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LOWER SEMICONTINUITY AND RELAXATION VIA YOUNG MEASURES FOR NONLOCAL VARIATIONAL PROBLEMS AND APPLICATIONS TO PERIDYNAMICS

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Abstract. We study nonlocal variational problems in L^p, like those that appear in peridynamics. The functional object of our study is given by a double integral. We establish characterizations of weak lower semicontinuity of the functional in terms of nonlocal versions of either a convexity notion of the integrand, or a Jensen inequality for Young measures. Existence results, obtained through the direct method of the Calculus of variations, are also established. We cover different boundary conditions, for which the coercivity is obtained from nonlocal Poincaré inequalities. Finally, we analyze the relaxation (that is, the computation of the lower semicontinuous envelope) for this problem when the lower semicontinuity fails. We state a general relaxation result in terms of Young measures and show, by means of two examples, the difficulty of having a relaxation in L^p in an integral form. At the root of this difficulty lies the fact that, contrary to what happens for local functionals, non-positive integrands may give rise to positive nonlocal functionals.

Key words. Lower semicontinuity, relaxation, Young measures, nonlocal variational problems, peridynamics

AMS subject classifications. 26B25, 34B10, 49J45, 74B20, 74G65

1. Introduction.

This paper studies functionals I of the form

$$I(u) = \int_\Omega \int_\Omega w(x, x', u(x), u(x'))\, \mathrm{d}x\, \mathrm{d}x',$$

where $\Omega \subset \mathbb{R}^n$ is an open subset, $u : \Omega \to \mathbb{R}^d$ is in some Lebesgue space L^p, and the integrand $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ has some measurability and continuity properties. This kind of functionals appears in many contexts in the mathematical modelling of some processes, whose common feature is their *nonlocal* nature; we mention here micromagnetics [38], phase transitions [4], peridynamics [39], pattern formation [25], image processing [27], population dispersal [20], diffusion [8] and optimal design [5]. It also has applications in the characterization of Sobolev spaces [16]. Finally, although not strictly relevant to the current work, functionals of the style of I share many features with the (linear and nonlinear) fractional Laplacian [18, 24]. Our main motivation, though, comes from peridynamics: in this context, Ω represents the body in its reference configuration, u is the deformation of the body and I is the energy of the deformation.

Apart from the nonlocality, another significant attribute of I is the absence of derivatives of u, which makes the Lebesgue space L^p a natural set of admissible functions. In fact, in most of the examples cited before, the nonlocal derivative-free modelling of I substitutes a more common local model involving derivatives.

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In this work we do not deal with the evolution problem associated to I but rather its equilibrium solutions, in particular, minimizers u of I, which is usually given the interpretation of an energy (the total *macroelastic* potential energy, in the context of peridynamics). We will carry out the direct method of the Calculus of variations in order to establish the existence of minimizers. The two main ingredients of this method are *coercivity* and *lower semicontinuity*; the topology chosen in L^p is the weak topology.

The issue of coercivity has been addressed in several papers [16, 17, 37, 6, 7, 1, 2, 29, 30, 11, 12], and, besides nonlocality, the main difficulty was that typically w vanishes in a great part of the domain, namely, in points $(x, x') \in \Omega \times \Omega$ for which $|x - x'| > \delta$ for some fixed $\delta > 0$ called the *horizon* (or *interaction*) distance. In those papers several nonlocal versions of Poincaré's inequality are given for different cases: Dirichlet, Neumann or mixed boundary conditions. Clarified formulations of Poincaré's inequality useful for our purposes were presented in [11], which will be recalled here for proving the existence results.

Different characterizations of the lower semicontinuity property were obtained in [23, 15, 11, 36] and, within a different context for functionals involving derivatives, in [34, 33]. In this paper we further explore those characterizations of lower semicontinuity, unify the previous approaches, establish their equivalence, and point out and fix misleading statements appearing in some of those references. More explicit formulations of this nonlocal convexity were obtained for the one-dimensional case ($n = 1$) in [32, 13, 19], although, even in this situation, lower semicontinuity is characterized through a nonlocal convexity notion, as shown in a counterexample in [11]. In this paper, weak lower semicontinuity of the functional I in $L^p(\Omega, \mathbb{R}^d)$, for $p \geq 1$, is characterized through two equivalent notions: one involving the convexity of certain integrals (already introduced in [23, 11]), and another one in terms of Young measures (first introduced for functionals depending on derivatives in [34]). We also cover the case $p = 1$, which is treated somewhat separately.

In the absence of lower semicontinuity, the existence of minimizers is not guaranteed and a usual approach is the *relaxation*, which consists in finding the lower semicontinuous envelope of I in the relevant topology. In the classical context of nonlinear elasticity, understanding the relaxation is capital to study the microstructure of the material [10]. Relaxation for nonlocal functionals similar to I but depending on ∇u was first studied in [34, 32, 13, 19]. In this paper, we first analyze the relaxation of the functional I in terms of Young measures. We proceed by extending L^p to the space of Young measures equipped with the narrow topology. We conclude with a relaxed formulation of the functional I in terms of Young measures, so providing a full characterization of the relaxation. In fact, Young measures appear throughout the paper as a useful tool to analyze both lower semicontinuity and relaxation. Good accounts on L^p Young measures can be found in [35, 9, 26].

The relaxation in L^p turns out to be a considerably difficult issue; in fact, the existence of a relaxed formulation in an integral form defined on L^p is not clear at all. In this respect, we construct an explicit example ruling out the natural candidate for the relaxed formulation in the homogeneous case (i.e., when the integrand w does not depend on the independent variables x, x'), namely, the functional in which the integrand w is replaced by its separately convex envelope. In addition, we give another example in which, assuming that there exists a relaxation in L^p of an integral form, we prove that the integrand must be a separately convex function which lies sometimes above and sometimes below the original integrand w. This unexpected fact makes it complicated the possible definition of an integrand of a relaxed formulation of integral form in L^p. This is the first incursion in this issue, but more work in the future will be needed to understand this interesting question.

One of the reasons for the difficulty of the L^p relaxation are the surprising facts appearing in nonlocal functionals. A first unexpected fact is that different integrands w may have the same functional I, which cannot happen in the local case; characterization of nonlocal integrands giving rise to the same functional was given in [23]. Our example for the relaxation in L^p mentioned in the previous paragraph is based on an integrand taking both positive and negative values in sets of positive measure that nevertheless gives rise to a positive functional.

This paper is organized as follows. In Section 2 we set the general notation. Section 3 explains the results on Young measures that will be used throughout the paper. Section 4 collects the results of other works about coercivity and, assuming weak lower semicontinuity of the functional, establishes the existence of minimizers. Section 5 is one of the central parts of this paper: it shows several necessary and sufficient conditions for the weak lower semicontinuity of I in L^p. Section 6 computes the relaxation of I in the space of Young measures. Finally, in Section 7 we make, by means of two examples, some remarks about the difficulty of computing the relaxation of I in $L^p(\Omega, \mathbb{R}^d)$.

2. Notation. In this section we set the general notation of the paper, most of which is standard.

Given $E \subset \mathbb{R}^n$, $C(E)$ is the set of continuous functions in E, while $C_0(E)$ is its subset of functions that vanish at infinity; in other words, a $u \in C(E)$ belongs to $C_0(E)$ whenever for every $\varepsilon > 0$ there exists a compact $K \subset E$ such that $|u(x)| < \varepsilon$ for all $x \in E \setminus K$. The subset of bounded functions in $C(E)$ is denoted by $C_b(E)$, and is endowed with the supremum norm $\|\cdot\|_\infty$.

For $1 \leq p < \infty$, the Lebesgue L^p space is defined in the usual way. This $p \geq 1$ will always be finite. In function spaces, we will indicate the domain and target sets, as in, for example, $L^p(E, \mathbb{R}^d)$, except if the target space is $\mathbb{R}$, in which case we will simply write $L^p(E)$; here E is a measurable set of $\mathbb{R}^n$. The norm in $L^p(E, \mathbb{R}^d)$ is denoted by $\|\cdot\|_{L^p(E,\mathbb{R}^d)}$.

We denote by $\mathcal{M}(E)$ the set of (positive) measures in E. A probablity measure in E is a $\mu \in \mathcal{M}(E)$ such that $\mu(E) = 1$. Given $a \in E$, the Dirac delta at a is denoted by δ_a.

Given $\mu_1 \in \mathcal{M}(E_1)$ and $\mu_2 \in \mathcal{M}(E_2)$, we denote by $\mu_1 \otimes \mu_2 \in \mathcal{M}(E_1 \times E_2)$ its product measure. Analogously, given two functions $u_1 : \Omega_1 \to E_1$ and $u_2 : \Omega_2 \to E_2$, its product $u_1 \otimes u_2 : \Omega_1 \times \Omega_2 \to E_1 \times E_2$ is defined as $(u_1 \otimes u_2)(x_1, x_2) := (u_1(x_1), u_2(x_2))$.

We will deal with two types of measurability: Lebesgue and Borel. Lebesgue measurability will be in a Lebesgue measurable subset Ω of $\mathbb{R}^n$, while Borel measurability will be in $\mathbb{R}^d$. The Lebesgue measure in any Lebesgue measurable subset Ω of $\mathbb{R}^n$ will be denoted by $\mathcal{L}^n$. When we just write *measurable* it means Lebesgue measurable, while when we say $\mathcal{B}^d$-measurable it means Borel measurable in $\mathbb{R}^d$. Likewise, $\mathcal{L}^n \otimes \mathcal{B}^d$-measurable means measurable in $\Omega \times \mathbb{R}^d$ with respect to the product measure. In fact, most of the paper deals with functions defined in $\Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d$ that are $\mathcal{L}^{2n} \otimes \mathcal{B}^{2d}$-measurable. For this kind of functions $w = w(x, x', y, y')$ we will often use expressions like "for a.e. $x, x' \in \Omega$ and all $y, y' \in \mathbb{R}^d$", which means "for all $(x, x', y, y') \in M \times \mathbb{R}^d \times \mathbb{R}^d$, for some $M \subset \Omega \times \Omega$ measurable with $\mathcal{L}^{2n}(\Omega \times \Omega) = \mathcal{L}^{2n}(M)$".

The characteristic function of a $B \subset \mathbb{R}^n$ is denoted by χ_B. The average integral

$\fint_B$ denotes the integral in B divided by $\mathcal{L}^n(B)$. The negative part of a function f is denoted by f^-. Given $A \subset \Omega$, we denote $A^c = \Omega \setminus A$.

Weak convergence in L^p is denoted by $\rightharpoonup$. We will also use *biting* convergence, defined as follows (see, e.g., [35, Sect. 6.4] or [26, Def. 2.65]). We say that $u_j \xrightharpoonup{b} u$ in $L^1(\Omega, \mathbb{R}^d)$ (convergence in the biting sense) when $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^1(\Omega, \mathbb{R}^d)$ and there exists a decreasing sequence $\{E_k\}_{k\in\mathbb{N}}$ of measurable subsets of Ω such that $\mathcal{L}^n(E_k) \to 0$ as $k \to \infty$, and, for any $k \in \mathbb{N}$,

$$u_j \rightharpoonup u \quad \text{in } L^1(\Omega \setminus E_k, \mathbb{R}^d) \qquad \text{as } j \to \infty.$$

Of course, weak convergence in L^1 implies biting convergence. Consequently, if a functional is lower semicontinuous with respect to the biting convergence then it also lower semicontinuous with respect to the weak convergence in L^1.

A function $g : \mathbb{R}^d \times \mathbb{R}^d$ is *separately convex* if $g(\cdot, y)$ and $g(y, \cdot)$ are convex for each $y \in \mathbb{R}^d$.

3. Young measures in L^p. In this section we briefly recall the definitions and results concerning Young measures that are needed in the paper; for the proofs and general expositions, we refer the reader to [40, 41, 35, 9, 26] as well as the references therein. We only provide a proof for those results for which we have not found a precise reference. In this section, we follow the exposition in [9, Sect. 4.3], which is based on Prokhorov's theorem, instead of other usual approaches based on the duality between $L^1(\Omega, C_0(\mathbb{R}^d))$ and $L^\infty(\Omega, \mathcal{M}(\mathbb{R}^d))$.

Let $\Omega \subset \mathbb{R}^n$ be an open bounded subset. A *Young measure* in $\Omega \times \mathbb{R}^d$, equipped with the $\mathcal{L}^n \otimes \mathcal{B}^d$-sigma algebra, is a measure ν in $\Omega \times \mathbb{R}^d$ such that for any measurable $E \subset \Omega$,

$$\nu(E \times \mathbb{R}^d) = \mathcal{L}^n(E).$$

We denote by $\mathcal{Y}(\Omega, \mathbb{R}^d)$ the set of Young measures in $\Omega \times \mathbb{R}^d$.

The procedure of *disintegration* (or *slicing*; see, e.g., [9, Th. 4.2.4]) allows us for an alternative description of Young measures. Accordingly, we can identify ν with a family $(\nu_x)_{x\in\Omega}$ of probability measures on $\mathbb{R}^d$ such that for all $f \in C_0(\Omega \times \mathbb{R}^d)$, the map

$$\Omega \ni x \mapsto \int_{\mathbb{R}^d} f(x, y) \, \mathrm{d}\nu_x(y)$$

is measurable and

$$\int_{\Omega\times\mathbb{R}^d} f(x, y) \, \mathrm{d}\nu(x, y) = \int_\Omega \left(\int_{\mathbb{R}^d} f(x, y) \, \mathrm{d}\nu_x(y) \right) \mathrm{d}x.$$

We write $\nu = (\nu_x)_{x\in\Omega}$, although a more proper notation would be $\nu = \mathcal{L}^n \otimes (\nu_x)_{x\in\Omega}$. That is why Young measures are also called *parametrized measures*. In the sequel, we will use both approaches.

The sets $\mathcal{M}(\Omega \times \mathbb{R}^d)$, and, hence, $\mathcal{Y}(\Omega, \mathbb{R}^d)$ can be given a variety of topologies (see, e.g., [14]). The most relevant to the current work is the narrow topology in $\mathcal{Y}(\Omega, \mathbb{R}^d)$: it is weakest topology that makes the maps

$$\nu \mapsto \int_{\Omega\times\mathbb{R}^d} \varphi(x, y) \, \mathrm{d}\nu(x, y)$$

continuous, for all $\mathcal{L}^n \otimes \mathcal{B}^d$-measurable $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that

$$\varphi(x, \cdot) \in C_b(\mathbb{R}^d) \text{ for a.e. } x \in \Omega \qquad \text{and} \qquad \int_\Omega \|\varphi(x, \cdot)\|_\infty \, \mathrm{d}x < \infty. \tag{1}$$

In particular, it induces the following convergence: a sequence $\{\mu^j\}_{j\in\mathbb{N}} \subset \mathcal{Y}(\Omega,\mathbb{R}^d)$ narrowly converges to a $\mu \in \mathcal{Y}(\Omega,\mathbb{R}^d)$, and write $\mu^j \overset{\text{nar}}{\rightharpoonup} \mu$ in $\mathcal{Y}(\Omega,\mathbb{R}^d)$ as $j\to\infty$, when for all $\mathcal{L}^n\otimes\mathcal{B}^d$-measurable $\varphi : \Omega\times\mathbb{R}^d\to\mathbb{R}$ with property (1), one has

$$\lim_{j\to\infty}\int_{\Omega\times\mathbb{R}^d}\varphi(x,y)\,\mathrm{d}\mu^j(x,y) = \int_{\Omega\times\mathbb{R}^d}\varphi(x,y)\,\mathrm{d}\mu(x,y).$$

Moreover, with the identification $\mu^j = \left(\mu^j_x\right)_{x\in\Omega}$ and $\mu = (\mu_x)_{x\in\Omega}$, we have that $\mu^j \overset{\text{nar}}{\rightharpoonup} \mu$ in $\mathcal{Y}(\Omega,\mathbb{R}^d)$ as $j\to\infty$ if and only if for all $g\in L^1(\Omega)$ and $h\in C_0(\mathbb{R}^d)$,

$$\lim_{j\to\infty}\int_{\Omega} g(x)\left(\int_{\mathbb{R}^d} h(y)\,\mathrm{d}\mu^j_x(y)\right)\mathrm{d}x = \int_{\Omega} g(x)\left(\int_{\mathbb{R}^d} h(y)\,\mathrm{d}\mu_x(y)\right)\mathrm{d}x$$

(see, e.g., [9, Th. 4.3.1], which states that narrow convergence of Young measures and weak convergence of their corresponding probability measures are equivalent). The narrow topology is not metrizable, but the relevance of working with sequences (instead of nets) will become clear in Theorem 3.4; we anticipate that convergence of sequences is enough for the purposes of this work.

The following concept is of central importance.

Definition 3.1. *A set $\mathcal{H}\subset\mathcal{Y}(\Omega,\mathbb{R}^d)$ is tight when for all $\varepsilon>0$ there exists a compact $K\subset\mathbb{R}^d$ such that*

$$\sup_{\nu\in\mathcal{H}}\nu\left(\Omega\times(\mathbb{R}^d\setminus K)\right)<\varepsilon.$$

Prokhorov's theorem states the relative compactness of bounded tight sets of measures (see, e.g., [14, Sect. 5]). When applied to sequences of Young measures, it can be stated as follows.

Theorem 3.2. *Let $\{\nu^j\}_{j\in\mathbb{N}}\subset\mathcal{Y}(\Omega,\mathbb{R}^d)$.*

a) If $\{\nu^j\}_{j\in\mathbb{N}}$ is tight, then there exist a subsequence (not relabelled) and a $\nu\in\mathcal{Y}(\Omega,\mathbb{R}^d)$ such that $\nu^j \overset{\text{nar}}{\rightharpoonup} \nu$ in $\mathcal{Y}(\Omega,\mathbb{R}^d)$ as $j\to\infty$.

b) If there exists $\nu\in\mathcal{Y}(\Omega,\mathbb{R}^d)$ such that $\nu^j \overset{\text{nar}}{\rightharpoonup} \nu$ in $\mathcal{Y}(\Omega,\mathbb{R}^d)$ as $j\to\infty$ then $\{\nu^j\}_{j\in\mathbb{N}}$ is tight.

Tightness can also be characterized by the following criterion, similar in spirit to de la Vallée-Poussin criterion for equiintegrability; see [40, Prop. 8] or [41, Comment 2, p. 369].

Proposition 3.3. *A set $\mathcal{H}\subset\mathcal{Y}(\Omega,\mathbb{R}^d)$ is tight if and only if there exists a function $h:[0,\infty)\to[0,\infty]$ such that*

$$\lim_{t\to\infty}h(t)=\infty \tag{2}$$

and

$$\sup_{\nu\in\mathcal{H}}\int_{\Omega\times\mathbb{R}^d}h(|y|)\,\mathrm{d}\nu(x,y)<\infty.$$

Property (2) is sometimes called *coercivity*. In our finite-dimensional context such h are also characterized by being *inf-compact*; see [9, Sect. 3.2.5].

Despite that fact that the narrow topology is not metrizable, the following result holds; it is a consequence of [40, Thms. 1, 2 and 11].

Theorem 3.4. *Let $\mathcal{H}$ be a tight subset of $\mathcal{Y}(\Omega,\mathbb{R}^d)$. Then the narrow topology on $\mathcal{H}$ is metrizable.*

Any measurable function $u : \Omega \to \mathbb{R}^d$ can be identified with the Young measure $\nu = (\nu_x)_{x\in\Omega}$ given by $\nu_x = \delta_{u(x)}$, i.e.,

$$\int_{\Omega\times\mathbb{R}^d} \varphi(x,y)\,\mathrm{d}\nu(x,y) = \int_\Omega \varphi(x,u(x))\,\mathrm{d}x$$

for all $\varphi \in C_0(\Omega\times\mathbb{R}^d)$. With a small abuse of notation, we can write $u \in \mathcal{Y}(\Omega,\mathbb{R}^d)$; analogously, we can talk about narrow convergence of a sequence of measurable functions, meaning narrow convergence of their associated Young measures. Thus, a sequence $\{u_j\}_{j\in\mathbb{N}}$ of measurable functions from Ω to $\mathbb{R}^d$ is tight if and only if for every $\varepsilon > 0$ there exists $M > 0$ such that

$$\sup_{j\in\mathbb{N}} \mathcal{L}^n\left(\{x\in\Omega : |u_j(x)| > M\}\right) < \varepsilon;$$

equivalently, in view of Proposition 3.3, there exists a function $h : [0,\infty) \to [0,\infty]$ with property (2) such that

$$\sup_{j\in\mathbb{N}} \int_\Omega h\left(|u_j(x)|\right)\,\mathrm{d}x < \infty.$$

In this case, Theorem 3.2 provides a version of the existence result for Young measures (see, e.g, [26, Th. 8.6]).

Proposition 3.5. *Let $\{u_j\}_{j\in\mathbb{N}}$ be a tight sequence of measurable functions from Ω to $\mathbb{R}^d$. Then there exist a subsequence (not relabelled) and a $\nu \in \mathcal{Y}(\Omega,\mathbb{R}^d)$ such that $u_j \overset{\text{nar}}{\rightharpoonup} \nu$ in $\mathcal{Y}(\Omega,\mathbb{R}^d)$ as $j\to\infty$.*

When $u_j \overset{\text{nar}}{\rightharpoonup} \nu$ in $\mathcal{Y}(\Omega,\mathbb{R}^d)$ as $j\to\infty$, we say that the sequence of functions $\{u_j\}_{j\in\mathbb{N}}$ *generates* the Young measure ν. Recall that it means that for all $\mathcal{L}^n\otimes\mathcal{B}^d$-measurable $\varphi : \Omega\times\mathbb{R}^d \to \mathbb{R}$ with property (1), one has

$$\lim_{j\to\infty} \int_\Omega \varphi(x,u_j(x))\,\mathrm{d}x = \int_{\Omega\times\mathbb{R}^d} \varphi(x,y)\,\mathrm{d}\nu(x,y). \tag{3}$$

In fact, the following continuity result shows that the above limit holds for a larger family of test functions (see, e.g., [9, Th. 4.3.3] or [26, Th. 8.6]).

Proposition 3.6. *Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of measurable functions from Ω to $\mathbb{R}^d$ narrowly converging to some ν in $\mathcal{Y}(\Omega,\mathbb{R}^d)$. Let $\varphi : \Omega\times\mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ be $\mathcal{L}^n\otimes\mathcal{B}^d$-measurable and satisfy*
a) for a.e. $x\in\Omega$, the function $\varphi(x,\cdot)$ is continuous;
b) the sequence of functions $\Omega\ni x\mapsto\varphi(x,u_j(x))$ is equiintegrable.
Then limit (3) holds.

Functions $\varphi : \Omega\times\mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ that are $\mathcal{L}^n\otimes\mathcal{B}^d$-measurable and satisfy *a)* of Proposition 3.6 are called *Carathéodory integrands* (see, e.g., [26, Def. 6.33] or [21, Def. 2.5]).

The following semicontinuity property also holds (see [9, Prop. 4.3.4] or [26, Th. 8.6]).

Proposition 3.7. *Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence of measurable functions from Ω to $\mathbb{R}^d$ narrowly converging to some ν in $\mathcal{Y}(\Omega,\mathbb{R}^d)$. Let $\varphi : \Omega\times\mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ be $\mathcal{L}^n\otimes\mathcal{B}^d$-measurable and satisfy*
a) for a.e. $x\in\Omega$, the function $\varphi(x,\cdot)$ is lower semicontinuous;

b) the sequence of functions $\Omega \ni x \mapsto \varphi^-(x, u_j(x))$ is equiintegrable.
Then

$$\int_{\Omega\times\mathbb{R}^d} \varphi(x,y)\,\mathrm{d}\nu(x,y) \le \liminf_{j\to\infty} \int_\Omega \varphi(x,u_j(x))\,\mathrm{d}x.$$

Functions $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ that are $\mathcal{L}^n \otimes \mathcal{B}^d$-measurable and satisfy *a)* of Proposition 3.7 are called *normal integrands* (see, e.g., [26, Def. 6.27 and Prop. 6.31] or [21, Def. 3.3]).

Both Propositions 3.6 and 3.7 are consequences of the following lower semicontinuity result for the narrow convergence of Young measures (see [9, Prop. 4.3.3]).

Proposition 3.8. *Let $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$ and let $\{\nu_j\}_{j\in\mathbb{N}} \subset \mathcal{Y}(\Omega, \mathbb{R}^d)$ narrowly converge to ν in $\mathcal{Y}(\Omega, \mathbb{R}^d)$. Let $\varphi : \Omega \times \mathbb{R}^d \to [0, \infty]$ be $\mathcal{L}^n \otimes \mathcal{B}^d$-measurable such that property a) of Proposition 3.7 holds. Then*

$$\int_{\Omega\times\mathbb{R}^d} \varphi(x,y)\,\mathrm{d}\nu(x,y) \le \lim_{j\to\infty} \int_{\Omega\times\mathbb{R}^d} \varphi(x,y)\,\mathrm{d}\nu^j(x,y). \tag{4}$$

As a consequence of the definition of narrow convergence, the following slight generalization of Proposition 3.8 follows.

Corollary 3.9. *Let $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$ and let $\{\nu_j\}_{j\in\mathbb{N}} \subset \mathcal{Y}(\Omega, \mathbb{R}^d)$ narrowly converge to ν in $\mathcal{Y}(\Omega, \mathbb{R}^d)$. Let $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be $\mathcal{L}^n \otimes \mathcal{B}^d$-measurable such that property a) of Proposition 3.7 holds, and, in addition, $\varphi \ge \psi$ for some $\mathcal{L}^n \otimes \mathcal{B}^d$-measurable $\psi : \Omega \times \mathbb{R}^d \to \mathbb{R}$ such that*

$$\psi(x,\cdot) \in C_b(\mathbb{R}^d) \text{ for a.e. } x \in \Omega \qquad \text{and} \qquad \int_\Omega \|\psi(x,\cdot)\|_\infty\,\mathrm{d}x < \infty.$$

Then inequality (4) holds.

Given $p \ge 1$, we call $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$ the set of $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$ such that

$$\int_{\Omega\times\mathbb{R}^d} |y|^p\,\mathrm{d}\nu(x,y) < \infty.$$

As a consequence of Hölder's inequality, $\mathcal{Y}^p(\Omega, \mathbb{R}^d) \subset \mathcal{Y}^q(\Omega, \mathbb{R}^d)$ if $1 \le q \le p$.

Note that $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$ is not closed in $\mathcal{Y}(\Omega, \mathbb{R}^d)$ under the narrow topology. Nevertheless, given a bounded sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega, \mathbb{R}^d)$, thanks to Proposition 3.5, for a subsequence, there exists $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$ such that $\{u_j\}_{j\in\mathbb{N}}$ generates ν; moreover, due to Proposition 3.7, $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$. The converse result also holds (see, e.g., [35, Th. 7.7]); we present both in the following proposition.

Proposition 3.10. *Let $p \ge 1$. If $\{u_j\}_{j\in\mathbb{N}}$ is a bounded sequence in $L^p(\Omega, \mathbb{R}^d)$ generating $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$, then $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$. Conversely, for any $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ there exists a bounded sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega, \mathbb{R}^d)$ generating ν such that $\{|u_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable.*

Proposition 3.10 can be restated in the following somewhat abstract way.

Proposition 3.11. *Let $p \ge 1$ and $M > 0$. Then the closure of $\{u \in L^p(\Omega, \mathbb{R}^d) : \|u\|^p_{L^p(\Omega,\mathbb{R}^d)} \le M\}$ in the narrow topology of $\mathcal{Y}(\Omega, \mathbb{R}^d)$ is the set of $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ such that*

$$\int_{\Omega\times\mathbb{R}^d} |y|^p\,\mathrm{d}\nu(x,y) \le M.$$

Proof. Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $L^p(\Omega,\mathbb{R}^d)$ such that $\|u_j\|^p_{L^p(\Omega,\mathbb{R}^d)} \leq M$ for all $j\in\mathbb{N}$ converging narrowly to a $\nu\in\mathcal{Y}(\Omega,\mathbb{R}^d)$. By Proposition 3.7,

$$\int_{\Omega\times\mathbb{R}^d} |y|^p \, \mathrm{d}\nu(x,y) \leq M.$$

Conversely, given a $\nu\in\mathcal{Y}(\Omega,\mathbb{R}^d)$ with $\int_{\Omega\times\mathbb{R}^d} |y|^p \, \mathrm{d}\nu(x,y) \leq M$, by Proposition 3.10 there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega,\mathbb{R}^d)$ converging narrowly to ν such that $\{|u_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable. By Proposition 3.6,

$$\lim_{j\to\infty} \|u_j\|^p_{L^p(\Omega,\mathbb{R}^d)} = \int_{\Omega\times\mathbb{R}^d} |y|^p \, \mathrm{d}\nu(x,y) \leq M.$$

Then the sequence $\{v_j\}_{j\in\mathbb{N}}$ defined by

$$v_j := \frac{\int_{\Omega\times\mathbb{R}^d} |y|^p \, \mathrm{d}\nu(x,y)}{\|u_j\|^p_{L^p(\Omega,\mathbb{R}^d)}} u_j, \qquad j\in\mathbb{N}$$

satisfies that $v_j - u_j$ converges to zero in $L^p(\Omega,\mathbb{R}^d)$, so in measure, hence (see, e.g., [9, Prop. 4.3.8]) $v_j \overset{\text{nar}}{\rightharpoonup} \nu$ as $j\to\infty$, and, in addition, $\|v_j\|^p_{L^p(\Omega,\mathbb{R}^d)} \leq M$ for all $j\in\mathbb{N}$. □

The following property is immediate (see, if necessary, the proof of [34, Prop. 2.3] or [23, Th. 11]).

Lemma 3.12. *If $\{u_j\}_{j\in\mathbb{N}}$ is a bounded sequence in $L^p(\Omega,\mathbb{R}^d)$ generating $\nu\in\mathcal{Y}(\Omega,\mathbb{R}^d)$, then $\{u_j\otimes u_j\}_{j\in\mathbb{N}}$ generates $\nu\otimes\nu$. Moreover, if $\{|u_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable then $\{|u_j\otimes u_j|^p\}_{j\in\mathbb{N}}$ is also equiintegrable.*

Given $\nu\in\mathcal{Y}^1(\Omega,\mathbb{R}^d)$, the *first moment* of ν is defined as the measurable function $u:\Omega\to\mathbb{R}^d$

$$u(x) := \int_{\mathbb{R}^d} y \, \mathrm{d}\nu_x(y), \quad \text{a.e. } x\in\Omega.$$

Jensen's inequality shows at once that $u\in L^p(\Omega,\mathbb{R}^d)$ whenever $\nu\in\mathcal{Y}^p(\Omega,\mathbb{R}^d)$ for a given $p\geq 1$. The following classical result shows the relationship between narrow convergence and weak convergence in $L^p(\Omega,\mathbb{R}^d)$ (see, e.g., [26, Th. 8.11] or [35, Th. 6.8]).

Lemma 3.13. *Let $p\geq 1$. Let $\{u_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $L^p(\Omega,\mathbb{R}^d)$ generating ν, and let u be the first moment of ν. Then $u\in L^p(\Omega,\mathbb{R}^d)$ and $u_j \overset{b}{\rightharpoonup} u$ as $j\to\infty$. If, in addition, $p>1$ then $u_j\rightharpoonup u$ in $L^p(\Omega,\mathbb{R}^d)$ as $j\to\infty$.*

4. Boundary conditions, coercivity and existence of minimizers. In this section we give conditions for the existence of minimizers of

$$I : L^p(\Omega,\mathbb{R}^d)\to\mathbb{R}\cup\{\infty\}, \qquad I(u) := \int_\Omega\int_\Omega w(x,x',u(x),u(x'))\,\mathrm{d}x\,\mathrm{d}x' \tag{5}$$

and its extended functional

$$\bar{I} : \mathcal{Y}^p(\Omega,\mathbb{R}^d)\to\mathbb{R}\cup\{\infty\}, \qquad \bar{I}(\nu) := \int_{\Omega\times\mathbb{R}^d}\int_{\Omega\times\mathbb{R}^d} w(x,x',y,y')\,\mathrm{d}\nu(x,y)\,\mathrm{d}\nu(x',y'). \tag{6}$$

The functional I is, of course, the main object of study in this work, whereas $\bar{I}$ turns out to be the relaxation of I in terms of Young measures, as will be shown in Section 6.

In this section, we assume that I is lower semicontinuous, while in Section 5 we will characterize this property. Thus, we ought to study the issue of coercivity in order to carry out the direct method of the Calculus of variations.

Typically, a lower bound for the integrand w together with some adequate boundary conditions yield the coercivity of the functional I, so we start explaining the type of boundary conditions normally used in nonlocal problems (see, e.g., [8, 28, 22, 11]), with the caveat that they are slightly different than in local problems, one of the reason being that L^p functions do not have traces on the boundary $\partial\Omega$.

First we establish the precise meaning of *translation invariance* of the functionals I and $\bar{I}$.

Definition 4.1. *The functional I is invariant under translations if $I(u) = I(u + a)$ for all $u \in L^p(\Omega, \mathbb{R}^d)$ and $a \in \mathbb{R}^d$.*

The functional $\bar{I}$ is invariant under translation if $\bar{I}(\nu) = \bar{I}(\nu^a)$ for all $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ and $a \in \mathbb{R}^d$, where ν^a is defined as the only $\mathcal{L}^n \otimes \mathcal{B}^d$-measure in $\Omega \times \mathbb{R}^d$ that satisfies, for any measurable $E \subset \Omega$ and any Borel $F \subset \mathbb{R}^d$,

$$\nu^a(E \times F) = \nu(E \times (F + a)),$$

where $F + a$ is the translated set of F by a.

It is immediate to check that $\nu^a \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ whenever $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$. Moreover,

$$\int_{\Omega \times \mathbb{R}^d} \varphi(x, y) \,\mathrm{d}\nu^a(x, y) = \int_{\Omega \times \mathbb{R}^d} \varphi(x, y - a) \,\mathrm{d}\nu(x, y) \tag{7}$$

for all $\mathcal{L}^n \otimes \mathcal{B}^d$-measurable $\varphi : \Omega \times \mathbb{R}^d \to \mathbb{R}$ satisfying condition (1). In fact, by monotone convergence, equality (7) also holds for any $\mathcal{L}^n \otimes \mathcal{B}^d$-measurable $\varphi : \Omega \times \mathbb{R}^d \to [0, \infty]$.

Note that a sufficient condition for I and $\bar{I}$ to be invariant under translations is that the integrand w depends on (x, x', y, y') through $(x, x', y - y')$, but the analysis of [23, Sect. 6] shows that there are more possibilities.

We now explain the nonlocal analogue of *Dirichlet* and *mixed* boundary conditions. We require the choice of a non-empty open set $\Omega_0 \subset \Omega$ (which plays the role of *nonlocal interior*) and a $\delta > 0$ such that $\Omega_0 + B(0, \delta) \subset \Omega$. In the context of peridynamics, this δ is also the *horizon distance*: particles $x, x' \in \Omega$ with $|x - x'| \geq \delta$ do not interact, although this condition is not required in the paper. Of course, $\Omega_0 + B(0, \delta)$ denotes the set of points in $\mathbb{R}^n$ that can be expressed as a sum of an element of Ω_0 plus an element of $B(0, \delta)$ (the open ball of centre 0 and radius δ). Pure Dirichlet conditions, in this context, prescribe the value of u in $\Omega \setminus \Omega_0$, while mixed Dirichlet–Neumann conditions prescribe the value of u in a measurable subset $\Omega_D \subset \Omega \setminus \Omega_0$ with $0 < \mathcal{L}^n(\Omega_D) < \mathcal{L}^n(\Omega \setminus \Omega_0)$; minimizers automatically satisfy a *nonlocal natural boundary condition* in $\Omega \setminus (\Omega_0 \cup \Omega_D)$. Pure *Neumann* conditions, which, again, are not imposed explicitly, require that the functional I is invariant under translations; in this case, the restriction $\int_\Omega u \,\mathrm{d}x = 0$ is made, so as to avoid that invariance. As before, minimizers of this problem satisfy a nonlocal natural boundary condition, which can be consulted in [11, Sect. 8]. Moreover, this kind of nonlocal boundary conditions can be given an interpretation of a nonlocal flux through the boundary, thus mimicking what happens for the local equations. This nonlocal calculus is developed in [28, 22, 3], to which we refer for further explanation.

The assumption that δ is finite in the above boundary conditions is typical in Solid Mechanics. Nevertheless, in other nonlocal problems, a *complement value* condition is imposed, which roughly corresponds to taking $\delta = \infty$ in the above approach. The analysis of this paper remains valid, with little modifications, if a Dirichlet datum is prescribed in $\mathbb{R}^n \setminus \Omega$. In this way, one can include kernels of the form

$$k(x, x') = \max\left\{1, \frac{1}{|x - x'|^{n+sp}}\right\}$$

for $0 < s < 1$, which, for $p = 2$, corresponds to a truncation of the kernel of the fractional Laplacian. A relevant work for complement value problems is [24].

The coercivity for the functional I was studied in [11] by collecting several *nonlocal Poincaré inequalities* that had appeared in the literature: they were suitable for integrands depending on (x, x', y, y') through $(x, x', y - y')$, as in the case of translation-invariant functionals I, and, in particular, in peridynamics. They also took into account the fact that the integrand $w(x, x', y, y')$ can vanish when $|x - x'| \geq \delta$.

The first nonlocal Poincaré inequality that we present is suitable for pure Dirichlet and mixed Dirichlet–Neumann conditions. It has been proved, within several contexts and with slightly different versions, in [7, Prop. 2.5], [1, Lemma 2.4], [2, Prop. 4.1], [29, Lemma 3.5] and [12, Prop. 8]. The current formulation is taken from [11, Cor. 4.4].

Proposition 4.2. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta > 0$ and let $p \geq 1$. Let Ω_0 be a non-empty open subset of Ω satisfying $\Omega_0 + B(0, \delta) \subset \Omega$. Let Ω_D be a measurable subset of $\Omega \setminus \Omega_0$ with positive measure. Then there exists $\lambda > 0$ such that for all $u \in L^p(\Omega, \mathbb{R}^d)$,*

$$\int_\Omega |u(x)|^p \, \mathrm{d}x \leq \lambda \int_\Omega \int_{\Omega \cap B(x,\delta)} |u(x) - u(x')|^p \, \mathrm{d}x' \, \mathrm{d}x + \lambda \int_{\Omega_D} |u(x)|^p \, \mathrm{d}x.$$

The second nonlocal Poincaré inequality that we show is adequate for Neumann conditions. Again, it has been proved, with different versions, in [16], [17, Th. 1], [37, Th. 1.1], [6, Prop. 4.1], [1, Cor. 3.4] and [30, Cor. 4.6]. The following formulation is taken from [11, Prop. 4.2].

Proposition 4.3. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta > 0$ and let $p \geq 1$. Then there exists $\lambda > 0$ such that for all $u \in L^p(\Omega, \mathbb{R}^d)$,*

$$\int_\Omega \left| u(x) - \fint_\Omega u \right|^p \mathrm{d}x \leq \lambda \int_\Omega \int_{\Omega \cap B(x,\delta)} |u(x) - u(x')|^p \, \mathrm{d}x' \, \mathrm{d}x.$$

In order to prove existence for $\bar{I}$, we will need the following versions of Propositions 4.2 and 4.3 for Young measures.

Proposition 4.4. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta > 0$ and let $p \geq 1$. Let Ω_0 be a non-empty open subset of Ω satisfying $\Omega_0 + B(0, \delta) \subset \Omega$. Let Ω_D be a measurable subset of $\Omega \setminus \Omega_0$ with positive measure. Then there exists $\lambda > 0$ such that for all $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$,*

$$\begin{aligned} &\int_{\Omega \times \mathbb{R}^d} |y|^p \, \mathrm{d}\nu(x, y) \\ &\leq \lambda \int_{\Omega \times \mathbb{R}^d} \int_{(\Omega \cap B(x,\delta)) \times \mathbb{R}^d} |y - y'|^p \, \mathrm{d}\nu(x', y') \, \mathrm{d}\nu(x, y) + \lambda \int_{\Omega_D \times \mathbb{R}^d} |y|^p \, \mathrm{d}\nu(x, y). \end{aligned}$$

Proof. Let $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$. By Proposition 3.10, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega,\mathbb{R}^d)$ generating ν such that $\{|u_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable. According to Proposition 4.2, there exists $\lambda>0$ such that for all $j\in\mathbb{N}$,

$$\int_\Omega |u_j(x)|^p\,\mathrm{d}x \le \lambda\int_\Omega\int_{\Omega\cap B(x,\delta)} |u_j(x)-u_j(x')|^p\,\mathrm{d}x'\,\mathrm{d}x + \lambda\int_{\Omega_D}|u_j(x)|^p\,\mathrm{d}x.$$

By Proposition 3.6,

$$\lim_{j\to\infty}\int_\Omega |u_j(x)|^p\,\mathrm{d}x = \int_{\Omega\times\mathbb{R}^d}|y|^p\,\mathrm{d}\nu(x,y)$$

and

$$\lim_{j\to\infty}\int_{\Omega_D} |u_j(x)|^p\,\mathrm{d}x = \int_{\Omega_D\times\mathbb{R}^d}|y|^p\,\mathrm{d}\nu(x,y).$$

Similarly, having in mind Lemma 3.12, we also obtain

$$\begin{aligned}&\lim_{j\to\infty}\int_\Omega\int_{\Omega\cap B(x,\delta)} |u_j(x)-u_j(x')|^p\,\mathrm{d}x'\,\mathrm{d}x\\ &= \int_{\Omega\times\mathbb{R}^d}\int_{(\Omega\cap B(x,\delta))\times\mathbb{R}^d}|y-y'|^p\,\mathrm{d}\nu(x',y')\,\mathrm{d}\nu(x,y).\end{aligned}$$

This concludes the proof. ◻

PROPOSITION 4.5. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta>0$ and let $p\ge 1$. Then there exists $\lambda>0$ such that for all $\nu\in\mathcal{Y}^p(\Omega,\mathbb{R}^d)$,*

$$\int_{\Omega\times\mathbb{R}^d}\left|y - ⨍_\Omega u\right|^p\mathrm{d}\nu(x,y) \le \lambda\int_{\Omega\times\mathbb{R}^d}\int_{(\Omega\cap B(x,\delta))\times\mathbb{R}^d}|y-y'|^p\,\mathrm{d}\nu(x',y')\,\mathrm{d}\nu(x,y),$$

where u is the first moment of ν.

Proof. Let $\nu\in\mathcal{Y}^p(\Omega,\mathbb{R}^d)$ be such that its first moment u satisfies $\int_\Omega u = 0$. As in Proposition 4.4, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega,\mathbb{R}^d)$ generating ν such that $\{|u_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable. By Lemma 3.13, $u_j \rightharpoonup u$ in $L^p(\Omega,\mathbb{R}^d)$ as $j\to\infty$; this also holds for $p=1$ because $\{|u_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable. In particular, $\int_\Omega u_j \to 0$ as $j\to\infty$. Define $v_j := u_j - ⨍_\Omega u_j$ for each $j\in\mathbb{N}$, which satisfies $\int_\Omega v_j = 0$. Then $v_j - u_j \to 0$ in measure as $j\to\infty$, and, hence (see, e.g., [9, Prop. 4.3.8]), $\{v_j\}_{j\in\mathbb{N}}$ generates ν. Moreover, $\{|v_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable as the sum of two equiintegrable sequences. Thanks to Proposition 4.3, there exists $\lambda>0$ such that for all $j\in\mathbb{N}$,

$$\int_\Omega |v_j(x)|^p\,\mathrm{d}x \le \lambda\int_\Omega\int_{\Omega\cap B(x,\delta)}|v_j(x)-v_j(x')|^p\,\mathrm{d}x'\,\mathrm{d}x.$$

Arguing as in Proposition 4.4, we obtain that

$$\int_{\Omega\times\mathbb{R}^d}|y|^p\,\mathrm{d}\nu(x,y) \le \lambda\int_{\Omega\times\mathbb{R}^d}\int_{(\Omega\cap B(x,\delta))\times\mathbb{R}^d}|y-y'|^p\,\mathrm{d}\nu(x',y')\,\mathrm{d}\nu(x,y),$$

which concludes the proof in this case.

Now let be given a general $\nu\in\mathcal{Y}^p(\Omega,\mathbb{R}^d)$ with first moment u, call $a = ⨍_\Omega u$ and consider the Young measure ν^a of Definition 4.1. Clearly, its first moment u^a satisfies $\int_\Omega u^a = 0$. By the first part of the proof,

$$\int_{\Omega\times\mathbb{R}^d}|y|^p\,\mathrm{d}\nu^a(x,y) \le \lambda\int_{\Omega\times\mathbb{R}^d}\int_{(\Omega\cap B(x,\delta))\times\mathbb{R}^d}|y-y'|^p\,\mathrm{d}\nu^a(x',y')\,\mathrm{d}\nu^a(x,y).$$

The result is concluded by noting that, thanks to (7),

$$\int_{\Omega\times\mathbb{R}^d} |y|^p \, \mathrm{d}\nu^a(x,y) = \int_{\Omega\times\mathbb{R}^d} |y-a|^p \, \mathrm{d}\nu(x,y)$$

and

$$\int_{\Omega\times\mathbb{R}^d} \int_{(\Omega\cap B(x,\delta))\times\mathbb{R}^d} |y-y'|^p \, \mathrm{d}\nu^a(x',y') \, \mathrm{d}\nu^a(x,y)$$
$$= \int_{\Omega\times\mathbb{R}^d} \int_{(\Omega\cap B(x,\delta))\times\mathbb{R}^d} |y-y'|^p \, \mathrm{d}\nu(x',y') \, \mathrm{d}\nu(x,y). \qquad \square$$

Finally, when the functional I is not invariant under translations, one can just impose a lower bound in $w(x,x',y,y')$ in terms of $|y|^p$ so as to obtain coercivity trivially, but this assumption is unrealistic in peridynamics.

Having proved the coercivity results, and assuming lower semicontinuity, we present the existence theorems: they generalize those of [11, Thms. 5.1 and 5.2]; in particular, the case $p=1$ is covered. First, we state the result for the functional I; we will show three variants: for no boundary conditions, for Dirichlet (or mixed Dirichlet–Neumann) conditions and for Neumann conditions.

THEOREM 4.6. *Let $p \geq 1$ and let $w : \Omega\times\Omega\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ be $\mathcal{L}^{2n}\otimes\mathcal{B}^{2d}$-measurable. Assume that I is lower semicontinuous with respect to the weak convergence in $L^p(\Omega,\mathbb{R}^d)$ if $p>1$, or the biting convergence if $p=1$. Then the following hold:*

a) *Assume there exist $c>0$ and $a \in L^1(\Omega\times\Omega)$ such that*

$$w(x,x',y,y') \geq c\,|y|^p + a(x,x'), \qquad \text{for a.e. } x,x'\in\Omega \text{ and all } y,y'\in\mathbb{R}^d.$$

Then there exists a minimizer of I in $L^p(\Omega,\mathbb{R}^d)$.

b) *Assume Ω is a Lipschitz domain, fix $\delta>0$ and let Ω_0 be a non-empty open subset of Ω satisfying $\Omega_0+B(0,\delta)\subset\Omega$. Let Ω_D be a measurable subset of $\Omega\setminus\Omega_0$ with positive measure. Let $u_0\in L^p(\Omega,\mathbb{R}^d)$. Assume that there exist $c>0$ and $a\in L^1(\Omega\times\Omega)$ such that*

$$w(x,x',y,y') \geq c\,\chi_{B(0,\delta)}(x-x')\,|y-y'|^p + a(x,x'), \tag{8}$$

for a.e. $x,x'\in\Omega$ and all $y,y'\in\mathbb{R}^d$. There there exists a minimizer of I in the set of $u\in L^p(\Omega,\mathbb{R}^d)$ such that $u=u_0$ in Ω_D.

c) *Assume Ω is a Lipschitz domain, fix $\delta>0$ and let Ω_0 be a non-empty open subset of Ω satisfying $\Omega_0+B(0,\delta)\subset\Omega$. Assume that there exist $c>0$ and $a\in L^1(\Omega\times\Omega)$ such that inequality (8) holds. If $p=1$, assume, in addition, that I is invariant under translations. Then there exists a minimizer of I in the set of $u\in L^p(\Omega,\mathbb{R}^d)$ such that $\int_\Omega u = 0$.*

Proof. We can assume that I is not identically infinity. Let $\{u_j\}_{j\in\mathbb{N}}$ be a minimizing sequence of I in the corresponding set of admissible functions. Then $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega,\mathbb{R}^d)$: this is immediate under assumption *a)*, it is a consequence of Proposition 4.2 under assumption *b)*, and it is a consequence of Proposition 4.3 under assumption *c)*. For a subsequence (not relabelled), $\{u_j\}_{j\in\mathbb{N}}$ converges to some $u\in L^p(\Omega,\mathbb{R}^d)$ weakly if $p>1$ and in the biting sense if $p=1$. Since we are assuming lower semicontinuity of I,

$$I(u) \leq \liminf_{j\to\infty} I(u_j).$$

Under assumption *b)*, it is easy to check that $u = u_0$ in Ω_D, in both cases $p > 1$ and $p = 1$. Under assumption *c)*, it is immediate that $\int_\Omega u = 0$ when $p > 1$. Therefore, u is a minimizer of I in set of admissible functions in all cases, except perhaps in *c)* when $p = 1$, where the minimizer is $u - \fint_\Omega u$, thanks to the translation-invariance of I. This concludes the proof. ∎

Now we study the existence of minimizers for $\bar{I}$. Analogously as before, we will use Propositions 4.4 and 4.5 to obtain coercivity for translation-invariant functionals, and Proposition 3.3 otherwise.

Theorem 4.7. *Let $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be $\mathcal{L}^{2n} \otimes \mathcal{B}^{2d}$-measurable. Assume that $\bar{I}$ is lower semicontinuous with respect to the narrow topology in $\mathcal{Y}(\Omega, \mathbb{R}^d)$. Then, the following hold:*

a) Assume there exist $h : [0, \infty) \to [0, \infty)$ and $a \in L^1(\Omega \times \Omega)$ such that $\lim_{t\to\infty} h(t) = \infty$ and

$$w(x, x', y, y') \geq h(|y|) + a(x, x'), \qquad \text{for a.e. } x, x' \in \Omega \text{ and all } y, y' \in \mathbb{R}^d.$$

Then there exists a minimizer of $\bar{I}$ in $\mathcal{Y}(\Omega, \mathbb{R}^d)$.

b) Let $p \geq 1$. Assume Ω is a Lipschitz domain, fix $\delta > 0$ and let Ω_0 be a non-empty open subset of Ω satisfying $\Omega_0 + B(0, \delta) \subset \Omega$. Let Ω_D be a measurable subset of $\Omega \setminus \Omega_0$ with positive measure. Let $u_0 \in L^p(\Omega, \mathbb{R}^d)$. Assume that there exist $c > 0$ and $a \in L^1(\Omega \times \Omega)$ such that inequality (8) holds. There there exists a minimizer of $\bar{I}$ in the set of $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ such that $\nu_x = \delta_{u_0(x)}$ for a.e. $x \in \Omega_D$.

c) Let $p \geq 1$. Assume Ω is a Lipschitz domain, fix $\delta > 0$ and let Ω_0 be a non-empty open subset of Ω satisfying $\Omega_0 + B(0, \delta) \subset \Omega$. Assume that there exist $c > 0$ and $a \in L^1(\Omega \times \Omega)$ such that inequality (8) holds. If $p = 1$, assume, in addition, that $\bar{I}$ is invariant under translations. Then there exists a minimizer of $\bar{I}$ in the set of $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ whose first moment u satisfies $\int_\Omega u = 0$.

Proof. We can assume that $\bar{I}$ is not identically infinity. Let $\{\nu^j\}_{j\in\mathbb{N}}$ be a minimizing sequence. Thanks to Theorem 3.2 and Proposition 3.3, for a subsequence (not relabelled) $\{\nu^j\}_{j\in\mathbb{N}}$ converges narrowly to some $\nu \in \mathcal{Y}(\Omega, \mathbb{R}^d)$: this is immediate under assumption *a)*, it is a consequence of Proposition 4.4 under assumption *b)*, and it is a consequence of Proposition 4.5 under assumption *c)*. As $\bar{I}$ is lower semicontinuous,

$$\bar{I}(\nu) \leq \liminf_{j\to\infty} \bar{I}(\nu^j).$$

Moreover, $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ under assumptions *b)* or *c)*, thanks to Proposition 3.8. Under assumption *b)*, it is easy to check that $\nu_x = \delta_{u_0(x)}$ for a.e. $x \in \Omega_D$. Under assumption *c)*, let u be the first moment of ν. If $p > 1$ we have that $\int_\Omega u = 0$ thanks to Lemma 3.13. Therefore, ν is a minimizer of $\bar{I}$ in the admissible set in all cases, except perhaps in *c)* with $p = 1$, where a minimizer is the Young measure ν^a of Definition 4.1, where $a := \fint_\Omega u$, thanks to the translation-invariance of $\bar{I}$. This concludes the proof. ∎

5. Necessary and sufficient conditions for weak lower semicontinuity.

In this section we study the lower semicontinuity of the functional I of (5) under the weak topology of $L^p(\Omega, \mathbb{R}^d)$ when $p > 1$, and the biting convergence when $p = 1$. We also study the lower semicontinuity of the functional $\bar{I}$ of (6) under the narrow topology.

In fact, Elbau [23, Th. 11] (see also [15, Prop. 8.8]) found the following necessary and sufficient condition, which we call *nonlocal convexity*, for the lower semicontinuity of I in terms of w:

(NC) For a.e. $x \in \Omega$ and every $v \in L^p(\Omega, \mathbb{R}^d)$, the function

$$\Phi_{x,v} : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}, \qquad \Phi_{x,v}(y) := \int_\Omega w(x, x', y, v(x')) \, \mathrm{d}x'$$

is convex.

This section aims to understand this condition and provide more characterizations of the lower semicontinuity, as well as an alternative proof to that of [23]. We will make use of the functional $\bar{I}$ of (6).

A generalization of condition (NC), already appearing in [23], is the following:

(NY) For a.e. $x \in \Omega$ and every $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$, the function

$$\Phi_{x,\nu} : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}, \qquad \Phi_{x,\nu}(y) := \int_{\Omega \times \mathbb{R}^d} w(x, x', y, y') \, \mathrm{d}\nu(x', y')$$

is convex.

Condition (NY) is called *nonlocal convexity for Young measures.* Note that, via the identification of a function $u \in L^p(\Omega, \mathbb{R}^d)$ with its associated Young measure, we have that $\Phi_{x,u}$ according to both definitions in (NC) and (NY) coincide, so condition (NY) is, in principle, stronger that condition (NC).

On the other hand, and in a slightly different context (namely, functionals involving derivatives), Pedregal [34, Prop. 3.1 and eq. (4.3)] showed a necessary and sufficient condition for the lower semicontinuity of the analogue of our functional I. It reads as follows:

(NJ) For any $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$, and letting u be its first moment, we have

$$I(u) \leq \bar{I}(\nu).$$

Condition (NJ) is called *nonlocal Jensen's inequality.*

In full rigour, conditions (NC), (NY) and (NJ) should be called $(\mathrm{NC})_p$ or with a similar symbol indicating its dependence on p. For simplicity of notation, we do not indicate that dependence and assume that the exponent p is fixed throughout the section. This dependence is especially significant when distinguishing the cases $p > 1$ and $p = 1$.

We first show a lemma on the equiintegrability of a sequence.

Lemma 5.1. *Let $p \geq 1$. Assume that $\{u_j\}_{j \in \mathbb{N}}$ is a bounded sequence in $L^p(\Omega, \mathbb{R}^d)$ and let $g : [0, \infty) \to [0, \infty)$ be continuous, strictly increasing such that*

$$\lim_{t \to \infty} \frac{g(t)}{t^p} = 0. \tag{9}$$

Then the sequence of functions $\{g(|u_j|)\}_{j \in \Omega}$ is equiintegrable.

Proof. As g is continuous and strictly increasing, it has an inverse g^{-1} defined on $[g(0), g(\infty))$, where $g(\infty)$ stands for $\lim_{t \to \infty} g(t)$. If $g(\infty) < \infty$ then $\{g(|u_j|)\}_{j \in \mathbb{N}}$ is bounded in $L^\infty(\Omega)$, so equiintegrable. If $g(\infty) = \infty$ then we define $h : [g(0), \infty) \to [0, \infty)$ as $h(s) := g^{-1}(s)^p$, which satisfies

$$\lim_{s \to \infty} \frac{h(s)}{s} = \lim_{t \to \infty} \frac{h(g(t))}{g(t)} = \lim_{t \to \infty} \frac{t^p}{g(t)} = \infty$$

and

$$\sup_{j \in \mathbb{N}} \int_\Omega h(g(|u_j|)) \, \mathrm{d}x = \sup_{j \in \mathbb{N}} \int_\Omega |u_j|^p \, \mathrm{d}x < \infty.$$

The conclusion follows from de la Vallée-Poussin criterion. $\square$

As remarked after Propositions 3.6 and 3.7, natural assumptions for an integrand to satisfy is that of being *normal* or of *Carathéodory* type. In addition, for integrands defined on $\Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d$, the *symmetry* assumption can be assumed without loss of generality, since, by Fubini's theorem, given any $\mathcal{L}^{2n} \otimes \mathcal{B}^{2d}$-measurable $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ with suitable integrability properties, the integrands w and

$$\frac{w(x, x', y, y') + w(x', x, y', y)}{2}$$

give rise to the same functional. The precise definitions are as follows.

DEFINITION 5.2. *Let* $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$.

a) We say that w is symmetric if $w(x, x', y, y') = w(x', x, y', y)$ *for a.e.* $x, x' \in \Omega$ *and all* $y, y' \in \mathbb{R}^d$.

b) We say w is normal if it is $\mathcal{L}^{2n} \otimes \mathcal{B}^{2d}$*-measurable and for a.e.* $x, x' \in \Omega$*, the function* $w(x, x', \cdot, \cdot)$ *is lower semicontinuous.*

c) We say w is Carathéodory if it is $\mathcal{L}^{2n} \otimes \mathcal{B}^{2d}$*-measurable and for a.e.* $x, x' \in \Omega$*, the function* $w(x, x', \cdot, \cdot)$ *is continuous.*

In the next result we establish the equivalence between condition (NJ) and the lower semicontinuity of I. Its proof is an adaptation of that of [34, Prop. 3.1] for functionals without derivatives and with p-growth conditions.

PROPOSITION 5.3. *Let* $p \geq 1$ *and let* $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ *be symmetric. The following statements hold:*

a) Assume w is Carathéodory and there exist $a \in L^1(\Omega \times \Omega)$ *and* $c > 0$ *such that*

$$|w(x, x', y, y')| \leq a(x, x') + c\,|y|^p\,, \qquad \text{for a.e. } x, x' \in \Omega \text{ and all } y, y' \in \mathbb{R}^d. \tag{10}$$

If

i) $p > 1$ *and I is lower semicontinuous in* $L^p(\Omega, \mathbb{R}^d)$ *with respect to the weak convergence; or*

ii) $p = 1$ *and I is lower semicontinuous in* $L^p(\Omega, \mathbb{R}^d)$ *with respect to the biting convergence,*

then condition (NJ) holds.

b) Assume w is normal, there exist $a \in L^1(\Omega \times \Omega)$ *and a continuous strictly increasing* $g : [0, \infty) \to [0, \infty)$ *with (9) such that*

$$w^-(x, x', y, y') \leq a(x, x') + g\left(|y|\right), \qquad \text{for a.e. } x, x' \in \Omega \text{ and all } y, y' \in \mathbb{R}^d. \tag{11}$$

If condition (NJ) holds then the functional I is lower semicontinuous in $L^p(\Omega, \mathbb{R}^d)$

i) with respect to the weak convergence if $p > 1$*; and*

ii) with respect to the biting convergence if $p = 1$.

Proof. We first prove *a)*. Consider $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$ and let $u \in L^p(\Omega, \mathbb{R}^d)$ be its first moment. By Proposition 3.10, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ generating ν such that $\{|u_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable. By (10), the sequence of functions

$$\Omega \times \Omega \ni (x, x') \mapsto w(x, x', u_j(x), u_j(x'))$$

is equiintegrable. Hence, by Proposition 3.6,

$$\lim_{j\to\infty} I(u_j) = \bar{I}(\nu). \tag{12}$$

On the other hand, we have by Lemma 3.13 that $u_j \rightharpoonup u$ in $L^p(\Omega, \mathbb{R}^d)$ as $j \to \infty$ if $p > 1$, and $u_j \overset{b}{\rightharpoonup} u$ if $p = 1$. Applying the lower semicontinuity of I, we find that

$$I(u) \leq \liminf_{j \to \infty} I(u_j). \tag{13}$$

Comparing (12) and (13), we obtain condition (NJ).

We now prove *b)*. Let $\{u_j\}_{j \in \mathbb{N}}$ be a bounded sequence in $L^p(\Omega, \mathbb{R}^d)$ weakly converging to u if $p > 1$, and $u_j \overset{b}{\rightharpoonup} u$ as $j \to \infty$ if $p = 1$. Passing to a subsequence, we can assume by Propositions 3.5 and 3.10 that $\{u_j\}_{j \in \mathbb{N}}$ generates a $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$. Moreover, by Lemma 3.13, u is the first moment of ν. By (11) and Lemma 5.1, the sequence of functions

$$\Omega \times \Omega \ni (x, x') \mapsto w^-(x, x', u_j(x), u_j(x'))$$

is equiintegrable. By (NJ) and Proposition 3.7,

$$I(u) \leq \bar{I}(\nu) \leq \liminf_{j \to \infty} I(u_j),$$

so I is lower semicontinuous. □

We will use several times the following auxiliary result, which is a version of Lebesgue's differentiation theorem for double integrals.

Lemma 5.4. *Let $h \in L^1_{\mathrm{loc}}(\Omega \times \Omega)$. Then, for a.e. $x_0 \in \Omega$,*

$$\lim_{r \searrow 0} \fint_{B(x_0,r)} \int_{B(x_0,r)} h(x, x') \, \mathrm{d}x' \, \mathrm{d}x = 0 \tag{14}$$

and

$$\begin{aligned} \lim_{r \searrow 0} \fint_{B(x_0,r)} \int_{\Omega \setminus B(x_0,r)} h(x, x') \, \mathrm{d}x' \, \mathrm{d}x &= \lim_{r \searrow 0} \fint_{B(x_0,r)} \int_{\Omega} h(x, x') \, \mathrm{d}x' \, \mathrm{d}x \\ &= \int_{\Omega} h(x_0, x') \, \mathrm{d}x'. \end{aligned} \tag{15}$$

Proof. By Fubini's theorem, $\int_\Omega h(\cdot, x') \, \mathrm{d}x' \in L^1_{\mathrm{loc}}(\Omega)$, hence by Lebesgue's differentiation theorem, for a.e. $x_0 \in \Omega$,

$$\lim_{r \searrow 0} \fint_{B(x_0,r)} \int_{\Omega} h(x, x') \, \mathrm{d}x' \, \mathrm{d}x = \int_{\Omega} h(x_0, x') \, \mathrm{d}x'.$$

This shows the second equality of (15); the first one follows from the second and (14). It, therefore, remains to prove (14).

By Lebesgue's differentiation theorem, for a.e. $(x_0, x_0') \in \Omega \times \Omega$,

$$\lim_{r \searrow 0} \fint_{B(x_0,r)} \fint_{B(x_0',r)} h(x, x') \, \mathrm{d}x' \, \mathrm{d}x = h(x_0, x_0');$$

in particular,

$$\lim_{r \searrow 0} \fint_{B(x_0,r)} \int_{B(x_0',r)} h(x, x') \, \mathrm{d}x' \, \mathrm{d}x = 0. \tag{16}$$

Fix $x_0 \in \Omega$ such that limit (16) holds for a.e. $x_0' \in \Omega$; note that a.e. $x_0 \in \Omega$ satisfies that. Observe now (e.g., by Lebesgue's dominated theorem) that the function

$$(x_1, r) \mapsto \fint_{B(x_0,r)} \int_{B(x_1,r)} h(x,x')\,\mathrm{d}x'\,\mathrm{d}x$$

is continuous in the set $\{(x_1,r) \in \Omega \times [0,\infty) : B(x_1,r) \subset \Omega\}$, hence uniformly continuous in compact subsets. Thus, given $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x_0' \in \bar{B}(x_0,\delta)$ and any $0 < r \le \delta$ one has

$$\left| \fint_{B(x_0,r)} \int_{B(x_0,r)} h(x,x')\,\mathrm{d}x'\,\mathrm{d}x - \fint_{B(x_0,r)} \int_{B(x_0',r)} h(x,x')\,\mathrm{d}x'\,\mathrm{d}x \right| < \varepsilon.$$

Taking, additionally, an $x_0' \in \Omega$ such that (16) holds, we obtain that there exists $r_0 \in (0,\delta]$ such that for any $0 < r < r_0$,

$$\left| \fint_{B(x_0,r)} \int_{B(x_0',r)} h(x,x')\,\mathrm{d}x'\,\mathrm{d}x \right| < \varepsilon,$$

so

$$\left| \fint_{B(x_0,r)} \int_{B(x_0,r)} h(x,x')\,\mathrm{d}x'\,\mathrm{d}x \right| < 2\varepsilon$$

and the lemma is proved. ∎

The main result in this section is the following theorem, which establishes the equivalence between (NC), (NY) and (NJ).

THEOREM 5.5. *Let $p \ge 1$. Let $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be symmetric and $\mathcal{L}^{2n} \otimes \mathcal{B}^{2d}$-measurable. The following implications hold:*

a) Assume w is Carathéodory and there exist $a \in L^1(\Omega \times \Omega)$ and $c > 0$ such that (10) holds. Then condition (NC) implies (NY).

b) Assume there exists $c > 0$ such that

$$\left| w^-(x,x',y,y') \right| \le a(x,x') + c\,|y|^p, \qquad \text{for a.e. } x,x' \in \Omega \text{ and all } y,y' \in \mathbb{R}^d.$$

Then condition (NY) implies (NJ).

c) Assume that there exist $a \in L^1(\Omega \times \Omega)$ and $c > 0$ such that (10) holds. Then condition (NJ) implies (NC).

Proof. We first prove a). Fix $x \in \Omega$ such that for every $v \in L^p(\Omega,\mathbb{R}^d)$ the function $\Phi_{x,v}$ is convex. Let $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$; by Proposition 3.10, there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega,\mathbb{R}^d)$ generating ν such that $\{|u_j|^p\}$ is equiintegrable. Fix $y \in \mathbb{R}^d$. Thanks to (10), the sequence of functions $\{f_{j,x,y}\}_{j\in\mathbb{N}}$ in Ω defined by

$$f_{j,x,y}(x') := w(x,x',y,u_j(x')), \qquad x' \in \Omega$$

is equiintegrable. Therefore, by Proposition 3.6,

$$\lim_{j\to\infty} \Phi_{x,u_j}(y) = \lim_{j\to\infty} \int_\Omega f_{j,x,y}(x')\,\mathrm{d}x' = \int_{\Omega\times\mathbb{R}^d} w(x,x',y,y')\,\mathrm{d}\nu(x',y') = \Phi_{x,\nu}(y).$$

Consequently, $\Phi_{x,\nu}$ is convex as a pointwise limit of convex funtions.

Now we prove *b)*. Take $\nu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$, and let $u \in L^p(\Omega, \mathbb{R}^d)$ be its first moment. Note that

$$
\begin{aligned}
&\bar{I}(\nu) = \int_{\Omega\times\mathbb{R}^d} \Phi_{x,\nu}(y)\,\mathrm{d}\nu(x,y), \qquad I(u) = \int_{\Omega} \Phi_{x,u}(u(x))\,\mathrm{d}x, \\
&\int_{\Omega} \Phi_{x,\nu}(u(x))\,\mathrm{d}x = \int_{\Omega\times\mathbb{R}^d} \Phi_{x,u}(y)\,\mathrm{d}\nu(x,y).
\end{aligned} \tag{17}
$$

Thanks to (NY), we can use Jensen's inequality to the convex function $\Phi_{x,\nu}$ and the probability measure ν_x, and obtain that

$$
\Phi_{x,\nu}(u(x)) \leq \int_{\mathbb{R}^d} \Phi_{x,\nu}(y)\,\mathrm{d}\nu_x(y), \qquad \text{a.e. } x \in \Omega,
$$

so

$$
\int_{\Omega} \Phi_{x,\nu}(u(x))\,\mathrm{d}x \leq \int_{\Omega\times\mathbb{R}^d} \Phi_{x,\nu}(y)\,\mathrm{d}\nu(x,y). \tag{18}
$$

Analogously,

$$
\int_{\Omega} \Phi_{x,u}(u(x))\,\mathrm{d}x \leq \int_{\Omega\times\mathbb{R}^d} \Phi_{x,u}(y)\,\mathrm{d}\nu(x,y). \tag{19}
$$

Putting together the relations (17), (18) and (19) we obtain

$$
\begin{aligned}
I(u) = \int_{\Omega} \Phi_{x,u}(u(x))\,\mathrm{d}x &\leq \int_{\Omega\times\mathbb{R}^d} \Phi_{x,u}(y)\,\mathrm{d}\nu(x,y) = \int_{\Omega} \Phi_{x,\nu}(u(x))\,\mathrm{d}x \\
&\leq \int_{\Omega\times\mathbb{R}^d} \Phi_{x,\nu}(y)\,\mathrm{d}\nu(x,y) = \bar{I}(\nu),
\end{aligned}
$$

as desired.

We now show *c)*. For this we follow the idea of the proof of [36, Th. 2.6]: we shall construct a family in $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$ such that, when (NJ) is imposed, we will arrive at (NC). Let $u \in L^p(\Omega, \mathbb{R}^d)$. Fix $y_0, y_1, y_2 \in \mathbb{R}^d$ and $\alpha \in [0,1]$ such that $y_0 = \alpha y_1 + (1-\alpha) y_2$. Consider a measurable subdomain $A \subset \Omega$, and define the parametrized measures ν and μ by

$$
\nu_x = \begin{cases} \delta_{y_0}, & \text{for } x \in A, \\ \delta_{u(x)}, & \text{for a.e. } x \in A^c, \end{cases} \qquad \mu_x = \begin{cases} \alpha\delta_{y_1} + (1-\alpha)\delta_{y_2}, & \text{for } x \in A, \\ \delta_{u(x)}, & \text{for a.e. } x \in A^c. \end{cases} \tag{20}
$$

Clearly, $\nu, \mu \in \mathcal{Y}^p(\Omega, \mathbb{R}^d)$. Furthermore, the parametrized measure

$$
\mu^t := (t\mu_x + (1-t)\nu_x)_{x\in\Omega}
$$

also belongs to $\mathcal{Y}^p(\Omega, \mathbb{R}^d)$ for each $t \in [0,1]$. Define the function $f : [0,1] \to \mathbb{R}$ as $f(t) := \bar{I}(\mu^t)$ and note that

$$
f(t) = t^2 \bar{I}(\mu) + 2t(1-t) \int_{\Omega\times\mathbb{R}^d} \int_{\Omega\times\mathbb{R}^d} w(x,x',y,y')\,\mathrm{d}\nu(x,y)\,\mathrm{d}\mu(x',y') + (1-t)^2 \bar{I}(\nu)
$$

and that all coefficients of the second-order polynomial f are finite.

The first moment $\tilde{u}$ of μ^t turns out to be independent of $t \in [0,1]$, and is

$$\tilde{u}(x) = \begin{cases} y_0, & \text{for } x \in A, \\ u(x), & \text{for a.e. } x \in A^c. \end{cases}$$

Imposing condition (NJ) to μ^t yields

$$f(t) \geq \int_\Omega \int_\Omega w(x, x', \tilde{u}(x), \tilde{u}(x'))\, \mathrm{d}x\, \mathrm{d}x', \qquad t \in [0,1], \tag{21}$$

We now observe that in (21) we have equality for $t = 0$, as is immediate to check. Therefore, f attains its minimum at $t = 0$, and, hence, $f'(0) \geq 0$, obtaining the inequality

$$\int_{\Omega\times\mathbb{R}^d} \int_{\Omega\times\mathbb{R}^d} w(x, x', y, y')\, \mathrm{d}\nu(x, y)\, \mathrm{d}\mu(x', y') \geq \bar{I}(\nu).$$

Having in mind expressions (20), and simplifying common terms in both sides, the last inequality becomes

$$\begin{aligned} &\int_A \int_A [\alpha\, w(x, x', y_1, y_0) + (1-\alpha)\, w(x, x', y_2, y_0)]\, \mathrm{d}x'\, \mathrm{d}x \\ &+ \int_A \int_{A^c} [\alpha\, w(x, x', y_1, u(x')) + (1-\alpha)\, w(x, x', y_2, u(x'))]\, \mathrm{d}x'\, \mathrm{d}x \\ &\geq \int_A \int_A w(x, x', y_0, y_0)\, \mathrm{d}x'\, \mathrm{d}x + \int_A \int_{A^c} w(x, x', y_0, u(x'))\, \mathrm{d}x'\, \mathrm{d}x. \end{aligned}$$

We now take A to be a ball, divide by $|A|$ and take limits when the radius of A goes to zero; by Lemma 5.4 we find that for a.e. $x \in \Omega$,

$$\int_\Omega [\alpha\, w(x, x', y_1, u(x')) + (1-\alpha)\, w(x, x', y_2, u(x'))]\, \mathrm{d}x' \geq \int_\Omega w(x, x', y_0, u(x'))\, \mathrm{d}x',$$

that is to say,

$$\alpha\, \Phi_{x,u}(y_1) + (1-\alpha)\, \Phi_{x,u}(y_2) \geq \Phi_{x,u}(y_0),$$

so $\Phi_{x,u}$ is convex. □

For ease of reference, we summarize the conclusions of Proposition 5.3 and Theorem 5.5.

Corollary 5.6. *Let $p \geq 1$ and let $w : \Omega\times\Omega\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ be Carathéodory and symmetric. Assume that there exist $a \in L^1(\Omega\times\Omega)$, a continuous strictly increasing $g : [0,\infty) \to [0,\infty)$ with (9), and a constant $c > 0$ such that (10) and (11) hold. Then, conditions (NC), (NY), (NJ) and the lower semicontinuity of I in the weak convergence of $L^p(\Omega, \mathbb{R}^d)$ if $p > 1$ (the biting convergence if $p = 1$) are equivalent.*

Contrarily to what is claimed in [33, Th. 5.1], it is not true that if $\Phi_{x,a}$ is convex for a.e. $x \in \Omega$ and all $a \in \mathbb{R}^d$ then condition (NC) holds, as the following example shows, which is a small adaptation to that in [11, Sect. 3] showing that (NC) is weaker than separate convexity. In that example the integrand is quartic in the dependent variables with sign-changing principal coefficient depending on the independent variables, so that it is not convex for the independent variables but it satisfies (NC). Such an example is not surprising taking into account the results in [31] proving existence for linear peridynamics models with sign changing kernels.

Example 5.7. *Let $n = d = 1$, $p \geq 4$ and $\Omega = (0,1)$. Let $h : (-1,1) \to \mathbb{R}$ be any bounded smooth function such that*

- *$h(t) = h(-t)$ for all $t \in (-1,1)$.*
- *$h > 0$ in $(-1+\delta, 1-\delta)$, and $h < 0$ in $(-1,-1+\delta) \cup (1-\delta,1)$, for some $0 < \delta < \frac{1}{2}$.*
- *$\int_{-1+t}^{t} h \geq 0$ for all $t \in (0,1)$.*

Define $w : \Omega \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as $w(x,x',y,y') := \frac{1}{12} h(x-x')(y-y')^4$. Then, for any $x \in \Omega$ and $a \in \mathbb{R}$, the function $\Phi_{x,a}$ is convex, but for all $x \in (1-\delta,1)$ there exists $u \in L^p(\Omega)$ such that $\Phi_{x,u}$ is not convex.

Proof. For any $x \in \Omega$ and $u \in L^p(\Omega)$,

$$\Phi''_{x,v}(y) = \int_0^1 h(x-x') \left(y - u(x')\right)^2 \mathrm{d}x'$$

so, for all $a \in \mathbb{R}$,

$$\Phi''_{x,a}(y) = (y-a)^2 \int_0^1 h(x-x') \, \mathrm{d}x' \geq 0,$$

hence $\Phi_{x,a}$ is convex. Now fix $x \in (1-\delta,1)$ and choose $u = b\chi_{(0,x-1+\delta)}$ for $b > 0$ big enough. Then

$$\Phi''_{x,u}(y) = (y-b)^2 \int_0^{x-1+\delta} h(x-x') \, \mathrm{d}x' + y^2 \int_{x-1+\delta}^1 h(x-x') \, \mathrm{d}x'.$$

Therefore, there exist $b > 0$ and $y \in \mathbb{R}$ such that $\Phi''_{x,u}(y) < 0$, so $\Phi_{x,u}$ is not convex. □

What is true, nevertheless, is that, given a dense subset D of $L^p(\Omega,\mathbb{R}^d)$, and assuming that w satisfies the assumptions of Theorem 5.5 *a)*, if $\Phi_{x,u}$ is convex for a.e. $x \in \Omega$ and all $u \in D$, then condition (NC) holds. The proof of this fact is similar (in fact, easier, since it does not involve Young measures) to the proof of Theorem 5.5 *a)*.

Condition (NC) (or, equivalently, (NY) and (NJ)) depends on the domain Ω; this is awkward in view of the applications in peridynamics, when one would expect that the lower semicontinuity property of the energy density of a material does not depend on the reference configuration of the body. The relevance of having the lower semicontinuity property for a collection of domains became more apparent in the work [12], where that assumption was needed in order to pass to the limit as the horizon tends to zero in the peridynamic model to obtain the classical nonlinearly elastic model. Recently, Pedregal [36, Th. 2.6] showed that condition (NC) for all domains Ω reduces to separate convexity. We provide a simpler proof of this fact.

Proposition 5.8. *Assume w is $\mathcal{L}^{2n} \otimes \mathcal{B}^{2d}$-measurable, symmetric, and that*

$$w(x,\cdot,y,y') \in L^1_{\mathrm{loc}}(\Omega) \quad \text{for a.e. } x \in \Omega \text{ and all } y,y' \in \mathbb{R}^d.$$

Suppose that for a.e. $x \in \Omega$, every $y' \in \mathbb{R}^d$ and every measurable $D \subset \Omega$, the function

$$\Phi_{x,y',D} : \mathbb{R}^d \to \mathbb{R}, \qquad \Phi_{x,y',D}(y) := \int_D w(x,x',y,y') \, \mathrm{d}x'$$

is convex. Then for a.e. $x,x' \in \Omega$, the function $w(x,x',\cdot,\cdot)$ is separately convex.

Proof. Fix $x \in \Omega$ such that for every $y' \in \mathbb{R}$ and every measurable $D \subset \Omega$, the function $\Phi_{x,y',D}$ is convex. Take $y \in \mathbb{R}^d$ and let x' be a Lebesgue point of $w(x,\cdot,y,y')$. Then

$$\lim_{r \searrow 0} \frac{1}{\mathcal{L}^n(B(x',r))} \Phi_{x,y',B(x',r)}(y) = w(x,x',y,y').$$

Hence $w(x,x',\cdot,y')$ is convex as a pointwise limit of convex functions. Thanks to the symmetry of w, the function $w(x,x',\cdot,\cdot)$ is separately convex. ∎

In contrast with the functional I, and as a consequence of Corollary 3.9, the functional $\bar{I}$ is lower semicontinuous for integrands with a suitable lower bound, without the need of any convexity assumption.

Proposition 5.9. *Let $w:\Omega\times\Omega\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}\cup\{\infty\}$ be normal, symmetric, and, additionally, $w\geq h$ for some $\mathcal{L}^{2n}\otimes\mathcal{B}^{2d}$-measurable $h:\Omega\times\Omega\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}\cup\{\infty\}$ such that*

$$h(x,x',\cdot,\cdot)\in C_b(\mathbb{R}^d\times\mathbb{R}^d) \text{ for a.e. } x,x'\in\Omega$$

and

$$\int_\Omega\int_\Omega \|h(x,x',\cdot,\cdot)\|_\infty \,\mathrm{d}x'\,\mathrm{d}x<\infty.$$

Let $\{\nu_j\}_{j\in\mathbb{N}}\subset\mathcal{Y}(\Omega,\mathbb{R}^d)$ narrowly converge to some ν in $\mathcal{Y}(\Omega,\mathbb{R}^d)$. Then

$$\bar{I}(\nu)\leq\liminf_{j\to\infty}\bar{I}(\nu_j).$$

6. Relaxation via Young measures. In this section, we calculate the relaxation of the functional I in the space of Young measures with the narrow topology. In general, the *relaxation* is the lower semicontinuous envelope in the chosen topology. We specify in the next paragraphs the precise definition in our case, and, in particular, the domain where the relaxation takes places.

On the one hand, condition (10) together with any of the coercivity inequalities of Theorem 4.6 imply at once that, for any measurable $u:\Omega\to\mathbb{R}^d$, the quantity $I(u)$ is well defined and finite if and only if $u\in L^p(\Omega,\mathbb{R}^d)$. In fact, given a subset $\mathcal{A}\subset L^p(\Omega,\mathbb{R}^d)$, we have that $\mathcal{A}$ is bounded in $L^p(\Omega,\mathbb{R}^d)$ if and only if $\{I(u):u\in\mathcal{A}\}$ is bounded. Similarly, if $\bar{I}$ is as in (6), we have that, for any $\nu\in\mathcal{Y}(\Omega,\mathbb{R}^d)$, the quantity $\bar{I}(\nu)$ is well defined and finite if and only if $\nu\in\mathcal{Y}^p(\Omega,\mathbb{R}^d)$, and that, and given a subset $\mathcal{A}\subset\mathcal{Y}(\Omega,\mathbb{R}^d)$, we have that

$$\sup_{\nu\in\mathcal{A}}\int_{\Omega\times\mathbb{R}^d}|y|^p\,\mathrm{d}\nu(x,y)<\infty \quad\text{if and only if}\quad \sup_{\nu\in\mathcal{A}}\bar{I}(\nu)<\infty. \tag{22}$$

Having in mind now Proposition 3.11 and Lemma 3.12, we conclude that the natural domain for extending I in terms of Young measures is $\mathcal{Y}^p(\Omega,\mathbb{R}^d)$.

Thus, considering the inclusion $L^p(\Omega,\mathbb{R}^d)\subset\mathcal{Y}^p(\Omega,\mathbb{R}^d)$ as explained in Section 3, we first extend I to a functional I_1 defined in $\mathcal{Y}^p(\Omega,\mathbb{R}^d)$ by setting I_1 to be ∞ in $\mathcal{Y}^p(\Omega,\mathbb{R}^d)\setminus L^p(\Omega,\mathbb{R}^d)$. What we relax is this functional I_1. In this way, the lower semicontinuous envelope $\bar{I}$ of I_1 in the narrow topology of $\mathcal{Y}^p(\Omega,\mathbb{R}^d)$ is the greatest lower semicontinuous function in $\mathcal{Y}^p(\Omega,\mathbb{R}^d)$ that is below I, i.e., for each $\nu\in\mathcal{Y}^p(\Omega,\mathbb{R}^d)$,

$$\begin{aligned}\bar{I}(\nu):=\sup\bigl\{J(\nu):\ J\colon\mathcal{Y}^p(\Omega,\mathbb{R}^d)\to\mathbb{R}\cup\{\infty\}&\text{ is lower semicontinuous}\\ &\text{in the narrow topology and } J\leq I_1\bigr\}.\end{aligned}$$

Because of the comments in the paragraph above, it suffices to consider bounded sets in $L^p(\Omega,\mathbb{R}^d)$ and, in general, sets $\mathcal{A}\subset\mathcal{Y}^p(\Omega,\mathbb{R}^d)$ for which any of the two equivalent conditions of (22) hold. In virtue of Proposition 3.3 and Theorem 3.4, the topology in those sets $\mathcal{A}$ is metrizable. In particular, for any $\nu\in\mathcal{Y}^p(\Omega,\mathbb{R}^d)$,

$$\bar{I}(\nu)=\inf\left\{\liminf_{j\to\infty}I_1(\nu^j):\ \{\nu^j\}_{j\in\mathbb{N}}\subset\mathcal{Y}^p(\Omega,\mathbb{R}^d)\ \text{ and }\ \nu^j\overset{\text{nar}}{\rightharpoonup}\nu \text{ as } j\to\infty\right\}$$

(see, e.g., [9, Th. 11.1.1] or [26, Prop. 3.12]). Moreover, having in mind that I_1 is the extension by infinity of I, we have that a functional $\bar{I}$ is the relaxation of I_1 if and only if:

(i) For any bounded sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega,\mathbb{R}^d)$ such that $u_j \overset{\text{nar}}{\rightharpoonup} \nu$ as $j\to\infty$ for some $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$, we have

$$\bar{I}(\nu) \le \liminf_{j\to\infty} I(u_j).$$

(ii) For any $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$ there exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega,\mathbb{R}^d)$ such that $u_j \overset{\text{nar}}{\rightharpoonup} \nu$ as $j\to\infty$ and

$$\bar{I}(\nu) = \lim_{j\to\infty} I(u_j).$$

See, e.g., [9, Prop. 11.1.1] for this equivalence. Note that, even though the two conditions above are sometimes taken as a definition of relaxation, we needed all the preliminaries about metrizability on tight sets to conclude that assertion.

We first present the relaxation result without boundary conditions; a similar result was proved in [34, Th. 5.1].

Theorem 6.1. *Let $p\ge 1$ and assume $w:\Omega\times\Omega\times\mathbb{R}^d\times\mathbb{R}^d\to\mathbb{R}\cup\{\infty\}$ is symmetric, Carathéodory and there exist $a,\alpha\in L^1(\Omega\times\Omega)$ and $c>0$ such that*

$$\alpha(x,x') + \frac{1}{c}\,|y|^p \le |w(x,x',y,y')| \le a(x,x') + c\,|y|^p\,, \tag{23}$$

for a.e. $x,x'\in\Omega$ and all $y,y'\in\mathbb{R}^d$. Define $I_1:\mathcal{Y}^p(\Omega,\mathbb{R}^d)\to\mathbb{R}\cup\{\infty\}$ and $\bar{I}:\mathcal{Y}^p(\Omega,\mathbb{R}^d)\to\mathbb{R}$ as

$$I_1(\nu) = \begin{cases} I(u) & \text{if } \nu = \left(\delta_{u(x)}\right)_{x\in\Omega} \text{ for some } u\in L^p(\Omega,\mathbb{R}^d),\\ \infty & \text{otherwise,}\end{cases}$$

$$\bar{I}(\nu) := \int_{\Omega\times\mathbb{R}^d}\int_{\Omega\times\mathbb{R}^d} w(x,x',y,y')\,\mathrm{d}\nu(x,y)\,\mathrm{d}\nu(x',y').$$

Then, the lower semicontinuous envelope of I_1 with respect to the narrow topology is $\bar{I}$.

Proof. By the discussion above, it suffices to prove (i) and (ii). Property (i) is a consequence of Proposition 5.9.

Let now $\nu\in\mathcal{Y}^p(\Omega,\mathbb{R}^d)$. By Proposition 3.10, there exists a bounded sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega,\mathbb{R}^d)$ generating ν such that $\{|u_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable. Due to (23), the sequence of functions

$$\Omega\times\Omega\ni(x,x')\mapsto w(x,x',u_j(x),u_j(x'))$$

is equiintegrable, hence, thanks to Proposition 3.6,

$$\bar{I}(\nu) = \lim_{j\to\infty} I_1(u_j)$$

with the identification of a function with its corresponding Young measure. □

Now we present the relaxation result for Dirichlet and mixed Dirichlet–Neumann conditions, as explained in Section 4. The conclusion is that the boundary conditions pass to the relaxation.

THEOREM 6.2. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta > 0$ and let $p \geq 1$. Let Ω_0 be a non-empty open subset of Ω satisfying $\Omega_0 + B(0,\delta) \subset \Omega$. Let Ω_D be a measurable subset of $\Omega \setminus \Omega_0$ with positive measure. Assume $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is symmetric, Carathéodory and there exist $a, \alpha \in L^1(\Omega \times \Omega)$ and $c > 0$ such that*

$$\alpha(x,x') + \frac{1}{c}\chi_{B(0,\delta)}(x-x')\,|y-y'|^p \leq |w(x,x',y,y')| \leq a(x,x') + c\,|y|^p\,, \tag{24}$$

for a.e. $x, x' \in \Omega$ and all $y, y' \in \mathbb{R}^d$. Let $u_0 \in L^p(\Omega_D, \mathbb{R}^d)$. Let $\mathcal{Y}^p_{u_0}(\Omega,\mathbb{R}^d)$ be the set of $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$ such that $\nu_x = \delta_{u_0(x)}$ for a.e. $x \in \Omega_D$. Define $I_1, \bar{I} : \mathcal{Y}^p(\Omega,\mathbb{R}^d) \to \mathbb{R} \cup \{\infty\}$ as

$$I_1(\nu) = \begin{cases} I(u) & \text{if } \nu = \left(\delta_{u(x)}\right)_{x\in\Omega} \text{ for some } u \in L^p(\Omega,\mathbb{R}^d) \text{ with } u|_{\Omega_D} = u_0 \text{ a.e.,} \\ \infty & \text{otherwise,} \end{cases}$$

$$\bar{I}(\nu) := \begin{cases} \displaystyle\int_{\Omega\times\mathbb{R}^d}\int_{\Omega\times\mathbb{R}^d} w(x,x',y,y')\,\mathrm{d}\nu(x,y)\,\mathrm{d}\nu(x',y') & \text{if } \nu \in \mathcal{Y}^p_{u_0}(\Omega,\mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases}$$

Then, the lower semicontinuous envelope of I_1 with respect to the narrow topology is $\bar{I}$.

Proof. Only two additional steps to those of the proof of Theorem 6.1 are needed.

Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $L^p(\Omega,\mathbb{R}^d)$ such that $u_j|_{\Omega_D} = u_0$ and $\sup_{j\in\mathbb{N}} I(u_j) < \infty$. Thanks to (24) and Proposition 4.2, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega,\mathbb{R}^d)$. Then, $\{u_j\}_{j\in\mathbb{N}}$ generates a $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$. It is immediate to see from the definition of narrow convergence (see, e.g., [9, Rk. 4.3.1]) that $u_j|_{\Omega_D} \overset{\text{nar}}{\rightharpoonup} \nu|_{\Omega_D}$ in $\mathcal{Y}^p(\Omega_D,\mathbb{R}^d)$. Since $u_j|_{\Omega_D}$ is identified with $\{\delta_{u_0(x)}\}_{x\in\Omega_D}$, we have that $\nu_x = \delta_{u_0(x)}$ a.e. $x \in \Omega_D$.

Conversely, assume that $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$ satisfies $\nu_x = \delta_{u_0(x)}$ for a.e. $x \in \Omega_D$, and let $\{u_j\}_{j\in\mathbb{N}}$ be any bounded sequence in $L^p(\Omega;\mathbb{R}^d)$ generating ν such that $\{|u_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable. As before, $\{u_j|_{\Omega_D}\}_{j\in\mathbb{N}}$ generates $\nu|_{\Omega_D}$ so (see, e.g., [9, Prop. 4.3.8]) $u_j|_{\Omega_D} \to u_0$ in measure. Define $v_j := u_j\chi_{\Omega\setminus\Omega_D} + u_0\chi_{\Omega_D}$. Then $v_j - u_j \to 0$ in measure and, hence (see again [9, Prop. 4.3.8]), $\{v_j\}_{j\in\mathbb{N}}$ generates ν. Moreover, $\{|v_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable as the sum of two equiintegrable sequences. This is enough to conclude that the proof of Theorem 6.1 can be adapted to this case. ∎

Finally, the relaxation result for Neumann conditions is as follows; in this case, we need $p > 1$ for the restriction $\int_\Omega u = 0$ to pass to the limit. We will deal in Theorem 6.4 with the case $p = 1$.

THEOREM 6.3. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta > 0$ and let $p > 1$. Assume $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is symmetric, Carathéodory and there exist $a, \alpha \in L^1(\Omega \times \Omega)$ and $c > 0$ such that (24) holds. Let $\mathcal{Y}^p_0(\Omega,\mathbb{R}^d)$ be the set of $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$ whose first moment u satisfies $\int_\Omega u = 0$. Define $I_1, \bar{I} : \mathcal{Y}^p(\Omega,\mathbb{R}^d) \to \mathbb{R} \cup \{\infty\}$ as*

$$I_1(\nu) = \begin{cases} I(u) & \text{if } \nu = \left(\delta_{u(x)}\right)_{x\in\Omega} \text{ for some } u \in L^p(\Omega,\mathbb{R}^d) \text{ with } \displaystyle\int_\Omega u = 0, \\ \infty & \text{otherwise,} \end{cases}$$

$$\bar{I}(\nu) := \begin{cases} \displaystyle\int_{\Omega\times\mathbb{R}^d}\int_{\Omega\times\mathbb{R}^d} w(x,x',y,y')\,\mathrm{d}\nu(x,y)\,\mathrm{d}\nu(x',y') & \text{if } \nu \in \mathcal{Y}^p_0(\Omega,\mathbb{R}^d), \\ \infty & \text{otherwise.} \end{cases}$$

Then, the lower semicontinuous envelope of I_1 with respect to the narrow topology is $\bar{I}$.

Proof. Only two additional steps to those of the proof of Theorem 6.1 are needed.

Let $\{u_j\}_{j\in\mathbb{N}}$ be a sequence in $L^p(\Omega,\mathbb{R}^d)$ such that $\int_\Omega u_j = 0$ for all $j \in \mathbb{N}$ and $\sup_{j\in\mathbb{N}} I(u_j) < \infty$. Thanks to (24) and Proposition 4.3, $\{u_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega,\mathbb{R}^d)$. Then, $\{u_j\}_{j\in\mathbb{N}}$ generates a $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$. Let u be the first moment of ν. By Lemma 3.13, $\{u_j\}_{j\in\mathbb{N}}$ converges to u weakly in $L^p(\Omega,\mathbb{R}^d)$, so $\int_\Omega u = 0$.

Conversely, assume that $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$ has a first moment u satisfying $\int_\Omega u = 0$, and let $\{u_j\}_{j\in\mathbb{N}}$ be any bounded sequence in $L^p(\Omega;\mathbb{R}^d)$ generating ν such that $\{|u_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable. Then $u_j \rightharpoonup u$ in $L^p(\Omega;\mathbb{R}^d)$, so $\int_\Omega u_j \to 0$ as $j \to \infty$. As in the proof of Proposition 4.5, define $v_j := u_j - \fint_\Omega u_j$ for each $j \in \mathbb{N}$, which satisfies $\int_\Omega v_j = 0$. Moreover, $\{v_j\}_{j\in\mathbb{N}}$ generates ν and $\{|v_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable. This concludes the proof. $\square$

As a consequence of a general abstract fact (see, e.g., [9, Th. 11.1.2]) and of the existence theorems of Section 4, we have that, in the context of any of Theorems 6.1, 6.2 or 6.3, we have the following facts:

a) The functional $\bar{I}$ has a minimizer, and its minimum coincides with the infimum of I.
b) Minimizing sequences of I converge narrowly, up to a subsequence, to a minimizer of $\bar{I}$.
c) Every minimizer of $\bar{I}$ is a narrow limit of a mimimizing sequence of I.

It only remains to deal with the case $p = 1$ with Neumann boundary conditions. In this case, the restriction $\int_\Omega u = 0$ does not pass to the relaxation, so we discard it. It was that restriction that allowed us to focus the attention on tight sets of Young measures. Without that restriction, we are still able to show that properties (i) and (ii) of the beginning of this section hold, and that minimizers of $\bar{I}$ are precisely the limit of suitable *translated* minimizing sequences of I, so mimicking properties a)–c) above. This is, in fact, a kind of relaxation result, but this itself does not show that $\bar{I}$ is the lower semicontinuous envelope of I because the narrow topology is not metrizable outside tight sets.

THEOREM 6.4. *Let Ω be a Lipschitz domain of $\mathbb{R}^n$, fix $\delta > 0$ and let $p = 1$. Assume $w : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ is symmetric, Carathéodory and there exist $a, \alpha \in L^1(\Omega\times\Omega)$ and $c > 0$ such that (24) holds. Define $I_1 : \mathcal{Y}^p(\Omega,\mathbb{R}^d) \to \mathbb{R}\cup\{\infty\}$ and $\bar{I} : \mathcal{Y}^p(\Omega,\mathbb{R}^d) \to \mathbb{R}$ as in Theorem 6.1. Then, properties (i) and (ii) hold. Moreover, under the assumption that I and $\bar{I}$ are invariant under translations, the following hold:*

a) Calling $m := \inf_{u\in L^p(\Omega,\mathbb{R}^d)} I(u)$, we have that $m = \min_{\nu\in\mathcal{Y}^p(\Omega,\mathbb{R}^d)} \bar{I}(\nu)$.

b) If $\{u_j\}_{j\in\mathbb{N}}$ is a sequence in $L^p(\Omega,\mathbb{R}^d)$ with $I(u_j) \to m$ as $j \to \infty$ then, for a subsequence (not relabelled), the sequence $\{v_j\}_{j\in\mathbb{N}}$ defined as

$$v_j := u_j - \fint_\Omega u_j, \qquad j \in \mathbb{N}$$

converges narrowly to some $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$, and any such ν satisfies $\bar{I}(\nu) = m$.

c) For every $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$ satisfying $\bar{I}(\nu) = m$ there exists an equiintegrable sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^p(\Omega,\mathbb{R}^d)$ generating ν such that $I(u_j) \to m$ as $j \to \infty$.

Proof. The proof of properties (i) and (ii) is identical to that of Theorem 6.1. In fact, the proof of (ii) shows property *c)* at once.

Let now $\{u_j\}_{j\in\mathbb{N}}$ and $\{v_j\}_{j\in\mathbb{N}}$ be as in *b)*. By (24) and Proposition 4.3, $\{v_j\}_{j\in\mathbb{N}}$ is bounded in $L^p(\Omega,\mathbb{R}^d)$, so by Proposition 3.10, after a subsequence, it converges narrowly to some $\nu \in \mathcal{Y}^p(\Omega,\mathbb{R}^d)$. Let ν be any such limit. By Proposition 5.9, $\bar{I}(\nu) \leq m$, while part *c)* allows us to conclude that $\bar{I}(\nu) = m$ as well as part *a)*, since, by Theorem 4.6, $\bar{I}$ has minimizers. ∎

7. Remarks on the relaxation in L^p. In this section we make some comments about the difficulty of computing the relaxation of I in the weak topology of $L^p(\Omega,\mathbb{R}^d)$ for $p > 1$. An analogous reasoning to that of the beginning of Section 6 allows us to write three equivalent definitions of $I^* : L^p(\Omega,\mathbb{R}^d) \to \mathbb{R}$, the lower semicontinuous envelope of I the weak topology of $L^p(\Omega,\mathbb{R}^d)$, provided that condition (10) and any of the coercivity inequalities of Theorem 4.6 hold. To be precise, we choose $\mathcal{A}$ as the closed subspace of $L^p(\Omega,\mathbb{R}^d)$ codifying the boundary conditions, according to the cases *a)*–*c)* of Theorem 4.6; in other words, $\mathcal{A} = L^p(\Omega,\mathbb{R}^d)$ in case *a)*, $\mathcal{A} = L^p_{u_0}(\Omega,\mathbb{R}^d)$ in case *b)* and $\mathcal{A} = L^p_0(\Omega,\mathbb{R}^d)$ in case *c)*, where

$$L^p_{u_0}(\Omega,\mathbb{R}^d) := \left\{u \in L^p(\Omega,\mathbb{R}^d) : u = u_0 \text{ a.e. in } \Omega_D\right\}$$

and

$$L^p_0(\Omega,\mathbb{R}^d) := \left\{u \in L^p(\Omega,\mathbb{R}^d) : \int_\Omega u = 0\right\}.$$

Then, for any $u \in \mathcal{A}$,

$$\begin{aligned} &I^*(u) \\ &:= \sup\{J(u)\colon\ J\colon \mathcal{A} \to \mathbb{R} \text{ is lower semicontinuous in the weak topology and } J \leq I\} \\ &= \inf\left\{\liminf_{j\to\infty} I(u_j) :\ \{u_j\}_{j\in\mathbb{N}} \subset \mathcal{A} \ \text{ and }\ u_j \rightharpoonup u \text{ as } j \to \infty\right\}. \end{aligned}$$

Moreover, I^* is characterized by the following two facts:

(i) For any sequence $\{u_j\}_{j\in\mathbb{N}}$ in $\mathcal{A}$ such that $u_j \rightharpoonup u$ in $L^p(\Omega,\mathbb{R}^d)$ as $j \to \infty$, we have
$$I^*(u) \leq \liminf_{j\to\infty} I(u_j).$$

(ii) There exists a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $\mathcal{A}$ such that $u_j \rightharpoonup u$ $L^p(\Omega,\mathbb{R}^d)$ as $j \to \infty$ and
$$I^*(u) = \lim_{j\to\infty} I(u_j).$$

Moreover, the abstract properties a)–c) of Section 6 also hold for I^* replacing $\bar{I}$, under any of the coercivity assumptions of Theorems 6.1, 6.2 or 6.3. In particular,

$$\min_{u\in\mathcal{A}} I^*(u) = \min_{\nu\in\mathcal{Y}^p(\Omega,\mathbb{R}^d)} \bar{I}(\nu) = \inf_{u\in L^p(\Omega,\mathbb{R}^d)} I(u). \tag{25}$$

We focus our attention on the simplest case of a $w : \mathbb{R}^d \times \mathbb{R}^d \to [0,\infty)$ with no spatial dependence. Then conditions (NC) and (NY) trivially reduce to the separate convexity of w, since w is symmetric. Thanks to Corollary 5.6, so do condition (NJ) and the lower semicontinuity of I in $L^p(\Omega,\mathbb{R}^d)$, provided that the hypotheses therein hold, which we assume. Therefore, $I^* = I$ if and only if w is separately convex. Let now I^{sc} be the functional associated to w^{sc}, the *separately convex hull* of w (i.e., w^{sc} is the greatest separately convex function that is below w). Then $I^{sc} \leq I^*$ since I^{sc} is lower semicontinuous and $I^{sc} \leq I$.

We first mention that Pedregal [36, Sect. 3] showed an example of a function $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ for which $I^{sc} \neq I^*$. We present here a simpler variant of his construction.

EXAMPLE 7.1. *Let $p \geq 1$. Take $a_1 < a_2 < a_3 < a_4$ and consider the points*

$$\begin{aligned} z_1 &:= (a_2, a_1), \quad & z_2 &:= (a_4, a_1), \quad & z_3 &:= (a_4, a_3), \\ z_4 &:= (a_3, a_4), \quad & z_5 &:= (a_1, a_4), \quad & z_6 &:= (a_1, a_2). \end{aligned}$$

Let $w : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be a continuous symmetric function such that $w(z) = 0$ if and only if $z \in \{z_1, \dots, z_6\}$, and there exists $c > 0$ for which

$$w(y, y') \geq c\, |y|^p - \frac{1}{c}, \qquad \text{for all } y, y' \in \mathbb{R}.$$

Then $I^{sc} \neq I^*$.

Proof. We shall show that $\min_{\nu \in \mathcal{Y}^p(\Omega)} \bar{I}(\nu) > 0$ and $\min_{u \in L^p(\Omega)} I^{sc}(u) = 0$: property (25) will conclude.

Suppose, for a contradiction, that $\bar{I}(\nu) = 0$ for some $\nu \in \mathcal{Y}^p(\Omega)$. Then

$$\int_{\Omega \times \mathbb{R}} \int_{\Omega \times \mathbb{R}} w(y, y') \,\mathrm{d}\nu(x, y) \,\mathrm{d}\nu(x', y') = 0,$$

hence $w(y, y') = 0$, equivalently, $(y, y') \in \{z_i\}_{i=1}^6$ for ν-a.e. $(x, y), (x', y') \in \Omega \times \mathbb{R}$. In particular, $\operatorname{supp} \nu_x \subset \{a_1, a_2, a_3, a_4\}$ for a.e. $x \in \Omega$, so $\nu_x = \sum_{i=1}^4 \lambda_i(x) \delta_{a_i}$ for some measurable $\lambda_i : \Omega \to [0, 1]$ $(i \in \{1, \dots, 4\})$ with $\sum_{i=1}^4 \lambda_i(x) = 1$, whence

$$\begin{aligned} \bar{I}(\nu) &= \int_\Omega \int_\Omega \sum_{i,j=1}^4 \lambda_i(x)\, \lambda_j(x')\, w(a_i, a_j) \,\mathrm{d}x \,\mathrm{d}x' \\ &\geq \int_\Omega \int_\Omega \sum_{i=1}^4 \lambda_i(x)\, \lambda_i(x')\, w(a_i, a_i) \,\mathrm{d}x \,\mathrm{d}x' > 0, \end{aligned} \qquad \square$$

a contradiction.

Now, w^{sc} satisfies $w^{sc} \geq 0$ and w^{sc} vanishes in $\{z_i\}_{i=1}^6$. As w^{sc} is separately convex, it vanishes in the separately convex hull of $\{z_i\}_{i=1}^6$, which is easily seen to include $[a_2, a_3] \times [a_2, a_3]$ (see Figure 1). Hence $I^{sc}(u) = 0$ for any $u \in L^\infty(\Omega)$ such that $a_2 \leq u(x) \leq a_3$ for a.e. $x \in \Omega$. In particular, $\min_{u \in L^p(\Omega)} I^{sc}(u) = 0$.

A question of capital importance is whether I^* admits an integral representation, i.e., whether there exists an $\mathcal{L}^{2n} \otimes \mathcal{B}^{2d}$-measurable function $w^* : \Omega \times \Omega \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ such that

$$I^*(u) = \int_\Omega \int_\Omega w^*(x, x', u(x), u(x')) \,\mathrm{d}x \,\mathrm{d}x' \qquad \text{for all } u \in L^p(\Omega, \mathbb{R}^d).$$

The usual theorems on integral representations (see, e.g., [26, Thms. 5.29 and 6.65]) cannot be applied since the functional I is not additive with respect to the set of integration Ω. In [36] it is claimed that if I^* admits an integral representation then the corresponding integrand must be w^{sc}, but its proof is dubious, since it seems to use that if a continuous symmetric integrand $\tilde{w} : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ is such that its corresponding functional $\tilde{I}$ satisfies $\tilde{I} \geq 0$ then $\tilde{w} \geq 0$, which is not true in general, as

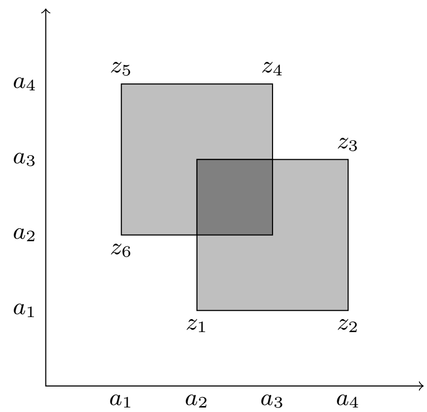

FIG. 1. *Corresponding to Example 7.1: in light grey, the separately convex hull of $\{z_i\}_{i=1}^6$; in dark grey, the square $[a_2, a_3] \times [a_2, a_3]$.*

we will see in Example 7.2. In fact, we are unable to answer the question of whether I^* admits an integral representation in Examples 7.1 and 7.2.

We now present an example of a different nature. Let us consider afunctional I given through an integrand $w : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $w(y, y') = f(y - y')$, and assume that its relaxation I^* is given through an integrand $w^* = w^*(y, y') = g(y - y')$ (and the function g can be assumed to be even because of the symmetry), then we show that g takes vales both above and below f. This shows the difficulty of computing I^*, since there is no natural candidate in integral form. To fix ideas, we consider homogeneous Neumann boundary conditions, i.e., we set $\mathcal{A} = L^p_0(\Omega, \mathbb{R}^d)$.

EXAMPLE 7.2. *Let Ω be a bounded open set of $\mathbb{R}^n$. Let $p = 4$ and fix $\alpha > 0$. Let $f : \mathbb{R} \to \mathbb{R}$ be the polynomial*

$$f(t) := -2\alpha\, t^2 + t^4.$$

Let $I : L^p_0(\Omega) \to \mathbb{R}$ be the nonlocal functional

$$I(u) := \int_\Omega \int_\Omega f(u(x) - u(x'))\,\mathrm{d}x\,\mathrm{d}x'.$$

Then I has minimizers, even though it is not weakly lower semicontinuous. Moreover, $I \geq -|\Omega|^2\alpha^2/2$, even though f takes values below $-|\Omega|^2\alpha^2/2$.

Let I^* be the lower semicontinuous envelope of I in $L^p_0(\Omega)$ with respect to the weak topology. Then any $u \in L^p_0(\Omega)$ such that $|u| \leq \sqrt{\alpha}/2$ is a minimizer of I^*. Moreover, if there exists a continuous even $g : \mathbb{R} \to \mathbb{R}$ such that

$$I^*(u) = \int_\Omega \int_\Omega g\,(u(x) - u(x'))\,\mathrm{d}x\,\mathrm{d}x' \qquad \textit{for all } u \in L^p_0(\Omega)$$

then g is convex and $g(t) = -\alpha^2/2$ for all $t \in [-\sqrt{\alpha}, \sqrt{\alpha}]$.

Proof. First we show the lower bound $I(u) \geq -|\Omega|^2\alpha^2/2$ for all $u \in L^p_0(\Omega)$. We have

$$\begin{aligned}\int_\Omega \int_\Omega (u(x) - u(x'))^2\,\mathrm{d}x\,\mathrm{d}x' &= \int_\Omega \int_\Omega \left(u(x)^2 - 2u(x)\,u(x') + u(x')^2\right)\mathrm{d}x\,\mathrm{d}x' \\ &= 2\,\|u\|^2_{L^2(\Omega)}\,|\Omega|,\end{aligned}$$

since $\int_\Omega u = 0$. Analogously,

$$\int_\Omega \int_\Omega (u(x) - u(x'))^4 \, \mathrm{d}x \, \mathrm{d}x' = \sum_{i=0}^{4} (-1)^i \binom{4}{i} \int_\Omega \int_\Omega u(x)^i \, u(x')^{4-i} \, \mathrm{d}x \, \mathrm{d}x'$$
$$= 2 \, \|u\|_{L^4(\Omega)}^4 \, |\Omega| + 6 \, \|u\|_{L^2(\Omega)}^4 .$$

Hölder's inequality ensures that $\|u\|_{L^2(\Omega)} \leq |\Omega|^{\frac{1}{4}} \, \|u\|_{L^4(\Omega)}$. Using this and the fact that the polynomial $t \to -4\alpha t|\Omega| + 8t^2$ attains its minimum at $t = \alpha|\Omega|/4$ and equals $-|\Omega|^2\alpha^2/2$, we obtain

$$\begin{aligned} I(u) &= -4\alpha \, \|u\|_{L^2(\Omega)}^2 \, |\Omega| + 2 \, \|u\|_{L^4(\Omega)}^4 \, |\Omega| + 6 \, \|u\|_{L^2(\Omega)}^4 \\ &\geq -4\alpha \, \|u\|_{L^2(\Omega)}^2 \, |\Omega| + 8 \, \|u\|_{L^2(\Omega)}^4 \geq -\frac{\alpha^2}{2} |\Omega|^2 . \end{aligned}$$

Moreover, the equality $I(u) = -|\Omega|^2 \frac{\alpha^2}{2}$ holds if and only if

$$\|u\|_{L^2(\Omega)} = |\Omega|^{\frac{1}{4}} \, \|u\|_{L^4(\Omega)} \quad \text{and} \quad \|u\|_{L^2(\Omega)}^2 = \frac{\alpha|\Omega|}{4} .$$

Now, the first equality holds if and only if the functions $|u|$ and 1 are parallel, i.e., there exists $c \geq 0$ such that $|u| = c$ a.e. Hence, u must be of the form $u = c(\chi_A - \chi_{A^c})$ for some measurable set $A \subset \Omega$. The restriction $\int_\Omega u = 0$ yields $|A| = \frac{1}{2}|\Omega|$. Finally, imposing $\|u\|_{L^2(\Omega)}^2 = \frac{\alpha|\Omega|}{4}$ leads to $c = \frac{1}{2}\sqrt{\alpha}$.

Now let $u \in L^p_0(\Omega)$ be such that $|u| \leq \sqrt{\alpha}/2$. Define $\lambda \in L^\infty(\Omega)$ and $\nu \in \mathcal{Y}(\Omega)$ as

$$\lambda(x) = \frac{1}{2} \left(1 - \frac{2}{\sqrt{\alpha}} \, u(x) \right) , \qquad \nu_x = \lambda(x) \, \delta_{-\sqrt{\alpha}/2} + (1 - \lambda(x)) \, \delta_{\sqrt{\alpha}/2}, \qquad \text{a.e. } x \in \Omega.$$

As $u \in L^p_0(\Omega)$, it is easy to check that $\nu \in \mathcal{Y}^p_0(\Omega)$. Moreover,

$$\begin{aligned} \bar{I}(\nu) &= \int_{\Omega\times\mathbb{R}} \int_{\Omega\times\mathbb{R}} f(y - y') \, \mathrm{d}\nu(x, y) \, \mathrm{d}\nu(x', y') \\ &= 2 \, f(\sqrt{\alpha}) \int_\Omega \lambda(x) \, \mathrm{d}x \int_\Omega (1 - \lambda(x)) \, \mathrm{d}x = -|\Omega|^2 \frac{\alpha^2}{2} . \end{aligned}$$

Let $\{u_j\}_{j\in\mathbb{N}}$ be a bounded sequence in $L^p_0(\Omega)$ generating ν such that $\{|u_j|^p\}_{j\in\mathbb{N}}$ is equiintegrable; this sequence was shown to exist in Proposition 4.5. Then

$$-|\Omega|^2 \frac{\alpha^2}{2} = \bar{I}(\nu) = \lim_{j\to\infty} I(u_j)$$

and $u_j \rightharpoonup u$ in $L^p(\Omega, \mathbb{R}^d)$, since u is the first moment of ν (recall Lemma 3.13), so

$$I^*(u) \leq \lim_{j\to\infty} I(u_j) = -|\Omega|^2 \frac{\alpha^2}{2} .$$

Hence, u is a minimizer of I^*.

Now, suppose that there exists a g as in the statement. Corollary 5.6 shows that g must be convex. Now, given $t \in \mathbb{R}$ and any measurable set $A \subset \Omega$ with $\mathcal{L}^n(A) = \frac{1}{2}\mathcal{L}^n(\Omega)$ we apply the inequalities $-|\Omega|^2 \frac{\alpha^2}{2} \leq I^* \leq I$ to

$$u = \frac{t}{2} \, (\chi_A - \chi_{A^c}) \in L^p_0(\Omega)$$

and obtain

$$
\begin{aligned}
-|\Omega|^2 \frac{\alpha^2}{2} &\leq \mathcal{L}^n(A)^2\, g(0) + 2\,\mathcal{L}^n(A)\,\mathcal{L}^n(A^c)\, g(t) + \mathcal{L}^n(A^c)^2 g(0) \\
&\leq \mathcal{L}^n(A)^2\, f(0) + 2\,\mathcal{L}^n(A)\,\mathcal{L}^n(A^c)\, f(t) + \mathcal{L}^n(A^c)^2\, f(0),
\end{aligned}
$$

which reduce to

$$-\alpha^2 \leq g(0) + g(t) \leq f(t).$$

As g is convex, we deduce that

$$-\alpha^2 \leq g(0) + g(t) \leq f^c(t),$$

where $f^c : \mathbb{R} \to \mathbb{R}$ is the *convexification* of f. Since f attains its minimum at $t = \pm\sqrt{\alpha}$ and equals $-\alpha^2$, we infer that $f^c(t) = -\alpha^2$ for all $t \in [-\sqrt{\alpha}, \sqrt{\alpha}]$. Therefore,

$$-\alpha^2 \leq g(0) + g(t) \leq -\alpha^2 \qquad \text{for all } t \in [-\sqrt{\alpha}, \sqrt{\alpha}].$$

which shows that $g(t) = -\alpha^2/2$ for all $t \in [-\sqrt{\alpha}, \sqrt{\alpha}]$. ◻

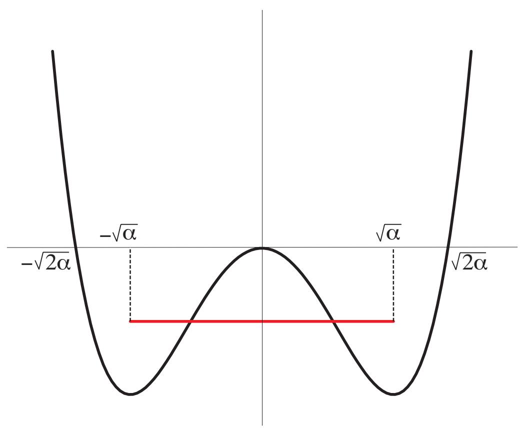

FIG. 2. *Functions f (black) and g (red) of Example 7.2.*

This example is very special in the sense that everything works out for this quartic polynomial and considering an energy density only depending on $y - y'$: this is because the two inequalities that were used (Hölder's and the minimum of a quadratic polynomial) happen to be equalities for the same function. We neither have other essentially different examples nor know what happens if we assume that the relaxed energy density is given by a g also depending on the independent variables x, x'.

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