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Quasi-local energy and compactification

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Abstract: Based on the quasi-local energy definition of Brown and York, we compute the integral of the trace of the extrinsic curvature over a codimension-2 hypersurface. In particular, we study the difference between the uncompactified Minkowski spacetime and the toroidal Kaluza-Klein compactification. For the latter, we find that this quantity interpolates between zero and the value for the uncompactified spacetime, as the size of the compact dimension increases. Thus, the quasi-local energy is able to discriminate between the two spacetimes.

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Contents

1 Introduction

One of the first problems encountered when trying to understand quantum effects in gravity (confer [1] and references therein) is that there is no available energetic argument in order to determine the ground state of the theory. This is one of the many aspects in which gravity differs from the other fundamental interactions, where there is a well-defined hamiltonian which is supposed to be minimized by the vacuum of the theory.

In the case of gravitation, any Ricci-flat spacetime is a priori a valid candidate to a ground state and it is believed that different asymptotics are in different energy sectors. Therefore, it does not have physical sense to compare the respective energies, even in the few cases in which they can be computed (essentially the ADM or the Bondi mass) [2, 3].

One would like to have a criteria to discriminate, for example, between a n-dimensional Ricci-flat spacetime from another n-dimensional Ricci-flat spacetime with some dimensions compactified; that is a Kaluza-Klein [4, 5] type of vacuum. E. Witten [6] has been able to show that the five-dimensional Kaluza-Klein vacuum is semiclassically unstable; but no general energetic argument is available.

Recently, however, a more general concept of gravitational energy has been proposed (see [7] for a recent review), namely *quasi-local energy* (QLE). The fact that it is a quasi-local quantity makes it suitable to compare spacetimes with different asymptotics. There are several definitions of QLE in the literature [8–14]. Here we follow the one by Brown and York [11]. The main idea is to associate to a given hypersurface of a spacetime, $\Sigma \hookrightarrow \mathcal{M}$, the integral of the trace of the second fundamental form. Schematically,[1]

$$Q(\Sigma) \equiv \int_{\Sigma} K - E_0, \tag{1.1}$$

where the zero-point energy E_0 is computed by an isometric embedding of the hypersurface in $\mathbb{R}^3$ (or else in M_4 in other versions [14]).

[1] We are working in units where $\kappa^2 = 1$.

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The aim of the present paper is to begin the exploration of the QLE in toroidal spacetimes and compare it to the corresponding uncompactified spacetime. In this preliminary investigation we are going to discuss very simple examples, for which we believe the discussion of the zero-point energy to be less relevant.

To set up our notation, consider a codimension-p hypersurface Σ embedded in an ambient spacetime $\mathcal{M}$ of dimension n and Lorentzian signature, $\Phi : \Sigma \to \mathcal{M}$. Let y^α and x^a be two coordinate systems on $\mathcal{M}$ and Σ, respectively, with $\alpha = 1, \ldots, n$ and $a = 1, \ldots, m$, where $m = n - p$ is the dimension of Σ. The embedding is defined by the equations

$$\Phi : y^\alpha = y^\alpha(x^a). \tag{1.2}$$

Denoting by $g_{\mu\nu}(y)$ the metric in the ambient manifold, the induced metric on the hypersurface is given by

$$h_{ab}(x) \equiv g_{\alpha\beta}\left(y\left(x\right)\right) \frac{\partial y^\alpha}{\partial x^a} \frac{\partial y^\beta}{\partial x^b}. \tag{1.3}$$

The m vectors on the tangent space to the ambient manifold, $\mathcal{T}(\mathcal{M})$, tangent to the hypersurface are given by

$$t_a^\alpha \equiv \frac{\partial y^\alpha}{\partial x^a}, \tag{1.4}$$

In the useful reference [15] it is proved the fact that if $h \equiv \det(h_{ab}) \neq 0$, then there are p (as many as the codimension of the hypersurface) real mutually orthogonal vectors normal to Σ, none of which are null. Let us denote them by $n_A \in \mathcal{T}(\mathcal{M})$, $A = 1, \ldots, p$.

$$n_A \cdot n_B = \epsilon(A)\ \delta_{AB} \tag{1.5}$$

where $\epsilon(A) = \pm 1$.[2] The generalization of the second fundamental form is the set of p symmetric tensors given by

$$K^A_{ab} \equiv n^A_\alpha t^\lambda_b \nabla_\lambda t^\alpha_a = -t^\lambda_b t^\alpha_a \nabla_\lambda n^A_\alpha, \tag{1.6}$$

where the orthogonality $n^A \cdot t_a = 0$ has been used.

We are interested in the integral

$$\int_\Sigma \sqrt{h}\ K^\alpha n_\alpha dS, \tag{1.7}$$

where dS is the surface element of the hypersurface Σ and

$$K^\alpha = h^{ab} K^\alpha_{ab} \equiv h^{ab} K^A_{ab} n^\alpha_A. \tag{1.8}$$

The paper is organized as follows. In the next section we compute (1.8) in the case of 5-dimensional flat spacetime. In section 3 we repeat the calculation for the compactified spacetime $M_4 \times S_1$. Then, in section 4 we study the stationary points of the QLE before we end up with some conclusions.

[2] Our metric conventions are $(+ - - - \ldots)$.

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2 Codimension-2 spheres in M_5

Consider a codimension-2 spacelike hypersurface in 5-dimensional flat space

$$ds^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta . \tag{2.1}$$

Let the hypersurface be a 3-sphere defined by the embedding

$$\begin{aligned} y_1 &= T, \\ \sum_{i=2}^{i=5} (y_i)^2 &\equiv L^2, \end{aligned} \tag{2.2}$$

where latin indices $i, j, \ldots$ denote spatial coordinates. The normal vectors are given by

$$n_A \equiv \left(\frac{\partial}{\partial t}, \frac{y^i}{L} \frac{\partial}{\partial y^i} \right). \tag{2.3}$$

In this setup, it is plain that the only normal vector with non-vanishing derivative is the last one

$$n \equiv n_2 = \frac{y^i}{L} \frac{\partial}{\partial y^i}. \tag{2.4}$$

It yields

$$\nabla_\beta n^\alpha = \frac{L^2 \delta^\alpha_\beta - y^\alpha y_\beta}{L^3}. \tag{2.5}$$

We have to project this on the tangent space using the tangent vectors t^α_a. In spherical coordinates, the hypersurface can be parametrized as follows

$$\begin{aligned} y_2 &= L \sin\,\theta_1 \,\sin\,\theta_2 \,\sin\,\theta_3 \\ y_3 &= L \sin\,\theta_1 \,\sin\,\theta_2 \,\cos\,\theta_3 \\ y_4 &= L \sin\,\theta_1 \cos\,\theta_2 \\ y_5 &= L \cos\,\theta_1 \end{aligned} \tag{2.6}$$

so that the induced metric reads

$$d\sigma^2 = -L^2 \left(d\theta_1^2 + \sin^2\,\theta_1 \, d\theta_2^2 + \sin^2\,\theta_1 \,\sin^2\,\theta_2 \, d\theta_3^2 \right) \tag{2.7}$$

It follows that the tangent vectors t^α_a take the form

$$\begin{aligned} t_{\theta_1} &= L \left(0, \cos\,\theta_1 \,\sin\,\theta_2 \,\sin\,\theta_3, \cos\,\theta_1 \,\sin\,\theta_2 \,\cos\theta_3, \cos\,\theta_1 \,\cos\theta_2, -\sin\theta_1 \right), \\ t_{\theta_2} &= L \left(0, \sin\,\theta_1 \,\cos\,\theta_2 \,\sin\,\theta_3, \sin\,\theta_1 \,\cos\,\theta_2 \,\cos\theta_3, -\sin\theta_1 \,\sin\theta_2, 0 \right), \\ t_{\theta_3} &= L \left(0, \sin\,\theta_1 \,\sin\,\theta_2 \,\cos\,\theta_3, -\sin\,\theta_1 \,\sin\,\theta_2 \,\sin\theta_3, 0, 0 \right), \end{aligned} \tag{2.8}$$

and satisfy

$$y_\alpha t^\alpha_a = 0. \tag{2.9}$$

Their norm $t^2 = \eta_{\alpha\beta} t^\alpha t^\beta$ is given by

$$t^2_{\theta_1} = -L^2\,; \quad t^2_{\theta_2} = -L^2 \,\sin^2\theta_1\,; \quad t^2_{\theta_3} = -L^2 \,\sin^2\theta_1 \,\sin^2\theta_2\,. \tag{2.10}$$

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Hence,[3]

$$K^A_{ab} = \left(0, -\frac{1}{L}\,\delta_{\alpha\beta} t^\alpha_a t^\beta_b\right) \tag{2.12}$$

The integrand of (1.8) is then given by

$$K^\alpha n_\alpha = h^{ab}\ K^\alpha_{ab}\ n_\alpha = \frac{3}{L}. \tag{2.13}$$

The integration measure in this coordinates takes the form

$$\sqrt{h}\ dS = \ L^3 \sin^2\,\theta_1\, \sin\,\theta_2\, d\theta_1 d\theta_2 d\theta_3, \tag{2.14}$$

so that the integral finally yields

$$Q_{M_5} = \int_\Sigma \sqrt{h}\ K^\alpha n_\alpha dS = 6\pi^2 L^2. \tag{2.15}$$

3 Codimension-2 spheres in $M_4 \times S_1$

The metric of the ambient space is now

$$ds^2 = \eta_{\mu\nu} dy^\mu dy^\nu. \tag{3.1}$$

where the last coordinate is compact and has periodicity

$$y_5 = y_5 + 2\pi l, \tag{3.2}$$

and l is the radius of the compact dimension. We assume the same algebraic surface as in the previous case, that is

$$\begin{aligned} y_1 &= T \\ \sum_{i=2}^{i=5} y_i^2 &= L^2. \end{aligned} \tag{3.3}$$

In this case, the spacelike normal vector depends on the compact coordinate, but its expression is the same as in the previous case

$$n = \frac{y_i}{L}\frac{\partial}{\partial y^i}. \tag{3.4}$$

Also, the covariant derivative of the normal vector can still be written as

$$\nabla_\beta n^\alpha = \frac{L^2 \delta^\alpha_\beta - y^\alpha y_\beta}{L^3}. \tag{3.5}$$

[3] The Gauss-Codazzi equations, which relate the ambient Riemann tensor $R_{\alpha\beta\gamma\delta}$ projected on the hypersurface with the Riemann tensor corresponding to the induced metric, $R_{abcd}[h]$,

$$t^\alpha_a t^\beta_b t^\gamma_c t^\delta_d\, R_{\alpha\beta\gamma\delta} = R_{abcd}[h] - \sum_{A=1}^{A=p} \epsilon(A)\, \left(K^A_{ac} K^A_{bd} - K^A_{ad} K^A_{bc}\right). \tag{2.11}$$

provide a useful check of our computations.

For this computation, we parametrize the hypersurface in cartesian coordinates

$$\begin{aligned}
y_2 &= x, \\
y_3 &= y, \\
y_4 &= z, \\
y_5 &= \sqrt{L^2 - x^2 - y^2 - z^2}.
\end{aligned} \tag{3.6}$$

The induced metric then reads

$$h_{ab} = \frac{1}{L^2 - x^2 - y^2 - z^2} \begin{pmatrix} L^2 - y^2 - z^2 & xy & xz \\ xy & L^2 - x^2 - z^2 & yz \\ xz & yz & L^2 - x^2 - y^2 \end{pmatrix} \tag{3.7}$$

The tangent vectors still obey

$$y_\alpha t_a^\alpha = 0. \tag{3.8}$$

Explicitly, they read

$$\begin{aligned}
t_x &= (0, 1, 0, 0, \frac{-x}{\sqrt{L^2 - x^2 - y^2 - z^2}}), \\
t_y &= (0, 0, 1, 0, \frac{-y}{\sqrt{L^2 - x^2 - y^2 - z^2}}), \\
t_z &= (0, 0, 0, 1, \frac{-z}{\sqrt{L^2 - x^2 - y^2 - z^2}}).
\end{aligned} \tag{3.9}$$

The second fundamental form K_{ab}^A then takes the form

$$K_{ab}^1 = 0 \; ; \qquad K_{ab}^2 = \frac{1}{L} \begin{pmatrix} \frac{x^2}{L^2-x^2-y^2-z^2} + 1 & \frac{xy}{L^2-x^2-y^2-z^2} & \frac{xz}{L^2-x^2-y^2-z^2} \\ \frac{xy}{L^2-x^2-y^2-z^2} & \frac{y^2}{L^2-x^2-y^2-z^2} + 1 & \frac{yz}{L^2-x^2-y^2-z^2} \\ \frac{xz}{L^2-x^2-y^2-z^2} & \frac{yz}{L^2-x^2-y^2-z^2} & \frac{z^2}{L^2-x^2-y^2-z^2} + 1 \end{pmatrix} . \tag{3.10}$$

The integrand of (1.8) is again given by

$$K^\alpha n_\alpha = h^{ab} \; K_{ab}^\alpha \; n_\alpha = \frac{3}{L}. \tag{3.11}$$

One has to be careful with the integration range over the compactified coordinate. For small 3-spheres that completly lie within the compact dimension, that is with $L < l\pi$, the integration is done over the full hypersurface, so that $-L \leq y_5 \leq L$. On the other hand, when $L > l\pi$, there are self intersections of the hypersurface, due to the periodicity of the compact dimension. Thus, the integration range is restricted to $-l\pi \leq y_5 \leq l\pi$, as can be seen in figure 1. We obtain

$$\begin{aligned}
Q_{M_4 \times S_1} &= 6\pi^2 L^2 && \text{for} \quad L \leq l\pi, \\
Q_{M_4 \times S_1} &= 12\pi^2 l \sqrt{L^2 - \pi^2 l^2} + 12\pi L^2 \tan^{-1}\left(\frac{\pi l}{\sqrt{L^2 - \pi^2 l^2}}\right) && \text{for} \quad L > l\pi.
\end{aligned} \tag{3.12}$$

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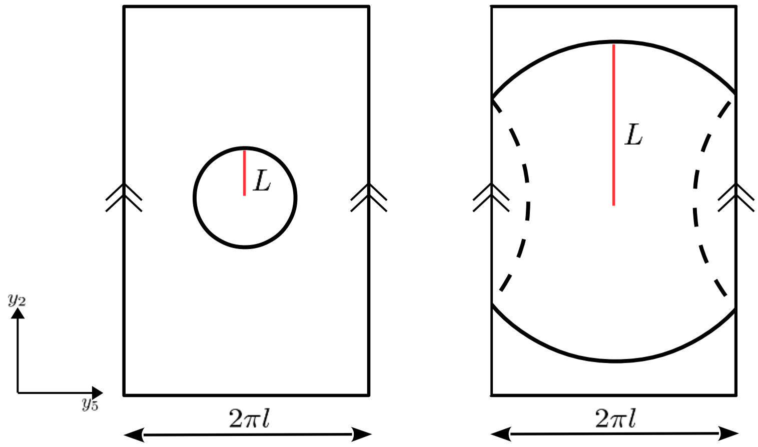

Figure 1. Different types of hypersurface depending on whether $L > \pi l$ or $L < \pi l$. For simplicity, we only show the compact dimension, y_5, and one extended dimension, y_2.

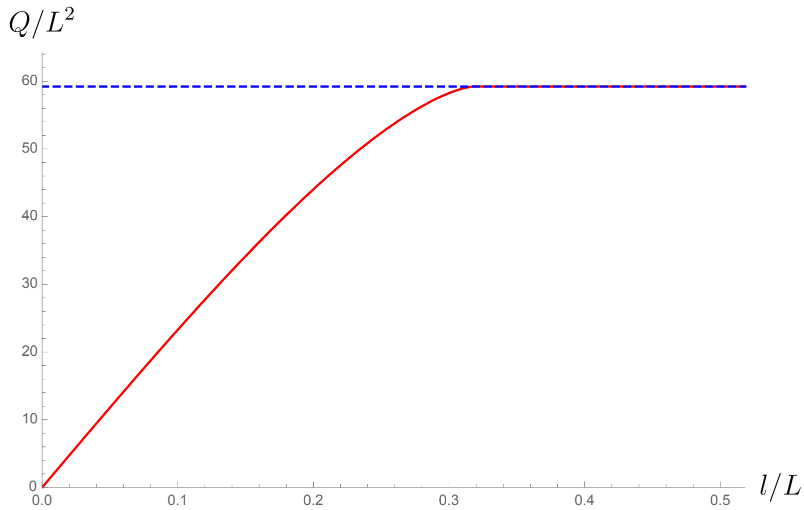

Figure 2. Comparison between the QLE for M_5 (dashed blue) and $M_4 \times S_1$ (red).

As expected, in the decompactification limit where $l \to \infty$ for any finite L, the QLE for $M_4 \times S_1$ is that of M_5. In fact, this happens whenever $L \leq l\pi$, since the hypersurface does not see the periodicity of the compact dimension; thus, the QLE cannot distinguish between $M_4 \times S_1$ and M_5. When $L > l\pi$, as can be seen in figure 2, the QLE monotonically decreases to zero as $l \to 0$.

4 Stationary points of the QLE

Let us study the stationary points of the QLE integral under variations of the spacetime ambient metric, keeping fixed the equation for the embedding

$$\delta Q \equiv \delta \int \sqrt{h}\, d^{n-2}x\, h^{ab}\, t_a^\alpha\, t_b^\beta\, \nabla_\alpha n_\beta = 0. \tag{4.1}$$

From the normalization of the normal vectors we have

$$g_{\alpha\beta}n_A^\alpha n_B^\beta = \eta_{AB} \Longrightarrow \delta n^\alpha = -g^{\alpha\gamma}\delta g_{\gamma\beta}n^\beta, \tag{4.2}$$

where from now onwards we will omit the label A in the normal vectors. Orthogonality between normal and tangent vectors implies

$$\delta\left(n^\alpha g_{\alpha\beta}t_a^\beta\right) = 0, \tag{4.3}$$

and note that

$$\delta n_\alpha = \delta\left(g_{\alpha\beta}n^\beta\right) = 0. \tag{4.4}$$

Let us define the auxiliary tensor

$$G^{\alpha\beta} \equiv h^{ab}t_a^\alpha t_b^\beta, \tag{4.5}$$

(remember that the tangent vectors we are using are not normalized), in such a way that

$$G^{\alpha\beta}g_{\alpha\beta} \equiv G_\alpha^\alpha = h^{ab}h_{ab} = n-2. \tag{4.6}$$

The determinant of the induced metric also varies

$$\delta h = hh^{ab}t_a^\alpha t_b^\beta \delta g_{\alpha\beta} = hG^{\alpha\beta}\delta g_{\alpha\beta}, \tag{4.7}$$

because

$$\delta h_{ab} = t_a^\alpha t_b^\beta \delta g_{\alpha\beta}\ ; \qquad \delta h^{ab} = -h^{ac}h^{bd}\delta h_{cd}. \tag{4.8}$$

Thus, the variation of the QLE reads

$$\delta Q = \int \sqrt{h}d^nx \left\{\frac{1}{2}\delta g_{\alpha\beta}G^{\alpha\beta}h^{ab}t_a^\mu t_b^\nu \nabla_\mu n_\nu - h^{ac}h^{bd}t_c^\mu t_d^\nu \delta g_{\mu\nu}t_a^\alpha t_b^\beta \nabla_\alpha n_\beta + \right.$$
$$\left. -\frac{1}{2}G^{\alpha\beta}g^{\gamma\delta}n_\gamma\left(-\nabla_\delta \delta g_{\alpha\beta} + \nabla_\alpha \delta g_{\beta\delta} + \nabla_\beta \delta g_{\alpha\delta}\right)\right\}. \tag{4.9}$$

It is not possible in general to integrate by parts, because

$$g \neq h. \tag{4.10}$$

It would be interesting to study classes of solutions to those integral-differential equations. In the particular case where the variation of the metric is assumed to be covariantly constant

$$\nabla_\gamma \delta g_{\alpha\beta} = 0, \tag{4.11}$$

the equations reduce to the much simpler condition

$$K_{ab}t^{a\alpha}t^{b\beta} = \frac{1}{2}KG^{\alpha\beta}, \tag{4.12}$$

where $K = K^\alpha n_\alpha$. For *umbilic surfaces* where the extrinsic curvature is proportional to the induced metric

$$K_{ab} = \lambda h_{ab}, \tag{4.13}$$

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Eq. (4.12) reduces to

$$\lambda h_{ab} t^{a\alpha} t^{b\beta} = \frac{1}{2} \lambda h^{ab} h_{ab} h_{cd} t^{c\alpha} t^{d\beta},$$
$$\lambda = \frac{\lambda}{2}(n-2), \tag{4.14}$$

so that it implies

$$\lambda = 0 \quad \text{or} \quad n = 4. \tag{4.15}$$

We leave the study of more complex hypersurfaces for further investigation.

5 Conclusions

We have begun to apply some preliminary ideas on quasi-local energy to the simplest instances of toroidal compactifications, and we have found somewhat surprisingly, that this observable can be sensitive to it. We take this fact as an encouragement to pursue this set of ideas, with the final objective in mind of being able to apply energetic arguments to the study of the ground state of fundamental physics including gravity.

In particular, when one dimension is allowed to compactify, we find that there is a runaway behavior of sorts, and the configuration that minimizes the QLE corresponds to this dimension disappearing completely. It is even possible that this behavior is not unrelated to the old problem of stabilization of extra dimensions in a Kaluza-Klein setting (confer [16, 17], and references therein).

We have also determined the equations that make stationary the QLE under arbitrary variations of the spacetime metric. They are quite complicated, but seem worthy of further consideration.

We are aware that we are exploring uncharted waters here. Ours are only preliminary ideas. The main difference between Brown and York's definition of quasi-local energy and the more evolved definitions (there are several of them) of Yau and collaborators [13, 14], lies in the term that is substracted to the extrinsic curvature. This is crucial in order to get positivity and to define the zero of energy in flat spacetimes. We have evaded the consideration of those thorny issues by working in flat spacetimes, but we will tackle them in future work. Previous definitions, for example the one of Hawking [8], have not been considered since they yield a non-vanishing energy for certain hypersurfaces in flat space [18].

There are several lines of further work that can be pursued. It would be interesting to analyze the cases where not all the compact dimensions are contained in the hypersurface, as well as to study more general compact geometries. This includes, in particular, the energetics of fluxes in non-trivial cycles [19] as compared with the same geometry without the fluxes. Although we considered a simple example, it should be straightforward to generalize it to higher dimensional compact spaces. The *holy grail* we are looking for is to estimate the definition of gravitational energy by just considering a small portion of the interior of the spacetime, without necessarily worrying about its asymptotic behavior. This will hopefully lead to a *quasi-local Wheeler-de Witt equation.* Work on those issues is also in progress.

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