 1.4.17. *Let $X \subset V^{(n)}$ be a d-dimensional variety over a field of characteristic zero, and $\mathcal{G}^{(n)}$ a Rees algebra over $V^{(n)}$ representing the multiplicity of X locally in an (étale) neigborhood of $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$. Then it is possible to find, in an (étale) neighborhood of ξ, a smooth projection $\beta : V^{(n)} \longrightarrow V^{(d)}$ inducing an elimination algebra $\mathcal{G}^{(d)}$ of $\mathcal{G}^{(n)}$. Moreover, the order*[3] $\mathrm{ord}_{\xi}^{(d)}(\mathcal{G}^{(n)}) := \mathrm{ord}_{\beta(\xi)}\mathcal{G}^{(d)}$ *does not depend on the choice of the embedding or of the algebra $\mathcal{G}^{(n)}$.*

Proof. This fact follows from [16, Section 21, Theorem 28.8, Theorem 28.10 and Example 28.2].

□

Example 1.4.18. Let us suppose that X is a hypersurface of dimension d, and consider the Rees algebra $\mathcal{G}_X^{(n)}$ representing the multiplicity of X, as in Example 1.4.1. There is a Rees algebra $\mathcal{G}_X^{(d)}$ over S, the elimination algebra of $\mathcal{G}_X^{(n)}$, given by

$$\mathcal{G}_X^{(d)} = \mathrm{Diff}(S[x][fW^b]) \cap S[W] \tag{1.28}$$

[3] For simplicity, we will sometimes write ξ when we refer to the image of $\xi^{(n)}$ by most of the maps we use here. In particular, we will often write ord_ξ meaning $\mathrm{ord}_{\xi^{(n)}}$ or $\mathrm{ord}_{\xi^{(d)}}$.

describing the image by (1.17) of $\underline{\text{Max}}\,\text{mult}(X)$. For a description of this elimination algebra see Lemma 1.4.20 below.

Remark 1.4.19. Let us go back to Example 1.4.5. It can be checked that $\mathcal{G}_X^{(d)}$ is invariant under translations of the variable x (see [79] and [80]), and hence the $S[x]$-Rees algebra generated by $fW^b \in S[x][W]$ and the $S[\tilde{x}]$-Rees algebra generated by $\tilde{f}W^b \in S[\tilde{x}][W]$ give equivalent elimination algebras $\text{Diff}(S[x][fW^b]) \cap S[W]$ and $\text{Diff}(S[\tilde{x}][\tilde{f}W^b]) \cap S[W]$ respectively. As a consequence, we may reduce the hypersurface case to the case of Example 1.4.1.

Lemma 1.4.20. *Let X be given by f as in Example 1.4.1. Then the elimination algebra of $\mathcal{G}_X^{(n)}$ relative to (1.17) is (up to integral closure):*

$$\mathcal{G}_X^{(d)} = \text{Diff}(S[B_{b-2}W^2, \ldots, B_iW^{b-i}, \ldots, B_0W^b]). \tag{1.29}$$

Proof. Considering the expression given by Lemma 1.4.2, (1.29) follows from the facts that $B_i \in S$ for $i = 0, \ldots, b-2$, and that $\mathcal{G}^{(n-e)} = \mathcal{G}^{(n)} \cap \mathcal{O}_{V^{(n-e)}}$ for any $1 \leq e \leq n-d$. □

Example 1.4.21. If we go back to examples in 1.3.26 and considering the differential closure of the algebras representing the multiplicity of the corresponding varieties, already computed in Example 1.3.35, we note that, if $\text{char}(k) = 0$, then

- For $\mathcal{G}_1$, the projection given by $\text{Spec}(k[x,y]) \longrightarrow \text{Spec}(k[y])$ gives an elimination algebra
$$\mathcal{G}_1^{(1)} = k[y][y^3W^2, y^2W] = \text{Diff}(k[y][y^3W^2]).$$
- For $\mathcal{G}_3$ the projection $\text{Spec}(k[x,y,z]) \longrightarrow \text{Spec}(k[x,y])$ induces an elimination
$$\mathcal{G}_3^{(2)} = k[x,y][(x^3 - xy^3)W, (y^5 - x^2y^2)W] = \text{Diff}(k[x,y][(x^2 - y^3)^2W^2]).$$
- For $\mathcal{G}_4$, the projection $\text{Spec}(k[x,y,z]) \longrightarrow \text{Spec}(k[y,z])$ gives the elimination
$$\begin{aligned}\mathcal{G}_4^{(2)} &= k[y,z][yz^2W, y^3W, y^2zW, y^2z^2W^2, y^3zW^2, y^3z^2W^3] = \\ &= \text{Diff}(k[y,z][y^3z^2W^3]).\end{aligned}$$
- For $\mathcal{G}_6$, the projection $\text{Spec}(k[x,y,z]) \longrightarrow \text{Spec}(k[y,z])$ gives the elimination
$$\begin{aligned}\mathcal{G}_6^{(2)} &= k[y,z][(yz,z^2)W, (yz^2,z^3)W^2, (yz^3)W^3] = \\ &= \text{Diff}(k[y,z][z^2W^2, (-yz^3 + z^5)W^3]).\end{aligned}$$

- Finally, for $\mathcal{G}_7$, the projection $\operatorname{Spec}(k[x,y,z]) \longrightarrow \operatorname{Spec}(k[y,z])$ gives the elimination
$$\mathcal{G}_7^{(2)} = k[y,z][yW, z^2W] = \overline{\operatorname{Diff}(k[y,z][yW, z^4W^2])}.$$
In this case, we can also consider the projection $\operatorname{Spec}(k[x,y,z]) \longrightarrow \operatorname{Spec}(k[z])$, which gives the elimination
$$\mathcal{G}_7^{(1)} = k[z][z^2W].$$

Remark 1.4.22. Going back to Example 1.4.1, one can check that $\mathcal{G}_X^{(n)}$ is the smallest $S[x]$-Rees algebra containing xW and $\mathcal{G}_X^{(d)}$. By abuse of notation, we will simply write
$$\mathcal{G}_X^{(n)} = S[x][xW] \odot \mathcal{G}_X^{(d)},$$
meaning that we extend both algebras to Rees algebras over the same ring and apply $\odot$ afterwards (see Definition 1.3.3).

Lemma 1.4.23. *Let X be a hypersurface given by f as in Example 1.4.1. Let $\mathcal{G}_X^{(d)}$ be the elimination algebra of $\mathcal{G}_X^{(n)}$ as in (1.28). Then for $\xi \in \operatorname{Sing}(\mathcal{G}_X^{(n)})$,*
$$\operatorname{ord}_\xi(\mathcal{G}_X^{(d)}) = \min_{i=0,\dots,b-2} \left\{ \frac{\nu_\xi(B_i)}{(b-i)} \right\}. \tag{1.30}$$

This follows from the fact that $\operatorname{ord}_\xi(\mathcal{G}_X^{(n)}) = \operatorname{ord}_\xi(\operatorname{Diff}(\mathcal{G}_X^{(n)}))$ for any $\xi \in \operatorname{Sing}(\mathcal{G}_X^{(n)})$ (see [10, Proposition 3.11]). We include the proof here since it will help us to know which terms are important for computing the orders of the algebras in Chapter 3.

Proof. By the expression of $\mathcal{G}_X^{(d)}$ given in Lemma 1.4.20, it is clear that it is enough to prove that, for any i, the element B_iW^{b-i} has lower order than any of its derivatives in ξ. The element B_iW^{b-i} has order
$$\operatorname{ord}\xi(B_iW^{b-i}) = \frac{\nu_\xi(B_i)}{b-i},$$
while for any diferential operator D^j of order $j < b-i$:
$$\operatorname{ord}_\xi(D^j(B_i)W^{b-i-j}) \geq \frac{\nu_\xi(B_i)-j}{b-i-j},$$
We only need that, for any pair of positive integers $A \geq A'$,
$$\frac{A}{A'} \leq \frac{A-k}{A'-k}$$
as long as $k < A'$. $\square$

Remark 1.4.24. Let X be a hypersurface given by f as in Example 1.4.5. Then the result in Lemma 1.4.23 can be applied after a variable change.

Example 1.4.25. If X is as in Example 1.4.7, for any $i \in \{1, \dots, n-d\}$, $f_i \in S[x_i]$ is the equation of a hypersurface H_i in a scheme of dimension $e = d+1$, $\mathrm{Spec}(S[x_i])$. Let us recall, that there is a commutative diagram

$$\begin{array}{lccccc}
\mathcal{G}_X^{(n)} & R = S[x_1,\dots,x_{n-d}] & \longrightarrow & S[x_1,\dots,x_{n-d}]/(f_1,\dots,f_{n-d}) & \longrightarrow & B \longrightarrow 0 \\
 & \uparrow & & \uparrow & \nearrow_{\beta^*_{H_i}} & \\
\mathcal{G}_{H_i}^{(e)} & S[x_i] & \longrightarrow & S[x_i]/(f_i) & & \\
 & \uparrow & \nearrow & & & \\
\mathcal{G}_X^{(d)} \supset \mathcal{G}_{H_i}^{(d)} & S & & & &
\end{array} \tag{1.31}$$

By Remark 1.4.22:

$$\mathcal{G}_{H_i}^{(e)} = \mathrm{Diff}(S[x_i][f_i W^{b_i}]) = S[x_i][x_i W] \odot \mathcal{G}_{H_i}^{(d)}.$$

By extending this algebra to $\mathcal{O}_{V^{(n)},\xi}$, we obtain

$$\mathcal{G}_{H_i}^{(n)} = \mathrm{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_i W^{b_i}]) = \mathcal{O}_{V^{(n)},\xi}[x_i W] \odot \mathcal{G}_{H_i}^{(d)}.$$

Hence, (1.21) can be written as

$$\mathcal{G}_X^{(n)} = \mathcal{G}_{H_1}^{(n)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(n)} = \mathcal{O}_{V^{(n)},\xi}[x_1 W, \dots, x_{n-d} W] \odot \mathcal{G}_{H_1}^{(d)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(d)}.$$

But also, by (1.31),

$$\mathcal{G}_X^{(n)} = \mathcal{G}_{H_1}^{(e)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(e)} \tag{1.32}$$

$$= S[x_1][x_1 W] \odot \dots \odot S[x_{n-d}][x_{n-d} W] \odot \mathcal{G}_X^{(d)}, \tag{1.33}$$

Thus, there is an easy expression for the elimination algebra of $\mathcal{G}_X^{(n)}$ relative to the projection

$$\mathrm{Spec}(S[x_1, \dots, x_{n-d}]) = V^{(n)} \longrightarrow V^{(d)} = \mathrm{Spec}(S)$$

namely

$$\mathcal{G}_X^{(d)} = \mathcal{G}_{H_1}^{(d)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(d)}. \tag{1.34}$$

An explanation of this elimination can be found in [16, Remark 16.10]. The elimination algebra $\mathcal{G}_X^{(d)}$ will be differentially closed (see [81, Proposition 5.1]). See Section 1.5 for the role of $\mathcal{G}_X^{(d)}$ in algorithmic resolution.

Example 1.4.26. For the example X_2 in 1.3.26, one can see that

$$\mathrm{Diff}(\mathcal{G}_2) = \mathrm{Diff}(k[x,z][(x^2 - z^5)W^2]) \odot \mathrm{Diff}(k[y,z][(y^3 - z^4)W^3])$$

gives the following elimination algebra via the projection

$$\mathrm{Spec}(k[x,y,z]) \longrightarrow \mathrm{Spec}(k[z])$$

$$\begin{aligned}\mathcal{G}_2^{(1)} =& k[z][z^4W^3, z^3W^2, z^2W] = \mathrm{Diff}(k[z][z^5W^2, z^4W^3]) = \\ & = \mathrm{Diff}(k[z][z^5W^2]) \odot \mathrm{Diff}(k[z][z^4W^3]).\end{aligned}$$

There is a different approach to attaching a d-dimensional Rees algebra to the maximum multiplicity of a variety: given X of dimension d defined over a perfect field and $\xi \in \underline{\mathrm{Max}}\ \mathrm{mult}(X)$, there is a canonical choice of an $\mathcal{O}_X$-Rees algebra of dimension d attached to the multiplicity of X locally in a neighborhood of ξ. This is a Rees algebra over X, and does not depend on the choice of an immersion up to integral closure (see [1, Definition 5.1 and Theorem 5.3]). However, note that this is not a Rees algebra over a smooth scheme, but over a singular one, and does not represent the multiplicity of X in the sense of Definition 1.3.23.

1.5 Constructive resolution and invariants

As it was already mentioned in the beginning of this chapter, given X there is not a unique resolution of its singularities. However, one can decide a criterion to find a particular one, by making some choices. A *constructive resolution* (or an *algorithmic resolution*) is an algorithm that stratifies each variety attending to the value that some upper semicontinuous function takes at each point. There are many possible choices for these functions (resolution functions), and each one gives a constructive resolution. Fixed an algorithm, it must satisfy some compatibility properties, which give a unique resolution of singularities for each variety. These properties include:

- If X_1 and X_2 are two varieties over a field k and $X_1 \xrightarrow{\phi} X_2$ is an isomorphism of varieties, then the resolution provided by the algorithm for each of them must be compatible with ϕ.

- A resolution of a variety X must induce a resolution in any open subset $U \subset X$.

Moreover, the resolution provided by the algorithm for any variety X must be compatible with algebraic action of groups on X (equivariance) (see [78]).

For the construction of an algorithm of resolution [34], consider a well ordered set $(\Lambda, \geq)$ and an upper semicontinuous function defined on varieties $F(X) = F_X$, $F_X : X \longrightarrow (\Lambda, \geq)$ such that for any X, $\underline{\mathrm{Max}}\, F_X \subset X$ is a closed and smooth subset, and F_X is constant on X if and only if X is smooth. Set $\underline{\mathrm{Max}}\, F_X$ as the center of the first blow up $X \xleftarrow{\pi_1} X_1$. The function F_X must satisfy $F_X(\xi) > F_{X_1}(\xi')$ whenever $\xi = \pi_1(\xi') \in \underline{\mathrm{Max}}\, F_X$. Given a variety X, the algorithm will give us a sequence of blow ups by iterating the process, that is,

$$X = X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} \dots \xleftarrow{\pi_r} X_r,$$

with π_i being the blow up at $\underline{\mathrm{Max}}\, F_{X_{i-1}}$ for $i = 1, \dots, r$.

Invariants

When it comes to the construction of the resolution function, we use invariants of the varieties in order to assign a value (in fact, a set of values) to each point reflecting the complexity of the singularities. Example 1.3.17 and Theorem 1.2.11 give upper semicontinuous functions which are often useful for this construction.

As a first coordinate of the resolution function F_X, we can consider the Hilbert-Samuel function or the multiplicity at each point. In particular, we will be interested in considering the multiplicity. However, $\underline{\operatorname{Max}} \operatorname{mult}(X)$ is not necessarily a regular subset (see X_3 in Example 1.2.7), so it is necessary to refine F_X by adding more coordinates. We will compare the values of F_X at different points using the lexicographical order, and this first coordinate will allow us to focus already on the stratum of maximum value of the multiplicity in X.

To each $\xi \in \underline{\operatorname{Max}} \operatorname{mult}(X)$, we know that we can attach a local presentation and an algebra $\mathcal{G}_X^{(n)}$ for the multiplicity.

Definition 1.5.1. Given a Rees algebra $\mathcal{G} = \oplus_{i \geq 0} I_i W^i$, *Hironaka's order* at $\xi \in \operatorname{Sing}(\mathcal{G})$ is $\operatorname{ord}_\xi(\mathcal{G})$. If $\mathcal{G}$ is, in particular, a Rees algebra representing the multiplicity of X locally in a neighborhood of ξ, then this order is defined at all points of $\operatorname{Sing}(\mathcal{G}) = \underline{\operatorname{Max}} \operatorname{mult}(X)$.

We will take this order as the second coordinate of F_X. Weak equivalence of Rees algebras ensures that this order does not depend on the choice of the presentation for the multiplicity of X (on the Rees algebra attached to the multiplicity of X), see Corollary 1.3.37.

Hironaka's order is the most important invariant in constructive resolution, and all other invariants derive from it.

If X is a d-dimensional variety, then it can be shown that there are suitable admissible projections to smooth $(n-i)$- dimensional schemes $V^{(n-i)}$, and elimination algebras $\mathcal{G}^{(n-i)}$, $i = 1, \ldots, n-d$. For the following coordinates, we will use the orders $\operatorname{ord}_\xi \mathcal{G}_X^{(n-i)}$ of the eliminations as in 1.4.16 (6), for $i = 1, \ldots, n-d$ (see 1.4.17):

$$F_X(\xi) = \Big(\operatorname{mult}_\xi(X),\ \operatorname{ord}_\xi \mathcal{G}_X^{(n)},\ \operatorname{ord}_\xi^{(n-1)} \mathcal{G}_X^{(n)}, \ldots,\ \operatorname{ord}_\xi^{(d+1)} \mathcal{G}_X^{(n)},\ \operatorname{ord}_\xi^{(d)} \mathcal{G}_X^{(n)}, \ldots\Big). \tag{1.35}$$

These invariants behave well under weak equivalence of Rees algebras, so they are really invariants. More precisely:

Remark 1.5.2. Two weakly equivalent Rees algebras $\mathcal{G}$ and $\mathcal{G}'$ over V share their resolution invariants and hence the constructive resolution of each of them induces the constructive resolution of the other one. This follows from the fact that all invariants that we consider for the construction or the resolution functions derive from Hironaka's order function ([13, 10.3],[34, 4.11, 4.15]) together with Corollary 1.3.37. In particular, this is the case for Rees algebras coming from different local presentations once we have fixed an immersion (see 1.3.38).

Among the orders in (1.35), the next theorem will tell us that $\operatorname{ord}_\xi^{(d)}\mathcal{G}_X^{(n)}$ is the first interesting one, since all the previous are necessarily equal to 1, and therefore this will be the coordinate we will focus on for our results.

Theorem 1.5.3. *[16, 16.7] Let X be a d-dimensional variety, and let $(V^{(n)}, \mathcal{G}^{(n)})$ be an n-dimensional pair attached to X at a point $\xi \in \underline{\operatorname{Max}} \operatorname{mult}(X)$. Then for any $e < n-d$ we have $\operatorname{ord}_\xi^{(n-e)}\mathcal{G}^{(n)} = 1$.*

Thus, F_X can actually be constructed as

$$F_X(\xi) = \left(\operatorname{mult}_\xi(X), \operatorname{ord}_\xi^{(d)}\mathcal{G}_X^{(n)}, \dots\right). \tag{1.36}$$

It follows from 1.4.16 that $\operatorname{ord}_\xi^{(d)}\mathcal{G}_X^{(n)} = \operatorname{ord}_\xi(\mathcal{G}_X^{(d)})$ does not depend on the choice of the elimination algebra $\mathcal{G}_X^{(d)}$. It neither depends on the immersion, by Theorem 1.4.17. For this reason, we will sometimes use the notation

$$\operatorname{ord}_\xi^{(d)}(X),$$

which does not depend on the particular choice of the Rees algebra attached to the maximum multiplicity, neither on the embedding. In fact, it can be proven that a resolution of the Rees algebra $\mathcal{G}_X^{(d)}$ induces a sequence of blow ups on X such that $\max \operatorname{mult}(X)$ decreases, so $\mathcal{G}_X^{(d)}$ is strongly related to the set $\underline{\operatorname{Max}} \operatorname{mult}(X)$.

Indeed, a different approach to the definition of this invariant, via projections, can be found in [2, Chapter 7]. There, given X of dimension d, for a suitable projection of generic rank $m = \max \operatorname{mult}(X)$ over a smooth scheme $V^{(d)}$ of dimension d over a field of characteristic zero, a Rees algebra $\mathcal{G}$ over $V^{(d)}$ is constructed ([2, Lemma 7.2.1]), so that it represents the maximum multiplicity in the following sense:

1. β induces a homemorphism between $\underline{\operatorname{Max}} \operatorname{mult}(X)$ and $\operatorname{Sing}(\mathcal{G})$;

2. Any $\mathcal{G}$-local sequence on V induces a compatible $\operatorname{mult}(X)$-local sequence $\pi_i : X_i \longrightarrow X_{i-1}$ for $i = 1, \dots, r$, $X_0 = X$, and compatible finite dominant morphisms of generic rank k $\beta_i : X_i \longrightarrow V_i$, for $i = 1, \dots, r$, $V_0 = V$:

$$\begin{array}{ccccccc} X_0 & \longleftarrow & X_1 & \longleftarrow & \dots & \longleftarrow & X_r \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta_1} & & & & \downarrow{\scriptstyle\beta_r} \\ V_0 & \longleftarrow & V_1 & \longleftarrow & \dots & \longleftarrow & V_r \end{array}$$

3. Moreover, for $i = 1, \dots, r$, β_i defines a homeomorphism between $\underline{\operatorname{Max}} \operatorname{mult}(X_i)$ and $\operatorname{Sing}(\mathcal{G}_i)$.

Our main result (Theorem 4.0.1) will show that, over fields of characteristic zero, this invariant, $\operatorname{ord}_\xi^{(d)}\mathcal{G}_X^{(n)}$, can be obtained from the arcs in X through ξ.

Example 1.5.4. In Examples 1.4.21 and 1.4.26, we already computed some elimination algebras, which show the invariant $\operatorname{ord}_{\xi}^{(d)}(X)$ for the corresponding varieties:

- For $X_1 = \mathbb{V}(x^2 - y^3)$ as in 1.4.21, $\operatorname{ord}_{\xi}^{(1)}(X_1) = 3/2$.
- For $X_2 = \mathbb{V}(x^2 - z^5, y^3 - z^4)$ as in 1.4.26, $\operatorname{ord}_{\xi}^{(1)}(X_2) = 4/3$.
- For $X_4 = \mathbb{V}(x^3 - y^3 z^2)$ as in 1.4.21, $\operatorname{ord}_{\xi}^{(2)}(X_4) = 5/3$.
- For $X_6 = \mathbb{V}(x^3 - xyz^2 - yz^3 + z^5)$ as in 1.4.21, $\operatorname{ord}_{\xi}^{(2)}(X_6) = 4/3$.
- For $X_7 = \mathbb{V}(xy - z^4)$ as in 1.4.21, $\operatorname{ord}_{\xi}^{(2)}(X_7) = 1$.

Chapter 2

Arc Spaces and Singularities

In the origin of the concepts of arc and jet spaces is the study of singularities of varieties by J. F. Nash ([69]). Later, many mathematicians have studied these objects from different points of view. Along this chapter, we will introduce jets and arcs, and point out how the spaces of arcs and jets happen to be the schemes representing interesting functors ([50],[8]). Then, we will show how they can be constructed from Hasse-Schmidt derivations ([83]), and afterwards stop at some properties of their topology ([56], [69], [45], [46], [28], [50]). We will also show the natural connection between arc spaces and valuations ([46], [24]). We list some results about the relation of singularities to the properties of their arc and jet schemes, due to various authors ([67], [48], [49]). The last section is devoted to introducing the Nash multiplicity sequence ([58], [40]), which will be the base of the new invariants that we shall define in Chapter 3.

2.1 Jet and Arc Schemes

Let X be a scheme of finite type over a field k.

Definition 2.1.1. For any k-algebra A and a fixed $m \in \mathbb{N}$, an *m-jet in X* is a morphism

$$\gamma_m : \operatorname{Spec}(A[t]/t^{m+1}) \longrightarrow X.$$

An *arc in X* is a morphism

$$\alpha : \operatorname{Spec}(A[[t]]) \longrightarrow X.$$

The *center* of an m-jet γ, respectively of an arc α, is the image by γ_m, resp. α, of the closed point of $\operatorname{Spec}(A[t]/(t^{m+1}))$, resp of $\operatorname{Spec}(A[[t]])$, that is,

$$\gamma_m(\langle t \rangle) \subset X, \text{ resp.}$$

$$\alpha(\langle t\rangle) \subset X.$$

We will sometimes refer to an m-jet or an arc with center at $\xi \in X$ as an m-jet or an arc *through* ξ.

Any jet γ_m in $X = \mathrm{Spec}(B)$ corresponds to a ring homomorphism

$$\begin{aligned}\gamma_m^* : B &\longrightarrow A[t]/(t^{m+1})\\ x &\longmapsto \gamma_m^*(x) = a_0 + a_1 t + \ldots + a_m t^m.\end{aligned}$$

Similarly, an arc in X corresponds to a ring homomorphism

$$\begin{aligned}\alpha^* : B &\longrightarrow A[[t]]\\ x &\longmapsto \alpha^*(x) = \sum_{i\in\mathbb{N}} a_i t^i.\end{aligned}$$

We will often use these expressions to give examples.

Definition 2.1.2. We define the *order of an arc* φ in X through $\xi \in X$,

$$\varphi : \mathcal{O}_{X,\xi} \longrightarrow K[[t]]$$

as the largest positive integer n such that $\varphi(\mathcal{M}_\xi) \subset (t^n)$, where $\mathcal{M}_\xi$ is the maximal ideal of the local ring $\mathcal{O}_{X,\xi}$, and denote it by $\mathrm{ord}(\varphi)$.

Proposition-Definition 2.1.3. *Let X be a scheme of finite type over k. For a fixed $m \in \mathbb{N}$, consider the functor from k-schemes to sets mapping*

$$Y \overset{J_m}{\longmapsto} \mathrm{Hom}_k(Y \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K[t]/t^{m+1}), X).$$

This functor is representable by a scheme of finite type over k ([50, Proposition 2.2]). We call this scheme the scheme of m-jets of X, and we denote it by $\mathcal{L}_m(X)$.

For any field extension $K \supset k$, the K-points of $\mathcal{L}_m(X)$ are the morphisms

$$\gamma_m : \mathrm{Spec}(K[t]/t^{m+1}) \longrightarrow X.$$

Examples 2.1.4. For a given X, we have:

- $\mathcal{L}_0(X) = X$.
- $\mathcal{L}_1(X) = \{l : l \text{ is a linear form through some point } \xi \in X\} = \bigcup_{\xi\in X} \{\text{tangent space of } X \text{ at } \xi\}$.

For every $p \geq m$, there exists a truncation morphism

$$A[t]/t^{p+1} \longrightarrow A[t]/t^{m+1}$$

and an induced projection

$$\pi_{p,m} : \mathcal{L}_p(X) \longrightarrow \mathcal{L}_m(X). \tag{2.1}$$

That is, each p-jet on X gives, by truncation, an m-jet via

$$\begin{array}{ccc} \operatorname{Spec}(A[t]/t^{p+1}) & \xrightarrow{\gamma_p} & X \\ \uparrow & \nearrow_{\gamma_m} & \\ \operatorname{Spec}(A[t]/t^{m+1}) & & \end{array} \tag{2.2}$$

The projections $\pi_{p,m}$, for $m \leq p$, give a projective system that allows us to define:

Definition 2.1.5. The *space of arcs* of X is the inverse limit

$$\mathcal{L}(X) := \varprojlim_{m \in \mathbb{N}} \mathcal{L}_m(X).$$

Theorem 2.1.6. *[8, Corollary 1.2] The space of arcs $\mathcal{L}(X)$ of X represents the functor from the category of k-schemes to the category of sets given by*

$$Y \overset{\mathcal{A}}{\longmapsto} \operatorname{Hom}_k \left(Y \times_{\mathbb{Z}} \operatorname{Spf}(\mathbb{Z}[[t]]), X\right).$$

The space of arcs of X is a scheme, which is not of finite type, and whose K-points for any field extension $K \supset k$ are the morphisms

$$\alpha : \operatorname{Spec}(K[[t]]) \longrightarrow X.$$

We shall use the same name for α as a K-point of $\mathcal{L}(X)$ and for it as a morphism.

The truncations $A[[t]] \longrightarrow A[t]/t^{m+1}$ for each $m \in \mathbb{N}$ induce natural projections

$$\pi_{X,m} : \mathcal{L}(X) \longrightarrow \mathcal{L}_m(X). \tag{2.3}$$

Remark 2.1.7. Every arc in X gives an m-jet for all $m \in \mathbb{N}$ via the truncation $\pi_{X,m}$. However, not every m-jet is the truncation of an arc. The conditions under which an m-jet can be lifted to an arc are a consequence of Artin approximation theorem ([4]).

Of special interest will be the projection

$$\begin{aligned} \pi_X = \pi_{X,0} : \mathcal{L}(X) &\longrightarrow X \\ \alpha &\longmapsto \alpha(0), \end{aligned}$$

where 0 denotes the closed point of $\operatorname{Spec}(A[[t]])$. Note that $\alpha(0) \subset X$ is the center of α in X.

Example 2.1.8. Consider the n-dimensional affine space over k,

$$\mathbb{A}_k^n = \operatorname{Spec}(k[x_1, \ldots, x_n]).$$

The m-jet scheme of $\mathbb{A}_k^n$ for a fixed $m \in \mathbb{N}$ is

$$\mathcal{L}_m(\mathbb{A}_k^n) = \operatorname{Spec}(k[x_1^{(0)}, \ldots, x_n^{(0)}, x_1^{(1)}, \ldots, x_n^{(1)}, \ldots, x_1^{(m)}, \ldots, x_n^{(m)}]),$$

and the arc space of $\mathbb{A}_k^n$ is the spectrum of an infinite dimensional k-algebra:

$$\mathcal{L}(\mathbb{A}_k^n) = \operatorname{Spec}(k[x_1^{(0)}, \ldots, x_n^{(0)}, x_1^{(1)}, \ldots, x_n^{(1)}, \ldots]).$$

A K-valued point of $\mathcal{L}_m(\mathbb{A}_k^n)$, say

$$\gamma_m = (a_1^{(0)}, \ldots, a_n^{(0)}, \ldots, a_1^{(m)}, \ldots, a_n^{(m)}) \in K^{nm}$$

corresponds to a homomorphism of rings

$$\begin{aligned} \gamma_m^* : k[x_1, \ldots, x_n] &\longrightarrow K[t]/(t^{m+1}) \\ x_i &\longmapsto a_i^{(0)} + a_i^{(1)} t + \ldots + a_i^{(m)} t^m, \;\; i = 1, \ldots, n. \end{aligned}$$

In a similar way, a K-valued point

$$\alpha = (a_1^{(0)}, \ldots, a_n^{(0)}, a_1^{(1)}, \ldots, a_n^{(1)}, \ldots) \in \mathcal{L}(\mathbb{A}_k^n)$$

corresponds to a homomorphism

$$\begin{aligned} \alpha^* : k[x_1, \ldots, x_n] &\longrightarrow K[[t]] \\ x_i &\longmapsto \sum_{j \in \mathbb{N}} a_i^{(j)} t^j, \;\; i = 1, \ldots, n. \end{aligned}$$

Let α be an arc in $\operatorname{Spec}(B)$, $\alpha^* : B \longrightarrow K[[t]]$. Then the preimage of the ideal (t) is a prime ideal $\mathcal{P}$ in B. This prime corresponds to the center of α.

2.2 Jets and Arcs via Hasse-Schmidt derivations

In order to understand the jet and arc spaces of affine algebraic varieties, we can consider the equations defining them as subsets of affine spaces (infinite dimensional affine spaces in the case of arcs). To see how these equations arise, we introduce now Hasse-Schmidt derivations. For more details about the constructions described along this section, we refer to [83].

Definition 2.2.1. Let A be a ring and let B and R be A-algebras, with $f : A \to B$. Fix $m \in \mathbb{N} \cup \{\infty\}$. A *Hasse-Schmidt derivation of order m from B to R over A* is a sequence

$$(D_0, D_1, \ldots, D_m)$$

where

$$D_0 : B \longrightarrow R$$

is a homomorphism of A-algebras and for $i = 1, \ldots, m$,

$$D_i : B \longrightarrow R$$

is a homomorphism of abelian groups, satisfying

1. For any $a \in A$, $D_i(f(a)) = 0$,
2. For any $x, y \in B$, $D_i(xy) = \sum_{j+k=i} D_j(x)D_k(y)$.

From now on, when we consider a Hasse-Schmidt derivation of order m, this m can be either a natural number or infinity.

Remark 2.2.2. If $D = (D_0, \ldots, D_m)$ is a Hasse-Schmidt derivation of order m from B to R over A, then R can be seen as a B-module via D_0, and:

- $D_1 : B \longrightarrow R$ is a derivation over A.
- For all $i = 1, \ldots, m$, $D_i : B \longrightarrow R$ is a differential operator of order at most i. That is, an A-linear map such that for any $b \in B$, $[b, D_i] = b \cdot D_i(\bullet) - D_i(b \cdot \bullet) : B \longrightarrow R$ is a differential operator of order at most $i - 1$ if $i > 0$, or a B-linear operator if $i = 0$. Indeed, it is easy to see that D_1 is, since for any $b \in B$,

$$[b, D_1]\,(c) = b \cdot D_1(c) - D_1(b \cdot c) = -D_1(b) \cdot c \;\forall c \in B,$$

so $[b, D_1] : B \longrightarrow R$ is a differential operator of order 0, and for any $n = 2, \ldots, m$, under the assumption of $D_{n-1} : B \longrightarrow R$ being a differential operator of order $n - 1$, we have that $[b, D_n] : B \longrightarrow R$ is also a differential operator of order $n - 1$ for any $b \in B$:

$$[b, D_n]\,(c) = b \cdot D_n(c) - D_n(b \cdot c) = b \cdot D_n(c) - \sum_{i=0}^{n} D_i(b)D_{n-i}(c) =$$

$$= -\sum_{i=1}^{n} D_i(b)D_{n-i}(c) \;\forall c \in B.$$

Example 2.2.3. Let A be a ring and let B and R be A-algebras with $\mathrm{char}(R) = 0$ and $R \subset B$. Assume that $D : B \longrightarrow R$ is a derivation over A. Then the sequence $(D_0, \ldots, D_m)$, where

$$D_i := \frac{1}{i!} D^i$$

for $i = 0, \ldots, m$, is a Hasse-Schmidt derivation of order m from B to R over A.

Example 2.2.4. Let $D = (D_0, \ldots, D_m)$ be a Hasse-Schmidt derivation from B to R over A of order m, and let $r \in R$. Then $D' = (D'_0, \ldots, D'_m)$ where

$$D'_i = r^i D_i \quad \text{for} \quad i = 0, \ldots, m$$

is also a Hasse-Schmidt derivation from B to R over A of order m. To see this, note that $D'_0 = D_0$, and that the D'_i are homomorphisms of abelian groups, and satisfy that

1. for any $a \in A$, $D'_i(f(a)) = r^i D_i(f(a)) = 0$, and

2. for any $x, y \in B$,

$$\begin{aligned} D'_i(xy) = r^i D_i(xy) = \sum_{j+k=i} r^{j+k} D_j(x) D_k(y) = \sum_{j+k=i} (r^j D_j(x))(r^k D_k(y)) = \\ = \sum_{j+k=i} D'_j(x) D'_k(y). \end{aligned}$$

Definition 2.2.5. Given a ring A and an A-algebra B, $f : A \to B$, let $B_m = B[x', x'', \dots, x^{(m)}]_{x \in B}$. Let us define for $i = 0, \dots, m$

$$\begin{aligned} d_i : B &\longrightarrow \mathrm{HS}^m_{B/A} := B_m/I \\ x &\longmapsto x^{(i)}, \end{aligned}$$

where I is the minimal ideal containing the sets

$$S_1 = \left\{(x+y)^{(i)} - x^{(i)} - y^{(i)} : x, y \in B\right\}_{i=1,\dots,m},$$

$$S_2 = \left\{f(a)^{(i)} : a \in A\right\}_{i=1,\dots,m},$$

$$S_3 = \left\{(xy)^{(i)} - \sum_{j+k=i} x^{(j)} y^{(k)} : x, y \in B\right\}_{i=1,\dots,m}.$$

The quotient by the elements in S_1 guarantees that the d_i are group homomorphisms, and the quotient by the elements in S_2 and S_3 respectively guarantee that the conditions (1) and (2) from Definition 2.2.1 hold. Hence,

$$(d_0, \dots, d_m)$$

defined in this way is a Hasse-Schmidt derivation of order m from B to B_m over A. We call it the *universal derivation.* Note that m can be infinity here.

Example 2.2.6. Let $A = k$, $B = k[x, y]$ and

$$R = k[x_0 = x, y_0 = y, x_1 = x', y_1 = y', x_2 = x'', y_2 = y'', \dots, x_m = x^{(m)}, y_m = y^{(m)}]/I$$

as above. Let us compute d_0, d_1 and d_2 for $x^2 - y^3 \in B$:

$$\begin{aligned} d_0(x^2 - y^3) &= d_0(x^2) - d_0(y^3) = d_0(x)d_0(x) - d_0(y)d_0(y^2) = \\ &= d_0(x)d_0(x) - d_0(y)(d_0(y)d_0(y)) = d_0(x)^2 - d_0(y)^3 = \\ &= x_0^2 - y_0^3 = x^2 - y^3 \end{aligned}$$

$$\begin{aligned} d_1(x^2 - y^3) &= d_1(x^2) - d_1(y^3) = 2d_0(x)d_1(x) - d_0(y)d_1(y^2) - d_1(y)d_0(y^2) = \\ &= 2d_0(x)d_1(x) - d_0(y)(2d_0(y)d_1(y)) - d_1(y)d_0(y)^2 = \\ &= 2x_0x_1 - 2y_0^2y_1 - y_1y_0^2 = 2x(x') - 3y^2(y') \end{aligned}$$

$$
\begin{aligned}
d_2(x^2-y^3) =& d_2(x^2)-d_2(y^3)= \\
&= [2d_0(x)d_2(x)+d_1(x)d_1(x)] - \Big[d_0(y)d_2(y^2)+d_1(y)d_1(y^2)+ \\
&\quad d_2(y)d_0(y^2)\Big] = \Big[2d_0(x)d_2(x)+d_1(x)^2\Big] - [d_0(y)(2d_0(y)d_2(y)+ \\
&\quad d_1(y)^2)+d_1(y)(2d_0(y)d_1(y))+d_2(y)(d_0(y)^2)\Big] = 2x_0x_2+x_1^2- \\
&\quad 2y_0^2y_2-y_0y_1^2-2y_0y_1^2-y_0^2y_2 = 2x(x'')+(x')^2-3y^2(y'')-3y(y')^2.
\end{aligned}
$$

Remark 2.2.7. With the setting from Definition 2.2.1, note that $\mathrm{HS}^m_{B/A}$ is a B-algebra, and also an A-algebra via f. Indeed, it is a graded algebra with

$$\deg(d_i(x)) = i \ \forall x \in B.$$

Remark 2.2.8. With the setting from Definition 2.2.1, let $\phi : R' \longrightarrow R$ be a homomorphism of A-algebras and $(D_0,\ldots,D_m)$ a Hasse-Schmidt derivation of order m from B to R' over A. Then, it induces a Hasse-Schmidt derivation of order m from B to R over A:

$$(\phi\circ D_0,\ldots,\phi\circ D_m) : B \longrightarrow R.$$

Let us denote by $\mathrm{Der}^m_A(B,R)$ the set of Hasse-Schmidt derivations of order m from B to R over A. We have a covariant functor $\mathrm{Der}^m_A(B,\bullet)$ from the category of A-algebras to the category of sets given by

$$R \longmapsto \mathrm{Der}^m_A(B,R).$$

On the other hand, given a Hasse-Schmidt derivation of order m from B to R over A, $(D_0,\ldots,D_m)$, there exists a unique A-algebra homomorphism

$$\phi : \mathrm{HS}^m_{B/A} \longrightarrow R$$

such that $(D_0,\ldots,D_m) = (\phi\circ d_0,\ldots,\phi\circ d_m)$. That is, there exists a unique ϕ such that for $i=1,\ldots,m$ the diagram

$$
\begin{array}{ccc}
B & \xrightarrow{\quad D_i \quad} & R \\
 & {\scriptstyle d_i}\searrow \quad \nearrow{\scriptstyle \phi} & \\
 & \mathrm{HS}^m_{B/A} &
\end{array}
\tag{2.4}
$$

commutes. Note that any element $\tilde{x} \in \mathrm{HS}^m_{B/A}$ is the class of an element $x^{(i)} \in B[x^{(0)}=x,x',x'',\ldots,x^{(m)}]_{x\in B}$, for some $x\in B$. It is enough to set

$$
\begin{aligned}
\tilde{\phi} : B[x^{(i)}]_{x\in B, i\in\mathbb{Z}_{\geq 0}} &\longrightarrow R \\
x^{(i)} &\longmapsto D_i(x)
\end{aligned}
$$

and define ϕ as the map induced by $\tilde{\phi}$ for the quotient of $B[x^{(i)}]_{x\in B, i\in\mathbb{Z}_{\geq 0}}$ by the ideal I in Definition 2.2.5. It is well defined since the definition of Hasse-Schmidt derivations guarantees that the D_i are such that I is contained in the kernel of $\tilde{\phi}$ defined as here.

We may conclude, by replacing R' in Remark 2.2.8 by $\mathrm{HS}^m_{B/A}$ and $(D_0, \ldots, D_m)$ by the universal derivation, that the map

$$\begin{aligned} \mathrm{Hom}_A(\mathrm{HS}^m_{B/A}, R) &\longrightarrow \mathrm{Der}^m_A(B, R) \\ \phi &\longmapsto (\phi \circ d_0, \ldots, \phi \circ d_m) \end{aligned} \tag{2.5}$$

is bijective:

Theorem 2.2.9. *The B-algebra $\mathrm{HS}^m_{B/A}$ together with $(d_0, \ldots, d_m)$ represent the functor $\mathrm{Der}^m_A(B, \bullet)$.*

Definition 2.2.10. Note that $\mathrm{HS}^m_{B/A}$ is a B-algebra. It is the *algebra of Hasse-Schmidt derivations of B/A of order m.*

It is not very difficult to observe a relation between Hasse-Schmidt derivations and m-jets. Let us develope here the tools to connect them for any variety. The first clue is given by the following Lemma:

Lemma 2.2.11. *Let A, $f : A \longrightarrow B$ be as before and let R be an A-algebra and fix $m \in \mathbb{N}$. Let $D = (D_0, \ldots, D_m) : B \longrightarrow R$ be a Hasse-Schmidt derivation over A of order m. It is possible to define a homomorphism of A-algebras from B to $R[t]/(t^{m+1})$ as follows:*

$$\begin{aligned} \varphi : B &\longrightarrow R[t]/(t^{m+1}) \\ x &\longmapsto D_0(x) + D_1(x)t + \ldots + D_m(x)t^m. \end{aligned}$$

Moreover, there is a bijection

$$\mathrm{Der}^m_A(B, R) \longrightarrow \mathrm{Hom}_A(B, R[t]/(t^{m+1})) \tag{2.6}$$

Proof. The definition of the D_i guarantees that the defined map is a homomorphism of A-algebras, and the map in (2.6) taking each higher order derivation to the homomorphism constructed in this way is obviously injective, since the coefficients of the image polynomial for each element in B are determined by the derivation. To see that it is surjective, let

$$\begin{aligned} \varphi : B &\longrightarrow R[t]/(t^{m+1}) \\ x &\longmapsto \varphi(x) = a_0^{(x)} + a_1^{(x)}t + \ldots + a_m^{(x)}t^m, \end{aligned}$$

and then define

$$\begin{aligned} \varphi_i : B &\longrightarrow R \\ x &\longmapsto \varphi_i(x) = a_i^{(x)}, \end{aligned}$$

for $i = 0, \dots, m$. It is clear that φ_0 is a homomorphism of A-algebras, since it can be seen as the composition of two homomorphisms of A-algebras: $\varphi_0 = \varphi \circ z$, where $z : R[t]/t^{m+1} \longrightarrow R$ maps t to 0 and is the identity on R. It is as well easy to check that the φ_i for $i = 1, \dots, m$ are homomorphisms of groups so, to see that

$$(\varphi_0, \dots, \varphi_m) : B \longrightarrow R$$

is a Hasse-Schmidt derivation over A of order m, it only lasts to observe that, for $i = 1, \dots, m$,

- since φ is a homomorphism of A-algebras, for all $a \in A$ one has $\varphi(f(a)) = a \cdot \varphi(1) = a$. But

$$\varphi(f(a)) = \varphi_0(f(a)) + \sum_{i=1}^{m} \varphi_i(f(a))t^i = a + \sum_{i=1}^{m} 0 \cdot t^i,$$

  so $\varphi_i(f(a)) = 0$ for any $a \in A$, $i = 1, \dots, m$;

- for any $x, y \in B$, if $\varphi(x) = \sum_{i=0}^{m} a_i t^i$ and $\varphi(y) = \sum_{i=0}^{m} b_i t^i$, then

$$\varphi(xy) = \varphi(x)\varphi(y) = \left(\sum_{i=0}^{m} a_i t^i\right)\left(\sum_{i=0}^{m} b_i t^i\right) = \sum_{i=0}^{m}\left(\sum_{i=j+k} a_j b_k\right) t^i,$$

  and hence $\varphi_i(xy) = \sum_{i=j+k} a_j b_k = \sum_{i=j+k} \varphi_j(x)\varphi_k(y)$.

□

Corollary 2.2.12. *The bijections in (2.5) and (2.6) lead to a new bijection through the universal derivation:*

$$\begin{aligned} \mathrm{Hom}_A(\mathrm{HS}^m_{B/A}, R) &\longrightarrow \mathrm{Hom}_A(B, R[t]/t^{m+1}) \\ \phi &\longmapsto \gamma^*_\phi, \end{aligned} \tag{2.7}$$

for any A-algebra R, given by

$$\begin{aligned} \gamma^*_\phi : B &\longrightarrow R[t]/(t^{m+1}) \\ x &\longrightarrow \phi(d_0(x)) + \phi(d_1(x))t + \ldots + \phi(d_m(x))t^m. \end{aligned}$$

Assume that $X = \mathrm{Spec}(B)$ is an affine scheme over $k = A$, and let $R = k$. The previous bijection assigns to each homomorphism ϕ from $\mathrm{HS}^m_{B/A}$ to R a Hasse-Schmidt derivation, by composing ϕ with the universal one, and then an m-jet γ_m, mapping each $x \in B$ to $\phi(d_0(x)) + \phi(d_1(x))t + \ldots + \phi(d_m(x))t^m$.

Remark 2.2.13. If R in Corollary 2.2.12 is also a B algebra, say

$$A \xrightarrow{f} B \xrightarrow{g} R,$$

$\operatorname{Hom}_A(\mathrm{HS}^m_{B/A}, R)$	$\operatorname{Der}^m_A(B, R)$	$\operatorname{Hom}_A(B, R[t]/t^{m+1})$
$\phi : \mathrm{HS}^m_{B/A} \to R$	$D : B \to R$ $D = (D_0, \dots, D_m)$	$\varphi : B \to R[t]/t^{m+1}$
$\phi(\left[x^{(i)}\right]) = D_i(x)$	$D(x) = (a_0^{(x)}, \dots, a_m^{(x)})$ $= (\phi \circ d_0(x), \dots, \phi \circ d_m(x))$	$\varphi(x) = \sum_{i=0}^m a_i^{(x)} t^i$ $= \sum_{i=0}^m D_i(x) t^i$ $= \sum_{i=0}^m \phi(d_i(x)) t^i$

Table 2.1: Correspondence

then (2.7) induces a bijection

$$\operatorname{Hom}_B(\mathrm{HS}^m_{B/A}, R) \longleftrightarrow \left\{\gamma \in \operatorname{Hom}_A(B, R[t]/t^{m+1}) : z \circ \gamma = g\right\},$$

where $z : R[t]/t^{m+1} \longrightarrow R$ is the identity on R and maps t to 0. Note that any $\phi \in \operatorname{Hom}_A(\mathrm{HS}^m_{B/A}, R)$ which happens to be also a homomorphism of B-algebras must satisfy $\phi \circ d_0 = g$ (by the correspondence in the table above). But the image of such a homomorphism by (2.7) is a ring homomorphism

$$x \overset{\gamma}{\longmapsto} \phi \circ d_0(x) + t \cdot q,$$

where $q \in R[t]/t^{m+1}$, so necessarily $z \circ \gamma(x) = \phi \circ d_0(x) = g(x)$ for all $x \in B$, and hence $z \circ \gamma = g$.

Let us go back to the B-algebras $\mathrm{HS}^m_{B/A}$ for each $m \in \mathbb{N}$. Via the truncation morphisms in (2.1) we have, for $0 \leq m \leq p \leq \infty$, homomorphisms of graded B-algebras $f_{m,p}$:

$$\begin{array}{ccc} \mathrm{HS}^m_{B/A} & \xrightarrow{f_{m,p}} & \mathrm{HS}^p_{B/A} \\ & \nwarrow \quad \nearrow & \\ & B & \end{array} \tag{2.8}$$

satisfying $f_{m,m} = \mathrm{Id}$ for any $m \in \mathbb{N} \cup \{\infty\}$, and

$$f_{m,p} = f_{k,p} \circ f_{m,k}$$

for any $0 \leq m \leq k \leq p \leq \infty$. They give a direct system, and we denote

$$\mathrm{HS}^\infty_{B/A} = \varinjlim_{m \in \mathbb{N}} \mathrm{HS}^m_{B/A}.$$

This fact gives rise to analogous bijections to (2.5), (2.6) and (2.7) for the context of arcs and Hasse-Schmidt derivations of infinite order. More precisely, each commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{D} & R \\ & {\scriptstyle d}\searrow \quad \nearrow{\scriptstyle \phi} & \\ & \mathrm{HS}^\infty_{B/A} & \end{array}$$

where $D = (D_0, D_1, \dots) \in \mathrm{Der}_A^{\infty}(B, R)$, d is the universal derivation of B/A, and ϕ is a homomorphism of A-algebras, corresponds to a homomorphism

$$\begin{aligned} \alpha_\phi^* : B &\longrightarrow R[[t]] \\ x &\longrightarrow \sum_{i\in\mathbb{N}} D_i(x)t^i = \sum_{i\in\mathbb{N}} \phi(d_i(x))t^i. \end{aligned}$$

That is, every Hasse-Schmidt derivation of B/A corresponds to an arc in $\mathrm{Spec}(B)$.

Fundamental exact sequences

Similarly to what happens for the module of first order differentials, one can construct exact sequences which yield interesting properties of the algebras of Hasse-Schmidt derivations. These will be exact sequences of A-algebras. Let

$$A \xrightarrow{f} B \xrightarrow{g} C$$

be a sequence of ring homomorphisms. Fix $m \in \mathbb{N} \cup \{\infty\}$. Consider the A-algebras of Hasse-Schmidt derivations of order m of B/A, C/A and C/B (via $h = g \circ f$) respectively: $\mathrm{HS}^m_{B/A}$, $\mathrm{HS}^m_{C/A}$ and $\mathrm{HS}^m_{C/B}$. Note that both $\mathrm{HS}^m_{C/A}$ and $\mathrm{HS}^m_{C/B}$ are quotients of the same C-algebra

$$C[x^{(i)}]_{x\in C; i=1,\dots,m}$$

as in Definition 2.2.5, by ideals $I_{C/A}$ and $I_{C/B}$ respectively,

$$I_{C/A} \subset I_{C/B} \subset C[x^{(i)}]_{x\in C; i=1,\dots,m}.$$

The difference between both ideals is due to the difference $\{g(b) : b \in B \setminus A\}$ between the generating sets S_2 from Definition 2.2.5 for $\mathrm{HS}^m_{C/A}$ and $\mathrm{HS}^m_{C/B}$.

Theorem 2.2.14. *[83, Theorem 2.1] (First fundamental exact sequence) The sequence of graded C-algebras*

$$0 \longrightarrow (\mathrm{HS}^m_{B/A})^+ \mathrm{HS}^m_{C/A} \longrightarrow \mathrm{HS}^m_{C/A} \longrightarrow \mathrm{HS}^m_{C/B} \longrightarrow 0, \tag{2.9}$$

where $(\mathrm{HS}^m_{B/A})^+$ is the irrelevant ideal, is exact.

Theorem 2.2.15. *[83, Theorem 2.2] (Second fundamental exact sequence) Assume that $B \xrightarrow{g} C$ is surjective, and denote $I := \mathrm{Ker}(g) \subset B$. The sequence*

$$0 \longrightarrow J \longrightarrow \mathrm{HS}^m_{B/A} \longrightarrow \mathrm{HS}^m_{C/A} \longrightarrow 0,$$

where $J := (x^{(i)})_{x\in I; i=0,\dots,m} \subset \mathrm{HS}^m_{B/A}$, is exact.

Base changes

We will show now how $\mathrm{HS}^m_{X/Y}$ can be constructed for an arbitrary morphism of schemes $X \longrightarrow Y$. The following result ensures that a sheaf $\mathrm{HS}^m_{X/Y}$ of $\mathcal{O}_X$-algebras can be defined by gluing the algebras of Hasse-Schmidt derivation of open affine subsets of X and Y. We will afterwards work locally on affine subsets, knowing that the jets and arcs schemes of affine schemes can be glued together to define the jet and arc schemes of the glued scheme.

Theorem 2.2.16. *[83, Theorem 4.3] Let $f : X \longrightarrow Y$ be a morphism of schemes, and let $m \in \mathbb{N} \cup \{\infty\}$. Then, there exists a quasi-coherent sheaf $\mathrm{HS}^m_{X/Y}$ of $\mathcal{O}_X$-algebras such that:*

1. *For each*
$$\mathrm{Spec}(A) \subset Y \quad \textit{and}$$
$$\mathrm{Spec}(B) \subset f^{-1}(\mathrm{Spec}(A)),$$
open affine subsets, we have an isomorphism of B-algebras
$$\Phi_{A/B} : (\Gamma(\mathrm{Spec}(B), \mathrm{HS}^m_{X/Y})) \xrightarrow{\cong} \mathrm{HS}^m_{B/A}.$$
2. *The pair*
$$\left(\mathrm{HS}^m_{X/Y}, (\Phi_{A/B})_{A,B}\right)$$
formed by the sheaf of $\mathcal{O}_X$-algebras and the collection of isomorphisms in 1. is unique up to unique isomorphism.

This is a consequence of two localization properties that Lemmas 2.2.17 and 2.2.18 below provide.

Assuming that Y is affine, $Y = \mathrm{Spec}(A)$, take $\mathrm{Spec}(B) \subset X$. The first localization property that we have is:

Lemma 2.2.17. *Let $f : A \longrightarrow B$ be an injective ring homomorphism, and fix $m \in \mathbb{N} \cup \{\infty\}$. Let $S \subset B$ be a multiplicative subset. Then there is an isomorphism*
$$S^{-1}\mathrm{HS}^m_{B/A} \xrightarrow{\Psi} \mathrm{HS}^m_{S^{-1}B/A},$$
induced by the map
$$B[x^{(i)}]_{x \in B, i=1,\dots,m} \longrightarrow S^{-1}B[y^{(i)}]_{y \in S^{-1}B, i=1,\dots,m}.$$

Proof. This lemma is a consequence of the fact that the localization map $B \longrightarrow S^{-1}B$ is formally étale. The fact that it is formally unramified gives, first, a chain of ring homomorphisms
$$\mathrm{Hom}_B\left(\mathrm{HS}^m_{S^{-1}B/B}, R\right) \xrightarrow{\cong} \mathrm{Hom}_B\left(S^{-1}B, R[t]/t^{m+1}\right) \xrightarrow{\cong} \mathrm{Hom}_B\left(S^{-1}B, R\right)$$

for any B-algebra R. For the first isomorphism one only needs Corollary 2.2.12, and the second one is given by $\phi \longmapsto z \circ \phi$, where z maps t to 0 and is the identity in R, which is obviously surjective, and whose injectivity comes from the formally unramification of $S^{-1}B$ over B. Hence,

$$\operatorname{Hom}_B\left(\mathrm{HS}^m_{S^{-1}B/B}, R\right) \cong \operatorname{Hom}_B\left(S^{-1}B, R\right)$$

for any R, so necessarily $\mathrm{HS}^m_{S^{-1}B/B} = S^{-1}B$. From this, we will show that the map $S^{-1}\mathrm{HS}^m_{B/A} \xrightarrow{\cong} \mathrm{HS}^m_{S^{-1}B/A}$ is surjective.
Since

$$A \longrightarrow B \longrightarrow S^{-1}B$$

is exact, the first fundamental sequence (2.9) given in this case a sequence of graded $S^{-1}B$-algebras

$$0 \to \left(\mathrm{HS}^m_{B/A}\right)^+ \mathrm{HS}^m_{S^{-1}B/A} \to \mathrm{HS}^m_{S^{-1}B/A} \to \mathrm{HS}^m_{S^{-1}B/A} \to \underset{\substack{\| \\ S^{-1}B}}{\mathrm{HS}^m_{S^{-1}B/B}} \to 0. \quad (2.10)$$

By looking at the part of degree 0, we have

$$0 \longrightarrow 0 \longrightarrow \mathrm{HS}^0_{S^{-1}B/A} = S^{-1}B \longrightarrow S^{-1}B \longrightarrow 0,$$

while the parts of degree i for each $i > 0$ give us exact sequences

$$0 \longrightarrow \left(\mathrm{HS}^l_{B/A}\right)^+ \mathrm{HS}^{i-l}_{S^{-1}B/A} \longrightarrow \mathrm{HS}^i_{S^{-1}B/A} \longrightarrow 0 \longrightarrow 0. \quad (2.11)$$

Let us prove that Ψ is surjective by showing that it is surjective in each degree $i = 0, \ldots, m$. It is clear for degree 0, since

$$S^{-1}\mathrm{HS}^0_{B/A} = S^{-1}B = \mathrm{HS}^0_{S^{-1}B/A}.$$

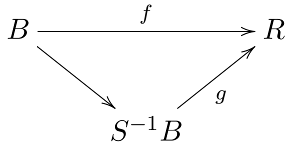
Assume that it is true for every degree smaller than i. Now, any element $\gamma \in \mathrm{HS}^i_{S^{-1}B/A}$ is a finite sum $\sum_\beta \alpha_\beta \beta$, where $\alpha_\beta \in \left(\mathrm{HS}^{i-l}_{B/A}\right)^+$, $\beta \in \mathrm{HS}^l_{S^{-1}B/A}$ for some $l < i$, by (2.11). We may actually assume that $\gamma = \alpha \cdot \beta$, where $\beta = d_l(x)$ for some $l < i$ and some $x \in B$. This implies that $\alpha \in \mathrm{HS}^{i-l}_{B/A}$, so α is necessarily the image of some $\tilde{\alpha} \in S^{-1}\mathrm{HS}^{i-l}_{B/A}$ by induction hypothesis. Therefore $\alpha \cdot \beta = \Psi(\tilde{\alpha} \cdot \beta)$ lies in the image of $S^{-1}\mathrm{HS}^{i-l+l}_{B/A}$ by Ψ, as we wanted to prove.
To prove the injectivity of Ψ, let us show that, in fact, there is a section

$$\Xi : S^{-1}\mathrm{HS}^m_{B/A} \longrightarrow \mathrm{HS}^m_{S^{-1}B/A},$$

satisfying $\Xi \circ \Psi = \mathrm{id}$. This is equivalent to showing that

$$\operatorname{Hom}_{S^{-1}B}(\mathrm{HS}_{S^{-1}B/A}, R) \longrightarrow \operatorname{Hom}_{S^{-1}B}(S^{-1}\mathrm{HS}^m_{B/A}, R)$$

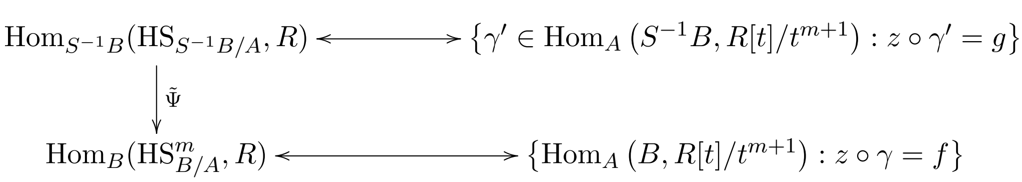
is surjective for any $S^{-1}B$-algebra R, say



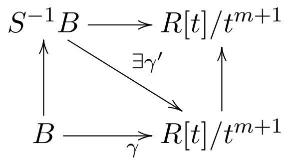
and then replacing R by $S^{-1}\mathrm{HS}^m_{B/A}$. But note that

$$\mathrm{Hom}_{S^{-1}B}(S^{-1}\mathrm{HS}^m_{B/A}, R) = \mathrm{Hom}_B(\mathrm{HS}^m_{B/A}, R).$$

Moreover, by Remark 2.2.13, finding Ψ is equivalent to having a map



Since $S^{-1}B$ is formally smooth, any $\gamma : B \longrightarrow R[t]/t^{m+1}$ is the image by $\tilde{\Psi}$ of some $\gamma' : S^{-1}B \longrightarrow R[t]/t^{m+1}$ via the diagram



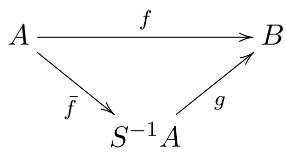
which proves the surjectivity of $\tilde{\Psi}$ and, as a consequence, the injectivity of Ψ. $\square$

The property described by Lemma 2.2.17 is functorial in S, and gives the gluing condition that we need to construct $\mathrm{HS}^m_{X/Y}$ if Y is affine. Now let Y be any scheme, take $\mathrm{Spec}(A) \subset Y$ and $\mathrm{Spec}(B) \subset f^{-1}(\mathrm{Spec}(A))$.

Lemma 2.2.18. *[83, Lemma 4.1] Let $f : A \longrightarrow B$ be a ring homomorphism, and fix $m \in \mathbb{N} \cup \{\infty\}$. Let $S \subset A$ be a multiplicative subset such that f factors through $S^{-1}A$. Then, there is an isomorphism*

$$\mathrm{HS}^m_{B/A} \xrightarrow{\cong} \mathrm{HS}^m_{B/S^{-1}A}.$$

Proof. The hypothesis on f implies that the elements $f(s) \in B$ are invertible in B for all $s \in S$, via



Note now that

$$\mathrm{HS}^m_{B/A} = \frac{B[x^{(i)}]_{x\in B, i=0,\dots,m}}{I_{B/A}} \text{ and } \mathrm{HS}^m_{B/S^{-1}A} = \frac{B[x^{(i)}]_{x\in B, i=0,\dots,m}}{I_{B/S^{-1}A}},$$

where $I_{B/S^{-1}A} \supset I_{B/A}$, being the difference generated by the elements of the form $(\bar{f}(s^{-1}a))^{(i)}$ for some $s \in S$, $a \in A$ and $i \in \{0, \dots, m\}$. However, we will see that in fact both ideals coincide. Let us show that for any $i = 0, \dots, m$, and for any $a \in A$ and $s \in S$, $(f(s^{-1}a))^{(i)} \in I_{B/A}$. We proceed by induction:

- $(f(s^{-1}))^{(0)} = f(s^{-1}a) = f(s^{-1})f(a)$, and note that $f(a) \in I_{B/A}$ and that $f(s)$ is invertible in B. Then, necessarily $f(s^{-1}a) \in I_{B/A}$.
- Assume $(f(a^{-1}a))^{(i-1)} \in I_{B/A}$. Then $(f(ss^{-1}a))^{(i)} = f(a)^{(i)} \in I_{B/A}$. But

$$(f(ss^{-1}a))^{(i)} = \sum_{i=j+k} (f(s))^{(j)}(f(s^{-1}a))^{(k)} =$$
$$= \sum_{i=j+k, k<i} (f(s))^{(j)}(f(s^{-1}a))^{(k)} + f(s)(f(s^{-1}a))^{(i)},$$

  where $(f(s^{-1}a))^{(k)} \in I_{B/A}$ for all $k < i$, and $f(s)$ is an invertible element in B. Hence, $(f(s^{-1}a))^{(i)} \in I_{B/A}$.

□

This property is functorial in S, and the combination of both lemmas give the following commutative diagram for any multiplicative sets $T \subset B$ and $S \subset A$ such that $A \longrightarrow B$ factors through $S^{-1}A$:

$$\begin{array}{ccc} T^{-1}\mathrm{HS}^m_{B/A} & \longrightarrow & T^{-1}\mathrm{HS}^m_{B/S^{-1}A} \\ \downarrow & & \downarrow \\ \mathrm{HS}^m_{T^{-1}B/A} & \longrightarrow & \mathrm{HS}_{T^{-1}B/S^{-1}A} \end{array}$$

These gluing properties yield the construction of the sheaf $\mathrm{HS}^m_{X/Y}$, which allows us to define the jet schemes of arbitrary schemes.

Definition 2.2.19. Let $X \longrightarrow Y$ be any morphism of schemes. The *scheme of m-jet differentials* of X over Y is defined as

$$J_m(X/Y) := \mathbf{Spec}(\mathrm{HS}^m_{X/Y}),$$

where $\mathrm{HS}^m_{X/Y}$ is a sheaf of algebras, so $\mathbf{Spec}(\mathrm{HS}^m_{X/Y})$ is constructed by gluing affine schemes. If $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$, then $J_m(\mathrm{Spec}(B)/\mathrm{Spec}(A)) = \mathrm{Spec}\left(\mathrm{HS}^m_{B/A}\right)$.

For $X \longrightarrow Y$, the morphisms $f_{m,p} : \mathrm{HS}^m_{B/A} \longrightarrow \mathrm{HS}^p_{B/A}$ in (2.8) for $A \longrightarrow B$ can be glued together, and there exist graded homomorphisms of $\mathcal{O}_X$-algebras

$$f_{m,p} : \mathrm{HS}^m_{X/Y} \longrightarrow \mathrm{HS}^p_{X/Y},$$

for $0 \leq m \leq p \leq \infty$, and

$$\mathrm{HS}^\infty_{X/Y} = \varinjlim_{m \in \mathbb{N}} \mathrm{HS}^m_{X/Y}.$$

These induce morphisms of schemes

$$\pi_{p,m} : J_p(X/Y) \longrightarrow J_m(X/Y), \ \ p \geq m,$$

satisfying $\pi_{m,m} = \mathrm{Id}$ for any $m \in \mathbb{N} \cup \{\infty\}$, and

$$\pi_{p,m} = \pi_{k,m} \circ \pi_{p,k}$$

for any $0 \leq m \leq k \leq p \leq \infty$.

Definition 2.2.20. The *scheme of arc differentials* of X over Y is defined as

$$J(X/Y) := \varprojlim_{m \in \mathbb{N}} J_m(X/Y).$$

Examples

Example 2.2.21. Let $Y = \mathrm{Spec}(k)$, and let $X \subset \mathbb{A}^n_k = \mathrm{Spec}(k[x_1, \ldots, x_n])$ be an affine variety over k defined by $X = \mathbb{V}(f)$ for some $f \in k[x_1, \ldots, x_n]$. The m-jet scheme of X, for a fixed $m \in \mathbb{N}$, is given as

$$\mathcal{L}_m(X) = \mathrm{Spec}\left(\frac{k[x_1^{(0)}, \ldots, x_n^{(0)}, \ldots, x_1^{(m)}, \ldots, x_n^{(m)}]}{(f, d_1(f), \ldots, d_m(f))}\right).$$

The arc space of X is

$$\mathcal{L}(X) = \mathrm{Spec}\left(\frac{k[x_1^{(0)}, \ldots, x_n^{(0)}, x_1', \ldots, x_n', \ldots]}{(f, d_1(f), \ldots)}\right).$$

Let us go back to Example 2.2.6. For $X_1 = \mathbb{V}(x^2 - y^3) \subset \mathrm{Spec}(k[x, y])$, we have

$$\mathcal{L}_0(X_1) = X_1 = \mathrm{Spec}\left(\frac{k[x, y]}{(x^2 - y^3)}\right),$$

$$\mathcal{L}_1(X_1) = \mathrm{Spec}\left(\frac{k[x, y, x', y']}{(x^2 - y^3, 2x(x') - 3y^2(y'))}\right),$$

$$\mathcal{L}_2(X_1) = \mathrm{Spec}\left(\frac{k[x, y, x', y', x'', y'']}{(x^2 - y^3, 2x(x') - 3y^2(y'), 2x(x'') + (x')^2 - 3y^2(y'') - 3y(y')^2)}\right),$$

$$\mathcal{L}_3(X_1) = \mathrm{Spec}\left(\frac{k[x, y, x', y', x'', y'', x''', y''']}{\begin{array}{c}(x^2 - y^3, 2x(x') - 3y^2(y'), 2x(x'') + (x')^2 - 3y^2(y'') - 3y(y')^2, \\ 2x(x') + 2(x')(x'') - 3y^2(y''') - 6x(y')(y'') - (y')^3)\end{array}}\right).$$

Example 2.2.22. Let $Y = \operatorname{Spec}(k)$ and let $X_8 = \mathbb{V}(xy - z^3) \subset \operatorname{Spec}(k[x,y,z])$. Then

$$\mathcal{L}_0(X_8) = X_8 = \operatorname{Spec}\left(\frac{k[x,y,z]}{(xy - z^3)}\right),$$

$$\mathcal{L}_1(X_8) = \operatorname{Spec}\left(\frac{k[x,y,z,x',y',z']}{(xy - z^3, x(y') + (x')y - 3z^2(z'))}\right),$$

$$\mathcal{L}_2(X_8) = \operatorname{Spec}\left(\frac{k[x,y,z,x',y',z',x'',y'',z'']}{\substack{(xy - z^3, x(y') + (x')y - 3z^2(z'),\\ x(y'') + (x')(y') + (x'')y - 3z^2(z'') - 3z(z')^2)}}\right),$$

$$\mathcal{L}_3(X_8) = \operatorname{Spec}\left(\frac{k[x,y,z,x',y',z',x'',y'',z'',x''',y''',z''']}{\substack{(xy - z^3, x(y') + (x')y - 3z^2(z'),\\ x(y'') + (x')(y') + (x'')y - 3z^2(z'') - 3z(z')^2,\\ x(y''') + (x')(y'') + (x'')(y') + (x''')y - 3z^2(z''') - 6z(z')(z'') - (z')^3)}}\right).$$

Example 2.2.23. Let $Y = \operatorname{Spec}(k)$ again and let now $X_9 = \mathbb{V}(xy - z^2) \subset \operatorname{Spec}(k[x,y,z])$. In this case

$$\mathcal{L}_0(X_9) = X_9 = \operatorname{Spec}\left(\frac{k[x,y,z]}{(xy - z^2)}\right),$$

$$\mathcal{L}_1(X_9) = \operatorname{Spec}\left(\frac{k[x,y,z,x',y',z']}{(xy - z^2, x(y') + (x')y - 2z(z'))}\right),$$

$$\mathcal{L}_2(X_9) = \operatorname{Spec}\left(\frac{k[x,y,z,x',y',z',x'',y'',z'']}{\substack{(xy - z^2, x(y') + (x')y - 2z(z'),\\ x(y'') + (x')(y') + (x'')y - 2z(z'') - (z')^2)}}\right).$$

$$\mathcal{L}_3(X_9) = \operatorname{Spec}\left(\frac{k[x,y,z,x',y',z',x'',y'',z'',x''',y''',z''']}{\substack{(xy - z^2, x(y') + (x')y - 2z(z'),\\ x(y'') + (x')(y') + (x'')y - 2z(z'') - (z')^2,\\ x(y''') + (x')(y'') + (x'')(y') + (x''')y - 2z(z''') - 2(z')(z''))}}\right).$$

Example 2.2.24. Let us consider now the m-jets for X which are centered at the origin. The arcs satisfying this condition for the previous examples correspond to the following:

- For X_1, the 1-jets with center the origin $(0,0)$ are

$$\operatorname{Spec}\left(\frac{k[x,y,x',y']}{(x,y)}\right),$$

  the 2-jets centered at $(0,0)$ are

$$\operatorname{Spec}\left(\frac{k[x,y,x',y',x'',y'']}{(x,y,(x')^2)}\right),$$

and the 3-jets with the same property are

$$\operatorname{Spec}\left(\frac{k[x,y,x',y',x'',y'',x''',y''']}{(x,y,(x')^2,(y')^3)}\right).$$

The k-3-jets of X will correspond then to morphisms:

$$\begin{aligned}\gamma_3^* : k[x,y] &\longrightarrow k[t]/t^4\\ x &\longmapsto at^2+bt^3\\ y &\longmapsto ct+dt^2+et^3,\end{aligned}$$

where $a,b,c,d,e \in k$ and either c or d must be zero.

- For X_8, the 1-jets with center the origin $(0,0,0)$ are

$$\operatorname{Spec}\left(\frac{k[x,y,z,x',y',z']}{(x,y,z)}\right),$$

the 2-jets with center the origin are

$$\operatorname{Spec}\left(\frac{k[x,y,z,x',y',z',x'',y'',z'']}{(x,y,z,(x')(y'))}\right),$$

and for the 3-jets we have

$$\operatorname{Spec}\left(\frac{k[x,y,z,x',y',z',x'',y'',z'',x''',y''',z''']}{(x,y,z,(x')(y'),(x')(y'')+(x'')(y')-(z')^3)}\right).$$

- As for X_9, the 1-jets with center the origin are

$$\operatorname{Spec}\left(\frac{k[x,y,z,x',y',z']}{(x,y,z)}\right),$$

the 2-jets through the origin are given by

$$\operatorname{Spec}\left(\frac{k[x,y,z,x',y',z',x'',y'',z'']}{(x,y,z,(x')(y')-(z')^2)}\right),$$

and the 3-jets

$$\operatorname{Spec}\left(\frac{k[x,y,z,x',y',z',x'',y'',z'',x''',y''',z''']}{(x,y,z,(x')(y')-(z')^2,(x')(y'')+(x'')(y')-2(z')(z''))}\right).$$

Remark 2.2.25. Let $X = \operatorname{Spec}(B)$ be a variety over a field k, and consider the map

$$\begin{aligned}\pi_X : \mathcal{L}(X) &\longrightarrow X\\ \alpha &\longmapsto \alpha(0),\end{aligned}$$

mapping each arc to its center. This corresponds to a ring homomorphism:

$$B \longrightarrow \mathrm{HS}^{\infty}_{X/\operatorname{Spec}(k)}$$

where, for any point (arc) α of $\mathcal{L}(X)$ corresponding to a prime ideal $\mathcal{Q}_\alpha \in \mathrm{HS}^{\infty}_{X/\operatorname{Spec}(k)}$, the preimage by this ring homomorphism is the prime ideal in B defining the center of the arc α.

2.3 Structure of Arc spaces

The distinction between thin and fat arcs arises in the study of constructible subsets of jet spaces and cylinders ([28]). It is related to the connection between arcs and valuations, and helpful in the study of the components of arc spaces of varieties.

Definition 2.3.1. Let X be an irreducible k-scheme of finite type. An K-arc α in X is *fat* if α does not factor through any proper closed subscheme $Z \subset X$. That is, $\alpha : \mathrm{Spec}(K[[t]]) \longrightarrow X$ is fat if for no proper closed subscheme $Z \subset X$, $\alpha \in \mathcal{L}(Z)$. Equivalently, α is fat if $\alpha(\eta) \in X$ is the generic point of X, where η is the generic point of $\mathrm{Spec}(K[[t]])$. That is, if $\overline{\mathrm{Im}(\alpha)} = X$. An arc which is not fat is called *thin.*

Fat and thin arcs, as points of the arc space, induce a notion of fat and thin subsets of the same space:

Definition 2.3.2. An irreducible (not necessarily closed) subset $C \subset \mathcal{L}(X)$ is *thin* if there exists a proper closed subset $Z \subset X$ such that $C \subset \mathcal{L}(Z)$. An irreducible $C \subset \mathcal{L}(X)$ is *fat* if it is not thin.

Proposition 2.3.3. *[47, 2.19] An irreducible, not necessarily closed, subset $C \subset \mathcal{L}(X)$ with generic point $\gamma \in C$ such that $\overline{\gamma} \supset C$ is fat if and only if γ is a fat arc.*

Example 2.3.4. Consider $X_{10} = \mathbb{V}((x^2 - y^3)^2 - z^6) \subset \mathrm{Spec}(\mathbb{C}[x, y, z])$. The $\mathbb{C}$-arc φ_1 given by

$$\begin{aligned} \varphi_1^* : \mathbb{C}[x, y, z] &\longrightarrow \mathbb{C}[[t]] \\ x &\mapsto t^3 \\ y &\mapsto t^2 \\ z &\mapsto 0 \end{aligned}$$

is a thin arc, since it factors through the subvariety $Z = \mathbb{V}(x^2 - y^3, z) \subset X_{10}$. On contrary, the $\mathbb{C}$-arc φ_2

$$\begin{aligned} \varphi_2^* : \mathbb{C}[x, y, z] &\longrightarrow \mathbb{C}[[t]] \\ x &\mapsto 2t^3 \\ y &\mapsto t^2 \\ z &\mapsto \sqrt[3]{3}t^2 \end{aligned}$$

does not factor through Z.

Example 2.3.5. The arc in $X_1 = \mathbb{V}(x^2 - y^3) \subset \mathrm{Spec}(k[x, y])$ given as[1]

$$(\varphi^*(x), \varphi^*(y)) = (t^2, t^3)$$

is fat.

[1]Let us use this abbreviate notation from now on.

Example 2.3.6. Consider $\bar{X}_1 = \mathbb{V}(x^2 - y^3) \subset \operatorname{Spec}(k[x,y,z])$. Any k-arc of the form

$$(\varphi^*(x), \varphi^*(y), \varphi^*(z)) = (t^2, t^3, a),$$

for $a \in k$ is thin, while the $k(\chi)$-arc

$$(\varphi^*(x), \varphi^*(y), \varphi^*(z)) = (t^2, t^3, \chi),$$

where χ is transcendent over k, is fat.

Proposition 2.3.7. *[46, Proposition 2.5] Let X be a variety over $\mathbb{C}$, and let $\alpha : \operatorname{Spec}(K[[t]]) \to X$ be a K-arc. Then α is fat if and only if the induced ring homomorphism $\alpha^* : \mathcal{O}_{X,\alpha(0)} \to K[[t]]$ is injective.*

The following property lets us study arcs through the resolution process:

Proposition 2.3.8. *[46, Proposition 2.5] Let X and Y be two varieties over k, and let $\varphi : Y \longrightarrow X$ be a proper birational map. If $\alpha \in \mathcal{L}(X)$ is a fat arc, then α lifts to an arc $\alpha \in \mathcal{L}(Y)$.*

2.4 Arcs vs. valuations

Let X be a reduced scheme of finite type over an algebraically closed field k. For any field extension $K \supset k$, an arc $\alpha : \operatorname{Spec}(K[[t]]) \longrightarrow X$ defines a ring homomorphism

$$\begin{aligned} \phi_\alpha : \mathcal{O}_{X,\alpha(0)} &\longrightarrow \mathbb{N} \cup \{\infty\} \\ f &\longmapsto \operatorname{ord}_t \alpha^*(f). \end{aligned}$$

This homomorphism can be extended to a valuation in the function field of X if and only if the image of the generic point of $\operatorname{Spec}(K[[t]])$, $\alpha(\eta)$ is dense in X. That is, if and only if α is a fat arc in X. To make this precise, let us start by a more general concept to that of valuation (from [75]).

Definition 2.4.1. Let R be a k-algebra. A (discrete) *semi-valuation* v in R is a map

$$v : R \longrightarrow \mathbb{N} \cup \{\infty\}$$

satisfying:

1. $v(fg) = v(f) + v(g)$ for any $f, g \in R$,
2. $v(f+g) \geq \min\{v(f), v(g)\}$ for any $f, g \in R$,
3. $v(0) = \infty$ and
4. $v(\lambda) = 0$ for any $\lambda \in k \setminus \{0\}$.

Note that condition (3) does not exclude other elements $g \in F$, $g \neq 0$ from verifying $v(g) = \infty$. The set

$$\mathfrak{a}_v = \{f \in R : v(f) = \infty\} \supset (0)$$

is a prime ideal called the *home* of v. If $\mathfrak{a}_v = (0)$, then v is a (discrete) *valuation*. This is always the case, for instance, if R is a domain. The semi-valuation v gives a valuation in $\mathfrak{Frac}(R/\mathfrak{a}_v)$.
The set

$$\mathcal{C}_v = \{f \in R : v(f) > 0\}$$

is also a prime ideal, called the *center* of v.

Indeed, if v is a semi-valuation in X, then it is a valuation in $Y \subset X$, defined by the home of v. Let $\mathcal{O}_v \subset K = \mathfrak{Frac}(\mathcal{O}_Y)$ be the valuation ring of v, and let $k_v = \mathcal{O}_v/\mathcal{C}_v$ its residue field. Then we have

$$\mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow \mathcal{O}_v \hookrightarrow \hat{\mathcal{O}}_v \xrightarrow[\cong]{\Lambda} k_v[[t]], \qquad \alpha_{v,\Lambda} : \mathcal{O}_X \longrightarrow k_v[[t]] \tag{2.12}$$

which, for each choice of Λ, gives an arc $\alpha_{v,\Lambda}$. Hence, there is not a unique arc attached to each semi-valuation.

Conversely, any arc $\alpha : \mathrm{Spec}(K[[t]]) \longrightarrow \mathrm{Spec}(R)$ induces a semi-valuation

$$\begin{aligned} v_\alpha : R &\longrightarrow \mathbb{N} \cup \{\infty\} \\ a &\longmapsto v_\alpha(a) = \mathrm{ord}_t \alpha^*(a) \end{aligned}$$

via

$$\begin{array}{ccc} R & \xrightarrow{\quad v_\alpha \quad} & \mathbb{N} \cup \{\infty\} \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle \mathrm{ord}_t} & \\ & K[[t]] & \end{array}$$

Remark 2.4.2. Each choice of Λ in (2.12) gives an arc. However, note that all possible arcs arising from v in this way give rise to the same semi-valuation.

Proposition 2.4.3. *[46, Definition 2.6],[24, 3.1] An arc $\alpha \in \mathcal{L}(X)$ is fat if and only if the induced semi-valuation*

$$\begin{aligned} v_\alpha : K(X)^* &\longrightarrow \mathbb{Z} \\ a &\longmapsto \mathrm{ord}_t \alpha^*(a) \end{aligned}$$

is a valuation.

This last proposition is a consequence of Proposition 2.3.7. For this reason, fat arcs are also called *valuative arcs.*

2.5 Properties of jet and arc spaces

Let us show here some properties of the schemes of arcs and jets that will lead to important results about singularities. The proofs and related results can be found in [50], [46], [47] and [23], among others.

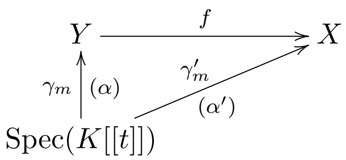
Proposition 2.5.1. *Let $f : Y \longrightarrow X$ be a morphism of schemes of finite type over a field k. Then, for any $m \in \mathbb{N}$, f induces morphisms between the respective schemes of m-jets*

$$\mathcal{L}_m(Y) \xrightarrow{f_m} \mathcal{L}_m(X),$$

and between the schemes of arcs

$$\mathcal{L}(Y) \xrightarrow{f_\infty} \mathcal{L}(X).$$

Proof. It suffices to note that the composition with f gives, for any m-jet γ_m (resp. arc α) in Y, an m-jet γ'_m (resp. arc α') in X, via the diagram



□

Remark 2.5.2. If $f : Y \longrightarrow X$ is proper and birational, the fat arcs in Y are in bijection with the fat arcs in X via f_∞, as a consequence of Proposition 2.5.1 and Proposition 2.3.8.

Proposition 2.5.3. *[50, Proposition 3.2] Let $f : Y \longrightarrow X$ be a proper birational morphism of schemes over k. Say that there exist closed subsets $U \subset X$ and $V \subset Y$ such that $Y \setminus V \cong X \setminus U$ via f. Then f_∞ induces a bijection from $\mathcal{L}(Y) \setminus \mathcal{L}(V)$ to $\mathcal{L}(X) \setminus \mathcal{L}(U)$.*

Proposition 2.5.4. *[50, Proposition 3.3] Let $f : Y \longrightarrow X$ be an étale morphism of schemes of finite type over k. Then, for all $m \in \mathbb{N}$,*

$$\mathcal{L}_m(Y) \cong \mathcal{L}_m(X) \times_X Y$$

and

$$\mathcal{L}(Y) \cong \mathcal{L}(X) \times_X Y.$$

Proposition 2.5.5. *[50, Proposition 3.4] Let X and Y be schemes of finite type over k. Then, for all $m \in \mathbb{N}$,*

$$\mathcal{L}_m(X \times Y) \cong \mathcal{L}_m(X) \times \mathcal{L}_m(Y)$$

and

$$\mathcal{L}(X \times Y) \cong \mathcal{L}(X) \times \mathcal{L}(Y).$$

Proposition 2.5.6. *[50, Proposition 3.5] Let X and Y be schemes of finite type over k. If $f : Y \longrightarrow X$ is an open (resp. closed) immersion, then the induced morphisms between their respective schemes of m-jets for all $m \in \mathbb{N}$ and between their respective schemes of arcs are also open (resp. closed).*

Remark 2.5.7. Having $f : Y \longrightarrow X$ surjective does not imply that f_m or f_∞ are surjective. Similarly, if f is closed, f_m and f_∞ do not need to be closed.

Theorem 2.5.8. *[69] The set of arcs of a variety X, defined over a field of characteristic zero, with center inside of the singular locus of X has a finite number of irreducible components.*

The following result from Kolchin ([56]) has been deeply studied, specially in order to understand the irreducible components of the space of arcs, for example, with the focus on the Nash problem:

Theorem 2.5.9. *[56, Chapter IV, Proposition 10] Let k be a field of characteristic zero. If X is a variety over k, then $\mathcal{L}(X)$ is irreducible.*

Remark 2.5.10. Irreducibility of $\mathcal{L}(X)$ is not guaranteed if X is a variety over a field k of positive characteristic. The theorem is also false for $\mathcal{L}_m(X)$, even if $\text{char}(k) = 0$.

Theorem 2.5.11. *[45, Corollary 3.3] If k is a field of arbitrary characteristic and X is a toric variety over k, then $\mathcal{L}(X)$ is irreducible.*

A large number of works explore the properties of arc and jet spaces of varieties, many of them focused on their singularities. Examples of this can be found in [38], [72], [60], [45], [73], [74], [65], [64], [53], [27], [59], [66]. Other applications of arc and jet schemes are shown, for instance, in [17].

2.6 Arcs and Singularities

Jet and arc spaces give some information about the singularities of varieties. We expose next a few results in this direction, to give a flavour of the interest in the tools developed along this chapter in the study of singularities. Afterwards, we will focus on some specific information that one can extract form the arc space of a variety: the so called Nash multiplicity sequence, around which the results of this thesis attend. We will show that it is actually connected to invariants of constructive resolution.

Theorem 2.6.1. *[67, Theorem 0.1] Let k be an algebraically closed field of characteristic zero, and let X be a complete intersection variety defined over k. Then $\mathcal{L}_m(X)$ is irreducible for all $m \geq 1$ if and only if X has rational singularities.*

The following results from S. Ishii relate smoothness of a variety X to the smoothness of its jet schemes:

Theorem 2.6.2. *[48, Corollary 1.2] Let k be a field of arbitrary characteristic. Let X be a scheme of finite type over k. Then X is smooth if and only if $\mathcal{L}_m(X)$ is smooth for some $m \in \mathbb{N}$.*

Theorem 2.6.3. *Let X be a scheme of finite type over an algebraically closed field k. Then:*

1. *If $\mathrm{char}(k) = 0$, then X is nonsingular if and only if there exist $m, m' \in \mathbb{Z}_{\geq 0}$ with $m < m'$ such that the truncation morphism $\pi_{m',m} : \mathcal{L}_{m'}(X) \longrightarrow \mathcal{L}_m(X)$ is flat. ([48, Theorem 1.3])*
2. *If $\mathrm{char}(k) > 0$ and X is reduced, then X is nonsingular if and only if there exist $m, m' \in \mathbb{Z}_{>0}$ with $m < m'$ such that the truncation morphism $\pi_{m',m} : \mathcal{L}_{m'}(X) \longrightarrow \mathcal{L}_m(X)$ is flat. ([48, Theorem 1.4])*

In [49], some geometrical properties of m-jets $\mathcal{L}_m(X)$ are shown to imply the same properties of X:

Theorem 2.6.4. *[49, Section 3] Let X be a variety over a field k. Then*

1. *$\mathcal{L}_m(X)$ reduced for some $m \in \mathbb{N} \Rightarrow X$ reduced.*
2. *$\mathcal{L}_m(X)$ connected for some $m \in \mathbb{N} \Leftrightarrow \mathcal{L}_m(X)$ connected for any $m \in \mathbb{N} \Leftrightarrow X$ connected.*
3. *$\mathcal{L}_m(X)$ irreducible for some $m \in \mathbb{N} \Rightarrow X$ irreducible.*
4. *$\mathcal{L}_m(X)$ locally integral for some $m \in \mathbb{N} \Rightarrow X$ locally integral.*
5. *$\mathcal{L}_m(X)$ locally integral and normal for some $m \in \mathbb{N} \Rightarrow X$ locally integral and normal.*
6. *$\mathcal{L}_m(X)$ locally complete intersection for some $m \in \mathbb{N} \Rightarrow X$ locally complete intersection.*

Moreover, when it comes to the complexity of the singularities of X, we have

Theorem 2.6.5. *[49, Theorems 3.10 and 3.11] Let X be a variety over a field k of characteristic zero. Then:*

1. *The existence of $m \in \mathbb{N}$ such that $\mathcal{L}_m(X)$ has at worst canonical/terminal/log terminal singularities implies that X has at worst canonical/terminal/log terminal singularities.*
2. *The existence of $m \in \mathbb{N}$ such that $\mathcal{L}_m(X)$ has at worst log canonical singularities implies that X has at worst log terminal singularities.*

In the same work, flatness of morphisms of schemes is also studied through flatness of the induced morphisms of jet schemes:

Theorem 2.6.6. *[49, Theorem 4.1] Let X and Y be two schemes over a field k, and let $f : X \longrightarrow Y$ be a morphism of schemes. If the induced morphism of m-jet schemes $f_m : \mathcal{L}_m(X) \longrightarrow \mathcal{L}_m(Y)$ is flat for some $m \in \mathbb{N}$, then f is also flat.*

2.7 The Nash multiplicity sequence

In [58], M. Lejeune-Jalabert introduced a sequence of positive integers attached to an arc in a germ of a hypersurface at a point, and she called it the *Nash multiplicity sequence*. This sequence is non increasing:

$$m_0 \geq m_1 \geq m_2 \geq \ldots \geq m_k \geq 1$$

and stabilizes for some $k \in \mathbb{N}$.

Later, in [40], M. Hickel generalized this sequence for varieties of higher codimension. The way in which he constructs the sequence, involves a sequence of blow ups determined by the chosen arc. For this construction, Hickel works with arcs inside of a germ of a variety at a point (analytic context). We will work with arcs inside of a local neighborhood of the variety at the point (local algebraic context). We will explain now this construction carefully, to show the computation of the Nash multiplicity sequence from this local algebraic point of view.

Let $X^{(d)}$ be an irreducible algebraic variety of dimension d over a perfect field k. Let ξ be a point contained in $\underline{\operatorname{Max}}\operatorname{mult}(X^{(d)})$, the closed set of points of maximum multiplicity of $X^{(d)}$.[2] For simplicity, we will assume that ξ is a closed point. This will allow us to consider the blow up at ξ, since ξ is a smooth center in this case. In case one wants to consider non closed points, one needs just to localize X at ξ before performing the sequences that we will construct in this Section.

Consider the product of $X^{(d)}$ with an affine line. We have a surjective morphism

$$X^{(d)} \xleftarrow{p} X_0^{(d+1)} = X^{(d)} \times \mathbb{A}_k^1, \tag{2.13}$$

given by the projection onto the first component. Let us write $\xi_0 = (\xi, 0)$, which is a point in $X_0^{(d+1)}$ dominating ξ.

Consider the blow up of $X_0^{(d+1)}$ at ξ_0, which we will denote by π_1. We will write $X_1^{(d+1)}$ for the transform of $X_0^{(d+1)}$ under π_1. After performing this blow up, we can choose a new point $\xi_1 \in X_1^{(d+1)}$, and call π_2 the blow up of $X_1^{(d+1)}$ at ξ_1.

Next, we will establish a criterion for the choice of each $\xi_i \in X_i^{(d+1)}$ using an arc, so that we can perform a sequence of blow ups at points in this way.

$$(X_0^{(d+1)}, \xi_0) \xleftarrow{\pi_1} (X_1^{(d+1)}, \xi_1) \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_r} (X_r^{(d+1)}, \xi_r). \tag{2.14}$$

Let $\varphi \in \mathcal{L}(X^{(d)})$ be an arc in $X^{(d)}$ through ξ. That is, we have a local homomorphism of local rings

$$\begin{aligned} \varphi^* : \mathcal{O}_{X^{(d)},\xi} &\longrightarrow K[[t]] \\ \mathcal{M}_\xi &\longrightarrow \langle t^n \rangle, \end{aligned}$$

[2]Note that we can always assume this situation for any $\xi \in X$, since one can always consider a neighborhood of ξ where this is true.

for some positive integer n or, equivalently, a morphism $\varphi : \mathrm{Spec}(K[[t]]) \longrightarrow X^{(d)}$, mapping the closed point to ξ. This, together with the inclusion map $i : k[t] \to K[[t]]$ gives an arc Γ_0 in $X_0^{(d+1)}$ through ξ_0

$$\begin{aligned} \Gamma_0^* : \mathcal{O}_{X_0^{(d+1)},\xi_0} &\overset{\varphi^*\otimes i}{\longrightarrow} K[[t]] \\ \mathcal{M}_{\xi_0} &\longmapsto \langle t \rangle \end{aligned}$$

where Γ_0 is the morphism given by the universal property of the fiber product:

$$\begin{array}{ccc} \mathrm{Spec}(K[[t]]) & \overset{i^*}{\longrightarrow} & \mathrm{Spec}(K[t]) \\ \Gamma_0 \searrow & & \\ (X^{(d)},\xi) \times_k \mathrm{Spec}(K[t]) = (X_0^{(d+1)},\xi_0) & \longrightarrow & \mathrm{Spec}(K[t]) \\ \downarrow & & \downarrow \\ (X^{(d)},\xi) & \longrightarrow & \mathrm{Spec}(k). \end{array} \quad (\text{with } \varphi : \mathrm{Spec}(K[[t]]) \to (X^{(d)},\xi)) \tag{2.15}$$

Note that Γ_0 is in fact the graph of φ.

Consider the blow up π_1 of $X_0^{(d+1)}$ at ξ_0. The initial *Nash multiplicity of X at ξ* is defined as

$$m = m_0 = \mathrm{mult}_{\xi_0}(X_0^{(d+1)}) = \mathrm{mult}_{\xi}(X^{(d)}),$$

where the last identity follows from the faithful flatness of (2.13).

After blowing up $X_0^{(d+1)}$ at ξ_0 (as in 2.14), the valuative criterion of properness ensures that we can lift Γ_0 to a unique arc in $X_1^{(d+1)}$, which we will denote by Γ_1. Now Γ_1 maps the closed point of $\mathrm{Spec}(K[[t]])$ to some closed point $\xi_1 \in X_1^{(d+1)}$. Furthermore, $\xi_1 \in E_1 = \pi_1^{-1}(\xi_0)$ and $\xi_1 \in \mathrm{Im}(\Gamma_1)$. This point ξ_1 will be the center of the blow up π_2. We iterate this process: for $i = 1, \ldots, r$, let Γ_i be the lifting of the arc $\Gamma_{i-1} \in \mathcal{L}(X_{i-1}^{(d+1)})$ through ξ_{i-1} by the blow up π_i of $X_{i-1}^{(d+1)}$ with center ξ_{i-1}. Then Γ_i is an arc in $\mathcal{L}(X_i^{(d+1)})$ through a point ξ_i in the exceptional divisor $E_i = \pi_i^{-1}(\xi_{i-1})$.

Definition 2.7.1. We will say that the sequence of transformations at points chosen in this way is the sequence *directed by* φ (or that the blow ups themselves are directed by φ), meaning that $\xi_0 = (\varphi(0), 0) = (\xi, 0)$ and $\xi_i = \mathrm{Im}(\Gamma_i) \cap E_i$ for $i = 1, \ldots, r$:

$$\begin{array}{ccccccc} (X_0^{(d+1)},\xi_0) & \overset{\pi_1}{\longleftarrow} & (X_1^{(d+1)},\xi_1) & \overset{\pi_2}{\longleftarrow} & \ldots & \overset{\pi_r}{\longleftarrow} & (X_r^{(d+1)},\xi_r) \\ \Gamma_0 \uparrow & & \Gamma_1 \uparrow & & & & \Gamma_r \uparrow \\ (\mathrm{Spec}(K[[t]]),0) & \overset{id}{\longleftarrow} & (\mathrm{Spec}(K[[t]]),0) & \overset{id}{\longleftarrow} & \ldots & \overset{id}{\longleftarrow} & (\mathrm{Spec}(K[[t]]),0). \end{array} \tag{2.16}$$

For this sequence, the multiplicity of $X_i^{(d+1)}$ at ξ_i, will be the *i-th Nash multiplicity*, m_i.

The sequence $m_0, m_1, \dots, m_r$ is non-increasing, because the blow up at regular equimultiple centers does not increase the multiplicity (see [40, Theorem 4.1] or [22]), and it eventually decreases whenever the generic point of the initial arc φ is not contained in $\underline{\operatorname{Max}}\operatorname{mult}(X)$.

Theorem 2.7.2. *[58, Theorem 5] Let φ be an arc in X centered at ξ. The limit of the Nash multiplicity sequence of φ at ξ is the multiplicity of X at the generic point of* $\operatorname{Im}(\varphi) \subset X$.

That is, if the generic point of φ is contained in the stratum of X of multiplicity m' but not totally contained in any stratum of multiplicity greater than m', then the sequence stabilizes at the value m'. Therefore, for our purpose, we need to choose arcs whose generic point is not contained in the set of points of highest multiplicity of X.

Thus, we can find some r so that for the diagram above the sequence of Nash multiplicities is such that $m_0 = \ldots = m_{r-1} > m_r$. This integer r will be an object of interest for us (see 3.1).

To conclude this chapter, let us give an idea about how the Nash multiplicity sequence is defined in [58]:

Consider a hypersurface given by $\mathbb{V}(f) = X \subset V^{(n)} = \operatorname{Spec}(k[x_1, \dots, x_n])$ locally in a neighborhood of a point $\xi \in X$. Assume that f has order m_0 (hence X has multiplicity m_0 at ξ). Then, as we showed in Example 2.2.21, the m-jet space of X is given, for $m \in \mathbb{N}$, as

$$\mathcal{L}_m(X) = \operatorname{Spec}\left(\frac{k[x_1^{(0)}, \dots, x_n^{(0)}, \dots, x_1^{(m)}, \dots, x_n^{(m)}]}{(f, d_1(f), \dots, d_m(f))}\right),$$

while the arc space of X is

$$\mathcal{L}(X) = \operatorname{Spec}\left(\frac{k[x_1^{(j)}, \dots, x_n^{(j)}]_{j\geq 0}}{(d_j(f))_{j\geq 0}}\right),$$

for $d_j f$ as in Definition 2.2.5. Note that

$$d_m f \in k[x_1^{(0)}, \dots, x_n^{(0)}, \dots, x_1^{(m)}, \dots, x_n^{(m)}]$$

for any $m \in \mathbb{N}$, so each $d_m f$ is an equation of $\mathcal{L}(X)$ in $\operatorname{Spec}\left(k[x_1^{(j)}, \dots, x_n^{(j)}]_{j\geq 0}\right)$ in the variables $(x_1^{(0)}, \dots, x_n^{(0)}, \dots, x_1^{(m)}, \dots, x_n^{(m)})$. Any K-arc in X can be described by

$$\begin{aligned} \varphi^* : k[x_1, \dots, x_n] &\longrightarrow K[[t]] \\ x_i &\longmapsto \sum_{j\geq 0} a_{i,j} t^j, \end{aligned} \tag{2.17}$$

for some $a_{i,j} \in K$, $i = 1, \dots, n$, $j \geq 0$. In order to be an arc in X, φ must satisfy an infinite set of equations:

$$f(a_{1,0}, \dots, a_{n,0}) = 0,$$

$$(d_m f)\left((a_{i,j})_{i=1,\ldots,n;j=0,\ldots,m}\right) = 0, \;\; \forall\, m \geq 0,$$

as a consequence of

$$f(\sum_{j\geq 0} a_{1,j}t^j, \ldots, \sum_{j\geq 0} a_{n,j}t^j) = 0.$$

However, the following idea can be interesting: one can start by looking at the equations defining m-jets, and then find equations for $(m+1)$-jets preserving the previous conditions, repeating this process and obtaining jets of higher order at each iteration. With this idea, one can consider the equations of the m-jets in X which can be lifted to arcs in X, less conditions can be enough. The Nash multiplicities give some information in this direction.

To begin with, assuming that ξ is the center of the arcs we are considering, we deduce the equations

$$x_1^{(0)} = \ldots = x_n^{(0)} = 0. \tag{2.18}$$

Observe now that $d_1 f$ involves the variables $x'_1, \ldots, x'_n$ but, as a consequence of 2.18,

$$d_1 f = \ldots = d_{m_0-1} f = 0$$

identically. Hence, to obtain a first condition on $x'_1, \ldots, x'_n$ we need to go as far as $d_{m_0} f$. How far we need to look in order to obtain a full set of equations determining the possible values of $x'_1, \ldots, x'_n$ so that φ is a 1-jet in X which can be lifted to an arc in X, depends on X.

In Example 2.2.21, where

$$f = x^2 - y^3,$$

if

$$x^{(0)} = y^{(0)}$$

is asked, then $d_1 f = 0$, so it gives no conditions for x', y'. From $d_2 f$ we obtain

$$(x')^2 = 0,$$

and $d_3 f$ yields

$$2(x')(x'') - (y')^3 = 0.$$

Computing $d_4 f$ we obtain also

$$(x')(x''') + (x'')^2 - 3(y')^2(y'') = 0.$$

In [58], M. Lejeune-Jalabert proved that the set

$$\rho_1(X) = \left\{(a_{1,1}, \ldots, a_{n,1}) \in K^n : \exists \varphi \in \mathcal{L}(X),\, \varphi_1 = \pi_{X,1}(\varphi) = (a_{1,1}t, \ldots, a_{n,1}t) \in \mathcal{L}_1(X)\right\},$$

where $\pi_{X,1}$ is as in (2.3), satisfies

$$\rho_1(X) \subset \cup_{1\leq \mu \leq m_0} H_{m_0,\mu},$$

for certain locally closed subsets $H_{m_0,\mu} \subset K^n$. Each of them is defined by a finite set of polynomials arising from the action of differential operators up to a certain order on the homogeneous parts of f up to a certain degree. More precisely: recall that we are considering $f \in k[x_1, \ldots, x_n]$, and let $\tilde{f}$ be its image in the completion $k[\widehat{x_1, \ldots, x_n}]_\xi$,

$$\tilde{f}(x_1, \ldots, x_n) = \sum_{i \geq m} f_i(x_1, \ldots, x_n)$$

for $f_i(x_1, \ldots, x_n)$ a homogeneous polynomial of order i, $i \geq m$. Then $H_{m_0,\mu}$ is defined as the set of points of K^n where the polynomials arising from applying to the homogeneous parts of degree $m+j$ for j up to $\mu - 1$ differential operators of order up to $\mu - j - 1$ (including order 0) are zero, while those coming from differential operators of order $\mu - j$ are not:

$$\begin{aligned} H_{m_0,\mu} = & \{a \in K^n : D_I f_{m+j}(a) = 0 \ \forall \ |I| + j < \mu\} \\ & \setminus \{a \in K^n : D_I f_{m+j}(a) \neq 0 \ \forall \ |I| + j = \mu, \, j < \mu\}, \end{aligned}$$

where the D_I are the usual differential operators in $k[x_1, \ldots, x_n]$, of order $|I|$. Observe that the $H_{m_0,\mu}$ are disjoint. The result in [58] asserts that any $a \in \rho_1(X)$ is thus contained in a $H_{m_0,\mu}$, for some $1 \leq \mu \leq m_0$. For a fixed arc φ in X centered at ξ as in (2.17), the 1-jet $\varphi_1 = \pi_{X,1}(\varphi)$

$$\begin{aligned} \varphi_1^* : k[x_1, \ldots, x_n] & \longrightarrow K[[t]] \\ x_i & \longmapsto a_{i,1} t, \end{aligned}$$

the second term of the Nash multiplicity sequence m_1 is the integer such that $(a_{1,1}, \ldots, a_{n,1}) \in H_{m_0,m_1}$. Similarly, the third term m_2 is prescribed in some way by the order of the equations determining $\pi_{X,2}(\varphi)$ as a 2-jet, and the same happens for m_i as an i-jet, $i > 2$.

Remark 2.7.3. The Nash multiplicity sequence can be regarded as a refinement of the usual multiplicity function in the following sense:

From a general point of view, for a given arc $\varphi \in \mathcal{L}(X)$ centered at ξ, each m_i is determined by the cancelation of a certain set of polynomials in

$$K[x_1^{(0)}, \ldots, x_n^{(0)}, \ldots, x_1^{(i)}, \ldots, x_n^{(i)}]$$

which arise from applying certain differential operators to f. In the case of m_0, these polynomials are the coefficients in $x_1^{(0)}, \ldots, x_n^{(0)}$ of the powers of t in

$$f(x_1^{(0)} + t y_1^{(0)}, \ldots, x_n^{(0)} + t y_n^{(0)}),$$

where $y_1^{(0)}, \ldots, y_n^{(0)}$ are variables. Note that the previous expression is nothing but the Taylor expansion of f at ξ. If φ is as in (2.17), then m_0 will be such that the coefficient in $x_1^{(0)}, \ldots, x_n^{(0)}$ corresponding to $t^{m_0 - 1}$ is 0 when evaluated in $a_{1,0}, \ldots, a_{n,0}$, but the one corresponding to t^{m_0} is not. This is a condition in the 0-jet $\pi_X(\varphi) = \xi$, and hence a condition in X, which turns out to be the same condition for X to have multiplicity equal to m_0 at ξ.

Indeed, the usual differential operators applied to f define a stratification of X into locally closed sets:

$$X = \cup_{1 \leq \mu \leq m} \mathcal{H}_\mu,$$

where

$$\begin{aligned} \mathcal{H}_\mu = &\{a \in X : D_I f(a) = 0 \;\forall\; |I| < \mu\} \\ &\setminus \{a \in X : D_I f(a) = 0 \;\forall\; |I| = \mu\} \end{aligned}$$

is the subset of X given by the points of multiplicity equal to μ.

The stratification of $\rho_1(X)$ given by $\cup_{1 \leq \mu_1 \leq \mu_0} (H_{\mu_0,\mu_1} \cap \rho_1(X))$ induces a stratification of $\pi_1(\mathcal{L}(X)) \subset \mathcal{L}_1(X)$,

$$\pi_1(\mathcal{L}(X)) = \cup_{1 \leq \mu_1 \leq \mu_0} \mathcal{H}_{\mu_0,\mu_1} \subset \mathcal{L}_1(X)$$

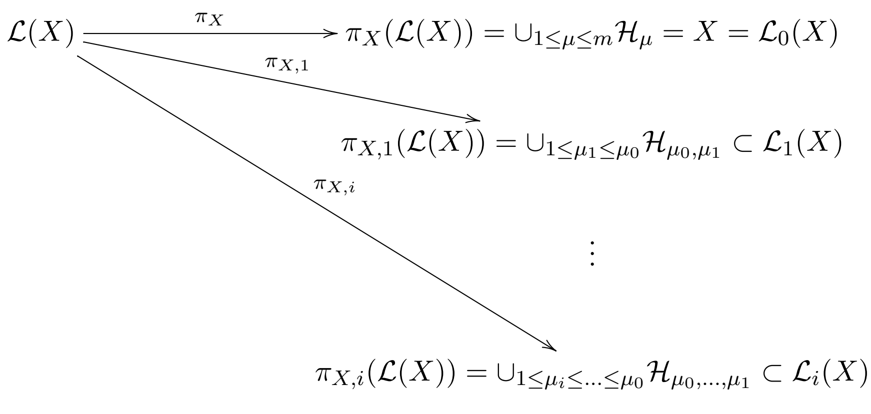
into locally closed subsets:

$$\begin{aligned} \mathcal{H}_{\mu_0,\mu_1} = &\{\varphi_1 \in \mathcal{L}_1(X) : \exists \varphi \in \mathcal{L}(X),\; \varphi_1 = \pi_{X,1}(\varphi) = (a_{1,0} + a_{1,1}t, \ldots, a_{n,0} + a_{n,1}t) \\ &(a_{1,0}, \ldots, a_{n,0}) \in H_{\mu_0},\; (a_{1,1}, \ldots, a_{n,1}) \in H_{\mu_0,\mu_1}\} \end{aligned}$$

Similarly, the results in [58] induce, for each $i \geq 1$, a stratification of $\pi_{X,i}(\mathcal{L}(X)) \subset \mathcal{L}_i(X)$ into locally closed subsets:

$$\pi_{X,i}(\mathcal{L}(X)) = \cup_{1 \leq \mu_i \leq \ldots \leq \mu_0} \mathcal{H}_{\mu_0,\ldots,\mu_i} \subset \mathcal{L}_i(X)$$

These stratifications are defined in terms of the cancellation of certain differential operators applied to f:



Given $\varphi \in \mathcal{L}(X)$, the sequence $(m_0, \ldots, m_i)$ indicates in which stratum of $\pi_{X,i}(\mathcal{L}(X))$ is $\pi_{X,i}(\varphi)$ contained. That is:

$$\begin{aligned} \mathcal{L}(X) &\xrightarrow{\pi_X} \pi_X(\mathcal{L}(X)) \\ \varphi &\longmapsto \pi_X(\varphi) = \xi = \varphi(\langle t \rangle) \in \mathcal{H}_{m_0} \end{aligned}$$

$$\begin{aligned} \mathcal{L}(X) &\xrightarrow{\pi_{X,1}} \pi_{X,1}(\mathcal{L}(X)) \\ \varphi &\longmapsto \pi_{X,1}(\varphi) \in \mathcal{H}_{m_0,m_1} \end{aligned}$$

$$\vdots$$

$$\begin{aligned} \mathcal{L}(X) &\xrightarrow{\pi_{X,i}} \pi_{X,i}(\mathcal{L}(X)) \\ \varphi &\longmapsto \pi_{X,i}(\varphi) \in \mathcal{H}_{m_0,\ldots,m_i} \end{aligned}$$

Therefore, one can consider the Nash multiplicity sequence $(m_0, \ldots, m_i, \ldots)$ of φ as the multiplicity of the arc space $\mathcal{L}(X)$ along the direction given by the arc φ.

Among the applications of this sequence, Lejeune-Jalabert describes a relation with the Artin β function. In the particular case of irreducible plane curves, an expression for the Nash multiplicity sequence in terms of the Puiseux pairs attached to the curve is given (see [58, Appendix]).

In Section 3.1, we compute the Nash multiplicity sequence for some examples.

Chapter 3

New invariants for Singularities

From the point of view of singularities, the results exposed in Chapter 2 reveal some of the motivation for the study of arc and jet spaces. The Nash multiplicity sequence, defined in section 2.7, is a construction which also reflects in some way the complexity of the singularities of an algebraic variety. Along this chapter, we define some invariants attached to points of maximum multiplicity which arise from this sequence. To do this, we observe how the Nash multiplicity sequences attached to different arcs centered at a singular point (of maximum multiplicity) behave.

The main invariants defined here for a fixed X, a point $\xi \in \underline{\mathrm{Max}}\,(X)$ and an arc in X centered at ξ are the *persistance* of the arc (Definitions 3.1.1 and 3.1.2), and the *order of contact* of the arc with the maximum multiplicity locus (Definitions 3.2.16 and 3.2.18). For the computation of the latter one, in Section 3.2 we construct the *algebra of contact*, a Rees algebra which is deeply related to the first steps of the Nash multiplicity sequence, in a sense that will be explained later. In Section 3.3, we will show that both invariants are strongly connected. We also define a set of rational numbers (Definition 3.2.20) which is an invariant for X and ξ, collecting all the information of the orders of contact of all arcs in X through ξ. This set will be useful for the interpretation of the results in the last two chapters, where we collect the conclusions from the study of these new invariants and their connections with constructive resolution of singularities.

From now on, we shall use the same notation for an arc (mostly φ) either if we mean a point of the arc space, the morphism of schemes or the induced ring homomorphism.

3.1 The Nash multiplicity sequence and the persistance of an arc

Along this section, X will always be a variety over a field k of characteristic zero, and we will always choose arcs whose generic point is not contained in the subset of maximum multiplicity of X. That is, φ will typically be chosen as an arc in X not factoring through $\underline{\operatorname{Max}}\operatorname{mult}(X)$:

$$\varphi \in \mathcal{L}(X) \setminus \mathcal{L}(\underline{\operatorname{Max}}\operatorname{mult}(X)).$$

Definition 3.1.1. Let φ be an arc in X through $\xi \in \underline{\operatorname{Max}}\operatorname{mult}(X)$. We denote by $\rho_{X,\varphi}$ the minimum number of blow ups directed by φ which are needed to lower the Nash multiplicity of X at ξ. That is, $\rho_{X,\varphi}$ is such that

$$m = m_0 = \ldots = m_{\rho_{X,\varphi}-1} > m_{\rho_{X,\varphi}}.$$

We will call $\rho_{X,\varphi}$ the *persistance of* φ *in* X. We denote by $\rho_X(\xi)$ the infimum, over all arcs in X centered at ξ, of the number of blow ups directed by the arc which are needed to lower the Nash multiplicity at ξ for the first time:

$$\begin{aligned} \rho_X : \underline{\operatorname{Max}}\operatorname{mult}(X) &\longrightarrow \mathbb{N} \\ \xi &\longmapsto \rho_X(\xi) = \min_{\varphi\in\mathcal{L}(X),\varphi(0)=\xi} \{\rho_{X,\varphi}\}, \end{aligned}$$

which is a function taking values in $\mathbb{N}$ as a consequence of Theorem 2.7.2.

To keep the notation as simple as possible, $\rho_{X,\varphi}$ does not contain a reference to the point ξ, even though it is clear that it is local. However, the point is determined by φ as its center, so it is implicit, although not explicit, in the notation. Similarly, we may refer to $\rho_X(\xi)$ as ρ_X once the point is fixed.

Let us define normalized versions of $\rho_{X,\varphi}$ and ρ_X in order to avoid the influence of the order of the arc in the number of blow ups needed to lower the Nash multiplicity. To understand the motivation for this, we refer to Example 3.2.1.

Definition 3.1.2. For a given arc φ in X, we will write

$$\bar{\rho}_{X,\varphi} = \frac{\rho_{X,\varphi}}{\operatorname{ord}(\varphi)},$$

and similarly, we will denote

$$\begin{aligned} \bar{\rho}_X : \underline{\operatorname{Max}}\operatorname{mult}(X) &\longrightarrow \mathbb{Q} \\ \xi &\longmapsto \bar{\rho}_X(\xi) = \inf_{\varphi\in\mathcal{L}(X),\varphi(0)=\xi} \{\bar{\rho}_{X,\varphi}\}. \end{aligned}$$

However, computing this number without performing the sequence of blow ups is not simple a piori. In the following section we show how this problem can be translated into a problem of resolution of Rees algebras.

3.2 The order of contact of an arc with the stratum of maximum multiplicity

Let X be a d-dimensional variety over k. As it was already explained in Section 1.2, locally in an étale neighborhood $\mathcal{U}_\xi$ of each point $\xi \in X$, we can find an immersion $\mathcal{U}_\xi \hookrightarrow V^{(n)}$, and a Rees algebra $\mathcal{G}_X^{(n)}$ over $\mathcal{O}_{V^{(n)},\xi}$ such that

$$\mathrm{Sing}(\mathcal{G}_X^{(n)}) = \underline{\mathrm{Max}}\,\mathrm{mult}(X), \tag{3.1}$$

and so that the equality is preserved by $\mathcal{G}_X^{(n)}$-local sequences over $V^{(n)}$ as long as the maximum multiplicity does not decrease (see [82]). In other words, the multiplicity is represented by this $\mathcal{G}_X^{(n)}$ (see Definition 1.3.23). Let us recall that $\mathcal{G}_X^{(n)}$ can be chosen to be differentially closed (see 1.21), by just choosing an appropriate representative of the weak equivalence class. For simplicity of the notation, we will also write X for this neighborhood $\mathcal{U}_\xi$ from now on.

Let us choose a point $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$. If we go back to (2.13), after the product $X^{(d)} \times \mathbb{A}^1_k$ we also have an immersion, and thus a commutative diagram

$$\begin{array}{ccc} V^{(n)} & \xleftarrow{\;p\;} & V_0^{(n+1)} = V^{(n)} \times \mathbb{A}^1_k \\ \uparrow & & \uparrow \\ X^{(d)} & \xleftarrow{\;p|_{X_0^{(d+1)}}\;} & X_0^{(d+1)} = X^{(d)} \times \mathbb{A}^1_k. \end{array} \tag{3.2}$$

In particular, p is a local sequence on $V^{(n)}$ and preserves (3.1). Thus, the smallest $\mathcal{O}_{V_0^{(n+1)},\xi_0}$-Rees algebra contaning $\mathcal{G}_X^{(n)}$ (the extended algebra) represents the function $\mathrm{mult}(X_0^{(d+1)})$. We will refer to this algebra as the $\mathcal{O}_{V_0^{(n+1)},\xi_0}$-Rees algebra $\mathcal{G}_{X_0}^{(n+1)}$.

Fix an arc $\varphi \in \mathcal{L}(X)$ through ξ not factoring through $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. The sequence of blow ups at points directed by φ defined in (2.16) induces a sequence[1] of blow ups for $V_0^{(n+1)}$:

$$\begin{array}{ccccccc} (V_0^{(n+1)},\xi_0) & \xleftarrow{\;\pi_1\;} & (V_1^{(n+1)},\xi_1) & \xleftarrow{\;\pi_2\;} & \dots & \xleftarrow{\;\pi_r\;} & (V_r^{(n+1)},\xi_r) \\ \uparrow & & \uparrow & & & & \uparrow \\ (X_0^{(d+1)},\xi_0) & \xleftarrow{\;\pi_1|_{X_1^{(d+1)}}\;} & (X_1^{(d+1)},\xi_1) & \xleftarrow{\;\pi_2|_{X_2^{(d+1)}}\;} & \dots & \xleftarrow{\;\pi_r|_{X_r^{(d+1)}}\;} & (X_r^{(d+1)},\xi_r) \\ \Gamma_0 \uparrow & & \Gamma_1 \uparrow & & & & \Gamma_r \uparrow \\ (\mathrm{Spec}(K[[t]]),0) & \xleftarrow{\;id\;} & (\mathrm{Spec}(K[[t]]),0) & \xleftarrow{\;id\;} & \dots & \xleftarrow{\;id\;} & (\mathrm{Spec}(K[[t]]),0). \end{array} \tag{3.3}$$

Let us give a few examples of how to compute the Nash multiplicity sequence:

[1]For simplicity of the notation, we will often identify the points ξ_i in $X_i^{(d+1)}$ with their images in $V_i^{(n+1)}$.

Example 3.2.1. Let $X = \mathbb{V}(x^2 - y^3) \subset \mathrm{Spec}(k[x,y]) = V^{(2)}$, as X_1 from Example 1.2.7. Choose a k-arc $\varphi \in \mathcal{L}(X)$ as the k-homomorphism

$$\begin{aligned} \varphi : k[x,y] &\longrightarrow k[[t]] \\ x &\mapsto t^3 \\ y &\mapsto t^2. \end{aligned}$$

Note that φ is centered at the origin $\xi = (0,0)$. We will denote $V_0^{(3)} = V^{(2)} \times \mathbb{A}^1$ and $X_0^{(2)} = X^{(1)} \times \mathbb{A}^1$. Let us construct here the Nash multiplicity sequence for φ at ξ. Let us denote by $V_i^{(3)}$ an affine chart of the transform of $V_{i-1}^{(3)}$ by π_i, which will be the most interesting for us:

$$\begin{array}{ccccc}
V_0^{(3)} = \mathrm{Spec}(k[x,y,w]) & \xleftarrow{\pi_1} & V_1^{(3)} & \xleftarrow{\pi_2} & V_2^{(3)} \\
 & & & & \\
X_0^{(2)} = \mathbb{V}(x^2 - y^3) & \longleftarrow & X_1^{(2)} \supset \mathbb{V}(x_1^2 - y_1^3 w_1) & \longleftarrow & X_2^{(2)} \supset \mathbb{V}(x_2^2 - y_2^3 w_2^2) \\
 & & & & \\
\Gamma_0 = (t^3, t^2, t) & \longleftarrow & \Gamma_1 = (t^2, t, t) & \longleftarrow & \Gamma_2 = (t, 1, t) \\
 & & & & \\
\xi_0 = (0,0,0) & \longleftarrow & \xi_1 = (0,0,0) & \longleftarrow & \xi_2 = (0,1,0) \\
 & & & & \\
m_0 = 2 & & m_1 = 2 & & m_2 = 2
\end{array}$$

After a linear change of coordinates, we may assume that

$$\begin{array}{ccc}
V_2^{(3)} & \xleftarrow{\pi_3} & V_3^{(3)} \\
 & & \\
X_2^{(2)} \supset \mathbb{V}(x_2^2 - (y_2+1)^3 w_2^2) & \longleftarrow & X_3^{(2)} \supset \mathbb{V}(x_3^2 - (y_3 w_3 + 1)^3) \\
 & & \\
\Gamma_2 = (t, 0, t) & \longleftarrow & \Gamma_3 = (1, 0, t) \\
 & & \\
\xi_2 = (0,0,0) & \longleftarrow & \xi_3 = (1,0,0) \\
 & & \\
m_2 = 2 & & m_3 = 1
\end{array}$$

The Nash multiplicity sequence is in this case

$$m_0 = 2 = m_1 = m_2 > m_3 = 1 = m_4 = \dots,$$

so the persistance of φ in $\underline{\text{Max}}$ mult(X) is

$$\rho_{X,\varphi} = 3.$$

On the other hand, for the k-homomorphism

$$\begin{aligned} \varphi_2 : k[x,y] &\longrightarrow k[[t]] \\ x &\mapsto t^6 \\ y &\mapsto t^4 \end{aligned}$$

and for

$$\begin{aligned} \varphi_3 : k[x,y] &\longrightarrow k[[t]] \\ x &\mapsto t^9 \\ y &\mapsto t^6, \end{aligned}$$

we have

$$\rho_{X,\varphi_2} = 6 \text{ and } \rho_{X,\varphi_3} = 9$$

respectively. However, note that these three arcs define the same curve in X, and that

$$\overline{\rho}_{X,\varphi} = \overline{\rho}_{X,\varphi_1} = \overline{\rho}_{X,\varphi_2} = 3/2.$$

Example 3.2.2. Let $X = \mathbb{V}(x^3 - xyz^2 - yz^3 + z^5) \subset \operatorname{Spec}(k[x,y,z])$ as X_6 from Example 1.3.26 and choose $\varphi \in \mathcal{L}(X)$ given by the k-homomorphism

$$\begin{aligned} \varphi : k[x,y,z] &\longrightarrow k[[t]] \\ x &\mapsto t^2 \\ y &\mapsto t^2 \\ z &\mapsto t. \end{aligned}$$

The Nash multiplicity sequence for φ at $\xi = \varphi((t)) = (0,0,0)$ is

$$m_0 = 3 > m_1 = 2 > m_2 = 1 = m_3 = \dots$$

and

$$\overline{\rho}_{X,\varphi} = 1.$$

Example 3.2.3. Let $X = \mathbb{V}(x^3 - y^3z^2) \subset \operatorname{Spec}(k[x,y,z])$, as X_4 from Example 1.2.7. Let $\varphi_1 \in \mathcal{L}(X)$ be given by the k-homomorphism

$$\begin{aligned} \varphi_1 : k[x,y,z] &\longrightarrow k[[t]] \\ x &\mapsto t^3 \\ y &\mapsto t \\ z &\mapsto t^3. \end{aligned}$$

The Nash multiplicity sequence for φ_1 at $\xi = \varphi_1((t)) = (0,0,0)$ is

$$m_0 = 3 = m_1 = m_2 > m_3 = 1 = m_4 = \dots$$

If we choose $\varphi_2 \in \mathcal{L}(X)$ instead, also centered at $(0,0,0)$, as the k-homomorphism

$$\begin{aligned} \varphi_2 : k[x,y,z] &\longrightarrow k[[t]] \\ x &\mapsto t^5 \\ y &\mapsto t^3 \\ z &\mapsto t^3, \end{aligned}$$

then the Nash multiplicity sequence is

$$m_0 = 3 = m_1 = m_2 = m_3 = m_4 > m_5 = 1 = m_6 = \dots$$

We have

$$\begin{aligned} \overline{\rho}_{X,\varphi_1} &= 3, \\ \overline{\rho}_{X,\varphi_2} &= 5/3. \end{aligned}$$

Example 3.2.4. Let $X = \mathbb{V}(xy - z^4) \subset \operatorname{Spec}(k[x,y,z])$ as X_7 from Example 1.3.26 and choose the k-arc in X with center at $\xi = (0,0,0)$ given by

$$\begin{aligned} \varphi_1 : k[x,y,z] &\longrightarrow k[[t]] \\ x &\mapsto t \\ y &\mapsto t^3 \\ z &\mapsto t. \end{aligned}$$

The Nash multiplicity sequence of φ_1 at ξ is in this case as simple as

$$m_0 = 2 > m_1 = 1 = m_2 = \dots$$

and

$$\overline{\rho}_{X,\varphi_1} = 1.$$

If we choose the k-arc φ_2, also centered at $\xi = (0,0,0)$ and given by

$$\begin{aligned} \varphi_2 : k[x,y,z] &\longrightarrow k[[t]] \\ x &\mapsto t^2 \\ y &\mapsto t^2 \\ z &\mapsto t, \end{aligned}$$

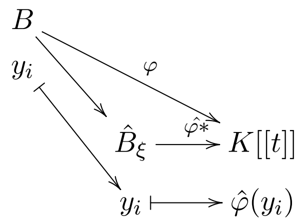
the Nash multiplicity sequence is

$$m_0 = 2 = m_1 > m_2 = 1 = m_3 = \dots$$

and

$$\overline{\rho}_{X,\varphi_2} = 2.$$

Remark 3.2.5. Let $\varphi : B \longrightarrow K[[t]]$ be a K-arc for an n-dimensional local regular ring B, with center $\xi \in \mathrm{Spec}(B)$, and let $K \supset k$. Then φ is determined by the images of a regular system of parameters $\{y_1, \ldots, y_n\}$ of B at ξ. This follows from the fact that φ is continuous and factorized by the completion map $B \longrightarrow \hat{B}_\xi$:



Algebra of contact

Consider now the ring $\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]]$, and the localization at $\xi_0 = (\xi, 0)$:[2]

$$\delta : \mathcal{O}_{V_0^{(n+1)},\xi_0} \longrightarrow (\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}, \tag{3.4}$$

and let us denote

$$\tilde{V}_0^{(n+1)} = \mathrm{Spec}(\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}$$

and

$$\tilde{X}_0^{(d+1)} = \mathrm{Spec}(\mathcal{O}_{X^{(d)},\xi} \otimes_k K[[t]])_{\xi_0}.$$

Observe that the morphism δ is faithfully flat.
Let us choose a regular system of parameters

$$y_1, \ldots, y_n \in \mathcal{O}_{V^{(n)},\xi},$$

so that $\{y_1, \ldots, y_n, t\}$ is a regular system of parameters in $(\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}$.

Let φ be an arc in X (and in $V^{(n)}$) through $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$ defined by the K-morphism

$$\begin{aligned} \varphi : \mathcal{O}_{V^{(n)},\xi} &\longrightarrow K[[t]] \\ y_i &\longmapsto \varphi_{y_i}, \text{ for } i = 1, \ldots, n \end{aligned}$$

Now Γ_0 can be described by the images of t and the classes $\overline{y_i}$ of the y_i in $\mathcal{O}_{X_0^{(d+1)},\xi_0}$, for $i = 1, \ldots, n$:

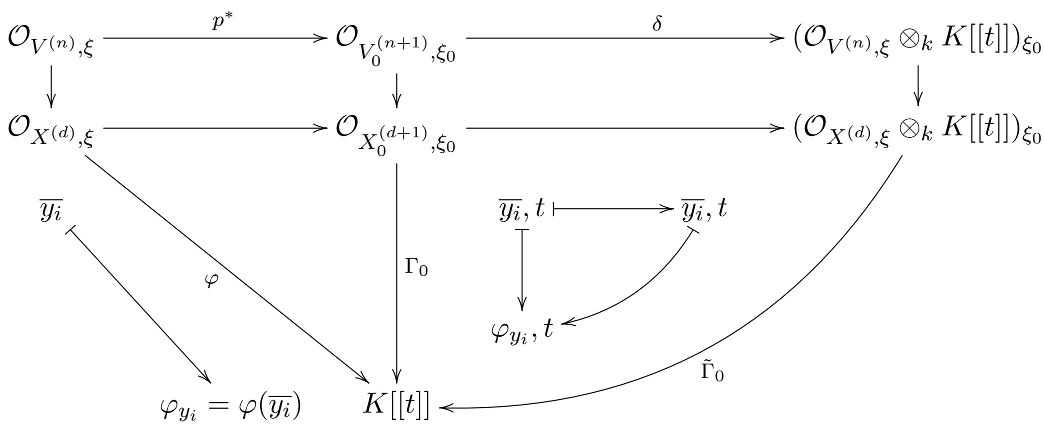
$$\begin{aligned} \Gamma_0 : \mathcal{O}_{X_0^{(d+1)},\xi_0} &\longrightarrow K[[t]] \\ \overline{y_i} &\longmapsto \varphi(\overline{y_i}) = \varphi_{y_i} \quad \text{i=1,...,n} \\ t &\longmapsto t. \end{aligned}$$

[2]We use the same notation ξ for the image of $\xi \in X^{(d)}$ in $V^{(n)}$. We will also write ξ instead of $\xi_0 \in V_0^{(n+1)}$ sometimes.

There is a k-morphism

$$\tilde{\Gamma}_0 : (\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0} \longrightarrow K[[t]]$$

which is completely determined by the images of the $\overline{y_i}$ and t. Since Γ_0 is in addition an arc in X_0, the map $\mathcal{O}_{X_0^{(d+1)},\xi} \longrightarrow K[[t]]$ factorizes through $(\mathcal{O}_{X^{(d)},\xi} \otimes_k K[[t]])_{\xi_0} \longrightarrow K[[t]]$. For simplicity, we shall denote such morphism by $\tilde{\Gamma_0}$. The following commutative diagram provides an overview of the situation:



(3.5)

Note that $\tilde{\Gamma}_0$ is an arc in $\tilde{X}_0^{(d+1)}$ defining a curve C_0 in $\tilde{V}_0^{(n+1)}$, given by the equations

$$h_1 = y_1 - \varphi_{y_1} = 0, \ldots, h_n = y_n - \varphi_{y_n} = 0, \tag{3.6}$$

where $\varphi_{y_i} \in K[[t]]$ for $i = 1, \ldots, n$. The curve C_0 is a smooth curve and the h_i defining it are elements in a local regular ring, so C_0 is a complete intersection. It is the closure of the image of

$$\tilde{\Gamma}_0 : \operatorname{Spec}(K[[t]]) \to \tilde{V}_0^{(n+1)},$$

induced by $\tilde{\Gamma}_0$. With other words, C_0 is the closure of the generic point of Γ_0. We get an analogous diagram to that in (3.3):

$$\begin{array}{ccccccc}
(\tilde{V}_0^{(n+1)}, \xi_0) & \xleftarrow{\tilde{\pi}_1} & (\tilde{V}_1^{(n+1)}, \xi_1) & \xleftarrow{\tilde{\pi}_2} & \ldots & \xleftarrow{\tilde{\pi}_r} & (\tilde{V}_r^{(n+1)}, \xi_r) \\
\uparrow & & \uparrow & & & & \uparrow \\
(\tilde{X}_0^{(d+1)}, \xi_0) & \xleftarrow{\tilde{\pi}_1|_{\tilde{X}_1^{(d+1)}}} & (\tilde{X}_1^{(d+1)}, \xi_1) & \xleftarrow{\tilde{\pi}_2|_{\tilde{X}_2^{(d+1)}}} & \ldots & \xleftarrow{\tilde{\pi}_r|_{\tilde{X}_r^{(d+1)}}} & (\tilde{X}_r^{(d+1)}, \xi_r) \\
\uparrow & & \uparrow & & & & \uparrow \\
(C_0, \xi_0) & \xleftarrow{\tilde{\pi}_1|_{C_1}} & (C_1, \xi_1) & \xleftarrow{\tilde{\pi}_2|_{C_2}} & \ldots & \xleftarrow{\tilde{\pi}_r|_{C_r}} & (C_r, \xi_r)
\end{array} \tag{3.7}$$

where we can see that the preimage $\tilde{E}_i$ of ξ_{i-1} by $\tilde{\pi}_i$ always intersects C_i at a single point. This point is ξ_i, the center of the blow up $\tilde{\pi}_{i+1}$.

Let us look now at the closed set $C_0 \subset \tilde{V}_0^{(n+1)}$ defined by the graph of the arc φ. We can find an $(\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}$-Rees algebra $\mathcal{G}_\varphi^{(n+1)}$ representing C_0 in the sense of Definition 1.3.23. That is, $\mathcal{G}_\varphi^{(n+1)}$ will satisfy

$$\operatorname{Sing}(\mathcal{G}_\varphi^{(n+1)}) = C_0 \subset \tilde{V}_0{}^{(n+1)},$$

and for any local sequence as in (1.9), $\operatorname{Sing}(\mathcal{G}_{\varphi,i}^{(n+1)}) = C_i \subset \tilde{V}_i^{(n+1)}$, where C_i is the transform of C_{i-1} (the strict transform by ϕ_i if it is a blow up at a smooth center, or the pullback if ϕ_i is a smooth morphism). It can be shown that, for the equations in 3.6:

$$\mathcal{G}_\varphi^{(n+1)} = \mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}[h_1 W, \ldots, h_n W]. \tag{3.8}$$

Consider now the closed set

$$Z_0 = C_0 \cap \left\{\eta \in \tilde{X}_0^{(d+1)} : \operatorname{mult}_\eta(\tilde{X}_0^{(d+1)}) = m\right\} \subset \tilde{V}_0^{(n+1)}. \tag{3.9}$$

For any local sequence

$$\tilde{V}_0^{(n+1)} \xleftarrow{\pi_1} \tilde{V}_1^{(n+1)} \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_r} \tilde{V}_r^{(n+1)}, \tag{3.10}$$

we define Z_i as the closed set

$$Z_i = C_i \cap \left\{\eta \in \tilde{X}_i^{(d+1)} : \operatorname{mult}_\eta(\tilde{X}_i^{(d+1)}) = m\right\}, \tag{3.11}$$

for $i = 1, \ldots, r$, where C_i is the transform of C_{i-1} by π_i and $\tilde{X}_i^{(d+1)}$ is the transform of $\tilde{X}_{i-1}^{(d+1)}$.

Remark 3.2.6. Note that, when X is multiplied by an affine line in (3.2), each point in the set of multiplicity $m = \max \operatorname{mult}(X)$ becomes a whole line of points of multiplicity m in X_0. Any sequence of blow ups directed by an arc $\varphi \in \mathcal{L}(X)$ centered at a point of multiplicity m consists only of closed point blow ups, so such a sequence will never give a resolution of singularities for X_0. Moreover, the upper-semicontinuity of the multiplicity function guarantees that it will not make the maximum multiplicity of X_0 decrease either. Therefore, none of the sets whose intersection defines Z_i in (3.11) can be empty. Hence, $Z_r = \emptyset$ implies necessarily that the arc Γ_r no longer intersects the subset of points of multiplicity m of X_r.

Definition 3.2.7. Let us suppose now that one can find an $(\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}$-Rees algebra $\mathcal{H}$ whose singular locus is Z_0, and such that this is preserved by local sequences as in (3.10) (and in particular for sequences of blow ups of $\tilde{X}_0^{(d+1)}$ directed by φ). We will say that such an algebra, if it exists, is an *algebra of contact of* φ *with* $\underline{\operatorname{Max}} \operatorname{mult}(X)$.

Remark 3.2.8. Lowering the Nash multiplicity of X at ξ, m, is therefore equivalent to resolving this $\mathcal{H}$, and consequently $\rho_{X,\varphi}$ as in Definition 3.1.1 is the number of induced transformations of this Rees algebra $\mathcal{H}$ which are necessary to resolve it (see Definition 1.3.11).

Remark 3.2.9. Note that, by the way in which it has been defined, the algebra of contact of φ with $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$, if it exists, is unique up to weak equivalence (see [2]).

Denote

$$\mathcal{G}^{(n+1)}_{X_0,\varphi} := \mathcal{G}^{(n+1)}_{\tilde{X}_0} \odot \mathcal{G}^{(n+1)}_{\varphi}, \tag{3.12}$$

where $\mathcal{G}^{(n+1)}_{\tilde{X}_0}$ is the extension of $\mathcal{G}^{(n)}_{X}$ to $(\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}$ (see (3.2) and (3.4)) and $\mathcal{G}^{(n+1)}_{\varphi}$ is as in (3.8). Note that $\mathcal{G}^{(n+1)}_{\varphi}$ and $\mathcal{G}^{(n+1)}_{\tilde{X}_0}$ are differentially closed (relative to $p^* \circ \delta$) by definition.

Example 3.2.10. We produce here an algebra as in (3.12) for some examples:

- Consider $X_1 \subset V_0 = \mathrm{Spec}(k[x,y])$ from Example 1.2.7. Using the computations in Example 1.3.35 and φ, φ_2 and φ_3 as in Example 3.2.1, we obtain

$$\begin{aligned} \mathcal{G}^{(3)}_{(\tilde{X}_1)_0,\varphi} =& \mathcal{O}_{\tilde{V}_0}[xW, y^2W, y^3W^2] \odot k[[t]][(x-t^3)W, (y-t^2)W] = \\ &= \mathcal{O}_{\tilde{V}_0}[xW, y^2W, y^3W^2, t^3W, (y-t^2)W], \end{aligned}$$

$$\mathcal{G}^{(3)}_{(\tilde{X}_1)_0,\varphi_2} = \mathcal{O}_{\tilde{V}_0}[xW, y^2W, y^3W^2, t^6W, (y-t^4)W],$$

$$\mathcal{G}^{(3)}_{(\tilde{X}_1)_0,\varphi_3} = \mathcal{O}_{\tilde{V}_0}[xW, y^2W, y^3W^2, t^9W, (y-t^6)W]$$

  respectively.

- Consider now $X_6 = \mathbb{V}(x^3 - xyz^2 - yz^3 + z^5) \subset V_0 = \mathrm{Spec}(k[x,y,z])$, as in Example 3.2.2, together with the arc φ used there. We have

$$\begin{aligned} \mathcal{G}^{(4)}_{(\tilde{X}_6)_0,\varphi} =& \mathcal{O}_{\tilde{V}_0}[(x,yz,z^2)W, (yz^2,z^3)W^2, yz^3W^3] \odot \\ & k[[t]][(x-t^2, y-t^2, z-t)W] = \\ =& \mathcal{O}_{\tilde{V}_0}[(x,yz,z^2,t^2,y-t^2,z-t)W, (yz^2,z^3)W^2, yz^3W^3]. \end{aligned}$$

- Let $X_4 \subset V_0 = \mathrm{Spec}(k[x,y,z])$ be as in Example 1.2.7, and consider φ_1 and φ_2 from Example 3.2.3. Then

$$\mathcal{G}^{(4)}_{(\tilde{X}_4)_0,\varphi_1} = \mathcal{O}_{\tilde{V}_0}[(x,yz^2,y^3,y^2z,t^3,y-t,z-t^3)W, (y^2z^2,y^3z)W^2, (y^3z^2)W^3]$$

  and

$$\mathcal{G}^{(4)}_{(\tilde{X}_4)_0,\varphi_2} = \mathcal{O}_{\tilde{V}_0}[(x,yz^2,y^3,y^2z,t^5,y-t^3,z-t^3)W, (y^2z^2,y^3z)W^2, (y^3z^2)W^3].$$

- Let now $X_7 = \mathbb{V}(xy - z^4) \subset V_0 = \mathrm{Spec}(k[x,y,z])$ and φ_1, φ_2 be as in Example 3.2.4. We obtain

$$\mathcal{G}^{(4)}_{(\tilde{X_7})_0,\varphi_1} = \mathcal{O}_{\tilde{V_0}}[xW, yW, z^2W, tW, t^3W, (z-t)W]$$

and

$$\mathcal{G}^{(4)}_{(\tilde{X_7})_0,\varphi_1} = \mathcal{O}_{\tilde{V_0}}[xW, yW, z^2W, t^2W, (z-t)W].$$

Remark 3.2.11. Note that if $\beta_X : X \to \mathrm{Spec}(S) = V^{(d)}$ is a finite morphism as in (1.20) then after the natural base extension, $\tilde{X}_0^{(d+1)} \to \tilde{V}_0^{(d+1)}$ is also a finite morphism. We will need this fact in the proof of Proposition 3.2.12 below.

Proposition 3.2.12. *Let X be a variety, let ξ be a point in* $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$*, and let φ be an arc in X through ξ. Then the Rees algebra $\mathcal{G}^{(n+1)}_{X_0,\varphi}$ from (3.12) is an algebra of contact of φ with* $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$*. Moreover, the restriction $\mathcal{G}^{(1)}_{X_0,\varphi}$ of the same Rees algebra to the curve C_0 defined by φ is also an algebra of contact of φ with* $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$*. In particular, resolving $\mathcal{G}^{(n+1)}_{X_0,\varphi}$ is equivalent to resolving $\mathcal{G}^{(1)}_{X_0,\varphi}$.*

Proof. By definition of $\mathcal{G}^{(n+1)}_{X_0,\varphi}$,

$$\mathcal{F}_{\tilde{V}_0}\left(\mathcal{G}^{(n+1)}_{X_0,\varphi}\right) = \mathcal{F}_{\tilde{V}_0}\left(\mathcal{G}^{(n+1)}_{\tilde{X}_0}\right) \cap \mathcal{F}_{\tilde{V}_0}\left(\mathcal{G}^{(n+1)}_{\varphi}\right)$$

(see Definition 1.3.21). Then, $\mathcal{G}^{(n+1)}_{X_0,\varphi}$ is an algebra of contact of φ with $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ as long as $\mathcal{G}^{(n+1)}_{\tilde{X}_0}$ represents $\underline{\mathrm{Max}}\,\mathrm{mult}(\tilde{X}_0)$ and $\mathcal{G}^{(n+1)}_{\varphi}$ represents C_0 in the sense of Definition 1.3.23[3]. For the latter, see (3.8). For the first assertion, we may assume locally that we are in the situation of Example 1.4.7, and with the notation there we have now that

$$S \otimes_k K[[t]] \subset B \otimes_k K[[t]] = S[\theta_1, \ldots, \theta_{n-d}] \otimes_k K[[t]]$$

is a finite extension of rings satisfying the properties in [82, 4.5], and therefore the argument in [82, Proposition 5.7] is also valid for them: $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(\tilde{X}_0)$ if and only if $\mathrm{ord}_\xi f_i \geq n_i$ for $i = 1, \ldots, n-d$, so the f_i are also the minimal polynomials of the θ_i over $L \otimes K[[t]]$, where L is the quotient field of S.

On the other hand, by [13, Proposition 6.6]

$$\mathcal{F}_{C_0}\left(\mathcal{G}^{(n+1)}_{\tilde{X}_0}\Big|_{C_0}\right) = \mathcal{F}_{\tilde{V}_0}\left(\mathcal{G}^{(n+1)}_{\tilde{X}_0}\right) \cap \mathcal{F}_{\tilde{V}_0}\left(\mathcal{G}^{(n+1)}_{\varphi}\right),$$

[3] Although $\mathcal{G}^{(n+1)}_{\tilde{X}_0}$ is not a Rees algebra over a k-algebra of finite type, one can check that the properties in Definition 1.3.23 still hold.

since $\mathcal{G}^{(n+1)}_{\tilde{X}_0}$ is differentially closed and C_0 is smooth. Hence, it is clear that the Rees algebra $\mathcal{G}^{(n+1)}_{\tilde{X}_0}\Big|_{C_0}$ defines the same tree of closed sets as $\mathcal{G}^{(n+1)}_{X_0,\varphi}$. In addition, the restriction of $\mathcal{G}^{(n+1)}_{X_0,\varphi}$ to C_0 defines the very same tree, since

$$\mathcal{F}_{\tilde{V}_0}\left(\mathcal{G}^{(1)}_{X_0,\varphi}\right) := \mathcal{F}_{\tilde{V}_0}\left(\mathcal{G}^{(n+1)}_{\tilde{X}_0}\Big|_{C_0} \odot \mathcal{G}^{(n+1)}_{\varphi}\Big|_{C_0}\right) =$$
$$= \mathcal{F}_{\tilde{V}_0}\left(\mathcal{G}^{(n+1)}_{\tilde{X}_0}\Big|_{C_0}\right) \cap \mathcal{F}_{\tilde{V}_0}\left(\mathcal{G}^{(n+1)}_{\varphi}\Big|_{C_0}\right) = \mathcal{F}_{C_0}\left(\mathcal{G}^{(n+1)}_{\tilde{X}_0}\Big|_{C_0}\right),$$

and the proposition is proved.

□

Let us show now how to compute the algebra $\mathcal{G}^{(1)}_{X_0,\varphi}$ that appears in the last Proposition.

Definition 3.2.13. Let $\mathcal{G}$ be a Rees algebra over $V^{(n)}$ given as

$$\mathcal{G} = \mathcal{O}_{V^{(n)},\xi}[g_1 W^{c_1}, \dots, g_s W^{c_s}]$$

locally in a neighborhood of ξ. Then, for any arc $\varphi \in \mathcal{L}(V^{(n)})$ through ξ, we define

$$\varphi(\mathcal{G}) = K[[t]][\varphi(g_1) W^{c_1}, \dots, \varphi(g_s) W^{c_s}] \subset K[[t]][W].$$

Remark 3.2.14. We may define the image by $\tilde{\Gamma}_0$ of the Rees algebra $\mathcal{G}^{(n+1)}_{X_0,\varphi}$ from (3.12). This image happens to be the restriction of the algebra $\mathcal{G}^{(n+1)}_{X_0,\varphi}$ to the curve C_0 defined by φ, and the proof of Proposition 3.2.12 shows that if $\mathcal{G}^{(n+1)}_{\tilde{X}_0} = \mathcal{O}_{V_0^{(n+1)}}[g_1 W^{c_1}, \dots, g_s W^{c_s}]$, then

$$\mathcal{G}^{(1)}_{X_0,\varphi} = \tilde{\Gamma}_0(\mathcal{G}^{(n+1)}_{X_0,\varphi}) = K[[t]][\varphi(g_1) W^{c_1}, \dots, \varphi(g_s) W^{c_s}] = \varphi(\mathcal{G}^{(n)}_X),$$

since $\tilde{\Gamma}_0(h_i) = 0$ for $i = 1, \dots, n$. Hence, sometimes we will denote $\mathcal{G}^{(1)}_{X_0,\varphi}$ by $\varphi(\mathcal{G}^{(n)}_X)$.

Example 3.2.15. For the examples in 3.2.10, we obtain the following restrictions:

- For X_1,
$$\mathcal{G}^{(1)}_{(\tilde{X}_1)_0,\varphi} = k[[t]][t^3 W],$$
$$\mathcal{G}^{(1)}_{(\tilde{X}_1)_0,\varphi_2} = k[[t]][t^6 W],$$
$$\mathcal{G}^{(1)}_{(\tilde{X}_1)_0,\varphi_3} = k[[t]][t^9 W].$$

- For X_6,
$$\mathcal{G}^{(1)}_{(\tilde{X}_6)_0,\varphi} = k[[t]][t^2 W, t^3 W^2].$$

- For X_4 and φ_1,
$$\mathcal{G}^{(1)}_{(\tilde{X}_4)_0,\varphi_1} = k[[t]][t^3W],$$
while for φ_2
$$\mathcal{G}^{(1)}_{(\tilde{X}_4)_0,\varphi_2} = k[[t]][t^5W].$$

- For X_7 and φ_1,
$$\mathcal{G}^{(1)}_{(\tilde{X}_7)_0,\varphi_1} = k[[t]][tW],$$
and for φ_2
$$\mathcal{G}^{(1)}_{(\tilde{X}_7)_0,\varphi_2} = k[[t]][t^2W].$$

Order of contact

Our goal now is to define an invariant for X, ξ and φ using the algebra of contact of φ with $\underline{\operatorname{Max}}\operatorname{mult}(X)$. However, Proposition 3.2.12 shows that it would also make sense to define it from the restriction $\mathcal{G}^{(1)}_{X_0,\varphi}$ to C_0. Indeed, from the way in which $\mathcal{G}^{(n+1)}_{X_0,\varphi}$ is constructed, we know that it has elements of order 1 in weight 1, and hence has order 1 itself at all points of its singular locus. In fact, $\mathcal{G}^{(n+1)}_{\varphi}$ has order one (see (3.8)). On contrary, the order of $\mathcal{G}^{(1)}_{X_0,\varphi}$ will be much more interesting, as we will see in Proposition 3.3.1.

Definition 3.2.16. Let X be a variety, and let φ be an arc in X through $\xi \in \underline{\operatorname{Max}}\operatorname{mult}(X)$. We define the *order of contact* of φ with $\underline{\operatorname{Max}}\operatorname{mult}(X)$ as the order[4] at ξ of the restriction $\mathcal{G}^{(1)}_{X_0,\varphi}$ to C_0 of the algebra of contact of φ with $\underline{\operatorname{Max}}\operatorname{mult}(X)$, and we write
$$r_{X,\varphi} = \operatorname{ord}_\xi(\mathcal{G}^{(1)}_{X_0,\varphi}) \in \mathbb{Q}.$$
We denote by r_X the infimum of the orders of contact of $\underline{\operatorname{Max}}\operatorname{mult}(X)$ with all arcs in X through ξ:
$$r_X = \inf_{\varphi\in\mathcal{L}_\xi(X)} \left\{\operatorname{ord}_\xi(\mathcal{G}^{(1)}_{X_0,\varphi})\right\} \in \mathbb{R}.$$

Remark 3.2.17. We have defined an invariant $r_{X,\varphi}$ for the pair (X,φ) and another invariant r_X for X: by Hironaka's trick (see [34, Section 7]), it can be shown that $r_{X,\varphi}$ depends only on X, ξ and φ, not on the choice of the algebra of contact (which is not unique). For the same reason, r_X depends only on X and on the point ξ we are looking at.

[4] As we have done before, we will write ξ for the image of ξ under most of the morphisms we use, as long as the identification between both points is clear.

Definition 3.2.18. Normalizing $r_{X,\varphi}$ and r_X by the order of the respective arcs we define:

$$\bar{r}_{X,\varphi} = \frac{\mathrm{ord}_\xi(\mathcal{G}^{(1)}_{X_0,\varphi})}{\mathrm{ord}(\varphi)} \in \mathbb{Q},$$

and

$$\bar{r}_X = \inf_{\varphi \in \mathcal{L}_\xi(X)} \left\{ \frac{\mathrm{ord}_\xi(\mathcal{G}^{(1)}_{X_0,\varphi})}{\mathrm{ord}(\varphi)} \right\} \in \mathbb{R}.$$

Example 3.2.19. For the examples in 3.2.15 we obtain the following:

- For $X_1 = \mathbb{V}(x^2 - y^3) \subset \mathrm{Spec}(k[x,y])$ and $\varphi(x,y) = (t^3, t^2)$, $\varphi_2(x,y) = (t^6, t^4)$ and $\varphi_3(x,y) = (t^9, t^6)$, we obtain

$$r_{X_1,\varphi} = 3,\ \ r_{X_1,\varphi_2} = 6,\ \ r_{X_1,\varphi_3} = 9,$$

  while

$$\overline{r}_{X_1,\varphi} = \overline{r}_{X_1,\varphi_2} = \overline{r}_{X_1,\varphi_3} = 3/2,$$

  so $\overline{r}_{X_1}$ is at most 3/2.

- For $X_6 = \mathbb{V}(x^3 - xyz^2 - yz^3 + z^5) \subset \mathrm{Spec}(k[x,y,z])$ and $\varphi = (t^2, t^2, t)$, the order of contact is

$$r_{X_6,\varphi} = 3/2 = \overline{r}_{X_6,\varphi}.$$

  Hence, $\overline{r}_{X_6} \leq 3/2$.

- For $X_4 = \mathbb{V}(x^3 - y^3 z^2) \subset \mathrm{Spec}(k[x,y,z])$

$$r_{X_4,\varphi_1} = 3 = \overline{r}_{X_4,\varphi_1},$$

  where $\varphi_1 = (t^3, t, t^3)$. For $\varphi_2 = (t^5, t^3, t^3)$,

$$r_{X_4,\varphi_2} = 5,\ \ \overline{r}_{X_4,\varphi_2} = 5/3,$$

  so $\overline{r}_{X_4} \leq 5/3$.

- For $X_7 = \mathbb{V}(xy - z^4) \subset \mathrm{Spec}(k[x,y,z])$ and $\varphi_1 = (t, t^3, t)$,

$$r_{X_7,\varphi_1} = 1 = \overline{r}_{X_7,\varphi_1},$$

  while for $\varphi_2 = (t^2, t^2, t)$

$$r_{X_7,\varphi_1} = 2 = \overline{r}_{X_7,\varphi_1}.$$

  In this case, necessarily $\overline{r}_{X_7} = 1$.

Definition 3.2.20. Let us denote

$$\Phi_{X,\xi} = \{\overline{r}_{X,\varphi}\}_\varphi \subset \mathbb{Q}_{\geq 1}, \tag{3.13}$$

where φ runs over all arcs in X through ξ.

Remark 3.2.21. Fixed X and $\xi \in \underline{\mathrm{Max}}\ \mathrm{mult}(X)$, the set $\Phi_{X,\xi}$ is an invariant of X at ξ. In some way, this set of rational numbers reflects how difficult it is for arcs centered at ξ to separate from the subset of maximum multiplicity. This will be explained in the next section. The infimum of this set

$$\overline{r}_X = \inf \Phi_{X,\xi}$$

is also an invariant of X at ξ. In Chapter 4 we will show that this invariant is strongly related to constructive resolution. This is one of the main results of this work.

3.3 Relations between $\rho_{X,\varphi}$ and $r_{X,\varphi}$

Along this chapter, we have defined two different but related invariants attached to a variety X at a point $\xi \in \underline{\mathrm{Max}}\ \mathrm{mult}(X)$ and an arc $\varphi \in \mathcal{L}(X) \setminus \mathcal{L}(\underline{\mathrm{Max}}\ \mathrm{mult}(X))$ centered at ξ. The second one, the order of contact $r_{X,\varphi}$ of φ with $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$, is the order of a Rees algebra representing a closed subset of $\underline{\mathrm{Max}}\ \mathrm{mult}(X) \times \mathbb{A}^1_k$ defined by the graph of the arc. The first one, the persistance $\rho_{X,\varphi}$ of φ in $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$, is the number of blow ups at carefully selected points that are necessary to resolve this algebra. It is a natural consequence that $r_{X,\varphi}$ is a refinement of $\rho_{X,\varphi}$, although they were defined in the opposite order. The following proposition shows that $\rho_{X,\varphi}$ may certainly be obtained from $r_{X,\varphi}$.

Proposition 3.3.1. *Let X be a variety, let ξ be a point in $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$ and let φ be an arc in X through ξ. Then*

$$\rho_{X,\varphi} = [r_{X,\varphi}]\,. \tag{3.14}$$

That is, the persistance of φ in X equals the integral part of the order of contact of φ with $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$.

Proof. Since $\mathcal{G}^{(1)}_{X_0,\varphi}$ is a Rees algebra over a smooth curve, it is defined over a regular local ring $\mathcal{O}_{C_0,\xi}$ of dimension one. If the maximal ideal of ξ in $\mathcal{O}_{C_0,\xi}$ is $\mathcal{M}_\xi = \langle T \rangle$ for some regular parameter T, then $\mathcal{G}^{(1)}_{X_0,\varphi}$ is necesarily generated by a finite set of elements of the form $T^\alpha W^{l_\alpha}$, where α, l_α are positive integers. Observe also that $\mathcal{G}^{(1)}_{X_0,\varphi}$ is integrally equivalent to a Rees algebra generated by JW^l for some principal ideal $J \subset \mathcal{O}_{C_0,\varphi}$ and some positive integer l, at least in a neighborhood of ξ (see [13, Lemma 1.7]). Therefore, we can suppose that $\mathcal{G}^{(1)}_{X_0,\varphi} = \mathcal{O}_{C_0,\xi}[T^\alpha W^l]$. In this case, the order of $\mathcal{G}^{(1)}_{X_0,\varphi}$ at ξ will be given by

$$\mathrm{ord}_\xi(\mathcal{G}^{(1)}_{X_0,\varphi}) = \frac{\alpha}{l}.$$

By the transformation law (1.5), the first transform of $\mathcal{G}^{(1)}_{X_0,\varphi}$ by blowing up at the closed point is

$$\mathcal{G}^{(1)}_{X_0,\varphi,1} = \mathcal{O}_{C_0,\xi}[T^{\alpha-l} W^l].$$

The order of the k-th transform will therefore be

$$\frac{\alpha - k \cdot l}{l},$$

and the number $\rho_{X,\varphi}$ of blow ups needed to resolve $\mathcal{G}^{(1)}_{X_0,\varphi}$ must satisfy:

$$0 \leq \alpha - \rho_{X,\varphi} \cdot l < l.$$

But this implies

$$0 \leq \frac{\alpha}{l} - \rho_{X,\varphi} < 1,$$

which means that $\rho_{X,\varphi}$ is the integral part of $\frac{\alpha}{l} = \mathrm{ord}_{\xi}(\mathcal{G}^{(1)}_{X_0,\varphi})$, the order of contact of φ with $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$. □

Corollary 3.3.2. *For any variety X,*

$$\rho_X = [r_X],$$

$$[\bar{r}_X] \leq \bar{\rho}_X \leq \bar{r}_X.$$

The proof follows solely from the definitions of r_X, $\bar{r}_X$, ρ_X and $\bar{\rho}_X$ together with Proposition 3.3.1, by means of algebraic manipulations of their integral parts.

Example 3.3.3.

- For $X_1 = \mathbb{V}(x^2 - y^3) \subset \mathrm{Spec}(k[x,y])$ and $\varphi = (t^3, t^2)$, we obtain

$$\rho_{X_1,\varphi} = 3.$$

- For $X_6 = \mathbb{V}(x^3 - xyz^2 - yz^3 + z^5) \subset \mathrm{Spec}(k[x,y,z])$ and $\varphi = (t^2, t^2, t)$, the persistance is

$$\rho_{X_6,\varphi} = [3/2] = 1.$$

- For $X_4 = \mathbb{V}(x^3 - y^3z^2) \subset \mathrm{Spec}(k[x,y,z])$, the persistance of $\varphi_1 = (t^3, t, t^3)$ in the subset of maximum multiplicity is

$$\rho_{X_4,\varphi_1} = 3,$$

  and the persistance of $\varphi_2 = (t^5, t^3, t^3)$,

$$\rho_{X_4,\varphi_2} = 5.$$

- For $X_7 = \mathbb{V}(xy - z^4) \subset \mathrm{Spec}(k[x,y,z])$ and $\varphi_1 = (t, t^3, t)$,

$$\rho_{X_7,\varphi_1} = 1$$

  and for $\varphi_2 = (t^2, t^2, t)$

$$\rho_{X_7,\varphi_1} = 2.$$

Chapter 4

The Nash multiplicity sequence and Hironaka's order in dimension d

The invariants defined in Chapter 3 for any point of maximum multiplicity of a variety (over fields of characteristic zero) codify information given by the arcs in the variety. In particular, they all derive from the Nash multiplicity sequences of arcs centered at each point. We already mentioned some properties and interpretations of this sequence in Section 2.7 and Chapter 3. In the present chapter, we expose some results which indicate that this information is strongly used in the construction of resolution functions, when one studies resolution of singularities of varieties (see Section 1.5). More precisely, we will see that for a d-dimensional variety X defined over a field k of characteristic zero and a point $\xi \in \underline{\operatorname{Max}} \operatorname{mult}(X)$, the invariant $\operatorname{ord}_{\xi}^{(d)} X$ from Section 1.5 can be read in the arc space of X. Indeed, for

$$\bar{r}_X = \inf \Phi_{X,\xi},$$

as defined in Section 3.2, we have the following theorem:

Theorem 4.0.1. *Let X be an algebraic variety of dimension d defined over a field k of characteristic zero and let ξ be a point in $\underline{\operatorname{Max}} \operatorname{mult}(X)$. Then, the infimum $\bar{r}_X$ is a minimum, and*

$$\bar{r}_X = \operatorname{ord}_{\xi}^{(d)} X \in \mathbb{Q}.$$

Equivalently, for every arc $\varphi \in \mathcal{L}(X)$ through ξ,

$$\bar{r}_{X,\varphi} \geq \operatorname{ord}_{\xi}^{(d)} X,$$

and in addition one can find an arc $\varphi_0 \in \mathcal{L}(X)$ through ξ such that

$$\bar{r}_{X,\varphi_0} = \operatorname{ord}_{\xi}^{(d)} X.$$

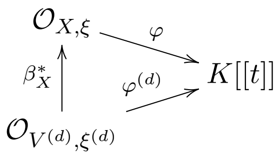
This is the main result in [11]. For the proof of Theorem 4.0.1, we proceed as follows: we first prove it for a particular hypersurface, assuming the situation of Example 1.4.1, in Section 4.1. In Section 4.2, we show how to reduce the general case to the previous one, and give the complete proof. We explain some consequences of this results in Section 4.3 and give some examples in Section 4.4.

In what follows, when talking about an arc, we will denote both the morphism of schemes and the ring homomorphism by φ, to ease the notation.

Remark 4.0.2. Let X be as in Example 1.4.7, and let $\xi \in \underline{\operatorname{Max}} \operatorname{mult}(X)$. Let us recall that we can find a transversal projection

$$\beta_X : X = \operatorname{Spec}(B) \longrightarrow \operatorname{Spec}(S) = V^{(d)},$$

locally in an étale neighborhood of ξ. Let φ be an arc in X through ξ which is not contained in $\underline{\operatorname{Max}} \operatorname{mult}(X)$. We may project φ to an arc $\varphi^{(d)}$ in $V^{(d)}$ through $\xi^{(d)}$ via β_X, that is: $\varphi^{(d)} = \varphi \circ \beta_X^*$. We obtain a commutative diagram



In particular, if $\mathcal{M}_\xi$ is the maximal ideal of ξ in $\mathcal{O}_{X,\xi}$ and $\mathcal{M}_{\xi^{(d)}}$ is the maximal ideal of $\xi^{(d)}$ in $\mathcal{O}_{V^{(d)},\xi^{(d)}}$, note that $\varphi(\mathcal{M}_\xi) \supset \varphi^{(d)}(\mathcal{M}_{\xi^{(d)}})$, so

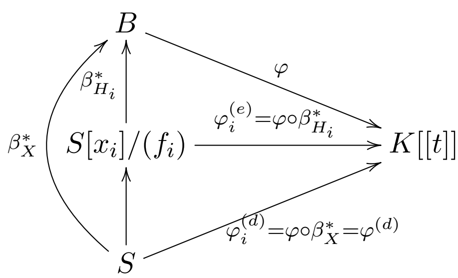
$$\operatorname{ord}(\varphi) = \operatorname{ord}_t(\varphi(\mathcal{M}_\xi)) \leq \operatorname{ord}_t(\varphi^{(d)}(\mathcal{M}_{\xi^{(d)}})) = \operatorname{ord}(\varphi^{(d)}). \tag{4.1}$$

Let $\mathcal{G}_X^{(n)} = \operatorname{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_1 W^{b_1}, \ldots, f_{n-d} W^{b_{n-d}}])$ be a Rees algebra attached to the maximum multiplicity of X locally in an (étale) neighborhood of ξ as in (1.21). Recall that diagram (1.31) yields a decomposition

$$\mathcal{G}_X^{(n)} = \mathcal{G}_{H_1}^{(d+1)} \odot \ldots \odot \mathcal{G}_{H_{n-d}}^{(d+1)},$$

where $H_i = \mathbb{V}(f_i) \subset \operatorname{Spec}(S[x_1, \ldots, x_{n-d}])$ and can also be regarded as a hypersurface in $\operatorname{Spec}(S[x_i])$ for $i = 1, \ldots, n-d$. In order to express $\varphi(\mathcal{G}_X^{(n)})$ by means of this decomposition, we may consider the projections of φ over the H_i, that we shall denote by $\varphi_i^{(e)}$, and which are actually arcs in the corresponding H_i through $\beta_{H_i}(\xi)$, because $f_i \in I(X)$ for $i = 1, \ldots, n-d$:

(4.2)



4.1 Hypersurfaces

Assume now that X is locally a hypersurface given, in an étale neighborhood of a point ξ, by an equation as in (1.18). Using this expression, we will prove Theorem 4.0.1 for the particular case of Example 1.4.1 by dividing it into two theorems: Theorem 4.1.10 states that $\operatorname{ord}_{\xi}^{(d)} X$ is a lower bound for $\bar{r}_{X,\varphi}$ for any arc $\varphi \in \mathcal{L}(X)$ through ξ, and Theorem 4.1.12 shows that in fact we can find an arc such that the equality holds. This implies that $\bar{r}_X$ is certainly a minimum.

For the proof of these two results, we will define diagonal arcs, which will help us analyzing the orders of contact in terms of the order $\operatorname{ord}_{\xi}^{(d)} X$ (see 1.4.16, 1 to 5, and Theorems 1.4.17 and 1.5.3 and the discussion that follows).

Setting

Throughout this section, we will always be under the following assumptions:
Let $X = X^{(d)}$ be a d-dimensional variety over a field k of characteristic zero. Let b be the maximum multiplicity of X, and let $\xi \in \underline{\operatorname{Max}}\operatorname{mult}(X)$. Let us suppose that X at ξ is locally a hypersurface, given by $\mathcal{O}_{X,\xi} \cong S[x]/(f)$ for a regular local k-algebra S and a variable x, as in Example 1.4.1. As we did in (1.18), we assume that f has an expression of the form

$$f(x) = x^b + B_{b-2}x^{b-2} + \ldots + B_i x^i + \ldots + B_0 \tag{4.3}$$

in some étale neighborhood of $\xi \in X$, with $B_0, \ldots B_{b-2} \in S$, and where we write $n = d+1$ for the dimension of the ambient space $V^{(n)} = \operatorname{Spec}(S[x])$. Consider the projection

$$\beta : V^{(n)} \longrightarrow V^{(d)} = \operatorname{Spec}(S)$$

(see Corollary 1.4.17), and let $\mathcal{G}_X^{(d)}$ be the elimination algebra of $\mathcal{O}_{V^{(n)},\xi}[fW^b]$ in $\mathcal{O}_{V^{(d)},\xi^{(d)}}$ induced by it, as the diagram shows:

$$\begin{array}{ccccc}
\mathcal{G}_X^{(n)} & \longrightarrow & \mathcal{G}_{X_0}^{(n+1)} & \longrightarrow & \mathcal{G}_{\tilde{X}_0}^{(n+1)} \\
& & & & \\
\mathcal{O}_{V^{(n)},\xi} & \xrightarrow{p^*} & \mathcal{O}_{V_0^{(n+1)},\xi_0} & \xrightarrow{\delta} & (\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0} \\
{\scriptstyle \beta^*}\big\uparrow & & \big\uparrow & & \big\uparrow \\
\mathcal{O}_{V^{(d)},\xi^{(d)}} & \longrightarrow & \mathcal{O}_{V_0^{(d+1)},\xi_0^{(d+1)}} & \longrightarrow & (\mathcal{O}_{V^{(d)},\xi_0^{(d)}} \otimes_k K[[t]])_{\xi_0^{(d+1)}} \\
& & & & \\
\mathcal{G}_X^{(d)} & \longrightarrow & \mathcal{G}_{X_0}^{(d+1)} & \longrightarrow & \mathcal{G}_{\tilde{X}_0}^{(d+1)}
\end{array} \tag{4.4}$$

where $\mathcal{G}_{X_0}^{(d+1)}$ is an elimination of $\mathcal{G}_{X_0}^{(n+1)}$. We have the following expression:

$$\begin{aligned} \mathcal{G}_X^{(n)} &= \operatorname{Diff}(\mathcal{O}_{V^{(n)},\xi}[fW^b]) = \mathcal{O}_{V^{(n)},\xi}[xW] \odot \mathcal{G}_X^{(d)} = \\ &= \mathcal{O}_{V^{(n)},\xi}[xW] \odot \operatorname{Diff}(S[B_{b-2}W^2, \ldots, B_i W^{b-i}, \ldots, B_0 W^b]) \end{aligned} \tag{4.5}$$

(see Lemma 1.4.20).

Let φ be an arc in X through ξ, not contained in $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. Suppose that for a regular system of parameters $\{z_1,\ldots,z_d\}\in S$ such that $\mathcal{M}_{\xi}=<x,z_1,\ldots,z_d>$, φ is given by

$$\begin{aligned}\varphi:\mathcal{O}_{X,\xi}&\longrightarrow K[[t]]\\ x&\longmapsto u_0t^{\alpha_0},\\ z_i&\longmapsto u_it^{\alpha_i},\end{aligned}$$

as in (3.5), where $u_0,\ldots,u_d$ are units in $K[[t]]$ and $\alpha_0,\ldots,\alpha_d$ are positive integers. This gives the following expressions for the algebra of contact of φ with $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ (as in Proposition 3.2.12):

$$\begin{aligned}\mathcal{G}^{(n+1)}_{X_0,\varphi}&=\mathrm{Diff}(\mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}[fW^b])\odot\mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}[(x-u_0t^{\alpha_0})W,(z_i-u_it^{\alpha_i})W;i=1,\ldots,d]=\\ &=\mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}[xW]\odot\mathcal{G}^{(d)}_X\odot\mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}[(x-u_0t^{\alpha_0})W,(z_i-u_it^{\alpha_i})W;i=1,\ldots,d].\end{aligned}\tag{4.6}$$

Let us recall that Properties 1.4.16, 1-4, guarantee that $\mathcal{G}^{(d)}_X$ represents the subset $\beta(\underline{\mathrm{Max}}\,\mathrm{mult}(X))$ in $V^{(d)}$. Consider then the elimination algebra $\mathcal{G}^{(d+1)}_{X_0}$ above. We can construct an algebra of contact of $\varphi^{(d)}=\varphi\circ\beta^*_X$ with $\beta(\underline{\mathrm{Max}}\,\mathrm{mult}(X))$ by an analogous construction to that in (3.12), using the fact that $\mathcal{G}^{(d)}_X$ represents $\beta(\underline{\mathrm{Max}}\,\mathrm{mult}(X))$. Then we obtain the $\mathcal{O}_{\tilde{V}^{(d+1)},\xi^{(d+1)}}$-Rees algebra

$$\mathcal{G}^{(d+1)}_{X_0,\varphi^{(d)}}=\mathcal{G}^{(d+1)}_{\tilde{X}_0}\odot\mathcal{G}^{(d+1)}_{\varphi^{(d)}}.\tag{4.7}$$

Let $\varphi^{(d)}(\mathcal{G}^{(d)}_X)$ be the restriction of $\mathcal{G}^{(d+1)}_{X_0,\varphi^{(d)}}$ to the image $C^{(d)}_0$ of C_0 in $\tilde{V}^{(d+1)}_0$, as in Remark 3.2.14. Note that

$$\varphi^{(d)}(\mathcal{G}^{(d)}_X)=\tilde{\Gamma}^{(d)}_0(\mathcal{G}^{(d+1)}_{X_0,\varphi^{(d)}}),$$

where

$$\tilde{\Gamma}^{(d)}_0:(\mathcal{O}_{V^{(d)},\xi^{(d)}}\otimes_k K[[t]])_{\xi^{(d)}_0}\to K[[t]]$$

is induced by $\varphi^{(d)}:\mathcal{O}_{V^{(d)},\xi^{(d)}}\to K[[t]]$ as in (3.5). With this notation, we can write

$$\begin{aligned}\mathcal{G}^{(n+1)}_{X_0,\varphi}&=\mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}[xW,t^{\alpha_0}W]\odot\mathcal{G}^{(d)}_X\odot\mathcal{G}^{(d+1)}_{\varphi^{(d)}}=\\ &=\mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}[xW]\odot K[[t]][t^{\alpha_0}W]\odot\mathcal{G}^{(d+1)}_{X_0,\varphi^{(d)}}\end{aligned}$$

using (4.6) and (4.7), and hence

$$\varphi(\mathcal{G}^{(n)}_X)=K[[t]][t^{\alpha_0}W]\odot\varphi^{(d)}(\mathcal{G}^{(d)}_X).$$

The order of contact of φ with $\underline{\mathrm{Max}}$ $\mathrm{mult}(X)$ can now be computed as

$$r_{X,\varphi}=\mathrm{ord}_{\xi}(\varphi(\mathcal{G}^{(n)}_X))=\min\left\{\alpha_0,\mathrm{ord}_{\xi}(\varphi^{(d)}(\mathcal{G}^{(d)}_X))\right\}.\tag{4.8}$$

Auxiliary results

The following Lemma shows that, in fact, α_0 is not relevant in expression (4.8).

Lemma 4.1.1. *Let X be as in the beginning of the section. Let $\xi \in \underline{\mathrm{Max}}\ \mathrm{mult}(X)$. Then for any arc $\varphi \in \mathcal{L}(X)$ through ξ:*

$$\mathrm{ord}_\xi(\varphi(\mathcal{G}_X^{(n)})) = \mathrm{ord}_\xi(\varphi^{(d)}(\mathcal{G}_X^{(d)})).$$

Proof. Assume that X is given by f as in (4.3). Let us suppose that φ is given by $(\varphi_x, \varphi_{z_1}, \ldots, \varphi_{z_d}) = (u_0 t^{\alpha_0}, u_1 t^{\alpha_1}, \ldots, u_d t^{\alpha_d})$, with $u_0, \ldots, u_d$ units in $K[[t]]$ and $\alpha_0, \ldots, \alpha_d$ positive integers, and recall that, since $\varphi \in \mathcal{L}(X)$,

$$\varphi(f) = \varphi\left(x^b + \sum_{i=0}^{b-2} B_i x^i\right) = 0. \tag{4.9}$$

By (4.8), it suffices to prove that

$$\alpha_0 \geq \mathrm{ord}_\xi(\varphi^{(d)}(\mathcal{G}_X^{(d)})). \tag{4.10}$$

On the other hand, from Lemma 1.4.20 and diagram (4.4) we know that

$$\mathcal{G}_{\tilde{X}_0}^{(d+1)} = \mathrm{Diff}(\mathcal{O}_{\tilde{V}_0^{(d+1)}, \xi_0^{(d+1)}}[B_i W^{b-i} : i = 0, \ldots, b-2]).$$

Denote

$$\mathcal{H} = \mathcal{O}_{\tilde{V}_0^{(d+1)}, \xi_0^{(d+1)}}[B_i W^{b-i} : i = 0, \ldots, b-2],$$

and note that

$$\mathcal{H} \subset \mathcal{G}_{\tilde{X}_0}^{(d+1)},$$

and that the inclusion also holds after restricting both algebras to $C_0^{(d)}$. Thus

$$\mathrm{ord}_\xi(\varphi^{(d)}(\mathcal{H})) \geq \mathrm{ord}_\xi(\varphi^{(d)}(\mathcal{G}_{\tilde{X}_0}^{(d+1)})) = \mathrm{ord}_\xi(\varphi^{(d)}(\mathcal{G}_X^{(d)})).$$

To finish, we will show that

$$\alpha_0 \geq \mathrm{ord}_\xi(\varphi^{(d)}(\mathcal{H})), \tag{4.11}$$

which implies (4.10). Let us suppose that, on the contrary,

$$\alpha_0 < \mathrm{ord}_\xi(\varphi^{(d)}(\mathcal{H})) = \min_{i=0,\ldots,b-2}\left\{\frac{\mathrm{ord}_t \varphi^{(d)}(B_i))}{b-i}\right\}.$$

That is,

$$\alpha_0 < \left(\frac{\mathrm{ord}_t(\varphi^{(d)}(B_i))}{b-i}\right), \text{ for } \quad i = 0, \ldots, b-2,$$

or equivalently

$$(b-i)\alpha_0 < \mathrm{ord}_t(\varphi^{(d)}(B_i))), \text{ for } \quad i = 0, \ldots, b-2. \tag{4.12}$$

Observe now that this implies

$$\begin{aligned} \mathrm{ord}_t(\varphi(f - x^b)) &= \mathrm{ord}_t(\sum_{i=0}^{b-2} \varphi^{(d)}(B_i) u_0^i t^{i\alpha_0}) \geq \\ &\geq \min_{i=0,\ldots,b-2} \left\{ \mathrm{ord}_t(\varphi^{(d)}(B_i))) + i \cdot \alpha_0 \right\} > b \cdot \alpha_0. \end{aligned}$$

But this contradicts (4.9), so necessarily (4.11) holds, concluding the proof of the Lemma. □

We know now that we can just focus on the projection of X over S for the computation of the order of contact. For this, we need to know how the induced projections of arcs (4.2) behave.

Definition 4.1.2. We say that an arc $\varphi \in \mathcal{L}(V^{(d)})$ through $\xi^{(d)} \in V^{(d)}$ is a *diagonal arc* if there exists a regular system of parameters $\{z_1, \ldots, z_d\} \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}$, units $u_1, \ldots, u_d \in K[[t]]$ and a positive integer α such that $\varphi(z_i) = u_i t^{\alpha}$ for $i = 1, \ldots, d$.

Remark 4.1.3. The following definition is equivalent to the previous one:
We say that an arc $\varphi \in \mathcal{L}(V^{(d)})$ through $\xi^{(d)} \in V^{(d)}$ is a diagonal arc if there exists a regular system of parameters $\{z_1, \ldots, z_d\} \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}$ inducing a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}(\varphi) & \longrightarrow & \mathcal{O}_{V^{(d)},\xi^{(d)}} & \overset{\varphi}{\searrow} & \\ & & & & \downarrow p^* & & \\ 0 & \longrightarrow & \mathrm{Ker}(\Gamma_0) & \longrightarrow & \mathcal{O}_{V_0^{(d+1)},\xi_0^{(d+1)}} & \xrightarrow{\Gamma_0} & K[[t]] \\ & & & & \downarrow \delta & \overset{\tilde{\Gamma}_0}{\nearrow} & \\ 0 & \longrightarrow & \mathrm{Ker}(\tilde{\Gamma}_0) & \longrightarrow & (\mathcal{O}_{V^{(d)},\xi} \otimes_k K[[t]])_{\xi_0} & & \end{array} \tag{4.13}$$

where the ideal

$$\mathrm{Ker}(\tilde{\Gamma}_0) = \mathrm{Ker}(\Gamma_0)(\mathcal{O}_{V^{(d)},\xi} \otimes_k K[[t]])_{\xi_0} \subset (\mathcal{O}_{V^{(d)},\xi} \otimes_k K[[t]])_{\xi_0}$$

is generated by elements of the form $(u_j z_i - u_i z_j)$, where the $u_l \in K[[t]]$ are units for $l = 1, \ldots, d$.

Remark 4.1.4. Let φ and φ' be two arcs in $\mathcal{L}(V^{(d)})$ through $\xi \in V^{(d)}$ whose respective graphs are Γ_0 and Γ'_0. If φ is diagonal and $\mathrm{Ker}(\Gamma_0) = \mathrm{Ker}(\Gamma'_0)$, then φ' is also diagonal. Moreover, since φ is given by $\varphi(z_i) = u_i t^{\alpha}$ for some regular system of parameters $\{z_1, \ldots, z_d\}$, where $u_1, \ldots, u_d$ are units in $K[[t]]$ and α is some positive integer, then φ' is given as $\varphi'(z_i) = u_i g'(t)$, $i = 1, \ldots, d$, for some $g'(t) \in \langle t \rangle \subset K[[t]]$.

Lemma 4.1.5. *Let X and $V^{(d)}$ be as in the beginning of the section, let $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$ and let $\varphi^{(d)}$ be an arc in $V^{(d)}$ through $\xi^{(d)} = \beta(\xi) \in V^{(d)}$. Then*

$$\mathrm{ord}_{\xi}(\varphi^{(d)}(\mathcal{G}_X^{(d)})) \geq \mathrm{ord}_{\xi}(\mathcal{G}_X^{(d)}) \cdot \mathrm{ord}(\varphi^{(d)}). \tag{4.14}$$

Proof. Suppose, contrary to our claim, that $\mathrm{ord}_\xi(\varphi^{(d)}(\mathcal{G}_X^{(d)})) < \mathrm{ord}_\xi(\mathcal{G}_X^{(d)}) \cdot \alpha$, where $\alpha = \mathrm{ord}(\varphi^{(d)})$. Let $\varphi^{(d)}$ be given by $\varphi^{(d)}(z_i) = u_i t^{\alpha_i}$ for some regular system of parameters $\{z_1, \ldots, z_d\}$ in $\mathcal{O}_{V^{(d)},\xi^{(d)}}$, units $u_1, \ldots, u_d \in K[[t]]$ and positive integers $\alpha_1, \ldots, \alpha_d \geq \alpha$. Then for some $qW^l \in \mathcal{G}_X^{(d)}$,

$$\frac{\mathrm{ord}_t(\varphi^{(d)}(q))}{l} < \mathrm{ord}_\xi(\mathcal{G}_X^{(d)}) \cdot \alpha. \tag{4.15}$$

But $\mathrm{ord}_t(\varphi^{(d)}(q)) \geq \alpha \cdot \mathrm{ord}_\xi(q)$, and hence

$$\frac{\mathrm{ord}_t(\varphi^{(d)}(q))}{l} \geq \frac{\alpha \cdot \mathrm{ord}_\xi(q)}{l} \geq \alpha \cdot \mathrm{ord}_\xi(\mathcal{G}_X^{(d)}),$$

leading to a contradiction, and proving the Lemma.

□

Note that in the Lemma $\varphi^{(d)}$ is, in principle, any arc in $\mathcal{L}(V^{(d)})$ through ξ, not necessarily the projection of an arc $\varphi \in \mathcal{L}(X)$ through ξ.

Definition 4.1.6. Let $\mathcal{G}^{(d)}$ be a Rees algebra over $V^{(d)}$. We say that an arc $\varphi^{(d)} \in \mathcal{L}(V^{(d)})$ through ξ is *generic for* $\mathcal{G}^{(d)}$ if

$$\mathrm{ord}_\xi\left(\left(\mathcal{G}^{(d)} \odot \mathcal{G}^{(d+1)}_{\varphi^{(d)}}\right)\Big|_{C_0^{(d)}}\right) = \mathrm{ord}(\varphi^{(d)}) \cdot \mathrm{ord}_\xi(\mathcal{G}^{(d)}).$$

If $\varphi^{(d)}$ is also diagonal, we say that it is *diagonal-generic*.

Remark 4.1.7. In the situation of Lemma 4.1.5, an arc for which (4.14) is an equality is a generic arc for $\mathcal{G}_X^{(d)}$. This is a consequence of

$$\mathcal{G}_X^{(d)} \odot \mathcal{G}^{(d+1)}_{\varphi^{(d)}}\Big|_{C_0^{(d)}} = \mathcal{G}^{(d+1)}_{X_0,\varphi^{(d)}}\Big|_{C_0^{(d)}} = \varphi^{(d)}(\mathcal{G}_X^{(d)}).$$

Note that such an arc can always be found by taking the following into account:

If $q \in R$ for a regular local ring R with maximal ideal $\mathcal{M}$, then we denote by $\mathrm{in}_\xi(q)$ the *initial part of* q *at the closed point* ξ, meaning the equivalence class of q in the quotient $\mathcal{M}^n/\mathcal{M}^{n+1}$, where n is such that $q \in \mathcal{M}^n$ but $q \notin \mathcal{M}^{n+1}$. Therefore $\mathrm{in}_\xi(q) \in \mathfrak{gr}_{R_\mathcal{M}} \cong k'[z_1, \ldots, z_d]$ is a homogeneous polynomial of degree n and k' is the residue field of R at $\mathcal{M}$.

Since k is infinite, it is possible to choose a diagonal arc $\varphi^{(d)}$ in $V^{(d)}$ through $\xi^{(d)} \in V^{(d)}$ given by $(u_1 t^\alpha, \ldots, u_d t^\alpha)$ for some regular system of parameters $\{z_1, \ldots, z_d\}$ and some positive integer α and units $u_1, \ldots, u_d \in k$ such that there exists some element $qW^l \in \mathcal{G}_X^{(d)}$ satisfying $\frac{\mathrm{ord}_\xi(q)}{l} = \mathrm{ord}_\xi(\mathcal{G}_X^{(d)})$, and $(\mathrm{in}_\xi(q))(u_1, \ldots, u_d) \neq 0$. For this arc,

$$\mathrm{ord}_t(\varphi^{(d)}(q)) = \alpha \cdot \mathrm{ord}_\xi(q),$$

and hence

$$\varphi^{(d)}(\mathcal{G}_X^{(d)}) \leq \frac{\mathrm{ord}_t(\varphi^{(d)}(q)))}{l} = \frac{\alpha \cdot \mathrm{ord}_{\xi}(q)}{l} = \alpha \cdot \mathrm{ord}_{\xi}(\mathcal{G}_X^{(d)}),$$

but Lemma 4.1.5 forces the last inequality to be an equality.

Even though in this section we are always under the assumption of X being locally a hypersurface, the following Lemma will be stated and proved for a variety of arbitrary codimension, since no extra work is needed and this generality will be necessary in the next section.

Lemma 4.1.8. *Let X be a variety of dimension d over a field k, and consider a transversal projection*

$$\beta_X : X \longrightarrow V^{(d)}$$

as in (1.20). Let $\mathcal{G}_X^{(d)}$ be the elimination via β_X of a Rees algebra $\mathcal{G}_X^{(n)}$ attached to the maximum multiplicity of X locally in an (étale) neighborhood of a point $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$. Let $\bar{\varphi}^{(d)}$ be a diagonal arc in $V^{(d)}$ through $\xi^{(d)} \in V^{(d)}$ which is diagonal-generic for $\mathcal{G}_X^{(d)}$. Then it is possible to find an arc $\varphi \in \mathcal{L}(X)$ through ξ whose projection $\varphi^{(d)}$ onto $V^{(d)}$ via β_X is a diagonal arc, which is also diagonal-generic for $\mathcal{G}_X^{(d)}$ and such that $\mathrm{Ker}(\varphi^{(d)}) = \mathrm{Ker}(\overline{\varphi}^{(d)})$.

Proof. Consider a local presentation as in Example 1.4.7 attached to the multiplicity of X at ξ. Let us recall that not every arc in $\mathbb{V}(f_1, \ldots, f_{n-d})$ is an arc in X, since

$$(f_1, \ldots, f_{n-d}) \subset I(X) \Longrightarrow X \subset \mathbb{V}(f_1, \ldots, f_{n-d}) \subset V^{(n)}.$$

Assume that $\bar{\varphi}^{(d)}(z_i) = u_i t^{\alpha}$, $i = 1, \ldots, d$ for some units $u_1, \ldots, u_d \in K[[t]]$ and some $\alpha \in \mathbb{Z}_{>0}$. We need to choose an arc φ such that $\varphi \in \mathcal{L}(\mathbb{V}(f))$ for all $f \in I(X)$, or equivalently an arc such that $\mathrm{Ker}(\varphi) \supset I(X)$. Consider the following diagram

$$\begin{array}{ccc}
\mathcal{O}_{X,\xi} \cong \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_1, \ldots, x_{n-d}]/I(X) & \longleftarrow & \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_1, \ldots, x_{n-d}] \\
 & {}_{\beta_X^*}\nwarrow \quad \nearrow_{\beta^*} & \\
 & \mathcal{O}_{V^{(d)},\xi^{(d)}} &
\end{array}$$

where β_X^* (induced by β_X) is a finite morphism. Let

$$\mathcal{P} = \mathrm{Ker}(\overline{\varphi}^{(d)}) \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}.$$

There is a prime ideal $\mathcal{Q} \subset \mathcal{O}_{X,\xi}$ such that $\mathcal{Q} \cap \mathcal{O}_{V^{(d)},\xi^{(d)}} = \mathcal{P}$. Note that $\mathcal{Q}$ is lifted to a unique ideal $\mathcal{Q}' \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_1, \ldots, x_{n-d}]$, with the property that $I(X) \subset \mathcal{Q}'$. We have the following diagram

$$\begin{array}{ccc}
\mathcal{Q} \subset \mathcal{O}_{X,\xi} & \longrightarrow & \mathcal{O}_{X,\xi}/\mathcal{Q} \\
\beta_X^* \uparrow & & \uparrow \\
\mathcal{P} \subset \mathcal{O}_{V^{(d)},\xi^{(d)}} & \longrightarrow & \mathcal{O}_{V^{(d)},\xi^{(d)}}/\mathcal{P}
\end{array}$$

where the left vertical arrow is a finite morphism, forcing the right vertical one to be also finite. Then, the two rings in the right side of the diagram have the same dimension, and thus $\mathcal{Q}$ defines a closed set of dimension 1 in X: C. There is an arc φ (different from the morphism 0) in C through ξ and, locally in a neighborhood of ξ, $\mathcal{Q} = \mathrm{Ker}(\varphi) \subset \mathcal{O}_{X^{(d)},\xi}$ where moreover

$$\mathrm{Ker}(\varphi) \cap \mathcal{O}_{V^{(d)},\xi^{(d)}} = \mathrm{Ker}(\varphi^{(d)}) = \mathrm{Ker}(\bar{\varphi}^{(d)}) = \mathcal{P},$$

so the projection of φ onto $V^{(d)}$, $\varphi^{(d)}$, is diagonal by Remark 4.1.4. To see that it is generic for $\mathcal{G}_X^{(d)}$, note that there exists some element $qW^l \in \mathcal{G}_X^{(d)}$ with $\frac{\mathrm{ord}_\xi(q)}{l} = \mathrm{ord}_\xi(\mathcal{G}_X^{(d)})$ for which $(\mathrm{in}_\xi(q))(u_1, \ldots, u_d) \neq 0$. By passing to the completion of $(\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}$ at its maximal ideal (see Remark 4.1.3) and by Remark 4.1.4 we also know that $\varphi^{(d)} = (u_1 g'(t), \ldots, u_d g'(t))$ for some $g'(t) \in \langle t \rangle \subset K[[t]]$, which implies that $\varphi^{(d)}$ is also generic for $\mathcal{G}_X^{(d)}$.

□

Remark 4.1.9. The arc obtained in Lemma 4.1.8 is given (as in (3.5)) by

$$\varphi = (g_1(t), \ldots, g_{n-d}(t), u_1 g'(t), \ldots, u_d g'(t)) \tag{4.16}$$

for some $g_1(t), \ldots, g_{n-d}(t), g'(t) \in \langle t \rangle \subset K[[t]]$ and $u_1, \ldots, u_d \in K[[t]]$, because

$$\mathrm{Ker}(\varphi) \cap \mathcal{O}_{V^{(d)},\xi^{(d)}} = \mathrm{Ker}(\bar{\varphi}^{(d)}) = \mathrm{Ker}(\varphi^{(d)})$$

and $\varphi^{(d)}$ is diagonal (see Remark 4.1.4).

Results for hypersurfaces

Now we return to the particular case of Example 1.4.1. Now we have enough tools to prove the following theorem:

Theorem 4.1.10. *Let X be a variety of dimension d which is a hypersurface locally at $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$, given as in Example 1.4.1. For any $\varphi \in \mathcal{L}(X)$ through ξ:*

$$\overline{r}_{X,\varphi} \geq \mathrm{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{4.17}$$

Proof. Let $\varphi = (u_0 t^{\alpha_0}, \ldots, u_d t^{\alpha_d})$ for some units $u_0, \ldots, u_d \in K[[t]]$ and some $\alpha_0, \ldots, \alpha_d \in \mathbb{Z}_{>0}$. Let us write $\alpha = \mathrm{ord}(\varphi) = \min\{\alpha_0, \ldots, \alpha_d\}$. From Lemma 4.1.5, for any diagonal arc $\tilde{\varphi}$ in $V^{(n)}$ through ξ, given as $(\tilde{u}_0 t^\alpha, \ldots, \tilde{u}_d t^\alpha)$

$$\alpha \cdot \mathrm{ord}_\xi(\mathcal{G}_X^{(d)}) \leq \mathrm{ord}_\xi(\tilde{\varphi}^{(d)}(\mathcal{G}_X^{(d)})).$$

It suffices to show that it is possible to choose units $\tilde{u}_i \in K[[t]]$ for $i = 0, \ldots, d$ so that

$$\mathrm{ord}_\xi(\tilde{\varphi}^{(d)}(\mathcal{G}_X^{(d)})) \leq \mathrm{ord}_\xi(\varphi^{(d)}(\mathcal{G}_X^{(d)})). \tag{4.18}$$

This, together with Lemma 4.1.1, would imply that

$$\alpha \cdot \mathrm{ord}_{\xi}(\mathcal{G}_X^{(d)}) \leq \mathrm{ord}_{\xi}(\varphi^{(d)}(\mathcal{G}_X^{(d)})) = \mathrm{ord}_{\xi}(\varphi(\mathcal{G}_X^{(n)})),$$

and complete the proof of the Theorem.

In order to prove (4.18), let us consider a finite set of generators of $\mathcal{G}_X^{(d)}$, $\left\{g_i W^{l_i}\right\}_{i=1,\dots,r}$. Since this set is finite and k is infinite, it is possible to choose units $\tilde{u}_1, \dots, \tilde{u}_d \in k$ in a way such that

$$\mathrm{in}_{\xi}(g_i)(\tilde{u}_1, \dots, \tilde{u}_d) \neq 0 \quad \text{for} \quad i = 1, \dots, r.$$

Let $\lambda_i = \mathrm{ord}_{\xi}(g_i)$ for $i = 1, \dots, r$. As $\mathrm{in}_{\xi}(g_i)$ is a homogeneous polynomial,

$$\mathrm{in}_{\xi}(\tilde{\varphi}^{(d)}(g_i)) = t^{\alpha \cdot \lambda_i} \cdot \mathrm{in}_{\xi}(g_i)(\tilde{u}_1, \dots, \tilde{u}_d)$$

and

$$\mathrm{ord}_t(\tilde{\varphi}^{(d)}(g_i)) = \alpha \cdot \lambda_i.$$

On the other hand, observe that

$$\varphi^{(d)}(g_i) \in \langle t^{\alpha \cdot \lambda_i} \rangle,$$

so

$$\mathrm{ord}_t(\varphi^{(d)}(g_i)) \geq \alpha \cdot \lambda_i = \mathrm{ord}_t(\tilde{\varphi}^{(d)}(g_i)). \tag{4.19}$$

Since (4.19) holds for all $i \in \{1, \dots, r\}$, and for some $k \in \{1, \dots, r\}$,

$$\frac{\mathrm{ord}_t(\varphi^{(d)}(g_k))}{l_k} = \mathrm{ord}_{\xi}(\varphi^{(d)}(\mathcal{G}_X^{(d)})),$$

it follows that

$$\mathrm{ord}_{\xi}(\varphi^{(d)}(\mathcal{G}_X^{(d)})) = \frac{\mathrm{ord}_t(\varphi^{(d)}(g_k)))}{l_k} \geq \frac{\mathrm{ord}_t(\tilde{\varphi}^{(d)}(g_k))}{l_k} \geq \mathrm{ord}_{\xi}(\tilde{\varphi}^{(d)}(\mathcal{G}_X^{(d)}))$$

concluding the proof of (4.18), and the proof of the Theorem.

□

To prove that there actually exists an arc giving an equality in (4.17), we will use the following Lemma:

Lemma 4.1.11. *Let X be as in Theorem 4.1.10, and let φ be an arc in X through $\xi \in \underline{\mathrm{Max}}\, \mathrm{mult}(X)$, where $\varphi(x) = g_1(t)$ and $\varphi(z_i) = u_i g'(t)$, u_i a unit in $K[[t]]$, for $i = 1, \dots, d$. Assume that φ is such that the projection $\varphi^{(d)}$ on $V^{(d)}$ is a diagonal-generic arc for $\mathcal{G}_X^{(d)}$.*[1] *If $\mathrm{ord}(\varphi) = \mathrm{ord}_t(g_1(t))$, then*

$$\overline{r}_{X,\varphi} = \mathrm{ord}_{\xi}(\mathcal{G}_X^{(d)}) = 1.$$

[1]We know that such an arc exists by Remark 4.1.9.

Proof. Let us suppose that $g'(t) = t^L$ for some positive integer L, that is,

$$\varphi_{z_i} = u_i t^L$$

for $i = 1, \ldots, d$. By Lemma 4.1.1,

$$\mathrm{ord}_\xi(\varphi(\mathcal{G}_X^{(n)})) = \mathrm{ord}_\xi(\varphi^{(d)}(\mathcal{G}_X^{(d)})),$$

and since $\varphi^{(d)}$ is generic for $\mathcal{G}_X^{(d)}$, Remark 4.1.7 yields

$$\mathrm{ord}_\xi(\varphi^{(d)}(\mathcal{G}_X^{(d)})) = L \cdot \mathrm{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{4.20}$$

It suffices to prove that

$$\mathrm{ord}_t(g_1(t)) \geq L \cdot \mathrm{ord}_\xi(\mathcal{G}_X^{(d)}), \tag{4.21}$$

since it implies

$$1 \leq \mathrm{ord}_\xi(\mathcal{G}_X^{(d)}) \leq \overline{r}_{X,\varphi} = \frac{L \cdot \mathrm{ord}_\xi(\mathcal{G}_X^{(d)})}{\mathrm{ord}_t(g_1(t))} \leq 1, \tag{4.22}$$

where we have used Theorem 4.1.10 for the second inequality and (4.20) together with the definition of $\overline{r}_{X,\varphi}$ for the equality. Hence $\mathrm{ord}_\xi(\mathcal{G}_X^{(d)}) = \overline{r}_{X,\varphi} = 1$, concluding the proof of the Lemma. In order to prove (4.21), let us suppose that our claim is false, that is:

$$\mathrm{ord}_t(g_1(t)) < L \cdot \mathrm{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{4.23}$$

Then, in particular,

$$\mathrm{ord}_t(g_1(t)) < L \cdot \frac{\mathrm{ord}_\xi(B_i)}{b-i} \leq \frac{\mathrm{ord}_t(\varphi^{(d)}(B_i))}{b-i} \quad \text{for} \quad i = 0, \ldots, b-2 \tag{4.24}$$

where the first inequality follows from the same argument used in the proof of Lemma 4.1.1. Therefore

$$\mathrm{ord}_t(\varphi^{(d)}(B_i)) > \mathrm{ord}_t(g_1(t))(b-i)$$

and

$$\begin{aligned} \mathrm{ord}_t(\varphi(f - x^b)) = \mathrm{ord}_t\left(\sum_{i=0}^{b-2} \varphi^{(d)}(B_i) g_1(t)^i\right) \geq \\ \geq \min_{i=0,\ldots,b-2} \left\{\mathrm{ord}_t(\varphi^{(d)}(B_i)) + i \cdot \mathrm{ord}_t(g_1(t))\right\} > \\ > \min_{i=0,\ldots,b-2} \{\mathrm{ord}_t(g_1(t))(b-i) + i \cdot \mathrm{ord}_t(g_1(t))\} = b \cdot \mathrm{ord}_t(g_1(t)), \end{aligned}$$

where (4.24) is needed for the second inequality. But this contradicts $\varphi(f) = 0$ and hence the fact that φ is an arc in X, so necessarily (4.21) holds, concluding the proof. □

Theorem 4.1.12. *Let X be a d-dimensional variety over a field k of characteristic zero which is locally a hypersurface in a neighborhood of $\xi \in \underline{\text{Max}}\,\text{mult}(X)$, and assume that we are moreover in the situation of Example 1.4.1. Then there exists some $\varphi \in \mathcal{L}(X)$ through ξ such that*

$$\overline{r}_{X,\varphi} = \operatorname{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{4.25}$$

Proof. Pick a diagonal-generic arc for $\mathcal{G}_X^{(d)}$ (see Remark 4.1.7 for the existence). By Lemma 4.1.8 this arc can be lifted to an arc φ in X through ξ whose projection $\varphi^{(d)}$ onto $V^{(d)}$ is diagonal generic for $\mathcal{G}_X^{(d)}$. Remark 4.1.9 shows that φ is given (as in (3.5)) by

$$(g(t), u_1 g'(t), \ldots, u_d g'(t)) \tag{4.26}$$

for some $g(t), g'(t) \in K[[t]]$ and $u_1, \ldots, u_d \in k$. We only need to check that for such an arc (4.25) holds. Let $N = \operatorname{ord}_t(g'(t))$. Note that, since $\varphi^{(d)}$ is generic for $\mathcal{G}_X^{(d)}$, $\operatorname{ord}_\xi(\varphi^{(d)}(\mathcal{G}_X^{(d)})) = N \cdot \operatorname{ord}_\xi(\mathcal{G}_X^{(d)})$. By Lemma 4.1.1,

$$\operatorname{ord}_\xi(\varphi(\mathcal{G}_X^{(n)})) = N \cdot \operatorname{ord}_\xi(\mathcal{G}_X^{(d)}). \tag{4.27}$$

Consider now two possible situations, depending on whether $\operatorname{ord}(\varphi) = \operatorname{ord}_t(g(t))$ or not. If $\operatorname{ord}(\varphi) = \operatorname{ord}_t(g(t))$, then Lemma 4.1.11 implies

$$1 = \operatorname{ord}_\xi(\mathcal{G}_X^{(d)}) = \overline{r}_{X,\varphi}.$$

Otherwise $\operatorname{ord}(\varphi) = N$, and from definition of $\overline{r}_{X,\varphi}$ and (4.27), we obtain

$$\overline{r}_{X,\varphi} = \frac{N \cdot \operatorname{ord}_\xi(\mathcal{G}_X^{(d)})}{N},$$

completing the proof. ☐

Remark 4.1.13. Under the assumptions of Theorem 4.1.12, let φ be the arc (4.26) given by the proof of the Theorem. For this arc

$$\operatorname{ord}(\varphi) = N. \tag{4.28}$$

To see this we observe that, since we have proved that $\overline{r}_{X,\varphi} = \operatorname{ord}_\xi(\mathcal{G}_X^{(d)})$, it follows easily from (4.27) that:

$$\operatorname{ord}_\xi(\mathcal{G}_X^{(d)}) = \overline{r}_{X,\varphi} = \frac{\operatorname{ord}_\xi(\varphi(\mathcal{G}_X^{(n)}))}{\operatorname{ord}(\varphi)} = \frac{N \cdot \operatorname{ord}_\xi(\mathcal{G}_X^{(d)})}{\operatorname{ord}(\varphi)} \Rightarrow \frac{N}{\operatorname{ord}(\varphi)} = 1.$$

4.2 The general case

As we have just done in the proof of Theorem 4.0.1 for the particular case of Example 1.4.1, we will use that we can find, in an étale neighborhood of each point ξ of X, a local presentation (as in Example 1.4.7) given by a collection of hypersurfaces (presented as those in Example 1.4.1) and integers. We will assume that each of these hypersurfaces is given by an expression as in Remark 1.4.6. As a consequence, for any arc φ in X through ξ we will be able to give an expression of the algebra of contact of φ with $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$ in terms of some algebras of contact of arcs with hypersurfaces. This will lead to an easy formula for $r_{X,\varphi}$. With these tools, we will prove in Theorem 4.2.4 that $\mathrm{ord}_\xi \mathcal{G}_X^{(d)}$ is again a lower bound for $\bar{r}_{X,\varphi}$ for any arc φ, and that $\bar{r}_X$ is also a minimum in this case in Theorem 4.2.6. They will be consequences of Theorems 4.1.10 and 4.1.12 respectively.

Setting

Let X be a variety of dimension d, and let ξ be a point in $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$. As in Example 1.4.7, we can find, in an étale neighborhood of ξ a local presentation for X attached to the multiplicity, meaning an immersion in $V^{(n)}$, elements $f_i \in \mathcal{O}_{V^{(n)},\xi} = \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_1,\ldots,x_{n-d}]$ and positive integers b_i for $i=1,\ldots,n-d$ as in (1.22), such that

$$\mathcal{G}_X^{(n)} = \mathrm{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_1W^{b_1},\ldots,f_{n-d}W^{b_{n-d}}]) \tag{4.29}$$

represents the function $\mathrm{mult}(X)$. Consider the differential closure of the $\mathcal{O}_{V_0^{(n+1)},\xi_0^{(n+1)}}$-Rees algebra generated by the f_i, $\mathcal{G}_{X_0}^{(n+1)}$. We already mentioned that f_i is the minimal polynomial of θ_i and has coefficients in $\mathcal{O}_{V^{(d)},\xi^{(d)}}$, where $\mathcal{O}_{X,\xi} = \mathcal{O}_{V^{(d)},\xi^{(d)}}[\theta_1,\ldots,\theta_{n-d}]$, and we can assume (by 1.4.5) that each f_i has the form:

$$f_i = x_i^{b_i} + B_{\{i\},b_i-2}x_i^{b_i-2} + \ldots + B_{\{i\},0} \in \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_i] \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_1,\ldots,x_{n-d}],$$

where $\{z_1,\ldots,z_d,t\}$ is a regular system of parameters in $\mathcal{O}_{V_0^{(d+1)},\xi_0}$ and

$$\{x_1,\ldots,x_{n-d},z_1,\ldots,z_d,t\}$$

a regular system of parameters in $(\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}$, $B_{\{i\},b_i-j} \in \mathcal{O}_{V^{(d)},\xi^{(d)}}$, and where $\mathrm{ord}_\xi(B_{\{i\},b_i-j}) \geq j$ for $j=2,\ldots,b_i$, $i=1,\ldots,n-d$.

We know from Example 1.4.7 that

$$\begin{aligned}\mathcal{G}_X^{(n)} &= \mathrm{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_1W^{b_1},\ldots,f_{n-d}W^{b_{n-d}}]) = \\ &=\mathrm{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_1W^{b_1}]) \odot \ldots \odot \mathrm{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_{n-d}W^{b_{n-d}}]),\end{aligned}$$

where each $\mathrm{Diff}(\mathcal{O}_{V^{(n)},\xi}[f_iW^{b_i}])$ is the smallest differentially closed $\mathcal{O}_{V^{(n)},\xi}$-Rees algebra with the property of containing the algebra $\mathrm{Diff}(\mathcal{O}_{V^{(d)},\xi^{(d)}}[x_i][f_iW^i])$, since $f_i \in \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_i]$. Therefore we can write

$$\mathcal{G}_X^{(n)} = \mathrm{Diff}(\mathcal{O}_{V^{(d)},\xi^{(d)}}[x_1][f_1W^{b_1}]) \odot \ldots \odot \mathrm{Diff}(\mathcal{O}_{V^{(d)},\xi^{(d)}}[x_{n-d}][f_{n-d}W^{b_{n-d}}]). \tag{4.30}$$

Observe that, for each f_i, $H_i = \{f_i = 0\}$ is a hypersurface in

$$V_i^{(e)} = \mathrm{Spec}(\mathcal{O}_{V^{(d)},\xi^{(d)}}[x_i]),$$

where $e = d+1$. By Remark 1.4.22, the Rees algebra

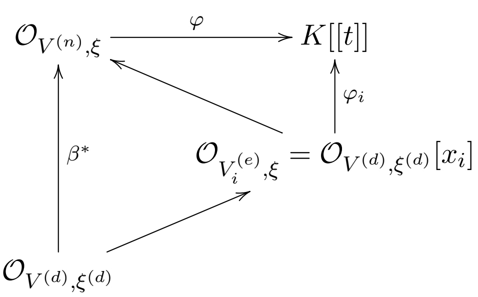
$$\mathcal{G}_{H_i}^{(e)} = \mathrm{Diff}(\mathcal{O}_{V^{(d)},\xi^{(d)}}[x_i][f_i W^{b_i}]) = \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_i][x_i W] \odot \mathcal{G}_{H_i}^{(d)} \tag{4.31}$$

represents $\mathrm{mult}(H_i)$.

Remark 4.2.1. Using (4.30) we can rewrite $\mathcal{G}_X^{(n)}$ in terms of the $\mathcal{G}_{H_i}^{(e)}$ for $i = 1, \ldots, n-d$:

$$\begin{aligned} \mathcal{G}_X^{(n)} &= \mathcal{G}_{H_1}^{(e)} \odot \ldots \odot \mathcal{G}_{H_{n-d}}^{(e)} = \\ &= \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_1][x_1 W] \odot \mathcal{G}_{H_1}^{(d)} \odot \ldots \odot \mathcal{O}_{V^{(d)},\xi^{(d)}}[x_{n-d}][x_{n-d} W] \odot \mathcal{G}_{H_{n-d}}^{(d)} = \\ &= \mathcal{O}_{V^{(n)},\xi}[x_1 W, \ldots, x_{n-d} W] \odot \mathcal{G}_{H_1}^{(d)} \odot \ldots \odot \mathcal{G}_{H_{n-d}}^{(d)}. \end{aligned} \tag{4.32}$$

If one goes back to diagram (4.4), using the factorization



(4.33)

one can consider also the Rees algebras $\mathcal{G}_{H_{i,0}}^{(e+1)}$ and $\mathcal{G}_{\tilde{H}_{i,0}}^{(e+1)}$ induced by $\mathcal{G}_{H_i}^{(e)}$ over $\mathcal{O}_{V_{i,0}^{(e+1)},\xi_0} = \mathcal{O}_{V_0^{(d+1)},\xi_0^{(d+1)}}[x_i]$ and $(\mathcal{O}_{V_{i,0}^{(e)},\xi} \otimes_k K[[t]])_{\xi_0}$ respectively.

Consider now an arc $\varphi \in \mathcal{L}(X)$ through ξ, and the $\mathcal{O}_{\tilde{V}_0^{(n+1)},\xi_0}$-Rees algebra of contact of φ with $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$, $\mathcal{G}_{X_0,\varphi}^{(n+1)}$. Let us suppose that φ is given by

$$(\varphi_{x_1}, \ldots, \varphi_{x_{n-d}}, \varphi_{z_1}, \ldots, \varphi_{z_d})$$

as in (3.5). At the same time, for $i = 1, \ldots, n-d$, the projection of φ onto $V_i^{(e)}$ by (4.33) is an arc φ_i given by

$$(\varphi_{x_i}, \varphi_{z_1}, \ldots, \varphi_{z_d})$$

in $\mathcal{L}(H_i)$. Therefore we can define

$$\mathcal{G}_{H_{i,0},\varphi_i}^{(e+1)} = \mathrm{Diff}((\mathcal{O}_{V^{(n)},\xi} \otimes_k K[[t]])_{\xi_0}[f_i W^{b_i}, h_i W, h_{n-d+1} W, \ldots, h_n W]) = \mathcal{G}_{H_i}^{(e)} \odot \mathcal{G}_{\varphi_i}^{(e+1)}, \tag{4.34}$$

where $h_i = x_i - \varphi_{x_i}$ for $i = 1, \dots, n-d$ and $h_{n-d+j} = z_j - \varphi_{z_j}$ for $j = 1, \dots, d$, and

$$\begin{aligned}\mathcal{G}_{\varphi}^{(n+1)} =&(\mathcal{O}_{V_{i,0}^{(e)},\xi} \otimes_k K[[t]])_{\xi_0}[h_1W, \dots, h_nW] = \\ =&(\mathcal{O}_{V_{i,0}^{(e)},\xi} \otimes_k K[[t]])_{\xi_0}[h_1W, h_{n-d+1}W, \dots, h_nW] \odot \dots \\ &\odot (\mathcal{O}_{V_{i,0}^{(e)},\xi} \otimes_k K[[t]])_{\xi_0}[h_{n-d}W, h_{n-d+1}W, \dots, h_nW] = \\ =&\mathcal{G}_{\varphi_1}^{(e+1)} \odot \dots \odot \mathcal{G}_{\varphi_{n-d}}^{(e+1)}.\end{aligned} \tag{4.35}$$

Now we can use the result for hypersurfaces in Theorem 4.1.10 to assert that, for $i = 1, \dots, n-d$,

$$\frac{\mathrm{ord}_\xi(\varphi_i(\mathcal{G}_{H_i}^{(e)}))}{\mathrm{ord}(\varphi_i)} \geq \mathrm{ord}_\xi(\mathcal{G}_{H_i}^{(d)}).$$

Note that

$$\mathrm{ord}(\varphi) = \min_{i=1,\dots,n-d} \{\mathrm{ord}(\varphi_i)\}. \tag{4.36}$$

The following is a key remark for the generalization of Theorem 4.1.10.

Remark 4.2.2. The Rees algebra $\mathcal{G}_{X_0,\varphi}^{(n+1)}$ can be written in terms of the $\mathcal{G}_{H_{i,0},\varphi_i}^{(e+1)}$, by means of (3.12), (4.32), (4.35) and (4.34):

$$\begin{aligned}\mathcal{G}_{X_0,\varphi}^{(n+1)} &= \mathcal{G}_{X_0}^{(n+1)} \odot \mathcal{G}_{\varphi}^{(n+1)} = \\ = \mathcal{O}_{V_0^{(n+1)},\xi_0^{(n+1)}}[x_1W, \dots, x_{n-d}W] &\odot \mathcal{G}_{H_1}^{(d)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(d)} \odot \mathcal{G}_{\varphi}^{(n+1)} = \\ = \mathcal{G}_{H_1}^{(e)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(e)} &\odot \mathcal{G}_{\varphi_1}^{(e+1)} \odot \dots \odot \mathcal{G}_{\varphi_{n-d}}^{(e+1)} = \\ =\mathcal{G}_{H_1}^{(e)} \odot \mathcal{G}_{\varphi_1}^{(e+1)} &\odot \dots \odot \mathcal{G}_{H_{n-d}}^{(e)} \odot \mathcal{G}_{\varphi_{n-d}}^{(e+1)} = \\ = \mathcal{G}_{H_{1,0},\varphi_1}^{(e+1)} &\odot \dots \odot \mathcal{G}_{H_{n-d,0},\varphi_{n-d}}^{(e+1)}.\end{aligned} \tag{4.37}$$

After expressing the algebras $\mathcal{G}_{X_0}^{(n+1)}$ and $\mathcal{G}_{X_0,\varphi}^{(n+1)}$ in terms of Rees algebras attached to hypersurfaces as we have done in (4.32) and (4.37), it is easy to establish a relation among the order of all Rees algebras involved in both cases, as the following Lemma states:

Lemma 4.2.3. *Let X be a d-dimensional variety, and let $\xi \in \underline{\mathrm{Max}}\ \mathrm{mult}(X)$:*

1. *Let $\mathcal{G}_X^{(n)}$ and $\mathcal{G}_{H_i}^{(e)}$ be as in (4.29) and (4.31). Let $\mathcal{G}_X^{(d)}$ and $\mathcal{G}_{H_i}^{(d)}$ be respectively the elimination Rees algebras associated to their projection over $V^{(d)}$. Then*

$$\mathcal{G}_X^{(d)} = \mathcal{G}_{H_1}^{(d)} \odot \dots \odot \mathcal{G}_{H_{n-d}}^{(d)}, \tag{4.38}$$

and thus

$$\mathrm{ord}_\xi(\mathcal{G}_X^{(d)}) = \min_{i=1,\dots,n-d} \left\{\mathrm{ord}_\xi(\mathcal{G}_{H_i}^{(d)})\right\}. \tag{4.39}$$

2. *Let* $\mathcal{G}^{(n+1)}_{X_0,\varphi}$ *and* $\mathcal{G}^{(e+1)}_{H_{i,0},\varphi_i}$ *be as in (4.37) and (4.34). Let* $\varphi(\mathcal{G}^{(n)}_X)$ *and* $\varphi_i(\mathcal{G}^{(e)}_{H_i})$ *be their restrictions to the curves defined by the arcs* $\varphi, \varphi_1, \ldots \varphi_{n-d}$ *respectively (as in Proposition 3.2.12). Then*

$$\varphi(\mathcal{G}^{(n)}_X) = \varphi_1(\mathcal{G}^{(e)}_{H_1}) \odot \ldots \odot \varphi_{n-d}(\mathcal{G}^{(e)}_{H_{n-d}}). \tag{4.40}$$

As a consequence

$$\mathrm{ord}_{\xi}(\varphi(\mathcal{G}^{(n)}_X)) = \min_{i=1,\ldots,n-d} \left\{ \mathrm{ord}_{\xi}(\varphi_i(\mathcal{G}^{(e)}_{H_i})) \right\}. \tag{4.41}$$

Proof. Part (1) follows from the elimination of $\mathcal{G}^{(n)}_X$ associated to the projection $V^{(n)} \longrightarrow V^{(d)}$, using the expression in (4.32). For (2), one must note, by looking at the expression in (4.37), that the restriction of $\mathcal{G}^{(n+1)}_{X_0,\varphi}$ to the curve defined by φ equals the smallest algebra containing, for $i = 1, \ldots, n-d$, the restrictions $\mathcal{G}^{(1)}_{H_{i,0},\varphi_i}$ of the

$$\mathcal{G}^{(e+1)}_{H_{i,0},\varphi_i} = \mathcal{O}_{V_0^{(n+1)},\xi_0}[x_i W] \odot \mathcal{G}^{(d)}_{H_i} \odot \mathcal{G}^{(e+1)}_{\varphi_i}$$

to the respective curves defined by the φ_i, since all the Rees algebras are differentially closed. □

Results for the general case

Theorem 4.2.4. *Let* X *be a variety as in the beginning of the Section, let* $\xi \in$ $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ *and let* φ *be an arc in* X *through* ξ *with the notation used there. Then*

$$\overline{r}_{X,\varphi} \geq \mathrm{ord}_{\xi}(\mathcal{G}^{(d)}_X). \tag{4.42}$$

Proof. From (4.41) we obtain

$$\overline{r}_{X,\varphi} = \frac{\mathrm{ord}_{\xi}(\varphi(\mathcal{G}^{(n)}_X))}{\mathrm{ord}(\varphi)} = \frac{\min_{i=1,\ldots,n-d}\left\{\mathrm{ord}_{\xi}(\varphi_i(\mathcal{G}^{(e)}_{H_i}))\right\}}{\mathrm{ord}(\varphi)}.$$

For every $i \in \{1, \ldots, n-d\}$, Theorem 4.1.10 gives

$$\frac{\mathrm{ord}_{\xi}(\varphi_i(\mathcal{G}^{(e)}_{H_i}))}{\mathrm{ord}(\varphi_i)} \geq \mathrm{ord}_{\xi}(\mathcal{G}^{(d)}_{H_i}),$$

and this together with (4.36) and (4.39) implies

$$\frac{\mathrm{ord}_{\xi}(\varphi_i(\mathcal{G}^{(e)}_{H_i}))}{\mathrm{ord}(\varphi)} \geq \frac{\mathrm{ord}_{\xi}(\varphi_i(\mathcal{G}^{(e)}_{H_i}))}{\mathrm{ord}(\varphi_i)} \geq \mathrm{ord}_{\xi}(\mathcal{G}^{(d)}_{H_i}) \geq \mathrm{ord}_{\xi}(\mathcal{G}^{(d)}_X),$$

for $i = 1, \ldots, n-d$. As a consequence, we get

$$\overline{r}_{X,\varphi} = \frac{\min_{i=1,\ldots,n-d}\left\{\mathrm{ord}_{\xi}(\varphi_i(\mathcal{G}^{(e)}_{H_i}))\right\}}{\mathrm{ord}(\varphi)} \geq \mathrm{ord}_{\xi}(\mathcal{G}^{(d)}_X),$$

concluding the proof of the Theorem. □

Remark 4.2.5. If k is a field of characteristic zero, it is always possible to find a diagonal arc $\bar{\varphi}^{(d)}$ which is diagonal-generic for $\mathcal{G}^{(d)}_{H_i}$ for $i = 1, \dots, n-d$. As we did in Remark 4.1.7, one needs only to consider for each $i \in \{1, \dots, n-d\}$, an element $p_i W^{l_i} \in \mathcal{G}^{(d)}_{H_i}$ such that $\operatorname{ord}_\xi(\mathcal{G}^{(d)}_{H_i}) = \frac{\operatorname{ord}_\xi(p_i)}{l_i}$ and find units $u_1, \dots, u_d \in k$ such that $\operatorname{in}_\xi(p_i)(u_1, \dots, u_d) \neq 0$ for $i = 1, \dots, n-d$. This is again possible because we are considering a finite set of elements $\{p_1, \dots, p_{n-d}\}$ and an infinite field k. Now, the arc $\bar{\varphi}^{(d)}$ given by $(u_1 t^\alpha, \dots, u_d t^\alpha)$, where α is some positive integer, is diagonal-generic for $\mathcal{G}^{(d)}_{H_i}$ for all $i = 1, \dots, n-d$. In particular, $\bar{\varphi}^{(d)}$ is diagonal-generic for $\mathcal{G}^{(d)}_X$ (this follows from Lemma 4.2.3). By Lemma 4.1.8, $\bar{\varphi}^{(d)}$ can be lifted to an arc φ in X and the projection of φ onto $V^{(d)}$ is diagonal-generic for every $\mathcal{G}^{(d)}_{H_i}$, $i = 1, \dots, n-d$, as well as for $\mathcal{G}^{(d)}_X$.

Theorem 4.2.6. *Let X be a variety as in the beginning of the section and let $\xi \in \underline{\operatorname{Max}} \operatorname{mult}(X)$. There exists an arc $\varphi \in \mathcal{L}(X)$ through ξ such that*

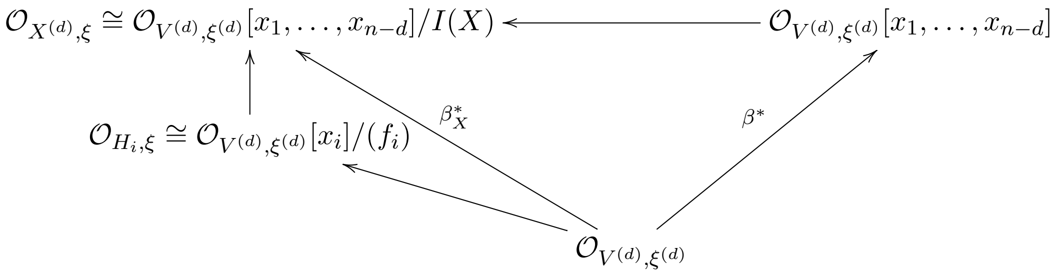
$$\overline{r}_{X,\varphi} = \operatorname{ord}_\xi(\mathcal{G}^{(d)}_X). \tag{4.43}$$

Proof. By Remark 4.2.5, we can choose a diagonal arc which is diagonal-generic for $\mathcal{G}^{(d)}_{H_1}, \dots, \mathcal{G}^{(d)}_{H_{n-d}}$ and $\mathcal{G}^{(d)}_X$. Let us denote it by $\bar{\varphi}^{(d)}$. We can lift $\bar{\varphi}^{(d)}$ to an arc $\varphi \in \mathcal{L}(X)$ through ξ. We know that φ is given (as in (3.5)) by

$$(g_1(t), \dots, g_{n-d}(t), u_1 g'(t), \dots, u_d g'(t))$$

for some $g_1(t), \dots g_{n-d}(t), g'(t) \in K[[t]]$ and some $u_1, \dots, u_d \in k$ due to Remark 4.1.9. By Remark 4.2.5, $\varphi^{(d)}$ is also generic for $\mathcal{G}^{(d)}_{H_i}$, $i = 1, \dots, n-d$. The proof will be complete by showing that any arc of this form satisfies (4.43).

Let us denote $N = \operatorname{ord}_t(g'(t))$. As in (1.23), β factorizes through $\mathcal{O}_{H_i,\xi}$ for $i = 1, \dots, n-d$:



(4.44)

and hence the projection φ_i of φ onto $V_i^{(d+1)}$ is, in particular, a lifting of $\bar{\varphi}^{(d)}$ to H_i, and the projection of each φ_i to $V^{(d)}$ is $\varphi^{(d)}$, which is diagonal-generic for $\mathcal{G}^{(d)}_{H_i}$. As a consequence, the result of Theorem 4.1.12 holds for each H_i, as well as Remark 4.1.13, and both together imply

$$\operatorname{ord}_\xi(\varphi_i(\mathcal{G}^{(e)}_{H_i})) = \operatorname{ord}(\varphi_i) \cdot \operatorname{ord}_\xi(\mathcal{G}^{(d)}_{H_i}) = N \cdot \operatorname{ord}_\xi(\mathcal{G}^{(d)}_{H_i})$$

for $i = 1, \dots, n-d$. By (4.36) we also know that $\text{ord}(\varphi) = N$. From this, together with Lemma 4.2.3, it follows that

$$\bar{r}_{X,\varphi} = \frac{\text{ord}_\xi(\varphi(\mathcal{G}_X^{(n)}))}{\text{ord}(\varphi)} = \frac{\min_{i=1,\dots,n-d}\left\{\text{ord}_\xi(\varphi_i(\mathcal{G}_{H_i}^{(e)}))\right\}}{N} =$$

$$= \frac{\min_{i=1,\dots,n-d}\left\{N \cdot \text{ord}_\xi(\mathcal{G}_{H_i}^{(d)})\right\}}{N} = \min_{i=1,\dots,n-d}\left\{\text{ord}_\xi(\mathcal{G}_{H_i}^{(d)})\right\} = \text{ord}_\xi(\mathcal{G}_X^{(d)}),$$

which completes the proof. □

4.3 Consequences

The result in Theorem 4.0.1, proven along Sections 4.1 and 4.2 relates, for a given X of dimension d over a field of characteristic zero, and any $\xi \in \underline{\text{Max}}\,\text{mult}(X)$, the invariant $\text{ord}_\xi^{(d)} X$ and the order of contact of the arcs in X centered at ξ with $\underline{\text{Max}}\,\text{mult}(X)$. However, as a consequence of this relation we may obtain some conclussions for the persistance of these arcs too:

Theorem 4.3.1. *Let X be a variety of dimension d. Let ξ be a point in $\underline{\text{Max}}\,\text{mult}(X)$. For any arc φ in X through ξ,*

$$\rho_{X,\varphi} \geq \left\lceil \text{ord}_\xi \mathcal{G}_X^{(d)} \right\rceil \cdot \text{ord}(\varphi),$$

where $\mathcal{G}_X^{(d)}$ is the elimination algebra described in Example 1.4.25. Moreover,

$$\min_{\varphi \in \mathcal{L}(X), \varphi(0)=\xi} \{[\bar{\rho}_{X,\varphi}]\} = \left\lceil \text{ord}_\xi \mathcal{G}_X^{(d)} \right\rceil.$$

One can find an arc φ_0 in X through ξ satisfying

$$\bar{\rho}_{X,\varphi_0} = \text{ord}_\xi \mathcal{G}_X^{(d)}.$$

Proof. For the first formula we use Proposition 3.3.1 and Theorem 4.2.4

$$\rho_{X,\varphi} = [r_{X,\varphi}] = \left\lceil \frac{r_{X,\varphi}}{\text{ord}(\varphi)} \cdot \text{ord}(\varphi) \right\rceil \geq \left\lceil \frac{r_{X,\varphi}}{\text{ord}(\varphi)} \right\rceil \cdot \text{ord}(\varphi) \geq \left\lceil \text{ord}_\xi \mathcal{G}_X^{(d)} \right\rceil \cdot \text{ord}(\varphi).$$

As a consequence,

$$\frac{r_{X,\varphi}}{\text{ord}(\varphi)} \geq \frac{[r_{X,\varphi}]}{\text{ord}(\varphi)} = \frac{\rho_{X,\varphi}}{\text{ord}(\varphi)} \geq \frac{\left\lceil \text{ord}_\xi \mathcal{G}_X^{(d)} \right\rceil \cdot \text{ord}(\varphi)}{\text{ord}(\varphi)} = \left\lceil \text{ord}_\xi \mathcal{G}_X^{(d)} \right\rceil.$$

That is,

$$\bar{r}_{X,\varphi} \geq \bar{\rho}_{X,\varphi} \geq \left\lceil \text{ord}_\xi \mathcal{G}_X^{(d)} \right\rceil,$$

where we may take integral parts and then the minimums over all arcs in X through ξ, obtaining

$$\min_{\varphi\in\mathcal{L}(X),\varphi(0)=\xi}\{[\bar{r}_{X,\varphi}]\} \geq \min_{\varphi\in\mathcal{L}(X),\varphi(0)=\xi}\{[\bar{\rho}_{X,\varphi}]\} \geq \left[\operatorname{ord}_\xi \mathcal{G}_X^{(d)}\right] = \min_{\varphi\in\mathcal{L}(X),\varphi(0)=\xi}\{[\bar{r}_{X,\varphi}]\},$$

proving the second formula of the Theorem.

Finally, for the third formula, let us go back to the proof of Theorem 4.43. It allows us to find an arc φ_1 in X through ξ satisfying $\bar{r}_{X,\varphi_1} = \operatorname{ord}_\xi \mathcal{G}_X^{(d)}$. This arc will be given by $(g_1(t),\dots,g_{n-d}(t),u_1g'(t),\dots,u_dg'(t))$ for some $g_1(t),\dots,g_{n-d}(t),g'(t)\in K[[t]]$ and some $u_1,\dots,u_d\in k$, and the projection $\varphi_1^{(d)}$ given by $(u_1g'(t),\dots,u_dg'(t))$ will be diagonal generic for $\mathcal{G}_X^{(d)}$. Let us choose φ_0 as the arc in X through ξ given by

$$(g_1(t^{b'}),\dots,g_{n-d}(t^{b'}),u_1g'(t^{b'}),\dots,u_dg'(t^{b'})),$$

where $b'\in\mathbb{Z}_{>0}$ is such that $\operatorname{ord}_\xi \mathcal{G}_X^{(d)} \in \frac{1}{b'}\cdot\mathbb{Z}_{>0}$, whose projection $\varphi_0^{(d)}$ is also diagonal generic for $\mathcal{G}_X^{(d)}$, so it is also valid for Theorem 4.43, having $\bar{r}_{X,\varphi_0} = \operatorname{ord}_\xi \mathcal{G}_X^{(d)}$. In particular, this implies that

$$\bar{r}_{X,\varphi_0} = \bar{r}_{X,\varphi_1}.$$

Note also that $\operatorname{ord}(\varphi_0) = \operatorname{ord}(\varphi_1)\cdot b'$. We have found an arc such that

$$r_{X,\varphi_0} = \operatorname{ord}_\xi \mathcal{G}_X^{(d)} \cdot \operatorname{ord}(\varphi_0),$$

and for which

$$r_{X,\varphi_0} = [r_{X,\varphi_0}] = \rho_{X,\varphi_0},$$

since $\operatorname{ord}(\varphi_0)\in b'\cdot\mathbb{Z}_{>0}$, concluding the proof. □

The following Corollary gives a characterization of $\operatorname{ord}_\xi^{(d)} X$ in terms of the $\bar{\rho}_{X,\varphi}$.

Corollary 4.3.2. *Let X be a variety of dimension d. Let ξ be a point in $\underline{\operatorname{Max}}\operatorname{mult}(X)$. Consider the subset $\mathcal{C}\subset\mathcal{L}(X)$ of all arcs φ through ξ satisfying $\bar{r}_{X,\varphi} = \operatorname{ord}_\xi^{(d)} X$. Then:*

$$\operatorname{ord}_\xi^{(d)} X = \max_{\varphi\in\mathcal{C}}\{\bar{\rho}_{X,\varphi}\}.$$

Proof. We know that $\mathcal{C}$ is nonempty by theorem 4.2.6 For any arc $\varphi\in\mathcal{C}$,

$$\rho_{X,\varphi} = [r_{X,\varphi}] = \left[\operatorname{ord}_\xi \mathcal{G}_X^{(d)}\cdot\operatorname{ord}(\varphi)\right].$$

It follows that

$$\frac{\rho_{X,\varphi}}{\operatorname{ord}(\varphi)} = \frac{\left[\operatorname{ord}_\xi \mathcal{G}_X^{(d)}\cdot\operatorname{ord}(\varphi)\right]}{\operatorname{ord}(\varphi)} \leq \operatorname{ord}_\xi \mathcal{G}_X^{(d)}.$$

Now, the result is a consequence of this together with Theorem 4.3.1. □

For every arc $\varphi \in \mathcal{L}(X)$ through ξ, we have the following relations:

Corollary 4.3.3. *For X as in Proposition 3.3.1, and for every arc $\varphi \in \mathcal{L}(X)$ through ξ:*

1. $\bar{r}_{X,\varphi} \geq \bar{\rho}_{X,\varphi}$
2. $\bar{\rho}_{X,\varphi} \geq [\mathrm{ord}_\xi \mathcal{G}_X^{(d)}]$
3. *Since* $\bar{r}_{X,\varphi} \geq \mathrm{ord}_\xi \mathcal{G}_X^{(d)}$ *and* $\bar{r}_{X,\varphi} \geq \bar{\rho}_{X,\varphi} \geq [\mathrm{ord}_\xi \mathcal{G}_X^{(d)}]$*, two possible situations can happen for* $\bar{\rho}_{X,\varphi}$ *and* $\mathrm{ord}_\xi \mathcal{G}_X^{(d)}$*, namely:*
   - $\bar{r}_{X,\varphi} \geq \mathrm{ord}_\xi \mathcal{G}_X^{(d)} \geq \bar{\rho}_{X,\varphi} \geq [\mathrm{ord}_\xi \mathcal{G}_X^{(d)}]$
   - $\bar{r}_{X,\varphi} \geq \bar{\rho}_{X,\varphi} > \mathrm{ord}_\xi \mathcal{G}_X^{(d)} \geq [\mathrm{ord}_\xi \mathcal{G}_X^{(d)}]$

Proof. 1. Follows from the definitions of $\bar{r}_{X,\varphi}$ and $\bar{\rho}_{X,\varphi}$ toghether with Proposition 3.3.1.

2. By Definition 3.1.2, Proposition 3.3.1, Theorem 4.2.4:

$$\bar{\rho}_{X,\varphi} = \frac{\rho_{X,\varphi}}{\mathrm{ord}\varphi} = \frac{[r_{X,\varphi}]}{\mathrm{ord}\varphi} \geq \frac{[\mathrm{ord}_\xi \mathcal{G}_X^{(d)} \cdot \mathrm{ord}\varphi]}{\mathrm{ord}\varphi} \geq [\mathrm{ord}_\xi \mathcal{G}_X^{(d)}].$$

3. This is just an observation which follows from 1. and 2..

□

Finally, let us show how the persistance can recover the invariant $\mathrm{ord}_\xi^{(d)}(X)$, proving that it is intrinsic to the variety:

Corollary 4.3.4. *For X as in Proposition 3.3.1, pick any arc $\varphi : \mathrm{Spec}(K[[t]]) \longrightarrow X$ centered at ξ. Consider the family of arcs given as*

$$\varphi_n = \varphi \circ i_n$$

for $i > 1$, where $i_n^* : K[[t]] \longrightarrow K[[t^n]]$ maps t to t^n. Then

$$\mathrm{ord}_\xi^{(d)}(X) = \inf_\varphi \left(\frac{1}{\mathrm{ord}(\varphi)} \cdot \lim_{n\to\infty} \frac{\rho_{X,\varphi_n}}{n} \right),$$

where φ runs over all arcs in X centered at ξ which are not contained in $\underline{\mathrm{Max}}\ \mathrm{mult}(X)$.

Proof. The statement is just a consequence of

$$r_{X,\varphi} = \lim_{n\to\infty} \frac{[n \cdot r_{X,\varphi}]}{n} = \lim_{n\to\infty} \frac{[r_{X,\varphi_n}]}{n} = \lim_{n\to\infty} \frac{\rho_{X,\varphi_n}}{n},$$

and this follows from the fact that $r_{X,\varphi_n} = n \cdot r_{X,\varphi}$ (which can be deduced from the definition of $r_{X,\varphi}$). □

4.4 Examples

Let us recover some examples from the previous chapters to show their invariants and the relation among them. We will use the computations from Examples 1.5.4, 3.2.19 and 3.3.3

Example 4.4.1.

- Let $X_1 = \mathbb{V}(x^2 - y^3) \subset \mathrm{Spec}(k[x,y])$ from Example 1.2.7. Using the arc $\varphi = (t^3, t^2)$, we observe that

$$\mathrm{ord}_{\xi}^{(1)} X_1 = 3/2 = \overline{r}_{X_1,\varphi} = \overline{\rho}_{X_1,\varphi}.$$

- Let $X_4 = \mathbb{V}(x^3 - y^3 z^2) \subset \mathrm{Spec}(k[x,y,z])$ from Example 1.2.7. Using the arc $\varphi_1 = (t^3, t, t^3)$, we have

$$\mathrm{ord}_{\xi}^{(2)} X_4 = 5/3 < \overline{r}_{X_4,\varphi_1} = 3 = \overline{\rho}_{X_4,\varphi_1}.$$

  The arc $\varphi_2 = (t^5, t^3, t^3)$, gives the order in dimension d:

$$\mathrm{ord}_{\xi}^{(2)} X_4 = 5/3 = \overline{r}_{X_4,\varphi_2} = \overline{\rho}_{X_4,\varphi_2}.$$

- Let $X_6 = \mathbb{V}(x^3 - xyz^2 - yz^3 + z^5) \subset \mathrm{Spec}(k[x,y,z])$ from Example 1.3.26. For the arc $\varphi = (t^2, t^2, t)$, we have

$$\overline{r}_{X_6,\varphi} = 3/2 > \mathrm{ord}_{\xi}^{(2)} X_6 = 4/3 > \overline{\rho}_{X_6,\varphi} = 1 = \left[\mathrm{ord}_{\xi}^{(2)} X_6\right].$$

- Let $X_7 = \mathbb{V}(xy - z^4) \subset \mathrm{Spec}(k[x,y,z])$ from Example 1.3.26. The arc $\varphi_1 = (t, t^3, t)$ gives

$$\mathrm{ord}_{\xi}^{(2)} X_7 = 1 = \overline{r}_{X_7,\varphi_1} = \overline{\rho}_{X_7,\varphi_1}.$$

  On the other hand arc $\varphi_2 = (t^2, t^2, t)$ gives

$$\mathrm{ord}_{\xi}^{(2)} X_7 = 1 < \overline{r}_{X_7,\varphi_2} = 2 = \overline{\rho}_{X_7,\varphi_2}.$$

Chapter 5

The Nash multiplicity sequence for isolated points of maximum multiplicity

Along this chapter, we study the isolation of points of maximum multiplicity of a variety from the point of view of the invariants from Chapter 3. In this line, we prove that the fact of being isolated for a point $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$ is related to how large $\overline{r}_{X,\varphi}$ (from 3.2.18) can be for the different arcs in the variety (which are centered at ξ). This is proven in Theorem 5.0.1 and Corollary 5.2.2. For points with maximal τ invariant, we give a precise upper bound for $\overline{r}_{X,\varphi}$ in Proposition 5.3.1. We state here the central result:

Theorem 5.0.1. *Let X be a variety over a field k of characteristic zero, and let ξ be a point in $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. Then, ξ is an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ if and only if*

$$\sup \Phi_{X,\xi} < \infty.$$

(See 3.2.20.)

Setting

Let us assume the setting from Section 4.2, that is:

Let X be a d-dimensional variety defined over a field k of characteristic zero. Let $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$ be a point of maximum multiplicity of X. Consider a local presentation for the multiplicity of X, locally in an étale neighborhood of ξ:

$$X^{(d)} \hookrightarrow V^{(n)}$$

$$\mathcal{G}_X^{(n)} = \operatorname{diff}\left(\mathcal{O}_{V^{(n)},\xi}\left[f_1 W^{n_1}, \ldots, f_{n-d} W^{(n-d)}\right]\right)$$

as in Example 1.4.7. Recall that we have a transversal projection $\beta_X : X^{(d)} \longrightarrow V^{(d)}$ inducing an elimination algebra $\mathcal{G}_X^{(d)}$ of $\mathcal{G}_X^{(n)}$. For any K-arc φ in X centered at ξ we obtain an arc in $V^{(d)}$ centered at $\beta_X(\xi)$ (see Remark 4.0.2). We define it as in Remark 3.2.5, by choosing a regular system of parameters $\{x_1, \ldots, x_{n-d}, z_1, \ldots, z_d\} \subset \mathcal{O}_{V^{(n)},\xi}$. This is explained in the diagram:

$$\begin{array}{cccc}
\mathcal{G}_X^{(n)} & X \subset V^{(n)} & & \\
& \Big\downarrow \beta & \overset{\varphi}{\nwarrow} & \\
& & & \operatorname{Spec}(K[[t]]) \\
& & \underset{\varphi^{(d)}}{\swarrow} & \\
\mathcal{G}_X^{(d)} & V^{(d)} & &
\end{array}$$

For the proof of Theorem 5.0.1 we will need some facts which are consequences of the results from Chapter 4. The following Lemma collects them:

Lemma 5.0.2. *Let X be as in Theorem 5.0.1 and let ξ be a point in $\underline{\operatorname{Max}}\operatorname{mult}(X)$. Let φ be an arc in X through ξ. With the notation from Section 4.2:*

1. $r_{X,\varphi} = \operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)}))$ *and*
2. $\operatorname{ord}_t(\varphi(x_i)) \geq \operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)}))$ *for* $i = 1, \ldots, n-d$.

Proof. It follows from (1.32) that

$$r_{X,\varphi} = \operatorname{ord}_t(\varphi(\mathcal{G}_X^{(n)})) = \min_{i=1,\ldots,n-d}\left\{\operatorname{ord}_t(\varphi_i^{(e)}(\mathcal{G}_{H_i}^{(e)}))\right\} =$$
$$= \min\left\{\operatorname{ord}_t(\varphi(x_1)), \ldots, \operatorname{ord}_t(\varphi(x_{n-d})), \operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)}))\right\} \leq \operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)})),$$

where $e = d+1$. On the other hand, for each i, by Lemma 4.1.1,

$$\operatorname{ord}_t(\varphi_i^{(e)}(\mathcal{G}_{H_i}^{(e)})) = \min\left\{\operatorname{ord}_t(\varphi_i^{(e)}(x_i)), \operatorname{ord}_t(\varphi_i^{(d)}(\mathcal{G}_{H_i}^{(d)}))\right\} = \operatorname{ord}_t(\varphi_i^{(d)}(\mathcal{G}_{H_i}^{(d)})), \quad (5.1)$$

so

$$r_{X,\varphi} = \min_{i=1,\ldots,n-d}\left\{\operatorname{ord}_t(\varphi_i^{(d)}(\mathcal{G}_{H_i}^{(d)}))\right\}.$$

But note that $\mathcal{G}_{H_i}^{(d)} \subset \mathcal{G}_X^{(d)}$ and $\varphi_i^{(d)} = \varphi^{(d)}$ (see (1.31) and (4.2)). Thus,

$$\varphi_i^{(d)}(\mathcal{G}_{H_i}^{(d)}) = \varphi^{(d)}(\mathcal{G}_{H_i}^{(d)}) \subset \varphi^{(d)}(\mathcal{G}_X^{(d)})$$

and

$$\operatorname{ord}_t(\varphi_i^{(d)}(\mathcal{G}_{H_i}^{(d)})) \geq \operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)})). \quad (5.2)$$

Consequently,

$$\operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)})) \geq r_{X,\varphi} \geq \operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)})),$$

proving 1. Now 2 is a consequence of (5.1), together with (5.2) and the fact that, for all $i = 1, \ldots, n-d$,

$$\varphi_i^{(d)}(x_i) = \varphi_i^{(e)}(x_i) = \varphi(x_i).$$

□

5.1 Isolated points

In order to prove Theorem 5.0.1, let us divide it in two one side implications, reformulated in Propositions 5.1.1 and 5.2.1 respectively, in a way that will be more convenient for their respective proofs. We first give a simple version of the proof of the easier one:

Proposition 5.1.1. *Let ξ be an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$. Then there exists a positive integer $Q \in \mathbb{Z}_{>0}$, depending only on X and ξ, such that for any arc φ in X through ξ,*

$$\bar{r}_{X,\varphi} \leq Q.$$

Proof. Consider the graded structure of a Rees algebra $\mathcal{G}_X^{(n)}$ representing the multiplicity of X in an étale neighborhood of ξ,

$$\mathcal{G}_X^{(n)} = \oplus_{i\geq 0} I_i W^i \subset \mathcal{O}_{V^{(n)}}[W].$$

Since we assume $\mathcal{G}_X^{(n)}$ to be differentially closed, the set $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ is determined by the zeros of the ideal I_1 (see Proposition 1.3.33). Therefore, $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ being of dimension 0 is equivalent to $\sqrt{I_1}$ being a maximal ideal. This is also equivalent to the fact that, for a (any) regular system of parameters $\{x_1, \ldots, x_{n-d}, z_1, \ldots, z_d\}$ in $\mathcal{O}_{X,\xi}$, I_1 contains some ideal of the form

$$(x_1^{a_1}, \ldots, x_{n-d}^{a_{n-d}}, z_1^{a_{n-d+1}}, \ldots, z_d^{a_n})$$

for some positive integers $a_1, \ldots, a_n$. Note that this implies that

$$\mathcal{G}_X^{(n)} \supset \mathcal{O}_{X,\xi}[x_1^{a_1}W, \ldots, x_{n-d}^{a_{n-d}}W, z_1^{a_{n-d+1}}W, \ldots, z_d^{a_n}W].$$

Therefore,

$$\varphi(\mathcal{G}_X^{(n)}) \supset \mathcal{O}_{X,\xi}[\varphi(x_1^{a_1})W, \ldots, \varphi(x_{n-d}^{a_{n-d}})W, \varphi(z_1^{a_{n-d+1}})W, \ldots, \varphi(z_d^{a_n})W],$$

and

$$\begin{aligned}\mathrm{ord}_t(\varphi(\mathcal{G}_X^{(n)})) \leq & \min\{a_1 \cdot \mathrm{ord}_t(\varphi(x_1)), \ldots, a_{n-d} \cdot \mathrm{ord}_t(\varphi(x_{n-d})),\\ & a_{n-d+1} \cdot \mathrm{ord}_t(\varphi(z_1)), \ldots, a_n \cdot \mathrm{ord}_t(\varphi(z_d))\}.\end{aligned}$$

Thus

$$\bar{r}_{X,\varphi} \leq a_j \in \mathbb{Z}_{>0}$$

for any $j \in \{1, \ldots, n-d\}$ such that $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(x_j))$ or any $j \in \{n-d+1, \ldots, n\}$ such that $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(z_{j-n+d}))$. □

The bound given by this proof is not optimal. In general, a rational number which will be smaller than the integer given by the a_j's can be found, yielding an optimal bound. Note that this rational number is an invariant of X at ξ.

Remark 5.1.2. For some arcs, we can say more about $\bar{r}_{X,\varphi}$: If φ is such that

$$\operatorname{ord}(\varphi) = \operatorname{ord}_t(\varphi(x_j))$$

for some $j \in \{1, \ldots, n-d\}$, then

$$\bar{r}_{X,\varphi} = 1.$$

Indeed,

$$a_1 = \ldots = a_{n-d} = 1$$

in the proof of Proposition 5.1.1, because $x_1, \ldots, x_{n-d} \in I_1$ (see (4.32)).

In the next section, a precise upper bound will be given under some special condition over X at ξ, in terms of orders of elimination algebras. This condition is related with the τ invariant of $\mathcal{G}_X^{(n)}$ at ξ (see Definition 1.4.9).

5.2 Non isolated points

We prove now the most delicate implication of Theorem 5.0.1. To make the proof easier to understand, we will deal separately with an easy case first, even though it of course follows from the general one, which we prove afterwards. For the techniques of resolution used in this proof, as well as definitions of strict and total transform of an ideal, we refer to [12, Section 7] or [36].

Proposition 5.2.1. *If ξ lies in a component of $\underline{\operatorname{Max}}\operatorname{mult}(X)$ of dimension greater than or equal to 1, then for any $q \in \mathbb{Q}$, one can find an arc φ in X through ξ such that*

$$\bar{r}_{X,\varphi} > q.$$

Proof. Since $r_{X,\varphi} = \operatorname{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)}))$ if $\varphi^{(d)} = \varphi \circ \beta_X^*$ (see Lemma 5.0.2), our strategy here will be choosing an arc $\bar{\varphi}^{(d)}$ in $V^{(d)}$ through $\xi^{(d)}$ which gives $\operatorname{ord}_t(\bar{\varphi}^{(d)}(\mathcal{G}_X^{(d)}))$ large enough first, and then lifting it via β_X to an arc φ in X through ξ, proving afterwards that it satisfies the statement in the Proposition.

Suppose first that there exists a smooth curve $\widetilde{C} \subset \underline{\operatorname{Max}}\operatorname{mult}(X)$ containing ξ. Then $C = \beta_X(\widetilde{C}) \subset V^{(d)}$ is a smooth curve containing $\xi^{(d)}$(see [82, Theorem 6.3]). Assume that C is defined by a prime ideal $J \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}$. Consider the family of arcs $\bar{\varphi}_N^{(d)}$ in $V^{(d)}$ through $\xi^{(d)}$, for $N \in \mathbb{Z}_{>0}$, given by

$$\begin{aligned} \bar{\varphi}_N^{(d)} : \mathcal{O}_{V^{(d)},\xi^{(d)}} &\longrightarrow K[[t]], \\ J &\longmapsto t^N, \\ \mathcal{M}_{\xi^{(d)}} &\longmapsto t. \end{aligned}$$

This can be done because we may assume that, in this situation, $J = (y_2, \ldots, y_d)$ for some regular system of parameters $\{y_1, \ldots, y_d\}$ of $\mathcal{O}_{V^{(d)},\xi^{(d)}}$. Then, such a family of arcs could be constructed by just defining $\bar{\varphi}_N^{(d)}(y_1) = t$ and $\bar{\varphi}_N^{(d)}(y_j) = t^N$ for $j = 2, \dots, d$. For any $N \in \mathbb{N}$, the arc $\bar{\varphi}_N^{(d)}$ can be lifted to an arc φ_N in X through ξ satisfying $\bar{r}_{X,\varphi_N} \geq N$ as follows:

Under the hypothesis $d \geq 2$, consider the ideal $\mathcal{P} = \mathrm{Ker}(\bar{\varphi}_N^{(d)}) \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}$. There exists a prime ideal $\mathcal{Q}$ in $\mathcal{O}_{X,\xi}$ dominating $\mathcal{P}$. We have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{Q} \subset \mathcal{O}_{X,\xi} & \xrightarrow{\mu} & \mathcal{O}_{X,\xi}/\mathcal{Q} \\ \uparrow \beta_X^* & & \uparrow \bar{\beta_X}^* \\ \mathcal{P} \subset \mathcal{O}_{V^{(d)},\xi^{(d)}} & \xrightarrow{\mu^{(d)}} & \mathcal{O}_{V^{(d)},\xi^{(d)}}/\mathcal{P} \end{array}$$

where the vertical arrows are finite morphisms, and both rings on the right side are 1-dimensional. The ideal $\mathcal{Q}$ defines a curve. One can find a nontrivial arc

$$\tilde{\varphi}_N : \mathcal{O}_{X,\xi}/\mathcal{Q} \longrightarrow K[[t]]$$

in $\mathbb{V}(\mathcal{Q})$ through $\mu(\xi)$. It induces another K-arc, where K is the residue field of $\mathcal{O}_{X,\xi}/\mathcal{Q}$ at $\mu(\xi)$:

$$\varphi_N = \tilde{\varphi_N} \circ \mu : \mathcal{O}_{X,\xi} \longrightarrow K[[t]]$$

in X through ξ, and a K-arc

$$\varphi_N^{(d)} = \varphi_N \circ \beta_X^* = \tilde{\varphi}_N \circ \bar{\beta_X}^* \circ \mu^{(d)} : \mathcal{O}_{V^{(d)},\xi^{(d)}} \longrightarrow K[[t]]$$

in $V^{(d)}$ through $\xi^{(d)}$, with

$$\mathrm{Ker}(\varphi_N^{(d)}) = \mathcal{P} = \mathrm{Ker}(\bar{\varphi}_N^{(d)}) = (y_2 - y_1^N, y_2 - y_j : 2 < j \leq d) \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}.$$

Since $C \subset \mathrm{Sing}(\mathcal{G}_X^{(d)})$,

$$\mathrm{ord}_C(I_i \mathcal{O}_{V^{(d)},\xi^{(d)}}) \geq i \quad \forall i \geq 0,$$

so $I_i \mathcal{O}_{V^{(d)},C} \subset J^i \mathcal{O}_{V^{(d)},C}$. But note that J is a regular prime in $\mathcal{O}_{V^{(d)},\xi^{(d)}}$ defining C, so $I_i \mathcal{O}_{V^{(d)},\xi^{(d)}} \subset J^i$ for all $i \geq 0$. Consequently,

$$\mathcal{G}_X^{(d)} \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}[JW],$$

and

$$\varphi_N^{(d)}(\mathcal{G}_X^{(d)}) \subset \varphi_N^{(d)}(\mathcal{O}_{V^{(d)},\xi^{(d)}}[JW]).$$

Hence, for $\varphi_N^{(d)}$ constructed as above,

$$\mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{G}_X^{(d)})) \geq \mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{O}_{V^{(d)},\xi^{(d)}}[JW])) = \mathrm{ord}_t(\varphi_N^{(d)}(J)).$$

Using also Lemma 5.0.2 and the fact that $\mathrm{ord}(\varphi_N) \leq \mathrm{ord}(\varphi_N^{(d)}) = \mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{M}_{\xi^{(d)}}))$ (see (4.1)), we arrive to

$$\bar{r}_{X,\varphi_N} = \frac{\mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{G}_X^{(d)}))}{\mathrm{ord}(\varphi_N)} \geq \frac{\mathrm{ord}_t(\varphi_N^{(d)}(J))}{\mathrm{ord}(\varphi_N)} \geq \frac{\mathrm{ord}_t(\varphi_N^{(d)}(J))}{\mathrm{ord}(\varphi_N^{(d)})} = \frac{\mathrm{ord}_t(\varphi_N^{(d)}(J))}{\mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{M}_{\xi^{(d)}}))}.$$

Assume that $\varphi_N^{(d)}(y_j) = u_j t^{\alpha_j}$ for $j = 1, \dots, d$ for some units u_j in $K[[t]]$ and some $\alpha_j \in \mathbb{Z}_{>0}$. Then

$$\varphi_N^{(d)}(y_2 - y_1^N) = 0 = \varphi_N^{(d)}(y_2) - \varphi_N^{(d)}(y_1)^N = u_2 t^{\alpha_2} - u_1^N t^{\alpha_1 \cdot N} \text{ and}$$
$$\varphi_N^{(d)}(y_2 - y_j) = 0 = \varphi_N^{(d)}(y_2) - \varphi_N^{(d)}(y_j) = u_2 t^{\alpha_2} - u_j t^{\alpha_j} \text{ for } 2 < j \leq d.$$

Necessarily

$$\alpha_2 = \alpha_1 \cdot N \text{ and}$$
$$\alpha_2 = \alpha_j \text{ for } 2 < j \leq d,$$

so

$$\bar{r}_{X,\varphi_N} \geq \frac{\mathrm{ord}_t(\varphi_N^{(d)}(J))}{\mathrm{ord}_t(\varphi_N^{(d)}(\mathcal{M}_{\xi^{(d)}}))} = \frac{\min_{i=2,\dots,d}\{\alpha_i\}}{\min_{j=1,\dots,d}\{\alpha_j\}} = \frac{\alpha_2}{\alpha_1} = N$$

which, for a fixed $q \in \mathbb{Q}$, can be greater than q by just choosing N big enough.

Suppose now that $\widetilde{C} \subset \underline{\mathrm{Max}}\ \mathrm{mult}(X)$ is not smooth. As before, assume that $C = \beta(\widetilde{C}) \subset V^{(d)}$ and denote $J = I(C) \subset \mathcal{O}_{V^{(d)},\xi^{(d)}}$. Consider the following sequence:

$$\begin{array}{ccccccc} V^{(d)} & = & V_0^{(d)} \xleftarrow{\pi_1} & V_1^{(d)} \xleftarrow{\pi_2} & \dots \xleftarrow{\pi_r} & V_r^{(d)} \\ & & \cup & \cup & \dots & \cup \\ C & = & C_0 & C_1' & \dots & C_r' \\ \xi^{(d)} & = & \xi_0^{(d)} & \xi_1^{(d)} & \dots & \xi_r^{(d)} \end{array} \tag{5.3}$$

where π_i is the blow up at the point $\xi_{i-1}^{(d)}$, and $\xi_i^{(d)} \in \pi_i^{-1}(\xi_{i-1}^{(d)}) \cap C_i'$ for $i = 1, \dots, r$, and such that the strict transform C_r' of C_0 by $\pi = \pi_1 \circ \dots \circ \pi_r$ is a smooth curve having normal crossings with the exceptional divisor at $\xi_r^{(d)}$. Such a sequence can always be found, being an embedded desingularization of C. Let us look now at the total transform $J_r = J\mathcal{O}_{V_r^{(d)}}$ of the ideal J by π, which will be, locally in a neighborhood of $\xi_r^{(d)}$, of the form

$$J_r = \mathscr{M} \cdot J_r',$$

where J_r' is contained in the ideal $I(C_r')$ defining the strict transform C_r' of C in $V_r^{(d)}$, and $\mathscr{M}$ is a locally a monomial. Let us choose a family of arcs $\bar{\varphi}_{N,r}^{(d)}$ in $V_r^{(d)}$ through $\xi_r^{(d)}$ for $N \in \mathbb{Z}_{>0}$ such that $\bar{\varphi}_{N,r}^{(d)}(I(C_r')) = t^N$ and $\bar{\varphi}_{N,r}^{(d)}(\pi^*(\mathcal{M}_{\xi^{(d)}})) = t^a$ for some $a \in \mathbb{Z}_{>0}$ constant, as we did for the case of C smooth. For this, note that locally in a neighborhood of $\xi_r^{(d)}$, one can consider a regular system of parameters in $\mathcal{O}_{V_r^{(d)},\xi_r^{(d)}}$ given by

$$\{\tilde{y}_1 = I(H_1), \tilde{y}_2, \dots, \tilde{y}_d\},$$

so that $I(C_r') = (\tilde{y}_2, \dots, \tilde{y}_d)$, and moreover

$$\pi^*(\mathcal{M}_{\xi^{(d)}}) = I(H_1)^a$$

for $a \in \mathbb{N}$, where $H_1 = \pi_r^{-1}(\xi_{r-1})$ is the exceptional divisor of π_r, because of the way in which the centers of the π_i are chosen. Consider $\bar{\varphi}^{(d)}_{N,r}$ given as

$$\begin{aligned} \bar{\varphi}^{(d)}_{N,r} : \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}} &\longrightarrow K[[t]], \\ \tilde{y}_1 &\longmapsto t, \\ \tilde{y}_j &\longmapsto t^N, \text{ for } j = 2, \dots, d, \end{aligned}$$

which satisfies the desired properties. Note that π induces a sequence of permissible transformations of X via β_X:

$$\begin{array}{ccccccccc} X & = & X_0 & & \xleftarrow{\quad\quad\quad \pi_X \quad\quad\quad} & & & & X_r \\ \downarrow{\scriptstyle \beta_X} & & & & & & & & \downarrow{\scriptstyle \beta_{X_r}} \\ V^{(d)} & = & V_0^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \xleftarrow{\pi_2} & \dots & \xleftarrow{\pi_r} & V_r^{(d)} \end{array}$$

For each $N \in \mathbb{Z}_{>0}$, $\bar{\varphi}^{(d)}_{N,r}$ can be lifted to an arc in X_r through $\xi_r^{(d)}$ via a diagram as in the regular case:

$$\begin{array}{ccc} \mathcal{Q} \subset \mathcal{O}_{X_r,\xi_r} & \xrightarrow{\mu} & \mathcal{O}_{X_r,\xi_r}/\mathcal{Q} \\ {\scriptstyle \beta^*_{X_r}}\uparrow & & \uparrow{\scriptstyle \bar{\beta}^*_{X_r}} \\ \mathcal{P} \subset \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}} & \xrightarrow{\mu^{(d)}} & \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}}/\mathcal{P} \end{array}$$

where $\mathcal{P} = \operatorname{Ker}(\bar{\varphi}^{(d)}_{N,r}) = \mathcal{Q} \cap \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}}$. As we did in the case of C a regular curve, we pick an arc

$$\tilde{\varphi}_{N,r} : \mathcal{O}_{X_r,\xi_r}/\mathcal{Q} \longrightarrow K[[t]],$$

where K is now the residue field of $\mathcal{O}_{X_r,\xi_r}/\mathcal{Q}$ at $\mu(\xi_r)$. We obtain

$$\varphi_{N,r} = \tilde{\varphi}_{N,r} \circ \mu : \mathcal{O}_{X_r,\xi_r} \longrightarrow K[[t]],$$

so that $\operatorname{Ker}(\bar{\varphi}^{(d)}_{N,r}) = \operatorname{Ker}(\varphi^{(d)}_{N,r})$, where

$$\varphi^{(d)}_{N,r} = \varphi_{N,r} \circ \beta^*_{X_r} : \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}} \longrightarrow K[[t]].$$

Note that $\operatorname{Ker}(\varphi^{(d)}_{N,r}) = (\tilde{y}_2 - \tilde{y}_1^N, \tilde{y}_2 - \tilde{y}_j : 2 < j \leq d)$, so

$$\operatorname{ord}_t(\varphi^{(d)}_{N,r}(\tilde{y}_2)) = \operatorname{ord}_t(\varphi^{(d)}_{N,r}(\tilde{y}_j)) = N \cdot \operatorname{ord}_t(\varphi^{(d)}_{N,r}(\tilde{y}_1))$$

for $2 < j \leq d$, and that

$$\operatorname{ord}(\varphi^{(d)}_{N,r}) = \operatorname{ord}_t(\varphi^{(d)}_{N,r}(\pi^*(\mathcal{M}_{\xi^{(d)}}))) = \operatorname{ord}_t(\varphi^{(d)}_{N,r}(\tilde{y}_1^a)) = a \cdot \operatorname{ord}_t(\varphi^{(d)}_{N,r}(\tilde{y}_1)),$$

so necessarily

$$\frac{\operatorname{ord}_t(\varphi^{(d)}_{N,r}(I(C'_r)))}{\operatorname{ord}_t(\varphi^{(d)}_{N,r}(\pi^*(\mathcal{M}_{\xi^{(d)}})))} = \left(\frac{1}{a}\right) \frac{\min_{j=2,\dots,d}\left\{\operatorname{ord}_t(\varphi^{(d)}_{N,r}(\tilde{y}_j))\right\}}{\min_{i=1,\dots,d}\left\{\operatorname{ord}_t(\varphi^{(d)}_{N,r}(\tilde{y}_i))\right\}} = \frac{N}{a}. \tag{5.4}$$

Finally, we obtain

$$\varphi_N : \mathcal{O}_{X,\xi} \longrightarrow K[[t]]$$

by composing $\varphi_{N,r} \circ \pi_X^*$, and we also obtain its projection to $V^{(d)}$ as

$$\varphi_N^{(d)} = \varphi_{N,r}^{(d)} \circ \pi^*.$$

Note that the sequence of transformations in (5.3) is such that the multiplicity of X_i along the curve does not decrease along the process, and hence

$$C_i' \subset \beta_{X_i}(\underline{\text{Max}}\ \text{mult}(X_i))$$

for $i = 0, \ldots, r$. As a consequence, it induces a sequence of permissible transformations of Rees algebras for $\mathcal{G}_X^{(d)}$ as in [80, Definition 6.1], since for all $i = 1, \ldots, r$, π_i is a blow up at a regular closed subset of $\text{Sing}(\mathcal{G}_{X,i-1}^{(d)})$:

$$\begin{array}{ccccccccc} V^{(d)} & = & V_0^{(d)} & \xleftarrow{\pi_1} & V_1^{(d)} & \xleftarrow{\pi_2} \cdots & \xleftarrow{\pi_r} & V_r^{(d)} \\ \mathcal{G}_X^{(d)} & = & \mathcal{G}_{X,0}^{(d)} = \oplus_{i\geq 0} I_i W^i & \longleftarrow & \mathcal{G}_{X,1}^{(d)} & \longleftarrow \cdots & \longleftarrow & \mathcal{G}_{X,r}^{(d)} = \oplus_{i\geq 0} I_{i,r} W^i \end{array} \tag{5.5}$$

where

$$I_i \mathcal{O}_{V_r^{(d)}} \subset I_{i,r}$$

for $i \geq 0$ (see (1.5)). In particular,

$$\mathcal{G}_X^{(d)} \mathcal{O}_{V_r^{(d)}} = \oplus_{i\geq 0} (I_i \mathcal{O}_{V_r^{(d)}}) W^i \subset \oplus_{i\geq 0} I_{i,r} W^i.$$

Moreover,

$$\varphi_N^{(d)}(\mathcal{G}_X^{(d)}) = \varphi_{N,r}^{(d)}(\oplus_{i\geq 0}(I_i \mathcal{O}_{V_r^{(d)}})W^i) \subset \varphi_{N,r}^{(d)}(\mathcal{G}_{X,r}^{(d)}),$$

so

$$\text{ord}_t(\varphi_N^{(d)}(\mathcal{G}_X^{(d)})) \geq \text{ord}_t(\varphi_{N,r}^{(d)}(\mathcal{G}_{X,r}^{(d)})).$$

Since $I(C_r')$ is a regular prime in $\mathcal{O}_{V_r^{(d)},\xi_r^{(d)}}$ defining a curve contained in $\text{Sing}(\mathcal{G}_{X,r}^{(d)})$,

$$\mathcal{G}_{X,r}^{(d)} \subset \mathcal{O}_{V_r^{(d)},\xi_r^{(d)}}[I(C_r')W],$$

and hence

$$\text{ord}_t(\varphi_N^{(d)}(\mathcal{G}_X^{(d)})) \geq \text{ord}_t(\varphi_{N,r}^{(d)}(\mathcal{G}_{X,r}^{(d)})) \geq \text{ord}_t(\varphi_{N,r}^{(d)}(I(C_r'))). \tag{5.6}$$

On the other hand,

$$\text{ord}(\varphi_N) = \text{ord}_t(\varphi_N(\mathcal{M}_\xi)) \leq \text{ord}_t(\varphi_N^{(d)}(\mathcal{M}_{\xi^{(d)}})) = \text{ord}(\varphi_N^{(d)}) = \text{ord}_t(\varphi_{N,r}^{(d)}(\pi^*(\mathcal{M}_{\xi^{(d)}}))).$$

This, together with Lemma 5.0.2, (5.4), and (5.6) implies, for each $N \in \mathbb{Z}_{>0}$,

$$\bar{r}_{X,\varphi_N} = \frac{\text{ord}_t(\varphi_N(\mathcal{G}_X^{(n)}))}{\text{ord}(\varphi_N)} \geq \frac{\text{ord}_t(\varphi_N^{(d)}(\mathcal{G}_X^{(d)}))}{\text{ord}(\varphi_N^{(d)})} \geq \frac{\text{ord}_t(\varphi_{N,r}^{(d)}(I(C_r')))}{\text{ord}_t(\varphi_{N,r}^{(d)}(\pi^*(\mathcal{M}_{\xi^{(d)}})))} = \frac{N}{a}.$$

Again, it is clear that for a fixed $q \in \mathbb{Q}$, we may choose N such that $\bar{r}_{X,\varphi_N} > q$. $\square$

As was stated in the beginning of the section, Theorem 5.0.1 means, in terms of the Nash multiplicity sequence, that ξ is an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ if and only if there exists an upper bound (not depending on the arc) for the number of blow ups directed by any arc φ in X through ξ which are needed before the Nash multiplicity sequence decreases for the first time (normalized by the order of φ):

Corollary 5.2.2. *Let X be a variety over a field k of characteristic zero. A point $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$ is an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ if and only if*

$$\sup_{\varphi}\left\{\frac{\rho_{X,\varphi}}{\mathrm{ord}(\varphi)}\right\} < \infty,$$

where the supremum is taken over all arcs φ in X through ξ.

Proof. The direct implication follows from Corollary 4.3.3. For the reverse one, assume that $\sup_{\varphi}\left\{\frac{\rho_{X,\varphi}}{\mathrm{ord}(\varphi)}\right\} = q \in \mathbb{Q}_{>0}$ and get to a contradiction: for $N = [q]+1 > q$, choose φ_{aN} as in the proof of Proposition 5.2.1, so that it satisfies $\bar{r}_{X,\varphi_{aN}} \geq N$. This implies

$$\rho_{X,\varphi_{aN}} = [r_{X,\varphi_{aN}}] \geq [N \cdot \mathrm{ord}(\varphi_{aN})] = N \cdot \mathrm{ord}(\varphi_{aN}).$$

But this is equivalent to

$$\frac{\rho_{X,\varphi_{aN}}}{\mathrm{ord}(\varphi_{aN})} \geq \frac{N \cdot \mathrm{ord}(\varphi_{aN})}{\mathrm{ord}(\varphi_{aN})} = N > q,$$

yielding a contradiction. □

5.3 One particular case

Assume now that $\tau_{\mathcal{G}_X^{(n)},\xi} = n-1$. Then, for some regular system of parameters $\{x'_1,\ldots,x'_{n-1},z\} \subset R = \mathcal{O}_{V^{(n)},\xi}$ for $\mathcal{G}_X^{(n)}$ differentially closed representing the multiplicity of X at ξ, we have

$$x'_1W,\ldots,x'_{n-1}W \subset \mathcal{G}_X^{(n)},$$

and one can find an admissible projection $V^{(n)} \xrightarrow{\beta_X^{(1)}} V^{(1)}$ and an elimination algebra $\mathcal{G}_X^{(1)}$. We may assume that, up to an étale extension, $R = S'[x'_1,\ldots,x'_{n-1}]$, where S' is a regular ring of dimension 1. Then

$$\mathcal{G}_X^{(n)} = R[x'_1W] \odot \ldots \odot R[x'_{n-1}W] \odot \mathcal{G}_X^{(1)} \tag{5.7}$$

where $\mathcal{G}_X^{(1)} \subset S'[W]$. Note that, in this situation:

$$\mathrm{ord}_{\xi}(\mathcal{G}_X^{(n)}) = 1 = \mathrm{ord}_{\xi^{(n-1)}}(\mathcal{G}_X^{(n-1)}) = \ldots = \mathrm{ord}_{\xi^{(2)}}(\mathcal{G}_X^{(2)}) < \mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}),$$

so $\mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)})$ is the first interesting resolution invariant in this case.

Under these hypotheses ξ is an isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, and hence Proposition 5.1.1 guarantees that $\Phi_{X,\xi}$ is upper bounded. It turns out that the additional condition on $\tau_{\mathcal{G}_X^{(n)},\xi}$ yields an improvement of that result:

Proposition 5.3.1. *If* $\tau_{\mathcal{G}_X^{(n)},\xi} = n-1$*, then for any arc* φ *in* X *through* ξ*:*

$$\bar{r}_{X,\varphi} \leq \mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}),$$

and this bound is sharp.

Proof. We may assume first that $\min_{i=1,\dots,n-1}\{\mathrm{ord}_t(\varphi(x'_i))\} = \mathrm{ord}_t(\varphi(x'_1))$. By (5.7), we obtain

$$r_{X,\varphi} \leq \min\left\{\mathrm{ord}_t(\varphi(x'_1)), \mathrm{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))\right\}, \tag{5.8}$$

where $\varphi^{(1)}$ is the projection of φ via the elimination map $\beta_X^{(1)} : \mathrm{Spec}(R) \longrightarrow \mathrm{Spec}(S')$. Note that, either $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(x'_1))$ or $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(z))$. In the first case,

$$1 \leq \bar{r}_{X,\varphi} \leq \frac{\min\left\{\mathrm{ord}_t(\varphi(x'_1)), \mathrm{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))\right\}}{\mathrm{ord}_t(\varphi(x'_1))} \leq 1,$$

which implies that

$$\bar{r}_{X,\varphi} = 1 < \mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}).$$

In the second case,

$$\bar{r}_{X,\varphi} \leq \frac{\mathrm{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))}{\mathrm{ord}_t(\varphi(z))}.$$

Note that $\mathrm{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)})) \geq \mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) \cdot \mathrm{ord}_t(\varphi(z))$ (see Lemma 4.1.5). But actually this inequiality is an equality here. This follows from the fact that $\mathcal{G}_X^{(1)} \subset S'[W]$ so, for all $gW^l \in \mathcal{G}_X^{(1)}$, we have that $\mathrm{ord}_t(\varphi(g)) = \mathrm{ord}_z(g)\cdot\mathrm{ord}_t(\varphi(z))$. One only needs to observe now that $\varphi^{(1)}(\mathcal{G}_X^{(1)}) = K[[t]][\varphi(g)W^l : gW^l \in \mathcal{G}_X^{(1)}]$, and the equality is clear. Hence

$$\bar{r}_{X,\varphi} \leq \frac{\mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) \cdot \mathrm{ord}_t(\varphi(z))}{\mathrm{ord}_t(\varphi(z))} = \mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}).$$

To see that $\mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)})$ is a sharp bound, consider an r.s.p.

$$\{x_1, \dots, x_{n-d}, z_1, \dots, z_{d-1}, z_d\} \subset R.$$

Since $\tau_{\mathcal{G}_X^{(n)},\xi} = n-1$, we may assume that

$$x_1W, \dots, x_{n-d}W, z_1W, \dots, z_{d-1}W \in \mathcal{G}_X^{(n)}.$$

We may choose an arc $\bar{\varphi}^{(d)}$ in $V^{(d)}$ through $\beta_X(\xi)$ such that $\bar{\varphi}^{(d)}(z_d) = t$ and $\bar{\varphi}^{(d)}(z_1) = \ldots = \bar{\varphi}^{(d)}(z_{d-1}) = t^a$, for some $a \in \mathbb{Z}_{>0}$, $a > \mathrm{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) > 1$. This arc can be lifted to an arc φ in X through ξ, for which

$$\bar{r}_{X,\varphi} = \frac{\mathrm{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)}))}{\mathrm{ord}(\varphi)} \geq \frac{\mathrm{ord}_t(\varphi^{(d)}(\mathcal{G}_X^{(d)}))}{\mathrm{ord}(\varphi^{(d)})} =$$

$$= \frac{\min\left\{\mathrm{ord}_t(\varphi^{(d)}(z_1)), \dots, \mathrm{ord}_t(\varphi^{(d)}(z_{d-1})), \mathrm{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))\right\}}{\mathrm{ord}(\varphi^{(d)})}$$

by Lemma 5.0.2 and (4.1), where

$$\varphi^{(d)} = \varphi \circ \beta_X^* \text{ and } \varphi^{(1)} = \varphi \circ (\beta_X^{(1)})^*.$$

Also,

$$\operatorname{Ker}(\varphi^{(d)}) = \operatorname{Ker}(\bar{\varphi}^{(d)}) = (z_d^a - z_1, \ldots, z_d^a - z_{d-1}),$$

so for $i = 1, \ldots, d-1$ it is clear that

$$\operatorname{ord}_t(\varphi^{(d)}(z_i)) = a \cdot \operatorname{ord}_t(\varphi^{(d)}(z_d)) >$$
$$\operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) \cdot \operatorname{ord}_t(\varphi^{(d)}(z_d)) = \operatorname{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)})) > \operatorname{ord}_t(\varphi^{(d)}(z_d)).$$

Thus,

$$\operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) \geq \bar{r}_{X,\varphi} \geq \frac{\operatorname{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))}{\operatorname{ord}_t(\varphi^{(d)}(z_d))} = \operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}).$$

□

However, under the hypothesis of Proposition 5.3.1, sometimes it is possible to find arcs such that $\operatorname{ord}_\xi(\mathcal{G}_X^{(d)}) = 1 < \bar{r}_{X,\varphi} < \operatorname{ord}_\xi(\mathcal{G}_X^{(1)})$. Let us show an example for this:

Example 5.3.2. Consider $X \hookrightarrow \operatorname{Spec}(k[x,y,z])$ defined by the equation $f = xy - z^5$ and $\xi = (0,0,0) = \underline{\operatorname{Max}}\operatorname{mult}(X)$, and let φ be the arc defined by $\varphi(x) = t^3$, $\varphi(y) = t^2$, $\varphi(z) = t$. Here

$$\mathcal{G}_X^{(3)} = \operatorname{Diff}(k[x,y,z][fW^2]) = k[x][xW] \odot \mathcal{G}_X^{(2)} = k[x,y][xW, yW] \odot \mathcal{G}_X^{(1)},$$

where $\mathcal{G}_X^{(2)} = k[y,z][yW, z^5W^2, z^4W]$ and $\mathcal{G}_X^{(1)} = k[z][z^5W^2, z^4W]$, so $\operatorname{ord}_{\xi^{(2)}}(\mathcal{G}_X^{(2)}) = 1$ and $\operatorname{ord}_{\xi^{(1)}}(\mathcal{G}_X^{(1)}) = 5/2$. Note that $\operatorname{ord}(\varphi) = \operatorname{ord}_t(\varphi(z)) = 1$. On the other hand,

$$r_{X,\varphi} = \operatorname{ord}_t(\varphi(\mathcal{G}_X^{(3)})) = \operatorname{ord}_t(\varphi^{(2)}(\mathcal{G}_X^{(2)})) =$$
$$= \min\left\{\operatorname{ord}_t(\varphi^{(2)}(y)), \operatorname{ord}_t(\varphi^{(1)}(\mathcal{G}_X^{(1)}))\right\} = \min\{2, 5/2\} = 2.$$

Hence, for this example $1 < \bar{r}_{X,\varphi} = 2 < 5/2$.

5.4 Examples

Let us end our discussion with a couple of illustrative examples for Propositions 5.1.1 and 5.2.1 respectively. The first one shows an isolated point of $\underline{\operatorname{Max}}\operatorname{mult}(X)$ for which $\Phi_{X,\xi}$ is upper bounded by 3:

Example 5.4.1. Let $X = \left\{x^2y^3 - z^3s^4 = 0\right\} \hookrightarrow \operatorname{Spec}(k[x,y,z,s])$ and let $\xi = (0,0,0,0) = \underline{\operatorname{Max}}\operatorname{mult}(X)$. We have

$$\mathcal{G}_X^{(4)} = \operatorname{Diff}(k[x,y,z,s][(x^2y^3 - z^3s^4)W^5]) =$$
$$= k[x,y,z,s][xW, yW, zsW, z^3W, s^2W, z^3sW^2, zs^2W^2,$$
$$zs^4W^3, z^2s^3W^3, z^3s^2W^3, z^3s^3W^4, z^3s^4W^5].$$

Observe now that $xW, yW, z^3W, s^2W \in \mathcal{G}_X^{(4)}$, and $\varphi(x)W, \varphi(y)W, \varphi(z)^3W, \varphi(s)^2W \in \varphi(\mathcal{G}_X^{(4)})$. Then

$$r_{X,\varphi} \leq \min\left\{\mathrm{ord}_t(\varphi(x)), \mathrm{ord}_t(\varphi(y)), 3 \cdot \mathrm{ord}_t(\varphi(z)), 2 \cdot \mathrm{ord}_t(\varphi(s))\right\}.$$

If $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(x))$ or $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(y))$, then $\bar{r}_{X,\varphi} = 1$. If $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(z))$, then $\bar{r}_{X,\varphi} \leq 3$, and if $\mathrm{ord}(\varphi) = \mathrm{ord}_t(\varphi(s))$, then $\bar{r}_{X,\varphi} \leq 2$. In any case

$$\bar{r}_{X,\varphi} \leq 3.$$

In the next example we construct, for a non isolated point of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, a family of arcs φ_N, $N \in \mathbb{Z}_{>0}$, for which $\bar{r}_{X,\varphi_N}$ equals a polynomial in N, namely $q(N) = N+2$, showing that $\Phi_{X,\xi}$ is not upper bounded:

Example 5.4.2. Let now $X = \left\{x^2y^3 - z^4s^5 = 0\right\}$, and let $\xi = (0,0,0,0)$ again. Now $\xi \subsetneq \underline{\mathrm{Max}}\,\mathrm{mult}(X)$. In this case,

$$\begin{aligned} \mathcal{G}_X^{(4)} &= \mathrm{Diff}(k[x,y,z,s][(x^2y^3 - z^4s^5)W^5]) = \\ &= k[x,y,z,s][xW, yW, zsW, s^5W, zs^5W^2, z^2s^5W^3, z^3s^5W^4, z^4s^5W^5]. \end{aligned}$$

Consider the following family of arcs through ξ parametrized by $N \in \mathbb{Z}_{>0}$:

$$\begin{aligned} \varphi_N : k[x,y,z,s]/(x^2y^3 - z^4s^5) &\longrightarrow K[[t]] \\ x &\longmapsto t^{2N+2}, \\ y &\longmapsto t^{2N+5}, \\ z &\longmapsto t, \\ s &\longmapsto t^{2N+3}. \end{aligned}$$

Now

$$\varphi(\mathcal{G}_X^{(4)}) = K[[t]][t^{2N+2}W]$$

and $\mathrm{ord}(\varphi_N) = 1$, so

$$\bar{r}_{X,\varphi} = 2N+2,$$

which grows with N.

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...flota
mientras tanto esta nota en algún pentagrama leve
y al compás de ese breve sonido un planeta gira
y una planta respira y el aire caliente sube
y el vapor de una nube destila una gota
que oscila un instante reacia...

J. Drexler

Reinventem les ruines velles i evitem tornar a perdre
Perqué caiguen els gegants
Invertim la gravetat

Ara eres tu qui canviarà la història
Eres la llum que brillarà en aquell parc
Ets com el fum entre la boira

Eres la veu que frena aquesta nòria
La flama que incèndia el meu cos
És el moment d'on neix l'eufòria... salvatge!

Smoking Souls