Universidad Autónoma de Madrid
Facultad de Ciencias
Departamento de Matemáticas

An integral representation and pointwise inequality for the fractional Laplace-Beltrami operator

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THESIS: AN INTEGRAL REPRESENTATION AND POINTWISE INEQUALITY FOR THE FRACTIONAL LAPLACE-BELTRAMI OPERATOR

ÁNGEL D. MARTÍNEZ

ABSTRACT: En esta tesis se presentan resultados relativos al operador fraccionario, definido espectralmente, sobre las funciones de una variedad riemanniana M. Los resultados obtenidos son herramientas que permiten probar teoremas sobre regularidad de soluciones de la ecuación cuasigeostrófica superficial (SQG) en la esfera dos dimensional. La motivación del estudio de dicha ecuación proviene de la meteorología y de sus similitudes analíticas con la ecuación de Euler tres dimensional. La tesis comienza con dos capítulos introductorios (castellano e inglés, respectivamente) en los que se describen los resultados expuestos y sus aplicaciones. Concluimos este resumen indicando brevemente el contenido del resto:

Chapter 3: se prueba un resultado fundamental para el operador de Dirichlet-Neumann, una desigualdad puntual que extiende la de Córdoba-Córdoba. La prueba se basa en una aplicación del lema de Hopf.

Chapter 4: se extiende la desigualdad de Córdoba-Córdoba (conocida en el espacio euclídeo con anterioridad) a variedades compactas arbitrarias mediante tcnicas de subordinacin de operadores.

Chapter 5: este capítulo trata el operador de Laplace-Beltrami y, en concreto, cmo ciertas cotas de las normas L^p de autofunciones están relacionadas con un problema de restricción a escalas pequeas para ciertas variedades que incluyen las superficies compactas de curvatura constante.

Chapter 6: se prueba una útil representación integral para el operador fraccionario en variedades.

INSTITUTO DE CIENCIAS MATEMÁTICAS (CSIC-UAM-UC3M-UCM) – DEPARTAMENTO DE MATEMÁTICAS (UNIVERSIDAD AUTÓNOMA DE MADRID), 28049 MADRID, SPAIN

E-mail address: angel.martinez@icmat.es

To my family
(whatever that means)

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Chapter 1

Introducción

1.1 Preámbulo

En este capítulo introductorio describimos los resultados que conforman esta tesis doctoral. Para ello hemos intentado motivar de dónde procedían nuestras aspiraciones a la hora de abordarlos, ubicándolos por tanto en un contexto y describiéndolos con precisión. Sin embargo, cada capítulo posee una introducción propia adecuada al mismo y algo más especializada. Las posibles redundancias que se derivan de esta decisión se han reducido en la medida de lo posible. También hemos incluido en lugar de apéndices discusiones sobre resultados que usamos y creemos no pertenecen al canon matemático actual. Finalmente, los capítulos pueden o no tener interrelaciones pero hemos tratado de hacerlos independientes.

1.2 El laplaciano fraccionario

En muchos modelos de la física matemática aparece el laplaciano, que en el espacio euclídeo $\mathbb{R}^n$ tiene la expresión

$$\Delta = \partial_1^2 + \cdots + \partial_n^2.$$

Este operador está presente en la teoría del potencial clásica y del calor. El flujo del calor tiene relación con cierto proceso estocástico: el movimiento browniano. Sin embargo, y por distintas razones, también se han estudiado los operadores fraccionarios $(-\Delta)^\alpha$ con $\alpha \in (0,1)$. En el campo de las ecuaciones diferenciales parciales los operadores fracionarios están relacionados con problemas finos sobre existencia o unicidad de soluciones a ecuaciones con disipación crítica en espacios de Sobolev. También son generadores infinitesimales de una familia de procesos estocásticos conocida como vuelos de Lévy. Desde el punto de vista más abstracto son un semigrupo interesante.

Otra conexión con estos operadores es la siguiente. Consideremos el operador de Dirichlet-Neumann, DN, que envía toda función razonable f (e.g. suave y con soporte compacto) a otra función g mediante la siguiente regla: dada f se considera el problema de Dirichlet

$$\begin{cases} \Delta u(x,y) = 0 & \text{si } (x,y) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x) & \text{si } x \in \mathbb{R}^n. \end{cases}$$

Por definición este operador manda el dato del problema de Dirichlet, f, al dato Neumann, $g = \partial_\nu u|_{\mathbb{R}^n \times \{0\}}$ (cf. sección 3.3 y 4.1). Es conocido (un sencillo ejercicio con la transformada de Fourier) que este operador concide con el laplaciano fraccionario $(-\Delta)^{\frac{1}{2}}$. Caffarelli y Silvestre probaron que, de hecho, existe para cada α un problema similar para el que la anterior interpretación del laplaciano fraccionaro en el espacio euclídeo es posible. De hecho no sólo el problema elíptico ha de cambiarse sino que la derivada normal se ha de sustituir por un límite adecuado.

Por otra parte es posible expresar estos operadores de la forma que sigue:

$$(-\Delta)^\alpha f(x) = c_{n,\alpha}\, \text{P.V.} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2\alpha}} dy,$$

siempre que f sea suficientemente regular y $0 < \alpha < 1$. La constante $c_{n,\alpha}$ resulta crucial para justificar el caso límite (cuando α se aproxima a 0). La fórmula integral se puede justificar empleando teoría de distribuciones o, más modestamente, análisis de Fourier e integración por partes (teorema de Gauss-Green-Stokes). Esto es posible porque $(-\Delta)^\alpha$ se corresponde, por definición, con el multiplicador $|\xi|^{2\alpha}$ que es una distribución homogénea.

1.3 Desigualdad puntual de Córdoba-Córdoba

La expresión integral anterior justifica que estos operadores sean no locales (salvo el propio laplaciano). Es decir, su valor puntual depende de los valores que toma la función en todo el espacio. Por eso es sorprendente que la siguiente desigualdad puntual generalice al caso no local:

$$f(x)(-\Delta)f(x) \geq \frac{1}{2}(-\Delta)(f^2)(x).$$

Ésta se puede justificar en este caso desarrollando el lado de la derecha como:

$$(-\Delta)(f^2)(x) = 2f(x)(-\Delta)f(x) - |\nabla f|^2(x)$$

y desestimando el último término, ya que tiene el signo adecuado.

En el caso fraccionario la observación se debe a A. Córdoba y D. Córdoba (cf. [23, 24]). Su prueba también es extremadamente simple: en este caso escribimos el lado de la izquierda, usando la expresión integral introducida, como sigue:

$$f(x)(-\Delta)^\alpha f(x) = c_{n,\alpha}\, f(x) \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2\alpha}} dy.$$

Empleando la identidad elemental:

$$X(X-Y) = \frac{1}{2}(X^2 - Y^2) + \frac{1}{2}(X-Y)^2$$

con $X = f(x)$ e $Y = f(y)$ se obtiene que lo anterior es igual a:

$$\frac{1}{2}\int_{\mathbb{R}^n} \frac{f(x)^2 - f(y)^2}{|x-y|^{n+2\alpha}} dy + \frac{1}{2}\int_{\mathbb{R}^n} \frac{(f(x) - f(y))^2}{|x-y|^{n+2\alpha}} dy$$

que, desestimando el último término que es positivo, es mayor que:

$$\frac{1}{2}\int_{\mathbb{R}^n} \frac{f(x)^2 - f(y)^2}{|x-y|^{n+2\alpha}} dy = \frac{1}{2}(-\Delta)^\alpha(f^2)(x)$$

donde la igualdad, de nuevo, no es otra cosa que la definición. La prueba en el caso del toro es similar. Esta sencilla prueba, presentada por A. Córdoba en un congreso celebrado en la universidad de Edimburgo, en el que también participaba E. M. Stein, les llevó a preguntarse si esto no sería verdad en mayor generalidad. A pesar de la simplicidad de los argumentos esgrimidos resulta claro que estamos empleando mucho conocimiento sobre el espacio euclídeo que no puede ser transferido con facilidad a otros operadores o espacios. Ésta era una de las dificultades a la hora de abordar generalizaciones de la desigualdad puntual. No obstante, la respuesta (positiva) llegó unos años más tarde. El siguiente resultado se presenta en el capítulo 4 de esta tesis doctoral

Teorema 1.1 (A. Córdoba, A. D. Martínez) *Sea (M, g) una variedad riemaniana compacta cuyo operador de Laplace-Beltrami asociado*

$$\Delta_g = div_g grad_g$$

permite definir espectralmente el operador fraccionario[1] *como el único operador lineal tal que $(-\Delta_g)^\alpha Y_k(x) = \lambda_k^{2\alpha} Y_k(x)$ donde Y_k es la k-ésima autofunción de $-\Delta_g$ con autovalor λ_k^2. Sea ϕ una función convexa, entonces*

$$\phi'(f(x))(-\Delta_g)^\alpha f(x) \geq (-\Delta_g)^\alpha(\phi(f))(x). \tag{1.3.1}$$

De la demostración se desprende que la desigualdad sería la contraria si ϕ fuera cóncava, sin embargo, en las aplicaciones se emplean funciones convexas, e.g. $\phi(x) = x^2$. Realicemos una digresión antes de proseguir describiendo los resultados. El operador de Laplace-Beltrami asociado a cierta métrica en una variedad es precisamente el operador que surge si uno deduce, por ejemplo, las ecuaciones del calor en un medio no homogéneo en el que las propiedades del material varian de punto a punto, o anisótropo, es decir, si varían según la dirección. Otra propiedad física relacionada es la conductancia que aparece, por ejemplo, en problemas inversos como el propuesto por Calderón. En este problema aparece el operador de Dirichlet-Neumann, del que tendremos ocasión de hablar de nuevo en un momento.

La prueba se basa en crear un problema auxiliar: un calor fraccionario. Para esto se precisa de cierta artillería: el teorema de Bernstein-Hausdorff-Widder para seguir una estrategia de subordinación introducida por Bochner. La idea, que no es nueva, surgió por analogía a la prueba del siguiente resultado que habíamos conseguido probar previamente:

Teorema 1.2 (A. Córdoba, A. D. Martínez) *Sea Ω un dominio $C^{1,\omega}$ en el espacio euclídeo, donde ω denota un módulo de continuidad de la clase de Dini (i.e. $\int_0^1 \frac{\omega(t)}{t} dt < \infty$). El operador de Dirichlet-Neumann DN satisface la desigualdad puntual*

$$f(x) DN(f)(x) \geq \frac{1}{2} DN(f^2)(x).$$

La prueba que publicamos aplica el principio de Hopf. Ésta y otras similares se pueden encontrar en el capítulo 3.

[1]Esta definición es intrínseca, no depende del espacio ambiente en el que podría estar inmersa la variedad en cuestión.

1.4 Representación integral

Sin embargo, para las aplicaciones que teníamos en mente no era suficiente la desigualdad anterior. En un trabajo conjunto con D. Alonso-Orán y A. Córdoba desempolvamos una de las ideas que habíamos desechado en el proceso: encontrar una representación integral similar a la clásica salvo por algún posible término de error admisible en las aplicaciones. Las ventajas de conseguir esta expresión incluyen probar desigualdades de Sobolev en el rango fraccionario (para los enteros ya lo había hecho Aubin en los setenta) y una mejora a la desigualdad debida a Constantin y Vicol. Sin embargo, ha de tenerse en cuenta que la desigualdad puntual que se obtiene con este método no es tan limpia debido, precisamente, a la presencia del término de error. Bajo las hipótesis descritas el resultado preciso es el siguiente:

Teorema 1.3 (D. Alonso-Orán, A. Córdoba, A. D. Martínez, [1]) *La siguiente representación integral es posible para toda función f suficientemente regular*

$$(-\Delta)^\alpha f(x) = c_{n,\alpha}\, P.V. \int_M \frac{f(x)-f(y)}{d_g(x,y)^{n+2\alpha}} k_N(x,y)\, dy + O(\|f\|_{H^{-N}(M)})$$

donde $d_g(x,y)$ denota la distancia geodésica en (M,g), el núcleo $k_N(x,y)$ está soportado en la diagonal y satisface

$$k_N(x,y) = (1 + O(d_g(x,y)))\chi(x,y).$$

El núcleo puede darse de forma semi explícita para cierta función suave χ, tal que se anula lejos de la diagonal. La prueba emplea la parametriz de Hadamard, el teorema del residuo de Cauchy y aparecen involucradas, por ejemplo, propiedades de las funciones de Bessel de segunda especie. En las aplicaciones es fundamental que el error suavice. Por otra parte, un resultado similar que no requiere la hipótesis de compacidad sobre la variedad parece ser posible. En ese caso el error es $O(\|f\|_\infty)$. La prueba utiliza la teoría de semigrupos y una parametriz para el núcleo del calor. Estos resultados se encuentran en el capítulo 6.

1.5 Aplicaciones, conclusión y resumen

Las desigualdades anteriormente expuestas resultan útiles en el estudio de ciertas ecuaciones con carácter no lineal y no local. En esta sección damos una breve descripción de algunas de las aplicaciones que, de hecho, motivaron su estudio.

Ecuaciones de transporte: algunos modelos de la física estudian cómo se desplaza cierta cantidad como puede ser la masa, el calor, etc. En el caso del calor además resulta natural combinar el desplazamiento con un efecto difusivo dependiente de las propiedades del material. Si el espacio ambiente está modelado por $\mathbb{R}^n$ la ecuación correspondiente tiene la forma

$$\begin{cases} \theta_t(x,t) + u(x,t)\cdot\nabla_x\theta(x,t) = -\kappa\Lambda^\alpha\theta(x,t) \\ \theta(x,0) = \theta_0(x) \end{cases}$$

donde el vector de velocidad u tiene divergencia cero, $\Lambda = (-\Delta)^{1/2}$ y κ denota una escalar dependiente del medio (e.g. difusividad térmica o viscosidad). La desigualdad puntual 1.3.1 es crucial para obtener el siguiente principio del máximo (cf. [23]):

$$\|\theta(\cdot,t)\|_p \leq \frac{\|\theta_0\|_p}{(1 + C\delta t\|\theta_0\|_p^{p\delta})^{1/p\delta}}$$

donde $\delta = \frac{\alpha}{2(p-1)}$, $C = C(\kappa, \alpha, \|\theta_0\|_1) > 0$ y $1 < p < \infty$.

Un caso especialmente relevante es el de la ecuación quasigeostrófica superficial en el que el campo de velocidades depende del escalar activo θ a través de las transformadas de Riesz de la siguiente forma $u = (-R_2\theta, R_1\theta)$. Este modelo mantiene cierto parentesco con las ecuaciones de Euler y Navier-Stokes. En [23] la desigualdad puntual se usa para probar que, bajo ciertas hipótesis, el problema de Cauchy anterior tiene solución para todo tiempo $t > 0$ siempre que el dato inicial θ_0 esté en el espacio de Sobolev $H^1(\mathbb{R}^2)$. También aparece, por ejemplo, en el trabajo de Caffarelli y Vasseur [16] donde los autores prueban resultados de regularidad para este tipo de ecuaciones en el espacio euclídeo $\mathbb{R}^n$ suponiendo que la difusión es crítica ($\alpha = 1/2$). Su resultado, en particular, implica el siguiente

Teorema 1.4 (L. Caffarelli, A. Vasseur) *Sea $\theta_0 \in L^2(\mathbb{R}^n)$ el dato inicial para el problema de Cauchy de la ecuación*

$$\partial_t + u\cdot\nabla\theta = -(-\Delta)^{1/2}\theta$$

donde $u = R^\perp\theta$. Entonces la solución $\theta(\cdot,t) \in C^\infty(\mathbb{R}^n)$ para todo $t > 0$.

Su intrincada prueba sigue los pasos que dió E. de Giorgi a la hora de resolver el decimonoveno problema de Hilbert a mediados del siglo XX.[2] No podemos dejar de mencionar el trabajo de Kiselev, Nazarov y Volberg [43] donde ya habían encontrado una prueba de un resultado ligeramente más débil mediante el estudio de la evolución del módulo de continuidad de la solución. Unos años más tarde una tercera forma de abordar esencialmente la misma cuestión fue descubierta por Constantin y Vicol [21] y retomada por Constantin, Vicol y Tarfulea [22]. Ésta aprovecha un principio del máximo no local que mejora la desigualdad puntual.

Evolución de interfases: la desigualdad 1.3.1 ha sido útil para estudiar problemas de frontera libre entre dos fluidos (e.g. water waves, el problema de Muskat o Hele-Shaw en un medio poroso). En estos casos hay una curva (en el plano) o una superficie (en el espacio euclídeo tridimensional) cuya evolución es estudiada. A partir de ciertas leyes fundamentales (ley de Bernoulli en el caso de water waves; ley de Darcy para medios porosos) uno puede asignar un campo de velocidades a la frontera móvil. Se obtiene así un sistema cerrado de ecuaciones diferenciales de cierta complejidad que no describiremos explícitamente aquí.

Sin embargo, pronto se observó que estos problemas están mal propuestos en la generalidad expuesta. En el caso del medio poroso, primero Rayleigh, y más tarde Taylor, observaron que la ecuación linearizada es inestable si cierta cantidad σ resulta ser negativa. Por tanto, para tener una teoría consistente se ha de controlar la evolución de esta cantidad, σ, asumiendo que es positiva al comienzo.

Eligiendo una parametrización adecuada (isoterma) de la frontera libre $\psi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ (donde $n = 1, 2$) el problema se reduce de forma natural a estimar la evolución de sus normas de Sobolev, esto es $\frac{d}{dt}\|\psi\|^2_{H^k}$. Resulta que tras ciertas manipulaciones algebraicas uno llega a la estimación:

$$\frac{d}{dt}\|\Lambda^k\psi\|^2_{L^2} \le -\sum_{j=1}^{n}\sum_{|\alpha|=k}\int \sigma(x,t)D^\alpha_x\psi_j(x,t)\Lambda D^\alpha_x\psi_j(x,t)\,dx + O(\|\Lambda^k\psi\|^p_{L^2})$$

para cierto $p > 0$. Una vez aquí la desigualdad 1.3.1 junto con la positividad del término de Rayleigh-Taylor, σ, permite controlar el único término

[2] De hecho, también fue resuelto por J. Nash en mayor generalidad. El problema original concierne ecuaciones elípticas mientras que J. Nash atacó directamente la ecuación parabólica con éxito.

realmente peligroso y, por tanto, probar que bajo éstas hipótesis la ecuación está bien propuesta. Los detalles se pueden encontrar en el artículo original [25].

Disipación crítica para la ecuación quasigeostrófica superficial en la esfera: como ya hemos mencionado Caffarelli y Vasseur probaron un teorema fortísimo en [16]. En trabajo realizado junto a D. Alonso-Orán y A. Córdoba [3] aplicamos la representación integral para probar un resultado existencia global de soluciones de la ecuación quasigeostrófica superficial en la esfera dos dimensional con la métrica g usual, i.e.

$$\partial_t \theta(x,t) + u(x,t) \cdot \nabla_g \theta(x,t) = -(-\Delta_g)^{1/2} \theta(x,t)$$

donde $x \in \mathbb{S}^2$ y u es un campo de velocidades dependiente de θ mediante la transformada de Riesz ortogonal $\nabla_g^{\perp}(-\Delta_g)^{1/2}$. La elección de esta ecuación no es casual. En efecto, la ecuación quasigeostrófica superficial proviene de modelos de la metereología y geofísica que tienen en cuenta la fuerza de coriolis. Sin embargo, se acaba modelando en un espacio ambiente que poco tiene que ver con la topología de la atmósfera. Esta dificultad extra junto con la no localidad de la difusión motivó el estudio de la anterior ecuación. Curiosamente nuestro estudio no se aplica a esferas que no tengan simetría o cuya dimensión sea superior a la que encontramos en la naturaleza.

Es aquí donde la representación integral tiene ventajas sobre la desigualdad puntual (cf. [23, 24]) ya que permite obtener cotas inferiores no lineales siguiendo la estrategia de Constantin y Vicol (cf. [21]), concretamente, uno puede probar desigualdades puntuales similares a

$$\nabla f(x) \cdot \Lambda^{\alpha} \nabla f(x) \geq \frac{1}{2} |\nabla f(x)|^2 + \frac{|\nabla f(x)|^{2+\alpha}}{c \|f\|^{\alpha}_{L^{\infty}(\mathbb{R}^n)}}.$$

Éstas resultan especialmente útiles a la hora de abordar el problema de existencia global de soluciones para la ecuación quasigesotrófica superficial en $\mathbb{R}^n$. Nuestro trabajo permite emular su estrategia gracias (entre otras cosas) a las simetrías de la esfera, de forma que:

Teorema 1.5 (A. Córdoba, D. Alonso-Orán, A. D. Martínez, [3]) *Considérese $u = \nabla_g^{\perp} \Lambda^{-1} \theta$ un campo vectorial y un dato inicial $\theta_0 \in C^{\infty}(\mathbb{S}^2)$. Entonces la única solución global débil al problema de Cauchy para la ecuación quasigeostrófica superficial, θ, es suave, i.e. $\theta \in C^{\infty}([0,\infty) \times \mathbb{S}^2)$.*

La prueba amalgama las ideas de De Giorgi siguiendo el trabajo de Caffarelli y Vasseur con el principio del máximo no lineal proveniente del trabajo de Constantin y Vicol (cf. [16, 21]). El hecho de que no existan coordenadas globales en la esfera provoca que aparezcan fenómenos y dificultades que no se observaban en los trabajos anteriormente citados y que, sorprendentemente, pueden ser circunvaladas en el caso de la esfera dos dimensional. Este último hecho contrasta con la generalidad del resultado de Caffarelli y Vasseur (cf. teorema 1.4). Los detalles se pueden encontrar en [2, 3] y constituirán parte de la tesis doctoral de D. Alonso-Orán.

1.6 Autofunciones del laplaciano

Otro modelo de la física con un comportamiento diferente a los ya descritos de transporte o disipación es el de la ecuación de ondas

$$\partial_{tt} f = \Delta f$$

Ésta modela cómo se transmite, por ejemplo, el sonido en un medio físico con ciertas propiedades hipotéticas. Los métodos del análisis de Fourier vuelven a ser fundamentales para su estudio. Utilizando el método de separación de variables uno puede estudiar con qué frecuencia vibrará una membrana (como un tambor). De esta forma aparece la ecuación de Helmholtz

$$-\Delta_g \psi_\lambda = \lambda^2 \psi_\lambda$$

donde Δ_g es el operador de Laplace-Beltrami asociado a la superficie de la membrana con la métrica g adecuada. En este contexto ψ_λ se conoce como autofunción y λ^2 como autovalor de la misma. Lo mismo aplica si se considera, en lugar de una membrana o superficie, una variedad riemanniana. Si la variedad es compacta estos valores $\lambda > 0$ son discretos y se corresponden con la frecuencia de la vibración del armónico ψ_λ. Casos particulares son las exponenciales $e^{2\pi i\nu\cdot x}$ en el toro $\mathbb{T}^n$ o los armónicos esféricos. A pesar de que no podemos profundizar aquí en la relación de estos objetos con la física no podemos evitar mencionar que la hipótesis de de Broglie afirma que la frecuencia y la energía son proporcionales. Uno puede entrever en este hecho la cuantización de la energía y la relación con la mecánica cuántica.

Volviendo de esta digresión un problema interesante sería el de estudiar, dada una membrana, cómo de altas pueden ser las vibraciones del material cuando la energía crece (que se correspondería con el límite a la física clásica). En lugar de esto uno puede estudiar qué sucede para los modos de vibración básicos que resultan corresponerse con las autofunciones ψ_λ que, evidentemente, han de estar convenientementemente normalizadas. En otras palabras, el problema es equivalente al de encontrar una cota para el supremo $\|\psi_\lambda\|_\infty$ suponiendo que (la densidad de probabilidad) está normalizada (i.e. $\|\psi_\lambda\|_2 = 1$). En 1968 Hörmander probó la cota

$$\|\psi_\lambda\|_\infty \leq C(M)\lambda^{\frac{n-1}{2}}$$

para cualquier variedad compacta (M, g). Este resultado es óptimo, como muestran ciertos ejemplos de armónicos esféricos que se concentran en geodésicas. En el capítulo 5 mostramos trabajo inédito que surgió de un intento de probar dicha cota por métodos alternativos a los existentes en la literatura. En su lugar, conseguimos probar una relación entre ésa y otras similares a las existentes en los problemas de restricción abordados por Zygmund y Stein, entre otros. En particular conseguimos probar que la cota de Hörmander es equivalente a la cota

$$\|\psi_\lambda\|_{L^2(\gamma)} \leq C(M)\|\psi_\lambda\|_2$$

donde γ es la frontera de una bola de radio inversamente proporcional a λ. El método que seguimos es una combinación de principios del máximo, aproximaciones WKB para la ecuación de Euler-Poisson-Darboux en ciertas variedades y propiedades específicas de ciertas funciones de Bessel. Más detalles acerca de éste y otros problemas relacionados se pueden encontrar en la introducción del capítulo correspondiente.

Chapter 2

Introduction

2.1 Preamble

In this introductory chapter we describe the results that compose this PhD dissertation. We have tried to highlight where our motivation comes from, describing how these questions originated and stating them with some precision. Nonetheless, each chapter has its own specialized introduction. Some amount of redundancy derives from this choice which we have tried to minimize. Instead of appendices we have integrated appropiate discussions about alien results that are no longer standard mathematical knowledge. Finally, chapters may or may not have interconnections but we have tried to make them independent.

2.2 The fractional laplacian

The laplacian appears naturally in a number of models from mathematical physics, in the euclidean space $\mathbb{R}^n$ it can be expressed as

$$\Delta = \partial_1^2 + \cdots + \partial_n^2.$$

This operator appears in classical potential theory as well as in heat diffusion. The latter is related to an specific stochastic process known as brownian motion. Nevertheless, for different reasons, fractional counterparts of it, $(-\Delta)^\alpha$ with $\alpha \in (0,1)$, have been studied. These are related with fine properties of solutions to certain partial differential equations in the scale of

Sobolev spaces. They are also infinitesimal generators of another family of stochastic processes known as Lévy flights. From an abstract point of view they generate an interesting semigroup.

Another connection of these operators is the following. Consider the Dirichlet-Neumann operator, DN, defined as the operator that sends any reasonable function (e.g. smooth and compactly supported) to another function g in the following way: given f consider the Dirichlet problem

$$\begin{cases} \Delta u(x,y) = 0 & \text{si } (x,y) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x) & \text{si } x \in \mathbb{R}^n. \end{cases}$$

By definition, this operator sends the Dirichlet problem datum, f, to the Neumann datum, $g = \partial_\nu u|_{\mathbb{R}^n \times \{0\}}$ (cf. section 3.3 and 4.1). It is well known, a Fourier transform easy exercise, that this operator turns out to be the fractional laplacian $(-\Delta)^{\frac{1}{2}}$. Caffarelli and Silvestre proved that, in fact, for any α there is a similar problem so that the above interpretation of the fractional laplacian is possible. To do so one has to change the elliptic problem and the normal derivative has to be replaced by an adecuate limit.

On the other hand it is possible to express these operators as follows:

$$(-\Delta)^\alpha f(x) = c_{n,\alpha}\,\text{P.V.}\int_{\mathbb{R}^n} \frac{f(x)-f(y)}{|x-y|^{n+2\alpha}}dy,$$

whenever f is smooth enough and $0 < \alpha < 1$. The constant behaviour, $c_{n,\alpha}$, is crucial to justify the limiting case (when α is near 0). The integral formula can be proven using distributions theory or, equivalently, Fourier analysis and integration by parts (Gauss-Green-Stokes theorem). The calculations are possible due to the relation of $(-\Delta)^\alpha$ to certain homogeneous distributions; indeed, recall its Fourier multiplier is $|\xi|^{2\alpha}$.

2.3 The Córdoba-Córdoba pointwise inequality

Our previous integral representation justifies the non local nature of this family of operators (with the exception of the laplacian). Indeed, observe that its values depend on the values the function takes on the whole space. It is then surprising that the following pointwise inequality generalizes to the non local setting:

$$f(x)(-\Delta)f(x) \geq \frac{1}{2}(-\Delta)(f^2)(x).$$

In this case it can be justified developing the right hand side as follows:

$$(-\Delta)(f^2)(x) = 2f(x)(-\Delta)f(x) - |\nabla f|^2(x)$$

and neglecting the last term, since it has the proper sign.

In the non local case this was observed by A. Córdoba and D. Córdoba (cf. [23, 24]). Their proof is extremely simple too: in this case we can write the left hand side, using the integral representation, as

$$f(x)(-\Delta)^\alpha f(x) = c_{n,\alpha}\, f(x) \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x-y|^{n+2\alpha}} dy.$$

Taking advantage of the identity:

$$X(X - Y) = \frac{1}{2}(X^2 - Y^2) + \frac{1}{2}(X - Y)^2$$

with $X = f(x)$ and $Y = f(y)$, one obtains that the above equals:

$$\frac{1}{2} \int_{\mathbb{R}^n} \frac{f(x)^2 - f(y)^2}{|x-y|^{n+2\alpha}} dy + \frac{1}{2} \int_{\mathbb{R}^n} \frac{(f(x) - f(y))^2}{|x-y|^{n+2\alpha}} dy$$

neglecting the last term, which is positive, the above is bigger than:

$$\frac{1}{2} \int_{\mathbb{R}^n} \frac{f(x)^2 - f(y)^2}{|x-y|^{n+2\alpha}} dy = \frac{1}{2}(-\Delta)^\alpha(f^2)(x)$$

where the equality, again, is nothing but the integral representation. A similar proof works in the periodic case. This simple proof, exposed by A. Córdoba in a congress hosted by Edimburgh University, also attended by E. M. Stein, led them to raise the question of whether this could be true in greater generality. Despite the simplicity of the given proofs it is clear they employ a lot of knowledge about the euclidean space that can not be easily transfered to other operators or spaces. This was one of the difficulties we confronted. Nevertheless, the (positive) answer come a number of years later. The following result is presented in Chapter 4 of this dissertation.

Theorem 2.3.1 (A. Córdoba, A. D. Martínez) *Let (M, g) be a compact riemannian manifold whose Laplace-Beltrami operator*

$$\Delta_g = div_g grad_g$$

allows to define the fractional operator spectrally[1] *as the linear operator such that $(-\Delta_g)^\alpha Y_k(x) = \lambda_k^{2\alpha} Y_k(x)$ where Y_k denotes the kth eigenfunction of $-\Delta_g$ with eigenvalue λ_k^2. Let ϕ be a convex function, then*

$$\phi'(f(x))(-\Delta_g)^\alpha f(x) \geq (-\Delta_g)^\alpha(\phi(f))(x). \tag{2.3.1}$$

It will be evident from the proof that the reverse inequality would hold if ϕ is supposed to be concave. In the applications the functions one considers are convex, e.g. $\phi(x) = x^2$. Let us digress now. The Laplace-Beltrami operator associated to a certain riemannian metric on the manifold is precisely the operator that arises if one deduces, for example, the heat equation on a non homogeneous media or anisotropic (i.e. whose properties varies from point to point or the conduction depends on the direction, respectively). Another related physical quantity is conductance, which appears, for example, in some inverse problems such as the one proposed by A. Calderón. In the latter problem the Dirichlet-Neumann operator arises, of which we will also talk in a moment.

The proof proceeds creating an auxiliary problem: a fractional heat equation. To achieve this one needs some tools: the Bernstein-Hausdorff-Widder theorem to follow a subordination strategy first introduced by S. Bochner. This idea, which is not new, came out by analogy with a prior proof of the following result:

Theorem 2.3.2 (A. Córdoba, A. D. Martínez) *Let Ω be a $C^{1,\omega}$ domain in the euclidean space, where ω denotes a continuity modulus in the Dini class (i.e. $\int_0^1 \frac{\omega(t)}{t} dt < \infty$). The Dirichlet-Neumann operator, DN, satisfies the following pointwise inequality*

$$f(x)DN(f)(x) \geq \frac{1}{2} DN(f^2)(x).$$

The proof we published employs Hopf's lemma. This and other results can be found in Chapter 3.

[1] Notice this definition is intrinsic, it does not depend on the ambient space where the manifold might be living in.

2.4 Integral representation

These inequalities were not enough for the applications we had in mind. In a joint work with D. Alonso-Orán and A. Córdoba we revisited some ideas we rejected earlier on the process, namely: to find an integral representation similar to the classical one up to an admisible error term. Taking advantage of this expression we were able to prove Sobolev inequalities in the fractional range (the integral case was already done by Aubin in the seventies) and an improvement of the pointwise inequality due to Constantin and Vicol. Nevertheless, it should be stressed that the pointwise inequality one obtains with this method has different features due, precisely, to the presence of the error term. Under the above hypothesis the precise result is:

Teorema 2.6 (D. Alonso-Orán, A. Córdoba, A. D. Martínez, [1]) *The following integral representation holds for any function f smooth enough*

$$(-\Delta)^\alpha f(x) = c_{n,\alpha}\, P.V. \int_M \frac{f(x)-f(y)}{d_g(x,y)^{n+2\alpha}}\, k_N(x,y)\, dy + O(\|f\|_{H^{-N}(M)})$$

where $d_g(x,ky)$ denotes the geodesic dinstance in (M,g) and the kernel $k_N(x,y)$ supported on the diagonal satisfies

$$k_N(x,y) = (1+O(d_g(x,y))\chi(x,y).$$

Furthermore the kernel can be computed semiexplicitly for any fixed cut-off χ, that vanishes far from the diagonal. The proof employs the Hadamard parametrix, Cauchy's residue theorem and there are involved, for example, some properties of Bessel functions of the second species. The regularization effect of the error term is crucial in the applications. On the other hand, it seems possible to remove the compacity hypothesis on the manifold weakening the error term estimate to be $O(\|f\|_\infty)$. The proof in this case employs semigroup theory and a heat kernel parametrix. This results are exposed in Chapter 6.

2.5 Applications, conclusions and summary

The aforementioned inequalities are useful in the study of certain nonlinear nonlocal partial differential equations. In this section we provide a brief description of some of their applications that, in fact, motivated their study.

Transport equations: these equations model the displacement of a certain quantity such as mass, heat, etc. In the heat case it is also natural to combine the media displacement with some diffusive effect depending on its material properties. If the ambient space is modelled by $\mathbb{R}^n$ the corresponding equation has the form

$$\begin{cases} \theta_t(x,t) + u(x,t) \cdot \nabla_x \theta(x,t) = -\kappa \Lambda^\alpha \theta(x,t) \\ \theta(x,0) = \theta_0(x) \end{cases}$$

where the velocity vector u is divergence free, $\Lambda = (-\Delta)^{1/2}$ and κ denotes a scalar quantity depending on the media (e.g. thermic diffusivity or viscosity). The pointwise inequality 2.3.1 is crucial to obtain the following maximum principle (cf.[23]):

$$\|\theta(\cdot,t)\|_p \leq \frac{\|\theta_0\|_p}{(1 + C\delta t \|\theta_0\|_p^{p\delta})^{1/p\delta}}$$

where $\delta = \frac{\alpha}{2(p-1)}$, $C = C(\kappa, \alpha, \|\theta_0\|_1) > 0$ and $1 < p < \infty$.

Among the equations of this form a particularly relevant one is the surface quasigeostrophic equation for which the velocity field depends on the active scalar θ through Riesz transforms so that $u = (-R_2\theta, R_1\theta)$. This is a toy model related with the Euler and Navier-Stokes equations. In [23] the pointwise inequality is employed to prove that, under certain hypothesis, the previous Cauchy problem has a global solution whenever the initial datum θ_0 belongs to the Sobolev space $H^1(\mathbb{R}^2)$. It also appears, for instance, in the work of Caffarelli and Vasseur [16]. There the authors prove regularity results for this type of equations in the euclidean space $\mathbb{R}^n$, asuming critical diffusion ($\alpha = 1/2$). Their result, in particular, implies the following

Theorem 2.5.1 (L. Caffarelli, A. Vasseur) *Let $\theta_0 \in L^2(\mathbb{R}^n)$ be the initial datum for the Cauchy problem associated to the equation*

$$\partial_t + u \cdot \nabla\theta = -(-\Delta)^{1/2}\theta$$

where $u = R^\perp\theta$. Then the weak solution $\theta(\cdot,t) \in C^\infty(\mathbb{R}^n)$ for any $t > 0$.

Their intrincate proof relies on a tecnique introduced by E. de Giorgi to solve Hilbert's nineteenth problem in 20th midcentury.[2] Let us conclue

[2]In fact, it was solved independently by J. Nash in greater generality. The original problem belongs to the theory of elliptic equations while J. Nash tackles the parabolic case succesfully.

mentioning the previous work of Kiselev, Nazarov and Volberg [43] where a slightly weaker result was obtained through a careful, and very original, study of the evolution of the continuity modulus of a solution. A few years later a third approach was discoverd by Constantin and Vicol [21] and improved by Constantin, Vicol and Tarfulea [22]. These makes crucial use of a nonlocal maximum principle that improves the Córdoba-Córdoba pointwise inequality.

Interphase evolution: inequality 2.3.1 has been useful to study free boundary problems between two fluids (e.g water waves, Muskat's problem or Hele-Shaw in a porous media). In this cases the evolution of a curve (in the plane) or a surface (in three dimensional euclidean space) is studied. One can asign a velocity field to the moving boundary provided one asumes that certain fundamental laws hold (Bernoulli's law in the case of water waves or Darcy's law in the porous media). We refrain from describing the closed differential system of equations that arises in this way.

Soon it was observed that this problems are not wellposed in the exposed generality. Indeed, in the porous media case, first Rayleigh, and later Taylor, observed that the linearized equation was unstable if a certain quantity σ turns out to be negative. As a consequence, to obtain a consistent theory one needs to control how this quantity evolves asuming it is positive at the beginning.

Choosing an (isothermal) parametrization of the free boundary, for example $\psi : \mathbb{R}^n \to \mathbb{R}^{n+1}$ (where $n = 1, 2$), the problem reduces in a natural way to that of controlling some of its Sobolev norms, that is $\frac{d}{dt}\|\psi\|^2_{H^k}$. It turns out, after some algebraic manipulations, that

$$\frac{d}{dt}\|\Lambda^k\psi\|^2_{L^2} \leq -\sum_{j=1}^{n}\sum_{|\alpha|=k}\int \sigma(x,t)D^\alpha_x\psi_j(x,t)\Lambda D^\alpha_x\psi_j(x,t)\,dx + O(\|\Lambda^k\psi\|^p_{L^2})$$

for certain $p > 0$. Inequality 2.3.1 together with the positivity of the Rayleigh-Taylor term, σ, allows to control the only dangerous term and, therefore, to prove that under this hypothesis the problem is wellposed. Further details can be found in the original article [25].

Critical surface quasigeostrophic equation on the sphere: as we mentioned Caffarelli and Vasseur proved a strong theorem in [16]. In joint work with D. Alonso-Orán and A. Córdoba [3] we apply the integral representation to obtain a global regularity result for solutions to the surface quasigeostrophic equation on the standard two dimensional sphere, i.e.

$$\partial_t\theta(x,t) + u(x,t)\cdot\nabla_g\theta(x,t) = -(-\Delta_g)^{1/2}\theta(x,t)$$

where $x \in \mathbb{S}^2$ and u is a velocity field that depends on θ through an orthogonal Riesz transform $\nabla_g^{\perp}(-\Delta_g)^{1/2}$. The equation's choice is not arbitrary. Indeed, the surface quasigestrophic equations stem from metereology and geophysical science models that take into account the coriolis force. Despite of this, one usually models it over a plane or euclidean space which is a good approximation at small scales but does not take into consideration the topological shape of the atmosphere. This diffulty, together with the non locality of the diffusion, motivated our study which, surpisingly, does not apply to spheres without symmetry or whose dimension is higher than the one we find in Nature.

Our procedure takes advantage of the integral representation, instead of the pointwise inequality (cf. [23, 24]), which allows us to obtain nonlinear lower bounds following the strategy of Constantin and Vicol (cf. [21]), concretely, one can prove pointwise inequalities similar to

$$\nabla f(x)\cdot\Lambda^\alpha\nabla f(x) \geq \frac{1}{2}|\nabla f(x)|^2 + \frac{|\nabla f(x)|^{2+\alpha}}{c\|f\|^\alpha_{L^\infty(\mathbb{R}^n)}}.$$

As we mentioned, this turned out to be particularly useful in the study of global existence of solutions to the surface quasigesotrophic equation on $\mathbb{R}^n$. Our work emulates their strategy thank (among other things) to the symmetries of the standard sphere, ultimately we are able to show that:

Theorem 2.5.2 (A. Córdoba, D. Alonso-Orán, A. D. Martínez, [3]) *Consider $u = \nabla_g^{\perp}\Lambda^{-1}\theta$ a vector field and an initial datum $\theta_0 \in C^\infty(\mathbb{S}^2)$. Then the unique global weak solution to the Cauchy problem for the surface quasigeostrophic equation, θ, is smooth, i.e. $\theta \in C^\infty([0,\infty)\times\mathbb{S}^2)$.*

In the proof we merge ideas from De Giorgi, following the work of Caffarelli and Vasseur, with the nonlinear maximum principle as in the work of Constantin and Vicol (cf. [16, 21]). Since there are no global coordinates in the sphere some difficulties not present in the euclidean context arise. Surprisingly, these can be avoided in the two dimensional case which is in contrast with the generality of Caffarelli and Vasseur's result (cf. theorem 2.5.1). Details can be found in [2, 3] which will be a part of a PhD thesis defended by D. Alonso-Orán.

2.6 Laplace eigenfunctions

Another model from physics with a completely different behaviour to the ones already introduced, is the wave equation

$$\partial_{tt} f = \Delta f.$$

It models, for example, how sound is transmited in a physical medium that satisfies certain uniformity hypotheses. One can think on a string or a membrane (resembling a guitar or drum). Fourier analysis methods are again fundamental in the first case. On the other hand, the method of separation of variables allows to study at which frequency a membrane might vibrate. In this way Helmoltz equation arises naturally

$$-\Delta_g \psi_\lambda = \lambda^2 \psi_\lambda$$

where Δ_g is the Laplace-Beltrami operator associated to the membrane surface with the appropiate metric g. In this context ψ_λ is known as its eigenfunction and λ^2 eigenvalue. The same applies if one considers, instead of a membrane or surface, a riemannian manifold. If the manifold is supposed to be compact the possible values $\lambda > 0$ are discrete and correspond to the frequencies of the harmonics ψ_λ. Particular cases are the exponential $e^{2\pi i\nu\cdot x}$ in the tori $\mathbb{T}^n$ and the spherical harmonics. Lack of space and time prevents us from exploring the deep relations of this objects with further physical theories. Let us mention, though, that de Broglie's hypothesis establishes a proportionality between frequencies and energies. One can foresee in this fact the quantization of energy and its relation with quantum mechanics.

Coming back from this digression, an interesing problem would be to study, given a membrane, how much can the membrane vibrate when the energy grows (which would corresponds with classical physics in the limit). Instead of this one can study the harmonics or eigenfunctions, ψ_λ, conveniently normalized. In other words, the problem is equivalent to that of finding a bound for $\|\psi_\lambda\|_\infty$ asuming that (the probability density) is normalized (i.e. $\|\psi_\lambda\|_2 = 1$). In 1968 Hörmander proved that

$$\|\psi_\lambda\|_\infty \leq C(M)\lambda^{\frac{n-1}{2}}$$

holds for any compact manifold (M, g). This result is sharp, as it is shown by spherical harmonics concentrating along geodesics. In Chapter 5 we show some unpublished work, an outgrowth of an attempt to prove this bound by different methods to the ones available in the literature. Instead of its truth we showed that it is related to a problem similar to the restriction problems present in the work of A. Zygmund and E. M. Stein, among others. In particular, we prove that Hörmander's bound is equivalent to

$$\|\psi_\lambda\|_{L^2(\gamma)} \leq C(M)\|\psi_\lambda\|_2$$

where γ is the boundary of a ball of radius inversely proportional to λ. The method we follow is a combination of maximum principles, WKB approximation for the Euler-Poisson-Daroux equation on certain manifolds and specific properties of Bessel functions. More details about this and other related problems can be found in the specific introduction of the corresponding chapter.

Chapter 3

Pointwise inequality for the Dirichet-to-Neumann operator

In this chapter we prove with different methods and in different degree of generality the same pointwise inequality. We follow a labourious path, similar to our own. We prove virtually the same fact several times in different guises. We hope this might be instructive for the interested reader.

Fix a domain $\Omega \subseteq \mathbb{R}^n$ with $C^{1,\omega}$ boundary, where ω is some Dini modulus of continuity, i.e. it satisfies

$$\int_0^1 \frac{\omega(t)}{t} dt < \infty.$$

One may consider the Dirichlet-Neumann operator DN_Ω acting on smooth enough functions f in the boundary $\partial\Omega$. This operator arises in several models of physics and mathematical questions about them (e.g. Calderón problem). Let a general elliptic operator be of the form

$$Lu = \sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2}{\partial x_i \partial x_k} u + \sum_i b_i \frac{\partial}{\partial x_i} u$$

The ellipticity means that pointwise the matrix $(a_{ij})_{i,j}$ is positive definite. The functions a_{ik} are supposed to be sufficiently regular. Notice it has no linear term. This nuisance can be refined to cover a more general elliptic operator but it is of no interest to us since the Laplace-Beltrami operator has this form. To define the DN_Ω consider the Dirichlet problem

$$\begin{cases} Lu(x) = 0 & \text{for } x \in \Omega \\ u(x) = f(x) & \text{for } x \in \partial\Omega \end{cases}$$

where L is a Laplace-Beltrami operator for some metric. Then the Dirichlet-to-Neumann operator acts sending f to $\partial_\nu u|_{\partial\Omega}$. We are now in position to state the main result of this section

Theorem 3.0.1 *In the context described above the following pointwise inequality holds*

$$\frac{1}{2m} DN_\Omega(f^{2m})(x) \leq f(x)^{2m-1} DN_\Omega f(x)$$

for any positive integer $m \geq 1$.

We will prove different instances of this result, starting with simple proofs in special cases which become more elaborate and finally a rather simple general proof which covers all of them.

3.1 The n-dimensional ball

This was the first proof we found which was not already available in the literature. We are concerned with the Dirichlet-to-Neumann map on the unit ball B that sends f to $\partial_\nu u|_{S^{n-1}}$ where

$$\begin{cases} \Delta u(x) = 0 & \text{for any } x \in B \\ u(x) = f(x) & \text{for any } x \in \partial B \end{cases}$$

Notice $\partial B = S^{n-1}$ where $B \subseteq \mathbb{R}^n$ denotes the ball of radius one centered at the origin. We assume f is smooth enough. We take advantage of the following explicit Poisson kernel (cf. [68] p. 43, theorem 1.10)

$$p(s, x) = \frac{1 - |x|^2}{|x - s|^n}.$$

This kernel allows to express u, the solution to the Dirichlet problem above, as the integral

$$u(x) = \int_{S^{n-1}} f(s) p(s, x) d\sigma(s)$$

where $d\sigma$ denotes the normalized area measure on the $(n-1)$-dimensional sphere.

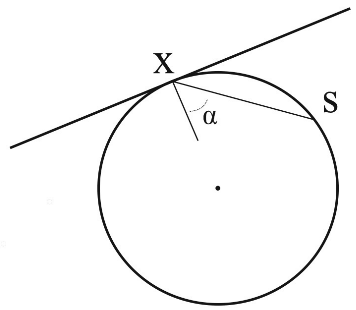


Figure 3.1: $\cos\alpha \geq 0$

As a consequence of this we can represent our Dirichlet-to-Neumann operator acting on f as follows

$$DN_B f(s') = \lim_{x\to s'} \int_{S^{n-1}} f(s) \frac{\partial}{\partial r} p(s,x) d\sigma(s)$$

where $|x| = r$ and the limit is radial, i.e. $r \to 1^-$ for $x = rs'$. Notice that

$$DN_B f(x) = \int_{S^{n-1}} f(s) \frac{\partial}{\partial r} p(s,x) d\sigma(s)$$

holds by definition for any $x \in B$ due to the imposed regularity on f, taking limits then as $x \to s' \in S^{n-1}$ one gets the above. The inequality

$$f(s') DN_B f(s') \geq \frac{1}{2} DN_B(f^2)(s')$$

is now equivalent to

$$\lim_{x\to s'} f(x)^2 \int_{S^{n-1}} k(s,x) d\sigma(s) \geq \lim_{x\to s'} \int_{S^{n-1}} (f(s) - f(x))^2 k(s,x) d\sigma(s)$$

where $k(s,x) = \partial_r p(s,x)$. Notice that for $x \in B$ the kernel

$$\int_{S^{n-1}} k(s,x) d\sigma(s) = \partial_\nu \left(\int_{S^{n-1}} p(s,x) d\sigma(s) \right) = \partial_\nu(1) = 0$$

by standard properties of the Poisson kernel. The limit in the right hand side can get inside the integral since we have extra cancellation killing the kernel's singularity due to the squared difference of the integrand. As a consequence one gets that our problem reduces to prove that the right hand side of the integral inequality is negative. This would be a consequence of $k(s, s') \le 0$ which follows easily from the explicit expression of the kernel provided above and the following observation

$$\frac{\partial}{\partial r}\frac{1}{|x-s|^n} = -n|x-s|^{-n-1}\langle\frac{x-s}{|x-s|}, \frac{x}{|x|}\rangle$$

which is negative, as a geometric argument shows.

3.2 Domains in the plane

In the previous section we proved by a rather explicit method that the pointwise inequality holds for balls in the n-dimensional euclidean space. One may then wish to reduce the study of this operator for simply connected domains to the case treated already by an appropiate change of variables, i.e. a diffeomorphism. This desideratum has some problems *a priori*. Indeed, one should be able to define a normal vector field to the surface, which requires some sort of regularity on the boundary. Furthermore, a diffeomorphism might change the equation to some other for which the Poisson kernel can not be explicitly computed as before.

The purpose of this section is to glimpse that in the two dimensional case this approach can lead to the following related result:

Theorem 3.2.1 *Let Ω be a simply connected domain in the plane whose boundary is a Jordan curve that can be parametrized by a function having Dini continuous derivatives. Let N denote a vector field parallel to the normal vector field, to be defined later, then the pointwise inequality*

$$\frac{1}{2}\partial_N(f^2)(x) \le f(x)\partial_N f(x)$$

holds.

The existence of a conformal map from a ball to the interior of Ω is a cornerstone of mathematical knowledge (cf. section 3.2.1 below). Conformality up to the boundary of Ω is possible (if one is willing to pay the price by imposing extra regularity on the boundary). We will explain these matters in the following sections. Before we proceed let us recall that the existence of a conformal map is known to be possible (only) in the two dimensional setting, in the realms of complex analysis (cf. Liouville's theorem, [18] p. 338 theorem 4). This section should be considered a glimpse of evidence for the pointwise inequalities to hold in rather more general situations, at least in the two dimensional case.

3.2.1 The Riemann mapping

In XIXth century Riemann proved, surely guided by strong physical intuition, the following remarkable

Theorem 3.2.2 *The interior of any simply connected region whose boundary contains more than one point can be mapped in a one-to-one conformal manner on the interior of the unit circle.*

The reader might find a proof in the classic book written by O. D. Kellog ([51], p. 367). Notice that the above assumes nothing about the regularity of the Jordan curve. The mapping provided by the above theorem, $\phi : \mathbb{D} \to \Omega$, is known as Riemann mapping in the literature. It has the remarkable property that

$$\Delta f = 0 \text{ if, and only if, } \Delta(f \circ \phi) = 0$$

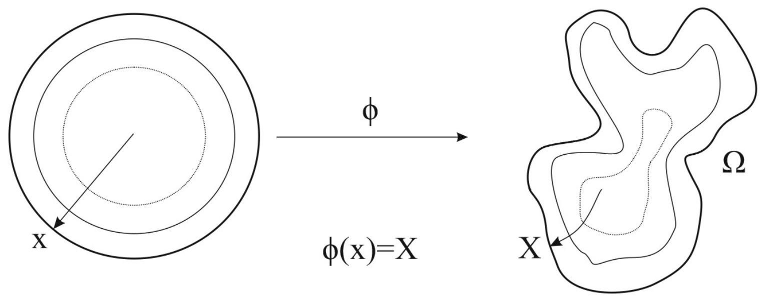
This solves one of our concerns but we still need to know something about the normal vector field to the boundary. The following theorem, due to O. D. Kellog himself, provides what we need

Theorem 3.2.3 (O. D. Kellog) *Consider a domain Ω bounded by a Jordan curve whose parametrization has Dini continuous first derivatives and let ϕ denote the corresponding Riemann mapping. Then ϕ' can be extended up to the boundary $\bar{\mathbb{D}}$ and*

$$\frac{\phi(\zeta) - \phi(z)}{\zeta - z} \to \phi'(z) \neq 0$$

for $\zeta \to z$ and $\zeta, z \in \bar{\mathbb{D}}$.

This is a refinement of a previous result, which deals with the smooth boundary case, obtained by Painlevé in 1888 (cf. [45]). For an account of the proof we refer the reader to [55] (theorem 3.5, p. 48) where they can find a proof of this result, and an extension of it due to Warschawski which involves higher derivatives. This theorem allows to define a normal vector field N (which might not be parametrized by arclength).



3.2.2 Proof of theorem 3.2.1

The theorem follows applying the Riemann mapping theorem to change coordinates to those of a two dimensional ball, so that the pointwise inequality is true if applied to $f \circ \phi$. The normal vector field N from the statement is taken to be the image of $\frac{\partial}{\partial r}$ by $d\phi$.

REMARK: if the distorsion near the boundary given by the Riemann mapping theorem in the normal direction does not degenerate one might even assert the same for the usual Dirichlet-to-Neumann operator.

3.3 Classical potential theory

Let us study the Dirichlet-to-Neumann operator for a $C^{1,\alpha}$ domain Ω. As in the previous section we will not be quite formal in the presentation of the results herein. This should be considered as an heuristic approximation to the result we stated in the introduction. The operator that we will denote as $DN_\Omega(f)$ asigns to any f defined on the boundary $\partial\Omega$ the outward normal derivative $\partial_\nu u$ of u solution of

$$\begin{cases} \Delta u(x) = 0 & \text{for any } x \in \Omega \\ u = f & \text{in } \partial\Omega \end{cases}$$

with certain decay at the infinity. This can be paraphrased in the following way: it sends the Dirichlet data to the Neumann data for the elliptic boundary problem satisfying the Laplace equation in the interior of the domain Ω.

A special case deserves some attention, namely

$$\Omega = \mathbb{H}^{n+1} = \{(x, y) : x \in \mathbb{R}^n, y > 0\}.$$

In this domain the Laplace equation $\partial_y^2 u(x,y) + \Delta u(x,y) = 0$, where Δ is understood as the Laplacian in the x variable, can be rewritten taking Fourier transform in the x variable as

$$\partial_y^2 \hat{u}(\xi,y) = |\xi|^2 \hat{u}(\xi,y)$$

from which one easily deduces that $\hat{u}(\xi,y) = e^{-y|\xi|}\hat{u}(\xi,0)$. Using this expression one observes that on the Fourier side

$$\partial_\nu \hat{u}(\xi,y)|_{y=0} = -\partial_y \hat{u}(\xi,y)|_{y=0} = |\xi|\hat{g}(\xi)$$

and the right hand side is precisely the Fourier multiplier of $(-\Delta)^{\frac{1}{2}}$. This establishes a way to link Dirichlet-to-Neumann operators with fractional laplacians. In fact, Caffarelli and Silvestre worked out a way to link any fractional laplacian on $\mathbb{R}^n$ and not just $\alpha = 1$ in a similar spirit (cf. section 4.1). It is thus quite natural to try to see what happens for similar operators in more general domains Ω than the halfspace.

First let us state the fact we want to discuss

Heuristic: Let $f \in C(\partial\Omega)$ and u satisfy

$$\begin{cases} \Delta u = 0 & \text{for } x \in \Omega \\ u = f & \text{for } x \in \partial\Omega \end{cases}$$

for some bounded $C^{1,\alpha}$ domain $\Omega \subset \mathbb{R}^n$, $\alpha > 0$. Then

$$f(x)DN(f)(x) \geq \frac{1}{2}DN(f^2)(x) + E(f)$$

pointwise.

Remark: the error E will be nice in a sense to be specified later but essentially it will be a smoother operator (in the sense that heuristically it gains derivatives). The philosophy of this result is that the error term should not present any trouble in the applications we have in mind (i.e. energy estimates). This is done in the sequel for rather general domains under certain regularity assumptions proving that, in fact, $E(f) = 0$.

The argument relies on the method of layer potentials (cf. chapter 3 of [52] or chapter XI of [51]). It can be proved (loc.cit.) that

$$u(x) = \mathcal{D}\left(\frac{1}{2}\mathrm{Id} + K\right)^{-1} f(x)$$

where $\mathcal{D}$ is the double layer potential

$$\mathcal{D}(\phi)(x) = \int_{\partial\Omega} \frac{-(x-y)\cdot\nu(y)}{\omega_n|x-y|^n}\phi(y)d\sigma(y)$$

for any $x \in \Omega$ and K is provided by the same formula but since its output belongs to $C(\partial\Omega)$ it should be defined for $x \in \partial\Omega$ (observe that this is necessary for the composition to make sense). It is known (loc. cit) that the operator K is compact, its spectra has modulus strictly less than $1/2$ and, as a consequence, the Fredholm theory or Neumann series allows to invert the expresion above.

In fact, if we are about to take limits as x tends to $\hat{x} \in \partial\Omega$ since

$$\int_{\partial\Omega} \frac{-(x-y)\cdot\nu(y)}{\omega_n|x-y|^n}d\sigma(y) = 1$$

we may write (it can be checked later that this limit makes sense):

$$\partial_\nu u(\hat{x}) = \lim_{x\to\hat{x}} \partial_\nu u(x) = \lim_{x\to\hat{x}} \partial_\nu \int_{\partial\Omega} \frac{-(x-y)\cdot\nu(y)}{\omega_n|x-y|^n}(\phi(y)-\phi(\hat{x}))d\sigma(y)$$

where the x in the limit approaches the boundary orthogonally and

$$\phi(x) = \left(\frac{1}{2}\mathrm{Id} + K\right)^{-1} f(x)$$

Since it is absolutely integrable by compactness of Ω and continuity of the integrand it is justified to interchange derivation and integrations so that the above equals

$$\lim_{x\to\hat{x}} \left(\int_{\partial\Omega} \frac{-\nu(x)\cdot\nu(y)}{\omega_n|x-y|^n}(\phi(y)-\phi(\hat{x})) \right.$$
$$\left. + \int_{\partial\Omega} \frac{-(\nu(y)\cdot(x-y))(\nu(x)\cdot(x-y))}{\omega_n|x-y|^{n+2}}(\phi(y)-\phi(\hat{x}))d\sigma(y) \right)$$

The second integral has the singularity around $\hat{x}$ of order (written in polar coordinates), it is roughly

$$\int_0^\varepsilon \frac{\rho^{1+\alpha}\rho^{1+\alpha}}{\rho^{n+2}}\rho\rho^{n-2}d\rho$$

hence $\int_0^\varepsilon \rho^{-1+2\alpha}d\rho$ which is integrable for any $\alpha > 0$. Let us state the (geometric) estimates employed above

Lemma 3.3.1 *If $\partial\Omega \in C^{1,\alpha}$ and $x, y \in \partial\Omega$ the following estimate holds true* $\nu(y) \cdot (x - y) = O(|x - y|^{1+\alpha})$.

A proof in the case of C^2 domains can be found in [52] (Ch. II 3.15). As a consequence by the dominated convergence one gets this equals it with $\hat{x}$ replacing x. Now the leading operator should be, heuristically, the first one of the Neumann series, which is proportional to f. The resulting integral might be localized cutting off around $\hat{x}$ in such a way that the inner product $\nu(x) \cdot \nu(y)$ inside the integration becomes positive (say, bigger than some fixed positive quantity) and as a consequence

$$\partial_\nu u(\hat{x}) = \lim_{x \to \hat{x}} \int_{N(\hat{x})} \frac{\nu(x) \cdot \nu(y)}{|x - y|^n} (f(y) - f(\hat{x})) d\sigma(y) + Ef(\hat{x})$$

where E collects a certain series which is expected to be a smoother. The argument would conclude applying the algebraic identity as in the Córdoba and Córdoba original proof. This argument has many defects though, at least, it shows the (pointwise) inequality to be plausible in any dimension provided we allow a (nice) error term.

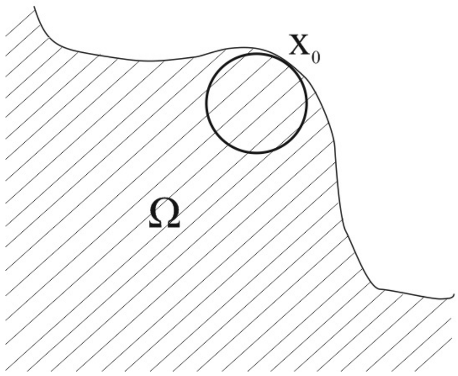
3.4 Hopf's lemma

Elliptic equations (cf. the introduction to this chapter) satisfy certain maximum principles. The following was proved by E. Hopf in 1952

Theorem 3.4.1 ([36]) *Suppose $u \in C^2(\Omega)$, $u \geq 0$ and $Lu \leq 0$ in Ω. Suppose the limit value of u at $x_0 \in \partial\Omega$ is zero, then either $u \equiv 0$ in Ω or $\partial_\nu u > 0$ at x_0.*

The hypothesis on Ω from the original proof imposes it to contain a ball B such that $x_0 \in \partial B$.

We refrain to give a proof here, instead we urge the reader to read Hopf's delightful three page paper. The proof uses this geometrical fact to construct an appropiate barrier function so that the ordinary maximum principle applies to a combination of the solution and the barrier. This allows to express the normal derivative as an increment quotient from which the statement follows easily. The interior ball condition is satisfied for example by C^2



domains. This is classical and can be found in any standard book dealing with partial differential equations (cf. [50], also [56]). Nonetheless recall we are dealing with $C^{1,\omega}$ domains for which, certainly, the interior ball condition fails in general. The proof can be done verbatimly but the existence of appropiate barriers is more involved (cf. [46]).

3.5 Proof of theorem 3.0.1

To begin with we propose the following Dirichlet problems in the domain

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = f & \text{in } \partial\Omega \end{cases}$$

and

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \\ v = f^{2m} & \text{in } \partial\Omega \end{cases}$$

Then $w = u^{2m} - v$ satisfies

$$\begin{cases} \Delta w = 2m(2m-1)|\nabla u|^2 u^{2m-2} & \text{in } \Omega \\ w = 0 & \text{in } \partial\Omega \end{cases}$$

that is, the subharmonic function w must be non-positive in Ω since it vanishes at $\partial\Omega$. Consequently, Hopf's lemma implies that $\frac{\partial}{\partial\nu}w(x) > 0$ for any $x \in \partial\Omega$ where ν is the exterior normal to the domain Ω. But this is exactly the desired inequality.

Remark: in fact this proof applies verbatimly to Ω being of class $C^{1,\alpha}$ taking into account that the Hopf lemma also holds in such a case (cf. [46]).

Chapter 4

Pointwise inequality for the fractional Laplace-Beltrami operators

The Laplace operator has extensions to the setting of Riemannian compact manifolds and the question of whether inequalities of the Córdoba-Córdoba type are true was raised by E. M. Stein several years ago. We undertake this now.

Let us introduce some notation first, if we denote the metric on a compact manifold M by $\sum_{j,k} g_{jk}(x)dx_j dx_k$ recall that the Laplace-Beltrami operator associated with it is given in local coordinates by

$$\Delta_g = \frac{1}{\sqrt{|g|}} \sum_{j,k} \frac{\partial}{\partial x_j} \left(\sqrt{|g|} g^{jk} \frac{\partial}{\partial x_k} \right)$$

where $(g^{jk}) = (g_{jk})^{-1}$ and dx denotes the associated volume form as usual. Then the eigenvalues λ_k^2 of $-\Delta_g$ are non-negative, numerable and one can find a basis given by the corresponding eigenfunctions $\{\phi_k\}$. This result stems from a conjecture raised at the beggining of the 20th century, and proved within a few months by the young H. Weyl. It affirms that the number of eigenvalues less than a given parameter x satisfy the following asymtotic

$$|\{k : \lambda_k^2 \leq x\}| = c_n \text{vol}(M) x^n + O(x^{n-1})$$

A proof can be found in [17]. The fractional powers of the laplacian $(-\Delta_g)^\alpha$, $0 \leq \alpha \leq 1$ can be described spectrally as the linear operator that satisfies

$(-\Delta_g)^\alpha \phi_k = \lambda_k^{2\alpha}\phi_k$ for any k. Now we can state the main result this chapter covers

Theorem 4.0.1 *Given $\alpha \in (0,1]$ and a convex function $\phi \in \mathcal{C}^1(\mathbb{R})$ the following pointwise inequality holds*

$$(-\Delta_g)^\alpha(\phi(f))(x) \leq \phi'(f(x))(-\Delta)_g^\alpha f(x)$$

for any $f \in \mathcal{C}^\infty(M)$.

4.1 Caffarelli-Silvestre extension

In this section we describe a result that links the Dirichlet-to-Neumann operator and the fractional laplacian. The main difference to keep in mind is that the first operator is local while the second is not. The result allows, nonetheless, to understand the non local fractional operator as a local one in one more spatial dimension. This has helped understanding certain non local questions by "localizing" them. The philosophy underneath this result makes our *parabolic proof* of theorem 4.0.1 rather natural. Indeed, our proof employs one more extra dimension, namely the *time variable* in the auxiliar equation. On the other hand employing such a parabolic equation was guided by the availability of certain maximum principles for that class of equations (cf. Hopf's principle proof in the preceding chapter). Let us state now the

Theorem 4.1.1 (Caffarelli and Silvestre, [14]) *Let $\alpha \in (0,1)$ and consider the following elliptic problem*

$$\begin{cases} \Delta_x u(x,y) + \frac{1-2\alpha}{y}u_y + u_{yy} = 0 & \text{for } (x,y) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}^n \end{cases}$$

Then the following inequality holds

$$(-\Delta)^\alpha f(x) = c_{n,\alpha} \lim_{y\to 0^+} \frac{u(x,y)-u(x,0)}{y^{2\alpha}}$$

Their proof relies on Fourier analysis. This generalizes a fact already mentioned since in the case $\alpha = \frac{1}{2}$ the right hand side reduces to the Dirichlet-to-Neumann operator in the half space (cf. section 3.3). Let us point out that Caffarelli and Sire had noticed that this extension problem can be used together with analogous Hopf principles to show these pointwise inequalities in certain cases (cf. [15]).

4.2 The Bernstein-Hausdorff-Widder theorem

In this section we will discuss a theorem concerning the representation of functions $f : [0, \infty) \to \mathbb{R}$ as Laplace-Stieltjes transforms of the measure given by a bounded non-decreasing function, α, i.e. functions of the form

$$f(t) = \int_0^\infty e^{-xt} d\alpha(x). \tag{4.2.1}$$

The material is contained and fully explained in Widder's exposition [70] to which we refer for further reference and complete proofs. Notice that if a function has an integral representation as (4.2.1) above, necessarily, one gets that for any $n \in \mathbb{N}$ and $t \geq 0$

$$(-1)^n \frac{d^n}{dt^n} f(t) \geq 0 \tag{4.2.2}$$

Any function that satisfies (4.2.2) is named completely monotonic in the literature. The Bernstein-Hausdorff-Widder theorem asserts that this condition is also sufficient. In order to thoroughly sketch its proof we will need to detour discussing the Hausdorff moment problem and Carlson uniqueness theorem. We will end the argument sketch stating without proof some well known properties of completely monotonic functions.

Remark: Widder's book contains several proofs of this theorem. There is a really short elementary proof due to H. Pollard [54] relying solely on Helly's compactness theorem (which already underlies the proof of the Hausdorff moment problem). It has the disadvantage of been rather tedious and not quite insightful. Fernando Chamizo has pointed out to me that a translation from russian of a short proof due to B. Korenblum can be found in [44].

4.2.1 The Hausdorff moment problem

A reasonable way to understand a bounded non-decreasing function α would be to understand its mass, mean, variance and more generally: its moments

$$\mu_n = \int x^n d\alpha(x) \tag{4.2.3}$$

for any $n \in \mathbb{N}$. An interesting problem is, provided a sequence $\{\mu_n\}_{n=0}^\infty$, to find (if possible) a bounded non-decreasing function α satisfying (4.2.3).

Depending whether the integration happens in $(-\infty, \infty)$, $(0, \infty)$ or $(0, 1)$ such a problem is known as Stieltjes, Hamburguer or Hausdorff problem respectively. We are interested in the latter, from now on we assume the integration to be over $[0, 1]$. Let us remark that this circle of ideas can be used for instance to prove the central limit theorem (specialy when Fourier analytic techniques do not seem to work) or the interated logarithm law.

If the sequence $\{\mu_n\}_{n=0}^{\infty}$ is given by (4.2.3) then certain linear combinations of those moments will have a positive sign, indeed

$$\int_0^1 x^n (1-x)^k d\alpha(x) \geq 0$$

for any choice of $n, k \in \mathbb{N}$. In order not to detour too far afield let us mention that a sequence of real numbers satisfying the resulting inequalities is known as completely monotonic. The relevant theorem due to Hausdorff claims that complete monotonicity is a necessary and sufficient condition for a sequence $\{\mu_n\}_{n=0}^{\infty}$ to have a representation (4.2.3) where α is some bounded non-decreasing function.

4.2.2 Carlson theorem (d'aprés G. H. Hardy)

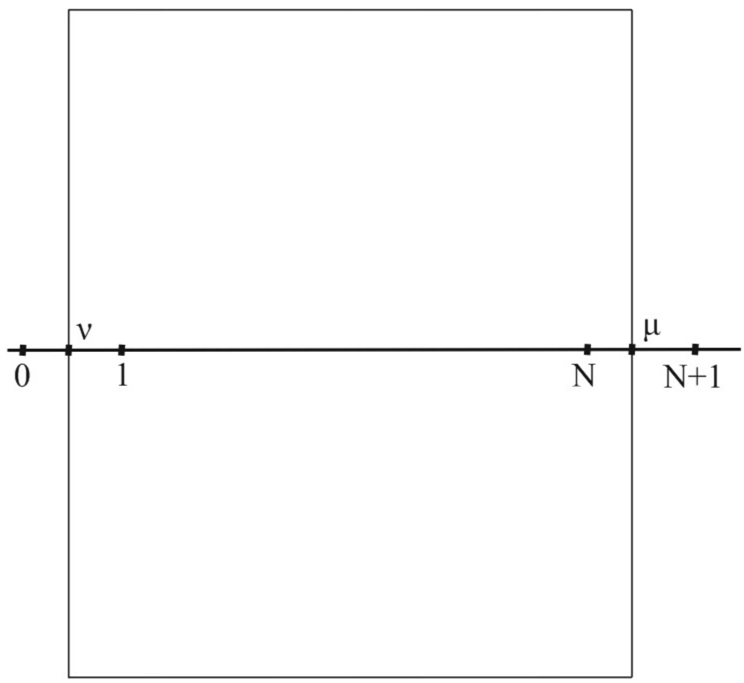
We devote this section to present a result whose flavour lies close to Liouville's theorem and the well-known fact that analytic functions are uniquely determined by their values in a set with accumulation points. We will state and prove a simple version we need for our purposes following Hardy's account [35]. We refer the interested reader to this paper and the references contained therein for further information.

Proposition 4.2.1 (Carlson) *Let f be a bounded holomorphic function defined in $\{z = x + iy \in \mathbb{C} : x > 0\}$ such that $f(n) = 0$ for any $n \in \mathbb{N}$. Then f is identically zero.*

Proof: The Cauchy residue theorem (cf. [48]) provides

$$0 = \sum_{n=1}^{N} f(n)(-w)^n = \frac{1}{2\pi i} \int_{\mu - i\infty}^{\mu + i\infty} \frac{\pi}{\sin(\pi z)} f(z) w^z dz - \int_{\nu - i\infty}^{\nu + i\infty} \frac{\pi}{\sin(\pi z)} f(z) w^z dz$$

where $\mu \in (N, N+1)$ and $\nu \in (0, 1)$. Notice that this follows integrating in a rectangle and letting the horizontal segments go to infinity, those are



neglected due to the boundedness hypothesis and the exponential decay the sine provides (the w^z term oscillates boundedly there).

We claim that for $w \in (0,1)$ the first integral vanishes as $\mu = N + \frac{1}{2}$ tends to infinity. Indeed one may bound it by

$$w^{N+\frac{1}{2}} \int_{-\infty}^{\infty} \frac{|f(N+\frac{1}{2}+iy)|}{\cosh(\pi y)} dy \leq C w^{N+\frac{1}{2}} \int_{-\infty}^{\infty} \exp(-\pi|y|) dy$$

for some $C > 0$. As a consequence of this

$$\frac{1}{2\pi i} \int_{\nu - i\infty}^{\nu + i\infty} \frac{\pi}{\sin(\pi z)} f(z) w^z dz = 0 \qquad \text{for any } w \in (0,1).$$

Notice that the function in the variable w defined by the left hand side is well defined holomorphic for $w > 0$. Now choose $\nu_1, \nu_2 \in (0,1)$ such that $\nu_1 < s < \nu_2$, then

$$\int_0^1 w^{-s-1} \frac{1}{2\pi i} \int_{\nu_2 - i\infty}^{\nu_2 + i\infty} \frac{\pi}{\sin(\pi z)} f(z) w^z dz dw = 0$$

and likewise

$$\int_1^\infty w^{-s-1}\frac{1}{2\pi i}\int_{\nu_1-i\infty}^{\nu_1+i\infty}\frac{\pi}{\sin(\pi z)}f(z)w^z dzdw = 0.$$

Both double integrals are absolutely convergent and hence we may interchange the order of their integrations. Adding them one readily gets

$$0 = \frac{1}{2\pi i}\int_{\nu_2-i\infty}^{\nu_2+i\infty}\frac{\pi}{\sin(\pi z)}\frac{f(z)}{z-s}dz - \frac{1}{2\pi i}\int_{\nu_1-i\infty}^{\nu_1+i\infty}\frac{\pi}{\sin(\pi z)}\frac{f(z)}{z-s}dz$$

which, using Cauchy residue theorem again amounts to

$$\frac{\pi}{\sin(\pi s)}f(s) = 0$$

for any $s \in (0,1)$. This proves that f vanishes on an interval and being analytic it follows that it should vanish everywhere.

REMARK: this works if $f(z) = O(e^{\gamma z})$ if $\gamma < \pi$ but it does not if $\gamma = \pi$ due to the counterexample $f(z) = \sin(\pi z)$ (cf. this with Phragmén-Lindelöf maximum principle).

4.2.3 Sketch of proof

In order to completely prove the Bernstein-Hausdorff-Widder theorem one has to know that completely monotonic functions are bounded analytic in halfplane and restriction of them to arithmetic progressions always provide completely monotonic sequences (cf. [70]). In particular, $\{f(n)\}_{n=0}^\infty$ is completely monotonic, hence representable for some bounded non-decreasing function α as its moments

$$f(n) = \int_0^1 x^n d\alpha(x).$$

A change of variables $x = e^{-t}$ shows

$$f(n) = \int_0^\infty e^{-nt}d\beta(t)$$

where $\beta(t) = -\alpha(e^{-t})$ is a bounded non-decreasing function. To conclude the argument one observes that

$$g(s) = \int_0^\infty e^{-st}d\beta(t)$$

is a bounded analytic function on a halfplane and it coincides with f in every $n \in \mathbb{N}$, Carlson theorem shows they should be equal.

4.3 Proof of theorem 4.0.1

Let us consider the following initial value problems

$$\begin{cases} \frac{\partial}{\partial t}u(x,t) + (-\Delta_g)^\alpha u(x,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

and

$$\begin{cases} \frac{\partial}{\partial t}v(x,t) + (-\Delta_g)^\alpha v(x,t) = 0 \\ v(x,0) = \phi(f(x)) \end{cases}$$

The solutions admit representations

$$u(x,t) = \int_M f(y)G_{2\alpha}(x,y,t)dy$$

and

$$v(x,t) = \int_M \phi(f(y))G_{2\alpha}(x,y,t)dy$$

respectively, where the kernel is given by

$$G_{2\alpha}(x,y,t) = \sum_k e^{-\lambda_k^{2\alpha}t}\phi_k(x)\overline{\phi_k(y)} \tag{4.3.1}$$

Observe that our searched inequality will be an inmediate consequence of the following estimate

$$\frac{\partial}{\partial t}\Big(v(x,t) - \phi(u(x,t))\Big)\Big|_{t=0} \geq 0$$

since $(v-\phi(u))|_{t=0} = 0$ it is enough to show that $v(x,t) - \phi(u(x,t)) \geq 0$ for any $x \in M$ and $t > 0$ (cf. the use of Hopf's principle in section 3.5). But this is a consequence of the positivity nature of the fractional heat kernels $G_{2\alpha}(x,y,t) \geq 0$ (see below), $\int_M G_{2\alpha}(x,y,t)dy = 1$ and applying Jensen's inequality together with the convexity hypothesis about ϕ one gets the desired inequality

$$\begin{aligned} \phi(u(x,t)) &= \phi\left(\int_M f(y)G_{2\alpha}(x,y,t)dy\right) \\ &\leq \int_M \phi(f(y))G_{2\alpha}(x,y,t)dy = v(x,t). \end{aligned}$$

To close our argument let us mention that the positivity of $G_{2\alpha}$ has been considered among others by S. Bochner (see [10]; cf. [9], [8]) using the so-called subordination principle. For the sake of completeness we sketch in the following the main lines of its proof.

The first step consists in proving a maximum principle. Suppose that a continuous function f in M is the initial data for a solution u to the heat equation $\frac{\partial}{\partial t}u - \Delta_g u = 0$ then for any positive integer m there is some positive constant $c_m > 0$ such that

$$\begin{aligned} \frac{\partial}{\partial t}\|u(\cdot,t)\|^{2m}_{L^{2m}(M)} &= 2m\int_M u^{2m-1}(x,t)\frac{\partial}{\partial t}u(x,t)dx \\ &= 2m\int_M u^{2m-1}(x,t)\Delta_g u(x,t)dx \\ &= -c_m\int_M u^{2m-2}(x,t)|\nabla_g u(x,t)|^2dx \le 0. \end{aligned}$$

From which one concludes $\|u(\cdot,t)\|_{L^{2m}} \le \|f\|_{L^{2m}}$ for any $m \ge 1$, taking limits as m tends to infinity we obtain

$$\|u(\cdot,t)\|_{L^\infty(M)} \le \|f\|_{L^\infty(M)}.$$

From this estimate we can deduce that $G_2(x,y,t) \ge 0$ by the following argument: assume $G_2(x_0,y_0,t_0) < 0$ the, since $\int_M G_2(x,y,t)dy = 1$ there would be an open set $U \subseteq M$ where

$$\int_U G_2(x_0,y,t_0)dy > 1.$$

Let ψ be a smooth bump function supported in U such that $0 \le \psi \le 1$ and such that it approximates the indicator function of U in such a way that

$$\int_M \psi(y)G_2(x_0,y,t_0)dy > 1$$

which contradicts the aforementioned maximum principle.

Now we are in position to extend the positivity of the kernel to the rest of values $\alpha \in (0,1)$ for which we will use (4.3.1) and invoke the Hausdorff-Bernstein-Widder theorem that characterizes representability of functions as a Laplace-Stieltjes transform that assures

$$e^{-\lambda_k^{2\alpha}t} = \int_0^\infty e^{-\lambda_k^2 t^{\frac{1}{\alpha}}s}d\gamma_\alpha(s)$$

where γ_α is a non-decreasing monotone function. Using this and 4.3.1 one is tempted to inmediately write

$$G_{2\alpha}(x, y, t) = \int_0^\infty G_2(x, y, t^{\frac{1}{\alpha}} s) d\gamma_\alpha(s)$$

from which the positivity of the fractional heat kernel $G_{2\alpha}$ would follow. To make this a rigorous argument one needs to control the L^∞-norm of the eigenfunctions of $-\Delta_g$ in terms of the corresponding eigenvalues (any polynomial growth[1] would be enough; cf. [64]) and use Weyl's law.

[1]A rather straightforward way to obtain polynomial growth is to apply Sobolev's embedding theorem.

Chapter 5

Eigenfunction maxima and spherical means

For any given compact Riemannian manifold (M, g) of dimension n with Laplace-Beltrami operator $-\Delta_g$ one wishes to understand its eigenfunctions $-\Delta_g \psi_\lambda = \lambda^2 \psi_\lambda$. Recall that due to the elliptic and selfadjoint nature of the operator the eigenfunctions will be smooth and the eigenvalues non-negative real numbers. The following estimate, due originally to L. Hörmander [37], holds:

$$\|\psi_\lambda\|_{L^\infty(M)} \leq C(M)\lambda^{\frac{n-1}{2}}\|\psi_\lambda\|_{L^2(M)}$$

It was extended to similar L^2-L^p estimates by C. D. Sogge, whose proof relies on stationary phase estimates for Bessel potentials ocurring in the so-called Hadamard parametrix (see [64, 65] for details and [30] for explicit geometrical dependence of the constant). A weaker inequality can be obtained using the Sobolev embedding but the above have the feature of being saturated in the sense that they are sharp when one considers the standard $(n-1)$-sphere S^{n-1}.

Our interest on Hörmander-(Sogge) estimate stems from our work [26]. The present work originated as an attempt to understand it geometrically without the use of Hadamard's parametrix. Our purpose is to relate the above type of estimate with the following restriction estimate

$$\|\psi_\lambda\|_{L^2(\gamma)} \leq C(M)\|\psi_\lambda\|_{L^2(M)}$$

where $\gamma \subset M$ is a geodesic sphere of radius $\approx 1/\lambda$. During the process we learnt about A. Reznikov work [58] which was superseded by different methods in the work of Burq-Gérard-Tzvetkov [13]. The former paper considers among other things restriction estimates to geodesic spheres while the latter studies restriction to hypersurfaces with improvements in the case of non-vanishing geodesic curvature. This encouraged us to better understand the relation between those two problems.

Let us state our result now.

Theorem 5.0.1 *Let (M, g) be such that the volume of a geodesic ball depends only on its radius and not on its center. Then the following two statements are equivalent:*

(a) The following estimate holds:

$$\|\psi_\lambda\|_{L^\infty(M)} \leq C(M,\lambda)\lambda^{\frac{n-1}{2}}\|\psi_\lambda\|_{L^2(M)}$$

(b) There exists a constant $\kappa = \kappa(M)$ such that for any geodesic sphere γ of radius $\kappa\lambda^{-1}$ the following estimate holds:

$$\|\psi_\lambda\|_{L^2(\gamma)} \leq C(M,\lambda)\|\psi_\lambda\|_{L^2(M)}$$

The direct implication is obvious, our efforts and presentation will focus in the converse direction. Notice also that the statement itself is quite local in spirit; the radius should be smaller than the injectivity radius to avoid further issues. The scale is *the correct one* according to [32]. One may try to understand the problem globally. As a by-product of Hörmander estimate one knows $C(M,\lambda) \leq C(M)$ but improvements might be possible. Since it is sharp for the sphere one can not wish better estimates in full generality. However, there has been some results in the negative curvature case; for example, Bérard proved a logarithmic improvement is possible (cf. [6]). Lot of research has appeared ever since trying to improve this though. Let us mention that there are results by J. Bourgain [12] asserting that the exponent of λ in Hörmander's bound can not be improved beyond a certain limit for a perturbation of the two dimensional flat torus. Flat tori are quite interesting since number theory gets into the picture. In the case of constant negative curvature there are considerable improvements on the exponent (cf. Iwaniec and Sarnak [41]), open conjectures (cf. Sarnak [61]) and counterexamples available for their analogues in dimension $n = 3$ (cf. Rudnick and Sarnak [60]) and $n \geq 5$ (cf. Donelly [31]).

Finally, let us mention as a curiosity that A. Zygmund's Theorem 4 [71] has the same flavour of those conjectures from the restriction side of our equivalence. Let us state it

Theorem 5.0.2 (Zygmund) *If* $\{a_\nu\} \in \ell^p(\mathbb{Z}^2)$, $1 \leq p < 4/3$ *and*

$$f(x) \sim \sum a_\nu e^{i\nu\cdot x}$$

then for $q = \frac{1}{3}p'$ *and any* $0 < \rho \leq \pi$ *we have*

$$\left(\int_{\{|x|=\rho\}} |f(x)|^q\right)^{1/q} \leq A_p \rho^{1/p'} \|a\|_p$$

Observe that this theorem holds for a larger class of functions and in this respect it is very different. Furthermore, notice that on the one hand a small geodesic circle is *as curved as it can be* in a hyperbolic space and even *more curved* than its euclidean counterpart; on the other hand the current philosophy on Fourier restriction estimates is that curvature lurks behind it: this might not be a coincidence.

5.1 A comparison principle

This section is a technicality, a minor elaboration of some results from [56] to our present interests. The reader willing to read the core of the proof may proceed to section 5.3 coming back here whenever they need to. We will consider the following second order differential operator

$$Lu = u''(x) + g(x)u'(x) + h(x)u(x)$$

acting on sufficiently smooth functions u defined on a fixed open interval; without loss of generality we will assume it to be $(0, 1)$. The functions g and h are supposed to be defined and bounded in any closed subinterval of $(0, 1)$. Maximum principles, as the celebrated Hopf's lemma [36] (cf. section 3.4), can be used to prove uniqueness results, our present interest relies on the following related result:

Proposition 5.1.1 *Let* u *be such that* $Lu \geq 0$ *in the interval and suppose the existence of some* $\varphi > 0$ *satisfying* $L\varphi \leq 0$. *Then* $\frac{u}{\varphi}$ *can not achieve a* *positive maximum.*

Furthermore, if $g(x)$ is positive with a singularity at zero of the type x^α for some $\alpha \geq -1$ then $u(x) \geq \varphi(x)$ for any $x \in (0,1)$ provided $u(0) = \varphi(0)$ and $u'(0) = \varphi'(0)$ are finite.

Proof: the first part can be found in [56], we include it here for the sake of completeness. We will not prove it in the most straightforward way so as to stress where the hypothesis are needed and how the proof elaborates over rather simple arguments. The second part is an extension suggested by the application we have in mind and new to the best of our knowledge.

If h is non positive there is no need to use φ. (In fact, $\varphi \equiv 1$ has the desired properties.) Under such restriction let us suppose, arguing by contradiction, that u is not constant and has a positive maximum at the point $x_0 \in (0,1)$; then, the first derivative vanishes at that point $u'(x_0) = 0$ and, finally, as a consequence of $Lu \geq 0$, one gets $u''(x_0) \geq -h(x)u(x_0) \geq 0$ from the hypothesis. This is in contradiction with the fact that it is a maximum since, in such a case, $u''(x_0) \leq 0$ unless $u''(x_0) = 0$. To overcome this difficulty and rule out the possibility that $u''(x_0) = 0$ one can use a barrier function as follows: first, notice that the above argument holds if one considers the maximum to be achieved at an endpoint x_0 in such a way that $u'(x_0) = 0$. (This can be understood as a one-dimensional Hopf's lemma.) We will use this view now: instead of considering u we consider $u+v$ on $(0, x_0)$ for some v satisfying $Lv \geq 0$, $v \leq 0$, $v(x_0) = 0$ and $v''(x_0) > 0$. This can be achieved taking $v(x) = e^{-M(x-x_0)} - 1$ for an appropiate choice of the constant M. Now the one-dimensional Hopf's lemma applied to $u+v$ provides the desired contradiction. (Notice $u+v$ has a maximum at x_0 if u does.)

Let us recall now that in our statement we had no positivity assumption on h. The way to overcome this reduces, precisely, to the existence of φ as in the statement. Indeed, if one considers $u = v\varphi$ it is easily proved that

$$0 \geq Lu = L(v\varphi) = \varphi v'' + (2\varphi' + g\varphi)v' + L\varphi \cdot v$$

which shows such a v (which is well-defined under our hypothesis) satisfies an equation of the same type with $\tilde{g} = 2\frac{\varphi'}{\varphi} + g$ and $\tilde{h} = \frac{1}{\varphi} L\varphi \leq 0$ and hence satisfying the special hypothesis for v.

Let us now go over the last part of the statement. We will proceed as before presenting first a more transparent version of the proof and elaborating it to the more refined one. Let $w = u - \varphi$ which will satisfy $Lw \geq 0$, $w(0) = 0$ and $w'(0) = 0$. One might deduce from the differential inequality, using Taylor's expansion and the hypothesis, that $w''(0) \geq 0$. If we consider the case when $w''(0) > 0$ and h is non positive we will be done,indeed, the solution would increase initially; as a consequence: $w > 0$ in $(0, \varepsilon)$ for some small $\varepsilon > 0$. This would end the argument from the non existence of a positive maximum as follows: if it becomes negative at some stage it should achieve a maximum meanwhile. Contradiction. But $w''(0)$ might vanish and h might be positive. To dispose of this generalities we will define instead $w_\delta = u - \varphi + \delta v$ for some small positive δ and a barrier v satisfying $Lv \geq 0$, $v \geq 0$ and $v'' > 0$. A function with this properties exist, e.g. $v(x) = e^x - x - 1$ as can be checked. (This is where the positivity of g is crucial.) Notice that for such a function $w''_\delta > 0$ holds and, as before, it increases for some $(0, \varepsilon(\delta))$. As a by-product it remains positive and the same is true for the quotient $w_\delta \varphi^{-1}$; which can not achieve a maximum due to the first part of our result, as a consequence, it remains positive $w_\delta \geq 0$ for any δ. Taking δ tend to zero concludes the argument.

5.2 The Euler-Poisson-Darboux equation

This section is a straightforward adaptation of F. John's account of the Euler-Poisson-Darboux equation (cf. [42], pp. 88-89) where it is deduced in the case of the euclidean space. We include it here for the reader's convenience. Let us introduce the following notation for the spherical means of a fixed function f around a point x

$$I(x, r) = \fint_{\partial B_r(x)} f(y) d\sigma(y)$$

(cf. section 5.3 for more details). Making some abuse of notation (understanding de integration in a fixed chart with normal coordinates):

$$\int_0^r I(x, \rho) h(\rho) d\rho = \int_{B_r(0)} f(\exp_x(y)) d\mathrm{vol}_g(y)$$

where h is the Radon-Nykodym derivative $\frac{d\mathrm{vol}_g}{d\rho}$. In the case of the n-euclidean space $h(\rho) = \omega_{n-1} \rho^{n-1}$. Notice $h(0) = 0$ in general since its infinitesimal equivalent is r^{n-1}. Furthermore it increases initially and is obviously positive. We may apply the Laplace-Beltrami operator in the variable x to the above equation and use of the divergence theorem as follows

$$\begin{aligned}\int_0^r \Delta_g I(x,\rho)h(\rho)d\rho &= \int_{B_r(0)} \Delta_g f(\exp_x(y))d\mathrm{vol}_g(y) \\ &= \int_{\partial B_r(0)} \frac{\partial}{\partial \nu_y} f(\exp_x(y))d\sigma(y) \\ &= h(r) \fint_{|z|=1} \frac{\partial}{\partial r} f(\exp_x(r\cdot z))d\sigma(z) \\ &= h(r)\frac{\partial}{\partial r} I(x,r)\end{aligned}$$

Taking derivatives in r one finally gets the Euler-Poisson-Darboux equation

$$\frac{d^2}{dr^2}I(x,r) + g(r)\frac{d}{dr}I(x,r) = \Delta_g I(x,r)$$

where $g(r)$ denotes the logarithmic derivative of $h(r)$ and hence $g(r) - \frac{n-1}{r}$ is a continuous function bounded at zero.

5.3 Proof of theorem 5.0.1

The proof will reduce to the study of certain spherical means and hence it is related to, but not subsummed in, Hadamard's parametrix method (cf. [65] and [42]). Let us sketch the argument first: we will study the spherical means of a smooth function f, namely:

$$I_f(x,r) = \fint_{\partial B_r(x)} f(y)d\sigma(y)$$

It satisfies an explicit second order ordinary differential inequality when $f = \psi_\lambda^2$ at the point $x = x_\lambda$ where it attains its maximum. This permits us to compare $I_\lambda(x_\lambda, 0) = |\psi_\lambda(x_\lambda)|^2 = \|\psi_\lambda\|^2_{L^\infty(M)}$ with $I(x_\lambda, \kappa\lambda^{-1})$ for certain fixed quantity κ. Nothing else is needed to prove the theorem since for small radius (i.e. $\lambda \gg 1$) one can compare the riemannian volume with the euclidean one. The crucial step is based on the comparison principle presented in section 5.1 which enables us to bound $I_\lambda(x_\lambda, r)$ below by some function satisfying an ordinary differential equation. The argument is rather involved since the ordinary differential equation that arises and, as a consequence, its solution depend on λ; one has to get rid of this dependence so as to find uniform bounds, this is done employing a WKB method for the equation at hand allowing to reduce the argument to a fixed Bessel function if λ is big enough. Before proceeding to the proof let us remark, leaving details to the reader, that it is enough to prove the estimate for γ a geodesic sphere around a point where the maximum of $|\psi_\lambda(x)|$ is achieved.

Given an eigenfunction ψ_λ one can consider $I_\lambda(r) = I(x_\lambda, r)$ the spherical means of its square, it satisfies the Euler-Poisson-Darboux differential equation (cf. section 5.2) which involves

$$\begin{aligned}
\Delta_g I_\lambda(x,r) &= \frac{1}{h(r)}\Delta_g \int_{\partial B_r(0)} \psi_\lambda(\exp_x(y))^2 d\sigma(y) \\
&= \fint_{\partial B_r(0)} \Delta_g(\psi_\lambda^2) d\sigma(y) \\
&= 2\fint_{\partial B_r(0)} (|\nabla_g \psi_\lambda|^2 - \lambda^2\psi_\lambda^2) d\sigma \\
&\geq -\lambda^2 I_\lambda(x,r)
\end{aligned}$$

as a consequence the following differential inequality is satisfied:

$$\frac{d^2}{dr^2} I_\lambda(r) + g(r)\frac{d}{dr} I_\lambda(r) + 2\lambda^2 I_\lambda(r) \geq 0$$

(See section 5.2 to learn more about g.) Using our comparison result from section 5.1 one can compare such a function with J_λ a solution of

$$\frac{d^2}{dr^2} J_\lambda(r) + g(r)\frac{d}{dr} J_\lambda(r) + 2\lambda^2 J_\lambda(r) = 0$$

satisfying $J_\lambda(0) = I_\lambda(0) = \|\psi_\lambda\|^2_{L^\infty(M)}$ and $J'_\lambda(0) = 0$. It will be enough to prove the existence of a constant $\kappa = \kappa(M)$, independent of λ, such that

$$J_\lambda(r) \geq \frac{1}{2} J_\lambda(0) \text{ for any } r \in (0, \kappa\lambda^{-1})$$

We will make now the change of variables $\rho = r\sqrt{2}\lambda$ and $\varepsilon = (\sqrt{2}\lambda)^{-1}$ to express the above in a more convenient form, namely

$$\frac{d^2}{d\rho^2} K_\lambda(\rho) + \frac{n-1}{\rho}\frac{d}{d\rho} K_\lambda(\rho) + K_\lambda(\rho) = \varepsilon k(\rho\lambda^{-1})\frac{d}{d\rho} K_\lambda(\rho)$$

where $K_\lambda(\rho) = J_\lambda(\rho(\sqrt{2}\lambda)^{-1})$ and k is a bounded function that is bounded at zero (cf. section 5.2). At least formally this admits a solution of the form

$$K_\lambda(\rho) = J_\lambda(0) \sum_{j=0}^{\infty} \varepsilon^j v_j(\rho)$$

where v_0 does not depend on λ and satisfies

$$\begin{cases} v_0''(\rho) + \frac{1}{\rho} v_0'(\rho) + v_0(\rho) = 0 \\ v_0(0) = 1 \\ v_0'(0) = 0 \end{cases}$$

and the rest of the expansion follow interatively from v_0:

$$\begin{cases} v_{j+1}''(\rho) + \frac{1}{\rho} v_{j+1}'(\rho) + v_{j+1}(\rho) = k(\varepsilon\rho) v_j'(\rho) \\ v_{j+1}(0) = 0 \\ v_{j+1}'(0) = 0 \end{cases}$$

we are abusing notation since there is a hidden dependence in λ due to its appearance in k. This will not be a problem since all the properties we need from such a function can be shown to be uniform in λ; namely, that it and its first derivative are uniformily bounded. To make this rigorous one needs to show v_j exist and that the series defining $K_\lambda(\rho)$ converges appropiately in an interval for some ε small enough. Once this is done it is clear that for large λ the parameter ε will be small and hence one may choose κ to be so small that $v_0(\rho) \geq \frac{3}{4}$ holds for any $\rho \in (0, \kappa)$ and then λ so large that the error term is smaller than, say, $\frac{1}{4}$. After a change of variables this would end the proof.

The last step will be a consequence of some well-known results from the theory of second order ordinary differential equations (cf. Ince and Sneddon [40], chapter 5 for further details). One can write a solution to a general second order ordinary differential equation $Lu = f$ with boundary values $u(0) = u'(0) = 0$ as

$$u(x) = y_2(x) \int_0^x \frac{y_1(t) f(t)}{W(y_1, y_2, t)} dt - y_1(x) \int_0^x \frac{y_2(t) f(t)}{W(y_1, y_2, t)} dt$$

where W denotes the wronskian of y_1, y_2 independent solutions of the homogeneous equation. In our case at hand those solutions are related to some Bessel functions, one of them singular at zero. It will be at zero where we will have to be more careful then. Our claim is that the operation $v_j \mapsto v_{j+1}$ is bounded from $L^\infty \to L^\infty$. This would be enough for our purposes and requires further understanding of y_1, y_2 and their wronskian near zero; namely, we will need that $y_1(x) \approx 1$,

$$y_2(x) \approx \begin{cases} \log(x) & \text{if } n = 2 \\ x^{2-n} & \text{if } n \geq 3 \end{cases}$$

and, finally, $W(y_1, y_2, x) \approx x^{1-n}$ for $x \approx 0$. This reduces to knowledge of Bessel functions of the first and second kind since a pair of independent solutions is provided by $y_1(x) = x^{-\frac{n-2}{2}} J_{\frac{n-2}{2}}(x)$ and $y_2(x) = x^{-\frac{n-2}{2}} Y_{\frac{n-2}{2}}(x)$.

The relevant properties of Bessel functions can be found in Watson's treatise [69], the standard reference for Bessel functions; in particular we refer to: §3 · 1(8), §3 · 51(3), §3 · 52(3) and §3 · 63(1), respectively.

The claim follows from the following equality:

$$\begin{aligned} v_{j+1}(x) &= y_2(x) \int_0^x \frac{y_1(t)k(\varepsilon t)v_j'(t)}{W(y_1, y_2, t)} dt - y_1(x) \int_0^x \frac{y_2(t)k(\varepsilon t)v_j'(t)}{W(y_1, y_2, t)} dt \\ &= \varepsilon y_1(x) \int_0^x \frac{y_2(t)k'(\varepsilon t)v_j(t)}{W(y_1, y_2, x)} dt - \varepsilon y_2(x) \int_0^x \frac{y_1(t)k'(\varepsilon t)v_j(t)}{W(y_1, y_2, t)} dt \\ &\quad + y_1(x) \int_0^x \frac{d}{dt} \left(\frac{y_2(t)}{W(y_1, y_2, t)} \right) k(\varepsilon t) v_j(t) dt \\ &\quad - y_2(x) \int_0^x \frac{d}{dt} \left(\frac{y_1(t)}{W(y_1, y_2, t)} \right) k(\varepsilon t) v_j(t) dt \end{aligned}$$

where we have used an integration by parts to get rid of v_j' and the aforementioned properties of Bessel functions. Once that equality is known it is straightforward to bound it to obtain a uniform bound $\|v_{j+1}\|_\infty \leq C\|v_j\|_\infty$. (Notice that ε is bounded since it tends to zero).

To end the argument one still needs to check that the series defining $K_\lambda(\rho)$ converges in such a way that it is a function of class C^2 in $(0, 1)$ and that it satisfies the second order ordinary differential equation that originated it. Let us say that this can be done if one could differentiate the series termwise. To justify it one might prove in a similar guise the estimates $\|v_{j+1}'\|_\infty \leq C\|v_j'\|_\infty$ and $\|v_{j+1}''\|_\infty \leq C\|v_j'\|_\infty$. We leave the details to the reader.

5.4 Some remarks about this chapter

During the congress Analysis and Beyond at the Institute for Advanced Studies in 2016, shortly after I came up with this result, Ch. D. Sogge kindly communicated to me how to use some of his estimates on L^p-estimates of eigenfunctions on small balls to come up with this result in full generality under certain hypothesis to settle one of my goals at the time with respect to this circle of ideas. In this section I will comment informally on this, on a rather trivial extension and on why I believe this strategy might distinguish between positive and non-positive curvature.

Let us now digress and introduce (without proof) two of Sogge's inequalities. Now we work with a general riemannian manifold (M, g) in contrast with the rest of the chapter. The first inequality controls decay of L^2 norm on small balls uniformily on $x \in M$

$$\lambda^{1/2}\|\psi_\lambda\|_{L^2(B_{1/2\lambda}(x))} \leq C$$

for some constant $C > 0$ (cf. equation (4.1) from [66]). The other one controls its height in terms of the previous quantity, namely

$$\|\psi_\lambda\|_{L^\infty(M)} \leq C\lambda^{\frac{n-1}{2}} \sup_{x\in M} \lambda^{1/2}\|\psi_\lambda\|_{L^2(B_{1/2\lambda}(x))}$$

for some constant $C > 0$ (cf. equation (3.3), [66]). Let us now introduce the following hypothesis, part of the essence of the proof above, namely

$$\|\psi_\lambda\|_{L^\infty(M)} \approx r^{\frac{1-n}{2}} \|\psi_\lambda\|_{L^2(\partial B_r(x_0))} \tag{5.4.1}$$

for any $r \approx \lambda^{-1}$ where $x_0 \in M$ is the point where the maxima is achieved. This chapter provides some evidence for this providing proof for a certain class of manifolds. Squaring (5.4.1) and integrating against $r^{n-1}dr$ one may obtain (at least heuristically)

$$\|\psi_\lambda\|_{L^\infty(M)} \approx \|\psi_\lambda\|_{L^2(C_{1/\lambda})}\lambda^{n/2}$$

where C_r denotes the corona $B_r(x_0) \setminus B_{r/2}(x_0)$. This together with Hörmander's bound and (5.4.1) implies the restriction estimate on small scales. On the other hand if the hypothesis (5.4.1) holds true, integrating the square of the restriction estimate for radii around λ^{-1} would imply

$$\|\psi_\lambda\|_{L^2(C_{1/\lambda})} \leq \frac{C}{\lambda^{1/2}}\|\psi_\lambda\|_{L^2(M)}$$

Now we can find a small ball $B_{1/2\lambda}(x') \subseteq C_{1/\lambda}$ and employ Sogge's second estimate to conclude our theorem. Proving the hypothesis (5.4.1) for general compact manifolds would therefore improve our result immediately.

We omit details for the reader's convenience but a rather more general statement of Theorem 5.0.1 is true, its proof is obtained from the given one just replacing the square spherical means of $|\psi_\lambda|^2$ by different powers. This leads to another equivalent assertion, namely

(c) There exist a constant $\kappa = \kappa(M, p)$ such that for any geodesic sphere γ of radius $\kappa\lambda^{-1}$ the following estimate holds:

$$\|\psi_\lambda\|_{L^p(\gamma)} \leq C(M, \lambda)\lambda^{\frac{(n-1)(p-2)}{2p}} \|\psi_\lambda\|_{L^2(M)}$$

which obviously reduces to (b) for $p = 2$.

We finish mentioning a remarkable property of spherical means on expanding geodesic spheres. In the case of flat tori and certain quotients of hyperbolic spaces it has been proven by B. Randol (cf. [57]) that the spherical means converge to the space integral on the manifold itself. Notice that this ergodicity property is not shared by standard spheres. One may dream to use this property together with some short of comparison theorem for suitable ordinary differential equations. Unfortunately, in this chapter we are able to exploit very little from this daydream. Indeed, we use small radii for a particular and rather simple minded geometrically inspired equation.

Chapter 6

Integral representation for fractional Laplace-Beltrami operators

As usual let (M, g) be a closed compact manifold of dimension $n \geq 2$ whose Laplace-Beltrami operator is denoted by $-\Delta_g$. The following is the main result of this chapter

Theorem 6.0.1 *Let f be smooth and $s \in (0,1)$, then for a sufficiently large parameter N_0 one has the representation*

$$(-\Delta_g)^s f(x) = P.V. \int_M \frac{f(x) - f(y)}{d_g(x,y)^{n+2s}} (c_{n,s}\chi u_0 + k_{N_0})(x,y) dvol_g(y) + O(\|f\|_{H^{-N_0}(M)}),$$

where $k_{N_0}(x,y) = O(d_g(x,y))$ is a smooth function, χ is a smooth cut off function equal to one on the diagonal and supported around it, the implicit constant depends on N_0, $c_{s,n} > 0$ is a constant independent of N_0 and u_0 is a smooth function such that $u_0(x,x) = 1$.

Notice that the norm in the error might be taken to be L^∞. The proof uses spectral calculus interwined with the Hadamard parametrix to provide an explicit integral representation, similar to a classical singular integral, with a harmless error. This is enough for the applications we have in mind and in order to establish it we will start with the following

Lemma 6.0.2 *For $s \in (-1,0)$ and under the hypothesis of the previous theorem*

$$(-\Delta_g)^s f(x) = \int_M \frac{f(y)}{d_g(x,y)^{n+2s}}(c_{n,s}\chi u_0 + k_{N_0})(x,y)dvol_g(y) + O(\|f\|_{H^{-N_0-2}(M)})$$

holds for any f orthogonal to the constants.

An integration by parts will then imply the theorem for any $s \in (0,1)$. The kernels from both statements are related but are not equal. Notice also that in the two dimensional setting for $s=-1$ a logarithm singularity must appear in the lemma, but here we will not consider this case. The explicit expression of the error is crucial for the applications, and one may compare it with similar expresions for the cases of the flat tori or euclidean space, as was pointed out in the introduction (cf. [23, 24, 64]). In the case of manifolds with boundary there is controlled degeneration of constants as one approaches the boundary.

In the non compact case, instead of Hadamard parametrix one might use the fundamental solution of the heat operator which, nevertheless, is related to the former by a Laplace transform (cf. [53]). One needs then to employ a different identity to begin with, together with some sharp heat kernel bounds whose validity demands geometrical restrictions on the Ricci curvature, which has to be bounded below. Also a strictly positive lower bound for the injectivity radius is required. Along the proof of the following result we will deal with the compact case only stressing where changes might be done to handle the non compact case.

Theorem 6.0.3 *Under the hypothesis of theorem 6.0.1 the following weaker statement holds*

$$(-\Delta_g)^s f(x) = P.V. \int_M \frac{f(x)-f(y)}{d_g(x,y)^{n+2s}}(c_{n,s}\chi U_0 + k_{N_0})(x,y)dvol_g(y) + O(\|f\|_\infty),$$

where $k_{N_0}(x,y) = O(d_g(x,y))$ is a smooth function, χ is a smooth diagonal cut off function, the implicit constant depends on N_0, $c_{s,n} > 0$ is a constant independent of N_0 and U_0 is a smooth function such that $U_0(x,x) = 1$.

Observe that the error term has a remarkable difference with the main result stated before (cf. Theorem 6.0.1). We will present this result first (although it is not quite as strong as the former). Afterwards we will introduce the needed notational convention together with the essentials of the Hadamard parametrix construction. Then we will end the chapter with a proof of the main result and, for the sake of completeness, the fractional Sobolev embedding theorem that was needed in the proof of Theorem 1.5.

Corollary 6.0.4 *Let (M, g) be a compact riemannian manifold of dimension n, $s \in (0, \frac{1}{2})$ and $p = \frac{2n}{n-2s}$. Then there exist a constant $C > 0$ such that*

$$\|f\|_p \leq C(\|f\|_2 + \|\Lambda^s f\|_2).$$

The cases when s is an integer are well known and can be found in [5, 4].

6.1 Proof of Theorem 6.0.3

In this and in subsequent sections we will make use of polar coordinates, a generalization of radial coordinates, which are adapted to the local geometry of the manifold. Some basic properties of these in the case of surfaces can be found in [27] or, with more generality, in [28]. Special attention deserves Gauss' lemma in the calculations involved in the Hadamard parametrix construction (cf. [65]).

In this proof we will take advantage of the following well known representation formula

$$(-\Delta_g)^s f(x) = \frac{1}{\Gamma(-s)} \int_0^\infty (f(x) - e^{-t\Delta_g} f(x)) \frac{dt}{t^{1+s}}.$$

The semigroup action might be expressed through the heat kernel $G(x, y, t)$ as follows

$$\int_0^\infty \int_M G(x, y, t)(f(x) - f(y)) d\text{vol}_g(y) \frac{dt}{t^{1+s}}.$$

We will split this integral in three parts: corresponding to large times, small times but far from the spatial singularity and, finally, small times near the singularity. In fact the latter is the main part contributing to the kernel in our expression while the other two will add to the error term in the statement. The first integral corresponds to

$$\int_1^\infty \int_M G(x, y, t)(f(x) - f(y)) d\text{vol}_g(y) \frac{dt}{t^{1+s}}$$

which is $O(\|f\|_\infty)$. The second integral has the form

$$\int_0^1 \int_{d_g(x,y)>1} G(x,y,t)(f(x)-f(y))d\mathrm{vol}_g(y)\frac{dt}{t^{1+s}}.$$

To control it we use the bound of Li and Yau (Corollary 3.1 from [47]). Here the conditions on the curvature arise, namely, the Ricci curvature bound assures the existence of a positive constant C such that

$$G(x,y,t) \le C\frac{1}{\mathrm{vol}_g^{1/2}(B_x(\sqrt{t}))\mathrm{vol}_g^{1/2}(B_y(\sqrt{t}))}e^{-C\kappa t}e^{-\frac{d_g(x,y)^2}{5t}}$$

where $\mathrm{Ric}_g \ge \kappa$, C is some positive constant and $B_x(r)$ denotes the ball of radius r centered at x. While, on the other hand, for any $z \in M$

$$ct^{n/2} \le \mathrm{vol}_g^{1/2}(B_z(\sqrt{t}))$$

where the constant $c > 0$ depends on the upper bound for the sectional curvature and the dimension (cf. Berger, theorem 54 [7]). It is now quite elementary to check that such an integral is bounded by $O(\|f\|_\infty)$. We are hence left with

$$\lim_{\varepsilon\to 0}\int_0^1 \int_{\varepsilon<d_g(x,y)<1} G(x,y,t)(f(x)-f(y))d\mathrm{vol}_g(y)\frac{dt}{t^{1+s}}.$$

In such a range we can use the heat kernel parametrix (which is actually closely related to Hadamard's parametrix, cf. [53]), namely:

$$G(x,y,t) = \frac{1}{(4\pi t)^{n/2}}e^{-\frac{d_g(x,y)^2}{4t}}\left(U_0(x,y)\chi(x,y) + \chi(x,y)\sum_{j=1}^{k}U_j(x,y)t^j + O(t^{k+1})\right),$$

where $U_0(x,x) = 1$, χ is a radial cut off function around x whose support is within a ball with radius smaller than the injectivity radius and k is big enough. We refer to [53] or [17] for further details. One may expand the O-term asymtotically where further powers of t will appear together with some smooth functions having an interesting geometric meaning. However, they are unnecessary for our current purposes. Plugging that information into the above integral one finds after the change of variable $t' = \frac{t}{4d_g(x,y)^2}$ it is equal to:

$$\lim_{\varepsilon\to 0}\int_{\varepsilon<d_g(x,y)<1}\frac{f(y)-f(x)}{d_g(x,y)^{n+2s}}\int_0^{1/d_g(x,y)^2}\frac{e^{-1/t'}(\chi(x,y)U_0(x,y)+O(t'))}{t'^{n/2+1+2s}}dt'd\mathrm{vol}_g(y).$$

One can now complete the range of the inner integral to the whole interval $(0,\infty)$ to obtain a constant, while noticing that the added part decays very rapidly to zero.

6.2 The Hadamard parametrix

In this section we present a brief description of the Hadamard parametrix construction which is suited for our purposes. A complete though rather technical treatment can be found in Hörmander's [38]. A more recent and accesible exposition can be found in [33]. We intermingle both here. Let us introduce F_0^z known as a Bessel potential of order zero which is a fundamental solution of $(-\Delta - z)F_0^z(x) = \delta_0(x)$. Fourier analysis can be employed to obtain the following representation formula (in the sense of distributions)

$$F_0^z(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi}(|\xi|^2 - z)^{-1} d\xi.$$

Notice that it is radial being the Fourier transform of another radial function. The Hadamard parametrix method introduces more potentials, $F_\nu^z(x)$, as part of the construction which can be expressed in terms of modified Bessel functions of the second kind as follows (cf. [33])

$$F_\nu^z(x) = c_\nu |x|^{-\frac{n}{2}+\nu+1} z^{\frac{n}{4}-\frac{\nu+1}{2}} K_{n/2-\nu-1}(\sqrt{z}|x|).$$

This can be found in [34] where they prove it for $n/2 < \nu + 1$ and extend its validity to other values by analytic continuation. The special functions involved satisfy the symmetry $K_\ell = K_{-\ell}$ and the following bounds for $\ell \geq 0$

$$|K_\ell(w)| \leq \begin{cases} C_0 \log|w|^{-1} & \text{if } |w| \leq 1 \text{ and } \ell = 0 \\ C_\ell |w|^{-\ell} & \text{if } |w| \leq 1 \text{ and } \ell \neq 0 \\ C_\ell e^{-\mathrm{Re}\,(w)} & \text{if } |w| > 1 \end{cases}$$

for some positive constant $C_\ell > 0$ (cf. [69], §7·23). They also satisfy several recursive relations among which we will use $-2\frac{\partial F_\nu}{\partial x} = xF_{\nu-1}$ for $\nu > 0$. Let u_0 be some function to be specified later

$$(-\Delta_g - z)(u_0 F_0^z) = u_0(0)\det(g^{ik})^{\frac{1}{2}}\delta_0 + (-\Delta_g u_0)F_0 + 2\left(hu_0 - 2\langle x, \frac{\partial u_0}{\partial x}\rangle\right)(F_0^z)'(|x|^2).$$

where $h(x) = \sum g_{jk}(x)b^j(x)x_k$ in normal coordinates. Here we are employing normal coordinates and a consequence of Gauss lemma implicitly. To get rid of the last term we will know choose u_0 to be a function such that $u_0(0) = 1$ and

$$hu_0 = 2\langle x, \frac{\partial u_0}{\partial x}\rangle.$$

For further approximations one proceeds similarly, where all the functions u_ν that appear in the process are smooth. We refer the reader to the aforementioned references for further details on the construction. Let $\chi(x, y)$ be a cut off function supported near the diagonal, it follows that the operator

$$\mathcal{P}_N^z f(x) = \int_M \chi(x, y) \sum_{\nu=0}^{N} u_\nu(x, y) F_\nu^z(d_g(x, y)) f(y) d\mathrm{vol}_g(y)$$

is a right parametrix of $(-\Delta_g - z)$ that is $(-\Delta_g - z)\mathcal{P}_N^z = \delta + R^z$ where

$$R_N^z f(x) = \int_M \chi(x, y) h_N(x, y) F_N^z(d_g(x, y)) f(y) d\mathrm{vol}_g(y)$$

where h_N is some specific smooth function.

6.3 Proof of Theorem 6.0.1

We will prove the Lemma 6.0.2, i.e. $s < 0$, from which the theorem follows. We may use spectral calculus to get

$$(-\Delta_g)^s f(x) = \frac{1}{2\pi i} \int_\gamma z^s (-\Delta_g - z)^{-1} f(x) dz$$

where γ is some appropiate contour enclosing the positive eigenvalues of the Laplace-Beltrami operator and such that z^s is holomorphic inside. The contour integral we choose is sketched in Figure 6.1 (cf. [63]). Notice that this is consistent with the spectral definition of the operator itself for functions orthogonal to the constants. This identity together with Hadamard's parametrix implies $(-\Delta_g)^s f(x)$ equals

$$\frac{1}{2\pi i} \int_\gamma z^s \mathcal{P}_N^z f(x) dz + \frac{1}{2\pi i} \int_\gamma (-\Delta_g - z)^{-1} R_N^z f(x) dz$$

where the first integral is of the form

$$\frac{1}{2\pi i} \int_\gamma z^s \int_M f(y) \chi(x, y) u_0(x, y) F_0^z(d_g(x, y)) d\mathrm{vol}_g(y) dz$$

plus lower order terms.

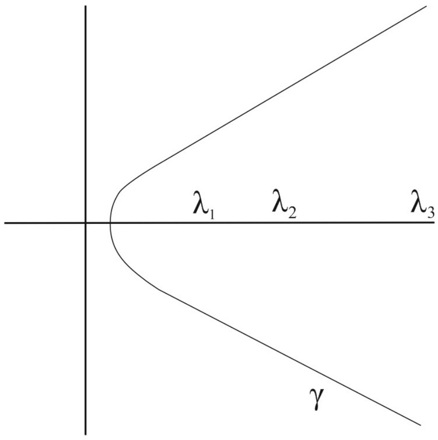


Figure 6.1: Contour γ.

Each summand from the first integral can be expressed as

$$\frac{1}{2\pi i}\int_\gamma\int_M z^{s+\frac{n}{4}-\frac{\nu+1}{2}}K_{n/2-\nu-1}(\sqrt{z}d_g(x,y))\chi u_\nu(x,y)f(y)d_g(x,y)^{-\frac{n}{2}+\nu+1}d\mathrm{vol}_g(y)dz.$$

Making the change of variables $w = zd_g(x,y)^2$ this double integral might be expressed as the product of a contour integral and a space integral. Indeed, one first checks absolute integrability using the appropiate bounds for the Bessel functions which allows to interchange the integration order. After this the aforementioned change of variables allows to get an integral over another path $\tilde{\gamma}$ depending on $d_g(x,y)$. Both paths γ and $\tilde{\gamma}$ provide the same integral because the integrand is holomorphic in the region contained between them. This proves the claim. Notice now that the contour integral is a constant not depending on the geometry, which coincides with the explicit one corresponding to the tori case. As a consequence the above double integral, up to a constant, is given by

$$\int_M f(y)d_g(x,y)^{-n-2s+2\nu}\chi(x,y)u_\nu(x,y)d\mathrm{vol}_g(y).$$

Let us now turn to the error term which has the form

$$\frac{1}{2\pi i}\int_\gamma z^s(-\Delta_g - z)^{-1}R_N^z f(x)dz.$$

Subtitution of the expressions above leads to

$$\frac{1}{2\pi i}\int_\gamma z^s(-\Delta_g - z)^{-1}\int_M \chi(x,y)h_N(x,y)F_N^z(d_g(x,y))f(y)d\mathrm{vol}_g(y)dz.$$

Let us denote by $H_0^m \subseteq H^m$ the subspace of functions orthogonal to constants. The claim follows combining the boundedness of

$$(-\Delta_g - z)^{-1} : H_0^m(M) \to H_0^m(M)$$

with constant $O((1+|z|)^{-1})$, Minkowski inequality, Sobolev's embedding $H^{n/2+\varepsilon} \hookrightarrow L^\infty$, the recursive relations satisfied by the Bessel potentials and the estimates for the modified Bessel function. Indeed, one can bound the above by

$$\int_\gamma |z|^s \left| (-\Delta_g - z)^{-1} \int_M (-\Delta_g^y)^\beta \left(\chi(x,y) h_N(x,y) F_N^z(d_g(x,y))\right) (-\Delta_g^y)^{-\beta} f(y) dy \right| d|z|$$

where $\beta > 0$ will be some large parameter and we have introduced the notation $(-\Delta_g^y)$ to mean the Laplace-Beltrami operator in the y variable to avoid future confussion. Notice we are using implicitly that f is orthogonal to the constant functions. Taking into account that the L^∞ bound in the x variable is bounded by Sobolev's embedding by some L^2 norm of a number of derivatives of the integral and using Cauchy-Schwarz inequality (twice) one gets the above is controlled by a number of terms, $\alpha > n/2$, of the form $C_s \|(-\Delta_g)^{-\beta+n/2+\varepsilon} f\|_{L^2}$ times the square root of

$$\int_\gamma \frac{|z|^s}{1+|z|} \int_M \int_M \left\{ (-\Delta_g^x)^\alpha (-\Delta_g^y)^\beta \left(\chi(x,y) h_N(x,y) F_N^z(d_g(x,y))\right) \right\}^2 dy dx d|z|$$

where $\|(-\Delta_g)^{-\beta+n/2+\varepsilon} f\|_{L^2}$ comes from an application of Sobolev's embedding in the integrand and $C_s > 0$ is a constant depending on s. (Observe this requires $N_0 \geq \beta - n/2 - \varepsilon$ in the statement.) All we need to justify is the uniform boundedness of the triple integral above. To do so we might expand the Laplace-Beltrami operators which might hit χ, h_N (both smooth functions) or Bessel potentials F_N^z (in local coordinates). This yields a number of summands. The former cases provide bounded functions while the latter will provide other Bessel potentials with lower index, F_{N-r}^z for some $0 \leq r \leq \alpha + \beta$ due to the aforementioned recurrence. If N is big enough (compared with the parameters α and β) one gets that $L = N - r$ can be considered to be a rather large positive number. This allows to bound any term of the above by a constant depending on χ, h_N, the dimension n and β times an integral of the form

$$\int_\gamma \frac{|z|^s}{1+|z|} \int_M \int_M |F_L^z(d_g(x,y))|^2 \chi(x,y) d\mathrm{vol}_g(y) d\mathrm{vol}_g(x) d|z|.$$

This leads using the relation of Bessel potentials and special functions alluded before to

$$\int_M \int_M \int_\gamma |K_{n/2-L-1}(\sqrt{z} d_g(x,y))|^2 \frac{|z|^{s+n/2-L-1}}{1+|z|} d|z| d_g(x,y)^{-n+2L+2} \chi(x,y) dy dx$$

Using the bounds for the Bessel functions one obtains that near zero, i.e. $\sqrt{z} d_g(x,y) \leq 1$, the integrand is bounded by $|z|^{s+n-2L-2}$ which provides a bound depending on n and L but integrates since $0 \notin \gamma$. On the other hand, far away it is bounded by

$$d_g(x,y)^{-n+2L+2} |z|^{s+n/2-L-1} \frac{1}{|z| d_g(x,y)^2} e^{-2\mathrm{Re}\,(\sqrt{z}) d_g(x,y)}.$$

One can neglect the exponential and bound it by one, the decay in z provided by L is enough to bound the inner integral. The two remaining integrals are bounded by compactness by a constant depending on the diameter of M and L.

For general $s \in (0,1)$ we apply the above formula to $-\Delta_g f(y)$ which coincide with $-\Delta_g(f(y) - f(x))$. This last trick allows some extra cancellation. The argument is classical in potential theory: first delete a ball of radius ε and integrate by parts, the boundary terms will be neglected as ε goes to zero. Finally, this provides an integral in the sense of a principal value.

At this point a comment is in order: the Sobolev embedding theorem mentioned above can be proved independently of our next section. Indeed

$$f(x) = (-\Delta_g)^{-n/2-\varepsilon} (-\Delta_g)^{n/2+\varepsilon} f(x) = \sum_\nu \frac{a_\nu}{\lambda_\nu^{n+\varepsilon}} Y_\nu(x)$$

where $(-\Delta_g)^{n/2+\varepsilon} f(x) = \sum_\nu a_\nu Y_\nu(x)$ is the eigenfunction decomposition with $-\Delta_g Y_k = \lambda_k^2 Y_k$. One may now apply Cauchy-Schwarz inequality, Weyl's law estimates and Plancherel to conclude $|f(x)| \leq C \|f\|_{H^{n/2+\varepsilon}}$.

Remark 6.3.1 (Manifolds with boundary) In this case one uses Dirichlet or Neumann eigenfunctions and define acordingly its fractional operator. The parametrix still works due to its local character but the cut off χ should be taken more carefully. It would be enough to take $\chi(x,y)$ to be supported in a ball inside the manifold, but as a consequence the bounds of its derivatives which appear in the constant within the error term degenerate as we approach the boundary. We leave the details to the reader.

Remark 6.3.2 (The Sobolev embedding theorem) It is well known for integral number of derivatives and, therefore, we may restrict ourselves without loss of generality to fractions of the Laplace-Beltrami operator $(-\Delta_g)^s$ where $s \in (0,1)$. The proof follows the usual lines (cf. Brascamp and Lieb treatise [49]). Indeed, for any f with zero mean

$$\|f\|_{L^p(M)} = \sup_{\|g\|_q=1} \left| \int_M f(y)g(y)dy \right| = \sup_{\|g\|_q=1} \left| \int \Lambda^s f \Lambda^{-s} g \right| \leq \|f\|_{H^s(M)} \|\Lambda^{-s} g\|_2$$

where $p^{-1} + q^{-1} = 1$ are conjugates. But the last L^2-norm is bounded since it equals $\int g(-\Delta_g)^{-s} g$ which, due to our formula, is susceptible to an application of the Hardy-Littlewood-Sobolev inequality. The error term introduced is even nicer. To show it does not affect the validity of our statement it is enough to interpolate between the $L^2 \to L^2$ and the $L^1 \to L^\infty$ bounds. The former has already been settled, let us show how to deal with the latter for which the following estimate holds

$$\begin{aligned} \|(-\Delta_g - z)^{-1} Ef\|_{L^\infty(M)} &\leq C \|(-\Delta_g - z)^{-1} Ef\|_{H^{n/2+\varepsilon}(M)} \\ &\leq C \|Ef\|_{H^{n/2+\varepsilon}(M)} \leq C \|f\|_{L^1(M)} \end{aligned}$$

where we are denoting by Ef the space integral in the error term, whose kernel is able to absorb derivatives without affecting integrability.

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