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Asymptotic techniques in the analysis of invariant manifolds of dynamical systems

Memoria presentada para optar al grado

de doctor por la Universidad Autónoma de Madrid

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Residencia de Estudiantes

ABSTRACT. RESUMEN

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Abstract

In this document, we address some questions concerning the structure of the invariant manifolds of vector (and also spinor) fields. We do so by studying the limiting properties of sequences of solutions to certain PDEs.

The first part of the document studies the geometric structure of smooth invariant manifolds of vector and spinor fields abiding to a PDE on certain compact manifolds. More precisely, the vector fields are steady fluid flows (stationary solutions of the Euler equations), while the spinor fields are Dirac spinors (eigenfunctions of the Dirac operator). The central problem is to understand how topologically intricate the invariant manifolds can get to be, while respecting the analytic constraints imposed by the PDEs.

We introduce a new set of techniques to show that there are solutions to the PDEs with arbitrarily complicated invariant manifolds (such as knotted and linked trajectories and invariant tori in the hydrodynamic case), and that these solutions are not rare, provided one looks for these topological structures at high energies (in the spectral sense) and small scales.

In the second part of the thesis we study a new promising framework to address problems of invariant manifolds of vector fields in dimension 3, discovered by C. Taubes. When the helicity of an exact vector field X is non-zero, new non-trivial invariant measures of X can be obtained through an asymptotic analysis of a perturbation of the *Seiberg-Witten equations*. Furthermore, some analytic properties of sequences of solutions to the Seiberg-Witten equations are tied to the dynamical properties of the invariant measures of the vector field: as a striking example, when solutions satisfy a suitable "finite energy condition", they yield measures supported on periodic orbits of X. We recast Taubes framework and we obtain a new related result concerning the measures arising as limits of sequences of solutions to the two dimensional vortex equations.

The final chapter of the thesis answers a question of V. Arnold and B. Khesin concerning the helicity of an exact vector field. One of the remarkable properties of the helicity is its invariance under the action of volume preserving diffeomorphisms. We show that, under some mild technical assumptions, any other integral functional with this symmetry must be a function of the helicity.

Resumen

En esta memoria resolvemos algunos problemas sobre la estructura de las variedades invariantes de campos vectoriales y espinoriales. Para ello, estudiamos las propiedades asintóticas de secuencias de soluciones de ciertas EDPs.

En la primera parte de la tesis, estudiamos la topología de las variedades invariantes suaves de campos vectoriales y espinoriales que satisfacen una EDP en ciertas variedades compactas. De forma más precisa, los campos vectoriales son soluciones estacionarias de la ecuación de Euler, mientras que los campos espinoriales son autofunciones del operador de Dirac. El problema principal es comprender cómo de intrincadas pueden llegar a ser las variedades invariantes de estos campos.

Presentamos una serie de técnicas que permiten demostrar que hay soluciones con variedades invariantes tan complicadas como se desee (por ejemplo, fluidos ideales estacionarios con torbellinos anudados); y que estas soluciones no son raras, siempre y cuando uno busque estas estructuras complicadas a altas energías.

En la segunda parte de este documento estudiamos un nuevo marco conceptual muy prometedor para el estudio de variedades invariantes de campos vectoriales exactos en dimensión tres, descubierto por C. Taubes. Cuando la helicidad de un campo exacto es distinta de cero, se pueden obtener nuevas medidas invariantes del campo como límites de secuencias de soluciones de las ecuaciones de Seiberg-Witten (perturbadas de manera adecuada). Revisamos el marco conceptual de Taubes, y presentamos un nuevo resultado sobre los límites de secuencias de soluciones a la ecuación del vórtice en dimensión dos (que son un modelo local de las ecuaciones de Seiberg-Witten).

El último captulo de este documento resuelve una conjetura de V. Arnold y B. Khesin a propósito de la helicidad. Demostramos que todo funcional integral en el espacio de campos exactos de clase C^1, invariante ante la acción de difeomorfismos que preservan el volumen, tiene que ser una función de la helicidad.

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Chapter 1

Introduction, summary, and conclussions. Introducción, resumen, y conclusiones

In recent years, new experimental techniques [39, 40] have enabled the controlled observation of topological structures emerging in a wide range of physical processes, from fluid dynamics to solid state physics. These topological structures take the form, for example, of knotted vortex tubes in fluid flows and plasmas, and of topological dislocations in superconductors. They provide a powerful visual tool to gain understanding of complex physical systems.

Mathematically, these physical phenomena are described by (vector valued or scalar) fields abiding to a system of Partial Differential Equations; and the emerging topological structures are instances of the so-called *invariant manifolds* of their solutions.

For example, in Fig. 1.1, we see a *vortex tube* of water with the form of a trefoil knot. If we describe the water flow by assigning, to each point x in the domain of the fluid and to each instant of time t, a vector $u(x,t)$ (the velocity field), this vortex tube is just the region enclosed by an *invariant torus* of the vorticity field at a fixed time t, $\omega(\cdot,t) = \operatorname{curl} u(\cdot,t)$. Other interesting examples of invariant manifolds of vector fields would be periodic orbits (like the one that runs through the core of the invariant torus) and more generally, supports of invariant measures of the flow.

In many other condensed matter phenomena the state of the system is described by a scalar field (or more generally, by a section of a complex vector bundle) and the invariant manifolds observed in experiments correspond to connected components of their zero set (also called nodal set).

The results coming from these experiments offer a very interesting challenge for mathematicians: to understand how topologically complex the invariant manifolds can get to be, while respecting the analytic constraints imposed by a

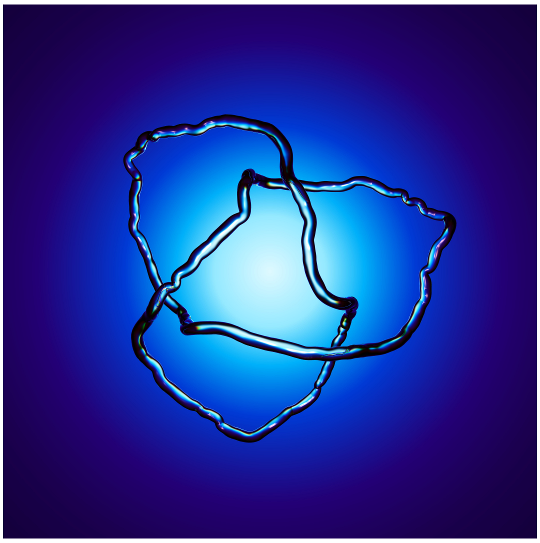

Figure 1.1: A knotted vortex tube of water obtained at the Irvine Lab in Chicago. Photo courtesy of W. Irvine

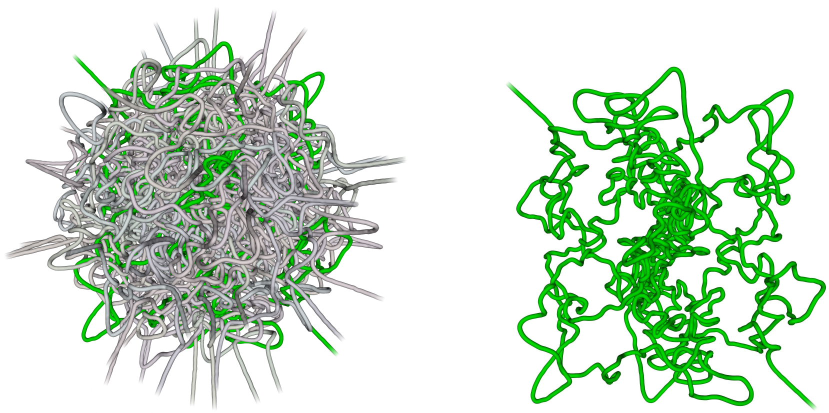

Figure 1.2: The zero set of an eigenfunction of the 3D quantum harmonic oscillator, and one of its knotted connected components. Simulation courtesy of Mark Dennis.

PDE. Addressing this problem requires to orchestrate concepts and techniques from analysis, differential geometry and mathematical physics; so it is a very good test of our current level of understanding of these fields.

1.1 Realization problems for vector and scalar fields

To illustrate the type of mathematical questions that spring from the above discussion, consider the following two problems, which would serve as prototypical examples of the problems that we will address in this thesis:

- Problem A (vector-type problem): Find a vector field u satisfying the stationary Euler equations of hydrodynamics

  $$u \times \omega = \nabla B, \;\; \mathrm{div} u = 0, \;\; \omega := \mathrm{curl}\, u,$$

  (where $B := \frac{|u|^2}{2} + p$ is the so called Bernoulli function, and p is the pressure) such that the vorticity field ω has an invariant torus (i.e, a vortex tube) diffeomorphic to a tubular neighborhood of a trefoil knot (or, more generally, diffeomorphic to a tubular neighborhood of any given knot). (That is, show that our ideal fluid models are compatible with what we see in Fig. 1.1)

- Problem B (scalar-type problem): Find a complex-valued function $\psi : \mathbb{R}^3 \to \mathbb{C}$, satisfying the Helmholtz equation

  $$\Delta\psi + \psi = 0,$$

  and whose zero set (which is a collection of curves, as long as zero is not a critical value) has a connected component diffeomorphic to a given knot L. This problem is a simplified version of the one coming from Fig 1.2: there, we see the zero set (and a knotted connected component of it) of a complex-valued function ψ satisfying $\Delta\psi - |x|^2\psi + \lambda\psi = 0$.

These problems are often called *realization problems*: one is asked to realize a given manifold as invariant manifold of a solution to a PDE.

Now, for the solutions to be relevant to the physical context in which the problems arise, one should add a further requirement to the problems above. The invariant manifold that one wishes to realize (be it a zero set or an invariant torus) must be *structurally stable*. This simply means that the invariant manifold must survive if one perturbs the solution a bit (in the C^m sense). Indeed, in an experiment, the structures that we observe must be robust under small perturbations for us to observe them at all. [1]

[1] This is not just because of the classical "observing perturbs the observed" dictum. When we observe a physical process, we are recording a finite amount of information —be it with our eyes or with a different data acquisition device—; that is, we observe at a finite resolution. So we can't actually see a solution to a PDE, just an open ϵ-neighborhood of a solution in the C^m topology.

Before disclosing how to address the problems above, let us put them in a more historical perspective (because in fact, as is often the case, these mathematical problems predate the recent experiments that we used to motivate them). Problem A was already posed by the renowned physicist William Thomson (Lord Kelvin) around the third quarter of the nineteenth century [70]. He was, however, not thinking in vortex knots in water, but in the (then ubiquitous) aether. While his physical motivation turned out to be misguided (he suggested atoms to be vortex knots of the aether fluid, of different knot-types) the mathematical problem stimulated the development of knot theory, and remained unsolved for more than a century despite efforts of later mathematicians.

As for Problem B, it was first studied by M. Berry and M. Dennis in [12]. There, Berry and Dennis devised a method to obtain solutions of the Helmholtz equation with a zero set L diffeomorphic to a torus knot, an conjectured that Problem B always has a solution, for any knot type L. Berry also considered ([13]) generalizations of Problem B for PDEs of the form $H\psi = \lambda\psi$, with H being a Schődinger operators $H := -\Delta + V$ (for example, the Hydrogen atom or the quantum harmonic oscillator), suspecting again that solutions exhibiting any knot should exist.

Analogs of Problem B, which concerns more generally the allowed shapes of zero sets of solutions to elliptic PDEs, also have some interesting history. In the case of the Poisson equation,

$$\Delta\psi = \rho$$

the problem can again be traced back to nineteenth century mathematical physicists, who were intrigued by it because of their interest in describing the surfaces of constant gravitational or electric potential (for example, to characterize the possible equilibrium shapes of a gravitating fluid). In the case of the Cauchy-Riemann equations in a complex manifold X

$$\bar{\partial}\psi = 0, \quad \psi : X \to \mathbb{C}$$

the problem actually corresponds to a weakened version of the celebrated second Cousin problem: what (real) codimension 2 submanifolds of a complex manifold can be the nodal set of a holomorphic function? As first shown by K. Oka [63] (in what became both one of the precursors of sheaf theoretical methods in algebraic geometry, and a first hint of M. Gromov's Oka Principle [34]), in the case of a Stein manifold, the only obstruction is the obvious one: it must be possible to realize the submanifold in question as the zero set of a continuous, complex-valued function.

1.1.1 The euclidean case: a general strategy

More recently, Alberto Enciso and Daniel Peralta-Salas introduced a very general strategy [27, 24] that yields flexibility results in the context of realization problems (in a similar spirit to the previous result by K. Oka, albeit through completely different techniques). More precisely, by flexibility we mean that the PDE does not impose any additional restriction to the invariant manifolds that can appear: one can find solutions realizing any conceivable manifold.

This strategy is in itself rather malleable, and although its adaptation to particular PDEs can be quite non-trivial (for example, in the case of the recent resolution of Problem A for vector fields in $\mathbb{R}^3$, also by Enciso and Peralta-Salas [24]), the main scheme can be readily grasped with a simple enough example. Problem B provides a good template for this purpose.

Recall that Problem B asks us to find a complex-valued function ψ in $\mathbb{R}^3$ that satisfies the Helmholtz equation $\Delta\psi + \psi = 0$, and such that the set $\psi^{-1}(0)$ has a connected component diffeomorphic to a given knot L.

Given such a problem, Enciso and Peralta-Salas' scheme proceeds as follows. First, one solves the problem *locally*, that is, one finds a solution $\widetilde{\psi}$ of the equation $\Delta\widetilde{\psi} + \widetilde{\psi} = 0$ *in a tubular neighborhood of the knot* L.

The local solution $\widetilde{\psi}$ must be such that, on the one hand, $\widetilde{\psi}|_L = 0$ (so that it realizes L as zero set), and on the other hand, the pair of vector fields $(\nabla\widetilde{\psi}_1, \nabla\widetilde{\psi}_2)$ at L spans the normal bundle of L. This last condition ensures that the zero set L is robust under small, C^1 perturbations of $\widetilde{\psi}$ (think, for example, on the zero set of a real function: if the gradient of the function does not vanish at the zero set, a small enough perturbation of the function will still have a zero set diffeomorphic to the original one). For second order elliptic PDEs such as the Helmholtz equation, a local existence theorem (such as the Cauchy-Kovalevskaya theorem, or a well-posed boundary value problem) can be used to ensure that such a local solution exists.

In the next step, one finds a *global solution* ψ to the PDE approximating the local one $\widetilde{\psi}$ to any desired precision. This local-to-global approximation is achieved by means of a Runge-type approximation theorem, such as the Lax-Malgrange theorem and its variants. Given some local solution to an elliptic PDE in some open set U, these theorems provide a function satisfying the PDE in the whole space and approximating the local solution in U arbitrarily well.

The condition of the normal derivatives of $\widetilde{\psi}$ at L that we required in the previous step makes the zero set of $\widetilde{\psi}$ robust under perturbation. Hence, the knot does not dissolve in the process of local-to-global approximation, and an almost-exact copy of it (obtained deforming L trough a diffeomorphism close to the identity) persists in the zero set of the global solution, thus Problem B is solved.

Both the local existence and global approximation theorems that we have described above are not specific to the Helmholtz equation; they hold in a wide variety of contexts (and when not, suitable analogs are often at reach). This makes the overall strategy flexible enough to be adapted to many different realization problems (albeit this adaptation can be highly non-trivial, as we already pointed out).

For example, Enciso and Peralta-Salas used it to show the existence of solutions to a very general class of second order elliptic PDEs with zero sets of arbitrarily complicated topology [27] (including the Helmholtz equation above). Later on, together with D. Hartley, they used it to give a positive answer to M. Berry conjecture about the existence of arbitrarily complicated knots in the zero sets of eigenfunctions of the quantum harmonic oscillator [22] and of the hydrogen atom [23]. Further still, the resolution of Enciso and Peralta-Salas of Kelvin's vortex atom conjecture [24] (on the existence of stationary fluid flows with arbitrarily knotted and linked vortex tubes) is based on a similar strategy.

But, however wide ranging, the strategy works only for PDEs in *open spaces*. Trying to carry it out in *compact manifolds* encounters a fundamental obstruction. As could be expected, this obstruction lies in the local-to-global approximation step.

It is exactly of the same nature that the obstruction we encounter when trying to extend a holomorphic function in an annulus to an entire function: a singularity is bound to appear at the disk enclosed by the annulus.

Indeed, Lax-Malgrange approximation Theorem requires the local solution to be defined in an open set U whose complement has no compact components (for example a ball, or a solid torus). If the complement of the set U has a compact component K, in general one can do no better than finding a global solution *with singularities in K*.

1.1.2 The compact case: inverse localization at small scales and large wavenumbers

In Part I of this thesis, we present a new strategy to bypass the obstruction explained in the previous subsection, and apply it to solving realization problems for invariant manifolds of vector and spinor fields in compact spaces. In particular, we extend Enciso and Peralta-Salas' existence result for stationary fluid flows with arbitrarily knotted and linked vortex tubes (which hold in euclidean space) to $\mathbb{S}^3$ and $\mathbb{T}^3$; and we study the topology of the nodal sets of Dirac spinors (eigenfunctions of the Dirac operator) on $\mathbb{S}^n$ and $\mathbb{T}^n$.

The interest of extending our grasp of realization problems to the compact case is more than purely technical. Indeed, notice, for example, that fluid flows in $\mathbb{T}^3$ and $\mathbb{S}^3$ are closer to the fluids in the real world than the ones in euclidean space that Enciso and Peralta-Salas originally considered in [24]. This might seem puzzling from a purely geometrical viewpoint, but it is not so from the analytical viewpoint. Real fluid flows have finite energy (mathematically, finite L^2 norm), as fluids in compact spaces do; whereas the solutions of the Euler equations in [24] can decay only as fast as the inverse of the distance to the origin. This finite energy considerations are much more relevant when studying some regimes of fluid motion than naive considerations about the geometry of the container.

The main idea that enables us to bypass the obstruction in compact manifolds is based on the asymptotic analysis of the eigenfunctions of a self-adjoint operator (be it the curl operator, when dealing with stationary fluid flows in Chapter 2, or the Dirac operator for spinors in Chapter 3) at very high energies (in the spectral sense) and correspondingly small scales.

Let M be a compact Riemannian manifold. Let us examine with a microscope the neighborhood of a point p, that is, we consider a ball of small radius $1/\sqrt{\lambda}$

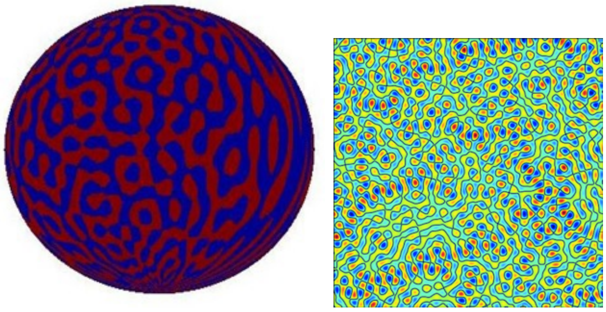

Figure 1.3: The nodal domains of high energy eigenfunctions of the laplacian on the sphere (left) and the torus (right)

centered at the point and rescale the geodesic coordinates by a factor $\sqrt{\lambda}$, so that the ball looks like the Euclidean ball of radius 1, with the point p at the origin. Now, an eigenfunction of a second order self-adjoint operator on M (for simplicity, think of eigenfunctions of the laplacian, as represented in Fig. 1.3) of large enough eigenvalue λ, looks under this microscope arbitrarily close to a solution of an euclidean PDE with constant coefficients (e.g the Helmholtz equation $\Delta\psi + \psi = 0$).

One could be tempted to jump from this observation to the conclusion that, if one goes to high enough energies and looks at small scales, it could be possible to find eigenfunctions having the desired invariant manifolds. After all, they look like euclidean eigenfunctions at these small scales, and euclidean eigenfunctions can be shown (by Enciso and Peralta-Salas techniques) to display invariant manifolds of arbitrary topology.

However, notice that the euclidean techniques only ensure that *some* particular solutions have the desired structures; since we cannot ensure, in general, that there is a high-energy eigenfunction on the compact manifold that approximates a given euclidean solution, the euclidean small scale behavior is a priori useless for our purposes. The key to our approach is precisely to revert this situation, that is, to ensure that, in certain compact manifolds, any given solution in euclidean space is indeed the small scale approximation of a sufficiently high-energy eigenfunction.

That is, if one examines the eigenfunctions of the relevant operator at increasingly high energies and correspondingly small scales, one ends up seeing, at any desired resolution, *any given solution* of energy 1 of the corresponding operator on euclidean space. We call this kind of result an *inverse localization theorem*, since it can be pictured as picking a euclidean solution in the unit ball, shrinking the ball, and localizing it in the compact manifold. A crucial ingredient in the proof is that the relevant operator has increasingly degenerate eigenvalues, that is, there is an increasingly large pool of eigenfunctions for us to combine as the eigenvalue increases. This is why our results work in spheres and tori.

This approximation result implies that *any property* that solutions of the euclidean operator display on compact sets is also exhibited at small scales by high energy eigenfunctions of the operator on the sphere and the torus, provided such property is robust under suitably small perturbations. The problem is thus reduced to the analog in euclidean space, where one can rely on Enciso and Peralta-Salas technique.

Let us end this introduction of the Part I of the thesis by exposing in more detail the contents of each chapter.

1.2 A closer look at the contents of Chapter 2: Realization of knotted vortex structures in high energy Beltrami fields

Chapter 2 addresses the topology of vortex lines and vortex tubes of stationary solutions to the Euler equations on the round sphere $\mathbb{S}^3$ and the flat torus $\mathbb{T}^3$. We will concentrate on a special family of solutions, Beltrami fields, which are in a sense the building blocks of fluid motion.

For us, a Beltrami field u is an eigefunction of the curl operator

$$\mathrm{curl} u = \lambda u.$$

(Eigenfunctions of the curl operator are sometimes called in the literature *strong* Beltrami fields, with the term "Beltrami field" left to include vector fields solving the above equation with a non-constant factor λ.)

Note that a Beltrami field is automatically divergence free, i.e, its flow preserves the volume form of the manifold. It is also easy to see that Beltrami fields are, in particular, stationary solutions of the Euler equations

$$u \times \omega = \nabla B, \quad \mathrm{div} u = 0, \omega = \mathrm{curl}\, u,$$

Finally, as eigenfunctions of the curl operator, Beltrami fields form a basis in the space of all exact divergence free vector fields. Our main result in Chapter 2 states that there are many Beltrami fields on $\mathbb{S}^3$ and $\mathbb{T}^3$ with vortex lines and vortex tubes of any given knot and link type, provided the eigenvalue is high enough and one looks at small scales.

We note that by vortex line we precisely mean an orbit of the vector field ω, while by vortex tube we mean an embedded solid torus in the manifold which is invariant under the flow of ω.

Theorem 1.2.1 (Arbitrarily knotted vortex estructures in high energy Beltrami fields)**.** *Let $\mathcal{S}$ be a finite union of (pairwise disjoint, but possibly knotted and linked) closed curves and tubes in $\mathbb{S}^3$ or $\mathbb{T}^3$. In the case of the torus, we also assume that $\mathcal{S}$ is contained in a contractible subset of $\mathbb{T}^3$. Then for any large enough odd integer λ there exists a Beltrami field u satisfying the equation* $\mathrm{curl}\, u = \lambda u$ *and a diffeomorphism Φ of $\mathbb{S}^3$ or $\mathbb{T}^3$ such that $\Phi(\mathcal{S})$ is a union of vortex lines and vortex tubes of u. Furthermore, this set is structurally stable.*

This theorem corresponds to Theorem 2.1.1 in Chapter 2. The proof is based on an inverse localization theorem for Beltrami fields at small scales, following the strategy described in the previous section. In order to state it more precisely, let us fix a point p on the manifold $\mathbb{M}^3$ ($\mathbb{S}^3$ or $\mathbb{T}^3$) and take a patch of normal geodesic coordinates $\Psi : \mathbb{B} \to B$ centered at p, where by B (resp. $\mathbb{B}$) we denote the ball in $\mathbb{R}^3$ (resp. the geodesic ball in $\mathbb{M}^3$) centered at the origin (resp. at p) and of radius 1. A vector field u under these coordinates reads

$$\Psi_* u(x) = \sum_{i=1}^{3} u^i(x)\, e_i\,,$$

where Ψ_* is the pushforward of Ψ and where $u^i(x)$ are the three components of $\Psi_* u$ in the Cartesian basis $\{e_i\}_{i=1}^3$ of $\mathbb{R}^3$. The small scale behavior of the vector field u is then described by the rescaled vector

$$\Psi_* u\Big(\frac{x}{\lambda}\Big) := \sum_{i=1}^{3} u^i\Big(\frac{x}{\lambda}\Big)\, e_i\,.$$

for $x \in B$.

The inverse localization theorem reads

Theorem 1.2.2 (Inverse localization for high energy Beltrami fieds)**.** *Let v be a Beltrami field in $\mathbb{R}^3$, satisfying* $\operatorname{curl} v = v$*. Fix any positive numbers ϵ and m. Then for any large enough odd integer λ there is a Beltrami field u, satisfying* $\operatorname{curl} u = \lambda u$ *in $\mathbb{M}^3$, such that*

$$\left\| \Psi_* u\Big(\frac{\cdot}{\lambda}\Big) - v \right\|_{C^m(B)} < \epsilon\,. \tag{1.2.1}$$

The effect of the diffeomorphism Φ in Theorem 1.2.1 becomes thus clear: it consists in shrinking a ball containing the set of tubes $\mathcal{S}$ into a ball of very small radius $1/\lambda$.

As a closing comment for this Chapter, we note that a particularly attractive aspect of our result from the physical viewpoint is that it correlates geometric intricacy of the fluid trajectories with energy and length scales. This is somehow related to turbulence, where one observes the emergence of complicated vortex structures at small scales and large wave-numbers.

1.3 A closer look at the contents of Chapter 3: the topology of the zero sets of high energy Dirac spinors

In Chapter 3, the objects of interest are Dirac spinors on n-dimensional spheres instead of vector fields, and the role of the invariant manifolds is played by their zero sets. [2]

[2]The results of Chapter 3 can also be proven with minor variations in the n-dimensional flat torus, and we include some comments about this in the final section of the chapter. The reason for

Recall that eigenfunctions of the Dirac operator are sections of a hermitian vector bundle S of complex rank $r(n) = 2^{\lfloor \frac{n}{2} \rfloor}$, called the spinor bundle. If the spinor bundle is trivial, as is the case in spheres, a choice of trivialization makes any section ψ of S into a collection $(\psi_1, ..., \psi_{r(n)})$ of $r(n)$ complex-valued functions. In a spin manifold of dimension 3 or higher, the regular zero sets of a spinor are empty (since $2r(n) > n$, the relevant codimension is negative), so we will focus our attention on the topology of the zero sets of the spinor components ψ_i. These spinor components have values in $\mathbb{C}$, so their zero sets are, generically, codimension 2 submanifolds of $\mathbb{S}^n$.

For example, in $\mathbb{S}^3$, a spinor can be decomposed in two components $\psi := (\psi_1, \psi_2)$. If ψ is an eigenfunction of the Dirac operator, ψ_1 and ψ_2 are tied by a first order partial differential relation. A subtler analog of Problem B can then be posed: if L_1 and L_2 are two disjoint closed curves, arbitrarily knotted and linked between them, is it possible to find an eigenfunction of the Dirac operator such that L_1 is a component of the zero set of ψ_1, and L_2 is a component of the zero set of ψ_2 ?

Our results in this Chapter imply that this is indeed the case. More precisely, we will show that there are many eigenfunctions of the Dirac operator with zero sets of arbitrarily complex topology, for any choice of trivialization of the spinor bundle. Here is a simplified statement of our main theorem (Theorem 3.1.1 in Chapter 3)

Theorem 1.3.1 (Realization theorem for high energy Dirac spinors in $\mathbb{S}^n$)**.** *In $\mathbb{S}^n$, for $n \geqslant 3$, let $\Sigma := \{\Sigma_1, ..., \Sigma_{r(n)}\}$ be any collection of codimension 2 smooth submanifolds of arbitrarily complicated topology ($r(n)$ being the complex dimension of the spinor bundle). There is always an eigenfunction $\psi = (\psi_1, ..., \psi_{r(n)})$ of the Dirac operator (in fact, infinitely many of them) such that the submanifold Σ_i, modulo ambient diffeomorphism, is a structurally stable component of the nodal set of the spinor component ψ_i. The result holds for any choice of trivialization of the spinor bundle.*

As was the case with Theorem 1.2.1, the emergence of these complicated structures takes place at small scales and sufficiently high energies.

The proof consists mainly of two ingredients. The first one is an inverse localization result, like in the previous chapter, this time for eigenfunctions of the Dirac operator (Theorem 3.2.1 in Chapter 3):

Theorem 1.3.2 (Inverse localization for high energy Dirac spinors)**.** *Let $\phi := (\phi_1, ..., \phi_{r(n)})$ be a $\mathbb{C}^{r(n)}$ valued function in $\mathbb{R}^n$, satisfying the Dirac equation $D_0\phi = \phi$. Fix an integer $m \geqslant 1$ and a positive constant δ. For any large enough positive integer k, there is an eigenfunction ψ of the Dirac operator D on $\mathbb{S}^n$ of eigenvalue $(\frac{n}{2} + k)$ such that*

$$\left\| \phi - \widehat{\Psi}_* \psi\left(\frac{\cdot}{k}\right) \right\|_{C^m(B)} < \delta \,.$$

not including here the results on the torus at the same level as those on the sphere (in contrast to what is done in Chapter 2) is that they are somewhat less general: the n-torus has a spin structure (and an associated Dirac operator) for each element in the first cohomology group $H^1(\mathbb{T}^n, \mathbb{Z})$ and our Theorem can only be stated for the trivial one.

Here, D_0 is the standard euclidean Dirac operator, and as in the previous subsection, we have a geodesic patch $\Psi : \mathbb{B} \to B$ defined on a ball $\mathbb{B}$ of radius 1 centered at an arbitrary point $p \in \mathbb{S}^n$; with the map $\widehat{\Psi}_* : S|_{\mathbb{B}} \to B \times \mathbb{C}^{r(n)}$ being an analog of the pushforward map for the spinor bundle.

This reduces the problem to the euclidean case. The second ingredient is the following euclidean result (Theorem 3.1.2 in Chapter 3)

Theorem 1.3.3 (Realization theorem for Dirac spinors in $\mathbb{R}^n$)**.** *Fix an integer $m \geqslant 1$, and an arbitrarily small real number $\epsilon > 0$. Inside the unit ball $B \subset \mathbb{R}^n$, consider a collection $\mathfrak{S} := \{\Sigma_a\}_{a=1}^{r(n)}$ of $r(n)$ closed, pairwise disjoint, smooth codimension 2 submanifolds. For any given $\lambda \in \mathbb{R}$, there is a $\mathbb{C}^{r(n)}$-valued function $\phi := (\phi_1, ..., \phi_{r(n)})$, satisfying the Dirac equation $D_0\phi = \lambda\phi$ on $\mathbb{R}^n$, and a diffeomorphism $\Phi_0 : B \to B$ satisfying $||\Phi_0 - Id||_{C^m} \leqslant \epsilon$, such that $\Phi_0(\Sigma_a)$ is a component of the nodal set of ϕ_a. Furthermore, these sets are structurally stable.*

The proof is an adaptation of the strategy in [27] that we introduced in section 1.1.1 to solve Problem B. However, this adaptation is not immediate and requires some further considerations. The main reason is that, when solving Problem B, robustness of the nodal set under perturbations was ensured by prescribing the gradient of the local solution on the manifold we wished to realize, and we could prescribe the gradient because we were solving a second order Cauchy problem. But the Dirac operator is of order 1, so we can only construct local solutions with prescribed 0-jet: robustness does not come out automatically, and a more involved construction is needed. [3]

1.4 Inverse localization for spherical harmonics: Chapter 4

Both inverse localization results, for the curl and Dirac operators, rely on an analogous result for eigenfunctions of the laplacian. This result and a further refinement of it (more precisely, a *multiple* inverse localization result, that finds a spherical harmonic with prescribed small scale behavior at distinct regions of the sphere) are proved in Chapter 4.

Theorem 1.4.1 (Inverse localization for spherical harmonics)**.** *Let ϕ be a $\mathbb{R}^q$-valued function in $\mathbb{R}^n$, satisfying $\Delta\phi + \phi = 0$. Fix a positive integer m and a positive constant δ'. For any large enough integer k, there is a $\mathbb{R}^q$-valued spherical harmonic $Y := (Y_1, ..., Y_q)$ on $\mathbb{S}^n$ with energy $k(n + k - 1)$ such that*

$$\left\| \phi - Y \circ \Psi^{-1}\left(\frac{\cdot}{k}\right) \right\|_{C^m(B)} \leqslant \delta' .$$

It is understood that each component Y_j of Y is a real valued spherical harmonic.

[3] One could object that the same problem appears in Chapter 2, with the curl operator, and that is indeed the case, but in Chapter 2 we could rely on the realization theorem for Beltrami fields in Euclidean space due to Enciso and Peralta-Salas [24].

The proofs of the localization results in Chapters 2 and 3 combine the proposition above with Weitzenbőck-type formulas for the curl and Dirac operators.

We also prove in this Chapter a refinement of the above inverse localization result. In order to state it, let $\{p_\alpha\}_{\alpha=1}^{\Lambda}$ be a set of points in $\mathbb{S}^n$, with Λ an arbitrarily large (but fixed throughout) integer. We denote by $\Psi_\alpha : \mathbb{B}_\rho(p_\alpha) \to B_\rho$ the corresponding geodesic patches on balls of radius ρ centered on the points p_α. We fix a radius ρ such that no two balls intersect, for example by setting

$$\rho := \frac{1}{2} \min_{\alpha \neq \beta} \operatorname{dist}_{\mathbb{S}^n}(p_\alpha, p_\beta) .$$

It would be also necessary for our purposes to pick the points $\{p_\alpha\}_{\alpha=1}^{\Lambda}$ so that no pair of points are antipodal in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, i.e, so that $p_\alpha \neq -p_\beta$ for all α, β. The reason is that spherical harmonics of energy $k(n+k-1)$ have parity $(-1)^k$, that is, $Y(p_\alpha) = (-1)^k Y(-p_\alpha)$ (they are the restriction to the sphere of real harmonic homogenous polynomials of degree k).

Proposition 1.4.2 (Multiple inverse localization for spherical harmonics)**.** *Let $\{\phi_\alpha\}_{\alpha=1}^{\Lambda}$ be a set of Λ $\mathbb{R}^q$-valued functions in $\mathbb{R}^n$, $\phi_\alpha := (\phi_{\alpha 1}, ..., \phi_{\alpha q})$, satisfying $\Delta\phi_\alpha + \phi_\alpha = 0$. Fix a positive integer m and a positive constant δ. For any large enough integer k, there is a $\mathbb{R}^q$-valued spherical harmonic $Y := (Y_1, ..., Y_q)$ on $\mathbb{S}^n$ with energy $k(n+k-1)$ verifying the bound*

$$\left\| \phi_\alpha - Y \circ \Psi_\alpha^{-1}\left(\frac{\cdot}{k}\right) \right\|_{C^m(B)} < \delta$$

for all $1 \leqslant \alpha \leqslant \Lambda$.

An analog of Proposition 1.4.1 can be proved for the torus $\mathbb{T}^n$ (this is done in Chapter 2 for $n = 3$, but the argument generalizes to any dimension). However, the proof of Proposition 1.4.2 makes use of very specific properties of Gegenbauer polynomials (the building blocks of spherical harmonics) that have no equivalent in the torus. This prevents us from translating the multiple inverse localization to the torus case.

To conclude with the description of Part I of the thesis, we note that Chapter 2 is based on the paper [29] of the author in collaboration with Alberto Enciso and Daniel Peralta-Salas, while Chapter 3 is based on [69].

1.5 A change in perspective: detecting invariant sets through asymptotic limits of the Seiberg-Witten equations

Part I of the thesis can be described thus: we take advantage of small-scale, highly oscillatory asymptotic phenomena (e^{ikx}, with $k \to \infty$) in the solutions of a PDE in a compact manifold to *prescribe* invariant sets of a vector (or scalar) field that *satisfies* the PDE. These invariant sets are, so to speak, *allowed by the*

PDE, and optional for its solutions: not every solution to the PDE must display them.

Part II of the thesis is also concerned with invariant manifolds of particular classes of vector fields, and it also uses asymptotic analysis of PDEs for their study, albeit the viewpoint is rather different.

This change in viewpoint could be described as follows. As in Part I, a space of vector fields abiding to some constraint is given. This time, the constraint could in principle range from the analytical (the space of solutions to a PDE) to the purely topological (the space of all smooth vector fields on a given manifold). What we want to know, given the constraint, is which invariant sets are *mandatory*, and which are *forbidden*.

To do this, we take advantage of concentration phenomena in asymptotic analysis. (This will become transparent in the subsequent paragraphs.) The idea is, first, to use a vector field to write a PDE depending on a parameter. The vector field itself *is not this time a solution to the PDE*, it is just used as datum to *define* it. The solutions to these PDEs concentrate on some very particular sets of the ambient manifold, often showing exponentially fast decays away from them, decays that become more pronounced as the parameter goes to infinity (e^{-kx}, $k \to \infty$). The PDEs are cleverly designed in such a way that these sets of the manifold where the solutions concentrate end up being *invariant sets* of the vector fields used as data, and the solutions themselves converge to *invariant measures* of the vector field. Hence we could describe this scheme as detecting invariant sets, rather than prescribing them.

To better appreciate this switch in the point of view, consider first the following statements. They range from easy to prove results to well known open conjectures, but they are all examples of the new perspective described above:

- 1) A simple statement: any (non identically zero) area-preserving vector field on $\mathbb{S}^2$ has a periodic orbit.
- 2) The Poincare-Hopf lemma: the Euler characteristic of a manifold is equal to the suitable counted number of zeroes of any vector field whose zeroes can be counted.
- 3) A classical corollary from Morse theory: in a closed Riemannian manifold M, a generic gradient vector field (i.e, the gradient of a function) has its number of zeroes of index k bounded from below by the k-th Betti number of M.
- 4) Two deep theorems: 4.1) a 3-dimensional vector field having non-zero topological entropy must have a periodic orbit (this is a result by Katok); 4.2) the Reeb vector field of a contact form in dimension 3 must always have a periodic orbit (this is the celebrated Weinstein conjecture in dimension 3, solved by C. Taubes).
- 5) Three well known open conjectures: Seifert conjecture in the smooth volume preserving case, which posits that a volume preserving vector field without zeroes in $\mathbb{S}^3$ must always have a periodic orbit. Gottschalk

conjecture, also in the volume preserving case, which posits that there are no volume preserving flows in $\mathbb{S}^3$ all whose orbits are dense. And, finally, the above mentioned Weinstein conjecture in any (odd, since we are dealing with contact forms) dimension.

In all of them, a constraint on the vector field (the topology of the underlying manifold, the fact that it preserves some measure, etc) imposes some of the properties of its invariant sets.

It is perhaps very surprising that the asymptotic analysis of PDEs and concentration phenomena have anything to do at all with some of the above problems. But it is indeed the case. This relationship could be traced back to Witten's account of Morse theory [74], which in particular yields an elegant interpretation of item 3) above.

Part II of this thesis will in particular be concerned with asymptotic analysis in relation to items 4.2) and 5).

1.5.1 A very simple example

Let us explain more precisely how the aforementioned concentration phenomena tend to manifest and how this yields a new approach to structural problems of invariant manifolds.

The simplest (but nonetheless bearing relevance to the upcoming discussion) instance of an asymptotic decay phenomenon happens for the following family of ODEs, labelled by a parameter $r > 0$

$$\frac{1}{r}\frac{d^2u_r}{dx^2} - u_r = 0;\ u_r(0) = 1\,;\ \lim_{x\to\infty} u_r(x) = 0 \tag{1.5.1}$$

on the half space $\mathbb{R}_{\geqslant} := [0,\infty)$. We are interested in the behavior of the solutions u_r as $r \to \infty$.

By setting the parameter to infinity we obtain, in a purely formal way, the equation $u_\infty = 0$. This equation, together with the original boundary conditions, is often called the *limiting problem*. A natural question to ask is what do the "solutions" to the limiting problem tell us about the behavior of sequences u_r of solutions as $r \to \infty$.

There is the caveat that the limiting problem has no continuous solutions. In this case it is obvious, but more generally, it could be expected from the fact that the order of the differential equation drops, so we can no longer expect to satisfy the original boundary conditions.

Still, some lesson can be drawn. Note that the unique solution to (1.5.1) is $u_r(x) = exp(-\sqrt{r}x)$. Note, on the other hand, that the function defined as $u_\infty(x) := 0$ if $x \in \mathbb{R}_{\geqslant} \setminus \{0\}$ and as $u_\infty(x) := 1$ in $x = 0$ can be considered to be a continuous solution to the limiting problem in $\mathbb{R}_{\geqslant} \setminus \{0\}$.

We observe that, as $r \to \infty$, u_r converges uniformly in compact sets of $\mathbb{R}_{\geqslant} \setminus \{0\}$ to $u_\infty = 0$, except for an ever narrower window $[0, l)$ (with $l \sim r^{-\frac{1}{2}}$) near the boundary (the simplest instance of a *boundary layer*). In this region, a sudden readjustment occurs for u to meet the boundary condition $u(0) = 1$.

This is the typical behavior in most asymptotic problems: the solutions display a kind of non-uniform convergence. That is, they converge uniformly to solutions of a simpler PDE, except in some regions where sudden transitions appear and where, in the limit, the derivatives of the solution blow up.

The presence of boundaries is not at all required for these phenomena to appear. The general pattern is always thus: on a closed manifold (or on a manifold with boundary, in which case we suppose appropriate boundary conditions) we have a PDE of the form

$$Lu_r + r\Phi(u_r, \partial u_r) = 0 \tag{1.5.2}$$

where L is an elliptic operator acting on sections of some vector bundle, and $\Phi(u, \partial u)$ is a function of u and its first derivatives. One knows that smooth solutions to Eq. (1.5.2) exist for any r. However, the space of continuous solutions to $\Phi(u, \partial u) = 0$ is either empty (for instance, because boundary conditions fail to be satisfied, as in the previous example); or "trivial": made up of solutions (for example, $u = 0$) that do not satisfy some reasonable feature that solutions to (1.5.2) all display (for example, being always uniformly bounded away from zero in some region of the manifold).

This unavoidable dilemma is ultimately resolved by the solutions quickly transitioning, and their derivatives ultimately blowing up, on certain subsets of M. This is the ultimate lesson of this type of asymptotic analysis.

1.5.2 C. Taubes solution of the Weinstein conjecture. Example 1.5.1 revisited

One can use this concentration phenomenon to *detect* invariant sets of a vector field X. The general scheme is thus: the vector field X is used to define a PDE

$$Lu_r + r\Phi(u_r, \partial u_r, X) = 0$$

whose solutions u_r, depending on the case, could be functions, p-forms, or spinors and connections; but in any case the catch is that, as r goes to infinity, *the solutions are forced to concentrate on an invariant set* of the vector field.

Witten was the first to use an approach of this kind, writing a PDE (a modified Laplace-Beltrami operator) whose solutions show a tendency to concentrate around the zeros of gradient vector fields, and using this as a starting point to construct a new approach to Morse Theory.

Clifford Taubes, building on his previous work on pseudoholomorphic curves in symplectic manifolds, used analogous principles to prove the Weinstein conjecture on dimension 3 [68]. (We recall that this conjecture states that the Reeb vector field of a contact form must always have a periodic orbit.) Later on, he generalized his construction to more general volume preserving vector fields [66]. Part II of this thesis is devoted at examining and expanding these ideas.

Let us introduce the general set-up. Our object of interest is a nowhere vanishing vector field X on a closed 3-manifold M, preserving a volume form μ. Given these data, we endow M with a Riemannian metric g *adapted* to X and μ: this means that $g(X,X) = 1$ and that μ is the volume form associated to the metric g.

(To alleviate the amount of necessary geometric background in what is to follow and focus on the analytic and dynamical aspects, we will suppose henceforth that M is diffeomorphic to $\mathbb{S}^3$ (note that the metric g depends on the field X, in particular it is not the round metric, with the exception of X being a Hopf vector field). All the operators that will appear (curl, gradient, divergence) are the ones defined by the metric g. Finally, we denote the scalar product of two vector fields by a dot, and integration will be understood with respect to the volume measure μ.)

With these data, that any volume preserving vector field X on $\mathbb{S}^3$ provides, Taubes strategy starts by defining the following system of PDEs (for which the unknowns are a vector field A_r and a function $\psi_r : \mathbb{S}^3 \to \mathbb{C}^2$):

$$\operatorname{curl} A_r = r(X - (\psi_r^\dagger \sigma_1 \psi_r)X - (\psi_r^\dagger \sigma_2 \psi_r)Y - (\psi_r^\dagger \sigma_3 \psi_r)Z) + v \tag{1.5.3}$$

$$D_{A_r}\psi_r := i\sum_k \sigma_k e_k \cdot (\nabla \psi_r - iA_r\psi_r) = 0 \tag{1.5.4}$$

where $\{e_1, e_2, e_3\} = \{X, Y, Z\}$ is a (global) orthonormal parallelization of the tangent bundle $T\mathbb{S}^3$ (in a general manifold M, this could only be done locally) and

$$\sigma_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \sigma_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix},\ \sigma_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.5.5}$$

are the Pauli matrices. The operator $\nabla_{A_r} := (\nabla - iA_r)$ can be seen as a covariant derivative on the trivial bundle $\mathbb{C}^2 \times \mathbb{S}^3$, and it must be understood as acting on each complex-valued component of $\psi_r = (\psi_{1r}, \psi_{2r})$ separately. The term v stands for a given divergence free vector field; it is a small perturbation of the equation that ensures the existence of solutions with suitable properties.

The above are a modified version of the 3-dimensional *Seiberg-Witten equations*, that we will call $SW(r,v)$-equations.

Let us first outline how this framework yields a proof to the Weinstein conjecture. First of all, the Reeb vector field X of a contact form α on a 3-manifold M can be shown to be equivalent to the data (X, μ, g), with X further satisfying

$$\operatorname{curl} X = X,$$

that is, in particular, X is a Beltrami field for the metric g.

Example (1.5.1) now serves as a good guiding model for what happens.

It is easy to check that, if one divides the equation (1.5.3) by r and, formally, sets $\frac{1}{r} = 0$, the analog of the "limiting solutions" Example (1.5.1) is

$$\psi_\infty = (\psi_{1\infty}, \psi_{2\infty}) = (1, 0).$$

It is not difficult to see that this "limiting solution" is basically unique, up to multiplication by a function $u : \mathbb{S}^3 \to \mathbb{C}$ with $|u| = 1$ (which can be interpreted as a choice of gauge.)

Moreover, there is also an analog of the boundary condition in (1.5.1). Taubes proves that certain sequences of solutions of the $SW(r, v)$-equations must satisfy

$$\sup(1 - |\psi_{1r}|^2) > \delta$$

as $r \to \infty$, with δ not depending on r. So, as in Example (1.5.1), the equations cannot converge everywhere to the naive limiting solution, because this solution fails to meet the condition above. Hence a readjustment in the value of $|\psi_{1r}|$ must take place *somewhere*, if we are to trust the analogy with Example (1.5.1) until the end

This somewhere is precisely a periodic orbit X. More exactly, Taubes proves that, as the parameter r tends to infinity, the quantity $u_r := (1 - |\psi_{1r}|^2)$ concentrates around one dimensional sets invariant by the flow of X, and decays to zero away from them. At the limit, the solutions do indeed converge to the naive limiting solution in all M except for a set of closed curves where $u_\infty = 1$, which are periodic orbits of the vector field. Since this phenomenon occurs without reference to any further particularity of the Reeb field that one uses as input, one concludes that all Reeb vector fields must have a periodic orbit.

Furthermore, the set-up can be defined for any non-vanishing volume preserving vector field X on $\mathbb{S}^3$, regardless of it being Reeb or not (and more generally, for any non-vanishing *exact* vector field on a 3-manifold M: exact means that the two form $i_X\mu$, whose closedness is equivalent to X being volume preserving, is also exact). In general, though, one cannot prove that solutions concentrate around closed curves, nor that they decay exponentially away from some other smooth invariant sets; but still one can prove that the quantity

$$\sigma_r := \frac{r(1 - |\psi_{1r}|^2)\mu}{\int \operatorname{curl} A_r \cdot X}$$

converges (maybe after passing to a subsequence) to an invariant measure of the vector field, *provided this vector field has positive* helicity,

$$\mathcal{H}(X) := \int X \cdot \operatorname{curl}^{-1} X > 0\,.$$

We will introduce the notion of helicity and the precise statement of the relevant results in the next section, which describes the exact contents of Chapter 5. For the moment, let us just say that the above relationship (between new invariant measures coming from the Seiberg-Witten equations and helicity) is quite remarkable, and constitutes a new promising framework to study volume preserving vector fields on $\mathbb{S}^3$. It is also worth emphasizing that the positivity condition on the helicity is actually equivalent to $\mathcal{H}(X) \neq 0$. The reason is that helicity changes sign under a volume preserving, but orientation reversing, diffeomorphism of the manifold, so that if X has $\mathcal{H}(X) < 0$, a change in the orientation brings us to the case $\mathcal{H}(X) > 0$, where Taubes framework applies.

1.5.2.1 A brief description of Chapter 5

Chapter 5 can be divided in two parts. In the first part, which accounts for most of the Chapter, we recast the essential ingredients in Taubes framework, and rederive the results coming from it. The important ideas are all already in Taubes papers [68, 66]; the novelty of the first part of this chapter rests rather in how we assemble and present them. In particular, we put more emphasis on the PDE and dynamical aspects, and some of our proofs differ. Our aim has been to make Taubes ideas more accessible to a wider audience of mathematicians, since we believe that these ideas could provide important insights into other areas of mathematics traditionally disconnected from the gauge theoretic mathematics in [68, 66].

By contrast, the second part of the Chapter presents a previously unknown result, to the best of our knowledge. In a work in progress, we are trying to extend Taubes framework and to extract further properties of these invariant measures in more general cases. In this context, we encountered a related, but simpler problem, concerning sequences of solutions to the two dimensional vortex equations. The final section of Chapter 5 is devoted to presenting this problem and our solution to it.

Let us present now the results of this chapter in more detail.

1.6 A closer look at the contents of Chapter 5: Seiberg-Witten equations and invariant measures

As the above discussion suggests, many aspects of the Seiberg-Witten equations are subtly tied to the properties of the vector field used as input.

The existence of certain types of solutions (and the nature of the invariant measures that they converge to) depends on a collection of global quantities, all of which are instances of *Hopf invariants*.

The Hopf invariant of a pair of volume preserving vector fields V, W in $(\mathbb{S}^3, \mu)$ is defined as

$$\mathcal{H}(V, W) := \int V \cdot \operatorname{curl}^{-1} W.$$

The operator curl^{-1} is well defined in the space of volume preserving vector fields in $\mathbb{S}^3$ (and more generally in the space of exact volume preserving vector fields in a closed manifold M): $\operatorname{curl}^{-1} W$ is the unique exact volume preserving vector field such that $\operatorname{curl}\operatorname{curl}^{-1} W = W$. It is easy to see that the Hopf invariant is symmetric, i.e, that $\mathcal{H}(V, W) = \mathcal{H}(W, V)$. [4]

The helicity of a vector field, that we defined above, is an example of Hopf invariant, with $\mathcal{H}(X) := \mathcal{H}(X, X)$. The other main quantity of interest for us is $\mathcal{H}(\operatorname{curl} A, \operatorname{curl} X)$,

$$\mathcal{H}(\operatorname{curl} A, \operatorname{curl} X) = \int \operatorname{curl} A \cdot X = r \int (1 - \psi^\dagger \sigma_1 \psi) + \int_M X \cdot v,$$

where we have used Eq. (1.5.3).

We will set $\mathcal{H}_A(X) := \mathcal{H}(\operatorname{curl} A, \operatorname{curl} X)$.

The following is the main result of the first part of Chapter 5:

Theorem 1.6.1 (Taubes 06 [68], Taubes 08 [66]). *Let X be a nowhere-vanishing vector field on $\mathbb{S}^3$ preserving a volume form μ and with positive helicity with respect to it, $\mathcal{H}(X) > 0$. Fix $\epsilon > 0$ arbitrarily small. There exists a sequence $\{r_n, \psi_{r_n} := (\psi_{1r_n}, \psi_{2r_n}), A_{r_n}\}$ of solutions to the associated $SW(r_n, v)-$equations, for some volume preserving vector field v of C^k norm less than ϵ, such that*

(i) if the sequence of Hopf invariants $\mathcal{H}_{A_{r_n}}(X)$ has a bounded subsequence, the vector field X has a periodic orbit.

(ii) if the sequence of Hopf invariants $\mathcal{H}_{A_{r_n}}(X)$ has no bounded subsequence, then the signed measure

$$\sigma_{r_n} := \frac{r_n(1 - |\psi_{1r_n}|^2)\mu}{\mathcal{H}_{A_{r_n}}(X)}$$

converges (maybe after passing to a subsequence) to an invariant probability measure σ_∞ of X. This measure satisfies $\sigma_\infty(X \cdot \operatorname{curl}^{-1}(X)) \leqslant 0$; as a consequence, it is not a multiple of μ.

This is Theorem 5.2.1 in Chapter 5.

The following Theorem, which implies that any Beltrami field with no zeroes in $\mathbb{S}^3$ has a periodic orbit, can be seen as a (non-trivial) consequence of Theorem 5.2.1

[4]Note that the Hopf invariant can be defined without reference to any metric in $\mathbb{S}^3$: since by definition $i_{\operatorname{curl} X}\mu = di_X g$, Stokes theorem implies that the above expression is equal to

$$\int i_V \mu \wedge \eta$$

for any η such that $d\eta = i_W \mu$.

Theorem 1.6.2 (Taubes 06 [68]). *Let X be a nowhere-vanishing vector field on $\mathbb{S}^3$ preserving a volume form μ. If we have that* $\operatorname{curl}^{-1} X = hX$, *with h a positive function in $\mathbb{S}^3$, the vector field X has a periodic orbit.*

This corresponds to Theorem 5.2.2. Its proof is given in Section 5.4.

The proofs of the Theorems rest on the following three main ideas:

- (A) *A priori behavior of the solutions* (Section 5.5): The Weitzenbőck formula for D_{A_r}, through standard elliptic bootstrapping, yields a priori estimates for the solutions of the Seiberg-Witten equations. Roughly speaking, the main lesson of these estimates is that for large r, the curl of A_r is mostly parallel to X, $\operatorname{curl} A_r \sim r(1-|\psi_{1r}|^2)X$, and that $|\psi_{1r}|$ can change its value very quickly in the transverse directions of X, but not in the direction of the flow.
- (B) *The existence of non-trivial solutions* (Section 5.6): The Monopole Floer Homology, as constructed by P. Kronheimer and T. Mrowka in [46], provides the foundation on which the existence of solutions rests. Very roughly speaking, this theory associates topological invariants to 3-manifolds, by constructing an appropriate chain complex (and associated homology groups) using as generators some classes of solutions to Seiberg-Witten like equations, in the same way one uses critical points in Morse Theory. The relevance of this construction from the PDE viewpoint is that, whenever an homology group is not trivial, we know that there must be some generators, that is, some solutions. Since the resulting homology groups are independent of the precise geometric or analytic data used to define the equations, once the groups are known to be non trivial in one case, we can infer the existence of solutions in many other cases. Finally, it is key to ensure also that those solutions have the desired properties: it is at this point that the helicity of X being non-zero plays a significant role, that we will outline in Section 5.6.
- (C) *The asymptotic properties of solutions at small scales* (Section 5.7): By virtue of item (A), the solutions to the the Seiberg-Witten equations for r big enough are shown to approximate, at small scales, solutions to the 2-dimensional vortex equations in the transverse directions of X. The properties of these equations are the key input for the proof of item (i) in Theorem 1.6.1.

1.6.1 A result on the rescaled vortex equations: the realization of any probability measure as the limit of sequences of renormalized solutions

An important open problem is to better understand the nature of the invariant sets where the measure σ_∞ in Theorem 1.6.1 concentrates when $\mathcal{H}_{A_r}(X)$ is unbounded, and also to uncover further properties of the invariant measure. For example, do the equations impose any condition in the type of invariant measure that can be obtained as a limit? In the final section of Chapter 5, Section 5.8, we present a new result on a related problem in dimension 2, that appears naturally when trying to address this question.

Let us present first the context in which our result is inscribed.

The behavior of sequences of solutions to the $SW(r,v)$-equations as $r \to \infty$ becomes more transparent when one looks at the solutions under an adapted rescaling, in a small enough flow-box of the vector field. More precisely, let p be any point in $\mathbb{S}^3$. Consider, for positive constants ϵ and ρ, a map

$$\Phi_p : (-\epsilon,\epsilon) \times \mathbb{D}_\rho \longrightarrow \mathbb{S}^3$$

where $\mathbb{D}_\rho := \{z \in \mathbb{C}, |z| \leqslant \rho\}$ is the disk of radius ρ, and Φ_p is defined as

$$\Phi_p(t,z) := \phi_X^t(\exp_p(xY(p) + yZ(p))$$

with $t \in (-\epsilon,\epsilon)$ and $z = x+iy \in \mathbb{D}_\rho$, and where ϕ_X^t is the flow of X and $\exp_p : T_p\mathbb{S}^3 \to \mathbb{S}^3$ is the exponential map. With ϵ and ρ small enough, Φ_p is a well defined diffeomorphism.

Denote by $C_p(\epsilon,\rho)$ the set $\Phi_p((-\epsilon,\epsilon)\times\mathbb{D}_\rho) \subset \mathbb{S}^3$. The *flow box chart at* $C_p(\epsilon,\rho)$ is the map $\Psi_p : C_p(\epsilon,\rho) \to (-\epsilon,\epsilon)\times\mathbb{D}_\rho$ defined as $\Psi_p := \Phi_p^{-1}$. We note that in these coordinates $(\Psi_p)_* X = \partial_t$

We define the rescaled coordinates $(t',z') = (\sqrt{r}t, \sqrt{r}z)$, which now take values in the stretched cylinder $\mathcal{C}_{\sqrt{r}} := (-\epsilon\sqrt{r}, \epsilon\sqrt{r}) \times \mathbb{D}_{\sqrt{r}\rho}$. The rescaled solutions are

$$\widetilde{\psi}_r(t',z') := \psi_r \circ \Phi_p\Big(\frac{t'}{\sqrt{r}}, \frac{z'}{\sqrt{r}}\Big)$$

and

$$\widetilde{A}_r(t',z') := \frac{1}{\sqrt{r}}(\Psi_p)_* A_r\Big(\frac{t'}{\sqrt{r}}, \frac{z'}{\sqrt{r}}\Big)$$

(Note the extra rescaling factor in $\widetilde{A}_r$. This factor is consistent with the interpretation of A_r as a connection: this way, the covariant derivative $\nabla - iA_r$ gets homogeneously rescaled when rescaling the coordinates.)

For ease of notation, in what follows we call (t,z) the rescaled coordinates (t',z'). By virtue of the a priori properties of solutions of the $SW(r,v)$-equations, the rescaled solutions $\widetilde{\psi}_r(t,z)$ and $\widetilde{A}_r(t,z)$ satisfy, in the rescaled coordinates, a PDE with the schematic form

$$\operatorname{curl}_0 \widetilde{A}_r = (1 - |\widetilde{\psi}_{1r}|^2)\partial_t + \frac{B_1}{r} \tag{1.6.1}$$

$$\bar{\partial}_{\widetilde{A}_r}\widetilde{\psi}_{1r} = \frac{B_3}{r} \tag{1.6.2}$$

$$(\partial_t - iA_{tr})\psi_{1r} = \frac{B_2}{r} \tag{1.6.3}$$

with $\overline{\partial}_{\widetilde{A}_r} = (\partial_x - i\widetilde{A}_{xr}) + i(\partial_y - i\widetilde{A}_{yr})$ and where the B_i are bounded on compact sets of $\mathcal{C}_{\sqrt{r}}$ (so, as $r \to \infty$, the terms with the B_i go to zero on compact sets). For r very large, note that the transverse or longitudinal components of the equation with respect to the vector field X decouple: the terms involving $\widetilde{A}_{xr}$, $\widetilde{A}_{yr}$, $\widetilde{\psi}_{1r}$ and their derivatives in the transverse directions of the flow (∂_x and ∂_y) stay of order one, while the others go to zero.

These equations look closer and closer to the well known *self dual vortex equations* on $\mathbb{C}$

$$da = \partial_x a_y - \partial_y a_x = (1 - |\phi|^2) \tag{1.6.4}$$

$$\overline{\partial_a}\phi = \overline{\partial_z}\phi - (a_x - ia_y)\phi = 0 \tag{1.6.5}$$

where $\overline{\partial_z} := (\partial_x + i\partial_y)$ is (twice) the Cauchy-Riemann operator, and the unknowns are a real one form $a = a_x dx + a_y dy$ (that we will also sometimes identify with its dual vector field through the euclidean metric) and a complex valued function $\phi = \phi_1 + i\phi_2$, the "Higgs field".

And indeed, one proves that, on compact sets $[-T, T] \times \mathbb{D}_R \subset \mathcal{C}_{\sqrt{r}}$, there is a family of solutions (a_t, ϕ_t) to the self dual vortex equations, each living on slices $\{t\} \times \mathbb{C}$ of constant $t \in [-T, T]$, such that, for any $\epsilon > 0$ as small as desired, we have

$$||a_t - \widetilde{A}_r||_{C^m([-T,T]\times\mathbb{D}_R)} \leqslant \epsilon \tag{1.6.6}$$

$$||\phi_t - \widetilde{\psi}_{1r}||_{C^m([-T,T]\times\mathbb{D}_R)} \leqslant \epsilon \tag{1.6.7}$$

for r large enough. Moreover, this family of solutions is *gauge equivalent* to a solution (a, ϕ) not depending on the t factor, meaning that for each t there is a smooth function $u_t : \mathbb{C} \to \mathbb{C}$ with $|u_t| = 1$ and such that

$$(a_t + \frac{1}{u_t}\nabla u_t, u_t\phi_t) = (a, \phi)$$

(note that (a, ϕ) is still a solution, as the equations are invariant under such kind of transformations). In particular, note that this means that $|\phi_t|$ does not depend on t.

Hence, the behavior of the vortex equations can offer good clues about the limiting behavior of solutions to the $SW(r, v)$-equations, at least locally.

The vortex equations are very well understood in the finite action case (see the classical monograph [38]), i.e, when the *action functional*

$$\mathcal{E}(a, \phi) = \int_{\mathbb{C}} (|da|^2 + |\nabla\phi - ia\phi|^2 + \frac{1}{4}(1 - |\phi|^2)^2) \tag{1.6.8}$$

is bounded. This is the classical situation in gauge theories (corresponding to the so-called instantons). A particularly relevant feature of these equations in this context is that the modulus of the complex field $|\phi|$ is either identically one, or has a finite number of zeroes and approaches 1 exponentially fast outside the region where these zeroes lay.

When the energy functional $\mathcal{H}_{A_r}(X)$ of the sequence of Seiberg-Witten solutions is bounded

$$\mathcal{H}_{A_r}(X) = \int X \cdot \operatorname{curl} A_r \leqslant C,$$

(with C independent of r as $r \to \infty$) the vortex solutions (a, ϕ) that appear as local limits of solutions $(A_r, (\psi_{1r}, \psi_{2r}))$ at small flow boxes are of the type above.

Thus the limiting vortex solution $|\phi|$ has a finite number of zeroes, and this zeroes are seen as one dimensional curves in [-T, T] $\times \mathbb{D}_R$, and, rescaling back, as curves in $[-\frac{T}{\sqrt{r}}, \frac{T}{\sqrt{r}}] \times \mathbb{D}_{\frac{1}{\sqrt{r}}}$. Near these curves the Seiberg-Witten solution $|\psi_{1r}|$ is also zero, by Eqs. (1.6.6)—(1.6.7). The exponentially fast approach of the vortex solution $|\phi|$ to 1 corresponds roughly to a $e^{-\sqrt{r}|x|}$ decay of the quantity $(1 - |\psi_{1r}|^2)$ outside the curves where $|\psi_{1r}| = 0$ (here $|x|$ represents the distance to the points with $|\psi_{1r}| = 0$).

As we have seen previously, in the limit, these curves are precisely periodic orbits of X.

That the functional $\mathcal{H}_{A_r}(X)$ can be shown to be bounded in the case of Reeb fields is something of a miracle, and it depends strongly on the fact that, in the adapted metric g, Reeb fields can be written as vector fields proportional to their curl. For more general vector fields, however, the functional cannot be assumed to stay bounded. The solutions still converge locally to solutions of the vortex equations, but these do not have finite action. The study of the vortex equations outside the finite action regime could thus shed some light on the nature of the more general invariant measures that arise from the $SW(r, v)$-equations.

Perhaps more importantly, it could single out which invariant measures are forbidden as invariant measures of general volume preserving vector fields. Indeed, if there were severe constraints in, say, the arrangements of the (possibly infinite number of) zeroes of the vortex solutions ϕ, one could hope that these constraints imply that some limiting measures σ_∞ are impossible. [5]

The problem is that in the infinite action regime, the equations are much less understood, and the number of vortices (the zeroes of the Higgs field ϕ) can grow indefinitely. Even the existence of solutions is problematic: solutions with an infinite number of vortices are only known to exist if those vortices are sufficiently far apart, or when they are distributed in a periodic arrangement [55]

Still, something can be said. In Section 5.8, we introduce a 2-dimensional "vortex analog" of our problem of finding restrictions to measures arising from sequences of Seiberg-Witten equations of unbounded $\mathcal{H}_{A_r}(X)$. We study sequences of solutions to the *rescaled vortex equations* in $\mathbb{C}$:

[5] Note that this is not completely clear, however, since the convergence of the Seiberg-Witten solutions to the vortex solutions involves a rescaling that blows up at the limit.

$$\star da_r = r(1 - |\phi_r|^2) \tag{1.6.9}$$

$$\overline{\partial}_{a_r}\phi = \overline{\partial_z}\phi_r - (a_{xr} - ia_{yr})\phi = 0 \tag{1.6.10}$$

where the unknowns are again, for $r > 0$ fixed, a real one form $a_r := a_{1r}dx + a_{2r}dy$ and a complex valued function $\phi_r : \mathbb{C} \to \mathbb{C}$. We are interested in sequences of solutions to these equations with $r \to \infty$ and

$$\mathcal{F}_{a_r} := \int da_r$$

unbounded.

Section 5.8 is devoted to the proof of the following result:

Theorem 1.6.3. *Let ν be a Borel probability measure on the open disk $\mathbb{D} \subset \mathbb{C}$. There is a sequence $\{(\phi_{r_n}, a_{r_n})\}$ of solutions to the r_n-rescaled vortex equations in $\mathbb{C}$, with $r_n \to \infty$, such that the 2-form*

$$\sigma_{r_n} = \frac{r_n(1 - |\phi_{r_n}|^2)dx \wedge dy}{\int_{\mathbb{D}} da_{r_n}}$$

converges to ν in the sense of measures on $\mathbb{D}$, and is zero elsewhere.

The measure being defined on $\mathbb{D}$ is not an important condition in the above theorem: as the proof will make clear, the result works as well with ν a Borel probability measure in any bounded set on $\mathbb{C}$.

In a work in progress we try to pass from this result to an analogous, local result for invariant measures arising as limits of sequences of three forms

$$\sigma_r := \frac{r(1 - |\psi_{1r}|^2)\mu}{\mathcal{H}_{A_r}(X)}$$

defined from solutions (A_r, ψ_r) to the Seiberg-Witten equations. This would mean that, in the case of the Seiberg-Witten equations, we do not have any *local* obstruction to the type of limiting invariant measures that can appear. Hence, if one wants to understand further properties of the measure σ_∞ for general volume preserving vector fields X, one should bring in the *global* aspects of the solutions.

1.6.2 Concluding remarks: further implications of the Seiberg-Witten framework for dynamics in dimension 3

Let us finish our review of the contents of Chapter 5 with two observations about their wider significance. These observations might help to appreciate them better from the dynamical systems viewpoint.

- *The role of the Hopf invariants*. The first observation concerns the important role played in Theorem 1.6.1 by the Hopf invariants of the vector fields A_r, X and their curls.

  A (seemingly) dynamical interpretation has been known for some time for the Hopf invariant $\mathcal{H}(V, W)$ of two exact volume preserving vector fields V and W preserving a volume form μ: the Hopf invariant measures a suitable average of the asymptotic linking number between pairs of orbits.

  More specifically, choose a point x in the manifold and track its path as it flows with the vector field V for a time T, then join the two endpoints of the resulting arc by a minimal length geodesic: this yields a closed curve. Do the same with a different point y and the vector field W. One gets two closed curves in the manifold.

  These two closed curves do not intersect in general. Therefore, they have a well defined linking number. Divide the linking number between the two curves by T^2, and consider the limit of the resulting quantity as T goes to infinity.

  Perhaps surprisingly, for almost all pairs of points (x, y) (in the sense of a.e for the measure $\mu \times \mu$) this limit is a well defined measurable function on $M \times M$: it is called the *asymptotic linking number*, $\Lambda(x, y)$. The integral of this function over $\mathbb{S}^3 \times \mathbb{S}^3$ coincides with $\mathcal{H}(V, W)$. This fact was discovered by Arnold [4, 5] (who offered a proof modulo some missing details) and was later rigorously proven by T. Vogel [72].

  However, despite its elegance, it was unclear whether the above result offered any further insight into the dynamics of divergence-free vector fields. The result has thus remained more or less disconnected from the rest of the body of knowledge about volume preserving dynamical systems.

  Theorems 1.6.1 and 1.8.4 are probably the firsts instances of such a connection, and maybe hint at some deeper structure. In fact, they already provide a new characterization of the class of *uniquely ergodic* vector fields. Here, by uniquely ergodic vector field we mean a vector field preserving a volume form and having no other invariant measure. By Theorem 1.6.1, such a vector field must have helicity zero (it is actually in these terms that C. Taubes presented his results in [66]).

- *The Gottschalk conjecture*. The second observation resumes some of our scattered comments in the previous pages about the relevance of the Seiberg-Witten perspective for Gottschalk conjecture. We recall that this conjecture, still open in the smooth volume preserving case, posits the non-existence of minimal (all orbits are dense) flows on $\mathbb{S}^3$.

  Theorem 1.6.1 offers a new perspective on the conjecture, which is all the more compelling because it is rather consistent with other traditional approaches to it.

  Traditional approaches to the conjecture proceed by contradiction: they try to prove that a minimal flow must be transverse or tangent to a 2-foliation; for if that were the case, since every 2-foliation on $\mathbb{S}^3$ has a compact leaf (a theorem of Novikov), the flow would either get trapped inside the leaf or be tangent to it, and thus could not have all of its orbits dense after all.

But as it is easy to see, a volume preserving flow with vanishing helicity density ($\mathrm{curl}^{-1} X \cdot X = 0$) is tangent to a (possibly singular) foliation. A first step is thus to show that minimal flows must have helicity zero, and here the above theorems offer a new approach. Indeed, since Theorems 1.6.1 associate an invariant measure σ_∞ to any vector field of non-vanishing helicity, one would prove that minimal flows have helicity zero by showing that the invariant measures σ_∞ cannot be the invariant measure of a minimal flow (for example, by proving that their support is nowhere dense).

1.7 A look at the contents of Chapter 6: helicity is the only invariant of divergence-free fields

The final chapter of Part II, Chapter 6, while of certain relevance to Chapter 5, can be read independently and contains a result of its own interest. It concentrates on the concept of helicity (or Hopf invariant) of a volume preserving vector field X which, as we have just seen, measures in some sense the average linking number of its orbits, and plays a very important role in the Seiberg-Witten framework.

We show that helicity is, in a certain sense, the only quantity of its kind one can rely on when it comes to describing the dynamics of volume preserving fields. This proves a conjecture that V. Arnold and B. Khesin presented in their well-known book on Topological Hydrodynamics [5].

Let us recall the context. Let X be a vector field on a 3-manifold M, preserving a volume form μ. The vector field is called exact if its flux through any closed surface is zero. This is a topological condition, equivalent to the (closed) two form $i_X\mu$ being exact (in particular, if M is a homology sphere, a volume preserving vector field is always exact). The helicity of X is defined as

$$\mathcal{H}(X) = \int_M i_X\mu \wedge \alpha$$

where α is any one form such that $d\alpha = i_X\mu$. Stokes theorem ensures that the helicity does not depend on the particular one form α. Alternatively, if one introduces a Riemannian metric compatible with the volume form μ, one recovers the expression

$$\mathcal{H}(X) = \int_M X \cdot \mathrm{curl}^{-1} X$$

that we introduced in Section 1.8.

It is easy to see that helicity is invariant under the action of volume preserving diffeomorphisms. More precisely, for any volume preserving diffeomorphism $\Phi : M \to M$ respecting the orientation of the volume form μ, we have

$$\mathcal{H}(\Phi_* X) = \mathcal{H}(X)\,,$$

and if Φ is orientation reversing, we have that $\mathcal{H}(\Phi_* X) = -\mathcal{H}(X)$.

Besides being invariant, helicity has other interesting properties: as we have seen before, it is an *asymptotic invariant*, being equal to the average asymptotic linking number between pairs of orbits. But it is also an *integral invariant*, which means that it can be expressed as the integral of a density (in this case a 2-point density):

$$\mathcal{H}(X) = \int_{M\times M} H(x, y, X(x), X(y))\, d\mu_x\, d\mu_y$$

This is because the inverse of the curl operator (which is well defined in the space of exact volume preserving vector fields) is an integral operator analogous to the classical Biot-Savart operator in euclidean space: this allows us to write

$$\mathcal{H}(X) = \int_M X \cdot \operatorname{curl}^{-1} X = \int_M X(x) \cdot \int_M B(x, y) \times X(y)\, d\mu_y\, d\mu_x$$

In view of the expression above, V. Arnold and B. Khesin asked whether there could be other integral invariants of volume preserving vector fields, with general form

$$\mathcal{I}(X) = \int_{M_1\times\ldots\times M_n} G(x_1, ..., x_n, X(x_1), ..., X(x_n))\, d\mu_1 ... d\mu_n, \tag{1.7.1}$$

and they conjectured that there were none. In Chapter 6 we prove this conjecture in the case of functionals in the space of C^1 volume preserving vector fields: we show that any integral invariant $\mathcal{I}$ verifying some mild technical assumptions (that include in particular the integral invariants considered by Arnold and Khesin) must be a function of the helicity.

More precisely, if we define $\mathfrak{X}^1_{\mathrm{ex}}$ to be the space of volume preserving, exact vector fields on (M, μ) of class C^1, we have

Definition 1.7.1. *Let $\mathcal{I} : \mathfrak{X}^1_{\mathrm{ex}} \to \mathbb{R}$ be a C^1 functional. We say that $\mathcal{I}$ is a* regular integral invariant *if:*

(i) It is invariant under volume-preserving transformations, i.e., $\mathcal{I}(w) = \mathcal{I}(\Phi_* w)$ for any diffeomorphism Φ of M that preserves volume.

(ii) At any vector field $w \in \mathfrak{X}^1_{\mathrm{ex}}$, the (Fréchet) derivative of $\mathcal{I}$ is an integral operator with continuous kernel, that is,

$$(D\mathcal{I})_w(u) = \int_M K(w) \cdot u\,,$$

for any $u \in \mathfrak{X}^1_{\mathrm{ex}}$, where $K : \mathfrak{X}^1_{\mathrm{ex}} \to \mathfrak{X}^1_{\mathrm{ex}}$ is a continuous map.

This definition includes, in particular, all integral invariants with the form in Eq. (1.8.1) with G a reasonably well-behaved function on M.

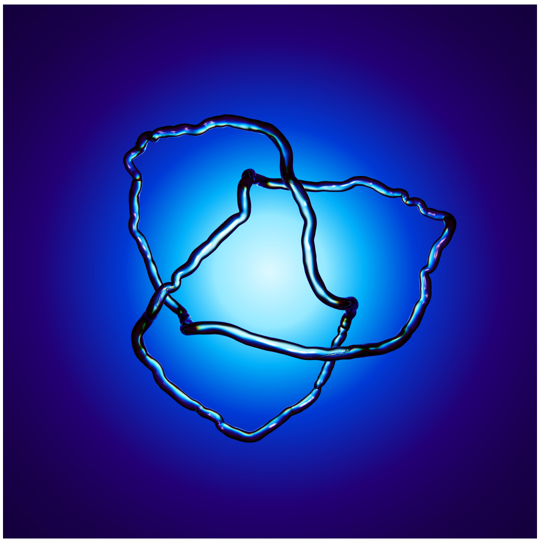

Figure 1.4: Un torbellino anudado en el agua. Cortesía de William Irvine.

The main theorem of Chapter 6 states

Theorem 1.7.2. *Let $\mathcal{I}$ be a regular integral invariant. Then $\mathcal{I}$ is a function of the helicity, i.e., there exists a C^1 function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{I} = f(\mathcal{H})$.*

The proof exploits an interesting property of the space of all volume preserving vector fields, unveiled by M. Bessa [14]: that topologically transitive ones (i.e, having at least one dense orbit) are dense in the C^1 topology.

This Chapter is based on the paper [28].

Remark 1.7.3. *Finally, let us say that we have aimed at making each Chapter reasonably self-contained, so that the reader does not need to refresh the notions and motivational context in this Introduction when going through each Chapter.*

1.8 Traducción: Introducción, resumen de los resultados, y conclusiones

Recientemente han aparecido nuevas técnicas experimentales [39, 40] que permiten observar en el laboratorio la emergencia de estructuras topológicas en diversos procesos físicos (dinámica de fluidos, óptica, física del estado sólido...). Estas estructuras topológicas se manifiestan, por ejemplo, en forma de torbellinos anudados en fluidos y plasmas, o en forma de dislocaciones en superconductores. Nos proporcionan una representación visual muy útil a la hora de estudiar sistemas complejos.

Desde el punto de vista matemático, estos fenómenos físicos vienen descritos por soluciones (ya sean vectoriales o escalares) de sistemas de ecuaciones en derivadas parciales; y las estructuras topológicas emergentes son ejemplos de variedades invariantes de estas soluciones.

Por ejemplo, en la imagen 1.4, vemos una fotografía de una de estas estructuras topológicas en el agua: se trata de un *torbellino* anudado (también llamado *tubo de vorticidad*), formando un nudo de trébol. Qué es, matemáticamente, tal torbellino? Si describimos el movimiento del agua asignando, a cada punto x del espacio que el fluido ocupa, y a cada instante t del intervalo de tiempo que nuestra película de su movimiento abarca, un vector $u(x,t)$ (el *campo de velocidad* del fluido), el torbellino es simplemente una región del fluido confinada en el interior de un *toro invariante* del campo de vorticidad $\omega(\cdot,t) = \operatorname{curl} u(\cdot,t)$, en un determinado instante t. Hay más ejemplos interesantes de variedades invariantes de campos vectoriales: órbitas periódicas (como por ejemplo aquella que recorre el núcleo del tubo de vorticidad) y, más generalmente, los soportes de medidas invariantes del fujo, que pueden ser muy complejos.

En muchos otros fenómenos, propios del estudio de la materia condensada, el estado del sistema se describe mediante un campo escalar (o, de forma más general, mediante una sección de un fibrado vectorial complejo). En tal caso, las estructuras topológicas que se manifiestan en los experimentos son, matemáticamente, regiones (conexas, es decir, de una pieza) del sistema en las que el campo escalar vale cero (técnicamnete, componentes conexas del llamado conjunto nodal).

Los resultados de estos experimentos ofrecen a los matemáticos un reto interesante: comprender cuán intrincadas (desde el punto de vista topológico) pueden llegar a ser las variedades invariantes de estos campos escalares y vectoriales, al tiempo que los campos satisfacen ciertas restricciones analíticas; restricciones que vienen impuestas por las ecuaciones en derivadas parciales que rigen su comportamiento.

Para abordar problemas de esta clase, uno tiene que conjugar conceptos y técnicas provenientes de áreas de las matemáticas diversas: análisis, geometría diferencial, física matemática...Los problemas que requieren de este esfuerzo integrador suele ser muy fructíferos, pues son una buena forma de comprobar hasta dónde llega nuestro discernimiento en estas áreas, y de detectar posibles lagunas en nuestra comprensión.

Problemas de realización para campos vectoriales y escalares

Para ilustrar la clase de cuestiones matemáticas que surgen de las consideraciones que preceden, considérense los siguientes problemas (que bien pueden servir de prototipos del tipo de cuestiones que trataremos más adelante en esta memoria):

- Problema A (problema vectorial): Encontrar una solución u de las *ecua-*

ciones de Euler estacionarias

$$u \times \omega = \nabla B, \ \ \mathrm{div} u = 0, \ \ \omega := \mathrm{curl}\, u,$$

(aquí $B := \frac{|u|^2}{2} + p$ es la llamada función de Bernoulli, y p es la presión) cuyo campo de vorticidad ω tenga un toro invariante (es decir, un torbellino) con forma de nudo de trébol (o, de forma más general, anudado como se desee). (Dicho de otra forma, muéstrese que nuestra descripción matemática de los fluidos es compatible con lo que se observa en la Figura 1.1).

- Problema B (problema escalar): Encontrar una función compleja $\psi : \mathbb{R}^3 \to \mathbb{C}$, que se avenga a la ecuación de Helmholtz

$$\Delta\psi + \psi = 0,$$

y cuyo conjunto nodal (que será genéricamente una colección de curvas, siempre y cuando el cero sea un valor no crítico) tenga una componente conexa difeomorfa a un nudo L dado.

A estos problemas se los denomina *problemas de realización*: el objetivo es realizar una determinada variedad como variedad invariante de la solución de una EDP.

Antes de revelar cómo se pueden abordar los problemas anteriores, conviene contemplarlos desde una perspectiva histórica. De hecho, como viene ocurriendo a menudo, estos problemas matemáticos preceden cronológicamente a los experimentos que les sirven de acicate. Así, el problema A fue originalmente propuesto por el físico William Thomson, Lord Kelvin, en torno al tercer cuarto del siglo XIX. Lord Kelvin no estaba considerando torbellinos anudados en el agua, sino en el (por entonces ubicuo) éter. Al decir de un célebre matemático de la época (que labios castellanos pueden pronunciar sin esfuerzo):

"Así el ilustre lord Kelvin ha buscado en los movimientos de los torbellinos una explicación mecánica del Universo. El Universo está lleno, según opinión de muchos sabios, por una materia continua, y lo que nosotros llamamos materia, materia propiamente dicha, no es más que un conjunto de torbellinos: átomos-torbellinos les llama el ilustre autor, átomos que, según Helmholtz, son indestructibles y eternos.

Verdad es, que según los descubrimientos de la radioactividad, el átomo puede destruirse y se destruye de hecho; pero en todo caso no hay más que correr la escala y suponer que es el electrón ese átomo-torbellino cuya existencia civil había usurpado el átomo de la Química. " [6]

Según Lord Kelvin, los átomo serían torbellinos anudados, y a cada tipo de nudo correspondería una especie de átomo (lo que hoy llamaríamos un elemento químico). Aunque la motivación física de Lord Kelvin acabó por mostrarse

[6] extracto de las notas del curso de Física Matemática impartido por José de Echegaray en el Ateneo de Madrid, ver *José de Echegaray: entre el teatro, la ciencia y la política*, de José Manuel Sánchez Ron, en Arbor CLXXIX, 707-708

errónea no mucho después (como ya se presiente en las palabras de Echegaray) el problema matemático que sugirió estimuló el desarrollo de la teoría de nudos, y permaneció sin resolver durante más de un siglo.

En cuanto al problema B, fue estudiado originalmente por M. Berry y M. Dennis [12]. Berry y Dennis idearon un método para construir soluciones a la ecuación de Helmholtz con un conjunto cero L difeomorfo a un nudo tórico dado, y conjeturaron que el Problema B tiene solución para cualquier tipo de nudo. Berry también estudió ([13]) generalizaciones del Problema B para EDPs de la forma $H\psi = \lambda\psi$, con H un operador de Schődinger, $H := -\Delta + V$ (por ejemplo, el átomo de hidrógeno o el oscilador armónico), y conjeturó de nuevo que existen soluciones en las que se forman nudos de cualquier clase.

Algunos análogos del Problema B, que estudian qué formas pueden adoptar los conjuntos cero de soluciones a EDPs elípticas, también tienen una biografía interesante. Si consideramos la ecuación de Poisson

$$\Delta\psi = \rho$$

podemos remontarnos de nuevo al siglo XIX: el problema ya intrigaba por entonces a los físicos-matemáticos, interesados como estaban en describir las superficies equipotenciales de la fuerza eléctrica o gravitatoria (por ejemplo, para caracterizar todas las posibles configuraciones de equilibrio que podía adoptar un fluido sometido a una fuerza gravitatoria).

En el caso de las ecuaciones de Cauchy-Riemann en una variedad compleja X

$$\bar{\partial}\psi = 0, \quad \psi : X \to \mathbb{C}$$

el problema es equivalente a una versión relajada del célebre segundo problema de Cousin: qué subvariedades de codimensión (real) 2 de una variedad compleja pueden aparecer como conjuntos nodales de una función holomorfa? Como demostró K. Oka (en un trabajo que fue a la vez precursor de los métodos de haces en geometría algebraica, y un primer indicio de la filosofía del H-principio de Gromov [34]), en el caso de una variedad de Stein, no hay más obstrucción que la topológica: la subvariedad en cuestión tiene que poder realizarse como conjunto cero de una función compleja continua.

El caso Euclídeo: una estrategia general

No hace mucho, Alberto Enciso y Daniel Peralta-Salas idearon una estrategia muy general para tratar problemas de realización de los tipos A y B. De esta estrategia se sigue que estas cuestiones exhiben una cierta *flexibilidad* (en el sentido del resultado de K. Oka descrito anteriormente, aunque por medio de técnicas completamente diferentes). Para ser más precisos, por flexibilidad entendemos lo siguiente: la ecuación en derivadas parciales que rige el comportamiento de las soluciones no impone ninguna restricción al tipo de variedad invariante que las soluciones pueden exhibir: se pueden encontrar soluciones que realicen cualquier subvariedad imaginable; siempre, claro está, que sea topológicamente posible.

La estrategia introducida por Enciso y Peralta-Salas es muy maleable, y puede adaptarse a problemas muy variados (aunque la adaptación a cada problema particular dista en muchas ocasiones de ser trivial: sirva como ejemplo la reciente resolución, también por Enciso y Peralta-Salas, del problema A para campos vectoriales en $\mathbb{R}^3$ [24]). El esquema general de la estrategia se puede ilustrar mediante ejemplos sencillos, y el problema B viene muy a mano para este fin.

Recordemos que el Problema B nos pedía hallar una función compleja ψ en $\mathbb{R}^3$, que resolviese la ecuación $\Delta\psi + \psi = 0$, y cuyo conjunto cero, $\psi^{-1}(0)$, tuviese una componente conexa difeomorfa a un nudo convenido L.

Ante tal problema, el esquema de Encisco y Peralta-Salas procede como sigue. En primer lugar, se resuelve el problema *localmente*: esto es, se construye una solución $\widetilde{\psi}$ de la ecuación $\Delta\widetilde{\psi} + \widetilde{\psi} = 0$ *en un entorno tubular del nudo L*.

Esta solución local tiene que cumplir, por una parte, que $\widetilde{\psi}|_L = 0$ (así, realiza el nudo como conjunto cero), y por otra parte, que la pareja de campos vectoriales gradiente $(\nabla\widetilde{\psi}_1, \nabla\widetilde{\psi}_2)$ en el conjunto L genera el fibrado normal de L. Esta última condición es necesaria para asegurar que el conjunto cero L es robusto, y sobrevive a perturbaciones de clase C^1 de la función $\widetilde{\psi}$ (considérese, por ejemplo, el conjunto cero de una función real: si el gradiente de la función no se anula en dicho conjunto, una perturbación suficientemente pequea de la función seguirá teniendo un conjunto cero difeomorfo al conjunto original). Si nos concierne, como es el caso, una EDP elíptica de segundo orden, basta un teorema de existencia local (como el teorema de Cauchy-Kovalevskaya, o un problema de contorno) para asegurarnos de que existe una solución local que cumple con los requisitos.

El siguiente paso consiste en encontrar una *solución global* ψ a la EDP que aproxime la solución local $\widetilde{\psi}$ con una resolución tan fina como se desee. Esto se consigue por medio de un teorema de aproximación la Runge, como el Teorema de Lax-Malgrange y sus variantes. Dada una solución local a una EDP elíptica en un abierto U, este tipo de teoremas aseguran la existencia de una solución global que aproxima a la solución local, en su dominio de definición U, con tanta precisión como se quiera.

El conjunto cero de $\widetilde{\psi}$ es robusto ante perturbaciones (es decir, también ante aproximaciones), gracias a la condición que se impuso a la derivadas de $\widetilde{\psi}$ en L en el paso previo. Así pues, el nudo L no se disuelve en el proceso de aproximación global, y una copia casi exacta de L (que se obtiene deformando L mediante un difeomorfismo muy parecido a no hacer nada) persiste en el conjunto cero de la solución global: el problema B queda resuelto.

Nótese que los dos ingredientes clave en la estrategia anterior (el teorema de existencia local, y el teorema de aproximación global) no son válidos solamente en el caso de la ecuación de Helmholtz: son aplicables en contextos mucho más generales (y cuando no es el caso, no es raro que haya análogos apropiados a nuestro alcance). Ahí reside la capacidad de adaptación de esta estrategia a problemas de realización muy variados (véase por ejemplo [27, 22, 23, 24]).

Sin embargo, por variadas que sean, las aplicaciones de esta estrategia se limitan a EDPs en espacios abiertos. Al intentar implementarla en variedades compactas uno se encuentra con una obstrucción fundamental. Como era de esperar, esta obstrucción se encuentra en el segundo paso: la aproximación global.

Es el mismo tipo de obstrucción con el que uno se topa al tratar de extender analíticamente a todo el plano complejo una función holomorfa definida en un anillo: en general, es ineludible que aparezca una singularidad en el disco delimitado por la circunferencia interior del anillo.

En efecto, el teorema de Lax-Malgrange requiere que la solución local esté definida en un abierto U cuyo complemento no tenga componentes compactas (por ejemplo, una bola, o un toro sólido). Si el complemento de U tiene una componente compacta K, lo más que se puede conseguir, en general, es encontrar una aproximación global a la solución local con singularidades en K.

Variedades compactas: localizacion inversa a pequea escala y altas energías

En la Parte I de esta memoria presentamos una estrategia para abordar problemas de realización en variedades compactas (superando así la obstrucción que acabamos de exponer en la sección anterior), y la aplicamos a problemas de realización para campos vectoriales y espinoriales en esferas y toros. En particular, resolvemos la conjetura de Kelvin en la 3-esfera y el 3-toro, y estudiamos la topología de de los conjuntos nodales de espinores de Dirac (autofunciones del operador de Dirac) en n-esferas y n-toros.

El interés por abordar problemas de realización en el caso compacto va más allá de lo puramente técnico. Nótese, por ejemplo, que un fluido en el 3-toro o la 3-esfera describe mucho mejor el comportamiento de los fluidos reales que un fluido en el espacio euclídeo del tipo que Enciso y Peralta-Salas estudiaron en [24]. Los fluidos reales tienen una cantidad finita energía (matemáticamente, su norma L^2 está acotada), y este es también el caso de los fluidos en espacios compactos; en cambio, las soluciones a la ecuación de Euler consideradas en [24] describen fluidos de infinita extensión, y el valor de su campo de velocidades decrece, en el mejor de los casos, tan rápido como el inverso de la distancia al origen de coordenadas. Estas consideraciones a propósito de la finitud de la energía suelen ser mucho más relevantes, a la hora comprender la dinámica de fluidos reales, que consideraciones sobre la geometría del espacio que contiene al fluido.

La idea principal que nos permite soslayar la obstrucción a la implementación de la estrategia previa en variedades compactas se basa en el análisis asintótico de las autofunciones de un operador autoadjunto (que será el operador rotacional, cuando tratemos fluidos estacionarios en el capítulo 2; o el operador de Dirac, cuando tratemos con espinores en el capítulo 3) a energías muy altas (en el sentido espectral) y a escalas muy pequeas

Veamos la idea en más detalle.

Sea M una variedad riemanniana compacta. Examinemos el entorno de un punto p al microscopio, es decir, consideremos una pequea bola de radio $1/\sqrt{\lambda}$ con centro en el punto p, y rescalemos las coordenadas geodésicas correspondientes por un factor $\sqrt{\lambda}$, de forma que la bola se nos aparezca en coordenadas rescaladas como la bola euclídea de radio 1, con el punto p en el origen de coordenadas. Bien: una autofunción de un operador autoadjunto de segundo orden en M (pensemos sencillamente en el laplaciano), cuyo autovalor λ sea suficientemente grande, se aparece a este microscopio como una solución de una EDP de coeficientes constantes en el espacio euclídeo (por ejemplo, en el caso del laplaciano, como una solución de la ecuación de Helmholtz $\Delta\psi + \psi = 0$).

A raíz de esta observación uno podría sentir la tentación de concluir que, siempre que se consideren altas energías y se busque en escalas pequeas, se pueden encontrar autofunciones de un operador que tengas variedades invariantes arbitrariamente intrincadas: después de todo, a esas pequeas escalas las autofunciones se comportan como soluciones de una EDP euclídea, y las técnicas de Enciso y Peralta-Salas muestran que para soluciones euclídeas la flexibilidad es la norma en problemas de realización.

Sin embargo, las técnicas euclídeas solamente aseguran que *algunas* soluciones muy particulares tienen estructuras topológicamente complicadas; y puesto que no podemos asegurar, en general, que haya una autofunción en la variedad compacta que aproxime una solución euclídea *dada*, el comportamiento euclídeo a pequea escala es inútil para nuestros fines.

La clave de nuestro método consiste precisamente en revertir la situación, es decir, en asegurarse de que, en algunas variedades compactas, *cualquier* solución euclídea es la aproximación a pequea escala de alguna autofunción de energía suficientemente elevada.

Esto es, si se analizan las autofunciones del operador de Dirac o del operador rotacional a energías cada vez más altas y a escalas cada vez más pequeas, uno termina por ver, a una resolución tan alta como se desee, *cualquier solución* de energía 1 del operador correspondiente en el espacio euclídeo. A un resultado de este tipo lo hemos llamado *teorema de localización inversa*.

Para probar los teoremas de localización inversa, un ingrediente clave es la dimensión creciente de los autoespacios del operador en la variedad compacta. Esto nos permite disponer de una gama cada vez mayor de autofunciones para combinar, conforme incrementamos la energía. Por este motivo, nuestros resultados valen para esferas y toros, con métricas de alta simetría.

El resultado de aproximación o localización inversa implica que *cualquier propiedad* que las soluciones de la EDP euclídea presenten en subconjuntos compactos de $\mathbb{R}^n$ es extensible a las autofunciones de alta energía del operador en esferas y toros, siempre y cuando tal propiedad sea robusta ante perturbaciones. La cuestión se reduce, por tanto, al caso euclídeo, en el que uno puede esperar resolver el problema de realización con las técnicas de Enciso y Peralta-Salas

A continuación expondremos con un poco más de detalle los resultados de los primeros dos capítulos de esta memoria.

Capítulo 2 : realización de torbellinos entrelazados en campos de Beltrami de altas energías

El capítulo 2 estudia la topología de los torbellinos (ya sean líneas, es decir, trayectorias de la vorticidad, o tubos, es decir, toros invariantes) presentes en soluciones estacionarias a las ecuaciones de Euler en $\mathbb{S}^3$ y $\mathbb{T}^3$. Nos ocuparemos en concreto de un tipo especial de soluciones, los llamados *campos de Beltrami*, que constituyen, en cierto sentido, los "átomos" del movimiento de los fluidos.

Para nosotros, un campo de Beltrami es una autofunción del operador rotacional

$$\mathrm{rot} u = \lambda u.$$

Nótese que un campo de Beltrami tiene divergencia cero, es decir, su flujo preserva la forma de volumen de la variedad. También es fácil caer en la cuenta de que los campos de Beltrami son soluciones estacionarias de las ecuaciones de Euler

$$u \times \omega = \nabla B, \quad \mathrm{div} u = 0, \omega = \mathrm{curl}\, u,$$

Finalmente, como autofunciones del operador rotacional, los campos de Beltrami forman una base en el espacio de todos los campos de divergencia cero exactos (un campo de divergencia cero es exacto si puede considerársele el campo de vorticidad de un campo de velocidades).

El resultado principal del Capítulo 2 establece que hay "bastantes" campos de Beltrami en $\mathbb{S}^3$ y $\mathbb{T}^3$ en cuyo seno podemos encontrar torbellinos anudados y entrelazados según cualquier tipo de nudo y enlace dado, siempre y cuando el autovalor sea lo suficientemente alto, y uno busque estos torbellinos a escalas muy pequeas.

(Recordamos que por *torbellino* entendemos tanto una trayectoria cerrada del campo de vorticidad, como un tubo invariante, es decir, un toro sólido embebido en la variedad e invariante bajo la acción del campo de vorticidad)

Theorem 1.8.1 (Torbellinos entrelazados en campos de Beltrami de altas energías)**.** *Sea $\mathcal{S}$ un conjunto finito de curvas cerradas y de tubos (disjuntos dos a dos, pero arbitrariamente anudados y entrelazados) en $\mathbb{S}^3$ o $\mathbb{T}^3$. En el caso del toro, suponemos también que el conjunto $\mathcal{S}$ está contenido en un subconjunto contráctil de $\mathbb{T}^3$. Para cualquier entero impar*

$$\lambda$$

suficientemente grande, existen un campo de Beltrami u (que satisface la ecuación $u = \lambda u$) y un difeomorfismo Φ de $\mathbb{S}^3$ o $\mathbb{T}^3$, tales que $\Phi(\mathcal{S})$ es un conjunto de torbellinos del campo u. Además, los torbellinos son estructuralmente estables. .

Este teorema se corresponde con el Teorema 2.1.1 del Capítulo 2. La prueba se basa en un teorema de localización inversa para campos de Beltrami, siguiendo un esquema similar al expuesto en la sección anterior.

Desde el punto de vista físico, un aspecto particularmente sugerente de nuestro resultado es que relaciona la complejidad de los torbellinos presentes en el fluido con las escalas de energía y de longitud del sistema. Esto puede tener tiene cierta relevancia en el estudio de los fenómenos de turbulencia, en los que se pueden observar la formación de torbellinos complejos a escalas pequeas y números de onda elevados.

Capítulo 3: topología de los conjuntos nodales de espinores de Dirac de energías elevadas

En el capítulo 3, los objetos que nos conciernen no son campos vectoriales, sino espinoriales, más concretamente autofunciones del operador de Dirac en n-esferas; y esta vez el papel de variedades invariantes lo juegan los conjuntos nodales.

Recordemos que las autofunciones del operador de Dirac son espinores, es decir, secciones de un fibrado hermítico S de rango (complejo) $r(n) = 2^{\lfloor \frac{n}{2} \rfloor}$, llamado fibrado espinorial. Cuando el fibrado espinorial es trivial (es el caso en esferas), la elección de una trivialización hace de cualquier sección ψ de S una colección $(\psi_1, ..., \psi_{r(n)})$ de $r(n)$ funciones con valores en $\mathbb{C}$. En una variedad espín de dimensión 3 o superior, los conjuntos cero regulares de una sección del fibrado espinorial están vacíos (porque $2r(n) > n$), así que en lo que estamos interesados es en la topología de los conjuntos cero de las *componentes* ψ_i del espinor. Estas últimas son $\mathbb{C}$-valuadas, por lo que sus conjuntos cero son (genéricamente) subvariedades de $\mathbb{S}^n$ de codimensión 2.

Por ejemplo, en $\mathbb{S}^3$, podemos descomponer un espinor en dos componentes, $\psi := (\psi_1, \psi_2)$. Cuando ψ es una autofunción del operador de Dirac, ψ_1 y ψ_2 están ligadas mediante una condición diferencial de primer orden. Podemos entonces plantear un análogo al problema B: dadas dos curvas cerradas L_1 y L_2, enlazadas y anudadas de cualquier manera, existe una función propia del operador de Dirac que tenga a L_1 por componente conexa del conjunto nodal de ψ_1, y a L_2 por componente conexa del conjunto nodal de ψ_2 ?

Nuestros resultados en este capítulo demuestran que sí. De forma un poco más precisa, demostraremos que hay muchas funciones propias del operador de Dirac con ceros de topologías arbitrariamente complicadas, sin importar la trivialización del fibrado espinorial que escojamos:

Theorem 1.8.2 (Teorema de realización para espinores de Dirac de alta energía). *En $\mathbb{S}^n$ (con $n \geqslant 3$), sea $\Sigma := \{\Sigma_1, ..., \Sigma_{r(n)}\}$ una colección de subvariedades de codimensión 2, con topologías tan complicadas como se quiera ($r(n)$ es aquí la dimensión compleja del fibrado espinorial). Existe siempre una autofunción $\psi = (\psi_1, ..., \psi_{r(n)})$ del operador de Dirac (de hecho, infinitas autofunciones) tal que Σ_i es difeomorfa a un* *subconjunto estructuralmente estable del conjunto nodal de la componente espinorial* ψ_i.

Este teorema es una versión simplificada del Teorema 3.1.1 del Capítulo 3. Como ocurría con el Teorema del Capítulo anterior, estas estructuras topológicamente complicadas aparecen a energías altas y escalas pequeas.

Dos teoremas sustentan a su vez el teorema anterior, el teorema 3.2.1 y el teorema 3.1.2. El primero es un resultado de localización inversa para autofunciones del operador de Dirac. Como ya se expuso en la sección anterior, este teorema reduce el problema al caso euclídeo. El segundo es, precisamente, un análogo del teorema principal en el caso euclídeo (en el caso de espinores de Dirac, el problema euclídeo presenta algunas sutilezas adicionales, y no basta con aplicar las técnicas que aparecen en [27] y que resumimos en la sección de introducción).

Los dos resultados de localización inversa (para el operador rotacional y el operador de Dirac), ingredientes fundamentales de los capítulos anteriores, requieren a su vez de un resultado de localización inversa para autofunciones del laplaciano. Este resultado y algunas variantes adicionales del mismo se demuestran en el Capítulo 4.

El capítulo 2 se basa en el artículo [29], del autor en colaboración con Alberto Enciso y Daniel Peralta-Salas; el capítulo 3 está basado en el artículo [69].

Un cambio de perspectiva: cómo detectar conjuntos invariantes mediante el estudio asintótico de las ecuaciones de Seiberg-Witten

La segunda parte de esta memoria también estudia variedades invariantes de campos vectoriales mediante el análisis asintótico de EDPs, pero la filosofía subyacente es distinta.

En la Parte I nos valíamos del comportamiento asintótico muy oscilatorio a escalas pequeas (e^{ikx}, con $k \to \infty$) de las soluciones de una EDP en una variedad compacta para *prescribir* la topología de los conjuntos invariantes de un campo vectorial (o espinorial) que *satisfacía* la EDP. Los conjuntos invariantes eran, por decirlo de alguna manera, emphconsentidos por la EDP, y facultativos para sus soluciones: no aparecían en todas ellas.

En esta parte, aprovechamos fenómenos de concentración asintótica, en lugar de fenómenos de oscilación. La idea es, en primer lugar, usar el campo vectorial cuyas variedades invariantes queremos estudiar para plantear una EDP. Esta vez, el campo vectorial *no es solución de la EDP en cuestión*, sino simplemente uno de los ingredientes que la *definen*. Las soluciones de esta EDP se concentran en ciertos conjuntos del espacio ambiente, y a menudo decaen exponencialmente a nada que nos alejemos un poco de ellos; decaimiento tanto más rápido cuanto mayor es cierto parámetro que hacemos tender a infinito (de ahí el adjetivo asintótica)

La EDP está astutamente diseada de tal forma que esos conjuntos, en los que las soluciones se concentran, acaban siendo precisamente conjuntos invariantes del campo vectorial; y las soluciones, por su parte, convergen a medidas invariantes del campo. Así, si en la Parte I hablábamos de prescripción de conjuntos invariantes, aquí podemos hablar de *detección*.

El esquema general es el siguiente: dado un campo vectorial X, definimos una EDP

$$Lu_r + r\Phi(u_r, \partial u_r, X) = 0$$

en la que L es un operador elíptico actuando sobre secciones de un fibrado vectorial u, y $\Phi(u, \partial u, X)$ es una función de u, sus derivadas, y X. Conforme r va a infinito, las soluciones u_r *se concentran en un conjunto invariante* de X.

Clifford Taubes se basó en este fenómeno para demostrar la conjetura de Weinstein en dimensión 3 [68]. (Recordemos que esta conjetura postula que todo campo de Reeb tiene una órbita periódica.) Poco después, generalizó estos resultados a campos exactos que preservan una forma de volumen [66].

La Parte II de esta memoria se basa a su vez en estas ideas de Taubes.

Antes de exponer en más detalle los resultados de esta parte de la tesis, presentemos el escenario más detalladamente. El objeto de interés esta vez es un campo vectorial X, sin ceros, en una variedad cerrada M de dimensión 3; este campo preserva una forma de volumen μ. Con estos datos, podemos definir en M una métrica riemanniana g *adaptada* al campo X y a la forma μ: esto se traduce sencillamente en que $g(X, X) = 1$ y en que μ es el volumen que corresponde a la métrica g.

(Supondremos que M es difeomorfa a $\mathbb{S}^3$, para simplificar la exposición y que destaquen las ideas analíticas subyacentes. Entenderemos además que los operadores que aparezcan (rotacional, gradiente, divergencia) son aquellos asociados a la métrica g. Finalmente, denotaremos el producto escalar de dos campos vectoriales por un punto, y la integración será siempre con respecto a la medida de volumen μ.)

Con los datos mencionados, que podemos obtener de cualquier campo vectorial X en $\mathbb{S}^3$ que preserve una forma de volumen, la estrategia de Taubes consiste, en primer lugar, en definir el siguiente sistema de EDPs (cuyas incógnitas son un campo vectorial A y una función $\psi_r : \mathbb{S}^3 \to \mathbb{C}^2$):

$$\operatorname{curl} A_r = r(X - (\psi_r^\dagger \sigma_1 \psi_r)X - (\psi_r^\dagger \sigma_2 \psi_r)Y - (\psi_r^\dagger \sigma_3 \psi_r)Z) + v \qquad (1.8.1)$$

$$D_{A_r}\psi_r := i\sum_k \sigma_k e_k \cdot (\nabla\psi_r - iA_r\psi_r) = 0 \qquad (1.8.2)$$

donde $\{e_1, e_2, e_3\} = \{X, Y, Z\}$ forma una paralelización ortogonal global del fibrado tangente $T\mathbb{S}^3$ (en una variedad M genérica, esto podría solamente hacerse de forma local) y

$$\sigma_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \sigma_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.8.3}$$

son las matrices de Pauli. El término v es una perturbación que asegura la existencia de soluciones apropiadas.

Las ecuaciones anteriores son una versión modificada de las ecuaciones de Seiberg-Witten en dimensión 3.

El Capítulo 5 consta de dos partes. En la primera parte, rederivamos los principales resultados que se siguen del análisis asintótico de las ecuaciones de Seiberg-Witten. Las ideas importantes están todas ya presentes en los artículos de Taubes [68, 66]; la novedad reside en cómo las estructuramos y las presentamos. Damos un énfasis especial a los aspectos dinámicos y analíticos, y algunas de nuestras demostraciones difieren de las de Taubes. Nuestro objetivo es hacer las ideas de Taubes accesibles a una audiencia de matemáticos más variada; consideramos que esto es importante porque hay áreas de las matemáticas, tradicionalmente alejadas de las matemáticas de [68, 66], en las que estas ideas pueden proporcionar una perspectiva original y muy interesante.

En la segunda parte del Capítulo, en cambio, presentamos un resultado hasta ahora desconocido, que explicaremos más adelante. Primero presentemos los resultados de la primera parte.

Para ello, es necesario introducir el concepto de *invariante de Hopf*. El invariante de Hopf de un par de campos vectoriales V, W que preserven la forma de volumen de $(\mathbb{S}^3, \mu)$ se define como

$$\mathcal{H}(V, W) := \int V \cdot \operatorname{curl}^{-1} W;$$

o, equivalentemente, como

$$\int i_V \mu \wedge \eta$$

para cualquier η tal que $d\eta = i_W \mu$.

La llamada helicidad de un campo vectorial es un ejemplo de invariante de Hopf, $\mathcal{H}(X) := \mathcal{H}(X, X)$. La otra magnitud de mayor interés en lo que sigue es $\mathcal{H}(\operatorname{curl} A, \operatorname{curl} X)$,

$$\mathcal{H}(\operatorname{curl} A, \operatorname{curl} X) = \int \operatorname{curl} A \cdot X = r \int (1 - \psi^\dagger \sigma_1 \psi) + \int_M X \cdot v,$$

que redefinimos como $\mathcal{H}_A(X) := \mathcal{H}(\operatorname{curl} A, \operatorname{curl} X)$.

El siguiente teorema es el resultado principal de la primera parte del Capítulo 5:

Theorem 1.8.3 (Taubes 06 [68], Taubes 08 [66])**.** *Sea X un campo sin ceros en $\mathbb{S}^3$ que preserva una forma de volumen μ y con helicidad positiva, $\mathcal{H}(X) > 0$. Fíjese un $\epsilon > 0$ tan pequeo como se desee. Existe una secuencia $\{r_n, \psi_{r_n} := (\psi_{1r_n}, \psi_{2r_n}), A_{r_n}\}$ de soluciones a las ecuaciones de Seiberg-Witten (para algún campo v que preserva el volumen y de norma C^k más pequea que ϵ) tal que*

(i) Si la secuencia de invariantes de Hopf $\mathcal{H}_{A_{r_n}}(X)$ está acotada, el campo X tiene una órbita periódica.

(ii) Si la secuencia de invariantes de Hopf $\mathcal{H}_{A_{r_n}}(X)$ no tiene ninguna subsecuencia acotada, entonces la medida

$$\sigma_{r_n} := \frac{r_n(1 - |\psi_{1r_n}|^2)\mu}{\mathcal{H}_{A_{r_n}}(X)}$$

converge a una medida de probabilidad invariante σ_∞ of X. Esta medida satisface además que $\sigma_\infty(X \cdot \operatorname{curl}^{-1}(X)) \leqslant 0$.

Este teorema se corresponde con el Teorema 5.2.1 en el Capítulo 5.

El siguiente teorema, que implica en particular que cualquier campo de Beltrami sin ceros en $\mathbb{S}^3$ tiene una órbita periódica, es una consecuencia (no trivial) del Teorema 5.2.1.

Theorem 1.8.4 (Taubes 06 [68])**.** *Sea X un campo sin ceros en $\mathbb{S}^3$ que preserva una forma de volumen μ. Si se tiene que $\operatorname{curl}^{-1} X = hX$, con h una función positiva en $\mathbb{S}^3$, entonces el campo vectorial X tiene una órbita periódica.*

Este es el Teorema 5.2.2, cuya prueba se expone en la Sección 5.4.

Las ecuaciones de vórtice: realización de cualquier medida de probabilidad en dimensión dos como límite de secuencias de soluciones renormalizadas

En un proyecto en curso, estamos intentando extender el marco conceptual creado por Taubes, para poder captar propiedades más finas de las medidas invariantes que emergen de las secuencias de soluciones a las ecuaciones de Seiberg-Witten. En este contexto, nos encontramos con un problema relacionado, pero más simple, que versaba sobre el comportamiento de las secuencias de soluciones a las ecuaciones del vórtice en dimensión 2. La segunda parte del Capítulo 5 presenta una solución a este problema.

Más concretamente, estudiamos secuencias de soluciones a las ecuaciones del vórtice rescaladas en $\mathbb{C}$:

$$\star da_r = r(1 - |\phi_r|^2) \tag{1.8.1}$$

$$\overline{\partial}_{a_r}\phi = \overline{\partial_z}\phi_r - (a_{xr} - ia_{yr})\phi = 0 \tag{1.8.2}$$

donde las incógnitas son, para $r > 0$ fijo, una 1-forma real $a_r := a_{1r}dx + a_{2r}dy$ y una función compleja $\phi_r : \mathbb{C} \to \mathbb{C}$. Nos interesa el comportamiento de las soluciones a estas ecuaciones a medida que $r \to \infty$ y que la secuencia

$$\mathcal{F}_{a_r} := \int da_r$$

no está acotada.

El resultado que obtenemos es el siguiente:

Theorem 1.8.5. *Sea ν una medida de probabilidad de Borel en el disco abierto $\mathbb{D} \subset \mathbb{C}$. Existe una secuencia $\{(\phi_{r_n}, a_{r_n})\}$ de soluciones a las ecuaciones del vórtice rescaladas en $\mathbb{C}$, con $r_n \to \infty$, tal que la 2-forma*

$$\sigma_{r_n} = \frac{r_n(1 - |\phi_{r_n}|^2)dx \wedge dy}{\int_{\mathbb{D}} da_{r_n}}$$

converge a ν (en el sentido de convergencia de medidas) en $\mathbb{D}$, y es idénticamente nula fuera.

En nuestro trabajo en curso, intentamos obtener un resultado análogo al anterior, pero en dimensión 3, local, y concerniente a la secuencia de medidas

$$\sigma_r := \frac{r(1 - |\psi_{1r}|^2)\mu}{\mathcal{H}_{A_r}(X)}$$

definidas a partir de soluciones (A_r, ψ_r) a las ecuaciones de Seiberg-Witten. Esto implicaría que, en el caso de las ecuaciones de Seiberg-Witten, no hay ninguna obstrucción local en cuanto al tipo de medida invariante que puede surgir en el límite asintótico. Así pues, para comprender mejor las propiedades de la medida límite σ_∞ en el caso campos vectoriales generales generales (exactos y que preservan el volumen), habría que recurrir a argumentos globales.

Capítulo 6: la helicidad como único invariante de campos que preservan el volumen

El último capítulo de la Parte II, aunque es relevante para los que lo preceden, se puede leer de forma independiente y contiene un resultado interesante en sí mismo.

En este capítulo nos centramos en el concepto de helicidad (o invariante de Hopf) de un campo vectorial X; concepto que, como ya hemos visto, tiene un papel crucial en el capítulo anterior.

Lo que demostramos en este capítulo es que la helicidad es, en un sentido preciso, la única cantidad que satisface ciertas propiedades naturales. Nuestro resultado resuelve en particular una conjetura de V. Arnold y B. Khesin, que aparece en su célebre monografía [5].

Recordemos qué es la helicidad. Sea X un campo vectorial en una 3-variedad M; supongamos que X preserva una forma de volumen μ. Un campo vectorial que preserva un volumen se denomina exacto si su flujo a través de cualquier superficie cerrada es cero. Es esta una condición topológica, equivalente a que la 2-forma $i_X\mu$ (que, notemos, es cerrada) sea exacta. La helicidad de un campo exacto X se define como

$$\mathcal{H}(X) = \int_M i_X\mu \wedge \alpha$$

donde α es cualquier 1-forma que verifique $d\alpha = i_X\mu$. Por el teorema de Stokes, $\mathcal{H}(X)$ no depende de la forma α que se escoja. Si consideramos sobre la variedad M una métrica compatible con la forma μ, podemos definir la helicidad de forma equivalente como

$$\mathcal{H}(X) = \int_M X \cdot \operatorname{curl}^{-1} X.$$

Es fácil comprobar que la helicidad es invariante ante la acción de difeomorfismos que preservan la forma de volumen. Más concretamente, si $\Phi : M \to M$ es un difeomorfismo que preserva μ y que respeta la orientación, se tiene que

$$\mathcal{H}(\Phi_* X) = \mathcal{H}(X)\,,$$

y si Φ invierte la orientación, tenemos $\mathcal{H}(\Phi_* X) = -\mathcal{H}(X)$.

Además de ser invariante, hay otras propiedades que hacen de la helicidad una magnitud interesante. Por una parte, la helicidad es un invariante asintótico: es igual a la media del número de entrelazado asintótico entre pares de órbitas del campo; por otra (y esto es más relevante para este capítulo), es un invariante integral, es decir, puede expresarse como la integral de una densidad:

$$\mathcal{H}(X) = \int_{M\times M} H(x, y, X(x), X(y))\, d\mu_x\, d\mu_y$$

Esto se debe a que la inversa del operador rotacional (que, en el espacio de campos exactos, está bien definida) es un operador integral:

$$\mathcal{H}(X) = \int_M X \cdot \operatorname{curl}^{-1} X = \int_M X(x) \cdot \int_M B(x, y) \times X(y)\, d\mu_y\, d\mu_x$$

En vista de la expresión anterior, V. Arnold y B. Khesin se preguntaron si podrían existir otros invariantes integrales de campos exactos, que pudiesen expresarse generalmente de la forma:

$$\mathcal{I}(X) = \int_{M_1 \times \ldots \times M_n} G(x_1, \ldots, x_n, X(x_1), \ldots, X(x_n))\, d\mu_1 \ldots d\mu_n, \qquad (1.8.1)$$

y conjeturaron que la helicidad era el único. En el capítulo 6 demostramos esta conjetura en el caso de funcionales en el espacio de campos exactos de regularidad C^1: demostramos que cualquier invariante integral $\mathcal{I}$ que satisfaga ciertas hipótesis técnicas naturales (hipótesis que satisfacen, entre otros, los invariantes integrales que Arnold y Khesin consideraban en su conjetura) ha de ser simplemente una función de la helicidad.

El ingrediente crucial de la demostración es una propiedad interesante de los campos que preservan el volumen, descubierta por M. Bessa [14]: que aquellos topológicamente transitivos son densos en la topología C^1.

Este capítulo esta basado en el artículo [28].

Part I

Beltrami fields and Dirac spinors

Chapter 2

Knotted structures in high energy Beltrami fields

Let $\mathcal{S}$ be a finite union of (pairwise disjoint but possibly knotted and linked) closed curves and tubes in the round sphere $\mathbb{S}^3$ or in the flat torus $\mathbb{T}^3$. In the case of the torus, $\mathcal{S}$ is further assumed to be contained in a contractible subset of $\mathbb{T}^3$. In this chapter, we show that for any sufficiently large odd integer λ there exists a Beltrami field on $\mathbb{S}^3$ or $\mathbb{T}^3$ satisfying $\operatorname{curl} u = \lambda u$ and with a collection of vortex lines and vortex tubes given by $\mathcal{S}$, up to an ambient diffeomorphism.

2.1 Introduction

An incompressible fluid flow in $\mathbb{R}^3$ is described by its velocity field $u(x,t)$, which is a time-dependent vector field satisfying the Euler equations

$$\partial_t u + (u \cdot \nabla) u = -\nabla P\,, \qquad \operatorname{div} u = 0$$

for some pressure function $P(x,t)$. When the velocity field does not depend on time, the fluid is said to be *stationary*. This chapter concerns stationary solutions of the Euler equations, which describe equilibrium configurations of the fluid.

A central topic in topological fluid mechanics, which can be traced back to Lord Kelvin in the XIX century [70], concerns the existence of knotted stream and vortex structures in stationary fluid flows. The most relevant of these structures are the stream lines, vortex lines and vortex tubes of the fluid. We recall that a *stream line* and a *vortex line* are simply a trajectory (or integral curve) of the velocity field u and the vorticity $\omega := \operatorname{curl} u$, respectively, while a *vortex tube* is the interior domain bounded by an invariant torus of the vorticity. The existence of topologically complicated stream and vortex lines is a central topic in the Lagrangian theory of turbulence and in magnetohydrodynamics, and has been studied extensively in the last decades (see e.g. [41, 58] for recent accounts of the subject).

Our understanding of the set of stationary states of the Euler equations in three dimensions is much more limited than in the two-dimensional situation [16, 60]. This is due to the fact that, in two dimensions, the vorticity is a scalar quantity, whereas in the three dimensional case it is a vector field, which can exhibit a much richer behavior. In particular, the existence of stationary solutions in $\mathbb{R}^3$ having stream lines, vortex lines and vortex tubes that are knotted and linked in arbitrarily complicated ways has been established only very recently [26, 24, 25]. Following a suggestion of Arnold [3, 5] related to his celebrated structure theorem, to prove these results one does not consider just any kind of solutions to the stationary Euler equations but a very particular class that are called Beltrami fields. A *Beltrami field* in $\mathbb{R}^3$ is a vector field satisfying the equation

$$\operatorname{curl} u = \lambda u \tag{2.1.1}$$

for some nonzero constant λ. Notice that stream lines and vortex lines coincide in the case of a Beltrami field, and that a Beltrami field is automatically smooth (even real analytic) by the elliptic regularity theory.

The stationary solutions in $\mathbb{R}^3$ that one can construct using the techniques in [26, 24] fall off at infinity as $1/|x|$, this decay being sharp for Beltrami fields but not fast enough for the velocity to be in the energy space $L^2(\mathbb{R}^3)$. In fact, the incompressibility condition ensures that there are no Beltrami fields in $\mathbb{R}^3$ with finite energy even if the proportionality factor λ is allowed to be nonconstant, as has been recently shown in [61, 15].

On the contrary, Beltrami fields in a closed Riemannian 3-manifold M (or a bounded domain of $\mathbb{R}^3$) are stationary solutions to the Euler equations that do have finite energy. If $\mathcal{S}$ is a union of (possibly knotted and linked) closed curves and embedded tori in the 3-sphere, in this setting one can use contact topology to show [30] that there is a Riemannian metric g on the sphere with an associated Beltrami field u having a collection of vortex lines and vortex tubes given precisely by $\mathcal{S}$. The main ideas of the proof are that the Reeb field of a contact form is in fact a Beltrami field in some adapted metric and that one can indeed construct contact forms on the sphere whose Reeb fields have the collection of periodic trajectories and invariant tori given by $\mathcal{S}$. Notice that, as it is a Reeb vector field, a Beltrami field obtained in this fashion does not vanish. Conversely, any nonvanishing Beltrami field on the sphere is the Reeb vector field of some contact form.

Our goal in this chapter is to establish the existence of knotted and linked vortex structures in Beltrami fields on compact manifolds with a *fixed* Riemannian metric. Specifically, we will consider Beltrami fields in the flat 3-torus $\mathbb{T}^3$ and in the unit 3-sphere $\mathbb{S}^3$; in fact, the former is the most fundamental space considered in the fluid mechanics literature other than $\mathbb{R}^3$ and the latter is perhaps the simplest example of a closed Riemannian 3-manifold from a geometric point of view.

It is worth emphasizing that, for a fixed Riemannian structure, the problem is much more rigid than when one can freely choose a metric adapted to the geometry of the set of lines and tubes that one aims to recover from the trajectories of a Beltrami field. An obvious reason is that, analytically, Beltrami fields in a closed Riemannian manifold arise as eigenfields of the curl operator, which defines a self-adjoint operator with discrete spectrum and a dense domain in the space of divergence-free L^2 fields. In the context of spectral theory, the proportionality constant λ, or rather its absolute value, can be thought of as the *energy* of the Beltrami field, although of course it is in no way related to the L^2 norm of the latter.

Our main theorem asserts that there are "many" Beltrami fields u in the sphere and in the torus with vortex lines and vortex tubes of any link type. Furthermore, these structures are *structurally stable* in the sense that any vector field on the torus or the sphere which is sufficiently close to u in the C^4 norm and which preserves some smooth volume measure will also have this collection of periodic trajectories and invariant tori, up to a diffeomorphism. To state this result precisely, let us call a *tube* the closure of a domain (in $\mathbb{S}^3$ or $\mathbb{T}^3$) whose boundary is an embedded torus. Throughout, diffeomorphisms are of class C^∞, curves are all assumed to be non-self-intersecting, and we will agree to say that an integer is large when it is large in absolute value.

Theorem 2.1.1 (Arbitrarily knotted and linked vortex tubes in high energy Beltrami fields)**.** *Let $\mathcal{S}$ be a finite union of (pairwise disjoint, but possibly knotted and linked) closed curves and tubes in $\mathbb{S}^3$ or $\mathbb{T}^3$. In the case of the torus, we also assume that $\mathcal{S}$ is contained in a contractible subset of $\mathbb{T}^3$. Then for any large enough odd integer λ there exists a Beltrami field u satisfying the equation* $\operatorname{curl} u = \lambda u$ *and a diffeomorphism Φ of $\mathbb{S}^3$ or $\mathbb{T}^3$ such that $\Phi(\mathcal{S})$ is a union of vortex lines and vortex tubes of u. Furthermore, this set is structurally stable.*

An important observation is that the proof of this theorem yields a reasonably complete understanding of the behavior of the diffeomorphism Φ, which is, in particular, connected with the identity. Oversimplifying a little, the effect of Φ is to uniformly rescale a contractible subset of the manifold that contains $\mathcal{S}$ to have a diameter of order $1/|\lambda|$. In particular, the control that we have over the diffeomorphism Φ allows us to prove an analog of this result for quotients of the sphere by finite groups of isometries (lens spaces). Notice that $\Phi(\mathcal{S})$ is not guaranteed to contain all vortex lines and vortex tubes of the Beltrami field. It is also worth mentioning that, if $\mathcal{S}$ only consists of curves, the condition that the perturbation of the Beltrami field be volume-preserving is not necessary for the structural stability of $\Phi(\mathcal{S})$, and the smallness in C^4 can be replaced by a C^1 condition.

In $\mathbb{S}^3$ and $\mathbb{T}^3$, Theorem 2.1.1 proves a conjecture of Arnold [3] asserting that there should be Beltrami fields having stream lines with complicated topology. Furthermore, it should be noticed that the helicity of the vorticity, that is, the quantity [57]

$$\mathcal{H}(\operatorname{curl} u) := \int_M u \cdot \operatorname{curl} u \,,$$

is proportional to its eigenvalue λ so the Beltrami fields constructed in the main theorem have very large helicity. More precisely, the scale-invariant quantity

$$\frac{\mathcal{H}(\operatorname{curl} u)}{\|u\|_{L^2}^2} \,,$$

which is given by λ in the case of a Beltrami field, becomes arbitrarily large. This is fully consistent with Moffatt's interpretation [57, 58] of helicity as a measure of the degree of knottedness of the vortex lines in the fluid flow.

The proof of the theorem involves an interplay between rigid and flexible properties of high-energy Beltrami fields. Indeed, rigidity appears because high-energy Beltrami fields in any 3-manifold behave, locally in sets of diameter $1/\lambda$, as Beltrami fields in $\mathbb{R}^3$ with parameter $\lambda = 1$ do in balls of diameter 1. The catch here is that, in general, one cannot check whether a given Beltrami field in $\mathbb{R}^3$ actually corresponds to a high-energy Beltrami field on the compact manifold. To prove a partial converse implication in this direction (Theorem 2.2.1), it is key to exploit some flexibility that arises in the problem as a consequence of the fact that large eigenvalues of the curl operator in the torus or in the sphere have increasingly high multiplicities. For this reason the proof does not work in a general Riemannian 3-manifold.

One should notice that the techniques introduced in [26, 24] to prove the existence of Beltrami fields in $\mathbb{R}^3$ with a prescribed set $\mathcal{S}$ of closed vortex lines and vortex tubes do not work for compact manifolds. The reason is that the proof is based on the construction of a local Beltrami field in a neighborhood of $\mathcal{S}$, which is then approximated by a global Beltrami field in $\mathbb{R}^3$ using a Runge-type global approximation theorem. For compact manifolds the complement of the set $\mathcal{S}$ is precompact, so we cannot apply the global approximation theorem obtained in [26, 24]. In fact, as is well known, this is not just a technical issue, but a fundamental obstruction in any approximation theorem of this sort. This invalidates the whole strategy followed in [26, 24] and makes it apparent that new tools are needed to prove the existence of Beltrami fields with geometrically complex vortex lines and vortex tubes in compact manifolds.

The chapter is organized as follows. In Section 2.2 we will prove the main theorem assuming that Theorem 2.2.1 holds. Theorem 2.2.1 will be proved in Section 2.3 in the case of the sphere (with the proof of some technical results relegated to Section 2.4 and to Chapter 4) and in Section 2.5 in the case of the torus. We conclude with some remarks that we present in Section 2.6, where in particular we prove an analog of the main theorem for lens spaces.

2.2 Proof of the main theorem: realization of knotted and linked vortex tubes at small scales and high energies

For the ease of notation, we shall write $\mathbb{M}^3$ to denote either $\mathbb{T}^3$ (the standard flat 3-torus, $(\mathbb{R}/2\pi\mathbb{Z})^3$) or $\mathbb{S}^3$ (the unit sphere in $\mathbb{R}^4$). A Beltrami field u in $\mathbb{M}^3$ is an eigenfield of the curl operator, which satisfies

$$\operatorname{curl} u = \lambda u \,,$$

for some nonzero constant λ. It is known (see e.g. [31]) that the spectrum of curl in the sphere are the integers of absolute value greater than or equal to 2. In the case of $\mathbb{T}^3$ it is easy to check using Fourier series that the spectrum consists of the real numbers of the form

$$\lambda = \pm |k|$$

for some $k \in \mathbb{Z}^3$. In particular, the spectrum of curl in $\mathbb{T}^3$ contains the set of integers. Here and in what follows, $|\cdot|$ denotes the usual Euclidean norm of a vector.

The following theorem, whose proof is presented in Section 2.3, shows that a Beltrami field v in $\mathbb{R}^3$ can be approximated, up to a suitable rescaling, by a high-energy Beltrami field u in $\mathbb{M}^3$. This fact is key to the proof of Theorem 2.1.1 as it implies that the dynamics of any Beltrami field of $\mathbb{R}^3$ in compact sets can be reproduced in a small ball of $\mathbb{M}^3$ by a high-energy Beltrami field on the manifold, provided that the dynamical properties under consideration are robust under suitably small perturbations. For concreteness, we will henceforth assume that λ is positive; the case of negative λ is completely analogous.

For the precise statement of the theorem, let us fix an arbitrary point $p_0 \in \mathbb{M}^3$ and take a patch of normal geodesic coordinates $\Psi : \mathbb{B} \to B$ centered at p_0. Here and in what follows, B_ρ (resp. $\mathbb{B}_\rho$) denotes the ball in $\mathbb{R}^3$ (resp. the geodesic ball in $\mathbb{M}^3$) centered at the origin (resp. at p_0) and of radius ρ, and we shall drop the subscript when $\rho = 1$. The theorem will be then stated in terms of the vector field $\Psi_* u$ on B, which is just the expression of the Beltrami field u in local normal coordinates. If $u^i(x)$ are the three components of $\Psi_* u$ in the Cartesian basis $\{e_i\}_{i=1}^3$ of $\mathbb{R}^3$, i.e.,

$$\Psi_* u(x) = \sum_{i=1}^{3} u^i(x)\, e_i \,,$$

we will make use of the rescaled vector field

$$\Psi_* u\Big(\frac{\cdot}{\lambda}\Big) := \sum_{i=1}^{3} u^i\Big(\frac{\cdot}{\lambda}\Big)\, e_i \,.$$

Theorem 2.2.1 (Inverse localization of Beltrami fields)**.** *Let v be a Beltrami field in $\mathbb{R}^3$, satisfying* $\operatorname{curl} v = v$*. Let us fix any positive numbers ϵ and m. Then for any large enough odd integer λ there is a Beltrami field u, satisfying* $\operatorname{curl} u = \lambda u$ *in $\mathbb{M}^3$, such that*

$$\left\| \Psi_* u\Big(\frac{\cdot}{\lambda}\Big) - v \right\|_{C^m(B)} < \epsilon \,. \tag{2.2.1}$$

Let us now show how this result can be exploited to prove the main theorem. For this, let Φ' be a diffeomorphism of $\mathbb{M}^3$ mapping the set $\mathcal{S}$ into the ball $\mathbb{B}_{1/2}$, and the ball $\mathbb{B}_{1/2}$ into itself. (In $\mathbb{S}^3$, the existence of such a diffeomorphism is trivial, while in the case of $\mathbb{T}^3$ it follows from the assumption that $\mathcal{S}$ is contained in a contractible set.) We can now define a set $\mathcal{S}'$ of finitely many closed curves and tubes in the ball $B_{1/2}$ as

$$\mathcal{S}' := (\Psi \circ \Phi')(\mathcal{S}) \,.$$

The following result is a straightforward consequence of the main theorem in [24]:

Theorem 2.2.2. *There is a Beltrami field v in $\mathbb{R}^3$ satisfying* $\operatorname{curl} v = v$ *and an orientation-preserving diffeomorphism Φ_0 of $\mathbb{R}^3$, which coincides with the identity in the complement of $B_{1/2}$, such that $\Phi_0(\mathcal{S}')$ is a union of vortex lines and vortex tubes of v. Furthermore, this set is structurally stable.*

Proof. It was shown in [24] that there is a Beltrami field $\tilde{v}$ in $\mathbb{R}^3$, satisfying

$$\operatorname{curl} \tilde{v} = \tilde{\lambda} \tilde{v}$$

for some small positive constant $\tilde{\lambda} < 1$, and an orientation-preserving diffeomorphism $\widetilde{\Phi}$ of $\mathbb{R}^3$ that is the identity in the complement of $B_{1/2}$ such that $\widetilde{\Phi}(\mathcal{S}')$ is a set of closed vortex lines and vortex tubes of $\tilde{v}$. The closed vortex lines are elliptic trajectories of $\tilde{v}$ and the boundaries of the vortex tubes are KAM-nondegenerate invariant tori of $\tilde{v}$. Given a positive number Λ let us denote the rescaling with factor Λ by $\Theta_\Lambda(x) := \Lambda x$. The theorem follows setting $v(x) := \tilde{v}(x/\tilde{\lambda})$, which satisfies the equation $\operatorname{curl} v = v$ in $\mathbb{R}^3$, and noticing that $(\Theta_{\tilde{\lambda}} \circ \widetilde{\Phi})(\mathcal{S}')$ is a set of closed vortex lines and vortex tubes of v. Since this set is contained in $B_{1/2}$ because $\tilde{\lambda} < 1$, it is standard that there exists a diffeomorphism Φ_0 of $\mathbb{R}^3$ mapping $\mathcal{S}'$ onto $\Theta_{\tilde{\lambda}} \circ \widetilde{\Phi}(\mathcal{S}')$ which is the identity in the complement of $B_{1/2}$. The closed vortex lines in the set $\Phi_0(\mathcal{S}')$ are structurally stable under C^1-small perturbations because they are elliptic [59, Section 2.1], while the vortex tubes are structurally stable under C^4-small volume-preserving perturbations by the KAM theorem. □

Let us now combine Theorems 2.2.1 and 2.2.2 to conclude the proof of Theorem 2.1.1. Theorem 2.2.1 guarantees that, for any large enough odd integer λ, the Beltrami field v constructed in Theorem 2.2.2 can be approximated in the sense of Eq. (2.2.1) by a Beltrami field u defined on $\mathbb{M}^3$. Then it is not hard to see that the structural stability of the set $\Phi_0(\mathcal{S}')$ of closed vortex lines and vortex tubes of v implies the existence of a diffeomorphism Φ_1 of $\mathbb{R}^3$, which is the identity in the complement of $B_{1/2}$, such that $\Phi_1(\mathcal{S}') \subset B_{1/2}$ is a set of structurally stable closed vortex lines and vortex tubes of the rescaled field

$$\Psi_* u\left(\frac{\cdot}{\lambda}\right). \tag{2.2.1}$$

Indeed, because of the ellipticity of the trajectories, this claim is immediate in the case of closed vortex lines provided that the number m appearing in the approximation estimate (2.2.1) is at least 1. For the case of vortex tubes one can use that the Beltrami field u is divergence-free in $\mathbb{M}^3$, which ensures that the field (2.2.1) preserves a smooth volume 3-form in B that is a small perturbation of the Euclidean one, namely

$$(\Psi_* \mu)\left(\frac{\cdot}{\lambda}\right) = \mu_0 + O(\lambda^{-1}).$$

Here μ and μ_0 respectively denote the canonical volume 3-forms of $\mathbb{M}^3$ and $\mathbb{R}^3$. Hence, taking $m \geqslant 4$ in the approximation estimate (2.2.1), this enables us to apply the KAM theorem for volume-preserving fields in $\mathbb{R}^3$, which ensures the existence of the aforementioned diffeomorphism Φ_1 yielding the desired set of vortex tubes of the rescaled field (2.2.1). (For the benefit of the reader let us recall that, in order to prove this KAM result, one takes a Poincaré section transversal to the tube of v under consideration, thereby reducing the problem to perturbations of a nondegenerate twist map of the annulus with the intersection property. It is then standard that one can apply a Moser-type twist theorem to guarantee the preservation of the invariant tori. The details, which go as in [24, Section 7.4], are omitted.)

It follows from the above discussion that the diffeomorphism Φ of $\mathbb{M}^3$ can be then defined as

$$\Phi(x) := \begin{cases} \Phi'(x) & \text{if } x \notin \Phi'^{-1}(\mathbb{B})\,, \\ (\Psi^{-1} \circ \widetilde{\Theta}_{1/\lambda} \circ \Phi_1 \circ \Psi \circ \Phi')(x) & \text{if } x \in \Phi'^{-1}(\mathbb{B})\,, \end{cases}$$

where $\widetilde{\Theta}_{1/\lambda}$ is a smooth diffeomorphism of $\mathbb{R}^3$ which is equal to the rescaling $\Theta_{1/\lambda}$ in the ball $B_{1/2}$ and is the identity in the complement of the ball $B_{3/4}$. This ensures that Φ is a smooth diffeomorphism of $\mathbb{M}^3$ such that the set $\Phi(\mathcal{S})$ is the union of structurally stable closed vortex lines and vortex tubes of the Beltrami field u, so the main theorem follows.

2.3 Proof of the inverse localization theorem in the sphere

In this section we show that for any Beltrami field v in $\mathbb{R}^3$ satisfying $\operatorname{curl} v = v$ there exists a Beltrami field u in $\mathbb{S}^3$ satisfying $\operatorname{curl} u = \lambda u$ whose dynamics in a ball of radius λ^{-1} is very close to the dynamics of v in the unit ball. The proof is largely based on the results in Chapter 4, and we divide it in three steps.

In the first step, the Beltrami field v, which is, in particular a solution of the Helmholtz equation

$$\Delta v + v = 0$$

(note that for divergence free fields, $\operatorname{curl}^2 = -\Delta$) is approximated in B by a field w that is a finite sum of spherical Bessel functions $j_0(|x - x_n|)$ centered at different points $x_n \in \mathbb{R}^3$ (Proposition 2.3.1). The field w is not a Beltrami field, however, but it still satisfies the Helmholtz equation $\Delta w + w = 0$.

In the second step we show that one can take three spherical harmonics Y_1, Y_2, Y_3 in $\mathbb{S}^3$ of energy $\lambda(\lambda - 2)$ whose behaviors in a ball of radius $1/\lambda$ respectively correspond to those of the three components of the field w in a ball of radius 1, provided that λ is large enough (Proposition 2.3.2). Finally, in the third step we construct a Beltrami field u in $\mathbb{S}^3$ of energy λ, using as key ingredients the spherical harmonics Y_k and a basis of Hopf fields, so that u approximates the field v in the sense of Eq. (2.6.1) (Proposition 2.3.3).

For notational convenience, in this section we will write $\Lambda := \lambda - 2$. Notice that Λ is then a large integer.

The first step is encoded by the following Proposition:

Proposition 2.3.1. *For any $\delta > 0$, there is a finite radius R and finitely many constants $\{c_n\}_{n=1}^N \subset \mathbb{R}^3$ and $\{x_n\}_{n=1}^N \subset B_R$ such that the field*

$$w := \sum_{n=1}^{N} c_n\, j_0(|x - x_n|)$$

approximates the Beltrami field v in the ball B as

$$\|v - w\|_{C^{m+2}(B)} < \delta\,.$$

This proposition is the particular case $q = 3$, $n = 3$ of Proposition 4.1.2 in Chapter 4. When comparing the above with the statement in Proposition 4.1.2, it is convenient to have in mind the following relationship between spherical Bessel functions and Bessel functions of the first kind:

$$j_0(t) = \sqrt{\frac{\pi}{2t}} J_{\frac{1}{2}}(r)\,.$$

For the second step, let us write the vector field w in terms of its components w^i in the Cartesian basis $\{e_i\}_{i=1}^3$ of $\mathbb{R}^3$:

$$w = \sum_{i=1}^{3} w^i e_i\,.$$

Each component w^i is a solution of the Helmholtz equation $\Delta w^i + w^i = 0$ in $\mathbb{R}^3$. We now show that for any large enough integer Λ, there exists a spherical harmonic Y_i on $\mathbb{S}^3$ with energy $\Lambda(\Lambda+2)$ that behaves in the ball $\mathbb{B}_{1/\Lambda}$ as w^i does in the unit ball.

Proposition 2.3.2. *Given any positive constant δ, for any large enough integer Λ there is a spherical harmonic Y_i on $\mathbb{S}^3$ with energy $\Lambda(\Lambda+2)$ such that*

$$\left\| w^i - Y_i \circ \Psi^{-1}\left(\frac{\cdot}{\Lambda}\right) \right\|_{C^{m+2}(B)} < \delta\,.$$

The proof of this result is the the particular case $q = 1$ and $n = 3$ of Proposition 4.1.3 in Chapter 4. We refer the reader to that chapter for details.

For the last step, let us consider the three positively oriented orthonormal Hopf vector fields in $\mathbb{S}^3$ that, in terms of the Cartesian coordinates of $\mathbb{R}^4$, are explicitly given by

$$\begin{aligned} h_1 &:= (-x_4, x_3, -x_2, x_1)\,, \\ h_2 &:= (-x_3, -x_4, x_1, x_2)\,, \\ h_3 &:= (-x_2, x_1, x_4, -x_3)\,. \end{aligned}$$

It is well known that they are curl eigenfields with eigenvalue 2, that is,

$$\operatorname{curl} h_i = 2h_i\,.$$

We have taken the the Cartesian basis e_i of $\mathbb{R}^3$ so that $\Psi_* h_i(0) = e_i$.

In the following proposition we show how to construct a Beltrami field on $\mathbb{S}^3$ using the spherical harmonics Y_i obtained in Proposition 2.3.2 and the Hopf fields h_i so that it approximates the Beltrami field v in a suitable sense.

Proposition 2.3.3. *The vector field on the sphere*

$$u := \frac{1}{2\Lambda^2} \operatorname{curl}(\operatorname{curl} + \Lambda)\,(Y_1 h_1 + Y_2 h_2 + Y_3 h_3)$$

is a Beltrami field satisfying $\operatorname{curl} u = (\Lambda+2)u$ and approximates v as

$$\left\| \Psi_* u\left(\frac{\cdot}{\Lambda}\right) - v \right\|_{C^m(B)} < C\delta\,,$$

provided that Λ is sufficiently large.

Here C is a constant depending on m but not on δ. Since rescaling $\Psi_* u$ by Λ is essentially equivalent to rescaling it by λ because

$$\frac{1}{\Lambda} = \frac{1}{\lambda}\left(1 + \frac{2}{\Lambda}\right),$$

Theorem 2.2.1 then follows from Proposition 2.3.3 provided Λ is sufficiently large and δ is chosen small enough for $C\delta$ not to be larger than $\epsilon/2$. The proof of Proposition 2.3.3 is given in Section 2.4.

2.4 Proof of Proposition 2.3.3

We start by defining a vector field $\widetilde{u}$ on $\mathbb{S}^3$ using the Hopf fields h_i as

$$\widetilde{u} := Y_1\, h_1 + Y_2\, h_2 + Y_3\, h_3\,,$$

where the functions Y_i are the spherical harmonics obtained in Proposition 2.3.2. In the following lemma we compute the action of the Laplacian on the vector field $\widetilde{u}$ using the properties of the Hopf fields. The Laplacian on vector fields that we need to consider is defined as the dual of the Hodge Laplacian on 1-forms, and can be computed as

$$\Delta := -\operatorname{curl}\operatorname{curl} + \nabla \operatorname{div}\,,$$

where ∇ and div are the gradient and divergence operators, respectively.

Lemma 2.4.1. *The Laplacian of the vector field $\widetilde{u}$ is*

$$-\Delta\widetilde{u} = \Lambda(\Lambda+2)\,\widetilde{u} + 2\operatorname{curl}\widetilde{u}\,.$$

Proof. The proof is simpler if we work with differential forms, so let us denote by $\widetilde{\beta}$ and α_i the 1-forms that are dual to $\widetilde{u}$ and h_i, respectively, with respect to the canonical metric on $\mathbb{S}^3$. We recall that the dual of $\operatorname{curl}\widetilde{u}$ is the 1-form $\star d\widetilde{\beta}$, with $\star$ being the Hodge star operator. The 1-form $\widetilde{\beta}$ is given by $\widetilde{\beta} = Y_i\, \alpha_i$, where summation over repeated indices is understood throughout. The Laplacian of $\widetilde{\beta}$ is then

$$-\Delta\widetilde{\beta} := dd^*\widetilde{\beta} + d^*d\widetilde{\beta} = -d\star d\star(Y_i\,\alpha_i) + \star d\star d(Y_i\,\alpha_i)\,.$$

Using that $\star d\alpha_i = 2\alpha_i$ because α_i is the dual 1-form of the Hopf field h_i, and that the differential of Y_i can be written as $dY_i = h_j(Y_i)\,\alpha_j$, where $h_j(Y_k)$ denotes the action of the vector field h_j on the scalar function Y_k, we readily obtain

$$d\star d\star(Y_i\,\alpha_i) = \frac{1}{2} d\star(h_j(Y_i)\,\alpha_j \wedge d\alpha_i)\,.$$

Observing that $\alpha_j \wedge d\alpha_i = 2\alpha_j \wedge \star\alpha_i = 2\delta_{jk}\,\mu$, where μ stands for the Riemannian volume 3-form on $\mathbb{S}^3$, it follows that

$$d\star d\star(Y_i\,\alpha_i) = d(h_i(Y_i)) = h_j h_i(Y_i)\,\alpha_j\,. \tag{2.4.1}$$

Analogously, a straightforward computation using that $\star(\alpha_j \wedge \alpha_i) = \varepsilon_{jil}\alpha_l$, where ε_{jil} stands for the Levi-Civita permutation symbol, and the identity $\varepsilon_{iml}\varepsilon_{jkl} = \delta_{ij}\delta_{mk} - \delta_{ik}\delta_{mj}$ yields

$$\star\, d(Y_i\,\alpha_i) = \varepsilon_{jil} h_j(Y_i)\,\alpha_l + 2Y_i\,\alpha_i\,, \tag{2.4.2}$$

$$\star\, d \star d(Y_i\,\alpha_i) = -h_j h_j(Y_i)\,\alpha_i + h_i h_j(Y_i)\,\alpha_j + 4\varepsilon_{jil} h_j(Y_i)\,\alpha_l + 4Y_i\,\alpha_i\,. \tag{2.4.3}$$

Finally, adding Eqs. (2.4.1) and (2.4.3) we obtain

$$\begin{aligned} -\Delta\widetilde{\beta} &= -h_j h_i(Y_i)\,\alpha_j + h_i h_j(Y_i)\,\alpha_j - h_j h_j(Y_i)\,\alpha_i + 4\varepsilon_{jil} h_j(Y_i)\,\alpha_l + 4Y_i\,\alpha_i \\ &= \Lambda(\Lambda+2)Y_i\,\alpha_i + 2\varepsilon_{jil} h_j(Y_i)\,\alpha_l + 4Y_i\,\alpha_i\,, \end{aligned}$$

where we have used that $\Delta Y_i = -\Lambda(\Lambda+2)Y_i$ and that the commutator of Hopf fields is $[h_i, h_j] = -2\varepsilon_{ijl}h_l$. The lemma then follows upon noticing that

$$2\varepsilon_{jil} h_j(Y_i)\,\alpha_l + 4Y_i\,\alpha_i = 2 \star d\widetilde{\beta}$$

by Eq. (2.4.2). □

Using this lemma, it is easy to check that

$$u := \frac{1}{2\Lambda^2}\operatorname{curl}(\operatorname{curl} + \Lambda)\widetilde{u}$$

is a Beltrami field with eigenvalue $\Lambda + 2$. Indeed, a straightforward computation shows that

$$\begin{aligned} \operatorname{curl} u = \frac{1}{2\Lambda^2}\operatorname{curl}\operatorname{curl}(\operatorname{curl} + \Lambda)\widetilde{u} &= \frac{1}{2\Lambda^2}\operatorname{curl}(-\Delta + \Lambda\operatorname{curl})\widetilde{u} \\ &= \frac{\Lambda+2}{2\Lambda^2}\operatorname{curl}(\operatorname{curl} + \Lambda)\widetilde{u} = (\Lambda+2)u\,. \end{aligned}$$

To prove the C^m estimate of the proposition, it is convenient to introduce the following auxiliary vector field in the unit ball B of $\mathbb{R}^3$

$$\bar{u}(x) := \widetilde{Y}_1(x)\,e_1 + \widetilde{Y}_2(x)\,e_2 + \widetilde{Y}_3(x)\,e_3\,,$$

where $x \in B$ and $\widetilde{Y}_i$ was defined in (4.1.4). There is no loss of generality in choosing the orthonormal basis e_i of $\mathbb{R}^3$ compatible with the Hopf fields h_i in the sense that $\Psi_*(h_i)(0) = e_i$. It is then easy to check that for $x \in B$ one has:

$$\begin{aligned} \Psi_*\widetilde{u}\Big(\frac{\cdot}{\Lambda}\Big) &= \bar{u} + \frac{G_1}{\Lambda}\bar{u}\,, \\ \Psi_*(\operatorname{curl}\widetilde{u})\Big(\frac{\cdot}{\Lambda}\Big) &= \Lambda\Big(\operatorname{curl}_0\bar{u} + \frac{G_2}{\Lambda}\bar{u} + \frac{G_3}{\Lambda}D\bar{u}\Big)\,, \\ \Psi_*(\operatorname{curl}\operatorname{curl}\widetilde{u})\Big(\frac{\cdot}{\Lambda}\Big) &= \Lambda^2\Big(\operatorname{curl}_0\operatorname{curl}_0\bar{u} + \frac{G_4}{\Lambda}\bar{u} + \frac{G_5}{\Lambda}D\bar{u} + \frac{G_6}{\Lambda}D^2\bar{u}\Big)\,. \end{aligned}$$

Here curl_0 denotes the Euclidean curl operator, acting on the variables x, and the functions $G_i(x,\Lambda)$ are (possibly matrix-valued) functions that depend smoothly on all their variables and whose derivatives are uniformly bounded as

$$\sup_{x\in B} |D_x^\alpha G_i(x,\Lambda)| < C_\alpha\,. \tag{2.4.4}$$

Here the constant C_α depends on the multiindex α but not on Λ.

These identities and the fact that $(\operatorname{curl}_0 \operatorname{curl}_0 + \operatorname{curl}_0)v = 2v$ then permits us to write

$$\begin{aligned}\left\|\Psi_* u\Big(\frac{\cdot}{\Lambda}\Big) - v\right\|_{C^m(B)} &\leqslant \left\|\frac{1}{2}(\operatorname{curl}_0 \operatorname{curl}_0 + \operatorname{curl}_0)(\bar{u} - v)\right\|_{C^m(B)} + \frac{C}{\Lambda}\|\bar{u}\|_{C^{m+2}(B)} \\ &\leqslant C\|\bar{u} - v\|_{C^{m+2}(B)} + \frac{C}{\Lambda}\|\bar{u} - w\|_{C^{m+2}(B)} \\ &\qquad + \frac{C}{\Lambda}\|v - w\|_{C^{m+2}(B)} + \frac{C}{\Lambda}\|v\|_{C^{m+2}(B)} .\end{aligned} \tag{2.4.5}$$

To conclude, notice that it stems from Propositions 2.3.1 and 2.3.2 that

$$\begin{aligned}\|v - w\|_{C^{m+2}(B)} &< \delta \\ \|\bar{u} - w\|_{C^{m+2}(B)} &< 3\delta ,\end{aligned}$$

so in particular

$$\|\bar{u} - v\|_{C^{m+2}(B)} \leqslant \|\bar{u} - w\|_{C^{m+2}(B)} + \|v - w\|_{C^{m+2}(B)} < 4\delta .$$

Hence the proposition follows from the estimate (2.4.5) upon noticing that v is a fixed vector field (so its norm is independent of Λ) and choosing Λ large enough, which also allows us to take δ as small as one wishes.

2.5 Proof of the inverse localization theorem in the torus

Arguing as in the proof of Proposition 2.3.1 we can readily show that for any $\delta > 0$, there exists a vector field v_1 on $\mathbb{R}^3$ that approximates the Beltrami field v in the ball B as

$$\|v_1 - v\|_{C^0(B)} < \delta , \tag{2.5.1}$$

and that can be represented as the Fourier transform of a distribution supported on the unit sphere of the form

$$v_1(x) = \int_{\mathbb{S}^2} f(\xi)\, e^{i\xi\cdot x}\, d\sigma(\xi) .$$

Again $\mathbb{S}^2$ denotes the unit sphere $\{\xi \in \mathbb{R}^3 : |\xi| = 1\}$ and f is a smooth $\mathbb{R}^3$-valued function on $\mathbb{S}^2$.

Let us now cover the sphere $\mathbb{S}^2$ by finitely many closed sets $\{U_n\}_{n=1}^N$ with piecewise smooth boundaries and pairwise disjoint interiors such that the diameter of each set is at most δ'. We can then repeat the argument used in the proof of Proposition 2.3.1 to infer that, if ξ_n is any point in U_n and we set

$$c_n := f(\xi_n)\, |U_n| ,$$

the field

$$w(x) := \sum_{n=1}^{N} c_n \, e^{i\xi_n \cdot x}$$

approximates the field v_1 uniformly with an error proportional to δ':

$$\|w - v_1\|_{C^0(B)} < C\delta' .$$

The constant C depends on δ but not on δ', so one can choose the maximal diameter δ' small enough so that

$$\|w - v_1\|_{C^0(B)} < \delta . \tag{2.5.2}$$

In turn, the uniform estimate

$$\|w - v\|_{C^0(B)} \leqslant \|w - v_1\|_{C^0(B)} + \|v - v_1\|_{C^0(B)} < 2\delta$$

can be readily promoted to the C^{m+2} bound

$$\|w - v\|_{C^{m+2}(B)} < C\delta . \tag{2.5.3}$$

This follows from standard elliptic estimates as both w (whose Fourier transform is supported on $\mathbb{S}^2$) and v satisfy the Helmholtz equation:

$$\Delta v + v = 0 , \qquad \Delta w + w = 0 .$$

Furthermore, replacing w by its real part if necessary, we can safely assume that the field w is real-valued.

Let us now observe that for any large enough odd integer Λ one can choose the points $\xi_n \in U_n \subset \mathbb{S}^2$ so that they have rational components (i.e., $\xi_n \in \mathbb{Q}^3$) and the rescalings $\Lambda\xi_n$ are actually integer vectors (i.e., $\Lambda\xi_n \in \mathbb{Z}^3$). This is because rational points $\xi \in \mathbb{S}^2 \cap \mathbb{Q}^3$ with $\Lambda\xi \in \mathbb{Z}^3$ are uniformly distributed on the unit sphere as $\Lambda \to \infty$ through odd values [20].

Choosing ξ_n as above, we are now ready to prove Theorem 2.2.1 in the torus. Without loss of generality, we will take the origin as the base point p, so that we can identify the ball $\mathbb{B}$ with B through the canonical 2π-periodic coordinates on the torus. In particular, the diffeomorphism $\Psi : \mathbb{B} \to B$ that appears in the statement of Theorem 2.2.1 can be understood to be the identity.

Since $\Lambda\xi_n \in \mathbb{Z}^3$, it follows that the vector field

$$\widetilde{u}(x) := \sum_{n=1}^{N} c_n e^{i\Lambda\xi_n \cdot x}$$

is 2π-periodic (that is, invariant under the translation $x \to x + 2\pi\, a$ for any vector $a \in \mathbb{Z}^3$). Therefore it descends to a well-defined vector field on the torus $\mathbb{T}^3 := \mathbb{R}^3/(2\pi\mathbb{Z})^3$, which we will still denote by $\widetilde{u}$.

Since the Fourier transform of $\widetilde{u}$ if now supported on the sphere of radius Λ, $\widetilde{u}$ then satisfies the Helmholtz equation on the flat torus $\mathbb{T}^3$ with energy Λ^2,

$$\Delta\widetilde{u} + \Lambda^2\widetilde{u} = 0 .$$

A straightforward calculation then reveals that the vector field on the torus

$$u := \frac{\operatorname{curl}\operatorname{curl}\widetilde{u} + \Lambda \operatorname{curl}\widetilde{u}}{2\Lambda^2}$$

satisfies the equation

$$\operatorname{curl} u = \Lambda u\,,$$

so it is a Beltrami field on $\mathbb{T}^3$ with eigenvalue $\lambda := \Lambda$.

Let us now notice that, with some abuse of notation,

$$\widetilde{u}\left(\frac{x}{\Lambda}\right) = w(x)$$

for all points x, say, in the ball B. In particular, as the derivatives of the rescaled vector field $\widetilde{u}(\cdot/\Lambda)$ behave as

$$\begin{aligned} \operatorname{curl}\widetilde{u}\left(\frac{\cdot}{\Lambda}\right) &= \Lambda \operatorname{curl} w\,, \\ \operatorname{curl}\operatorname{curl}\widetilde{u}\left(\frac{\cdot}{\Lambda}\right) &= \Lambda^2 \operatorname{curl}\operatorname{curl} w\,, \end{aligned}$$

it then follows that

$$\begin{aligned} \left\| u\left(\frac{\cdot}{\Lambda}\right) - v \right\|_{C^m(B)} &= \left\| \frac{\Lambda^2 \operatorname{curl}\operatorname{curl} w + \Lambda^2 \operatorname{curl} w}{2\Lambda^2} - v \right\|_{C^m(B)} \\ &= \left\| \frac{\operatorname{curl}\operatorname{curl}(w-v) + \operatorname{curl}(w-v)}{2} \right\|_{C^m(B)} \\ &\leqslant C\|w - v\|_{C^{m+2}(B)} \\ &< C\delta\,, \end{aligned}$$

where we have used the identity $\operatorname{curl}\operatorname{curl} v + \operatorname{curl} v = 2v$ to pass to the second equality and the estimate (2.5.3) to derive the last inequality. The theorem then follows provided that δ is chosen small enough for $C\delta < \epsilon$.

2.6 Concluding remarks

To conclude, let us make a few simple observations about our main result that follow from its proof:

There are many Beltrami fields with closed vortex lines and tubes of a given link type

Indeed, since our construction works for any large enough odd integer λ and Beltrami fields corresponding to different eigenvalues are L^2 orthogonal, there are many non-proportional Beltrami fields with closed vortex lines and tubes realizing any given link.

In the sphere, the result holds true for any large enough eigenvalue λ

Indeed, the fact that Λ is odd was never used in the proof of Theorem 2.2.1 in $\mathbb{S}^3$ (cf. Section 2.3), so it stems that, given any finite union of closed curves and tubes $\mathcal{S}$, for any integer λ with $|\lambda|$ greater than certain constant $\Lambda_0(\mathcal{S})$ there is a Beltrami field with eigenvalue λ having a structurally stable set of vortex lines and vortex tubes diffeomorphic to $\mathcal{S}$.

In our Beltrami fields on the sphere, knots and links appear in pairs

In fact, using the Hopf basis $\{h_i\}_{i=1}^3$ introduced in Section 2.3, any Beltrami field u on $\mathbb{S}^3$ with eigenvalue $\lambda := \Lambda + 2$, with Λ a nonnegative integer, can be written as

$$u = F_1\, h_1 + F_2\, h_2 + F_3\, h_3\,,$$

where F_i are smooth functions on the sphere. It is then easy to check using Eq. (2.4.2) that F_i must be a spherical harmonic of energy $\Lambda(\Lambda+2)$. Since such a spherical harmonic is known to have parity $(-1)^\Lambda$, in the sense that

$$F_i(-p) = (-1)^\Lambda\, F_i(p)$$

for all points p in the unit sphere $\mathbb{S}^3$, and the Hopf fields h_i are odd (i.e., $h_i(-p) = -h_i(p)$), we conclude that a Beltrami field on the sphere with eigenvalue λ has parity $(-1)^{\lambda+1}$, so it is either even or odd. Therefore, the fact that $\Phi(\mathcal{S})$ is a set of vortex lines and vortex tubes of the Beltrami field u diffeomorphic to $\mathcal{S}$ and contained in a ball of small radius $1/\lambda$ automatically implies that so is the antipodal set $-\Phi(\mathcal{S})$.

The result carries over to lens spaces

In order to see why, the key is that in the sphere the statement of Theorem 2.2.1 can be refined to include localizations around different points of the sphere. More precisely, let us fix l points $P_1, \dots, P_l$ in $\mathbb{S}^3$, none of which are antipodal to another (that is, $P_j \neq -P_k$), and denote by $\Psi_j : \mathbb{B}(P_j, R_0) \to B_{R_0}$ a patch of normal geodesic coordinates centered at the point P_j. Here $\mathbb{B}(P_j, R_0)$ denotes the geodesic ball in the sphere of center P_j and radius

$$R_0 := \frac{1}{2} \min_{j \neq k} \operatorname{dist}_{\mathbb{S}^3}(P_j, P_k)\,.$$

The approximation theorem can then be stated as follows:

Theorem 2.6.1. *Let $\{v_j\}_{j=1}^l$ be Beltrami fields in $\mathbb{R}^3$, satisfying* $\operatorname{curl} v_j = v_j$. *Let us fix any positive numbers ϵ and m. Then for any large enough integer λ there is a Beltrami field u, satisfying* $\operatorname{curl} u = \lambda u$ *in $\mathbb{S}^3$, such that*

$$\left\| (\Psi_j)_* u\left(\frac{\cdot}{\lambda}\right) - v_j \right\|_{C^m(B)} < \epsilon \tag{2.6.1}$$

for all $1 \leqslant j \leqslant l$.

For the proof, it is enough to argue exactly as in Section 2.3, but using, instead of Proposition 2.3.2, a refinement of it, Proposition 4.2.1, that is proven in Chapter 4.

In particular, this yields the existence of Beltrami fields in the sphere having prescribed sets of closed vortex lines and tubes (modulo diffeomorphism) around any finite number of points $P_1, \dots, P_l$. These lines and tubes are contained in balls of radius $1/\lambda$. This line of reasoning also allow us to prove an analog of Theorem 2.1.1 in any lens space $L(p,q)$:

Theorem 2.6.2. *Let $\mathcal{S}$ be a finite union of (pairwise disjoint, but possibly knotted and linked) closed curves and tubes contained in a contractible subset of a three-dimensional lens space $L(p,q)$. Then for any large enough even integer λ there exists a Beltrami field u satisfying the equation* $\operatorname{curl} u = \lambda u$ *and a diffeomorphism Φ of $L(p,q)$ such that $\Phi(\mathcal{S})$ is a union of vortex lines and vortex tubes of u. Furthermore, this set is structurally stable.*

Proof. The lens space can be written as

$$L(p,q) = \mathbb{S}^3/G\,,$$

where G is a finite isometry group isomorphic to $\mathbb{Z}_p$. We can assume that G is generated by certain isometry g. Let us now fix a point $p_0 \in \mathbb{S}^3$ and set

$$P_j := g^j \cdot p_0$$

for $0 \leqslant j \leqslant p-1$. If Ψ is a patch of normal geodesic coordinates around p_0, we will also set $\Psi_j(x) := \Psi(g^{-j} \cdot x)$. Notice that if p is odd there are not any points in the set $\{P_j\}_{j=0}^{p-1}$ that are antipodal to each other, while for p even P_j and P_k are antipodal if and only if $|j-k| = \frac{p}{2}$.

Let us fix a Beltrami field v in $\mathbb{R}^3$ as in Theorem 2.2.2. Theorem 2.6.1 then ensures the existence of a Beltrami field $\widetilde{u}$ in $\mathbb{S}^3$ such that

$$\left\| (\Psi_j)_* \widetilde{u}\Big(\frac{\cdot}{\lambda}\Big) - v_j \right\|_{C^m(B)} < \epsilon\,,$$

where $0 \leqslant j \leqslant p'-1$ with $p' := p$ if p is odd and $p' := \frac{p}{2}$ if p is even. Here $v_0 := v$ and $v_j := 0$ for $1 \leqslant j \leqslant p'-1$. Notice that, as λ is even, we saw in the previous remark that $\widetilde{u}$ is odd, i.e., $\widetilde{u}(x) = -\widetilde{u}(-x)$, so that $\widetilde{u}$ is equivariant under the isometry $x \mapsto -x$. Hence, by construction, the vector field

$$u := \sum_{j=0}^{p'-1} (g^j)_* \widetilde{u}$$

is G-equivariant, and therefore it defines a vector field in the quotient space $L(p,q) = \mathbb{S}^3/G$ that we still denote by u with some abuse of notation. Arguing exactly as in the proof of the main theorem one can show that the vector field u on $L(p,q)$ indeed has the desired properties, so the statement then follows. □

In the torus, the distribution of rational points on the 2-sphere is key

The proof that we have given holds provided that the eigenvalue λ is an odd integer of sufficiently large absolute value. It does not say anything about even integers, or about eigenvalues that are not integers. This assertion can be refined a little, however. We have seen that for any eigenvalue λ of the curl operator in $\mathbb{T}^3$ there is a set of points $\{\xi_n\}_{n=1}^N$ lying on the unit sphere $\mathbb{S}^2$ of $\mathbb{R}^3$ such that $\lambda\xi_n \in \mathbb{Z}^3$ (this is obvious from the fact that one can write $\lambda = |k|$ with $k \in \mathbb{Z}^3$). Therefore, in the proof of Theorem 2.2.1 for the torus (cf. Section 2.5) one can substitute the collection of odd integers Λ by any subset of eigenvalues λ for which there is a set of points $\{\xi_n\}_{n=1}^N \subset \mathbb{S}^2$ (depending on λ and such that the rescalings $\lambda\xi_n$ are in $\mathbb{Z}^3$) that becomes dense in the sphere as $|\lambda| \to \infty$ along this subset of eigenvalues. In particular, replacing the density condition by the more stringent assumption that $\{\xi_n\}$ becomes equidistributed on the sphere, it turns out that the characterization of the numbers λ that satisfy this property is somehow related to the celebrated Linnik problem in number theory. In particular, since the aforementioned equidistribution property holds for any eigenvalue for which the integer λ^2 is square-free [19], we immediately infer that the statement of Theorem 2.1.1 also holds for any large enough eigenvalue λ of curl (possibly even or non-integer) for which λ^2 is square-free.

Another interesting point to consider is whether the above methods apply to tori of the form $\mathbb{T}^3_{\mathcal{L}} := \mathbb{R}^3/(2\pi\mathcal{L})$, where $\mathcal{L}$ is a general lattice in $\mathbb{R}^3$. As above, the key in this case is to have a density or equidistribution result at disposal, but this time not for rational points on the sphere and integer eigenvalues, but for a set of points $\{\xi_n\}_{n=1}^N \subset \mathbb{S}^2$ and real eigenvalues λ such that $\lambda\xi_n \in \mathcal{L}'$, where $\mathcal{L}'$ is the so called reciprocal lattice to $\mathcal{L}$, which is defined as the set of points $k \in \mathbb{R}^3$ such that $k \cdot x \in \mathbb{Z}$ for all $x \in \mathcal{L}$. In some very particular cases, e.g. for lattices of the form $\mathcal{L} := a\mathbb{Z}^3$ with $a \in \mathbb{R}\backslash\{0\}$, the previous equidistribution results directly hold for a sequence of eigenvalues of the form λ/a with λ an odd integer. In general, however, the authors are not aware of any results in this direction.

Chapter 3

Geometric structures in the eigenfunctions of Dirac operators

In spheres of dimension $n \geqslant 3$, let $\Sigma := \{\Sigma_1, ..., \Sigma_N\}$ be any collection of codimension 2 smooth submanifolds of arbitrarily complicated topology (N being the complex dimension of the spinor bundle). In this chapter we show that there is always an eigenfunction $\psi = (\psi_1, ..., \psi_N)$ of the Dirac operator (in fact, infinitely many of them) such that the submanifold Σ_i, modulo ambient diffeomorphism, is a structurally stable nodal set of the spinor component ψ_i. The result holds for any choice of trivialization of the spinor bundle. The emergence of these complicated structures takes place at small scales and sufficiently high energies.

3.1 Introduction

Regarding the spectral properties of elliptic operators on a compact manifold, a problem of much physical and mathematical significance is to understand the ultraviolet regime, that is, which patterns emerge as the eigenvalues get larger.

The paradigmatic example of such a pattern is Weyl's law on the growth of the number of eigenvalues of the Laplace operator, which first appeared as a heuristic derivation of the energy distribution of black body radiation (the so-called Rayleigh-Jeans law, at the heart of the ultraviolet catastrophe). A more modern example, to this day the object of intense investigation, is S. T. Yau's conjecture [75] on the growth of the Hausdorff measure of the zero sets of eigenfunctions: the total hypersurface measure is expected to be proportional to the square root of the eigenvalue, regardless of the manifold. In the case of real analytic Riemannian metrics, this conjecture was proved by H. Donnelly and C. Fefferman [18]; in the general case, some very recent breakthroughs have been made by A. Logunov [52, 53] and A. Logunov and E. Malinnikova [54].

These asymptotic laws suggest a certain universality of the ultraviolet behavior. Indeed, leaving the constants aside, they do not depend on the metric used to define the elliptic operator, nor do they detect the topology of the underlying manifold. Even if the magnitudes of interest are global, they seem to be controlled only by the small scale behavior. The intuition is that, at sufficiently high energies, the characteristic scales are very small with respect to those relevant to the geometry and topology of the manifold, hence the behavior of eigenfunctions should not be very sensible to these.

Our aim in this chapter is to investigate, from the above perspective, the *topology* of the nodal sets of eigenfunctions of *Dirac operators*.

Dirac operators have gained a central role in geometry, by virtue of their analytic properties (especially the index theorem and the Weitzenbőck formula), and how these relate to the geometry and topology of the manifold (see e.g [35, 73]). What can be learned from their spectral properties has also interested mathematicians and physicists alike [17, 9, 71].

On the other hand, the problem of the allowed shapes of zero sets of solutions to elliptic PDEs also has some interesting history. If, instead of the Dirac operator, one considers the Cauchy-Riemann operator, it corresponds to a weakened version of the second Cousin problem: what codimension 2 submanifolds of a complex manifold can be (maybe up to diffeomorphism) the nodal set of a holomorphic function? As first shown by K. Oka [63] (in what became both one of the precursors of sheaf theoretical methods in algebraic geometry, and a first hint of M. Gromov's Oka Principle [34]), in the case of a Stein manifold, the only obstruction is the obvious one: it must be possible to realize the submanifold in question as the zero set of a continuous, complex-valued function.

In a n-dimensional spin manifold M, we can formulate an analogous problem for eigenfunctions of the Dirac operator. Recall that eigenfunctions of the Dirac operator are sections of a hermitian vector bundle S of complex rank $r(n) = 2^{\lfloor \frac{n}{2} \rfloor}$, called the spinor bundle. In a spin manifold of dimension 3 or higher, the regular zero sets of a spinor are empty (because $2r(n) > n$), so we will focus our attention on the topology of the zero sets of the spinor components. These are complex-valued, so their regular zero sets are, as in the holomorphic case, codimension 2 submanifolds (interestingly, by a result of C. Bằr [10], critical level sets of a spinor are also of codimension at most 2).

To be more precise, if the spinor bundle is trivial, a choice of trivialization makes any section of S a collection of $r(n)$ complex-valued functions; if the section is an eigenfunction of the Dirac operator, these complex-valued functions are related by a first order partial differential relation, and this relation might impose restrictions on how topologically intricate the zero sets of the functions can get to be. For example, in $\mathbb{S}^3$, a spinor can be decomposed in two components (ψ_1, ψ_2). If L_1 and L_2 are two disjoint closed curves, arbitrarily knotted and linked, one might ask whether it is possible to find an eigenfunction of the Dirac operator such that L_1 is a nodal set of ψ_1, and L_2 is a nodal set of ψ_2, possibly up to an ambient diffeomorphism (henceforth, by a nodal set of a function we mean a union of connected components of its zero set, not necessarily the whole zero set).

More generally, given a collection of $r(n)$ codimension 2 submanifolds, $\mathfrak{S} := \{\Sigma_a\}_{a=1}^{r(n)}$, we will say that a section of the spinor bundle ψ *realizes* $\mathfrak{S}$ if each submanifold Σ_a, possibly modulo an ambient diffeomorphism Φ, is a nodal set of the corresponding spinor component ψ_a, $\Phi(\Sigma_a) \subset \psi_a^{-1}(0)$. The main result in this chapter is that, in the case of the round n-dimensional sphere, any collection can be realized, for any given trivialization, by Dirac eigenfunctions *of high enough energy*:

Theorem 3.1.1. *Let $\mathfrak{S} := \{\Sigma_a\}_{a=1}^{r(n)}$ be a collection of $r(n)$ closed, pairwise disjoint, smooth codimension 2 submanifolds in $\mathbb{S}^n$, for $n \geqslant 3$. Fix an integer $m \geqslant 1$. For any large enough positive integer k, and for any choice of orthonormal basis of Killing spinors trivializing the spinor bundle, there is a C^m-open set of eigenfunctions of the Dirac operator of eigenvalue $\pm(\frac{n}{2}+k)$ realizing $\mathfrak{S}$.*

The proof exhibits the interplay between, on one side, the flexibility of such questions in euclidean space (or more generally, in open manifolds); and on the other side, a certain notion of universality at high energies and corresponding small scales that reduces the problem to the euclidean case.

Flexibility in euclidean space is captured by the following result, whose proof will be given in Section 3.4. If we denote by D_0 the standard Dirac operator on $\mathbb{R}^n$, it reads:

Theorem 3.1.2. *Fix an integer $m \geqslant 1$, and an arbitrarily small real number $\epsilon > 0$. Inside the unit ball $B \subset \mathbb{R}^n$, consider a collection $\mathfrak{S} := \{\Sigma_a\}_{a=1}^{r(n)}$ of $r(n)$ closed, pairwise disjoint, smooth codimension 2 submanifolds. For any given $\lambda \in \mathbb{R}$, there is a $\mathbb{C}^{r(n)}$-valued function $\phi := (\phi_1, ..., \phi_{r(n)})$, satisfying the Dirac equation $D_0\phi = \lambda\phi$ on $\mathbb{R}^n$, and a diffeomorphism $\Phi_0 : B \to B$ satisfying $||\Phi_0 - Id||_{C^m} \leqslant \epsilon$, such that $\Phi_0(\Sigma_a)$ is a nodal set of ϕ_a. Further, these nodal sets are structurally stable.*

The nodal sets being structurally stable means that any other $\mathbb{C}^{r(n)}$-valued function $\varphi := (\varphi_1, ..., \varphi_{r(n)})$ that is close enough to $\phi = (\phi_1, ..., \phi_{r(n)})$ in the C^m topology will have the same collection of nodal sets, modulo a diffeomorphism arbitrarily close to the identity (more precisely, once one choses how close to the identity the diffeomorphism is to be, there exists a bound on how close to ϕ a function φ needs to be). This stability is a consequence of the fact that, by our construction, the derivatives $d_x\phi_a$ will have full rank for all $x \in \Phi(\Sigma_a)$ (the relation between the full rank of the derivatives and the stability of the structures is often called Thom's isotopy lemma, see e.g [1]).

In [27], A. Enciso and D. Peralta-Salas introduced a general strategy to tackle similar realization questions for level sets of solutions of second order elliptic PDEs in euclidean space. It is based on finding local solutions with prescribed 1-jet on the submanifold one wants to realize (which can be done by means of Cauchy-Kovalevskaya theorem, or solving a well chosen boundary value problem), and then extending these local solutions to global ones by means of a Runge-type approximation theorem. That we have prescribed the first derivative at the submanifold becomes crucial in this last step, to ensure the structural stability of the level set.

The proof of our euclidean result adapts this strategy to the case of the Dirac operator, but this requires some further considerations. The main difficulty is that now we can only prescribe the 0-jet on the local Cauchy problem, because the Dirac operator is of order one, so structural stability does not come out automatically and a more involved construction is needed.

On compact manifolds, however, the whole strategy above is guaranteed to fail from the start. The reason for this failure is of a fundamental nature: on a compact manifold, one cannot use Runge-type approximations without creating singularities in the global solutions.

In contrast, the idea of the proof of Theorem 3.1.1 is to regain some flexibility by exploiting the increasing dimension of the space of eigenfunctions of the Dirac operator. This is analogous to the strategy used in [29] in the context of invariant sets of eigenfunctions of the curl operator. Specifically, we prove that for any given eigenfunction ϕ of the euclidean Dirac operator of eigenvalue 1, there are many Dirac spinors in the sphere of high enough eigenvalue $\frac{n}{2} + k$ whose behavior (in the sense of C^m norm) in a ball of radius k^{-1} is very close to the behavior of ϕ in the unit ball.

This implies that any property that eigenfunctions of the euclidean Dirac operator exhibit on compact sets is also exhibited at small scales by high energy eigenfunctions of the Dirac operator on the sphere, provided such property is robust under suitably small perturbations. Theorem 3.1.2 precisely ensures that the realization of any arbitrarily complicated collection of submanifolds $\mathfrak{S}$ is a property of this kind.

The proof of the theorem yields as well a rather precise understanding of the ambient diffeomorphism Φ through which the structure $\mathfrak{S}$ is realized. It basically consists In rescaling an open subset containing $\mathfrak{S}$ so that it gets inside a ball of radius proportional to k^{-1}. In principle, we have no control on the nodal sets outside the small ball. Nonetheless, the analysis can be refined so that one is able to construct eigenfunctions with prescribed nodal sets inside a given number (as large as we want) of small enough balls.

The chapter is organized as follows. In Section 3.2 we will prove Theorem 3.1.1, assuming the euclidean realization theorem (Theorem 3.1.2) and the key inverse localization result (Theorem 3.2.1). The inverse localization result is then proved in Section 3.3, assuming as input the key inverse localization for spherical harmonics that is the main concern of the Chapter 4. The euclidean realization theorem is proved in Section 3.4. In Section 3.5 we adapt the previous results to prescribe the nodal sets at multiple regions at once. We conclude with some comments about the analog of Theorem 3.1.1 in the torus case, which can be proved with minor adaptations of the ideas in this chapter, together with the ones in Section 2.5 of Chapter 2.

3.2 Proof of the main theorem

To set the stage, consider the sphere $\mathbb{S}^n$ ($n \geqslant 3$) endowed with the round metric g of constant sectional curvature 1. We will denote by S the spinor bundle, whose sections $\psi : \mathbb{S}^n \to S$ we call spinors. The canonical covariant derivative on S associated with the Levi-Civita connection on $T\mathbb{S}^n$ will be denoted by ∇^S. The Dirac operator D can be written locally as

$$D = \sum_{i=1}^{n} \rho(e_i) \nabla^S_{e_i},$$

where ρ is the Clifford multiplication map, $\rho : T\mathbb{S}^n \to \mathrm{End}(S)$, and $\{e_i\}_{i=1}^n$ is an orthonormal basis of $T\mathbb{S}^n$.

The spectrum of the Dirac operator on $\mathbb{S}^n$ is well studied (see e.g [8]). In particular, eigenvalues are of the form $\pm(\frac{n}{2} + k)$, for $k \in \mathbb{N}$, and the linear space of eigenfunctions of fixed k has complex dimension

$$D(n,k) = r(n) \binom{n+k-1}{k}.$$

A very explicit characterization of the spinor bundle of the n-dimensional sphere can be provided in terms of *Killing spinors*. Killing spinors are solutions of the equation

$$\nabla^S \chi = \lambda \rho(\cdot) \chi,$$

for some fixed constant λ, called the Killing number of the Killing spinor.

Note that a Killing spinor is either identically zero, or it has no zeroes at all. Indeed, the Killing equation implies, on one hand, that a Killing spinor is also an eigenfunction of the Dirac operator, so that it satisfies the strong unique continuation property; and on the other hand, that in a point where a Killing spinor vanishes, all of its derivatives vanish too.

It is well known (see e.g [8]) that for $\lambda = \pm\frac{1}{2}$ one can define a global orthonormal frame $\{\chi_a\}_{a=1}^{r(n)}$ of S consisting on λ-Killing spinors. Thus, any section ψ of S can be specified by the choice of $r(n)$ complex functions $\psi_a : \mathbb{S}^n \to \mathbb{C}$,

$$\psi = \sum_{a=1}^{r(n)} \psi_a \chi_a.$$

Let us now introduce the notations needed to describe the small scale behavior of spinors. Fix an arbitrary point $p_0 \in \mathbb{S}^n$ and take a patch of normal geodesic coordinates $\Psi : \mathbb{B} \to B$ centered at p_0. We will denote by B_ρ (resp. $\mathbb{B}_\rho$) the ball in $\mathbb{R}^n$ (resp. the geodesic ball in $\mathbb{S}^n$) of radius ρ and centered at the origin (resp. at p_0). We omit the subscript when $\rho = 1$. Via the diffeomorphism Ψ one defines a map on the spinor bundle

$$\widehat{\Psi}_* : S|_{\mathbb{B}} \longrightarrow B \times \mathbb{C}^{r(n)};$$

and a spinor field ψ will be expressed in local coordinates x on B as

$$\widehat{\Psi}_*\psi(x) = \sum_{a=1}^{r(n)} \widehat{\psi}_a(x)\,\xi_a\,,$$

where $\{\xi_a\}_{a=1}^{r(n)}$ is the standard basis on $\mathbb{C}^{r(n)}$. Throughout the rest of the chapter, unless we explicitly say otherwise, we assume the choice of a basis of Killing spinors χ_a, and we choose the basis ξ_a accordingly, that is, with $\widehat{\Psi}_*\chi_a(0) = \xi_a$.

The small scale behavior of the spinor ψ on a ball of radius k^{-1} around p_0 is thus captured by the rescaled field

$$\widehat{\Psi}_*\psi\Big(\frac{\cdot}{k}\Big) := \sum_{a=1}^{r(n)} \widehat{\psi}_a\Big(\frac{\cdot}{k}\Big)\,\xi_a\,.$$

Note that, in general, the expression of the a-th spinor component ψ_a in normal coordinates, $\psi_a \circ \Psi^{-1}$, does correspond exactly to the euclidean component $\widehat{\psi}_a$. Still, one has

$$\psi_a \circ \Psi^{-1}\Big(\frac{\cdot}{k}\Big) = \widehat{\psi}_a\Big(\frac{\cdot}{k}\Big) + \sum_b A_{ab}\Big(\frac{\cdot}{k}\Big)\widehat{\psi}_b\Big(\frac{\cdot}{k}\Big) \tag{3.2.1}$$

with smooth functions A_{ab} verifying

$$\Big\|A_{ab}\Big(\frac{\cdot}{k}\Big)\Big\|_{C^m(B)} \leqslant \frac{C_m}{k},$$

with constants C_m not depending on k. Finally, we will denote the standard Dirac operator on $\mathbb{R}^n$ by D_0.

We are now ready to state the following approximation theorem, which is the main ingredient in the proof. It makes precise the statement in the introduction to the effect that the increasing degeneracy of the spectrum of the Dirac operator introduces the exact amount of flexibility we need in our problem. For the sake of concreteness, we will concentrate henceforth on the case of positive eigenvalues, the negative case being completely analogous:

Theorem 3.2.1. *Let $\phi := (\phi_1, ..., \phi_{r(n)})$ be a $\mathbb{C}^{r(n)}$ valued function in $\mathbb{R}^n$, satisfying the Dirac equation $D_0\phi = \phi$. Fix an integer $m \geqslant 1$ and a positive constant δ. For any large enough positive integer k, there is an eigenfunction ψ of the Dirac operator D on $\mathbb{S}^n$ of eigenvalue $(\frac{n}{2} + k)$ such that*

$$\Big\|\phi - \widehat{\Psi}_*\psi\Big(\frac{\cdot}{k}\Big)\Big\|_{C^m(B)} < \delta\,.$$

Note that it is the converse statement which is trivially true, not only in spheres but in any spin manifold: an eigenfunction of the Dirac operator of high enough eigenvalue λ always behaves, at scales of order λ^{-1}, as an eigenfunction of the euclidean Dirac operator of eigenvalue 1; however, this is not very revealing, since in principle we have no information whatsoever as to which particular eigenfunction we are converging to, and its properties.

By contrast, the above theorem ensures that any property that can be exhibited by solutions to the euclidean Dirac equation $D_0\phi = \phi$ on Euclidean space is also exhibited, at small scales, by eigenfunctions of D of high enough eigenvalue, provided such property is robust under C^m perturbations.

Now, let Φ' be a diffeomorphism of $\mathbb{S}^n$ mapping the collection $\{\Sigma_a\}$ into the ball $\mathbb{B}_{1/2}$, and the ball $\mathbb{B}_{1/2}$ into itself. Consider the collection of codimension two submanifolds $\{\Sigma'_a\}$ in $B_{1/2}$, defined as

$$\Sigma'_a := (\Psi \circ \Phi')(\Sigma_a)\,.$$

The euclidean realization theorem (Theorem 3.1.2) yields a $\mathbb{C}^{r(n)}$-valued function $\phi = (\phi_1, ..., \phi_r(n))$ and a diffeomorphism Φ_0, very close to simply being the identity, such that $\Phi_0(\Sigma'_a) \subset \phi_a^{-1}(0)$.

Besides, Theorem 3.2.1 allows us to find, for any large enough integer k, a spinor ψ verifying $D\psi = (\frac{n}{2}+k)\psi$ and approximating ϕ in the $C^m(B)$ norm as much as we want. In particular, we can choose k so that each component ψ_a of the spinor approximates ϕ_a as much as we want. In view of equation (3.2.1), we have

$$\left\|\phi_a - \psi_a \circ \Psi^{-1}\left(\frac{\cdot}{k}\right)\right\|_{C^m(B)} \leqslant \left\|\phi_a - \widehat{\psi}_a\left(\frac{\cdot}{k}\right)\right\|_{C^m(B)} + \frac{C}{k}\left\|\phi - \widehat{\Psi}_*\psi\left(\frac{\cdot}{k}\right)\right\|_{C^m(B)} + \frac{C}{k}||\phi||_{C^m(B)},$$

so, choosing k big enough, we get

$$\left\|\phi_a - \psi_a \circ \Psi^{-1}\left(\frac{\cdot}{k}\right)\right\|_{C^m(B)} < \delta$$

for $\delta > 0$ as small as we want.

The structural stability property ensures then the existence of a diffeomorphism $\Phi_1 : \mathbb{R}^n \to \mathbb{R}^n$ very close to the identity, such that $\Phi_1(\Phi_0(\Sigma'_a)) \subset \psi_a \circ \Psi^{-1}(\frac{\cdot}{k})$. Therefore, each ψ_a has the corresponding submanifold $\Psi^{-1}(\Phi_1(\Phi_0(\Sigma'_a))$ as a nodal set, and we conclude that ψ realizes $\mathfrak{S}$. Every other eigenfunction close enough to ψ will also realize $\mathfrak{S}$, through a slightly perturbed diffeomorphism.

3.3 Inverse localization: proof of Theorem 3.2.1

On any spin manifold, the Weitzenbőck identity relates two natural second order elliptic operators arising from the spinor connection: the square of the Dirac operator D^2, and the covariant laplacian $\Delta_S := \nabla^{S*}\nabla^S$ (here ∇^{S*} is the L^2 adjoint of the covariant derivative on the spinor bundle). It reads

$$D^2 = \Delta_S + \frac{s}{4}\,, \tag{3.3.1}$$

where s is the scalar curvature.

In the n-sphere, the Weitzenbőck formula provides a link between Dirac spinors and spherical harmonics. To see this, first consider the twisted connection

$$\nabla^{\chi} := \nabla^{S} + \frac{1}{2}\rho(\cdot)$$

whose covariantly constant sections are precisely the Killing spinors of Killing number $-\frac{1}{2}$. The corresponding laplacian $\Delta_\chi := \nabla^{\chi *}\nabla^{\chi}$ has a transparent interpretation when written in an orthonormal frame $\{\chi_a\}_{a=1}^{r(n)}$ of $-\frac{1}{2}$-Killing spinors: a straightforward computation yields

$$\Delta_\chi \psi = \sum_{a=1}^{r(n)} (\Delta_{\mathbb{S}^n} \psi_a)\chi_a \,, \tag{3.3.2}$$

where $\Delta_{\mathbb{S}^n}$ is the Laplace-Beltrami operator acting on functions.

On the other hand, consider the twisted Dirac operator

$$\not{D} := D - \frac{1}{2}.$$

From the Weitzenbőck formula we obtain that the couple $\not{D}^2$ and Δ_χ satisfies the identity

$$\not{D}^2 = \Delta_\chi + \frac{(n-1)^2}{4}\,, \tag{3.3.3}$$

where we have used that the scalar curvature of the round n-sphere is $n(n-1)$. Now, let ψ be an eigenfuntion of D of eigenvalue $\frac{n}{2} + k$. It verifies

$$\Big(D - \frac{1}{2}\Big)^2 \psi = \Big(\frac{n-1}{2} + k\Big)^2 \psi = k(n+k-1)\psi + \frac{(n-1)^2}{4}\psi, \tag{3.3.4}$$

hence Equation (3.3.3) allows us to conclude that

$$\Delta_\chi \psi = k(n+k-1)\psi$$

and so, in view of equation (3.3.2), the components ψ_a of ψ are complex spherical harmonics of eigenvalue $k(n+k-1)$.

The following proposition captures the small scale behavior of spherical harmonics:

Proposition 3.3.1. *Let ϕ be a complex-valued function in $\mathbb{R}^n$, satisfying $\Delta\phi + \phi = 0$. Fix a positive integer m and a positive constant δ'. For any large enough integer k, there is a complex-valued spherical harmonic Y on $\mathbb{S}^n$ with energy $k(n+k-1)$ such that*

$$\Big\| \phi - Y \circ \Psi^{-1}\Big(\frac{\cdot}{k}\Big) \Big\|_{C^{m+2}(B)} \leqslant \delta' \,.$$

Here and in what follows, by Δ we denote the euclidean negative Laplacian, $\Delta = \sum \partial_\mu^2$. For the proof of Proposition 3.3.1 we refer the reader to Chapter 4, where the same result is proven when $\phi := (\phi_1, ..., \phi_q)$ is a $\mathbb{R}^q$-valued functions in $\mathbb{R}^n$, and hence with $Y := (Y_1, ..., Y_q)$ a $\mathbb{R}^q$-valued spherical harmonic.

Now, let $\phi := (\phi_1, ..., \phi_r(n))$ be a $\mathbb{C}^{r(n)}$-valued function on $\mathbb{R}^n$ satisfying $D_0\phi = \phi$. Since $D_0^2 = -\Delta$, the complex functions ϕ_a satisfy the Helmholtz equation $\Delta\phi_a + \phi_a = 0$. From Proposition 3.3.1, we obtain a collection of spherical harmonics Y_a, with $a = 1, ..., r(n)$, locally approximating the corresponding functions ϕ_a. The spinor $\widetilde{\psi}$ defined as

$$\widetilde{\psi}_a := Y_a$$

is an eigenfunction of the operator $\not{D}^2$ of eigenvalue $k(n + k - 1) + \frac{(n-1)^2}{4}$ (by equation (3.3.3)). It satisfies

$$\left\|\phi - \widehat{\Upsilon}_* \widetilde{\psi}\Big(\frac{\cdot}{k}\Big)\right\|_{C^{m+2}(B)} \leqslant \delta' + Ck^{-1}||\phi||_{C^{m+2}(B)},$$

where the last term on the right hand side of the inequality comes from the mismatch between euclidean and spherical trivializations (as in equation (3.2.1)).

Since ϕ is fixed, by choosing δ' in Proposition 3.3.1 small enough and k large enough, we get that $\widetilde{\psi}$ satisfies the bound

$$\left\|\phi - \widehat{\Upsilon}_* \widetilde{\psi}\Big(\frac{\cdot}{k}\Big)\right\|_{C^{m+2}(B)} < \delta.$$

for a given δ as small as we want.

Given such a $\widetilde{\psi}$, the following lemma provides us with an eigenfunction of D that still approximates the euclidean spinor ϕ:

Lemma 3.3.2. *Let ϕ and $\widetilde{\psi}$ be as above. The spinor ψ defined as*

$$\psi := \frac{\not{D}(\not{D}\widetilde{\psi} + (\frac{n-1}{2} + k)\widetilde{\psi})}{2(\frac{n-1}{2} + k)^2}$$

is an eigenfunction of the Dirac operator D of eigenvalue $\frac{n}{2} + k$, satisfying

$$\left\|\phi - \widehat{\Upsilon}_* \psi\Big(\frac{\cdot}{k}\Big)\right\|_{C^{m}(B)} < C\delta + Ck^{-1}||\phi||_{C^{m+2}(B)}.$$

with constants C not depending on k.

Proof. In view of equation (3.3.4) , one can easily check that ψ satisfies the identity

$$\not{D}\psi = \Big(\frac{n-1}{2} + k\Big)\psi,$$

and since $\not{D} = D - \frac{1}{2}$, the spinor ψ is an eigenfunction of D with eigenvalue $(\frac{n}{2} + k)$. With regards to the C^m bound, first let us note that we have

$$\widehat{\Upsilon}_*(\not{D}\widetilde{\psi}) = k\Big(D_0\widehat{\Upsilon}_*\widetilde{\psi} + \frac{G_1}{k}\partial\widehat{\Upsilon}_*\widetilde{\psi} + \frac{G_2}{k}\widehat{\Upsilon}_*\widetilde{\psi}\Big)$$

$$\widehat{\Psi}_*(\not{D}^2\widetilde{\psi}) = k\Big(D_0^2\widehat{\Psi}_*\widetilde{\psi} + \frac{G_3}{k}\partial^2\widehat{\Psi}_*\widetilde{\psi} + \frac{G_4}{k}\partial\widehat{\Psi}_*\widetilde{\psi} + \frac{G_5}{k^2}\widehat{\Psi}_*\widetilde{\psi}\Big)$$

where the G_i are smooth matrix-valued functions satisfying the uniform bounds

$$||G_i||_{C^m(B)} \leqslant C_m$$

and $\partial\widehat{\Psi}_*\widetilde{\psi}$ (resp. $\partial^2\widehat{\Psi}_*\widetilde{\psi}$) is a matrix whose entries are first (resp. second) order derivatives of the components of $\widehat{\Psi}_*\widetilde{\psi}$.

Further, since $2\phi = D_0(D_0+1)\phi$, we have

$$\begin{aligned}\Big\|\phi - \widehat{\Psi}_*\psi\Big(\frac{\cdot}{k}\Big)\Big\|_{C^m(B)} &\leqslant \Big\|\frac{1}{2}(D_0^2 + D_0)(\phi - \widehat{\Psi}_*\widetilde{\psi}\Big(\frac{\cdot}{k}\Big))\Big\|_{C^m(B)} + \frac{C}{k}\Big\|\widehat{\Psi}_*\widetilde{\psi}\Big(\frac{\cdot}{k}\Big)\Big\|_{C^{m+2}(B)} \\ &\leqslant C\Big\|\phi - \widehat{\Psi}_*\widetilde{\psi}\Big(\frac{\cdot}{k}\Big)\Big\|_{C^{m+2}(B)} + \frac{C}{k}\Big\|\phi - \widehat{\Psi}_*\widetilde{\psi}\Big(\frac{\cdot}{k}\Big)\Big\|_{C^{m+2}(B)} \\ &\quad + \frac{C}{k}\|\phi\|_{C^{m+2}(B)} \leqslant C\delta + \frac{C}{k}. \qquad (3.3.1)\end{aligned}$$

and the lemma follows. □

Finally, Theorem 3.2.1 follows upon choosing δ as small and correspondingly k as large as needed.

3.4 Flexibility in euclidean space: proof of Theorem 3.1.2

Let us recall the setting: inside the unit n-dimensional ball $B \subset \mathbb{R}^n$, we are given a collection $\mathfrak{S} := \{\Sigma_a\}_{a=1}^{r(n)}$ of $r(n)$ closed, pairwise disjoint, codimension 2 smooth submanifolds. Our aim in this section is to find functions $\phi_a : \mathbb{R}^n \to \mathbb{C}$, with $a = 1, ..., r(n)$, and a diffeomorphism $\Phi : B \to B$, as close to the identity in the C^m norm as we want, such that, on the one hand, each $\Phi(\Sigma_a)$ is a structurally stable nodal set of the corresponding ϕ_a, and on the other hand, the spinor $\phi := (\phi_1, ..., \phi_{r(n)})$ satisfies the standard Dirac equation in $\mathbb{R}^n$

$$D_0\phi = \sum_{\mu=1}^{n} \rho(e_\mu)\partial_\mu\phi = \lambda\phi$$

where $\{e_\mu\}_{\mu=1}^n$ is an orthonormal basis on $\mathbb{R}^n$ and $\rho(e_\mu)$ denotes the standard Clifford multiplication. We will henceforth fix $\lambda = 1$, the general case being completely analogous.

To find and ϕ and Φ, we will adapt the strategy introduced in [27] for analogous realization problems in the case of second order elliptic PDEs. To begin with, one finds local solutions to the PDE under consideration that vanish at the Σ_a: this is achieved by means of some local existence theorem (we will use Cauchy-Kovalevskaya theorem). Then, one promotes these local solutions to global ones by means of a Runge-type approximation theorem (we will use Lax-Malgrange theorem). Through the process, one must ensure that the zero sets are structuraly stable, and this is done by prescribing the normal derivatives of the local solutions at the nodal sets and applying Thom's isotopy lemma.

Still, the application of the above scheme to our problem requires additional considerations, the reason being, first of all, that the Dirac operator is of order 1, and thus we lose the capacity to prescribe the normal derivatives on the Cauchy problem; and further, that the Dirac operator mixes all components ϕ_a of ϕ, so that we cannot solve the problem for each component separately.

Let us begin with our construction. First of all, we note that, to use the Cauchy-Kovalevskaya theorem to find local solutions, our submanifolds Σ_a should be real analytic. This can be achieved by an arbitrarily small perturbation of the original submanifolds, because of the density of analytic functions in the space of smooth functions. Indeed, since the normal bundle of each Σ_a is of rank 2, it is always trivial (by a theorem of W. S. Massey [56]), and therefore we can always find a smooth complex-valued function F_a vanishing at Σ_a and whose differential has full rank at Σ_a. The full rank condition makes the zero set Σ_a of F_a structurally stable, so a real analytic function approximating F_a well enough is guaranteed to have as zero set a very small perturbation of Σ_a (in the sense of being diffeomorphic by an ambient diffeomorphism arbitrarily close to the identity).

We will keep denoting by Σ_a the analytic submanifolds obtained after small perturbation, and by F_a the collection of complex-valued, real analytic functions realizing such submanifolds as zero sets.

For each $a = 1, ..., r(n)$, let M_a be an analytic hypersurface containing Σ_a, and such that, for $a \neq b$, M_a and M_b are disjoint. For example, we can choose M_a to be

$$M_a := \{x \in \mathbb{R}^n, \text{dist}(x, \Sigma_a) \leqslant \epsilon_a, \text{Im}(F_a)(x) = 0\}$$

for ϵ_a small enough.

We will denote by n_a the normal vector field to $M_a \subset \mathbb{R}^n$, and by ν_a the normal vector field to $\Sigma_a \subset M_a$, when considering Σ_a as a hypersurface of M_a.

The next lemma will play an important role in what is to follow, providing the initial condition for the Cauchy problem that is compatible with structural stability:

Lemma 3.4.1. *There is a collection of real-analytic functions $g_b : \Sigma_a \to \mathbb{C}$, with $b \in \{1, ..., r(n)\} \setminus \{a\}$, such that the spinor $\varphi := (g_1, g_2, ..., \nu_a \cdot \nabla F_a, ..., g_{r(n)})$ verifies the condition*

$$Im\ [\rho(n_a)\rho(\nu_a)\varphi(x)]_a > 0$$

for all $x \in \Sigma_a$. Here, by $[\cdot]_a$ we denote the a-th component of a spinor.

Proof. Let $\Gamma(x) = \{\Gamma_{cd}(x)\}$, with c, $d = 1, ..., r(n)$, denote the matrix $\rho(n_a)\rho(\nu_a)(x)$. Note that at each point $x \in \Sigma_a$, $\Gamma(x)$ is an antihermitian matrix, so $\Gamma_{aa}(x)$ is purely imaginary or zero. The above condition can be written as

$$-i\Gamma_{aa}(x)\nu_a \cdot \nabla \mathrm{Re}(F_a) + \mathrm{Im} \sum_{b=1}^{r(n)} \Gamma_{ab}(x) g_b(x) > 0. \tag{3.4.1}$$

The matrix $\Gamma(x)$ always has non-zero determinant, so the functions $\Gamma_{ab}(x)$, for $b = 1, ..., r(n)$, cannot all vanish at once. Besides, note that the term $\nu_a \cdot \nabla \mathrm{Re} F_a(x)$ cannot vanish either, because $\mathrm{rank}(dF_a|_{\Sigma_a})$ is 2 while $\nabla \mathrm{Im} F_a(x)|_{M_a}$ vanishes. Thus, at any point $x_0 \in \Sigma_a$, it is easy to see that one can find a collection of complex numbers w_{b0} such that condition 3.4.1 holds. Further, since all the coefficients in condition 3.4.1 are analytic functions, and the condition is open, on a sufficiently small neighborhood of the point x_0 we can find complex valued real analytic functions $g_{b0}(x)$, with $g_{b0}(x_0) = w_{b0}$, and satisfying 3.4.1, .

Thus, the above discussion grants the existence of an open cover $\{U_\alpha\}$ of Σ_a and an associated collection of complex analytic functions $g_{b\alpha} : U_\alpha \to \mathbb{C}$ satisfying condition 3.4.1 on each U_α. Therefore, if $\{\psi_\alpha, U_\alpha\}$ is a partition of unity, it is easy to check that the smooth functions

$$g_b(x) := \sum_\alpha \psi_\alpha(x) g_{b\alpha}(x)$$

satisfy condition 3.4.1 for all $x \in \Sigma_a$. These functions fail to be analytic, because they are defined through a partition of unity; nevertheless, since the condition is open and Σ_a is compact, we can find analytic functions close enough to the g_b so that 3.4.1 is still satisfied. □

Now, let $N_a \subset M_a$ be a tubular neighborhood of Σ_a (to define the tubular neighborhood, Σ_a is being considered as hypersurface of M_a), and let X be the projection of the vector field $\nabla \mathrm{Re}(F_a)$ to the tangent space of M_a. Provided N_a is small enough, the vector field X is never zero on N_a. Thus, any point $y \in N_a$ can be written uniquely as $y = \Phi_X^t(x)$, for some $x \in \Sigma_a$ and $t \in \mathbb{R}$. The functions f_b on N_a defined as

$$f_b(y) = f_b(\Phi_X^t(x)) = t g_b(x),$$

are real-analytic, vanish on Σ_a, and verify that $\nu_a \cdot \nabla f_b = g_b$. With these functions as input, we pose the following Cauchy problem for the Dirac operator on a neighborhood of the hypersurfaces N_a:

$$\begin{cases} D_0\phi = \phi \\ \phi_b|_{N_a} = f_b \text{ for } b \in \{1, ..., r(n)\} \setminus \{a\} \\ \phi_a|_{N_a} = F_a \end{cases} \tag{3.4.2}$$

The Cauchy-Kovalevskaya theorem yields a solution ϕ to (3.4.2) on a small enough tubular neighborhood U_a of N_a (it is understood that we have one such solution ϕ^a for each submanifold Σ_a, but we will not make this explicit for ease of notation).

Let us analyze the properties of this local solution. First of all, Σ_a is a nodal set of the component ϕ_a of ϕ, because we have imposed that $\phi_a|_{\Sigma_a} = F_a|_{\Sigma_a} = 0$.

Further, by virtue of the properties of the functions f_b, we are going to see that this nodal set is structurally stable (note that, in fact, Σ_a is also a nodal set of the rest of the components ϕ_b, but it need not be structurally stable for them).

To get structural stability, we need to ensure that the real and imaginary parts of $\nabla\phi_a$ are linearly independent on Σ_a. Firstly, note that we can write

$$\nabla\phi_a|_{\Sigma_a} = (n_a \cdot \nabla\phi_a)n_a + \nabla_{N_a}\phi_a|_{\Sigma_a} = (n_a \cdot \nabla\phi_a)n_a + X|_{\Sigma_a}$$

because the projection of $\nabla\mathrm{Im}(F_a)|_{\Sigma_a}$ into the tangent space of N_a is zero. In addition, X is real and does not vanish. Hence, for $\nabla\mathrm{Re}(\phi_a)$ and $\nabla\mathrm{Im}(\phi_a)$ to be linearly independent, it is necessary and sufficient that the term Im $(n_a \cdot \nabla\phi_a)$ does not vanish on Σ_a.

Next, note that the Dirac equation on Σ_a can be written as

$$n_a \cdot \nabla\phi|_{\Sigma_a} = \rho(n_a)\rho(\nabla_M\phi)|_{\Sigma_a},$$

because $\phi|_{\Sigma_a} = 0$ and $\rho(n_a)\rho(n_a) = -\mathrm{Id}$. It is at this point that the properties of the functions f_b are important: since $f_b|_{\Sigma_a} = 0$ and $\nu_a \cdot \nabla f_b = g_b$, we have that

$$\rho(n_a)\rho(\nabla_M\phi)|_{\Sigma_a} = \rho(n_a)\rho(\nu_a)\varphi$$

with $\varphi = (g_1, ..., \nabla\mathrm{Re}(F_a), ..., g_{r(n)})$. Hence, by Lemma 3.4.1, the term

$$\mathrm{Im}\ [\rho(n_a)\rho(\nu_a)\varphi]_a = \mathrm{Im}\ n_a \cdot \nabla\phi|_{\Sigma_a}$$

is greater than zero, and structural stability follows.

To sum up, what we get after carrying out the same construction for each submanifold Σ_a is a collection of local solutions $\{\phi^a = (\phi^a_1, ..., \phi^a_{r(n)})\}_{a=1}^{r(n)}$ to the Dirac equation, each one being defined on a small enough open neighborhood U_a of the corresponding submanifold Σ_a (small enough so that the open sets can be assumed to be pairwise disjoint) and verifying that $\Sigma_a = (\phi^a_a)^{-1}(0)$, with the structural stability property. Since the complement of the union of all the open sets U_a has no relatively compact components, the Lax-Malgrange approximation theorem (see e.g [62]) ensures the existence, for any given constant $\delta > 0$ and integer $m \geqslant 1$, of a global solution ϕ' to the Dirac equation on $\mathbb{R}^n$ satisfying

$$\|\phi^a - \phi'\|_{C^m(U_a)} \leqslant \delta.$$

In particular, the components verify $\|\phi^a_a - \phi'_a\|_{C^m(U_a)} \leqslant \delta$. Since $\mathrm{rank}(d\phi'_a)|_{\Sigma_a}$ is 2, by Thom's isotopy lemma we can choose an appropriate δ so that, for a given $\epsilon > 0$ as small as desired, there is a diffeomorphism $\Phi : B \to B$ verifying $\|\Phi - \mathrm{Id}\|_{C^m(U_a)} \leqslant \epsilon$ and such that $\Phi(\Sigma_a) \subset \phi'^{-1}_a(0)$.

3.5 Prescribing nodal sets at different regions at once

To set the stage, consider a collection of structures $\{\mathfrak{S}^\alpha\}_{\alpha=1}^{\Lambda}$, where Λ is a positive integer that we can choose as large as we want, and where each $\mathfrak{S}^\alpha$ is itself a collection of $r(n)$ codimension 2 submanifolds, $\mathfrak{S}^\alpha = \{\Sigma_1^\alpha, ..., \Sigma_{r(n)}^\alpha\}$.

Following the notation in Section 3.2, for a set of points $\{p_\alpha\}_{\alpha=1}^{\Lambda}$ in $\mathbb{S}^n$ we denote by $\Psi_\alpha : \mathbb{B}_\rho(p_\alpha) \to B_\rho$ the corresponding geodesic patches on balls of radius ρ centered on the points p_α. We fix a radius ρ such that no two balls intersect, for example by setting

$$\rho := \frac{1}{2} \min_{\alpha \neq \beta} \operatorname{dist}_{\mathbb{S}^n}(p_\alpha, p_\beta) .$$

Our goal is to find a Dirac spinor realizing each structure $\mathfrak{S}^\alpha$ within the ball $\mathbb{B}_\rho(p_\alpha)$.

Remark 3.5.1. *It is necessary to choose the set $\{p_\alpha\}_{\alpha=1}^{\Lambda}$ so that no pair of points are antipodal in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, i.e, so that $p_\alpha \neq -p_\beta$ for all α, β. The reason is that if a structure $\mathfrak{S}$ is realized by the Dirac spinor ψ of eigenvalue $\frac{n}{2} + k$ in the ball $\mathbb{B}_{k^{-1}}(p)$, $\mathfrak{S}$ is also automatically realized by ψ in the antipodal ball $\mathbb{B}_{k^{-1}}((-1)^k p)$. Indeed, the components ψ_a have parity $(-1)^k$, that is*

$$\psi_a(-p) = (-1)^k \psi_a(p),$$

because, being complex spherical harmonics of energy $k(n + k - 1)$, they are the restriction to the sphere of a couple of real harmonic homogenous polynomials of degree k.

The realization of many different structures at once is a direct consequence of the following proposition:

Proposition 3.5.2. *Let $\{\phi_\alpha\}_{\alpha=1}^{\Lambda}$ be a set of Λ complex-valued functions in $\mathbb{R}^n$, satisfying $\Delta\phi_\alpha + \phi_\alpha = 0$. Fix a positive integer m and positive constant δ. For any large enough integer k, there is a spherical harmonic Y on $\mathbb{S}^n$ with energy $k(n + k - 1)$ verifying the bound*

$$\left\| \phi_\alpha - Y \circ \Psi_\alpha^{-1}\left(\frac{\cdot}{k}\right) \right\|_{C^{m+2}(B_\rho)} < \delta$$

for all $1 \leqslant \alpha \leqslant \Lambda$.

The proof is given in Section 4.2 of Chapter 4.

Given this proposition, by arguing exactly as in Sections 3.2 and 3.3, we find a spinor ψ and a diffeomorphism Φ such that $\Phi(\Sigma_a^\alpha) \subset \psi_a^{-1}(0) \cap \mathbb{B}(p_\alpha, \rho)$, for $\alpha = 1, ..., \Lambda$.

3.6 Concluding remarks: the case of the torus

We conclude by sketching how to prove an analog of Theorem 3.1.1 on the torus $\mathbb{T}^n$.

The spin structures on the n-dimensional torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ are indexed by elements in the first cohomology class $H^1(\mathbb{T}^n, \mathbb{Z}_2) \cong \mathbb{Z}_2^n$. The Dirac operator corresponding to the zero cohomology class can be regarded as the one inherited from the standard Dirac operator on $\mathbb{R}^n$.

For this Dirac operator, an inverse localization theorem (as Theorem 3.2.1) can be proven following a similar strategy as the one in Section 3.3 and Section 2.5 of Chapter 2. To begin with, Weitzenbőck formulas (and hence the relation between spinors and eigenfunctions of the Laplace-Beltrami operator) are trivial. Next, for the proof of the torus analog of Proposition 3.3.1, it suffices to argue as in Section 2.5, that is: the role of the shifted Bessel functions centered at points x_j is now played by trigonometric polynomials of frequencies ξ_j, with $|\xi_j| = 1$, and instead of spherical harmonics expressed as sums of ultraspherical polynomials, we have trigonometric polynomials of frequencies $k\xi_j$ (with k an eigenvalue of the Dirac operator in the torus); with the further requirement that k and ξ_j verify that $k\xi_j \in \mathbb{Z}^n$ (so that the trigonometric polynomials define eigenfunctions of the laplacian on the torus). These results in hand, the reasoning in Section 3.2 can be directly transplanted to the torus case.

We encounter here thus the same number-theoretical subtlety that we remarked in Section 2.6 of Chapter 2. We will nevertheless recall it here: in the torus analog of Proposition 4.1.2, one would like to approximate the complex-valued function ϕ_1 (in the notation of Chapter 4)

$$\phi_1(x) = \int_{\mathbb{S}^{n-1}} f_1(\xi)\, e^{ix\cdot\xi}\, d\sigma(\xi)\,,$$

by a sum of the form

$$\varphi(x) = \sum_{j=1}^{N} c_j e^{i\xi_j\cdot x}$$

with $|\xi_j| = 1$ and such that $k\xi_j \in \mathbb{Z}^n$ for some high enough k. One can always approximate the above integral as much as one wishes by a discretized sum centered at some points $\xi_j \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$; the problem is that we have a very specific requirement on the points ξ_j. The approximation would hold whenever, for an increasing sequence of eigenvalues k, there are corresponding sets of points $\{\xi_j\}_{j=1}^{N}$, depending on k and with $k\xi_j \in \mathbb{Z}^n$, that become dense in $\mathbb{S}^{n-1}$ as k increases.

However, not all sequences of eigenvalues might verify this property. As a matter of fact, if we restrict our attention to sequences of integer eigenvalues, and we substitute the density property by the (stronger) property of equidistribution of the corresponding sets of rational points, the problem is related to the celebrated Linnik problem in number theory (when k is an integer, we say that the frequencies ξ_j with $\xi_j k \in \mathbb{Z}^n$ are rational points on $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ of *height* k). For instance, in dimension 3, only certain sequences of integers are known to verify the equidistribution property [19, 20], for example, sequences of odd integers, or sequences of integers whose squares are square-free. To take this phenomenon into account, one has to modify accordingly the statement of Theorem 3.1.1, so that it reflects that it is the eigenfunctions with, say, sufficiently high *odd* eigenvalue (in the case of $\mathbb{T}^3$) that realize the given structure $\mathfrak{S}$.

Chapter 4

Inverse localization for spherical harmonics

Given any function $\phi : \mathbb{R}^n \to \mathbb{R}^q$ satisfying $\Delta\phi + \phi = 0$ (where the Laplacian is understood as acting component-wise), we show that there is a $\mathbb{R}^q$-valued spherical harmonic $Y := (Y_1, ..., Y_q)$ in $\mathbb{S}^n$ (each Y_j being an ordinary real-valued spherical harmonic), satisfying $\Delta_{\mathbb{S}^n} Y = k(n+k-1)Y$, whose behavior in a small ball of radius k^{-1} is very close to that of ϕ in the euclidean unit ball B. This provides a converse to the general fact that high energy eigenfunctions of an elliptic operator on a compact manifold behave, at small scales, like eigenfunctions of eigenvalue 1 of an euclidean elliptic operator with constant coefficients.

This result provides some of the key Propositions needed both in Chapters 2 and 3. In the case of Chapter 2, which deals with vector fields in $\mathbb{S}^3$, Propositions 2.3.1 and 2.3.2 in Section 2.3 follow from this Chapter by setting $n = 3$ and $q = 3$. As for Chapter 3, Proposition 3.3.1 correspond to the case $q = 2$ and arbitrary $n \geqslant 3$.

4.1 Inverse localization in a small ball

Theorem 4.1.1. *Let ϕ be a $\mathbb{R}^q$-valued function in $\mathbb{R}^n$, satisfying $\Delta\phi + \phi = 0$. Fix a positive integer m and a positive constant δ'. For any large enough integer k, there is a $\mathbb{R}^q$-valued spherical harmonic $Y := (Y_1, ..., Y_q)$ on $\mathbb{S}^n$ with energy $k(n+k-1)$ such that*

$$\left\| \phi - Y \circ \Psi^{-1}\left(\frac{\cdot}{k}\right) \right\|_{C^{m+2}(B)} \leqslant \delta' .$$

We will proceed in two successive approximation steps. First, we will approximate the function $\phi(x)$ in B by a function $\varphi(x)$ that consists in a finite sum of terms of the form

$$\frac{c_j}{|x-x_j|^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|x-x_j|)$$

with $c_j \in \mathbb{R}^q$ and $x_j \in \mathbb{R}^n$, $j = 1, ..., N$, for N big enough (Proposition 4.1.2). In the second step, we show that there is a collection of q spherical harmonics $Y := (Y_1, ..., Y_q)$ in $\mathbb{S}^n$ of energy $k(n+k-1)$ which, when considered in a ball of radius k^{-1} with coordinates rescaled to the euclidean ball of radius 1, approximate $\varphi := (\varphi_1, ..., \varphi_q)$, provided that k is large enough. The proof does not change if one sets $q = 1$. We recall that the dimension of the space of spherical harmonics on the n-sphere of eigenvalue $k(n+k-1)$ is given by

$$d(k,n) := \binom{n+k-1}{k} \frac{n+2k-1}{n+k-1}.$$

Proposition 4.1.2. *Given any $\delta > 0$, there is a finite radius R and finitely many $\mathbb{R}^q$-valued constants $\{c_j\}_{j=1}^N$ and points $\{x_j\}_{j=1}^N \subset B_R$ such that the function*

$$\varphi := \sum_{j=1}^{N} c_j \frac{1}{|x-x_j|^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|x-x_j|)$$

approximates the function ϕ in the ball B as

$$\|\phi - \varphi\|_{C^{m+2}(B)} < \delta\,.$$

Proof. It will be more convenient to work with complex-valued functions: we set $\widetilde{\phi} := \phi + i\phi$. First, we notice that, since $\widetilde{\phi}$ is also a solution of the Helmholtz equation, it can be written in the ball B_2 as an expansion

$$\widetilde{\phi} = \sum_{l=0}^{\infty} \sum_{j=1}^{d(n-1,l)} b_{lj}\, j_l(r)\, Y_{lj}(\omega), \tag{4.1.1}$$

where $r := |x| \in \mathbb{R}^+$ and $\omega := x/r \in \mathbb{S}^{n-1}$ are spherical coordinates in $\mathbb{R}^n$, Y_{lj} are a basis of spherical harmonics of eigenvalue $l(l+n-2)$, j_l are n-dimensional hyperspherical Bessel functions and $b_{lj} \in \mathbb{C}^q$ are constant coefficients.

The series in (4.1.1) is convergent in the L^2 sense, so for any $\delta' > 0$, we can truncate the sum at some integer L

$$\phi_1 := \sum_{l=0}^{L} \sum_{j=1}^{d(n-1,l)} b_{lj}\, j_l(r)\, Y_{lj}(\omega) \tag{4.1.2}$$

so that it approximates $\widetilde{\phi}$ as

$$\|\phi_1 - \widetilde{\phi}\|_{L^2(B_2)} < \delta'\,. \tag{4.1.3}$$

Every component of the function ϕ_1 decays as $|\phi_{1i}(x)| \leqslant C/|x|^{\frac{n-1}{2}}$ for large enough $|x|$ (because of the decay properties of the spherical Bessel functions).

Hence, Herglotz's theorem (see e.g. [37, Theorem 7.1.27]) ensures that we can write

$$\phi_1(x) = \int_{\mathbb{S}^{n-1}} f_1(\xi)\, e^{ix\cdot\xi}\, d\sigma(\xi)\,, \tag{4.1.4}$$

where $d\sigma$ is the area measure on $\mathbb{S}^{n-1} := \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ and f_1 is a $\mathbb{C}^q$-valued function in $L^2(\mathbb{S}^{n-1})$.

We now choose a smooth $\mathbb{C}^q$-valued function f_2 approximating f_1 as

$$\|f_1 - f_2\|_{L^2(\mathbb{S}^{n-1})} < \delta'\,,$$

which is always possible since smooth functions are dense in $L^2(\mathbb{S}^{n-1})$. The function defined as the inverse Fourier transform of f_2,

$$\phi_2(x) := \int_{\mathbb{S}^{n-1}} f_2(\xi)\, e^{ix\cdot\xi}\, d\sigma(\xi)\,, \tag{4.1.5}$$

approximates ϕ_1 uniformly: by the Cauchy–Schwarz inequality, we get

$$|\phi_2(x) - \phi_1(x)| = \left| \int_{\mathbb{S}^{n-1}} (f_2(\xi) - f_1(\xi))\, e^{ix\cdot\xi}\, d\sigma(\xi) \right| \leqslant C\|f_2 - f_1\|_{L^2(\mathbb{S}^{n-1})} < C\delta' \tag{4.1.6}$$

for any $x \in \mathbb{R}^n$.

Our next objective is to approximate the function f_2 by a trigonometric polynomial: for any given δ', we will find a radius $R > 0$ and finitely many points $\{x_j\}_{j=1}^N \subset B_R$ and constants $\{c_j\}_{n=1}^N \subset \mathbb{C}^q$ such that the smooth function in $\mathbb{R}^n$

$$f(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{j=1}^{N} c_j\, e^{-ix_j\xi},$$

when restricted to the unit sphere, approximates f_2 in the C^0 norm,

$$\|f - f_2\|_{C^0(\mathbb{S}^{n-1})} < \delta'\,. \tag{4.1.7}$$

In order to do so, we begin by extending f_2 to a smooth function $g : \mathbb{R}^n \to \mathbb{C}^q$ with compact support,

$$g(\xi) := \chi(|\xi|)\, f_2\left(\frac{\xi}{|\xi|}\right),$$

where $\chi(s)$ is a real-valued smooth bump function, being 1 when, for example, $|s-1| < \frac{1}{4}$, and vanishing for $|s-1| > \frac{1}{2}$. The Fourier transform $\widehat{g}$ of g is Schwartz, so it is easy to see that, outside some ball B_R, the L^1 norm of $\widehat{g}$ is very small,

$$\int_{\mathbb{R}^n\setminus B_R} |\widehat{g}(x)|\, dx < \delta'\,,$$

and therefore we get a very good approximation of g by just considering its Fourier representation with frequencies within the ball B_R, that is,

$$\sup_{\xi\in\mathbb{R}^n} \left| g(\xi) - \int_{B_R} \widehat{g}(x)\, e^{-ix\cdot\xi}\, dx \right| < \delta'\,. \tag{4.1.8}$$

Next, we can approximate the integral

$$\int_{B_R} \widehat{g}(x)\, e^{-ix\cdot\xi}\, dx$$

by the sum

$$f(\xi) := \frac{1}{(2\pi)^{\frac{n}{2}}} \sum_{j=1}^{N} c_j\, e^{-ix_j\cdot\xi} \tag{4.1.9}$$

with constants $c_j \in \mathbb{C}^q$ and points $x_j \in B_R$, so that we have the bound

$$\sup_{\xi\in\mathbb{S}^{n-1}} \left| \int_{B_R} \widehat{g}(x)\, e^{-ix\cdot\xi}\, dx - f(\xi) \right| < \delta' . \tag{4.1.10}$$

To see this, consider a covering of the ball B_R by closed sets $\{U_j\}_{j=1}^N$, with piecewise smooth boundaries, pairwise disjoint interiors, and diameters not exceeding δ''. Since the function $e^{-ix\cdot\xi}\, \widehat{g}(x)$ is smooth, we have that for each $x, y \in U_j$

$$\sup_{\xi\in\mathbb{S}^{n-1}} |\widehat{g}(x)\, e^{-ix\cdot\xi} - \widehat{g}(y)\, e^{-iy\cdot\xi}| < C\delta'' ,$$

with the constant C depending on $\widehat{g}$ (and therefore on δ') but not on δ''. If x_j is any point in U_j and we set $c_j := (2\pi)^{\frac{n}{2}} \widehat{g}(x_j)\, |U_j|$ in (4.1.9), we get

$$\begin{aligned}\sup_{\xi\in\mathbb{S}^{n-1}} \left| \int_{B_R} \widehat{g}(x)\, e^{-ix\cdot\xi}\, dx - f(\xi) \right| &\leqslant \sum_{j=1}^{N} \int_{U_j} \sup_{\xi\in\mathbb{S}^{n-1}} \left|\widehat{g}(x)\, e^{-ix\cdot\xi} - \widehat{g}(x_j)\, e^{-ix_j\cdot\xi}\right| dx \\ &\leqslant C\delta'' ,\end{aligned}$$

with C depending again on δ' and R but not on δ'' or N. By taking δ'' so that $C\delta'' < \delta'$, the estimate (4.1.10) follows.

Now, in view of (4.1.8) and (4.1.10), one has

$$\|f - g\|_{C^0(\mathbb{S}^{n-1})} < C\delta' ,$$

where C does not depend on δ'. The estimate (4.1.7) follows upon noticing that the function f_2 is the restriction to $\mathbb{S}^{n-1}$ of the function g.

To conclude, set

$$\begin{aligned}\widetilde{\varphi}(x) := \int_{\mathbb{S}^{n-1}} f(\xi)\, e^{ix\cdot\xi}\, d\sigma(\xi) = \sum_{j=1}^{N} \frac{1}{(2\pi)^{\frac{n}{2}}} c_j \int_{\mathbb{S}^{n-1}} e^{i(x-x_j)\cdot\xi}\, d\sigma(\xi) = \\ = \sum_{j=1}^{N} c_j\, \frac{1}{|x-x_j|^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|x-x_j|) ,\end{aligned}$$

from Equation (4.1.7) we infer that

$$\|\widetilde{\varphi} - \phi_2\|_{C^0(\mathbb{R}^n)} \leqslant \int_{\mathbb{S}^{n-1}} |f(\xi) - f_2(\xi)|\, d\sigma(\xi) < C\delta' ,$$

and from Equations (4.1.3) and (4.1.6) we get the L^2 estimate

$$\|\widetilde{\phi}-\widetilde{\varphi}\|_{L^2(B_2)} \leqslant C\|\varphi-\phi_2\|_{C^0(\mathbb{R}^n)} + C\|\phi_2-\phi_1\|_{C^0(\mathbb{R}^n)} + \\ + \|\phi_1-\widetilde{\phi}\|_{L^2(B_2)} < C\delta' . \quad (4.1.11)$$

Furthermore, both $\widetilde{\varphi}$ and $\widetilde{\phi}$ are $\mathbb{C}^q$-valued functions satisfying the Helmholtz equation in $\mathbb{R}^n$ (note that the Fourier transform of φ is supported on $\mathbb{S}^{n-1}$), so by standard elliptic estimates we have

$$\|\widetilde{\phi}-\widetilde{\varphi}\|_{C^{m+2}(B)} \leqslant C\|\widetilde{\phi}-\widetilde{\varphi}\|_{L^2(B_2)} < C\delta' .$$

This in particular implies that

$$\|\phi - \mathrm{Re}\,\widetilde{\varphi}\|_{C^{m+2}(B)} < C\delta' ,$$

and taking δ' small enough so that $C'\delta' < \delta$, resetting $c_j := \mathrm{Re}\, c_j$, and defining $\varphi := \mathrm{Re}\,\widetilde{\varphi}$, the proposition follows. □

The next step consists in showing that, for any large enough integer k, we can find a $\mathbb{R}^q$-valued spherical harmonic Y on $\mathbb{S}^n$ with eigenvalue $k(n+k-1)$ that approximates, in the ball $\mathbb{B}_{1/k}$ and when rescaled, the function φ in the unit ball. The proof is based on asymptotic expansions of ultraspherical polynomials, and uses the representation of φ as sum of shifted Bessel functions which we just obtained as a key ingredient.

Proposition 4.1.3. *Given a constant $\delta > 0$, for any large enough positive integer k there is a $\mathbb{R}^q$-valued spherical harmonic Y on $\mathbb{S}^n$ with energy $k(n+k-1)$ satisfying*

$$\left\| \varphi - Y \circ \Psi^{-1}\left(\frac{\cdot}{k}\right)\right\|_{C^{m+2}(B)} < \delta .$$

Theorem 4.1.1 follows from this proposition, provided k is large enough and δ is chosen small enough for 2δ not to be larger than δ'.

Proof. Consider the ultraspherical polynomial of dimension $n+1$ and degree k, $C_k^n(t)$, which is defined as

$$C_k^n(t) := \frac{\Gamma(k+1)\Gamma(\frac{n}{2})}{\Gamma(k+\frac{n}{2})} P_k^{(\frac{n}{2}-1,\frac{n}{2}-1)}(t) , \quad (4.1.1)$$

where $\Gamma(t)$ is the gamma function and $P_k^{(\alpha,\beta)}(t)$ are the Jacobi polynomials (see e.g [65, Chapter IV, Section 4.7]). We have included a normalizing factor so that $C_k^n(1) = 1$ for all k .

Let p, q be two points in $\mathbb{S}^n$, considered as the set $\{|p| = 1\}$ of $\mathbb{R}^{n+1}$. The addition theorem for ultraspherical polynomials ensures that $C_k^n(p \cdot q)$ (where $p \cdot q$ denotes the scalar product in $\mathbb{R}^{n+1}$ of the vectors p and q) can be written as

$$C_k^n(p\cdot q) = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \frac{1}{d(k,n)} \sum_{j=1}^{d(k,n)} Y_{kj}(p)\, Y_{kj}(q) , \quad (4.1.2)$$

with $\{Y_{kj}\}_{j=1}^{d(k,n)}$ being an arbitrary orthonormal basis of spherical harmonics with eigenvalue $k(k+n-1)$.

We recall that the function φ was expressed as the finite sum

$$\varphi(x) = \sum_{j=1}^{N} c_j \frac{1}{|x-x_j|^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|x-x_j|)\,,$$

with coefficients $c_j \in \mathbb{R}^q$ and points $x_j \in B_R$. With these c_j and x_j we define, for any $p \in \mathbb{S}^n$, the function

$$Y(p) := \sum_{j=1}^{N} c_j \frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} C_k^n(p \cdot p_j)\,,$$

where $p_j := \Psi^{-1}(\frac{x_j}{k})$. As long as $k > R$, p_j is well defined. In view of Equation (4.1.2) it is clear that Y is a spherical harmonic with eigenvalue $k(n+k-1)$.

Our aim is to study the asymptotic properties of the spherical harmonic Y. To begin with, note that if we consider points $p := \Psi^{-1}(\frac{x}{k})$ with $k > R$ and $x \in B_R$, we have

$$p \cdot p_j = \cos\left(\mathrm{dist}_{\mathbb{S}^n}(p, p_j)\right) = \cos\left(\frac{|x-x_j| + O(k^{-1})}{k}\right), \tag{4.1.3}$$

as $k \to \infty$. The last equality comes from $\Psi : \mathbb{B} \to B$ being a patch of normal geodesic coordinates (by $\mathrm{dist}_{\mathbb{S}^n}(p, p_j)$ we mean the distance between p and p_j considered on the sphere $\mathbb{S}^n$). From now on we set

$$\widetilde{Y}(x) := Y \circ \Psi^{-1}\left(\frac{x}{k}\right). \tag{4.1.4}$$

When k is large, one has

$$\frac{\Gamma(k+1)}{\Gamma(k+\frac{n}{2})} = k^{1-\frac{n}{2}} + O(k^{-\frac{n}{2}})\,,$$

so from Equation (4.1.3) we infer

$$C_k^n(p \cdot p_j) = \left(\Gamma\left(\frac{n}{2}\right) k^{1-\frac{n}{2}} + O(k^{-\frac{n}{2}})\right) P_k^{(\frac{n}{2}-1,\frac{n}{2}-1)}\left(\cos\left(\frac{|x-x_j| + O(k^{-1})}{k}\right)\right). \tag{4.1.5}$$

By virtue of Darboux's formula for the Jacobi polynomials [65, Theorem 8.1.1], we have the estimate

$$\frac{1}{k^{\frac{n}{2}-1}} P_k^{(\frac{n}{2}-1,\frac{n}{2}-1)}\left(\cos\frac{t}{k}\right) = 2^{\frac{n}{2}-1} \frac{J_{\frac{n}{2}-1}(t)}{t^{\frac{n}{2}-1}} + O(k^{-1})\,,$$

uniformly in compact sets (e.g., for $|t| \leqslant 2R$). Hence, in view of Equation (4.1.5), $\widetilde{Y}$ can be written as

$$\begin{aligned}\widetilde{Y}(x) &= \sum_{j=1}^{N} c_j \frac{1}{2^{\frac{n}{2}-1}\Gamma(\frac{n}{2})} C_k^n\left(\cos\left(\frac{|x-x_j| + O(k^{-1})}{k}\right)\right)\\ &= \sum_{j=1}^{N} c_j \frac{1}{|x-x_j|^{\frac{n}{2}-1}} J_{\frac{n}{2}-1}(|x-x_j|) + O(k^{-1})\,,\end{aligned}$$

for k big enough and $x, x_j \in B_R$. From this we get the uniform bound

$$\|\varphi - \widetilde{Y}\|_{C^0(B)} < \delta' \tag{4.1.6}$$

for any $\delta' > 0$ and all k large enough.

It remains to promote this bound to the C^{m+2} estimate. For this, note that, since the spherical harmonic Y has eigenvalue $k(n+k-1)$, the rescaled function $\widetilde{Y}$ verifies on B the equation

$$\Delta \widetilde{Y} + \widetilde{Y} = \frac{1}{k} A \widetilde{Y},$$

with

$$A\widetilde{Y} := -(n-1)\widetilde{Y} + G_1\, \partial \widetilde{Y} + G_2\, \partial^2 \widetilde{Y},$$

and where $\partial \widetilde{Y}$ is a matrix whose entries are first order derivatives of $\widetilde{Y}$, and $G_i(x,k)$ are smooth matrix-valued functions with uniformly bounded derivatives, i.e.,

$$\sup_{x \in B} |\partial_x^\alpha G_i(x,k)| \leqslant C_\alpha . \tag{4.1.7}$$

with constants C_α independent of k.

Since φ satisfies the Helmholtz equation $\Delta\varphi + \varphi = 0$, the difference $\varphi - \widetilde{Y}$ satisfies

$$\Delta(\varphi - \widetilde{Y}) + (\varphi - \widetilde{Y}) = \frac{1}{k} A \widetilde{Y},$$

and, considering the estimates (4.1.6) and (4.1.7), by standard elliptic estimates we get

$$\begin{aligned}\|\varphi - \widetilde{Y}\|_{C^{m+2,\alpha}(B)} &< C\|\varphi - \widetilde{Y}\|_{C^0(B)} + \frac{C}{k}\|A\widetilde{Y}\|_{C^{m,\alpha}(B)} \\ &< C\delta' + \frac{C}{k}\|\varphi - \widetilde{Y}\|_{C^{m+2,\alpha}(B)} + \frac{C}{k}\|\varphi\|_{C^{m+2,\alpha}(B)},\end{aligned}$$

so we conclude that, for k big enough and δ' small enough,

$$\|\varphi - \widetilde{Y}\|_{C^{m+2}(B)} \leqslant C\delta' + \frac{C\|\varphi\|_{C^{m+2,\alpha}}}{k} < \delta$$

and the proposition follows. □

4.2 Inverse localization in multiple regions

The results of the previous section can be refined to include inverse localizations around different points of the sphere. This way, we get a spherical harmonic that approximates several given solutions of the Helmholtz equation. Behind this multiple localization is the fast decay of ultraspherical polynomials of high degree outside the balls where they behave as shifted Bessel functions. Notice that, in contrast, trigonometric polynomials do not exhibit this decay, hence the lack of an analog of the following multiple realization results (Theorem in Chapter 2, for Beltrami fields, and Theorem in Chapter 3, for Dirac fields) in the case of the torus.

Let $\{p_\alpha\}_{\alpha=1}^{\Lambda}$ be a set of points in $\mathbb{S}^n$, with Λ an arbitrarily large (but fixed throughout) integer. We denote by $\Psi_\alpha : \mathbb{B}_\rho(p_\alpha) \to B_\rho$ the corresponding geodesic patches on balls of radius ρ centered on the points p_α. We fix a radius ρ such that no two balls intersect, for example by setting

$$\rho := \frac{1}{2} \min_{\alpha \neq \beta} \operatorname{dist}_{\mathbb{S}^n}(p_\alpha, p_\beta) .$$

We further choose the points $\{p_\alpha\}_{\alpha=1}^{\Lambda}$ so that no pair of points are antipodal in $\mathbb{S}^n \subset \mathbb{R}^{n+1}$, i.e, so that $p_\alpha \neq -p_\beta$ for all α, β. The reason is that spherical harmonics of energy $k(n+k-1)$ have parity $(-1)^k$

$$Y(p_\alpha) = (-1)^k Y(-p_\alpha)$$

(they are the restriction to the sphere of real harmonic homogenous polynomials of degree k); so that prescribing the behavior of a spherical harmonic in a ball around the point p_α automatically determines its behavior in the antipodal ball.

Proposition 4.2.1. *Let $\{\phi_\alpha\}_{\alpha=1}^{\Lambda}$ be a set of Λ $\mathbb{R}^q$-valued functions in $\mathbb{R}^n$, $\phi_\alpha := (\phi_{\alpha 1}, ..., \phi_{\alpha q})$, and satisfying $\Delta\phi_\alpha + \phi_\alpha = 0$. Fix a positive integer m and positive constant δ. For any large enough integer k, there is a $\mathbb{R}^q$-valued spherical harmonic Y on $\mathbb{S}^n$ with energy $k(n+k-1)$ verifying the bound*

$$\left\| \phi_\alpha - Y \circ \Psi_\alpha^{-1}\left(\frac{\cdot}{k}\right) \right\|_{C^{m+2}(B_\rho)} < \delta$$

for all $1 \leqslant \alpha \leqslant \Lambda$.

Proof. Applying Theorem 4.1.1 to each ϕ_α we obtain, for high enough k, $\mathbb{R}^q$-valued spherical harmonics $\{Y_\alpha\}_{\alpha=1}^{\Lambda}$ satisfying the bound

$$\left\| \phi_\alpha - Y_\alpha \circ \Psi_\alpha^{-1}\left(\frac{\cdot}{k}\right) \right\|_{C^{m+2}(B)} < \delta .$$

The $\mathbb{R}^q$-valued spherical harmonics $Y_\alpha(p)$ are linear combinations (with coefficients in $\mathbb{R}^q$) of ultraspherical polynomials $C_k^n(p \cdot q)$, with $\operatorname{dist}_{\mathbb{S}^n}(p_\alpha, q)$ proportional to k^{-1}. Recall that ultraspherical polynomials verified the asymptotic formula

$$C_k^n(p \cdot q) = \frac{\Gamma(\frac{n}{2})}{k^{\frac{n}{2}-1}} P_k^{(\frac{n}{2}-1, \frac{n}{2}-1)}(\cos(\operatorname{dist}_{\mathbb{S}^n}(p, q))) + O(k^{-\frac{n}{2}}) ,$$

so considering the fact that the Jacobi polynomials behave as (see [65, Theorem 7.32.2])

$$k^{\frac{n}{2}-1} P_k^{(\frac{n}{2}-1, \frac{n}{2}-1)}(\cos t) = \frac{O(k^{-1})}{t} ,$$

uniformly for $k^{-1} < t < \pi - k^{-1}$, we can conclude that the $C_k^n(p \cdot q)$ are uniformly bounded as

$$|C_k^n(p \cdot q)| \leqslant \frac{C_\rho}{k}$$

for any points p and q verifying

$$\mathrm{dist}_{\mathbb{S}^n}(p,q) \geqslant \rho \quad \text{and} \quad \mathrm{dist}_{\mathbb{S}^n}(p,-q) \geqslant \rho\,,$$

and where C_ρ is a constant depending only on ρ. The same decay is thus also exhibited by the spherical harmonics Y_α,

$$\|Y_\alpha\|_{C^0(\mathbb{S}^n\setminus(\mathbb{B}(p_\alpha,\rho)\cup\mathbb{B}(-p_\alpha,\rho))} \leqslant \frac{C'_\rho}{k}$$

since they are just normalized linear combinations of ultraspherical polynomials (here C'_ρ depends also on the particular coefficients in the expansion of Y_α, that is, on ϕ_α and δ).

Now, if we define the $\mathbb{R}^q$-valued spherical harmonic

$$Y := \sum_{\alpha=1}^{\Lambda} Y_\alpha$$

and we choose k large enough, and ρ small enough so that the sets $\mathbb{B}(p_\alpha,\rho) \cup \mathbb{B}(-p_\alpha,\rho)$ are disjoint for all α, the desired bound automatically follows in the C^0 norm. By standard elliptic estimates, we promote it to the C^{m+2} norm, and the proposition follows. □

Part II

Helicity, adiabatic limit of the Seiberg-Witten equations, and invariant measures of volume preserving vector fields

Chapter 5

The asymptotic analysis of the Seiberg-Witten equations and invariant measures

Let X be a nowhere vanishing volume preserving vector field on the 3-sphere (or more generally, an exact volume preserving vector field on a closed 3-manifold M). C. Taubes discovered that, when the helicity of X

$$\mathcal{H}(X) = \int i_X\mu \wedge \alpha \quad \text{(for any } \alpha \text{ with } d\alpha = i_X\mu)$$

is non-zero, new non-trivial invariant measures of X can be obtained through an asymptotic analysis of the so-called *Seiberg-Witten equations* in dimension 3. Furthermore, some analytic properties of sequences of solutions to the Seiberg-Witten equations are tied to the dynamical properties of the invariant measures of the vector field: as a striking example, when solutions satisfy a suitable "finite energy condition", they yield measures supported on periodic orbits of X.

Sections 2-7 of this Chapter give an account of the above phenomenon. Our aim is to make this circle of ideas more accessible to mathematicians that might be not conversant in the gauge-theoretic mathematics involved, but who work in fields related to PDEs and dynamical systems where Taubes discoveries could provide deep new insights. Thus, the contribution of those sections is on the expository side: we provide an alternative account of the original ideas of Taubes in [68, 66], putting more emphasis on the PDE and dynamical aspects.

Section 8 of the Chapter presents a previously unknown result, to the best of our knowledge. Its relation to the above is as follows: in a work in progress, we are trying to extend Taubes framework and to extract further properties of these invariant measures in more general cases; in this context we encountered a related, but simpler problem, concerning the measures arising as limits of sequences of solutions to the two dimensional vortex equations. Section 8 is devoted to presenting this problem and our solution to it.

5.1 Introduction

Asymptotic analysis can be defined as the study of the properties of solutions differential equations depending on a parameter, as this parameter goes to infinity. Alternatively, it can be said to deal with the emergence of discontinuities in solutions to PDEs.

These techniques are extremely important in physics, where they account for a wide range of phenomena: from the shadows casted by an object as light passes by, to shock waves in a stream of gas, and to the emergence of classical behavior from quantum mechanics.

All these phenomena have the paradoxical property of displaying a discontinuous nature, while being described by PDEs whose solutions one –rightly– expects to be smooth. What happens is that these discontinuities appear only to the naked eye: in reality, they are just quick transitions in the value of the solutions in certain regions; transitions that become stepper and stepper (until asymptotically the derivatives blow up) as the relevant parameter goes to infinity.

The application of Seiberg-Witten theory to dynamical systems that we are to present falls into a more general class of methods in geometry, whose common denominator is their taking advantage of the asymptotic techniques above not to explain some physical phenomenon, but to gain geometrical information. The idea is to analyze the asymptotics of a judiciously chosen PDE to detect, through the emerging discontinuities, *subsets of the ambient space with geometric or dynamical meaning*. In our case, these sets will be invariant sets of a vector field. As was the case with the traditional methods of asymptotic analysis, these new geometric approaches also have their origin in theoretical physics.

5.1.1 An example: detecting critical points of a gradient vector field

An illustrative example of how to use asymptotic analysis to detect relevant geometric data is provided by Witten's approach to Morse Theory.

The setting is a closed Riemannian manifold (M, g), where we are given a smooth function f. Let

$$C := \{p \in M, \nabla f(p) = 0\}$$

denote the set of critical points of f.

Witten introduces a deformation of the exterior derivative associated to f

$$d_r := e^{-rf} d_r e^{rf} := d + r df \wedge \cdot \ ,$$

whose adjoint is readily seen to be

$$d_r^* = d^* - r i_{\nabla f}.$$

With these operators one can define a twisted version of the Laplace-Beltrami operator on M acting on k-forms, namely

$$\Delta_r := \frac{1}{r}(d_r^* d_r + d_r d_r^*),$$

which can be more explicitely written as

$$\Delta_r = \frac{1}{r}\Delta + r|\nabla f|^2 + V, \tag{5.1.1}$$

where $|\nabla f|$ denotes the norm (defined with the metric g) of the gradient of f, and the action of V on any k-form α_r can be locally written as

$$V\alpha_r = \sum_i e_i \cdot Hf(e_j)(e^i \wedge i_{e_j}\alpha_r - i_{e_j}(e^i \wedge \alpha_r))$$

where Hf is the Hessian of f, and $\{e_i\}_{i=1}^n$ is a basis of orthonormal vectors of TM, with $\{e^i\}$ representing the dual one-forms.

The point now is that sequences of k-forms solving the PDE

$$\Delta_r \alpha_r = 0 \tag{5.1.2}$$

with $r \to \infty$, concentrate around the critical points of f. More precisely

Example 5.1.1. *Let α_r be a solution to Eq.* (5.1.2)*, suitably normalized so that $||\alpha_r||_{L^2(M)} = 1$. Given any $\epsilon > 0$ and $\rho > 0$ arbitrarily small, we have that, for all r large enough, the L^2 norm of α_r outside the set $\{p \in M, dist(p, C) \leqslant \rho\}$ is smaller than ϵ.*

Proof. First, note the following heuristic: as r grows, the equation $\Delta_r\alpha_r = 0$ "tends" to the equation $|\nabla f|^2\alpha_\infty = 0$, that is, a "solution" α_∞ must be zero except, possibly, at the critical points of f.

This heuristic argument points towards the right direction: one can see that for r large, the L^2 norm of the forms in the kernel of Δ_r is concentrated around the critical points of f.

Indeed, normalize α_r so that it has unit L^2 norm, and consider the quantity $g(\alpha_r, \Delta_r\alpha_r) = \Delta_r\alpha_r \wedge \star\alpha_r$. An integration by parts yields

$$\frac{1}{r}\int_M (|d\alpha_r|^2 + |d^*\alpha_r|^2) + r\int_M |\nabla f|^2|\alpha_r|^2 + \int_M V\alpha_r \wedge \star\alpha_r = 0$$

hence

$$\int_M |\nabla f|^2|\alpha_r|^2 \leqslant \frac{1}{r}\left|\int_M V\alpha_r \wedge \star\alpha_r\right| \leqslant \frac{C}{r}||\alpha_r||_{L^2}^2 = \frac{C}{r}$$

From this we infer that, if we set $C(\rho) := \{p \in M, \text{dist}(p, C) \leqslant \rho\}$,

$$\int_{M\setminus C(\rho)} |\nabla f|^2|\alpha_r|^2 \leqslant c\rho\int_{M\setminus C(\rho)} |\alpha_r|^2 \leqslant \frac{C}{r}||\alpha_r||_{L^2}^2$$

where c is a constant depending on the second derivatives of f. For a fixed ρ, taking r large enough, the statement in the Example follows. $\square$

(An argument of this kind, which rests on a Weitzenbőck type formula, will appear later when considering the concentration sets of the Seiberg-Witten solutions, and it is also behind our result on the rescaled vortex equations in the last section.)

5.2 The Seiberg-Witten approach to dynamics: Setting and statement of the main Theorems

We will show in the following sections how to use the asymptotic concentration of solutions of another PDE, the so-called Seiberg-Witten equations, to detect invariant sets and construct invariant measures of a vector field X.

Both the Seiberg-Witten equations and their applications that we are to present can be stated in any manifold M, and in fact have deep roots in three (and four) dimensional geometry. Nevertheless, we will focus on the case of the 3-sphere $\mathbb{S}^3$, as this allows us to lighten up the amount of geometric background and to concentrate on the underlying analysis.

Statement of the equations.

Our main object of interest is a nowhere vanishing vector field X preserving a volume form μ on the 3-sphere $\mathbb{S}^3$. This amounts to having a closed 2-form $i_X\mu$; this form is, in addition, exact (since the homology is trivial). To define the relevant version of the Seiberg-Witten equations, we endow $\mathbb{S}^3$ with a metric g compatible with the volume form μ (this means, in particular, that for any vector field W, $i_W g = \star i_W \mu$, where $\star$ is the Hodge dual defined by the metric); and such that $g(X, X) = 1$. With these data we can always find vector fields Y and Z such that $\{X, Y, Z\}$ defines an orthonormal parallelization of the tangent bundle $T\mathbb{S}^3$ (in a general three manifold M it would only be possible to define Y and Z locally, the reason being that the plane bundle Ker $i_X g$ might be nontrivial). It will be often convenient to refer to $\{X, Y, Z\}$ as $\{e_1, e_2, e_3\}$.

We will denote by a dot the scalar product defined by g, and by $|W|^2 := W \cdot W$ the norm squared of a vector field W. Throughout this chapter, each time that we are given a pair (X, μ) in $\mathbb{S}^3$, we automatically assume as well that we have endowed the manifold with an adapted metric g. Integration will be always understood with respect to the volume form μ.

Finally, let us introduce the Pauli matrices

$$\sigma_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \sigma_2 := \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \sigma_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The Seiberg-Witten equations associated to the above data are

$$\operatorname{curl} A_r = r(X - \psi_r^\dagger \sigma \psi_r) + v \tag{5.2.1}$$

$$D_{A_r}\psi_r := i\sum_k \sigma_k e_k \cdot (\nabla\psi_r - iA_r\psi_r) = 0 \tag{5.2.2}$$

where, for each fixed value of r, the unknowns are a vector field A_r and a function $\psi_r : \mathbb{S}^3 \to \mathbb{C}^2$. The term $\psi_r^\dagger\sigma\psi_r$ represents the vector field

$$\psi_r^\dagger\sigma\psi_r := (\psi_r^\dagger\sigma_1\psi_r)X + (\psi_r^\dagger\sigma_2\psi_r)Y + (\psi_r^\dagger\sigma_3\psi_r)Z, \tag{5.2.3}$$

and ∇ and curl are the gradient and curl operators defined by the metric g. The operator $\nabla_A := (\nabla - iA_r)$ can be seen as a covariant derivative on the trivial bundle $\mathbb{C}^2 \times \mathbb{S}^3$, and it must be understood as acting on each complex-valued component of ψ_r separately.

Finally, v is a given divergence-free vector field, which can be chosen to have arbitrarily small C^k norm: it acts as a perturbation to the equation that ensures the existence of solutions with desired properties. For a fixed r and v, Eqs (5.2.1) and (5.2.2) will be referred to as $SW(r,v)$-equations.

An important feature of the equations is that they are *gauge invariant* under the action of the group $U(1)$. This means that, given a solution (A_r, ψ_r) and any smooth function $u : \mathbb{S}^3 \to \mathbb{C}$ with $|u| = 1$, the pair $u(A_r, \psi_r) := (A + u_1\nabla u_2 - u_2\nabla u_1, u\psi_r)$ is also a solution (here u_1 and u_2 are such that $u := u_1 + iu_2$). Note that the vector field $(u_1\nabla u_2 - u_2\nabla u_1)$ is curless (because $u_1^2 + u_2^2 = 1$) and so, since $H^1(\mathbb{S}^3) = 0$, it must be the gradient of a function. Consequently, we can always write a gauge transformation as $u = e^{if}$ for a well defined real function $f : \mathbb{S}^3 \to \mathbb{R}$, and we have $u(A_r, \psi_r) = (A_r + \nabla f, e^{if}\psi_r)$. This symmetry of the equations is important in many arguments, as it grants us some extra freedom when dealing with their analytical properties.

Main results.

Before stating the main theorems, we have to introduce the notion of *Hopf invariant*: the Hopf invariant of a pair of volume preserving vector fields V, W in $(\mathbb{S}^3, \mu)$ is defined as

$$\mathcal{H}(V,W) = \int i_V\mu \wedge \alpha \quad \text{(for any } \alpha \text{ with } d\alpha = i_W\mu\text{)}.$$

Stokes theorem guarantees that the Hopf invariant is well defined regardless of the form α one choses. In particular, one can choose α with $d \star \alpha = 0$, and this α is unique, since there are no non-trivial harmonic functions. In terms of vector fields and a metric g compatible with μ, this α is the dual of a divergence free vector field, that we denote by $\operatorname{curl}^{-1} W$: $i_{\operatorname{curl}^{-1} W}\, g = \alpha$. The operator curl^{-1} is thus a well defined one-to-one map in the space of volume preserving vector fields in $\mathbb{S}^3$, with $\operatorname{curl}^{-1} W$ being the unique volume preserving vector field such that $\operatorname{curl}\operatorname{curl}^{-1} W = W$. In these terms, we can express the Hopf invariant as

$$\mathcal{H}(V,W) := \int V \cdot \operatorname{curl}^{-1} W.$$

Finally, let us note that the Hopf invariant is symmetric, i.e, that $\mathcal{H}(V,W) = \mathcal{H}(W,V)$, as can be easily checked with an integration by parts.

An important role in what is to follow will be played by the *helicity* of a vector field, which is defined as the Hopf invariant of X with itself, $\mathcal{H}(X) := \mathcal{H}(X, X)$. The other main quantity of interest for us will be $\mathcal{H}(\operatorname{curl} A_r, \operatorname{curl} X)$,

$$\mathcal{H}(\operatorname{curl} A_r, \operatorname{curl} X) = \int \operatorname{curl} A_r \cdot X = r \int (1 - \psi_r^\dagger \sigma_1 \psi_r) + \int X \cdot v,$$

We will set $\mathcal{H}_{A_r}(X) := \mathcal{H}(\operatorname{curl} A_r, \operatorname{curl} X)$.

We are now ready to estate the main two theorems that we will prove in the first part of this Chapter. These results are originally due to Clifford Taubes in [68, 66] and, in fact, he proves them with greater generality: not only in $\mathbb{S}^3$, but on any closed 3-manifold. In particular, Item (i) in Theorem 5.2.1, together with Theorem 5.2.2, follow from Taubes paper [68], which is the first paper examining this framework; the rest of Theorem 5.2.1 appeared later in [66].

Theorem 5.2.1. *Let X be a nowhere-vanishing vector field on $\mathbb{S}^3$ preserving a volume form μ, and such that its helicity $\mathcal{H}(X)$ is positive. Fix $\epsilon > 0$ arbitrarily small. There exists a sequence $\{r_n, \psi_{r_n} := (\psi_{1r_n}, \psi_{2r_n}), A_{r_n}\}$ of solutions to the associated $SW(r, v)$-equations, for some volume preserving vector field v of C^k norm less than ϵ, such that*

- *(i) if the sequence of Hopf invariants $\mathcal{H}_{A_{r_n}}(X)$ has a bounded subsequence, the vector field X has a periodic orbit.*
- *(ii) if the sequence of Hopf invariants $\mathcal{H}_{A_{r_n}}(X)$ has no bounded subsequence, then the signed measure*

$$\sigma_{r_n} := \frac{r_n(1 - |\psi_{1r_n}|^2)\mu}{\mathcal{H}_{A_{r_n}}(X)}$$

converges (maybe after passing to a subsequence) to an invariant probability measure σ_∞ of X. This measure satisfies $\sigma_\infty(X \cdot \operatorname{curl}^{-1}(X)) \leqslant 0$.

We will prove this Theorem in Section 5.3. The following Theorem can be seen as a (non-trivial) consequence of Theorem 5.2.1

Theorem 5.2.2. *Let X be a nowhere-vanishing vector field on $\mathbb{S}^3$ preserving a volume form μ. If we have that $\operatorname{curl}^{-1} X = hX$, with h a positive function in $\mathbb{S}^3$, the vector field X has a periodic orbit.*

Indeed, this Theorem follows from Theorem 5.2.1 once one establishes that a rescaling $\tilde{X}$ of the vector field X has an associated sequence of solutions to the $SW(r, v)$-equations with bounded $\mathcal{H}_{A_r}(\tilde{X})$. This will be proven in Section 5.4.

Let us disclose the main ingredients behind the proof of Theorem 5.2.1. Each item below has a corresponding section in this Chapter:

- (A) *A priori behavior of the solutions* (Section 5.5): The Weitzenbőck formula for D_{A_r}, through standard elliptic bootstrapping, yields a priori estimates for the solutions of the Seiberg-Witten equations. Roughly speaking, the main lessons of these estimates are: that for large r, ψ_{2r} goes to zero everywhere; that the curl of A_r is mostly parallel to X, curl $A_r \sim r(1 - |\psi_{1r}|^2)X$; and that the derivatives of $|\psi_{1r}|$ can change quickly in the transverse directions of X, but not in the direction of the flow.

- (B) *The existence of non-trivial solutions* (Section 5.6) The Monopole Floer Homology, as constructed by P. Kronheimer and T. Mrowka in [46], provides the foundation on which the existence of solutions rests. Very roughly speaking, this theory associates topological invariants to 3-manifolds by constructing an appropriate chain complex (and associated homology groups) using as generators solutions to Seiberg-Witten-like equations, in a similar spirit one uses critical points in Morse Theory. The relevance of this construction from the PDE viewpoint is as follows: whenever an homology group is not trivial, we know that there must be some generators, that is, some solutions. Better still, the resulting homology groups are independent of the precise geometric or analytic data used to define the equations, so once the groups are known to be non trivial in some case, we can infer the existence of solutions in many other cases. Finally, it is key to ensure as well that those solutions have the desired properties: it is at this point that the helicity of X being non-zero plays a significant role that we will explain in more detail in Section 5.6.

- (C) *The asymptotic properties of solutions at small scales* (Section 5.7): By virtue of item (A), the solutions to the Seiberg-Witten equations for r big enough are shown to approximate, at small scales, solutions to the 2-dimensional vortex equations in the transverse directions of X. The properties of these 2-dimensional equations are the key input for the proof of item (i) in Theorem 5.2.1.

An important open problem is to understand better the invariant sets where the measure σ_∞ in item (ii) of Theorem 5.2.1 concentrates (i.e in the case of unbounded sequences $\mathcal{H}_{A_r}(X)$). For example, do the equations impose any condition on the type of invariant measure that can be obtained as a limit? In the final section of this Chapter (Section 5.8) we present a new result on a related, prototype problem in dimension 2. This problem appears naturally when trying to address this question, and concerns the *rescaled vortex equations* in $\mathbb{C}$:

$$\star da_r = r(1 - \phi_r^2) \tag{5.2.1}$$

$$\overline{\partial}_{a_r}\phi = \overline{\partial}_z\phi_r - i(a_{xr} - ia_{yr})\phi = 0 \tag{5.2.2}$$

where the unknowns are, for $r > 0$ fixed, a real one form $a_r := a_{1r}dx + a_{2r}dy$ in $\mathbb{R}^2$ (that we identify with $\mathbb{C}$) and a complex valued function $\phi_r : \mathbb{C} \to \mathbb{C}$.

The problem is to understand whether sequences of solutions to the above equations, with increasing r, could converge to any conceivable probability measure on $\mathbb{C}$. Section 5.8 is devoted to the proof of the following:

Theorem 5.2.3. *Let ν be a Borel probability measure on the open disk $\mathbb{D} \subset \mathbb{C}$. There is a sequence $\{(\phi_{r_n}, a_{r_n})\}$ of solutions to the r_n-rescaled vortex equations in $\mathbb{C}$ (with $r_n \to \infty$) such that the 2-form*

$$\sigma_{r_n} = \frac{r_n(1 - |\phi_{r_n}|^2) dx \wedge dy}{\int da_{r_n}}$$

converges to ν in the sense of measures on $\mathbb{D}$, and is zero elsewhere.

(The measure being defined on $\mathbb{D}$ is not an important requirement in the above. As the proof will make clear, the result works as well with ν any Borel probability measure on $\mathbb{C}$.)

In work in progress, we try to pass from this result to an analogous result for invariant measures arising as limits of sequences of three forms of the type

$$\sigma_r := \frac{r(1 - |\psi_{1r}|^2)\mu}{\mathcal{H}_{A_r}(X)},$$

coming from solutions (A_r, ψ_r) to the Seiberg-Witten equations.

The chapter is organized as follows. Section 5.3 is devoted to the proof of Theorem 5.2.1, and Section 5.4 gives the proof of Theorem 5.2.2. The proof of some key propositions that are needed in the two first sections is postponed to Sections 5.5, 5.6, and 5.7 (concerning, respectively, the a priori bounds, the existence of solutions, and the small scale behavior). We end the chapter with the proof of Theorem 5.2.3 in Section 5.8.

5.3 Proof of Theorem 5.2.1

We will divide the proof into two parts, corresponding to the proof of item (ii) in Theorem 5.2.1 (Subsection 5.3.1) and then item (i) (Subsection 5.3.2). Before engaging with the proof of each item, we state two Propositions that are crucially needed for both, and which are concerned with, on the one hand, the a priori properties of solutions, and on the other hand, with the existence of solutions.

The first Proposition describes the a priori behavior that *any* solution to the $SW(r, v)$ equations displays:

Proposition 5.3.1 (A priori bounds)**.** *Let (A_r, ψ_r) be a solution to the $SW(r, v)$-equations, with v a fixed C^∞ divergence free vector field. We have the following uniform estimates for all $m \geqslant 1$:*

$$||\psi_{1r}||^2_{C^0(S^3)} \leqslant 1 + \frac{c_0}{r}; \; ||\psi_{2r}||^2_{C^0(S^3)} \leqslant \frac{c_0(1 - |\psi_{1r}|^2)}{r} + \frac{c_0}{r^2} \tag{5.3.1}$$

$$||(\nabla - iA_r)\psi_{1r}||_{C^m(S^3)} \leqslant c_m r^{\frac{m+1}{2}} \tag{5.3.2}$$

$$||(\nabla - iA_r)\psi_{2r}||_{C^m(\mathbb{S}^3)} \leqslant c_m r^{\frac{m}{2}}. \tag{5.3.3}$$

where all the constants depend only on the metric and v.

The proof of this Proposition is given in Section 5.5.

The next Proposition yields the existence of a solutions with additional properties, suitable for our ends.

Proposition 5.3.2 (Existence of solutions)**.** *Let X be a nowhere-vanishing vector field on $\mathbb{S}^3$ preserving a volume form μ and* with non-zero helicity. *There always exists a smooth perturbing volume preserving vector field v, of arbitrarily small C^m norm, for which the associated $SW(r,v)$-equations have a sequence $\{r_n,\, \psi_{r_n} := (\psi_{1r_n}, \psi_{2r_n}),\, A_{r_n}\}$ of solutions such that*

- *A) $\sup\,(1 - |\psi_{r_n}|^2) > \delta$, for all r_n and some $\delta > 0$,*
- *B) Either the sequence $\mathcal{H}_{A_{r_n}}(X)$ is bounded or the sequences of functionals $\mathcal{H}_{A_{r_n}}(X)$ and $\mathcal{H}_{A_{r_n}}(\operatorname{curl}^{-1} X)$ verify*

$$\frac{\mathcal{H}_{A_{r_n}}(\operatorname{curl}^{-1} X)}{\mathcal{H}_{A_{r_n}}(X)} \leqslant \varepsilon$$

  for any $\varepsilon > 0$ as small as we want, provided r_n is large enough.

The proof of this Proposition is given in Section 5.6. We recall that the functionals $\mathcal{H}_{A_r}(X)$ and $\mathcal{H}_{A_r}(\operatorname{curl}^{-1} X)$ were defined as

$$\mathcal{H}_{A_r}(X) = \int \operatorname{curl} A_r \cdot X\,, \quad \mathcal{H}_{A_r}(\operatorname{curl}^{-1} X) = \int \operatorname{curl} A_r \cdot \operatorname{curl}^{-1} X\,.$$

Granted the above Propositions, we proceed with the proof of Theorem 5.2.1.

5.3.1 Proof of item (ii) in Theorem 5.2.1

Let $\{r,\, \psi_r := (\psi_{1r}, \psi_{2r}),\, A_r\}$ be a sequence as in Proposition 5.3.2 (we will drop henceforth the subscript n). Suppose first that the sequence $\mathcal{H}_{A_r}(X)$ is *not bounded*. This is the relevant case for item (ii) in Theorem 5.2.1. Our goal is to prove that the 3-form

$$\sigma_r := \frac{r(1 - |\psi_{1r}|^2)\mu}{\mathcal{H}_{A_r}(X)}$$

converges (maybe after passing to a subsequence) to an invariant measure of X.

We divide the proof into three steps: the first step shows that σ_r converges to a probability measure, i.e, there is a σ_∞ with $\sigma_\infty(\mathbb{S}^3) = 1$. The second step shows that this measure is indeed invariant under the flow of X. Finally, the third step shows that $\sigma_\infty(X \cdot \operatorname{curl}^{-1} X) \leqslant 0$ (hence, in particular, σ_∞ cannot be a multiple of the volume form).

Step 1: Subsequences of σ_r converge to a probability measure σ_∞

Our first goal is to prove that the sequence σ_r is bounded from above, which will imply that there is a weakly convergent subsequence and hence, at least, a limiting measure σ_∞.

To that end, we first notice the following straightforward consequence of the bounds in (5.3.1)

Lemma 5.3.3. *Let (A_r, ψ_r) be any solution to the $SW(r,v)$-equations. We have that, on the one hand*

$$r\int |1-|\psi_{1r}|^2| \leqslant \mathcal{H}_{A_r}(X) + C$$

and on the other hand

$$\mathcal{H}_{A_r}(X) \leqslant r\int |1-|\psi_{1r}|^2| + C$$

for C a positive constant depending on the metric and v, but not on r.

Proof of Lemma 5.3.3. Indeed, we have

$$\mathcal{H}_{A_r}(X) = r\int (1-|\psi_{1r}|^2) + r\int |\psi_{2r}|^2 + \int X\cdot v\,.$$

The third integral in the right hand side is bounded by the volume of $\mathbb{S}^3$ (which we assume to be normalized to one) times the C^0 norm of v. The second integral can be bounded, by virtue of (5.3.1), by

$$r\int |\psi_{2r}|^2 \leqslant c\int |1-|\psi_{1r}|^2| \leqslant c$$

for some constant c. Finally, note that the bound (5.3.1) also implies that

$$r\int (1-|\psi_{1r}|^2) = r\int |1-|\psi_{1r}|^2| + O(1)\,.$$

where by $O(1)$ we denote a term bounded by an r-independent constant (and that, in this case, is negative). Hence, we have that

$$\mathcal{H}_{A_r}(X) = r\int |1-|\psi_{1r}|^2| + O(1)\,.$$

which readily implies both bounds in Lemma 5.3.3. □

Now, using the first inequality in Lemma 5.3.3, we have that, for any continuous function φ

$$\sigma_r(\varphi) = \frac{r\int \varphi(1-|\psi_{1r}|)^2}{\mathcal{H}_{A_r}(X)} \leqslant \frac{||\varphi||_{C^0(\mathbb{S}^3)}\, r\int |1-|\psi_{1r}|^2|}{\mathcal{H}_{A_r}(X)} \leqslant$$
$$\leqslant ||\varphi||_{C^0(\mathbb{S}^3)}(1+\frac{C}{\mathcal{H}_{A_r}(X)}) \leqslant ||\varphi||_{C^0(\mathbb{S}^3)}(1+\varepsilon)$$

where ε can be chosen as small as desired as long as r is big enough. Hence, the sequence of measures has a weakly convergent subsequence (by the Banach-Alaoglu theorem), and we get a limiting measure σ_∞. Note that the total mass of this measure is bounded above by 1.

Furthermore, the limit σ_∞ of any convergent subsequence is never zero, because the sequence of measures is also bounded from below: it easily follows from the second inequality in Lemma 5.3.3 that

$$\sigma_r(1) \geqslant \frac{\int r(1-|\psi_{1r}|^2)}{\int r(1-|\psi_{1r}|^2)+C} \geqslant 1+\varepsilon.$$

again with ε as small as desired as long as r_n is big enough. In particular, the above expression shows that the limiting measure σ_∞ has total mass one, $\sigma_\infty(\mathbb{S}^3) = 1$. Finally, the limiting measures are always non-negative, because

$$\sigma_r(U) \geqslant \frac{\int_U r|1-|\psi_{1r}|^2| - C}{\int_{\mathbb{S}^3} r|1-|\psi_{1r}|^2| + C} \geqslant -\epsilon$$

for $\epsilon > 0$ as small as we want, as long as r is large enough. So any limiting measure σ_∞ is a probability measure.

5.3.1.1 Step 2: Any limiting probability measure σ_∞ is invariant under the flow of X

Now let us demonstrate that the flow of X leaves σ_∞ invariant, i.e, that for any continuous function φ, we have that

$$\sigma_\infty(\varphi \circ \phi_X^t) = \sigma_\infty(\varphi).$$

There is no loss of generality in taking f to be C^∞ (since smooth functions approximate arbitrarily well continuous ones).

Differentiating the above expression with respect to the parameter t, we get the equivalent condition for invariance:

$$\sigma_\infty(X \cdot \nabla f) = 0. \tag{5.3.1}$$

For finite r, we can integrate by parts $\sigma_r(X \cdot \nabla f)$ to get

$$\sigma_r(X \cdot \nabla \varphi) = \frac{r \int \varphi \operatorname{div}(1-|\psi_{1r}|^2)X}{\mathcal{H}_{A_r}(X)} \tag{5.3.2}$$

Lemma 5.3.4. *The quantity div* $(1-|\psi_{1r}|^2)X$ *satisfies, for any* C^1 *function* φ

$$\int \varphi \operatorname{div}(1-|\psi_{1r}|^2)X \leqslant ||\nabla\varphi||_{C^0(\mathbb{S}^3)} \Big(\frac{C}{r}\int |1-|\psi_{1r}|^2| + \frac{C}{\sqrt{r}}\int |1-|\psi_{1r}|^2|^{\frac{1}{2}}\Big)$$

with constant C depending on v and the metric g, but not on r.

Proof of Lemma 5.3.4. It is convenient to introduce the following notation: the second of the Seiberg-Witten equations, that is

$$\sigma \cdot \nabla \psi = i\sigma \cdot A\psi\,, \tag{5.3.1}$$

can be written component-wise as

$$X \cdot \nabla_A \psi_2 = \overline{\partial}_A \psi_1 \tag{5.3.2}$$

$$X \cdot \nabla_A \psi_1 = -\partial_A \psi_2 \tag{5.3.3}$$

where (we recall that, in our conventions, $Z = e_3$ and $Y = e_2$)

$$\partial_A := Z \cdot \nabla_A + iY \cdot \nabla_A = (\nabla_3 - iA_3) + i(\nabla_2 - iA_2) \tag{5.3.4}$$

$$\overline{\partial}_A := Z \cdot \nabla_A - iY \cdot \nabla_A = (\nabla_{e_3} - iA_3) + i(\nabla_2 - iA_2) \tag{5.3.5}$$

The notation is justified thus: the operator $\overline{\partial}_A$ (resp. ∂_A) can be understood as a (twisted) Cauchy-Riemann operator in the transverse planes to X. We further notice the the general identity

$$\int \overline{\psi}_1 \partial_A \psi_2 + \overline{\partial_A \psi_2} \psi_1 = -\int \overline{\overline{\partial}_A \psi_1} \psi_2 - \overline{\psi}_2 \overline{\partial}_A \psi_1\,.$$

Given the above notations, a straightforward calculation yields that

$$\int \varphi \text{ div } (1 - |\psi_{1r}|^2) X = \int |\psi_{2r}|^2 \nabla_X \varphi + 2\int \text{Re } (\overline{\psi}_{2r} \psi_{1r}) \nabla_Z \varphi + 2\int \text{Im } (\overline{\psi}_{2r} \psi_{1r}) \nabla_Y \varphi \tag{5.3.6}$$

where c.c stands for the complex conjugate of the term just preceding it. From this we clearly get the following bound

$$\int \varphi \text{div } (1 - |\psi_{1r}|^2) X \leqslant ||\nabla \varphi||_{C^0} \int |\psi_{2r}|^2 + ||\nabla \varphi||_{C^0} \int |\psi_{2r}|\,.$$

Now, recall that Proposition 5.3.1 grants that:

$$|\psi_{2r}|^2 \leqslant \frac{c_0 |1 - |\psi_{1r}|^2|}{r} + \frac{c_0}{r^2}$$

and therefore, we conclude that

$$\int \varphi \text{ div } (1 - |\psi_{1r}|^2) X \leqslant C ||\nabla \varphi||_{C^0} \Big(\frac{1}{r} \int |1 - |\psi_{1r}|^2| + \frac{1}{\sqrt{r}} \int |1 - |\psi_{1r}|^2|^{\frac{1}{2}} \Big) \tag{5.3.7}$$

with constants depending on the metric and v, and on the C^1 norm of φ, and the Lemma follows. □

From this, we clearly have that

$$\sigma_r(X \cdot \nabla\varphi) \leqslant \frac{c}{\mathcal{H}_{A_r}(X)} + \frac{c\sqrt{r}\int |1 - |\psi_{1r}|^{\frac{1}{2}}|}{\mathcal{H}_{A_r}(X)} \tag{5.3.8}$$

Since $\mathcal{H}_{A_r}(X)$ has no convergent subsequence, the first term clearly goes to zero as r increases. Now for the second term, recall that the first inequality in Lemma 5.3.3 gives the bound

$$\mathcal{H}_{A_r}(X) \geqslant r\int |1 - |\psi_{1r}|^2| - C$$

Further still, Lemma 5.3.3 also implies that if $\mathcal{H}_{A_r}(X)$ is unbounded, $r\int |1 - |\psi_{1r}|^2|$ is also unbounded, and therefore

$$\mathcal{H}_{A_r}(X) \geqslant cr\int |1 - |\psi_{1r}|^2|\,,$$

for any desired constant $c < 1$ as long as r is large enough. This permits to bound from above the second term in the right hand side of Eq (5.3.7):

$$\frac{c\sqrt{r}\int |1 - |\psi_{1r}|^2||^{\frac{1}{2}}}{\mathcal{H}_{A_r}(X)} \leqslant c\frac{||\sqrt{r}|1 - |\psi_{1r}|^2|^{\frac{1}{2}}||_{L^1}}{||\sqrt{r}|1 - |\psi_{1r}|^2|^{\frac{1}{2}}||^2_{L^2}}$$

where c is a new constant depending again on g, v, and φ.

By the Cauchy-Schwarz inequality, the term in the numerator is bounded by

$$||\sqrt{r}|1 - |\psi_{1r}|^2|^{\frac{1}{2}}||_{L^1} \leqslant ||\sqrt{r}|1 - |\psi_{1r}|^2|^{\frac{1}{2}}||_{L^2}\,.$$

Therefore, we finally get

$$\sigma_r(X \cdot \nabla f) \leqslant \frac{c}{\mathcal{H}_{A_r}(X)} + \frac{c}{(r\int |1 - |\psi_{1r}|^2|)^{\frac{1}{2}}}$$

The above expression goes to zero as $r \to \infty$, hence for any converging sequence, the limit verifies $\sigma_\infty(X \cdot \nabla f) = 0$, as we wanted to show.

Now the only thing left to prove is the key fact that $\sigma_\infty(X \cdot \operatorname{curl}^{-1} X) \leqslant 0$.

5.3.1.2 Step 3: $\sigma_\infty(X \cdot \operatorname{curl}^{-1} X) \leqslant 0$

The key input for this step is given by item B in Proposition 5.3.2: since we have a sequence of r's such that the sequence $\mathcal{H}_{A_r}(X)$ has no bounded subsequence, our solutions verify

$$\frac{\int \operatorname{curl} A_r \cdot \operatorname{curl}^{-1} X}{\int \operatorname{curl} A_r \cdot X} \leqslant \varepsilon \tag{5.3.9}$$

for any ε as small as we want, and long as r is large enough.

The above statement is actually pretty much equivalent to what we want to prove. Indeed, since the A_r solve the $SW(r,v)$-equations, we have:

$$\frac{\int \operatorname{curl} A_r \cdot \operatorname{curl}^{-1} X}{\int \operatorname{curl} A_r \cdot X} = \frac{r\int (1-|\psi_{1r}|^2) X \cdot \operatorname{curl}^{-1} X}{\mathcal{H}_{A_r}(X)} + G \tag{5.3.10}$$

where G can be bounded (using again the bound (5.3.1) in Proposition 5.3.1) as

$$\frac{\sqrt{r}|1-|\psi_{1r}|^2|^{\frac{1}{2}}}{\mathcal{H}_{A_r}(X)}.$$

Note that the first term in Eq. (5.3.10) is exactly equal to

$$\sigma_r(X \cdot \operatorname{curl}^{-1} X)$$

As for the second term G, arguing as in Step 2 we can bound it as

$$c\frac{||\sqrt{r}|1-|\psi_{1r}|^2|^{\frac{1}{2}}||_{L^1}}{||\sqrt{r}|1-|\psi_{1r}|^2|^{\frac{1}{2}}||^2_{L^2}} \leqslant \frac{c}{(r\int|1-|\psi_{1r}|^2|)^{\frac{1}{2}}},$$

which goes to zero. Thus, we infer from Eq. (5.3.9) that, for any $\varepsilon > 0$ and r sufficiently large, there is a $\delta > 0$ such that

$$\frac{\mathcal{H}_{A_r}(\operatorname{curl}^{-1} X)}{\mathcal{H}_{A_r}(X)} = \sigma_r(X \cdot \operatorname{curl}^{-1} X) + \delta \leqslant \varepsilon,$$

and so we get that $\sigma_\infty(X \cdot \operatorname{curl}^{-1} X) \leqslant 0$, as desired. Step 3 is done and so is the proof of item (ii) in Theorem 5.2.1.

A remarkable consequence of Step 3 is that σ_∞ is not the volume measure μ, or some multiple of it. Indeed, the helicity of X (that is, $\mu(X \cdot \operatorname{curl}^{-1} X)$) is positive by assumption. Furthermore, $\sigma_\infty(U) \geqslant 0$ for any open set U, so the possibility that σ_∞ is a negative rescaling of μ is also discarded.

5.3.2 Proof of item (i) of Theorem 5.2.1: convergence to a set of periodic orbits in the bounded energy case

Through this subsection, we assume that

$$\mathcal{H}_{A_r}(X) = r\int(1-|\psi_{1r}|^2) + r\int|\psi_{2r}|^2 + \int X \cdot v \leqslant C$$

for some constant C not depending on r; and we take from Proposition 5.3.2 a sequence of solutions (A_r, ψ_r) with $r \to \infty$ such that

$$\sup\,(1-|\psi_{1r}|^2) > \delta > 0 \tag{5.3.11}$$

with δ not depending on r.

For notational convenience, throughout the rest of this subsection we set $u_r := (1-|\psi_{1r}|^2)$.

Note that Eq. (5.3.11) grants the existence of a sequence of points $\{p_r\}$ in $\mathbb{S}^3$ with $u_r(p_r) > \delta$ as $r \to \infty$. The next thing that we will settle is that we can actually choose a fixed point p with $u_r(p) > \delta$ (for a sequence of r big enough) rather than a sequence of points p_r.

This is not a priori evident. Granted, by compactness of $\mathbb{S}^3$, we can always choose a subsequence of points p_r where the supremum of u_r is attained, and converging to a fixed point p. But this just means that there certainly exists a point q such that, for any ε as small as we wish, there is some point p_r with $u(p_r) > \delta$ within a distance ε from q. Nevertheless, since u_r could have derivatives as big as $\sqrt{r}$, we could have $\varepsilon >> \frac{1}{\sqrt{r}}$ (we have no control on the relation between the convergence of p_r to q, and the growth of r) and thus we cannot ensure that q has also $u_r(q) > \delta$.

Still, we can argue by contradiction that a point p with $u_r(p) > \delta'$, for some $\delta' > 0$ and an infinite sequence of $r \to \infty$, must exists. Indeed, suppose that there was no such point. Then, for any point p, we should have that $u_r(p) \leqslant \epsilon$, for any $\epsilon > 0$ as small as we want, as long as r is big enough. But then, by compactness of $\mathbb{S}^3$, we would know that, for any given ϵ, for r big enough we have that $u_r(p) \leqslant \epsilon$ for every point p. This clearly contradicts (5.3.11).

So we can indeed choose a fixed point p with

$$u(p) > \delta'$$

for some $\delta' > 0$. In what follows, we reset $\delta' = \delta$.

The key to the proof of item (i) in Theorem 5.2.1 rests on the local behavior of the solutions to the $SW(r,v)$-equations in small flow boxes of X, and in particular, in flow boxes around points p as above.

In order to properly describe this local behavior, let us define, at any point $p \in \mathbb{S}^3$, a "flow-box chart" adapted to the field X around p.

Definition 5.3.5 (Adapted flow-box and flow-box chart at a point p)**.** *Let p be any point in $\mathbb{S}^3$. Consider, for positive constants ϵ and ρ, the map*

$$\Phi_p : (-\epsilon, \epsilon) \times \mathbb{D}_\rho \longrightarrow \mathbb{S}^3$$

with $\mathbb{D}_\rho := \{z \in \mathbb{C}, |z| \leqslant \rho\}$ being the disk of radius ρ, and

$$\Phi_p(t,z) := \phi_X^t(exp_p(xY(p) + yZ(p))$$

with $t \in (-\epsilon, \epsilon)$ and $z = x + iy \in \mathbb{D}_\rho$, and where $exp_p : T_p\mathbb{S}^3 \to \mathbb{S}^3$ is the exponential map. With ϵ and ρ small enough, Φ_p is a well defined diffeomorphism.

Denote by $C_p(\rho,\epsilon)$ the set $\Phi_p((-\epsilon,\epsilon) \times \mathbb{D}_\rho) \subset \mathbb{S}^3$. This is an adapted flow box *at p. The* flow box chart *at $C_p(\rho,\epsilon)$ is the map $\Psi_p : C_p(\rho,\epsilon) \to (-\epsilon,\epsilon) \times \mathbb{D}_\rho$ defined as $\Psi_p := \Phi_p^{-1}$.*

Note that in the flow box chart coordinates, we have that $(\Psi_p)_* X = \partial_t$.

With a slight abuse of notation, we will denote by $\psi(t,z)$ the function $\psi \circ \Psi_p^{-1}(t,z)$, and follow this convention for all the other variables. The metric g in the flow-box coordinates, i.e, $(\Psi_p)_* g$, will also keep being referred to as g; as well as the vector field $A(t,z) := (\Psi_p)_* A$.

We begin by noticing the following direct consequence of Proposition 5.3.1 together with the second of the Seiberg-Witten equations, that will play an important role all through the proof:

Lemma 5.3.6. *Let (A_r, ψ_r) be a solution to the $SW(r,v)$-equations. At any point $p \in \mathbb{S}^3$, we have*

$$|X \cdot \nabla u_r|(p) < \widehat{C}$$
$$|Y \cdot \nabla u_r|(p) < \widehat{C}\sqrt{r}$$
$$|Z \cdot \nabla u_r|(p) < \widehat{C}\sqrt{r}$$

where $\widehat{C}$ is a constant depending on the metric and on v, but not on r.

Proof. First, note that, for any vector field e_k

$$|e_k \cdot \nabla(1-|\psi_{1r}|^2)| = |\nabla_k |\psi_{1r}|^2| = |\mathrm{Re}\ (\psi_{1r}^\dagger(\nabla_k - iA_k)\psi_{1r})| \leqslant |\psi_{1r}||(\nabla_k - iA_k)\psi_{1r}|$$

and so, by virtue of Proposition 5.3.1, we have

$$|\nabla(1-|\psi_{1r}|^2)| \leqslant (1+\frac{c_0}{r})c_1\sqrt{r} \leqslant C\sqrt{r}$$

which already yields the last two bounds in the Lemma. For the bound in the X direction, recall that the second of the $SW(r,v)$-equations reads component wise (we write the relevant component only)

$$X \cdot (\nabla - iA)\psi_{1r} = -Z \cdot (\nabla - iA)\psi_{2r} - iY \cdot (\nabla - iA)\psi_{2r}$$

and hence, by the last bound in Proposition 5.3.1

$$|X \cdot \nabla(1-|\psi_{1r}|^2)| \leqslant |X \cdot (\nabla - iA)\psi_{1r}| \leqslant |(\nabla - iA)\psi_{2r}| \leqslant C.$$

□

Remark 5.3.7. *We will henceforth fix the value of the constant $\widehat{C}$ as above; any other constant appearing under some of the usual labels (C, c_0...) may change its value from one appearance to the next, and depend on some data to be specified at each appearance, but it is always understood that it does not depend on r.*

Lemma 5.3.8. *Let p be any point $p \in \mathbb{S}^3$ with $u_r(p) > \delta$, and let $C_p(\frac{R}{\sqrt{r}}, \epsilon)$ be an adapted flow box, for some constant R and ϵ verifying*

$$R \leqslant \frac{\delta}{2\widehat{C}},\ \epsilon \leqslant \frac{\delta}{2\widehat{C}}$$

where $\widehat{C}$ is the same constant as in Lemma 5.3.6. Then

$$r\int_{C_p(\frac{R}{\sqrt{r}},\epsilon)} u_r > C'$$

where C' is a constant depending only on the metric, on δ, and on $\widehat{C}$ (which in turn depend just on the upper bounds for $r^{-\frac{1}{2}}|(\nabla - iA_r)\psi_{1r}|$ and $|(\nabla - iA_r)\psi_{2r}|$ in Proposition 5.3.1).

Proof of Lemma 5.3.8. The proof is a straightforward consequence of Lemma 5.3.6: the above choice of R and ϵ ensures that

$$u|_{C_p(\frac{R}{\sqrt{r}},\epsilon)} > \frac{\delta}{2},$$

therefore,

$$r\int_{C_p(\frac{R}{\sqrt{r}},\epsilon)} u_r \geqslant c\delta R^2\epsilon$$

□

The following Proposition is the last ingredient that will enable us to detect periodic orbits of X:

Proposition 5.3.9. *Let p be a point in $\mathbb{S}^3$ with $u_r(p) > \delta$ as $r \to \infty$ for some $\delta > 0$ independent of r. Let $C_p(\rho,\epsilon)$ be a flow box around the point p, with ρ and ϵ small enough. There are two large enough constants R_1 and R_2 (possibly depending on the point, and with $R_1 < R_2$) such that, for any given $\varepsilon > 0$ as small as desired, and any given $R' > R_2$ as large as desired, we have*

(i)

$$\sup_{C_p(\frac{R'}{\sqrt{r}},\frac{\varepsilon}{\hat{C}})} u_r < 2\varepsilon$$

(ii)

$$\sup_{C_p(\frac{R_1}{\sqrt{r}},\epsilon)} u_r > 1-\varepsilon$$

provided r is large enough.

Remark 5.3.10. *In fact, as the proof will make manifest, the supremum in item (ii) is achieved at some point q which is in the same slice of the flow fox as the given point p, i.e, with flow-chart coordinates $\Psi_p(q) = (z,0)$.*

We will prove this Proposition at the end of this subsection.

From Proposition 5.3.9 together with Lemma 5.3.6 we readily get the following key corollary:

Corollary 5.3.11. *Let p be a point in $\mathbb{S}^3$ with $u_r(p) > \delta$ as $r \to \infty$ for some $\frac{1}{2} > \delta > 0$ independent of r. Fix $\varepsilon > 0$ such that $\frac{1-\varepsilon}{2} > \delta$. There is a point q_r, with dist $(p,q_r) \leqslant 2R_1\frac{1}{\sqrt{r}}$, such that*

$$u(\phi_X^t(q_r)) > \frac{1-\varepsilon}{2} > \delta$$

for all t such that $|t| \leqslant \frac{1-\varepsilon}{2\hat{C}}$.

With these ingredients stablished, we are ready to show that the vector field X has a periodic orbit. More precisely, the following construction finds a measure λ_r whose limit λ_∞ is an invariant measure of the field, supported on a periodic orbit.

To start with the construction, fix a point p_1 with $u(p_1) > \delta$ for some δ. We agree on δ being less than 1/2 throughout. Consider a flow box $C_{p_1}(\frac{\rho}{\sqrt{r}}, \tau)$, with $\rho := \frac{\delta}{2\widehat{C}}$ and $\tau := \frac{\delta}{2\widehat{C}}$.

Define λ_{1r} to be the measure given (in flow-chart coordinates in $\mathbb{D}_{\frac{\rho}{\sqrt{r}}} \times [-\tau, \tau]$) as $\lambda_{1r} := \delta(z) \otimes dt$. This completes the first step. We note in passing that, because of Lemma 5.3.8, we have

$$r \int_{C_{p_1}(\frac{\rho}{\sqrt{r}}, \tau)} u_r \geqslant c$$

with c depending only on δ and $\widehat{C}$. The main idea now is to repeat this construction at different points p_{kr}, in such a way that the union of the flow boxes end up converging to a periodic orbit of X, and the sum of the measures λ_{kr} converges to a measure supported on the orbit.

For the second step, we first find a point p_{2r}, with $u_r(p_{2r}) > \delta$, and which is, roughly speaking, at a fixed distance 2τ from p_1 in the direction of the flow, and at a distance of order $O(r^{-1})$ in the transverse directions. This we can do thanks to Corollary 5.3.11.

Indeed, choose (with the notations of Corollary 5.3.11 with $p = p_1$) a point $p_{2r} := \phi_X^t(q_r)$ with $|t| \leqslant \frac{1-\varepsilon}{2\widehat{C}}$. As is estated in the corollary, by choosing ε small enough, we can rest assured that the point p_{2r} has $u_r(p_{2r}) > \frac{1-\varepsilon}{2} > \delta$. Furthermore, since by construction $2\tau = \frac{\delta}{\widehat{C}} \leqslant \frac{1-\varepsilon}{2\widehat{C}}$, we can set $t := 2\tau$.

We now define a flow-box $C_{p_{2r}}(\frac{\rho}{\sqrt{r}}, \tau)$ at p_{2r}. We again have the lower bound

$$r \int_{C_{p_{2r}}(\frac{\rho}{\sqrt{r}}, \tau)} u_r \geqslant c$$

and moreover, (note that the closures of the two flow boxes intersect, at most, on a disk)

$$r \int_{C_p(\frac{\rho}{\sqrt{r}}, \tau) \cup C_{p_{2r}}(\frac{\rho}{\sqrt{r}}, \tau)} u_r \geqslant 2c \tag{5.3.1}$$

As we did in the first step, we define a measure $\lambda_{2r} := \delta(z) \otimes dt$, supported now in the core of the flow box $C_{p_{r2}}$.

Again, thanks to Corollary 5.3.11 we find a point p_{3r} with the same relation to p_{2r} as the one that p_{2r} has with p_1, and we repeat the construction. This scheme is iterated until the flow box $C_{p_{Nr}}(\frac{\rho}{\sqrt{r}}, \tau)$, centered at some point p_{Nr}, intersects (in a set of non-zero volume) the original flow box $C_{p_1}(\frac{\rho}{\sqrt{r}}, \tau)$ centered at p_1. That this must happen after a finite number of steps N is a consequence of the boundedness of $\mathcal{H}_{A_r}(X)$. Indeed, if after N steps, the flow box $C_{p_{Nr}}(\frac{\rho}{\sqrt{r}}, \tau)$ does not intersect the flow box we started with, we have, by virtue of Eq. 5.3.1

$$Nc \leqslant r \int_{C_p(\frac{\rho}{\sqrt{r}},\tau)\cup\ldots\cup C_{p_{Nr}}(\frac{\rho}{\sqrt{r}},\tau)} u_r$$

but, on the other hand,

$$r \int_{C_p(\frac{\rho}{\sqrt{r}},\tau)\cup\ldots\cup C_{p_{Nr}}(\frac{\rho}{\sqrt{r}},\tau)} u_r \leqslant r \int_{S^3} |u| \leqslant \mathcal{H}_{A_r}(X) + c'$$

so the maximum number of steps N must always stay smaller than some constant. (Note that, as long as r is big enough, the flow-boxes are as thin as we want, so we can always discard the possibility that the process ends with a flow box around a point p_{Nr} intersecting the flow box centered at a point p_{kr} distinct from p_1.)

We thus get, at the end of the process some measure

$$\lambda_r := \sum_{i=1}^{N} \lambda_{ir}$$

which by construction is supported in N arcs of orbits of the vector field X. This measure is clearly bounded from above (by the total length of the arcs, $N\tau$), so it converges to some measure λ_∞.

Furthermore, it is easy to see (since the supporting arcs of λ_r are within a distance $O(\frac{1}{\sqrt{r}})$ from each other), that λ_∞ is invariant under the flow of X: it is thus supported on a closed orbit of X. This completes the proof of item (i) in 5.2.1.

Proof of Proposition 5.3.9. Let $\Psi_p : C_p(\rho, \epsilon) \to (-\epsilon, \epsilon) \times \mathbb{D}_\rho$ be an adapted flow-box chart at p, as in Definition 5.3.5. It is convenient at this point to define the rescaled coordinates $(t', z') = (\sqrt{r}t, \sqrt{r}z)$, which now take values in the stretched cylinder $\mathcal{C}_{\sqrt{r}} := (-\epsilon\sqrt{r}, \epsilon\sqrt{r}) \times \mathbb{D}_{\sqrt{r}\rho}$. The variables in rescaled coordinates will be denoted by their namesakes with a tilde, e.g, $\widetilde{\psi}(t', z') := \psi(\frac{t'}{\sqrt{r}}, \frac{z'}{\sqrt{r}})$; with the exception of the vector field A, which gets an extra rescaling factor:

$$\widetilde{A}(t', z') := \frac{1}{\sqrt{r}} A(\frac{t'}{\sqrt{r}}, \frac{z'}{\sqrt{r}})$$

(note that this rescaling factor is consistent with A being considered a connection: the covariant derivative $\nabla - iA$ gets this way homogeneously rescaled when rescaling the coordinates).

Let us first prove the bound (i) in Proposition 5.3.9. This amounts to proving that $\widetilde{u}_r$ must be ever smaller outside disks of ever bigger radius R.

Suppose it were not the case. Then, we could find an increasing sequence of points z'_k, separated from each other by a distance at least some fixed R, and such that $\widetilde{u}(0, z'_k) = u_r(0, \frac{z'_k}{\sqrt{r}}) > \delta'$ for some $\delta' > 0$. But then, around each of these points, we could repeat the argument in the proof of Lemma 5.3.8 to find that each point contributes to the energy by at least some constant c depending only on δ', R, and $\widehat{C}$. But this would imply that

$$r \int_{\mathbb{S}^3} |u|$$

is not bounded. However, by Lemma 5.3.3, the boundedness of $\mathcal{H}_{A_r}(X)$ implies that the above quantity is also bounded, so we get a contradiction.

Thus we find, for any ε as small as we want, a radius R_2 with $\widetilde{u}_r(z', 0) \leqslant \varepsilon$ for any $|z'| \geqslant R_2$. This together with Lemma 5.3.6 implies the first bound in the Proposition.

The second bound is subtler. It rests on the properties of the vortex equations, a system of PDEs which captures very well the behavior of solutions to the $SW(r, v)$-equations at small scales. More precisely, sequences of solutions of the $SW(r, v)$-equations with increasing r converge, in the rescaled coordinates, to solutions of the *self-dual vortex equations* on slices $t \times \mathbb{D}_{\sqrt{r}}$ of constant t:

Definition 5.3.12 (Self-dual vortex equations)**.** *The self dual vortex on $\mathbb{C}$ are the system of equations*

$$\partial_x a_y - \partial_y a_x = (1 - |\phi|^2)$$
$$\overline{\partial}_a \phi = \partial_z \phi - (a_x - i a_y)\phi = 0$$

where $\overline{\partial_z} := (\partial_x + i\partial_y)$ is the Cauchy-Riemann operator, and the unknowns are a real one form $a = a_x dx + a_y dy$ (that we will also sometimes identify with its dual vector field through the euclidean metric) and a complex valued function $\phi = \phi_1 + i\phi_2$, the "Higgs field". As is the case with the Seiberg-Witten equations, these equations are invariant under $U(1)$-gauge transformations: if (a, ϕ) is a solution to the equations, then $g(a, \phi) = (a - g^{-1}dg, g\phi)$ is also a solution, for any smooth function $g : \mathbb{C} \to \mathbb{C}$ with $|g| = 1$.

Lemma 5.3.13. *Let (A_r, ψ_r) be a family of solutions to the SW(r, v)-equations. Given a point $p \in \mathbb{S}^3$ and a sequence of values of r going to infinity, and given a compact set $[-T, T] \times \mathbb{D}_R \subset \mathbb{R} \times \mathbb{C}$, there always exists a subsequence r_i such that the rescaled fields*

$$\widetilde{A}_i(t', z') := \frac{1}{\sqrt{r_i}} (\Psi_{p_i})_* A_{r_i} \Big(\frac{t'}{\sqrt{r_i}}, \frac{z'}{\sqrt{r_i}} \Big),\ \widetilde{\psi}_{1i}(t', z') := \psi_{1r_i} \circ \Psi_{p_i}^{-1} \Big(\frac{t'}{\sqrt{r_i}}, \frac{z'}{\sqrt{r_i}} \Big)$$

converge in $[-T, T] \times \mathbb{D}_R$ in the C^m norm to a smooth family of solutions $(a_{t'}(z'), \phi_{t'}(z'))$ of the vortex equations on $t' \times \mathbb{C}$, with $t' \in [-T, T]$. Furthermore, all members of this family are gauge equivalent, that is, there is a smooth function $g : [-T, T] \times \mathbb{D}_R \to \mathbb{C}$ with $|g| = 1$ and a solution to the vortex equation $(a(z'), \phi(z'))$ such that $g(t', \cdot)_*(a(t', \cdot), \phi(t', \cdot)) = (a(\cdot), \phi(\cdot))$.

The proof of this Lemma is given in section 5.7, Subsection 5.7.1.

The above Lemma ensures that, for any choice of $R > 0$, $T > 0$ and $\varepsilon' > 0$, we have, for r large enough,

$$\sup_{(z', t') \in \mathbb{D}_R \times [-T, T]} |\widetilde{u}_r - (1 - |\phi|^2|)| < \varepsilon' \qquad (5.3.1)$$

$$\sup_{(z',t')\in\mathbb{D}_R\times[-T,T]} |(\nabla - i\widetilde{A}_r)\widetilde{\psi}_r - (\nabla - ia)\phi| < \varepsilon' \tag{5.3.2}$$

with (ϕ, a) being solution to the self-dual vortex equations on $\mathbb{C}$, understood here as a solution to the equations on $\mathbb{C}\times\mathbb{R}$ invariant in the $\mathbb{R}$-direction. Furthermore, Proposition 5.3.1 implies that $|\phi| \leqslant 1$ everywhere.

Equation (5.3.1) together with the first bound in Proposition 5.3.9 (which we already proved) implies that the solution of the vortex equations we are converging to satisfies $(1 - |\phi(z')|^2) \to 0$ as $|z'| \to \infty$.

This in turn yields the desired proof of bound (ii) in Proposition 5.3.9, via the following lemma:

Lemma 5.3.14. *Let (a, ϕ) be a solution to the vortex equations. Then*

(i) The set $|\phi|^{-1}(0)$ is either empty or consists on a set of isolated points.

(ii) If $|\phi| \leqslant 1$, then either $|\phi| = 1$ everywhere or $|\phi| < 1$. Further, if $|\phi| < 1$, any local minima of $|\phi|$ has $|\phi| = 0$.

This lemma is proved in Section 5.7, Subsection 5.7.2.

For the proof of (ii) in Proposition 5.3.9, note that, since $u_r(p) = \widetilde{u}_r(0,0) > \delta$, the limiting vortex solution ϕ cannot be identically 1, by item (ii) in Lemma 5.3.14, so we have that $|\phi| < 1$ everywhere. Further, since $|\phi(z')|^2 \to 1$ as $|z'| \to \infty$, item (i) in Lemma 5.3.14 implies that the function $|\phi|$ has a finite number of zeroes.

Hence, there is at least one point $z'_* \in \mathbb{D}_{R_1}$, for some $R_1 < R_2$, with $|\phi(z'_*)| = 0$. Now for r large enough, (5.3.1) ensures that $\widetilde{u}_r(z'_*, 0) < 1 - \varepsilon$, so item (ii) of Proposition 5.3.9 follows.

□

5.4 Proof of Theorem 5.2.2

We start with the following

Proposition 5.4.1. *Let (X, μ, g) be a nowhere-vanishing vector field on $\mathbb{S}^3$ preserving a volume form μ and with adapted metric g; and such that $\operatorname{curl}^{-1} X = X$. There always exists a smooth volume preserving vector field v, of arbitrarily small C^m norm, for which there is a sequence $\{r_n,\ \psi_{r_n} := (\psi_{1r_n}, \psi_{2r_n}),\ A_{r_n}\}$ of solutions to the associated $SW(r, v)$-equations such that*

(i) sup $(1 - |\psi_{r_n}|^2) > \delta$, for all r_n and some $\delta > 0$,

(ii) The sequence $\mathcal{H}_{A_{r_n}}(X)$ is bounded

Subsection 5.6.2 in Section 5.6 is devoted to proving the Proposition above.

With this Proposition granted, the proof of theorem 5.2.2 is just a consequence of the following Lemma:

Lemma 5.4.2. *Let (X, μ, g) be a nowhere-vanishing vector field on $\mathbb{S}^3$ preserving a volume form μ and with adapted metric g; and such that* $\mathrm{curl}^{-1} X = hX$*, with h a positive function in $\mathbb{S}^3$. Then, the vector field $\widetilde{X} := hX$ preserves the volume form $\tilde{\mu} = \frac{1}{h}\mu$ and, with respect to the adapted metric $\tilde{g} := \frac{1}{h^2}g$, satisfies* $\mathrm{curl}_{\tilde{g}}^{-1} \widetilde{X} = \widetilde{X}$.

Proof. It is easy to see that $\tilde{\mu}$ and $\tilde{g}$ are an invariant volume form and an adapted metric of the vector field $\widetilde{X}$. As for the fact that $\mathrm{curl}_{\tilde{g}}^{-1} \widetilde{X} = \widetilde{X}$, note that curl^{-1} can be defined as

$$i_X \mu = d i_{\mathrm{curl}^{-1} X} g$$

and hence it is straightforward to see that

$$i_{\widetilde{X}} \tilde{\mu} = i_X \mu = d i_{\mathrm{curl}^{-1} X} g = d i_{h^{-2}\widetilde{X}} g = d i_{\widetilde{X}} \tilde{g}$$

□

Now, granted Lemma 5.4.2 and Proposition 5.4.1, Theorem 5.2.2 is just a corollary of Theorem 5.2.1.

5.5 A priori estimates: Proof of Proposition 5.3.1 and of Lemma 5.6.5

In this section we prove the a priori bounds in Proposition 5.3.1, which are needed as input for the proof of Theorem 5.2.1 in Section 5.3; and we further establish a corollary of those bounds, Lemma 5.6.5, that is needed as input in the proof of Proposition 5.6.4 in Section 5.6.

We recall that, for any solution (A_r, ψ_r) of the $SW(r, v)$-equations (with v a fixed C^∞ divergence free vector field), these bounds read:

$$||\psi_{1r}||^2_{C^0(\mathbb{S}^3)} \leqslant 1 + \frac{c_0}{r}\,; \quad ||\psi_{2r}||^2_{C^0(\mathbb{S}^3)} \leqslant \frac{c_0(1 - |\psi_{1r}|^2)}{r} + \frac{c_0}{r^2} \tag{5.5.1}$$

$$||(\nabla - iA_r)\psi_{1r}||_{C^m(\mathbb{S}^3)} \leqslant c_m r^{\frac{m+1}{2}} \tag{5.5.2}$$

$$||(\nabla - iA_r)\psi_{2r}||_{C^m(\mathbb{S}^3)} \leqslant c_m r^{\frac{m}{2}}. \tag{5.5.3}$$

with all the constants depending only on the metric and v. The corollary needed in Section 5.6 asserts that the following inequality holds

$$|\mathcal{H}(\mathrm{curl}\, A_r)| \leqslant C r^{\frac{2}{3}} (|\mathcal{H}_{A_r}(X)|^{\frac{4}{3}} + C). \tag{5.5.4}$$

for some constant C independent of r, and where $\mathcal{H}(\text{curl}\, A_r)$ is the helicity of the vector field curl A_r.

5.5.1 Proof of the bounds (5.5.1)–(5.5.3)

The proof of the bounds above is just an application of the Weitzenbőck formula, the maximum principle and standard elliptic estimates.

The Weitzenbőck formula states that the square of the Dirac operator D_A differs from the ordinary twisted Laplacian associated to ∇_A by a zero order multiplication operator, which depends on the scalar curvatures of the manifold and the curvature of the connection A (i.e the curl of the vector field A). More precisely, we have

$$D_A^2 = \nabla_A^* \nabla_A + \frac{s}{4} - \sigma \cdot \text{curl}\, A \tag{5.5.5}$$

where $\nabla_A^* = \text{div} + iA$ is the L^2-adjoint of the covariant derivative $\nabla - iA$, s stands for the scalar curvature of the metric g, and by $\sigma \cdot W$ we denote the Pauli matrix in the direction of the vector field W

$$\sigma \cdot W = \sum_k \sigma_k W_k$$

We will denote the operator $\nabla_A^* \nabla_A$ by $-\Delta_A$. In explicit form, this operator reads

$$-\Delta_A = -\Delta + 2iA \cdot \nabla + \text{div}A + |A|^2.$$

Notice that for us the laplacian is a negative operator, i.e, we set $\Delta := \text{div}\, \nabla = \sum_k e_k \cdot \nabla(e_k \cdot \nabla)$ (in terms of the orthonormal basis e_k).

If (A_r, ψ_r) satisfies the $SW(r, v)$-equations, the Weitzenbők formula becomes

$$-\Delta_{A_r} \psi + \frac{s}{4} \psi_r - r\sigma_1 \psi_r + r|\psi_r|^2 \psi_r - \sigma \cdot v\psi_r = 0 \tag{5.5.6}$$

where we have used that

$$\sigma \cdot \psi^\dagger \sigma \psi = |\psi|^2 \psi.$$

Multiplying Eq. (5.5.6) by $\psi_r^\dagger$ and taking the real part of the resulting expression, it is easy to see that we get

$$-\frac{1}{2}\Delta|\psi_r|^2 + |\nabla\psi_r - iA_r\psi_r|^2 + \frac{s}{4}|\psi_r|^2 - r(|\psi_{1r}|^2 - |\psi_{2r}|^2 + |\psi_r|^4) - \text{Re}(\psi_r^\dagger \sigma \cdot v\psi_r) = 0$$

where we have used that

$$\text{Re}(\psi^\dagger \Delta\psi) = \frac{1}{2}\Delta|\psi|^2 + |\nabla\psi|^2$$

and

$$\text{Re}(2\psi^\dagger iA \cdot \nabla\psi) + |\psi|^2 \text{div}A + |A|^2|\psi|^2 = |\nabla\psi - iA\psi|^2 - |\nabla\psi|^2.$$

Discarding the positive terms and taking into account that

$$\mathrm{Re}(\psi^\dagger \sigma \cdot v\psi) \leqslant |v||\psi|^2 ,$$

and that $|\psi_{1r}|^2 - |\psi_{2r}|^2 \leqslant |\psi_r|^2$, we readily have the inequality

$$-\frac{1}{2}\Delta|\psi_r|^2 + r|\psi_r|^2\Big(\frac{s' - |v|}{r} - 1 + |\psi_r|^2\Big) \leqslant 0 \tag{5.5.7}$$

where $s' = s/4$.

By the maximum principle, at the point where $|\psi_r|^2$ achieves its maximum value, we have that $\Delta|\psi_r|^2 \leqslant 0$, and hence

$$\Big(\frac{s' - |v|}{r} - 1 + |\psi_r|^2\Big) \leqslant 0 .$$

Therefore, for any solution, at any point of the manifold, we have

$$|\psi_r|^2 \leqslant 1 + \frac{c_0}{r}$$

with $c_0 := \sup v - \min s'$. The bound for ψ_{1r} in Eq. (5.5.1) follows.

The bound for ψ_{2r} is derived following an analogous (though more calculationally involved) route; this time departing from the component-wise versions of Eq. (5.5.6), multiplying each component separately with the corresponding component of $\psi^\dagger$, and (taking into account the bound already obtained for ψ_{1r}) applying the maximum principle to an appropriate linear combination of both equations. We refer to [66] for the details.

In order to derive the bounds for the derivatives, Eqs. (5.5.2) and (5.5.3), it is convenient to work in the Coulomb gauge, so that the Seiberg-Witten system becomes elliptic. More precisely, the idea is to consider the $SW(r,v)$-equations as a PDE on the space of *divergence free vector fields*, that is, to impose that $\mathrm{div}A = 0$. The convenience comes from the fact that, in this case, the curl operator has elliptic symbol (its square is minus the Laplace-Beltrami operator) and trivial kernel (note that in $\mathbb{S}^3$ there are no harmonic vector fields, because the first cohomology group is trivial).

There is no problem in imposing the extra equation $\mathrm{div}A = 0$; indeed, any solution to the $SW(r,v)$-equations is gauge-equivalent to a solution with $\mathrm{div}A = 0$. More precisely, given any (A', ψ') solving the equations, finding a gauge equivalent solution $(A,\psi) = (A' + \nabla f, e^{if}\psi')$ with divergence free A amounts to finding f such that $\Delta f = -\mathrm{div}A'$. Since there is always exactly one f solving $\Delta f = -\mathrm{div}A'$ up to constants (note that $\int \mathrm{div}A' = 0$), there is exactly one divergence free A in each gauge equivalent class of solutions.

We will work now on local coordinates in the neighborhood of some point p, rescaling lengths by an $\sqrt{r}$ factor. For this purpose, let $\Psi : \mathbb{B}(p,\delta) \to B(0,\delta) \subset \mathbb{R}^3$ be a patch of normal geodesic coordinates centered at a point p. We will drop the point subscript and just write B_δ for the Euclidean ball of radius δ centered at the origin. For the ball of unit radius we will just write B.

We set

$$\widetilde{\psi}_r(x) := \psi_r \circ \Psi^{-1}\Big(\frac{x}{\sqrt{r}}\Big)$$
$$\widetilde{A}_r(x) := \frac{1}{\sqrt{r}}(\Psi_* A_r)\Big(\frac{x}{\sqrt{r}}\Big)$$
$$\tilde{v}(x) := (\Psi)_* v(\frac{x}{\sqrt{r}})$$

where the coordinates x now take values in the rescaled ball $B_{\sqrt{r}\delta)}$. Note that in these rescaled coordinates, if $x \in B_R$ for fixed R (i.e not depending on r) the metric $\tilde{g} := \Psi_* g$ is euclidean up to terms of order $O(r^{-1})$. We will use the convention of indexing the euclidean coordinates by greek indices $\mu = 1, ..., 3$, and we construct the patch so that $(\Psi_p)_* e_k(0) = \delta_{\mu k}\partial_\mu$, where $\delta_{\mu k}$ is the Kroenecker delta. Hence, we have that

$$(\Psi)_* e_k(\frac{x}{\sqrt{r}}) = \delta_{\mu k}\partial_\mu + O(1/\sqrt{r})$$

on compact sets (e.g in B_R), so that the e_k and ∂_μ (with $k = \mu$) components of any vector field coincide up to terms of the form $\frac{G(x)}{\sqrt{r}}$, with G uniformly bounded in all derivatives.

In these coordinates, the equations read

$$\operatorname{curl}_{\tilde{g}} \widetilde{A}_r = \Big(\partial_1 + \frac{G}{\sqrt{r}} - \widetilde{\psi}_r^\dagger \sigma \widetilde{\psi}_r\Big) + \frac{\tilde{v}}{r} \tag{5.5.8}$$

$$i \sum_\mu \sigma_\mu(x)\partial_\mu \widetilde{\psi}_r = -\sum_\mu \sigma_\mu(x)\widetilde{A}_{\mu r}\widetilde{\psi}_r \tag{5.5.9}$$

$$\operatorname{div}_{\tilde{g}} \widetilde{A}_r = 0 \tag{5.5.10}$$

where $\operatorname{curl}_{\tilde{g}}$ and $\operatorname{div}_{\tilde{g}}$ are the curl and divergence operators defined by the rescaled metric (notice that they are a very small perturbation of their euclidean counterparts), and where $\sigma_\mu(x) = \delta_{\mu i}\sigma_i + \frac{1}{\sqrt{r}}\sum_j G_j(x)\sigma_j$. Finally, we have that both G and the G_j are uniformly bounded in compact sets

$$||G_j||_{C^m(B_R)} \leqslant c_m.$$

The bound (5.5.2) follows from standard elliptic regularity estimates starting from the fact that, by virtue of Eq. (5.5.1), $|\widetilde{\psi}|$ is bounded independently of r.

In order to see this, we first perform a (local) gauge transformation: we define $\bar{A} = \widetilde{A}_r - \nabla_{\tilde{g}} f$ with f such that, on the unit ball B

$$\Delta_{\tilde{g}} f = 0\,, \;\; x \cdot \nabla_{\tilde{g}} f = x \cdot \widetilde{A}_r \text{ on } \partial B\,,$$

and outside B we extend f smoothly (note that both f and $\bar{A}$ depend on r, but we drop the subscript). This ensures that $\bar{A}$ is divergence-free and tangent to the boundary of the unit ball B. When such is the case, it is standard that we have the following inequality (see e.g the main Theorem in [21])

$$||\bar{A}||_{W^{k+1,p}(B)} \leqslant C(g,k)||\operatorname{curl} \bar{A}||_{W^{k,p}(B)}$$

Furthermore, the pair $(\bar{A}, u\widetilde{\psi})$, with $u := e^{-if}$, is a solution to (5.5.8)–(5.5.10), so we get the inequalities

$$||\bar{A}||_{W^{k+1,p}(B)} \leqslant C||\,|\widetilde{\psi}|^2||_{W^{k,p}(B)} + \frac{1}{r}||\tilde{v}||_{W_{k,p}(B)} \tag{5.5.11}$$

and

$$||u\widetilde{\psi}||_{W^{k+1,p}(B)} \leqslant \left\| \sum_\mu \sigma_\mu(x)\partial_\mu(u\widetilde{\psi}) \right\|_{W^{k,p}(B)} + ||\widetilde{\psi}||_{L^p(B)} \leqslant C||\bar{A}u\widetilde{\psi}||_{W^{k,p}(B)} + ||\widetilde{\psi}||_{L^p(B)}\,. \tag{5.5.12}$$

with constants C depending on the metric g and k, p. Now, starting from the pointwise bound on $\widetilde{\psi}$, the standard elliptic bootstrapping argument applied to Eqs. (5.5.11) and (5.5.12) yields that both $\bar{A}$ and $u\widetilde{\psi}$ are bounded in $W^{k,p}(B)$ for all $k \geqslant 0, p > 1$. Thus, from Sobolev embedding, one readily gets

$$||\bar{A}||_{C^m(B)} \leqslant C'_m \tag{5.5.13}$$

$$||u\widetilde{\psi}||_{C^m(B)} \leqslant C'_m \tag{5.5.14}$$

where the constants depend on v and g, but not on r. To get the bounds in the statement of the Lemma, note that (5.5.14) implies that

$$||\nabla(u\widetilde{\psi})||_{C^m(B)} \leqslant C'_{m+1}$$

but on the other hand we have that

$$|\nabla(u\widetilde{\psi})| = |\nabla\widetilde{\psi} - i(\nabla f)\widetilde{\psi}| = |\nabla\widetilde{\psi} - i\widetilde{A}_r + i\bar{A}\widetilde{\psi}|$$

and hence, in view of the bound (5.5.13), we deduce

$$||\nabla\widetilde{\psi}_r - i\widetilde{A}_r||_{C^m(B)} \leqslant C_m.$$

Rescaling back, we get the estimates (recall the extra $\sqrt{r}$ rescaling of $\widetilde{A}_r$)

$$||\nabla\psi_r - iA||_{C^m(\mathbb{B}_{\frac{1}{\sqrt{r}}})} \leqslant C_m r^{\frac{m+1}{2}}$$

from which estimates (5.5.2) follow by compactness of $\mathbb{S}^3$.

Finally, to derive the bound (5.5.3) on the derivatives of ψ_{2r}, we use a similar argument departing from the second component of Eq. (5.5.6), which, under rescaled coordinates, reads

$$-\Delta_{\tilde{g}}\widetilde{\psi}_{2r} + 2i\widetilde{A}_r \cdot \nabla_{\tilde{g}}\widetilde{\psi}_{2r} + (1 + \frac{f_1}{r})\widetilde{\psi}_{2r} = -\frac{f_2}{r}\widetilde{\psi}_{1r} - \frac{|\psi|^2}{r}\widetilde{\psi}_{2r}$$

where f_1 is a function depending linearly on the scalar curvature and the components of v, and f_2 just on the components of v.

5.5.2 Proof of the Lemma 5.6.5

Our objective in this subsection is to prove the Lemma 5.6.5, which establishes the bound

$$|\mathcal{H}(\operatorname{curl} A_r)| \leqslant Cr^{\frac{2}{3}}(|\mathcal{H}_{A_r}(X)|^{\frac{4}{3}} + C)\,.$$

We recall that

$$\mathcal{H}(\operatorname{curl} A_r) = \int \operatorname{curl} A_r \cdot A_r\,, \quad \mathcal{H}_{A_r}(X) = \int \operatorname{curl} A_r \cdot X$$

The above bound is a crucial ingredient in the proof of Propostion 5.3.2 in Subsection 5.6.1.

To this end, first note that the bounds (5.5.1) imply that

$$|\operatorname{curl} A_r| \leqslant r|1 - |\psi_{1r}|^2| + G \tag{5.5.15}$$

where G is a function pointwise bounded as

$$|G| \leqslant C(g, v).$$

and depending on terms of the form $\psi_{1r}\psi_{2r}$ and on v.

Indeed, this follows upon noticing that

$$|X - \psi_r^\dagger \sigma \psi_r|^2 \leqslant |1 - |\psi_{1r}|^2|^2 + \frac{|\psi_{1r}||1 - |\psi_{1r}||^2}{r}$$

and hence

$$|X - \psi_r^\dagger \sigma \psi_r| \leqslant |(1 - |\psi_{1r}|^2)| + \frac{|\psi_{1r}|^2}{2r}.$$

Furthermore, we also deduce (recall that the negative part of $(1 - |\psi_r|^2)$ is of order $O(r^{-1})$) :

$$\int |\operatorname{curl} A_r| \leqslant c + r\int |1 - |\psi_{1r}|^2| \leqslant c' + r\int (1 - |\psi_{1r}|^2) \leqslant |\mathcal{H}_{A_r}(X)| + C \tag{5.5.16}$$

where the constants are positive and depend again only on g and v.

Thus, we can write that

$$|\mathcal{H}(\operatorname{curl} A_r)| \leqslant ||A_r||_{C^0(\mathbb{S}^3)} \int |\operatorname{curl} A_r| \leqslant ||A_r||_{C^0(\mathbb{S}^3)}(|\mathcal{H}_{A_r}(X)| + C)$$

The following lemma yields thus the desired bound (5.5.4):

Lemma 5.5.1. *In the Coulomb gauge, $divA = 0$, the following inequality holds*

$$||A_r||_{C^0(\mathbb{S}^3)} \leqslant Cr^{\frac{2}{3}}(|\mathcal{H}_{A_r}(X)|^{\frac{1}{3}} + C)$$

Proof. The proof follows easily from the remarks above. First, notice that the absence of harmonic vector fields in the sphere implies that, on the space of divergence free fields, the curl operator has a well defined inverse (if there were harmonic vector fields, we will further need to ensure that we are working with divergence free fields whose harmonic part is zero). Thus, by fixing the gauge to Coulomb, we can write

$$A(x) = \int B(x, y) \cdot \operatorname{curl} A(y)\, d\,\mu(y)\,,$$

where $B(x,y)$ is a matrix valued kernel (the Green function of the curl). This kernel is the analog of the Biot-Savart operator (see e.g [21, 72]) in $\mathbb{R}^3$. In particular, it has a singularity of the form $B(x,y) \sim (\mathrm{dist}(x,y))^{-2}$.

We can decompose the above integral as:

$$|A(x)| \leqslant c \int_{B(x,\rho)} \frac{|\operatorname{curl} A|}{\mathrm{dist}(x,y))^2} + \int_{M\setminus B(x,\rho)} \frac{|\operatorname{curl} A|}{\mathrm{dist}(x,y)^2}.$$

The first integral has an integrable singularity in its kernel, and can be bounded by

$$\int_{B(x,\rho)} \frac{|\operatorname{curl} A|}{\mathrm{dist}(x,y)^2} \leqslant Cr\rho$$

where the constant C depends on the metric and the C^0 norm of v. The second integral is readily seen by virtue of (5.5.16) to be bounded as

$$\int_{M\setminus B(x,\epsilon)} \frac{|\operatorname{curl} A|}{\mathrm{dist}(x,y))^2} \leqslant C\rho^{-2}(|\mathcal{H}_{A_r}(X)| + C) + C.$$

Now choosing $\rho^3 := r^{-1}|\mathcal{H}_{A_r}(X) + C|$ we finally get

$$|A(x)| \leqslant Cr^{\frac{2}{3}}(|\mathcal{H}_A(X)|^{\frac{1}{3}} + C),$$

which is the desired bound. □

5.6 Existence of solutions: Proof of Propositions 5.3.2 and 5.4.1

The existence of solutions meeting our goals (i.e, the construction of non-trivial invariant measures) rests on the following Proposition. Before stating it, let us define

Remark 5.6.1. *Let $\mathfrak{X}^\infty(\mathbb{S}^3)$ be the space of smooth vector fields in $\mathbb{S}^3$. Set $\mathfrak{M} := \mathfrak{X}^\infty(\mathbb{S}^3)\times C^\infty(\mathbb{S}^3,\mathbb{C}^2)$. Solutions to the $SW(r,v)$-equations are critical points of the functional $\mathcal{S}_r : \mathfrak{M} \to \mathbb{R}$ defined as*

$$\mathcal{S}_r(A,\,\psi) := \frac{1}{2}\mathcal{H}(\operatorname{curl} A) - r\mathcal{H}_A(\operatorname{curl}^{-1}(X)) - \int_{\mathbb{S}^3} \operatorname{curl}^{-1} v \cdot \operatorname{curl} A + ir \int_{\mathbb{S}^3} \psi^\dagger \sigma \cdot (\nabla - iA)\psi. \tag{5.6.1}$$

Note that this functional is invariant under gauge transformations $(A,\psi) \to (A + \nabla f,\, e^{if}\psi)$

Proposition 5.6.2. *Let X be a nowhere vanishing volume preserving vector field in $\mathbb{S}^3$ with $\mathcal{H}(X) > 0$. There is a real number $\delta > 0$ and an unbounded from below set of integers $\Lambda \subset \mathbb{Z}$ such that, for each fixed $\lambda \in \Lambda$, we have:*

(i) For any chosen $\epsilon > 0$, $m \geqslant 0$, there is a smooth divergence-free vector field v with $||v||_{C^m(\mathbb{S}^3)} \leqslant \epsilon$ such that the $SW(r,v)$-equations have a non trivial

solution for all $r \in \bigcup_{k=1}^{\infty} (\rho_k, \rho_{k+1})$*, where* $\{\rho_k\}_1^{\infty} \subset [1, \infty)$ *is a set of positive real numbers, with no accumulation points, and depending on all the previous data. These solutions are said to have* degree λ.

(ii) On each interval (ρ_k, ρ_{k+1})*, the family of solutions* $(A(r), \psi(r))$ *varies smoothly with r.*

(iii) The solutions (A_r, ψ_r) *of degree* $\lambda \neq 0$ *satisfy that*

$$sup\ (1 - |\psi_r|) > \delta$$

for the constant $\delta > 0$.

(iv) The solutions (A_r, ψ_r) *of any degree* λ *satisfy*

$$|\lambda - \mathcal{H}(\mathrm{curl}\, A)| \leqslant r^{2-\delta^*}$$

for a constant δ^* *with* $\delta^* > \frac{1}{16}$.

(v) There is a continuous map $\widehat{\mathcal{S}} : [\rho_1, \infty) \to \mathbb{R}$ *whose restriction to each interval* (ρ_k, ρ_{k+1})*, is equal to* $\mathcal{S}_r(A(r), \psi(r))$.

This Proposition is a combination of several deep results in Sections 3, 4 and 5 of C. Taubes' [68]. The proof is highly involved, and rests ultimately on P. Kronheimer and T. Mrowka's construction of Monopole Floer Homology in [46]. We will not attempt at reproducing it here, but we will give a rough sketch in subsection 5.6.3, whose only aim is to explain how the Helicity of X has anything to do with the existence of solutions.

5.6.1 Proof of Proposition 5.3.2

We recall that our goal is to prove the following:

Let X be a nowhere-vanishing vector field on $\mathbb{S}^3$, preserving a volume form μ and with non-zero helicity. There always exists a smooth volume preserving vector field v, of arbitrarily small C^m norm, for which there is a sequence $\{r_n, \psi_{r_n} := (\psi_{1r_n}, \psi_{2r_n}), A_{r_n}\}$ of solutions to the associated $SW(r,v)$-equations satisfying

- A) sup $(1 - |\psi_{r_n}|^2) > \delta$, for all r_n and some $\delta > 0$,
- B) Either the sequence $\mathcal{H}_{A_{r_n}}(X)$ is bounded or the sequences of functionals $\mathcal{H}_{A_{r_n}}(X)$ and $\mathcal{H}_{A_{r_n}}(\mathrm{curl}^{-1} X)$ are such that

$$\frac{\mathcal{H}_{A_{r_n}}(\mathrm{curl}^{-1} X)}{\mathcal{H}_{A_{r_n}}(X)} \leqslant \varepsilon$$

  for any $\varepsilon > 0$ as small as we want, provided r_n is large enough.

Proof. From Proposition 5.6.2, we choose, for r large enough, a piecewise smooth family of solutions (A_r, ψ_r) to the $SW(r, v)$-equations (for some v smooth and with very small C^m norm), with *fixed* degree $\lambda \neq 0$. Any sequence of solutions of such a family already satisfies item A,

$$\sup\,(1 - |\psi_{r_n}|^2) > \delta$$

(by virtue of Item (iii) in Proposition 5.6.2). For item B, it remains to prove that, assuming a sequence $\mathcal{H}_{A_{r_n}}(X)$ of the family has no bounded subsequence, then we can find a subsequence with

$$\frac{\mathcal{H}_{A_{r_n}}(\mathrm{curl}^{-1}\, X)}{\mathcal{H}_{A_{r_n}}(X)} \leqslant \varepsilon$$

Suppose then that the family $\mathcal{H}_{A_r}(X)$ as no bounded subsequence. We will split the proof into two cases. To alleviate the notation, we define

$$\mathcal{X}(r) := \mathcal{H}_{A_r}(X)$$

and

$$\mathcal{X}^*(r) := \mathcal{H}_{A_r}(\mathrm{curl}^{-1}\, X)$$

(Note that $\mathcal{X}$ and $\mathcal{X}^*$ are well-defined and smooth in $(\rho_1, \infty) \setminus \{\rho_k\}_{k=2}^{\infty}$.)

First suppose that there is a subsequence of values r_n with

$$\mathcal{X}^*(r_n) \leqslant 1$$

Since by assumption $\mathcal{X}(r_n)$ is unbounded, then it is clear that

$$\frac{\mathcal{X}^*(r_n)}{\mathcal{X}(r_n)} \leqslant \varepsilon$$

for any large enough r_n (recall that $\mathcal{X}(r)$ is always positive for r large enough, as can be inferred from Lemma 5.5).

The remaining possibility is that there is no such subsequence with $\mathcal{X}^*(r_n) \leqslant 1$, i.e, there exists an r' such that

$$\mathcal{X}^*(r) \geqslant 1$$

for all $r > r'$ (it is understood we are avoiding the points ρ_k, where $\mathcal{X}^*$ is not well-defined).

Introduce now two constants κ and α, with $\kappa > 0$ and $\alpha \in (0, \frac{\delta^*}{16}]$ (where δ^* is the constant in item (iv) of Porposition 5.6.2). The reason for that choice of the range of α will become evident as we go along. Suppose first that there is a subsequence r_n of the family with

$$\mathcal{X}(r_n) \geqslant \kappa r^{\alpha} \mathcal{X}^*(r_n)$$

it is clear then that item B follows, no matter the precise value of α and κ.

If there is no such subsequence, what remains then to prove is that item B holds when

$$\mathcal{X}(r) \leqslant \kappa r^{\alpha} \mathcal{X}^*(r)$$

for $r > r'$, for some r' big enough.

This follows immediately from the subsequent lemma:

Lemma 5.6.3. *With the notations above, suppose there is $\kappa > 0$ and $\alpha \in (0, \frac{\delta^*}{16}]$ such that, for all r large enough*

$$\mathcal{X}(r) \leqslant \kappa r^{\alpha} \mathcal{X}^*(r)\,.$$

Then

$$\mathcal{X}^*(r) \leqslant C$$

for some C independent of r.

Granted this lemma, whose prove we give below, it is clear that

$$\frac{\mathcal{X}^*(r_n)}{\mathcal{X}(r_n)} \longrightarrow 0$$

as $r_n \to \infty$, for any sequence of r_n avoiding the constants ρ_k. Hence, Proposition 5.3.2 is proved. □

5.6.1.1 Proof of Lemma 5.6.3

It is at this stage that the particular items (ii) and (v) in Proposition 5.6.2 become important. Recall that those items ensure that the family of solutions (A_r, ψ_r) is piecewise smooth with respect to the parameter r, and that there is a continuous function $\widehat{\mathcal{S}}(r)$ which, in the intervals $\{(\rho_k, \rho_{k+1})\}_{k=1}^{\infty}$ where (A_r, ψ_r) is smooth, is equal to

$$\widehat{\mathcal{S}}(r) = \frac{1}{2}\mathcal{H}(\operatorname{curl} A_r) - r\mathcal{H}_{A_r}(\operatorname{curl}^{-1} X) - \int_{S^3} \operatorname{curl}^{-1} v \cdot \operatorname{curl} A_r$$

Note that in particular, this means that $\mathcal{S}(r)$ is smooth in the intervals (ρ_k, ρ_{k+1}). Note also that the term in $\mathcal{S}$

$$\int_{S^3} \psi_r^{\dagger} \sigma \cdot (\nabla - iA_r)\psi_r.$$

is zero for a solution (A_r, ψ_r).

We define the functions

$$\mathcal{A}(r) := \mathcal{H}(\operatorname{curl} A_r) = \int \operatorname{curl} A_r \cdot A_r\,,$$

and

$$\mathcal{V}(r) := \int_{S^3} \operatorname{curl}^{-1} v \cdot \operatorname{curl} A_r\,.$$

They are also piece-wise smooth. We then have that

$$\widehat{\mathcal{S}}(r) = \frac{1}{2}\mathcal{A}(r) - r\mathcal{X}^*(r) - \mathcal{V}(r).$$

Now, it is not hard to see that, on the intervals $\{(\rho_k, \rho_{k+1})\}_{k=1}^{\infty}$, we have

$$\widehat{\mathcal{S}}'(r) = -\mathcal{X}^*(r) \tag{5.6.1}$$

where by $\widehat{\mathcal{S}}'$ we denote the derivative of $\widehat{\mathcal{S}}$. (Note that as a consequence, $-\widehat{\mathcal{S}}$ is increasing for r big enough.)

Indeed, let (B, η, s) denote an element in the tangent space $T_{(A_r,\psi_r,r)}\mathfrak{M} \times \mathbb{R} \simeq \mathfrak{M} \times \mathbb{R}$. The (Frechet) derivative of $\mathcal{S}(A_r, \psi_r, r)$ in the direction of (B, η, s) is easily seen to be given by

$$(D\mathcal{S})_{(A_r,\psi_r,r}((B,\eta,s)) = \int B \cdot (\operatorname{curl} A_r - rX - v) - s\mathcal{H}_{A_r}(\operatorname{curl}^{-1} X) +$$

$$+ir\int \eta^\dagger\sigma \cdot (\nabla - iA)\psi + ir\psi^\dagger\sigma \cdot (\nabla - iA)\eta + \int \psi^\dagger\sigma \cdot B\psi$$

Now, the pair (B, η) is tangent to the space of solutions of the $SW(r, v)$-equations at the configuration (A_r, ψ_r) when it satisfies the linearized equations

$$\operatorname{curl} B = \eta^\dagger\sigma\psi_r + \psi_r^\dagger\sigma\eta \tag{5.6.2}$$

$$i\sigma \cdot (\nabla - iA_r)\eta + \sigma \cdot B\psi = 0 \tag{5.6.3}$$

In view of the above equations one gets that, if (B, η, s) is tangent to the space of solutions at (A_r, ψ_r, r)

$$(D\mathcal{S})_{(A_r,\psi_r,r}((B,\eta,s)) = s\mathcal{H}_{A_r}(\operatorname{curl}^{-1} X)$$

and Eq. (5.6.1) follows.

Notice in particular that Eq. (5.6.1) implies that the quantity

$$\mathcal{E}(r) := \frac{-2\widehat{\mathcal{S}}(r)}{r} = \frac{-\mathcal{A}}{r} + 2\mathcal{X}^* + \frac{2\mathcal{V}}{r} \tag{5.6.4}$$

satisfies

$$\mathcal{E}'(r) = \frac{\mathcal{A}(r)}{r^2} - \frac{2\mathcal{V}}{r^2} \tag{5.6.5}$$

where the term $\frac{\mathcal{V}}{r^2}$ is of order $O(\frac{1}{r})$. Our next goal is to show that $\mathcal{E}(r)$ is bounded as $r \to \infty$, and that this implies that $\mathcal{X}^*$ is also bounded, hence proving Lemma 5.6.3.

As can be suspected by inspecting Eqs. (5.6.4) and (5.6.5), the key to get some bounds on $\mathcal{E}$ and $\mathcal{X}^*$ is in the behavior of the function $\mathcal{A}(r)$.

Lemma 5.6.4. *Let δ^* be the constant in item (iv) of Proposition 5.6.2, i.e, the constant in the bound*

$$|\lambda(A_r, \psi_r) - \mathcal{H}(\operatorname{curl} A_r)| \leqslant cr^{2-\delta^*}.$$

where $\lambda(A_r, \psi_r)$ is the degree of the solution (A_r, ψ_r). For any $d < \frac{\delta^*}{4}$ we have, in the notations above

$$\mathcal{A}(r) \leqslant \kappa r^{1-d}\mathcal{X}(r)$$

Proof of Lemma 5.6.4. Suppose it were not the case, that is, that we had an increasing, unbounded sequence of values of r for which

$$\mathcal{A}(r_n) \geqslant \kappa r_n^{1-d} \mathcal{X}(r_n). \tag{5.6.1}$$

for some $d < \frac{\delta^*}{4}$

The following lemma provides a contradiction between Eq. (5.6.1) and item (iv) of Proposition 5.6.2

Lemma 5.6.5. *Let (A_r, ψ_r) be any solution to the $SW(r,v)$-equations. We have the bound*

$$|\mathcal{H}(\operatorname{curl} A)| \leqslant C r^{\frac{2}{3}} (|\mathcal{H}_A(X)|^{\frac{4}{3}} + C)\,.$$

This lemma is proved in Subsection 5.5.2 of Section 5.5. By the above lemma, we have that, if the bound in Eq. (5.6.1) holds, then

$$\kappa r_n^{1-d} \mathcal{X}(r) \leqslant \mathcal{A}(r_n) \leqslant C r_n^{\frac{2}{3}} \mathcal{X}^{\frac{4}{3}}(r_n)\,,$$

and hence we get

$$\mathcal{X}(r_n) \geqslant c r_n^{1-3d}$$

but then, again assuming that bound (5.6.1) holds, we get that there is a sequence of values of r_n such that

$$\mathcal{A}(r_n) \geqslant c r_n^{2-4d}\,.$$

Now, if $d < \frac{\delta^*}{4}$, this yields a contradiction with the fact that our solutions have fixed degree for all r. Indeed, by item (iv) in Proposition 5.6.2, our solutions must satisfy

$$\mathcal{A}(r) \leqslant c r^{2-\delta^*}.$$

□

Since by assumption we have that $\mathcal{X} \leqslant \kappa r^{\alpha} \mathcal{X}^*$, Proposition 5.6.4 implies that

$$\mathcal{A} \leqslant \kappa r^{1-d+\alpha} \mathcal{X}^*$$

and since by assumption we have $\alpha \leqslant \frac{\delta^*}{16}$, for $\frac{\delta^*}{4} > d \geqslant \frac{\delta *}{8}$ we have

$$\mathcal{A} \leqslant \kappa r^{1-\alpha} \mathcal{X}^*\,. \tag{5.6.1}$$

Introducing this information in Eq. (5.6.4) we get that

$$\mathcal{E} \geqslant (2 - \kappa r^{-\alpha})\mathcal{X}^* + \frac{\mathcal{V}}{r} \geqslant (2-\kappa)\mathcal{X}^* - c \tag{5.6.2}$$

where c is a constant coming from the $\mathcal{V}$ term. Since $\mathcal{X}^*(r) \geqslant 1$ by assumption, we infer that $\mathcal{X}^*$ is bounded if $\mathcal{E}$ is bounded. Furthermore, we can assume that, actually

$$\mathcal{E} \geqslant c'(2 - \kappa r^{-\alpha})\mathcal{X}^*$$

for some fixed constant c' since, if that were not the case, then we would have that $\mathcal{X}^*$ is already bounded.

Now, notice that taking into account combining the above bound (5.6.1) with Eq. (5.6.5) we get the differential inequality

$$\mathcal{E}'(r) \leqslant cr^{-1-\alpha}\mathcal{X}^* + \frac{c'}{r^2} \leqslant c'r^{-1-\alpha}\mathcal{X}^* \leqslant c''r^{-1-\alpha}\mathcal{E}(r)$$

Integrating the above readily yields that $\mathcal{E}$ is bounded, hence $\mathcal{X}^*$ is bounded and we have proved Lemma 5.6.3.

5.6.2 Proof of Proposition 5.4.1

We first recall the statement of Proposition 5.4.1:

Let (X, μ, g) be a nowhere-vanishing vector field on $\mathbb{S}^3$ preserving a volume form μ and with adapted metric g; and such that $\operatorname{curl}^{-1} X = X$. There always exists a smooth volume preserving vector field v, of arbitrarily small C^m norm, for which there is a sequence $\{r_n,\ \psi_{r_n} := (\psi_{1r_n}, \psi_{2r_n}),\ A_{r_n}\}$ of solutions to the associated $SW(r_n, v)$-equations such that

(i) $\sup\,(1 - |\psi_{r_n}|^2) > \delta$, for all n and some $\delta > 0$,

(ii) The sequence $\mathcal{H}_{A_{r_n}}(X)$ is bounded

As in the previous subsection, we first use Proposition 5.6.2 to find a piece-wise smooth family of solutions (A_r, ψ_r) to the $SW(r, v)$-equations of fixed degree $\lambda \neq 0$, so that the condition

$$\sup\,(1 - |\psi_{r_n}|^2) > \delta$$

is already satisfied for any subsequence of r_n. It remains to show that this family has a subsequence with bounded $\mathcal{H}_{A_{r_n}}(X)$.

In the notation from the previous subsection, note that the condition $\operatorname{curl}^{-1} X = X$ implies that $\mathcal{X}(r) = \mathcal{X}^*(r)$ and hence Eq. (5.6.4) reads in this case

$$\mathcal{E}(r) := \frac{-2\widehat{\mathcal{S}}}{r} = \frac{-\mathcal{A}}{r} + 2\mathcal{X}(r) + 2\frac{\mathcal{V}}{r} \tag{5.6.3}$$

and we have the same expression for the derivative of $\mathcal{E}$:

$$\mathcal{E}'(r) = \frac{\mathcal{A}}{r^2} - 2\frac{\mathcal{V}}{r^2} \tag{5.6.4}$$

Notice that an analog of Lemma 5.6.4 also applies here, since the argument in the proof of Lemma 5.6.4 uses only the general results from Proposition 5.6.2 and the a priori bound in Lemma 5.6.5. Thus, we have that for $d < \frac{\delta^*}{4}$ and some fixed constant κ (that we can choose to be small enough):

$$\mathcal{A}(r) \leqslant \kappa r^{1-d}\mathcal{X}(r). \tag{5.6.5}$$

for r large enough.

From Eqs. (5.6.5) and (5.6.4) we get, on the one hand, that

$$\mathcal{E}'(r) \leqslant \kappa r^{-1-d}\mathcal{X}(r) - 2\frac{\mathcal{V}}{r^2} \leqslant c\kappa r^{-1-d}\mathcal{X}(r) \tag{5.6.6}$$

(note that the last inequality would be false only if $\mathcal{X}$ had already a bounded subsequence, since by virtue of Proposition 5.3.1, $\mathcal{X}$ is already bounded from below by a constant and hence it cannot have sequences going to $-\infty$). On the other hand, from Eq. (5.6.3) and (5.6.5) we get

$$\mathcal{E}(r) \geqslant (2 - \kappa r^{-d})\mathcal{X}(r) + \frac{\mathcal{V}}{r}\,;$$

and again, we can assume that there is a constant c' such that

$$\mathcal{E}(r) \geqslant c'(2 - \kappa r^{-d})\mathcal{X}(r)$$

or otherwise we would already know that $\mathcal{X}$ has a bounded subsequence. In view of Eq. (5.6.6) we conclude that

$$\mathcal{E}'(r) \leqslant c'' r^{-1-d}\mathcal{E}$$

and integrating we get that $\mathcal{E}$ is bounded, hence so is $\mathcal{X}$.

5.6.3 The role of Helicity in Proposition 5.6.2

Helicity is important to ensure that we can choose sequences of solutions with

$$|\mathcal{H}(\operatorname{curl} A_r)| \leqslant r^{2-\delta_*}\,.$$

This inequality is a key input in the proofs of both Propositions 5.3.2 and 5.4.1 in Subsections 5.6.1 and 5.6.2.

Let us explain how helicity enters the picture. First, note that, actually, the $SW(r,v)$-equations always have a solution with

$$\sup\,(1 - |\psi_{1r}|^2) > \delta\,,$$

as item A in Proposition 5.3.2 requires. Indeed, this is the solution corresponding to $\psi = 0$ and

$$A_r = r\operatorname{curl}^{-1} X + \operatorname{curl}^{-1} v\,,$$

which is unique modulo gauge transformation, i.e, adding a gradient field to A_r.

Definition 5.6.6 (Reducible solution vs irreducible solution)**.** *The solution* $(A_r, \psi_r) := (r\operatorname{curl}^{-1} X + \operatorname{curl}^{-1} v, 0)$ *to the* $SW(r,v)$ *equations is called the* reducible solution*. The other solutions are called* irreducible solutions*.*

However, it is clear that the sequence of reducible solutions $(A_r, 0)$ is not the sequence of solutions that we want in Theorem 5.2.1, since its associated sequence of measures is just

$$\sigma_r = \frac{r\mu}{r\mathrm{vol}(\mathbb{S}^3) + c}$$

which clearly converges to the volume measure μ.

So we must ensure that other solutions do exist.

The following lemma is the main input form Kronheimer and Mrowka's Monopole Floer Homology, and it is also where helicity plays its role, to ensure the existence of irreducible solutions. It is a very rough distillation of some of the results in Sections 2 and 3 of [68]:

Lemma 5.6.7. *Given an integer λ, a constant $\epsilon > 0$, and $m \geqslant 0$, there is always an integer $\lambda' \leqslant \lambda$ and a positive number r_k such that there always exists an irreducible solution to the $SW(r, v)$ equations of degree λ', for some smooth divergence free field v with $||v||_{C^m(\mathbb{S}^3)} \leqslant \epsilon$.*

Idea of the proof. This Lemma rests ultimately on two facts:

- A version of the Monopole Floer Homology groups $HF(M)$ of a 3-manifold, as defined by Kronheimer and Mrowka, is non-zero for an infinite set of degrees λ' unbounded from below.
- There is a correspondence between generators of the chain complex giving rise to the homology groups $HF_{\lambda'}(M)$ and *both the reducible and irreducible solutions* to the $SW(r, v)$ equations of degree λ', for some sufficiently large r and some $v \in \Omega$.

Now, since the $HF_{\lambda'}(M)$ is non trivial, generators must exists. And hence, by the second item above, solutions to $SW(r, v)$ must exists also. The problem is that *those generators might, in principle, correspond to the reducible solution*.

It is at this point that the Helicity of X becomes key, via the following lemma:

Lemma 5.6.8. *Let $(A = r\,\mathrm{curl}^{-1}X + v, \psi = 0)$ be the reducible solution to the $SW(r, v)$ − equations. Then,*

$$degree(A, \psi) = -cr^2\mathcal{H}(X) + O(r^{2-\delta})$$

for a positive constant c and $\delta > 0$.

This lemma follows from Section 5 in [68]. Our statement actually incurs in a slight abuse of terminology: in rigor, the degree cannot be directly defined for the reducible solution (it can for the irreducible ones). However, the conclusion of this lemma still holds: the actual generators corresponding to the reducible solution in $HF(M)$ have degree $\lambda' \sim -r^2$ when $\mathcal{H}(X) > 0$.

Hence, for λ' fixed and sufficiently large r, all the generators in $HF_{\lambda'}(M)$ must correspond to irreducible solutions, and Lemma 5.6.7 follows.

□

5.7 The vortex equations: Proof of Lemma 5.3.13 and Lemma 5.3.14

We divide this section into two subsections. Subsection 5.7.1 proves Lemma 5.3.13, and Subsection 5.7.2 proves Lemma 5.3.14 (both are needed for the proof in 5.3 of the detection of periodic orbits in the bounded $\mathcal{H}_A(X)$ case).

We recall that the self dual vortex equations on $\mathbb{C}$ are the system of PDEs

$$da = \partial_x a_y - \partial_y a_x = (1 - |\phi|^2) \tag{5.7.1}$$

$$\overline{\partial}_a \phi = \overline{\partial}_z \phi - i(a_x - i a_y)\phi = 0 \tag{5.7.2}$$

where $\overline{\partial}_z := \partial_x + i\partial_y$, and the unknowns are a real one form $a = a_x dx + a_y dy$ (that we will identify when needed with its dual vector field through the euclidean metric) and a complex valued function $\phi = \phi_1 + i\phi_2$. These equations are invariant under $U(1)$-gauge transformations: if (a, ϕ) is a solution to Eqs. 5.7.1 and 5.7.2, then $g(a, \phi) = (a - g^{-1}dg, g\phi)$ is also a solution, for any smooth function $g : \mathbb{C} \to \mathbb{C}$ with $|g| = 1$.

5.7.1 Proof of Lemma 5.3.13

Let $\Psi_p : C_p(\rho, \epsilon) \to (-\epsilon, \epsilon) \times \mathbb{D}_\rho$ be an adapted flow-box chart at p, as in Definition 5.3.5. We will drop here the subscript indexing the point since no confusion can arise. We recall that Lemma 5.3.13 states:

Let (A_r, ψ_r) be a family of solutions to the SW(r, v)-equations. Given a point $p \in \mathbb{S}^3$ and a sequence of values of r going to infinity, and given a compact set $[-T, T] \times \mathbb{D}_R \subset \mathbb{R} \times \mathbb{C}$, there always exists a subsequence r_i such that the rescaled fields

$$\widetilde{A}_i(t, z) := \frac{1}{\sqrt{r_i}}(\Psi)_* A_{r_i}\Big(\frac{t}{\sqrt{r_i}}, \frac{z}{\sqrt{r_i}}\Big),\ \widetilde{\psi}_{1i}(t', z') := \psi_{1r_i} \circ \Psi^{-1}\Big(\frac{t}{\sqrt{r_i}}, \frac{z}{\sqrt{r_i}}\Big)$$

converge in $[-T, T] \times \mathbb{D}_R$ in the C^m norm to a smooth family of solutions $(a_t(z), \phi_t(z))$ of the vortex equations on $t \times \mathbb{C}$, with $t \in [-T, T]$. Furthermore, all members of these family are gauge equivalent, that is, there is a smooth function $g : [-T, T] \times \mathbb{D}_R \to \mathbb{C}$ with $|g| = 1$ and a solution to the vortex equation $(a(z), \phi(z))$ such that $g(t, \cdot)_*(a(t, \cdot), \phi(t, \cdot)) = (a(\cdot), \phi(\cdot))$.

Proof. Note that from now we incur in a slight abuse of the notation with respect to the notations in Section 5.3 by renaming the rescaled euclidean coordinates (t', z') as (t, z). Recall that these rescaled coordinates take values in the stretched cylinder $\mathcal{C}_{\sqrt{r}} := (-\epsilon\sqrt{r}, \epsilon\sqrt{r}) \times \mathbb{D}_{\sqrt{r}\rho}$. We will use the convention of indexing the rescaled euclidean coordinates by greek indices $\mu = 1, 2, 3$, as opposed to latin indices $k = 1, 2, 3$ for the orthonormal basis in $\mathbb{S}^3$. In these rescaled coordinates, the metric is almost euclidean and it is easy to see that we have

$$\Psi_* X\Big(\frac{\cdot}{\sqrt{r}}\Big) = \partial_t$$

$$\Psi_* Y\Big(\frac{\cdot}{\sqrt{r}}\Big) = \partial_x + \frac{1}{\sqrt{r}} \sum_{\mu=1}^{3} G_\mu(\cdot)\partial_\mu$$

$$\Psi_* Z\Big(\frac{\cdot}{\sqrt{r}}\Big) = \partial_y + \frac{1}{\sqrt{r}} \sum_{\mu=1}^{3} G'_\mu(\cdot)\partial_\mu$$

where the indices $\{1, 2, 3\}$ correspond to $\{t, x, y\}$ and where G_μ and G'_μ are smooth functions, uniformly bounded in compact sets $[-T, T] \times \mathbb{D}_R \subset \mathcal{C}_{\sqrt{r}}$ as

$$\Big\| G_\mu \Big\|_{C^m([-T,T]\times\mathbb{D}_R)} \leqslant C_m,$$

where the C_m are constants not depending on r.

Hence, we have that, under these coordinates

$$(D_A\psi) \circ \Psi^{-1}\Big(\frac{\cdot}{\sqrt{r}}\Big) = \sqrt{r} i \sum_{\mu=1}^{3} \sigma_\mu(\partial_\mu - i\widetilde{A}_\mu)\widetilde{\psi} + \frac{G_4(\widetilde{A}, \psi, \nabla\widetilde{\psi})}{\sqrt{r}}$$

where we set $\sigma_\mu := \sum_k \delta_{k\mu}\sigma_k$; and

$$\Psi_*(\operatorname{curl} A)\Big(\frac{\cdot}{\sqrt{r}}\Big) = r\Big(\operatorname{curl}_{\mathbb{R}^3} \widetilde{A} + \frac{G_5(\widetilde{A}, \partial\widetilde{A})}{\sqrt{r}}\Big)$$

here, again, G_4 and G_5 are uniformly bounded on compact sets as

$$\Big\| G_4(\widetilde{A}, \widetilde{\psi}, \partial\widetilde{\psi}) \Big\|_{C^m([-T,T]\times\mathbb{D}_R)} \leqslant C_m \Big\| \nabla\widetilde{\psi} + \widetilde{A}\psi \Big\|_{C^m([-T,T]\times\mathbb{D}_R)},$$

and

$$\Big\| G_5(\widetilde{A}, \partial\widetilde{A}) \Big\|_{C^m([-T,T]\times\mathbb{D}_R)} \leqslant C_m \Big\| \nabla\widetilde{A} + \widetilde{A} \Big\|_{C^m([-T,T]\times\mathbb{D}_R)}.$$

Furthermore, by virtue of the a priori bounds obtained in Lemma 5.3.1, we have that $\widetilde{A}$ and $\widetilde{\psi}$ are bounded with all their derivatives, i.e,

$$\|\widetilde{\psi}_1\|_{C^0(\mathcal{C}(\sqrt{r}))} \leqslant 1 + \frac{C}{r}; \;\; \|\widetilde{\psi}_2|\|_{C^0(\mathcal{C}(\sqrt{r}))} \leqslant \frac{C}{\sqrt{r}} \tag{5.7.3}$$

$$\|(\nabla - i\widetilde{A})\widetilde{\psi}_1\|_{C^m(\mathcal{C}(\sqrt{r}))} \leqslant C_m; \;\; \|(\nabla - i\widetilde{A})\widetilde{\psi}_2\|_{C^m(\mathcal{C}(\sqrt{r}))} \leqslant \frac{C_m}{\sqrt{r}}; \tag{5.7.4}$$

Hence, in the rescaled coordinates the Seiberg-Witten equations take the form

$$\operatorname{curl}_0 \widetilde{A} = (1 - |\widetilde{\psi}_1|^2)\partial_t + \frac{B_1}{r} \tag{5.7.5}$$

$$\overline{\partial}_{\widetilde{A}}\widetilde{\psi}_1 = \frac{B_3}{r} \tag{5.7.6}$$

$$(\partial_t - i\widetilde{A}_t)\widetilde{\psi}_{1r} = \frac{B_2}{r} \tag{5.7.7}$$

where $\overline{\partial}_{\widetilde{A}} := (\partial_x + i\partial_y) - i(\widetilde{A}_x - i\widetilde{A}_y)$ and the B_i are smooth functions depending on $\widetilde{A}$, $\widetilde{\psi}$ and their derivatives, but whose C^m norms on compact sets $[-T,T] \times \mathbb{D}_\rho$ do not depend on r.

In a similar fashion as was done in the proof of Lemma 5.3.1, one can, locally on compact sets $[-T,T] \times \mathbb{D}_\rho$, apply a gauge transformation to $(\widetilde{A}, \widetilde{\psi})$ so that the following bounds are satisfied

$$\|\nabla\widetilde{\psi}_1\|_{C^m([-T,T]\times\mathbb{D}_\rho)} \leqslant C_m; \;\; \|\nabla\widetilde{\psi}_2\|_{C^m([-T,T]\times\mathbb{D}_\rho)} \leqslant \frac{C_m}{\sqrt{r}}; \;\; \|A\|_{C^m([-T,T]\times\mathbb{D}_\rho)} \leqslant C_m\,. \tag{5.7.8}$$

Notice how these bounds apply now only on compact subsets of the rescaled flow-box $\mathcal{C}(\sqrt{r})$, as opposed to the a priori bounds (5.7.4), which come from global bounds on $\mathbb{S}^3$ and are thus valid in the whole flow-box $\mathcal{C}(\sqrt{r})$

Eqs. (5.7.5)–(5.7.6) together with the uniform bounds (5.7.3) and (5.7.8) ensure that a subsequence of $(\widetilde{A}_r, \widetilde{\psi}_r)$ uniformly converges on $[-T,T] \times \mathbb{D}_R$ towards *a smooth family* of solutions $(a_t(z), \phi_t(z))$ of (5.7.1)-(5.7.2).

It remains to show that this family is gauge equivalent to a unique, $\mathbb{R}$-invariant solution (a, ϕ). This is granted by Equation (5.7.7): it implies that the limiting solution $(a(t,z), \phi(t,z))$ must verify

$$(\partial_t - ia_t)\phi = 0,$$

and hence, in particular, we have that

$$\partial_t|\phi|^2 = \mathrm{Re}(\overline{\phi}(\partial_t - ia_t)\phi) = 0$$

The complex field can then be written as $\phi(t,z) = |\phi|(z)u(t,z)$, with $|u| = 1$. Note that u is not well-defined on the points where $|\phi|$ vanishes (and $|du|$ grows indefinitely nearby), so in principle u does not define a good Gauge transformation and we cannot define the g in the statement of the Lemma as, say, $g := u^{-1}$, or $g := \phi(t_0,z)/\phi(t,z)$ for any chosen t_0, to conclude that the solutions are gauge equivalent.

Nevertheless, by virtue of Lemma 5.3.14, the zeroes of any given solution ϕ are isolated, and moreover, near an isolated zero z_0 the field ϕ can be written as

$$\phi(z) = h(z)(z - z_0)^{n_0}$$

with $h(z)$ a non-vanishing function. (We will prove this in the next Subsection). It is then clear that if two solutions of the vortex equations have the same modulus, $|\phi_1| = |\phi_2|$, the quotient $u_{12} := \phi_1/\phi_2$ is a well defined smooth function $u_{12} : \mathbb{C} \to \mathbb{C}$ with $|u_{12}| = 1$, and ϕ_1 and ϕ_2 are gauge equivalent. Back to our case, by choosing any reference solution $(a(t_0,z), \phi(t_0,z))$ for some $t_0 \in [-T,T]$, we can define g as $g(t,z) := \phi(t_0,z)/\phi(t,z)$. □

5.7.2 Proof of Lemma 5.3.14

In this subsection we prove that any solution (a, ϕ) to the vortex equations verifies that

(i) The set $|\phi|^{-1}(0)$ is either empty or consists of a set of isolated points.

(ii) If $|\phi| \leqslant 1$, then either $|\phi| = 1$ everywhere or $|\phi| < 1$. Further, if $|\phi| < 1$ any local minima of $|\phi|$ has $|\phi| = 0$.

Let us note at this point that the reader can find a throughout treatment of the vortex equations (including the lemma above) in the monograph [38].

We begin with the proof of item (i). This follows from the lemma

Lemma 5.7.1. *On any disk $\mathbb{D} \subset \mathbb{C}$, a smooth solution (a, ϕ) to the vortex equations can be written as*

$$\phi = e^{-f} h$$

where f is a smooth complex-valued function bounded on $\mathbb{D}$ and h is a holomorphic function on the disk.

The above lemma implies that at in the neighborhood of any zero, ϕ is locally of the form $g(z)(z - z_k)^{n_k}$, with g a non-vanishing function, so item (i) follows.

Proof of Lemma 5.7.1. Set $\alpha := a_x - i a_y$. The Lemma is just a consequence of the properties of the Cauchy kernel. If we set

$$f(z) := \int_{\mathbb{D}} \frac{\alpha}{2\pi(z - w)} dw \wedge d\overline{w}. \tag{5.7.1}$$

then f satisfies on $\mathbb{D}$ the equation

$$\overline{\partial}_z f = i\alpha \tag{5.7.2}$$

and standard arguments show that if α (that is, the connection a) is bounded with all derivatives on the disk, so is f. Now, define $h := e^f \phi$. Equation (5.7.2) together with the second vortex equation implies that h is holomorphic,

$$\overline{\partial}_z h = (\overline{\partial}_z e^f)\phi + e^f \overline{\partial}_z \phi = e^f (\overline{\partial}_z \phi - i\alpha\phi) = 0\,.$$

and the lemma follows. □

For the proof of item (i), note that the vortex equations imply that (a, ϕ) satisfies the second order PDE

$$- \star d_a \star d_a \phi = (1 - |\phi|^2)\phi$$

with $d_a = d - ia$ and $\star$ the standard euclidean Hodge operator. Multiplying the above expression by $\overline{\phi}$, extracting the real part, and taking into account that

$$\mathrm{Re}(-\bar{\phi} \star d_a \star d_a \phi) = 2|d_a \phi|^2 - \Delta|\phi|^2$$

we get

$$-\Delta|\phi|^2 + 2|d_a\phi|^2 = (1 - |\phi|^2)|\phi|^2 \tag{5.7.3}$$

Item (ii) follows from standard applications of the maximum principle to the above equation. More precisely, the first statement follows from the proposition:

Proposition 5.7.2 (see for example [33], Theorem 3.5)**.** *Let u be a $C^2(\Omega)$ function on an open set Ω (which does not need to be bounded), and let $\mathcal{L} := L + f$ with L a uniformly elliptic operator with continuous coefficients (if the region has a boundary, then continuous on the boundary too) and f a continuous, non-positive function on Ω. If $\mathcal{L}u \leqslant 0$ on Ω, then any interior minimum of u must be positive, or else u is constant.*

By applying the above Proposition to $u = (1 - |\phi|^2)$, $\mathcal{L} = \Delta - |\phi|^2$ and $\Omega = \mathbb{C}$ the first statement in item (ii) follows. Note that since we have by assumption that $|\phi| \leqslant 1$, standard elliptic estimates applied to the vortex equations imply that ϕ is bounded uniformly with all derivatives in $\mathbb{C}$ and hence $(1 - |\phi|^2)$ satisfies the hypotheses of the above Proposition.

For the second statement in item (ii), we use a very particular property of the self-dual vortex equations: that from Eq. (5.7.3) we can get a PDE on $|\phi|^2$ only. First, note that the second vortex equation can be written as

$$\bar{\partial}_a\phi = 0 \longleftrightarrow d_a\phi = i \star d_a\phi \tag{5.7.1}$$

The above relationship and the fact that the connection is unitary, that is, that a is real, allows us to deduce

$$\nabla|\phi|^2 = 2\mathrm{Re}(\bar{\phi}\nabla_a\phi) = 2 \star \mathrm{Im}(\bar{\phi}\nabla_a\phi) \tag{5.7.2}$$

and so

$$|\nabla|\phi|^2|^2 = 2|\phi|^2|\nabla_a\phi|^2 \tag{5.7.3}$$

where we have identified the 1-form $d_a\phi$ with its dual vector field $\nabla_a\phi$. Notice that if the relation (5.7.1) did not hold, the equality in Eq. (5.7.3) would be an inequality: a trivial instance of the so-called Kato's inequality for connections compatible with the metric (note that in our context, compatibility with the metric is equivalent to the connection being unitary, i.e, to a being real).

From Equations (5.7.3) and (5.7.3), we get an equation for the modulus of ϕ:

$$-\Delta|\phi|^2 + \frac{|\nabla|\phi|^2|^2}{|\phi|^2} = (1 - |\phi|^2)|\phi|^2 \tag{5.7.4}$$

The second statement in item (ii) follows from the application of the maximum principle to the above. Indeed, if a local minimum of $|\phi|$ is not zero and $|\phi| < 1$, then the right hand side of Eq. (5.7.4) is strictly bigger than zero, while the left hand side is equal to $-\Delta|\phi|^2$ and thus negative (or zero) at a minimum.

5.8 Probability measures as the renormalized limit of the vortex equations: Proof of Theorem 5.2.3

We begin by recalling that the rescaled vortex equations on $\mathbb{C}$, which are our main concern in this Section, read

$$\star da_r = r(1 - |\phi_r|^2) \tag{5.8.1}$$

$$\overline{\partial}_{a_r}\phi = \overline{\partial}_z\phi_r - i(a_{xr} - ia_{yr})\phi = 0 \tag{5.8.2}$$

for some fixed parameter $r > 0$. Our aim in this section is to prove the following realization theorem for sequences of solutions to the rescaled vortex equations (Theorem 5.2.3 in the Section 5.2):

Let ν be a Borel probability measure on the disk $\mathbb{D} \subset \mathbb{C}$. There is a sequence $\{(\phi_{r_n}, a_{r_n}), N_{r_n}\}_{n=1}^{\infty}$ of solutions to the r_n-rescaled vortex equations in $\mathbb{C}$ with N_{r_n} zeroes, with $r_n \to \infty$, such that the 2-form

$$\sigma_{r_n} = \frac{r_n(1 - |\phi_{r_n}|^2)dx \wedge dy}{\int da_{r_n}}$$

converges to ν in the sense of measures on $\mathbb{D}$.

Proof. The proof of Theorem 5.2.3 rests mainly in the following Proposition:

Proposition 5.8.1. *Given a fixed finite set of points $\mathcal{P} := \{z_j\}_{j=1}^k \subset \mathbb{D}$ and an associated collection of positive integers $\{m_j\}_{j=1}^k$, with $\sum m_j = N$, for each $r > 0$, there is a solution (a_r, ϕ_r) to the rescaled vortex equations on $\mathbb{C}$ with $|\phi_r|^{-1}(0) = \mathcal{P}$ and with each zero z_j having multiplicity m_j. Furthermore, for r large enough, we have*

- *a) For any $z \in \mathbb{D} \setminus \mathcal{P}$ there is a constant $\rho_z > 0$ such that*

$$|1 - |\phi_r|^2(z)| \leqslant e^{-c\sqrt{r}\rho_z}$$

$$|\nabla|\phi_r|^2(z)| \leqslant \frac{\sqrt{r}}{c}e^{-\sqrt{r}\rho_z}$$

- *b) The family of measures*

$$\sigma_r := \frac{r(1 - |\phi_r|^2)dx \wedge dy}{\int_{\mathbb{D}} da_r}$$

converge weakly as $r \to \infty$ to the probability measure

$$\delta_{\mathcal{P}} = \frac{1}{N}\sum_{j=1}^{k} m_j\delta(z - z_j)$$

Given this proposition, that we prove in Subsection 5.8.1, Theorem 5.2.3 follows from the following Lemma:

Lemma 5.8.2. *Given a signed finite Borel measure ρ on the open disk, there is a sequence of positive integerts $\{k_n\}_{n=1}^{\infty}$ going to infinity, and associated sequences of sets of points $\{z_{nj}\}_{j=1}^{k_n} \subset \mathbb{D}$ and constants $\{c_{nj}\}_{j=1}^{k_n}$, such that for any $\epsilon > 0$ and any bounded continuous function φ on $\mathbb{D}$, we have*

$$|\rho(\varphi) - \sum_j c_{nj}\delta_{z_{nj}}(\varphi)| < \epsilon$$

for all k_n large enough, and where $\delta_{z_{nj}} := \delta(z - z_{nj})$ is the Dirac measure centered at the point z_{nj}

We prove this Lemma at the end of this subsection. Granted the lemma, choose a sequence of measures $\delta_{\mathcal{P}_n}$ of the form

$$\delta_{\mathcal{P}_n} := \sum_{z_{nj} \in \mathcal{P}_n} c_{nj}\delta(z - z_{nj})$$

(where for each n, $\mathcal{P}_n$ is a finite set of points in $\mathbb{D}$) such that $\delta_{\mathcal{P}_n}$ converges, in the weak-sense, to ν. Since ν is a probability measure, it is not hard to see that the coefficients c_{nj} can be chosen positive and such that

$$\sum_j c_{nj} = 1\,,$$

and by density, it is also clear that the c_{nj} can be chosen rational. Then, for each fixed index n we can set

$$c_{nj} = \frac{m_{nj}}{N_n}$$

for $m_{nj} \in \mathbb{N}$ (choosing N_n to be the smallest integer such that $N_n c_{nj} \in \mathbb{N}$); and if they add up to one, we have that $N_n = \sum_j m_{nj}$, so we get a sequence of probability measures of the form

$$\delta_{\mathcal{P}_n} := \frac{1}{N_n} \sum_{z_{nj} \in \mathcal{P}_n} m_{nj}\delta(z_{nj})$$

converging to ν, and we notice that these measures are of the same type as the Dirac measures in item (b) of Proposition 5.8.1.

Thus, Proposition 5.8.1 grants the existence, for each fixed n above, of a sequence $\{r_{in}, a_{in}, \phi_{in}\}_{i=1}^{\infty}$ of increasing real numbers r_{in} and of solutions (a_{in}, ϕ_{in}) to the r_{in}-rescaled vortex equations with $|\phi_{in}|^{-1}(0) = \mathcal{P}_n$ and, moreover, with

$$\sigma_n^i := \frac{r_{in}(1 - |\phi_{in}|^2)dx \wedge dy}{\int_{\mathbb{D}} da_{in}}$$

converging weakly to $\delta_{\mathcal{P}_n}$. Now let φ a bounded continuous function on the disk: we have, on the one hand, that

$$|\sigma_n^i(\varphi) - \delta_{\mathcal{P}_n}(\varphi)| \leqslant \frac{\epsilon}{2}$$

for any ϵ as small as we want, provided i is large enough; and on the other hand

$$|\nu(\varphi) - \delta_{\mathcal{P}_n}(\varphi)| \leqslant \frac{\epsilon}{2}$$

provided n is large enough, hence

$$|\nu(\varphi) - \sigma_n^i(\varphi)| \leqslant \epsilon$$

for ϵ as small as we want as long as (i, n) are large enough. Hence, there is a sequence of pairs (i, n) such that σ_n^i converges weakly to ν. Redefining $r_{in} := r_n$, we get the statement in Theorem 5.2.3.

Proof of Lemma 5.8.2. Let $\mathcal{M}(\mathbb{D})$ be the vector space of signed measures on the disk, and let $\mathcal{D}(\mathbb{D})$ the linear subspace formed by signed measures which are finite linear combinations of Dirac measures on points of $\mathbb{D}$.

The statement of the lemma is equivalent to $\mathcal{D}$ being dense in $\mathcal{M}$ in the weak-$*$ topology, because $\mathcal{M}$ is in the continuous dual of the space of bounded continuous functions on the disk.

Now the Hahn-Banach theorem implies that a linear subspace of $\mathcal{M}(\mathbb{D})$ is dense if and only if the only continuous linear form vanishing on it is 0. In the weak star topology, the continuous linear forms on $\mathcal{M}$ are precisely the bounded continuous functions, acting on measures in the obvious way. Since all Dirac masses δ_z, for $z \in \mathbb{D}$, are in $\mathcal{D}$, a continuous function vanishing on $\mathcal{D}$ must vanish on all points in $\mathbb{D}$: it must be 0. So $\mathcal{D}$ is dense in $\mathcal{M}$. □

5.8.1 Proof of Proposition 5.8.1

The following Theorem is the starting point for the proof of Proposition 5.8.1.

Theorem 5.8.3 (Taubes 79 [67, 38]). *Let $\mathcal{P} := \{z_j\}_{j=1}^k$ be an arbitrarily chosen, finite set of points $z_j \in \mathbb{C}$, and let $\{m_j\}_{j=1}^k$ be an associated set of non-negative integers. There is a solution (a, ϕ) to the vortex equations such that $\phi^{-1}(0) = \mathcal{P}$, and such that the zero z_j of ϕ has multiplicity m_j. Furthermore, we have that*

(i) $|\phi| \leqslant 1$ *and* $|\phi| \to 1$ *as* $|z| \to \infty$

(ii)

$$\int_{\mathbb{C}} |da|^2 + |d_a\phi|^2 + \frac{1}{4}(1 - |\phi|^2)^2 = \int_{\mathbb{C}} da = \int_{\mathbb{C}} (1 - |\phi|^2) = 2\pi \sum_j m_j = 2\pi N$$

(iii) There is a constant C, not depending on the particular configuration of points, such that

$$|\nabla|\phi|^2(z)| \leqslant |\nabla_a\phi| \leqslant C$$

(iv) Let $\Omega^-(\phi)$ denote the set of points in $\mathbb{C}$ where $|\phi|^2 \leqslant \frac{1}{2}$. There is a constant $c \in (0,1)$, not depending on the particular configuration of points, such that, for any $z \in \mathbb{C}$ with dist $(z, \Omega^-(\phi)) > c^{-1}$

$$|1 - |\phi(z)|^2| \leqslant e^{-c dist\,(z,\Omega^-(\phi))} \tag{5.8.1}$$

$$|\nabla|\phi|^2(z)| \leqslant |\nabla_a \phi| \leqslant \frac{1}{c} e^{-c dist\,(z,\Omega^-(\phi))} \tag{5.8.2}$$

We refer the reader to the monograph [38] for the proof. From the above Theorem we can immediately gather the following important corollary, which already proves the first statement (existence) of Proposition 5.8.1:

Corollary 5.8.4. *Let $\mathcal{P} := \{z_j\}_{j=1}^k$ be an arbitrarily chosen, finite set of points $z_j \in \mathbb{D}$, and let $\{m_j\}_{j=1}^k$ be an associated set of non-negative integers. For each $r > 0$, there is a solution (a_r, ϕ_r) to the rescaled vortex equations on $\mathbb{C}$ with $|\phi_r|^{-1}(0) = \mathcal{P}$ and with each zero z_j having multiplicity m_j. Furthermore, the solution (a_r, ϕ_r) to the rescaled equations thus obtained is such that $|\phi_r| \leqslant 1$, and satisfies that*

(i)

$$|\nabla|\phi_r|^2(z)| \leqslant |\nabla_{a_r}\phi_r| \leqslant C\sqrt{r}$$

(ii)

$$r\int_{\mathbb{C}} (1 - |\phi_r|^2) = 2\pi \sum_j m_j = 2\pi N$$

(iii) Furthermore, if we define

$$\Omega_r^- := \{z \in \mathbb{C} \text{ such that } |\phi_r|^2(z) \leqslant \frac{1}{2}\},$$

there is a constant c such that, if dist $(z, \Omega_r^-) \geqslant \frac{1}{c\sqrt{r}}$, we have

$$|1 - |\phi_r|^2(z)| \leqslant e^{-c\sqrt{r} dist\,(z,\Omega_r^-)}$$

and

$$|\nabla|\phi_r|^2(z)| \leqslant c^{-1}\sqrt{r} e^{-c\sqrt{r} dist\,(z,\Omega_r^-)}$$

Indeed, if $(a(z), \phi(z))$ is a solution to the original vortex equations (i.e, with $r = 1$) on $\mathbb{C}$ with zeros at $\{z_j\}_{j=1}^k$, then $(a_r(z), \phi_r(z)) := (\sqrt{r}a(\sqrt{r}z), \phi(\sqrt{r}z))$ is a solution to the rescaled vortex equations with zeroes in $\{\frac{z_j}{\sqrt{r}}\}_{j=1}^k$, so all items follow from rescaling the coordinates.

Now, notice that item a) in Proposition 5.8.1 does not follow automatically from item (iii) in Corollary 5.8.4. The reason is that the set of zeroes of the rescaled vortex equations is fixed for all r, and hence the set of zeroes of the solutions $\phi^r(z) := \phi_r(\frac{z}{\sqrt{r}})$ to the original vortex equations for $r = 1$, $\mathcal{P}_r := \{\sqrt{r}z_j\}$ (to which Theorem 5.8.3 applies) is changing with r and becoming more and more spread. Although the constant c in item (iii) of Theorem 5.8.3 does not depend on the particular configuration of points, we do lose control of the set of points with dist $(z, \Omega^-(\phi^r)) \geqslant \frac{1}{c}$, i.e, of the set dist $(z, \Omega_r^-) \geqslant \frac{1}{c\sqrt{r}}$.

Our first goal is to remedy the above situation. The following lemma meets this goal. In order to state it, consider a sequence of solutions (a_r, ϕ_r) to the rescaled vortex equations obtained from Corollary 5.8.4, with $r \to \infty$, and with $|\phi_r|^{-1}(0) = \mathcal{P} \subset \mathbb{D}$, for $\mathcal{P}$ a fixed set of points, with fixed associated multiplicities. We have:

Lemma 5.8.5. *For any p_r and q_r in the same connected component of Ω_r^-, there is a constant C such that dist$(p_r, q_r) \leqslant \frac{C}{\sqrt{r}}$.*

Proof. Let γ_r be a smooth embedded curve inside Ω_r^- joining the points p_r and q_r. It is clear that all points laying in γ_r have $|\phi_r|^2 \leqslant \frac{1}{2}$. Moreover, recall that from Corollary 5.8.4 we have that

$$|\nabla|\phi_r|^2(z)| \leqslant C\sqrt{r}\,,$$

thus, there is a constant C such that all points within a distance of $\frac{\epsilon}{C\sqrt{r}}$ of γ_r satisfy $|\phi_r|^2 \leqslant \frac{1}{2} + \epsilon$; and this for $\epsilon > 0$ as small as desired.

Denote by U_r the set of points $z \in \mathbb{C}$ at a distance smaller or equal to $\frac{\epsilon}{C\sqrt{r}}$ from γ_r. The area of U_r is bounded from above by

$$\int_{U_r} dx \wedge dy \leqslant |\gamma_r| \frac{\epsilon}{C\sqrt{r}}$$

where by $|\gamma_r|$ we denote the length of the curve γ_r. Hence,

$$r\int_{U_r}(1 - |\phi_r|^2)dx \wedge dy \geqslant r\Big(\frac{1}{2} - \epsilon\Big)|\gamma_r|\frac{\epsilon}{C\sqrt{r}} \geqslant c'\sqrt{r}|\gamma_r|$$

for some constant c' independent of r. On the other hand, note that, by Corollary 5.8.4,

$$r\int_{U_r}(1 - |\phi_r|^2)dx \wedge dy \leqslant r\int_{\mathbb{C}}(1 - |\phi_r|^2)dx \wedge dy = 2\pi N$$

where N is the number of zeroes of ϕ_r counted with multiplicities. Therefore, for the above bound to hold, the length of γ_r must be of order $O(\frac{1}{\sqrt{r}})$. Since the length of γ_r is always greater or equal than the distance between p_r and q_r, the Lemma follows. ☐

With this Lemma in hand, it is easy to see that, for any point $z \in \mathbb{D} \setminus \mathcal{P}$,

$$\text{dist}\ (z, \Omega_r^-) \geqslant \rho_z > 0$$

for r large enough (where "large enough" depends on z), and with ρ_z a constant depending on the point but not on r. Indeed, we have that

$$\text{dist}\ (z, \mathcal{P}) \leqslant \text{dist}\ (z, \Omega_r^-) + \text{dist}\ (p_r, \mathcal{P}) \tag{5.8.1}$$

where p_r is some point in Ω_r^- such that dist $(z, \Omega_r^-) =$ dist (z, p_r). Now, note that p_r must be in the connected component of some point in the set of zeroes $\mathcal{P}$; otherwise, the function $|\phi_r|$ would have local minima other than zero, contradicting Lemma 5.3.14. But then, by lemma 5.8.5, dist $(p_r, \mathcal{P}) = O(\frac{1}{\sqrt{r}})$. Since the distance of a point z to any zero in $\mathcal{P}$ is always fixed, we have that dist (z, Ω_r^-) must be bounded from below by some constant ρ_z for the bound (5.8.1) to hold.

The above discussion ensures that for any $z \in \mathbb{D} \setminus \mathcal{P}$, and for r big enough, we always have that dist $(z, \Omega_r^-) \geqslant \rho_z \geqslant \frac{1}{c\sqrt{r}}$. Item (a) in Proposition 5.8.1 follows.

For the proof of item (b), it is convenient to work in terms of the function u_r defined as $e^{u_r} := |\phi_r|^2$. Note that, since $|\phi_r| \leqslant 1$, $u_r \leqslant 0$. It is not hard to check, taking into account Lemma 5.7.1, that if ϕ_r is a solution to the rescaled vortex equations with zeroes at $z_j \in \mathcal{P}$ and multiplicities m_j, the function u_r abides to the PDE

$$\Delta u_r - r(e^{u_r} - 1) = 2\pi \sum_{z_j \in \mathcal{P}} m_j \delta(z - z_j). \tag{5.8.2}$$

The measure σ_r can be written in these terms as

$$\sigma_r = \frac{r(1 - e^{u_r}) dx \wedge dy}{\int_{\mathbb{D}} da_r}.$$

Note first that the exponential decay granted by item (a) implies that the integral in the denominator

$$\int_{\mathbb{D}} da_r = \int_{\mathbb{D}} r(1 - |\phi|^2)$$

converges, as $r \to \infty$, to its value on the whole complex plane, that is, to $2\pi N$. As for the numerator, Eq. (5.8.2) implies that, for any continuous function $\varphi : \mathbb{D} \to \mathbb{R}$ we have

$$r \int_{\mathbb{D}} (1 - e^{u_r}) \varphi = - \int \varphi \Delta u_r + 2\pi \sum_{z_j \in \mathcal{P}} m_j \delta(z - z_j)(\varphi) \,.$$

Hence, item (b) would follow if we manage to prove that, as $r \to \infty$

$$\int_{\mathbb{D}} \varphi \Delta u_r \to 0.$$

To this end, assume that the function φ is C^∞ with bounded derivatives in $\mathbb{D}$ (there is no loss in generality in doing so, because of the density of C^∞ functions in C^0). An integration by parts yields

$$\int_{\mathbb{D}} \varphi \Delta u_r = - \int_{\mathbb{D}} \nabla u_r \cdot \nabla \varphi + \int_{\partial \mathbb{D}} \varphi \nabla u_r \cdot n d\theta \tag{5.8.3}$$

where n is the outward pointing normal vector to the boundary of the disk.

Item (a) implies that for any point in the boundary of the disk, we have

$$|\nabla u_r|(z) = |\frac{1}{|\phi_r(z)|^2} \nabla |\phi_r(z)|^2| \leqslant \epsilon$$

for ϵ as small as desired, provided r is big enough. Hence,

$$\int_{\partial\mathbb{D}} \varphi \nabla u_r \cdot n d\theta \to 0.$$

As for the remaining term in (5.8.3), integrating by parts again we get

$$-\int_{\mathbb{D}} \nabla u_r \cdot \nabla \varphi = \int_{\mathbb{D}} u_r \Delta \varphi - \int_{\partial\mathbb{D}} u_r \nabla \varphi \cdot n d\theta$$

The rightmost term clearly converges to zero, because when $z \in \partial\mathbb{D}$, we have that $u_r(z) \to 0$ as $r \to \infty$. Finally

$$-\int_{\mathbb{D}} u_r \Delta \varphi \leqslant ||\varphi||_{C^2(\mathbb{D})} \int_{\mathbb{D}} |u_r|$$

It remains to show that

$$\int_{\mathbb{D}} |u_r| = -\int_{\mathbb{D}} u_r \longrightarrow 0$$

To see this, set $u_r'(\sqrt{r}z) = u_r(z)$, so that $u_r'(z)$ is a finite energy solution to the original vortex equations with $r = 1$. Note that, for z sufficiently far away from the zero set, u_r' decays exponentially:

$$|u_r'|(z) \leqslant |\log(|\phi^r|^2)(z)| \leqslant |\log(1 - e^{-c\text{dist}(z,\Omega^-(\phi^r))})| \leqslant c' e^{-c\text{dist}(z,\Omega^-(\phi^r))}$$

and near the zero set u_r' is integrable since it consists on a logarithmic singularity. Hence, we have that

$$-\int_{\mathbb{D}(\sqrt{r})} u_r' \leqslant C$$

so that, rescaling,

$$-\int_{\mathbb{D}} u_r \leqslant C/r.$$

and we conclude that

$$\int_{\mathbb{D}} \varphi \Delta u_r \to 0\,.$$

Therefore

$$r\int_{\mathbb{D}} (1 - e^{u_r})\varphi = 2\pi \sum_{z_j \in \mathcal{P}} m_j \delta(z - z_j)(\varphi) + \epsilon$$

for ϵ as small as desired as $r \to \infty$. Item (b) in the Proposition follows. □

Chapter 6

Helicity is the only invariant of volume preserving vector fields

Helicity is a remarkable conserved quantity that is fundamental to all the natural phenomena described by a vector field whose evolution is given by volume-preserving transformations. This is the case of the vorticity of an inviscid fluid flow or of the magnetic field of a conducting plasma. The topological nature of the helicity was unveiled by Moffatt, but its relevance goes well beyond that of being a new conservation law. Indeed, the helicity defines an integral invariant under any kind of volume-preserving diffeomorphisms. A well-known open problem is whether there exist any integral invariants other than the helicity. In this chapter we answer this question by showing that, under some mild technical assumptions, the helicity is the only integral invariant.

6.1 Introduction

Incompressible inviscid fluids are modeled by the three-dimensional Euler equations, which assert that the velocity field $u(x,t)$ of the fluid flow must satisfy the system of differential equations

$$\partial_t u + (u \cdot \nabla) u = -\nabla p \,, \qquad \operatorname{div} u = 0 \,.$$

Here the scalar function $p(x,t)$ is another unknown of the problem, which physically corresponds to the pressure of the fluid.

It is customary to introduce the vorticity $\omega := \operatorname{curl} u$ to simplify the analysis of these equations, as it enables us to get rid of the pressure function. In terms of the vorticity, the Euler equations read as

$$\partial_t \omega = [\omega, u] \,, \tag{6.1.1}$$

where $[\omega, u] := (\omega \cdot \nabla)u - (u \cdot \nabla)w$ is the commutator of vector fields and u can be written in terms of ω using the Biot–Savart law

$$u(x) = \operatorname{curl}^{-1} \omega(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(y) \times (x-y)}{|x-y|^3}\, dy\,, \tag{6.1.2}$$

at least when the space variable is assumed to take values in the whole space $\mathbb{R}^3$.

The transport equation (6.1.1) was first derived by Helmholtz, who showed that the meaning of this equation is that the vorticity at time t is related to the vorticity at initial time t_0 via the flow of the velocity field, provided that the equation does not develop any singularities in the time interval $[t_0, t]$. More precisely, if ϕ_{t,t_0} denotes the (time-dependent) flow of the divergence-free field u, then the vorticity at time t is given by the action of the push-forward of the volume-preserving diffeomorphism ϕ_{t,t_0} on the initial vorticity:

$$\omega(\cdot, t) = (\phi_{t,t_0})_* \, \omega(\cdot, t_0)\,.$$

The phenomenon of the transport of vorticity gives rise to a new conservation law of the three-dimensional Euler equations. Moffatt coined the term *helicity* for this conservation law in his influential paper [57], and exhibited its topological nature. Indeed, defining the helicity of a divergence-free vector field w in $\mathbb{R}^3$ as

$$\mathcal{H}(w) := \int_{\mathbb{R}^3} w \cdot \operatorname{curl}^{-1} w \, dx\,,$$

it turns out that the helicity of the vorticity $\mathcal{H}(\omega(\cdot, t))$ is a conserved quantity for the Euler equations. In fact, helicity is also conserved for the compressible Euler equations provided the fluid is barotropic (i.e. the pressure is a function of the density).

It is well known that the relevance of the helicity goes well beyond that of being a new (non-positive) conserved quantity for the Euler equations. On the one hand, the helicity appears in other natural phenomena that are also described by a divergence-free field whose evolution is given by a time-dependent family of volume-preserving diffeomorphisms [58]. For instance, the case of magnetohydrodynamics (MHD), where one is interested in the helicity of the magnetic field of a conducting plasma, has attracted considerable attention. On the other hand, it turns out that the helicity does not only correspond to a conserved quantity for evolution equations such as Euler or MHD, but in fact defines an integral invariant for vector fields under any kind of volume-preserving diffeomorphisms [5].

It is important to emphasize that conserved quantities of the Euler or MHD equations (e.g., the kinetic energy and the momentum) are not, in general, invariant under arbitrary volume-preserving diffeomorphisms, but they are invariant only under the very particular diffeomorphism defined by the flow of the velocity field of the fluid or conducting plasma. Perhaps the key feature of the helicity, which distinguishes it from other conserved quantities of Euler or MHD, is its invariance under any kind of volume-preserving transformations (in particular, it is invariant under the transport of the vorticity or the magnetic field by an arbitrary divergence-free vector field), so let us elaborate on this property.

Helicity is often analyzed in the context of a compact 3-dimensional manifold M without boundary, endowed with a Riemannian metric. The simplest case would be that of the flat 3-torus, which corresponds to fields on Euclidean space with periodic boundary conditions. To define the helicity in a general compact 3-manifold, let us introduce some notation. We will denote by $\mathfrak{X}^1_{\mathrm{ex}}$ the vector space of exact divergence-free vector fields on M of class C^1, endowed with its natural C^1 norm. We recall that a divergence-free vector field w is *exact* if its flux through any closed surface is zero (or, equivalently, if there exists a vector field v such that $w = \operatorname{curl} v$). This is a topological condition, and in particular when the first homology group of the manifold is trivial (e.g., in the 3-sphere) every divergence-free field is automatically exact.

As is well known, the reason to consider exact fields in this context is that, on exact fields, the curl operator has a well defined inverse $\operatorname{curl}^{-1} : \mathfrak{X}^1_{\mathrm{ex}} \to \mathfrak{X}^1_{\mathrm{ex}}$. The inverse of curl is a generalization to compact 3-manifolds of the Biot–Savart operator (6.1.2), and can also be written in terms of a (matrix-valued) integral kernel $k(x,y)$ as

$$\operatorname{curl}^{-1} w(x) = \int_M k(x,y)\, w(y)\, dy\,, \tag{6.1.3}$$

where dy now stands for the Riemannian volume measure. Using this integral operator, one can define the helicity of a vector field w on M as

$$\mathcal{H}(w) := \int_M w \cdot \operatorname{curl}^{-1} w\, dx\,.$$

Here and in what follows the dot denotes the scalar product of two vector fields defined by the Riemannian metric on M. The helicity is then invariant under volume-preserving transformations, that is, $\mathcal{H}(w) = \mathcal{H}(\Phi_* w)$ for any diffeomorphism Φ of M that preserves volume.

In view of the expression (6.1.3) for the inverse of the curl operator, it is clear that the helicity is an *integral invariant*, meaning that it is given by the integral of a density of the form

$$\mathcal{H}(w) = \int G(x,y,w(x),w(y))\, dx\, dy\,.$$

Arnold and Khesin conjectured [5, Section I.9] that, in fact, the helicity is the only integral invariant, that is, there are no other invariants of the form

$$\mathcal{I}(u) := \int G(x_1,\dots,x_n,u(x_1),\dots,u(x_n))\, dx_1 \cdots dx_n \tag{6.1.4}$$

with G a reasonably well-behaved function. Here all variables are assumed to be integrated over M.

Our objective in this chapter is to show, under some natural regularity assumptions, that the helicity is indeed the only integral invariant under volume-preserving diffeomorphisms. To this end, let us define a regular integral invariant as follows:

Definition 6.1.1. *Let $\mathcal{I} : \mathfrak{X}^1_{\mathrm{ex}} \to \mathbb{R}$ be a C^1 functional. We say that $\mathcal{I}$ is a* regular integral invariant *if:*

(i) *It is invariant under volume-preserving transformations, i.e.,* $\mathcal{I}(w) = \mathcal{I}(\Phi_* w)$ *for any diffeomorphism* Φ *of* M *that preserves volume.*

(ii) *At any point* $w \in \mathfrak{X}^1_{\mathrm{ex}}$*, the (Fréchet) derivative of* $\mathcal{I}$ *is an integral operator with continuous kernel, that is,*

$$(D\mathcal{I})_w(u) = \int_M K(w) \cdot u \,,$$

for any $u \in \mathfrak{X}^1_{\mathrm{ex}}$*, where* $K : \mathfrak{X}^1_{\mathrm{ex}} \to \mathfrak{X}^1_{\mathrm{ex}}$ *is a continuous map.*

In the above definition and in what follows, we omit the Riemannian volume measure under the integral sign when no confusion can arise. Observe that any integral invariant of the form (6.1.4) is a regular integral invariant provided that the function G satisfies some mild technical assumptions. In particular, the helicity is a regular integral invariant.

The following theorem, which is the main result of this chapter, shows that the helicity is essentially the only regular integral invariant in the above sense. The proof of this result is presented in Section 6.2, and is an extension to any closed 3-manifold of a theorem of Kudryavtseva [48], who proved an analogous result for divergence-free vector fields on 3-manifolds that are trivial bundles of a compact surface with boundary over the circle, which admit a cross section and are tangent to the boundary. Kudryavtseva's theorem is based on her work on the uniqueness of the Calabi invariant for area-preserving diffeomorphisms of the disk [47]. We observe that our main result does not imply the aforementioned theorem because we consider manifolds without boundary.

Let $\mathcal{I}$ be a regular integral invariant. Then $\mathcal{I}$ is a function of the helicity, i.e., there exists a C^1 function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathcal{I} = f(\mathcal{H})$.

We would like to remark that this theorem does not exclude the existence of other invariants of divergence-free vector fields under volume-preserving diffeomorphisms that are not C^1 or whose derivative is not an integral operator of the type described in the definition above. For example, the KAM-type invariants recently introduced in [43] are in no way related to the helicity, but they are not even continuous functionals on $\mathfrak{X}^1_{\mathrm{ex}}$.

Other type of invariants that have attracted considerable attention are the asymptotic invariants of divergence-free vector fields [1, 32, 42, 6, 7, 2, 45]. These invariants are of non-local nature because they are defined in terms of a knot invariant (e.g., the linking number) and the flow of the vector field. In some cases, it turns out that the asymptotic invariant can be expressed as a regular integral invariant, as happens with the asymptotic linking number for divergence-free vector fields [4], the asymptotic signature [32] and the asymptotic Vassiliev invariants [7, 45] for ergodic divergence-free vector fields. In these cases, the authors prove that the corresponding asymptotic invariant is a function of the helicity, which is in perfect agreement with our main theorem.

The so-called higher order helicities [11, 51, 44] are also invariants under volume-preserving diffeomorphisms. However, they are not defined for any divergence-free vector field, but just for vector fields supported on a disjoint union of solid tori. This property is, of course, not even continuous in $\mathfrak{X}^1_{\text{ex}}$, so these functionals do not fall in the category of the regular integral invariants considered in this chapter.

Our main theorem is reminiscent of Serre's theorem [64] showing that any conserved quantity of the three-dimensional Euler equations that is the integral of a density depending on the velocity field and its first derivatives,

$$\mathcal{I}(u) := \int_{\mathbb{R}^3} G(u(x,t), Du(x,t))\, dx\,,$$

is a function of the energy, the momentum and the helicity. From a technical point of view, the proof of our main theorem is totally different to the proof of Serre's theorem, which is purely analytic, only holds in the Euclidean space, and is based on integral identities that the density G must satisfy in order to define a conservation law of the Euler equations.

Even more importantly, from a conceptual standpoint it should be emphasized that Serre's theorem applies to conserved quantities of the Euler equations, while our theorem concerns the existence of functionals that are invariant under any kind of volume-preserving diffeomorphisms, which is a much stronger requirement, as explained in a previous paragraph. In particular, the fact that the energy and the momentum are not functions of the helicity does not contradict our main theorem, because they are conserved by the evolution determined by the Euler equations but they are not invariant under the flow of an arbitrary divergence-free vector field. Accordingly, our theorem does not mean that there are no other integrals of motion of the Euler (or MHD) equations.

It is worth noticing that one can construct well-behaved integral invariants of Lagrangian type that are invariant under general volume-preserving diffeomorphisms but which are not functions of the helicity. These functional arise in a natural manner in the analysis of the Euler or MHD equations especially when one considers integrable fields, that is, fields whose integral curves are tangent to a family of invariant surfaces. For example, one can define a partial helicity as the helicity integral taken over the region Ω bounded by an invariant surface of the field. In this context, if f is any well-behaved function (e.g., a smooth function supported on the region Ω covered by invariant surfaces) which is assumed to be transported under the action of the diffeomorphism group, the functional

$$\mathcal{F}(f,w) := \int_M f\, w \cdot \operatorname{curl}^{-1} w\, dx$$

is invariant under volume-preserving diffeomorphisms (and it is not a function of the helicity). The key point here is that the assumption that f is transformed in a Lagrangian way means that the action of the volume-preserving diffeomorphism group is not the one considered in this chapter, which would be

$$\Phi \cdot \mathcal{F}(f,w) := \mathcal{F}(f, \Phi_* w)\,,$$

but the one given by

$$\Phi \cdot \mathcal{F}(f,w) := \mathcal{F}(f \circ \Phi^{-1}, \Phi_* w)\,.$$

In this sense, this new action is defined on functionals mapping a function and a vector field (rather than just a vector field) to a number, so it does not fall within the scope of our theorem. In the context of the partial helicity defined above, this action means that not only the vector field w, but also the function f and the region Ω where it is supported, are transported by the fluid flow.

6.2 Proof of the main theorem

We divide the proof of the main theorem in five steps. The idea of the proof, which is inspired by Kudryavtseva's work on the uniqueness of the Calabi invariant [47], is that the invariance of the functional $\mathcal{I}$ under volume-preserving diffeomorphisms implies the existence of a continuous first integral for each exact divergence-free vector field. Since a generic vector field in $\mathfrak{X}^1_{\text{ex}}$ is not integrable, we conclude that the aforementioned first integral is a constant (that depends on the field), which in turn implies that $\mathcal{I}$ has the same value for all vector fields in a connected component of the level sets of the helicity. Since these level sets are path connected, the theorem will follow.

Step 1: For each vector field $w \in \mathfrak{X}^1_{\text{ex}}$, either $\operatorname{curl} K(w) = fw$ on $M\backslash w^{-1}(0)$ for some function $f \in C^0(M\backslash w^{-1}(0))$ or the field w admits a nontrivial first integral (that is, $\nabla F \cdot w = 0$ for some nonconstant function $F \in C^1(M)$).

We first notice that the flow ϕ_t of any divergence-free vector field u is a 1-parameter family of volume-preserving diffeomorphisms, so the functional $\mathcal{I}$ must take the same values on w and its push-forward $(\phi_t)_* w$, i.e.

$$\mathcal{I}((\phi_t)_* w) = \mathcal{I}(w)$$

for all $t \in \mathbb{R}$. Taking derivatives with respect to t in this equation and evaluating at $t = 0$, we immediately get

$$0 = \frac{d}{dt}\mathcal{I}((\phi_t)_* w) = (D\mathcal{I})_w([w,u]) = \int_M K(w) \cdot [w,u] \,. \tag{6.2.1}$$

The identity $[w,u] = \operatorname{curl}(u \times w)$ for divergence-free fields allows us to write the integral above as

$$\begin{aligned}\int_M K(w) \cdot [w,u] &= \int_M K(w) \cdot \operatorname{curl}(u \times w) \\ &= \int_M \operatorname{curl} K(w) \cdot (u \times w) \\ &= \int_M u \cdot (w \times \operatorname{curl} K(w))\end{aligned}$$

where we have integrated by parts to obtain the second equality. Hence Eq. (6.2.1) implies that for each pair of vector fields $u, w \in \mathfrak{X}^1_{\text{ex}}$ we have

$$\int_M u \cdot (w \times \operatorname{curl} K(w)) = 0 \,.$$

It then follows that the vector field $w \times \operatorname{curl} K(w)$ is L^2-orthogonal to all the divergence-free vector fields on M, and hence the Hodge decomposition theorem implies that there exists a C^1 function F on M such that $w \times \operatorname{curl} K(w) = \nabla F$. Then $w \cdot \nabla F = 0$, so F is a first integral of w.

In the case that F is identically constant, we have that $w \times \operatorname{curl} K(w) = 0$, so $\operatorname{curl} K(w)$ is proportional to w at any point of M where the latter does not vanish. Since $\operatorname{curl} K(w)$ is a continuous vector field on M because, by assumption, $K(w) \in \mathfrak{X}^1_{\mathrm{ex}}$, it follows that there is a continuous function f such that

$$\operatorname{curl} K(w) = f w \tag{6.2.2}$$

in $M \backslash w^{-1}(0)$, as we wanted to prove.

Step 2: The function $f \in C^0(M \backslash w^{-1}(0))$ is a continuous first integral of w.

The flow box theorem ensures that for any point in the complement of the zero set $w^{-1}(0)$ there is a neighborhood U and a diffeomorphism $\Phi : U \to [0,1] \times D$ such that $\Phi_* w = \partial_z$. Here $D := \{x \in \mathbb{R}^2 : |x| \leqslant 1\}$ is the closed unit 2-disk, and $[0,1] \times D$ is endowed with the natural Cartesian coordinates $x \in D$ and $z \in [0,1]$. Using the notation $\mathcal{D}_s := \Phi^{-1}(\{s\} \times D)$ and $\mathcal{S} := \Phi^{-1}([0,1] \times \partial D)$, it is obvious from the definition of the flow box that

$$\partial U = \mathcal{D}_0 \cup \mathcal{D}_1 \cup \mathcal{S}\,,$$

and that the integral curves of w are tangent to the cylinder $\mathcal{S}$ and transverse to the disks $\mathcal{D}_0$ and $\mathcal{D}_1$.

Taking the negative orientation for the surface ∂U (i.e., choosing a unit normal vector ν on ∂U that points inward), we can compute the flux of fw across ∂U as

$$\int_{\partial U} f w \cdot \nu \, d\sigma = \int_{\mathcal{D}_0} f w \cdot \nu_0 \, d\sigma - \int_{\mathcal{D}_1} f w \cdot \nu_1 \, d\sigma\,,$$

where $d\sigma$ denotes the induced surface measure and ν_s denotes the unit normal on $\mathcal{D}_s$ pointing in the direction of w (that is, $w \cdot \nu_s > 0$).

Using Eq. (6.2.2), the flux of fw can also be written as

$$\int_{\partial U} f w \cdot \nu \, d\sigma = \int_{\partial U} \operatorname{curl} K(w) \cdot \nu \, d\sigma = 0\,,$$

with the integral vanishing by Stokes' theorem. Therefore we conclude that the fluxes through the caps $\mathcal{D}_0$ and $\mathcal{D}_1$ must be equal, that is,

$$\int_{\mathcal{D}_0} f w \cdot \nu_0 \, d\sigma = \int_{\mathcal{D}_1} f w \cdot \nu_1 \, d\sigma\,. \tag{6.2.3}$$

Suppose now that f is not constant along the integral curves of w. Then we can take a point $x_0 \in D$ such that the function f takes different values at the points $p_s := \Phi^{-1}(s, x_0) \in \mathcal{D}_s$, with $s = 0, 1$. For concreteness, let us assume that

$$f(p_0) < f(p_1)\,, \tag{6.2.4}$$

the case $f(p_0) > f(p_1)$ being completely analogous. By the continuity of f, we can then take the flow box narrow enough (i.e. with $\mathcal{D}_0$ and $\mathcal{D}_1$ having very small diameters) such that $c_0 < c_1$, where

$$c_0 := \max_{x\in\mathcal{D}_0} f(x)\,, \qquad c_1 := \min_{x\in\mathcal{D}_1} f(x)\,.$$

Therefore, since $w \cdot \nu_s > 0$ on $\mathcal{D}_s$, we have the bound

$$\int_{\mathcal{D}_0} f w \cdot \nu_0 \, d\sigma \leqslant c_0 \int_{\mathcal{D}_0} w \cdot \nu_0 \, d\sigma < c_1 \int_{\mathcal{D}_1} w \cdot \nu_1 \, d\sigma \leqslant \int_{\mathcal{D}_1} f w \cdot \nu_1 \, d\sigma\,,$$

where to obtain the second inequality we have used that, as w is divergence-free, Stokes' theorem implies that

$$\int_{\mathcal{D}_0} w \cdot \nu_0 \, d\sigma = \int_{\mathcal{D}_1} w \cdot \nu_1 \, d\sigma\,.$$

This inequality above contradicts Eq. (6.2.3), so we conclude that f must be constant along the integral curves of w, thus proving that f is a continuous first integral of w on $M\backslash w^{-1}(0)$, as we had claimed.

Step 3: There exists a continuous functional $\mathcal{C}$ on $\mathfrak{X}^1_{\mathrm{ex}}\backslash\{0\}$ such that derivatives of the invariant $\mathcal{I}$ and of the helicity $\mathcal{H}$ are related by $(D\mathcal{I})_w = \mathcal{C}(w)(D\mathcal{H})_w$

Let us start by noticing that Steps 1 and 2 imply that either w has a nontrivial first integral $F \in C^1(M)$ or the function f defined in Step 1 is a continuous first integral of w in the complement of its zero set. Now we observe that there exists a residual set $\mathcal{R}$ of vector fields in $\mathfrak{X}^1_{\mathrm{ex}}$ such that any $w \in \mathcal{R}$ is topologically transitive and its zero set consists of finitely many hyperbolic points. (We recall that a set is *residual* if it is the intersection of countably many open dense sets. In particular, a residual set is always dense but not necessarily open.) This theorem was proved in [14] for divergence-free C^1 vector fields, not necessarily exact. However, it is not difficult to prove that the same result holds true for exact divergence free vector fields. Indeed, the proof of [14] consists in perturbing a divergence-free vector field w to obtain another divergence-free vector field $\tilde{w}$ of the form

$$\tilde{w} = w + \sum_{i=1}^{N} v_i\,,$$

where each v_i is a C^1 divergence-free vector field supported in a contractible set. Each vector field v_i is necessarily exact because any divergence-free vector field supported in a contractible set is, so the resulting perturbed field $\tilde{w}$ is exact too. With this observation, the main theorem in [14] automatically applies to the class of exact divergence-free C^1 vector fields, $\mathfrak{X}^1_{\mathrm{ex}}$.

Hence let us take a vector field $w \in \mathcal{R}$. Since it is topologically transitive, it has an integral curve that is dense in M, so any continuous first integral of w must be a constant. Accordingly, Steps 1 and 2 imply that $\operatorname{curl} K(w) = fw$ in $M\backslash w^{-1}(0)$, with f a first integral of w, and therefore the function f is a constant c_w (depending on w) in the complement of the zero set $w^{-1}(0)$. Since this set consists of finitely many points, c_w is the unique continuous extension of f to the whole manifold M. As $\operatorname{curl} K(w)$ is a continuous vector field, for any $w \in \mathcal{R}$ it follows that

$$\operatorname{curl} K(w) = c_w w \tag{6.2.5}$$

in M, so $\operatorname{curl} K(w) \times w = 0$.

Since the kernel K is a continuous map $\mathfrak{X}^1_{\text{ex}} \to \mathfrak{X}^1_{\text{ex}}$, the fact that $\operatorname{curl} K(w) \times w = 0$ for all w in the residual set $\mathcal{R} \subset \mathfrak{X}^1_{\text{ex}}$ implies that $\operatorname{curl} K(w) \times w = 0$ for all $w \in \mathfrak{X}^1_{\text{ex}}$. Therefore for any $w \in \mathfrak{X}^1_{\text{ex}} \backslash \{0\}$ we can define a function $f \in C^0(M \backslash w^{-1}(0))$ by setting

$$f := \frac{w \cdot \operatorname{curl} K(w)}{|w|^2}\,,$$

such that

$$\operatorname{curl} K(w) = f w$$

on $M \backslash w^{-1}(0)$. In view of the expression for f, the mapping $w \to f$ is continuous on $\mathfrak{X}^1_{\text{ex}} \backslash \{0\}$ due to the continuity of the kernel $K : \mathfrak{X}^1_{\text{ex}} \to \mathfrak{X}^1_{\text{ex}}$. Since f is given by a w-dependent constant c_w whenever w lies in the residual set $\mathcal{R}$ of $\mathfrak{X}^1_{\text{ex}}$, we conclude that this must also be the case for all $w \in \mathfrak{X}^1_{\text{ex}} \backslash \{0\}$, so the map $w \mapsto \frac{1}{2} c_w$ defines a continuous functional $\mathcal{C} : \mathfrak{X}^1_{\text{ex}} \backslash \{0\} \to \mathbb{R}$. (The factor $\frac{1}{2}$ has been included for future notational convenience.) The continuous functionals $\operatorname{curl} K(w)$ and $2\mathcal{C}(w)\, w$ coinciding in a residual set, it stems that for any $w \in \mathfrak{X}^1_{\text{ex}} \backslash \{0\}$ one has

$$\operatorname{curl} K(w) = 2\,\mathcal{C}(w)\, w$$

in all M.

Since the curl operator is invertible on $\mathfrak{X}^1_{\text{ex}}$ and $\mathcal{C}(w)$ is just a constant, we can use the above equation for $\operatorname{curl} K(w)$ to write the derivative of $\mathcal{I}$ at w as

$$(D\mathcal{I})_w(u) = 2\,\mathcal{C}(w) \int_M \operatorname{curl}^{-1} w \cdot u\,.$$

The claim of this step then follows upon recalling that the differential of the helicity is given by

$$(D\mathcal{H})_w(u) = 2 \int_M \operatorname{curl}^{-1} w \cdot u\,.$$

Step 4: The level sets of the helicity, $\mathcal{H}^{-1}(c)$, are path connected subsets of $\mathfrak{X}^1_{\text{ex}}$.

Let w_0 and w_1 be two vector fields in $\mathfrak{X}^1_{\text{ex}}$ with the same helicity:

$$\mathcal{H}(w_0) = \mathcal{H}(w_1) = c\,.$$

For concreteness, let us assume that c is positive. It is easy to see that the path connectedness of the level set $\mathcal{H}^{-1}(c)$ is immediate if one can prove the existence of a path of positive helicity connecting w_0 and w_1, i.e., a continuous map $w : [0,1] \to \mathfrak{X}^1_{\text{ex}}$ such that $w(0) = w_0$, $w(1) = w_1$ and $\mathcal{H}(w(t)) > 0$ for all $t \in [0,1]$. Indeed, one can then set

$$\tilde{w}(t) := \left(\frac{c}{\mathcal{H}(w(t))}\right)^{\frac{1}{2}} w(t)$$

to conclude that $\tilde{w} : [0,1] \to \mathfrak{X}^1_{\text{ex}}$ is a continuous path connecting w_0 and w_1 of helicity c: $\tilde{w}(0) = w_0$, $\tilde{w}(1) = w_1$ and $\mathcal{H}(\tilde{w}(t)) = c$ for all $t \in [0,1]$.

To show the existence of a path of positive helicity connecting w_0 and w_1, we first observe that the curl defines a self-adjoint operator with dense domain on the space of exact divergence-free L^2 fields (see e.g. [36]), so we can take an orthonormal basis of eigenfields $\{v_n^+, v_n^-\}_{n=1}^\infty$ satisfying $\operatorname{curl} v_n^\pm = \lambda_n^\pm v_n^\pm$. Here we are denoting by λ_n^+ and λ_n^- the positive and negative eigenvalues of the curl, respectively.

Given any vector field $v \in \mathfrak{X}^1_{\text{ex}}$, we can expand v in this orthonormal basis as

$$v = \sum_{n=1}^\infty (c_n^+ v_n^+ + c_n^- v_n^-)\,.$$

This series converges in the Sobolev space H^1. As $\operatorname{curl}^{-1} v_n^\pm = v_n^\pm / \lambda_n^\pm$, the helicity of the field v can be written in terms of the coefficients of the series expansion as

$$\mathcal{H}(v) = \sum_{n=1}^\infty \left(\frac{(c_n^+)^2}{\lambda_n^+} - \frac{(c_n^-)^2}{|\lambda_n^-|} \right). \tag{6.2.6}$$

We shall denote by $c_{j,n}^\pm$ the coefficients of the eigenfunction expansion corresponding to w_j, with $j = 0, 1$. Let us fix two integers n_j for which the coefficient c_{j,n_j}^+ is nonzero (notice that the coefficients corresponding to positive eigenvalues cannot be all zero because of the formula (6.2.6) for the helicity, which is positive in the case of w_j).

We can now construct the desired continuous path $w : [0,1] \to \mathfrak{X}^1_{\text{ex}}$ of positive helicity connecting w_0 and w_1 by setting

$$w(t) := \begin{cases} 8t\, c_{0,n_0}^+ v_{n_0}^+ + (1-4t)\, w_0 & \text{if } 0 \leqslant t \leqslant \frac{1}{4}\,, \\ 2\cos(\pi t - \frac{\pi}{4})\, c_{0,n_0}^+ v_{n_0}^+ + 2\sin(\pi t - \frac{\pi}{4})\, c_{1,n_1}^+ v_{n_1}^+ & \text{if } \frac{1}{4} \leqslant t \leqslant \frac{3}{4}\,, \\ (8-8t)\, c_{1,n_1}^+ v_{n_1}^+ + (4t-3)\, w_1 & \text{if } \frac{3}{4} \leqslant t \leqslant 1\,. \end{cases}$$

Notice that $w(t) \in \mathfrak{X}^1_{\text{ex}}$ for all t because both w_j and the eigenfields $v_{n_j}^+$ are in $\mathfrak{X}^1_{\text{ex}}$ (recall that the eigenfields of curl are automatically smooth because they are also eigenfields of the Hodge Laplacian acting on vector fields). It is also obvious that $w(0) = w_0$ and $w(1) = w_1$. Furthermore, one can see that w is a path of positive helicity. For this, it is enough to use the formula (6.2.6) for the helicity in terms of the coefficients of the eigenfunction expansion. Indeed, since $\mathcal{H}(w_j) = c$, an elementary computation then yields

$$\mathcal{H}(w(t)) = \begin{cases} 16t \frac{(c_{0,n_0}^+)^2}{\lambda_0^+} + (1-4t)^2 c & \text{if } 0 \leqslant t \leqslant \frac{1}{4}\,, \\ \frac{4(c_{0,n_0}^+)^2}{\lambda_{n_0}^+} \cos^2(\pi t - \frac{\pi}{4}) + \frac{4(c_{1,n_1}^+)^2}{\lambda_1^+} \sin^2(\pi t - \frac{\pi}{4}) & \text{if } \frac{1}{4} \leqslant t \leqslant \frac{3}{4}\,, \\ 16(1-t) \frac{(c_{1,n_1}^+)^2}{\lambda_{n_1}^+} + (4t-3)^2 c & \text{if } \frac{3}{4} \leqslant t \leqslant 1\,, \end{cases}$$

provided that $n_0 \neq n_1$, so $\mathcal{H}(w(t)) > 0$. When $n_0 = n_1$, the only change in the formula above is that the value of $\mathcal{H}(w(t))$ is

$$\frac{4\Big(\cos(\pi t - \frac{\pi}{4})c^+_{0,n_0} + \sin(\pi t - \frac{\pi}{4})c^+_{1,n_1}\Big)^2}{\lambda^+_{n_0}}$$

if $\frac{1}{4} \leqslant t \leqslant \frac{3}{4}$, which is also positive. This proves the connectedness of $\mathcal{H}^{-1}(c)$ when $c > 0$.

The case where the constant c is negative is completely analogous so, in order to finish the proof of the claim, it only remains to show that the zero level set $\mathcal{H}^{-1}(0)$ is path connected too. This is immediate because two vector fields $w_0, w_1 \in \mathfrak{X}^1_{\text{ex}}$ with $\mathcal{H}(w_0) = \mathcal{H}(w_1) = 0$ can be joined through the continuous path of zero helicity $w : [0,1] \to \mathfrak{X}^1_{\text{ex}}$ given by

$$w(t) := \begin{cases} (1-2t)w_0 & \text{if } 0 \leqslant t \leqslant \frac{1}{2}\,, \\ (2t-1)w_1 & \text{if } \frac{1}{2} \leqslant t \leqslant 1\,. \end{cases}$$

Obviously $w(0) = w_0$, $w(1) = w_1$ and $\mathcal{H}(w(t)) = 0$ for all t, so the claim follows.

Step 5: The regular integral invariant $\mathcal{I}$ is a function of the helicity.

We have shown in Step 3 that the derivatives of the functional $\mathcal{I}$ and the helicity $\mathcal{H}$ are related by $(D\mathcal{I})_w = \mathcal{C}(w)(D\mathcal{H})_w$ at any $w \in \mathfrak{X}^1_{\text{ex}}\backslash\{0\}$. In particular, this implies that $\mathcal{I}$ is constant on each path connected component of the level set $\mathcal{H}^{-1}(c)\backslash\{0\}$. If $c \neq 0$, since 0 is not contained in $\mathcal{H}^{-1}(c)$, the aforementioned level set is path connected as proved in Step 4. The level set $\mathcal{H}^{-1}(0)$ of zero helicity contains the 0 vector field, so the set $\mathcal{H}^{-1}(0)\backslash\{0\}$ does not need to be connected. However, since any component of $\mathcal{H}^{-1}(0)\backslash\{0\}$ is path connected with 0 as shown in the last paragraph of Step 4, the continuity of the functional $\mathcal{I}$ in $\mathfrak{X}^1_{\text{ex}}$ implies that it takes the same constant value on any connected component of $\mathcal{H}^{-1}(0)\backslash\{0\}$, so it is constant on the path connected level set $\mathcal{H}^{-1}(0)$. We conclude that there exists a function $f : \mathbb{R} \to \mathbb{R}$ which assigns a value of $\mathcal{I}$ to each value of the helicity, i.e., $\mathcal{I} = f(\mathcal{H})$. Moreover, f is of class C^1 because $\mathcal{I}$ is a C^1 functional. The main theorem is then proved.

The only part of the proof where it is crucially used that the regularity of the vector fields is C^1 is in Step 3, when we invoke Bessa's theorem for generic vector fields in $\mathfrak{X}^1_{\text{ex}}$. To our best knowledge, it is not known if there is a residual subset of the space $\mathfrak{X}^k_{\text{ex}}$ of exact divergence-free vector fields of class C^k, with $1 < k \leqslant \infty$, whose elements do not admit a C^{k-1} first integral. In particular, for $k > 3$ the KAM theorem [43] implies that there is no a residual subset of $\mathfrak{X}^k_{\text{ex}}$ whose elements are topologically transitive vector fields, thus showing that Bessa's theorem does not hold for these spaces and hence it cannot be used to address the problem of the existence of a first integral for a generic vector field. Apart from the topological transitivity, we are not aware of other properties of a dynamical system implying that a vector field does not admit a (nontrivial) continuous first integral. The lack of results in this direction prevents us from extending the main theorem to regular integral invariants acting on $\mathfrak{X}^k_{\text{ex}}$ with $k > 1$.

Acknowledgements

This thesis would not be the same without the generous and enlightening guidance of Daniel Peralta-Salas. He has been always unsparing in his time and energy, and has shared with me many insights. To him I would like to express, first and foremost, my gratitude.

I am also very thankful to Alberto Enciso, with whom I had the privilege to work during these years, and from whom I learned a lot.

A third person whose generosity towards me, and whose manifold influences on this work I want to ackowledge, is Francisco Presas.

I want also to express my gratitude to Manuel de León, the director of the ICMAT for most of the duration of this thesis, for his support and for making the ICMAT what it is now. I wish to thank also Oscar García-Prada, who first hosted me at ICMAT and who was very supportive and generous with me. Many thanks to Ana Torres, Eduardo Frechilla, and all the rest of the administrative staff working here as well: they have been always eager to help me, and their efficiency and kindness made my arrival at ICMAT and my work much easier.

My colleagues at ICMAT have also taught me many things. I will like to thank Álvaro del Pino, Miguel Vaquero, David Hartley, Juan Ángel Rojo, Carlos Pastor, Angel David Martínez, María de los Angeles García-Ferrero and Bruno Vergara, to mention a few.

During the third year of my PhD I profited from a fruitful stay in the West Coast of the U.S. I want to thank Clifford Taubes for providing me with the opportunity to spend the spring semester at Harvard University Mathematics Department. I want to thank also Simon Donaldson for giving me the chance to visit the SCGP, for taking an interest in my work and for sharing with me his insights. I am also very grateful to Dennis Sullivan, who has been very generous to me and with whom I have had several enlightening discussions. I also enjoyed very much a visit to Stefan Steinerberger at Yale University. I am very grateful to him for sharing with me some very interesting results and conjectures, and for his generous support and advice.

I acknowledge the financial support of the European Research Council, through the ERC Starting Grant 335079.

This thesis is also the outcome of my three years of residence at the Residencia de Estudiantes, under a Fellowship from the Madrid City Council. The Residencia provided an exceptional and stimulating environment for the pursue of my research; a great debt is owed to all the people behind this institution. I want to thank its director, Alicia Gómez-Navarro, and the rest of the management team, Pablo Martín Acea and Rosario Romero. I want also to express my gratitude towards Anabel Hernández and Beatriz Pablos, and towards all the rest of the staff, colleagues and friends that I made there.

I also want to say thank you to Paula.

Finalmente, quiero dar las gracias a mi madre, a mi padre, a mi hermano y a mis abuelos. Esta tesis se la dedico a ellos.

Bibliography

[1] R. Abraham and J. Robbin, *Transversal mappings and flows*, Benjamin, New York, 1967.

[2] P.M. Akhmetev, Quadratic helicities and the energy of magnetic fields. Proc. Steklov Inst. Math. 278 (2012) 10–21.

[3] V.I. Arnold, Sur la topologie des écoulements stationnaires des fluides parfaits, C. R. Acad. Sci. Paris 261 (1965) 17–20.

[4] V.I. Arnold, The asymptotic Hopf invariant and its applications. Selecta Math. Soviet. 5 (1986) 327–345.

[5] V.I. Arnold, B. Khesin, *Topological methods in hydrodynamics*, Springer, New York, 1999.

[6] S. Baader, Asymptotic concordance invariants for ergodic vector fields. Comment. Math. Helv. 86 (2011) 1–12.

[7] S. Baader, J. Marché, Asymptotic Vassiliev invariants for vector fields. Bull. Soc. Math. France 140 (2012) 569–582.

[8] C. Băr, The Dirac operator on space forms of positive curvature, J. Math. Soc. Japan 48 (1996) 1, 69–83.

[9] C. Băr, The Dirac operator on hyperbolic manifolds of finite volume, J. Differential Geom. 54 (2000) 3 439–488.

[10] C. Băr, Zero sets of solutions to semilinear elliptic systems of first order. Invent. Math. 138 (1999) 1, 183–202.

[11] M.A. Berger, Third-order link integrals. J. Phys. A 23 (1990) 2787–2793.

[12] M. Berry, M. Dennis, Knotted and linked phase singularities in monochromatic waves. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. 457 (2001), 2013, 2251–2263.

[13] M. Berry, Knotted zeros in the quantum states of hydrogen. Found. Phys. 31 (2001) 659–667

[14] M. Bessa, A generic incompressible flow is topological mixing. C. R. Math. Sci. Paris 346 (2008) 1169–1174.

[15] D. Chae, P. Constantin, Remarks on a Liouville-type theorem for Beltrami flows, Int. Math. Res. Notices 20 (2015) 10012–10016.

[16] A. Choffrut, V. Sverak, Local structure of the set of steady-state solutions to the 2D incompressible Euler equations, Geom. Funct. Anal. 22 (2012) 136–201.

[17] H. Dlubek, Th. Friedrich, Spektraleigenschaften des Dirac-Operators–die Fundamentallősung seiner Wărmeleitungsgleichung und die Asymptotenentwicklung der Zeta-Funktion, J. Differential Geom. 15 (1980) 1 1–26.

[18] H. Donnelly and C. Fefferman, Nodal sets for eigenfunctions of the Laplacian on surfaces, J. Amer. Math. Soc. 3 (2) (1990) 333–353.

[19] W. Duke, Hyperbolic distribution problems and half-integral weight Maass forms, Invent. Math. 92 (1988) 73–90.

[20] W. Duke, Rational points on the sphere, Ramanujan J. 7 (2003) 235–239.

[21] A. Enciso, M. de los Angeles Garcia-Ferrero, D. Peralta-Salas, The Biot-Savart operator of a bounded domain, arXiv:1702.04327

[22] A. Enciso, D. Hartley, D. Peralta-Salas, A problem of Berry and knotted zeros in the eigenfunctions of the harmonic oscillator, J. Eur. Math. Soc. 20 2 (2018) 301–314.

[23] A. Enciso, D. Hartley, D. Peralta-Salas, Dislocations of arbitrary topology in Coulomb eigenfunctions, arXiv:1603.03214v2

[24] A. Enciso, D. Peralta-Salas, Existence of knotted vortex tubes in steady Euler flows, Acta Math. 214 (2015) 61–134.

[25] A. Enciso, D. Peralta-Salas, Knotted vortex lines and vortex tubes in stationary fluid flows, EMS Newsletter 96 (2015) 26–33.

[26] A. Enciso, D. Peralta-Salas, Knots and links in steady solutions of the Euler equation, Ann. of Math. 175 (2012) 345–367.

[27] A. Enciso, D. Peralta-Salas, Submanifolds that are level sets of solutions to a second-order elliptic PDE, Adv. Math. 249 (2013) 204–249.

[28] A. Enciso, D. Peralta-Salas, F. Torres de Lizaur, Helicity is the only integral invariant of volume-preserving transformations, Proc. Natl. Acad. Sci. 113 (2016) 2035–2040.

[29] A. Enciso, D. Peralta-Salas, F. Torres de Lizaur, Knotted structures in high-energy Beltrami fields on the torus and the sphere, Ann. Sci. Ec. Norm. Sup. 50 (2017) 4, 995–1016.

[30] J. Etnyre, R. Ghrist, Contact topology and hydrodynamics III, Knotted orbits, Trans. Amer. Math. Soc. 352 (2000) 5781–5794.

[31] G.B. Folland, Harmonic analysis of the de Rham complex on the sphere, J. Reine Angew. Math. 398 (1989) 130–143.

[32] J.M. Gambaudo, E. Ghys, Signature asymptotique d'un champ de vecteurs en dimension 3. Duke Math. J. 106 (2001) 41–79.

[33] D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*. Grundlehren der Mathematischen Wissenschaften, Vol. 224. Springer-Verlag, Berlin-New York, 1977.

[34] M. Gromov, *Partial differential relations*. Ergebnisse der Mathematik und ihrer Grenzgebiete (3) 9. Springer-Verlag, Berlin,1986.

[35] M. Gromov, H. Blaine Lawson, Positive scalar curvature and the Dirac operator on complete Riemannian manifolds, Inst. Hautes tudes Sci. Publ. Math. 58 (1983) 83–196.

[36] Y. Giga, Z. Yoshida, Remarks on spectra of operator rot. Math. Z. 204 (1990) 235–245.

[37] L. Hörmander, *The analysis of linear partial differential operators I*, Springer, Berlin, 2003.

[38] A. Jaffe, C. Taubes, Vortices and monopoles. Structure of static gauge theories, Progress in Physics 2 Birkhäuser, Boston, Mass., 1980.

[39] D. Kleckner and W.T.M. Irvine, Creation and dynamics of knotted vortices in fluid, Nature Physics 9 (2013) 253–258.

[40] D. Kleckner, L. H. Kauffman, and W.T.M. Irvine, How superfluid vortex knots untie, Nature Physics 12, 7 (2016) 650–655.

[41] B. Khesin, Topological fluid dynamics, Notices Amer. Math. Soc. 52 (2005) 9–19.

[42] B. Khesin, Geometry of higher helicities. Moscow Math. J. 3 (2003) 989–1011.

[43] B. Khesin, S. Kuksin, D. Peralta-Salas, KAM theory and the 3D Euler equation. Adv. Math. 267 (2014) 498–522.

[44] R. Komendarczyk, The third order helicity of magnetic fields via link maps. Commun. Math. Phys. 292 (2009) 431–456.

[45] R. Komendarczyk, I. Volic, On volume-preserving vector fields and finite-type invariants of knots. Ergo. Theor. Dynam. Syst., in press.

[46] P. Kronheimer, T. Mrowka, *Monopoles and Three-manifolds*, Cambridge University Press, 2007.

[47] E.A. Kudryavtseva, Conjugation invariants on the group of area-preserving diffeomorphisms of the disk. Math. Notes 95 (2014) 877–880.

[48] E.A. Kudryavtseva, Helicity is the only invariant of incompressible flows whose derivative is continuous in C^1-topology. Math. Notes 99 (2016) 626–630; ArXiv:1511.03746v2.

[49] G. Kuperberg, A volume-preserving counterexample to the Seifert conjecture, Comment. Math. Helv. 71 (1996) 1 70–97.

[50] K. Kuperberg, A smooth counterexample to the Seifert conjecture, Ann. of Math. 2 140 (1994) 3 723–732.

[51] P. Laurence, E. Stredulinsky, Asymptotic Massey products, induced currents and Borromean torus links. J. Math. Phys. 41 (2000) 3170–3191.

[52] A. Logunov, Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure, arxiv:1605.02587

[53] A. Logunov, Nodal sets of Laplace eigenfunctions: proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture, arxiv:1605.02589.

[54] A. Logunov, E. Malinnikova, Nodal sets of Laplace eigenfunctions: estimates of the Hausdorff measure in dimension two and three, arxiv:1605.02595

[55] M. Macri, M. Nolasco, T. Ricciardi, Asymptotics for selfdual vortices on the torus and on the plane: a gluing technique, S.I.A.M. J. Math. Anal. 37 (2005) 1 1–16

[56] W. S. Massey, On the normal bundle of a sphere imbedded in Euclidean space, Proc. Amer. Math. Soc. 10 (1959) 959–964.

[57] H.K. Moffatt, The degree of knottedness of tangled vortex lines, J. Fluid Mech. 35 (1969) 117–129.

[58] K. Moffatt, Helicity and singular structures in fluid dynamics, Proc. Natl. Acad. Sci. 111 (2014) 3663-3670.

[59] J. Moser, E. Zehnder, *Notes on dynamical systems*, AMS, New York, 2005.

[60] N. Nadirashvili, On stationary solutions of two-dimensional Euler equation, Arch. Rat. Mech. Anal. 209 (2013) 729–745.

[61] N. Nadirashvili, Liouville theorem for Beltrami flow, Geom. Funct. Anal. 24 (2014) 916–921.

[62] R. Narasimhan, *Analysis on real and complex manifolds*, North-Holland Publishing Co., Amsterdam, 1985.

[63] K. Oka, Sur les fonctions analytiques de plusieurs variables III: Deuxième problème de Cousin, Journ. Sci. Hiroshima Univ. 9 (1939) 7–19 (translated by R. Narasimhan in: *K. Oka, Collected papers*, Springer-Verlag, Berlin, 1984).

[64] D. Serre, Les invariants du premier ordre de l'equation d'Euler en dimension trois. Phys. D 13 (1984) 105–136.

[65] G. Szegő, *Orthogonal polynomials*, AMS, Providence, 1975.

[66] C. Taubes, An observation concerning uniquely ergodic vector fields on 3-manifolds, J. Gőkova Geom. Topol. 3 (2009) 9–21.

[67] C. Taubes, Arbitrary N-vortex solutions to the first order Ginzburg-Landau equations, Comm. Math. Phys. 72 3 (1980) 277–292.

[68] C. Taubes, The Seiberg-Witten equations and the Weinstein conjecture, Geom. Topol. 11 (2007) 2117–2202.

[69] F. Torres de Lizaur, Geometric structures in eigenfunctions of Dirac operators, arXiv:1712.10310.

[70] W. Thomson (Lord Kelvin), On vortex atoms, Proc. Roy. Soc. Edin. 6 (1867) 94–105 (reprinted in: *Mathematical and physical papers IV*, Cambridge University Press, Cambridge, 2011).

[71] C. Vafa, E. Witten, Eigenvalue inequalities for fermions in gauge theories, Comm. Math. Phys. 95 (1984) 3 257–276.

[72] T. Vogel, On the asymptotic linking number, Proc. Amer. Math. Soc. 131 (2003), 2289–229

[73] E. Witten, A new proof of the positive energy theorem, Comm. Math. Phys. 80 (1981) 3 381–402.

[74] E. Witten, Supersymmetry and Morse theory. J. Differential Geom. 17 (1982) 4, 661–692.

[75] S.T. Yau, Survey on partial differential equations in differential geometry. Seminar on Differential Geometry, Ann. of Math. Stud., 102, pp 3–71, Princeton Univ. Press, Princeton, N.J., 1982.