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Nonlinear and nonlocal diffusion equations. Qualitative theory and asymptotic behaviour

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Nonlinear and nonlocal diffusion equations. Qualitative theory and asymptotic behaviour

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To Juan Luis and Susanna

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Contents

Introduction and summary of the results

This Ph.D. thesis is structured in two independent parts: the first one is devoted to the study of reaction equations with doubly nonlinear diffusion, while the second one to the analysis of the nodal set of solutions to a nonlocal parabolic equation. In the next paragraphs we introduce some basic concepts and the main results of both parts.

Before moving on, we would like to stress that both parts are centred on the research topic of the *diffusion equations*, which is a crucial issue in PDEs. A wide variety of natural phenomena can be mathematically described by diffusion processes and they have been intensively studied in the last 200 years. Still nowadays, they are the subject of considerable research.
The foundation of the entire diffusion theory is the Heat Equation

$$\partial_t u = \Delta u \qquad x \in \mathbb{R}^N,\ t \geq 0 \tag{HE}$$

introduced by Fourier in 1822, in his work: *Théorie analytique de la chaleur*, [104]. As the name of the equation suggests, Fourier's goal was to describe the time variation of a heat density in $\mathbb{R}^N$, assuming to know its distribution at time $t = 0$. Later, the same equation have been employed in other fields of applied mathematics in connection with diffusive phenomena like dynamics of populations, fluid dynamics, elasticity and so on. Due to its historical importance and in order to bring the reader into the diffusion framework from the very beginning, we devote some paragraphs to review some well-known but crucial concepts that, as we will see, influence the majority of the methods and techniques employed in this treatise to study more advanced and complex models.

So, from the PDEs point of view, we are interested in describing the analytical properties of equation (HE), such as existence, uniqueness, regularity and long time behaviour of its solutions. These are encoded in a special solution called *fundamental solution*. The idea is to look for solutions in self-similar form

$$U(x,t) = t^{-\alpha} F(x t^{-\alpha/N}), \qquad x \in \mathbb{R}^N,\ t > 0,$$

for some exponent $\alpha > 0$. In the case of the Heat Equation, the fundamental solution is the time-dependent Gaussian function

$$G_N(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}},$$

with $\alpha = N/2$, and $F(\xi) = e^{-\xi^2/4}$ with $\xi = |x|t^{-1/2}$, which is the most common density employed in Probability and Statistics (in the case of nonlinear and nonlocal diffusion we will find different special solutions that will play a decisive role in both settings). In addition to the self-similar form, the Gaussian $G_N = G_N(x,t)$ possesses remarkable properties like symmetry with respect to the spacial variable $x \in \mathbb{R}^N$, conservation of the initial mass, and strict positivity. Moreover, it converges to a Dirac's Delta when $t \to 0$:

$$G_N(x,t) \to \delta_0(x), \qquad \text{as } t \to 0, \text{ in the sense of distributions.}$$

This last property is crucial in the study of equation (HE) with initial datum $u_0(x) = u(x,0)$, $x \in \mathbb{R}^N$ and $u_0 \in L^1(\mathbb{R}^N)$ (this last assumption can be relaxed, but for now it is unimportant, see for instance

[193, 208]). Indeed, we have the convolution formula

$$u(x,t) = (G_N * u_0)(x,t) := \int_{\mathbb{R}^N} G(x-y,t)\, u_0(y)\, dy, \tag{CFHE}$$

which gives an analytic expression for the solution of (HE) in terms of the convolution of the Gaussian with initial datum $u_0(\cdot)$ (this property will not be available for the nonlinear diffusion models that we will study in the first part and we will have to use techniques based on comparison with different self-similar solutions). From the previous formula, it is possible to deduce existence, uniqueness, regularity and asymptotic behaviour of the solutions, for a wide class of initial data.

Finally, we must highlight the probabilistic interpretation of formula (CFHE), discovered in the work of Einstein [90], which links the solution of (HE) with initial datum $u(x,0) = u_0(x)$, with the Brownian motion which is the stochastic process most used in the applications. Indeed, it is simple to see that

$$u(x,t) = (G_N * u_0)(x,t) = \mathbb{E}_x(u_0(W_t)),$$

where $\{W_t\}_{t\geq 0}$ is a Brownian motion and $\mathbb{E}_x(\cdot)$ is the expectation w.r.t. the distribution of a Brownian motion starting at x, i.e., the Gaussian distribution found earlier. The previous formula is probably the first significative result which has strongly connected the fields of parabolic equations and stochastic processes, and it is basically equivalent to the fact that the Laplacian is the generator of the Brownian motion:

$$-\Delta u(x) = 2\lim_{h\to 0} \frac{u(x) - \mathbb{E}_x[u(W_h)]}{h},$$

for all $u = u(x)$ smooth enough (cfr. for instance with Chapter 1 of [164]). We highlight that the diffusion equations studied in this thesis have interesting connections with the probability field, too. However, we will focus on nonlinear and nonlocal models that describe *non-Gaussian* processes (cfr. for instance with [14, 108, 195, 197, 198] for and their references for models and connections with the probability field) which frequently emerge in many applied sciences.

Some of the facts reviewed in the above paragraphs have serious consequences (from both the theoretical and applicative point of view) in both topics if this thesis. For what concerns the first part, in which we focus on the study of the long time behaviour of solutions to reaction-diffusion equations with doubly nonlinear diffusion, we prove the existence of special wave fronts which describe the asymptotic behaviour for large times of a much larger class of solutions (extending the classical works [12, 13] to the doubly nonlinear setting).
In to the second part, we will extend some of the results proved in [63, 122] (for the local case) to a class of solutions to a nonlocal parabolic equation. We find a class of parabolically homogeneous polynomials of Hermite and Laguerre type, which will be employed to characterize the nodal points of more general solutions. In both cases the analysis of some *special/fundamental* solutions (wave fronts and homogeneous polynomials) will be the key to describe the properties of general solutions. We can now pass to the presentation of the two main parts of the thesis.

Reaction equations with doubly nonlinear diffusion

The first part of this manuscript essentially contains the results proved in the papers [17, 18] written in collaboration with Prof. Juan Luis Vázquez and the preprint [15] by the author. It is devoted to the study of the long-time behaviour of solutions to the reaction-diffusion initial value problem

$$\begin{cases} \partial_t u = \Delta_p u^m + f(u) & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{RDNL}$$

for a wide class of initial data $u_0 = u_0(x)$ ($u_0 \in C_c(\mathbb{R}^N)$ with $0 \leq u_0 \leq 1$), different ranges of the parameters $m > 0$ and $p > 1$, and different reaction terms $f(\cdot)$.

The doubly nonlinear operator is defined by

$$\Delta_p u^m := \Delta_p(u^m) = \nabla \cdot (|\nabla(u^m)|^{p-2}\nabla(u^m)), \qquad m > 0,\ p > 1,$$

where ∇ is the spatial gradient while $\nabla\cdot$ is the spatial divergence. We call it doubly nonlinear since it can be seen as the composition of the m-th power and the p-Laplacian (cfr. with [59, 81, 94, 131, 144, 197, 198] and their references, for some physical models). Note that we recover the Porous Medium operator choosing $p = 2$ or the p-Laplacian operator choosing $m = 1$. Of course, choosing $m = 1$ and $p = 2$ we obtain the classical Laplacian. It thus clear that the diffusion strongly depends on the parameter $m > 0$ and $p > 1$. We will divide the analysis in three different ranges $m > 0$ and $p > 1$ depending on the sign/value of the quantity:

$$m(p-1) - 1 > 0, \qquad m(p-1) - 1 = 0, \qquad -\frac{p}{N} < m(p-1) - 1 < 0,$$

called slow, pseudo-linear and fast ranges, respectively. The function $f(\cdot)$ will essentially be of two types, modeled on two different classical reaction terms:

$$f(u) = u(1-u), \qquad f(u) = u(1-u)(u-a),$$

where $0 < a < 1$ is fixed. The first one is known as Fisher-KPP reaction [103, 135] or reaction of type A [31], whilst the second one bistable reaction or reaction of type C [31], see also the works [12, 13, 102, 159]. The first important step is to establish the existence/non-existence of Travelling Wave (TW) fronts for the equation in (RDNL). They are special solutions to (RDNL) (here we take spatial dimension $N = 1$) with the form

$$u(x,t) = \varphi(\xi), \qquad \xi = x - ct, \quad c > 0,$$

where the constant $c > 0$ is the wave's *speed of propagation*, and $\varphi(\cdot)$ is its *profile* (sometimes the variable ξ is called moving coordinate). More precisely, we consider wave profiles with the properties

$$0 \le \varphi \le 1, \quad \varphi(-\infty) = 0,\ \varphi(\infty) = 1 \quad \text{and} \quad \varphi' \le 0,$$

and we refer to them as *admissible* TWs. Finally, an admissible TW is said *finite* if $\varphi(\xi) = 0$ for $\xi \le \xi_0$, or *positive* if $\varphi(\xi) > 0$, for all $\xi \in \mathbb{R}$. Note that a finite TW has a *free boundary*: $x = \xi_0 - ct$.

TW solutions are truly valuable tools in the study of the asymptotic behaviour of solutions to (RDNL), and the study of their existence/non-existence leads us to a nonstandard ODEs phase-plane analysis that we present in Chapter 1. The main results of that chapter are the following (cfr. with Theorem 1.1 and Theorem 1.2):

• Take $m > 0$ and $p > 1$ such that $m(p-1) - 1 \ge 0$ (slow and pseudo-linear diffusion) and $f(\cdot)$ of the Fisher-KPP type. Then there exists a unique $c_* = c_*(m,p,f) > 0$ such that equation (RDNL) possesses a unique admissible TW for all $c \ge c_*$ and does not have admissible TWs for $0 < c < c_*$.
Moreover, if $m(p-1) - 1 > 0$ the TW corresponding to the value $c = c_*$ is finite (i.e., it vanishes in an infinite half-line), while the TWs corresponding to the values $c > c_*$ are positive everywhere. If $m(p-1) - 1 = 0$ any admissible TW profile is positive everywhere.

• Take $m > 0$ and $p > 1$ such that $m(p-1) - 1 \ge 0$ and $f(\cdot)$ of the type C. Then there exists a unique $c_* = c_*(m,p,f) > 0$ such that equation (RDNL) possesses a unique admissible TW for $c = c_*$ and does not have admissible TWs for $0 \le c \ne c_*$.
Again, if $m(p-1) - 1 > 0$ the TW corresponding to the value $c = c_*$ is finite. If $m(p-1) - 1 = 0$ the unique admissible TW profile is positive everywhere.

The corresponding result in the linear case $m = 1$ and $p = 2$, was shown by Aronson and Weinberger in [12] (in that case all the TWs are positive). The Fisher-KPP case with Porous Medium diffusion

($m > 1$ and $p = 2$) was studied by Aronson [10, 11] and later by De Pablo and Vázquez in [79, 80] (see also [145] and very recent work [87] for more general equations), while for the p-Laplacian diffusion ($m = 1$ and $p > 2$) there were some partial results in [91, 114]. Note that for what concerns reactions of type C there were no results (at least to our knowledge) for nonlinear diffusion except the work of Jin, Yin and Zheng [129] in which the authors worked with delayed reactions and Porous Medium diffusion. W.r.t. to the classical case, in the slow diffusion range, TWs exhibit *free boundaries*. This property is an important feature of Porous Medium and p-Laplacian diffusion, which is extended also to solutions to (RDNL).

Passing to Chapter 2, in which the PDEs part begins, we show the following two asymptotic behaviour theorems, which are the main results for slow and pseudo-linear diffusion (cfr. with Theorem 2.1 and Theorem 2.2):

• Take $m > 0$ and $p > 1$ such that $m(p-1) - 1 \geq 0$ (slow and pseudo-linear diffusion), $f(\cdot)$ of the Fisher-KPP type, and $u_0 \in C_c(\mathbb{R}^N)$ with $0 \leq u_0 \leq 1$. Then the solution $u = u(x,t)$ to (RDNL) satisfies:

$$u(x,t) \to \begin{cases} 1 & \text{uniformly in } \{|x| \leq ct\} \text{ for all } c < c_* \\ 0 & \text{uniformly in } \{|x| \geq ct\} \text{ for all } c > c_* \end{cases} \qquad \text{as } t \to \infty.$$

Moreover, if $m(p-1)-1 > 0$ and $c > c_*$, $u \equiv 0$ in $\{|x| \geq ct\}$ for any t large enough (in particular $u = u(x,t)$ has a *free boundary*).

• Take $m > 0$ and $p > 1$ such that $m(p-1)-1 \geq 0$, $f(\cdot)$ of type C. Then the solution $u = u(x,t)$ to (RDNL) satisfies the following three assertions:

(i) There are initial data $u_0 \in C_c(\mathbb{R}^N)$ with $0 \leq u_0 \leq 1$ such that

$$u(x,t) \to 0 \text{ point-wise in } \mathbb{R}^N, \quad \text{as } t \to +\infty.$$

(ii) There are initial data $u_0 \in C_c(\mathbb{R}^N)$ with $0 \leq u_0 \leq 1$ such that

$$u(x,t) \to 1 \text{ point-wise in } \mathbb{R}^N, \quad \text{as } t \to +\infty.$$

(iii) For the same class of initial data of (ii)

$$u(x,t) \to \begin{cases} 1 & \text{uniformly in } \{|x| \leq ct\} \text{ for all } c < c_* \\ 0 & \text{uniformly in } \{|x| \geq ct\} \text{ for all } c > c_* \end{cases} \qquad \text{as } t \to \infty.$$

Again, if $m(p-1)-1 > 0$ and $c > c_*$, $u \equiv 0$ in $\{|x| \geq ct\}$ for any t large enough. We must briefly comment these two PDEs results.

From a dynamical point of view, it follows that the both steady states $u = 0$ and $u = 1$ are attractors (part (i) and (ii)) for the space of nontrivial initial data $u_0 \in C_c(\mathbb{R}^N)$, $0 \leq u_0 \leq 1$, whilst the stationary solution $u = a$ is unstable. This is in contrast with what stated for Fisher-KPP framework, where the steady state $u = 1$ is globally stable, whilst $u = 0$ is unstable (which means, from the point of view of the applications, that the density $u = u(x,t)$ saturates all the available space with constant speed of propagation $c = c_*$ for large times).

The different asymptotic behaviour for reactions of type C is known in literature as *threshold effect*, i.e., the initial datum must be large enough to avoid the finite time extinction of the corresponding solution. As we will explain deeper later, threshold phenomena for reaction diffusion equations are known since [12, 13] and much work have been done also in recent years (cfr. for instance with [88, 157, 166] and their references). We stress that except for some particular cases [88], there are not *sharp* threshold results, i.e., at least for nonlinear diffusion, there is not a complete characterization of the classes of initial data for which (i) and (ii) hold true. This problem seems to be very challenging also in the classical diffusion framework. In our nonlinear setting, we restrict ourselves to prove the validity of a threshold effect by using the TWs studied in first chapter.

As we have mentioned before, it is important to stress that the PDEs part strongly relies on the ODEs one. In some sense, the TWs studied in Chapter 1 play the role (for what concerns the asymptotic behaviour) of the fundamental solution for the Heat Equation (HE).

Finally, in Chapter 3 we focus on the fast diffusion range $-p/N < m(p-1)-1 < 0$ and Fisher-KPP reactions. The main fact in this framework is that solutions do not propagate with constant speed of propagation for large times, but exponentially fast in space for large times. In particular (cfr. with Theorem 3.1), defining

$$\sigma_* := \frac{\widehat{\gamma}}{p} f'(0),$$

we can prove that the solution $u = u(x,t)$ to (RDNL) (with $m > 0$ and $p > 1$ are such that $-p/N < m(p-1)-1 < 0$ and $f(\cdot)$ is of the Fisher-KPP type) satisfies

$$u(x,t) \to \begin{cases} 1 & \text{uniformly in } \{|x| \le e^{\sigma t}\} \text{ for all } \sigma < \sigma_* \\ 0 & \text{uniformly in } \{|x| \ge e^{\sigma t}\} \text{ for all } \sigma > \sigma_* \end{cases} \qquad \text{as } t \to \infty.$$

This is the fast diffusion version of the asymptotic theorem proved in the slow and pseudo-linear range, but now the speed of propagation is infinite and, more precisely, exponential w.r.t. the time variable, for large times. We point out that exponential propagation was also found by Cabré and Roquejoffre for fractional diffusion [47] and by Hamel and Roques [121] for linear diffusion ($m = 1$, $p = 2$) and slow decaying initial data. Finally, the above convergence was *formally* proved by King and McCabe in [133], in the Porous Medium setting, $(N-2)_+/N < m < 1$ and $p = 2$, and $f(u) = u(1-u)$. Our result generalizes and completes it for doubly nonlinear diffusion (note that we also work with more general reaction terms).

Finally, in the fast diffusion range, we prove precise bounds for the level sets of general solutions to problem (RDNL) with the classical Fisher-KPP reaction $f(u) = u(1-u)$. We show that for all level $0 < \omega < 1$, there exist a large constant $C_\omega > 0$ and a large time $t_\omega > 0$ such that

$$E_\omega(t) = \{u(x,t) = \omega\} \subset \{C_\omega^{-1} e^{\sigma_* t} \le |x| \le C_\omega e^{\sigma_* t}\}, \quad \text{for all } t \ge t_\omega.$$

The above inclusion is stated in Theorem 3.2. As always, we are not imposing restrictions on the dimension $N \ge 1$, and $m > 0$ and $p > 1$ are taken in the fast diffusion range. Taking spacial logarithmic coordinates, we can re-write the previous expression as

$$E_\omega(t) \subset \{-\ln C_\omega \le \ln|x| - \sigma_* t \le \ln C_\omega\}, \quad \text{for all } t \ge t_\omega.$$

This last formulation is particularly meaningful since it allows us to compare our result with the classical case, see the work of Bramson [44, 45] and the more recent papers [120, 160, 161]. Indeed, in the linear case, it turns out that the location of the level sets is linear (with coefficient $c_* = 2$) up to a (time variable) logarithmic shift and a bounded interval of uncertainty. Taking logarithmic spacial coordinates we obtain a linear propagation in time with coefficient $\sigma_* = \widehat{\gamma}/p$, but not a time shift, for large times. We stress that these bounds are new for both Porous Medium setting and the p-Laplacian one and, possibly, is the most original result contained in the first part. We point out that similar bounds were found in [47] for the half-Laplacian $(-\Delta)^{1/2}$ and dimension $N = 1$. Finally, we quote the very recent preprints [7] in which different bounds for level sets (propagating exponentially fast and not) are studied in the case of Porous Medium diffusion.

Nodal properties of solutions to a nonlocal parabolic equation

In the second part we present the preprint [16] written in collaboration with Prof. Susanna Terracini. The main goal is to describe as precisely as possible the nodal set of solutions to the nonlocal parabolic equation

$$(\partial_t - \Delta)^s u = 0 \quad \text{in } \mathbb{R}^N \times (-T, 0), \tag{NLHE}$$

where $0 < s < 1$ and $0 < T < \infty$ are fixed. We stress from the beginning that, w.r.t. the first part, the view-point and the main goals strongly change. In Part I we study nonnegative solutions to parabolic reaction-diffusion equations, whilst in Part II we are focus on the local properties of solutions near their nodal set.

Fractional powers of the Laplacian have a long history (cfr. with the works of Riesz [169, 170]) and a wide number of applications (cfr. for instance with Athanasopoulos et al. [14], Berestycki et al. [33], Caffarelli and Vázquez [55, 56, 57, 58], Danielli et al. [71], Figalli et al. [21, 48], Metzler and Klafter [153], and the less recent work [89]). We finally quote some very recent works of Nyström and Sande [163], Stinga and Torrea [185], and Banerjee and Garofalo [19], to which our work is strictly related. In the first two, the authors present a parabolic extension method for equation (NLHE), together with the proof of smoothness of solutions, while in the third one new monotonicity formulae and strong unique continuation properties for a larger class of solutions are proved.

As for the elliptic setting (cfr. with [54]), nonlocal operators are often defined in terms of their Fourier transform. In our case, introducing the heat operator $H := \partial_t - \Delta$, we define

$$\widehat{H^s u}(\eta, \vartheta) := (i\vartheta + |\eta|^2)^s\, \widehat{u}(\eta, \vartheta),$$

for any $0 < s < 1$ and for all functions $u = u(x,t)$ belonging to the domain

$$\mathrm{dom}(H^s) := \left\{ u \in L^2(\mathbb{R}^{N+1}) : (i\vartheta + |\eta|^2)^s\, \widehat{u} \in L^2(\mathbb{R}^{N+1}) \right\}.$$

On the other hand, there are different representations of H^s that do not involve Fourier transform. A very significative one uses hypersingular integrals and has been found in Theorem 1.1 of [185] (and observed in [163]),

$$H^s u(x,t) = \frac{1}{|\Gamma(-s)|} \int_{-\infty}^{t} \int_{\mathbb{R}^N} [u(x,t) - u(z,t')]\, \frac{G_N(x-z, t-t')}{(t-t')^{1+s}} dz dt',$$

for all $u \in \mathcal{S}(\mathbb{R}^{N+1})$, where $G_N = G_N(x,t)$ is the standard Gaussian probability density introduced before. From the above formula we deduce that the value of $H^s u$ at a point (x,t) depends on all the past values of $u = u(x,t)$ and so equation (NLHE) is nonlocal both in space and time.

A key tool for studying such nonlocal operators is the extension method (cfr. with the work of Caffarelli and Silvestre [54]). We briefly recall a parabolic version of it, studied in [163, 185]. If $u \in \mathrm{dom}(H^s)$ and $a := 1 - 2s$, we define its extension $U = U(x,y,t)$ to the extended space $\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}$ as

$$U(x,y,t) := \int_0^{\infty} \int_{\mathbb{R}^N} u(x-z, t-t') P_y^a(z,t') dz dt',$$

where the Poisson kernel is defined by

$$P_y^a(x,t) = \frac{1}{2^{1-a}\Gamma(\frac{1-a}{2})} G_N(x,t)\, \frac{y^{1-a}}{t^{1+\frac{1-a}{2}}} e^{-\frac{y^2}{4t}} \qquad (x,y) \in \mathbb{R}^{N+1}_+,\ t > 0.$$

Then $U = U(x,y,t)$ solves

$$\begin{cases} \partial_t U - y^{-a}\nabla \cdot (y^a \nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+ \times (-\infty, \infty), \\ U(x,0,t) = u(x,t), \end{cases} \qquad \text{with} \qquad -c_a \partial_y^a U(x,t) = H^s u(x,t),$$

where $\partial_y^a U(x,t) := \lim_{y \to 0^+} y^a \partial_y U(x,y,t)$ and $c_a > 0$ is a suitable constant. The main idea is thus to investigate solutions to equation (NLHE) by studying the *local* problem in the extended space (note the change of variables $t \to -t$)

$$\begin{cases} \partial_t U + y^{-a}\nabla \cdot (y^a \nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+ \times (0,T) \\ -\partial_y^a U = 0 & \text{in } \mathbb{R}^N \times \{0\} \times (0,T), \end{cases} \tag{NP}$$

and to recover information on $u = u(x,t)$ passing to the trace $U(x,0,t) = u(x,t)$. This is also called Dirichlet-to-Neumann approach (we anticipate we will work in a more general setting, but, for simplicity, in this introduction we focus on problem (NP)). This approach is crucial since it allows to introduce an Almgren-Poon type quotient

$$N(t,U) := \frac{tI(t,U)}{H(t,U)} = \frac{t\int_{\mathbb{R}^{N+1}_+} |\nabla U|^2(x,y,t)\,d\mu_t(x,y)}{\int_{\mathbb{R}^{N+1}_+} U^2(x,y,t)\,d\mu_t(x,y)}, \tag{APQ}$$

where $\{d\mu_t\}_{t>0}$ is a family of probability measures on $\mathbb{R}^{N+1}_+$, defined by

$$d\mu_t(x,y) := \frac{1}{2^a\Gamma(\frac{1+a}{2})} G_N(x,t) \frac{1}{t^{\frac{1+a}{2}}} e^{-\frac{y^2}{4t}} dxdy,$$

and prove the function $t \to N(t,U)$ is monotone nondecreasing (for a suitable class of solutions to (NP)). The time monotonicity of the Almgren-Poon quotient have been established for smooth solutions by Stinga and Torrea in [185, Theorem 1.15] and for a larger class by Banerjee and Garofalo in [19, Theorem 8.3]. We recall that this class of quotients was introduced by Poon [167] for the parabolic case, to study strong unique continuation properties of solutions. We will review the proofs in Chapter 4 for completeness, adapting them to our setting.

The second main goal of Chapter 4 is establishing for what class of solutions the quotient (APQ) is *constant* in time. The main fact is that the function $t \to N(t,U)$ is constant if and only if $U = U(x,y,t)$ is parabolically homogeneous of degree $\kappa \in \mathbb{R}$, i.e.

$$U(\delta x, \delta y, \delta^2 t) = \delta^{2\kappa} U(x,y,t), \quad \text{for any } \delta > 0,$$

and some $\kappa \in \mathbb{R}$, which is equivalent to say that $U = U(x,y,t)$ satisfies the problem

$$\begin{cases} t\partial_t U + \frac{(x,y)}{2} \cdot \nabla U = \kappa U & \text{in } \mathbb{R}^{N+1}_+ \times (0,T) \\ -\partial_y^a U = 0 & \text{in } \mathbb{R}^N \times \{0\} \times (0,T). \end{cases}$$

The above problem has an equivalent and (from our point of view) clearer form, which is obtained by passing to the re-scaled version $\widetilde{U}(x,y,t) = U(\sqrt{t}x, \sqrt{t}y, t)$, which satisfies the Ornstein-Uhlenbeck eigenvalue problem type

$$\begin{cases} -y^{-a}\nabla \cdot (y^a \nabla \widetilde{U}) + \frac{(x,y)}{2} \cdot \nabla \widetilde{U} = \kappa \widetilde{U} & \text{in } \mathbb{R}^{N+1}_+ \\ -\partial_y^a \widetilde{U} = 0 & \text{in } \mathbb{R}^N \times \{0\}, \end{cases} \tag{OU}$$

for all $0 < t < T$. We are thus lead to solve an eigenvalue problem and carry out a complete spectral analysis. We find (cfr. with Theorem 4.1) that the eigenvalues to (OU) are the half-integers

$$\widetilde{\kappa}_{n,m} = \frac{n}{2} + m, \qquad m, n \in \mathbb{N},$$

and the corresponding eigenfunctions are all possible linear combinations of Hermite and Laguerre polynomials of the type:

$$V(x,y) = \sum_{(\alpha,m)\in\widetilde{J}_0} \widetilde{v}_{\alpha,m} V_{\alpha,m}(x,y) = \sum_{(\alpha,m)\in\widetilde{J}_0} \widetilde{v}_{\alpha,m} H_\alpha(x) L_{(\frac{a-1}{2}),m}(y^2/4),$$

where $\widetilde{J}_0$ is a finite set of indexes and $\alpha \in \mathbb{Z}^N_{\geq 0}$ is a multi-index of order $n \in \mathbb{N}$. This has significative consequences in what follows.

In Chapter 5 the blow-up analysis begins and we study the asymptotic behaviour of the normalized blow-up family

$$U_{p_0,\lambda}(x,y,t) = \frac{U(x_0+\lambda x, \lambda y, t_0+\lambda^2 t)}{\sqrt{H(\lambda^2, U)}} \qquad \lambda > 0,$$

as $\lambda \to 0^+$ (where $H(\cdot, U_{p_0})$ is defined in (APQ) and $p_0 = (x_0, 0, t_0)$). The main convergence results we prove are (cfr. with Theorem 5.1 and Theorem 5.3):

$$U_{p_0,\lambda} \to \widetilde{\Theta}_{p_0} \quad \text{in } L^2_{loc}([0,\infty); H^1_{\mu_t}) \quad \text{and in} \quad L^\infty_{loc}(\overline{\mathbb{R}^{N+1}_+} \times (0,\infty)), \tag{BUC}$$

as $\lambda \to 0^+$, where the blow-up limit $\widetilde{\Theta}_{p_0} = \widetilde{\Theta}_{p_0}(x,y,t)$ is defined in terms of re-scaled eigenfunctions:

$$\widetilde{\Theta}_{p_0}(x,y,t) = t^{\widetilde{\kappa}_{n,m}} \sum_{(\alpha,m)\in\widetilde{J}_0} \widetilde{v}_{\alpha,m} H_\alpha\left(\frac{x}{\sqrt{t}}\right) L_{(\frac{a-1}{2}),m}\left(\frac{y^2}{4t}\right),$$

and $H^1_{\mu_t}$ is a suitable H^1 Gaussian type space. Together with the above convergence properties we obtain that the limit of the Almgren-Poon quotient (APQ) must be an eigenvalue of problem (OU), namely

$$\lim_{t\to 0^+} N(t,U) \in \widetilde{\mathcal{K}} := \{\widetilde{\kappa}_{n,m}\}_{n,m\in\mathbb{N}}.$$

Now, the convergence results in (BUC) and the fact that the limit of the Almgren-Poon quotient can assume only a countable number of values, are crucial in the analysis of the nodal set of solutions to (NLHE). Indeed, the local uniform convergence combined with Federer's reduction principle implies a first bound on the *parabolic Hausdorff dimension* of the nodal set $\Gamma(u) := u^{-1}(0)$ of a nontrivial solution $u = u(x,t)$ (cfr. with Theorem 5.7):

$$\dim_{\mathcal{P}}(\Gamma(u)) \le N+1,$$

where $\dim_{\mathcal{P}}(E)$ denotes the *parabolic Hausdorff dimension* of a set $E \subset \mathbb{R}^N \times \mathbb{R}$ (cfr. with Subsection 5.5.1 for the definition). On the other hand, writing $\Gamma(u)$ as the (disjoint) union

$$\Gamma(u) = \mathcal{R}(u) \cup \mathcal{S}(u),$$

(regular and singular part of the nodal set) where

$$\mathcal{R}(u) := \left\{p_0 = (x_0, 0, -\tau_0) \in \Gamma(u) : \lim_{t\to 0^+} N(t, U_{p_0}) = \frac{1}{2}\right\},$$

and $\mathcal{S}(u) := \Gamma(u)\setminus\mathcal{R}(u)$, we prove (cfr. with Theorem 5.8) that $\mathcal{R}(u)$ is indeed regular, i.e. $|\nabla_x u| \neq 0$ at any point of $\mathcal{R}(u)$. From this fact it will follow that it is a locally C^1-manifold of Hausdorff dimension N. Let us mention that these same results where proved by Han and Lin in [122] for solutions $u = u(x,t)$ to some quite general *local* parabolic equations and by Chen [63] for systems of parabolic inequalities. They also gave a dimensional estimate of the singular set $\mathcal{S}(u) = u^{-1}(0) \cap |\nabla_x u|^{-1}(0)$ (we recall here the Chen's one)

$$\dim_{\mathcal{P}}(\mathcal{S}(u)) \le N. \tag{SSLE}$$

The main novelty is that the above estimate seems not hold for solutions to the *nonlocal* equation (NLHE). The most interesting fact here is is that the non-locality of the operator H^s only affects the local behaviour of solutions near their *singular* nodal points, whilst, in some sense, leaves invariant the regular part $\mathcal{R}(u)$. We conclude this introduction by anticipating that, in place of (SSLE), we will prove a structure of the singular set theorem together with the sharp regularity of solutions at their nodal points (cfr. with Theorem 5.9 and Theorem 5.12). These are the main results of the second part and are based on quite recent techniques based on the works of Garofalo et al. [71, 112]. However, their statements require some advanced notations that we prefer not to introduce here, and we refer the reader to the introduction of Chapter 5.

Introducción y presentación de los resultados

Esta tesis doctoral está estructurada en dos partes independientes: la primera está dedicada al estudio de ecuaciones de reacción con difusión doblemente no lineal, mientras que la segunda, al análisis del conjunto nodal de soluciones de ecuaciones parabólicas no locales. En los párrafos siguientes introducimos algunos conceptos básicos y los resultados más relevantes de ambas partes.

Antes de esto, queremos enfatizar que las dos partes están centradas en el mismo tema de investigación: *ecuaciones de difusión*, un asunto esencial en EDPs. Una amplia variedad de fenómenos naturales puede ser matemáticamente descrita a través de procesos de difusión que, por lo tanto, han sido estudiados intensivamente en los últimos 200 años y, también hoy en día, son el foco de considerable atención matemática.
El fundamento de la teoría difusiva es la Ecuación del Calor

$$\partial_t u = \Delta u \qquad x \in \mathbb{R}^N,\ t \geq 0 \tag{HE}$$

introducida por Fourier en el año 1822, en su trabajo: *Théorie analytique de la chaleur*, [104]. Como el nombre de la ecuación sugiere, el objetivo de Fourier consistía en describir la variación temporal de una densidad de calor en $\mathbb{R}^N$, asumiendo que se conoce su distribución inicial en el tiempo $t = 0$. Más tarde, la misma ecuación ha sido empleada en otros campos de la matemática aplicada relacionados con fenómenos difusivos como la dinámica de poblaciones, dinámica de fluidos, elasticidad. . . Debido a su importancia histórica y con el fin de introducir al lector en el tema de la teoría de difusión, dedicamos unos párrafos a revisar algunos conceptos bien conocidos que todavía tienen influencia en la mayoría de los métodos y técnicas empleadas en este trabajo, donde se estudian modelos más avanzados y complejos.

Desde el punto de vista de las EDPs, estamos interesados en describir las propiedades analíticas de (HE), como la existencia, unicidad, regularidad y el comportamiento a largo plazo de sus soluciones. Estos aspectos están codificados en una solución especial llamada *solución fundamental*. La idea es buscar soluciones en forma auto-similar

$$U(x,t) = t^{-\alpha} F(xt^{-\alpha/N}), \qquad x \in \mathbb{R}^N,\ t > 0,$$

para un exponente $\alpha > 0$ adecuado. En el caso de la Ecuación del Calor, la solución fundamental es la función Gaussiana (en las variables auto-similares)

$$G_N(x,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}},$$

donde $\alpha = N/2$ y $F(\xi) = e^{-\xi^2/4}$ siendo $\xi = |x|t^{-1/2}$, que es la densidad más comúnmente empleadada en Probabilidad y Estadística (en el caso de difusión no lineal y no local encontraremos soluciones especiales que jugarán un papel decisivo en ambos casos). Además, la Gaussiana $G_N = G_N(x,t)$ tiene notables propriedades como la simetría respecto a la variable espacial $x \in \mathbb{R}^N$, la conservación de la masa inicial y la positividad estricta. $G_N = G_N(x,t)$ converge a la Delta de Dirac cuando $t \to 0$:

$$G_N(x,t) \to \delta_0(x), \qquad \text{con } \ t \to 0, \ \text{ en sentido distribucional.}$$

Esta última propiedad es crucial en el estudio de la ecuación (HE) con dato inicial $u_0(x) = u(x,0)$, $x \in \mathbb{R}^N$ y $u_0 \in L^1(\mathbb{R}^N)$ (esta última suposición se puede relajar pero no es importante en este momento, ver por ejemplo [193, 208]). De hecho, existe una fórmula de convolución

$$u(x,t) = (G_N * u_0)(x,t) := \int_{\mathbb{R}^N} G(x-y,t)\, u_0(y)\, dy, \tag{CFHE}$$

que da una expresión analítica para la solución de (HE) en términos de la convolución de la Gaussiana y del dato inicial $u_0(\cdot)$ (esta propiedad no será aplicable a los modelos con difusión no lineal estudiados en la primera parte y tendremos que usar técnicas de comparación con diferentes soluciones auto-similares). Queremos destacar que, de la fórmula anterior, es posible deducir existencia, unicidad, regularidad y comportamiento asintótico de las soluciones para una amplia clase de datos iniciales.

Por último, tenemos que subrayar la interpretación probabilística de la fórmula (CFHE), descubierta por Einstein [90], que conecta la solución de (HE) con dato inicial $u(x,0) = u_0(x)$, con el movimiento Browniano (el proceso estocástico más usado en la práctica). De hecho, es sencillo ver que

$$u(x,t) = (G_N * u_0)(x,t) = \mathbb{E}_x(u_0(W_t)),$$

donde $\{W_t\}_{t\geq 0}$ es un movimiento Browniano y $\mathbb{E}_x(\cdot)$ es la esperanza matemática respecto a la distribución de un movimiento Browniano que empieza en x, es decir, la distribución Gaussiana encontrada antes. La fórmula anterior es probablemente el primer resultado significativo que conectó fuertemente el campo de las ecuaciones parabólicas con los procesos estocásticos, y es básicamente equivalente al hecho de que el Laplaciano es el generador del movimiento Browniano:

$$-\Delta u(x) = 2 \lim_{h\to 0} \frac{u(x) - \mathbb{E}_x[u(W_h)]}{h},$$

para cada $u = u(x)$ suficientemente regular (cfr. por ejemplo el Capítulo 1 de [164]). Subrayamos que también las ecuaciones de difusión estudiadas en esta tesis tienen interesantes conexiones con el campo de la probabilidad. Sin embargo nos enfocaremos en modelos no lineales y no locales que describen procesos *no Gaussianos* (cfr. por ejemplo [14, 108, 195, 197, 198] y sus referencias para modelos y conexiones con el campo probabilístico) que aparecen frecuentemente en varias ciencias aplicadas.

Algunos de los hechos revisados en los párrafos anteriores tienen importantes consecuencias (desde el punto de vista teórico y aplicado) en ambos temas de esta tesis. En lo que concierne a la primera parte, en la cual nos enfocamos en el estudio del comportamiento asintótico (para tiempos grandes) de las soluciones de ecuaciones de reacción-difusión con difusión doblemente no lineal, probamos la existencia de frentes de onda especiales que describen el comportamiento asintótico para tiempos grandes de una clase de soluciones más amplia (extendiendo así los trabajos clásicos [12, 13] al caso de difusión doblemente no linal).

En la segunda parte, extendemos unos resultados demostrados en [63, 122] (para el caso local) a una clase de soluciones de una ecuación parabólica no local. Demostramos la existencia de una clase de polinomios parabólicamente homogéneos de tipo Hermite y Laguerre que serán empleados para caracterizar los puntos nodales de soluciones más generales. En los dos casos, el análisis de soluciones *especiales/fundamentales* (ondas viajeras y polinomios homogéneos) será la clave para describir las propriedades de soluciones generales. Pasamos ahora a la presentación de las dos partes de la tesis.

Ecuaciones de reacción con difusión doblemente no lineal

La primera parte de este manuscrito contiene esencialmente los resultados demostrados en los artículos [17, 18] escritos en colaboración con el Profesor Juan Luis Vázquez y el preprint [15] por el autor y está dedicada al estudio del comportamiento para tiempos grandes de las soluciones del problema de reacción difusión con datos iniciales

$$\begin{cases} \partial_t u = \Delta_p u^m + f(u) & \text{en } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x) & \text{en } \mathbb{R}^N, \end{cases} \tag{RDNL}$$

para una amplia clase de datos iniciales $u_0 = u_0(x)$ ($u_0 \in C_c(\mathbb{R}^N)$ con $0 \le u_0 \le 1$), diferentes intervalos de los parámetros $m > 0$ y $p > 1$, y diferentes términos de reacción $f(\cdot)$.

El operador doblemente no lineal está definido por

$$\Delta_p u^m := \Delta_p(u^m) = \nabla \cdot (|\nabla(u^m)|^{p-2}\nabla(u^m)), \qquad m > 0,\ p > 1,$$

donde ∇ es el gradiente (espacial) mientras que $\nabla\cdot$ es el operador de divergencia (espacial). Lo llamamos doblemente no lineal porque se puede ver como la composición de la potencia m-ésima y el operador p-Laplaciano (cfr. [59, 81, 94, 131, 144, 197, 198] y sus referencias, para unos modelos físicos). Cabe destacar que recuperamos el operador de los Medios Porosos cogiendo $p = 2$ o el p-Laplaciano cogiendo $m = 1$. Claramente, cogiendo $m = 1$ y $p = 2$ obtenemos el Laplaciano clásico. Está entonces claro que la difusión depende fuertemente de los parámetros $m > 0$ y $p > 1$. Dividiremos el análisis en tres intervalos diferentes para $m > 0$ y $p > 1$ dependiendo del valor de la cantidad:

$$m(p-1) - 1 > 0, \qquad m(p-1) - 1 = 0, \qquad -\frac{p}{N} < m(p-1) - 1 < 0,$$

llamados intervalos de difusión lenta, pseudo-lineal y rápida, respectivamente. La función $f(\cdot)$ será esencialmente de dos tipos, modelizados sobre dos términos diferentes de reacción clásicos:

$$f(u) = u(1-u), \qquad f(u) = u(1-u)(u-a),$$

donde $0 < a < 1$ está fijo. El primero es conocido como reacción de tipo Fisher-KPP [103, 135] o reacción de tipo A [31], mientras el segundo como reacción biestable o reacción de tipo C [31], veanse también los trabajos [12, 13, 102, 159]. El primer paso consiste en establecer la existencia/no-existencia de Ondas Viajeras (Travelling Wave, TW) para la ecuación en (RDNL). Las TWs son soluciones especiales de (RDNL) (ahora consideramos dimensión espacial $N = 1$) de la forma

$$u(x,t) = \varphi(\xi), \qquad \xi = x + ct, \quad c > 0,$$

donde la constante $c > 0$ es la *velocidad de propagación* de la onda, y $\varphi(\cdot)$ es su *perfil* (a veces la variable ξ también se llama coordenada móvil). De manera más precisa, consideramos perfiles de onda que satisfacen

$$0 \le \varphi \le 1, \quad \varphi(-\infty) = 0,\ \varphi(\infty) = 1 \quad \text{y} \quad \varphi' \ge 0,$$

y los llamamos TWs *admisibles*. En particular, una TW admisible se llama *finita* si $\varphi(\xi) = 0$ para $\xi \le \xi_0$, o *positiva* si $\varphi(\xi) > 0$, para cada $\xi \in \mathbb{R}$. Notamos que una TW admisible tiene una *frontera libre*: $x = \xi_0 - ct$.

Las soluciones de tipo TW son herramientas importantes en el estudio del comportamiento asintótico de las soluciones de (RDNL), y el estudio de su existencia/no-existencia nos conduce a un análisis de EDOs en un plano de fase no estándar que presentamos en el Capítulo 1. Los resultados más relevantes de este capítulo son los siguientes (cfr. el Teorema 1.1 y el Teorema 1.2):

• Sean $m > 0$ y $p > 1$ tales que $m(p-1) - 1 \ge 0$ (difusión lenta y pseudo-lineal) y $f(\cdot)$ de tipo Fisher-KPP. Entonces existe un único $c_* = c_*(m, p, f) > 0$ tal que la ecuación (RDNL) tiene una única TW admisible

para cada $c \geq c_*$ y no tiene TWs admisibles para $0 < c < c_*$.
Además, si $m(p-1)-1 > 0$, la TW correspondiente al valor $c = c_*$ es finita (es decir, es cero en toda una semirrecta), mientras que las TWs correspondientes a los valores $c > c_*$ son siempre positivas. Si $m(p-1)-1 = 0$ cada perfil de TW admisible es siempre positivo.

• Sean $m > 0$ y $p > 1$ tales que $m(p-1)-1 \geq 0$ y $f(\cdot)$ de tipo C. Entonces existe un único $c_* = c_*(m, p, f) > 0$ tal que la ecuación (RDNL) tiene una única TW admisible para cada $c \geq c_*$ y no tiene TWs admisibles para $0 \leq c \neq c_*$.
De nuevo, si $m(p-1)-1 > 0$, la TW correspondiente al valor $c = c_*$ es finita. Si $m(p-1)-1 = 0$, el único perfil TW admisible es siempre positivo.

El resultado correspondiente en el caso lineal $m = 1$ y $p = 2$, fue demostrado por Aronson y Weinberger en [12] (en este caso todas las TWs son positivas). El caso Fisher-KPP con difusión de tipo Medios Porosos ($m > 1$ y $p = 2$) fue estudiado por Aronson [10, 11] y luego por De Pablo y Vázquez en [79, 80] (véanse también [145] y el muy reciente artículo [87] que trata ecuaciones más generales), mientras que para la difusión p-Laplaciana ($m = 1$ y $p > 2$) existen resultados parciales en [91, 114]. Cabe destacar que en lo que respecta a las reacciones de tipo C no había resultados (por lo menos hasta donde sabemos) para difusión no lineal excepto el trabajo de Jin, Yin y Zheng [129] donde los autores trabajaron con reacciones retrasadas y difusión de tipo Medios Porosos. Respecto al caso clásico, en el intervalo de difusión lenta, las TWs muestran *fronteras libres*. Esta propiedad es una característica de la difusión no lineal (Medios Porosos y p-Laplaciano), y se extiende también a las soluciones de (RDNL).

Pasando al capítulo 2, donde empieza la parte de EDPs, demostramos los dos siguientes teoremas de comportamiento asintótico, que son los resultados más relevantes para los casos de difusión lenta y pseudo-lineal (cfr. el Teorema 2.1 y el Teorema 2.2):

• Sean $m > 0$ y $p > 1$ tales que $m(p-1)-1 \geq 0$ (difusión lenta y pseudo-lineal), $f(\cdot)$ de tipo Fisher-KPP, y $u_0 \in C_c(\mathbb{R}^N)$ con $0 \leq u_0 \leq 1$. Entonces la solución $u = u(x,t)$ de (RDNL) cumple:

$$u(x,t) \to \begin{cases} 1 & \text{uniformemente en } \{|x| \leq ct\} \text{ para cada } c < c_* \\ 0 & \text{uniformemente en } \{|x| \geq ct\} \text{ para cada } c > c_* \end{cases} \qquad \text{cuando } t \to \infty.$$

Además, si $m(p-1)-1 > 0$ y $c > c_*$, $u \equiv 0$ en $\{|x| \geq ct\}$ para cada t suficientemente grande (en particular $u = u(x,t)$ tine una *frontera libre*).

• Sean $m > 0$ y $p > 1$ tales que $m(p-1)-1 \geq 0$, $f(\cdot)$ de tipo C. Entonces la solución $u = u(x,t)$ de (RDNL) satisface lo siguiente:

(i) Existen datos iniciales $u_0 \in C_c(\mathbb{R}^N)$ con $0 \leq u_0 \leq 1$ tales que

$$u(x,t) \to 0 \text{ puntualmente en } \mathbb{R}^N, \quad \text{cuando } t \to +\infty.$$

(ii) Existen datos iniciales $u_0 \in C_c(\mathbb{R}^N)$ con $0 \leq u_0 \leq 1$ tales que

$$u(x,t) \to 1 \text{ puntualmente en } \mathbb{R}^N, \quad \text{cuando } t \to +\infty.$$

(iii) Para la misma clase de datos iniciales que en (ii)

$$u(x,t) \to \begin{cases} 1 & \text{uniformemente en } \{|x| \leq ct\} \text{ para cada } c < c_* \\ 0 & \text{uniformemente en } \{|x| \geq ct\} \text{ para cada } c > c_* \end{cases} \qquad \text{cuando } t \to \infty.$$

De nuevo, si $m(p-1)-1 > 0$ y $c > c_*$, $u \equiv 0$ en $\{|x| \geq ct\}$ para cada t suficientemente grande. Comentamos brevemente estos dos resultados de EDPs.

Desde un punto de vista dinámico, se sigue que ambos estados estacionarios $u = 0$ y $u = 1$ son atractores (parte (i) y (ii)) para el espacio de datos iniciales no triviales $u_0 \in C_c(\mathbb{R}^N)$, $0 \leq u_0 \leq 1$, mientras que la solución estacionaria $u = a$ es inestable. Esto es una contradicción con el caso Fisher-KPP, donde el estado estacionario $u = 1$ es globalmente estable, mientras que $u = 0$ es inestable (que significa, desde el punto de vista de las aplicaciones, que la densidad $u = u(x,t)$ satura todo el espacio disponible con velocidad de propagación constante $c = c_*$ para tiempos grandes).

Por otro lado, el comportamiento asintótico para reacciones de tipo C es conocido en la literatura como *threshold effect*, es decir, el dato inicial tiene que ser suficientemente grande para evitar la extinción en tiempo finito de la solución correspondiente. Como explicaremos con más detalles luego, fenómenos de threshold para ecuaciones de reacción y difusión son conocidos desde [12, 13] y un gran número de artículos han sido escritos en años recientes (cfr. por ejemplo [88, 157, 166] y sus referencias). Subrayamos que, excepto unos casos particulares (véanse [88]), no existen resultados de threshold *sharp*, es decir, por lo menos para el caso de difusión no lineal, no hay una caracterización completa de las clases de datos iniciales para los cuales tenemos (i) y (ii). Este problema parece ser muy desafiante también en el caso de la difusión clásica . En el caso no lineal, nos limitamos a probar la validez de un efecto threshold empleando una vez más las TWs estudiadas en el primer capítulo.
Como hemos mencionado antes, es importante destacar que la parte de EDPs está fuertemente basada en la de ODEs. Las TWs estudiadas en el Capítulo 1 juegan el papel (en lo que concierne al comportamiento asintótico) de la solución fundamental de la Ecuación del Calor (HE).

Finalmente, en el Capítulo 3 nos enfocamos en el intervalo de difusión rápida $-p/N < m(p-1)-1 < 0$ y reacciones de tipo Fisher-KPP. El hecho más importante en este caso es que las soluciones no se propagan con velocidad constante para tiempos grandes, sino de manera exponencialmente rápida en el espacio para tiempos grandes. En particular (cfr. el Teorema 3.1), definiendo

$$\sigma_* := \frac{\widehat{\gamma}}{p} f'(0),$$

demostramos que la solución $u = u(x,t)$ de (RDNL) (cuando $m > 0$ y $p > 1$ son tales que $-p/N < m(p-1) - 1 < 0$ y $f(\cdot)$ es de tipo Fisher-KPP) cumple

$$u(x,t) \to \begin{cases} 1 & \text{uniformemente en } \{|x| \le e^{\sigma t}\} \text{ para cada } \sigma < \sigma_* \\ 0 & \text{uniformemente en } \{|x| \ge e^{\sigma t}\} \text{ para cada } \sigma > \sigma_* \end{cases} \qquad \text{cuando } t \to \infty.$$

Esta es la versión "rápida" del teorema asintótico demostrado para los casos de difusión lenta y pseudo-lineal. En este caso la velocidad de propagación es infinita, más precisamente, exponencial respecto a la variable temporal, para tiempos grandes. Señalamos que también ha sido observada propagación exponencial ha sido también observada por Cabré y Roquejoffre para difusión fraccionaria [47] y por Hamel y Roques [121] para difusión lineal ($m = 1$, $p = 2$) y datos con decaimiento lento. Finalmente, la convergencia de arriba fue demostrada *formalmente* por King y McCabe en [133] en el caso de los Medios Porosos, $(N-2)_+/N < m < 1$ y $p = 2$, y $f(u) = u(1-u)$. Nuestro resultado lo generaliza y lo extiende al caso de difusión doblemente no lineal (nótese que trabajamos también con términos de reacción más generales).

Para terminar, en el caso de difusión rápida, probamos límites precisos para los conjuntos de nivel de soluciones generales de (RDNL) con reacción de tipo Fisher-KPP clásica $f(u) = u(1-u)$. Demostramos que para cada nivel $0 < \omega < 1$, existe una constante $C_\omega > 0$ y un tiempo $t_\omega > 0$ tales que

$$E_\omega(t) = \{u(x,t) = \omega\} \subset \{C_\omega^{-1} e^{\sigma_* t} \le |x| \le C_\omega e^{\sigma_* t}\}, \quad \text{para cada } t \ge t_\omega.$$

La inclusión de arriba está presentada en el Teorema 3.2. Como siempre, no ponemos restricciones sobre la dimensión $N \ge 1$, y $m > 0$ y $p > 1$ se eligen en el intervalo de difusión rápida. En coordenadas espaciales logarítmicas, podemos rescribir la expresión anterior como

$$E_\omega(t) \subset \{-\ln C_\omega \le \ln|x| - \sigma_* t \le \ln C_\omega\}, \quad \text{para cada } t \ge t_\omega.$$

Esta última formulación es particularmente significativa porque nos permite comparar nuestro resultado con el caso clásico, véanse los trabajos de Bramson [44, 45] y los artículos más recientes

[120, 160, 161]. De hecho, en el caso lineal, se descubre que la posición de los conjuntos de nivel crece linealmente (con coeficiente $c_* = 2$) a menos de un desplazamiento logarítmico en la variable temporal y un intervalo acotado de incertidumbre. Cogiendo coordenadas espaciales logarítmicas obtenemos propagación lineal en el tiempo con coeficiente $\sigma_* = \widehat{\gamma}/p$, pero no aparece desplazamiento temporal para tiempos grandes. Subrayamos que este resultado es nuevo para el caso de los Medios Porosos y p-Laplaciano y, posiblemente, es el resultado más original de esta primera parte. Este tipo de límites han sido demostrados en [47] para el 1/2-Laplaciano $(-\Delta)^{1/2}$ y dimensión $N = 1$. Finalmente, citamos el reciente preprint [7] donde se estudian diferentes límites para conjuntos de nivel (tanto para los que se propagan exponencialmente rápido como para los que no) en el caso de los Medios Porosos.

Propiedades nodales de soluciones de una ecuación parabólica no local

En la segunda parte presentamos el preprint [16] escrito en colaboración con la Profesora Susanna Terracini. El objetivo principal es describir de la manera más precisa posible el conjunto nodal de las soluciones de la ecuación parabólica no local

$$(\partial_t - \Delta)^s u = 0 \quad \text{en } \mathbb{R}^N \times (-T, 0), \tag{NLHE}$$

donde $0 < s < 1$ y $0 < T < \infty$ están fijos. Subrayamos desde el primer momento que, respecto a la primera parte, el punto de vista y los objetivos más importantes cambian fuertemente. En la Parte I estudiamos soluciones no negativas de ecuaciones de reacción-difusión parabólicas, mientras que en la Parte II nos enfocamos en las propiedades nodales de las soluciones cerca de sus conjuntos nodales.

Las potencias fraccionarias del Laplaciano tienen una larga historia (cfr. los trabajos de Riesz [169, 170]) y un gran número de aplicaciones (cfr. por ejemplo Athanasopoulos et al. [14], Berestycki et al. [33], Caffarelli y Vázquez [55, 56, 57, 58], Danielli et al. [71], Figalli et al. [21, 48], Metzler y Klafter [153], y el trabajo menos reciente [89]). Finalmente, citamos los trabajos muy recientes de Nyström y Sande [163], Stinga y Torrea [185], y Banerjee y Garofalo [19], con los cuales nuestro trabajo está estrechamente relacionado. En los dos primeros, los autores presentan un método de extensión parabólico para la ecuación (NLHE), junto con la prueba de la regularidad de las soluciones, mientras que, en el tercero, los autores prueban una nueva fórmula de monotonía y propiedades de continuación única para una amplia clase de soluciones.

Como para el caso elíptico (cfr. [54]), los operadores están comúnmente definidos en términos de sus transformadas de Fourier. En nuestro caso, introduciendo el operador del calor $H := \partial_t - \Delta$, definimos

$$\widehat{H^s u}(\eta, \vartheta) := (i\vartheta + |\eta|^2)^s\, \widehat{u}(\eta, \vartheta),$$

para cada $0 < s < 1$ y para cada función $u = u(x,t)$ que partenece al dominio

$$\mathrm{dom}(H^s) := \left\{ u \in L^2(\mathbb{R}^{N+1}) : (i\vartheta + |\eta|^2)^s\, \widehat{u} \in L^2(\mathbb{R}^{N+1}) \right\}.$$

Por otra parte, existen representaciones diferentes de H^s que no implican el uso de la transformada de Fourier. Un ejemplo muy significativo (véanse el Teorema 1.1 de [185] y también [163]) que emplea integrales hipersingulares es

$$H^s u(x,t) = \frac{1}{|\Gamma(-s)|} \int_{-\infty}^{t} \int_{\mathbb{R}^N} [u(x,t) - u(z,t')] \frac{G_N(x-z, t-t')}{(t-t')^{1+s}} dz dt',$$

para cada $u \in \mathcal{S}(\mathbb{R}^{N+1})$, donde $G_N = G_N(x,t)$ es la densidad de probabilidad Gaussiana estándar introducida previamente. De la fórmula anterior se deduce que el valor de $H^s u$ en un punto (x,t) depende de todo el pasado de $u = u(x,t)$ y entonces la ecuación (NLHE) es no local en espacio y tiempo.

Una herramienta técnica para estudiar estos operadores no locales es el método de extensión (cfr. el trabajo de Caffarelli y Silvestre [54]). Revisamos a continuación una versión parabólica, estudiada en [163, 185]. Si $u \in \mathrm{dom}(H^s)$ y $a := 1-2s$, definimos su extensión $U = U(x,y,t)$ en el espacio extendido $\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}$ como

$$U(x,y,t) := \int_0^\infty \int_{\mathbb{R}^N} u(x-z,t-t')P_y^a(z,t')dzdt',$$

donde el núcleo de Poisson está definido por

$$P_y^a(x,t) = \frac{1}{2^{1-a}\Gamma(\frac{1-a}{2})} G_N(x,t)\, \frac{y^{1-a}}{t^{1+\frac{1-a}{2}}} e^{-\frac{y^2}{4t}} \qquad (x,y) \in \mathbb{R}_+^{N+1},\ t > 0.$$

Entonces $U = U(x,y,t)$ cumple

$$\begin{cases} \partial_t U - y^{-a}\nabla\cdot(y^a\nabla U) = 0 & \text{en } \mathbb{R}_+^{N+1}\times(-\infty,\infty), \\ U(x,0,t) = u(x,t), \end{cases} \qquad \text{con} \qquad -c_a\partial_y^a U(x,t) = H^s u(x,t),$$

donde $\partial_y^a U(x,t) := \lim_{y\to 0^+} y^a\partial_y U(x,y,t)$ y $c_a > 0$ es una constante adecuada. La idea más importante es entonces investigar las soluciones de la ecuación (NLHE) estudiando el problema *local* en el espacio extendido (nótese el cambio de variable $t \to -t$)

$$\begin{cases} \partial_t U + y^{-a}\nabla\cdot(y^a\nabla U) = 0 & \text{en } \mathbb{R}_+^{N+1}\times(0,T) \\ -\partial_y^a U = 0 & \text{en } \mathbb{R}^N\times\{0\}\times(0,T), \end{cases} \tag{NP}$$

y obtener información sobre $u = u(x,t)$ pasando a la traza $U(x,0,t) = u(x,t)$. Esto se conoce también como método Dirichlet-to-Neumann (anticipamos que trabajaremos en un escenario más general, pero en esta introducción nos enfocamos en el problema (NP)). Este método es crucial porque permite introducir un cociente de tipo Almgren-Poon

$$N(t,U) := \frac{tI(t,U)}{H(t,U)} = \frac{t\int_{\mathbb{R}_+^{N+1}} |\nabla U|^2(x,y,t)\,d\mu_t(x,y)}{\int_{\mathbb{R}_+^{N+1}} U^2(x,y,t)\,d\mu_t(x,y)}, \tag{APQ}$$

donde $\{d\mu_t\}_{t>0}$ es una familia de medidas de probabilidad sobre $\mathbb{R}_+^{N+1}$, definidas por

$$d\mu_t(x,y) := \frac{1}{2^a\Gamma(\frac{1+a}{2})} G_N(x,t)\frac{1}{t^{\frac{1+a}{2}}} e^{-\frac{y^2}{4t}}dxdy,$$

y probar que la función $t \to N(t,U)$ es monótona no decreciente (para una clase adecuada de soluciones de (NP)). La monotonía temporal del cociente de Almgren-Poon ha sido demostrada para soluciones regulares por Stinga y Torrea en [185, Teorema 1.15] y para una clase más amplia por Banerjee y Garofalo en [19, Teorema 8.3]. Recordamos que esta clase de cocientes fue introducida por Poon [167] para el caso parabólico, con el fin de estudiar propriedades de continuación única fuerte de las soluciones. Revisaremos las demostraciones en el Capítulo 4 para completitud, adaptándolas a nuestro contexto.

El segundo objetivo principal del Capítulo 4 es establecer para qué clase de soluciones el cociente (APQ) es *constante* en el tiempo. La idea principal es que la función $t \to N(t,U)$ es constante si y sólo si $U = U(x,y,t)$ es parabólicamente homogénea de grado $\kappa \in \mathbb{R}$, es decir

$$U(\delta x, \delta y, \delta^2 t) = \delta^{2\kappa} U(x,y,t), \qquad \text{para cada } \delta > 0,$$

y $\kappa \in \mathbb{R}$ adecuado, que es equivalente a decir que $U = U(x,y,t)$ resuelve el problema

$$\begin{cases} t\partial_t U + \frac{(x,y)}{2}\cdot\nabla U = \kappa U & \text{en } \mathbb{R}_+^{N+1}\times(0,T) \\ -\partial_y^a U = 0 & \text{en } \mathbb{R}^N\times\{0\}\times(0,T). \end{cases}$$

El problema anterior tiene una forma equivalente y (desde nuestro punto de vista) más clara, que se obtiene pasando a la versión reajustada $\widetilde{U}(x,y,t) = U(\sqrt{t}x, \sqrt{t}y, t)$, que resuelve el problema de los autovalores de tipo Ornstein-Uhlenbeck

$$\begin{cases} -y^{-a}\nabla\cdot(y^a\nabla\widetilde{U}) + \frac{(x,y)}{2}\cdot\nabla\widetilde{U} = \kappa\widetilde{U} & \text{en } \mathbb{R}^{N+1}_+ \\ -\partial^a_y\widetilde{U} = 0 & \text{en } \mathbb{R}^N\times\{0\}, \end{cases} \tag{OU}$$

para cada $0 < t < T$. Entonces tenemos que resolver un problema de autovalores y realizar un análisis espectral completo. Descubrimos (cfr. el Teorema 4.1) que los autovalores de (OU) son los semi-enteros

$$\widetilde{\kappa}_{n,m} = \frac{n}{2} + m, \qquad m, n \in \mathbb{N},$$

y las autofunciones correspondientes son todas las posibles combinaciones lineales de polinómios del tipo:

$$V(x,y) = \sum_{(\alpha,m)\in\widetilde{J}_0} \widetilde{v}_{\alpha,m}V_{\alpha,m}(x,y) = \sum_{(\alpha,m)\in\widetilde{J}_0} \widetilde{v}_{\alpha,m}H_\alpha(x)L_{(\frac{a-1}{2}),m}(y^2/4),$$

donde $\widetilde{J}_0$ es un conjunto finito de índices y $\alpha \in \mathbb{Z}^N_{\geq 0}$ es un multi-índice de orden $n \in \mathbb{N}$. Este hecho tiene consecuencias significativas más adelante.

En el Capítulo 5 empieza el análisis de blow-up y estudiamos el comportamiento asintótico de la familia de blow-up normalizada

$$U_{p_0,\lambda}(x,y,t) = \frac{U(x_0+\lambda x, \lambda y, t_0 + \lambda^2 t)}{\sqrt{H(\lambda^2, U)}} \qquad \lambda > 0,$$

cuando $\lambda \to 0^+$ (donde $H(\cdot, U_{p_0})$ está definido en (APQ) y $p_0 = (x_0, 0, t_0)$). Los resultados de convergencia que probamos (cfr. el Teorema 5.1 y el Teorema 5.3) son los siguientes:

$$U_{p_0,\lambda} \to \widetilde{\Theta}_{p_0} \quad \text{en } L^2_{loc}([0,\infty); H^1_{\mu_t}) \quad \text{y en} \quad L^\infty_{loc}(\overline{\mathbb{R}^{N+1}_+}\times(0,\infty)), \tag{BUC}$$

cuando $\lambda \to 0^+$, donde el límite blow-up $\widetilde{\Theta}_{p_0} = \widetilde{\Theta}_{p_0}(x,y,t)$ se define en terminos de las autofunciones reajustadas:

$$\widetilde{\Theta}_{p_0}(x,y,t) = t^{\widetilde{\kappa}_{n,m}} \sum_{(\alpha,m)\in\widetilde{J}_0} \widetilde{v}_{\alpha,m}H_\alpha\left(\frac{x}{\sqrt{t}}\right)L_{(\frac{a-1}{2}),m}\left(\frac{y^2}{4t}\right),$$

y $H^1_{\mu_t}$ es un espacio H^1 adecuado de tipo Gaussiano. Junto a las propiedades de convergencia de arriba, obtenemos que el límite del conciente de Almgren-Poon (APQ) tiene que ser un autovalor del problema (OU), concretamente

$$\lim_{t\to 0^+} N(t,U) \in \widetilde{\mathcal{K}} := \{\widetilde{\kappa}_{n,m}\}_{n,m\in\mathbb{N}}.$$

Los resultados de convergencia en (BUC) y el hecho de que el límite del cociente de Almgren-Poon solo puede tomar una cantidad numerable de valores son características cruciales en el análisis del conjunto nodal de soluciones de (NLHE). De hecho, la convergencia uniforme local combinada con el principio de reducción de Federer implican un primer resultado sobre la *dimensión parabólica de Hausdorff* del conjunto nodal $\Gamma(u) := u^{-1}(0)$ de una solución no trivial $u = u(x,t)$ (cfr. el Teorema 5.7):

$$\dim_{\mathcal{P}}(\Gamma(u)) \leq N + 1,$$

donde $\dim_{\mathcal{P}}(E)$ denota la *dimensión parabólica de Hausdorff* de un conjunto $E \subset \mathbb{R}^N\times\mathbb{R}$ (cfr. la Subsección 5.5.1 para la definición). Por otra parte, escribiendo $\Gamma(u)$ como la unión (disjunta)

$$\Gamma(u) = \mathcal{R}(u) \cup \mathcal{S}(u),$$

(parte regular y singular del conjunto nodal) donde

$$\mathcal{R}(u) := \left\{ p_0 = (x_0, 0, -\tau_0) \in \Gamma(u) : \lim_{t \to 0^+} N(t, U_{p_0}) = \frac{1}{2} \right\},$$

y $\mathcal{S}(u) := \Gamma(u) \setminus \mathcal{R}(u)$, demostramos (cfr. el Teorema 5.8) que $\mathcal{R}(u)$ es regular, es decir, $|\nabla_x u| \neq 0$ en cada punto de $\mathcal{R}(u)$. De este hecho se sigue que $\mathcal{R}(u)$ es una variedad localmente C^1 con dimensión de Hausdorff N. Estos mismos resultados fueron probados por Han y Lin en [122] para soluciones $u = u(x,t)$ de ecuaciones parabólicas *locales* bastante generales y por Chen [63] para sistemas de desigualdades parabólicas. Dieron también una estimación dimensional para el conjunto singular $\mathcal{S}(u) = u^{-1}(0) \cap |\nabla_x u|^{-1}(0)$ (aquí recordamos el resultado de Chen)

$$\dim_{\mathcal{P}}(\mathcal{S}(u)) \leq N. \tag{SSLE}$$

La novedad más relevante es que esta desigualdad parece no ser válida para soluciones de la ecuación *no local* (NLHE). El hecho más interesante es que la no localidad del operador H^s afecta únicamente al comportamiento asintótico de las soluciones cerca de sus puntos nodales *singulares*, mientras que, en algun sentido, deja invariante la parte regular $\mathcal{R}(u)$. Terminamos esta introducción anticipando que, en lugar de (SSLE), probaremos un teorema de estructura del conjunto singular junto con la regularidad óptima de las soluciones cerca de sus puntos nodales (cfr. el Teorema 5.9 y el Teorema 5.12). Estos son los resultados principales de esta segunda parte y están basados en técnicas bastante recientes, introducidas en los trabajos de Garofalo et al. [71, 112]. Sin embargo, sus enunciados necesitan unas nociones más avanzadas que preferimos no introducir aquí, y referimos al lector a la introducción del Capítulo 5.

Part I

Long time behaviour for reaction equations with doubly nonlinear diffusion

Introduction

In this first part, we study the doubly nonlinear reaction-diffusion (RDNL) problem posed in the whole Euclidean space

$$\begin{cases} \partial_t u = \Delta_p u^m + f(u) & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

for a wide class of initial data $u_0 = u_0(x)$, different ranges of the parameters $m > 0$ and $p > 1$, and different reaction terms $f(\cdot)$.

We stress from the beginning the the main goal of our work consists in establishing the asymptotic behaviour of the solutions $u = u(x, t)$ to problem (1) for large times. Other important issues like existence, uniqueness and regularity of the solutions are not studied here, but briefly presented with the related bibliography.

As the reader can easily imagine, the asymptotic behaviour of solutions to (1) will depend on the choice of the initial datum, on the diffusion parameters ($m > 0$ and $p > 1$), and on the reaction $f(\cdot)$. For this reason, we have decided to dedicate this introduction to giving a clear and methodical presentation of the basic concepts needed in the rest of the first part. Furthermore, we will recall some known facts and preliminaries, essential in the rest of the treatise (as the definition of Travelling Wave (TW) to whom we dedicate a short section) and, finally, we will briefly sketch the contents of each chapter.

"Slow", "Fast" and "Pseudo-linear" diffusion

We recall that the p-Laplacian is a nonlinear operator defined by the formula

$$\Delta_p v := \nabla \cdot (|\nabla v|^{p-2} \nabla v),$$

and we consider the more general diffusion term

$$\Delta_p u^m := \Delta_p(u^m) = \nabla \cdot (|\nabla(u^m)|^{p-2} \nabla(u^m)), \qquad m > 0, \ p > 1,$$

that we call "doubly nonlinear"operator. Here, ∇ is the spatial gradient while $\nabla \cdot$ is the spatial divergence. The doubly nonlinear operator (which can be though as the composition of the m-th power and the p-Laplacian) is much used in the elliptic and parabolic literature (see [59, 81, 94, 131, 144, 197, 198] and their references) and allows to recover the Porous Medium operator choosing $p = 2$ or the p-Laplacian operator choosing $m = 1$. Of course, choosing $m = 1$ and $p = 2$ we obtain the classical Laplacian.

In order to fix the notations and avoid cumbersome expressions in the rest of the of the first part, we define the constant

$$\gamma := m(p-1) - 1, \qquad m > 0, \ p > 1,$$

which will play an important role in our study. The importance of the constant γ is related to the properties of the fundamental solutions to the "purely diffusive" doubly nonlinear parabolic equation and we refer the reader to [197] and to the section dedicated to the preliminaries on doubly nonlinear diffusion, for a more detailed explanation. We will consider parameters $m > 0$ and $p > 1$ such that

$$\gamma \geq 0 \qquad \text{or} \qquad -\frac{p}{N} < \gamma < 0, \qquad m > 0,\ p > 1.$$

We refer to the range $\gamma > 0$ as "slow diffusion" assumption, "pseudo-linear" assumption when we consider $\gamma = 0$, whilst "fast diffusion" assumption when $-p/N < \gamma < 0$. In Figure 1 the corresponding ranges in the $(m, p-1)$-plane are reported.

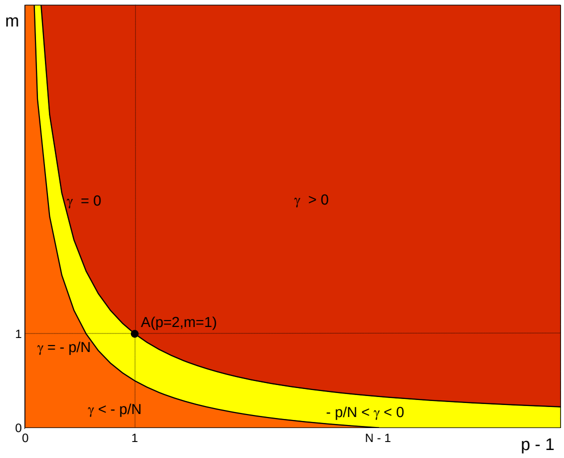

Figure 1: The "slow diffusion" region (red area), "fast diffusion" region (yellow area) and the "pseudo-linear" line in the $(m, p-1)$-plane. The range $\gamma \leq -p/N$ (orange area) has peculiar features and it is not studied in this treatise.

Reaction terms and initial data

The function $f(\cdot)$ will be of three different types. The first one is a reaction term modeled on the famous references by Fisher [103], and Kolmogorov-Petrovski-Piscounoff [135] in their seminal works on the existence of travelling wave propagation. The classical example is the logistic term $f(u) = u(1-u)$, $0 \leq u \leq 1$. More generally, we will assume that

$$\begin{cases} f(0) = f(1) = 0, & 0 < f(u) \leq f'(0)u \ \text{ in } (0,1) \\ f \in C^1([0,1]), & f'(0) > 0,\ f'(1) < 0 \\ f(\cdot) \text{ has a unique critical point in } (0,1), \end{cases} \tag{2}$$

see [12, 13, 101, 103, 135, 158] for a more complete description of the model. We will refer to reactions satisfying (2) as Fisher-KPP reactions or reactions of type A.

The second one is modeled on the function $f(u) = u(1-u)(u-a)$, where $0 < a < 1$ is a fixed parameter and $0 \leq u \leq 1$. More precisely, we assume

$$\begin{cases} f(0) = f(a) = f(1) = 0, & f(u) < 0 \text{ in } (0,a), \quad f(u) > 0 \text{ in } (a,1) \\ f \in C^1([0,1]), & f'(0) < 0,\ f'(a) > 0,\ f'(1) < 0 \\ \int_0^1 u^{m-1} f(u)du > 0 \\ f(\cdot) \text{ has a unique critical point in } (0,a) \text{ and a unique critical point in } (a,1). \end{cases} \tag{3}$$

Note that the classical reaction $f(u) = u(1-u)(u-a)$ with $0 < a < 1/2$ satisfies (3) in the case $m = 1$, and the last assumption implies that the unique critical point of $f(\cdot)$ in $(0, a)$ is a (local) minimum while the unique critical point in $(a, 1)$ is a (local) maximum. Differently from the reactions of the Fisher-KPP type (or type A) ([103, 135]), there is not a standard way to indicate them: Fitzhugh-Nagumo model or Nagumo's equation in [43, 102, 150, 159], "heterozygote inferior" reaction in [12], reaction of type C in [31], or Allen-Cahn reaction [149], for reaction terms like (3). We will refer to them following the notation proposed in [31], i.e., reaction of type C.

In the study of the third one, we assume

$$\begin{cases} f(0) = f(a) = f(1) = 0, & 0 < f(u) \le f'(0)u \ \text{ in } (0,a), \ f(u) < 0 \text{ in } (a,1) \\ f \in C^1([0,1]), & f'(0) > 0, \ f'(a) < 0, \ f'(1) > 0 \\ f(\cdot) \text{ has a unique critical point in } (0,a) \text{ and a unique critical point in } (a,1). \end{cases} \tag{4}$$

We point out that in this second case, the basic model for the reaction is $f(u) = u(1-u)(a-u)$, $0 \le u \le 1$ and $0 < a < 1$ is again a fixed parameter (note that w.r.t. the previous setting, our assumptions imply that the unique critical point of $f(\cdot)$ in $(0, a)$ is (local) maximum while the unique critical point in $(a, 1)$ is a (local) minimum). We point out from the beginning that the last assumptions in (3) and (4) will be mostly employed in the ODEs analysis. According to the previous choice, we will refer to a function satisfying (4) as reaction of type C′, even though it was proposed as "heterozygote superior" in [12].

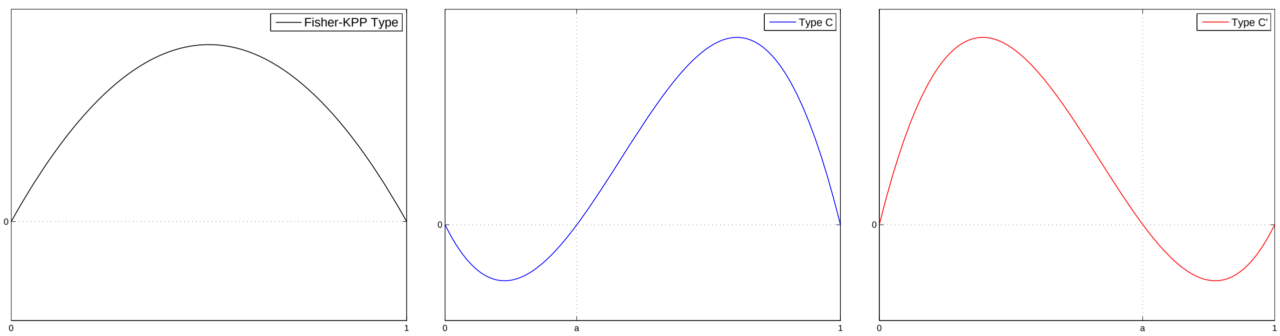

Figure 2: Qualitative representation of the reactions of Fisher-KPP type, type C and type C′, respectively.

It is the least studied of the three models. This is possibly due to the fact that reactions satisfying (4) are Fisher-KPP reactions (or reaction of type A) on the interval $[0, a]$, i.e., they satisfy

$$\begin{cases} f(0) = f(a) = 0, & f(u) > 0, \text{ in } (0,a) \\ f \in C^1([0,a]), & f'(0) > 0, \ f'(a) < 0, \end{cases}$$

and so, part of the theory concerning reactions (4) is similar to the study of models with Fisher-KPP reactions type. Let us see this fact through a scaling technique. Let us fix $0 < a < 1$ and let us suppose for a moment that $u = u(x, t)$ satisfies the equation

$$\partial_t u = \Delta_p u^m + f(u) \quad \text{in } \mathbb{R}^N \times (0, \infty),$$

where now $f(\cdot)$ is of the Fisher-KPP type (or type A), i.e.

$$\begin{cases} f(0) = f(1) = 0, & f(u) > 0, \text{ in } (0,1) \\ f \in C^1([0,1]), & f'(0) > 0, \ f'(1) < 0. \end{cases}$$

Then the re-scaled $u_a = u_a(y,s)$ of $u = u(x,t)$ defined by

$$u(x,t) = a^{-1}u_a(y,t), \quad \text{with } \; y = a^{\gamma/p}x,$$

satisfies the equation

$$\partial_t u_a = \Delta_p u_a^m + f_a(u_a) \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

where $f_a(u_a) := af(a^{-1}u_a)$ is of type C′ in $[0,a]$. This property will be very helpful both in the ODEs and PDEs analysis, where we will highlight the connections and the significant differences between the type C′ setting and the Fisher-KPP one.

Finally, typical assumptions on the initial datum are

$$\begin{cases} u_0 : \mathbb{R}^N \to \mathbb{R} \text{ is continuous with compact support: } u_0 \in C_c(\mathbb{R}^N) \\ u_0 \not\equiv 0 \;\text{ and }\; 0 \le u_0 \le 1. \end{cases} \tag{5}$$

We point out that majority of our results hold true even for a larger class of initial data. However, out of clarity in the exposition, we have decided to keep (5) as "basic" assumption on the initial datum and be more specific when we will give the precise statement of our theorems.

Preliminaries on doubly nonlinear diffusion

Now we present some basic results concerning the Barenblatt solutions of the "purely diffusive" doubly nonlinear parabolic equation which are essential to develop our study in the next sections (the reference for this issue is [197]). Moreover, we recall some basic facts on existence, uniqueness, regularity and Comparison Principles for the solutions of problem (1).

Barenblatt solutions

Fix $m > 0$ and $p > 1$ such that $\gamma > -p/N$ and consider the "purely diffusive" doubly nonlinear problem:

$$\begin{cases} \partial_t u = \Delta_p u^m & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(t) \to M\delta_0 & \text{in } \mathbb{R}^N \text{ as } t \to 0, \end{cases} \tag{6}$$

where $M\delta_0(\cdot)$ is the Dirac's function with mass $M > 0$ at the origin of $\mathbb{R}^N$ and the convergence has to be intended in the sense of measures.

Case $\gamma > 0$. It has been proved (see [197]) that problem (6) admits continuous weak solutions in self-similar form $B_M(x,t) = t^{-\alpha}F_M(xt^{-\alpha/N})$, called Barenblatt solutions, where the *profile* $F_M(\cdot)$ is defined by the formula:

$$F_M(\xi) = \left(C_M - k|\xi|^{\frac{p}{p-1}}\right)_+^{\frac{p-1}{\gamma}}$$

where

$$\alpha = \frac{1}{\gamma + p/N}, \quad k = \frac{\gamma}{p}\left(\frac{\alpha}{N}\right)^{\frac{1}{p-1}}$$

and $C_M > 0$ is determined in terms of the mass choosing $M = \int_{\mathbb{R}^N} B_M(x,t)dx$ (see [197] for a complete treatise). It will be useful to keep in mind that we have the formula

$$B_M(x,t) = MB_1(x, M^\gamma t)$$

which describes the relationship between the Barenblatt solution of mass $M > 0$ and mass $M = 1$. We remind the reader that the solution has a *free boundary* which separates the set in which the solution is positive from the set in which it is identically zero ("slow" diffusion case).

Case $\gamma = 0$. Again we have Barenblatt solutions in self-similar form. The new profile can be obtained passing to the limit as $\gamma \to 0$:

$$F_M(\xi) = C_M \exp\left(- k|\xi|^{\frac{p}{p-1}}\right),$$

where $C_M > 0$ is a free parameter and it is determined fixing the mass, while now $k = (p-1)p^{-p/(p-1)}$. Note that, in this case the constant $\alpha = N/p$ and for the values $m = 1$ and $p = 2$, we have $\alpha = N/2$ and $F_M(\cdot)$ is the Gaussian profile. The main difference with the case $\gamma > 0$ is that now the Barenblatt solutions have no *free boundary* but are always positive.

Case $-p/N < \gamma < 0$. Even in this range there are Barenblatt solutions, with *profile* $F_M(\cdot)$:

$$F_M(\xi) = \left[C_M + k|\xi|^{\frac{p}{p-1}}\right]^{-\frac{p-1}{\widehat{\gamma}}},$$

where we set $\widehat{\gamma} := -\gamma$ and now

$$\alpha = \frac{1}{p/N - \widehat{\gamma}}, \quad k = \frac{\widehat{\gamma}}{p}\left(\frac{\alpha}{N}\right)^{\frac{1}{p-1}}, \quad C_M > 0,$$

where the constants α, k, and C_M are generally different from the ones of the range $\gamma \geq 0$. We point out that the relation between Barenblatt solutions of different masses still holds true, replacing γ by $-\widehat{\gamma}$:

$$B_M(x,t) = MB_1(x, M^{-\widehat{\gamma}}t).$$

Moreover, we have some estimates of the profile corresponding to the Barenblatt solution of mass $M > 0$:

$$K_2(1 + |\xi|^{p/\widehat{\gamma}})^{-1} \leq F_M(\xi) \leq K_1|\xi|^{-p/\widehat{\gamma}} \quad \text{for all } \xi \in \mathbb{R}^N$$

for suitable positive constants K_1 and K_2 depending on $M > 0$.

Existence, Uniqueness, Regularity and Comparison Principles

Before presenting the main results of this paper, we briefly discuss the basic properties of solutions to problem (1). Results about existence of weak solutions of the pure diffusive problem and its generalizations, can be found in the survey [131] and the large number of references therein. The problem of uniqueness was studied later (see for instance [3, 83, 84, 141, 147, 192, 198, 210]). The classical reference for the regularity of nonlinear parabolic equations is [137], followed by a wide literature. For the Porous Medium case ($p = 2$) we refer to [197, 198], while for the p-Laplacian case we suggest [81, 144] and the references therein. Finally, in the doubly nonlinear setting, we refer to [128, 168, 204] and, for the "pseudo-linear" case, [136]. The results obtained in these works showed the Hölder continuity of the solution of problem (1). We mention [81, 198, 210, 212] for a proof of the Comparison Principle. Finally, we suggest [4] and [181] for more work on the "pure diffusive" doubly nonlinear equation and the asymptotic behaviour of its solutions.

Travelling Waves

They are special solutions with remarkable applications, and they will play an essential role in the first two chapters of this first part. Let us review the main concepts and definitions.

Fix $m > 0$ and $p > 1$ such that $\gamma \geq 0$, and assume that we are in space dimension 1 (note that when $N = 1$, the DNL operator has the simpler expression $\Delta_p u^m = \partial_x\left(|\partial_x u^m|^{p-2}\partial_x u^m\right)$. A TW solution to the equation in (1):

$$\partial_t u = \partial_x\left(|\partial_x u^m|^{p-2}\partial_x u^m\right) + f(u) \quad \text{in } \mathbb{R} \times [0, \infty),$$

is a solution of the form $u(x,t) = \varphi(\xi)$, where $\xi = x - ct$, $c > 0$ and the *profile* $\varphi(\cdot)$ is a real function. In our reaction-diffusion setting, we will need the profile to satisfy

$$0 \le \varphi \le a, \quad \varphi(-\infty) = a, \ \varphi(\infty) = 0 \quad \text{and} \quad \varphi' \le 0, \tag{7}$$

for some $0 < a \le 1$. In the case in which $a = 1$ we say that $u(x,t) = \varphi(\xi)$ is an *admissible* TW solution, whilst if $0 < a < 1$, we will talk about *a-admissible* TW solution. Similarly, one can consider TWs of the form $u(x,t) = \varphi(\xi)$ with $\xi = x + ct$, φ nondecreasing and such that $\varphi(-\infty) = 0$ and $\varphi(\infty) = a$. It is easy to see that these two options are equivalent, since the profile of the second one can be obtained by reflection of the first one, and it moves in the opposite direction of propagation. In the rest of the paper, we will prevalently use the first kind of *admissible/a-admissible*, (7).

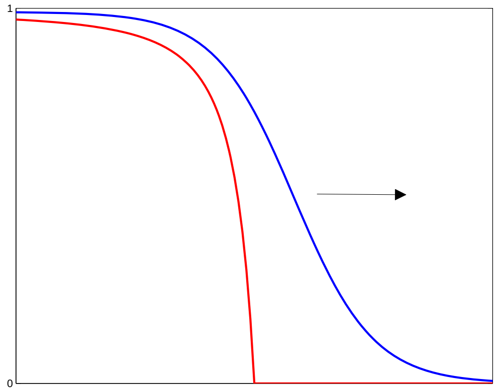

Figure 3: Examples of admissible TWs: Finite and Positive types

Finally, an *admissible/a-admissible* TW is said *finite* if $\varphi(\xi) = 0$ for $\xi \ge \xi_0$ and/or $\varphi(\xi) = 1$ for $\xi \le \xi_1$, or *positive* if $\varphi(\xi) > 0$, for all $\xi \in \mathbb{R}$. The line $x = \xi_0 + ct$ that separates the regions of positivity and vanishing of $u(x,t)$ is then called the *free boundary*. Same name would be given to the line $x = \xi_1 + ct$ and $\varphi(\xi) = 1$ for $\xi \ge \xi_1$ with x_1 finite, but this last situation will not happen.

Before moving forward, we point that sometimes it will be useful to work with "normalized" reactions, i.e., functions $f(\cdot)$ satisfying one of (2), (3), or (4) and, furthermore, $|f'(0)| = 1$, without losing in generality. This reduction will be very useful to make the reading easier and it is strictly related to the speed of propagation of the travelling wave solutions.
To see this equivalence, let us fix $A > 0$. We define the function $u_A(y,\tau) = u(x,t)$, where $x = Ay$, $t = A^p\tau$, and $u = u(x,t)$ is a solution of the equation in (1):

$$\partial_t u = \Delta_{p,x} u^m + f(u) \quad \text{in } \mathbb{R}^N \times (0,\infty),$$

and $\Delta_{p,x}$ is the p-Laplacian with respect to the spacial variable $x \in \mathbb{R}^N$. Thus, it is simple to see that $u_A = u_A(y,\tau)$ solves the same equation where we replace $\Delta_{p,x}$ with $\Delta_{p,y}$ (the p-Laplacian with respect to the spacial variable $y \in \mathbb{R}^N$) and the function $f(\cdot)$ with the reaction $f_A(\cdot) = A^p f(\cdot)$. Hence, to have $|f_A'(0)| = 1$, it is sufficient to choose $A = |f'(0)|^{-1/p}$. We then recover the properties of a solution of the equation with $|f'(0)| \ne 1$ by means of the formula

$$u(x,t) = u_A\left(|f'(0)|^{1/p}x, |f'(0)|t\right).$$

We point out that our transformation changes both the space and the time variables. Consequently, we have to take into account the change of variable of the initial datum $u_0(x) = u_{A0}(|f'(0)|^{1/p}x)$ when we consider the initial-value problem associated with our equation. Moreover, notice that the speed of propagation changes when we consider TW solutions $u(x,t) = \varphi(x - ct)$, $c > 0$. Indeed, if $c > 0$ is the speed of propagation of a TW of the equation in (1) with $|f'(0)| \neq 1$ and $\nu > 0$ is the same speed when $|f'(0)| = 1$, we have

$$c = \frac{x}{t} = A^{1-p}\,\frac{y}{\tau} = |f'(0)|^{\frac{p-1}{p}}\,\nu.$$

This last formula will be useful since it clearly shows how the propagation speed changes when we change the derivative $|f'(0)|$. Thus, from now on, we will try to make explicit the dependence of the propagation speeds on the quantity $|f'(0)|$ by using the previous formula. This fact allows us to state our results in general terms and, at the same time, in a standard and clear way.

Organization of the chapters

The first part of the thesis is organized in three chapters. Here we just introduce the main ideas and the distinct "block of work", whilst we will present all the details and the precise statements of our results at the beginning of each chapter.

In Chapter 1 we consider the range of parameters $m > 0$ and $p > 1$ such that $\gamma \geq 0$ ("slow" and "pseudo-linear" diffusion) and we study the existence/non-existence of *admissible/a-admissible* TW solutions to the equation in (1) for reaction terms of the three different type introduced before (cfr. with (2), (3), and (4)) through a phase plane analysis.
We find that *admissible/a-admissible* TWs exist depending on the wave's propagation speed $c > 0$ and, of course, on the reaction term $f(\cdot)$. This strong dependence is a very well-known fact for both linear diffusion (see for instance [12, 13, 102, 103, 135, 150]) and nonlinear diffusion (see [61, 79, 91]), and we extend it to the doubly nonlinear setting.

In Chapter 2 the PDEs analysis begins. Again we consider the range of parameters $m > 0$ and $p > 1$ such that $\gamma \geq 0$ and we employ the TWs analyzed in the previous chapter together with *a priori* PDEs lemmas to study the asymptotic behaviour of solutions to problem (1). Again, our results depend on the reaction term $f(\cdot)$ which, as always, is assumed to satisfy one of (2), (3), or (4). The nature of the reaction influences and modifies the stability of the steady states $u = 0$, $u = a$ and $u = 1$ for large times, so that we will obtain "saturation/non-saturation" and "threshold effect" phenomena depending on $f(\cdot)$. Moreover, employing TWs as barriers, we will prove that for a large class of initial data the solutions to (1) propagate in space with constant speed for large times. This is the main result of the first two chapters (even in this case, it extends some very well-known facts for linear diffusion).

Finally, in Chapter 3 we study the asymptotic behaviour of solutions to (1) in the "fast diffusion" range $-p/N < \gamma < 0$ and for Fisher-KPP type reactions (i.e. satisfying (2)). This framework turns out to be very interesting since TWs cannot describe the asymptotic behaviour in time, for more general solutions. Indeed, we find that solutions to (1) propagate exponentially fast in space for large times, with a strong deviance w.r.t. the "slow" and "pseudo-linear" setting (see also [47, 121, 133, 182] for previous and related works). Moreover, we prove precise bounds for the level sets of the solutions, getting very powerful information on the front location.

Chapter 1

TWs for "slow" diffusion

This first chapter is devoted to the study of the existence/non-existence of travelling wave solutions to the equation in (1) (that we rename for simplifying the reading):

$$\partial_t u = \partial_x(|\partial_x u^m|^{p-2}\partial_x u^m) + f(u) \quad \text{in } \mathbb{R}\times(0,\infty), \tag{1.1}$$

where $N = 1$ and $\gamma := m(p-1) - 1 \geq 0$ (cfr. with Figure 1.1). Of course, the function $f(\cdot)$ will always satisfy one of (2), (3), or (4).

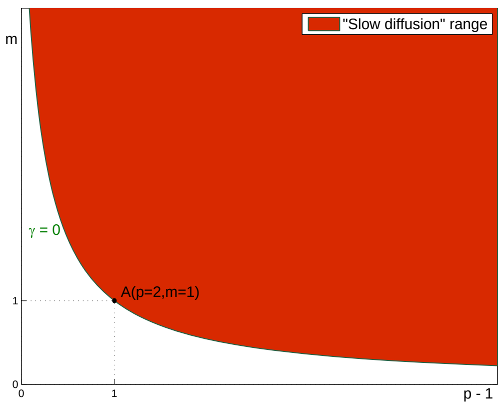

Figure 1.1: The "slow diffusion" region (red area) and the "pseudo-linear" line (green line) in the $(m, p-1)$-plane.

We recall that a TW solution to equation (1.1) is a solution of the form $u(x,t) = \varphi(\xi)$, where $\xi = x - ct$, $c > 0$ and the *profile* $\varphi(\cdot)$ is a real function. We will mainly focus on *a-admissible* TWs, i.e., TWs satisfying (7):

$$0 \leq \varphi \leq a, \quad \varphi(-\infty) = a, \ \varphi(\infty) = 0 \quad \text{and} \quad \varphi' \leq 0,$$

for some $0 < a \leq 1$. As explained in the introduction to Part I, when $a = 1$ we abbreviate *1-admissible* to *admissible* TW solutions, and we will talk about *finite* TW if $\varphi(\xi) = 0$ for $\xi \geq \xi_0$, while *positive* TW if $\varphi(\xi) > 0$, for all $\xi \in \mathbb{R}$. The line $x = \xi_0 + ct$ that separates the regions of positivity and vanishing of $u(x,t)$ is called the *free boundary*.
Finally, we stress the TWs are essentially one dimensional objects (a natural extension to several dimensions consists in looking for solutions in the form $u(x,t) = \varphi(x \cdot n - ct)$ where $x \in \mathbb{R}^N$ and n is a unit vector of $\mathbb{R}^N$ which direction coincides with the direction of the wave propagation) and that there is an equivalent definition in which we consider "reflected" TWs satisfying $u(x,t) = \varphi(\xi)$ with $\xi = x + ct$, φ nondecreasing and such that $\varphi(-\infty) = 0$ and $\varphi(\infty) = a$. These TWs move in the opposite direction of propagation and can be obtained be the others by reflection.

1.1 Main results

As we have mentioned in the introduction, in this Chapter we perform an accurate ODEs analysis to study the existence of TWs to equation (1.1) when $\gamma := m(p-1) - 1 \geq 0$. Out of clarity, we divide the "slow" diffusion range $\gamma > 0$ to the "pseudo-linear" one $\gamma = 0$, highlighting the interesting differences between the two ranges.

Theorem 1.1. *(cfr. with Theorem 2.1 of [17] and Theorem 1.1 of [15])*
Fix $N = 1$, and $m > 0$ and $p > 1$ such that $\gamma > 0$.

(i) If the reaction $f(\cdot)$ is of Fisher-KPP type (or type A), i.e., it satisfies (2)*, then there exists a unique $c_* = c_*(m, p, f) > 0$ such that equation* (1.1) *possesses a unique admissible TW for all $c \geq c_*$ and does not have admissible TWs for $0 < c < c_*$. Moreover, the TW corresponding to the value $c = c_*$ is finite (i.e., it vanishes in an infinite half-line), while the TWs corresponding to the values $c > c_*$ are positive everywhere.*

(ii) If the reaction $f(\cdot)$ is of type C, i.e., it satisfies (3)*, then there exists a unique $c_* = c_*(m, p, f) > 0$ such that equation* (1.1) *possesses a unique admissible TW for $c = c_*$ and does not have admissible TWs for $0 \leq c \neq c_*$. Again, the TW corresponding to the value $c = c_*$ is finite.*

(iii) If the reaction $f(\cdot)$ is of type C′, i.e., it satisfies (4)*, then there exists a unique $c_* = c_*(m, p, f) > 0$ such that equation* (1.1) *possesses a unique a-admissible TW for all $c \geq c_*$ and does not have a-admissible TWs for $0 < c < c_*$. The TWs corresponding to values $c > c_*$ are positive everywhere while, the TW corresponding to the value $c = c_*$ is finite.*

In all parts (i), (ii), and (iii) the uniqueness of the TW is understood up to reflection and horizontal displacement. Moreover, we point out that not to exceed in notations, we have used the same symbol for $c_* = c_*(m, p, f)$ even if the critical speeds of part (i), (ii), and (iii) generally differ from each other.

The existence/non-existence of travelling wave solutions for reaction-diffusion equations has been widely studied and still nowadays it is an important field of research. Due to this fact, a bibliographical survey is now in order. In the linear setting ($m = 1$ and $p = 2$), the first version of Theorem 1.1 was proved in the pioneer work of Aronson and Weinberger in [12, 13], and by Fife and McLeod in [102]. Before these works, wave fronts were introduced by Fisher [103], Kolmogorov-Petrovsky-Piscounoff [135], and McKean in [150] with different techniques. However, in the linear setting, all the *admissible/a-admissible* TWs are always *positive* and solutions with *free boundaries*, which are the fundamental novelties respect to the classical case, are not admitted (cfr. with Theorem 1.2).

Passing to the nonlinear diffusion setting, the existence of *free boundaries* was already observed by DePablo and Vázquez in the Porous Medium setting $p = 2$ in [79] for Fisher-KPP reactions and only more recently in [138] and, later, in [129] for reactions of type C with time delay.
More precisely, part (i) of Theorem 1.1 generalizes the ODEs theorem proved in [79], whilst part (ii) extends the results of [138, 129] to the doubly nonlinear setting with reaction satisfying (3) (recall that here we do not consider reactions with time delay). For reactions of the Fisher-KPP type and p-Laplacian diffusion we also quote [91], where the existence of TW solution was already been proved with different techniques, and the more recent paper [113] where an ODEs analysis is carried out for Fisher-KPP reactions an (p, q)-Laplacian diffusion.

As mentioned above, TW solutions appear in other kind of reaction-diffusion equations. We mention the fundamental works of [28, 29, 30] for reactions equations in non homogeneous media, [6, 32, 115] for equations with linear diffusion and "non-local reactions", whilst [2, 47, 117, 152] for reaction equations with "non-local" diffusion of Fractional Laplacian type and [182] with "non-local and nonlinear" diffusion.

In the next theorem we treat the "pseudo-linear" range $\gamma = 0$ (for related bibliography, see the references given above). We anticipate that the proof of the following theorem is strictly related to the one of Theorem 1.1, which is the main result of this chapter. Nevertheless, the range $\gamma = 0$ shows different phenomena and significative technical details that need to be presented separately.

Theorem 1.2. *(cfr. with Theorem 2.2 of [17] and Theorem 1.1 of [15])*
Fix $N = 1$, and $m > 0$ and $p > 1$ such that $\gamma = 0$.

(i) If the reaction $f(\cdot)$ is of Fisher-KPP type (or type A), i.e., it satisfies (2), *then there exists a unique critical speed*

$$c_{0*} := p(m^2 f'(0))^{\frac{1}{mp}} \tag{1.2}$$

such that equation (1.1) *possesses a unique admissible TW for all $c \geq c_{0*}$ while it does not possess admissible TWs for $0 < c < c_{0*}$. All the admissible TWs are positive everywhere.*

(ii) If the reaction $f(\cdot)$ is of type C, i.e., it satisfies (3), *then there exists a unique $c_{0*} = c_{0*}(m, p, f) > 0$ such that equation* (1.1) *possesses a unique admissible TW for $c = c_{0*}$ and does not have admissible TWs for $0 \leq c \neq c_{0*}$. Again, the TW corresponding to the value $c = c_{0*}$ is positive everywhere.*

(iii) If the reaction $f(\cdot)$ is of type C′, i.e., it satisfies (4), *then there exists a unique $c_{0*} = c_{0*}(m, p, f) > 0$ such that equation* (1.1) *possesses a unique a-admissible TW for all $c \geq c_{0*}$ and does not have a-admissible TWs for $0 < c < c_{0*}$. All the a-admissible TWs are positive everywhere.*

As in Theorem 1.1, in all parts (i), (ii), and (iii) the uniqueness of the TW is understood up to reflection and horizontal displacement. Again we have employed the same symbol $c_{0*} = c_{0*}(m, p, f)$ to indicate the critical speed of propagation. In this case we will show that the critical speeds found in part (i) and (iii) coincide.

Theorem 1.2 is the "pseudo-linear" version of Theorem 1.1 and, as mentioned above, generalizes the results proved in [12, 13, 102] for the linear case ($m = 1$ and $p = 2$), to the all "pseudo-linear" range $\gamma = 0$. As one could have guessed solutions with *free boundaries* disappear and only *admissible/a-admissible* TWs are left. This turns out to be a really interesting fact, since it means that the positivity of the *admissible/a-admissible* TWs is not due to the linearity of the diffusion operator but depends on its homogeneity, i.e., on the relations between the diffusive parameters $m > 0$ and $p > 1$. Finally, we stress that for reactions of Fisher-KPP type (2) and type C′ (4) we have an explicit formula for the critical speed given by formula (1.2), which generalizes the classical one $c_{0*} = \sqrt{4f'(0)}$, found in [12], it agrees with the scaling for the critical speed:

$$c_*(m, p) = f'(0)^{\frac{p-1}{p}} \nu_*(m, p),$$

where $\nu_*(m, p)$ is the critical speed when and $f'(0) = 1$ (cfr. with the introduction).

In this chapter, we lastly prove that the critical speed $c_* = c_*(m, p, f)$ (when $\gamma > 0$), found in Theorem 1.1, converges to the critical speed $c_{0*} = c_{0*}(m, p, f)$ (when $\gamma = 0$), found in Theorem 1.2, as $\gamma \to 0$.

Theorem 1.3. *(cfr. with Theorem 2.3 of [17])*
Consider the region $\mathcal{R} = \{(m, p) : \gamma = m(p-1) - 1 > 0\}$ and let $c_*(m, p)$ and $c_{0*}(m, p)$ be the critical speeds of propagation found in Theorem 1.1 and Theorem 1.2, respectively, with reaction $f(\cdot)$ satisfying one of (2), (3), *or* (4). *Then the function*

$$(m, p) \quad \to \quad \begin{cases} c_*(m, p) & \text{if } \gamma > 0 \\ c_{0*}(m, p) & \text{if } \gamma = 0 \end{cases} \tag{1.3}$$

is continuous on the closure $\overline{\mathcal{R}} = \{(m, p) : \gamma = m(p-1) - 1 \geq 0\}$.

The previous theorem is proved in two steps (for Fisher-KPP reactions only since the other two cases are similar). We first show the continuity of the function (1.3) in the region $\mathcal{R}$ and then we extend the continuity to its closure $\overline{\mathcal{R}}$. Theorem 1.3 will allows us to unify the notations $c_* = c_*(m, p, f)$ and $c_{0*} = c_{0*}(m, p, f)$ in Chapter 2, when the PDEs part begins. In the current chapter we keep them separate not generate confusion.

Finally, we end the chapter with some extensions, comments and open problems.

1.2 Proof of Theorem 1.1

We fix $N = 1$, and $m > 0$ and $p > 1$ such that $\gamma > 0$ and we look for *admissible/a-admissible* TWs to equation (1.1). So, substituting the ansatz $u(x,t) = \varphi(\xi)$, $\xi = x - ct$ into (1.1), we obtain the equation of the profile

$$[|(\varphi^m)'|^{p-2}(\varphi^m)']' + c\varphi' + f(\varphi) = 0 \quad \text{in } \mathbb{R}, \tag{1.4}$$

where the notation φ' indicates the derivative of φ with respect to the variable ξ.

Now the standard approach consists in performing the change of variables $X = \varphi$ and $Y = \varphi'$, transforming the second-order ODE (1.4) into the system of two first-order ODEs

$$\frac{dX}{d\xi} = Y, \qquad (p-1)m^{p-1}X^{\mu}|Y|^{p-2}\frac{dY}{d\xi} = -cY - f(X) - \mu m^{p-1}X^{\mu-1}|Y|^p,$$

where we set for simplicity $\mu := (m-1)(p-1)$. It can be re-written as the less singular system

$$\frac{dX}{ds} = (p-1)m^{p-1}X^{\mu}|Y|^{p-2}Y, \qquad \frac{dY}{ds} = -cY - f(X) - \mu m^{p-1}X^{\mu-1}|Y|^p, \tag{1.5}$$

where we used the re-parametrization $d\xi = (p-1)m^{p-1}X^{\mu}|Y|^{p-2}ds$. Note that both system are equivalent for $Y \neq 0$ but, at least in the case $\mu > 1$, the second one has two critical points $(0,0)$ and $(1,0)$.
This setting seems to be convenient since proving the existence of an admissible TW for (1.1) corresponds to showing the existence of a trajectory in the region of the (X,Y)-plane where $0 \le X \le 1$ and $Y \ge 0$, and joining the critical points $(0,0)$ and $(1,0)$. More precisely, the desired trajectories must "come out" of $(0,0)$ and "going into" $(1,0)$. Let us point out that, contrary to the linear case (see for example [11, 12] and [135]), we face a more complicated problem. Indeed, it turns out that in the nonlinear diffusion case, the Lyapunov linearization method ([75], Chapter 8) used to analyze the local behaviour of the trajectories near the critical points cannot be directly applied to system (1.5), due to its heavy nonlinear features.

The previous observation suggests to change our approach. So, inspired by the methods used in the Porous Medium framework (see [79]), we introduce the new variables

$$X = \varphi \qquad \text{and} \qquad Z = -\left(\frac{m(p-1)}{\gamma}\varphi^{\frac{\gamma}{p-1}}\right)' = -mX^{\frac{\gamma}{p-1}-1}X'. \tag{1.6}$$

These variables correspond to the density and the derivative of the pressure profile (see [94]). Assuming only $X \ge 0$, we obtain the first-order ODE system

$$-m\frac{dX}{d\xi} = X^{1-\frac{\gamma}{p-1}}Z, \qquad -m(p-1)X^{\frac{\gamma}{p-1}}|Z|^{p-2}\frac{dZ}{d\xi} = cZ - |Z|^p - mX^{\frac{\gamma}{p-1}-1}f(X), \tag{1.7}$$

that again can be re-written as the non-singular system

$$\frac{dX}{d\tau} = (p-1)X|Z|^{p-2}Z, \qquad \frac{dZ}{d\tau} = cZ - |Z|^p - f_{m,p}(X), \tag{1.8}$$

where we have used the re-parametrization $d\xi = -m(p-1)X^{\frac{\gamma}{p-1}}|Z|^{p-2}d\tau$ and we have defined

$$f_{m,p}(X) = mX^{\frac{\gamma}{p-1}-1}f(X).$$

System (1.8) seems more reasonable than (1.5) and the procedure employed to get it does not change if the reaction $f(\cdot)$ satisfies (2), (3), or (4). However, in what follows, the function $f(\cdot)$ will play an important role. We will thus separate the proofs according to the nature of $f(\cdot)$, starting with the case of Fisher-KPP reactions (2).

Proof of Theorem 1.1: Part (i). So, let $f(\cdot)$ be a Fisher-KPP reaction satisfying (2):

$$\begin{cases} f(0) = f(1) = 0, \quad 0 < f(u) \le f'(0)u \;\text{ in } (0,1) \\ f \in C^1([0,1]), \qquad f'(0) > 0, \; f'(1) < 0 \\ f(\cdot) \text{ has a unique critical point in } (0,1), \end{cases}$$

Note that since $f'(0) > 0$, we deduce that $f_{m,p}(\cdot)$ is well-defined and continuous in $[0,1]$. Moreover, it is nonnegative, with a unique maximum point in the same interval and satisfies $f_{m,p}(0) = 0 = f_{m,p}(1)$.

The first important observation is that, since $f_{m,p}(0) = 0 = f_{m,p}(1)$, the critical points of system (1.8) are now three:

$$S = (1,0), \qquad O = (0,0) \qquad \text{and} \qquad R_c = (0, c^{1/(p-1)}).$$

It is then clear that the change of variables (1.6) "splits" the critical point $(0,0)$ of system (1.5) into two critical points: O and R_c of the new system. The idea is that formula (1.6) is the unique change of variables that allows us to separate the orbits corresponding to finite and positive TWs. In fact, we will show that the connections between O and S correspond to positive TW (they exist only if $c > c_*$ where c_* is the critical speed of propagation in the statement of Theorem 1.1) while the connection between R_c and S corresponds to a finite TW ($c = c_*$).

So, in order to carry out the plan just introduced, we focus on the range $0 \le X \le 1$ and we study the *equation of the trajectories*

$$\frac{dZ}{dX} = \frac{cZ - |Z|^p - f_{m,p}(X)}{(p-1)X|Z|^{p-2}Z} := H(X, Z; c), \tag{1.9}$$

obtained by eliminating the parameter τ, looking for solutions defined for $0 \le X \le 1$ and linking the critical points R_c and S for some $c > 0$. The main difference with respect to the Porous Medium and the linear case (see [12] and [79]) is that the critical points are all degenerate and it is impossible to describe the local behaviour of the trajectories by linearizing the system (1.8) around S. Consequently, in what follows, we study some local and global properties of the equation (1.9) with more qualitative ODEs methods to obtain a clear view of the graph of the trajectories. The proof is divided in some steps as follows.

Step1: Study of the null isoclines. Firstly, we study the *null isoclines* of system (1.8), i.e., the set of the points (X,Z) with $0 \le X \le 1$ and $Z \ge 0$ such that $H(X,Z;c) = 0$. We thus have to solve the equation

$$c\widetilde{Z}(X) - \widetilde{Z}^p(X) = f_{m,p}(X) \quad \text{in } [0,1] \times [0,\infty). \tag{1.10}$$

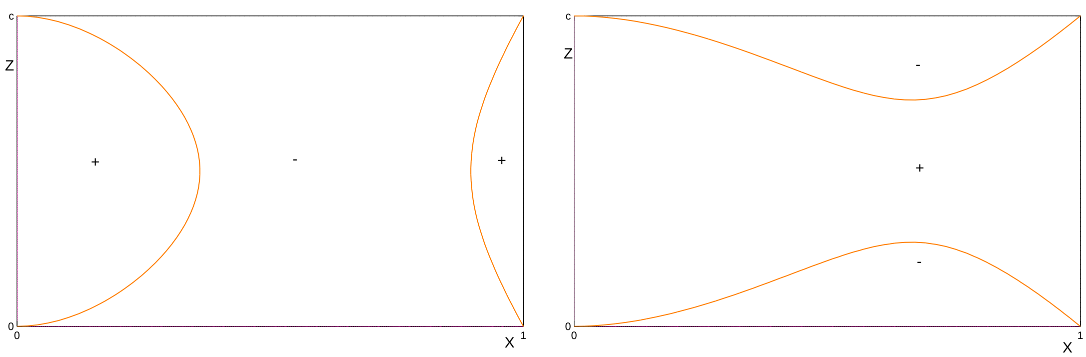

Figure 1.2: Fisher-KPP reactions, range $\gamma > 0$. Null isoclines in the (X,Z)-plane for $f(u) = u(1-u)$, in the cases $0 < c < c_0$ and $c > c_0$, respectively.

If $X_{m,p}$ is the (unique) maximum point of $f(\cdot)$ in $[0,1]$ and $F_{m,p} := f_{m,p}(X_{m,p})$, defining

$$c_0 = c_0(m,p,f) := p\left(\frac{F_{m,p}}{p-1}\right)^{(p-1)/p},$$

it is simple to see that for $c < c_0$ the graph of the isoclines is composed by two branches joining the point O with R_c and the point $(1,0)$ with $(1, c^{1/(p-1)})$, while for $c > c_0$ the branches link the point O with S and the point R_c with $(1, c^{1/(p-1)})$. As c approaches the value c_0 the branches move nearer and they touch in the point $(X_{m,p}, (c_0/p)^{1/(p-1)})$ for $c = c_0$. Note that the value c_0 is critical in the study of the isoclines and it is found by imposing

$$\max_{\widetilde{Z}\in[0,c^{1/(p-1)}]}\left\{c\widetilde{Z} - \widetilde{Z}^p\right\} = F_{m,p}. \tag{1.11}$$

For $c < c_0$, the trajectories in the region between the two branches have negative slopes, while those in the region between the Z-axis and the left branch and between the right branch and the line $X = 1$ have positive slopes. Conversely, for $c > c_0$ the trajectories in the region between the two branches have positive slopes while, those in the region between the bottom-branch and the X-axis and between the line $Z = c^{1/(p-1)}$ and the top-branch have negative slopes. Finally, for $c = c_0$, it is simple to see that in the regions between the bottom-branch and the X-axis and between the line $Z = c_0^{1/(p-1)}$ and the top-branch the trajectories have negative slopes and they have positive slopes in the rest of the rectangle $[0,1]\times[0, c_0^{1/(p-1)}]$. We conclude this paragraph noting that for all $c > 0$ the trajectories have negative slopes for $Z > c^{1/(p-1)}$ and positive slopes for $Z < 0$ (note that that above we have crucially used the last assumption in (2)).

Step2: Local analysis of $S(1,0)$. In this second step, we prove the existence and the uniqueness of solutions of (1.9) "coming into" the point S. We divide the proof in three subcases: $p = 2$, $p > 2$ and $1 < p < 2$.

Case $p = 2$. If $p = 2$, it is not difficult to linearize system (1.8) through Lyapunov method and showing that the point S is a saddle type critical point. Moreover, it follows that there exists exactly one locally linear trajectory $T_c = T_c(X)$ in the region $[0,1]\times[0,\infty)$ "coming into" S with slope $\lambda_S = (c - \sqrt{c^2 - 4mf'(1)})/2$.

Case $p > 2$. Substituting the expression $Z = \lambda(1-X)$, $\lambda > 0$ in the equation of trajectories (1.9) and taking $X \sim 1$ we get

$$-\lambda = H(X, \lambda(1-X)) \sim \frac{c\lambda(1-X) + mf'(1)(1-X)}{(p-1)\lambda^{(p-1)}(1-X)^{p-1}}, \quad \text{for } X \sim 1$$

which can be rewritten as

$$-(p-1)\lambda^p(1-X)^{p-2} \sim c\lambda + mf'(1), \quad \text{for } X \sim 1.$$

Since the left side goes to zero as $X \to 1$, the previous relation is satisfied only if $\lambda = -mc^{-1}f'(1) := \lambda_S^+ > 0$ (note that this coefficient coincides with the slope of the null isocline near $X = 1$). Hence, for $p > 2$, there exists at least one trajectory $T_c = T_c(X)$ "going into" the point S and it is linear near this critical point. Note that the approximation $T_c(X) \sim \lambda_S^+(1-X)$ as $X \sim 1$ can be improved with high order terms. However, we are basically interested in proving the existence of a trajectory "coming into" S and we avoid to present technical computations which can be performed by the interested reader.
To prove the uniqueness, we show that the trajectory T_c (satisfying $T_c(X) \sim \lambda_S^+(1-X)$ as $X \sim 1$) is "repulsive" near $X = 1$. It is sufficient to prove that the partial derivative of the function H with respect to the variable Z is strictly positive when it is calculated on the trajectory T_c and $X \sim 1$. It is

straightforward to see that

$$\frac{\partial H}{\partial Z}(X, \lambda_S^+(1-X)) \sim \frac{1}{D(X)}[-c\lambda_S^+(p-2)(1-X) - m(p-1)f'(1)(1-X)]$$
$$\sim -\frac{mf'(1)(1-X)}{D(X)} = -\frac{mf'(1)}{(p-1)(\lambda_S^+)^p(1-X)^{p-1}} \gg 0, \quad \text{for } X \sim 1.$$

Hence, our trajectory is "repulsive" near the point S, i.e., there are no other *locally linear* trajectories which "enter" into the point S with slope λ_S^+. Now, to show that T_c is the unique trajectory "coming into" the point S, we define the one-parameter family of curves

$$Z_a(x) = \lambda_S^+(1-X)^a, \qquad a > 1,$$

and we use an argument with invariant regions. We compute the derivative

$$\frac{dZ_a/dX}{dZ/dX}(X, Z_a(X); c) = \frac{dZ_a/dX(X)}{H(X, Z_a(X); c)} \sim \frac{-a(p-1)(\lambda_S^+)^p}{mf'(1)}(1-X)^{ap-2} \sim 0 \quad \text{for } X \sim 1,$$

for all $a > 1$ (since $p > 2$). This means that the "flux" derivative (in absolute value) along the curve $Z_a = Z_a(X)$ is infinitely larger with respect to the derivative of the curve when $X \sim 1$, i.e., the trajectories have vertical slopes on the curve $Z_a = Z_a(X)$ for $X \sim 1$. So, since $a > 1$ is arbitrary, it is possible to conclude the uniqueness of our trajectory "coming into" S.

Case $1 < p < 2$. Proceeding as before, it is not difficult to prove that when $1 < p < 2$, there exists a trajectory "coming into" S with local behaviour $T_c(X) \sim \lambda_S^-(1-X)^{2/p}$ for $X \sim 1$ and $\lambda_S^- := \{-pmf'(1)/[2(p-1)]\}^{1/p}$. Note that in this case the trajectory is not locally linear but presents a power-like local behaviour around S with power greater than one. Moreover, exactly as in the case $p > 2$, it is simple to see that this trajectory is "repulsive". Thus it remains to prove that our trajectory is the unique "coming into" S. We proceed using invariant regions as before. This time, the one-parameter family of curves is

$$Z_a(x) = \lambda_S^-(1-X)^a, \qquad 1 < a \neq 2/p$$

Again we compute the derivative

$$\frac{dZ_a/dX}{dZ/dX}(X, Z_a(X); c) = \frac{dZ_a/dX(X)}{H(X, Z_a(X); c)} \sim \frac{-a(p-1)(\lambda_S^-)^p}{mf'(1)}(1-X)^{ap-2},$$

and so

$$\frac{dZ_a/dX}{dZ/dX}(X, Z_a(X); c) \sim \begin{cases} +\infty, & \text{if } 1 < a < 2/p \\ 0, & \text{if } a > 2/p \end{cases} \qquad \text{for } X \sim 1.$$

Consequently, we conclude the uniqueness of our trajectory "coming into" S exactly as in the case $p > 2$.

Step3: Monotonicity of $T_c(\cdot)$ *w.r.t.* $c > 0$. In this crucial step we prove that

$$\text{for all } 0 < c_1 < c_2 \text{ then } T_{c_2}(X) < T_{c_1}(X), \text{ for all } 0 < X < 1$$

where, of course, T_c is the trajectory "coming into" in $S(1,0)$. So, fix $0 < c_1 < c_2$. First of all, we note that

$$\frac{\partial H}{\partial c}(X, Z; c) = \frac{1}{X} \geq 1 > 0, \quad \text{for all } 0 < X \leq 1, \ Z \geq 0, \ c > 0, \tag{1.12}$$

which implies $H(X, Z; c_1) < H(X, Z; c_2)$.

Now, assume by contradiction T_{c_1} and T_{c_2} touch in a point $(X_0, T_{c_1}(X_0) = T_{c_2}(X_0))$, with $0 < X_0 < 1$. Since $dT_{c_1}(X_0)/dX < dT_{c_2}(X_0)/dX$ by (1.12), we have that T_{c_2} stays above T_{c_1} in a small right-neighbourhood

I_0 of X_0 and so, by the continuity of the trajectories, there exists at least another "contact point" $X_0 < X_0^+ < 1$, with $T_{c_1}(X_0^+) = T_{c_2}(X_0^+)$. Consequently, for $h > 0$ small enough, we have

$$\frac{T_{c_2}(X_0^+) - T_{c_2}(X_0^+ - h)}{h} \leq \frac{T_{c_1}(X_0^+) - T_{c_1}(X_0^+ - h)}{h}$$

and taking the limit as $h \to 0$, we get the contradiction $dT_{c_2}(X_0^+)/dX \leq dT_{c_1}(X_0^+)/dX$. Our assertion follows from the arbitrariness of $0 < X_0 < 1$.

Step4: Existence and uniqueness of a critical speed $c = c_*$. We are ready to prove that there exists a unique value $c = c_*$ and a unique trajectory T_{c_*} linking S and R_{c_*}.
Before proceeding, we have to introduce some notations. Let Γ_1 be the left-branch of the isoclines in the case $0 < c < c_0$ (note that the study of the isoclines carried out in *Step1* tell us that there are no trajectories linking the points S and R_c for $c \geq c_0$), define $\Gamma_2 := \{(X, c^{1/(p-1)}) : 0 < X \leq 1\}$ and consider $\Gamma_c := \Gamma_1 \cup R_c \cup \Gamma_2$.
Now, fix $0 < \bar{c} < c_0$ and let $T_{\bar{c}}$ be the linear trajectory corresponding to the value $\bar{c}$ from the point S. Since $T_{\bar{c}}$ come from the point S and it lies in the region in which the slope is negative, it must join S with $\Gamma_{\bar{c}}$, i.e., there exists a point $(\overline{X}, \overline{Z}) \in \Gamma_{\bar{c}} \cap T_{\bar{c}}$. We have the following possibilities:

- If $(\overline{X}, \overline{Z}) = R_{\bar{c}}$, then we have $c_* = \bar{c}$ and the uniqueness follows by (1.12) and *Step3*.
- If $(\overline{X}, \overline{Z}) \in \Gamma_1$, using (1.12) it follows that there exists $0 < \bar{c}_2 < \bar{c}$ and a corresponding point $(\overline{X}_2, \overline{Z}_2) \in \Gamma_2 \cap T_{\bar{c}_2}$. Moreover, we have that for all $\bar{c} < c < c_0$ the trajectory T_c links S with Γ_1 and for all $0 < c < \bar{c}_2$ the trajectory T_c links S with Γ_2. Hence, from the continuity of the trajectories with respect to the parameter c, there exists $\bar{c}_2 < c_* < \bar{c}$ such that T_{c_*} joins S with R_{c_*} and the uniqueness follows from the strict monotonicity (1.12).
- If $(\overline{X}, \overline{Z}) \in \Gamma_2$, using (1.12) it follows that there exists $\bar{c} < \bar{c}_1 < c_0$ and a corresponding point $(\overline{X}_1, \overline{Z}_1) \in \Gamma_1 \cap T_{\bar{c}_1}$. Moreover, we have that for all $\bar{c}_1 < c < c_0$ the trajectory T_c links S with Γ_1 and for all $0 < c < \bar{c}$ the trajectory T_c links S with Γ_2. Hence, there exists a unique $\bar{c} < c_* < \bar{c}_1$ such that T_{c_*} joins S with R_{c_*}.

Thus, we can conclude that there exists exactly one value $c = c_* < c_0$ with corresponding trajectory T_{c_*} joining the points S and R_{c_*}. Moreover, since in *Step2* we have showed that the trajectory from the point S is unique, it follows that the trajectory T_{c_*} is unique too. We underline that our argument is completely qualitative and based on a topological observation: we proved the existence of two numbers $0 < \bar{c}_2 < \bar{c}_1 < c_0$ such that for all $0 < c < \bar{c}_2$, T_c links S with Γ_2, while for all $\bar{c}_1 < c < c_0$, T_c links S with Γ_1. Hence, since the trajectories are continuous respect with the parameter c it follows the existence of a value c_* such that T_{c_*} links S with $\Gamma_1 \cap \Gamma_2 = R_{c_*}$.
Finally, it remains to prove that the TW corresponding to c_* is finite i.e., it reaches the point $u = 0$ in finite "time" while the point $u = 1$ in infinite "time" (here the time is intended in the sense of the profile, i.e. the "time" is measured in terms of the variable ξ). In order to see this, it is sufficient to integrate the first equation in (1.7) by separation of variable between X_0 and X_1:

$$\xi_1 - \xi_0 = -m \int_{X_0}^{X_1} \frac{dX}{X^{1-\frac{\gamma}{p-1}} T_{c_*}(X)}. \tag{1.13}$$

So, since $\gamma > 0$ and $T_{c_*}(X) \sim c_*^{1/(p-1)}$ for $X \sim 0$ we have that for all $0 < X_1 < 1$, the integral of the right member is finite when $X_0 \to 0$ and so, our TW reaches the steady state $u = 0$ in finite time. More precisely, as $X_0 \to 0$, it holds

$$\xi - \xi_0 = -m \int_0^X \frac{dX}{X^{1-\frac{\gamma}{p-1}} T_{c_*}(X)} \sim -mc_*^{-\frac{1}{p-1}} \int_0^X X^{\frac{\gamma}{p-1}-1} dX = -c_*^{-\frac{1}{p-1}} \frac{m(p-1)}{\gamma} X^{\frac{\gamma}{p-1}},$$

and so the critical travelling wave profile satisfies

$$\varphi^{\frac{\gamma}{p-1}}(\xi) = X^{\frac{\gamma}{p-1}}(\xi) \sim c_*^{\frac{1}{p-1}} \frac{\gamma}{m(p-1)}(\xi_0 - \xi), \quad \text{for } \xi \sim \xi_0, \tag{1.14}$$

according to the Darcy law (see for instance [94, 198]). This is a crucial computation that shows that *finite propagation* holds in this case. Conversely, recalling the behaviour of the trajectory T_{c_*} near the point S (see *Step2*), it is not difficult to see that for all $0 < X_0 < 1$, the integral is infinite when $X_1 \to 1$ and so, the TW gets to the steady state $u = 1$ in infinite time.

We conclude this step with a brief analysis of the remaining trajectories. This is really simple once one note that the trajectory T_{c_*} acts as barrier dividing the set $[0,1] \times \mathbb{R}$ into subsets, one below and one above T_{c_*}. The trajectories in the subset below link the point O with $R_{-\infty} := (0, -\infty)$ (the same methods we use in *Step6* apply here), have slope zero on the left branch of the isoclines, are concave for $Z > 0$ and increasing for $Z < 0$. We name these trajectories CS-TWs (change sign TWs) of type 1. On the other hand, the trajectories above T_{c_*} recall "parabolas" (see *Step4*) connecting the points $R_\infty := (0, \infty)$ and $S_Z = (1, Z)$ for some $Z > 0$ and having slope zero on the right branch of the isoclines. Finally, note that, in this last case, the trajectories lying in the region $[0,1] \times [c_*^{1/(p-1)}, \infty)$ are always decreasing.

Step5: Non existence of admissible TWs for $0 < c < c_*$. In the next paragraph, we show that there are no admissible TW solutions when $0 < c < c_*$.
So, we assume by contradiction that there exists $0 < c < c_*$ such that the corresponding trajectory T_c joins S and $R_{\overline{Z}}$ for some $\overline{Z} \geq 0$. Then, by (1.12) and *Step3*, it must be $\overline{Z} > c_*^{1/(p-1)}$. Moreover, since $H(X, Z; c) < 0$ for all $Z > c^{1/(p-1)}$, there exists a "right neighbourhood" of $R_{\overline{Z}}$ in which the solution $Z = Z(X)$ corresponding to the trajectory T_c is invertible and the function $X = X(Z)$ has derivative

$$\frac{dX}{dZ} = \frac{(p-1)XZ^{p-1}}{cZ - Z^p - f_{m,p}(X)} := K(X, Z; c). \tag{1.15}$$

Choosing the neighbourhood $B_\delta(R_{\overline{Z}}) = \{(X, Z) : \ X^2 + (Z - \overline{Z})^2 < \delta, \ X \geq 0\}$, where $\delta > 0$ is small enough (for example, $\delta \leq \min\{1, \overline{Z} - C_*\}$), it is simple to obtain

$$\left| \frac{K(X, Z; c)}{X} \right| \leq I \quad \text{in } B_\delta(R_{\overline{Z}}),$$

where $I = I_{\overline{Z},\delta,m,p} > 0$ depends on $\overline{Z}$, δ, m and p. The previous inequality follows from the fact the quantity $|cZ - Z^p - f_{m,p}(X)|$ is greater than a positive constant in $B_\delta(R_{\overline{Z}})$ (see *Step1*) and Z is (of course) bounded in $B_\delta(R_{\overline{Z}})$. This means that the function $K(X, Z)$ is sub-linear respect with the variable X, uniformly in Z in $B_\delta(R_{\overline{Z}})$ and this is sufficient to guarantee the local uniqueness of the solution near $R_{\overline{Z}}$. Since the null function solves (1.15) with initial datum 0, from the uniqueness of this solution, it follows that $X = X(Z)$ is identically zero too and it cannot be invertible. This contradiction assures there are no TW solutions for $c < c_*$.

As we did in the previous step, we explain the qualitative properties of the trajectories. In the case $c < c_*$, the "zoo" of the trajectories is more various. First of all, the previous analysis shows that we have a connection between the points S and R_∞ always decreasing. Moreover, below this connection, there is a family of CS-TWs of type 1 linking the points O and $R_{-\infty}$ and family of CS-TWs of type 2, i.e., trajectories linking R_∞ with $R_{-\infty}$, decreasing for $Z > 0$ and increasing for $Z < 0$. Furthermore, for topological reasons, there exists a trajectory between the family of CS-TWs of type 1 and CS-TWs of type 2 which link the critical point R_c and $R_{-\infty}$. Finally, we find again the "parabolas" described in *Step3*.

Step6: Existence of admissible positive TWs for $c > c_*$. Finally, we focus on the case $c > c_*$. We follow the procedure used in *Step5*, assuming by contradiction that there exists $c > c_*$ such that the

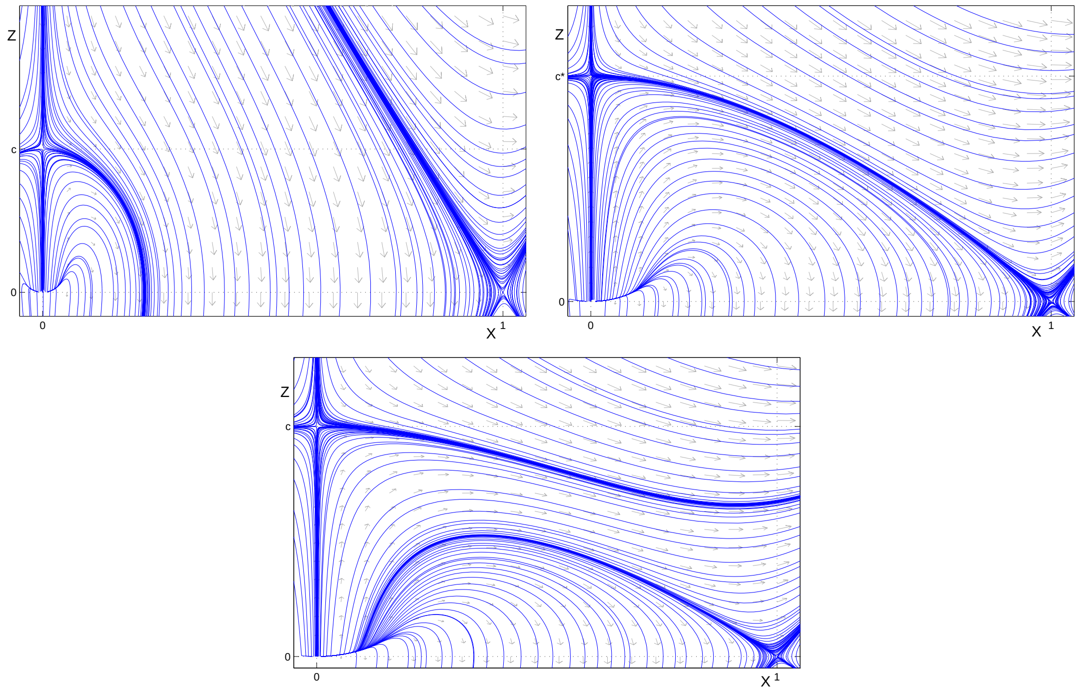

Figure 1.3: Fisher-KPP reactions, range $\gamma > 0$. Qualitative behaviour of the trajectories in the (X, Z)-plane for $f(u) = u(1-u)$. Cases $c < c_*$, $c = c_*$ and $c > c_*$, respectively.

corresponding trajectory T_c joins S and $R_{\overline{Z}}$ for some $\overline{Z} > 0$ (in this case it must be $\overline{Z} < c_*^{1/(p-1)}$). Note that, to be precise, we should treat separately the cases $c_* < c < c_0$ and $c \geq c_0$ since the phase plane changes markedly, but, however, our argument works independently of this distinction.
Again, we want to prove the sub-linearity of the function $K(X, Z; c)$ respect with the variable X, uniformly in Z in a “right” neighbourhood of $R_{\overline{Z}}$. Define

$$\mathcal{R} := \left\{(X, Z) : H(X, Z; c) > 0 \text{ for } 0 \leq X \leq 1,\ 0 \leq Z \leq c_*^{1/(p-1)}\right\}$$
$$B_\delta(R_{\overline{Z}}) = \left\{(X, Z) :\ X^2 + (Z - \overline{Z})^2 \leq \delta,\ X \geq 0\right\},$$

where $\delta > 0$ is taken small enough such that $B_\delta(R_{\overline{Z}}) \subset \mathcal{R}$. Hence, proceeding as in the previous step, we can state that, in $B_\delta(R_{\overline{Z}})$, the quantity $|cZ - Z^p - f_{m,p}(X)|$ is greater than a positive constant and so, it is simple to get

$$\left|\frac{K(X, Z; c)}{X}\right| \leq I \quad \text{in } B_\delta(R_{\overline{Z}}),$$

where $I = I_{\overline{Z},\delta,m,p} > 0$ depends only on $\overline{Z}$, δ, m and p. Hence, reasoning as before, we obtain that the trajectories cannot “touch” the Z-axis. This means that the point O “attracts” the trajectories and so for all $c > c_*$ there exists a connection between the points S and O, i.e., a TW solution.
Now, in order to prove that these TWs are positive, we show that all trajectories approach the branch of the isoclines near O given by equation (1.10). First of all, from the equation of the null isoclines (1.10), it is simple to see that the branch of the isoclines satisfies $\widetilde{Z}(X) \sim \widetilde{\lambda} X^{\gamma/(p-1)}$ as $X \sim 0$, where $\widetilde{\lambda} := mf'(0)/c$. Now, as we did in *Step2*, we use an argument with invariant regions. Consider the one-parameter family

$$Z_a(X) = aX^{\frac{\gamma}{p-1}}, \qquad \widetilde{\lambda} < a < \infty.$$

With straightforward computations as in *Step2*, it is simple to obtain

$$\frac{dZ_a/dX}{dZ/dX}(X,Z_a(X);c)=\frac{dZ_a/dX(X)}{H(X,Z_a(X);c)}\sim\frac{\gamma a^p}{c(a-\widetilde{\lambda})}X^{\gamma}\sim 0\quad\text{as } X\sim 0,$$

i.e., for all $a>\widetilde{\lambda}$ and for small values of the variable X, the derivative of the trajectories along the curve $Z_a=Z_a(X)$ is infinitely larger than the derivative of the same curve. This fact implies that for all $a>\widetilde{\lambda}$, all trajectories $Z=Z(X)$ satisfy $\widetilde{Z}(X)\le Z(X)\le Z_a(X)$ as $X\sim 0$ and so, since a can be chosen arbitrarily near to $\widetilde{\lambda}$ it follows $Z(X)\sim\widetilde{Z}(X)$ as $X\sim 0$.
We may now integrate the first differential equation in (1.7) by separation of variables. Since $\widetilde{Z}(X)\sim\widetilde{\lambda}X^{\gamma/(p-1)}$, we have that $\widetilde{Z}(X)X^{1-\frac{\gamma}{p-1}}\sim\widetilde{\lambda}X$ for $X\sim 0$. Hence, using the fact that $Z(X)\sim\widetilde{Z}(X)$ for $X\sim 0$, we get

$$\xi_1-\xi_0=-m\int_{X_0}^{X_1}\frac{dX}{X^{1-\frac{\gamma}{p-1}}Z(X)}=-m\int_{X_0}^{X_1}\frac{dX}{X^{1-\frac{\gamma}{p-1}}\widetilde{Z}(X)}\sim-\frac{c}{f'(0)}\int_{X_0}^{X_1}\frac{dX}{X}\quad\text{for } X_0\sim 0.$$

Thus, since the last integral is divergent in $X_0=0$, we deduce that the admissible TWs reach the level $u=0$ at the time $\xi=+\infty$, i.e., they are positive.
Moreover, we can describe the exact decaying of the profiles corresponding to these TWs when $X\sim 0$. Indeed, from the previous formula we have that $\xi_1-\xi\sim(c/f'(0))\ln(X_1/X)$ for $X\sim 0$, which can be easily re-written as

$$\varphi(\xi)=X(\xi)\sim a_1\exp\left(-\frac{f'(0)}{c}\xi\right)=a_1\exp\left(-\frac{f'(0)^{1/p}}{\nu}\xi\right),\quad\text{for }\xi\sim+\infty \tag{1.16}$$

where a_1 is a fixed positive constant (depending on ξ_1) and $\nu>0$ is the speed with $f'(0)=1$.

We conclude this long analysis, describing the "zoo" of the trajectories obtained in the (X,Z)-plane. As we observed in the case $c=c_*$ the TW joining S and O represents a barrier and divide the region $[0,1]\times\mathbb{R}$ in two subsets. We obtain again the CS-TWs of type 1 in the region below the positive TW and the "parabolas" in the region above it. Moreover, if $c_*<c<c_0$, we have a family of trajectories which "come" from O, are decreasing in the region between the branches of the isoclines and increasing in the regions between the Z-axis and the left branch and between the right branch and the line $X=1$. Performing the complete phase plane analysis it is possible to see that these trajectories are increasing for $X\ge 1$ and go to infinity as $X\to\infty$. We name them infinite TWs of type 1. If $c\ge c_0$, the analysis is similar except for the fact that there exist a family (or exactly one trajectory in the case $c=c_0$) of infinite TWs of type 2, i.e., trajectories "coming" from O and increasing for all $X\ge 0$. Finally, we have a trajectory from the point R_c trapped between the family of infinite TWs and the family of "parabolas" which goes to infinity as $X\to\infty$. □

Proof of Theorem 1.1: Part (ii). Now, let $f(\cdot)$ be a reaction term of type C satisfying (3):

$$\begin{cases} f(0)=f(a)=f(1)=0, & f(u)<0 \text{ in } (0,a),\ \ f(u)>0 \text{ in } (a,1)\\ f\in C^1([0,1]), & f'(0)<0,\ f'(a)>0,\ f'(1)<0\\ \int_0^1 u^{m-1}f(u)du>0 & \\ f(\cdot) \text{ has a unique critical point in } (0,a) \text{ and a unique critical point in } (a,1). & \end{cases}$$

As before, we consider system (1.8) and the *equation of the trajectories* (1.9):

$$\frac{dZ}{dX}=\frac{cZ-|Z|^p-f_{m,p}(X)}{(p-1)X|Z|^{p-2}Z}:=H(X,Z;c).$$

Since now $f(\cdot)$ satisfies (3), we get that the function $f_{m,p}(X) := mX^{\frac{\gamma}{p-1}-1}f(X)$ is well-defined and continuous in $[0,1]$, with $f_{m,p}(0) = f_{m,p}(a) = f_{m,p}(1) = 0$ and $f_{m,p}(X) < 0$ for $0 < X < a$, while $f_{m,p}(X) > 0$ for $a < X < 1$. Consequently, system (1.8) has four critical points

$$O(0,0), \qquad S(1,0), \qquad A(a,0) \qquad \text{and} \qquad R_c(0,c^{1/(p-1)}).$$

As we did in the proof of the first part, we prove the existence of a special speed $c_* = c_*(m,p,f)$ with corresponding trajectory linking $S(1,0)$ and $R_{c_*}(0,c_*^{1/(p-1)})$ and lying in the strip $[0,1]\times[0,+\infty)$ of the (X,Z)-plane. Again this connection is the *finite* TW we are looking for.
So, following the ideas of Part (i), we have to understand the qualitative behaviour of the trajectories of system (1.8) (or, equivalently, the solutions of equation (1.9)) in dependence of the parameter $c > 0$. Respect to the previous case, we start by considering the simpler case $c = 0$, which is fundamental to exclude the existence of *admissible* TWs for small speeds of propagation. The assumption

$$\int_0^1 u^{m-1}f(u)du > 0,$$

(cfr. with (3)) plays an important role here. Then, as before, we will need information on the local behaviour of the trajectories near the critical points and on the global monotonicity properties of the trajectories w.r.t. the speed $c > 0$. Once more, we employ them to show the existence or non-existence of trajectories linking the critical points $S(1,0)$ and $R_c(0,c^{1/(p-1)})$, which correspond to a *finite* TW.

Step0: Case $c = 0$. As we have explained in the previous paragraph, we begin by taking $c = 0$ an we show that for the null speed, there are not *admissible* TW profiles. With this choice, system (1.8) and equation (1.9) become

$$\begin{cases}\dot{X} = (p-1)X|Z|^{p-2}Z,\\ \dot{Z} = -|Z|^p - f_{m,p}(X),\end{cases} \qquad \text{and} \qquad \frac{dZ}{dX} = -\frac{|Z|^p + f_{m,p}(X)}{(p-1)X|Z|^{p-2}Z} = H(X,Z;0),$$

respectively (here $\dot{X}$ means $dX/d\tau$). The critical points are $O(0,0)$, $A(a,0)$, and $S(1,0)$ (note that the point R_c "collapses" to $O(0,0)$).
Respect to the linear case, our system does not conserve the energy along the solutions (see [12]). Consequently, excluding the existence of a trajectory, contained in the strip $(0,1)\times(0,\infty)$ in the (X,Z)-plane and linking $O(0,0)$ and $S(1,0)$, is done by studying more qualitative properties of the trajectories in the (X,Z)-plane.
So, we begin by analyzing the *null isoclines* $\widetilde{Z} = \widetilde{Z}(X)$ of our system, i.e. the solutions of the equation:

$$|\widetilde{Z}|^p + mX^{\frac{\gamma}{p-1}-1}f(X) = 0, \qquad 0 \le X \le 1.$$

They are composed by two branches linking the points $O(0,0)$ and $A(a,0)$, lying in the strip $[0,a]\times(0,\infty)$ and $[0,a]\times(-\infty,0)$, respectively (this follows by the last assumption in (3)) and they satisfy

$$\widetilde{Z}(X) \sim \pm\sqrt[p]{-mf'(0)}X^{\frac{\gamma}{p(p-1)}}, \qquad \text{for } X \sim 0.$$

Now, there are two symmetric trajectories (one positive, and one negative in a right-neighbourhood of $O(0,0)$) "leaving" $O(0,0)$ (this follows from study of the *null isoclines* and the sign of the derivative dZ/dX in the (X,Z)-plane). Moreover, since $H(X,-Z;0) = -H(X,Z;0)$, the two trajectories coincide and we obtain a unique trajectory linking $O(0,0)$ with itself. Now, let us focus on the part lying in $[0,1)\times[0,\infty)$, $T^+ = T^+(X)$ and let $T_0 = T_0(X)$ be the trajectory "going into" $S(1,0)$ (see *Step1* below). If $T^+ = T^+(X)$ and $T_0 = T_0(X)$ touch at a point, they coincide in $[0,1]$ and the resulting trajectory has the shape of an *admissible* profile. In the next paragraphs, we show that T^+ and T_0 must be two distinct trajectories and the just described case cannot happen.

As first observation, since the solution $T^+ = T^+(X)$ stays below the positive branch $\widetilde{Z} = \widetilde{Z}(X)$ for $X \sim 0$, a simple approximation argument (see also *Step2* of the proof of Part (i)) shows that

$$T^+(X) \sim \sqrt[p]{-\frac{mpf'(0)}{\gamma+p}} X^{\frac{\gamma}{p(p-1)}}, \quad \text{for } X \sim 0.$$

Hence, substituting it in the first equation of system (1.7), we obtain (up to a multiplicative constant):

$$-\frac{dX}{d\xi} \sim X^{1-\frac{\gamma}{p-1}} T^+(X) \quad \Leftrightarrow \quad X(\xi) = \varphi(\xi) \sim (\xi_0 - \xi)^{\frac{p}{\gamma}}, \quad \text{for } \xi \sim \xi_0^-,$$

which contradicts then Darcy law of the *free boundary* (see [198], Chapter 4 for the Porous Medium case). Consequently, if $T^+ = T^+(X)$ and $T_0 = T_0(X)$ coincide, we immediately conclude that the resulting trajectory linking $O(0,0)$ and $S(1,0)$ cannot be an *admissible finite* TW and we conclude the non-existence *admissible* TWs for $c = 0$. The qualitative behaviour of the trajectories in the (X,Z)-plane is shown in Figure 1.4.
However, in what follows, we will need to exclude the case in which the trajectory $T_0 = T_0(X)$ "going into" $S(1,0)$ has either a closed curve or $S(1,0)$ as *negative* limit set, or crosses at some point the negative half-line $X = 1$ (cfr. with the right picture of Figure 1.4). To achieve this, we will show that $T_0 = T_0(X) \sim +\infty$ as $X \sim 0$, using our assumption on the reaction term (see (3)):

$$\int_0^1 u^{m-1} f(u)\, du > 0. \tag{1.17}$$

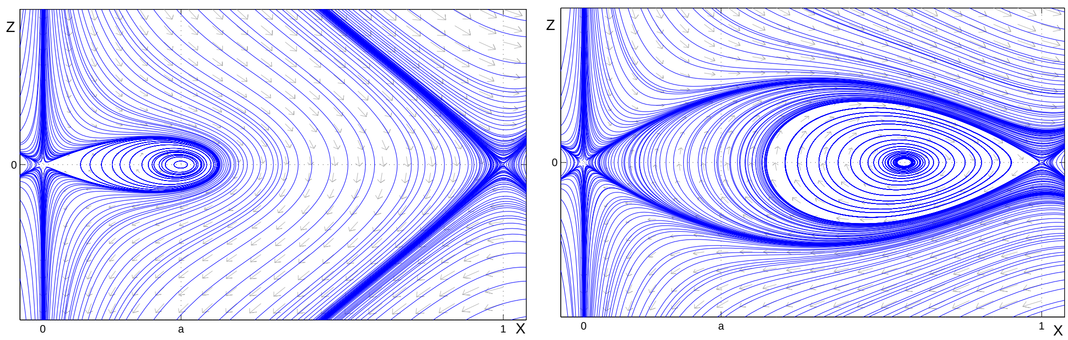

Figure 1.4: Reactions of type C, range $\gamma > 0$, case $c = 0$. Qualitative behaviour of the trajectories in the (X,Z)-plane for $f(u) = u(1-u)(u-a)$, $a = 0.3, 0.7$. The second case is excluded by the assumption $\int_0^1 u^{m-1} f(u) du > 0$.

For $0 < X < 1$ and $Z > 0$, the equation of the trajectories can be re-written as

$$\frac{dZ}{dX} = -\frac{Z^p + mX^{\frac{\gamma}{p-1}-1} f(X)}{(p-1)XZ^{p-1}} \quad \Leftrightarrow \quad pX^{2-\frac{\gamma}{p-1}} Z^{p-1} \frac{dZ}{dX} = -\frac{pX^{1-\frac{\gamma}{p-1}} Z^p + mpf(X)}{(p-1)}$$

Using that

$$\frac{d}{dX}\left(X^{2-\frac{\gamma}{p-1}} Z^p\right) = \left(2 - \frac{\gamma}{p-1}\right) X^{1-\frac{\gamma}{p-1}} Z^p + pX^{2-\frac{\gamma}{p-1}} Z^{p-1} \frac{dZ}{dX},$$

and the previous equation, we deduce that $S(X) := X^{2-\frac{\gamma}{p-1}} Z^p$ satisfies the equation

$$\frac{dS}{dX} = \frac{1-m}{X} S - \frac{mp}{p-1} f(X), \quad 0 < X < 1,$$

where we have used the definition of $\gamma := m(p-1)-1$. Now, assume for a moment $m \neq 1$. It is simple to integrate the previous equation obtaining

$$S(X) = X^{1-m}\left[k - \frac{mp}{p-1}\int_0^X u^{m-1}f(u)du\right], \quad 0 < X < 1,$$

where k is a free parameter. Now, coming back to the function $Z = Z(X)$, we get

$$Z(X) = X^{-\frac{1}{p-1}}\left[k - \frac{mp}{p-1}\int_0^X u^{m-1}f(u)du\right]^{\frac{1}{p}}, \quad 0 < X < 1, \tag{1.18}$$

and, thanks to our assumption (1.17), we can take

$$\int_0^1 u^{m-1}f(u)du := \frac{p-1}{mp}h > 0.$$

Furthermore, choosing $k = h$ in (1.18), we deduce $Z(1) = 0$. Hence, if $T_0 = T_0(X)$ is the trajectory "going into" in $S(1,0)$, we have by uniqueness of this solution

$$T_0(X) = X^{-\frac{1}{p-1}}\left[h - \frac{mp}{p-1}\int_0^X u^{m-1}f(u)du\right]^{\frac{1}{p}} \sim +\infty, \quad \text{for } X \sim 0,$$

proving our claim (cfr. with the left diagram shown in Figure 1.4). The case $m = 1$ is very similar and formula (1.18) holds with $m = 1$.
We end this paragraph pointing out that, thanks to the continuous dependence of the solutions w.r.t. to the parameter $c \geq 0$, we deduce that there are not *admissible* TWs for values of $c > 0$ small enough.

Step1: Local analysis of $S(1,0)$*.* From now on, we consider $c > 0$. This step coincide with *Step2* of Part (i). We recall that we have proved that there exists a unique trajectory $T_c = T_c(X)$ "coming into" $S(1,0)$ and is asymptotic behaviour near $X = 1$ is

$$T_c(X) \sim \begin{cases} \lambda_S^-(1-X)^{2/p} & \text{if } 1 < p < 2 \\ \lambda_S(1-X) & \text{if } p = 2 \\ \lambda_S^+(1-X) & \text{if } p > 2 \end{cases} \qquad \text{for } X \sim 1, \tag{1.19}$$

for suitable positive numbers λ_S^-, λ_S, and λ_S^+. The local analysis of the point $A(a,0)$ is less important in this setting and we skip it. In the Porous Medium case $p = 2$ and $m > 1$, it is not difficult to see that $A(a,0)$ is a *focus unstable* if $c < \sqrt{4ma^{m-1}f'(a)}$, while a *node unstable* if $c \geq \sqrt{4ma^{m-1}f'(a)}$ (cfr. with Figure 1.6).

Step2: Study of the null isoclines. Again, to obtain a qualitative picture of the trajectories of the system, we study the *null isoclines* of system (1.8), i.e., the curve $\widetilde{Z} = \widetilde{Z}(X)$ satisfying (1.10):

$$c\widetilde{Z} - |\widetilde{Z}|^p = f_{m,p}(X), \qquad \text{in } [0,1]\times(-\infty,\infty).$$

Now, let $F_{m,p} > 0$ be the maximum of $f_{m,p}(\cdot)$ in $[0,1]$, and take $c_0 > 0$ so that

$$\max_{\widetilde{Z}\in[0,c^{1/(p-1)}]}\left\{c\widetilde{Z} - |\widetilde{Z}|^p\right\} = F_{m,p} \qquad \text{i.e.} \qquad c_0 = c_0(m,p,f) := p\left(\frac{F_{m,p}}{p-1}\right)^{(p-1)/p}.$$

Then it is not difficult to see that for $0 < c < c_0$, the *null isocline* is composed of two disjoint branches: the left one, linking the points $O(0,0)$, $A(a,0)$, $(a, c^{1/(p-1)})$ and $R_c(0, c^{1/(p-1)})$, and the right one, connecting $S(1,0)$ and $(1, c^{1/(p-1)})$. The two branches approach as $c \to c_0$, until they touch at the point

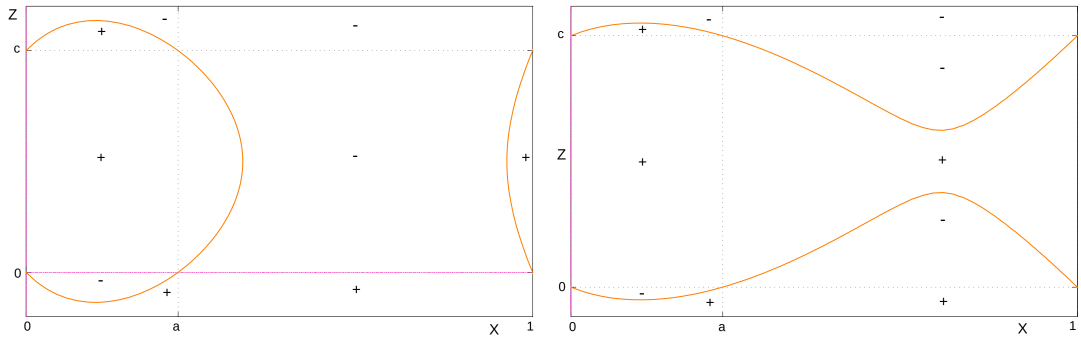

Figure 1.5: Reactions of type C, range $\gamma > 0$. Null isoclines in the (X, Z)-plane for $f(u) = u(1-u)(u-a)$, $a = 0.3$, in the cases $0 < c < c_0$ and $c > c_0$, respectively.

$(X_{m,p}, (c_0/p)^{1/(p-1)})$ when $c = c_0$, where $f_{m,p}(X_{m,p}) = F_{m,p}$. Finally, when $c > c_0$, there are again two disjoint branches: the upper one linking $R_c(0, c^{1/(p-1)})$, $(a, c^{1/(p-1)})$ and $(1, c^{1/(p-1)})$, whilst the lower one joining $O(0,0)$, $A(a,0)$ and $S(1,0)$. A qualitative representation is shown in Figure 1.5. From this analysis it is clear that if there exists a critical speed c_*, then it must be $c_* < c_0$ (again we employed the last assumption in (3)).

Step3: Monotonicity of $T_c(\cdot)$ w.r.t. $c > 0$. This step coincide with *Step3* of Part (i). We just mention that formula (1.12) holds true even in this setting and it is employed to show that

$$\text{for all } \; 0 < c_1 < c_2 \; \text{ then } \; T_{c_2}(X) < T_{c_1}(X), \; \text{ for all } \; a < X < 1$$

whereT_c is the trajectory "coming into" $S(1,0)$. Note that for $0 \le X \le a$, $T_c(\cdot)$ is not in general a function of X, so that we have to restrict our "comparison interval" to $(a, 1)$. However, our statement holds true on the interval of definition of $T_c = T_c(X)$.

Step4: Existence and uniqueness of a critical speed $c = c_*$. In *Step0*, we have shown that for $c = 0$ there are not *admissible* TWs, and, in particular, the trajectory $T_0 = T_0(X)$ "coming into" $S(1,0)$ stays above the trajectories "leaving" the origin $O(0,0)$.
Consequently, thanks to the continuity of the trajectories w.r.t. the parameter c we can conclude the same, for *small* values of $c > 0$, i.e., naming $T_c^+ = T_c^+(X)$ and $T_c^- = T_c^-(X)$ the trajectories from $R_c(0, c)$ and $O(0,0)$, respectively, we have that $T_c(X)$ is above $T_c^+(X)$ and $T_c^-(X)$ in $[0,1]$ (note that for $c = 0$, $R_0 = O$ and both T_0^+ and T_0^- "leave" O).
In particular, the study of the *null isoclines* carried out in *Step2* shows that $T_c^+(X = a) > c^{1/(p-1)}$ for all $c > 0$, and so, using the monotonicity of T_c w.r.t. $c > 0$ proved in *Step3*, we conclude that for $c > 0$ large enough it must be $T_c(X = a) < T_c^+(X = a)$, which means that for large $c > 0$, $T_c(X)$ stays below $T_c^+(X)$ in $[0,1]$ by uniqueness of the trajectories. This means that there exists a critical speed $c_* = c_*(m, p, f)$ such that $T_{c_*}(X) = T_{c_*}^+(X)$ for all $X \in [0,1]$, and the uniqueness of c_* follows the strict inequality in (1.12). In other words, the trajectories T_c^+ and T_c approach as $c < c_*$ grows until they touch (i.e. they coincide) for $c = c_*$, while for $c > c_*$ they are ordered in the opposite way w.r.t. the range $c < c_*$, i.e. $T_c^+(X) > T_c(X)$ in $[0,1]$ for all $c > c_*$.
We conclude this step by pointing out that the last part of *Step4* of Part (i) is still valid even in this setting. Indeed, we can integrate the first equation in(1.7) along $T_{c_*} = T_{c_*}(X)$ obtaining (1.13) and showing that this trajectory corresponds to an *admissible* TW *profile* $X(\xi) = \varphi(\xi)$ of a *finite* TW $u(x,t) = \varphi(x - ct)$, i.e., $\varphi(-\infty) = 1$ and $\varphi(\xi) = 0$, for all $\xi \ge \xi_0$, for some $-\infty < \xi_0 < +\infty$. More precisely, we still have formula (1.14):

$$\varphi^{\frac{\gamma}{p-1}}(\xi) = X^{\frac{\gamma}{p-1}}(\xi) \sim c_*^{\frac{1}{p-1}} \frac{\gamma}{m(p-1)}(\xi_0 - \xi), \quad \text{for } \xi \sim \xi_0,$$

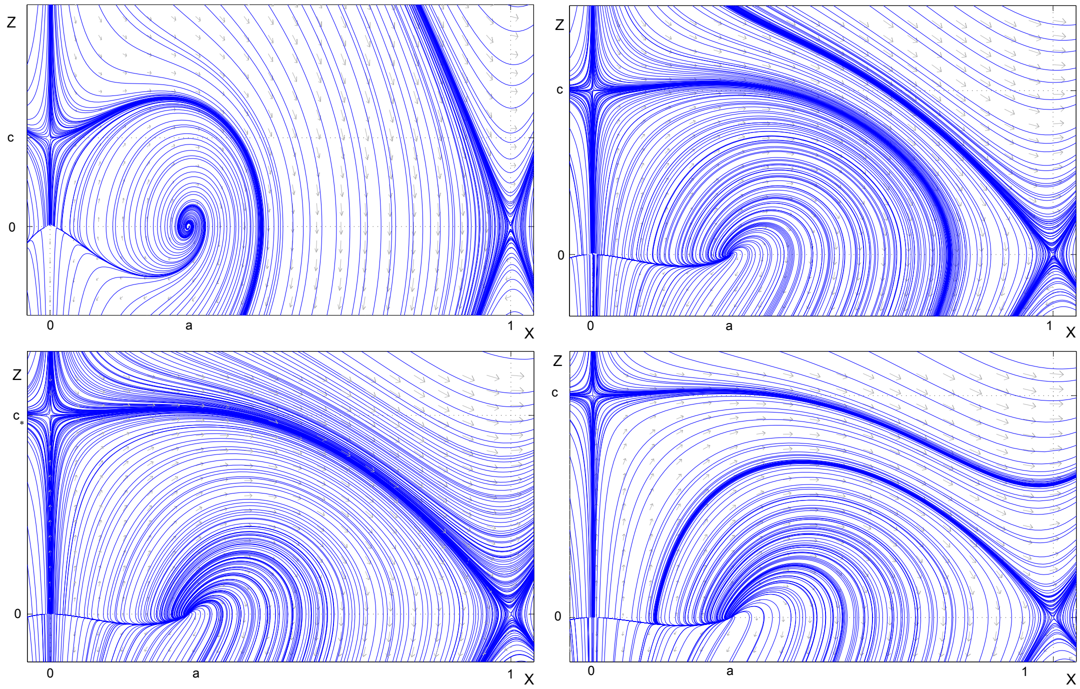

Figure 1.6: Reactions of type C, range $\gamma > 0$. Qualitative behaviour of the trajectories in the (X, Z)-plane for $f(u) = u(1-u)(u-a)$, $a = 0.3$. The first two pictures show the case $0 < c < c_*$, while the others the cases $c = c_*$ and $c > c_*$, respectively.

which gives the behaviour of the *finite* TW near the *free boundary* point $-\infty < \xi_0 < +\infty$.

Step5: Non existence of admissible TWs for $c > c_*$. We are left to prove that there are not *admissible* TW solutions when $c > c_*$. This follows from the fact that if the trajectory T_c joins $O(0,0)$ and $S(1,0)$, then the resulting connection is not *admissible* since the derivative of the corresponding *profile* changes sign. Indeed, using the continuity of the trajectory w.r.t. the speed of propagation, we know that for all $c > 0$, there exists a unique trajectory $T_c^- = T_c^-(X)$ "leaving" $O(0,0)$ (see *Step0*) and simple computations shows that

$$T_c^-(X) \sim \frac{mf'(0)}{c} X^{\frac{\gamma}{p-1}}, \quad \text{for } X \sim 0^+.$$

Hence, if T_c links $O(0,0)$ and $S(1,0)$, it must coincide with T_c^- and so, the derivative of its profile must change sign, i.e., it is not an *admissible* profile. A qualitative representation of the trajectories in the (X, Z)-plane is given in Figure 1.6.

Proof of Theorem 1.1: Part (iii). We lastly consider reaction terms $f(\cdot)$ of type C′, i.e., satisfying (4):

$$\begin{cases} f(0) = f(a) = f(1) = 0, & 0 < f(u) \le f'(0)u \ \text{ in } (0,a), \ f(u) < 0 \text{ in } (a,1) \\ f \in C^1([0,1]), & f'(0) > 0, \ f'(a) < 0, \ f'(1) > 0 \\ f(\cdot) \text{ has a unique critical point in } (0,a) \text{ and a unique critical point in } (a,1). \end{cases}$$

System (1.8) and the *equation of the trajectories* (1.9)

$$\frac{dZ}{dX} = \frac{cZ - |Z|^p - f_{m,p}(X)}{(p-1)X|Z|^{p-2}Z} := H(X, Z; c)$$

do not formally change, but this time $f_{m,p}(X) := mX^{\frac{\gamma}{p-1}-1} f(X)$ satisfies $f_{m,p}(0) = f_{m,p}(a) = f_{m,p}(1) = 0$, with $f_{m,p}(X) > 0$ for $0 < X < a$, while $f_{m,p}(X) < 0$ for $a < X < 1$. So, as for the reactions of type C, system

(1.8) has the four critical points

$$O(0,0), \qquad S(1,0), \qquad A(a,0) \qquad \text{and} \qquad R_c(0,c^{1/(p-1)}).$$

In this part, we study the existence of *a-admissible* TW solutions for equation (1.1) with reaction satisfying (4). Respect to the previous case, our proof strongly relies on the proof of Part (i). Indeed, as we have mentioned in the introduction, reaction terms satisfying (4) are of the Fisher-KPP type if we restrict them to the interval $[0,a] \subset [0,1]$, in the sense that

$$\begin{cases} f(0) = f(a) = 0, & f(u) > 0, \text{ in } (0,a) \\ f \in C^1([0,a]), & f'(0) > 0, \; f'(a) < 0. \end{cases}$$

For this reason, it follows that the qualitative behaviour of the trajectories in the strip $[0,a] \times (-\infty,\infty)$ of the (X,Z)-plane is the same of the ones studied Part (i) in the larger strip $[0,1] \times (-\infty,\infty)$, where the Fisher-KPP case has been analyzed. In this way, it is easily seen that the study of the trajectories corresponding to *a-admissible* TW solutions of equation (1.1) (with reaction of type C′) is reduced to the study of *admissible* TWs for equation (1.1) with a reaction of Fisher-KPP type (or type A). In view of this explanation, parts of the following paragraphs coincide with the ones of Part (i), so that, for the reader's convenience, we will try to report the most important ideas, quoting the specific paragraphs of the proof of Part(i) for each technical detail.

Step1: Local analysis of $A(a,0)$ and $S(1,0)$. Let us take $c > 0$. Proceeding as in the proof of Part (i) (see *Step2*), and recalling that now $f'(1) > 0$, while $f'(a) < 0$, we deduce that $A(a,0)$ is a *saddle* type critical point, and formulas (1.19) hold replacing $f'(1)$ with $f'(a)$.
For what concerns the point $S(1,0)$, we can conclude it has a *focus/node* nature from the study of the *null isoclines* we perform in *Step2*.

Now, let $T_c = T_c(X)$ be the trajectory entering in $A(a,0)$ with $T_c(X) > 0$ for all $0 < X < a$. In the next paragraphs, following the proof of Part (i), we show that there exists a unique $c_* = c_*(m,p,f)$ such that T_{c_*} links $R_{c_*}(0,c_*^{1/(p-1)})$ and $A(a,0)$ and we prove that this trajectory corresponds to a *finite* TW profile. Secondly, we show that for all $c > c_*$, T_c joins $O(0,0)$ and $A(a,0)$, and it corresponds to a *positive* TW profile. Finally, we prove that there are not connections of the type $A(a,0) \leftrightsquigarrow R_c(0,c^{1/(p-1)})$ and/or $A(a,0) \leftrightsquigarrow O(0,0)$ for $c < c_*$, i.e. there are not any *a-admissible* TW profiles for $c < c_*$.

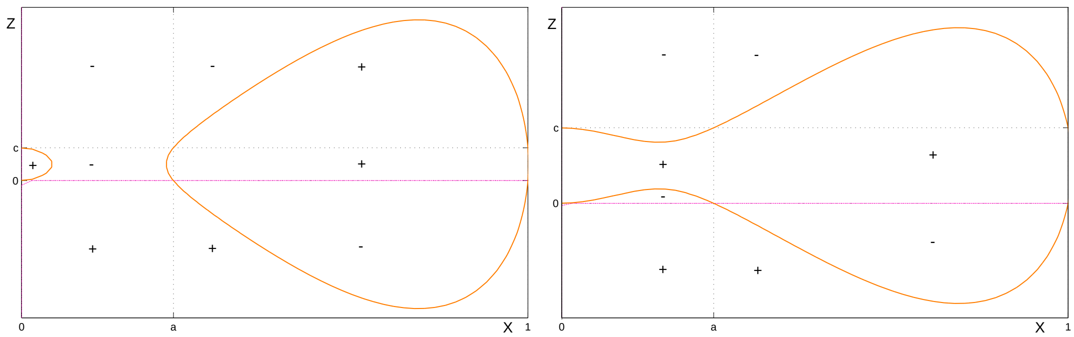

Figure 1.7: Reactions of type C′, range $\gamma > 0$. Null isoclines in the (X,Z)-plane for $f(u) = u(1-u)(a-u)$, $a = 0.3$, in the cases $0 < c < c_0$ and $c > c_0$, respectively.

Step2: Study of the null isoclines. We study the *null isoclines* of system (1.8), i.e., the curve $\widetilde{Z} = \widetilde{Z}(X)$ satisfying

$$c\widetilde{Z} - |\widetilde{Z}|^p = f_{m,p}(X),$$

exactly as in *Step1* of the proof of Part (i). We proceed as before and we obtain that there exists $c_0 > 0$ such that for $0 < c < c_0$, the *null isocline* is composed of two disjoint branches: the left one, linking the points $O(0,0)$ and $R_c(0, c^{1/(p-1)})$, and the right one, connecting $S(1,0)$, $A(a,0)$, $(a, c^{1/(p-1)})$ and $(1, c^{1/(p-1)})$. For $c > c_0$, we have again two branches: the upper one linking $R_c(0, c^{1/(p-1)})$, $(a, c^{1/(p-1)})$ and $(1, c^{1/(p-1)})$, whilst the lower one joining $O(0,0)$, $A(a,0)$ and $S(1,0)$. As before, the two branches approach as $c \to c_0$, and they touch at a point when $c = c_0$. We point out that for $c < c_0$ we obtain symmetric *null isoclines* respect to the previous case (cfr. with Figure 1.5). Again we see that if our c_* exists, then it has to be $c_* < c_0$. The qualitative shape of the *null isoclines* for reactions of type C′ in the cases $0 < c < c_0$ and $c > c_0$ is reported in Figure 1.7. We stress that the shape of *null isocline* in the rectangle $[0,a] \times [0, c^{1/(p-1)}]$ is (of course) the same of the one found for Fisher-KPP reactions in the rectangle $[0,1] \times [0, c^{1/(p-1)}]$ (cfr. with *Step1* of Part (i)).

Step3: Existence and uniqueness of a critical speed $c = c_*$. As we have explained in *Step1*, we have to prove the existence and the uniqueness of a speed $c_* = c_*(m,p)$ such that T_{c_*} links $R_{c_*}(0, c_*^{1/(p-1)})$ and $A(a,0)$ with corresponding TW profile vanishing in a half-line. Consequently, the proof of this fact coincides with what proved in *Step4* of Part (i), substituting the point $S(1,0)$ with $A(a,0)$.

Step4: The cases $0 < c < c_*$ and $c > c_*$. We have to show that for $0 < c < c_*$, the are not *a-admissible* TW, while to each $c > c_*$, it corresponds exactly one *a-admissible* TW and it is *positive*. Again it is sufficient to adapt *Step5* and *Step6* of Part (i) and we conclude the proof. A qualitative representation of the trajectories for $c < c_*$, $c = c_*$ and $c > c_*$ is shown in Figure 1.8. □

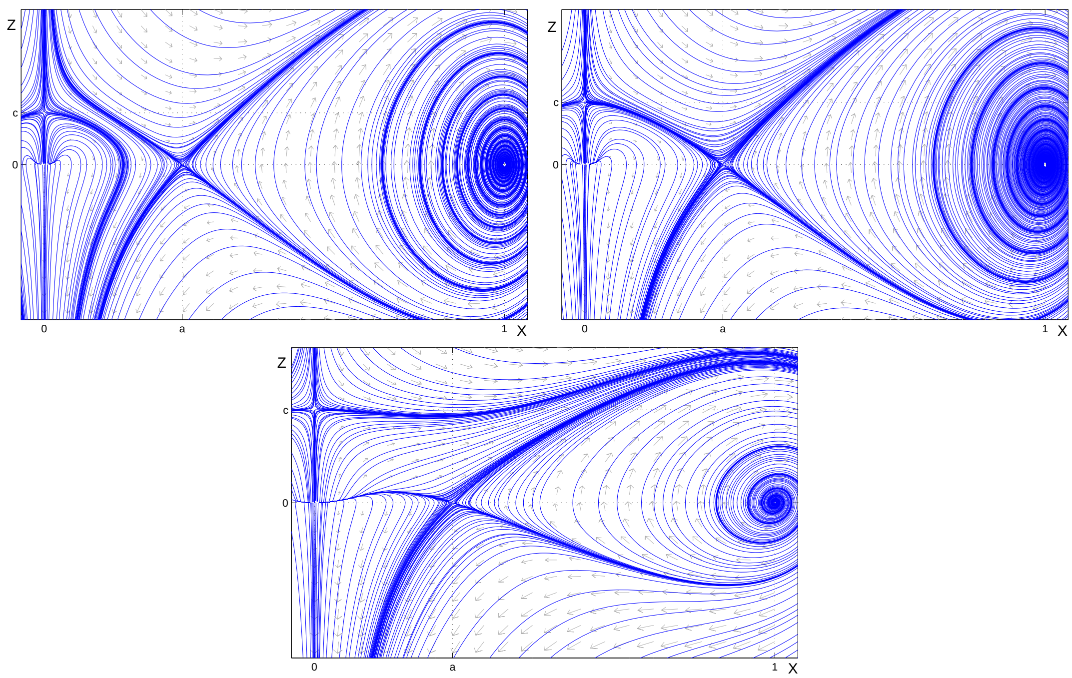

Figure 1.8: Reactions of type C′, range $\gamma > 0$. Qualitative behaviour of the trajectories in the (X,Z)-plane for $f(u) = u(1-u)(a-u)$, $a = 0.3$, in the ranges $0 < c < c_*$, $c = c_*$ and $c > c_*$, respectively.

1.2.1 Fisher-KPP reactions. Analysis of some special trajectories

In *Step5* of the proof of Theorem 1.1 (Part (i)), we have shown the existence of a family of CS-TWs of type 2 which link the points R_∞ and $R_{-\infty}$ in the (X,Z)-plane. These particular TWs will play a really important role in the study of more general solutions (see Chapter 2). In particular, we want to analyze their behaviour in the "real" plane $(\xi, \varphi(\xi)) = (\xi, X(\xi))$ near their global positive maximum (for example in $\xi = 0$) and near the two change sign points $\xi_0 < 0 < \xi_1$ where $\varphi(\xi_0) = 0 = \varphi(\xi_1)$.

Behaviour near the maximum point. The existence of a positive global maximum follows from the analysis performed in the (X, Z)-plane and it corresponds to the value of ξ such that $Z(\xi) = 0$ (we will always assume $\xi = 0$). However, in the proof of the asymptotic behaviour Theorem 2.1, we will need more information on the CS-TWs near their maximum point when $0 < m < 1$ and $p > 2$ (see relation (1.21)). So, with this choice of parameters, we differentiate with respect to the variable ξ the first equation in (1.7):

$$\frac{d^2X}{d\xi^2} = \frac{1}{m}\Big[\Big(1 - \frac{\gamma}{p-1}\Big)X^{-\gamma/(p-1)}\frac{dX}{d\xi}Z + X^{1-\gamma/(p-1)}\frac{dZ}{d\xi}\Big]. \tag{1.20}$$

Then, substituting the expressions in (1.7) and taking Z small, it is not difficult to see that

$$\frac{d^2X}{d\xi^2} \sim -\frac{X^{-\frac{\gamma}{p-1}}f(X)}{m(p-1)}|Z|^{2-p}, \quad \text{for } Z \sim 0.$$

In particular, for all $0 < X < 1$ fixed, it is straightforward to deduce the relation we are interested in:

$$\left|\frac{dX}{d\xi}\right|^{p-2}\left|\frac{d^2X}{d\xi^2}\right| \sim \frac{m^{2-p}}{m(p-1)}X^{p-2-\gamma}f(X) > 0, \quad \text{for } Z \sim 0, \tag{1.21}$$

near the maximum point of $\varphi = \varphi(\xi)$.

Behaviour near the change sign points. For what concerns the profile's behaviour near zero, we can proceed formally considering the equation of the trajectories (1.9) and observe that if $X \sim 0$ and $Z \sim \infty$ we have $cZ - Z^p - f_{m,p}(X) \sim -Z^p$ and so

$$\frac{dZ}{dX} \sim -\frac{1}{(p-1)}\frac{Z}{X} \qquad \text{i.e.} \qquad Z \sim a'X^{-\frac{1}{p-1}}, \quad \text{for } X \sim 0$$

for some $a' > 0$. Now, from the first differential equation in (1.7), we get

$$\frac{dX}{d\xi} \sim aX^{1-m}, \quad \text{for } X \sim 0 \tag{1.22}$$

where we set $a = a'/m$ and integrating it between $X = 0$ and $X = \varphi$ by separation of variables, we get that the profile $X = \varphi$ satisfies

$$\varphi(\xi) \sim ma(\xi - \xi_0)^{1/m} \quad \text{for } \xi \sim \xi_0 \tag{1.23}$$

which not only explains us that the CS-TWs get to the level zero in finite time, but also tells us that they cannot be weak solutions of problem (1) since they violate the Darcy law: $\varphi(\xi) \sim (\xi - \xi_0)^{\gamma/(p-1)}$ near the *free boundary* (see [94], or [198] for the Porous Medium equation). For this reason they cannot describe the asymptotic behaviour of more general solutions, but they turn out to be really useful when employed as "barriers" from below in the PDEs analysis. We ask the reader to note that, since the same procedure works when $X \sim 0$ and $Z \sim -\infty$, we obtain the existence of the second point $0 < \xi_1 < \infty$ such that $\varphi(\xi_1) = 0$ and, furthermore, the local analysis near ξ_1 is similar.
Finally, using that $Z \sim aX^{-1/(p-1)}$ for $X \sim 0$, we can again obtain information on the behaviour of the second derivative near the change sign point $\xi = \xi_0$. In particular, we get the relation

$$X^{2m-1}\frac{d^2X}{d\xi^2} \sim \frac{a^2(p-2-\gamma)}{m^2(p-1)} \quad \text{for } X \sim 0 \text{ and } Z \sim \infty, \tag{1.24}$$

near the "free boundary point" of $\varphi = \varphi(\xi)$. We will need this estimate in the proof of Theorem 2.1 in the case $0 < m < 1$ and $p > 2$.

Detailed derivation of relation (1.22). Before, we deduced relation (1.22) in a too formal way and we decided to end this section with a more complete proof. Let $Z = Z(X)$ be a branch of the trajectory of the CS-TW of type 2 and suppose $Z \geq 0$ (the case $Z \leq 0$ is similar).

We start proving that $Z(X) \geq aX^{-1/(p-1)}$ for $X \sim 0$ and some $a > 0$. Since, $f_{m,p}(X) \geq 0$ we have

$$\frac{dZ}{dX} \leq \frac{cZ - Z^p}{(p-1)XZ^{p-1}}$$

which can be integrated by variable separation and gives $Z^{p-1}(X) \geq -X_0(c - Z_0)^{p-1}X^{-1}$, for all $0 < X \leq X_0 < 1$ (X_0 is the initial condition and $Z_0 = Z(X_0)$). Hence, taking $X_0 \sim 0$ and, consequently, $Z_0 \sim \infty$, we have that $-X_0(c - Z_0)^{p-1} \sim X_0 Z_0^{p-1}$ and we deduce

$$Z(X) \geq X_0^{\frac{1}{p-1}} Z_0 X^{-\frac{1}{p-1}} \quad \text{for } X \sim 0.$$

Now, we show $Z(X) \leq aX^{-1/(p-1)}$ for $X \sim 0$. Using the fact that $f_{m,p}(X) \leq F_{m,p}$ and $Z \geq 0$, we get the differential inequality

$$\frac{dZ}{dX} \geq -\frac{Z^p + F_{m,p}}{(p-1)XZ^{p-1}}.$$

Proceeding as before, it is straightforward to deduce $Z^p \leq X_0^{p/(p-1)}(Z_0 + F)X^{-p/(p-1)}$, where X_0 and Z_0 are taken as before. Thus, since $X_0^{p/(p-1)}(Z_0 + F) \sim X_0^{p/(p-1)}Z_0$ for $X_0 \sim 0$, we obtain

$$Z(X) \leq X_0^{\frac{1}{p-1}} Z_0 X^{-\frac{1}{p-1}} \quad \text{for } X \sim 0,$$

which allows us to conclude $Z(X) \sim aX^{-\frac{1}{p-1}}$ for $X \sim 0$ and $a = X_0^{\frac{1}{p-1}} Z_0$ and, consequently, (1.22).

1.2.2 Reactions of type C. Analysis of some special trajectories

For what concerns the case of reactions of type C, it easily follows from the phase plane analysis that the family of CS-TWs of type 2 exist for all $0 \leq c < c_*$ also in this different framework and moreover, these special solutions have the same properties of the previous setting. In particular, formulas (1.21), (1.22), and (1.24) hold true. In this case, the important difference is that the maximum of the profile (that we can assume is attained at $\xi = 0$) is always greater than $0 < a < 1$ (cfr. with *Step0* and *Step2* of the proof of Part (ii)). More precisely, we can have

$$\varphi(0) = a + \delta,$$

where $\underline{\delta}_c \leq \delta < 1 - a$, and $\underline{\delta}_c > 0$ depends on $0 \leq c < c_*$ and satisfies $\underline{\delta}_c \to 1 - a$ as $c \to c_*$, while $\underline{\delta}_c \to \underline{\delta}_0$, as $c \to 0$, for some $0 < \underline{\delta}_0 < 1 - a$, thanks to the monotonicity of the trajectory $T_c = T_c(X)$ w.r.t $c \geq 0$.

In study of the so called "threshold" results for problem (1), we will employed other two important families of TWs, found in the ODEs analysis. The first one is composed by TW profiles $\varphi(\xi) = \varphi(x - ct)$ with the following properties:

$$\varphi(0) = 1 - \varepsilon, \qquad \varphi(\xi_0) = 0, \qquad \varphi(\xi_1) = a, \qquad \varphi(\xi) > a, \quad \text{for all } \; 0 < \xi < \xi_1,$$

for some $\xi_0 < 0$, $\xi_1 > 0$, $\varepsilon > 0$ small and $c \geq c_*$. The property $\varphi(\xi_0) = 0$ is obtained exploiting again the fact that $Z(X) \sim \pm X^{-\frac{1}{p-1}}$, for $X \sim 0$, $Z \sim \pm\infty$, in the (X, Z)-plane, while the others come from the analysis of the null isoclines (cfr. with Figure 1.6). We will call them "0-to-a" TW. As always, for any $c \geq c_*$, we can consider the reflections $\psi(\xi) = \psi(x + ct)$ satisfying

$$\psi(0) = 1 - \varepsilon, \qquad \psi(\xi_0) = a, \qquad \psi(\xi_1) = 0, \qquad \psi(\xi) > a, \quad \text{for all } \; \xi_0 < \xi < 0,$$

for some $\xi_0 < 0$, $\xi_1 > 0$, $\varepsilon > 0$ small, to which will refer as "a-to-0" TW.

1.2.3 Reactions of type C′. Analysis of some special trajectories

Passing to the reactions of type C′ (satisfying (4)), we will consider TW profiles $\varphi(\xi) = \varphi(x - ct)$ with the following properties:

$$\varphi(-\infty) = a, \qquad \varphi(\xi_0) = 1, \qquad \varphi'(\xi) > 0 \quad \text{for all } \xi \le \xi_0, \tag{1.25}$$

where $0 < a < 1$ with $f(a) = 0$, $\xi_0 \in \mathbb{R}$ is suitably chosen, and $c > 0$. The existence of these TW profiles follows from the analysis in the (X, Z)-plane (see part (i) and (iii) of Theorem 1.1). Indeed, the study of the null isoclines and local behaviour of the critical point $A(a, 0)$ show the existence of two trajectories "coming from" $A(a, 0)$ (it is an hyperbolic type critical point) and crossing the line $X = 1$ in the (X, Z)-plane. The first one, lying in the strip $[a, 1] \times (-\infty, 0]$ satisfies (1.25). The second one, lying in $[a, 1] \times [0, +\infty)$, has symmetric properties but less significative for our purposes. We will call "increasing a-to-1" TWs the profiles satisfying (1.25). These special solutions and their reflections will be used to prove that solutions to problem (1) converge to the steady state $u = a$ as $t \to +\infty$.

Finally, we point out that there CS-TWs of type 2 even in this setting, but now they satisfies

$$\varphi(0) = \delta,$$

where $0 < \underline{\delta}_0 \le \delta < 1$, $c < c_*$ and suitable $\xi_0 < \xi_1$, and $\underline{\delta}_0 > 0$. Their existence follows by analysis in the (X, Z)-plane or, as always, recalling the scaling property that links problem (1) with reaction of type C′ to the same problem with reaction of Fisher-KPP type to the one with reaction of type C′.

1.3 Proof of Theorem 1.2

As we have done for the "slow" diffusion range $\gamma > 0$, we fix $N = 1$, and $m > 0$ and $p > 1$ such that $\gamma = 0$ and we look for *admissible/a-admissible* TWs to equation (1.1), substituting $u(x, t) = \varphi(\xi)$, $\xi = x - ct$ into (1.1) and obtaining the equation of the profile (1.4):

$$[|(\varphi^m)'|^{p-2}(\varphi^m)']' + c\varphi' + f(\varphi) = 0 \quad \text{in } \mathbb{R},$$

where, as always, the notation φ' indicates the derivative of φ with respect to the variable ξ.

Now, for $\gamma = 0$, the change of variables (1.6):

$$X = \varphi \qquad \text{and} \qquad Z = -\left(\frac{m(p-1)}{\gamma}\varphi^{\frac{\gamma}{p-1}}\right)' = -mX^{\frac{\gamma}{p-1}-1}X',$$

becomes

$$X = \varphi \qquad \text{and} \qquad Z = -m(\log X)' = -mX^{-1}X'. \tag{1.26}$$

Hence, we get the system

$$-m\frac{dX}{d\xi} = XZ, \qquad -|Z|^{p-2}\frac{dZ}{d\xi} = cZ - |Z|^p - mX^{-1}f(X), \tag{1.27}$$

which can be re-written (after the re-parametrization $d\xi = |Z|^{p-2}d\tau$) as the non-singular system

$$\frac{dX}{d\tau} = (p-1)X|Z|^{p-2}Z, \qquad \frac{dZ}{d\tau} = cZ - |Z|^p - F(X), \tag{1.28}$$

where we set

$$F(X) := mX^{-1}f(X), \qquad X \in [0, 1],$$

which the "pseudo-linear" version of $f_{m,p}(\cdot)$ introduced in the range $\gamma > 0$. Also in this case we will give different proofs, depending on the reaction term $f(\cdot)$ and starting from the Fisher-KPP framework.

Proof of Theorem 1.2: Part (i). So, let $f(\cdot)$ be a Fisher-KPP reaction satisfying (2):

$$\begin{cases} f(0) = f(1) = 0, & 0 < f(u) \le f'(0)u \ \text{ in } (0,1) \\ f \in C^1([0,1]), & f'(0) > 0,\ f'(1) < 0 \\ f(\cdot) \text{ has a unique critical point in } (0,1), \end{cases}$$

Note that $F(\cdot)$ is well-defined and continuous in $[0,1]$. Moreover, $F(0) = mf'(0)$, $F(1) = 0$, $F(X) \ge 0$ and $F'(X) \le 0$ for all $0 \le X \le 1$.

System (1.28) possesses the critical point $S = (1,0)$ for all $c \ge 0$. Now, define

$$c_{0*}(m,p) := p(m^2 f'(0))^{\frac{1}{mp}},$$

as in (1.2). Then it is not difficult to prove:
• If $c < c_{0*}(m,p)$, then there are no other critical points for system (1.28).
• If $c = c_{0*}(m,p)$, then system (1.28) has another critical point $R_{\lambda_*} := (0, \lambda_*)$, where we define for simplicity

$$\lambda_* := (c_{0*}(m,p)/p)^m = (m^2 f'(0))^{1/p},$$

according to the definition of $c_{0*}(m,p)$.
• If $c > c_{0*}$, then system (1.28) has also two critical points $R_{\lambda_i} = (0, \lambda_i)$, $i = 1,2$ where $0 < \lambda_1 < \lambda_* < \lambda_2 < c^m$.
This follows from the fact that if $X = 0$ then $dZ/d\tau = 0$ if and only if $cZ - |Z|^p - mf'(0) = 0$ and the number of solutions of this equation depends on the parameter $c > 0$ following our classification. Equivalently, one can write the equation of the trajectories

$$\frac{dZ}{dX} = \frac{cZ - |Z|^p - F(X)}{(p-1)X|Z|^{p-2}Z} \tag{1.29}$$

and study the null isoclines imposing (exactly as we did at the beginning of the proof of Theorem 1.1):

$$\max_{\widetilde{Z} \in [0, c^m]} \{c\widetilde{Z} - |\widetilde{Z}|^p\} = F(0).$$

Solving the previous equation, we get the same value for $c_{0*}(m,p)$. In particular, for $c = c_{0*}(m,p)$ we have $c_{0*}\lambda_* - \lambda_*^p - mf'(0) = 0$ and $c_{0*} - p\lambda_*^{p-1} = 0$. When $c > c_{0*}(m,p)$, we have $c\lambda_i - \lambda_i^p - mf'(0) = 0$, for $i = 1,2$, while $c - p\lambda_1^{p-1} > 0$ and $c - p\lambda_2^{p-1} < 0$.

Step1: Study of the null isoclines. We can classify the null isoclines according to the ranges of the parameter $c > 0$: remembering the properties of the function $F(\cdot)$, it is not difficult to show that for the value $c = c_{0*}(m,p)$, we have a continuous isocline curve recalling a "horizontal parabola" with vertex in R_{λ_*} and linking this point with S and $(1, c_{0*}^m)$. Moreover, the trajectories are increasing in the area between the isocline and the line $X = 1$ whilst decreasing in the remaining part of the rectangle $[0,1] \times [0, c_{0*}^m]$. If $c < c_{0*}(m,p)$, the branch of the null isocline is again a "horizontal parabola", but, in this case, it does not have intersections with the axis $X = 0$. So, the trajectories have negative slope at the left of this curve while positive at the right. Finally, in the case $c > c_{0*}(m,p)$, the isoclines are composed by two branches: one in lower position linking the points R_{λ_1} and S and one in higher position linking R_{λ_2} and $(1, c^m)$. In the area between these two branches the trajectories are increasing while in the rest of the rectangle $[0,1] \times [0, c^m]$ are decreasing. Note that for all $c \ge 0$, the slopes of the trajectories are negative when $Z \ge c^m$ while positive $Z \le 0$.

Step2: Local analysis of $S(1,0)$. It is no difficult to prove the existence and the uniqueness of a trajectory "coming into" the critical point S. This trajectory satisfies the same properties of the

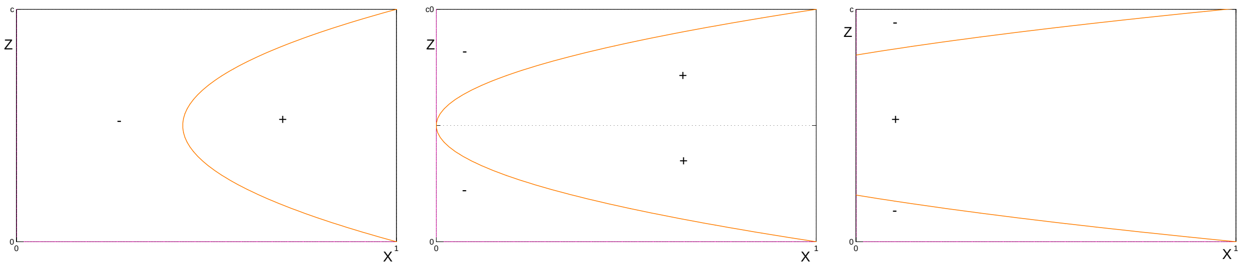

Figure 1.9: Fisher-KPP reactions, range $\gamma = 0$. Null isoclines in the (X, Z)-plane for $f(u) = u(1-u)$, in the cases $c < c_{0*}$, $c = c_{0*}$ and $c > c_{0*}$, respectively.

case $\gamma > 0$ (the fact that the same techniques used when $\gamma > 0$ apply here is not surprising since $f_{m,p}(X) \sim F(X) \sim mf(X)$ as $X \sim 1$).

Step3: Existence and uniqueness of a critical speed $c = c_{0*}$. In this case, it suffices to observe that the trajectory "coming into" S has to cross the axis $X = 0$ in a point $(0, \overline{Z})$ with $0 < \overline{Z} \le \lambda_*$. Nevertheless, proceeding as in *Step5* and *Step6* of the proof of Theorem 1.1 (Part (i)) it is simple to see that the only possibility is $\overline{Z} = \lambda_*$. Hence we have proved the existence of a connection $R_{\lambda_*} \leftrightarrow S$. To show that this admissible TW is positive we consider the first differential equation in (1.27) and we integrate between $X = 0$ and $X = 1$ the equivalent differential relation $d\xi = (XZ)^{-1}dX$. It is straightforward to see that the correspondent integral is divergent both near $X = 0$ and $X = 1$, which means that the TW is positive.

Now we would like to describe the exact shape of this TW (with speed $c_{0*}(m,p) := p(m^2 f'(0))^{1/(mp)}$) as $\xi \sim +\infty$. In *Step5* of Theorem 1.1, in the case $\gamma > 0$, we have given an analytic representation of the (finite) TWs corresponding to the value $c = c_*(m,p)$ nea its free boundary point (see (1.14)). This has been possible since we have been able to describe the asymptotic behaviour of the trajectories in the (X, Z)-plane near $X = 0$. The case $\gamma = 0$ is more complicated and we devote the entire Section 1.3.1 to the detailed analysis. Here, we simply report the asymptotic behaviour of our TW $\varphi = \varphi(\xi)$ with critical speed $c_{0*}(m,p)$:

$$\varphi(\xi) \sim a_0 |\xi|^{\frac{2}{p}} e^{-\frac{\lambda_*}{m}\xi} = a_0 |\xi|^{\frac{2}{p}} \exp\left(-m^{\frac{2-p}{p}} f'(0)^{\frac{1}{p}} \xi\right) \quad \text{for } \xi \sim +\infty, \tag{1.30}$$

where $\lambda_* := (c_{0*}(m,p)/p)^m = (m^2 f'(0))^{1/p}$ and $a_0 > 0$.

We conclude this paragraph with a brief description of the remaining trajectories. Below the positive TW, we have a family of trajectories linking R_{λ_*} with $R_{-\infty} = (0, -\infty)$ while above it, there are a family of "parabolas" (exactly as in the case $\gamma > 0$). Between these two families, there are trajectories from the point R_{λ_*} which cross the line $X = 1$.

Step4: Non existence of admissible TWs for $c < c_{0*}$. In this case there are no admissible TWs. Indeed, using again the same methods of *Step5* and *Step6* (see the proof of Theorem 1.1 (Part (i))) it is simple to show that the orbit "coming into" S cannot touch the axis $X = 0$, i.e., it links $R_\infty = (0, \infty)$ and S and so it is not an admissible TW. Below this trajectory there are a family of CS-TWs of type 2 and above it we have a family of "parabolas". We stress that, with the same techniques used for the case $\gamma > 0$, the CS-TWs of type 2 satisfy $Z(X) \sim a' X^{-1/(p-1)} = a' X^{-m}$ for $X \sim 0$ and a suited positive constant a'. Hence, exactly as we did in the proof of Theorem 1.1 (Part (i)), it is possible to show that the profile $X = \varphi$ reaches the level zero in two points $-\infty < \xi_0 < \xi_1 < \infty$. Indeed, exactly as we did before, using the first differential equation in (1.27), we get the estimate

$$\frac{dX}{d\xi} \sim aX^{1-m} \quad \text{for } X \sim 0,$$

from which we can estimate the (finite) times ξ_0 and ξ_1 (note that $a = a'/m$).

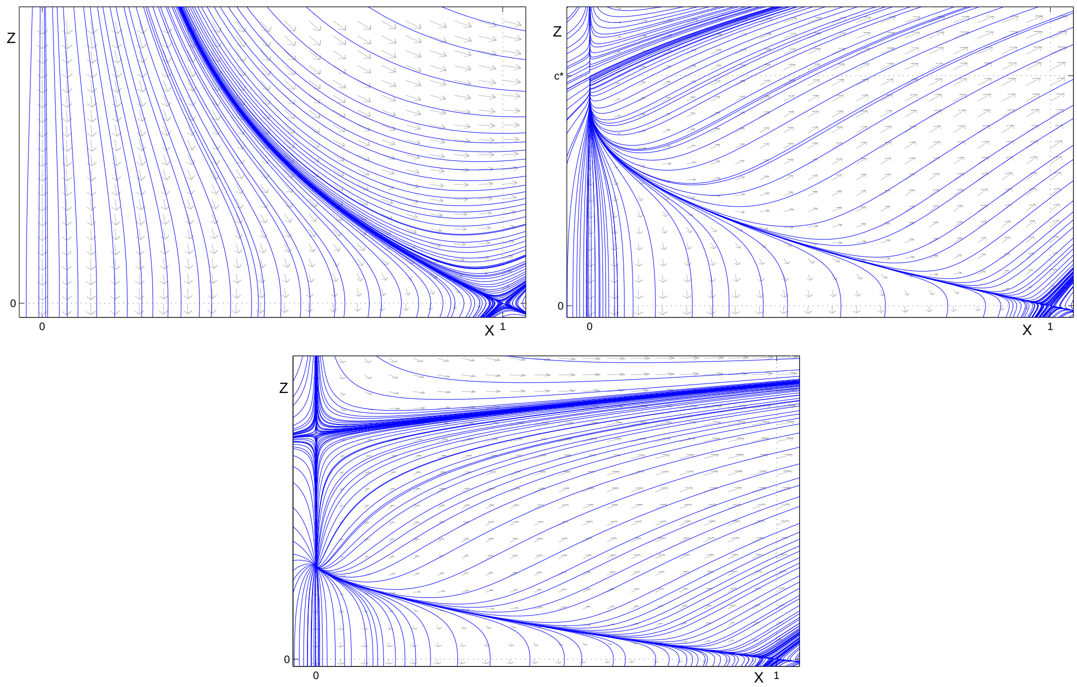

Figure 1.10: Fisher-KPP reactions, range $\gamma = 0$. Qualitative behaviour of the trajectories in the (X, Z)-plane for $f(u) = u(1-u)$, in the cases $0 < c < c_{0*}$, $c = c_{0*}$ and $c > c_{0*}$, respectively.

Step4: Existence of admissible TWs for $c > c_{0*}$. To show the existence of a connection $R_{\lambda_1} \leftrightarrow S$ and to prove that this TW is positive, it is sufficient to apply the methods used in the case $c = c_{0*}(m,p)$. We recall that for all $c > c_{0*}(m,p)$, $\lambda_1 = \lambda_1(c) < \lambda_*$, it solves $c\lambda_1 - \lambda_1^p - mf'(0) = 0$, and $c - p\lambda_1^{p-1} > 0$.
Now, by using the Lyapunov method, it is possible to linearize system (1.28) around the critical point $R_{\lambda_1} = (0, \lambda_1)$, and it is straightforward to see that its Jacobian matrix (calculated in $(0, \lambda_1)$) has two positive eigenvalues $\nu_1 = (p-1)\lambda_1^{p-1}$ and $\nu_2 = c - p\lambda_1^{p-1}$. This means that R_{λ_1} is a *node unstable*. Hence, we deduce that the trajectory $Z = Z(X)$ from $R_{\lambda_1} = (0, \lambda_1)$ satisfies $Z(X) \sim \lambda_1 - \nu_1 X$, for $X \sim 0$ and some $\nu_1 > 0$. Thus, we can re-write the first equation in (1.27) as

$$\frac{dX}{d\xi} \sim (1/m)X(\lambda_1 - \nu_1 X) \quad \text{for } X \sim 0,$$

which is a first order *logistic* type ODE, and so we easily obtain that the profile $X(\xi) = \varphi(x - ct)$ has the exponential decay

$$X(\xi) \sim a_0 e^{-\frac{\lambda_1}{m}\xi}, \quad \text{for } \xi \sim +\infty, \tag{1.31}$$

where $a_0 > 0$ is a fixed constant.
Below this connection we have a family of trajectories joining R_{λ_1} and $R_{-\infty}$. Above it, there are trajectories from R_{λ_2} and crossing the line $X = 1$, one trajectory from R_{λ_2} crossing the line $X = 1$ and, finally, a family of "parabolas" as in the other cases. See Figure 1.10 for a qualitative representation. □

Proof of Theorem 1.2: Part (ii). Let $f(\cdot)$ be a reaction term of type C satisfying (3):

$$\begin{cases} f(0) = f(a) = f(1) = 0, & f(u) < 0 \text{ in } (0,a), \;\; f(u) > 0 \text{ in } (a,1) \\ f \in C^1([0,1]), & f'(0) < 0, \;\; f'(a) > 0, \;\; f'(1) < 0 \\ \int_0^1 u^{m-1} f(u)du > 0 \\ f(\cdot) \text{ has a unique critical point in } (0,a) \text{ and a unique critical point in } (a,1). \end{cases}$$

As before, we consider system (1.28) and the *equation of the trajectories* (1.29):

$$\frac{dZ}{dX} = \frac{cZ - |Z|^p - F(X)}{(p-1)X|Z|^{p-2}Z} := H(X, Z; c),$$

where now the function $F(X) := mX^{-1}f(X)$ satisfies $F(0) = mf'(0) < 0$, $F(a) = F(1) = 0$, with $F(X) < 0$ and $F'(X) > 0$ in $(0, a)$. The critical points are now four for any $c > 0$:

$$S(1,0), \qquad A(a,0), \qquad R_{\lambda_1}(0,\lambda_1), \qquad \text{and} \qquad R_{\lambda_2}(0,\lambda_2),$$

where $\lambda_1 = \lambda_1(c) < 0 < c^m < \lambda_2 = \lambda_2(c)$ are the solutions to the equation

$$cZ - |Z|^p = F(0), \qquad c > 0.$$

In the next paragraph we look for trajectories in the strip $[0,1] \times [0,\infty)$ connecting $S(1,0) \leftrightsquigarrow R_{\lambda_2}(0,\lambda_2)$ for a specific speed of propagation $c = c_{0*}(m,p)$.

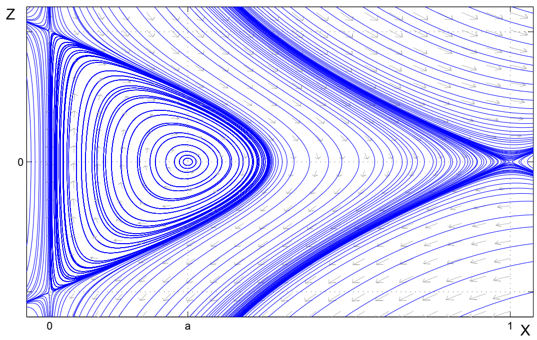

Figure 1.11: Reactions of type C, range $\gamma = 0$, case $c = 0$. Qualitative behaviour of the trajectories in the (X, Z)-plane for $f(u) = u(1-u)(u-a)$, $a = 0.3$.

Step0: Case $c = 0$. If $c = 0$, the *equation of the trajectories* reads

$$\frac{dZ}{dX} = -\frac{|Z|^p + F(X)}{(p-1)X|Z|^{p-2}Z} := H(X, Z; 0).$$

The *null isoclines* are composed by two branches, the upper one linking $R_{\lambda_2}(0,\lambda_2)$ and $A(a,0)$, and the lower one joining $R_{\lambda_1}(0,\lambda_1)$ and $A(a,0)$, where in this easier case

$$\lambda_1 = \lambda_1(0) = -\sqrt[p]{-mf'(0)}, \qquad \lambda_2 = \lambda_2(0) = \sqrt[p]{-mf'(0)}.$$

Employing the Lyapunov linearization method, it is not difficult to prove that $R_{\lambda_1}(\lambda_1, 0)$ and $R_{\lambda_2}(\lambda_2, 0)$ are two *saddle* points. So, there are exactly two trajectories $T^- = T^-(X)$ and $T_+ = T_+(X)$ "coming from" $R_{\lambda_1}(\lambda_1, 0)$ and $R_{\lambda_2}(\lambda_2, 0)$, respectively, lying in the strip $[0,1] \times (-\infty,\infty)$ in the (X, Z)-plane. Moreover, since

$$H(X, -Z; 0) = -H(X, Z; 0), \qquad \text{for all } 0 \le X \le 1,\ Z \in \mathbb{R},$$

we deduce that $T^- \equiv T^+$. At the same time, exactly as in the case $\gamma > 0$ we have a trajectory $T_0 = T_0(X) > 0$ "entering" in $S(1,0)$ (see *Step1* of the proof of Theorem 1.1, Part (ii)). Assuming (1.17), i.e., $\int_0^1 u^{m-1}f(u)\,du > 0$, it follows that $T_+(X) < T_0(X)$ for all $0 \le X \le 1$, with $T_0(X) \sim +\infty$ for $X \sim 0$. This follows by using the same technique of the case $\gamma > 0$. In particular, it is simple to see that the same construction works if we take $\gamma = 0$ and formula (1.18) holds. Consequently, there are not *admissible* TWs for $c = 0$.

Step1: Local analysis of $S(1,0)$. This step coincides with *Step1* of Part (i), since the nature of the critical point $S(1,0)$ does not change if we take $\gamma = 0$. This can be easily seen noting that $F(X) \sim f_{m,p}(X) \sim -mf'(1)(1-X)$ for $X \sim 1$.

Step2: Study of the null isoclines. We proceed as before by studying the solutions of the equation

$$c\widetilde{Z} - |\widetilde{Z}|^p = mX^{-1}f(X), \qquad \text{in } [0,1]\times(-\infty,\infty).$$

As before, we find that there exists $c_0 > 0$ such that for $0 < c < c_0$ the *null isoclines* are composed by two branches: the left one, linking the points $R_{\lambda_1}(0,\lambda_1)$, (a,c^m), $(a,0)$ and $R_{\lambda_2}(0,\lambda_2)$, whilst the second linking $(1,c^m)$ and $S(1,0)$. The two branches approach as $c \to c_0$ until they touch for $c = c_0$. Finally, for $c > c_0$, we again two branches: the upper one, linking $R_{\lambda_1}(0,\lambda_1)$, (a,c^m) and $(1,c^m)$, while the lower one joining $R_{\lambda_2}(0,\lambda_2)$, $(a,0)$ and $(1,0)$.

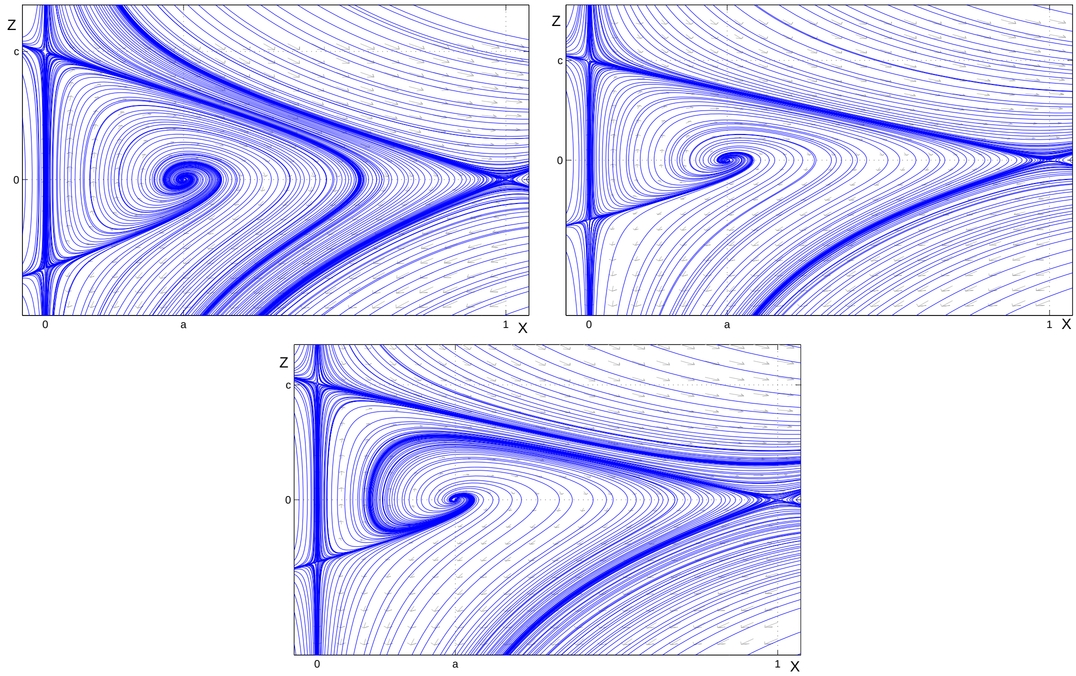

Figure 1.12: Reactions of type C, range $\gamma = 0$. Qualitative behaviour of the trajectories in the (X,Z)-plane for $f(u) = u(1-u)(u-a)$, $a = 0.3$. The first picture show the case $0 < c < c_{0*}$, while the others the cases $c = c_{0*}$ and $c > c_{0*}$, respectively.

Step3: Existence and uniqueness of a critical speed $c = c_{0*}$. The existence of a unique critical speed $c_{0*} = c_{0*}(m,p)$ with corresponding trajectory linking $S(1,0)$ and $R_{\lambda_2}(\lambda_2(c_*),0)$ follows exactly as in the case $\gamma > 0$, see Theorem 1.1, Part (ii). The unique (important) difference is the fact that the the TW is positive everywhere. Indeed, integrating the first equation in (1.27) along the trajectory $T_{c_{0*}} = T_{c_{0*}}(X) \sim \lambda_2(c_{0*})$ for $X \sim 0$, we obtain

$$\xi_0 - \xi_1 = m\int_{X_0}^{X_1} \frac{1}{XT_{c_{0*}}(X)}dX \sim m\lambda_2(c_{0*})\int_{X_0}^{X_1} X^{-1}dX \qquad \text{for } X_0 \sim 0,$$

from which we deduce $\varphi(+\infty) = X(+\infty) = 0$, i.e., the TW profile $X(\xi) = \varphi(\xi)$ reaches the level $u = 0$ in infinite time.

Step4: Non existence of admissible TWs for $c > c_{0*}$. Proving the non existence of *admissible* TW profiles is easier than the case $\gamma > 0$, since from the study of the critical points and the *null isoclines* it follows that there cannot exist nonnegative trajectories linking $S(1,0)$ and $O(0,0)$. The qualitative behaviour of the trajectories in the (X,Z)-plane is reported in Figure 1.12. □

Proof of Theorem 1.2: Part (iii). Finally we consider reaction terms $f(\cdot)$ of type C', i.e., satisfying (4):

$$\begin{cases} f(0) = f(a) = f(1) = 0, \quad 0 < f(u) \le f'(0)u \ \text{ in } (0,a), \ f(u) < 0 \text{ in } (a,1) \\ f \in C^1([0,1]), \qquad\qquad f'(0) > 0, \ f'(a) < 0, \ f'(1) > 0 \\ f(\cdot) \text{ has a unique critical point in } (0,a) \text{ and a unique critical point in } (a,1). \end{cases}$$

As always, we consider system (1.28) and the *equation of the trajectories* (1.29):

$$\frac{dZ}{dX} = \frac{cZ - |Z|^p - F(X)}{(p-1)X|Z|^{p-2}Z} := H(X, Z; c),$$

where $F(X) := mX^{-1}f(X)$, but this time it satisfies $F(0) = mf'(0) > 0$, $F(a) = F(1) = 0$, with $F(X) > 0$ and $F'(X) < 0$ in $(0,a)$. Note that for all $c > 0$, system (1.28) has the two critical points

$$S(1,0) \qquad \text{and} \qquad A(a,0).$$

Moreover, exactly as in the proof of Part (i), if

$$c_{0*}(m,p) := p(m^2 f'(0))^{\frac{1}{mp}},$$

as in (1.2), then there are no other critical points for system (1.28), when $c < c_{0*}(m,p)$, there is exactly one more critical point $R_{\lambda_*} := (0, \lambda_*)$, where

$$\lambda_* := (c_{0*}(m,p)/p)^m = (m^2 f'(0))^{1/p},$$

if f $c = c_{0*}(m,p)$, whilst if $c > c_{0*}$, system (1.28) has two more critical points $R_{\lambda_i} = (0, \lambda_i)$, $i = 1,2$ where $0 < \lambda_1 < \lambda_* < \lambda_2 < c^m$. Recall that λ_i, $i = 1,2$ are the solutions to $cZ - |Z|^p = F(0)$.

Step1: Local analysis of $A(a,0)$ *and* $S(1,0)$. This step coincides with *Step1* of 1.1, Part (iii).

Step2: Study of the null isoclines. As always, we study the solutions of the equation

$$c\widetilde{Z} - |\widetilde{Z}|^p = F(X), \qquad c > 0,$$

finding that for $0 < c < c_*$, the *null isoclines* are composed by a unique branch linking the points $(1, c^m)$, (a, c^m), $A(a,0)$, and $S(1,0)$, while when $c = c_{0*}$ the branch crosses the Z-axis at the point $R_{\lambda_*}(0, \lambda_*)$. Finally, for $c > c_{0*}$, there are two branches: the upper one, linking $R_{\lambda_2}(0, \lambda_2)$, (a, c^m) and $(1, c^m)$, whilst the lower one joining $R_{\lambda_1}(0, \lambda_1)$, $A(a,0)$ and $S(1,0)$.

Step3: Existence and uniqueness of a critical speed $c = c_{0*}$. In this step, we have to prove the existence of a trajectory $T_{c_{0*}}$ linking $A(a,0)$ with $R_{\lambda_*}(0, \lambda_*)$, corresponding to an *a-admissible positive* TW profile. This easily follows remembering the scaling property we explained before and substituting $S(1,0)$ with $A(a,0)$ in the proof of Part (i).

We stress that even in this case the "critical" TW has the following asymptotic behaviour (cfr. with (1.30)):

$$\varphi(\xi) \sim a_0|\xi|^{\frac{2}{p}} e^{-\frac{\lambda_*}{m}\xi} = a_0|\xi|^{\frac{2}{p}} \exp\Big(- m^{\frac{2-p}{p}} f'(0)^{\frac{1}{p}} \xi \Big) \quad \text{for } \ \xi \sim +\infty,$$

where as before $\lambda_* := (c_*/p)^m$ and $a_0 > 0$ is a suitable constant.

Step4: The cases $0 < c < c_{0*}$ *and* $c > c_{0*}$. If $0 < c < c_{0*}$, there are not *a-admissible* TW. The proof of this fact easily follows from the study of the *null isoclines* and from the non existence of critical points on the Z-axis.

To the other hand, at each $c > c_*$, it corresponds exactly one *a-admissible* TW and it is *positive*. This is proved by showing the existence of a trajectory T_c linking $A(a,0)$ and $R_{\lambda_1}(0, \lambda_1)$ corresponding to an *a-admissible positive* TW profile. Again we refer to the proof of Part (i) for any technical detail. See Figure 1.13 for a qualitative representation of the trajectories in the (X,Z)-plane. □

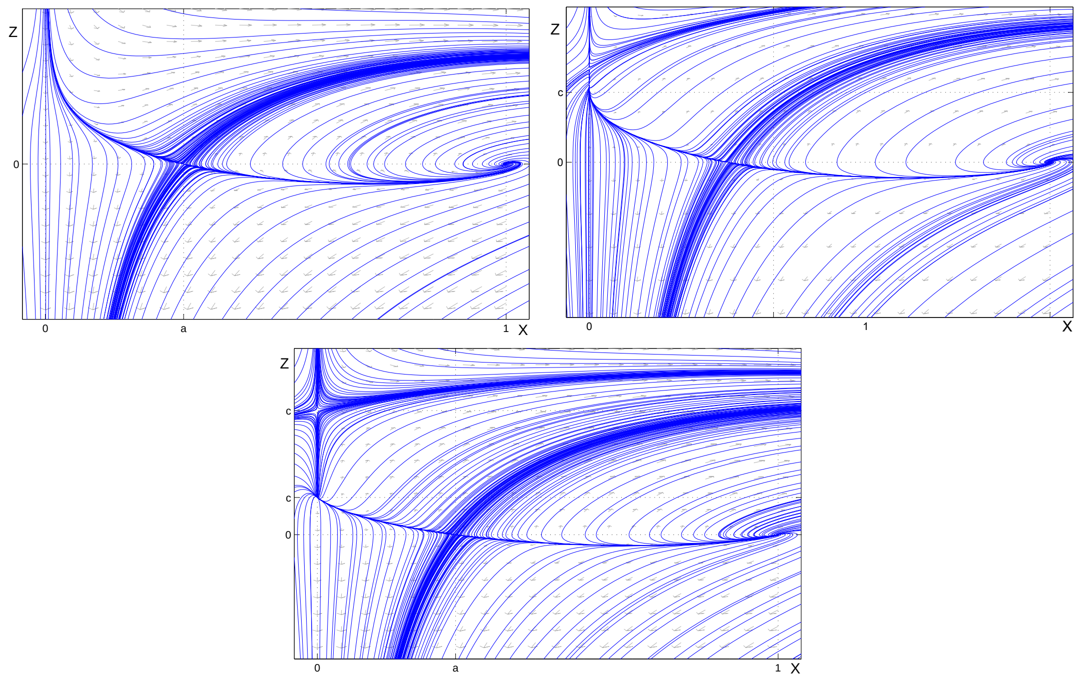

Figure 1.13: Reactions of type C′, range $\gamma = 0$. Qualitative behaviour of the trajectories in the (X, Z)-plane for $f(u) = u(1-u)(a-u)$, $a = 0.3$, in the ranges $0 < c < c_*$, $c = c_*$ and $c > c_*$, respectively.

Important Remarks. First of all, we point out that the existence of "special trajectories" like Change-Sign TWs found in the range $\gamma > 0$ can be easily proved also in the range $\gamma = 0$. In particular, all what we have explained in Subsections 1.2.1, 1.2.2, and 1.2.3 hold true even in the "pseudo-linear" range $\gamma = 0$.

Secondly, we want to comment the asymptotic behaviour of the function $c_{0*} = c_{0*}(m, p)$ (cfr. with (1.2)) for Fisher-KPP reactions and reactions of type C′. Since we assume $m(p-1) = 1$, it is simple to re-write the critical speed as a function of $m > 0$ or $p > 1$:

$$c_{0*}(m) = (1+m)m^{\frac{1-m}{1+m}} f'(0)^{\frac{1}{m+1}} \quad \text{or} \quad c_{0*}(p) = p(p-1)^{-\frac{2(p-1)}{p}} f'(0)^{\frac{p-1}{p}}.$$

Then we have:

$$\lim_{m\to 0} c_{0*}(m) = \lim_{p\to\infty} c_{0*}(p) = 0 \quad \text{and} \quad \lim_{m\to\infty} c_{0*}(m) = \lim_{p\to 1} c_{0*}(p) = 1.$$

These limits allow us to guess if in the cases $m \to 0$, $p \to \infty$ and $m \to \infty$, $p \to 1$ such that $m(p-1) = 1$, there exist TWs or not. It seems natural to conjecture that the answer is negative in the first case, while admissible TWs could exist in the second case.
Finally, it is not difficult to calculate the derivative of $c_{0*}(m)$ (or $c_{0*}(p)$) in the case $f(u) = u(1-u)$:

$$c'_{0*}(m) = (m+1)^{-1} m^{-\frac{2m}{m+1}} (m + 1 - 2m \log m)$$

and conclude that the maximum of the critical speed is assumed for a critical value of $m = m_*$, with $2 < m_* < 3$ (of course, we can repeat this procedure with the function $c_{0*}(p)$ finding a critical value $4/3 < p_* < 3/2$). This simple calculation assures the linear case $m = 1$ and $p = 2$ is not critical for the function $c_{0*}(m, p)$ while the maximal speed of propagation is found choosing $m = m_*$ and/or $p = p_*$.

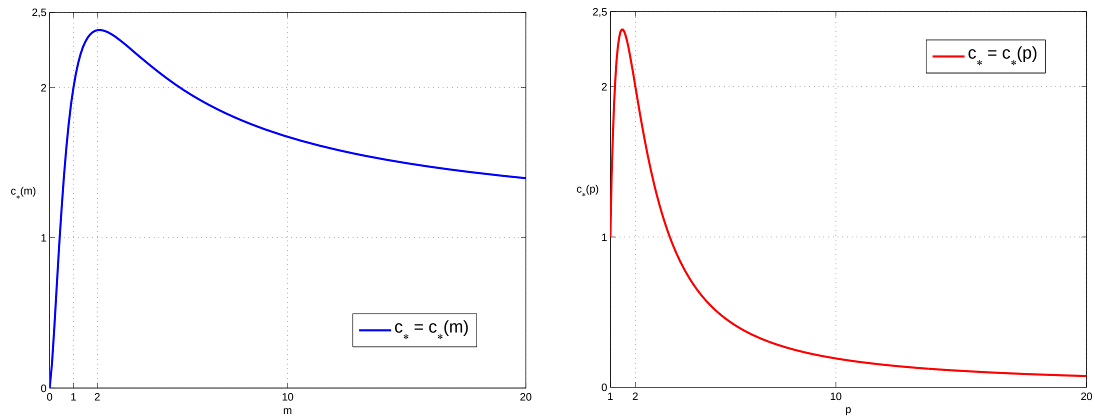

Figure 1.14: Pseudo-linear case: the graphics of $c_{0*} := c_*$ in dependence of $m > 0$ and $p > 1$.

1.3.1 Proof of formula (1.30)

As we explained in the proof of Theorem 1.2, Part (i), studying the asymptotic behaviour of the positive TW with critical speed $c_{0*}(m,p) := p(m^2 f'(0))^{1/(mp)}$, when $\gamma = 0$, is more complicated than the case $\gamma > 0$. In what follows, we give the detailed proof of formula (1.30).

Before proceeding with the analysis, let's recall some facts proved above. We have worked in the (X,Z)-plane, where $X = \varphi$ and $Z = -mX^{-1}X'$ are defined in (1.26) and we showed the existence of a trajectory linking $R_{\lambda_*} \leftrightarrow S$, where $R_{\lambda_*} = (\lambda_*, 0)$, $S = (1,0)$, and $\lambda_* := (c_{0*}(m,p)/p)^m = (m^2 f'(0))^{1/p}$. This trajectory corresponds to an admissible positive TW with critical speed $c_{0*}(m,p)$.

We are now ready to begin with the proof of (1.30). Let us suppose $f \in C^2([0,1])$ and let $Z = Z(X)$ be the analytic expression of the trajectory of this TW. We consider the approximation $Z \sim \lambda_* - \zeta_0(X)$ as $X \sim 0$, where $\zeta_0(X) \sim 0$ as $X \sim 0$. Our goal is to compute the remainder $\zeta_0 = \zeta_0(X)$ as $X \sim 0$. Let us consider the equation of the trajectories (1.29). Since

$$
\begin{aligned}
X^{-1} f(X) &\sim f'(0) + \frac{f''(0)}{2} X, \quad \text{for } X \sim 0, \\
(\lambda_* - \zeta_0(X))^p &\sim \lambda_*^p - p\lambda_*^{p-1}\zeta_0(X) + \frac{p(p-1)}{2}\lambda_*^{p-2}\zeta_0(X)^2 \quad \text{for } X \sim 0,
\end{aligned}
\tag{1.32}
$$

we have that equation (1.29) with $c = c_{0*}$ becomes

$$
\frac{d\zeta_0}{dX} \sim -\frac{c_{0*}\lambda_* - \lambda_*^p - mf'(0) - (c_{0*} - p\lambda_*^{p-1})\zeta_0 - \frac{p(p-1)}{2}\lambda_*^{p-2}\zeta_0^2 - m\frac{f''(0)}{2}X}{(p-1)\lambda_*^{p-1}X} \sim \frac{p}{2\lambda_*}\frac{\zeta_0^2}{X} + b_0, \quad \text{for } X \sim 0
$$

where $b_0 := mf''(0)/[2(p-1)\lambda_*^{p-1}] < 0$. We point out that the second approximation holds since both the quantities $c_{0*}\lambda_* - \lambda_*^p - mf'(0)$ and $c_{0*} - p\lambda_*^{p-1}$ are zero. Now, consider the equation

$$
\frac{d\widetilde{\zeta}_0}{dX} = \frac{p}{2\lambda_*}\frac{\widetilde{\zeta}_0^2}{X},
$$

which is obtained by taking $b_0 = 0$ in the previous equation. We have $\widetilde{\zeta}_0(X) \sim -(2\lambda_*/p)\ln^{-1}(X)$ for $X \sim 0$ while, for all small $\varepsilon > 0$, it is simple to see that the functions

$$
\underline{\zeta}_0(X) \sim -\frac{2\lambda_* + \varepsilon}{p}\ln^{-1}(X) \quad \text{and} \quad \overline{\zeta}_0(X) \sim -\frac{2\lambda_* - \varepsilon}{p}\ln^{-1}(X)
$$

are sub-solution and super-solution of the equation $d\zeta_0/dX = \frac{p}{2\lambda_*}\frac{\zeta_0^2}{X} + b_0$ for $X \sim 0$, respectively. Thus, using the arbitrariness of $\varepsilon > 0$ and Comparison Principle, we deduce that

$$\zeta_0(X) \sim \widetilde{\zeta}_0(X) \sim -\frac{2\lambda_*}{p}\ln^{-1}(X), \quad \text{for } X \sim 0.$$

Now, we iterate the previous procedure to compute high order terms. So let $Z \sim \lambda_* - \zeta_1(X)$, for $X \sim 0$, where $\zeta_1(X) = \zeta_0(X) + lh.o.t.$ We suppose for a moment to have $f \in C^3([0,1])$ and, proceeding as in (1.32), we compute the Taylor expansions:

$$X^{-1}f(X) \sim f'(0) + \frac{f''(0)}{2}X + \frac{f'''(0)}{6}X^2, \quad \text{for } X \sim 0,$$
$$(\lambda_* - \zeta_1(X))^p \sim \lambda_*^p - p\lambda_*^{p-1}\zeta_1(X) + \frac{p(p-1)}{2}\lambda_*^{p-2}\zeta_1(X)^2 - \frac{p(p-1)(p-2)}{6}\lambda_*^{p-3}\zeta_1(X)^3 \quad \text{for } X \sim 0,$$

and we substitute in the equation of the trajectories (1.29) to find

$$\frac{d\zeta_1}{dX} \sim \frac{p}{2\lambda_*}\frac{\zeta_1^2}{X} - \frac{p(p-2)}{6\lambda_*^2}\frac{\zeta_1^3}{X} + b_0 - b_1X, \quad \text{for } X \sim 0,$$

where $b_1 := mf'''(0)/[6(p-1)\lambda_*^{p-1}]$. Now, we know the less accurate approximation $\zeta_1(X) \sim \zeta_0(X) \sim -\frac{2\lambda_*}{p}(\ln X)^{-1}$, for $X \sim 0$. So, substituting it in the previous equation we obtain

$$\frac{d\zeta_1}{dX} \sim \frac{2\lambda_*}{p}\frac{1}{X\ln^2(X)} + \frac{4(p-2)\lambda_*}{3p^2}\frac{1}{X\ln^3(X)} + b_0 - b_1X, \quad \text{for } X \sim 0,$$

and once we integrate with respect to the variable X, we get

$$\begin{aligned}\zeta_1(X) &\sim -\frac{2\lambda_*}{p}\frac{1}{\ln(X)} - \frac{2(p-2)\lambda_*}{3p^2}\frac{1}{\ln^2(X)} + b_0X - \frac{b_1}{2}X^2 \\ &= \zeta_0(X) - \frac{2(p-2)\lambda_*}{3p^2}\frac{1}{\ln^2(X)} + b_0X - \frac{b_1}{2}X^2, \quad \text{for } X \sim 0.\end{aligned}$$

Using this information, the differential equation $-mX' = XZ$ in (1.27) becomes

$$-mX' \sim \lambda_*X\left[1 + \frac{2}{p}\frac{1}{\ln(X)} + \frac{2(p-2)}{3p^2}\frac{1}{\ln^2(X)} - \frac{b_0}{\lambda_*}X + \frac{b_1}{2\lambda_*}X^2\right], \quad \text{for } X \sim 0. \tag{1.33}$$

As the reader can easily see, a first approximation of the solution of (1.33) is given by

$$\ln X(\xi) \sim -\frac{\lambda_*}{m}\xi, \quad \text{for } \xi \sim +\infty.$$

Consequently, by substituting the previous expression in the square parenthesis of (1.33) we have

$$\frac{X'}{X} \sim -\frac{\lambda_*}{m} + \frac{2}{p}\frac{1}{\xi} - \frac{2m(p-2)}{3\lambda_*p^2}\frac{1}{\xi^2} - \frac{b_0}{m}e^{-\frac{\lambda_*}{m}\xi} + \frac{b_1}{2m}e^{-\frac{2\lambda_*}{m}\xi}, \quad \text{for } \xi \sim +\infty,$$

which, once integrated, can be re-written as

$$\ln(X) \sim -\frac{\lambda_*}{m}\xi + \frac{2}{p}\ln(|\xi|) + \frac{2m(p-2)}{3\lambda_*p^2}\frac{1}{\xi} + \frac{b_0}{\lambda_*}e^{-\frac{\lambda_*}{m}\xi} - \frac{b_1}{4\lambda_*}e^{-\frac{2\lambda_*}{m}\xi}, \quad \text{for } \xi \sim +\infty.$$

Hence, we have shown that equation (1.33) is satisfied by taking

$$X(\xi) = \varphi(\xi) \sim a_0|\xi|^{\frac{2}{p}} e^{-\frac{\lambda_*}{m}\xi} = a_0|\xi|^{\frac{2}{p}} \exp\left(-m^{\frac{2-p}{p}} f'(0)^{\frac{1}{p}}\xi\right) \quad \text{for } \xi \sim +\infty,$$

for some constant $a_0 > 0$, which is exactly (1.30). With the previous formula we complete the study of the asymptotic behaviour of the TW with critical speed for $\xi \sim +\infty$. We end this paragraph pointing out that neither the assumption $f \in C^2([0,1])$ nor $f \in C^3([0,1])$ is needed since the terms involving $f''(0)$ and $f'''(0)$ do not influence formula (1.30) and that this estimate is consistent with the results known in the linear case. Indeed, when $m = 1$ and $p = 2$, the TW $\varphi = \varphi(\xi)$ with critical speed $c_{0*}(m = 1, p = 2) := c'_{0*} = 2\sqrt{f'(0)}$ satisfies

$$\varphi(\xi) \sim |\xi| e^{\frac{c'_{0*}}{2}\xi} = a_0|\xi| e^{\sqrt{f'(0)}\xi}, \quad \text{for } \xi \sim -\infty,$$

for some $a_0 > 0$. See for instance [119] and the references therein.

1.4 Proof of Theorem 1.3

We now prove Theorem 1.3 for reactions of Fisher-KPP type (2). As the reader can easily see the proof in the case of reactions of type C and/or C′ is very similar and we skip it. To facilitate the reading, we divide the proof in two main steps: in the first one, we show the continuity of $c_* = c_*(m,p)$ in the range $\gamma > 0$ of the (m,p)-plane and then we extend the continuity to the all range $\gamma \geq 0$.

1.4.1 Continuity of the function c_* for $\gamma > 0$

We will see in a moment that the continuity of the critical speed strongly depends on the stability of the orbit "coming into" the point $S = (0,1)$ (recall that we proved its existence and its uniqueness for all $c > 0$ in *Step2* of Part (i) of Theorem 1.1). Before proceeding, we need to introduce the following notations:
- $Z_j = Z_j(X)$ stands for the analytic representation of the trajectory $T_{c_*(m_j,p_j)}$ (as a function of X) for the values $m_j > 0$ and $p_j > 1$ such that $\gamma_j = m_j(p_j - 1) - 1 > 0$ and $j = 0,1$.
- $A_j = A_j(X)$ will indicate the analytic representation of the trajectories "above" $T_{c_*(m_j,p_j)}$, $j = 0,1$.
- $B_j = B_j(X)$ will indicate the analytic representation of the trajectories "below" $T_{c_*(m_j,p_j)}$, $j = 0,1$.

The following lemma proves that the orbit T_{c_*} is continuous with respect to the parameters $m > 0$ and $p > 1$.

Lemma 1.4. *Let $c = c_*$. Then the orbit T_{c_*} linking $R_{c_*} = (0, c_*^{1/(p-1)})$ and $S = (1,0)$ is continuous with respect to the parameters $m > 0$ and $p > 1$ (with $\gamma > 0$) uniformly on $[0,1]$.*
This means that for all $m_0 > 0$ and $p_0 > 1$ with $\gamma_0 > 0$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|Z_0(X) - Z_1(X)| \leq \varepsilon \quad \textit{for all } |m_0 - m_1| + |p_0 - p_1| \leq \delta \textit{ with } \gamma_1 > 0$$

and for all $0 \leq X \leq 1$.

Proof. Fix $m_0 > 0$ and $p_0 > 1$ with $\gamma_0 > 0$ and note that the proof is trivial if $X = 1$.

Step1. First of all, we show that for all $\varepsilon > 0$ and for all $0 < \overline{X} < 1$, there exists $\delta > 0$ such that

$$|Z_0(\overline{X}) - Z_1(\overline{X})| \leq \varepsilon \quad \text{for all } |m_0 - m_1| + |p_0 - p_1| \leq \delta \text{ with } \gamma_1 > 0.$$

So, fix $0 < \overline{X} < 1$ and $\varepsilon > 0$. We consider the trajectories $A_0 = A_0(X)$ and $B_0 = B_0(X)$ with

$$A_0(\overline{X}) = Z_0(\overline{X}) + \varepsilon \quad \text{and} \quad B_0(\overline{X}) = Z_0(\overline{X}) - \varepsilon, \tag{1.34}$$

where $Z_0 = Z_0(X)$, as we explained before, is the analytic expression for the trajectory T_{c_*} with parameters m_0 and p_0. Since we proved that T_{c_*} is "repulsive" near S (see *Step2* of Theorem 1.1), we have that $A_0(\cdot)$ has to cross the line $Z = 1$ in some point with positive height while $B_0(\cdot)$ crosses the X-axis in a point with first coordinate in the interval $(\overline{X}, 1)$. Hence, we can apply the continuity of the trajectories with respect to the parameters m and p outside the critical points and deduce the existence of $\delta > 0$ such that for all m_1 and p_1 satisfying $|m_0 - m_1| + |p_0 - p_1| \le \delta$, the trajectories $A_1 = A_1(X)$ and $B_1 = B_1(X)$ with

$$A_1(\overline{X}) = Z_0(\overline{X}) + \varepsilon \quad \text{and} \quad B_1(\overline{X}) = Z_0(\overline{X}) - \varepsilon \tag{1.35}$$

satisfy $|A_0(X) - A_1(X)| \le \varepsilon$ and $|B_0(X) - B_1(X)| \le \varepsilon$ for all $\overline{X} \le X \le 1$. In particular, $A_1(\cdot)$ crosses the line $Z = 1$ in a point with positive height and $B_1(\cdot)$ has to cross the X-axis in a point with first coordinate in the interval $(\overline{X}, 1)$. Consequently, since $B_1(X) \le Z_1(X) \le A_1(X)$ for all $\overline{X} \le X \le 1$, we deduce that $|Z_0(\overline{X}) - Z_1(\overline{X})| \le \varepsilon$.

Step2. Now, in order to show that

$$|Z_0(X) - Z_1(X)| \le \varepsilon \quad \text{for all } \overline{X} \le X \le 1,$$

we suppose by contradiction that there exists a point $\overline{X} < \overline{X}' < 1$ such that $|Z_0(\overline{X}') - Z_1(\overline{X}')| > \varepsilon$. Without loss generality, we can suppose $Z_0(\overline{X}') > Z_1(\overline{X}') + \varepsilon$. Then, we can repeat the procedure carried out before by taking $A_0(\cdot)$ and $B_0(\cdot)$ satisfying (1.34) with $\overline{X} = \overline{X}'$. Hence, the continuity of the trajectories with respect to the parameters m and p (outside the critical points) assures us the existence of $A_1(\cdot)$ and $B_1(\cdot)$ satisfying (1.35) with $\overline{X} = \overline{X}'$. Hence, since $B_0(\cdot)$ crosses the X-axis in point with first coordinate in the interval $(\overline{X}', 1)$ and $B_1(\cdot)$ has to behave similarly (by continuity), we have that the trajectory described by $B_1(\cdot)$ and $Z_1(\cdot)$ have to intersect, contradicting the uniqueness of the solutions. We ask to the reader to note that, at this point, we have showed the continuity of the trajectory T_{c_*} with respect to the parameters m and p uniformly in the interval $[\overline{X}, 1]$, where $0 < \overline{X} < 1$ is arbitrary.

Step3. Finally, to conclude the proof, it is sufficient to check the continuity in $X = 0$, i.e., we have to prove that for all $m_0 > 0$ and $p_0 > 1$ such that $\gamma_0 > 0$, for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|Z_0(0) - Z_1(0)| \le \varepsilon \quad \text{for all } |m_0 - m_1| + |p_0 - p_1| \le \delta \text{ with } \gamma_1 > 0.$$

Arguing by contradiction again, we suppose that there exist $\varepsilon > 0$ such that for all $\delta > 0$, we can find m_1 and p_1 with $|m_0 - m_1| + |p_0 - p_1| \le \delta$ and $\gamma_1 > 0$ such that it holds $|Z_0(0) - Z_1(0)| > \varepsilon$.
Hence, since the trajectories are continuous with respect to the variable X, we deduce the existence of a small $0 < \overline{X} < 1$ such that $|Z_0(\overline{X}) - Z_1(\overline{X})| > \varepsilon/2$ and so, thanks to the result from the *Step1* and *Step2*, we can take $\delta > 0$ small so that $|Z_0(\overline{X}) - Z_1(\overline{X})| \le \varepsilon/2$, obtaining the desired contradiction. □

A direct consequence of the previous lemma is the continuity of the function $c_* = c_*(m, p)$ in the region $\mathcal{R} := \{(m, p) : \gamma = m(p - 1) - 1 > 0$, that we enunciate in the following corollary.

Corollary 1.5. *The function $c_* = c_*(m, p)$ is continuous in the region $\mathcal{R}$, i.e., for all m_0 and p_0 with $\gamma_0 > 0$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that it holds $|c_*(m_0, p_0) - c_*(m, p)| \le \varepsilon$, for all m and p with $\gamma > 0$ and satisfying $|m - m_0| + |p - p_0| \le \delta$.*

1.4.2 Continuity of the function c_* for $\gamma \ge 0$

In these last paragraphs, we complete the proof of Theorem 1.3 showing the following lemma.

Lemma 1.6. *The function defined in* (1.3) *is continuous in $\overline{\mathcal{R}}$, i.e., for all m_0 and p_0 with $\gamma_0 \ge 0$ and for all $\varepsilon > 0$, there exists $\delta > 0$ such that it holds $|c_{0*}(m_0, p_0) - c_*(m, p)| \le \varepsilon$, for all m and p with $\gamma \ge 0$ and satisfying $|m - m_0| + |p - p_0| \le \delta$.*

Proof. We divide the proof in some short steps as follows.

Step1. First of all, we observe that, thanks to Corollary 1.5 and since the function $c_{0*}(m,p)$ is continuous in the set $\{\gamma = 0\}$, it is sufficient to check the continuity along the boundary of the region $\mathcal{R}$. More precisely, we will show that for all m_0 and p_0 with $\gamma_0 = 0$, for all $\varepsilon > 0$, there exists $\delta > 0$ such that it holds $|c_{0*}(m_0,p_0) - c_*(m,p)| \le \varepsilon$, for all m and p with $\gamma > 0$ and satisfying $|m-m_0| + |p-p_0| \le \delta$.

Step2. Now, we ask the reader to note that, with the same notations and techniques used in the proof of Lemma 1.4, it is possible to prove the continuity of the trajectory "coming into" $S = (1,0)$ with respect to the parameters m and p with $\gamma \ge 0$ uniformly in the variable X in all sets $[\overline{X},1]$, where $\overline{X}$ is fixed in $(0,1]$. This fact can be easily checked since, as we explained in the proof of Theorem 1.2, the local behaviour of the trajectories near the point S is the same for $\gamma > 0$ and $\gamma = 0$.

Step3. Before, proceeding with the proof we need to recall a last property. In the proof of Theorem 1.1, we showed that $c_*(m,p) < c_0(m,p)$ for all m and p with $\gamma > 0$, where

$$c_0(m,p) := p\left(\frac{F_{m,p}}{p-1}\right)^{(p-1)/p},$$

and $F_{m,p}$ is the maximum of the function $f_{m,p}(X) = mX^{\frac{\gamma}{p-1}-1}f(X)$. Since $f_{m,p}(X) \to mX^{-1}f(X)$ as $\gamma \to 0$ for all $0 < X \le 1$ and the limit function is decreasing, we have that the maximum point of $f_{m,p}(\cdot)$ converges to zero as $\gamma \to 0$. Consequently, we obtain that $F_{m,p} \to mf'(0)$ as $\gamma \to 0$ and we deduce

$$c_0(m,p) \to p(m^2 f'(0))^{1/(m+1)} = c_{0*}(m,p) \quad \text{as } \gamma \to 0.$$

Now we have all the elements needed for completing the proof by proving the assertion stated in *Step1*.

Step4. Suppose by contradiction that there exists m_0 and p_0 with $\gamma_0 = 0$, $\varepsilon > 0$ and a sequence $(m_j,p_j) \to (m_0,p_0)$ as $j \to \infty$ with $\gamma_j > 0$, such that $|c_{0*}(m_0,p_0) - c_*(m_j,p_j)| \ge \varepsilon$ for all $j \in \mathbb{N}$.
Hence, there exists a subsequence that we call again $c_*(m_j,p_j)$ converging to a value $c \ne c_{0*}(m_0,p_0)$. Note that, since we proved $c_*(m_j,p_j) < c_0(m_j,p_j) \to c_{0*}(m_0,p_0)$, it has to be $c < c_{0*}(m_0,p_0)$ (see *Step3*).
Now, since $\gamma_j > 0$, we have that the trajectory $T_{c_*(m_j,p_j)}$ joins the point $S = (1,0)$ with the point $R_{c_*(m_j,p_j)} = (0, c_*(m_j,p_j)^{1/(p_j-1)})$ for all $j \in \mathbb{N}$ and $R_{c_*(m_j,p_j)} \to R_c = (0, c^{1/(p_0-1)})$ as $j \to \infty$. In particular, it follows that the sequence of trajectories $T_{c_*(m_j,p_j)}$ has to be bounded. However, thanks to the analysis done in the proof of Theorem 1.2, we know that if $c < c_{0*}(m_0,p_0)$, the trajectory "coming into" the point S is unbounded and joins the previous point with $R_\infty = (0,\infty)$. Consequently, applying the continuity of the trajectory "coming into" S with respect to the parameters m and p (stated in *Step2*) we obtain the desired contradiction and we conclude the proof. □

1.5 Extensions, comments and open problems

We end the chapter with some extensions, comments and open problems.

1.5.1 Models with "strong" reaction

In order to give to reader a wider vision of the work we have carried out, we focus on a model with a "strong" reaction term. Following [79] and [80], we shortly present an extension of Theorem 1.1 and Theorem 1.2, in framework of Fisher-KPP reactions, studying the existence of TWs for the equation

$$\partial_t u = \partial_x(|\partial_x u^m|^{p-2}\partial_x u^m) + u^n(1-u) \quad \text{in } \mathbb{R}\times(0,\infty), \tag{1.36}$$

where $m > 0$ and $p > 1$ such that $\gamma \ge 0$ and $n \in \mathbb{R}$ (note that we get the Fisher-KPP equation with doubly nonlinear diffusion choosing $n = 1$). The problem consists in understanding if equation (1.36) has admissible TWs (in the sense of definition (7)) when we replace the usual smooth reaction term $f(u) = u(1-u)$ with a non-smooth one, called "strong" reaction for the "singularity" in the point $u = 0$ (when $n < 1$). We start with the case $\gamma > 0$ and then we discuss the case $\gamma = 0$.

Theorem 1.7. *Let $m > 0$ and $p > 1$ such that $\gamma > 0$, $n \in \mathbb{R}$ and $q := [\gamma + (n-1)(p-1)]/(p-1)$. Then there exist admissible TWs for equation* (1.36) *if and only if $q \geq 0$.*
Moreover, for all $q \geq 0$, there exists a critical speed $c_* = c_*(m,p,n) > 0$ such that equation (1.36) *possesses a unique admissible TW for all $c \geq c_*(m,p,n)$ and does not have admissible TWs for $0 < c < c_*(m,p,n)$. The TW corresponding to the speed $c_*(m,p,n)$ is finite.*
Finally, we have:
- *If $q = 0$, each TW is finite;*
- *If $q > 0$, the TWs corresponding to the values $c > c_*(m,p,n)$ are finite if and only if $0 < n < 1$.*

Proof. The proof is very similar to the one of Part (i) of Theorem 1.1 and we sketch it quickly. We start writing the equation of the profile getting to the system

$$\frac{dX}{d\tau} = (p-1)X|Z|^{p-2}Z, \qquad \frac{dZ}{d\tau} = cZ - |Z|^p - mX^q(1-X)$$

where X and Z are defined as in (1.6) and the equation of the trajectories

$$\frac{dZ}{dX} = \frac{cZ - |Z|^p - mX^q(1-X)}{(p-1)X|Z|^{p-2}Z}.$$

Now, using the same methods of the proof of Theorem 1.1 (*Step5* of Part (i)), it is not difficult to see that if $q < 0$, there are no trajectories linking the saddle point $S = (1,0)$ with a point of the type $(0,\lambda)$ for all $\lambda \geq 0$ and so, there are no admissible TWs.

If $q = 0$, i.e. $\gamma = (1-n)(p-1)$ (note that this expression makes sense only if $n < 1$), we have null isoclines qualitatively equal to the ones found in Theorem 1.2 (Part (i)) and so, we can show the existence of a critical speed $c_* = c_*(m,p,n)$ such that there are no TWs for $c < c_*$ and there exists exactly one TW for all $c \geq c_*$. Moreover, since we know that $n < 1$ and using the same methods of *Step4* of Theorem 1.1 (Part (i)), it is simple to see these TWs are finite.

If $q > 0$, the analysis is very similar to the one done in Theorem 1.1. The unique significant difference appears when we study the local behaviour of the trajectories from the critical point $O = (0,0)$ in the case $c > c_*$. Indeed, it is simple to see that the trajectories from O satisfies $Z(X) \sim (m/c)X^q$ for $X \sim 0$. Hence, integrating the differential equation $-mX' = X^{1-\frac{\gamma}{p-1}}Z$ (cfr. with (1.7)), we get that the time (measured respect with the variable ξ) in which the profile approaches the level 0 depends on n:

$$\xi_1 - \xi_0 = -m\int_{X_0}^{X_1} \frac{dX}{X^{1-\frac{\gamma}{p-1}}Z(X)} \sim -c\int_{X_0}^{X_1} \frac{dX}{X^{1-\frac{\gamma}{p-1}+q}} = -c\int_{X_0}^{X_1} \frac{dX}{X^n},$$

where $0 < X_1 < 1$ is fixed. Then, letting $X_0 \to 0$, it follows that the TW is finite if and only if $0 < n < 1$. □

Remarks. (i) We have showed that the existence of admissible TWs depends on the sign of the value $q := [\gamma + (n-1)(p-1)]/(p-1)$ which, in some sense, describes the interaction between the diffusion and the reaction terms. In particular, we have proved a really interesting fact: if $q \geq 0$ and $0 < n < 1$, then all the admissible TWs for equation (1.36) are finite. This represents a very important difference with the case $\gamma > 0$ and $n = 1$. Indeed, when $\gamma > 0$ and $n = 1$ only the TW corresponding to the critical value c_* is finite.

(ii) We point out that the same procedure can be followed when $\gamma = 0$ and $n \in \mathbb{R}$. In this case, we have $q = q(n) = n - 1$ and system (1.28) has the form

$$\frac{dX}{d\tau} = (p-1)X|Z|^{p-2}Z, \qquad \frac{dZ}{d\tau} = cZ - |Z|^p - mX^{n-1}(1-X).$$

Following the proof of Part (i) of Theorem 1.1 and Theorem 1.2, it is not difficult to show that for all $n \geq 1$, there exists a critical speed of propagation $c_*(n) > 0$, such for all $c \geq c_*(n)$, there exists a unique positive TW for equation (1.36), while there are no admissible TWs for $c < c_*(n)$. Moreover, if $n < 1$ equation (1.36) does not possess TW solutions.
This means that when $\gamma = 0$, a "weak/strong" modification of the reaction term is not sufficient to have finite TWs. So, the previous observations and Theorem 1.7 explain us the exact combination of slow diffusion ($\gamma > 0$) and strong reaction needed in order to "generate" *only* finite TW solutions: we need to have $q \geq 0$ and $0 < n < 1$. Consequently, we can conclude that the method of balancing the doubly nonlinear slow diffusion with a strong reaction allows us to separate quantitatively the cases in which all the TWs are positive, there exists at least one finite TW and all the TWs are finite.

(iii) Note that Theorem 1.7 extends the result of DePablo and Vźquez proved in [79] for the Porous Medium case (i.e., $p = 2$ and $m > 1$).

(iv) We point out that the study of the super-level sets and the asymptotic study can be repeated when $f(u) = u^n(1-u)$ and $0 < n < 1$. Indeed, since $u^n(1-u) \geq u(1-u)$, we can employ the solutions of the problem with $n = 1$ as sub-solutions for the problem with $0 < n < 1$. The significative open problem in the study of problem (1) with a strong reaction term is that the solutions are not unique since the reaction term is not Lipschitz continuous (see also [78]).

1.5.2 An interesting limit case

In Part (i) of Theorem 1.2, we have studied the "pseudo-linear" case ($\gamma = 0$, i.e., $m(p-1) = 1$) finding an explicit formula for the function c_* and we wrote it as a function of $m > 0$ and/or $p > 1$. In particular, we found

$$\lim_{m\to\infty} c_*(m) = \lim_{p\to 1} c_*(p) = 1.$$

This fact allows us to conjecture the existence of admissible TWs for the limit case $m \to \infty$ and $p \to 1$. Now, we can formally compute the limit of the doubly nonlinear operator:

$$\begin{aligned}\Delta_p u^m &= m^{p-1}\nabla\cdot(u^{\mu}|\nabla u|^{p-2}\nabla u)\\ &= m^{1/m}\nabla\cdot(u^{2-p}|\nabla u|^{p-2}\nabla u) \to \nabla\cdot\left(u\frac{\nabla u}{|\nabla u|}\right) \quad \text{as } m\to\infty \text{ and } p\to 1,\end{aligned}$$

keeping $m(p-1) = 1$. We ask the reader to note that, assuming $m(p-1) = \theta$ with $\theta > 0$, we can repeat the previous computations and deduce that

$$\Delta_p u^m \to \nabla\cdot\left(u^{\theta}\frac{\nabla u}{|\nabla u|}\right) \quad \text{as } m\to\infty \text{ and } p\to 1,$$

keeping $m(p-1) = \theta$ fixed. Consequently, a very interesting open problem is the study of the existence of admissible TWs for the equation

$$\partial_t u = \partial_x\Big(u^{\theta}|\partial_x u|^{-1}\partial_x u\Big) + f(u) \quad \text{in } \mathbb{R}\times(0,\infty) \tag{1.37}$$

for different values of the parameter $\theta \geq 0$. The case $\theta = 1$ has been studied by Andreu et al. in [9], where the authors showed the existence of discontinuous TWs for equation (1.37) (with $\theta = 1$). Hence,

it seems reasonable to conjecture that in the limit case $m \to \infty$ and $p \to 1$ with $m(p-1) = \theta > 1$ there are admissible TWs but with less regularity.
Finally, we recommend the papers [60, 61, 62] for more work on TW solutions (in general discontinuous) to a nonlinear flux limited Fisher-KPP equation, which seem to be related with the limit TWs of our work.

Chapter 2

Long time behaviour for "slow" diffusion

In this chapter the PDEs part begins and we show our main results for the "slow" and the "pseudo-linear" diffusion ranges $\gamma > 0$ and $\gamma = 0$, respectively (cfr. with Figure 2.1). As explained in the introduction to Part I, we are concerned with the asymptotic behaviour for large times of the solutions to problem (1):

$$\begin{cases} \partial_t u = \Delta_p u^m + f(u) & \text{in } \mathbb{R}^N \times (0, \infty) \\ u(x, 0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

with reaction term satisfying on of (2), (3), or (4), and initial datum satisfying (5). We will see how different reaction terms influence the asymptotic behaviour of the solutions and the stability of the steady states, and why travelling waves solutions are so essential for our purposes.

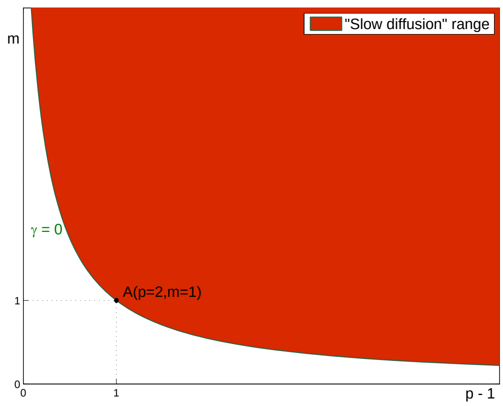

Figure 2.1: The "slow diffusion" region (red area) and the "pseudo-linear" line (green line) in the $(m, p-1)$-plane.

In what follows, we will need the concept of radial solutions to problem (1). So we say that $u = u(x,t)$ is a radial solution to problem (1), if $u(x,t) = u(r,t)$, with $r = |x|$ and it satisfies

$$\begin{cases} \partial_t u = \Delta_{p,r} u^m + f(u) & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u(r, 0) = u_0(r) & \text{in } \mathbb{R}_+, \end{cases} \tag{2.1}$$

where $\mathbb{R}_+$ is the set of positive real numbers and

$$\Delta_{p,r} u^m := r^{1-N} \partial_r \left(r^{N-1} |\partial_r u^m|^{p-2} \partial_r u^m \right) = \partial_r \left(|\partial_r u^m|^{p-2} \partial_r u^m \right) + \frac{N-1}{r} |\partial_r u^m|^{p-2} \partial_r u^m$$

is the "radial doubly nonlinear" operator. Finally, we assume that the initial datum $u_0(x) = u_0(r)$ satisfies (5) and it is radially decreasing. Note that when $N = 1$, it is equivalent to study problem (1) or (2.1) (with an even reflection).

2.1 Main results

As we did in Chapter 1, we report here the main statements we will prove in the next sections. Newly, we will keep separate the proofs depending on the reaction term $f(\cdot)$, starting with the Fisher-KPP case (2). In this treatise, it is the most significative between the three type of reactions, and we will give it more relevance.

Theorem 2.1. *(cfr. with Theorem 2.6 of [17])*
Let $m > 0$ and $p > 1$ such that $\gamma \geq 0$, and let $N \geq 1$. Let $u = u(x,t)$ be a radial solution to the initial-value problem (1) *(with radially decreasing initial datum* (5)*) and reaction of Fisher-KPP type (satisfying* (2)*). Then:*

(i) For all $0 < c < c_*$,

$$u(x,t) \to 1 \text{ uniformly in } \{|x| \leq ct\} \quad \text{as } t \to \infty.$$

(ii) For all $c > c_*$ it satisfies,

$$u(x,t) \to 0 \text{ uniformly in } \{|x| \geq ct\} \quad \text{as } t \to \infty.$$

where $c_* = c_*(m,p,f)$ is the critical speed found in Theorem 1.1, 1.2, Part (i). Moreover, if $\gamma > 0$ and $c > c_*$, then $u = u(x,t)$ has a free boundary and, in particular, $u(x,t) = 0$ in $\{|x| \geq ct\}$ as $t \to +\infty$.

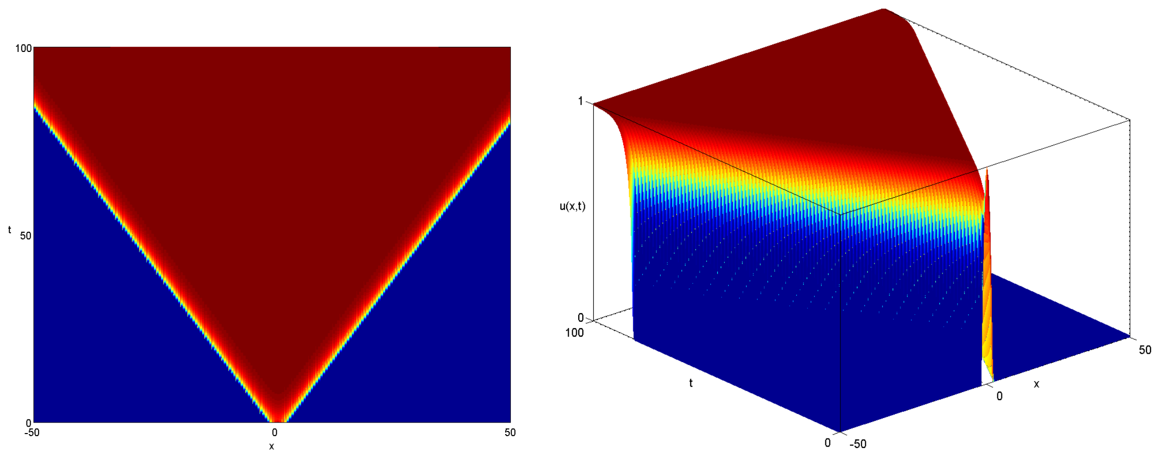

Figure 2.2: Fisher-KPP reactions, range $\gamma \geq 0$. Qualitative long time behaviour (convergence to 1 in the "inner" sets $\{|x| \leq ct\}$, for $c < c_*$) of the solutions for $f(u) = u(1-u)$.

Theorem (2.1) was proved by Aronson and Weinberger (see [12, 13]) in the linear setting $m = 1$ and $p = 2$. Here we extend it to the all range of parameters $m > 0$ and $p > 1$ such that $\gamma \geq 0$. It is important to mention that a similar asymptotic behaviour theorem was proved in [78] for Porous Medium diffusion and "strong reactions" (see also [79, 80]).

From the point of view of the applications, it explains that the density of population $u = u(x,t)$ invades all the available space with speed of propagation c_*. In other words, this means that the steady state $u = 0$ is unstable, while $u = 1$ is stable and the rate of convergence of general solutions to $u = 1$ is constant for large times.

Furthermore, as stated before, in the case $\gamma > 0$, we will prove that not only $u = u(x,t)$ converges to zero in the "outer" sets $\{|x| \geq ct\}$ for large values of $t > 0$ (for $c > c_*$), but also it will turn out that $u = u(x,t)$

is identically zero in $\{|x| \geq ct\}$ for large times. This means that when $\gamma > 0$, the general solutions have a *free boundary* and it represents a significant difference with respect to the "pseudo-linear" case ($\gamma = 0$), in which all solutions are positive everywhere. We stress that this is due to the fact that when $\gamma > 0$ there exists a finite TW corresponding to the value $c = c_*$.

Theorem 2.1 is proved in three main steps. The first one consists in showing that the solutions to problem (1) converges point-wise to 1 on every compact sets of $\mathbb{R}^N$ for times large enough (see Section 2.2). This is a technical and quite long part, based on a priori "lifting-up" lemmas, to whom we dedicate two sections depending on $\gamma > 0$ or $\gamma = 0$ (cfr. with Subsection 2.2.1 and Subsection 2.2.2). Then, using the TW solutions found in Theorem 1.1, 1.2 as sub-solutions and super-solutions, we prove a one dimensional version of Theorem 2.1. The last step consists in studying radial solutions (see (2.1)) by using solutions to the one-dimensional problem as barriers from above and below and suitable comparison principles. This second part is carried out in Section 2.3.

Theorem 2.2. *(cfr. with Theorem 1.2 of [15])*
Let $m > 0$ and $p > 1$ such that $\gamma \geq 0$, and let $N \geq 1$. Let $u = u(x,t)$ a radial solution to problem (1) *with reaction of type C (satisfying* (3)*). Then:*

(i) There are initial data satisfying (5) *such that*

$$u(x,t) \to 0 \textit{ point-wise in } \mathbb{R}^N, \quad \textit{as } t \to +\infty.$$

(ii) There are initial data satisfying (5) *such that*

$$u(x,t) \to 1 \textit{ point-wise in } \mathbb{R}^N, \quad \textit{as } t \to +\infty.$$

(iii) Asymptotic behaviour:

• *For the same class of initial data of (ii) and for all $0 < c < c_*$, it holds*

$$u(x,t) \to 1 \textit{ uniformly in } \{|x| \leq ct\}, \quad \textit{as } t \to +\infty.$$

• *For all radially decreasing initial data satisfying* (5) *and for all $c > c_*$ it holds*

$$u(x,t) \to 0 \textit{ uniformly in } \{|x| \geq ct\}, \quad \textit{as } t \to +\infty.$$

Moreover, if $\gamma > 0$ and $c > c_*$, then $u(x,t) = 0$ in $\{|x| \geq ct\}$ as $t \to +\infty$. Here $c_* = c_*(m,p,f)$ is the critical speed found in Theorem 1.1,1.2, Part (ii).

The previous statement is very significant in terms of stability/instability of the steady states $u = 0$, $u = a$, and $u = 1$, since it explains that the both $u = 0$ and $u = 1$ are "attractors" (part (i) and (ii)) for the space of nontrivial initial data $u_0 \in C_c(\mathbb{R}^N)$, $0 \leq u_0 \leq 1$. This is an important difference respect to the Fisher-KPP setting, where the steady state $u = 1$ is globally stable, whilst $u = 0$ is unstable (cfr. with Theorem 2.1). We ask the reader to note the part (ii) not only asserts that $u = 1$ is an "attractor" for a suitable class of initial data, but also gives the rate of convergence $c_* = c_*(m,p,f)$ of the solutions to this steady state, for large times. The precise classes of initial data in part (i) and (ii) will be given later (cfr. with Definition 2.13 and Definition 2.14).

Even threshold properties of reaction diffusion equations have been largely investigated since the first results proved in [13]. We quote the quite recent works [88, 157, 166] for the proof of sharp threshold theorems in the case of linear diffusion. As the reader can see, Theorem 2.2 is not sharp, but we will see how some special kind of TW solutions found in the fine ODEs analysis carried out in Chapter 1 can be employed as barriers to show the existence of a threshold effect, which is known in the linear setting but not a priori in the nonlinear one. We stress that, at least to our knowledge, in the case of nonlinear or non-local diffusion sharp threshold results are not known.

Even if in this framework the proof is easier, it will be divided depending on the spacial dimension $N = 1$ or $N \geq 2$ to simplify the reading. Let us state our last PDEs result:

Theorem 2.3. *(cfr. with Theorem 1.3 of [15])*
Let $m > 0$ and $p > 1$ such that $\gamma \geq 0$, and let $N \geq 1$. Let $u = u(x,t)$ be a radial solution to the initial-value problem (1) *(with radially decreasing initial datum* (5)*) and reaction of type C′ (satisfying* (4)*). Then:*

(i) For all $0 < c < c_*$,

$$u(x,t) \to a \text{ uniformly in } \{|x| \leq ct\} \quad \text{as } t \to \infty.$$

As always $0 < a < 1$ and satisfies $f(a) = 0$.

(ii) For all $c > c_*$ it satisfies,

$$u(x,t) \to 0 \text{ uniformly in } \{|x| \geq ct\} \quad \text{as } t \to \infty.$$

where $c_* = c_*(m,p,f)$ is the critical speed found in Theorem 1.1, 1.2, Part (iii). Moreover, if $\gamma > 0$ and $c > c_*$, then $u = u(x,t)$ has a free boundary and, in particular, $u(x,t) = 0$ in $\{|x| \geq ct\}$ as $t \to +\infty$.

Even in this setting, the previous theorem gives relevant information on the stability/instability of the steady states $u = 0$, $u = a$ and $u = 1$. Possibly, the most important one is that the state $u = a$ is globally stable w.r.t. the class of initial data $u_0 \in C_c(\mathbb{R}^N)$, $0 \leq u_0 \leq 1$, whilst both $u = 0$ and $u = 1$ are unstable. This is a strong departure from the previous case of reaction of Type C and of the Fisher-KPP type.
Note that the statement of Theorem 2.3 coincides with the one of Theorem 2.1 for Fisher-KPP reactions and $a = 1$ and was proved for the linear case in [13], together with a so called "hair-trigger effect" results that we do not study in this treatise. So, Theorem 2.3 and Theorem 2.1 can unified and give the asymptotic behaviour of a wider class of mono-stable reaction equations of type C′ (4) with $0 < a \leq 1$.

Important remark. In order to simplify the reading, we have decided to state Theorem 2.1, 2.2, and 2.3 for radial solutions to problem (1) (generated by radially decreasing initial data). A simple comparison with "sub" and "super" initial data shows the the three theorems hold true for initial data satisfying (5). Indeed, if $u_0 = u_0(x)$ satisfies (5), there are $\underline{u}_0 = \underline{u}_0(|x|)$ and $\overline{u}_0 = \overline{u}_0(|x|)$ radially decreasing satisfying (5) such that $\underline{u}_0 \leq u_0 \leq \overline{u}_0$ in $\mathbb{R}^N$. Consequently, if $\underline{u} = \underline{u}(x,t)$ and $\overline{u} = \overline{u}(x,t)$ are radial solutions to problem (1) with initial data $\underline{u}_0$ and $\overline{u}_0$, respectively, it follows $\underline{u}(x,t) \leq u(x,t) \leq \overline{u}(x,t)$ for all $x \in \mathbb{R}^N$ and $t > 0$, thanks to the comparison principle. So, since Theorem 2.1, 2.2, and 2.3 hold for $\underline{u} = \underline{u}(x,t)$ and $\overline{u} = \overline{u}(x,t)$, they will hold for $u = u(x,t)$, too.

2.2 Fisher-KPP reactions, range $\gamma \geq 0$. Convergence to 1 on compact sets

In this section, we show that the steady state $u = 1$ is stable, i.e., the solutions to problem (1), (5) with reaction term $f(\cdot)$ satisfying (2):

$$\begin{cases} f(0) = f(1) = 0, \quad 0 < f(u) \leq f'(0)u \text{ in } (0,1) \\ f \in C^1([0,1]), \qquad f'(0) > 0, \; f'(1) < 0 \\ f(\cdot) \text{ has a unique critical point in } (0,1) \end{cases}$$

converge to 1 on compact sets of $\mathbb{R}^N$, for large time (see Theorem 2.4). It will be crucial in the proof of Theorem 2.1. In what follows, we will make the additional assumption $u_0(0) = \max_{x\in\mathbb{R}^N} u_0(x)$ without loosing in generality (this choice is admissible up to a translation of the x-axis), that will allows us to avoid some tedious technical work.

Theorem 2.4. *Let $m > 1$ and $p > 1$ such that $\gamma \geq 0$ and let $N \geq 1$. Let $u = u(x,t)$ be a solution to problem* (1) *with initial datum satisfying* (5) *and Fisher-KPP reaction term* (2)*. Then, for all $\varepsilon > 0$, there exist $0 < \widetilde{a}_\varepsilon < 1$ and $\widetilde{\varrho}_\varepsilon > 0$ (which depend on $\varepsilon > 0$) such that for all $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_\varepsilon$, there exists $t_1 > 0$ (depending on $\widetilde{\varrho}_1$, ε) such that it holds*

$$u(x,t) \geq 1 - \varepsilon \quad \text{in } \{|x| \leq \widetilde{a}_\varepsilon \widetilde{\varrho}_1\}, \text{ for all } t \geq t_1.$$

As we have anticipated in the previous section, the proof is based on a priori "lifting up" results (see Proposition 2.5 for the case $\gamma > 0$ and Proposition 2.2.2 for the case $\gamma = 0$) that we prove in Subsection 2.2.1 and Subsection 2.2.2, respectively. Then, in the last subsection, we prove Theorem 2.4 (see Subsection 2.2).

2.2.1 Fisher-KPP reactions, range $\gamma > 0$. A priori "lifting up" results

In this section, we study problem (1), assuming that $f(\cdot)$ satisfies (2) with the following choice of the initial datum:

$$\widetilde{u}_0(x) := \begin{cases} \widetilde{\varepsilon} & \text{if } |x| \le \widetilde{\varrho}_0 \\ 0 & \text{if } |x| > \widetilde{\varrho}_0, \end{cases} \tag{2.2}$$

where $\widetilde{\varepsilon}$ and $\widetilde{\varrho}_0$ are positive real numbers. The choice of $\widetilde{u}_0(\cdot)$ in (2.2) is related to the finite propagation of the Barenblatt solutions in the case $\gamma > 0$ (see Section the preliminaries on doubly nonlinear diffusion reported in the introduction to Part I) and its usefulness will be clear in the next sections. We devote the all section to the proof of the following proposition.

Proposition 2.5. *Let $m > 0$ and $p > 1$ such that $\gamma > 0$ and let $N \ge 1$. Then, for all $0 < \widetilde{\varepsilon} < 1$, all $\widetilde{\varrho}_0 > 0$ and all $\widetilde{\varrho}_1 \ge \widetilde{\varrho}_0$, there exists $\underline{\widetilde{\varepsilon}} > 0$ and $t_0 > 0$, such that the solution $u(x,t)$ to problem* (1) *with initial datum* (2.2) *satisfies*

$$u(x,t) \ge \underline{\widetilde{\varepsilon}} \quad \text{in } \{|x| \le \widetilde{\varrho}_1/2\} \text{ for all } t \ge t_0.$$

This proposition asserts that for any initial data "small enough", the solutions to problem (1) are strictly greater than a fixed positive constant on every compact set of $\mathbb{R}^N$ for large times. This property will be essential in the study of the stability of the steady state $u = 1$.

We begin our study with an elementary lemma. Even though it has a quite simple proof, its meaning is important: it assures that the solution $u = u(x,t)$ does not extinguish in finite time, but, on the contrary, remains larger than a (small) positive level (depending on time), on all compact sets of $\mathbb{R}^N$.

Lemma 2.6. *Let $m > 0$ and $p > 1$ such that $\gamma > 0$ and let $N \ge 1$. Then for all $0 < \widetilde{\varepsilon} < 1$, for all $\widetilde{\varrho}_0 > 0$ and for all $\widetilde{\varrho}_1 \ge \widetilde{\varrho}_0$, there exists $t_1 > 0$ and $n_1 \in \mathbb{N}$ such that the solution $u(x,t)$ of problem* (1) *with initial datum* (2.2) *satisfies*

$$u(x,t_1) \ge \widetilde{\varepsilon}/n_1 \quad \text{in } \{|x| \le \widetilde{\varrho}_1\}.$$

Proof. Fix $0 < \widetilde{\varepsilon} < 1$ and $0 < \widetilde{\varrho}_0 \le \widetilde{\varrho}_1$.
We start with constructing a Barenblatt solution with positive parameters M_1 and θ_1 such that $B_{M_1}(x,\theta_1) \le \widetilde{u}_0(x)$ in $\mathbb{R}^N$. Since the profile of the Barenblatt solution is decreasing, we can choose $M_1, \theta_1 > 0$ such that $B_{M_1}(0,\theta_1) = \widetilde{\varepsilon}$ and $B_{M_1}(x,\theta_1)|_{|x|=\widetilde{\varrho}_0} = 0$. Thus, it is simple to obtain the relation

$$\left(M_1^{\gamma}\theta_1\right)^{\alpha} = (k/C_1)^{\frac{(p-1)N}{p}}\,\widetilde{\varrho}_0^{N} \tag{2.3}$$

and the constants

$$\theta_1 = k^{p-1}\widetilde{\varrho}_0^{p}\widetilde{\varepsilon}^{-\gamma} \qquad \text{and} \qquad M_1 = C_1^{-\frac{p-1}{\gamma}}(k/C_1)^{\frac{(p-1)N}{p}}\,\widetilde{\varrho}_0^{N}\widetilde{\varepsilon}, \tag{2.4}$$

where C_1 is the constant corresponding to the profile $F_1(\cdot)$ and

$$\alpha = \frac{1}{\gamma + p/N}, \quad k = \frac{\gamma}{p}\left(\frac{\alpha}{N}\right)^{\frac{1}{p-1}}$$

are defined in the section on doubly nonlinear diffusion preliminaries of the introduction to Part I. Then, we consider the solution of the problem

$$\begin{cases} \partial_t w = \Delta_p w^m & \text{in } \mathbb{R}^N \times (0,\infty) \\ w(x,0) = B_{M_1}(x,\theta_1) & \text{in } \mathbb{R}^N, \end{cases}$$

i.e., $w(x,t) = B_{M_1}(x,\theta_1 + t)$, which satisfies $u(x,t) \geq w(x,t)$ in $\mathbb{R}^N \times (0,\infty)$ thanks to the Comparison Principle (recall that by construction we have $w(x,0) \leq \widetilde{u}_0(x)$).
Now, we take $t_1 > 0$ and $n_1 \in \mathbb{N}$ satisfying

$$t_1 \geq 2^{\frac{N(p-1)}{\alpha p}} \theta_1 \left(\frac{\widetilde{\varrho}_1}{\widetilde{\varrho}_0}\right)^{\frac{N}{\alpha}} \qquad \text{and} \qquad n_1 \geq 2^{\frac{p-1}{\gamma}} \left(1 + \frac{t_1}{\theta_1}\right)^{\alpha}. \tag{2.5}$$

Thus, since the profile of the Barenblatt solutions is decreasing, in order to have $u(x,t_1) \geq \widetilde{\varepsilon}/n_1$ in $\{|x| \leq \widetilde{\varrho}_1\}$, it is sufficient to impose

$$w(x,t_1)|_{|x|=\widetilde{\varrho}_1} = B_{M_1}(x,\theta_1 + t_1)|_{|x|=\widetilde{\varrho}_1} \geq \widetilde{\varepsilon}/n_1. \tag{2.6}$$

Now, using the relations in (2.3) and (2.4), it is not difficult to compute

$$\begin{aligned} & B_{M_1}(x,\theta_1 + t_1)|_{|x|=\widetilde{\varrho}_1} = M_1 B_1(x, M_1^{\gamma}(\theta_1 + t_1))|_{|x|=\widetilde{\varrho}_1} \\ &= \frac{M_1}{(M_1^{\gamma}\theta_1)^{\alpha}} \left(\frac{\theta_1}{\theta_1 + t_1}\right)^{\alpha} \left\{ C_1 - k\widetilde{\varrho}_1^{\frac{p}{p-1}} \left[(M_1^{\gamma}\theta_1)^{-\frac{\alpha}{N}} \left(\frac{\theta_1}{\theta_1 + t_1}\right)^{\frac{\alpha}{N}} \right]^{\frac{p}{p-1}} \right\}_+^{\frac{p-1}{\gamma}} \\ &= \widetilde{\varepsilon} \left(\frac{\theta_1}{\theta_1 + t_1}\right)^{\alpha} \left[1 - \left(\frac{\widetilde{\varrho}_1}{\widetilde{\varrho}_0}\right)^{\frac{p}{p-1}} \left(\frac{\theta_1}{\theta_1 + t_1}\right)^{\frac{\alpha p}{N(p-1)}} \right]_+^{\frac{p-1}{\gamma}}. \end{aligned}$$

So, requiring (2.6) is equivalent to

$$\left[1 - \left(\frac{\theta_1}{\theta_1 + t_1}\right)^{\frac{\alpha p}{N(p-1)}} \left(\frac{\widetilde{\varrho}_1}{\widetilde{\varrho}_0}\right)^{\frac{p}{p-1}} \right]_+^{\frac{p-1}{\gamma}} \geq \frac{1}{n_1}\left(1 + \frac{t_1}{\theta_1}\right)^{\alpha}. \tag{2.7}$$

Since the first condition in (2.5) assures that the term on the left side of the previous inequality is larger than $2^{-(p-1)/\gamma}$, we have that a sufficient condition so that (2.7) is satisfied is

$$n_1 \geq 2^{\frac{p-1}{\gamma}} \left(1 + \frac{t_1}{\theta_1}\right)^{\alpha},$$

which is our second assumption in (2.5), and so our proof is complete. □

We proceed in our work, proving that, for all $\widetilde{\varepsilon} > 0$ small enough, the super-level sets $E^+_{\widetilde{\varepsilon}}(t) := \{x \in \mathbb{R}^N : u(x,t) \geq \widetilde{\varepsilon}\}$ of the solution $u = u(x,t)$ of problem (1) with initial datum (2.2) do not contract in time for t large enough and, in particular, we will show that for all $m > 0$ and $p > 1$ such that $\gamma > 0$ and for all $\widetilde{\varrho}_0 > 0$, it holds

$$\{|x| \leq \widetilde{\varrho}_0/2\} \subset E^+_{\widetilde{\varepsilon}}(t), \quad \text{for large times.}$$

This result highlights the role of the reaction term $f(\cdot)$. Indeed, the solution of the "pure diffusive" equation converges to zero as $t \to \infty$, whilst the presence of the function $f(\cdot)$ is sufficient to guarantee the strict positivity of the solution in a (small) compact set for large times.

Lemma 2.7. *Let $m > 0$ and $p > 1$ such that $\gamma > 0$ and let $N \geq 1$. Then, for all $\widetilde{\varrho}_0 > 0$, there exist $t_2 > 0$ and $0 < \widetilde{\varepsilon}_0 < 1$ which depend only on m, p, N, f and $\widetilde{\varrho}_0$, such that for all $0 < \widetilde{\varepsilon} \leq \widetilde{\varepsilon}_0$, the solution $u(x,t)$ of problem* (1) *with initial datum* (2.2) *satisfies*

$$u(x, jt_2) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq \widetilde{\varrho}_0/2\}, \text{ for all } j \in \mathbb{N}_+ = \{1,2,\ldots\}.$$

Proof. We prove the assertion by induction on $j \in \mathbb{N}_+$ and assuming firstly that $f(\cdot)$ is concave in $(0,1)$. We ask the reader to note that we follow the ideas previously used by Cabré and Roquejoffre [47] and, later, in [182] adapting them to our (quite different) setting.

Step0: Basic definitions. In this step, we introduce some basic definitions and quantities we will use during the proof. We fix $j = 1$ and $\widetilde{\varrho}_0 > 0$. Moreover, let $0 < \delta < 1$ and set $\lambda := f(\delta)/\delta$. Then take t_2 large enough such that

$$e^{\lambda t_2} \geq 2^{\frac{p-1}{\gamma}} \left(1 + \frac{\tau(t_2)}{\widetilde{C}_1}\right)^{\alpha} \tag{2.8}$$

where $\alpha > 0$ and $k > 0$ are defined in Section I (see also the beginning of the proof of Lemma 2.6). The constant $\widetilde{C}_1$ is defined by the formula

$$\widetilde{C}_1 := k^{p-1}\widetilde{\varrho}_0^{\,p}$$

and the function $\tau(\cdot)$ is defined as follows:

$$\tau(t) = \frac{1}{\gamma\lambda}\left[e^{\gamma\lambda t} - 1\right], \quad \text{for } t \geq 0. \tag{2.9}$$

Then, we set $\widetilde{\varepsilon}_0 := \delta e^{-f'(0)t_2}$ and, finally, we fix $0 < \widetilde{\varepsilon} \leq \widetilde{\varepsilon}_0$. Note that the choices in (2.8) are admissible since $\gamma\alpha < 1$ and $t_2 > 0$ does not depend on $\widetilde{\varepsilon}$.

Step1: Construction of a sub-solution. We construct a sub-solution to problem (1), (2.2) in $\mathbb{R}^N \times [0, t_2]$. First of all, as we did at the beginning of the proof of Lemma 2.6, we construct a Barenblatt solution of the form $B_{M_1}(x, \theta_1)$ such that $B_{M_1}(x, \theta_1) \leq \widetilde{u}_0(x)$ for all $x \in \mathbb{R}^N$. Evidently, we obtain the same formulas for $M_1 > 0$ and $\theta_1 > 0$ (see (2.3) and (2.4)). Before proceeding, we note that, using (2.4) and the fact that $\widetilde{\varepsilon} < 1$, it is simple to get

$$\theta_1 \geq \widetilde{C}_1 > 0. \tag{2.10}$$

Now, consider the change of time variable $\tau = \tau(t)$ defined in (2.9) and the "linearized" problem

$$\begin{cases} \partial_t w = \Delta_p w^m + \lambda w & \text{in } \mathbb{R}^N \times (0, \infty) \\ w(x, 0) = \widetilde{u}_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{2.11}$$

Then, the function $\widetilde{w}(x, \tau) = e^{-\lambda t} w(x, t)$ is a solution to

$$\begin{cases} \partial_\tau \widetilde{w} = \Delta_p \widetilde{w}^m & \text{in } \mathbb{R}^N \times (0, \infty) \\ \widetilde{w}(x, 0) = \widetilde{u}_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

Since $B_{M_1}(x, \theta_1) \leq \widetilde{u}_0(x) \leq \widetilde{\varepsilon}$ for all $x \in \mathbb{R}^N$, from the Comparison Principle we get

$$B_{M_1}(x, \theta_1 + \tau) \leq \widetilde{w}(x, \tau) \leq \widetilde{\varepsilon} \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{2.12}$$

Hence, using the concavity of f and the second inequality in (2.12) we get

$$w(x, t) = e^{\lambda t}\widetilde{w}(x, \tau) \leq \widetilde{\varepsilon} e^{f'(0)t} \leq \widetilde{\varepsilon}_0 e^{f'(0)t_2} = \delta, \quad \text{in } \mathbb{R}^N \times [0, t_2]$$

and so, since $w \leq \delta$ implies $f(\delta)/\delta \leq f(w)/w$, we have that w is a sub-solution to problem (1), (2.2) in $\mathbb{R}^N \times [0, t_2]$. Finally, using the first inequality in (2.12), we obtain

$$u(x, t) \geq e^{\lambda t}\widetilde{w}(x, \tau) \geq e^{\lambda t} B_{M_1}(x, \theta_1 + \tau), \quad \text{in } \mathbb{R}^N \times [0, t_2]. \tag{2.13}$$

Step2: Conclusion of $t = t_2$. In this step, we verify that the assumptions (2.8) on $t_2 > 0$ (depending only on m, p, N, f and $\widetilde{\varrho}_0$) are sufficient to prove $u(x, t_2) \geq \widetilde{\varepsilon}$ in the set $\{|x| \leq \widetilde{\varrho}_0/2\}$.

First of all, we note that, from the second inequality in (2.13) and since the profile of the Barenblatt solution is decreasing, it is clear that it is sufficient to have t_2 such that

$$e^{\lambda t_2} B_{M_1}(x, \theta_1 + \tau_2)|_{|x|=\widetilde{\varrho}_0/2} \geq \widetilde{\varepsilon}, \tag{2.14}$$

where $\tau_2 := \tau(t_2)$. Using the relations in (2.3) and (2.4) and proceeding as in the proof of Lemma 2.6, we get

$$e^{\lambda t_2} B_{M_1}(x, \theta_1 + \tau_2)|_{|x|=\widetilde{\varrho}_0/2} = \widetilde{\varepsilon}\, e^{\lambda t_2} \left(\frac{\theta_1}{\theta_1 + \tau_2}\right)^{\alpha} \left[1 - 2^{-\frac{p}{p-1}} \left(\frac{\theta_1}{\theta_1 + \tau_2}\right)^{\frac{\alpha p}{N(p-1)}}\right]_+^{\frac{p-1}{\gamma}}$$

Hence, we have that (2.14) is equivalent to

$$e^{\lambda t_2} \left[1 - 2^{-\frac{p}{p-1}} \left(\frac{\theta_1}{\theta_1 + \tau_2}\right)^{\frac{\alpha p}{N(p-1)}}\right]_+^{\frac{p-1}{\gamma}} \geq \left(1 + \frac{\tau_2}{\theta_1}\right)^{\alpha}. \tag{2.15}$$

Since for all fixed $\tau > 0$, the function $\theta/(\theta + \tau)$ satisfies

$$\frac{\theta}{\theta + \tau} \leq 1 \leq 2^{\frac{N}{\alpha p}}, \qquad \text{for all } \theta \geq 0$$

and since $\theta_1 \geq \widetilde{C}_1$ (see (2.10)), it is simple to deduce that a sufficient condition so that (2.15) is satisfied is

$$e^{\lambda t_2} \geq 2^{\frac{p-1}{\gamma}} \left(1 + \frac{\tau_2}{\widetilde{C}_1}\right)^{\alpha}$$

Note that it does not depend on $0 < \widetilde{\varepsilon} \leq \widetilde{\varepsilon}_0$ and it is exactly the assumption (2.8) on $t_2 > 0$. The proof of the case $j = 1$ is completed.

However, before studying the iteration of this process, we need to do a last effort. Let $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_0/2$ be such that

$$e^{\lambda t_2} B_{M_1}(x, \theta_1 + \tau_2)|_{|x|=\widetilde{\varrho}_1} = \widetilde{\varepsilon}$$

and introduce the function

$$v_0(x) := \begin{cases} \widetilde{\varepsilon} & \text{if } |x| \leq \widetilde{\varrho}_1 \\ e^{\lambda t_2} B_{M_1}(x, \theta_1 + \tau_2) & \text{if } |x| > \widetilde{\varrho}_1. \end{cases}$$

A direct computation (which we leave as an exercise for the interested reader) shows that condition (2.8) is sufficient to prove

$$u(x, t_2) \geq v_0(x) \geq B_{M_1}(x, \theta_1) \quad \text{in } \mathbb{R}^N. \tag{2.16}$$

Iteration. We suppose to have proved that the solution of problem (1), (2.2) satisfies

$$u(x, jt_2) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq \widetilde{\varrho}_0/2\}, \text{ for some } j \in \mathbb{N}_+$$

with the property

$$u(x, jt_2) \geq v_0(x) \geq B_{M_1}(x, \theta_1) \quad \text{in } \mathbb{R}^N \tag{2.17}$$

which we can assume since it holds in the case $j = 1$ (see (2.16)) and we prove

$$u(x, (j+1)t_2) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq \widetilde{\varrho}_0/2\}. \tag{2.18}$$

Thanks to (2.17), it follows that the solution $v(x, t)$ of the problem

$$\begin{cases} \partial_t v = \Delta_p v^m + f(v) & \text{in } \mathbb{R}^N \times (0, \infty) \\ v(x, 0) = v_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{2.19}$$

satisfies $u(x, t + jt_2) \geq v(x, t)$ in $\mathbb{R}^N \times [0, \infty)$ by the Comparison Principle. Consequently, we can work with the function $v(x, t)$ and proceeding similarly as before.

Step1'. We construct a sub-solution to problem (2.19) in $\mathbb{R}^N \times [0, t_2]$. This step is identical to *Step1* (case $j = 1$). However, we ask the reader to keep in mind that now we are building a sub-solution of the function $v = v(x, t)$. This means that we consider the "linearized" problem

$$\begin{cases} \partial_t w = \Delta_p w^m + \lambda w & \text{in } \mathbb{R}^N \times (0, \infty) \\ w(x, 0) = v_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

where $\lambda := f(\delta)/\delta$ and, using again the change of variable (2.9), we deduce that the function $\widetilde{w}(x, \tau) = e^{-\lambda t} w(x, t)$ satisfies the problem

$$\begin{cases} \partial_\tau \widetilde{w} = \Delta_p \widetilde{w}^m & \text{in } \mathbb{R}^N \times (0, \infty) \\ \widetilde{w}(x, 0) = v_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

Since $B_{M_1}(x, \theta_1) \leq v_0(x) \leq \widetilde{\varepsilon}$ for all $x \in \mathbb{R}^N$, from the Comparison Principle we get again

$$B_{M_1}(x, \theta_1 + \tau) \leq \widetilde{w}(x, \tau) \leq \widetilde{\varepsilon} \quad \text{in } \mathbb{R}^N \times (0, \infty)$$

and, moreover, $w(x, t) \leq \delta$ in $\mathbb{R}^N \times [0, t_2]$ which allows us to conclude that $w = w(x, t)$ is a sub-solution to problem (2.19) in $\mathbb{R}^N \times [0, t_2]$. In particular, it holds

$$u(x, (j + 1)t_2) \geq v(x, t_2) \geq w(x, t_2) \geq e^{\lambda t_2} B_{M_1}(x, \theta_1 + \tau_2).$$

Step2'. This step is identical to *Step2*, since we have to verify that

$$e^{\lambda t_2} B_{M_1}(x, \theta_1 + \tau_2)|_{|x| = \widetilde{\varrho}_0/2} \geq \widetilde{\varepsilon}.$$

Since we showed in *Step2* that (2.8) is as sufficient condition so that the previous inequality is satisfied, we obtain (2.18) and we conclude the proof.
However, in order to be precise and be sure that our iteration actually works, we have to prove that

$$u(x, (j + 1)t_2) \geq B_{M_1}(x, \theta_1) \quad \text{in } \mathbb{R}^N,$$

but this follows from the fact that $w(x, t_2) \geq v_0(x) \geq B_{M_1}(x, \theta_1)$ in $\mathbb{R}^N$.

Last Step. We conclude the proof for reactions $f(\cdot)$ satisfying (2) without the assumption of concavity. For such $f(\cdot)$, we can infer as in Remark 3.5 of [47], taking a new reaction $\widetilde{f} = \widetilde{f}(u)$ defined as the primitive of

$$h(u) := \min_{v \in [0,u]} f'(v),$$

satisfying $\widetilde{f}(0) = 0$. It easily seen that $\widetilde{f}(\cdot)$ satisfies

$$\begin{cases} \widetilde{f}(0) = \widetilde{f}(\theta) = 0, & f(u) \geq \widetilde{f}(u) > 0 \text{ in } (0, \theta) \\ \widetilde{f} \in C^1([0, \theta]), & (\widetilde{f})'(0) = f'(0) \\ \widetilde{f}(\cdot) \text{ is concave in } (0, \theta), & \end{cases}$$

for some $0 < \theta < 1$, cfr. with formula (3.20) of [47]. Now, since the above proof concerns only the "small" level sets of $u = u(x, t)$ (i.e. $\widetilde{\varepsilon}_0 > 0$ can be taken smaller), we can substitute $f(\cdot)$ with $\widetilde{f}(\cdot)$) (which is now concave) and the argue by comparison, since $f \geq \widetilde{f}$ in $(0, \theta)$. This conclude the proof of the Lemma. □

Corollary 2.8. *Let $m > 0$ and $p > 1$ such that $\gamma > 0$ and let $N \geq 1$. Then, there exist $t_2 > 0$ and $0 < \widetilde{\varepsilon}_0 < 1$ which depend only on m, p, N, f and $\widetilde{\varrho}_0$ such that, for all $0 < \widetilde{\varepsilon} \leq \widetilde{\varepsilon}_0$, the solution $u(x, t)$ to problem* (1) *with initial datum* (2.2) *satisfies*

$$u(x, t) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq \widetilde{\varrho}_0/2\} \text{ for all } t \geq t_2.$$

Proof. The previous lemma states that for all $\widetilde{\varrho}_0 > 0$ and for the sequence of times $t_j = (jt_2)_{j\in\mathbb{N}_+}$, the solution to problem (1), (2.2) reaches a positive value $\widetilde{\varepsilon}$ in the set $\{|x| \le \widetilde{\varrho}_0/2\}$, i.e.

$$u(x, jt_2) \ge \widetilde{\varepsilon} \quad \text{in } \{|x| \le \widetilde{\varrho}_0/2\}, \text{ for all } \; j \in \mathbb{N}_+,$$

for all $0 < \widetilde{\varepsilon} \le \widetilde{\varepsilon}_0 = \delta e^{-f'(0)t_2}$. We can improve this result choosing a smaller $\widetilde{\varepsilon}_0 > 0$.
Indeed, since conditions (2.8):

$$e^{\lambda t_2} \ge 2^{\frac{(p-1)}{\gamma}} \left(1 + \frac{\tau(t_2)}{\widetilde{C}_1}\right)^{\alpha}$$

are satisfied for all $t_2 \le t \le 2t_2$, we can repeat the same proof of Lemma 2.7, modifying the value of $\widetilde{\varepsilon}_0$ and choosing a different value $\underline{\widetilde{\varepsilon}}_0 = \delta e^{-2f'(0)t_2} > 0$, which is smaller but strictly positive for all $t_2 \le t \le 2t_2$. Hence, it turns out that for all $0 < \widetilde{\varepsilon} \le \underline{\widetilde{\varepsilon}}_0$, it holds

$$u(x,t) \ge \widetilde{\varepsilon} \quad \text{in } \{|x| \le \widetilde{\varrho}_0/2\}, \text{ for all } \; t_2 \le t \le 2t_2.$$

Now, iterating this procedure as in the proof of Lemma 2.7, we do not have to change the value of $\underline{\widetilde{\varepsilon}}_0$ when $j \in \mathbb{N}_+$ grows and so, for all $0 < \widetilde{\varepsilon} \le \underline{\widetilde{\varepsilon}}_0$, we obtain

$$u(x,t) \ge \widetilde{\varepsilon} \quad \text{in } \{|x| \le \widetilde{\varrho}_0/2\}, \text{ for all } \; j \in \mathbb{N}_+ \; \text{ and for all } \; jt_2 \le t \le (j+1)t_2.$$

Hence, using the arbitrariness of $j \in \mathbb{N}_+$, we end the proof of this corollary. □

Now, in order to prove Proposition 2.5), we combine Lemma 2.6 and Corollary 2.8.

Proof of Proposition 2.5 (case $\gamma > 0$). Fix $\widetilde{\varrho}_0 > 0$, $\widetilde{\varrho}_1 \ge \widetilde{\varrho}_0$ and consider the solution $u = u(x,t)$ to problem (1) with initial datum (2.2). So, thanks to Lemma 2.6, for all $0 < \widetilde{\varepsilon}_1 < 1$, there exist $t_1 > 0$ and $n_1 \in \mathbb{N}$ such that

$$u(x,t_1) \ge \widetilde{\varepsilon} := \widetilde{\varepsilon}_1/n_1 \quad \text{in } \{|x| \le \widetilde{\varrho}_1\}.$$

Now, define the function

$$\widetilde{v}_0(x) := \begin{cases} \widetilde{\varepsilon} & \text{if } |x| \le \widetilde{\varrho}_1 \\ 0 & \text{if } |x| > \widetilde{\varrho}_1 \end{cases}$$

and note that $u(x,t_1) \ge \widetilde{v}_0(x)$ in $\mathbb{R}^N$. Hence, the solution to the problem

$$\begin{cases} v_t = \Delta_p v^m + f(v) & \text{in } \mathbb{R}^N \times (0,\infty) \\ v(x,0) = \widetilde{v}_0(x) & \text{in } \mathbb{R}^N \end{cases}$$

satisfies $u(x,t_1+t) \ge v(x,t)$ in $\mathbb{R}^N \times (0,\infty)$. Finally, applying Corollary 2.8, we get

$$u(x,t) \ge \widetilde{\varepsilon} \quad \text{in } \{|x| \le \widetilde{\varrho}_1/2\} \; \text{ for all } t \ge t_0$$

with $t_0 := t_1 + t_2$ (note that, since $n_1 \in \mathbb{N}$ can be chosen larger, $\widetilde{\varepsilon} > 0$ is arbitrarily small and we "end up" in the hypotheses of Corollary 2.8). □

2.2.2 Fisher-KPP reactions, range $\gamma = 0$. A priori "lifting up" results

Following Subsection 2.2.1, we study problem (1), with reaction term $f(\cdot)$ satisfying (2) and with initial datum:

$$\widetilde{u}_0(x) := \begin{cases} \widetilde{\varepsilon} & \text{if } |x| \le \widetilde{\varrho}_0 \\ a_0 e^{-b_0|x|^{\frac{p}{p-1}}} & \text{if } |x| > \widetilde{\varrho}_0, \end{cases} \qquad a_0 := \widetilde{\varepsilon} e^{b_0 \widetilde{\varrho}_0^{\frac{p}{p-1}}} \tag{2.20}$$

where $\widetilde{\varepsilon}$, $\widetilde{\varrho}_0$ and b_0 are positive numbers (note that $\widetilde{u}_0 \in L^1(\mathbb{R}^N)$). We ask the reader to note that, alternatively, we can fix the constants $\widetilde{\varrho}_0$, a_0 and b_0 and obtaining $\widetilde{\varepsilon} > 0$ by the "inverse" of the second formula in (2.20).
The different choice of the initial datum is due to the different shape of the profile of the Barenblatt solutions in the case $\gamma = 0$ (see the introduction of Part I). In particular, the new datum has not compact support, but "exponential" tails. Now, our goal is to prove the "pseudo-linear" version of Proposition 2.5 stated below.

Proposition 2.9. *Let $m > 0$ and $p > 1$ such that $\gamma = 0$ and let $N \geq 1$. Then, for all $0 < \widetilde{\varepsilon} < 1$, all $\widetilde{\varrho}_0 > 0$ and all $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_0$, there exists $\underline{\widetilde{\varepsilon}} > 0$ and $t_0 > 0$, such that the solution $u(x,t)$ to problem* (1) *with initial datum* (2.20) *satisfies*

$$u(x,t) \geq \underline{\widetilde{\varepsilon}} \quad \text{in } \{|x| \leq \widetilde{\varrho}_1\} \text{ for all } t \geq t_0.$$

Even in this setting, we proceed with a series of shorter steps. Even though Proposition 2.9 is a version of Proposition 2.5 when $\gamma = 0$, we decided to separate their proofs since there are significant deviances in the techniques employed to show them.

Lemma 2.10. *Let $m > 0$ and $p > 1$ such that $\gamma = 0$ and let $N \geq 1$. Then for all $\widetilde{\varrho}_0 > 0$, there exist $t_1 > 0$, $a_0 > 0$ and $b_0 > 0$ such that the solution $u(x,t)$ to problem* (1) *with initial datum* (5) *satisfies*

$$u(x,t_1) \geq \widetilde{u}_0(x) \quad \text{in } \mathbb{R}^N$$

where $\widetilde{u}_0(\cdot)$ is defined in (2.20).

Proof. Let $u = u(x,t)$ the solution of problem (1), (5) and consider the solution $v = v(x,t)$ to the "pure diffusive" Cauchy problem:

$$\begin{cases} \partial_t v = \Delta_p v^m & \text{in } \mathbb{R}^N \times (0,\infty) \\ v(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

which satisfies $v(x,t) \leq u(x,t)$ in $\mathbb{R}^N \times [0,\infty)$ thanks to the Comparison Principle.

Step1. We begin by proving that for all $\tau > 0$, there exist $\varepsilon > 0$, $M > 0$ such that for all $\tau < t_1 < \tau + \varepsilon$ it holds

$$u(x,t_1) \geq B_M(x,t_1 - \tau) \quad \text{in } \mathbb{R}^N$$

where $B_M(x,t)$ is the Barenblatt solution in the "pseudo-linear" case (see the introduction to Part I) with exponential form

$$B_M(x,t) = C_M t^{-\frac{N}{p}} \exp\left(-k|xt^{-\frac{1}{p}}|^{\frac{p}{p-1}}\right),$$

where $C_M > 0$ is chosen depending on the mass M and $k = (p-1)p^{-p/(p-1)}$.
Fix $\tau > 0$, $\varrho > 0$ and let $\varepsilon > 0$ (for the moment arbitrary). Furthermore, take a mass $M > 0$ such that $C_M \leq 1$. We want to compare the general solution $v = v(x,t)$ with the "delayed" Barenblatt solution $B_M = B_M(x,t-\tau)$ in the strip

$$S = [\tau, \tau + \varepsilon] \times \{x \in \mathbb{R}^N : |x| \geq \varrho\},$$

in order to deduce $v(x,t) \geq B_M(x,t-\tau)$ in S. Hence, from the Comparison Principle, is sufficient to check this inequality on the parabolic boundary of S. Note that we have $v(x,\tau) \geq B_M(x,0) = 0$ for all $|x| \geq \varrho$. Now, in order to check that $v(\varrho,t) \geq B_M(\varrho,t-\tau)$ for all $\tau \leq t \leq \tau + \varepsilon$, we compute the derivative of the function $b(t) := B_M(\varrho,t-\tau)$ deducing that

$$b'(t) \geq 0 \quad \text{if and only if} \quad t \leq \tau + \left[\frac{pk}{N(p-1)}\right]^{p-1} \varrho^p.$$

Thus, choosing $\varepsilon \leq \{(pk)/[N(p-1)]\}^{p-1}\varrho^p$ and noting that $v(\varrho, t) \geq \underline{v} > 0$ for all $\tau \leq t \leq \tau + \varepsilon$ and some constant $\underline{v} > 0$ (the strict positivity follows from the Harnack Inequality proved in [134]), we have that it is sufficient to prove $\underline{v} \geq B_M(\varrho, \varepsilon)$. Since, $B_M(\varrho, \varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain the required inequality by taking eventually $\varepsilon > 0$ smaller. We ask to the reader to note that the assumption $C_M \leq 1$ guarantees that the choice of $\varepsilon > 0$ does not depend on $M > 0$.
Now, take $\tau < t_1 < \tau + \varepsilon$. We know $v(x, t_1) \geq B_M(x, t_1 - \tau)$ if $|x| \geq \varrho$ and so, by taking $C_M > 0$ smaller (depending on $\varepsilon > 0$) and using again the positivity of the solution $v = v(x, t)$, it is straightforward to check that the same inequality holds for all $|x| \leq \varrho$, completing the proof of the first step.

Step2. Fix $\widetilde{\varrho}_0 > 0$ and let $t_1 > 0$ be fixed as in the previous step. To end the proof, it is sufficient to prove

$$B_M(x, t_1 - \tau) \geq \widetilde{u}_0(x) \quad \text{in } \{|x| \geq \widetilde{\varrho}_0\},$$

since the profile of the Barenblatt solution is radially decreasing. A direct computation shows that the previous inequality is satisfied by taking

$$a_0 \leq C_M(t_1 - \tau)^{-N/p} \quad \text{and} \quad b_0 \geq k(t_1 - \tau)^{-1/(p-1)}$$

and so, the proof is complete. □

Following the ideas of the previous section (see Lemma 2.7), we prove that for all $\widetilde{\varepsilon} > 0$ small enough, the super-level sets $E^{+}_{\widetilde{\varepsilon}}(t)$ of the solution $u = u(x, t)$ to problem (1) with initial datum (2.20) expand in time for t large enough. More precisely, we will show that for all $\widetilde{\varrho}_0 > 0$ and for all $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_0$, it holds

$$\{|x| \leq \widetilde{\varrho}_1\} \subset E^{+}_{\widetilde{\varepsilon}}(t), \quad \text{for large times.}$$

This will be slightly simpler with respect to the case $\gamma > 0$, since now we work with strictly positive Barenblatt solutions (see the section of preliminaries on doubly nonlinear diffusion in the introduction of Part I). For instance, we do not need a version of Lemma 2.6, which seems to be essential in the case $\gamma > 0$.

Lemma 2.11. *Let $m > 0$ and $p > 1$ such that $\gamma = 0$ and let $N \geq 1$. Then, for all $\widetilde{\varrho}_0 > 0$ and for all $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_0$, there exist $t_0 > 0$ and $0 < \widetilde{\varepsilon}_0 < 1$ which depend only on m, p, N, f, $\widetilde{u}_0$ and $\widetilde{\varrho}_1$, such that for all $0 < \widetilde{\varepsilon} \leq \widetilde{\varepsilon}_0$, the solution $u(x, t)$ to problem* (1) *with initial datum* (2.20) *satisfies*

$$u(x, jt_0) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq \widetilde{\varrho}_1\}, \text{ for all } \; j \in \mathbb{N}_+ = \{1, 2, \dots\}.$$

Proof. We prove the assertion by induction on $j = 1, 2, \dots$, assuming again that $f(\cdot)$ is concave in $(0, 1)$. The general case follows exactly as the range $\gamma > 0$.

Step0: Basic definitions. As in the proof of Lemma 2.7, this "first" step is devoted to the introduction of basic definitions and quantities we need during the proof.
We fix $j = 1$, $\widetilde{\varrho}_0 > 0$ and $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_0$. Moreover, let $0 < \delta < 1$, set $\lambda := f(\delta)/\delta$ and fix $0 < \lambda_0 < \lambda$. Then take t_0 large enough such that

$$t_0 \geq k^{p-1}\widetilde{\varrho}_1^{\,p}, \qquad e^{\lambda_0 t_0} \geq \left(\frac{\theta_1 + t_0}{\theta_1}\right)^{N/p} \quad \text{and} \quad t_0 \geq \frac{1}{\lambda - \lambda_0}, \tag{2.21}$$

where (we ask the reader to note that we defined k in Section I, case $\gamma = 0$):

$$k := (p-1)p^{-p/(p-1)} \quad \text{and} \quad \theta_1 := \left(\frac{k}{b_0}\right)^{p-1}.$$

Then, we set $\widetilde{\varepsilon}_0 := \delta e^{-f'(0)t_0}$ and, finally, we fix $0 < \widetilde{\varepsilon} \leq \widetilde{\varepsilon}_0$.

Step1: Construction of a sub-solution. As before, we construct a sub-solution to problem (1), (2.20) in $\mathbb{R}^N \times [0, t_0]$.
First of all, as we did at the beginning of the proof of Lemma 2.6 and Lemma 2.7, we construct a Barenblatt solution of the form $B_{M_1}(x, \theta_1)$ such that $B_{M_1}(x, \theta_1) \leq \widetilde{u}_0(x)$ for all $x \in \mathbb{R}^N$. Note that in the case $\gamma = 0$, the Barenblatt solutions have exponential profile (see Section I). Imposing again $B_{M_1}(0, \theta_1) = \widetilde{\varepsilon}$ we get the first relation $C_{M_1}\theta_1^{-N/p} = \widetilde{\varepsilon}$ which is sufficient to have $B_{M_1}(x, \theta_1) \leq \widetilde{u}_0(x)$ in $\{|x| \leq \widetilde{\varrho}_0\}$. On the other hand, for all $|x| \geq \widetilde{\varrho}_0$, we need to have:

$$B_{M_1}(x, \theta_1) \leq a_0 e^{-b_0|x|^{\frac{p}{p-1}}}, \qquad |x| \geq \widetilde{\varrho}_0.$$

Using the relation $C_{M_1}\theta_1^{-N/p} = \widetilde{\varepsilon}$ and the fact that $\widetilde{\varepsilon} \leq a_0$, it is not difficult to see that a possible choice of parameters such that the previous inequality is satisfied too is

$$\theta_1 = \left(\frac{k}{b_0}\right)^{p-1} \qquad \text{and} \qquad C_{M_1} = \widetilde{\varepsilon}\left(\frac{k}{b_0}\right)^{\frac{N(p-1)}{p}}. \tag{2.22}$$

Note that, with respect to the case $\gamma > 0$, θ_1 does not depend on $\widetilde{\varepsilon} > 0$. Now, as in the proof of Lemma 2.7, we consider the "linearized" problem

$$\begin{cases} \partial_t w = \Delta_p w^m + \lambda w & \text{in } \mathbb{R}^N \times (0, \infty) \\ w(x, 0) = \widetilde{u}_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{2.23}$$

and we deduce that the function $\widetilde{w}(x, t) = e^{-\lambda t} w(x, t)$ satisfies the problem (note that when when $\gamma = 0$ the doubly nonlinear operator has homogeneity 1)

$$\begin{cases} \partial_t \widetilde{w} = \Delta_p \widetilde{w}^m & \text{in } \mathbb{R}^N \times (0, \infty) \\ \widetilde{w}(x, 0) = \widetilde{u}_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

Again we have that $B_{M_1}(x, \theta_1) \leq \widetilde{u}_0(x) \leq \widetilde{\varepsilon}$ for all $x \in \mathbb{R}^N$ and so, by comparison we deduce

$$B_{M_1}(x, \theta_1 + t) \leq \widetilde{w}(x, t) \leq \widetilde{\varepsilon} \quad \text{in } \mathbb{R}^N \times (0, \infty). \tag{2.24}$$

Hence, using the concavity of f and the second inequality in (2.24) we get

$$w(x, t) = e^{\lambda t}\widetilde{w}(x, t) \leq \widetilde{\varepsilon} e^{f'(0)t} \leq \widetilde{\varepsilon}_0 e^{f'(0)t_0} = \delta, \quad \text{in } \mathbb{R}^N \times [0, t_0]$$

and, consequently, $w = w(x, t)$ is a sub-solution to problem (1), (2.20) in $\mathbb{R}^N \times [0, t_0]$. Finally, using the first inequality in (2.24), we obtain

$$u(x, t) \geq e^{\lambda t} B_{M_1}(x, \theta_1 + t), \quad \text{in } \mathbb{R}^N \times [0, t_0]. \tag{2.25}$$

Step2. Conclusion for $t = t_0$. In this step, we verify that the assumptions (2.21) on $t_0 > 0$ are sufficient to prove $u(x, t_0) \geq \widetilde{\varepsilon}$ in the set $\{|x| \leq \widetilde{\varrho}_1\}$.
First of all, we note that, from the inequality in (2.25) and since the profile of the Barenblatt solution is decreasing, it is clear that it is sufficient to have t_0 such that

$$e^{\lambda t_0} B_{M_1}(x, \theta_1 + t_0)|_{|x| = \widetilde{\varrho}_1} \geq \widetilde{\varepsilon}. \tag{2.26}$$

Using the relations, we compute

$$e^{\lambda t_0} B_{M_1}(x, \theta_1 + t_0)|_{|x| = \widetilde{\varrho}_1} = \widetilde{\varepsilon}\left(\frac{\theta_1}{\theta_1 + t_0}\right)^{N/p} \exp\left[\lambda t_0 - k\left(\frac{\widetilde{\varrho}_1^{\,p}}{\theta_1 + t_0}\right)^{1/(p-1)}\right].$$

Hence, we have that (2.26) is equivalent to

$$\left(\frac{\theta_1}{\theta_1+t_0}\right)^{N/p}\exp\left[\lambda t_0 - k\left(\frac{\widetilde{\varrho}_1^{\,p}}{\theta_1+t_0}\right)^{1/(p-1)}\right] \geq 1. \tag{2.27}$$

Now, using the first and the second relation in (2.21) it is not difficult to see that a sufficient condition so that (2.27) is satisfied is

$$e^{(\lambda-\lambda_0)t_0-1} \geq 1,$$

which is equivalent to the third assumption in (2.21) and so, we have $u(x,t_0) \geq \widetilde{\varepsilon}$ in $\{|x| \leq \widetilde{\varrho}_1\}$, i.e. the thesis for $j = 1$.

As we did in Lemma 2.7, we conclude the analysis of the case $j = 1$ by showing that $u(x,t_0) \geq B_{M_1}(x,\theta_1)$ in $\mathbb{R}^N$. So let $\widetilde{\varrho}_2 \geq \widetilde{\varrho}_1$ be such that

$$e^{\lambda t_0}B_{M_1}(x,\theta_1+t_0)|_{|x|=\widetilde{\varrho}_2} = \widetilde{\varepsilon}.$$

Evidently, defining the function:

$$v_0(x) := \begin{cases} \widetilde{\varepsilon} & \text{if } |x| \leq \widetilde{\varrho}_2 \\ e^{\lambda t_0}B_{M_1}(x,\theta_1+t_0) & \text{if } |x| > \widetilde{\varrho}_2, \end{cases}$$

it is sufficient to prove $v_0(x) \geq B_{M_1}(x,\theta_1)$ in $\{|x| > \widetilde{\varrho}_2\}$, i.e.

$$e^{\lambda t_0}B_{M_1}(x,\theta_1+t_0) \geq B_{M_1}(x,\theta_1), \qquad \text{for } |x| > \widetilde{\varrho}_2.$$

This last inequality can be easily written as

$$e^{\lambda t_0}\left(\frac{\theta_1}{\theta_1+t_0}\right)^{\frac{N}{p}} \geq \exp\left[k|x|^{\frac{p}{p-1}}(\theta_1+t_0)^{-\frac{1}{p-1}} - k|x|^{\frac{p}{p-1}}\theta_1^{-\frac{1}{p-1}}\right], \qquad \text{for } |x| > \widetilde{\varrho}_2.$$

and, using the second relation in (2.21), it is simple to see that the above inequality holds provided

$$\exp\left[(\lambda-\lambda_0)t_0\right] \geq \exp\left[k|x|^{\frac{p}{p-1}}(\theta_1+t_0)^{-\frac{1}{p-1}} - k|x|^{\frac{p}{p-1}}\theta_1^{-\frac{1}{p-1}}\right], \qquad \text{for } |x| > \widetilde{\varrho}_2,$$

which is satisfied since the exponent of the left-hand-side is always positive while the exponent of the right-hand-side is always negative.

Iteration. We suppose to have proved that the solution of problem (1), (2.20) satisfies

$$u(x,jt_0) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq \widetilde{\varrho}_1\}, \text{ for some } j \in \mathbb{N}_+$$

with the property

$$u(x,jt_0) \geq v_0(x) \geq B_{M_1}(x,\theta_1) \quad \text{in } \mathbb{R}^N \tag{2.28}$$

and we prove

$$u(x,(j+1)t_0) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq \widetilde{\varrho}_1\}.$$

As in the proof of Lemma 2.7, we have that (2.28) implies that the solution $v = v(x,t)$ to the problem

$$\begin{cases} \partial_t v = \Delta_p v^m + f(v) & \text{in } \mathbb{R}^N \times (0,\infty) \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{2.29}$$

satisfies $u(x,t+jt_0) \geq v(x,t)$ in $\mathbb{R}^N \times [0,\infty)$ by the Comparison Principle and this allows us to work with the function $v = v(x,t)$. The rest of the proof is almost identical to the case $\gamma > 0$ (see Lemma 2.7) and we leave the details to the interested reader. □

Proof of Proposition 2.9 (case $\gamma = 0$). The proof is identical to the proof of Corollary 2.8, substituting condition (2.8) with conditions (2.21), $\widetilde{\varrho}_0/2$ with $\widetilde{\varrho}_1$ and t_2 with t_0. □

2.2.3 Fisher-KPP reactions, range $\gamma \geq 0$. Proof of Theorem 2.4

We begin by assuming $\gamma > 0$. We fix $\varepsilon > 0$ and proceed in some steps as follows.

Step1. In this step, we construct a sub-solution of problem (1) on a domain of the form $\Omega \times (t_1, \infty)$, where Ω is a ball in $\mathbb{R}^N$ and $t_1 > 0$.
First of all, we fix $\widetilde{\varrho}_2 = 2\widetilde{\varrho}_1 > 0$ arbitrarily large and we apply Proposition 2.5 deducing the existence of a value $\widetilde{\varepsilon} > 0$ and a time $t_0 > 0$ such that

$$u(x,t) \geq \widetilde{\varepsilon} \quad \text{in } \Omega \times [t_0, \infty)$$

where $\Omega := \{|x| \leq \widetilde{\varrho}_1\}$. Hence, since $\widetilde{\varepsilon} \leq u \leq 1$ and using the assumptions on $f(\cdot)$, we can deduce a linear bound from below for the reaction term

$$f(u) \geq q(1-u) \quad \text{in } \Omega \times [t_0, \infty), \quad \text{where } q := q(\widetilde{\varepsilon}).$$

Now, for all $\widetilde{\varrho}_1 > 0$, we consider a time $t_1 \geq t_0$ (note that t_1 is now "almost" arbitrary and its precise value depending on $\varepsilon > 0$ will be specified later). Thus, the solution of the problem

$$\begin{cases} \partial_t v = \Delta_p v^m + q(1-v) & \text{in } \Omega \times (t_1, \infty) \\ v = \widetilde{\varepsilon} & \text{in } \partial\Omega \times (t_1, \infty) \\ v(x, t_1) = \widetilde{\varepsilon} & \text{in } \Omega \end{cases} \tag{2.30}$$

satisfies $\widetilde{\varepsilon} \leq v(x,t) \leq u(x,t) \leq 1$ in $\Omega \times [t_1, \infty)$ by the Comparison Principle. Furthermore, since for all fixed $\tau > 0$, the function $w(x,t) = v(x, \tau + t)$ satisfies the equation in (2.30) and $w(x,t_1) \geq v(x,t_1)$, it follows $v(x, \tau + t) \geq v(x,t)$ in $\Omega \times [t_1, \infty)$, i.e., for all $x \in \Omega$, the function $v(x, \cdot)$ is non-decreasing. Consequently, since v is uniformly bounded, there exists the uniform limit $v_\infty(x) := \lim_{t\to\infty} v(x,t)$ and it solves the elliptic problem

$$\begin{cases} -\Delta_p v_\infty^m = q(1 - v_\infty) & \text{in } \Omega \\ v_\infty = \widetilde{\varepsilon} & \text{in } \partial\Omega \end{cases} \tag{2.31}$$

in the weak sense.

Step2. In this step, we define the constants a_ε and $\widetilde{\varrho}_\varepsilon$ and we complete the proof of the lemma. The value of these constants comes from the construction of a particular sub-solution of the elliptic problem (2.31). Since our argument is quite technical, we try to sketch it skipping some computations that can be verified directly by the reader.

We look for a sub-solution of the elliptic problem (2.31) in the form

$$w^m(r) = a\left[e^{g(r)} - 1\right],$$

where $r = |x|$, $x \in \mathbb{R}^N$ while the function $g(\cdot)$ and the constant $a > 0$ are taken as follows:

$$g(r) := 1 - \left(\frac{r}{\widetilde{\varrho}_1}\right)^\lambda, \quad \text{with } \lambda := \frac{p}{p-1} \quad \text{and} \quad \frac{(1-\varepsilon/2)^m}{e-1} < a < \frac{1}{e-1}.$$

Note that the radially decreasing function $w(\cdot)$ is well defined in $[0, \widetilde{\varrho}_1]$ and, moreover, $w(r) = 0$ on the boundary $\partial\Omega = \{|x| = \widetilde{\varrho}_1\}$.
Now, we define the value $\widetilde{\varrho}_\varepsilon$ (note it is well defined and positive thanks to the assumption on a) with the formula

$$\widetilde{\varrho}_\varepsilon^p := \frac{N(ae\lambda)^{p-1}}{q\{1 - [a(e-1)]^{1/m}\}}$$

and we show that for all $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_\varepsilon$, $w(\cdot)$ is a sub-solution of the equation in (2.31). Note that, since $w(\cdot)$ is radially decreasing, it is sufficient to consider our operator (the p-Laplacian) in radial coordinates and verify that for all $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_\varepsilon$ and $0 \leq r \leq \widetilde{\varrho}_1$, it holds

$$-\Delta_{p,r} w^m := -r^{1-N}\partial_r\left(r^{N-1}|\partial_r w^m|^{p-2}\partial_r w^m\right) \leq q(1-w). \tag{2.32}$$

A direct computation shows that

$$-\Delta_{p,r} w^m = \left(\frac{a\lambda}{\widetilde{\varrho}_1^{\lambda}}\right)^{p-1}\left[N - p\,(r/\widetilde{\varrho}_1)^{\lambda}\right] e^{(p-1)g(r)}$$

and that sufficient condition so that inequality (2.32) is satisfied is

$$N(ae\lambda)^{p-1}\widetilde{\varrho}_1^{\,-p} \leq q\left\{1 - [a(e-1)]^{1/m}\right\},$$

which is equivalent to say $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_\varepsilon$. Finally, we define $\widetilde{a}_\varepsilon$ with the formula:

$$\widetilde{a}_\varepsilon^{\lambda} := 1 - \log\left[\frac{a + (1-\varepsilon/2)^m}{a}\right].$$

It is not difficult to see that $w(r) \geq 1 - \varepsilon/2$ in $\{r \leq \widetilde{a}_\varepsilon\widetilde{\varrho}_1\}$ and that again our assumption on a guarantee the well definition of $\widetilde{a}_\varepsilon$.

Hence, if we suppose $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_\varepsilon$, we can apply the elliptic Comparison Principle (recall that $w = 0$ on the boundary $\partial\Omega$) deducing

$$v_\infty(x) \geq 1 - \varepsilon/2 \quad \text{in } \{|x| \leq \widetilde{a}_\varepsilon\widetilde{\varrho}_1\}.$$

So, since $v(x,t) \to v_\infty(x)$ as $t \to \infty$, a similar inequality holds for the function $v = v(x,t)$ and large times:

$$v(x,t) \geq 1 - \varepsilon \quad \text{in } \{|x| \leq \widetilde{a}_\varepsilon\widetilde{\varrho}_1\}, \text{ for all } t \geq t_1$$

where $t_1 > 0$ is chosen large enough (depending on $\varepsilon > 0$) and the same conclusion is true for the solution $u = u(x,t)$ of problem (1) since it holds $v(x,t) \leq u(x,t)$ in $\Omega \times [t_1, \infty)$.
As the reader can easily check, the proof when $\gamma = 0$ is identical to the case $\gamma > 0$ except for the fact the we apply Proposition 2.9 instead of Proposition 2.5 (see also the following remark). □

Remark 1. At the beginning of *Step1* (case $\gamma > 0$), we have applied Proposition 2.5, even though the assumptions (5) on u_0 are not sufficient to guarantee its hypotheses. However, it is really simple to see that for an initial datum $u_0(\cdot)$ satisfying (5) and $u_0(0) = \max_{x\in\mathbb{R}^N} u_0(x)$ there exist $\widetilde{\varrho}_0 > 0$ and $\widetilde{\varepsilon} > 0$ such that the function $\widetilde{u}_0 = \widetilde{u}_0(x)$ defined in (2.2) satisfies $\widetilde{u}_0(x) \leq u_0(x)$ in $\mathbb{R}^N$. Hence, using the Comparison Principle, it follows that the solution $u(x,t)$ of the problem (1) with initial datum (5) is greater than the solution of the problem (1) with initial datum (2.2). Consequently, we deduce that $u(x,t)$ satisfies the assertion of Proposition 2.5 applying the Comparison Principle again.
If $\gamma = 0$, we can proceed similarly. However, this time, we have to start applying Lemma 2.10 which guarantees the existence of a time $t_1' > 0$ large enough so that we can place an initial datum with form (2.20) under the solution $u = u(x,t)$ at the time $t = t_1'$. Then, we can apply Proposition 2.9 and proceed with the proof of Lemma 2.4.
We ask the reader to note that, in both cases, the key point consists in deducing that for any ball $\Omega \subset \mathbb{R}^N$ there exists a time $t_0 > 0$ such that

$$u(x,t) \geq \widetilde{\varepsilon} \quad \text{in } \Omega \times [t_0, \infty).$$

If $\gamma > 0$, we get the previous relation noting that we can always find a function satisfying (2.2) which can be placed under an initial datum satisfying (5) and applying Proposition 2.5, while, when $\gamma = 0$, we can repeat this procedure but we need both Lemma 2.10 and Proposition 2.9.

Remark 2. In above proof we have employed that for any $0 < \widetilde{\varepsilon} < 1$,

$$u \geq \widetilde{\varepsilon} \quad \Rightarrow \quad f(u) \geq q(1-u).$$

for some suitable $q = q(\widetilde{\varepsilon})$. To see this, let us fix $0 < \widetilde{\varepsilon} < 1$ and take

$$q_{\widetilde{\varepsilon}} := \frac{f(\widetilde{\varepsilon}/n)}{1-\widetilde{\varepsilon}} > 0,$$

for some integer $n \geq 1$ large enough. Since $f(\cdot)$ is increasing in $[0, \overline{u})$ (where $\overline{u}$ is the maximum point of $f(\cdot)$), we easily deduce that $f(u) \geq q_{\widetilde{\varepsilon}}(1-u)$ in $[\widetilde{\varepsilon}, \overline{u}]$, for any integer $n \geq 1$. We are left to prove that the same inequality holds in $(\overline{u}, 1)$ for a suitable choice of $n \geq 1$ large. So, assume by contradiction that for any integer $n \geq 1$, there exist $u_n \in (\overline{u}, 1)$, such that

$$f(u_n) \leq \frac{f(\widetilde{\varepsilon}/n)}{1-\widetilde{\varepsilon}}(1-u_n).$$

Assume $\overline{u} < u_n < 1-\sigma$ for some small $\sigma > 0$ independent of large values of $n \geq 1$. We can thus take the limit as $n \to +\infty$ in the above inequality to obtain that the l.h.s. remains strictly positive (since $\overline{u} < u_n < 1-\sigma$), while the r.h.s. converges to zero since $f(0) = 0$. Now, if $u_n \to 1^-$, we have $f(u_n) \sim -f'(1)(1-u_n)$ and, since $f'(1) < 0$, we get the same contradiction and we complete the proof of the claim.

2.3 Proof of Theorem 2.1

We now focus on to the study of the asymptotic behaviour of the solutions to problem (1), (5) with Fisher-KPP reactions (2):

$$\begin{cases} f(0) = f(1) = 0, & 0 < f(u) \leq f'(0)u \text{ in } (0,1) \\ f \in C^1([0,1]), & f'(0) > 0, \ f'(1) < 0 \\ f(\cdot) \text{ has a unique critical point in } (0,1), \end{cases}$$

As we mentioned in Section 2.1, we divide the proof depending on the spacial dimension $N = 1$ or $N \geq 2$. The case $N = 1$ is an important step for two reasons. First of all, we are going to understand the importance of the TWs we found in Theorem 1.1 and Theorem 1.2, since we employ them as sub-solutions and super-solutions of the general solution to problem (1), (5). Secondly, we will see that the one-dimensional solutions plays an important role in the study of higher-dimension solutions.

2.3.1 Proof of Theorem 2.1, case $N = 1$

Step1: Proof of Part (i). We fix $\gamma \geq 0$ and we prove that for all $\varepsilon > 0$ and for all $0 < c < c_*$, there exists $t_2 > 0$ such that

$$u(x,t) \geq 1-\varepsilon \quad \text{in } \{|x| \leq ct\} \text{ for all } t \geq t_2.$$

First of all, fix $0 < c < c_*$ arbitrarily and $\varepsilon > 0$ such that $c + \varepsilon < c_*$. Now, consider a *CS-TW solution of type 2*: $\varphi = \varphi(\xi)$, where $\xi = x - (c+\varepsilon)t$. Up to a translation of the ξ-axis, we can assume that $\max_{\xi \in \mathbb{R}} \varphi(\xi) = \varphi(0) = 1-\varepsilon$ and $\varphi(\xi_0) = 0 = \varphi(\xi_1)$ for some $\xi_0 < 0 < \xi_1$.
We define the function

$$\underline{\varphi}(\xi) = \begin{cases} 1-\varepsilon & \text{if } \xi \leq 0 \\ \varphi(\xi) & \text{if } 0 \leq \xi \leq \xi_1 \\ 0 & \text{otherwise.} \end{cases}$$

Computing the derivative with respect to the variable $\xi = x - (c+\varepsilon)t$, it not difficult to verify that

$$-c\underline{\varphi}' - \varepsilon\underline{\varphi}' \le [|(\underline{\varphi}^m)'|^{p-2}(\underline{\varphi}^m)']' + f(\underline{\varphi}), \quad \text{for all } \xi \in \mathbb{R}. \tag{2.33}$$

We proceed proving that $\underline{u}(x,t) = \underline{\varphi}(x - ct)$ is a sub-solution of $u = u(x,t)$ in $\mathbb{R}_+ \times [0,\infty)$, i.e., we verify that

$$\partial_t \underline{u} \le \partial_x(|\partial_x \underline{u}^m|^{p-2}\partial_x \underline{u}^m) + f(\underline{u}) \quad \text{in } \mathbb{R}_+ \times [0,\infty),$$

where $\mathbb{R}_+$ is the set of the positive real numbers. Imposing this condition and using (2.33), it is simple to obtain that a sufficient condition so that the previous inequality is satisfied is $\underline{\varphi}' \le 0$ for a.e. $\xi \in \mathbb{R}$, which is true by construction (note that in this case $\xi = x - ct$, but we do not introduce other variables to avoid weighting down our presentation).

Now, we fix $\widetilde{\varrho}_1 \ge \widetilde{\varrho}_\varepsilon$ large enough such that $\xi_1 \le \widetilde{a}_\varepsilon \widetilde{\varrho}_1$ where $\widetilde{a}_\varepsilon$ and $\widetilde{\varrho}_\varepsilon$ are the values found in Theorem 2.4 (of course they refer to a general solution $u = u(x,t)$ of problem (1) with initial datum (5)). The function $\underline{u} = \underline{u}(x,t)$ satisfies $u(x,t_1) \ge \underline{u}(x,0)$ in $\mathbb{R}_+$ thanks to Theorem 2.4 and $u(0,t_1+t) \ge 1-\varepsilon \ge \underline{u}(0,t)$ for all $t \ge 0$ (this follows from the construction of $\underline{u}$). Hence, we obtain

$$u(x,t_1+t) \ge \underline{u}(x,t) \quad \text{in } \mathbb{R}_+ \times [0,\infty).$$

In particular, we deduce $u(x,t_1+t) \ge \underline{u}(x,t) \ge 1-\varepsilon$ for all $0 < x \le ct$ and for all $t \ge 0$. We ask the reader to note that we can conclude that $u(x,t_1+t) \ge 1-\varepsilon$ for all $-ct < x < 0$ and for all $t \ge 0$, simply constructing a sub-solution in the set $\mathbb{R}_- \times [0,\infty)$ ($\mathbb{R}_-$ denotes the set of non-positive real numbers), considering "reflected" CS-TWs of type 2 and proceeding similarly as we did previously. Hence, we can assume

$$u(x,t_1+t) \ge 1-\varepsilon \quad \text{in } \{|x| \le ct\} \quad \text{for all } t \ge 0.$$

Finally, fix $0 < \widetilde{c} < c$ and $t_2 := ct_1/(c - \widetilde{c})$. Then it holds $u(x,t) \ge 1-\varepsilon$ in $\{|x| \le \widetilde{c}t\}$ for all $t \ge t_2$, and the thesis follows from the arbitrariness of $0 < \widetilde{c} < c$ and $0 < c < c_*$.

Step2: Proof of Part (ii). We begin with the case $\gamma > 0$. We construct a super-solution which is identically zero on the set $\{x \ge ct\}$ for all $c > c_*$ and t sufficiently large (note that the same construction can be repeated by "reflection" in the set $\{x \le -ct\}$).

Case $u_0(0) = \max_{x\in\mathbb{R}} u_0(x) < 1$. Set $\xi = x - c_*t$ and consider the function

$$\overline{u}(x,t) := \varphi(\xi) \quad \text{in } \mathbb{R} \times [0,\infty),$$

where now $\varphi = \varphi(\xi)$ is the profile of the *finite* TW found in Theorem 1.1. We showed that there exists $-\infty < \xi_0 < +\infty$ such that $\varphi(\xi) = 0$ for all $\xi \ge \xi_0$, $\varphi' \le 0$ and $\varphi(-\infty) = 1$. Hence, up to a translation of the ξ-axis we can assume $\overline{u}(x,0) \ge u_0(x)$ for all $x \in \mathbb{R}^N$. Consequently, applying the Comparison Principle we deduce

$$\overline{u}(x,t) \ge u(x,t) \quad \text{in } \mathbb{R} \times [0,\infty).$$

Let x_0 be the *free boundary* point of $\varphi = \varphi(x)$, i.e., $x_0 := \min\{x > 0 : \varphi(x) = 0\}$. Then, it is simple to deduce that

$$\overline{u}(x,t) \ne 0 \quad \text{if and only if} \quad x \le x_0 + c_*t.$$

Now, fix an arbitrary $c > c_*$ and define $t_2 := x_0/(c - c_*)$. Then for all $t \ge t_2$ we have $ct \ge x_0 + c_*t$ and so

$$\overline{u}(x,t) \equiv 0 \quad \text{in } \{x \ge ct\} \quad \text{for all } t \ge t_2$$

which implies $u(x,t) \equiv 0$ in $\{x \ge ct\}$ for all $t \ge t_2$ and, since $c > c_*$ was taken arbitrarily we get assertion (ii) in the case $u_0(0) < 1$.

Case $u_0(0) = \max_{x\in\mathbb{R}} u_0(x) = 1$. Now assume $u_0(0) = 1$. In this case, the initial datum cannot be "placed" under an admissible TW. Then, fix $\delta > 0$ and consider the function

$$w(y,\tau) = (1+\delta)u(x,t) \quad \text{where } \tau(t) = (1+\delta)t \text{ and } y(x) = (1+\delta)^{m(p-1)/p}x$$

which satisfies the equation

$$\partial_\tau w = \partial_y(|\partial_y w^m|^{p-2}\partial_y w^m) + f((1+\delta)^{-1}w). \tag{2.34}$$

Thus, Theorem 1.1 assures that equation (2.34) possesses a finite TW solution $\varphi = \varphi(\xi)$, with $\varphi(\xi) = 0$ for all $\xi \geq \xi_0$, $\varphi' \leq 0$ and $\varphi(-\infty) = 1+\delta$. Note that in this case both the moving coordinate $\xi = x - c_* t$ and $c_* = c_*(m,p,\delta)$ depend on $\delta > 0$. Now, since $\varphi(-\infty) = 1+\delta$, we can suppose $u_0(x) \leq \varphi(x)$ for all $x \in \mathbb{R}$ and we can repeat the same analysis we did in the case $u_0(0) < 1$.
Consequently, having that $c_*(m,p,\delta) \to c_*(m,p)$ as $\delta \to 0$ (this follows from the fact that the the system of ODEs of the re-scaled equation converges to system (1.8) which is derived from the standard equation), we get the assertion (ii) in the case $\gamma > 0$.

We finally consider the range $\gamma = 0$. Again we assume $u_0(0) < 1$ (the case $u_0(0) = 1$ can be treated as we did previously). Fix $\varepsilon > 0$ and let $\varphi = \varphi(\xi)$ be the profile of the positive TW with critical speed $c_* = c_*(m,p)$ found in Theorem 1.2, where $\xi = x - c_* t$. We have $\varphi > 0$, $\varphi' < 0$ and $\varphi(-\infty) = 1$, $\varphi(\infty) = 0$ and, in particular, $\varphi(\xi) < \varepsilon$ for all $\xi \geq \xi_\varepsilon$, where ξ_ε is chosen large enough depending on $\varepsilon > 0$. Now, define $\overline{u}(x,t) = \varphi(\xi)$ and note that we can suppose $\underline{u}(x,0) \geq u_0(x)$ in $\mathbb{R}$ (we ask the reader not to confuse the function $\overline{u}$ with the one used in part (ii)). Consequently, it follows $\overline{u}(x,t) \geq u(x,t)$ in $\mathbb{R}\times[0,\infty)$ and, furthermore, we obtain

$$u(x,t) \leq \varepsilon \quad \text{in } \{x \geq \xi_\varepsilon + c_* t\} \quad \text{for all } t \geq 0.$$

Now, for all fixed $c > c_*$ and $t \geq t_2 := \xi_\varepsilon/(c - c_*)$ we have $ct \geq \xi_\varepsilon + c_* t$ and so it follows

$$u(x,t) \leq \varepsilon \quad \text{in } \{x \geq ct\} \quad \text{for all } t \geq t_2$$

i.e., the thesis. Since the same procedure can be repeated for the "reflected" TWs and $c > c_*$ is arbitrary, we end the proof of (ii) case $\gamma = 0$. □

Remark. As previously mentioned, when $\gamma > 0$ ("slow diffusion" assumption) the general solutions of problem (1) exhibit free boundaries. This fact follows from the proof of the previous theorem (part (ii) case $\gamma > 0$). Indeed, we showed that the solution is identically zero when $c > c_*$ in the outer set $\{|x| \geq ct\}$ as $t \to \infty$. This fact represents the significant difference with respect to the case $\gamma = 0$ ("pseudo-linear" assumption) in which the general solutions are positive everywhere. Hence, we can conclude that for all $\gamma \geq 0$, the general solutions of problem (1) expand linearly in time (for large times) with a critical speed $c_* > 0$ but, in the case $\gamma > 0$, for all fixed time, they are identically zero outside a ball with radius large enough, whilst, when $\gamma = 0$, they are positive everywhere. This is true when $N = 1$ and, in the next section, we will see that it is possible to extend the previous assertions for all $N \geq 1$.

2.3.2 Proof of Theorem 2.1, case $N \geq 2$

As explained before, we focus on radial solutions to problem (1), i.e., on solutions $u(x,t) = u(r,t)$, with $r = |x|$, to problem (2.1):

$$\begin{cases} \partial_t u = \Delta_{p,r} u^m + f(u) & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u(r,0) = u_0(r) & \text{in } \mathbb{R}_+, \end{cases}$$

where

$$\Delta_{p,r} u^m := \partial_r\left(|\partial_r u^m|^{p-2}\partial_r u^m\right) + \frac{N-1}{r}|\partial_r u^m|^{p-2}\partial_r u^m$$

is the "radial doubly nonlinear" operator and the initial datum $u_0(x) = u_0(r)$ satisfies (5) and it is radially decreasing. We prove separately part (i) and (ii). In (i), we firstly consider the case $\gamma > 0$ and then $\gamma = 0$.

Proof of Theorem 2.1: Part (i), range $\gamma > 0$. So, let $\gamma > 0$, fix $0 < c < c_*$ and take $0 < \varepsilon < 1$ such that $c + \varepsilon < c_*$ (this choice does not represent a problem since $\varepsilon > 0$ will be taken arbitrary small). First of all, we construct a sub-solution for problem (2.1) in a set of the form $\mathbb{R}_+ \times [t_\varepsilon, \infty)$ for $t_\varepsilon > 0$ large enough.
As we showed in Theorem 1.1 and recalled in the proof of Theorem 2.3.1in the case $N = 1$, the one-dimensional equation

$$\partial_t v = \partial_y(|\partial_y v^m|^{p-2}\partial_y v^m) + f(v) \quad \text{in } \mathbb{R} \times [0, \infty)$$

admits *CS-TWs of type 2* in the form $v(y,t) = \varphi(y-(c+\varepsilon)t)$, $y \in \mathbb{R}$ and $t \ge 0$, with speed $c+\varepsilon$. In particular, we are considering the TW solution moving to the right direction with $\varphi(0) = \max_{\xi\in\mathbb{R}} \varphi(\xi) = 1 - \varepsilon$, $\varphi(\xi_0) = 0 = \varphi(\xi_1)$ for some $\xi_0 < 0 < \xi_1$. Now, we define

$$\underline{\varphi}(\xi) = \begin{cases} 1-\varepsilon & \text{if } \xi \le 0 \\ \varphi(\xi) & \text{if } 0 \le \xi \le \xi_1 \\ 0 & \text{otherwise} \end{cases}$$

and we observe that the profile $\underline{\varphi}(\cdot)$ satisfies the differential inequality (2.33) (the we rename for simplicity):

$$-c\underline{\varphi}' - \varepsilon\underline{\varphi}' \le [|(\underline{\varphi}^m)'|^{p-2}(\underline{\varphi}^m)']' + f(\underline{\varphi}), \quad \text{for all } \xi \in \mathbb{R} \tag{2.35}$$

where we computed the derivative with respect to the variable $\xi = y - (c+\varepsilon)t$, i.e., $\underline{\varphi}'(\xi) = d\underline{\varphi}(\xi)/d\xi$.

Step1: Range $m > 1$ and $p > 1$. We consider the function $\underline{u}(r,t) = \underline{\varphi}(\delta^{1/p}r - c\delta t)$, where $r = |x|$, $x \in \mathbb{R}^N$, $\delta = 1 - \varepsilon$, and we use (2.35) to show that , $\underline{u} = \underline{u}(r,t)$ is a sub-solution for problem (2.1) if t is large enough, i.e., it satisfies

$$\partial_t \underline{u} \le \Delta_{p,r}\underline{u}^m + f(\underline{u}), \quad \text{for } t \gg 0. \tag{2.36}$$

The inequality in (2.36) can be easily re-written as

$$-c\underline{\varphi}' \le [|(\underline{\varphi}^m)'|^{p-2}(\underline{\varphi}^m)']' + \frac{N-1}{\xi + c(1-\varepsilon)t}|(\underline{\varphi}^m)'|^{p-2}(\underline{\varphi}^m)' + \frac{1}{1-\varepsilon}f(\underline{\varphi}),$$

where $\xi = \delta^{1/p}r - c\delta t$. Now, using (2.35), we get that it suffices to verify

$$\varepsilon|\underline{\varphi}'| \ge \frac{b_\varepsilon}{\delta^{-1}\xi + ct}\,\underline{\varphi}^{(m-1)(p-1)}|\underline{\varphi}'|^{p-1} - \frac{\varepsilon}{1-\varepsilon}f(\underline{\varphi}), \tag{2.37}$$

where $b_\varepsilon := m^{p-1}(N-1)/(1-\varepsilon)$.

Case $\xi \sim 0$. For all $t \ge t'_{1\varepsilon} \gg 0$, the previous inequality is trivially satisfied when $\xi \sim 0$. Indeed, we have $\underline{\varphi} \sim 1 - \varepsilon$, $|\underline{\varphi}'|^{p-1} \sim 0$ (since $p > 1$), and $f(1-\varepsilon) > 0$.

Case $\xi \sim \xi_1$. On the other hand, when $\xi \sim \xi_1$, we have $\underline{\varphi} \sim 0$, $|\underline{\varphi}'| \sim a\underline{\varphi}^{1-m} \sim +\infty$ (since $m > 1$) and $\underline{\varphi}^{(m-1)(p-1)}|\underline{\varphi}'|^{p-1} \sim a^{p-1}$. Thus we can deduce that (2.37) is satisfied for all $t \ge 0$ (of course when $\xi \sim \xi_1$).

Case $0 < \xi < \xi_1$ with $\xi \nsim 0$ and $\xi \nsim \xi_1$. Finally, when $0 < \xi < \xi_1$ with $\xi \nsim 0$ and $\xi \nsim \xi_1$, it is possible to note that the condition

$$\varepsilon ct \ge b_\varepsilon\underline{\varphi}^{(m-1)(p-1)}|\underline{\varphi}'|^{p-2}$$

is sufficient to guarantee (2.37) and so, since with the current assumptions on ξ we have that $|\varphi'|$ is bounded from above and below, we deduce the existence of a value $t''_{1\varepsilon}$ such that (2.36) is satisfied for all $t \ge t''_{1\varepsilon}$ when $0 < \xi < \xi_1$ with $\xi \nsim 0$ and $\xi \nsim \xi_1$. Hence, we have that when $m > 1$ and $p > 1$, (2.36) is satisfied for all $t \ge t_{1\varepsilon} := \max\{t'_{1\varepsilon}, t''_{1\varepsilon}\}$.

Step2: Range $0 < m < 1$ *and* $p > 2$. This is the most delicate case. We define $\underline{u}(r,t) = \underline{\varphi}(r - c\delta t)$, with $\delta = (1+\varepsilon^n)^{-1}$ and we impose condition (2.36) (the value of $n \in \mathbb{N}$ is not important now and will be specified later). Using inequality (2.35) and carrying out some tedious computations, we obtain that a sufficient condition so that (2.36) is satisfied is

$$\varepsilon^{1-n} m^{1-p} |\underline{\varphi}'| \geq |\mu| \underline{\varphi}^{\mu-1} |\underline{\varphi}'|^p - (p-1)\underline{\varphi}^{\mu} |\underline{\varphi}'|^{p-2} \underline{\varphi}'' + b_\varepsilon (\xi + c\delta t)^{-1} \underline{\varphi}^{\mu} |\underline{\varphi}'|^{p-1} - m^{1-p} f(\underline{\varphi}), \tag{2.38}$$

where this time $b_\varepsilon := (1+\varepsilon^n)(N-1)/\varepsilon^n$ and $\mu := (m-1)(p-1) < 0$ since $m < 1$.

Case $\xi \sim 0$. If $\xi \sim 0$, we have $\underline{\varphi} \sim 1 - \varepsilon$ and $|\underline{\varphi}'| \sim 0$. Hence, recalling relation (1.21) and applying it with $X = 1 - \varepsilon$, it is not difficult to see that for all $t \geq t_{2\varepsilon} \gg 0$, (2.38) is equivalent to

$$m^{2-p} f(1-\varepsilon) \geq m^{2-p} (1-\varepsilon)^{\mu+p-2-\gamma} f(1-\varepsilon),$$

which is satisfied (the equality holds) since $\mu = \gamma + 2 - p$.

Case $\xi \sim \xi_1$. When $\xi \sim \xi_1$, we have again $\underline{\varphi} \sim 0$ and $\underline{\varphi}' \sim a\underline{\varphi}^{1-m}$ and, as the reader can easily check, (2.38) is equivalent to

$$0 \geq a^{p-2} \underline{\varphi}^{-m} \Big[a^2 |\mu| - (p-1) \underline{\varphi}^{2m-1} \underline{\varphi}'' \Big] + a^{p-1} b_\varepsilon (\xi_1 + c\delta t)^{-1}.$$

Using relation (1.24), it is simple to re-write the previous inequality as

$$0 \geq -a^p d_{m,p} \underline{\varphi}^{-m} + b_\varepsilon (\xi_0 + c\delta t)^{-1},$$

where $d_{m,p} := (p-2-\gamma)/m^2 + \mu = |\mu|/m^2 - |\mu|$. Since $0 < m < 1$, we have $d_{m,p} > 0$ and so, using that $\underline{\varphi}^{-m} \sim \infty$ as $\underline{\varphi} \sim 0$, we obtain that for all fixed $t \geq 0$, the last inequality is satisfied and we conclude the analysis of the case $\xi \sim \xi_0$.

Case $0 < \xi < \xi_1$ *with* $\xi \nsim 0$ *and* $\xi \nsim \xi_1$. It is not difficult to see that (2.38) is equivalent to

$$|\underline{\varphi}'| + \varepsilon^{n-1} f(\underline{\varphi}) \geq -\varepsilon^{n-1} (|(\underline{\varphi}^m)'|^{p-2} (\underline{\varphi}^m)')' + \varepsilon^{n-1} b_\varepsilon (\xi + c\delta t)^{-1} \underline{\varphi}^{\mu} |\underline{\varphi}'|^{p-1} \tag{2.39}$$

and, furthermore, we have that $\underline{\varphi}$ and $|\underline{\varphi}'|$ are bounded from above and below by positive constants. Moreover, combining (1.7) and (1.20), it is straightforward to see that the "second order term" $(|(\underline{\varphi}^m)'|^{p-2} (\underline{\varphi}^m)')'$ has not definite sign but is bounded too (from below and above). Moreover, since $t \geq 0$, we have that a sufficient condition so that (2.39) is satisfied is

$$|\underline{\varphi}'| + \varepsilon^{n-1} f(\underline{\varphi}) \geq -\varepsilon^{n-1} (|(\underline{\varphi}^m)'|^{p-2} (\underline{\varphi}^m)')' + \varepsilon^{n-1} b_\varepsilon \xi^{-1} \underline{\varphi}^{\mu} |\underline{\varphi}'|^{p-1}. \tag{2.40}$$

Hence, we take $n \in \mathbb{N}$ large enough (independently of $t \geq 0$) to make the terms in the right side of the previous inequality smaller and proving the validity of the inequality (2.40). Consequently, we can state that also in this last case, there exists $t_{2\varepsilon} > 0$ large enough such that $\underline{u} = \underline{u}(r,t)$ satisfies (2.36) for all $t \geq t_{2\varepsilon}$.

Hence, for all $\gamma > 0$, we constructed a sub-solution $\underline{u} = \underline{u}(r,t)$ in the set $\mathbb{R}_+ \times [t_\varepsilon, \infty)$, where $t_\varepsilon = \max\{t_{1\varepsilon}, t_{2\varepsilon}\}$. Now, we proceed with the proof of the case $m > 1$ and $p > 1$. We can treat the other range of parameters with identical methods.

Step3: Comparison and conclusion. Define $\underline{w}(r,t) := \underline{u}(r, t_\varepsilon + t)$ and let $r_0 > 0$ be the "free boundary point" of $\underline{w}(r,0)$, i.e., $r_0 := \min\{r > 0 : \underline{w}(r,0) = 0\}$. From the analysis done previously, we have that the sub-solution $\underline{w} = \underline{w}(r,t)$ satisfies

$$\begin{cases} \partial_t \underline{w} \leq \Delta_{p,r} \underline{w}^m + f(\underline{w}) & \text{in } \mathbb{R}_+ \times [0,\infty) \\ \underline{w}(r,t) = \underline{\varphi}(\delta^{1/p} r - c\delta(t_\varepsilon + t)) & \text{in } \{r = 0\} \times [0,\infty) \\ \underline{w}(r,0) = \underline{\varphi}(\delta^{1/p} r - c\delta t_\varepsilon) & \text{in } \mathbb{R}_+, \end{cases}$$

where $\delta = 1-\varepsilon$. Now, let $u = u(r,t)$ be a radial solution of problem (1) and fix $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_\varepsilon$ large enough such that $r_0 \leq \widetilde{a}_\varepsilon \widetilde{\varrho}_1$, where $0 < \widetilde{a}_\varepsilon < 1$ and $\widetilde{\varrho}_\varepsilon > 0$ are the values found in Lemma 2.4. Consider the function $w(r,t) = u(r,t_1+t)$ in $\mathbb{R}_+ \times (0,\infty)$, where $t_1 > 0$ is chosen depending on $\varepsilon > 0$ and $\widetilde{\varrho}_1 > 0$ as in Lemma 2.4. Thanks to Lemma 2.4, we have that $w(r,0) \geq \underline{w}(r,0)$ for all $r \geq 0$ and, since $1-\varepsilon \geq \underline{w}(r,t)$ by construction, it holds $w(0,t) \geq \underline{w}(0,t)$ for all $t \geq 0$. So, we can apply the Comparison Principle deducing

$$u(r,t_1'+t) \geq \underline{u}(r,t) \quad \text{in } \mathbb{R}_+ \times [0,\infty).$$

where $t_1' = t_1 - t_\varepsilon$. In particular, we have $u(r,t_1+t) \geq \underline{u}(r,t) = 1-\varepsilon$ for all $r \leq c(1-\varepsilon)^{(p-1)/p} t$ and all $t \geq 0$. Finally, proceeding as in the final lines of the proof of Theorem 2.1 in dimension $N = 1$ (part (i) case $\gamma > 0$) and using the arbitrariness of $\varepsilon > 0$, we complete the proof of Theorem 2.1 part (i) case $\gamma > 0$. Note that the case $0 < m < 1$ and $p > 2$ can be treated similarly. The unique difference is that we employ $\underline{u}(r,t) = \underline{\varphi}(r - c\delta t)$ instead of $\underline{u}(r,t) = \underline{\varphi}(\delta^{1/p} r - c\delta t)$.

Proof of Theorem 2.1: Part (i), range $\gamma = 0$. The analysis is very similar to the case $\gamma > 0$ since the TWs we use now have the same properties of the ones in the case $\gamma > 0$ (cfr. with Subsection 1.2.1). Note that the procedure carried out in the case $m > 1$, $p > 1$ ($\gamma > 0$) can be easily adapted to the linear case $m = 1$ and $p = 2$ (with $\gamma = 0$), which is the only one missing.

Proof of Theorem 2.1: Part (ii). The proof of Part (ii) is easier. Since we can use solutions to problem (1) posed in spacial dimension $N = 1$ as barriers from above for solutions to problem (2.1). Let us see why, with a comparison technique.

We firstly show that if $u = u(x,t)$ is a solution to

$$\begin{cases} \partial_t u = \partial_x \left(|\partial_x u^m|^{p-2} \partial_x u^m \right) + f(u) & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}, \end{cases}$$

with initial data $u_0 = u_0(x)$ satisfying (5), $u_0(x) = u_0(-x)$ and $u_0(\cdot)$ non-increasing for all $x \geq 0$, then $u(\cdot,t)$ is non-increasing w.r.t. $x \geq 0$, for all $t > 0$. So, fix $h > 0$ and let $v = v(x,t)$ be the solution to the problem

$$\begin{cases} \partial_t v = \partial_x \left(|\partial_x v^m|^{p-2} \partial_x v^m \right) + f(v) & \text{in } \mathbb{R} \times (0,\infty) \\ v(x,0) = v_0(x) := u_0(x+h) & \text{in } \mathbb{R}. \end{cases}$$

Hence, since $v_0(x) \leq u_0(x)$ we deduce $v(x,t) \leq u(x,t)$ and, by uniqueness of the solutions, it follows $v(x,t) = u(x+h,t)$. Hence, we obtain that $u(\cdot,t)$ is non-increasing for all $t \geq 0$ thanks to the arbitrariness of $x \geq 0$ and $h \geq 0$.

Now, assume $N \geq 2$ and consider radial solutions to problem (1), i.e., solutions $u = u(r,t)$ to problem (2.1):

$$\begin{cases} \partial_t u = \partial_r \left(|\partial_r u^m|^{p-2} \partial_r u^m \right) + \frac{N-1}{r} |\partial_r u^m|^{p-2} \partial_r u^m + f(u) & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u(r,0) = u_0(r) & \text{in } \mathbb{R}_+ \times \{0\}, \end{cases}$$

where $r = |x|$, $x \in \mathbb{R}^N$, and $u_0(\cdot)$ is a radially decreasing initial datum satisfying (5). Moreover, let $\overline{u} = \overline{u}(r,t)$ be a solution to the problem

$$\begin{cases} \partial_t \overline{u} = \partial_r \left(|\partial_r \overline{u}^m|^{p-2} \partial_r \overline{u}^m \right) + f(\overline{u}) & \text{in } \mathbb{R}_+ \times (0,\infty) \\ \overline{u} = u & \text{in } \{0\} \times (0,\infty) \\ \overline{u}(r,0) = u_0(r) & \text{in } \mathbb{R}_+ \times \{0\}. \end{cases}$$

For what explained before, we have $\partial_r u(r,t) \leq 0$ in $\mathbb{R}_+ \times (0,\infty)$, and so $\overline{u} = \overline{u}(r,t)$ is a super-solution to the radial problem (2.1) and satisfies $\overline{u}(r,t) \geq u(r,t)$ in $\mathbb{R}_+ \times (0,\infty)$.

Consequently, since $\overline{u} = \overline{u}(r,t)$ is a solution to the one-dimensional equation we obtain assertion (ii) from the case $N = 1$. □

We end this section with the following corollary, which allows us to relax the assumptions on the initial data.

Corollary 2.12. *Let $m > 0$ and $p > 1$ such that $\gamma \geq 0$, and let $N \geq 1$. Consider a Lebesgue-measurable initial datum $u_0 : \mathbb{R}^N \to \mathbb{R}$ with $u_0 \not\equiv 0$ and $0 \leq u_0 \leq 1$ and satisfying*

$$u_0(x) \leq a_0 \exp\Big(- \nu_*^{-1} f'(0)^{\frac{1}{p}} |x| \Big) \quad \text{for } |x| \sim \infty \qquad \text{if } \gamma > 0, \tag{2.41}$$

or

$$u_0(x) \leq a_0 |x|^{\frac{2}{p}} \exp\Big(- m^{\frac{2-p}{p}} f'(0)^{\frac{1}{p}} |x| \Big) \quad \text{for } |x| \sim \infty \qquad \text{if } \gamma = 0, \tag{2.42}$$

where $\nu_* = \nu_*(m,p)$ is the critical speed for $f'(0) = 1$, and $a_0 > 0$. Then the statements (i) and (ii) of Theorem 2.1 hold for the solution of the initial-value problem (1), (2.41) *if $\gamma > 0$, or* (1), (2.42) *if $\gamma = 0$, respectively.*

Proof. We divide the proof in four short steps.

Step1. In this first step we show assertion (i) of Theorem 2.1. We simply note that for all initial data $u_0(\cdot)$ satisfying (2.41) and/or (2.42), there exists a "sub-initial datum" $\underline{u}_0(\cdot)$ satisfying (5), i.e. with compact support, such that $\underline{u}_0(x) \leq u_0(x)$ for all $x \in \mathbb{R}^N$. Hence, the solution $\underline{u}(x,t)$ of problem (1) with initial datum $\underline{u}_0(\cdot)$ satisfies $\underline{u}(x,t) \leq u(x,t)$ in $\mathbb{R}^N \times (0,\infty)$ by the Comparison Principle. Here, $u(x,t)$ stands for the solution of problem (1) with initial data (2.41) or (2.42) depending on $\gamma \geq 0$. Thus, since $\underline{u}(x,t)$ satisfies statement (i) of Theorem 2.1, we deduce that $u(x,t)$ has to satisfy it too.

Step2. We show assertion (ii) of Theorem 2.1 when $\gamma > 0$ and $N = 1$. For all $c > c_*(m,p)$, we can place an admissible TW $\varphi(x - ct)$ above an initial datum $u_0(x)$ satisfying (2.41). This is possible thanks to formula (1.16), which can be re-written for $t = 0$ as

$$\varphi(x) \sim a_0 \exp\Big(- c^{-1} f'(0) x \Big) = a_0 \exp\Big(- \nu^{-1} f'(0)^{1/p} x \Big), \quad \text{for } x \sim \infty, \quad \text{for some } a_0 > 0.$$

We recall that $0 < \nu < \nu_*$ is the speed with $f'(0) = 1$. Consequently, proceeding as in the parts (ii) and (iii) of the proof of Theorem 2.1, case $N = 1$, we get our statement and we complete this step.

Step3. Take now $\gamma = 0$ and $N = 1$. We fix $c > c_{0*}(m,p) = p(m^2 f'(0))^{\frac{1}{mp}}$ (cfr. with (1.2)), and we proceed as in *Step2*. However, in this case, we consider an admissible TW $\varphi(x - ct)$ satisfying for $t = 0$

$$\varphi(x) \sim a_0 \exp\Big(- \frac{\lambda_1}{m} x \Big) \quad \text{for } x \sim \infty, \quad \text{for some } a_0 > 0,$$

according to (1.31). Recall that $\lambda_1 = \lambda_1(c)$ and $\lambda_1 < \lambda_* = (c_{0*}/p)^m = (m^2 f'(0))^{1/p}$ for all $c > c_{0*}(m,p)$. This fact implies that we can actually place the TW above an initial datum satisfying (2.42). Indeed, in (2.42) we have assumed

$$u_0(x) \leq a_0 \, x^{\frac{2}{p}} \exp\Big(- m^{\frac{2-p}{p}} f'(0)^{\frac{1}{p}} x \Big) = a_0 \, x^{\frac{2}{p}} \exp\Big(- \frac{\lambda_*}{m} x \Big) \quad \text{for } x \sim \infty.$$

Once we can employ the TW $\varphi(\cdot)$ as a barrier from above, we conclude the proof of this step with the same procedure carried out before.

Step4. We end the proof with some comments. First of all, when $N = 1$, we can repeat the previous analysis when $x \sim -\infty$ by using "reflected" TWs with the form $\varphi(x + ct)$ (note that we used $|x|$ in conditions (2.41) and (2.42)). Secondly, in order to prove our statements for $N \geq 2$, it is sufficient to repeat the arguments of the proof of part (ii) of Theorem 2.1 in which we have studied the asymptotic behaviour of non-increasing radial solutions. Finally, we stress that if the initial datum satisfies $\max_{x \in \mathbb{R}^N} u_0(x) = 1$, we can proceed as in the proof of part (ii) of Theorem 2.1, case $N = 1$. □

2.4 Proof of Theorem 2.2

This section is devoted to the proof of Theorem 2.2, which concerns the asymptotic behaviour of solutions to problem (1) with initial data satisfying (5) and reaction terms of type C, i.e., satisfying (3):

$$\begin{cases} f(0)=f(a)=f(1)=0, & f(u)<0 \text{ in } (0,a), \ \ f(u)>0 \text{ in } (a,1) \\ f\in C^1([0,1]), & f'(0)<0,\ f'(a)>0,\ f'(1)<0 \\ \int_0^1 u^{m-1}f(u)du>0 & \\ f(\cdot) \text{ has a unique critical point in } (0,a) \text{ and a unique critical point in } (a,1). & \end{cases}$$

and, as anticipated, on the stability/instability of the steady states $u=0$, $u=1$, depending on the initial data. Thus, before starting with the proof, we introduce two classes of initial data which generate solutions to problem (1) evolving to $u=0$ or $u=1$, respectively.

Definition 2.13. *We divide this definition depending on the dimension $N=1$ or $N\geq 2$.*
• *Let $N=1$. An initial data $u_0=u_0(x)$ satisfying* (5) *is called "not-reacting" if there are $c_1,c_2\geq c_*$ such that*

$$u_0(x)\leq \min\{\overline{\varphi},\overline{\psi}\}(x), \quad \textit{for all } x\in\mathbb{R},$$

where

$$\overline{\varphi}(\xi):=\begin{cases} 0 & \textit{if } \xi\leq\xi_0^{c_1} \\ \varphi_{c_1}(\xi) & \textit{if } \xi_0^{c_1}<\xi<\xi_1^{c_1} \\ a & \textit{if } \xi\geq\xi_1^{c_1}\end{cases} \qquad \overline{\psi}(\xi):=\begin{cases} a & \textit{if } \xi\leq\xi_0^{c_2} \\ \psi_{c_2}(\xi) & \textit{if } \xi_0^{c_2}<\xi<\xi_1^{c_2} \\ 0 & \textit{if } \xi\geq\xi_1^{c_2},\end{cases}$$

and $\varphi_{c_1}=\varphi_{c_1}(x-ct)$ is a "0-to-a" TW corresponding to c_1 and $\psi_{c_2}=\psi_{c_2}(x+ct)$ is a "a-to-0" TW corresponding to c_2 (see Subsection 1.2.2).
• *Let $N\geq 2$. An initial data $u_0=u_0(x)$ satisfying* (5) *is called "not-reacting" if*

$$u_0(x)\leq\widetilde{u_0}(|x|), \quad \textit{for all } x\in\mathbb{R}^N,$$

where $\widetilde{u_0}=\widetilde{u_0}(y)$ ($y\in\mathbb{R}$) is a radial "not-reacting" initial datum in $N=1$.

Definition 2.14. *Again we separate the cases $N=1$ or $N\geq 2$.*
• *Let $N=1$. An initial data $u_0=u_0(x)$ satisfying* (5) *is called "reacting" if there is $0<\widetilde{c}<c_*$ such that for all $0\leq c\leq\widetilde{c}$, it holds*

$$u_0(x)\geq\max\{\underline{\varphi},\underline{\psi}\}(x), \quad \textit{for all } x\in\mathbb{R},$$

where

$$\underline{\varphi}(\xi):=\begin{cases}\varphi_c(\xi) & \textit{if } \xi_0^c\leq\xi\leq\xi_1^c \\ 0 & \textit{otherwise}\end{cases} \qquad \underline{\psi}(\xi):=\begin{cases}\psi_c(\xi) & \textit{if } \xi_0^{c'}\leq\xi\leq\xi_1^{c'} \\ 0 & \textit{otherwise}\end{cases}$$

and $\varphi_c=\varphi_c(x-ct)$ is a "change-sign" TW (of type 2) corresponding to c and $\psi_c=\psi_c(x+ct)$ is its reflection (see Subsection 1.2.1 and 1.2.2).
• *Let $N\geq 2$. An initial data $u_0=u_0(x)$ satisfying* (5) *is called "reacting" if there is $0<c<c_*$ such that it holds*

$$u_0(x)>\underline{\varphi}(|x|-c\bar{t}) \quad \textit{for all } x\in\mathbb{R}^N,$$

where

$$\underline{\varphi}(\xi):=\begin{cases}\varphi(0) & \textit{if } \xi\leq 0 \\ \varphi(\xi) & \textit{if } 0\leq\xi\leq\xi_1 \\ 0 & \textit{otherwise}\end{cases} \qquad \textit{with } \varphi(0)=\max\varphi(\xi),$$

$\bar{t}>0$ is large enough and $\varphi=\varphi(\xi)$ is a "change-sign" TW (of type 2) corresponding to c (the value of the time $\bar{t}>0$ will be specified in the proof of Theorem 2.2).

We are now ready to prove Theorem 2.2.

Proof of Theorem 2.2: Part (i). We take a "non-reacting" initial datum $u_0 = u_0(x)$ (see Definition 2.13) and we prove that the solution $u = u(x,t)$ to problem (3) satisfy

$$u(x,t) \to 0 \text{ uniformly in } \mathbb{R}^N, \quad \text{as } t \to +\infty.$$

Let us firstly consider the case $N = 1$. Since $u_0 = u_0(x)$ is "not-reacting" there are $c_1, c_2 \geq c_*$ such that $u_0(x) \leq \min\{\overline{\varphi}, \overline{\psi}\}(x)$ in $\mathbb{R}$, as in Definition 2.13.
Note that both $\overline{\varphi}(\xi) = \overline{\varphi}(x - c_1 t)$ and $\overline{\psi}(\xi) = \overline{\psi}(x + c_2 t)$ are solutions to the equation in (3), and at time $t = 0$, we have $u_0(x) \leq \overline{\varphi}(x)$ and $u_0(x) \leq \overline{\psi}(x)$ for all $x \in \mathbb{R}$. Consequently, from the Comparison Principle we deduce $u(x,t) \leq \overline{\varphi}(x - c_1 t)$ and $u(x,t) \leq \overline{\psi}(x + c_2 t)$ for all $x \in \mathbb{R}$ and $t > 0$, and, since $\overline{\varphi}(x - c_1 t) = 0$ for all $x \leq \xi_0^{c_1} + c_1 t$ and $\overline{\psi}(x + c_2 t) = 0$ for all $x \geq \xi_1^{c_2} - c_2 t$, we deduce that there is a time $t_{c_1,c_2} > 0$, such that $u(x,t) = 0$ for all $t \geq t_{c_1,c_2}$. This conclude the proof for the case $N = 1$.
Before moving forward, we show that if $N = 1$ and $u = u(x,t)$ is a solution to

$$\begin{cases} \partial_t u = \partial_x \left(|\partial_x u^m|^{p-2} \partial_x u^m \right) + f(u) & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}, \end{cases}$$

with initial data $u_0 = u_0(x)$ satisfying (5), $u_0(x) = u_0(-x)$ and $u_0(\cdot)$ non-increasing for all $x \geq 0$, then $u(\cdot,t)$ is non-increasing w.r.t. $x \geq 0$, for all $t > 0$. So, fix $h > 0$ and let $v = v(x,t)$ be the solution to the problem

$$\begin{cases} \partial_t v = \partial_x \left(|\partial_x v^m|^{p-2} \partial_x v^m \right) + f(v) & \text{in } \mathbb{R} \times (0,\infty) \\ v(x,0) = v_0(x) := u_0(x+h) & \text{in } \mathbb{R}. \end{cases}$$

Hence, since $v_0(x) \leq u_0(x)$ we deduce $v(x,t) \leq u(x,t)$ and, by uniqueness of the solutions, it follows $v(x,t) = u(x+h,t)$. Hence, we obtain that $u(\cdot,t)$ is non-increasing for all $t \geq 0$ thanks to the arbitrariness of $x \geq 0$ and $h \geq 0$.

Now, assume $N \geq 2$ and consider radial solutions to problem (3), i.e., solutions $u = u(r,t)$ to the problem

$$\begin{cases} \partial_t u = \partial_r \left(|\partial_r u^m|^{p-2} \partial_r u^m \right) + \frac{N-1}{r} |\partial_r u^m|^{p-2} \partial_r u^m + f(u) & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u(r,0) = u_0(r) & \text{in } \mathbb{R}_+ \times \{0\}, \end{cases} \tag{2.43}$$

where $r = |x|$, $x \in \mathbb{R}^N$, and $u_0(\cdot)$ is a radially decreasing "not-reacting" initial datum. Moreover, let $\overline{u} = \overline{u}(r,t)$ be a solution to the problem

$$\begin{cases} \partial_t \overline{u} = \partial_r \left(|\partial_r \overline{u}^m|^{p-2} \partial_r \overline{u}^m \right) + f(\overline{u}) & \text{in } \mathbb{R}_+ \times (0,\infty) \\ \overline{u} = u & \text{in } \{0\} \times (0,\infty) \\ \overline{u}(r,0) = u_0(r) & \text{in } \mathbb{R}_+ \times \{0\}. \end{cases}$$

For what explained before, we have $\partial_r u(r,t) \leq 0$ in $\mathbb{R}_+ \times (0,\infty)$, and so $\overline{u} = \overline{u}(r,t)$ is a super-solution to (2.43). But $\overline{u} = \overline{u}(r,t)$ is a solution of the one-dimensional equation with "not-reacting" initial data, and so, from the case $N = 1$, it follows

$$\overline{u}(r,t) = 0 \text{ in } \mathbb{R}_+ \cup \{0\}, \quad \text{for all } t \geq t_{c_1,c_2},$$

and by the comparison, we deduce the same for $u = u(r,t)$, concluding the proof of Part (i). □

Proof of Theorem 2.2: Part (ii), case $N = 1$. We fix $N = 1$ and we proceed in two steps.

Step1: Propagation of minimal super-level sets. Consider a "reacting" initial datum $u_0 = u_0(x) \geq \max\{\underline{\varphi}, \underline{\psi}\}(x)$, $x \in \mathbb{R}$, where $\underline{\varphi}(\xi) = \underline{\varphi}(x - ct)$ and $\underline{\psi}(\xi) = \underline{\psi}(x + ct)$ are defined in Definition 2.14. From the ODEs analysis (see in particular Subsection 1.2.2), we can assume that for all $0 \leq c \leq \widetilde{c}$, it holds

$$\underline{\varphi}(0) = \underline{\psi}(0) = a + \underline{\delta}_c < 1, \qquad \underline{\delta}_c \geq \underline{\delta}_0,$$

for some $0 < \underline{\delta}_0 < 1 - a$. Now, since $u_0(x) \geq \underline{\varphi}(x)$ and $u_0(x) \geq \underline{\psi}(x)$, we deduce by comparison $u(x,t) \geq \underline{\varphi}(x-ct)$ and $u(x,t) \geq \underline{\psi}(x+ct)$ for all $x \in \mathbb{R}^N$ and $t > 0$. Hence, by the arbitrariness of $0 \leq c \leq \widetilde{c}$, we obtain that

$$u(x,t) \geq a + \underline{\delta}_0 \quad \text{in } \{|x| \leq \widetilde{c}t\}, \ \text{ for all } t > 0.$$

Step2: Convergence to 1 on compact sets. Now, fix $\varepsilon > 0$ small and $\widetilde{\varrho} > 0$ arbitrarily large. Then, we have

$$u(x,t) \geq a + \underline{\delta}_0 \quad \text{in } \{|x| \leq \widetilde{\varrho}\}, \ \text{ for all } t \geq t_{\widetilde{\varrho,c}} := \widetilde{\varrho}/\widetilde{c},$$

which implies

$$f(u) \geq q_{\underline{\delta}_0}(1-u) \quad \text{in } \{|x| \leq \widetilde{\varrho}\} \times [t_{\widetilde{\varrho,c}}, \infty), \qquad \text{with } q_{\underline{\delta}_0} = \frac{f(\underline{\delta})}{1-\underline{\delta}_0}, \tag{2.44}$$

for some $0 < \underline{\delta} \leq \underline{\delta}_0$ small enough. Thus, the solution $\underline{u} = \underline{u}(x,t)$ to the problem

$$\begin{cases} \partial_t \underline{u} = \partial_x \left(|\partial_x \underline{u}^m|^{p-2} \partial_x \underline{u}^m \right) + q_{\underline{\delta}_0}(1-\underline{u}) & \text{in } \{|x| \leq \widetilde{\varrho}\} \times [t_{\widetilde{\varrho,c}}, \infty) \\ \underline{u}(x,t) = a + \underline{\delta}_0 & \text{in } \partial\{|x| \leq \widetilde{\varrho}\} \times [t_{\widetilde{\varrho,c}}, \infty) \\ \underline{u}(x, t_{\widetilde{\varrho,c}}) = a + \underline{\delta}_0 & \text{in } \{|x| \leq \widetilde{\varrho}\} \end{cases} \tag{2.45}$$

is a sub-solution to problem (3) in $\{|x| \leq \widetilde{\varrho}\} \times [t_{\widetilde{\varrho,c}}, \infty)$, and so by the Comparison Principle, we obtain $\underline{u}(x,t) \leq u(x,t)$ in $\{|x| \leq \widetilde{\varrho}\} \times [t_{\widetilde{\varrho,c}}, \infty)$. Then, since $\widetilde{\varrho} > 0$ can eventually be taken larger, we can repeat the proof of Theorem 2.4, to show that there exist $t_1 > 0$ depending on $\varepsilon > 0$, such that

$$u(x,t) \geq \underline{u}(x,t) \geq 1 - \varepsilon \quad \text{in } \{|x| \leq \widetilde{a}_\varepsilon \widetilde{\varrho}\} \ \text{ for all } t \geq t_1,$$

where $0 < \widetilde{a}_\varepsilon < 1$ is a factor depending only on $\varepsilon > 0$. This conclude the proof in the case $N = 1$, since $\widetilde{\varrho} > 0$ can be chosen arbitrarily large. □

Proof of Theorem 2.2: Part (ii), case $N \geq 2$. We fix $N \geq 2$ and, proceeding as in part (i), we consider the radial problem

$$\begin{cases} \partial_t u = \partial_r \left(|\partial_r u^m|^{p-2} \partial_r u^m \right) + \frac{N-1}{r} |\partial_r u^m|^{p-2} \partial_r u^m + f(u) & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u(r,0) = u_0(r) & \text{in } \mathbb{R}_+ \times \{0\}, \end{cases}$$

where $r = |x|$, $x \in \mathbb{R}^N$, and $u_0(\cdot)$ is a radially decreasing "reacting" initial datum. By definition, we can assume that for any fixed $\varepsilon > 0$ (small), there is $0 < c < c_*$, such that

$$u_0(r) > \underline{\varphi}(r - (c+\varepsilon)\bar{t}) \quad \text{for all } r > 0,$$

where $\underline{\varphi} = \varphi(\xi)$ is as in Definition 2.14 part (ii) and $\bar{t} > 0$ is large enough and will be chosen later. Now, setting $\delta := 1 - \varepsilon$, we define

$$\begin{cases} \underline{u}(r,t) = \underline{\varphi}(\delta^{1/p} r - c\delta t) & \text{if } m > 1, \ p > 1 \ (\gamma > 0) \\ \underline{u}(r,t) = \underline{\varphi}(r - c\delta t) & \text{if } 0 < m < 1, \ p > 2 \ (\gamma > 0), \end{cases}$$

as in the proof of Theorem 2.1. Repeating the same procedure of that proof, it is easily seen that $\underline{u} = \underline{u}(r,t)$ is a sub-solution to the equation in problem (2.43) in $\mathbb{R}_+ \times [\widetilde{t}, \infty)$ where $\widetilde{t} > 0$ is suitably chosen (large enough). Consequently, taking $\bar{t} := \widetilde{t} \gg 0$ and noting that we can assume $u_0(r) \geq \underline{u}(r, \bar{t})$ for any $r > 0$ (since $\delta = 1 - \varepsilon$ and $\varepsilon > 0$ is arbitrarily small), we conclude by comparison

$$u(r,t) \geq \underline{u}(r, t + \bar{t}) \geq \varphi(0) = a + \underline{\delta}_0 \quad \text{in } \{r = |x| \leq \widetilde{c}t\}, \ \text{ for all } t > 0,$$

for some $\underline{\delta}_0 > 0$ and $\widetilde{c} := c\delta$ (see also Lemma 5.1 of [13] for the linear setting). Moreover, exactly as in *Step2* of the case $N = 1$, we have

$$f(u) \geq q_{\underline{\delta}_0}(1-u) \quad \text{in } \{|x| \leq \widetilde{\varrho}\} \times [t_{\widetilde{\varrho,c}}, \infty), \qquad \text{with } q_{\underline{\delta}_0} = \frac{f(\underline{\delta}_0)}{1-\underline{\delta}_0},$$

for any $\widetilde{\varrho} > 0$ and $t_{\widetilde{\varrho,c}} > 0$ large enough, and so we can newly repeat the construction of Theorem 2.4 to show that for all $\varepsilon > 0$ (small) and $\widetilde{\varrho} > 0$ (large), there exists $t_1 > 0$ such that

$$u(r,t) \geq 1 - \varepsilon \quad \text{in } \{r = |x| \leq \widetilde{a}_\varepsilon \widetilde{\varrho}\} \ \text{ for all } t \geq t_1,$$

where $0 < \widetilde{a}_\varepsilon < 1$ is as in the case $N = 1$, concluding the proof of the case $N \geq 2$. □

Remark. The above proof strongly relies on the fact that the function

$$\begin{cases} \underline{u}(r,t) = \underline{\varphi}(\delta^{1/p} r - c\delta t) & \text{if } m > 1,\ p > 1\ (\gamma > 0) \\ \underline{u}(r,t) = \underline{\varphi}(r - c\delta t) & \text{if } 0 < m < 1,\ p > 2\ (\gamma > 0), \end{cases}$$

defined depending on the value $m > 0$ and $p > 1$ such that $\gamma > 0$, is a sub-solution to problem (2.43) for large times $t \gg 0$. As we have mentioned before, this fact can be easily showed by repeating the proof of Theorem 2.1 for the Fisher-KPP framework. This parallelism is due to the fact that the main difficulty is to study the sign of the quantity

$$\partial_t \underline{u} - \partial_r \left(|\partial_r \underline{u}^m|^{p-2} \partial_r \underline{u}^m \right) - \frac{N-1}{r} |\partial_r \underline{u}^m|^{p-2} \partial_r \underline{u}^m - f(\underline{u})$$

near the points in which $\underline{u} = 0$, i.e. $\varphi = 0$ (here φ denotes the profile of a "change-sign" TW. The behaviour of φ near the "change-sign" points is completely understood and is the same for both reactions of type C and Fisher-KPP reactions (cfr. with Subsection 1.2.1 and 1.2.2).

Proof of Theorem 2.2: Part (iii). Let us prove that for all radially decreasing initial data $u_0 = u_0(x)$ satisfying (5) and for all $c > c_*(m,p,f)$ it holds

$$u(x,t) \to 0 \text{ uniformly in } \{|x| \geq ct\}, \qquad \text{as } t \to +\infty.$$

This part is the easiest and it actually coincides with Theorem 2.4. Here we just re-view the main ideas. If $N = 1$, we fix $c > c_*$, $\varepsilon > 0$, and we consider the functions

$$\overline{v}(x,t) := \varphi(x - c_* t), \qquad \overline{w}(x,t) := \psi(x + c_* t),$$

where $\varphi = \varphi(\xi)$ is the *finite admissible* TW studied in Theorem 1.1, part (ii), with its reflection $\psi(\xi) = \psi(x + c_* t)$. Since $u_0 = u_0(x)$ satisfies (5), we can assume $u_0(x) \leq \varphi(x)$ and $u_0(x) \leq \psi(x)$ for all $x \in \mathbb{R}$, and so, thanks to the Comparison Principle, we obtain both $u(x,t) \leq \overline{v}(x,t)$ and $u(x,t) \leq \overline{w}(x,t)$. Thus, since $\overline{v}(x,t) \leq \varepsilon$ for $x \geq c_* t + \xi_0$ and $\overline{w}(x,t) \leq \varepsilon$ for $x \leq -c_* t + \xi_0$ and $c > c_*$, we deduce that $u(x,t) \leq \varepsilon$ in $\{|x| \geq ct\}$ for $t > 0$ large enough.
We point out that if $\gamma > 0$ then $\overline{v}(x,t) = 0$ for $x \geq c_* t + \xi_0$ and $\overline{w}(x,t) = 0$ for $x \leq -c_* t + \xi_0$ which implies that $u = u(x,t)$ has a *free boundary*, too, whilst this does not happen when $\gamma = 0$, since the TW solutions are positive everywhere.

When $N \geq 2$, we follow the proof of Part (i), using that solutions of problem (1) with $N = 1$ are super-solution for radial solutions of the same problem and so, by comparison, the thesis follows.

Now, we show that for all "reacting" initial data $u_0 = u_0(x)$ and for all $0 < c < c_*(m,p,f)$, it holds

$$u(x,t) \to 1 \text{ uniformly in } \{|x| \leq ct\}, \qquad \text{as } t \to +\infty.$$

Let us consider the case $N = 1$. From part (ii) we obtain the for all $\varepsilon > 0$, $\widetilde{\varrho} > 0$, and all "reacting" initial data $u_0 = u_0(x)$, there exist $t_1 > 0$, such that

$$u(x,t) \geq 1 - \varepsilon \quad \text{in } \{|x| \leq \widetilde{\varrho}\} \text{ for all } t \geq t_1.$$

Hence, for all $0 \leq c < c_*$, taking eventually $\widetilde{\varrho} > 0$ larger, there is a "change-sign" TW $\overline{\varphi}(\xi) = \overline{\varphi}(x - ct)$ and its reflection $\overline{\psi}(\xi) = \overline{\psi}(x + ct)$ (we ask the reader not to confuse these TWs with the ones employed in *Step1*, part (ii)) such that

$$\underline{\varphi}(0) = \underline{\psi}(0) = 1 - \varepsilon \quad \text{and} \quad u(x,t_1) \geq \underline{\varphi}(x), \quad u(x,t_1) \geq \underline{\psi}(x) \quad x \in \mathbb{R}.$$

Consequently, by comparison we have $u(x, t_1 + t) \geq \underline{\varphi}(x - ct)$ and $u(x, t_1 + t) \geq \underline{\psi}(x + ct)$ for all $x \in \mathbb{R}^N$ and $t > 0$, and the level $1 - \varepsilon$ propagate with speed c. Hence, using again the arbitrariness of $0 \leq c < c_*$, we deduce

$$u(x,t) \geq 1 - \varepsilon \quad \text{in } \{|x| \leq ct\} \text{ for all } t \geq t_2,$$

for some $t_2 = t_2(\varepsilon, c)$ large enough. This shows our statement, since $\varepsilon > 0$ has been chosen arbitrarily small.

Finally, when $N \geq 2$, following the proof of part (ii), case $N \geq 2$ and using again the sub-solution constructed in the proof of Theorem 2.4 with speed $c < c_*$ and $\underline{\varphi}(0) = 1 - \varepsilon$, we conclude as in the case $N = 1$. □

2.5 Proof of Theorem 2.3

In this section we prove Theorem 2.3. We then consider reactions of type C′, i.e. satisfying (4):

$$\begin{cases} f(0) = f(a) = f(1) = 0, \quad 0 < f(u) \leq f'(0)u \text{ in } (0,a), \ f(u) < 0 \text{ in } (a,1) \\ f \in C^1([0,1]), \qquad f'(0) > 0, \ f'(a) < 0, \ f'(1) > 0 \\ f(\cdot) \text{ has a unique critical point in } (0,a) \text{ and a unique critical point in } (a,1). \end{cases}$$

and the radial problem (2.1):

$$\begin{cases} \partial_t u = \Delta_{p,r} u^m + f(u) & \text{in } \mathbb{R}_+ \times (0,\infty) \\ u(r,0) = u_0(r) & \text{in } \mathbb{R}_+. \end{cases}$$

As in the ODEs part, some of our proofs rely on the methods used to show Theorem 2.1 (Fisher-KPP framework) and can be recovered by scaling (see the beginning of the introduction to Part I).

This time, Part (ii) (of Theorem 2.3) is the most delicate one and there are significant differences between the range $\gamma > 0$ and $\gamma = 0$. As always, we divide the proof depending on the spacial dimension $N = 1$ or $N \geq 2$. As we will see in a moment, in the case $N = 1$, we construct a super-solution to prove that $u = u(x,t)$ reaches the level $0 < a < 1$ in finite time and a second super-solution combined to a scaling technique, to show that $u = u(x,t)$ converges uniformly to zero in the "outer sets" $\{|x| \geq ct\}$ as $t \to +\infty$. Part (i) is based on the proof of Theorem 2.1.

Proof of Theorem 2.3: Case $N = 1$, range $\gamma > 0$. Fix $m > 0$ and $p > 1$ such that $\gamma > 0$. We begin with two preliminary steps, crucial in the rest of the proof.

Step0. We first prove that for all $\varepsilon > 0$, there exists a waiting time $t_\varepsilon > 0$ such that

$$u(x,t) \leq a + \varepsilon, \quad \text{for all } x \in \mathbb{R}, \quad t \geq t_\varepsilon.$$

To do this we employ the “increasing a-to-1” TWs and their reflections, found in Theorem 1.1, cfr. with Subsection 1.2.3. To be more specific, we fix $c = 1$ and we consider a TW profile $\varphi(\xi) = \varphi(x - t)$ moving toward the right direction, satisfying

$$\varphi(-\infty) = a, \qquad \varphi(\xi_0) = 1, \qquad \varphi'(\xi) > 0 \quad \text{for all } \xi \le \xi_0,$$

and its “reflection” $\psi(\xi) = \psi(x + t)$, moving toward the left direction, satisfying

$$\psi(+\infty) = a, \qquad \psi(\xi_1) = 1, \qquad \psi'(\xi) < 0 \quad \text{for all } \xi \ge \xi_1,$$

for some $\xi_0, \xi_1 \in \mathbb{R}$ (cfr with formula (1.25)). Defining

$$\overline{\varphi}(\xi) := \begin{cases} \varphi(\xi) & \text{if } \xi \le \xi_0 \\ 1 & \text{if } \xi \ge \xi_0, \end{cases} \qquad \overline{\psi}(\xi) := \begin{cases} 1 & \text{if } \xi \le \xi_1 \\ \psi(\xi) & \text{if } \xi \ge \xi_1, \end{cases}$$

and recalling that $u_0 \in C_c(\mathbb{R})$ with $0 \le u_0 \le 1$ we can assume both $u_0(x) \le \overline{\varphi}(x)$ and $u_0(x) \le \overline{\psi}(x)$ for all $x \in \mathbb{R}$.
Now, we fix $\varepsilon > 0$ small, such that $1 - \varepsilon > 0$. Defining the function $\overline{v}(x,t) := \overline{\varphi}(x - (1-\varepsilon)t)$ and using the definition of $\overline{\varphi} = \overline{\varphi}(\xi)$, we get that

$$\begin{aligned} &\partial_t \overline{v} - \partial_x \left(|\partial_x \overline{v}^m|^{p-2} \partial_x \overline{v}^m \right) - f(\overline{v}) \\ &= -(1-\varepsilon)\overline{\varphi}' - \left[|\left(\overline{\varphi}^m\right)'|^{p-2} \left(\overline{\varphi}^m\right)' \right]' - f(\overline{\varphi}) = \varepsilon\overline{\varphi}' - \overline{\varphi}' - \left[|\left(\overline{\varphi}^m\right)'|^{p-2} \left(\overline{\varphi}^m\right)' \right]' - f(\overline{\varphi}) \\ &= \varepsilon\overline{\varphi}' \ge 0, \quad \text{for all } \xi \le \xi_0, \end{aligned}$$

where $\overline{\varphi}'$ stands for the derivative of $\overline{\varphi}(\cdot)$ w.r.t. ξ. Note that when $\xi \ge \xi_0$, $\overline{v}(x,t) = 1$, i.e., it is just a stationary state of the equation in (1) and the equality holds in the last inequality for $\xi \ge \xi_0$. In particular, it follows that the function $\overline{v} = \overline{v}(x,t)$ is a super-solution for the equation in (1).
Similarly, one can define $\overline{w}(x,t) = \overline{\psi}(x - (1+\varepsilon)t)$ and prove it is a super-solution too. In this case the function $\overline{w} = \overline{w}(x,t)$ is wave moving toward the left direction.
Hence, thanks to the comparison principle and remembering that $u_0(x) \le \overline{\varphi}(x)$ and $u_0(x) \le \overline{\psi}(x)$, we deduce

$$u(x,t) \le \overline{v}(x,t), \quad \text{and} \quad u(x,t) \le \overline{w}(x,t), \quad \text{in } \mathbb{R}^N \times [0,\infty).$$

Moreover, thanks to the properties of $\varphi = \varphi(\xi)$ and $\psi = \psi(\xi)$, we deduce the existence of $\xi_\varepsilon > 0$, such that

$$\begin{aligned} \overline{v}(x,t) &\le a + \varepsilon, \quad \text{for all } x \le -\xi_\varepsilon + (1-\varepsilon)t \\ \overline{w}(x,t) &\le a + \varepsilon, \quad \text{for all } x \le \xi_\varepsilon - (1+\varepsilon)t. \end{aligned}$$

Thus, we get $u(x,t) \le a + \varepsilon$ in $\mathbb{R}^N$ if

$$-\xi_\varepsilon + (1-\varepsilon)t \ge \xi_\varepsilon - (1+\varepsilon)t,$$

i.e. $t \ge t_\varepsilon := \xi_\varepsilon$.

Step1. In this step, we construct a global super-solution to problem (1) to show that our solution $u = u(x,t)$ propagates with *finite* speed of propagation, i.e., $u = 0$ outside an interval of $\mathbb{R}$ with radius expanding in time. Consider the solution to the problem

$$\begin{cases} \partial_t \overline{u} = \partial_x \left(|\partial_x \overline{u}^m|^{p-2} \partial_x \overline{u}^m \right) + f'(0)\overline{u} & \text{in } \mathbb{R} \times (0,\infty) \\ \overline{u}(x,0) = u_0(x) & \text{in } \mathbb{R}. \end{cases}$$

Observe that $u(x,t) \le \overline{u}(x,t)$ in $\mathbb{R} \times (0,\infty)$ since $f(u) \le f'(0)u$, thanks to the first assumption in (4) and the comparison principle. Furthermore, the function defined by

$$\widetilde{u}(x,\tau) = e^{-f'(0)t}\overline{u}(x,t), \qquad \text{with } \tau(t) = \frac{1}{f'(0)\gamma}\left(e^{f'(0)\gamma t} - 1\right), \quad t \ge 0,$$

satisfies the purely diffusive equation

$$\begin{cases} \partial_\tau \widetilde{u} = \partial_x \left(|\partial_x \widetilde{u}^m|^{p-2} \partial_x \widetilde{u}^m \right) & \text{in } \mathbb{R} \times (0,\infty) \\ \widetilde{u}(x,0) = u_0(x) & \text{in } \mathbb{R}. \end{cases}$$

Consequently, since $\widetilde{u} = \widetilde{u}(x,\tau)$ has *finite* speed of propagation (see for instance [197, 198]), we deduce the same for $\overline{u} = \overline{u}(x,t)$, and so for $u = u(x,t)$. We conclude this step pointing out that the same procedure can be adapted (with obvious changes) to the case $N \ge 1$.

Step2. In this part of the proof, we show that for all $c > c_*(m,p,f)$, there exists $t_1 = t_1(c) > 0$ such that

$$u(x,t) = 0, \quad \text{in } \{|x| \ge ct\}, \quad \text{for all } t \ge t_1.$$

So, fix $\varepsilon > 0$ small and $c > c_*(m,p,f)$. We assume for a moment that $0 \le u(x,t_\varepsilon) < a$ for all $x \in \mathbb{R}$, where $t_\varepsilon > 0$ is the one found in *Step0*. Moreover, we know that $u(x,t_\varepsilon) = 0$ outside an interval of $\mathbb{R}$ of radius large enough (see *Step1*). Hence, we define

$$\overline{v}(x,t) := \varphi(x - c_* t), \qquad \overline{w}(x,t) := \psi(x + c_* t),$$

where $\varphi = \varphi(\xi)$ is the *finite a-admissible* TW studied in Theorem Theorem 1.1 (part (iiI), range $\gamma > 0$), satisfying $\varphi(-\infty) = a$, $\varphi(\xi) = 0$ for all $\xi \ge \xi_0$, and $\psi = \psi(\xi)$ is its "reflection". Consequently, up to a left-right shift, we can assume $u(x,t_\varepsilon) \le \varphi(x)$ and $u(x,t_\varepsilon) \le \psi(x)$, and so, by the comparison principle we deduce

$$u(x,t+t_\varepsilon) \le \overline{v}(x,t), \qquad u(x,t+t_\varepsilon) \le \overline{w}(x,t), \quad \text{in } \mathbb{R}^N \times (0,\infty).$$

Thus, since $\overline{v}(x,t) = 0$ for $x \ge c_* t + \xi_0$ and $\overline{w}(x,t) = 0$ for $x \le -c_* t + \xi_0$ and $c > c_*$, we deduce that $u(x,t) = 0$ in $\{|x| \ge ct\}$ for large times exactly as in the proof of Theorem 2.2, Part (iii).
Now, if $u(x,t_\varepsilon) \ge a$ for some $x \in \mathbb{R}$, it must be $u(x,t_\varepsilon) \le a + \varepsilon$ in $\mathbb{R}$, from what proved in *Step0*. We consider the re-scaling of $u = u(x,t)$ defined by

$$u_\varepsilon(y,\tau) = a^{-1}(a+2\varepsilon)u(x,t), \quad \text{where } y = \left[a^{-1}(a+2\varepsilon)\right]^{\frac{m(p-1)}{p}} x, \quad \tau = a^{-1}(a+2\varepsilon)t,$$

which satisfies the equation

$$\partial_\tau u_\varepsilon = \partial_y \left(|\partial_y u_\varepsilon^m|^{p-2} \partial_y u_\varepsilon^m \right) + f\left(a(a+2\varepsilon)^{-1} u_\varepsilon \right), \quad \text{in } \mathbb{R}^N \times (0,\infty). \tag{2.46}$$

Note that now $f_\varepsilon(u_\varepsilon) := f\left(a(a+2\varepsilon)^{-1}u_\varepsilon\right)$ satisfies $f_\varepsilon(a+2\varepsilon) = f(a) = 0$. Hence, from Theorem 1.1 (part (iiI), range $\gamma > 0$), there exists a critical speed $c_*^\varepsilon = c_*(m,p,\varepsilon) > 0$ and a corresponding $(a+2\varepsilon)$-*admissible* TW with *finite* profile $\varphi_\varepsilon = \varphi_\varepsilon(\xi)$, and $\xi = x - c_*^\varepsilon t$:

$$\varphi_\varepsilon(-\infty) = a + 2\varepsilon, \quad \varphi_\varepsilon(\xi) = 0 \text{ for all } \xi \ge \xi_0^\varepsilon,$$

for some $\xi_0^\varepsilon \in \mathbb{R}$, satisfying equation (2.46). Thus, if $\widetilde{u}_\varepsilon = \widetilde{u}_\varepsilon(y,\tau)$ denotes the solution to equation (2.46) with $\widetilde{u}_\varepsilon(y,0) = u(y,t_\varepsilon) \le a + \varepsilon$ and $\overline{u}_\varepsilon(y,\tau) = \varphi_\varepsilon(y - c_*^\varepsilon \tau)$, we can repeat the comparison procedure with the assumption $u(x,t_\varepsilon) \le a$, since we can now assume $u(y,t_\varepsilon) \le \varphi_\varepsilon(y)$ and so, $\widetilde{u}_\varepsilon(y,\tau) \le \overline{u}_\varepsilon(y,\tau)$. We finally obtain $u(x,t) = 0$ in $\{|x| \ge ct\}$ for large times for the arbitrariness of $\varepsilon > 0$.

Step3. In this final step, we prove that for all for all $0 < \varepsilon < a$ and for all $0 < c < c_*(m,p,f)$, there exists $t_1' = t_1'(\varepsilon, c) > 0$ such that the solution $u = u(x,t)$ satisfies

$$u(x,t) \geq a - \varepsilon, \quad \text{in } \{|x| \leq ct\}, \quad \text{for } t \geq t_1'.$$

This follows by considering the solution $\underline{u} = \underline{u}(x,t)$ to the problem

$$\begin{cases} \partial_t \underline{u} = \partial_x \left(|\partial_x \underline{u}^m|^{p-2} \partial_x \underline{u}^m \right) + f(\underline{u}) & \text{in } \mathbb{R} \times (0,\infty) \\ \underline{u}(x,0) = \underline{u}_0(x) & \text{in } \mathbb{R}, \end{cases}$$

where $\underline{u}_0 \in C_c(\mathbb{R})$ is defined by $\underline{u}_0(x) := \min\{a, u_0(x)\}$. Consequently, we deduce $\underline{u}(x,t) \leq u(x,t)$ and $0 \leq \underline{u}(x,t) \leq a$ in $\mathbb{R} \times (0,\infty)$ thanks to the comparison principle, and, furthermore:

$$\underline{u}(x,t) \geq a - \varepsilon, \quad \text{in } \{|x| \leq ct\}, \quad \text{for } t \text{ large enough.}$$

This last property easily follows by applying Proposition 2.5, Theorem 2.4 and Theorem 2.1 to $\underline{u} = \underline{u}(x,t)$ and remembering the scaling property quoted at the beginning of this section. We could have also repeat the construction done in the Fisher-KPP framework using the "change sign" TWs introduced in Subsection 1.2.1. Note that this procedure applies to higher dimensions $N \geq 1$ too, as explained in Theorem 2.1. □

Proof of Theorem 2.3: Case $N = 1$, range $\gamma = 0$. Fix $m > 0$ and $p > 1$ such that $\gamma = 0$. The proof in this range is similar to the previous one, with some modifications.

Step0'. This step coincides with *Step0* of the range $\gamma > 0$.

Step1'. In this step we proceed as in *Step1* of the range $\gamma > 0$, considering the super-solution given by the problem

$$\begin{cases} \partial_t \overline{u} = \partial_x \left(|\partial_x \overline{u}^m|^{p-2} \partial_x \overline{u}^m \right) + f'(0)\overline{u} & \text{in } \mathbb{R} \times (0,\infty) \\ \overline{u}(x,0) = u_0(x) & \text{in } \mathbb{R}, \end{cases}$$

and the function $\widetilde{u}(x,t) = e^{-f'(0)t}\overline{u}(x,t)$ satisfying

$$\begin{cases} \partial_t \widetilde{u} = \partial_x \left(|\partial_x \widetilde{u}^m|^{p-2} \partial_x \widetilde{u}^m \right) & \text{in } \mathbb{R} \times (0,\infty) \\ \widetilde{u}(x,0) = u_0(x) & \text{in } \mathbb{R}. \end{cases}$$

This time $\widetilde{u} = \widetilde{u}(x,t)$ does not generally propagate with finite speed of propagation, but it is everywhere positive for all $t > 0$. In the next paragraphs, we provide a bound from above for $\widetilde{u} = \widetilde{u}(x,t)$ which will be enough for our purposes. The main tool are the Barenblatt solutions presented in the introduction. Indeed, since u_0 has compact support, there are a mass $M > 0$ large enough and delay $\theta > 0$ such that $u_0(x) \leq B_M(x,\theta)$ for all $x \in \mathbb{R}$. Thus, from the Comparison Principle, we obtain $\widetilde{u}(x,t) \leq B_M(x, t+\theta)$ for all $x \in \mathbb{R}$ and $t > 0$. Coming back to the solution $u = u(x,t)$, this gives

$$\begin{aligned} u(x,t) \leq \overline{u}(x,t) = e^{f'(0)t}\widetilde{u}(x,t) \leq e^{f'(0)t} B_M(x,t+\theta) = e^{f'(0)t}(t+\theta)^{-1/p} F_M(x(t+\theta)^{-1/p}) \\ = C_M (t+\theta)^{-1/p} \exp\left[f'(0)t - k(t+\theta)^{-1/(p-1)} |x|^{p/(p-1)} \right], \end{aligned}$$

where $k = (p-1)p^{-p/(p-1)}$ (cfr. with the introduction, range $\gamma = 0$). In particular, we obtain

$$u(x,t) \leq C_{M,\theta}(t) \exp\left[-k_\theta(t) |x|^{p/(p-1)} \right], \quad \text{in } \mathbb{R} \times (0,\infty), \tag{2.47}$$

where $C_{M,\theta}(t) = C_M(t+\theta)^{-1/p} e^{f'(0)t}$ and $k_\theta(t) = k(t+\theta)^{-1/(p-1)}$, $t > 0$. Again, this bound can be easily extended to the case $N \geq 1$, with minor changes in the functions $C_{M,\theta}(\cdot)$ and $k_\theta(\cdot)$.

Step2'. As before, in this step we prove that for all $\varepsilon > 0$ and for all $c > c_*(m, p, f)$, there exists $t_1 = t_1(\varepsilon, c) > 0$ such that

$$u(x,t) \le \varepsilon, \quad \text{in } \{|x| \ge ct\}, \quad \text{for all } t \ge t_1.$$

So, we fix $\varepsilon > 0$, $c > c_*(m, p, f)$, and we consider $t_\varepsilon > 0$ given by *Step1'-Step1*. As before, we can assume $u(x, t_\varepsilon) < a$, since the scaling technique exploited in *Step2* of the range $\gamma > 0$ works also in the present setting. Again, we consider

$$\overline{v}(x,t) := \varphi(x - c_*t), \qquad \overline{w}(x,t) := \psi(x + c_*t),$$

where $\varphi = \varphi(\xi)$ is the *positive a-admissible* TW studied in Theorem 1.1 (part (iii), range $\gamma = 0$), satisfying $\varphi(-\infty) = a$, $\varphi(+\infty) = 0$, and $\psi = \psi(\xi)$ is its "reflection". The main difference respect to the range $\gamma > 0$ is neither $u = u(x, t)$ nor $\overline{v} = \overline{v}(x, t)$ (resp. $\overline{w} = \overline{w}(x, t)$) have compact support in $\mathbb{R}$, and so we cannot immediately conclude $u(x, t_\varepsilon) \le \overline{v}(x, 0)$ (resp. $u(x, t_\varepsilon) \le \overline{w}(x, 0)$) in $\mathbb{R}$ (up to a right/left shift) of the profile $\varphi = \varphi(x)$ (resp. $\psi = \psi(x)$).
However, we known that asymptotic behaviour of the tales of $\varphi = \varphi(x)$ and $\psi = \psi(x)$ (cfr. with formula (1.30)):

$$\varphi(x) \sim a_0|x|^{\frac{2}{p}} e^{-\frac{\lambda_*}{m}x}, \quad \text{for } x \sim +\infty, \qquad \psi(x) \sim a_0|x|^{\frac{2}{p}} e^{-\frac{\lambda_*}{m}|x|}, \quad \text{for } x \sim -\infty,$$

where $\lambda_* := (c_*/p)^m$, $a_0 > 0$, and, at the same time,

$$u(x, t_\varepsilon) \le C_{M,\theta}(t_\varepsilon) \exp\left[-k_\theta(t_\varepsilon)|x|^{p/(p-1)}\right], \quad x \in \mathbb{R} \times (0, \infty),$$

from the global bound (2.47) of *Step1'*. Consequently, since $p > 1$, $u(x, t_\varepsilon)$ decays faster than $\varphi(x)$ and $\psi(x)$ when $|x| \sim \infty$, and so we can now assume $u(x, t_\varepsilon) \le \overline{v}(x, 0)$ and $u(x, t_\varepsilon) \le \overline{w}(x, 0)$ for all $x \in \mathbb{R}$ and applying the Comparison Principle to have $u(x, t + t_\varepsilon) \le \overline{v}(x, t)$ and $u(x, t + t_\varepsilon) \le \overline{w}(x, t)$ for all $x \in \mathbb{R}$ and $t > 0$. Thus, using that $\overline{v}(x, t) \le \varepsilon$ for $x \ge c_*t + \xi_0$ and $\overline{w}(x, t) \le \varepsilon$ for $x \le -c_*t + \xi_0$ and $c > c_*$, we deduce that $u(x, t) \le \varepsilon$ in $\{|x| \ge ct\}$ for large times (see also the proof of Theorem 2.2, Part (iii)). □

Proof of Theorem 2.3: Case $N \ge 2$. Fix $m > 0$ and $p > 1$ such that $\gamma > 0$ (the range $\gamma = 0$ is almost identical and we skip it). Again, we focus on radial solutions to problem (1), i.e., solutions $u = u(r, t)$ to problem (2.43):

$$\begin{cases} \partial_t u = \partial_r \left(|\partial_r u^m|^{p-2}\partial_r u^m\right) + \frac{N-1}{r}|\partial_r u^m|^{p-2}\partial_r u^m + f(u) & \text{in } \mathbb{R}_+ \times (0, \infty) \\ u(r, 0) = u_0(r) & \text{in } \mathbb{R}_+ \times \{0\}, \end{cases}$$

where $r = |x|$, $x \in \mathbb{R}^N$, and $u_0(\cdot)$ is a radial decreasing initial datum.

Step1: Convergence to zero in "outer" sets. Proceeding as in the proof of Theorem 2.2 (Part (i)), we can assume $\partial_r u^m \le 0$ in $\mathbb{R}_+ \times (0, \infty)$. Consequently, the solution $\overline{u} = \overline{u}(r, t)$ to the problem

$$\begin{cases} \partial_t \overline{u} = \partial_r \left(|\partial_r \overline{u}^m|^{p-2}\partial_r \overline{u}^m\right) + f(\overline{u}) & \text{in } \mathbb{R}_+ \times (0, \infty) \\ \overline{u} = u & \text{in } \{0\} \times (0, \infty) \\ \overline{u}(r, 0) = u_0(r) & \text{in } \mathbb{R}_+ \times \{0\}, \end{cases}$$

is a super-solution to (2.43) and, at the same time, it is a solution of the one-dimensional equation with compactly supported initial data. Thus, for all $c > c_*(m, p, f)$, it follows

$$u(r, t) = 0 \text{ uniformly in } \{r \ge ct\}, \quad \text{as } t \to +\infty,$$

and by the comparison, we deduce the same for $u = u(r, t)$. Of course, if $\gamma = 0$, the solutions are always positive and it holds $u(r, t) \le \varepsilon$ uniformly in $\{r \ge ct\}$ for large times $t > 0$.

We ask the reader to note that with the same comparison technique we can prove that for all $\varepsilon > 0$, it holds

$$u(x,t) \le a + \varepsilon, \quad \text{for all } x \in \mathbb{R}^N, \quad t \ge t_\varepsilon, \tag{2.48}$$

for some suitable waiting time $t_\varepsilon > 0$ (as we have seen before, this property holds for the case $N = 1$).

Step2: Convergence to a in "inner" sets. In this second step, we have to prove that for all $\varepsilon > 0$ and $0 < c < c_*(m, p, f)$, the solutions to problem (2.43) satisfies

$$u(r,t) \ge a - \varepsilon, \text{ uniformly in } \{r \le ct\}, \quad t \to +\infty. \tag{2.49}$$

Following the procedure of the Fisher-KPP framework, we have to proceed in three main steps. In the first one we have to show that the solution $u = u(r,t)$ does not extinguish and actually lifts-up to a small level $\widetilde{\varepsilon} > 0$ in compact sets of $\mathbb{R}^N$ for large times. This follows from Theorem 2.4 of the Fisher-KPP setting and recalling the scaling property linking reactions of type C′ to Fisher-KPP reactions.
Then following the proof of Theorem 2.2, Part (ii), case $N = 1$, we have

$$f(u) \ge q_{\widetilde{\varepsilon}}(a-u) \quad \text{in } \{|x| \le \widetilde{\varrho}\} \times [t_{\widetilde{\varrho}}, \infty), \quad \text{with } q_{\widetilde{\varepsilon}} = \frac{f(\widetilde{\varepsilon})}{1-\widetilde{\varepsilon}},$$

for any $\widetilde{\varrho} > 0$ and $t_{\widetilde{\varrho}} > 0$ large enough. We point out that the previous inequality holds true only when $\widetilde{\varepsilon} \le u \le a$, which is an assumption we can make thanks to (2.48) and the scaling technique employed in *Step2* of proof of the case $N = 1$ (see range $\gamma > 0$). Thus, exactly as before, we get that for all $\varepsilon > 0$ (small) and $\widetilde{\varrho} > 0$ (large)

$$u(r,t) \ge a - \varepsilon \quad \text{in } \{r = |x| \le \widetilde{\varrho}\} \text{ for all } t \ge t_1,$$

for some (large) $t_1 > 0$. Finally, we get (2.49) by constructing a sub-solution to problem (2.43) through "change sign" TWs (cfr. with the proof of Theorem 2.1). Recalling the scaling property linking reactions of type C′ to Fisher-KPP reactions, we consider a barrier (from below) built with the function

$$\underline{\varphi}(\xi) := \begin{cases} a-\varepsilon & \text{if } \xi \le 0 \\ \varphi_c(\xi) & \text{if } 0 \le \xi \le \xi_1^c \\ 0 & \text{otherwise} \end{cases} \qquad \text{with} \quad a - \varepsilon = \max \varphi_c(\xi),$$

where $\varphi_c = \varphi_c(x - (c+\varepsilon)t)$ is a "change-sign" TW (of type 2) corresponding to the speed $0 < c < c_*$ (see Subsection 1.2.1). Thus, the barrier propagate level $a - \varepsilon$ with speed c, and so, using the arbitrariness of $0 < c < c_*$ obtain (2.49) (cfr. with the proof of Theorem 2.1 for all the details). □

2.6 Extensions, comments and open problems

We end the chapter with some extensions, comments and open problems.

2.6.1 On the exact location of the propagation front

Theorems 2.1, 2.2, and 2.3 are the "basic" results describing the wave propagation of solutions to problem (1) with reactions (2), (3), and (4), respectively. However, they do not give precise information about the properties of the solution $u = u(x,t)$ on the moving coordinate $x = \xi - c_* t$, where c_* is the critical speed. Important steps forward have been made for linear diffusion and reactions of the Fisher-KPP type. For example, Bramson showed in [44] and [45], using probabilistic techniques, an interesting property of the level sets $E_\omega(t) = \{x > 0 : u(x,t) = \omega\}$, $\omega \in (0,1)$, of the solution $u = u(x,t)$ to problem (1)-(2) in the case $N = 1$, $m = 1$, and $p = 2$. In particular, he proved that for all $\omega \in (0,1)$ there exist constants x_ω, $a > 0$ and $C_\omega > 0$ such that

$$E_\omega(t) \subset \Big[c_* t - \frac{3}{2\omega_*}\ln t - x_\omega - \frac{a}{\sqrt{t}} - \frac{C_\omega}{t}, c_* t - \frac{3}{2\omega_*}\ln t - x_\omega - \frac{a}{\sqrt{t}} + \frac{C_\omega}{t}\Big] \tag{2.50}$$

for t large enough, where $\omega_* = c_*/2$. The previous formula is interesting since it allows to estimate the "delay" of the solution $u = u(x, t)$ from the positive TW with critical speed $c = c_*$ which, according to (2.50), grows in time and consists in a logarithmic deviance. More recently, a similar result have been proved in [120] with PDEs techniques. In particular, the authors showed that there exists a constant $C \geq 0$ such that

$$E_\omega(t) \subset \Big[c_*t - \frac{3}{2\omega_*}\ln t - C, c_*t - \frac{3}{2\omega_*}\ln t + C] \quad \text{for } t \text{ large enough,}$$

which is less precise than (2.50) but, anyway, it contains the most important information on the "delay" of the solution, i.e., the logarithmic shift (for more work on this issue see [139, 172, 194]). Moreover, they gave a interesting proof of the uniform convergence of the general solutions to equation to problem (1)-(2) (in the case $N = 1$, $m = 1$, and $p = 2$) to the TW solution with critical speed c_* (see Theorem 1.2 of [120]).
It seems quite natural to conjecture that, at least in the "pseudo-linear" case, the level sets of the solution of (1) satisfy similar properties. Nevertheless, studying the problem of the exact location of the propagation front for the doubly nonlinear diffusion seems a really difficult task and we pose it as an interesting open problem.

2.6.2 Reactions of type B

In the literature, other kind of reactions have been intensively investigated. Possibly, the most famous are the so called reactions of type B

$$\begin{cases} f(0) = f(a) = f(1) = 0, & f(u) \leq 0 \text{ in } (0, a), \quad f(u) > 0 \text{ in } (a, 1) \\ f \in C^1([0, 1]), & f'(1) < 0, \end{cases} \tag{2.51}$$

which emerge from combustion models (see for instance the famous works [31, 171] and the interesting survey [180]). We have not considered this framework in this paper but we want to point out, thanks to a simple comparison with reaction of type C, that part (ii) of Theorem 2.2 hold even for reaction of type B. Also part (i) holds if we take initial data $0 \leq u_0 \leq a$ (this is true even for reactions of type C thanks to a straightforward comparison technique, but we have not insisted on it since it goes out of our purposes).

2.6.3 Sharp threshold results

As we have pointed out in the presentation of the results of this paper, Theorem 2.2 has not a sharp threshold statement. As already explained, the problem has been studied and solved in dimension $N = 1$ and very general reaction terms by Du and Matano [88]. In this work, it is essential the existence of nontrivial solutions to

$$-\partial_{xx}u = f(u) \quad \text{in } \mathbb{R},$$

which correspond to stationary solutions to the corresponding parabolic problem and eventual limit configurations (see for instance Theorem 1.1 of [88]). The study of these stationary solutions is clearly more complicated in the doubly nonlinear framework and seems to be a very challenging open problem (see [157, 166] for the case $N \geq 1$).

Chapter 3

The Fisher-KPP problem with "fast" diffusion

In the last chapter of Part I, we study the asymptotic behaviour for large times of the solutions to problem (1):

$$\begin{cases} \partial_t u = \Delta_p u^m + f(u) & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

for reaction terms of the Fisher-KPP type (2):

$$\begin{cases} f(0) = f(1) = 0, & 0 < f(u) \le f'(0)u \;\text{ in } (0,1) \\ f \in C^1([0,1]), & f'(0) > 0, \; f'(1) < 0 \\ f(\cdot) \text{ has a unique critical point in } (0,1), \end{cases}$$

and in the "fast" diffusion range, i.e., for parameters $m > 0$ and $p > 1$ such that $-p/N < \gamma < 0$, where we recall that $\gamma := m(p-1) - 1$ (cfr. with Figure 3.1). We will see that the last assumption in (2) is not needed in this "fast" diffusion setting (this is due to the fact that this time we are not going to perform an ODEs analysis). However, in order to simplify notations, we have decided to not introduce new assumptions on $f(\cdot)$, so that we will work with reactions satisfying (2).

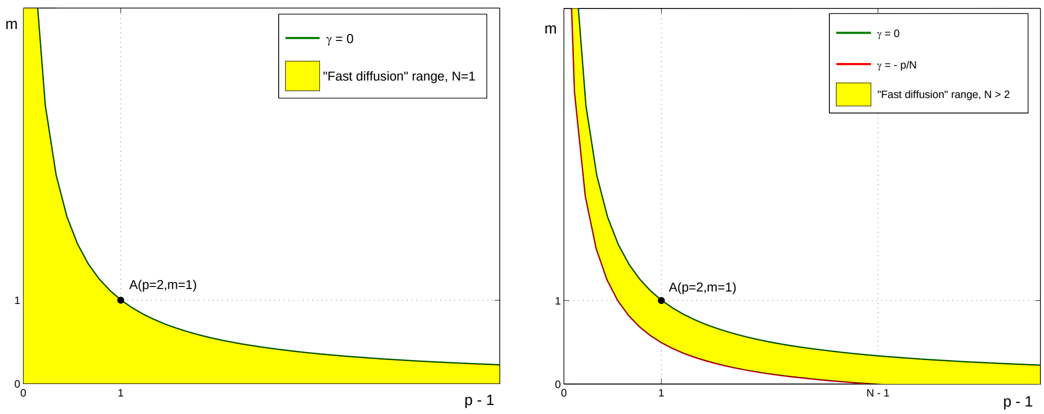

Figure 3.1: The "fast" diffusion range in the $(m, p-1)$-plane: cases $N = 1$ and $N > 2$, respectively.

From the beginning, we introduce the notation

$$\widehat{\gamma} := -\gamma,$$

so that the "fast" diffusion assumption $-p/N < \gamma < 0$ reads (in terms of $\widehat{\gamma}$):

$$0 < \widehat{\gamma} < p/N. \tag{3.1}$$

The positivity of $\widehat{\gamma}$ will be very useful for simplifying the reading. Finally, we will assume that the initial datum is a Lebesgue-measurable function and satisfies

$$\begin{cases} u_0(x) \le C|x|^{-p/\widehat{\gamma}}, & \text{for some } C > 0 \\ u_0 \not\equiv 0 \text{ and } 0 \le u_0 \le 1. \end{cases} \tag{3.2}$$

Note that the previous assumption is much more general than the one done in Chapter 2 and the polynomial decaying of the initial data is the same of Barenblatt solutions in the "fast" diffusion range (cfr. with the section of preliminaries on doubly nonlinear diffusion in the introduction of Part I). Moreover note that, since $0 < \widehat{\gamma} < p/N$, all data satisfying (3.2) are automatically integrable, $u_0 \in L^1(\mathbb{R}^N)$.

3.1 Main results

We present now the two most significative results proved in this third chapter. The first one, is "fast" diffusion counterpart of Theorem 2.1 for "slow" diffusion and present some very interesting features w.r.t. the "slow"/"pseudo-linear" diffusion framework. Before giving the precise statements, let us introduce the critical exponent

$$\sigma_* := \frac{\widehat{\gamma}}{p} f'(0), \tag{3.3}$$

which, as the reader we will easily note, will play an important role in what follows.

Theorem 3.1. *(cfr. with Theorem 1.1 and Theorem 1.2 of [18])*
Let $m > 0$ and $p > 1$ such that $0 < \widehat{\gamma} < p/N$, and let $N \ge 1$. Let $u = u(x,t)$ be a solution to the initial-value problem (1) *with initial datum* (3.2) *and reaction of Fisher-KPP type (satisfying* (2)*). Then:*

(i) For all $0 < \sigma < \sigma_*$,

$$u(x,t) \to 1 \text{ uniformly in } \{|x| \le e^{\sigma t}\} \quad \text{as } t \to \infty.$$

(ii) For all $\sigma > \sigma_*$ it satisfies,

$$u(x,t) \to 0 \text{ uniformly in } \{|x| \ge e^{\sigma t}\} \quad \text{as } t \to \infty,$$

where $\sigma_* = \sigma_*(m,p,f)$ is the critical exponent defined in (3.3).

For all $\sigma > \sigma_*$, we call $\{|x| \ge e^{\sigma t}\}$ "exponential outer set" or, simply, "outer set", while "inner set" $\{|x| \le e^{\sigma t}\}$ for $\sigma < \sigma_*$. The previous theorem shows that, for large times, the solution $u = u(x,t)$ converges to one in the "inner set", whilst to zero in the "outer set" and represent the most interesting difference w.r.t. the travelling wave behaviour for large times, found in Theorem 2.1 for "slow" and "pseudo-linear" diffusion (cfr. with Figure 3.2). Here the solutions do not spread in space with constant speed of propagation ($c_* = c_*(m,p,f)$) for large times, but exponentially fast with exponential rate $\sigma_* = \sigma_*(m,p,f)$. In terms of stability, the steady state $u = 1$ is stable, while $u = 0$ is unstable and the asymptotic stability/instability can be measured in terms of speed of convergence of the solution which, in this case, is asymptotically exponential in distance of the front location as function of time.

Exponential propagation was already observed before. In particular, we quote the paper of King and McCabe [133] in which they study the Porous Medium case, obtained by taking $p = 2$ in the equation in (1):

$$\begin{cases} \partial_t u = \nabla \cdot (u^{-(1-m)} \nabla u) + u(1-u) & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{3.4}$$

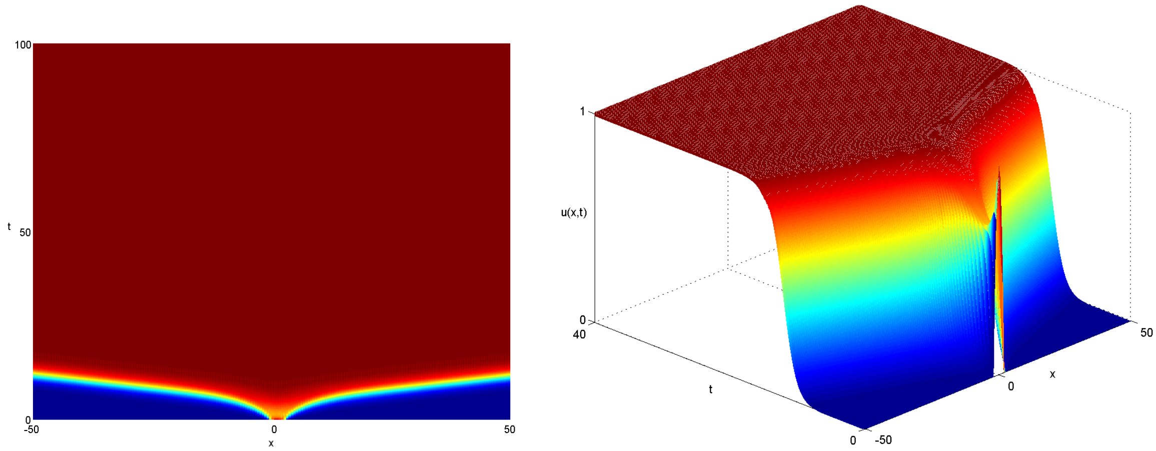

Figure 3.2: Fisher-KPP reactions, range $-p/N < \gamma < 0$. Qualitative long time behaviour (convergence to 1 in the "inner" sets $\{|x| \le e^{\sigma t}\}$, for $\sigma < \sigma_*$) of the solutions for $f(u) = u(1-u)$.

in the fast diffusion range, $0 < m < 1$ (note that we absorbed a factor m by using a simple change of variables). They considered non-increasing radial initial data $u_0 \in L^1(\mathbb{R}^N)$ decaying faster than $r^{-2/(1-m)}$ as $r = |x| \sim \infty$ and studied radial solutions to problem (3.4). We will see how to generalize their technique in the following sections.
They showed that when $(N-2)_+/N := m_c < m < 1$, the radial solutions $u = u(r,t)$ to the previous equation converge point-wise to 1 for large times with exponential rate $r(t) \sim e^{\sigma t}$, for $\sigma < (1-m)/2$. Their approach is based on a powerful heuristic which allows to compute the approximate solution to (3.4) for large times

$$u(r,t) \sim \frac{\left(\kappa r^{-2} e^{(1-m)t}\right)^{\frac{1}{m-1}}}{1 + \left(\kappa r^{-2} e^{(1-m)t}\right)^{\frac{1}{1-m}}}, \quad t \sim \infty \quad \text{and} \quad r = O\left(e^{\frac{1-m}{2}t}\right),$$

where $\kappa = 2[2-(1-m)N]/(1-m)^2$. Note that "our" critical exponent σ_* generalizes the value $(1-m)/2$ to the case $p > 1$ and to more general reaction terms than $f(u) = u(1-u)$.

Exponential propagation happens also with fractional diffusion, both linear and nonlinear, see for instance [47, 182] and the references therein. Finally, we recall that infinite speed of propagation depends not only on the diffusion operator but also on the initial datum. In particular, in [121], Hamel and Roques found that the solutions of the Fisher-KPP problem with linear diffusion i.e., ($m = 1$ and $p = 2$) propagate exponentially fast for large times if the initial datum has a power-like spatial decay at infinity.

Following the procedure used to prove Theorem 2.1, we will need some a priori "lifting-up" lemmas, which will be proved in Section 3.2. Moreover, in the proof of Part (i) (of Theorem 3.1) we will employ a Comparison Principle in non cylindrical domains and some known results about p-Laplacian diffusion with non-integrable initial data, that we recall in two separate sections (cfr. with Section 3.5 and Section 3.6) at the end of the chapter.

In the second main result we consider the classical reaction term $f(u) = u(1-u)$. We find interesting bounds for the level sets of the solution of problem (1), (3.2). In particular, we prove that the information on the level sets of the general solutions is contained, up to a multiplicative constant, in the set $|x| = e^{\sigma_* t}$, for large times.

Theorem 3.2. *(cfr. with Theorem 1.4 of [18])*
Fix $N \ge 1$. Let $m > 0$ and $p > 1$ such that $0 < \widehat{\gamma} < p/N$, and take $f(u) = u(1-u)$. Then for all $0 < \omega < 1$, there exists a constant $C_\omega > 0$ and a time $t_\omega > 0$ large enough, such that the solution of problem (1) *with initial*

datum (3.2) *and reaction* $f(u) = u(1-u)$ *satisfies*

$$\{|x| > C_\omega e^{\sigma_* t}\} \subset \{u(x,t) < \omega\} \quad \text{and} \quad \{|x| < C_\omega^{-1} e^{\sigma_* t}\} \subset \{u(x,t) > \omega\} \tag{3.5}$$

for all $t \ge t_\omega$. *In particular, we have*

$$E_\omega(t) = \{x \in \mathbb{R}^N : u(x,t) = \omega\} \subset \{x \in \mathbb{R}^N : C_\omega^{-1} e^{\sigma_* t} \le |x| \le C_\omega e^{\sigma_* t}\} \quad \text{for all } t \ge t_\omega.$$

An important feature of this result is that for all $0 < \omega < 1$, the set $\{C_\omega^{-1} e^{\sigma_* t} \le |x| \le C_\omega e^{\sigma_* t}\}$ does not depend on some $\sigma \neq \sigma_*$, while in Theorem 3.1 the "outer sets" and the "inner sets" depend on $\sigma > \sigma_*$ and $\sigma < \sigma_*$, respectively. Moreover, taking a *spatial logarithmic scale* we can write the estimate

$$E_\omega(t) := \{x \in \mathbb{R}^N : u(x,t) = \omega\} \subset \{x \in \mathbb{R}^N : -\ln C_\omega \le \ln|x| - \sigma_* t \le \ln C_\omega\},$$

for t large enough. Actually, this result was not known for "fast" nonlinear diffusion neither for the Porous Medium case, nor for the p-Laplacian case. However, it was proved by Cabré and Roquejoffre for the fractional Laplacian $(-\Delta)^{1/2}$ in [47], in dimension $N = 1$.

In order to fully understand the importance of Theorem 3.2, we need to compare it with the linear case $m = 1$ and $p = 2$, see formula (2.50):

$$E_\omega(t) \subset \Big[c_* t - \frac{3}{2\omega_*}\ln t - x_\omega - \frac{a}{\sqrt{t}} - \frac{C_\omega}{t}, c_* t - \frac{3}{2\omega_*}\ln t - x_\omega - \frac{a}{\sqrt{t}} + \frac{C_\omega}{t}\Big]$$

for t large enough, where $\omega_* = c_*/2$, and x_ω, C_ω, a are suitable positive constant. This means that in the linear case the location of the level sets is given by a main linear term in t with a logarithmic shift for large times, see [44, 45, 120] and [106] for an extension of these results to more general reaction equations. In other words, the propagation of the front is linear "up to" a logarithmic correction, for large times. Now, Theorem 3.2 asserts that this correction does not occur in the "fast diffusion" range. Using the logarithmic scale, we can compare the behaviour of our level sets with the ones of formula (2.50) for linear diffusion, noting that there is no logarithmic deviation, but the location of the level sets is approximately linear for large times (in spatial logarithmic scale, of course), and moreover there is a bounded interval of uncertainty on each level set location.

3.2 Fisher-KPP reactions, range $0 < \widehat{\gamma} < p/N$. A priori "lifting up" results

In this section, we study problem (1), with Fisher-KPP reaction term $f(\cdot)$ satisfying (2) and with the following choice of the initial datum:

$$\widetilde{u}_0(x) := \begin{cases} \widetilde{\varepsilon} & \text{if } |x| \le \widetilde{\varrho}_0 \\ a_0 |x|^{-p/\widehat{\gamma}} & \text{if } |x| > \widetilde{\varrho}_0, \end{cases} \tag{3.6}$$

where $\widetilde{\varepsilon}$ and $\widetilde{\varrho}_0$ are positive real numbers, and $a_0 := \widetilde{\varepsilon}\,\widetilde{\varrho}_0^{p/\widehat{\gamma}}$. Note that $\widetilde{u}_0(\cdot)$ has "tails" which are asymptotic to the profile of the Barenblatt solutions for $|x|$ large (see the introduction to Part I). The choice (3.6) will be clear in the next sections, where we will show the convergence of the solution to problem (1), (3.2) to the steady state $u = 1$. We devote the all section to the proof of the following crucial proposition (which is the "fast" diffusion version of Proposition 2.5 and Proposition 2.9 proved in Chapter 2 in the "slow" and "pseudo-linear" settings).

Proposition 3.3. *Fix* $N \ge 1$. *Let* $m > 0$ *and* $p > 1$ *such that* $0 < \widehat{\gamma} < p/N$ *and let* $0 < \sigma < \sigma_*$. *Then there exist* $t_0 > 0$, $\widetilde{\varepsilon} > 0$ *and* $\widetilde{\varrho}_0 > 0$ *such that the solution* $u = u(x,t)$ *of problem* (1) *with initial datum* (3.6) *satisfies*

$$u(x,t) \ge \widetilde{\varepsilon} \quad \text{in } \{|x| \le e^{\sigma t}\} \text{ for all } t \ge t_0.$$

This result asserts that for all initial data (3.6) "small enough" and for all $\sigma < \sigma_*$, the solution of problem (1) is strictly greater than a fixed positive constant on the "exponential inner sets" (or "inner sets") $\{|x| \le e^{\sigma t}\}$ for large times. Hence, it proves the non existence of travelling wave solutions (TWs) since "profiles" moving with constant speed of propagation cannot describe the asymptotic behaviour of more general solutions (cfr. with Chapter 1 and Chapter 2).
Moreover, this property will be really useful for the construction of sub-solutions of general solutions since, as we will see, it is always possible to place an initial datum with the form (3.6) under a general solution of (1) and applying the Comparison Principle (see the next lemma).

Lemma 3.4. *Fix $N \ge 1$ and let $m > 0$ and $p > 1$ such that $0 < \widehat{\gamma} < p/N$. Then for all $\theta > 0$, there exist $t_1 > \theta$, $\widetilde{\varepsilon} > 0$, and $\widetilde{\varrho}_0 > 0$, such that the solution $u = u(x,t)$ to problem* (1) *with nontrivial initial datum $0 \le u_0 \in L^1(\mathbb{R}^N)$ satisfies*

$$u(x,t_1) \ge \widetilde{u}_0(x) \quad \text{in } \mathbb{R}^N$$

where $\widetilde{u}_0(\cdot)$ is defined in (3.6).

Proof. Let $u = u(x,t)$ the solution to problem (1) with nontrivial initial datum $0 \le u_0 \in L^1(\mathbb{R}^N)$ and consider the solution $v = v(x,t)$ to the purely diffusive Cauchy problem:

$$\begin{cases} \partial_t v = \Delta_p v^m & \text{in } \mathbb{R}^N \times (0,\infty) \\ v(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

It satisfies $v(x,t) \le u(x,t)$ in $\mathbb{R}^N \times [0,\infty)$ thanks to the Comparison Principle.

Let $\theta > 0$. Since $v(\cdot,\theta)$ is continuous in $\mathbb{R}^N$ and non identically zero (the mass of the solution is conserved in the "good" exponent range $0 < \widehat{\gamma} < p/N$), we have that it is strictly positive in a small ball $\overline{B}_\varrho(x_0)$, $x_0 \in \mathbb{R}^N$ and $\varrho > 0$. Without loss of generality, we may take $x_0 = 0$. So, by continuity, we deduce $v(x,t+\theta) \ge \delta$ in $\overline{B}_\varrho \times [0,\tau]$, for some small $\delta > 0$ and $\tau > 0$.

Now, let us consider the function $v_\theta(x,t) := v(x,t+\theta)$ and the exterior cylinder

$$S := \{|x| \ge \varrho\} \times (0,\tau).$$

We compare $v_\theta(x,t)$ with a "small" Barenblatt solution $B_M(x,t)$ (we mean that M is small) at time $t = 0$ and on the boundary of S. Recall that Barenblatt solutions have the self-similar form

$$B_M(x,t) = t^{-\alpha}\left[C_M + k\left|xt^{-\frac{\alpha}{N}}\right|^{\frac{p}{p-1}}\right]^{-\frac{p-1}{\widehat{\gamma}}},$$

where α and k are positive constants defined in Subsection I, and $C_M > 0$ depends on the mass.

The comparison at time $t = 0$ is immediate since $v_\theta(x,0) \ge 0$ and $B_M(x,0) = 0$ for all $|x| \ge \varrho$. Now, let us take $|x| = \varrho$. We want to show that $v_\theta(|x| = \varrho, t) \ge B_M(|x| = \varrho, t)$ for all $0 \le t < \tau$. A simple computation shows that we can rewrite the Barenblatt solution as

$$B_M(|x| = \varrho, t) = \frac{t^{\frac{1}{\widehat{\gamma}}}}{\left[C_M t^{\frac{\alpha p}{N(p-1)}} + k\varrho^{\frac{p}{p-1}}\right]^{\frac{p-1}{\widehat{\gamma}}}} \le \left(\frac{\tau}{k^{p-1}\varrho^p}\right)^{\frac{1}{\widehat{\gamma}}},$$

since $0 \le t < \tau$. Thus, since $v_\theta(|x| = \varrho, t) \ge \delta$, it is sufficient to have

$$\left(\frac{\tau}{k^{p-1}\varrho^p}\right)^{\frac{1}{\widehat{\gamma}}} \le \delta, \quad \text{i.e.} \quad \tau \le \delta^{\widehat{\gamma}} k^{p-1}\varrho^p.$$

This condition is satisfied taking $\tau > 0$ small enough. We may now use the Comparison Principle to obtain the conclusion $v_\theta(x,t) \geq B_M(x,t)$ in the whole of S.

In particular, we evaluate the comparison at $t = \tau$, and we have

$$\begin{cases} v(x,\tau+\theta) \geq \delta & \text{if } |x| \leq \varrho \\ v(x,\tau+\theta) \geq B_M(x,\tau) & \text{if } |x| \geq \varrho. \end{cases} \Leftrightarrow \begin{cases} v(x,t_1) \geq \delta & \text{if } |x| \leq \varrho \\ v(x,t_1) \geq B_M(x,t_1-\theta) & \text{if } |x| \geq \varrho, \end{cases}$$

where we set $t_1 = \tau + \theta$. Let us fix $\widetilde{\varrho}_0 := \varrho$, $0 < \widetilde{\varepsilon} \leq \delta$, and $a_0 := \widetilde{\varepsilon}\,\widetilde{\varrho}_0^{\,p/\widehat{\gamma}}$. By taking $\widetilde{\varepsilon} > 0$ smaller, we can assume $k^{p-1}a_0^{\widehat{\gamma}} < \tau = t_1 - \theta$. Now, we verify that

$$B_M(x,t_1-\theta) \geq a_0|x|^{-p/\widehat{\gamma}}, \quad \text{for all } |x| \geq \widetilde{\varrho}_0,$$

and some suitable constant $C_M > 0$. Writing the expression for the Barenblatt solutions, the previous inequality reads:

$$C_M \leq \frac{(t_1-\theta)^{\frac{1}{p-1}} - k a_0^{\frac{\widehat{\gamma}}{p-1}}}{\left[a_0^{\widehat{\gamma}}(t_1-\theta)^{\frac{\alpha p}{N}}\right]^{\frac{1}{p-1}}}|x|^{\frac{p}{p-1}} := K|x|^{\frac{p}{p-1}}, \quad \text{for all } |x| \geq \widetilde{\varrho}_0.$$

Note that the coefficient K of $|x|^{p/(p-1)}$ is positive thanks to our assumptions on $\widetilde{\varepsilon} > 0$. Now, since $|x| \geq \widetilde{\varrho}_0$, we deduce that a sufficient condition so that the previous inequality is satisfied is $C_M \leq K\widetilde{\varrho}_0^{\,p/(p-1)}$. Consequently, we have shown that for all $\theta > 0$, there exist $t_1 > \theta$, $\widetilde{\varepsilon} > 0$, and $\widetilde{\varrho}_0 > 0$, such that

$$u(x,t_1) \geq v(x,t_1) \geq \widetilde{u}_0(x), \quad \text{for all } x \in \mathbb{R}^N,$$

which is our thesis. □

We ask the reader to note that improved global positivity estimates were proved in [124, 196] and [42] for the Porous Medium Equation. Now, with the next crucial lemma, we prove that the expansion of the super-level sets of the solution $u = u(x,t)$ of problem (1) with initial datum (3.6) is exponential for all $\sigma < \sigma_*$ and large times. Before proceeding, we recall two important properties of the Barenblatt solutions we will need in the next proof (cfr. with the introduction to Part I). The first one is the relation between the Barenblatt solution with mass $M > 0$ and mass 1:

$$B_M(x,t) = MB_1(x, M^{-\widehat{\gamma}}t), \tag{3.7}$$

while the second are the estimates on the profile corresponding to the Barenblatt solution of mass $M > 0$:

$$K_2(1 + |\xi|^{p/\widehat{\gamma}})^{-1} \leq F_M(\xi) \leq K_1|\xi|^{-p/\widehat{\gamma}} \quad \text{for all } \xi \in \mathbb{R}^N \tag{3.8}$$

for suitable positive constants K_1 and K_2 depending on $M > 0$.

Lemma 3.5. *Fix $N \geq 1$. Let $m > 0$ and $p > 1$ such that $0 < \widehat{\gamma} < p/N$, and let $0 < \sigma < \sigma_*$. Then there exist $t_0 > 0$ and $0 < \widetilde{\varepsilon}_0 < 1$ which depend only on m, p, N and f, such that the following hold. For all $0 < \widetilde{\varepsilon} \leq \widetilde{\varepsilon}_0$, there exists $\widetilde{\varrho}_0 > 0$ (large enough depending on $\widetilde{\varepsilon} > 0$), such that the solution $u = u(x,t)$ to problem* (1) *with initial datum* (3.6) *satisfies*

$$u(x, jt_0) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq \widetilde{\varrho}_0 e^{\sigma j t_0}\}, \text{ for all } j \in \mathbb{N}_+ = \{1, 2, \dots\}.$$

Proof. We prove the assertion of the thesis by induction on $j = 1, 2, \dots$, assuming $f(\cdot)$ to be concave in $(0, 1)$. The case $f(\cdot)$ follows exactly as in the case of "slow" diffusion (cfr. with Lemma 2.7). Again, we follow the ideas presented by Cabré and Roquejoffre in [47] and, later, in [182], for fractional diffusion.

Step0: Basic definitions. We set $j = 1$, $0 < \sigma < \sigma_*$ and introduce some basic definitions and quantities we will use during the proof. First of all, let C_1 be the constant corresponding to the profile $F_1(\cdot)$ (see the introduction to Part I) and let K_1 and K_2 be defined as in (3.8) with $M = 1$. In order to avoid huge expressions in the following of the proof, we introduce the constants

$$\overline{K} := \left(C_1^{(p-1)/\widehat{\gamma}} K_1^{-\alpha\widehat{\gamma}}\right)^{N/(\alpha p)} \quad \text{and} \quad \widetilde{K} := \frac{K_2}{2} C_1^{\frac{p-1}{\widehat{\gamma}}}. \tag{3.9}$$

We fix $0 < \delta < 1$ sufficiently small such that

$$\frac{\widehat{\gamma}}{p}\lambda > \sigma, \qquad \lambda := f(\delta)/\delta. \tag{3.10}$$

Then, we consider t_0 sufficiently large such that

$$\widetilde{K}e^{\lambda t_0} \geq 2^{\alpha} \quad \text{and} \quad \frac{K_2}{2K_1}e^{\lambda t_0} \geq e^{\frac{p}{\widehat{\gamma}}\sigma t_0} \tag{3.11}$$

(note that such a t_0 exists thanks to (3.10)) and we define $\widetilde{\varepsilon}_0 := \delta e^{-\lambda t_0}$. Finally, fix $0 < \widetilde{\varepsilon} \leq \widetilde{\varepsilon}_0$ and choose $\widetilde{\varrho}_0$ large enough such that

$$\widetilde{\varrho}_0^{\,p} \geq \frac{K_1^{\widehat{\gamma}}}{\lambda\widehat{\gamma}\,\widetilde{\varepsilon}^{\widehat{\gamma}}}. \tag{3.12}$$

The choice of the subtle conditions (3.10), (3.11) and (3.12) will be clarified during the proof.

Step1: Construction of a sub-solution. We construct a sub-solution to problem (1), (3.6) in $\mathbb{R}^N \times [0, t_0]$. First of all, we construct a Barenblatt solution of the form $B_{M_1}(x, \theta_1)$ such that

$$B_{M_1}(x, \theta_1) \leq \widetilde{u}_0(x) \quad \text{in } \mathbb{R}^N. \tag{3.13}$$

Since the profile of the Barenblatt solution is decreasing, we impose $B_{M_1}(0, \theta_1) = \widetilde{\varepsilon}$ in order to satisfy (3.13) in the set $\{|x| \leq \widetilde{\varrho}_0\}$. Moreover, using (3.8) and noting that $1 + \alpha\widehat{\gamma} = \alpha p/N$, it simple to get

$$B_{M_1}(x, \theta_1) \leq K_1\theta_1^{\frac{1}{\widehat{\gamma}}}|x|^{-\frac{p}{\widehat{\gamma}}} \quad \text{in } \mathbb{R}^N$$

and so, it is sufficient to require $K_1\theta_1^{\frac{1}{\widehat{\gamma}}} = a_0$, so that (3.13) is valid in $\{|x| \geq \widetilde{\varrho}_0\}$. Thus, it is simple to obtain the relations

$$M_1 = \overline{K}\widetilde{\varrho}_0^{\,N}\widetilde{\varepsilon} \qquad \text{and} \qquad \theta_1 = K_1^{-\widehat{\gamma}}\widetilde{\varrho}_0^{\,p}\widetilde{\varepsilon}^{\widehat{\gamma}} \tag{3.14}$$

Now, consider the linearized problem

$$\begin{cases} \partial_t w = \Delta_p w^m + \lambda w & \text{in } \mathbb{R}^N \times (0, \infty) \\ w(x, 0) = \widetilde{u}_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{3.15}$$

and the change of variable

$$\tau(t) = \frac{1}{\lambda\widehat{\gamma}}\left[1 - e^{-\lambda\widehat{\gamma}t}\right], \quad \text{for } t \geq 0. \tag{3.16}$$

Note that $0 \leq \tau(t) \leq \tau_\infty := \frac{1}{\lambda\widehat{\gamma}}$ and the function $\widetilde{w}(x, \tau) = e^{-\lambda t}w(x, t)$ satisfies the "purely" diffusive problem

$$\begin{cases} \partial_\tau\widetilde{w} = \Delta_p\widetilde{w}^m & \text{in } \mathbb{R}^N \times (0, \tau_\infty) \\ w(x, 0) = \widetilde{u}_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{3.17}$$

Since $B_{M_1}(x,\theta_1) \le \widetilde{u}_0(x) \le \widetilde{\varepsilon}$ for all $x \in \mathbb{R}^N$, from the Comparison Principle we get

$$B_{M_1}(x,\theta_1+\tau) \le \widetilde{w}(x,\tau) \le \widetilde{\varepsilon} \quad \text{in } \mathbb{R}^N \times (0,\tau_\infty). \tag{3.18}$$

Hence, using the concavity of f and the second inequality in (3.18) we get

$$w(x,t) = e^{\lambda t}\widetilde{w}(x,\tau) \le \widetilde{\varepsilon}_0 e^{\lambda t_0} = \delta, \quad \text{in } \mathbb{R}^N \times [0,t_0]$$

and so, since $w \le \delta$ implies $f(\delta)/\delta \le f(w)/w$, we have that w is a sub-solution to problem (1), (3.6) in $\mathbb{R}^N \times [0,t_0]$. Finally, using the first inequality in (3.18), we obtain

$$u(x,t) \ge e^{\lambda t}\widetilde{w}(x,\tau) \ge e^{\lambda t}B_{M_1}(x,\theta_1+\tau) \quad \text{in } \mathbb{R}^N \times [0,t_0]. \tag{3.19}$$

Step2: Conclusion for $t = t_0$ In this step, we show that the choices made in (3.10), (3.11) and (3.12) allow us to find positive numbers $\widetilde{\varrho}_1$ and a_1 such that $u(x,t_0) \ge \widetilde{u}_1(x)$ for all $x \in \mathbb{R}^N$, where

$$\widetilde{u}_1(x) := \begin{cases} \widetilde{\varepsilon} = a_1\widetilde{\varrho}_1^{-p/\widehat{\gamma}} & \text{if } |x| \le \widetilde{\varrho}_1 \\ a_1|x|^{-p/\widehat{\gamma}} & \text{if } |x| > \widetilde{\varrho}_1 \end{cases} \quad \text{and} \quad \widetilde{\varrho}_1 \ge \widetilde{\varrho}_0 e^{\sigma t_0},$$

which implies the thesis for $j = 1$. Now, in order to find $\widetilde{\varrho}_1$ and a_1 we proceed with the chain of inequalities in (3.19) for the values $t = t_0$, $\tau_0 = \tau(t_0)$ and $|x| = \widetilde{\varrho}_1$. Imposing

$$\widetilde{\varrho}_1\left[M_1^{-\widehat{\gamma}}(\theta_1+\tau_0)\right]^{-\alpha/N} \ge 1, \tag{3.20}$$

using (3.8) and observing that $(1+z)^{-1} \ge (2z)^{-1}$ for all $z \ge 1$, we look for $\widetilde{\varrho}_1$ and a_1 such that

$$\begin{aligned} u(x,t_0)|_{|x|=\widetilde{\varrho}_1} &\ge e^{\lambda t_0}B_{M_1}(x,\theta_1+\tau_0)|_{|x|=\widetilde{\varrho}_1} \\ &\ge e^{\lambda t_0}K_2M_1^{1+\alpha\widehat{\gamma}}(\theta_1+\tau_0)^{-\alpha}\left\{1+\widetilde{\varrho}_1^{\frac{p}{\widehat{\gamma}}}\left[M_1^{-\widehat{\gamma}}(\theta_1+\tau_0)\right]^{-\frac{\alpha p}{N\widehat{\gamma}}}\right\}^{-1} \\ &\ge \frac{K_2}{2}e^{\lambda t_0}M_1^{1+\alpha\widehat{\gamma}}(\theta_1+\tau_0)^{-\alpha}\left\{\widetilde{\varrho}_1\left[M_1^{-\widehat{\gamma}}(\theta_1+\tau_0)\right]^{-\frac{\alpha}{N}}\right\}^{-\frac{p}{\widehat{\gamma}}} \\ &= \frac{K_2}{2}e^{\lambda t_0}(\theta_1+\tau_0)^{\frac{1}{\widehat{\gamma}}}\widetilde{\varrho}_1^{-\frac{p}{\widehat{\gamma}}} \\ &\ge \widetilde{\varepsilon} = a_1\widetilde{\varrho}_1^{-\frac{p}{\widehat{\gamma}}}. \end{aligned}$$

Thus, we get $u(x,t_0)|_{|x|=\widetilde{\varrho}_1} \ge \widetilde{\varepsilon}$ taking, for instance, $\widetilde{\varrho}_1 > 0$ such that

$$\frac{K_2}{2}e^{\lambda t_0}(\theta_1+\tau_0)^{\frac{1}{\widehat{\gamma}}}\widetilde{\varrho}_1^{-\frac{p}{\widehat{\gamma}}} = \widetilde{\varepsilon}. \tag{3.21}$$

Note that this choice of $\widetilde{\varrho}_1 > 0$ performs the equality at the end of the previous chain and the value $a_1 = \frac{K_2}{2}e^{\lambda t_0}(\theta_1+\tau_0)^{1/\widehat{\gamma}}$ is determined too.

Remark 1. Note that the conditions $e^{\lambda t_0}B_{M_1}(x,\theta_1+\tau_0)|_{|x|=\widetilde{\varrho}_1} \ge \widetilde{\varepsilon}$ and (3.20) are sufficient to assure $u(x,t_0) \ge u_1(x)$ in $\mathbb{R}^N$. Indeed, it is guaranteed in the set $\{|x| \le \widetilde{\varrho}_1\}$ since the profile $F_1(\cdot)$ is non-increasing. On the other hand, if $|x| \ge \widetilde{\varrho}_1$ we have $|x|\left[M_1^{-\widehat{\gamma}}(\theta_1+\tau_0)\right]^{-\alpha/N} \ge 1$ by (3.20) and so, following the chain of inequalities as before, we get

$$u(x,t_0) \ge \frac{K_2}{2}e^{\lambda t_0}(\theta_1+\tau_0)^{\frac{1}{\widehat{\gamma}}}|x|^{-\frac{p}{\widehat{\gamma}}} = a_1|x|^{-\frac{p}{\widehat{\gamma}}} = \widetilde{u}_1(x) \quad \text{in } \{|x| \ge \widetilde{\varrho}_1\}.$$

Now, in order to conclude the proof of the case $j = 1$, we must check that the conditions (3.20) and (3.21) actually represent a possible choice and the value of t_0, defined at the beginning, performs their compatibility. The compatibility between (3.20) and (3.21) can be verified imposing

$$\frac{K_2}{2}e^{\lambda t_0}(\theta_1 + \tau_0)^{\frac{1}{\gamma}} = \widetilde{\varepsilon}\,\widetilde{\varrho}_1^{\frac{p}{\gamma}} \geq \widetilde{\varepsilon}\Big[M_1^{-\widehat{\gamma}}(\theta_1 + \tau_0)\Big]^{\frac{\alpha p}{N\widehat{\gamma}}},$$

which can be rewritten using the definitions (3.14) as

$$\widetilde{K}e^{\lambda t_0} \geq \left(1 + \frac{\tau_0}{\theta_1}\right)^{\alpha}, \tag{3.22}$$

Now, it is simple to verify that condition (3.12) implies $\tau_\infty \leq \theta_1$ and so it holds $\tau_0 \leq \theta_1$ too. Hence, a sufficient condition so that (3.22) is satisfied and does not depend on $\widetilde{\varepsilon} > 0$ is

$$\widetilde{K}e^{\lambda t_0} \geq 2^{\alpha},$$

i.e., our initial choice of t_0 in (3.11) which proves the compatibility between (3.20) and (3.21).

Remark 2. Rewriting formula (3.21) using the definition of θ_1, it is simple to deduce

$$\left(\frac{\widetilde{\varrho}_1}{\widetilde{\varrho}_0}\right)^{\frac{p}{\gamma}} = \frac{K_2}{2K_1}e^{\lambda t_0}\left(1 + \frac{\tau_0}{\theta_1}\right)^{\frac{1}{\gamma}} \tag{3.23}$$

and, using the second hypothesis on t_0 in (3.11), it is straightforward to obtain $\widetilde{\varrho}_1 \geq \widetilde{\varrho}_0 e^{\sigma t_0}$. In particular, we have shown

$$u(x, t_0) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq \widetilde{\varrho}_0 e^{\sigma t_0}\},$$

i.e., the thesis for $j = 1$.

Iteration. Set $t_j := (j + 1)t_0$, $\widetilde{\varrho}_j := \widetilde{\varrho}_0 e^{\sigma j t_0}$ and $a_j := \widetilde{\varepsilon}\,\widetilde{\varrho}_j^{p/\widehat{\gamma}}$ for all $j \in \mathbb{N}$ and define

$$\widetilde{u}_j(x) = \begin{cases} \widetilde{\varepsilon} & \text{if } |x| \leq \widetilde{\varrho}_j \\ a_j|x|^{-p/\widehat{\gamma}} & \text{if } |x| > \widetilde{\varrho}_j. \end{cases} \tag{3.24}$$

We suppose to have proved that the solution of problem (1), (3.6) satisfies

$$u(x, t_{j-1}) \geq \widetilde{u}_j(x) \quad \text{in } \mathbb{R}^N, \quad \text{for some } j \in \mathbb{N}_+$$

and we show $u(x, t_j) \geq \widetilde{u}_{j+1}(x)$ in $\mathbb{R}^N$ for the values $\widetilde{\varrho}_{j+1}$ and a_{j+1}. From the induction hypothesis, we have that the solution $v(x, t)$ of the problem

$$\begin{cases} \partial_t v = \Delta_p v^m + f(v) & \text{in } \mathbb{R}^N \times (t_{j-1}, \infty) \\ v(x, t_{j-1}) = \widetilde{u}_j(x) & \text{in } \mathbb{R}^N \end{cases} \tag{3.25}$$

is a sub-solution of problem (1), (3.6) in $\mathbb{R}^N \times [t_{j-1}, \infty)$ which implies $u(x, t) \geq v(x, t)$ in $\mathbb{R}^N \times [t_{j-1}, \infty)$ and so, it is sufficient to prove $v(x, t_j) \geq \widetilde{u}_{j+1}(x)$ in $\mathbb{R}^N$. Since we need to repeat almost the same procedure of the case $j = 1$, we only give a brief sketch of the induction step.

Step1′. Construction of a sub-solution of problem (3.25), (3.24), in $\mathbb{R}^N \times [t_{j-1}, t_j]$. With the same techniques used in *Step1*, we construct a Barenblatt solution $B_{M_{j+1}}(x, \theta_{j+1}) \leq \widetilde{u}_j(x)$ in $\mathbb{R}^N$ with parameters

$$M_{j+1} = \overline{K}\widetilde{\varrho}_j^N\widetilde{\varepsilon} \qquad \text{and} \qquad \theta_{j+1} = K_1^{-\widehat{\gamma}}\widetilde{\varrho}_j^p\widetilde{\varepsilon}^{\widehat{\gamma}} \tag{3.26}$$

and a sub-solution of problem (3.25), (3.24): $w(x,t) = e^{\lambda(t-t_{j-1})}\widetilde{w}(x,\widetilde{\tau})$ in $\mathbb{R}^N \times [t_{j-1}, t_j]$, where

$$\widetilde{\tau}(t) = \frac{1}{\lambda\widetilde{\gamma}}\Big[1 - e^{-\lambda\widetilde{\gamma}(t-t_{j-1})}\Big], \quad \text{for } t \geq t_{j-1}.$$

In particular, note that $\theta_{j+1} \geq \theta_j \geq \ldots \geq \theta_1$, $0 \leq \widetilde{\tau}(t) \leq \tau_\infty$, $\widetilde{\tau}(t_j) := \widetilde{\tau}_j = \tau_0$ and

$$v(x,t_j) \geq e^{\lambda t_0} B_{M_{j+1}}(x, \theta_{j+1} + \widetilde{\tau}_j).$$

Step2′. We have to study a chain of inequalities similar to the one carried out in *Step2* verifying that

$$e^{\lambda t_0} B_{M_{j+1}}(x, \theta_{j+1} + \widetilde{\tau}_j)|_{|x|=\widetilde{\varrho}_{j+1}} \geq \widetilde{\varepsilon}.$$

Thus, imposing conditions similar to (3.20) and (3.21) and requiring their compatibility, we have to check the validity of the inequality

$$\widetilde{K}e^{\lambda t_0} \geq \left(1 + \frac{\tau_0}{\theta_{j+1}}\right)^{\alpha}.$$

Since $\theta_1 \leq \theta_{j+1}$, we have $\tau_0 \leq \theta_{j+1}$ and so, a sufficient condition so that the previous inequality is satisfied is $\widetilde{K}e^{\lambda t_0} \geq 2^{\alpha}$, which is guaranteed by the initial choice of t_0. Finally, following the reasonings of the case $j = 1$ it is simple to obtain the relation

$$\left(\frac{\widetilde{\varrho}_{j+1}}{\widetilde{\varrho}_j}\right)^{\frac{p}{\gamma}} \geq \frac{K_2}{2K_1} e^{\lambda t_0}\left(1 + \frac{\tau_0}{\theta_{j+1}}\right)^{\frac{1}{\gamma}}$$

which implies

$$\widetilde{\varrho}_{j+1} \geq \widetilde{\varrho}_j e^{\sigma t_0} \geq \ldots \geq \widetilde{\varrho}_0 e^{\sigma j t_0},$$

and we complete the proof. □

Proof of Proposition 3.3. The previous lemma proves that for the sequence of times $t_j = (jt_0)_{j\in\mathbb{N}_+}$ and for any choice of the parameter $0 < \sigma < \sigma_*$, the solution of problem (1), (3.6) reaches a positive value $\widetilde{\varepsilon}$ in the sequence of sets $\{|x| \leq \widetilde{\varrho}_0 e^{\sigma j t_0}\}$ where $\widetilde{\varrho}_0 > 0$ is chosen large enough (in particular, we can assume $\widetilde{\varrho}_0 \geq 1$).
Actually, we obtained a more useful result. First of all, note that, for all $0 < \sigma < \sigma_*$, Lemma 3.5 implies

$$u(x, jt_0) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq e^{\sigma j t_0}\}, \text{ for all } \ j \in \mathbb{N}_+,$$

for all $0 < \widetilde{\varepsilon} \leq \widetilde{\varepsilon}_0 = \delta e^{-f'(0)t_0}$. Moreover, since conditions (3.11) are satisfied for all $t_0 \leq t_1 \leq 2t_0$, we can repeat the same proof of Lemma 3.5, modifying the value of $\widetilde{\varepsilon}_0$ and choosing a different value $\underline{\widetilde{\varepsilon}}_0 = \delta e^{-2f'(0)t_0} > 0$, which is smaller but strictly positive for all $t_0 \leq t_1 \leq 2t_0$. Hence, it turns out that for all $0 < \widetilde{\varepsilon} \leq \underline{\widetilde{\varepsilon}}_0$, it holds

$$u(x,t) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq e^{\sigma t}\}, \text{ for all } \ t_0 \leq t \leq 2t_0.$$

Now, iterating this procedure as in the proof of Lemma 3.5, it is clear that we do not have to change the value of $\underline{\widetilde{\varepsilon}}_0$ when $j \in \mathbb{N}_+$ grows and so, for all $0 < \widetilde{\varepsilon} \leq \underline{\widetilde{\varepsilon}}_0$, we obtain

$$u(x,t) \geq \widetilde{\varepsilon} \quad \text{in } \{|x| \leq e^{\sigma t}\}, \text{ for all } \ j \in \mathbb{N}_+ \ \text{ and for all } \ jt_0 \leq t \leq (j+1)t_0.$$

Then, using the arbitrariness of $j \in \mathbb{N}_+$, we complete the proof. □

Remark. Note that, to be precise, in the proof of Proposition 3.3, we have to combine Lemma 3.4 with Lemma 3.5 as follows. Let $u = u(x,t)$ the solution of problem (1) with initial datum (3.2). We wait a time $t_1 > 0$ given by Lemma 3.4, in order to have

$$u(x,t_1) \geq \widetilde{u}_0(x) \quad \text{in } \mathbb{R}^N,$$

for all $\widetilde{\varrho}_0 > 0$ and some $\widetilde{\varepsilon} > 0$ depending on t_1. Now, thanks to the Comparison Principle, we deduce $u(x,t+t_1) \geq \widetilde{u}(x,t)$ in $\mathbb{R}^N \times [0,\infty)$, where we indicate with $\widetilde{u} = \widetilde{u}(x,t)$ the solution of problem (1) with initial datum $\widetilde{u}_0 = \widetilde{u}_0(x)$. In this way, we deduce the statement of Lemma 3.5 for more general initial data satisfying (3.2) and we can prove Proposition 3.3.

3.3 Proof of Theorem 3.1

We know focus on Theorem 3.1. We first prove Part (i) and then Part (ii). As the reader will see, the proof of Part (i) strongly relies on some technical and non standard comparison techniques and on known results concerning p-Laplacian type equations (see Section 3.5 and Section 3.6 for a short review on these issues).

3.3.1 Proof of Theorem 3.1, Part (i)

Fix $0 < \widehat{\gamma} < p/N$, $0 < \sigma < \sigma_*$ and set $\underline{w} := 1 - u^m$ in $\mathbb{R}^N \times (0,\infty)$. We will prove that for all $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$\underline{w}(x,t) \leq \varepsilon \quad \text{ in } \{|x| \leq e^{\sigma t}, t \geq t_\varepsilon\},$$

which is equivalent to the assertion of the thesis.

Step1: Reduction to a p-Laplacian type equation. Fix $\sigma < \nu < \sigma_*$ and consider the inner set $\Omega_I := \{|x| \leq e^{\nu t}, t \geq t_1\}$, where $t_1 > 0$ is initially arbitrary. We recall that Proposition 3.3 assures the existence of $\widetilde{\varepsilon} > 0$ and $t_0 > 0$ such that $u \geq \widetilde{\varepsilon}$ in the set $\{|x| \leq e^{\nu t}, t \geq t_0\}$. In particular, for all $t_1 \geq t_0$, we have that $u = u(x,t)$ is bounded from below and above in the inner set:

$$\widetilde{\varepsilon} \leq u \leq 1 \quad \text{ in } \Omega_I. \tag{3.27}$$

Moreover, it is not difficult to see that, setting $a(x,t) = (1/m)u^{1-m}$ and $c(x,t) = f(u)/\underline{w}$, the function $\underline{w} = 1 - u^m$ solves the problem

$$\begin{cases} a(x,t)\partial_t \underline{w} - \Delta_p \underline{w} + c(x,t)\underline{w} = 0 & \text{in } \mathbb{R}^N \times (t_1,\infty) \\ \underline{w}(x,t_1) = 1 - [u(x,t_1)]^m & \text{in } \mathbb{R}^N. \end{cases} \tag{3.28}$$

Using (3.27), it is simple to see that

$$a_0 \leq a(x,t) \leq a_1 \quad \text{in } \Omega_I$$

where

$$a_0 := \begin{cases} (1/m)\widetilde{\varepsilon}^{1-m} & \text{if } 0 < m < 1 \\ 1/m & \text{if } m \geq 1 \end{cases} \qquad a_1 := \begin{cases} 1/m & \text{if } 0 < m < 1 \\ (1/m)\widetilde{\varepsilon}^{1-m} & \text{if } m \geq 1. \end{cases}$$

For what concerns $c(x,t)$, it is bounded from below in Ω_I:

$$c(x,t) \geq c_0 \quad \text{in } \Omega_I,$$

where $c_0 > 0$ and depends on $\widetilde{\varepsilon}$ and m. Indeed, if $0 < m < 1$ we have that $c(x,t) = f(u)/(1-u^m) \geq f(u)/(1-u)$ for all $0 \leq u \leq 1$. Hence, we get our bound from below recalling (3.27) and noting that

$$\frac{f(u)}{1-u} \sim -f'(1) > 0 \quad \text{as } u \sim 1.$$

If $m \geq 1$, we have the formula

$$c(x,t) = \frac{f(u)}{1-u^m} = \frac{f(u)}{(1-u)(1+u+\ldots+u^{m-1})},$$

and so, since $u \leq 1$ and arguing as in the case $0 < m < 1$, we deduce

$$c(x,t) \geq (1/m)\frac{f(u)}{1-u} \geq c_0 \quad \text{in } \Omega_I$$

for some $c_0 > 0$ depending on $\widetilde{\varepsilon}$ and m. In particular, it follows that $\underline{w} = \underline{w}(x,t)$ satisfies

$$a(x,t)\partial_t \underline{w} - \Delta_p \underline{w} + c_0 \underline{w} \leq 0 \quad \text{in } \Omega_I, \tag{3.29}$$

i.e., $\underline{w} = \underline{w}(x,t)$ is a sub-solution for the equation in problem (3.28) in the set Ω_I.

Step2: Construction of a super-solution. In this step, we look for a super-solution $\overline{w} = \overline{w}(x,t)$ of problem (3.28) with $\partial_t \overline{w} \leq 0$ in $\mathbb{R}^N \times (t_1,\infty)$. We consider the solution of the problem

$$\begin{cases} a_1 \partial_t \overline{w} - \Delta_p \overline{w} + c_0 \overline{w} = 0 & \text{in } \mathbb{R}^N \times (t_1,\infty) \\ \overline{w}(x,t_1) = 1 + |x|^\lambda & \text{in } \mathbb{R}^N. \end{cases} \tag{3.30}$$

According to the resume presented in Section 3.6, problem (3.30) is well posed if $0 < \lambda < p/(p-2)$ when $p > 2$. Further assumptions are not needed when $1 < p \leq 2$. Furthermore, since $c_0 > 0$ can be chosen smaller and $a_1 > 0$ larger, we make the additional assumption

$$\frac{c_0}{a_1} = \nu\lambda. \tag{3.31}$$

Now, we define the function

$$\tau(t) := \begin{cases} \frac{1}{c_0(2-p)}\left[e^{(c_0/a_1)(2-p)(t-t_1)} - 1\right] & \text{if } 1 < p < 2 \\ \frac{1}{a_1}(t-t_1) & \text{if } p = 2 \\ \frac{1}{c_0(p-2)}\left[1 - e^{-(c_0/a_1)(p-2)(t-t_1)}\right] & \text{if } p > 2. \end{cases}$$

Note that $\tau = \tau(t)$ is increasing with $\tau(t_1) = 0$ for all $p > 1$. Moreover, we define the limit of $\tau(t)$ as $t \to \infty$ with the formula

$$\tau_\infty := \begin{cases} \infty & \text{if } 1 < p \leq 2 \\ [c_0(p-2)]^{-1} & \text{if } p > 2. \end{cases}$$

Then, the function $\widetilde{w}(x,\tau) := e^{(c_0/a_1)(t-t_1)}\overline{w}(x,t)$ (with $\tau = \tau(t)$) solves the "pure diffusive" problem

$$\begin{cases} \partial_\tau \widetilde{w} = \Delta_p \widetilde{w} & \text{in } \mathbb{R}^N \times (0,\tau_\infty) \\ \widetilde{w}(x,0) = 1 + |x|^\lambda & \text{in } \mathbb{R}^N. \end{cases}$$

As we explained in Section 3.6, for all $\tau_1 \geq 0$ the problem

$$\begin{cases} \partial_\tau U = \Delta_p U & \text{in } \mathbb{R}^N \times (\tau_1,\infty) \\ U(x,\tau_1) = |x|^\lambda & \text{in } \mathbb{R}^N \end{cases} \tag{3.32}$$

admits self-similar solutions $U(x,\tau+\tau_1) = \tau^{-\alpha_\lambda}F(|x|(\tau+\tau_1)^{-\beta_\lambda})$, with self-similar exponents

$$\alpha_\lambda = -\frac{\lambda}{(1-\lambda)p + 2\lambda} \quad \text{and} \quad \beta_\lambda = \frac{1}{(1-\lambda)p+2\lambda},$$

and profile $F(\xi) \geq 0$ with $F'(\xi) \geq 0$ for all $\xi \geq 0$, where we set $\xi = |x|(\tau+\tau_1)^{-\beta_\lambda}$. Note that since we assumed $0 < \lambda < p/(p-2)$ when $p > 2$, the self-similar exponents are well defined with $\alpha_\lambda < 0$ and $\beta_\lambda > 0$ for all $p > 1$. Finally, recall that it is possible to describe the spacial "decay" of the self-similar solutions for large values of the variable $\xi = |x|(\tau+\tau_1)^{-\beta_\lambda}$, with the bounds

$$H_2|x|^\lambda \leq U(x,\tau+\tau_1) \leq H_1|x|^\lambda, \quad \text{for all } |x|(\tau+\tau_1)^{-\beta_\lambda} \geq h \tag{3.33}$$

for a constant $h \gg 0$ large enough, see formula (3.57). Now, it is not difficult to see that $\widetilde{w}(x,\tau) = 1 + U(x,\tau+\tau_1)$, for all fixed delays $\tau_1 \geq 0$. Moreover, we compute the time derivative:

$$\begin{aligned}\partial_t\overline{w}(x,t) &= \partial_t\left\{e^{-\frac{c_0}{a_1}(t-t_1)}[1+U(x,\tau+\tau_1)]\right\}\\ &= -(\tau+\tau_1)^{-\alpha_\lambda-1}e^{-\frac{c_0}{a_1}(t-t_1)}\left\{\frac{c_0}{a_1}(\tau+\tau_1)^{\alpha_\lambda+1}+\left[\frac{c_0}{a_1}(\tau+\tau_1)+\alpha_\lambda\tau'\right]F(\xi)+\beta_\lambda\tau'\xi F'(\xi)\right\},\end{aligned}$$

where $\xi = |x|(\tau+\tau_1)^{-\beta_\lambda}$ and τ' stands for the derivative respect with the variable $t \geq 0$. Let's set

$$Q(t) := (c_0/a_1)(\tau+\tau_1) + \alpha_\lambda\tau'.$$

Since, $F(\cdot)$, $F'(\cdot)$, and $\tau'(\cdot)$ are non-negative and $\beta_\lambda > 0$, in order to have $\partial_t\overline{w}(x,t) \leq 0$, it is sufficient to show $Q(t) \geq 0$ for all $t \geq t_1$ and a suitable choice of $\tau_1 > 0$.

If $p = 2$, this follows from a direct and immediate computation, choosing $\tau_1 > 0$ large enough.

If $1 < p < 2$, we may proceed similarly. It is simple to see that condition $Q(t) \geq 0$ for $t \geq t_1$ reads

$$[1+\alpha_\lambda(2-p)]e^{(c_0/a_1)(2-p)(t-t_1)} \geq 1 - \frac{\tau_1}{\tau_\infty}.$$

Consequently, since $1+\alpha_\lambda(2-p) \geq 0$, it is sufficient to choose $\tau_1 \geq \tau_\infty$.

Finally, when $p > 2$ it holds $\tau'(t) \leq 1/a_1$ for all $t \geq t_1$. Hence, it is simple to see that the choice $\tau_1 \geq -\alpha_\lambda/c_0$ is a sufficient condition so that $Q(t) \geq 0$ for all $t \geq t_1$. We stress that the choice of $\tau_1 > 0$ is independent of $t_1 > 0$.

Now, using the fact that $\partial_t\overline{w}(x,t) \leq 0$ in $\mathbb{R}^N \times (t_1,\infty)$ and that $0 \leq a(x,t) \leq a_1$ in Ω_I, it is straightforward to see that

$$a(x,t)\partial_t\overline{w} - \Delta_p\overline{w} + c_0\overline{w} \geq 0 \quad \text{in } \Omega_I. \tag{3.34}$$

Step3: Comparison and conclusion. Now we compare the functions $\underline{w}$ and $\overline{w}$, applying the Comparison Principle of Section 3.5. Hence, we have to check that the assumptions in Proposition 3.8 are satisfied.

(A1). It is simple to see that it holds $\underline{w}(x,t_1) \leq \overline{w}(x,t_1)$ in $\mathbb{R}^N$. Indeed, we have $\overline{w}(x,t_1) \geq 1$ while $\underline{w}(x,t_1) = 1 - [u(x,t_1)]^m \leq 1$.

(A2). We have to check that $\underline{w} \leq \overline{w}$ on the boundary of Ω_I, i.e., on the set $\{|x| = e^{\nu t}, t \geq t_1\}$. We use the first estimate in (3.33):

$$\begin{aligned}\overline{w} &= e^{-(c_0/a_1)(t-t_1)}\widetilde{w}(x,\tau) = e^{-(c_0/a_1)(t-t_1)}[1+U(x,\tau+\tau_1)]\\ &\geq e^{-(c_0/a_1)(t-t_1)}(1+H_2|x|^\lambda) = e^{-(c_0/a_1)(t-t_1)}(1+H_2e^{\nu\lambda t})\\ &= e^{(c_0/a_1)t_1}(H_2 + e^{-(c_0/a_1)t}) \geq 1 \geq \underline{w} \quad \text{in } \{|x| = e^{\nu t}, t \geq t_1\}.\end{aligned}$$

First of all, we point out that the last equality in the preceding chain is satisfied thanks to the first assumption in (3.31), i.e., $c_0/a_1 = \nu\lambda$.

Secondly, we note that the first inequality holds only if $|x|(\tau+\tau_1)^{-\beta_\lambda} \geq h$, which means

$$e^{\nu t} \geq h(\tau+\tau_1)^{\beta_\lambda}. \tag{3.35}$$

As the reader can easily check, when $p = 2$, (3.35) is satisfied by taking $t_1 \ge t_0$ so that $e^{\nu t_1} \ge h\sqrt{\tau_1}$. If $p > 2$, it is sufficient to fix $t_1 \ge t_0$ to have $e^{\nu t_1} \ge h(\tau_\infty + \tau_1)^{\beta_\lambda}$.
The case $1 < p < 2$ is a little bit subtle. First of all, set $b_\lambda := 1/\beta_\lambda = (1-\lambda)p + 2\lambda$ and note that, thanks to assumption (3.31), we have that (3.35) is automatically satisfied if

$$e^{\nu b_\lambda t} \ge h^{b_\lambda}\Big\{[c_0(2-p)]^{-1}e^{\nu\lambda(2-p)(t-t_1)} + \tau_1\Big\},$$

which, since $b_\lambda = p + \lambda(2-p)$, is equivalent to

$$e^{\nu\lambda(2-p)t}\Big\{e^{\nu pt} - [c_0(2-p)]^{-1}h^{b_\lambda}\Big\} \ge h^{b_\lambda}\Big\{[c_0(2-p)]^{-1}e^{-\nu\lambda(2-p)t_1} + \tau_1\Big\}.$$

Finally, it is straightforward to see that the last inequality is satisfied for all $t \ge t_1 \ge t_0$ so that

$$e^{\nu\lambda p t_1} \ge h^{b_\lambda}\Big\{2[c_0(2-p)]^{-1} + \tau_1\Big\}.$$

Hence, we have that condition (3.35) is satisfied for all $p > 1$ when $t_1 \ge t_0$ is taken large enough.

(A3). To check this third assumption it is sufficient to combine (3.29) and (3.34), restricting they validity to the set $\{|x| \le e^{\nu t}, t \ge t_1\}$.
Hence, we deduce $\underline{w} \le \overline{w}$ in $\{|x| \le e^{\nu t}, t \ge t_1\}$ by applying Proposition 3.8. So we have

$$\begin{aligned}\underline{w}(x,t) \le \overline{w}(x,t) = e^{-\frac{c_0}{a_1}(t-t_1)}\,[1 + U(x,\tau+\tau_1)] &\le e^{-\frac{c_0}{a_1}(t-t_1)}\Big[1 + U(e^{\nu t},\tau+\tau_1)\Big]\\ \le e^{-\frac{c_0}{a_1}(t-t_1)}\Big[1 + H_1|x|^\lambda\Big] &\le e^{-\frac{c_0}{a_1}(t-t_1)}\Big[1 + H_1 e^{\nu\lambda t}\Big]\end{aligned}$$

in the set $\{|x| \le e^{\nu t}, t \ge t_1\}$, thanks to (3.35).
Now, let us fix $\varepsilon > 0$ and take a time $t'_\varepsilon > 0$, and a constant $H_\varepsilon > 0$ such that

$$t'_\varepsilon \ge t_1 - \frac{a_1\ln(\varepsilon/2)}{c_0} \qquad \text{and} \qquad H_\varepsilon^\lambda \le \frac{\varepsilon}{2H_1}e^{-\frac{c_0}{a_1}t_1}.$$

These choices combined with the previous chain of inequalities give us

$$\underline{w}(x,t) \le e^{-\frac{c_0}{a_1}(t_\varepsilon - t_1)} + H_\varepsilon^\lambda H_1 e^{+\frac{c_0}{a_1}t_1} \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \le \varepsilon,$$

in the set $\{|x| \le H_\varepsilon e^{\nu t}, t \ge t'_\varepsilon\}$. Finally, noting that $\{|x| \le e^{\sigma t}, t \ge t_\varepsilon\} \subset \{|x| \le H_\varepsilon e^{\nu t}, t \ge t_\varepsilon\}$ for all $\sigma < \nu$ and for some $t_\varepsilon > 0$ large enough, we complete the proof of the theorem using the arbitrariness of $\sigma < \nu < \sigma_*$. □

Remarks. We end this section with two remarks. First of all, we ask the reader to note that at the beginning of the previous proof, we have made the change of variable $\underline{w} = 1 - u^m$ in order to obtain problem (3.28), which has non-constant coefficients, but the diffusion operator simplifies to a p-Laplacian. This is a considerable advantage since we can employ the well known theory of p-Laplacian diffusion and non-integrable initial data (see Section 3.6 for more details and references) to construct the super-solution given by problem (3.30). A different approach could be studying the existence, uniqueness and regularity of solutions for the doubly nonlinear equation and non-integrable initial data in the fast diffusion range $0 < \widehat{\gamma} < p/N$. Up to our knowledge, this theory has not been developed yet.
Secondly, we point out that in the previous proof we have showed a slightly different result too, that we state in the following corollary. It will be very useful in Section 3.4, where we will study the behaviour of the solution $u = u(x,t)$ on the set $|x| = e^{\sigma_* t}$.

Corollary 3.6. *Let $m > 0$ and $p > 1$ such that $0 < \widehat{\gamma} < p/N$ and let $u = u(x,t)$ be the solution of the problem* (1) *with initial datum* (3.2). *Suppose that there exist $\nu > 0$, $\underline{\varrho} > 0$, $\overline{\varepsilon} > 0$ and $t_0 > 0$ such that*

$$u(x,t) \geq \overline{\varepsilon} \quad \text{in } \{|x| \leq \underline{\varrho} e^{\nu t}\} \text{ for all } \; t \geq t_0.$$

Then, for all $0 < \omega < 1$, there exist $C_\omega > 0$ large enough and $t_\omega \geq t_0$ such that

$$u(x,t) \geq \omega \quad \text{in } \{|x| \leq C_\omega^{-1} e^{\nu t}\} \text{ for all } \; t \geq t_\omega.$$

The proof coincides with the one of Theorem 3.1. Notice indeed that we have begun by assuming that $u \geq \overline{\varepsilon} := \widetilde{\varepsilon}$ in $\{|x| \leq e^{\nu t}, t \geq t_1\}$ and for all $\varepsilon > 0$, we proved the existence of $t'_\varepsilon > 0$ and $H_\varepsilon > 0$ such that

$$u(x,t) \geq 1 - \varepsilon, \quad \text{in } \{|x| \leq H_\varepsilon e^{\nu t}, t \geq t'_\varepsilon\}.$$

We point out that in the previous statement the value of the exponent $\nu > 0$ does not change. In the proof of Theorem 3.1, we need to take $\sigma < \nu$ to obtain a "convergence inner set" not depending on H_ε. Indeed, in the second one we prove the convergence of the solution $u = u(x,t)$ to the steady state 1 in the set $\{|x| \leq e^{\sigma t}\}$ for all $\sigma < \sigma_*$, while now the exponent $\nu > 0$ is arbitrary.

3.3.2 Proof of Theorem 3.1, Part (ii)

This part is easier. Fix $N \geq 1$, $0 < \widehat{\gamma} < p/N$, and $\sigma > \sigma_*$. First of all, we construct a super-solution for problem (1), (3.2) using the hypothesis on the function $f(\cdot)$. Indeed, since $f(u) \leq f'(0)u$ for all $0 \leq u \leq 1$, the solution of the linearized problem

$$\begin{cases} \partial_t \overline{u} = \Delta_p \overline{u}^m + f'(0)\overline{u} & \text{in } \mathbb{R}^N \times (0,\infty) \\ \overline{u}(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

gives the super-solution we are interested in and, by the Comparison Principle, we deduce $u(x,t) \leq \overline{u}(x,t)$ in $\mathbb{R}^N \times (0,\infty)$. Now, consider the change of the time variable

$$\tau(t) = \frac{1}{f'(0)\widehat{\gamma}}\left[1 - e^{-f'(0)\widehat{\gamma}t}\right], \quad \text{for } t \geq 0,$$

with $0 \leq \tau(t) \leq \tau_\infty := \frac{1}{f'(0)\widehat{\gamma}}$. Then the function $\overline{v}(x,\tau) = e^{-f'(0)t}\overline{u}(x,t)$ solves the problem

$$\begin{cases} \partial_\tau \overline{v} = \Delta_p \overline{v}^m & \text{in } \mathbb{R}^N \times (0,\tau_\infty) \\ \overline{v}(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases}$$

From the properties of the profile of the Barenblatt solutions and the hypothesis on the initial datum (3.2), it is evident that there exist positive numbers M and θ such that $u_0(x) \leq B_M(x,\theta)$ in $\mathbb{R}^N$ and so, by comparison, we obtain

$$\overline{v}(x,\tau) \leq B_M(x,\theta+\tau) \quad \text{in } \mathbb{R}^N \times (0,\tau_\infty).$$

Now, since the profile of Barenblatt solutions satisfies $F_1(\xi) \leq K_1|\xi|^{-\frac{p}{\widehat{\gamma}}}$ for some constant $K_1 > 0$ and for all $\xi \in \mathbb{R}^N$ (see (3.8)), we can perform the chain of upper estimates

$$\begin{aligned} u(x,t) &\leq \overline{u}(x,t) = e^{f'(0)t}\overline{v}(x,\tau) \\ &\leq e^{f'(0)t}B_M(x,\theta+\tau) = e^{f'(0)t}M^{1+\alpha\widehat{\gamma}}(\theta+\tau)^{-\alpha}F_1\left(x(M^{-\widehat{\gamma}}(\theta+\tau))^{-\alpha/N}\right) \\ &\leq e^{f'(0)t}K_1M^{1+\alpha\widehat{\gamma}}(\theta+\tau)^{-\alpha}(M^{-\widehat{\gamma}}(\theta+\tau))^{\frac{\alpha p}{N\widehat{\gamma}}}|x|^{-p/\widehat{\gamma}} \\ &\leq Ke^{f'(0)t}|x|^{-p/\widehat{\gamma}}, \end{aligned}$$

where we set $K := K_1(2\tau_\infty)^{1/\widehat{\gamma}}$ and we used the first relation in (3.7) in the third inequality. Note that we used that $1 + \alpha = \alpha p/N$, too. Now, supposing $|x| \geq e^{\sigma t}$ in the last inequality, we get

$$u(x,t) \leq Ke^{(f'(0)-p\sigma/\widehat{\gamma})t} \to 0 \quad \text{in } \{|x| \geq e^{\sigma t}\} \text{ as } t \to \infty,$$

since we have chosen $\sigma > \sigma_*$, completing the proof. □

3.4 Case $f(u) = u(1-u)$: Proof of Theorem 3.2

We devote this section to the proof of Theorem 3.2. A similar result was showed in [47] for the Fisher-KPP equation with fractional diffusion. In particular, they studied the case of the fractional Laplacian $(-\Delta)^{1/2}$ and worked in dimension $N = 1$, see Theorem 1.6 of [47] for more details. In our setting, we consider the classical reaction term $f(u) = u(1-u)$ and the problem

$$\begin{cases} \partial_t u = \Delta_p u^m + u(1-u) & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N \end{cases} \tag{3.36}$$

where $u_0(\cdot)$ satisfies (3.2), i.e., $u_0(x) \leq C|x|^{-p/\widehat{\gamma}}$ and $0 \leq u_0 \leq 1$, for some constant $C > 0$. As always, we do not pose restrictions on the dimension $N \geq 1$, and we will work in the "fast diffusion" range $0 < \widehat{\gamma} < p/N$.

However, before proceeding with the proof, we dedicate some paragraphs to some heuristic computations which are the "doubly nonlinear" version to the ones done by King and McCabe in [133] for Porous Medium "fast" diffusion. So, we fix $0 < \widehat{\gamma} < p/N$ and we consider radial solutions to the equation in (3.36), which means

$$\partial_t u = r^{1-N}\partial_r\Big(r^{N-1}|\partial_r u^m|^{p-2}\partial_r u^m\Big) + u(1-u), \quad r > 0,\ t > 0.$$

Note that the authors of [133] worked with a slightly different equation (they absorbed the multiplicative factor m^{p-1} with a simple change of variables). We linearize the reaction term and we assume that $u = u(r,t)$ satisfies

$$\partial_t u = r^{1-N}\partial_r\Big(r^{N-1}|\partial_r u^m|^{p-2}\partial_r u^m\Big) + u, \quad \text{for } r \sim \infty. \tag{3.37}$$

Now, we look for a solution to (3.37) in the form $u(r,t) \sim r^{-p/\widehat{\gamma}}G(t)$ for $r \sim \infty$ which agrees with the assumption (3.2) on the initial datum and with the linearization (3.37). It is straightforward to see that for such solution, the function $G = G(t)$ has to solve the equation

$$\frac{dG}{dt} = G + \kappa G^{1-\widehat{\gamma}}, \quad t \geq 0 \qquad \kappa := \frac{(p-\widehat{\gamma}N)(mp)^{p-1}}{\widehat{\gamma}^p}. \tag{3.38}$$

Note that since $0 < \widehat{\gamma} < p/N$, we have that κ is well defined and positive, while $1 - \widehat{\gamma} = m(p-1) > 0$. Equation (3.38) belongs to the famous Bernoulli class and can be explicitly integrated:

$$G(t) = \Big(ae^{\widehat{\gamma}t} - \kappa\Big)^{\frac{1}{\widehat{\gamma}}}, \quad a \geq \kappa, \quad t \geq 0.$$

Hence, for all fixed $t \geq 0$, we obtain the asymptotic expansion for our solution

$$u(r,t) \sim r^{-\frac{p}{\widehat{\gamma}}}\Big(ae^{\widehat{\gamma}t} - \kappa\Big)^{\frac{1}{\widehat{\gamma}}}, \quad r \sim \infty. \tag{3.39}$$

Now, for all fixed $r > 0$, we consider a solution $\zeta_0 = \zeta_0(r,t)$ to the *logistic* equation

$$\partial_t \zeta_0 = \zeta_0(1-\zeta_0), \quad t \geq 0,$$

which describes the state in which there is not diffusion and the dynamics is governed by the reaction term. We assume to have

$$u(r,t) \sim \zeta_0(r,t) \quad \text{for } t \sim \infty,$$

where the leading-order term $\zeta_0 = \zeta_0(r,t)$ satisfies

$$\zeta_0(r,t) \sim \frac{\phi(r)e^t}{1+\phi(r)e^t}, \quad \text{for } t \sim \infty, \quad r \sim \infty, \tag{3.40}$$

for some unknown function $\phi = \phi(r)$, with $\phi(r) \to 0$, as $r \to \infty$. Now, matching (3.39) with (3.40) for t large and $r \sim \infty$, we easily deduce

$$\phi(r) \sim (ar^{-p})^{1/\widehat{\gamma}}, \quad \text{for } r \sim \infty.$$

Thus, substituting $\phi(r) \sim (ar^{-p})^{1/\widehat{\gamma}}$ in (3.40) and taking $r \sim e^{\widehat{\gamma}/pt}$ for $t \sim \infty$, we have

$$u(r,t) \sim \frac{(ar^{-p}e^{\widehat{\gamma}t})^{1/\widehat{\gamma}}}{1+(ar^{-p}e^{\widehat{\gamma}t})^{1/\widehat{\gamma}}} = \frac{\widehat{a}e^t}{r^{p/\widehat{\gamma}}+\widehat{a}e^t} \quad \text{for } t \sim \infty, \quad r \sim e^{\widehat{\gamma}/pt}, \tag{3.41}$$

where $\widehat{a} = a^{1/\widehat{\gamma}} \geq \kappa^{1/\widehat{\gamma}}$. The previous formula corresponds to a "similarity reduction" (see [133], pag. 2533) of the logistic equation with $\zeta_0 = \zeta_0(r/e^{\widehat{\gamma}/pt})$.

Note that taking $r \geq e^{\sigma t}$ and $\sigma > \widehat{\gamma}/p$, we have $u(r,t) \sim 0$ for $t \sim \infty$ while if $r \leq e^{\sigma t}$ and $\sigma < \widehat{\gamma}/p$ we have $u(r,t) \sim 1$ for $t \sim \infty$. This means that setting $\sigma_* = \widehat{\gamma}/p$, $r(t) \sim e^{\sigma_* t}$ is a "critical" curve, in the sense that it separates the region in which the solution $u = u(r,t)$ converges to $u = 0$ to the one which converges to $u = 1$. This is the formal proof of Theorem 3.1.

Let us now come back to the proof of Theorem 3.2. We divide the proof in two main parts. The first one is devoted to prove the "upper bound" i.e., the first inclusion (3.5). In the second one, we show the "lower bound" (the second inclusion (3.5)) which is the most difficult part. We point out that in this part, we have to give to separate proofs for the ranges $\widehat{\gamma} \leq p-1$ and $\widehat{\gamma} > p-1$.

3.4.1 Proof of Theorem 3.1: Upper bound

We begin to prove the first inclusion in (3.5), i.e., for all $0 < \omega < 1$, there exists a constant $C_\omega > 0$ large enough such that it holds

$$\{x \in \mathbb{R}^N : |x| > C_\omega e^{\sigma_* t}\} \subset \{x \in \mathbb{R}^N : u(x,t) < \omega\} \quad \text{for all } t \geq 0,$$

where $\sigma_* = \widehat{\gamma}/p$ (recall that in this setting $f'(0) = 1$).

We need to construct special super-solution. Thus, we repeat the computations carried out above, by looking for radial solutions to the equation

$$\partial_t \overline{u} = r^{1-N}\partial_r\Big(r^{N-1}|\partial_r \overline{u}^m|^{p-2}\partial_r \overline{u}^m\Big) + \overline{u}, \quad r = |x| > 0,$$

with separate variables

$$\overline{u}(r,t) = r^{-p/\widehat{\gamma}}G(t), \quad r > 0,\ t \geq 0.$$

We have seen that the previous ansatz gives us (cfr with formula (3.39)) an explicit solution

$$\overline{u}(x,t) = |x|^{-\frac{p}{\widehat{\gamma}}}\Big(ae^{\widehat{\gamma}t} - \kappa\Big)^{\frac{1}{\widehat{\gamma}}},$$

for all $a > \kappa$ (the value of $\kappa > 0$ is given in (3.38)). We have thus built our super-solution to the equation in (3.36). Now, in order to apply the Comparison Principle, we show $\overline{u}(x,0) \geq u_0(x)$ in $\mathbb{R}^N$. We consider the function

$$\overline{u}_0(x) = \begin{cases} 1 & \text{if } |x| \leq C^{\widehat{\gamma}/p} \\ C|x|^{-p/\widehat{\gamma}} & \text{if } |x| \geq C^{\widehat{\gamma}/p}, \end{cases}$$

where $C > 0$ is taken as in (3.2), see also the beginning of this section. It is simple to see that we have $\overline{u}_0(x) \geq u_0(x)$ for all $x \in \mathbb{R}^N$. So, taking $a \geq k + C^{\widehat{\gamma}}$, we have

$$\begin{aligned} \overline{u}(x,0) &= (a-\kappa)^{\frac{1}{\gamma}}|x|^{-\frac{p}{\gamma}} \geq C|x|^{-\frac{p}{\gamma}} = \overline{u}_0(x) \quad &\text{if } |x| \geq C^{\widehat{\gamma}/p}, \\ \overline{u}(x,0) &= (a-\kappa)^{\frac{1}{\gamma}}|x|^{-\frac{p}{\gamma}} \geq 1 = \overline{u}_0(x) \quad &\text{if } |x| \leq C^{\widehat{\gamma}/p}. \end{aligned}$$

Consequently, we have $\overline{u}(x,0) \geq u_0(x)$ in $\mathbb{R}^N$ and we obtain $\overline{u}(x,t) \geq u(x,t)$ in $\mathbb{R}^N \times [0,\infty)$, by applying the Comparison Principle.

We are ready to prove the first inclusion in (3.5). Thus, let us fix $0 < \omega < 1$ and consider $x \in \mathbb{R}^N$ satisfying $u(x,t) > \omega$. Using the super-solution $\overline{u}(x,t)$ constructed before, we have

$$\omega < \overline{u}(x,t) = |x|^{-\frac{p}{\gamma}}\left(ae^{\widehat{\gamma}t} - \kappa\right)^{\frac{1}{\gamma}} \quad \Rightarrow \quad |x|^{\frac{p}{\gamma}} < \frac{1}{\omega}\left(ae^{\widehat{\gamma}t} - \kappa\right)^{\frac{1}{\gamma}} < \frac{a^{1/\widehat{\gamma}}}{\omega}e^t, \quad t \geq 0,$$

which, setting $C_\omega = \left(a^{1/\widehat{\gamma}}/\omega\right)^{\widehat{\gamma}/p}$, implies

$$|x| < C_\omega e^{\frac{\widehat{\gamma}}{p}t} = C_\omega e^{\sigma_* t}, \quad t \geq 0,$$

and we conclude the proof of the first inclusion in (3.5).

3.4.2 Proof of Theorem 3.1: Lower bound

We want to show the second inclusion in (3.5): for all $0 < \omega < 1$, there exist $C_\omega > 0$ and t_ω large enough, such that

$$\{x \in \mathbb{R}^N : |x| < C_\omega^{-1}e^{\sigma_* t}\} \subset \{x \in \mathbb{R}^N : u(x,t) > \omega\} \quad \text{for all } t \geq t_\omega.$$

The idea consists in constructing a sub-solution $\underline{u} = \underline{u}(x,t)$ that will act as a barrier from below and then employ Corollary 3.6. Even though this idea was firstly used in [47], we stress that our construction is completely independent by the previous one, and the comparison is not done in the whole space, but only on a sub-region. Moreover, we will divide the range $0 < \widehat{\gamma} < p/N$ in two sub-ranges: $\widehat{\gamma} \leq p-1$ and $\widehat{\gamma} > p-1$. Of course, these two new ranges are always accompanied with the "fast" diffusion assumption $0 < \widehat{\gamma} < p/N$. We will specify when we will need to distinguish between these different cases. We divide the proof in some steps.

Step1. In this first step, we fix some important notations and we define the candidate sub-solution. We know the following two facts:
• First, for all $\varepsilon > 0$ (small) and for all $r_0 > 0$ (large), there exists $t_{\varepsilon,r_0} > 0$ (large enough) such that

$$u(x,t) \geq 1 - \varepsilon \quad \text{in } \{|x| \leq r_0\}, \quad \text{for all } t \geq t_{\varepsilon,r_0}.$$

This is a direct consequence of Theorem 3.1.
• Secondly, by applying Lemma 3.4, we have that for all $\theta > 0$, there exist $t_1 > \theta$, $\widetilde{\varepsilon} > 0$, and $\widetilde{\varrho}_0 > 0$ satisfying

$$u(x,t_1) \geq \widetilde{u}_0(x) := \begin{cases} \widetilde{\varepsilon} & \text{if } |x| \leq \widetilde{\varrho}_0 \\ a_0|x|^{-p/\widehat{\gamma}} & \text{if } |x| > \widetilde{\varrho}_0 \end{cases} \quad \text{in } \mathbb{R}^N,$$

where $a_0 := \widetilde{\varepsilon}\,\widetilde{\varrho}_0^{\,p/\widehat{\gamma}}$, see (3.6).
Now, for all $m > 0$ and $p > 1$ satisfying $0 < \widehat{\gamma} < p/N$, and $N \geq 1$, we consider the constants

$$d_1 := p/\widehat{\gamma}^2(p - \widehat{\gamma}N), \quad d_2 := p/\widehat{\gamma}^2[(p-1)(p-\widehat{\gamma}) + \widehat{\gamma}(N-1)], \quad d_3 := (p/\widehat{\gamma})^{2-p}m^{1-p}. \tag{3.42}$$

Note that the assumption $0 < \widehat{\gamma} < p/N$ guarantees that d_i are positive for all $i = 1, 2, 3$. The importance of these constants will be clear later. Now, they are simply needed to choose $r_0 > 0$. We take r_0 large (depending only on m, p, N and ε), satisfying

$$r_0^p \geq \frac{d_2^{\widehat{\gamma}+1}(1-\varepsilon)^{-\widehat{\gamma}}}{d_1^{\widehat{\gamma}} d_3(\widehat{\gamma}+1)}\varepsilon^{-1} \quad \text{and} \quad r_0^p \geq \frac{d_2^2(d_1 + d_1/d_2)^{\widehat{\gamma}-(p-1)}}{p d_1 d_3}\varepsilon^{-1/\widehat{\gamma}}. \tag{3.43}$$

The first assumption will be needed in the range $\widehat{\gamma} \leq p-1$, while the second when $\widehat{\gamma} > p-1$. Then we take $\theta := t_{\varepsilon,r_0}$ and $t_0 := t_1 > t_{\varepsilon,r_0}$ and we fix $\widetilde{\varepsilon} > 0$ and $\widetilde{\varrho}_0 > 0$ corresponding to the value of $t_0 > 0$.

We finally define the candidate sub-solution. We consider the function

$$\underline{u}(r,t) = \frac{e^t}{b\psi(r) + ce^t}, \quad r \geq r_0,\ t \geq t_0, \ \text{ and } \ \psi(r) = r^{p/\widehat{\gamma}},$$

where, of course, $r = |x|$. Note that $\underline{u} = \underline{u}(r,t)$ is a modified version of the approximated solution (3.41) computed at the beginning of Section 3.4. Finally, we fix

$$c = (1-\varepsilon)^{-1} > 1. \tag{3.44}$$

The parameter $b > 0$ will be chosen in the next step, independently from c. We ask the reader to note that c depends only on $\varepsilon > 0$. The fact that $c > 1$ will be important later.

Step2. Now, we consider the region $\mathcal{R}_0 := \{|x| \geq r_0\} \times [t_0, \infty)$ and, we show that

$$\underline{u}(x,t_0) \leq u(x,t_0) \quad \text{for } |x| \geq r_0 \qquad \text{and} \qquad \underline{u}(|x| = r_0, t) \leq u(|x| = r_0, t) \quad \text{for } t \geq t_0,$$

in order to assure that $\underline{u}(x,t)$ and $u(x,t)$ are well-ordered at time $t = t_0$ and on the boundary of $\mathcal{R}_0$.

• Comparison in $\{|x| \geq r_0\}$ at time $t = t_0$. Since both the parameters b and c are positive, we have

$$\underline{u}(x,t_0) \leq \begin{cases} b^{-1}e^{t_0}|x|^{-p/\widehat{\gamma}} & \text{for all } x \in \mathbb{R}^N \\ b^{-1}e^{t_0}r_0^{-p/\widehat{\gamma}} & \text{for all } |x| \geq r_0, \end{cases}$$

and so, comparing with $\widetilde{u}_0 = \widetilde{u}_0(x)$, we obtain

$$u(x,t_0) \geq \widetilde{u}_0(x) \geq \underline{u}(x,t_0) \quad \text{for all } |x| \geq r_0,$$

by taking the parameter $b > 0$ large enough depending on $r_0 > 0$ (but not on c):

$$b \geq (e^{t_0}/\widetilde{\varepsilon}) \max\{\widetilde{\varrho}_0^{-p/\widehat{\gamma}}, r_0^{-p/\widehat{\gamma}}\}.$$

• Comparison on the boundary $\{|x| = r_0\} \times \{t \geq t_0\}$. This part is simpler. Indeed, we have

$$\underline{u}(|x| = r_0, t) \leq 1/c = 1 - \varepsilon \leq u(|x| = r_0, t), \quad \text{for all } t \geq t_0,$$

thanks to our assumptions on $t_0 > 0$ and $c > 0$, see (3.44).

Step3. In this step we prove that $\underline{u} = \underline{u}(r,t)$ is a sub-solution of the equation in (3.36) in the region $\mathcal{R}_0$. For the reader convenience, we introduce the expression

$$A(r,t) := b\psi(r) + ce^t \quad \Rightarrow \quad \underline{u}(r,t) = e^t A(r,t)^{-1}.$$

We proceed by carrying out some computations. We have

$$\partial_t \underline{u} = be^t\psi(r)A(r,t)^{-2}, \qquad -\underline{u}(1-\underline{u}) = -e^t\Big[b\psi(r) + (c-1)e^t\Big]A(r,t)^{-2}. \tag{3.45}$$

Now, we need to compute the radial p-Laplacian of $\underline{u}^m$, which is given by the formula

$$-\Delta_{p,r}\underline{u}^m := -r^{1-N}\partial_r\Big(r^{N-1}|\partial_r\overline{u}^m|^{p-2}\partial_r\overline{u}^m\Big).$$

First of all, setting $B(t) := (mb)^{p-1}e^{(1-\widehat{\gamma})t} > 0$ and using the fact that $(m+1)(p-1) = p-\widehat{\gamma}$, it is not difficult to obtain

$$|\partial_r\underline{u}^m|^{p-2}\partial_r\underline{u}^m = -B(t)|\psi'(r)|^{p-2}\psi'(r)A(r,t)^{\widehat{\gamma}-p},$$

where $\psi' = d\psi/dr$. Consequently, we have

$$\begin{aligned}-\Delta_{p,r}\underline{u}^m &= B(t)r^{1-N}\partial_r\Big[r^{N-1}|\psi'(r)|^{p-2}\psi'(r)A(r,t)^{\widehat{\gamma}-p}\Big]\\ &= B(t)|\psi'(r)|^{p-2}A(r,t)^{\widehat{\gamma}-p-1}\Big\{\Big[\frac{N-1}{r}\psi'(r) + (p-1)\psi''(r)\Big]A(r,t) - b(p-\widehat{\gamma})(\psi'(r))^2\Big\}.\end{aligned}$$

Combining the last quantity with the ones in (3.45) and multiplying by $B(t)^{-1}|\psi'(r)|^{2-p}A(r,t)^{1+p-\widehat{\gamma}}$, we obtain

$$\begin{aligned}&B(t)^{-1}|\psi'(r)|^{2-p}A(r,t)^{1+p-\widehat{\gamma}}\Big[\partial_t\underline{u} - \Delta_{p,r}\underline{u}^m - \underline{u}(1-\underline{u})\Big]\\ &\quad= \frac{c-1}{(mb)^{p-1}}e^{(1+\widehat{\gamma})t}|\psi'(r)|^{2-p}A(r,t)^{p-1-\widehat{\gamma}} + \Big[(p-1)\psi''(r) + \frac{N-1}{r}\psi'(r)\Big]A(r,t) +\\ &\quad - b(p-\widehat{\gamma})(\psi'(r))^2.\end{aligned}$$

Let us take $\psi(r) = r^{p/\widehat{\gamma}}$ with $\psi'(r) = (p/\widehat{\gamma})r^{\frac{p}{\widehat{\gamma}}-1}$ and $\psi''(r) = (p/\widehat{\gamma})\big(p/\widehat{\gamma}-1\big)r^{\frac{p}{\widehat{\gamma}}-2}$. Since

$$\begin{aligned}(p-1)\psi''(r)+\frac{N-1}{r}\psi'(r) &= d_2\, r^{p/\widehat{\gamma}-2},\\ (\psi'(r))^2 = (p/\widehat{\gamma})^2r^{2(p/\widehat{\gamma}-1)}, &\qquad |\psi'(r)|^{2-p} = (p/\widehat{\gamma})^{2-p}r^{(2-p)(p/\widehat{\gamma}-1)}\end{aligned}$$

and recalling that $A(r,t) = br^{p/\widehat{\gamma}} + ce^t$, we substitute in the previous equation deducing

$$\begin{aligned}&B(t)^{-1}|\psi'(r)|^{2-p}A(r,t)^{1+p-\widehat{\gamma}}\Big[\partial_t\underline{u} - \Delta_{p,r}\underline{u}^m - \underline{u}(1-\underline{u})\Big]\\ &\quad= -d_3\frac{c-1}{b^{p-1}}e^{(1+\widehat{\gamma})t}r^{(2-p)(p/\widehat{\gamma}-1)}\Big(br^{p/\widehat{\gamma}} + ce^t\Big)^{p-1-\widehat{\gamma}} + d_2ce^tr^{p/\widehat{\gamma}-2} - bd_1r^{2(p/\widehat{\gamma}-1)},\end{aligned}$$

where $d_i > 0$, $i = 1,2,3$ are chosen as in (3.42):

$$d_1 := p/\widehat{\gamma}^2(p-\widehat{\gamma}N),\quad d_2 := p/\widehat{\gamma}^2[(p-1)(p-\widehat{\gamma}) + \widehat{\gamma}(N-1)],\quad d_3 := (p/\widehat{\gamma})^{2-p}m^{1-p}.$$

Now, multiplying by $r^{-2(p/\widehat{\gamma}-1)}$ and setting $\xi = e^tr^{-p/\widehat{\gamma}} > 0$, it is not difficult to obtain

$$\begin{aligned}&B(t)^{-1}|\psi'(r)|^{2-p}r^{-2(p/\widehat{\gamma}-1)}A(r,t)^{1+p-\widehat{\gamma}}\Big[\partial_t\underline{u} - \Delta_{p,r}\underline{u}^m - \underline{u}(1-\underline{u})\Big]\\ &\quad= -d_3\frac{c-1}{b^{p-1}}\,r^p\,\xi^{1+\widehat{\gamma}}\Big(b+c\xi\Big)^{p-1-\widehat{\gamma}} + d_2c\xi - bd_1\\ &\quad\le -d_3\frac{c-1}{b^{p-1}}\,r_0^p\,\xi^{1+\widehat{\gamma}}\Big(b+c\xi\Big)^{p-1-\widehat{\gamma}} + d_2c\xi - bd_1 := -C_{r_0}(\xi),\end{aligned}$$

for all $r \ge r_0$.

Case $\widehat{\gamma} \le p-1$. To prove that $\underline{u} = \underline{u}(r,t)$ is a sub-solution, it is sufficient to check that

$$C_{r_0}(\xi) = d_3 \frac{c-1}{b^{p-1}}\, r_0^p\, \xi^{1+\widehat{\gamma}}\Big(b + c\xi\Big)^{p-1-\widehat{\gamma}} - d_2 c\xi + bd_1 \ge 0, \tag{3.46}$$

for all $\xi > 0$. We will prove the previous inequality in two separate intervals $0 \le \xi \le \xi_0$ and $\xi \ge \xi_0$, where $\xi_0 > 0$ will be suitably chosen.
Suppose $0 \le \xi \le \xi_0$. In this interval we have $C_{r_0}(\xi) \ge -d_2 c\xi_0 + bd_1$ and so, a sufficient condition so that (3.48) is satisfied (for $0 \le \xi \le \xi_0$) is

$$c \le (d_1/d_2)\, b\, \xi_0^{-1}. \tag{3.47}$$

Suppose $\xi \ge \xi_0$ and assume (3.47) to be true. Since we are in the range $\widehat{\gamma} \le p-1$, we have $(b+c\xi)^{p-1-\widehat{\gamma}} \ge b^{p-1-\widehat{\gamma}}$, and so

$$C_{r_0}(\xi) \ge C_{1,r_0}(\xi) := d_3 \frac{c-1}{b^{\widehat{\gamma}}}\, r_0^p\, \xi^{1+\widehat{\gamma}} - d_2 c\xi + bd_1.$$

Now, we note that condition (3.47) not only implies $C_{r_0}(\xi) \ge 0$, but also $C_{1,r_0}(\xi) \ge 0$ for all $0 \le \xi \le \xi_0$. Hence, in order to prove that $C_{1,r_0}(\xi) \ge 0$ for all $\xi \ge \xi_0$, it is sufficient to show that the minimum point of $C_{1,r_0}(\cdot)$ is attained for some $0 < \xi_m \le \xi_0$. It is straightforward to compute the minimum point ξ_m of $C_{1,r_0}(\cdot)$:

$$\xi_m^{\widehat{\gamma}} = \frac{d_2}{d_3(1+\widehat{\gamma})r_0^p} \frac{c\, b^{\widehat{\gamma}}}{(c-1)}.$$

For our purpose we may choose

$$\xi_0^{\widehat{\gamma}} = \xi_m^{\widehat{\gamma}} = \frac{d_2}{d_3(1+\widehat{\gamma})r_0^p} \frac{c\, b^{\widehat{\gamma}}}{(c-1)}.$$

Now, since ξ_0 depends on c, we need to check that our choice of $\xi_0 > 0$ is compatible with (3.47), which we have assumed to be true. Thus, substituting the value of ξ_0 in (3.47), we obtain that the parameter c has to satisfy the inequality

$$\frac{d_2^{\widehat{\gamma}+1}}{d_1^{\widehat{\gamma}} d_3(\widehat{\gamma}+1) r_0^p}\, c^{1+\widehat{\gamma}} \le c-1.$$

The crucial fact is that the previous expressions do not depend on b. Indeed, taking $c = (1-\varepsilon)^{-1}$ as in (3.44), we can rewrite the previous inequality as

$$r_0^p \ge \frac{d_2^{\widehat{\gamma}+1}(1-\varepsilon)^{-\widehat{\gamma}}}{d_1^{\widehat{\gamma}} d_3(\widehat{\gamma}+1)} \varepsilon^{-1},$$

which exactly our first assumption in (3.43) on $r_0 > 0$. This proves that for all $\widehat{\gamma} \le p-1$ and $0 < \widehat{\gamma} < p/N$, the function $\underline{u} = \underline{u}(r,t)$ is a sub-solution for the equation in (3.36) in the region $\mathcal{R}_0 = \{|x| \ge r_0\} \times [t_0, \infty)$.

Case $\widehat{\gamma} > p-1$. In this range the proof is similar, but there are some technical changes that have to be highlighted. We rewrite $C_{r_0}(\cdot)$ as

$$\begin{aligned} -C_{r_0}(\xi) :&= -d_3 \frac{c-1}{b^{p-1}}\, r_0^p\, \xi^{1+\widehat{\gamma}}\Big(b+c\xi\Big)^{p-1-\widehat{\gamma}} + d_2 c\xi - bd_1 \\ &= -d_3 \frac{c-1}{b^{p-1}}\, r_0^p \left(\frac{\xi}{b+c\xi}\right)^{\widehat{\gamma}-(p-1)} \xi^p + d_2 c\xi - bd_1. \end{aligned}$$

So, in order to show that $\underline{u} = \underline{u}(r,t)$ is a sub-solution, we can verify that

$$C_{r_0}(\xi) = d_3 \frac{c-1}{b^{p-1}} r_0^p \left(\frac{\xi}{b+c\xi}\right)^{\widehat{\gamma}-(p-1)} \xi^p - d_2 c\xi + bd_1 \geq 0, \tag{3.48}$$

for all $\xi > 0$. Again we will pick a "good" $\xi_0 > 0$ and prove (3.48) in the intervals $0 \leq \xi \leq \xi_0$ and $\xi \geq \xi_0$.
Suppose $0 \leq \xi \leq \xi_0$. As before, in this interval we have $C_{r_0}(\xi) \geq -d_2 c\xi_0 + bd_1$ and so, taking again c as in (3.47), i.e.

$$c \leq (d_1/d_2)\, b\, \xi_0^{-1},$$

then (3.48) is automatically satisfied (for $0 \leq \xi \leq \xi_0$).
Now, suppose $\xi \geq \xi_0$ and assume again (3.47) to be true. Since we are in the range $\widehat{\gamma} > p-1$, the function

$$\xi \to \left(\frac{\xi}{b+c\xi}\right)^{\widehat{\gamma}-(p-1)}$$

is increasing (in ξ), we have

$$\begin{aligned} C_{r_0}(\xi) &\geq d_3 \frac{c-1}{b^{p-1}} r_0^p \left(\frac{\xi_0}{b+c\xi_0}\right)^{\widehat{\gamma}-(p-1)} \xi^p - d_2 c\xi + bd_1 \\ &\geq d_3 \frac{c-1}{b^{p-1}} r_0^p \left(\frac{\xi_0}{b+(d_1/d_2)b}\right)^{\widehat{\gamma}-(p-1)} \xi^p - d_2 c\xi + bd_1 \\ &= \frac{d_3(c-1)}{(1+d_1/d_2)^{\widehat{\gamma}-(p-1)} b^{p-1}} \xi_0^{\widehat{\gamma}-(p-1)} r_0^p \xi^p - d_2 c\xi + bd_1 := \widetilde{C}_{1,r_0}(\xi), \end{aligned}$$

where we used (3.47) in the second inequality. Exactly as in the previous case, condition (3.47) implies both $C_{r_0}(\xi) \geq 0$ and $\widetilde{C}_{1,r_0}(\xi) \geq 0$ for all $0 \leq \xi \leq \xi_0$. Hence, we show that the minimum point of $\widetilde{C}_{1,r_0}(\cdot)$ is attained for $\xi_m = \xi_0$ and this gives us $\widetilde{C}_{1,r_0}(\xi) \geq 0$ for all $\xi \geq \xi_0$. The minimum point ξ_m of $\widetilde{C}_{1,r_0}(\cdot)$ is given by the formula:

$$\xi_m^{p-1} = \frac{d_2(1+d_1/d_2)^{\widehat{\gamma}-(p-1)}}{pd_3\, \xi_0^{\widehat{\gamma}-(p-1)} r_0^p} \frac{c\, b^{\widehat{\gamma}}}{c-1}.$$

So we ask $\xi_m = \xi_0$, i.e.:

$$\xi_0^{\widehat{\gamma}} = \frac{d_2(1+d_1/d_2)^{\widehat{\gamma}-(p-1)}}{pd_3\, r_0^p} \frac{c\, b^{\widehat{\gamma}}}{c-1}.$$

Again we must check the compatibility between our choice of $\xi_0 > 0$ and (3.47). So, we substitute the value of ξ_0 in (3.47) and we obtain the inequality

$$\frac{d_2^2(1+d_1/d_2)^{\widehat{\gamma}-(p-1)}}{pd_1 d_3\, r_0^p} c^{\frac{1+\widehat{\gamma}}{\widehat{\gamma}}} \leq (c-1)^{1/\widehat{\gamma}},$$

in the parameter c. Also in this this case is is really important that the previous expressions do not depend on b. We take $c = (1-\varepsilon)^{-1}$ as in (3.44) and we rewrite the previous inequality as

$$r_0^p \geq \frac{d_2^2(1+d_1/d_2)^{\widehat{\gamma}-(p-1)}}{pd_1 d_3} \varepsilon^{-1/\widehat{\gamma}} (1-\varepsilon)^{(1+2\widehat{\gamma})/[\widehat{\gamma}(1+\widehat{\gamma})]}.$$

Since $1-\varepsilon \leq 1$ the last inequality is satisfied thanks to the assumption on $r_0 > 0$ in (3.43). Hence, we have showed that $\underline{u} = \underline{u}(r,t)$ is a sub-solution for the equation in (3.36) in the region $\mathcal{R}_0 = \{|x| \geq r_0\} \times [t_0, \infty)$, for the range $\widehat{\gamma} > p-1$, too.

Consequently, for all $0 < \widehat{\gamma} < p/N$, we obtain

$$u(x,t) \geq \underline{u}(x,t) \quad \text{in } \{|x| \geq r_0\} \times [t_0, \infty),$$

thanks to the comparison at time $t = t_0$ and on the boundary of $\mathcal{R}_0$ done in *Step2*. Note that the Comparison Principle can be applied since $0 \leq \underline{u}(r,t) \leq 1/c = 1 - \varepsilon$ in $\mathbb{R}^N \times [0,\infty)$ and $f(u) = u(1-u)$ can be re-defined outside $[0, 1-\varepsilon]$ to be Lipschitz continuous.

Step4. In this last step, we conclude the proof. The following procedure holds for all $0 < \widehat{\gamma} < p/N$ (see also [47]). Thanks to Corollary 3.6, to deduce the second inclusion in (3.5):

$$\forall\, 0 < \omega < 1, \ \exists\, t_\omega, C_\omega \gg 0 : \ \{|x| < C_\omega^{-1} e^{\sigma_* t}\} \subset \{u(x,t) > \omega\}, \ \ \forall t \geq t_\omega,$$

it is sufficient to prove $u(x,t) \geq \overline{\varepsilon}$ in $\{|x| \leq e^{\sigma_* t} = e^{\widehat{\gamma} t/p}\} \times [t_0, \infty)$, for some $\overline{\varepsilon} > 0$. So, in the set $\{r_0 \leq |x| \leq e^{\sigma_* t}\} \times [t_0, \infty)$, we have

$$u(x,t) \geq \underline{u}(x,t) = \frac{e^t}{b|x|^{p/\widehat{\gamma}} + ce^t} \geq \frac{1}{b+c} := \overline{\varepsilon}.$$

Note that the bound $u(x,t) \geq \overline{\varepsilon}$ can be extended to the region $\{|x| \leq r_0\} \times [t_0, \infty)$, thanks to our assumption on t_0 and Theorem 3.1. Consequently, applying Corollary 3.6 with $\nu = \sigma_*$, $\overline{\varepsilon} = 1/(b+c)$ and $\overline{\varrho} = 1$, we end the proof of the theorem. □

3.5 A Comparison Principle in non-cylindrical domains

In this brief section, we give the proof of a Comparison Principle for a certain class of parabolic equations with p-Laplacian diffusion. As mentioned in the introduction, a similar result have been introduced in [47], but proved with different techniques. This comparison principle is crucial in the study of the asymptotic behaviour of the general solutions of the Fisher-KPP problem, see Theorem 3.1.

Before proceeding we need to introduce some definitions. First of all, let $r \in C^1([0,\infty); \mathbb{R})$ be a positive and non-decreasing function, and consider the "inner-sets"

$$\Omega_I^T := \{(x,t) \in \mathbb{R}^N \times [0,T) : |x| \leq r(t)\}, \quad 0 < T \leq \infty, \qquad \text{with } \Omega_I^\infty := \Omega_I.$$

Now, for all $p > 1$, we consider the equation

$$a(x,t)\partial_t u - \Delta_p u + c_0 u = 0 \quad \text{in } \mathbb{R}^N \times (0,\infty) \tag{3.49}$$

where $a = a(x,t)$ is a continuous function in $\mathbb{R}^N \times (0,\infty)$, with $0 < a_0 \leq a(x,t) \leq a_1 < \infty$ in Ω_I, $c_0 > 0$ and $u_0 \in L^1(\mathbb{R}^N)$. The next definition is given following [40, 81]. See also [198], Chapter 8 for the Porous Medium setting.

Definition 3.7. *A nonnegative function $\overline{u} = \overline{u}(x,t)$ is said to be a "local strong" super-solution to equation* (3.49) *in Ω_I^T if*
(i) $\overline{u} \in C_{loc}(0,T : L^2_{loc}(\mathbb{R}^N)) \cap L^p_{loc}(0,T : W^{1,p}_{loc}(\mathbb{R}^N))$, and $\partial_t \overline{u} \in L^2_{loc}(\mathbb{R}^N \times (0,\infty))$;
(ii) $\overline{u} = \overline{u}(x,t)$ satisfies

$$\int_{\Omega_I^T} [a(x,t)\partial_t \overline{u} + c_0 \overline{u}]\eta + |\nabla \overline{u}|^{p-2} \nabla \overline{u}\, \nabla \eta \geq 0,$$

for all test function $\eta \in C^1_c(\Omega_I^T)$, $\eta \geq 0$.

A nonnegative function $\underline{u} = \underline{u}(x,t)$ is said to be a "local strong" sub-solution to equation (3.49) *in Ω_I^T if*
(i) $\underline{u} \in C_{loc}(0,T : L^2_{loc}(\mathbb{R}^N)) \cap L^p_{loc}(0,T : W^{1,p}_{loc}(\mathbb{R}^N))$, and $\partial_t \underline{u} \in L^2_{loc}(\mathbb{R}^N \times (0,\infty))$;

(ii) $\underline{u} = \underline{u}(x,t)$ *satisfies*

$$\int_{\Omega_I^T} [a(x,t)\partial_t \underline{u} + c_0 \underline{u}]\eta + |\nabla \underline{u}|^{p-2}\nabla \underline{u}\, \nabla \eta \leq 0,$$

for all test function $\eta \in C_c^1(\Omega_I^T)$, $\eta \geq 0$.

Proposition 3.8. *Consider two functions* $\overline{u} = \overline{u}(x,t)$ *and* $\underline{u} = \underline{u}(x,t)$ *defined and continuous in* $\mathbb{R}^N \times (0,\infty)$. *Assume that:*
(A1) $\overline{u}(x,0) \geq \underline{u}(x,0)$ *in* $\mathbb{R}^N$.
(A2) $\overline{u}(x,t) \geq \underline{u}(x,t)$ *in* $\partial\Omega_I = \{(x,t) \in \mathbb{R}^N \times (0,\infty) : |x| = r(t)\}$.
(A3) Finally, assume that $\overline{u} = \overline{u}(x,t)$ *is a "local strong" super-solution and* $\underline{u} = \underline{u}(x,t)$ *is a "local strong" sub-solution to equation* (3.49) *in* Ω_I^T.

Then $\overline{u} \geq \underline{u}$ *in* Ω_I^T.

Proof. Let's fix $0 < T \leq \infty$. For all $0 < t < T$, we define the subset of $\mathbb{R}^N$

$$\Omega_{I,t} := \{x \in \mathbb{R}^N : |x| \leq r(t)\}.$$

We show that for all $t > 0$, it holds

$$\left\|[\underline{u}(t) - \overline{u}(t)]_+\right\|_{L^1(\Omega_{I,t})} \leq \left\|[\underline{u}(0) - \overline{u}(0)]_+\right\|_{L^1(\mathbb{R}^N)}, \tag{3.50}$$

where $[\cdot]_+$ stands for the positive part. Consequently, we deduce the thesis thanks to assumption (A1). We proceed with a standard argument, see for instance Chapter 8 of [198] for the Porous Medium equation.
Let's consider a function $p \in C^1(\mathbb{R})$ such that

$$0 \leq p \leq 1, \qquad p(s) = 0 \text{ for } s \leq 0, \qquad p'(s) > 0 \text{ for } s > 0,$$

and a sequence $w_j \in C^1(\Omega_I^T)$ such that $w_j \to \underline{u} - \overline{u}$ as $j \to \infty$ in $L^p_{loc}(0, T : W^{1,p}_{loc}(\mathbb{R}^N))$. Note that we can assume

$$w_j \leq 0 \text{ on } \partial\Omega_I^T = \{(x,t) \in \mathbb{R}^N \times (0,T) : |x| = r(t)\}$$

thanks to assumption (A2). Hence, if $h \in C_0^1([0,T])$ with $0 \leq h \leq 1$, we can take as test function

$$\eta_j = p(w_j)h(t), \quad j = 1, 2, \ldots$$

Thus, by the definition of sub- and super-solutions, it is simple to deduce

$$\int_{\Omega_I^T} \Big[a(x,t)\partial_t(\underline{u} - \overline{u}) + c_0(\underline{u} - \overline{u})\Big]p(w_j)h + \Big\langle |\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla \overline{u}|^{p-2}\nabla \overline{u}, \nabla w_j \Big\rangle p'(w_j)h \, dxdt \leq 0.$$

The second integral converges to

$$\int_{\Omega_I^T} \Big\langle |\nabla \underline{u}|^{p-2}\nabla \underline{u} - |\nabla \overline{u}|^{p-2}\nabla \overline{u}, \nabla \underline{u} - \nabla \overline{u} \Big\rangle p'(\underline{u} - \overline{u})h \, dxdt \geq 0,$$

thanks to the fact that $\Big\langle |b|^{p-2}b - |a|^{p-2}a, b - a\Big\rangle \geq 0$ for all $a, b \in \mathbb{R}^N$ and $p > 1$, see the last section of [144]. Hence, taking the limit in the second integral we deduce

$$\int_{\Omega_I^T} \Big[a(x,t)\partial_t(\underline{u} - \overline{u}) + c_0(\underline{u} - \overline{u})\Big]p(\underline{u} - \overline{u})h \, dxdt \leq 0,$$

and, letting $p(\cdot) \to sign_+(\cdot) := [sign]_+(\cdot)$, we obtain

$$\int_{\Omega_I^T} a(x,t)\partial_t(\underline{u}-\overline{u})sign_+(\underline{u}-\overline{u})h + c_0(\underline{u}-\overline{u})sign_+(\underline{u}-\overline{u})h\, dxdt \le 0.$$

Now, we have

$$\frac{d}{dt}[\,\underline{u}-\overline{u}\,]_+ = \partial_t(\underline{u}-\overline{u})sign_+(\underline{u}-\overline{u}),$$

and, since $[s]_+ = s \cdot sign_+(s) \ge 0$, $a(x,t) \ge a_0 > 0$, and $c_0 > 0$ we easily get

$$\int_0^T \Big(\int_{\Omega_{I,t}} \partial_t[\,\underline{u}(t) - \overline{u}(t)\,]_+ dx \Big) h(t)\, dt \le 0 \quad \text{ for all } \; h \in C_c^1([0,T]),\; 0 \le h \le 1.$$

Thus, thanks to arbitrariness of h, we deduce that

$$\int_{\Omega_{I,t}} \partial_t[\,\underline{u}(t) - \overline{u}(t)\,]_+ dx \le 0,$$

for all $t > 0$. Using assumption (A2) again, it is not difficult to deduce

$$\frac{d}{dt}\Big(\int_{\Omega_{I,t}} [\,\underline{u}(t) - \overline{u}(t)\,]_+ dx \Big) \le 0,$$

which implies

$$\big\|[\underline{u}(t)-\overline{u}(t)]_+\big\|_{L^1(\Omega_{I,t})} \le \big\|[\underline{u}(0)-\overline{u}(0)]_+\big\|_{L^1(\Omega_{I,0})} \le \big\|[\underline{u}(0)-\overline{u}(0)]_+\big\|_{L^1(\mathbb{R}^N)},$$

i.e., the thesis. □

Remark. We point out that the functions we use in the proof of Theorem 3.1 satisfy the assumptions of regularity required in the statement of Proposition 3.8, as we have remarked in the introduction. See also the bibliography reported in the next section.

3.6 Self-similar solutions for increasing initial data

In this section, we recall some basic facts about the existence of Barenblatt solutions for the Cauchy problem

$$\begin{cases} \partial_t u = \Delta_p u & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{3.51}$$

where $p > 1$. In particular, we focus on the specific initial datum

$$u_0(x) = |x|^\lambda, \quad \lambda > 0. \tag{3.52}$$

A more complete analysis of the self-similarity of the p-Laplacian Equation can be found in [126]. We have decided to dedicate an entire section to this topic since solutions to problem (3.51) play a main role in the proof of Theorem 3.1. Moreover, we think it facilitates the reading and gives us the occasion to present the related bibliography. Before proceeding with our analysis, we need to recall some important properties about problem (3.51).

Case $p = 2$. The existence and uniqueness of solutions to the Heat Equation for continuous non-integrable initial has been largely studied, see Tychonov [193] and the references therein. In particular, he proved that if the initial datum satisfies

$$|u_0(x)| \le b \exp\left(a|x|^2\right), \quad \text{for } |x| \sim \infty, \tag{3.53}$$

for some positive a and b, then problem (3.51), (3.53) admits a unique (classical) solution defined in $\mathbb{R}^N \times (0, 1/(4a))$. More work on this issue can be found in [208].

Case $p > 2$. This range was studied in [84], by DiBenedetto and Herrero. The authors showed that, under the assumptions

$$u_0 \in L^1_{loc}(\mathbb{R}^N) \quad \text{and} \quad u_0(x) \le C|x|^\lambda, \text{ as } |x| \to \infty \tag{3.54}$$

for some $C > 0$ and $\lambda < p/(p-2)$, there exists a unique weak solution to problem (3.51), (3.54) defined in $\mathbb{R}^N \times (0, \infty)$ (see Theorem 1, Theorem 2, and Theorem 4 of [84]). Furthermore, they proved that $\partial_t u \in L^2_{loc}(\mathbb{R}^N \times (0, \infty))$ (i.e. u is a "local strong solution") and the function $(x, t) \to \nabla u(x, t)$ is locally Hölder continuous in $\mathbb{R}^N \times (0, \infty)$ (see also [82]).

Case $1 < p < 2$. The same authors (see [83]) considered problem (3.51) with $1 < p < 2$ and nonnegative initial data

$$u_0 \in L^1_{loc}(\mathbb{R}^N) \quad \text{and} \quad u_0(x) \ge 0 \text{ in } \mathbb{R}^N \tag{3.55}$$

without any assumption on the decay at infinity of $u_0(\cdot)$. First of all, they show existence and the uniqueness of weak solutions to problem (3.51), (3.55) by using the Benilan-Crandall regularizing effect, see [26]. Then they posed their attention on the regularity of these solutions when the initial datum is a non-negative σ-finite Borel measure in $\mathbb{R}^N$, in the range $2N/(N+1) := p_c < p < 2$. In particular, they showed the existence and the uniqueness of a locally Hölder continuous weak solution in $\mathbb{R}^N \times (0, \infty)$, with $\partial_t u \in L^2_{loc}(\mathbb{R}^N \times (0, \infty))$ (i.e. they are "local strong solutions"), with $(x, t) \to \nabla u(x, t)$ locally Hölder continuous in $\mathbb{R}^N \times (0, \infty)$.
The sub-critical range $1 < p \le p_c := 2N/(N+1)$ was studied later by Bonforte, Iagar and Vázquez in [40]. They proved new local smoothing effects when the initial datum is taken in $L^r_{loc}(\mathbb{R}^N)$ and p sub-critical, and special energy inequalities which are employed to show that bounded local weak solutions are indeed "local strong solutions", more precisely $\partial_t u \in L^2_{loc}(\mathbb{R}^N \times (0, \infty))$. Then, thanks to the mentioned smoothing effect and known regularity theory ([81] and [85]) they found that the local strong solutions are locally Hölder continuous.

Barenblatt solutions for problem (3.51), (3.52). From now on we take $U_0(x) = |x|^\lambda$, $\lambda > 0$. We do not make any other assumptions on $\lambda > 0$ if $1 < p \le 2$, whilst when $p > 2$ we assume $0 < \lambda < p/(p-2)$, according to the theory developed in [84], and presented before. As mentioned before, the assumptions on the parameter λ guarantees the existence, the uniqueness and the Hölder regularity of the solution of problem (3.51), (3.52), for all $p > 1$.
We look for solutions in *self-similar* form

$$U(x, t) = t^{-\alpha_\lambda} F(|x| t^{-\beta_\lambda}),$$

where α_λ and β_λ are real numbers and $F(\cdot)$ is called profile of the solution. Let $\xi = |x|t^{-\beta_\lambda}$ and write $F' = dF/d\xi$. It is not difficult to compute

$$\partial_t U = -t^{-\alpha_\lambda - 1}(\alpha_\lambda F(\xi) + \beta_\lambda \xi F'(\xi)), \qquad \Delta_p U = t^{-(\alpha_\lambda+\beta_\lambda)(p-1)-\beta_\lambda} \xi^{1-N}\left(\xi^{N-1}|F'(\xi)|^{p-2}F'(\xi)\right)'$$

and, by taking

$$2\alpha_\lambda + 1 = (\alpha_\lambda + \beta_\lambda)p, \tag{3.56}$$

we have $\alpha_\lambda + 1 = (\alpha_\lambda + \beta_\lambda)(p-1) + \beta_\lambda$, and so we obtain the equation of the profile

$$\xi^{1-N}\Big(\xi^{N-1}|F'(\xi)|^{p-2}F'(\xi)\Big)' + \beta_\lambda \xi F'(\xi) + \alpha_\lambda F(\xi) = 0.$$

Furthermore, since (3.56) guarantees that the equation in (3.51) is invariant under the transformation $U_k(x,t) = k^{\alpha_\lambda}U(k^{\beta_\lambda}x, kt)$, $k > 0$, we use the uniqueness of the solution of problem (3.51), (3.52) to deduce

$$k^{\alpha_\lambda+\lambda\beta_\lambda}|x|^\lambda = U_k(x,0) = U(x,0) = |x|^\lambda, \quad \text{for all } k > 0.$$

Hence, we get $\alpha_\lambda + \lambda\beta_\lambda = 0$ and, combining it with (3.51), we obtain the precise expressions for the self-similar exponents

$$\alpha_\lambda = -\frac{\lambda}{(1-\lambda)p + 2\lambda}, \qquad \beta_\lambda = \frac{1}{(1-\lambda)p+2\lambda}.$$

We point out that, thanks to the assumption $0 < \lambda < p/(p-2)$ when $p > 2$, we have $(1-\lambda)p + 2\lambda > 0$ for all $p > 1$, and so $\alpha_\lambda < 0$ while $\beta_\lambda > 0$.

Properties of the Barenblatt solutions. We are going to prove that the profile $F(\cdot)$ of the Barenblatt solutions is positive and monotone non-decreasing by applying the Aleksandrov's Symmetry Principle. Later, we show some asymptotic properties of the profile $F(\cdot)$.
Let $U_0(x) = |x|^\lambda$, with $0 < \lambda < p/(p-2)$ and, for all $j \in \mathbb{N}$, consider the approximating sequence of initial data

$$U_{0j}(x) := \begin{cases} |x|^\lambda & \text{if } |x| \le j \\ j^\lambda & \text{if } |x| \ge j. \end{cases}$$

Note that $U_{0j}(\cdot)$ are both radial non-decreasing and bounded in $\mathbb{R}^N$. Now, consider the sequence of initial data

$$v_{0j}(x) := j^\lambda - U_{0j}(x) \in C_c(\mathbb{R}^N) \ \text{ and radial non-increasing,}$$

and the sequence of solutions $v_j(x,t)$ to problem (3.51) with initial data $v_{0j}(\cdot)$, for all $j \in \mathbb{N}$. Hence, by applying the Aleksandrov's Symmetry Principle, we deduce that for all times $t > 0$, the solutions $v_j(\cdot,t)$ are radially non-increasing in space too. Finally, we define the sequence $U_j(x,t) = j^\gamma - v_j(x,t)$ which are radially non-decreasing in space and satisfy problem (3.51) with initial data U_{0j}, for all $j \in \mathbb{N}$. Hence, passing to the limit as $j \to \infty$, we have $U_j(x,t) \to U(x,t)$ and the limit $U(x,t)$, solution to problem (3.51) with initial datum $U_0(\cdot)$, inherits the same radial properties of the sequence $U_j(x,t)$.
Now, we show the existence of two constants $0 < H_2 < H_1$ such that the following asymptotic bounds hold

$$H_2|x|^\lambda \le U(x,t) \le H_1|x|^\lambda, \quad \text{for } |x|t^{-\beta_\lambda} \sim \infty. \tag{3.57}$$

Estimates (3.57) follow directly from that fact that $U(x,t) \to |x|^\lambda$ as $t \to 0$. Indeed, for all fixed $0 \ne x \in \mathbb{R}^N$, we have that

$$\Big|U(x,t) - |x|^\lambda\Big| = t^{-\alpha_\lambda}\Big|F(\xi) - \xi^\lambda\Big| = |x|^\lambda\left|\frac{F(\xi)}{\xi^\lambda} - 1\right|, \quad \text{where } \xi = |x|t^{-\beta_\lambda}.$$

Since, the left expression converges to 0 as $t \to 0$, we deduce that $F(\xi)/\xi^\lambda \to 1$, as $\xi \to \infty$ and, from this limit, we get (3.57). □

Aleksandrov's Symmetry Principle. The Aleksandrov-Serrin symmetry method was firstly introduced in [5] and [177] to show monotonicity of solutions of both (eventually nonlinear) elliptic and parabolic equations. Here, following [198], we give a short proof for the case of the "purely" diffusive p-Laplacian equation in (3.51), for all $p > 1$.
Before proceeding with the statement, we fix some notations. Let H be an hyperplane in $\mathbb{R}^N$, Ω_1 and Ω_2 the two half-spaces "generated" by H, and $\Pi : \Omega_1 \to \Omega_2$ the reflection with respect to the hyperplane H.

Theorem 3.9. *Let $u \geq 0$ be a solution of the initial-value problem* (3.51) *with initial datum $u_0 \in L^1(\mathbb{R}^N)$. Suppose that*

$$u_0(x) \geq u_0(\Pi(x)) \quad \textit{for all } x \textit{ in } \Omega_1.$$

Then, for all times $t > 0$ it holds

$$u(x,t) \geq u(\Pi(x),t) \quad \textit{for all } x \textit{ in } \Omega_1.$$

In particular, radial initial data generate radial solutions.

Proof. First of all, thanks to the rotation invariance of the equation in (3.51), we can assume $H = \{x \in \mathbb{R}^N : x_1 = 0\}$ and $\Pi(x_1, x_2, \dots, x_N) = (-x_1, x_2, \dots, x_N)$. Moreover, it follows that $\widehat{u}(x,t) = u(\Pi(x),t)$ satisfies problem (3.51) in $\mathbb{R}^N \times (0,\infty)$ with initial datum $\widehat{u_0}(x) = u(\Pi(x),0)$.
Now, we have $u_0(x) \geq u_0(\Pi(x))$ in Ω_1 and $u(x,t) = \widehat{u}(x,t)$ in $H \times (0,\infty) = \partial\Omega_1 \times (0,\infty)$. Hence, since the solution is continuous, we get the thesis by applying the Comparison Principle. Note that, to be precise, we should consider solutions of the Cauchy-Dirichlet problem posed in the ball $B_R(0)$ with zero boundary data. These solutions approximate $u = u(x,t)$ and $\widehat{u} = \widehat{u}(x,t)$. Consequently, we can apply the Comparison Principle to these approximate solutions and, finally, pass to the limit as $R \to \infty$. See Chapter 9 of [198] for more details.
If $u_0(\cdot)$ is radial, we can apply the statement for all hyperplane H passing through the origin of $\mathbb{R}^N$ and deducing that for all times $t > 0$, the solution $u(\cdot,t)$ is radial respect with the spacial variable too. □

3.7 Extensions, comments and open problems

We end the paper by discussing some open problems. Moreover, we present some final comments and remarks to supplement our work.
As we have mentioned in the introduction, nonlinear evolution processes give birth to a wide variety of phenomena. Indeed we have seen that solutions of problem (1) exhibit a travelling wave behaviour for large times when $\gamma \geq 0$, i.e. $\widehat{\gamma} \leq 0$, while infinite speed of propagation when $0 < \widehat{\gamma} < p/N$. It is natural to ask ourselves what happens in the range of parameters $\widehat{\gamma} \geq p/N$ that we call "very fast" diffusion assumption.
However, respect to the Porous Medium and the p-Laplacian case, we have to face the problem of lack of literature and previous works related to the doubly nonlinear operator (in this range of parameters). For this reason, in the next paragraphs we will briefly discuss what is known for the Porous Medium and the p-Laplacian case, trying to guess what could happen in the presence of the doubly nonlinear operator. We stress that our approach is quite *formal*, but can be interesting since it gives a more complete vision of the fast diffusion range, and allows us to explain what are (or could be) the main differences respect to the range $0 < \widehat{\gamma} < p/N$.

The critical case $\widehat{\gamma} = p/N$. This critical case was firstly studied by King [132] and later in [107] for the Porous Medium setting, i.e. $p = 2$ and $m = m_c := (N-2)_+/N$, with $N \geq 3$. When $N = 1, 2$ it follows $m = 0$, choice of parameter which goes out of our range and we avoid it. King studied the asymptotic behaviour of radial solutions to the "purely" diffusive equation

$$\partial_t \underline{u} = r^{1-N}\partial_r \left(r^{N-1} |\partial_r \underline{u}^m|^{p-2} \partial \underline{u}^m \right) \quad \text{in } \mathbb{R}^+ \times [0,\infty) \tag{3.58}$$

with $p = 2$, $0 < m \le m_c$, and $N \ge 3$. Actually, he considered a slightly different equation absorbing a factor m^{p-1} in the time varible and he studied the cases $N = 2$ and $m = 0$, too. Note that the choice $\widehat{\gamma} = p/N$ corresponds to $m = m_c$ when $p = 2$.
In [132], the author described the asymptotic behaviour of radial solutions to equation (3.58), given by the formula

$$\underline{u}(r,t) \sim \left(\frac{(N-2)t}{r^2 \ln r}\right)^{\frac{N}{2}}, \quad \text{as } t \sim \infty \quad \text{and} \quad t^{-N/(N-2)} \ln r \ge \eta_0,$$

where η_0 is a constant depending on N and on the initial datum (see formula (2.34) of [132]). In particular, it follows that the solutions of (3.58) have spacial power like decay r^{-N} "corrected" by a logarithmic term for $r \sim \infty$. We are interested in seeing that an analogue decay holds when $p > 1$ and $\widehat{\gamma} = p/N$ in the doubly nonlinear setting. We proceed as in [132], see Section 2.

Asymptotic behaviour for large r. Let's take for a moment $0 < \widehat{\gamma} < p/N$. Seeking solutions $\underline{u} = \underline{u}(r,t)$ to equation (3.58) in separate form as in *Step1* of Theorem 3.2, it is simple to see that if the initial datum satisfies (3.2), then for all $t > 0$ we have

$$\underline{u}(r,t) \sim a\, t^{\frac{1}{\widehat{\gamma}}}\, r^{-\frac{p}{\widehat{\gamma}}}, \quad \text{for } r \sim \infty, \tag{3.59}$$

for some suitable constant $a > 0$. Note that it corresponds to fix $t > 0$ and take the limit as $r \to \infty$ in the formula of the Barenblatt solutions, see Subsection I.
Now, motivated by the previous analysis, we fix $N > p$ (in order to remain in the ranges $m > 0$ and $p > 1$), $\widehat{\gamma} = p/N$, and we look for solutions to equation (3.58) in the form

$$\underline{u}(r,t) \sim a\, t^{N/p}\, r^{-N} F(r), \quad \text{for } r \sim \infty,$$

for some correction function $0 \le F(r) \to 0$ as $r \to \infty$ and some constant $a > 0$. In what follows we ask $rF'(r) = o(F(r))$ as $r \to \infty$, too. It is simple to compute

$$\begin{aligned}
\partial_t \underline{u} &\sim (aN/p) t^{\frac{N}{p}-1} r^{-N} F(r) \\
\partial_r \underline{u}^m &= m a^m t^{\frac{Nm}{p}} r^{-Nm-1} F(r)^{m-1}(-NF(r) + rF'(r)) \sim -Nma^m\, t^{\frac{Nm}{p}}\, r^{-Nm-1} F(r)^m \\
|\partial_r \underline{u}^m|^{p-2} \partial \underline{u}^m &\sim -(Nm)^{p-1} a^{\frac{N-p}{N}}\, t^{\frac{N-p}{p}}\, r^{1-N} F(r)^{\frac{N-p}{N}}
\end{aligned}$$

as $r \sim \infty$, where we have used the fact that $m(p-1) = 1 - \widehat{\gamma} = 1 - p/N$. Hence, it is simple to see that $\underline{u} = \underline{u}(r,t)$ satisfies (3.58) if and only if

$$(mN)^{p-1} a^{\frac{N-p}{N}}\, r \left[F(r)^{\frac{N-p}{N}}\right]' + a(N/p)F(r) = 0.$$

Now, it is clear that a possible choice is $F(r) = (\ln r)^{-b}$, for some $b > 0$, and a straightforward computation shows that the previous equation is satisfied by taking

$$a^{\frac{p}{N}} = m^{p-1}(N-p)N^{p-2} \quad \text{and} \quad b = \frac{N}{p},$$

so that for all $t > 0$, we obtain

$$\underline{u}(r,t) \sim \left(\frac{a^{p/N} t}{r^p \ln r}\right)^{\frac{N}{p}}, \quad \text{for } r \sim \infty, \tag{3.60}$$

which generalizes the case $p = 2$ and $m = m_c$.

Barenblatt solutions for $\widehat{\gamma} = p/N$. As was observed in [132] (see pag. 346), (3.60) does not respect the self-similarity reduction of equation (3.58). Indeed, it admits "pseudo-Barenblatt" solutions which, following the notation used in the introduction of Part I, can be written in the form $B_D(x,t) = R(t)^{-N}F_D(xR(t)^{-1})$ where

$$F_D(\xi) = \left[D + (1/N)|\xi|^{\frac{p}{p-1}}\right]^{-\frac{(p-1)}{p}N}, \qquad \xi = xR(t)^{-1}, \qquad R(t) = e^t,$$

and $D > 0$ is a free parameter (cfr. with [199] for the case $p = 2$). We point out that the profile $F_D(\cdot)$ satisfies the inequalities in (3.8) with $\widehat{\gamma} = p/N$. However, these self-similar solutions (also called "of Type III", see [197]) are quite different from the ones in the range $0 < \widehat{\gamma} < p/N$. In particular, they are eternal, i.e. defined for all $t \in \mathbb{R}$ and they do not converge to a Dirac Delta as $t \to 0$ (see also [46]). Finally, for all fixed $t \in \mathbb{R}$, these self-similar solutions are *not integrable* respect with to the spacial variable and show the spacial decay

$$B_D(x,t) \sim N^{\frac{(p-1)N}{p}}|x|^{-N}, \quad \text{for } |x| \sim \infty.$$

Taking into account these facts, when $\widehat{\gamma} = p/N$ it seems reasonable to study problem (1) with nontrivial initial datum satisfying

$$0 \le u_0(x) \le 1 \quad \text{and} \quad u_0(x) \le C\left(|x|^p \ln|x|\right)^{-\frac{N}{p}} \quad \text{for } |x| \sim \infty,$$

for some constant $C > 0$, and trying to extend the techniques used for the range $0 < \widehat{\gamma} < p/N$, to the this critical case.
First of all, we can define $\sigma_* := f'(0)/N$ by continuity (with the range $0 < \widehat{\gamma} < p/N$). Thus, it is possible to repeat the proof of Part (ii) of Theorem 3.1 by using "pseudo-Barenblatt" solutions instead of the usual ones. In this way, for all $\sigma > \sigma_*$, we show the convergence of the solutions to 0 in the "outer sets" $\{|x| \ge e^{\sigma t}\}$, as $t \to \infty$.
Moreover, thanks to the asymptotic expansion (3.60) it should be possible to prove a version of Lemma 3.4 with

$$\widetilde{u}_0(x) := \begin{cases} \widetilde{\varepsilon} & \text{if } |x| \le \widetilde{\varrho}_0 \\ a_0\left(|x|^p \ln|x|\right)^{-\frac{N}{p}} & \text{if } |x| > \widetilde{\varrho}_0, \end{cases} \tag{3.61}$$

for $a_0 := \widetilde{\varepsilon}\left(\widetilde{\varrho}_0^p \ln \widetilde{\varrho}_0\right)^{N/p}$ and some $0 < \widetilde{\varepsilon} < 1$ and $\widetilde{\varrho}_0 > 1$.
However, it is clear that the methods employed for showing Proposition 3.3 cannot be used in this case too. Indeed, in the range $0 < \widehat{\gamma} < p/N$, this crucial proposition has been proved by constructing barriers from below with Barenblatt solutions. This has been possible since the initial datum $\widetilde{u}_0 = \widetilde{u}_0(x)$ in (3.6) shares the same spacial decay of these self-similar solutions. In the critical case $\widehat{\gamma} = p/N$, this property would not be preserved as (3.60) suggests. In particular, "pseudo-Barenblatt" solutions cannot be placed under an initial datum satisfying (3.61) and so the validity of Proposition 3.3 in this critical case remains an open problem.

The range $\widehat{\gamma} > p/N$. Before discussing the doubly nonlinear diffusion, let us recall what is known in the Porous Medium setting in the corresponding range of parameters, $0 < m < m_c := (N-2)_+/N$, $p = 2$, and $N \ge 3$. Consider the Porous Medium Equation

$$\begin{cases} \partial_t v = \Delta v^m & \text{in } \mathbb{R}^N \times (0,\infty) \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

where $v_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. It has been proved that the corresponding solution $v = v(x,t)$ extinguishes in finite time (see for instance [27, 132, 197] and the references therein). In other words, there exists a critical "extinction time" $0 < t_c < \infty$ such that $v(\cdot, t) = 0$, for all $t \geq t_c$. Again, the cases $m = 0$, $N = 1$ and $m = 0$, $N = 2$ are critical and we refer to [197], Chapters 5 to 8.

Barenblatt solutions for $\widehat{\gamma} > p/N$. So, even though there is not literature on the subject (at least to our knowledge), it seems reasonable to conjecture that the doubly nonlinear diffusion shows a similar property in the range $\widehat{\gamma} > p/N$, with $N > p$. In particular, also in this case we have "pseudo-Barenblatt" solutions written in the form

$$B_D(x,t) = R(t)^{-N}\left[D + (\widehat{\gamma}/p)\left|xR(t)^{-1}\right|^{\frac{p}{p-1}}\right]^{-\frac{p-1}{\widehat{\gamma}}}, \qquad (3.62)$$

where $D \geq 0$ and, with a strong departure from the range $0 < \widehat{\gamma} < p/N$,

$$R(t) = [(N/|\alpha|)(t_c - t)]^{-\frac{|\alpha|}{N}},$$

where $t_c > 0$ is fixed and stands for the "extinction time"(cfr. with [197] pag. 194 or [199] for the case $p = 2$, and with the introduction to Part I for the range $0 < \widehat{\gamma} < p/N$). The existence of this kind of self-similar solutions (also said in [197] "of Type II") strengthen the idea that a larger class of solutions have an extinction time, i.e. they vanish in finite time.

Application to the Fisher-KKP equation. In Part (ii) of Theorem 3.1 we have seen that the linearized problem

$$\begin{cases} \partial_t \overline{u} = \Delta_p \overline{u}^m + f'(0)\overline{u} & \text{in } \mathbb{R}^N \times (0,\infty) \\ \overline{u}(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases}$$

gives a super-solution to the Fisher-KPP problem (1) with nontrivial initial datum $u_0 \in L^1(\mathbb{R}^N)$, $0 \leq u_0 \leq 1$. Again, with the change of variable

$$\tau(t) = \frac{1}{f'(0)\widehat{\gamma}}\left[1 - e^{-f'(0)\widehat{\gamma}t}\right], \quad \text{for } t \geq 0,$$

we deduce that the function $\overline{v}(x,\tau) = e^{-f'(0)t}\overline{u}(x,t)$ satisfies the problem

$$\begin{cases} \partial_\tau \overline{v} = \Delta_p \overline{v}^m & \text{in } \mathbb{R}^N \times (0,\tau_\infty) \\ \overline{v}(x,0) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \qquad (3.63)$$

Now, set $\tau_\infty := \frac{1}{f'(0)\widehat{\gamma}}$ and note that $0 \leq \tau(t) \leq \tau_\infty$. Now, let $\tau_c > 0$ be the "extinction time" of the solution of problem (3.63). Thus, we deduce $v(\cdot,\tau) = 0$, for all $\tau \geq \tau_c$ and, if $\tau_c < \tau_\infty$, it follows

$$0 \leq u(\cdot,t) \leq \overline{u}(\cdot,t) = e^{f'(0)t}\overline{v}(\cdot,\tau) = 0, \quad \text{for all } \tau \geq \tau_c,$$

which implies $u(\cdot,t) = 0$ for all $t \geq \tau_\infty \ln[\tau_\infty/(\tau_\infty - \tau_c)]$, and so the solution $u = u(x,t)$ of the Fisher-KPP problem (1) with initial datum u_0 extinguishes in finite time, too. This conclusion holds under the assumption $\tau_c < \tau_\infty$, which should be guaranteed if the the initial datum is "small enough" (in terms of the mass), see [197] Chapter 5, for the Porous Medium setting. The analysis of the case in which the initial mass is infinite is an interesting open problem.

Asymptotics for non-Fisher-KPP reactions. The problem of the long time behaviour of solutions to problem (1) in the "fast" diffusion range can be posed for different kind of reactions, like reactions of type C 3 or type C′ 4. It seems reasonable to conjecture that even in these different settings a "fast" diffusion version of Theorem 2.2 and Theorem 2.2 can be proved for reactions of type C and type C′, respectively. However, it is possibly harder to employ the proofs done in the Fisher-KPP setting w.r.t. the "slow" and the "pseudo-linear" frameworks, where we have seen (cfr. with Chapter 2) that the techniques used in Fisher-KPP case play a very important role in the others settings too.

Part II

Nodal properties of solutions to a nonlocal parabolic equation

Introduction

The second part of thesis is devoted to the analysis of solutions to the space-time nonlocal equation

$$(\partial_\tau - \Delta)^s u = 0 \quad \text{in } \mathbb{R}^N \times (-T, 0), \tag{1}$$

where $0 < s < 1$ and $0 < T < \infty$ are fixed from the beginning. As explained in [19], fractional powers of the Laplacian were introduced in the 40's by Riesz (cfr. with [169, 170]) and have interesting applications to physics and applied mathematics such as elasticity, fluid dynamics and finance. We quote the works [14, 33, 71, 89, 153] in which some very interesting examples of applications are presented.

Our main goal is to describe, as precisely as possible, the nodal properties of solutions to equation (1). This is a significative difference w.r.t. the previous part, in which we studied *nonnegative* solutions to parabolic reaction-diffusion equations. From now one, we will be mostly interested in *sign-changing* solutions and the way in which these solutions approach their zero level set.

Nonlocal equations have been intensively studied in the last two decades and so, it is important to relate our work to the existent literature. In particular, our results rely on some very recent papers of Nyström and Sande [163], Stinga and Torrea [185], and Banerjee and Garofalo [19]. In the first two, the authors came out with a "parabolic extension method" and proved smoothness of solutions, while in the third one new monotonicity formulae and strong unique continuation properties of solutions to equation (1) with *potential* were proved (we will come back on these references later).
As we did in Part I, in the following paragraphs we introduce the main concepts and some definitions needed in the rest of the treatise, together with some important bibliographical review. The main results and their proofs will be presented chapter by chapter.

A parabolic extension method

We now resume the extension method for solutions to equation (1). As in the elliptic setting (see for instance Caffarelli and Silvestre [54]), the main idea is to extend functions defined on the space-time $\mathbb{R}^N \times \mathbb{R}$ to a new space-time $\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}$, so that the "extensions" depend on $N + 1 + 1$ variables. The significative fact is that these extensions satisfy a *local* parabolic equation on $\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}$ and their boundary Neumann derivative equals the nonlocal operator of the extended function. In the next paragraphs we clarify this method following the papers [163, 185], and Section 2 and 3 of [19].

As in the above references, we introduce the Heat operator $H := \partial_t - \Delta$. For any $0 < s < 1$, its fractional power can be defined in terms of its Fourier's transform

$$\widehat{H^s u}(\eta, \vartheta) := (i\vartheta + |\eta|^2)^s \, \widehat{u}(\eta, \vartheta),$$

for all functions $u = u(x, t)$ belonging to the natural domain

$$\text{dom}(H^s) := \left\{ u \in L^2(\mathbb{R}^{N+1}) : (i\vartheta + |\eta|^2)^s \, \widehat{u} \in L^2(\mathbb{R}^{N+1}) \right\}.$$

As showed in Theorem 1.1 of [185] and observed in [163], the first important fact is that the operator H^s can be written for all $(x,t) \in \mathbb{R}^{N+1}$ as the parabolic hypersingular integral

$$H^s u(x,t) = \frac{1}{|\Gamma(-s)|} \int_0^\infty \int_{\mathbb{R}^N} [u(x,t) - u(x-z, t-t')] \frac{G_N(z,t')}{(t')^{1+s}} dz dt',$$

for all $u \in \mathrm{dom}(H^s)$ smooth enough (for instance $u \in \mathcal{S}(\mathbb{R}^{N+1})$), where $G_N = G_N(x,t)$ is the standard Gaussian probability density

$$G_N(x,t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}} \qquad x \in \mathbb{R}^N, \ t > 0.$$

From the above representation formula it is immediately seen that (1) is a nonlocal equation. Indeed, changing variables $x \to x - z$ and $t' \to t - t'$, we get

$$H^s u(x,t) = \frac{1}{|\Gamma(-s)|} \int_{-\infty}^t \int_{\mathbb{R}^N} [u(x,t) - u(z,t')] \frac{G_N(x-z, t-t')}{(t-t')^{1+s}} dz dt',$$

from which we deduce that the value of $H^s u$ at a point (x,t) depends on all the past values of $u = u(x,t)$. Moreover, as explained in Corollary 1.4 of [185], the above integral formulation shows (without using Fourier's transform) that for functions $u = u(x)$ not depending on $t \in \mathbb{R}$, the operator H^s is the Fractional Laplacian $(-\Delta)^s$, namely

$$H^s u(x) = (-\Delta)^s u(x) = P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(z)}{|x-z|^{N+2s}} dz,$$

up to a multiplicative constant, while if $u = u(t)$ does not depend on $x \in \mathbb{R}^N$ we obtain the Marchaud's derivative (or extended Caputo's derivative see for instance [148, 176] and the more recent work [34]):

$$H^s u(t) = (\partial_t)^s u(t) = \frac{1}{|\Gamma(-s)|} \int_{-\infty}^t \frac{u(t) - u(t')}{(t-t')^{1+s}} dt'.$$

It is thus clear that equation (1) significantly differs from another interesting model of fractional diffusion:

$$\partial_t u + (-\Delta)^s u = 0,$$

and its generalizations (see for instance the works of Vázquez et al. [41, 201] for the linear setting, while [76, 77, 202] or [38, 39] for the nonlinear one). See also the papers of Figalli et al. [21, 48], where the obstacle problem for the above nonlocal parabolic equation is studied and the recent paper of Fernández-Real and Ros-Oton [100]. Possibly, the most important diversity is that our equation it is nonlocal in both space and time, and "the random jumps are coupled with the random waiting times" (cfr. with page 3894 of [185]). This is in contrast with the equation above, in which jumps and waiting times are independent.

As we have anticipated, the second main issue concerns an extension property for functions $u \in \mathrm{dom}(H^s)$ proved in Theorem 1.7 of [185] and Theorem 1 of [163] (see also Theorem 3.1 of [19]). We recall it briefly adapting them to our setting and notations. Let $u \in \mathrm{dom}(H^s)$ and define the "extension" of $u = u(x,t)$ as

$$U(x,y,t) := \int_0^\infty \int_{\mathbb{R}^N} u(x-z, t-t') P_y^s(z,t') dz dt', \tag{2}$$

where the "Poisson kernel" is defined by

$$P_y^s(x,t) := \frac{1}{4^s \Gamma(s)} G_N(x,t) \frac{y^{2s}}{t^{1+s}} e^{-\frac{y^2}{4t}} \qquad (x,y) \in \mathbb{R}_+^{N+1}, \ t > 0,$$

and, setting $a = 1 - 2s$, it can be re-written (with a little abuse of notations) as

$$P_y^a(x,t) = \frac{1}{2^{1-a}\Gamma(\frac{1-a}{2})} G_N(x,t)\, \frac{y^{1-a}}{t^{1+\frac{1-a}{2}}} e^{-\frac{y^2}{4t}} \qquad (x,y) \in \mathbb{R}^{N+1}_+,\ t > 0. \tag{3}$$

Then $U = U(x,y,t)$ satisfies $U(\cdot,\cdot,y) \in C([0,\infty), L^2(\mathbb{R}^{N+1}))$ and solves

$$\begin{cases} \partial_t U - y^{-a}\nabla\cdot(y^a \nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+ \times (-\infty,\infty), \\ U(x,0,t) = u(x,t) & \text{in } L^2(\mathbb{R}^{N+1}), \end{cases} \tag{4}$$

and, furthermore, $U = U(x,y,t)$ is smooth in $\mathbb{R}^{N+1}_+ \times (-\infty,\infty)$ and

$$-\frac{2^{2s-1}\Gamma(s)}{\Gamma(1-s)} \lim_{y\to 0^+} y^{1-2s}\partial_y U(x,y,t) = H^s u(x,t) \quad \text{in } L^2(\mathbb{R}^{N+1}). \tag{5}$$

With convention $a := 1 - 2s$, we will write

$$\partial_y^a U(x,t) := \lim_{y\to 0^+} y^a \partial_y U(x,y,t) = (1-a) \lim_{y\to 0^+} \frac{U(x,y,t) - U(x,0,t)}{y^{1-a}},$$

where the above limits are intended in the $L^2(\mathbb{R}^{N+1})$ sense. So, as we have mentioned above, the first main fact is that $U = U(x,y,t)$ is defined on the "extended space" $\mathbb{R}^N \times \mathbb{R}_+ \times \mathbb{R}$ but it satisfies a well-known *local* parabolic equation (cfr. for instance with [65, 118]). On the other hand, we recover the value $H^s u(x,t)$ as a singular limit of the "boundary Neumann derivative" defined in (5). The crucial idea is thus to get information on the solutions $u = u(x,t)$ to the nonlocal equation (1) (posed in the all space):

$$H^s u = 0 \quad \text{in } \mathbb{R}^N \times (-\infty,\infty),$$

by studying the local "extended problem"

$$\begin{cases} \partial_t U - y^{-a}\nabla\cdot(y^a \nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+ \times (-\infty,\infty), \\ U(x,0,t) = u(x,t) & \text{in } L^2(\mathbb{R}^{N+1}) \\ \lim_{y\to 0^+} y^a \partial_y U(x,y,t) = 0 & \text{in } L^2(\mathbb{R}^{N+1}), \end{cases} \tag{6}$$

Working with the "extensions" has some technical advantages and, in particular, it allows us to bypass the non-locality of our equation. The drawbacks are the fact that we are forced to work with the fictitious variable $y \in \mathbb{R}_+$ (i.e. in a higher dimensional space) and then coming back to nonlocal through the (nontrivial) limit (5).

A parabolic equation in the whole "extended" space

In view of what explained in the above paragraphs, we will study some asymptotic properties of a class of solutions $U = U(x,y,t)$ to the *backward* parabolic equation pose in the whole $\mathbb{R}^{N+1}$

$$\partial_t U + |y|^{-a}\nabla\cdot(|y|^a \nabla U) = 0 \quad \text{in } \mathbb{R}^{N+1} \times (0,T), \tag{7}$$

where $-1 < a < 1$ and $0 < T < \infty$ are fixed and $\nabla = \nabla_{x,y}$ denotes the spacial gradient, while $\nabla\cdot = \nabla_{x,y}\cdot$ the spacial divergence. The choice of working with a backward Heat Equation-type is purely formal and will help us to keep the notations as simpler as possible. This is due to the fact that $0 < T < \infty$ is fixed and the time-inversion $t \to -t$ transforms equation (7) into its more standard version:

$$\partial_t U - |y|^{-a}\nabla\cdot(|y|^a \nabla U) = 0 \quad \text{in } \mathbb{R}^{N+1} \times (-T,0).$$

Of course things heavily change when $T = +\infty$ (we will see how in Chapter 5). Equation (7) deviates from standard diffusion because of its criticality near the characteristic hyperplane $\{y = 0\}$ given by the singular coefficient of the drift-term:

$$|y|^{-a}\nabla\cdot(|y|^a\nabla U) = \Delta_{x,y}U + \frac{a}{y}\partial_y U.$$

However, since the weight $w(y) = y^a$ belongs to the $\mathcal{A}_2$ Muckenhoupt class (see for instance [20, 156]), it is included in a wider group of problems studied in the 80's by Chiarenza and Serapioni [64, 65] (see also [118, 127], and [97] for the elliptic setting). We quote also [36, 37, 86, 116] and the references therein for more work on parabolic equations with weighted and nonlinear diffusion.
In those works the main goal was to establish a Harnack inequality for "weak solutions" to (7), together with their Hölder regularity. W.r.t. to the existing literature, in the present work we are mostly interested in giving a classification of the "blow-up profiles" of solutions to (7) in terms of rescaled eigenfunctions to a suitable related eigenvalue problem. This kind of classification is essential for two main reasons: first of all, it is useful to establish the optimal Hölder regularity (which, at least to our knowledge, is an open problem) and for giving qualitative information about the geometry and regularity of the nodal set of solutions to (1).

W.r.t. [19, 185], a key feature of our work, is that our main result concerning the "blow-up classification" of solutions to equation (7) will be obtained as a byproduct of the analysis of the solutions to the boundary value Neumann problem type

$$\begin{cases}\partial_t U + y^{-a}\nabla\cdot(y^a\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+\times(0,T)\\ -\partial^a_y U = 0 & \text{in } \mathbb{R}^N\times\{0\}\times(0,T),\end{cases}\tag{8}$$

where we recall that $\partial^a_y U := \lim_{y\to0^+} y^a\partial_y U(x,y,t)$, and the Dirichlet type one

$$\begin{cases}\partial_t U + y^{-a}\nabla\cdot(y^a\nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+\times(0,T)\\ U = 0 & \text{in } \mathbb{R}^N\times\{0\}\times(0,T),\end{cases}\tag{9}$$

respectively, in the sense that the separate study of problems (8) and (9) (together with suitable "reflection techniques") will enable us deduce some important features of solutions to (7). The deviances in the behaviour of solutions to problem (8) and (9) show themselves the behaviour of solutions to corresponding spectral problems. This important fact implies that solutions to (7) possess a wider class of possible "blow-up behaviours".

On the other hand, as we have seen before, problem (8) is strictly connected to equation (1) through the extension method. Consequently, in the last part of Chapter 5, we will focus on (1)-(8) to give a detailed description on the nodal set of solutions to (1) finding some interesting deviances from the classical results of Han and Lin (cfr. with [122, 143]) that we briefly present below out of completeness.

On the nodal set of local parabolic equations

In [122], Han and Lin studied the nodal set of solutions $u = u(x,t)$ to

$$\partial_t u = a_{ij}(x,t)\partial_{ij}u + b_i(x,t)\partial_i u + c(x,t)u \quad \text{in } Q_1\subset\mathbb{R}^N\times\mathbb{R},$$

where ∂_i and ∂_{ij} indicate the first and second partial derivatives of U w.r.t. to the spacial variables (cfr. with formula (1.1) of [122]), assuming the uniform ellipticity of the diffusion matrix $a_{ij} = a_{ij}(x,t)$ on Q_1, and the ν-Hölder regularity of the coefficients in Q_1:

$$\|a_{ij}\|_{C^{2\nu,\nu}(Q_1)} + \|b_i\|_{C^{2\nu,\nu}(Q_1)} + \|c\|_{C^{2\nu,\nu}(Q_1)} < +\infty,\tag{10}$$

for all $i, j = 1, \dots, N$ and some $0 < \nu < 1/2$ (cfr. with formulas (1.2) and (1.3) of Han and Lin's paper). Moreover, they worked with smooth solutions satisfying the "doubling property"(cfr. with formula (1.5)): there exist $C = C(N) > 0$ and a positive integer d, such that, setting

$$\|u\|^2_{(x,t),r} := \frac{1}{r^{N+2}} \int_{Q_r(x,t)} u^2, \qquad Q_r(x,t) := B_r(x) \times (t - 7r^2/8, t + r^2/8),$$

it holds

$$\frac{\|u\|_{(x,t),r}}{\|u\|_{(x,t),r/2}} \le C^d, \quad \text{for all } (x,t),\ r > 0 \ \text{ with } Q_r(x,t) \subset Q_1. \tag{11}$$

This is a non-degeneracy assumption needed by the authors to exclude solutions vanishing on nonempty open subset of $\mathbb{R}^N \times \mathbb{R}$ (cfr. with Jones's example [130]). Then, for this class of functions they proved the following estimates on the Hausdorff dimension of the nodal set of u (cfr. with Theorem 1.1 of [122]):

$$\begin{aligned} &\dim_{\mathcal{H}}(\Gamma(u)) \le N \\ &\dim_{\mathcal{H}}(\mathcal{S}(u)) \le N - 1, \end{aligned} \tag{12}$$

where we have set

$$\begin{aligned} \Gamma(u) &:= \{(x,t) \in \mathbb{R}^N \times (0,T) : u(x,t) = 0\} = u^{-1}(\{0\}) \\ \mathcal{S}(u) &:= \{(x,t) \in \Gamma(u) : |\nabla_x u| = 0\} = u^{-1}(\{0\}) \cap |\nabla_x u|^{-1}(\{0\}), \end{aligned}$$

It thus follows that $\Gamma(u) \cap Q_1$ is composed by the union of the locally C^1 manifold $\Gamma(u) \cap Q_1 \cap \{|\nabla_x u| > 0\}$ of Hausdorff dimension N and the (closed) set $\mathcal{S}(u) \cap Q_1$ of Hausdorff dimension not greater than $N - 1$.

In what follows we prevalently use the so-called "parabolic Hausdorff dimension" (cfr. for instance with [63]). In its remarkable work, Chen [63] considered more general differential parabolic inequalities (which contains the class of equations studied by Han and Lin) and proved estimates corresponding to (12), in terms of "parabolic Hausdorff dimension", i.e.

$$\begin{aligned} &\dim_{\mathcal{P}}(\Gamma(u)) \le N + 1 \\ &\dim_{\mathcal{P}}(\mathcal{S}(u)) \le N \\ &\dim_{\mathcal{H}}(\mathcal{Z}_t(u)) \le N - 1, \end{aligned} \tag{13}$$

where $\mathcal{Z}_t(u) := \{x \in \mathbb{R}^N : u(x,t) = 0\}$, $t > 0$. One of the main goals of this second part is to establish whether or not estimates like (12)-(13) hold for solutions to (1).

Organization of the chapters

The second part of the thesis is slightly shorter and organized in two chapters. As for the first part, we give below a vague idea of its structure, devoting the first section of each chapter to the detailed statements.

Chapter 4 contains the "preparatory material" on which the main results of the last chapter rely on. In particular, we introduce the definitions of solutions to equation (7), and problems (8) and (9), together with the associated functional setting. Moreover, we give the proof of an Almgren-Poon type monotonicity formula for the class of solutions we consider. It is important to stress that this monotonicity formula was already proved for a larger class of functions by Banerjee and Garofalo in [19] (cfr. with Section 7) and formally derived by Stinga and Torrea in [185] (cfr. with Theorem 1.15). We have decided to propose the proof for completeness.

As we will see, the study of the monotonicity of the Almgren-Poon quotient leads us to a Ornstein-Uhlenbeck eigenvalue problem type. More precisely, the class of parabolically 2κ-homogeneous functions makes the Almgren-Poon quotient constantly equal to $\kappa \in \mathbb{R}$, and κ must be an eigenvalue of the just mentioned problem. This fact will be crucial in the last chapter and the blow-up procedure. Consequently, a large part of the first chapter is devoted to the spectral analysis, from which we will finally derive a *sharp* Gaussian-Poincaré type inequality, which will turn out to be a very useful tool in the remaining part of the work.

Chapter 5 is the core of this second and final part. We start with a blow-up procedure in which we prove that the "normalized" blow-up sequences converge in some suitable energy norms to a linear combination of re-scaled eigenfunctions studied in Chapter 4. Then, through a second blow-up procedure based on the validity of some Liouville type theorem (that we prove by using the Gaussian-Poincaré inequality), we show that the "normalized" blow-up sequences converge to the same limit locally uniformly on $\mathbb{R}^{N+1} \times (0, \infty)$. This last property is fundamental in the final part of the chapter, where we focus on solutions to equation (1) and we study the properties of their nodal sets. In particular, we will give information on the Hausdorff dimension of the "regular" and "singular" part of the nodal set, together with an asymptotic expansion of the solutions near their nodal points.

Chapter 4

Almgren-Poon monotonicity formulas and spectral analysis

In this chapter we introduce the basic definitions and notations used in the rest of the treatise and we present some Almgren-Poon monotonicity formulas of parabolic type (already proved by Banerjee and Garofalo [19], and Stinga and Torrea [185]). To be more concrete, we will see that the Almgren-Poon quotient

$$t \to N(t,U) := \frac{t\int_{\mathbb{R}^{N+1}} |\nabla U|^2(x,y,t)\, d\mu_t(x,y)}{\int_{\mathbb{R}^{N+1}} U^2(x,y,t)\, d\mu_t(x,y)},$$

is a non-decreasing function along a certain class of solutions to equation (7). Here $d\mu_t = d\mu_t(x,y)$ is a suitable family of probability measures that will be introduced later (cfr. with formula (4.15)). Moreover, it will follow that $t \to N(t,U)$ is constant if and only if $U = U(x,y,t)$ is parabolically homogeneous of some degree $\kappa \in \mathbb{R}$, i.e., $U(x,y,t) = \delta^{2\kappa} U(\delta x, \delta y, \delta^2 t)$, $\delta > 0$, which is equivalent to say that the "re-scaled version" $\widetilde{U}(x,y,t) = U(\sqrt{t}x, \sqrt{t}y, t)$ satisfies the Ornstein-Uhlenbeck eigenvalue problem type

$$-|y|^{-a}\nabla\cdot(|y|^a\nabla\widetilde{U}) + \frac{(x,y)}{2}\cdot\nabla\widetilde{U} = \kappa\widetilde{U} \quad \text{in } \mathbb{R}^{N+1},$$

for all $0 < t < T$. The study of this eigenvalue problem is the main part of this chapter.

4.1 Main result

As explained above, the study of the monotonicity of the Almgren-Poon quotient naturally leads us to investigate the spectral properties of some Ornstein-Uhlenbeck type problems. In particular, we will deal with the following three eigenvalue problems:

$$\begin{cases} -y^{-a}\nabla\cdot(y^a\nabla V) + \frac{(x,y)}{2}\cdot\nabla V = \kappa V & \text{in } \mathbb{R}^{N+1}_+ \\ -\partial_y^a V = 0 & \text{in } \mathbb{R}^N\times\{0\}, \end{cases} \tag{4.1}$$

$$\begin{cases} -y^{-a}\nabla\cdot(y^a\nabla V) + \frac{(x,y)}{2}\cdot\nabla V = \kappa V & \text{in } \mathbb{R}^{N+1}_+ \\ V = 0 & \text{in } \mathbb{R}^N\times\{0\}, \end{cases} \tag{4.2}$$

and

$$-|y|^{-a}\nabla\cdot(|y|^a\nabla V) + \frac{(x,y)}{2}\cdot\nabla V = \kappa V \quad \text{in } \mathbb{R}^{N+1}, \tag{4.3}$$

where $V = V(x,y)$ will belong to a suitable functional space. Note that in both (4.1)/(4.2) and (4.3) we obtain the classical Ornstein-Uhlenbeck eigenvalue problem by taking $a = 0$. As we will see, an interesting fact is that the study of the eigenvalue problems (4.1) and (4.2) will give us enough information for describing the set of eigenvalues and eigenfunctions to the third one. In the following theorem we completely characterize the spectrum of the above problems and it is the main result of this chapter.

Theorem 4.1. *Fix $-1 < a < 1$. Then the following three assertions hold:*

(i) The set of eigenvalues of the homogeneous Neumann problem (4.1) *is*

$$\{\widetilde{\kappa}_{n,m}\}_{n,m\in\mathbb{N}}, \qquad \text{where} \qquad \widetilde{\kappa}_{n,m} := \frac{n}{2} + m, \quad n, m \in \mathbb{N},$$

with finite geometric multiplicity. For any $n_0, m_0 \in \mathbb{N}$, the eigenspaces are given by

$$\mathcal{V}_{n_0,m_0} = span\left\{V_{\alpha,m}(x,y) = H_\alpha(x)L_{(\frac{a-1}{2}),m}(y^2/4) : (\alpha, m) \in \widetilde{J}_0\right\},$$

where

$$\widetilde{J}_0 := \left\{(\alpha, m) \in \mathbb{Z}^N_{\geq 0} \times \mathbb{N} : |\alpha| = n \in \mathbb{N} \text{ and } \widetilde{\kappa}_{n,m} = \widetilde{\kappa}_{n_0,m_0}\right\},$$

while $H_\alpha(\cdot)$ is a N-dimensional Hermite polynomial of order $|\alpha|$, while $L_{(\frac{a-1}{2}),m}(\cdot)$ is the m^{th} Laguerre polynomial of order $(a-1)/2$. Furthermore, the set of eigenfunctions $\{V_{\alpha,m}\}_{\alpha,m}$ is an orthogonal basis of $L^2(\mathbb{R}^{N+1}_+, d\mu)$.

(ii) The set of eigenvalues of the homogeneous Dirichlet problem (4.2) *is*

$$\{\widehat{\kappa}_{n,m}\}_{n,m\in\mathbb{N}}, \qquad \text{where} \qquad \widehat{\kappa}_{n,m} := \frac{n}{2} + m + \frac{1-a}{2}, \quad n, m \in \mathbb{N},$$

with finite geometric multiplicity. For all $n_0, m_0 \in \mathbb{N}$, the eigenspaces are given by

$$\mathcal{V}_{n_0,m_0} = span\left\{V_{\alpha,m}(x,y) = H_\alpha(x)\, y^{1-a} L_{(\frac{1-a}{2}),m}(y^2/4) : (\alpha, n) \in J_0\right\},$$

where

$$\widehat{J}_0 := \left\{(\alpha, m) \in \mathbb{Z}^N_{\geq 0} \times \mathbb{N} : |\alpha| = n \in \mathbb{N} \text{ and } \widehat{\kappa}_{n,m} = \widehat{\kappa}_{n_0,m_0}\right\},$$

while now $L_{(\frac{1-a}{2}),m}(\cdot)$ is the m^{th} Laguerre polynomial of order $(1-a)/2$. Again, the set of eigenfunctions $\{V_{\alpha,m}\}_{\alpha,m}$ is an orthogonal basis of $L^2(\mathbb{R}^{N+1}_+, d\mu)$.

(iii) The set of eigenvalues of problem (4.3) *is*

$$\{\kappa_{n,m}\}_{n,m\in\mathbb{N}} = \{\widehat{\kappa}_{n,m}\}_{n,m\in\mathbb{N}} \cup \{\widetilde{\kappa}_{n,m}\}_{n,m\in\mathbb{N}},$$

with finite geometric multiplicity ($\widetilde{\kappa}_{n,m}$ and $\widehat{\kappa}_{n,m}$ are defined in part (i) and (ii), respectively). For any $n_0, m_0 \in \mathbb{N}$, the eigenspaces corresponding to $\widehat{\kappa}_{n_0,m_0}$ and $\widetilde{\kappa}_{n_0,m_0}$ are given by

$$\widetilde{\mathcal{V}}_{n_0,m_0} = span\left\{\widetilde{V}_{\alpha,m}(x,y) = H_\alpha(x)L_{(\frac{a-1}{2}),m}(y^2/4) : (\alpha, m) \in \widetilde{J}_0\right\},$$
$$\widehat{\mathcal{V}}_{n_0,m_0} = span\left\{\widehat{V}_{\alpha,m}(x,y) = H_\alpha(x)\, y|y|^{-a} L_{(\frac{1-a}{2}),m}(y^2/4) : (\alpha, m) \in \widehat{J}_0\right\},$$

respectively, where $\widetilde{J}_0$ and $\widehat{J}_0$ are defined in part (i) and (ii), respectively. Finally, similarly to the previous cases, the set $\{\widetilde{V}_{\alpha,m}(x,y)\}_{(\alpha,m)} \cup \{\widehat{V}_{\alpha,m}(x,y)\}_{(\alpha,m)}$ is an orthogonal basis of $L^2(\mathbb{R}^{N+1}, d\mu)$.

The proof of the above statement is based on some known results about Hermite and Laguerre polynomials and the idea of "separating variables" (cfr. also with [99] from which we borrow some ideas) and will be crucial to characterize the the "blow-up profiles" studied in the next chapter.

Finally, we mention that we will obtain some Gaussian-Poincaré type inequalities as almost immediate consequences of this spectral analysis. This class of inequalities was known from long time (cfr. with [22, 156, 187, 191]). However, since they will play an important role in the rest part of the work and we give versions of them with *optimal* constants, we have decided to devote to them an entire section.

Before proceeding with the proof, we need to introduce some important technical notions and the Almgren-Poon type monotonicity formulas.

4.2 "Fundamental solution", functional spaces and definitions

We devote this section to the detailed derivation of a "fundamental solution", the functional setting and basic definitions. Since the last two are strongly related to the first one, we begin with the study of the "fundamental solution".

4.2.1 Derivation of the "fundamental solution"

For $-1 < a < 1$ fixed, we consider the diffusion problem

$$\begin{cases} \partial_t U - \mathcal{L}_a U = 0 & \text{in } \mathbb{R}^{N+1}_+ \times (0,\infty) \\ -\partial^a_y U = 0 & \text{in } \mathbb{R}^N \times \{0\} \times (0,\infty), \end{cases} \tag{4.4}$$

cfr. with problem (8), where we have introduced the definitions

$$\mathcal{L}_a U := y^{-a}\nabla \cdot (y^a \nabla U), \qquad \partial^a_y U := \lim_{y\to 0^+} y^a \partial_y U.$$

As always, $\nabla\cdot = \nabla_{x,y}\cdot$ is the spacial divergence, $\nabla = \nabla_{x,y}$ is the spacial gradient. We look for solutions to problem (4.4) in the form

$$\mathcal{G}_a(x,y,t) = G_N(x,t)G_{a+1}(y,t). \tag{4.5}$$

We are going to obtain a special solution to (4.4) already found in [185] (see Theorem 1.8, formula (1.10)) by using different techniques. Here, inspired by [109], we present another approach based on the ansatz (4.5). We stress from the beginning that the idea of "separating variables" is crucial in the rest of the work.

Our procedure works as follows. Assume $\mathcal{G}_a = \mathcal{G}_a(x,y,t)$ is smooth and has form (4.5). So, since

$$\nabla\cdot(y^a\nabla\mathcal{G}_a) = \nabla_{x,y}\cdot\left\{y^a\nabla_{x,y}\left[G_N(x,t)G_{a+1}(y,t)\right]\right\} = y^a G_{a+1}\Delta_x G_N + G_N\partial_y\left(y^a\partial_y G_{a+1}\right),$$

assuming $y > 0$ and substituting in (4.4), we obtain that

$$G_{a+1}\left(\partial_t G_N - \Delta_x G_N\right) = G_N\left[\partial_t G_{a+1} - y^{-a}\partial_y\left(y^a\partial_y G_{a+1}\right)\right],$$

which is automatically satisfied if $G_N = G_N(x,t)$ and $G_{a+1} = G_{a+1}(y,t)$ satisfy

$$\begin{cases} \partial_t G_N - \Delta_x G_N = 0 & \text{in } \mathbb{R}^N \times (0,\infty) \\ G_N > 0, \end{cases} \tag{4.6}$$

and, setting $\mathbb{R}_+ := (0,\infty)$,

$$\begin{cases} \partial_t G_{a+1} - y^{-a}\partial_y\left(y^a\partial_y G_{a+1}\right) = 0 & \text{in } \mathbb{R}_+ \times (0,\infty) \\ G_{a+1} > 0, \end{cases} \tag{4.7}$$

respectively. Equation (4.6) is the famous Heat Equation which possesses the self-similar solution

$$G_N(x,t) = \frac{1}{(4\pi t)^{N/2}} e^{-\frac{|x|^2}{4t}}, \tag{4.8}$$

well-defined and positive in $\mathbb{R}^N \times (0,\infty)$, with $\int_{\mathbb{R}^N} G_N(x,t)\,dx = \int_{\mathbb{R}^N} G_N(x,1)\,dx = 1$, for all $t > 0$.

Let us look for self-similar solutions to problem (4.7), i.e., solutions in the form

$$G_{a+1}(y,t) = t^{-\alpha} g(t^{-\beta} y),$$

for some positive exponents α and β, and a profile $g > 0$. We have

$$\partial_t G_{a+1} = t^{-\alpha-1}[-\alpha g(\xi) - \beta \xi g'(\xi)],$$

where $\xi = t^{-\beta} y$. Similarly, it is easy to compute

$$y^{-a}\partial_y \left(y^a \partial_y G_{a+1} \right) = t^{-\alpha-2\beta}\xi^{-a}(\xi^a g'(\xi))',$$

and so, taking $\beta = 1/2$, we can match we the previous equation to obtain the equation of the profile

$$\xi^{-a}(\xi^a g'(\xi))' + \frac{1}{2}\xi g'(\xi) + \alpha g(\xi) = 0, \qquad \xi > 0. \tag{4.9}$$

Note that if $\alpha = (a+1)/2$, we can re-write equation (4.9) as

$$\left(\xi^a g'(\xi) + \frac{1}{2}\xi^{a+1} g(\xi) \right)' = 0,$$

which possesses the family of solutions

$$g(\xi) = Ce^{-\frac{\xi^2}{4}}, \quad C > 0.$$

We will take

$$C = \frac{1}{\int_0^\infty y^a e^{-y^2/4} dy} = \frac{1}{2^a \Gamma(\frac{1+a}{2})}, \tag{4.10}$$

in order to have $\int_0^\infty y^a G_{a+1}(y,t) dy = \int_0^\infty y^a G_{a+1}(y,1) dy = 1$ for all $t > 0$. Hence, we deduce the expression for the special solution to the equation in (4.4):

$$\mathcal{G}_a(x,y,t) = C_{N,a} t^{-\frac{N+a+1}{2}} e^{-\frac{|x|^2+y^2}{4t}} = \frac{1}{2^a \Gamma(\frac{1+a}{2})} G_N(x,t) \frac{1}{t^{\frac{1+a}{2}}} e^{-\frac{y^2}{4t}}, \tag{4.11}$$

where

$$C_{N,a} := \frac{1}{2^a \Gamma(\frac{1+a}{2})(4\pi)^{N/2}}.$$

Note that $\partial_y^a \mathcal{G}_a = 0$ in $\mathbb{R}^N \times \{0\} \times (0,\infty)$ and so $\mathcal{G}_a$ satisfies the boundary condition of (4.4), too. From now on, we will refer to $\mathcal{G}_a = \mathcal{G}_a(x,y,t)$ as "fundamental solution" to problem (4.4). Note that our choice of the normalization constant $C_{N,a} > 0$ is different from the one in [185] (see Theorem 1.8). This is due to the choice in (4.10) which will be convenient later. As pointed out in Remark 1.9 of [185] we get the "Poisson kernel" (3) by taking the co-normal derivative of the "fundamental solution" with parameter $-a$:

$$-y^{-a}\partial_y \mathcal{G}_{-a}(x,y,t) = P_y^a(x,t) \quad \text{in } \mathbb{R}_+^{N+1} \times (0,\infty),$$

see also formula (2.3) of [54] for the elliptic case. Before moving on, we point out that the "fundamental solution" $\mathcal{G}_a = \mathcal{G}_a(x,y,t)$ has two remarkable features which will have substantial importance in the rest of the present work. The first one is the scaling property:

$$\mathcal{G}_a(x,y,t) = t^{-\frac{N+a+1}{2}} \mathcal{G}_a\left(t^{-1/2}x, t^{-1/2}y, 1 \right),$$

form which we deduce the second important feature which is a sort of conservation of "mass":

$$\int_{\mathbb{R}_+^{N+1}} y^a \mathcal{G}_a(x,y,t)\, dxdy = \int_{\mathbb{R}_+^{N+1}} y^a \mathcal{G}_a(x,y,1)\, dxdy = 1,$$

Note that our choice of the constant $C_{N,a} > 0$ is necessary to normalize the above integrals. From now on, we denote with

$$d\mu_t = d\mu_t(x,y) := y^a \mathcal{G}_a(x,y,t)\,dxdy, \tag{4.12}$$

the probability measure on $\mathbb{R}^{N+1}_+$, associated to the family of probability densities

$$(x,y) \to y^a \mathcal{G}_a(x,y,t), \quad \text{for all } t > 0,$$

abbreviating, for $t = 1$, $d\mu := d\mu_1$. Finally, we anticipate that in some cases (cfr. with Section 4.4), we will use the notations

$$d\mu_x := G_N(x,1)dx = \frac{1}{(4\pi)^{N/2}} e^{-\frac{|x|^2}{4}} dx, \qquad d\mu_y := \frac{1}{2^a \Gamma(\frac{1+a}{2})} y^a e^{-\frac{y^2}{4}} dy,$$

to denote the "marginals" of the measure $d\mu = d\mu(x,y)$ w.r.t. to $x \in \mathbb{R}^N$ and $y > 0$, respectively.

Important Remark. Since $\partial_y^a \mathcal{G}_a = 0$ in $\mathbb{R}^N \times \{0\} \times (0,\infty)$ it follows that the function

$$\widetilde{\mathcal{G}}_a(x,y,t) := \frac{C_{N,a}}{2} t^{-\frac{N+a+1}{2}} e^{-\frac{|x|^2+y^2}{4t}}, \tag{4.13}$$

is a smooth solution to equation

$$\partial_t U - \mathcal{L}_a U = 0 \quad \text{in } \mathbb{R}^{N+1} \times (0,\infty), \tag{4.14}$$

where now, with some abuse of notation, we set $\mathcal{L}_a U := |y|^{-a} \nabla \cdot (|y|^a \nabla U)$ (cfr. with Lemma 5.1 of [185] or check it directly by repeating the above procedure for $y < 0$). We will call it "fundamental solution" to equation (4.14) and, with more abuse of notation, we will denote it with $\mathcal{G}_a = \mathcal{G}_a(x,y,t)$. Of course, it satisfies the same properties of the fundamental solution in the half-space $\mathbb{R}^{N+1}_+$. We will set again

$$d\mu_t = d\mu_t(x,y) := |y|^a \mathcal{G}_a(x,y,t)\,dxdy, \tag{4.15}$$

which is now a probability measure on $\mathbb{R}^{N+1}$, for all $t > 0$. Again we will abbreviate with $d\mu$ instead of $d\mu_1$. Also in this case we will indicate with

$$d\mu_y := \frac{1}{2^{1+a} \Gamma(\frac{1+a}{2})} |y|^a e^{-\frac{y^2}{4}} dy,$$

the "marginal" of the measure $d\mu = d\mu(x,y)$ w.r.t. to $y > 0$. We have decided not to distinguish between the to kinds of "fundamental solutions" and related probability measures not to exceed in notations. It will be always clear from the context in which framework we will work.

4.2.2 Functional setting and "strong solutions"

We have seen that the "fundamental solution" $\mathcal{G}_a = \mathcal{G}_a(x,y,t)$ naturally defines a family of probability measures on $\mathbb{R}^{N+1}_+$. We thus define the family of Hilbert spaces

$$L^2(\mathbb{R}^{N+1}_+, d\mu_t) := \left\{ V = V(x,y) \text{ measurable: } \int_{\mathbb{R}^{N+1}_+} V^2(x,y)\,d\mu_t(x,y) < +\infty \right\}, \qquad t > 0,$$

where $d\mu_t = d\mu_t(x,y)$ is defined in (4.12), endowed with the natural L^2 type norm. We anticipate that, we will often simplify the notation to

$$L^2_{\mu_t} := L^2(\mathbb{R}^{N+1}_+, d\mu_t) \qquad \text{and} \qquad L^2_\mu := L^2(\mathbb{R}^{N+1}_+, d\mu).$$

Then, we introduce the family of weighted Sobolev norms

$$\|V\|^2_{H^1(\mathbb{R}^{N+1}_+,d\mu_t)} := \int_{\mathbb{R}^{N+1}_+} V^2(x,y)\,d\mu_t(x,y) + t\int_{\mathbb{R}^{N+1}_+} |\nabla V|^2(x,y)\,d\mu_t(x,y),$$

and the Hilbert spaces

$$H^1(\mathbb{R}^{N+1}_+,d\mu_t) \qquad \text{and} \qquad H^1_0(\mathbb{R}^{N+1}_+,d\mu_t),$$

obtained as the closure of the spaces $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ and $C^\infty_c(\mathbb{R}^{N+1}_+)$ w.r.t. the norm $\|\cdot\|_{H^1(\mathbb{R}^{N+1}_+,d\mu_t)}$ defined above, respectively. Again, we will abbreviate

$$H^1_{\mu_t} := H^1(\mathbb{R}^{N+1}_+,d\mu_t), \quad H^1_{0,\mu_t} := H^1_0(\mathbb{R}^{N+1}_+,d\mu_t), \quad H^1_\mu := H^1(\mathbb{R}^{N+1}_+,d\mu), \quad H^1_{0,\mu} := H^1_0(\mathbb{R}^{N+1}_+,d\mu).$$

Finally, we will need some space-time L^2-Sobolev type spaces:

$$L^2(0,T;L^2_{\mu_t}) := \left\{V = V(x,y,t) \text{ measurable} : \int_0^T \|V(t)\|^2_{L^2_{\mu_t}}\,dt < +\infty\right\}$$
$$L^2(0,T;H^1_{\mu_t}) := \left\{V = V(x,y,t) \text{ measurable} : \int_0^T \|V(t)\|^2_{H^1_{\mu_t}}\,dt < +\infty\right\}$$
$$L^2(0,T;H^1_{0,\mu_t}) := \left\{V = V(x,y,t) \text{ measurable} : \int_0^T \|V(t)\|^2_{H^1_{0,\mu_t}}\,dt < +\infty\right\},$$

which are again Hilbert spaces with the natural scalar product and associated norm.
In the same way, we consider the family of spaces

$$L^2(\mathbb{R}^{N+1},d\mu_t) := \left\{V = V(x,y) \text{ measurable: } \int_{\mathbb{R}^{N+1}} V^2(x,y)\,d\mu_t(x,y) < +\infty\right\}, \qquad t>0,$$

where $d\mu_t = d\mu_t(x,y)$ is defined in (4.15), for all $t>0$. Again we will use the simplified notations

$$L^2_{\mu_t} := L^2(\mathbb{R}^{N+1},d\mu_t) \qquad \text{and} \qquad L^2_\mu := L^2(\mathbb{R}^{N+1},d\mu).$$

As before, considering the family of weighted Sobolev norms

$$\|V\|^2_{H^1(\mathbb{R}^{N+1},d\mu_t)} := \int_{\mathbb{R}^{N+1}} V^2(x,y)\,d\mu_t(x,y) + t\int_{\mathbb{R}^{N+1}} |\nabla V|^2(x,y)\,d\mu_t(x,y),$$

we can define the family of spaces $H^1(\mathbb{R}^{N+1},d\mu_t)$ obtained as the closure of the space $C^\infty_c(\mathbb{R}^{N+1})$ w.r.t. the above norm. Also in this case we will abbreviate $H^1_{\mu_t} := H^1(\mathbb{R}^{N+1},d\mu_t)$ and $H^1_\mu := H^1(\mathbb{R}^{N+1},d\mu)$. Finally, the Hilbert spaces $L^2(0,T;L^2_{\mu_t})$ and $L^2(0,T;H^1_{\mu_t})$ are defined as above substituting $\mathbb{R}^{N+1}_+$ with $\mathbb{R}^{N+1}$. We are now ready to introduce the definition of "strong solution" to equation (7), and problems (8) and (9).

Definition 4.2. *A function $U = U(x,y,t)$ is said to be a "strong solution" to equation* (7) *if the following properties hold true:*

- *$U \in L^2(0,T;H^1_{\mu_t})$, with $t\partial_t U + \frac{(x,y)}{2}\cdot\nabla U$, $\mathcal{L}_a U \in L^2_{loc}(0,T;L^2_{\mu_t})$.*
- *The identity $\partial_t U + \mathcal{L}_a U = 0$ is satisfied a.e. in $\mathbb{R}^{N+1}\times(0,T)$.*

Definition 4.3. *A function $U = U(x,y,t)$ is said to be a "strong solution" to problem* (8) *if the following properties hold true:*

- $U \in L^2(0,T;H^1_{\mu_t})$, *with* $t\partial_t U + \frac{(x,y)}{2} \cdot \nabla U$, $\mathcal{L}_a U \in L^2_{loc}(0,T;L^2_{\mu_t})$.
- *The identity* $\partial_t U + \mathcal{L}_a U = 0$ *is satisfied a.e. in* $\mathbb{R}^{N+1}_+ \times (0,T)$.
- $-\partial^a_y U(x,t) := -\lim_{y\to 0^+} y^a \partial_y U(x,y,t) = 0$ *in* $L^2(\mathbb{R}^N \times (0,T))$.

Definition 4.4. *A function* $U = U(x,y,t)$ *is said to be a "strong solution" to problem* (9) *if the following properties hold true:*

- $U \in L^2(0,T;H^1_{0,\mu_t})$, *with* $t\partial_t U + \frac{(x,y)}{2} \cdot \nabla U, \mathcal{L}_a U \in L^2_{loc}(0,T;L^2_{\mu_t})$.
- *The identity* $\partial_t U + \mathcal{L}_a U = 0$ *is satisfied a.e. in* $\mathbb{R}^{N+1}_+ \times (0,T)$.
- $U = 0$ *in* $\mathbb{R}^N \times (0,T)$ *in the sense of traces.*

Some important comments about Definitions 4.2, 4.3 and 4.4. The concept of "strong solution" is well-known in parabolic equations (see for instance Chapter 7 of [142] for linear diffusion, or Chapter 9 of [198] for nonlinear diffusion). However, before moving forward, we must clarify some aspects of the above definitions. We mainly focus on "strong solutions" to equation (7), specifying which are the main differences w.r.t. the other two cases.

• **On "strong solutions".** First of all, we point out that classical solutions (satisfying the first requirements in Definition 4.2) are "strong solutions", while smooth "strong solutions" are in fact classical solutions. This easily follows from the definition. On the other hand, "strong solutions" are "weak solutions" in the sense of Definition 2.1 of [65] (see also Lemma 5.1 of [185] or Definition 4.3 of [19]). More precisely, the substantial difference between "weak" and "strong" solutions is that in the second case the time derivative and the diffusion operator of solutions $U = U(x,y,t)$ are required to belong to the space $L^2_{loc}(0,T;L^2_{\mu_t})$. This is not the case in the "weak solutions" framework, where these derivatives are allowed to be distributions. To see this, note that multiplying the relation $\partial_t U + \mathcal{L}_a U = 0$ by the test function

$$t|y|^a \mathcal{G}_a(x,y,t)\eta(x,y,t),$$

where η belongs to $L^2(0,T;H^1_{\mu_t})$, and integrating over $\mathbb{R}^{N+1} \times (t_1,t_2)$ (for any choice $0 < t_1 < t_2 < T$), we get the identity

$$\int_{t_1}^{t_2} \int_{\mathbb{R}^{N+1}} \left[t\partial_t U + \frac{(x,y)}{2} \cdot \nabla U \right] \eta \, d\mu_t(x,y) dt = \int_{t_1}^{t_2} t \int_{\mathbb{R}^{N+1}} \nabla U \cdot \nabla \eta \, d\mu_t(x,y) dt \tag{4.16}$$

holds for all $\eta \in L^2(0,T;H^1_{\mu_t})$. Note that it suffices to take $\eta \in L^2_{loc}(0,T;H^1_{\mu_t})$, which is important to widen the set of test functions. However, we have decided to follow the choice $\eta \in L^2(0,T;H^1_{\mu_t})$ not to weight down the presentation.

In performing the above integration by parts, we have crucially used the remarkable property of the "fundamental solution"

$$\nabla_{x,y}\mathcal{G}_a(x,y,t) = -\frac{(x,y)}{2t}\mathcal{G}_a(x,y,t), \qquad (x,y,t) \in \mathbb{R}^{N+1}_+ \times (0,\infty).$$

From the above integral formulation it is thus clear the reason for the requirement $t\partial_t U + \frac{(x,y)}{2} \cdot \nabla U \in L^2_{loc}(0,T;L^2_{\mu_t})$. Conversely, having $\mathcal{L}_a U \in L^2_{loc}(0,T;L^2_{\mu_t})$ too, and "integrating back" in (4.16), we easily obtain $\partial_t U + \mathcal{L}_a U = 0$ a.e. in $\mathbb{R}^{N+1} \times (0,T)$ by the Test Lemma (note that we have strongly used the fact that $0 < t_1 < t_2 < T$ are arbitrarily fixed). Note that this implies that we could have replaced the second requirement in Definition 4.2 with the integral formulation (4.16). In the same way, we could have asked that the identity

$$\int_{\mathbb{R}^{N+1}} \left[\partial_t U + \frac{(x,y)}{2t} \cdot \nabla U \right] \eta \, d\mu_t(x,y) = \int_{\mathbb{R}^{N+1}} \nabla U \cdot \nabla \eta \, d\mu_t(x,y), \tag{4.17}$$

holds for a.e. $0 < t < T$ and all $\eta \in L^2(0,T;H^1_{\mu_t})$ (also in this case the choice $\eta \in L^2_{loc}(0,T;H^1_{\mu_t})$ would be enough). This last identity will be very important in the proof of the monotonicity formulae (cfr. with Lemma 4.5 and 4.6). The are versions of (4.16) and (4.17) for solutions to problems (8) and (9), which are obtained in the same way, but by integrating on $\mathbb{R}^{N+1}_+$ instead of $\mathbb{R}^{N+1}$ (cfr. with (4.27)). Note that for solutions (9), the test functions η are required to belong to $L^2(0,T;H^1_{0,\mu_t})$

• **Regularity and growth conditions.** As we have seen above, "strong solutions" are "weak solutions" and so, by Theorem 2.1 of [65] (see also the all Section 5 of [19], or [118, 127]) we deduce that any "strong solution" is Hölder continuous in $\mathbb{R}^{N+1}\times(0,T)$ (or $\overline{\mathbb{R}^{N+1}_+}\times(0,T)$) for some suitable exponent $0 < \nu < 1$. Moreover, note that the use of Gaussian type spaces in the integral formulations (4.16) and (4.17) is not standard, but it has the advantage to be very natural in the study of the monotonicity of the Almgren-Poon type quotient (cfr. with Section 4.3):

$$t \to N(t,U) := \frac{t\int_{\mathbb{R}^{N+1}}|\nabla U|^2(x,y,t)\,d\mu_t(x,y)}{\int_{\mathbb{R}^{N+1}}U^2(x,y,t)\,d\mu_t(x,y)},$$

since for any nontrivial function $U = U(x,y,t)$ with $U(\cdot,\cdot,t) \in H^1(\mathbb{R}^{N+1},d\mu_t)$, $0 < t < T$, the above quotient is well-defined for a.e. $0 < t < T$.

• **Scaling and re-normalized equation/problems.** Let $U = U(x,y,t)$ be a "strong solution" to equation (7). In the proof of the monotonicity formulae and the blow-up procedure, it will be useful to consider the re-scaled version of $U = U(x,y,t)$, defined by

$$\widetilde{U}(x,y,t) := U(\sqrt{t}x,\sqrt{t}y,t), \qquad (x,y)\in\mathbb{R}^{N+1},\ t>0. \tag{4.18}$$

It is straightforward to see that, defining the Ornstein-Uhlenbeck type operator

$$-O_a\widetilde{U} := -\frac{1}{|y|^a\mathcal{G}_a}\nabla\cdot(|y|^a\mathcal{G}_a\nabla\widetilde{U}) = -|y|^{-a}\nabla\cdot(|y|^a\nabla\widetilde{U}) + \frac{(x,y)}{2}\cdot\nabla\widetilde{U},$$

then $\widetilde{U} = \widetilde{U}(x,y,t)$ is a "strong solution" to

$$t\partial_t\widetilde{U} + O_a\widetilde{U} = 0 \quad \text{in } \mathbb{R}^{N+1}\times(0,T), \tag{4.19}$$

in the sense that $\widetilde{U} \in L^2(0,T;H^1_\mu)$ with $t\partial_t\widetilde{U}$, $O_a\widetilde{U} \in L^2_{loc}(0,T;L^2_\mu)$ and equation (4.19) is satisfied a.e. $\mathbb{R}^{N+1}\times(0,T)$. Note that, proceeding as before, we obtain the equivalent integral formulations for (4.19). In the first one, we integrate both in space and time, to deduce that for each choice of of times $0 < t_1 < t_2 < T$, the identity

$$\int_{t_1}^{t_2} t\int_{\mathbb{R}^{N+1}}\partial_t\widetilde{U}\eta\,d\mu(x,y)dt = \int_{t_1}^{t_2}\int_{\mathbb{R}^{N+1}}\nabla\widetilde{U}\cdot\nabla\eta\,d\mu(x,y)dt \tag{4.20}$$

holds for all $\eta \in L^2(0,T;H^1_\mu)$ (or $\eta \in L^2_{loc}(0,T;H^1_{\mu_t})$). In the second one, we integrate in space to obtain that $\widetilde{U} = \widetilde{U}(x,y,t)$ satisfies

$$t\int_{\mathbb{R}^{N+1}}\partial_t\widetilde{U}\eta\,d\mu(x,y) = \int_{\mathbb{R}^{N+1}}\nabla\widetilde{U}\cdot\nabla\eta\,d\mu(x,y), \tag{4.21}$$

for a.e. $0 < t < T$ and all $\eta \in L^2(0,T;H^1_\mu)$ (or as always $\eta \in L^2_{loc}(0,T;H^1_{\mu_t})$). It is much important to observe that testing with $\eta \equiv 1$, we obtain $\int_{\mathbb{R}^{N+1}}\partial_t\widetilde{U}\,d\mu(x,y) = 0$ for all $0 < t < T$, i.e., the "weighted mean" of $\widetilde{U} = \widetilde{U}(x,y,t)$ is constant in time

$$t \to \int_{\mathbb{R}^{N+1}}\widetilde{U}(x,y,t)\,d\mu(x,y) \equiv C, \tag{4.22}$$

for some suitable constant $C \in \mathbb{R}$.

Now, if $U = U(x, y, t)$ is a "strong solution" to problem (8) (resp. (9)), it follows that its re-scaling $\widetilde{U} = \widetilde{U}(x, y, t)$ satisfies $\widetilde{U} \in L^2(0, T; H^1_\mu)$ (resp. $\widetilde{U} \in L^2(0, T; H^1_{0,\mu})$), with $t\partial_t\widetilde{U}$, $O_a\widetilde{U} \in L^2_{loc}(0, T; L^2_\mu)$, and it satisfies the problem

$$\begin{cases} t\partial_t\widetilde{U} + O_a\widetilde{U} = 0 & \text{in } \mathbb{R}^{N+1}_+ \times (0, T) \\ -\partial^a_y\widetilde{U} = 0 \ \ (\text{resp. } \widetilde{U} = 0) & \text{in } \mathbb{R}^N \times \{0\} \times (0, T), \end{cases} \tag{4.23}$$

in the sense that the equation is satisfied a.e. in $\mathbb{R}^{N+1}_+ \times (0, T)$, while the boundary conditions in the sense of Definitions 4.3 and 4.4. Note that in this case, the Ornstein-Uhlenbeck type operator is

$$-O_a\widetilde{U} := -\frac{1}{y^a\mathcal{G}_a}\nabla \cdot (y^a\mathcal{G}_a\nabla\widetilde{U}) = -y^{-a}\nabla \cdot (y^a\nabla\widetilde{U}) + \frac{(x, y)}{2} \cdot \nabla\widetilde{U}.$$

Again, we have that for each choice of of times $0 < t_1 < t_2 < T$, the identity

$$\int_{t_1}^{t_2} t \int_{\mathbb{R}^{N+1}_+} \partial_t\widetilde{U}\eta \, d\mu(x, y)dt = \int_{t_1}^{t_2} \int_{\mathbb{R}^{N+1}_+} \nabla\widetilde{U} \cdot \nabla\eta \, d\mu(x, y)dt \tag{4.24}$$

holds for all $\eta \in L^2(0, T; H^1_\mu)$ and, similarly,

$$t \int_{\mathbb{R}^{N+1}_+} \partial_t\widetilde{U}\eta \, d\mu(x, y) = \int_{\mathbb{R}^{N+1}_+} \nabla\widetilde{U} \cdot \nabla\eta \, d\mu(x, y), \tag{4.25}$$

for a.e. $0 < t < T$ and all $\eta \in L^2(0, T; H^1_\mu)$. For what concerns the time conservation of the "weighted mean", note that the function (cfr. with (4.22))

$$t \to \int_{\mathbb{R}^{N+1}_+} \widetilde{U}(x, y, t) \, d\mu(x, y),$$

is constant only for "strong solutions" to (8) (testing with constant is not allowed in the definition of "strong solutions" to (9)). We now have all the elements necessary to move forward with our study. In the next section we prove some Almgren-Poon type monotonicity formulae for "strong solutions" to equation (7) and problems (8), (9).

4.3 Almgren-Poon type formulas and some applications

In this section, we will derive two parabolic Almgren-Poon type formulae which will be employed later for showing the asymptotic behaviour of "strong solutions" to equation (7), and problems (8), (9), respectively. We stress from the beginning that a similar monotonicity formula was firstly proved for more regular functions by Stinga and Torrea in [185] and an "averaged" version of it by Banerjee and Garofalo in [19] for non-smooth functions and in the more general setting of "weak solutions". It is important to stress that in the context of "strong solutions", the proofs of the monotonicity formulae are much easier than in the case of "weak solutions", where very hard work is required to obtain partial regularity of the "weak solutions" (cfr. with Section 5 and 6 of [19]). The choice of studying "strong solutions" is intended to avoid this hard technical work, while focusing on the asymptotic behaviour of blow-up sequences.

4.3.1 Poon-Almgren formula for solutions to problems (8) and (9)

Let $-1 < a < 1$ and $0 < T < \infty$ be fixed, and let $U = U(x, y, t)$ be a "strong solution" to problem (8) (cfr. with Definition 4.3):

$$\begin{cases} \partial_t U + \mathcal{L}_a U = 0 & \text{in } \mathbb{R}^{N+1}_+ \times (0, T), \\ -\partial^a_y U = 0 & \text{in } \mathbb{R}^N \times \{0\} \times (0, T), \end{cases}$$

or (9) (cfr. with Definition 4.4):

$$\begin{cases} \partial_t U + \mathcal{L}_a U = 0 & \text{in } \mathbb{R}^{N+1}_+ \times (0,T), \\ U = 0 & \text{in } \mathbb{R}^N \times \{0\} \times (0,T). \end{cases}$$

Following [167, 185], we consider the quantities

$$H(t,U) := \int_{\mathbb{R}^{N+1}_+} y^a U^2(x,y,t) \mathcal{G}_a(x,y,t)\, dxdy,$$
$$I(t,U) := \int_{\mathbb{R}^{N+1}_+} y^a |\nabla U|^2(x,y,t) \mathcal{G}_a(x,y,t)\, dxdy,$$

and the quotient

$$N(t,U) := \frac{tI(t,U)}{H(t,U)} = \frac{t \int_{\mathbb{R}^{N+1}_+} |\nabla U|^2(x,y,t)\, d\mu_t(x,y)}{\int_{\mathbb{R}^{N+1}_+} U^2(x,y,t)\, d\mu_t(x,y)}, \tag{4.26}$$

for all $t > 0$ such that $H(t,U) \neq 0$ (cfr. with Corollary 4.9). Note that all the integrals are well defined for "strong solutions" $U = U(x,y,t)$ to problems (8) and/or (9), for a.e. $0 < t < T$.
Passing to the re-scaled version $\widetilde{U} = \widetilde{U}(x,y,t)$ defined in (4.18) and using the intrinsic scaling of the "fundamental solution", it is easy to see that

$$H(t,U) = \int_{\mathbb{R}^{N+1}_+} y^a \widetilde{U}^2(x,y,t) \mathcal{G}_a(x,y,1)\, dxdy = \int_{\mathbb{R}^{N+1}_+} \widetilde{U}^2(x,y,t)\, d\mu(x,y) := H(1,\widetilde{U}),$$
$$I(t,U) = \frac{1}{t} \int_{\mathbb{R}^{N+1}_+} y^a |\nabla \widetilde{U}|^2(x,y,t) \mathcal{G}_a(x,y,1)\, dxdy = \frac{1}{t} \int_{\mathbb{R}^{N+1}_+} |\nabla \widetilde{U}|^2(x,y,t)\, d\mu(x,y) := \frac{1}{t} I(1,\widetilde{U}),$$

and so the quotient has a scaling too:

$$N(t,U) = \frac{tI(t,U)}{H(t,U)} = \frac{I(1,\widetilde{U})}{H(1,\widetilde{U})} := N(1,\widetilde{U}).$$

Let us proceed to the proof of the Almgren-Poon monotonicity type formula. Note that w.r.t. the proof given in [185], our proof works for less regular solutions and it is presented from a point of view based on scaling, which will be crucial in the rest of the paper (see in particular Section 4.4).

Lemma 4.5. *(cfr. with Theorem 8.3 of [19] and Theorem 1.15 of [185]) Let $U = U(x,y,t)$ be a function satisfying*

- *$U \in L^2(0,T;H^1_{\mu_t})$, with $t\partial_t U + \frac{(x,y)}{2} \cdot \nabla U \in L^2_{loc}(0,T;L^2_{\mu_t})$.*
- *The identity*

$$\int_{\mathbb{R}^{N+1}_+} \left[\partial_t U + \frac{(x,y)}{2t} \cdot \nabla U \right] \eta\, d\mu_t(x,y) = \int_{\mathbb{R}^{N+1}_+} \nabla U \cdot \nabla \eta\, d\mu_t(x,y), \tag{4.27}$$

  holds for a.e. $0 < t < T$ and all $\eta \in L^2(0,T;H^1_{\mu_t})$,

and let $N(t,U)$ be defined as in (4.26)*. Then the function*

$$t \to N(t,U)$$

is non-decreasing for a.e. $0 < t < T$. Moreover, the quotient $t \to N(t,U)$ is constant if and only if $U = U(x,y,t)$ is parabolically homogeneous of degree κ, for some $\kappa \in \mathbb{R}$.

Remark. First of all, we point out that the same statement holds for functions $U = U(x, y, t)$ satisfying

- $U \in L^2(0, T; H^1_{0,\mu_t})$, with $t\partial_t U + \frac{(x,y)}{2} \cdot \nabla U \in L^2_{loc}(0, T; L^2_{\mu_t})$.
- Identity (4.27) holds for a.e. $0 < t < T$ and all $\eta \in L^2(0, T; H^1_{0,\mu_t})$.

We will give proof only for functions satisfying the assumptions in the statement of the lemma, since the other case is almost identical. Secondly, we anticipate that during the proof we will pass to the re-scaled version $\widetilde{U} = \widetilde{U}(x, y, t)$, which satisfies $\widetilde{U} \in L^2(0, T; H^1_\mu)$, with $t\partial_t\widetilde{U} \in L^2_{loc}(0, T; L^2_\mu)$ and the integral relation (4.27) is transformed into (4.25). Finally, note that we could have decided to study the monotonicity of the Almgren-Poon quotient $t \to N(t, U)$ for "strong solutions" to problems (8) and (9) which, as we have seen, is a class of functions satisfying the assumptions of Lemma 4.5. However, in the blow-up procedure we will employ compactness techniques that allow to work with *integral* formulations like (4.27) instead of *point-wise* ones. This is the motivation of our non standard choice in the hypotheses of Lemma 4.5.

Proof. The proof is divided in four essential steps as follows.

Step1: Derivative of $H(t, U)$. Let us fix $0 < t_1 < t_2 < T$ and let us pass for a moment to the re-scaled version $\widetilde{U} = \widetilde{U}(x, y, t)$. Since both $\widetilde{U}$ and $\partial_t\widetilde{U}$ belong to $L^2_{loc}(0, T; L^2_\mu)$, it follows that the function $t \to \widetilde{U}(\cdot, \cdot, t)$ is absolutely continuous on $[t_1, t_2]$ (cfr. with Chapter 5 of [96]) and, furthermore, it holds

$$\frac{d}{dt}\|\widetilde{U}(t)\|^2_{L^2_\mu} = 2\langle\partial_t\widetilde{U}(t), \widetilde{U}(t)\rangle_{L^2_\mu},$$

in the weak sense. Consequently, using that $H(t, U) = H(1, \widetilde{U}) = \|\widetilde{U}(t)\|^2_{L^2_\mu}$, we deduce

$$\begin{aligned} H(t_2, U) - H(t_1, U) &= \int_{t_1}^{t_2} \frac{d}{dt}\|\widetilde{U}(t)\|^2_{L^2_\mu} dt = 2\int_{t_1}^{t_2} \langle\partial_t\widetilde{U}(t), \widetilde{U}(t)\rangle_{L^2_\mu} dt \\ &= 2\int_{t_1}^{t_2}\int_{\mathbb{R}^{N+1}_+} \partial_t\widetilde{U}\,\widetilde{U}\,d\mu(x, y)dt = 2\int_{t_1}^{t_2}\frac{1}{t}\int_{\mathbb{R}^{N+1}_+} |\nabla\widetilde{U}|^2\,d\mu(x, y)\,dt \\ &= 2\int_{t_1}^{t_2}\frac{1}{t}I(1, \widetilde{U})\,dt = 2\int_{t_1}^{t_2} I(t, U)\,dt, \end{aligned}$$

where we have employed the integral relation (4.21) in the fourth equality with test $\eta = \widetilde{U}$, and the fact that $tI(t, U) = I(1, \widetilde{U})$, in the last one. We have thus obtained

$$H'(t, U) = 2I(t, U) \qquad \text{or equivalently} \qquad H'(1, \widetilde{U}) = \frac{2}{t}I(1, \widetilde{U})$$

in the weak sense. Note that since $H'(1, \widetilde{U}) = 2\int_{\mathbb{R}^{N+1}_+} \widetilde{U}\,\partial_t\widetilde{U}\,d\mu(x, y)$, we easily deduce

$$I(1, \widetilde{U}) = t\int_{\mathbb{R}^{N+1}_+} \widetilde{U}\,\partial_t\widetilde{U}\,d\mu(x, y). \tag{4.28}$$

Step2: Derivative of $I(t, U)$. Now, testing the equation of $\widetilde{U} = \widetilde{U}(x, y, t)$ (cfr. with (4.24)) with test $\eta = \partial_t\widetilde{U}$, we see that

$$\int_{t_1}^{t_2} t\int_{\mathbb{R}^{N+1}_+} (\partial_t\widetilde{U})^2\,d\mu(x, y)dt = \int_{t_1}^{t_2}\int_{\mathbb{R}^{N+1}_+} y^a\nabla\widetilde{U}\cdot\nabla(\partial_t\widetilde{U})\mathcal{G}_a\,dxdydt,$$

from which we deduce that the r.h.s. is finite thanks to our integrability assumptions on $\partial_t\widetilde{U}$ (note that testing with $\eta = \partial_t\widetilde{U}$ is not admissible by definition but one can proceed by suitable approximations). On the other hand, from Fubini's Theorem and well known properties of weak derivatives, we have

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\mathbb{R}^{N+1}_+} y^a\nabla\widetilde{U}\cdot\nabla(\partial_t\widetilde{U})\mathcal{G}_a\,dxdydt &= \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^{N+1}_+}\frac{\partial}{\partial t}\left(|\nabla\widetilde{U}|^2\right)y^a\mathcal{G}_a\,dxdydt\\ &= \frac{1}{2}\left[\int_{\mathbb{R}^{N+1}_+}\left(|\nabla\widetilde{U}|^2(x,y,t_2)-|\nabla\widetilde{U}|^2(x,y,t_1)\right)d\mu(x,y)\right]\\ &= \frac{1}{2}\left[I(1,\widetilde{U}(t_2))-I(1,\widetilde{U}(t_1))\right].\end{aligned}$$

Consequently, we have obtained that

$$I'(1,\widetilde{U}) = \frac{2}{t}\int_{\mathbb{R}^{N+1}_+}(t\partial_t\widetilde{U})^2\,d\mu(x,y),$$

in the weak sense.

Step3: Derivative of $N(t,U)$. We are now ready to prove that the function $t\to N(t,U)$ is monotone non-decreasing. Using (4.28) and the expressions for $H'(1,\widetilde{U})$ and $I'(1,\widetilde{U})$, we deduce

$$\begin{aligned}&I'(1,\widetilde{U})H(1,\widetilde{U}) - I(1,\widetilde{U})H'(1,\widetilde{U})\\ &= \frac{2}{t}\left[\int_{\mathbb{R}^{N+1}_+}\widetilde{U}^2d\mu(x,y)\int_{\mathbb{R}^{N+1}_+}(t\partial_t\widetilde{U})^2\,d\mu(x,y) - \left(\int_{\mathbb{R}^{N+1}_+}\widetilde{U}\,t\partial_t\widetilde{U}\,d\mu(x,y)\right)^2\right]\geq 0,\end{aligned}$$

thanks to Cauchy-Schwartz inequality.

Step4: Final remarks. We are left to prove that the function $t\to N(t,U)$ is constant if and only if $U = U(x,y,t)$ is parabolically homogeneous of some degree.
Now, from the Cauchy-Schwartz inequality and the above inequality it follows that

$$N'(t,U)\equiv 0 \quad \text{for a.e. } 0<t<T \qquad \text{if and only if} \qquad t\partial_t\widetilde{U} = \kappa(t)\widetilde{U} \quad \text{for a.e. } 0<t<T,$$

for some real function $\kappa = \kappa(t)$. Consequently, for any $0<t_0<T$ fixed, $\widetilde{U} = \widetilde{U}(x,y,t)$ must have the form

$$\widetilde{U}(x,y,t) = e^{K(t)}\widetilde{U}_0(x,y), \tag{4.29}$$

for a.e. $0<t_0\leq t<T$, where $K(t) := \int_{t_0}^t\kappa(\tau)/\tau\,d\tau$, $\widetilde{U}_0(x,y) := \widetilde{U}(x,y,t_0)$. On the other hand, taking $t\partial_t\widetilde{U} = \kappa(t)\widetilde{U}$ in (4.25), we deduce that $\widetilde{U} = \widetilde{U}(x,y,t)$ satisfies

$$\kappa(t)\int_{\mathbb{R}^{N+1}_+}\widetilde{U}\eta\,d\mu(x,y) = \int_{\mathbb{R}^{N+1}_+}\nabla\widetilde{U}\cdot\nabla\eta\,d\mu(x,y), \tag{4.30}$$

for $0<t_0<t<T$ and all $\eta\in L^2(0,T;H^1_\mu)$ and so, the same equation is satisfied by $\widetilde{U}_0$. Hence, since $\widetilde{U}_0$ is constant w.r.t. the time variable t, it follows

$$\kappa(t)\equiv\kappa \quad \text{for all } 0<t_0\leq t<T,$$

for some constant $\kappa\in\mathbb{R}$ and, by the arbitrariness of $0<t_0<T$, we get that $\kappa(t)\equiv\kappa$ in $(0,T)$. Note that from (4.29), this implies that the function

$$t\to\frac{U(\sqrt{t}x,\sqrt{t}y,t)}{t^\kappa} \tag{4.31}$$

is constant in $(0,T)$. Moreover, (4.30) holds with κ instead of $\kappa(t)$, and so using (4.25):

$$\kappa\int_{\mathbb{R}^{N+1}_+}\widetilde{U}\eta\,d\mu(x,y)=\int_{\mathbb{R}^{N+1}_+}\nabla\widetilde{U}\cdot\nabla\eta\,d\mu(x,y)=t\int_{\mathbb{R}^{N+1}_+}\partial_t\widetilde{U}\eta\,d\mu(x,y),$$

for all $\eta\in L^2(0,T;H^1_\mu)$. Changing variables $x\to\sqrt{t}x$ and $y\to\sqrt{t}y$ in the first and the last integral, we obtain

$$\int_{\mathbb{R}^{N+1}_+}\left[t\partial_t U+\frac{(x,y)}{2}\cdot\nabla U-\kappa U\right]\eta\,d\mu_t=0,$$

for all $\eta\in L^2(0,T;H^1_{\mu_t})$, and so, by the Test Lemma, it follows

$$t\partial_t U+\frac{(x,y)}{2}\cdot\nabla U=\kappa U\quad\text{ a.e. in }\mathbb{R}^{N+1}_+\times(0,T),\tag{4.32}$$

which is equivalent to say that $U=U(x,y,t)$ is parabolically κ-homogeneous: $U(\delta x,\delta y,\delta^2 t)=\delta^{2\kappa}U(x,y,t)$, for any $\delta>0$ (cfr. with [19, 185]). Note that we would have got relation (4.32) by differentiating (w.r.t. time) the constant function (4.31).
Of course, the same homogeneity property holds also for $\widetilde{U}=\widetilde{U}(x,y,t)$ which satisfies the problem

$$\begin{cases}-O_a\widetilde{U}=\kappa\widetilde{U} & \text{in }\mathbb{R}^{N+1}_+\times(0,T),\\ -\partial^a_y\widetilde{U}=0 & \text{in }\mathbb{R}^N\times\{0\}\times(0,T)\end{cases}$$

where $-O_a\widetilde{U}:=-\mathcal{L}_a\widetilde{U}+\frac{(x,y)}{2}\cdot\nabla\widetilde{U}$ (it is the Ornstein-Uhlenbeck type operator introduced in Subsection 4.2.2), in the sense that the identity

$$\int_{\mathbb{R}^{N+1}_+}\nabla\widetilde{U}\cdot\nabla\eta\,d\mu(x,y)=\kappa\int_{\mathbb{R}^{N+1}_+}\widetilde{U}\eta\,d\mu(x,y),$$

is satisfied for a.e. $0<t<T$ and all $\eta\in L^2(0,T;H^1_\mu)$. Note that, w.r.t. (4.32), the above equation does not involve time derivatives but second order spacial derivatives. Consequently, the study of the parabolically κ-homogeneous profiles is equivalent to the study of the Ornstein-Uhlenbeck type problem introduced before, which we will present in detail in the next section. □

4.3.2 Poon-Almgren formula for solutions to equation (7)

Let $-1<a<1$ and $0<T<\infty$ be fixed. We now repeat the analysis carried out before, for "strong solutions" to equation (7):

$$\partial_t U+\mathcal{L}_a U=0\quad\text{ in }\mathbb{R}^{N+1}\times(0,T),$$

where we recall that in this setting $\mathcal{L}_a U:=|y|^{-a}\nabla\cdot(|y|^a\nabla U)$. As before, we consider the quantities

$$H(t,U):=\int_{\mathbb{R}^{N+1}}|y|^a U^2(x,y,t)\mathcal{G}_a(x,y,t)\,dxdy,$$
$$I(t,U):=\int_{\mathbb{R}^{N+1}}|y|^a|\nabla U|^2(x,y,t)\mathcal{G}_a(x,y,t)\,dxdy,$$

and the quotient

$$N(t,U):=\frac{tI(t,U)}{H(t,U)},\tag{4.33}$$

for a.e. $0 < t < T$ such that $H(t, U) \neq 0$. Again, $H(t, U)$ and $I(t, U)$ satisfy the scaling properties:

$$H(t,U) = \int_{\mathbb{R}^{N+1}} |y|^a \widetilde{U}^2(x,y,t)\mathcal{G}_a(x,y,1)\,dxdy = \int_{\mathbb{R}^{N+1}} \widetilde{U}^2(x,y,t)\,d\mu(x,y) := H(1,\widetilde{U}),$$
$$I(t,U) = \frac{1}{t}\int_{\mathbb{R}^{N+1}} |y|^a |\nabla\widetilde{U}|^2(x,y,t)\mathcal{G}_a(x,y,1)\,dxdy = \frac{1}{t}\int_{\mathbb{R}^{N+1}} |\nabla\widetilde{U}|^2(x,y,t)\,d\mu(x,y) := \frac{1}{t}I(1,\widetilde{U}),$$

where $\widetilde{U} = \widetilde{U}(x,y,t)$ is defined in (4.18), and so the quotient has a scaling too:

$$N(t,U) = \frac{tI(t,U)}{H(t,U)} = \frac{I(1,\widetilde{U})}{H(1,\widetilde{U})} := N(1,\widetilde{U}).$$

Lemma 4.6. *Let $U = U(x,y,t)$ be a function satisfying*

- $U \in L^2(0,T;H^1_{\mu_t})$, *with* $t\partial_t U + \frac{(x,y)}{2}\cdot\nabla U \in L^2_{loc}(0,T;L^2_{\mu_t})$.
- *Identity* (4.17)*:*

$$\int_{\mathbb{R}^{N+1}} \left[\partial_t U + \frac{(x,y)}{2t}\cdot\nabla U\right]\eta\,d\mu_t(x,y) = \int_{\mathbb{R}^{N+1}} \nabla U\cdot\nabla\eta\,d\mu_t(x,y),$$

  holds for a.e. $0 < t < T$ *and all* $\eta \in L^2(0,T;H^1_{\mu_t})$,

and let $N(t,U)$ be defined as in (4.33)*. Then the function*

$$t \to N(t,U)$$

is non-decreasing for a.e. $0 < t < T$. Moreover, the quotient $t \to N(t,U)$ is constant if and only if $U = U(x,y,t)$ is parabolically homogeneous of degree κ, for some $\kappa \in \mathbb{R}$. (We recall that in this framework we use the convention $L^2_{\mu_t} = L^2(\mathbb{R}^{N+1}, d\mu_t)$ and $H^1_{\mu_t} = H^1(\mathbb{R}^{N+1}, d\mu_t)$).

Proof. The proof is very similar to the previous one and we report it by completeness.

Step1: Derivative of $H(t,U)$. This step formally coincides with *Step1* of the proof of Lemma 4.5. We just have to replace $L^2(\mathbb{R}^{N+1}_+, d\mu_t)$ by $L^2(\mathbb{R}^{N+1}, d\mu_t)$, and using identity (4.17) instead of (4.27). Consequently, it follows newly $H'(t,U) = 2I(t,U)$ (i.e $H'(1,\widetilde{U}) = 2t^{-1}I(1,\widetilde{U})$) in the weak sense, and from the fact that $H'(1,\widetilde{U}) = 2\int_{\mathbb{R}^{N+1}} \widetilde{U}\,\partial_t\widetilde{U}\,d\mu(x,y)$, we have again

$$I(1,\widetilde{U}) = t\int_{\mathbb{R}^{N+1}} \widetilde{U}\,\partial_t\widetilde{U}\,d\mu(x,y).$$

Step2: Derivative of $I(t,U)$. Proceeding as before, we test equation (4.17) with $\eta = \partial_t\widetilde{U}$ and we compute

$$\begin{aligned}\int_{t_1}^{t_2} t\int_{\mathbb{R}^{N+1}} (\partial_t\widetilde{U})^2\,d\mu(x,y)dt &= \int_{t_1}^{t_2}\int_{\mathbb{R}^{N+1}} |y|^a\nabla\widetilde{U}\cdot\nabla(\partial_t\widetilde{U})\mathcal{G}_a\,dxdydt\\ &= \frac{1}{2}\int_{t_1}^{t_2}\int_{\mathbb{R}^{N+1}_+} \frac{\partial}{\partial t}\left(|\nabla\widetilde{U}|^2\right) y^a\mathcal{G}_a\,dxdydt\\ &= \frac{1}{2}\left[I(1,\widetilde{U}(t_2)) - I(1,\widetilde{U}(t_1))\right],\end{aligned}$$

using Fubini Theorem. This implies again

$$I'(1,\widetilde{U}) = \frac{2}{t}\int_{\mathbb{R}^{N+1}} (t\partial_t\widetilde{U})^2\,d\mu(x,y),$$

in the weak sense.

Step3: Derivative of $N(t,U)$. Exactly as before it is straightforward to compute

$$I'(1,\widetilde{U})H(1,\widetilde{U}) - I(1,\widetilde{U})H'(1,\widetilde{U})$$
$$= \frac{2}{t}\left[\int_{\mathbb{R}^{N+1}} \widetilde{U}^2 d\mu(x,y) \int_{\mathbb{R}^{N+1}} (t\partial_t\widetilde{U})^2\, d\mu(x,y) - \left(\int_{\mathbb{R}^{N+1}} \widetilde{U}\, t\partial_t\widetilde{U}\, d\mu(x,y)\right)^2\right] \geq 0,$$

thanks to Cauchy-Schwartz inequality.

Step4: Final remarks. This last step coincides with *Step4* of Lemma 4.5. In the same methods, we obtain that $N'(t,U) \equiv 0$ if and only if

$$t\partial_t U + \frac{(x,y)}{2}\cdot\nabla U = \kappa U \quad \text{a.e. in } \mathbb{R}^{N+1}\times(0,T),$$

which implies that $U = U(x,y,t)$ is parabolically homogeneous of degree $\kappa \in \mathbb{R}$. This time the equation for $\widetilde{U} = \widetilde{U}(x,y,t)$ is

$$-\mathcal{O}_a\widetilde{U} = \kappa\widetilde{U} \quad \text{in } \mathbb{R}^{N+1}\times(0,T)$$

for some $\kappa \in \mathbb{R}$, where $-\mathcal{O}_a\widetilde{U} := -\mathcal{L}_a\widetilde{U} + \frac{(x,y)}{2}\cdot\nabla\widetilde{U}$, in the sense that the identity

$$\int_{\mathbb{R}^{N+1}} \nabla\widetilde{U}\cdot\nabla\eta\, d\mu(x,y) = \kappa\int_{\mathbb{R}^{N+1}} \widetilde{U}\eta\, d\mu(x,y),$$

is satisfied for a.e. $0 < t < T$ and all $\eta \in L^2(0,T;H^1_\mu)$. Note that again the study of profiles making constant the Almgren-Poon quotient is equivalent the study of the above eigenvalue problem. □

4.3.3 Some immediate applications of the Poon-Almgren monotonicity formula

We now focus on some immediate applications of the monotonicity formula proved in Lemma 4.5 and Lemma 4.6. We will mostly show our results in the setting of Lemma 4.5 since the proof in the other framework is almost identical.

We recall that we have proved that if $U = U(x,y,t)$ satisfies the assumptions of Lemma 4.5, then the function

$$t \to N(t,U) = \frac{t\,I(t,U)}{H(t,U)} = \frac{t\int_{\mathbb{R}^{N+1}_+} y^a|\nabla U|^2(x,y,t)\mathcal{G}_a(x,y,t)\,dxdy}{\int_{\mathbb{R}^{N+1}_+} y^a U^2(x,y,t)\mathcal{G}_a(x,y,t)\,dxdy}$$

is monotone non-decreasing for all $t > 0$ (and a similar result hold for functions $U = U(x,y,t)$ satisfying the assumptions in Lemma 4.6).

Corollary 4.7. *Let $U = U(x,y,t)$ be a nontrivial function satisfying the assumptions of Lemma 4.5. Then both limits*

$$\lim_{t\to 0^+} N(t,U) = \kappa \geq 0, \qquad \lim_{t\to 0^+} t^{-2\kappa}H(t,U) \geq 0,$$

exist and are finite (of course, $N(t,U)$ and $H(t,U)$ are defined as in (4.26)*).*

Proof. The limit $\kappa \in [0,+\infty]$ exists since $N(t,U)$ is non-decreasing. Moveover, if $\kappa = +\infty$, then $N(t,U) \equiv +\infty$ for any $0 < t < T$ and so, since

$$tI(t,U) = t\int_{\mathbb{R}^{N+1}_+} y^a|\nabla U|^2(x,y,t)\mathcal{G}_a(x,y,t)\,dxdy < +\infty \quad \text{for a.e. } 0 < t < T,$$

it follows that $H(t,U) = 0$ for a.e. $0 < t < T$, i.e. $U \equiv 0$ in $\mathbb{R}^{N+1}_+ \times (0,T)$, which is a contradiction. For what concerns the second limit, we have

$$\frac{d}{dt}\left(t^{-2\kappa}H(t,U)\right) = 2t^{-2\kappa-1}H(t,U)\,[N(t,U)-\kappa] \geq 0, \quad t > 0, \tag{4.34}$$

and so the limit $\lim_{t\to 0^+} t^{-2\kappa}H(t,U)$ exists. The fact that it is finite follows as before. □

Corollary 4.8. *Let $U = U(x,y,t)$ be a nontrivial function satisfying the assumptions of Lemma 4.5. Then for any $0 < t_0 < T$, the following formulas hold*

$$H(t,U) \geq H(t_0,U)\left(\frac{t}{t_0}\right)^{2N(t_0,U)} \quad \textit{for all } 0 < t \leq t_0, \tag{4.35}$$

$$H(t,U) \leq H(t_0,U)\left(\frac{t}{t_0}\right)^{2\kappa} \quad \textit{for all } 0 < t \leq t_0, \tag{4.36}$$

where as before $\kappa = \lim_{t\to 0^+} N(t,U)$.

Proof. Remembering that $H'(t,U) = 2I(t,U)$ (see the proof of Lemma 4.5):

$$\frac{H'(t,U)}{H(t,U)} = \frac{2I(t,U)}{H(t,U)} = \frac{2N(t,U)}{t} \leq \frac{2N_{t_0}(U)}{t}, \quad \text{for all } 0 < t \leq t_0,$$

where $0 < t_0 < T$ is arbitrarily fixed. Integrating the previous inequality between t and t_0, it is straightforward to obtain (4.35). Similarly,

$$\frac{H'(t,U)}{H(t,U)} = \frac{2I(t,U)}{H(t,U)} = \frac{2N(t,U)}{t} \geq \frac{2\kappa}{t}, \quad \text{for all } t > 0,$$

Integrating between t and t_0 as before, we get (4.36), too. From another view point, we can note that proving (4.36) is equivalent to prove that the function

$$t \to t^{-2\kappa}H(t,U)$$

is nondecreasing, and this fact easily follows from (4.34). □

The following corollary (of Lemma 4.5) is classical and we follow the proof done in [99, 167]. See also the end of Section 6 of [19].

Corollary 4.9. *(Weak unique continuation property w.r.t. to t) Let $U = U(x,y,t)$ be a nontrivial function satisfying the assumptions of Lemma 4.5. If there exists $0 < t_0 < T$ such that*

$$U(\cdot,\cdot,t_0) \equiv 0 \quad \textit{a.e. in } \mathbb{R}^{N+1}_+,$$

then $U \equiv 0$ a.e. in $\mathbb{R}^{N+1}_+ \times (0,T)$.

Proof. We begin the proof by showing that for any $0 < T < \infty$ and any $U = U(x,y,t)$ satisfying our assumptions, it holds

$$U \not\equiv 0 \ \text{ a.e } \quad \text{in } \mathbb{R}^{N+1}_+ \times (0,T) \quad \Rightarrow \quad H(t,U) > 0 \quad \text{for all } 0 < t < T. \tag{4.37}$$

We proceed in three short steps as follows.

Step1. First, we note that

$$H(t_0,U) = 0 \quad \text{for some } t_0 > 0 \quad \Rightarrow \quad H(t,U) = 0 \quad \text{for all } 0 < t \leq t_0.$$

This easily follows from the fact that $H'(t,U) \geq 0$ for all $t > 0$.

Step2. Now, we show that

$$H(\cdot,U) \not\equiv 0 \quad \Rightarrow \quad H(t,U) > 0 \quad \text{for all } t > 0.$$

Since $H(\cdot,U) \not\equiv 0$ there is $t_0 > 0$ such that $H(t_0,U) > 0$, which implies $H(t,U) > 0$ for all $t \geq t_0$. Now, let

$$\underline{t}_0 := \inf\{t_0 > 0 : H(t_0,U) > 0\}.$$

If by contradiction $\underline{t}_0 > 0$, we deduce

$$0 = H(\underline{t}_0,U) \geq H(t_0,U)\left(\frac{\underline{t}_0}{t_0}\right)^{2N(t_0,U)} > 0,$$

for some $\underline{t}_0 < t_0$, which rises the desired contradiction.

Step3. Let $T > 0$. We show that

$$U \not\equiv 0 \ \text{ a.e} \quad \text{in } \mathbb{R}^{N+1}_+ \times (0,T) \quad \Rightarrow \quad H(t,U) > 0 \quad \text{for all } 0 < t < T.$$

Assume $U \not\equiv 0$ a.e. in $\mathbb{R}^{N+1}_+ \times (0,T)$. Hence, there is $0 < t_0 < T$ such that $U(\cdot,\cdot,t_0) \not\equiv 0$ and so $H(t_0,U) > 0$. From the previous step, we conclude $H(t,U) > 0$ for all $0 < t < T$.

Step4: Conclusions. If $U(\cdot,\cdot,t_0) \equiv 0$ a.e in $\mathbb{R}^{N+1}_+$ it obviously follows that $H(t_0,U) = 0$, and this contradicts (4.37), unless $U \equiv 0$ in $\mathbb{R}^{N+1}_+ \times (0,T)$. □

Important Remark. The statements of Corollary 4.7, Corollary 4.8 and Corollary 4.9 hold true for functions $U = U(x,y,t)$ satisfying the assumptions of Lemma 4.6, where $N(t,U)$ and $H(t,U)$ are defined in (4.33).

4.4 Proof of Theorem 4.1

As explained in the introduction, at the end of the proof of Lemma 4.5 (resp. Lemma 4.6), we have found that the quotients in (4.26) (resp. (4.33)) are constant in time if and only if it holds

$$\begin{cases} -\mathcal{O}_a\widetilde{U} = \kappa\widetilde{U} & \text{in } \mathbb{R}^{N+1}_+ \times (0,T) \\ -\partial^a_y\widetilde{U} = 0 \quad \text{or} \quad \widetilde{U} = 0 & \text{in } \mathbb{R}^N \times \{0\} \times (0,T), \end{cases} \qquad (\text{resp. } -\mathcal{O}_a\widetilde{U} = \kappa\widetilde{U} \quad \text{in } \mathbb{R}^{N+1} \times (0,T)),$$

for some $\kappa \in \mathbb{R}$ in a weak sense, where $-\mathcal{O}_a\widetilde{U} := -\mathcal{L}_a\widetilde{U} + \frac{(x,y)}{2}\cdot\nabla\widetilde{U}$ and $\widetilde{U}(x,y,t) := U\left(\sqrt{t}x, \sqrt{t}y, t\right)$, and $U = U(x,y,t)$ satisfies the assumptions of Lemma 4.5, and so we are taken back to study problems (4.1)/(4.2) and (4.3).

In what follows it will be useful to keep in mind that (4.3) is equivalent to (abbreviating $\mathcal{G}_a = \mathcal{G}_a(x,y,1)$)

$$-\frac{1}{|y|^a\mathcal{G}_a}\nabla\cdot(|y|^a\mathcal{G}_a\nabla V) = \kappa V \quad \text{in } \mathbb{R}^{N+1}, \tag{4.38}$$

and similarly for problems (4.1) and (4.2). The importance of the formulation in (4.38) is clearer in the following fundamental definition.

Definition 4.10. *We proceed with three different definitions.*

• *A nontrivial function $V \in H^1(\mathbb{R}^{N+1}_+, d\mu)$ is said to be a "weak eigenfunction" to problem* (4.1) *with eigenvalue $\kappa \in \mathbb{R}$ if*

$$\int_{\mathbb{R}^{N+1}_+}\nabla V\cdot\nabla\eta\,d\mu(x,y) = \kappa\int_{\mathbb{R}^{N+1}_+}V\eta\,d\mu(x,y), \quad \textit{for all } \eta \in H^1(\mathbb{R}^{N+1}_+, d\mu). \tag{4.39}$$

• *A nontrivial function* $V \in H_0^1(\mathbb{R}_+^{N+1}, d\mu)$ *is said to be a "weak eigenfunction" to problem* (4.2) *with eigenvalue* $\kappa \in \mathbb{R}$ *if*

$$\int_{\mathbb{R}_+^{N+1}} \nabla V \cdot \nabla \eta \, d\mu(x,y) = \kappa \int_{\mathbb{R}_+^{N+1}} V \eta \, d\mu(x,y), \quad \textit{for all } \eta \in H_0^1(\mathbb{R}_+^{N+1}, d\mu). \tag{4.40}$$

• *A nontrivial function* $V \in H^1(\mathbb{R}^{N+1}, d\mu)$ *is said to be a "weak eigenfunction" to problem* (4.3) *with eigenvalue* $\kappa \in \mathbb{R}$ *if*

$$\int_{\mathbb{R}^{N+1}} \nabla V \cdot \nabla \eta \, d\mu(x,y) = \kappa \int_{\mathbb{R}^{N+1}} V \eta \, d\mu(x,y), \quad \textit{for all } \eta \in H^1(\mathbb{R}^{N+1}, d\mu). \tag{4.41}$$

Before moving forward, we stress that using the fact that $\nabla \mathcal{G}_a(x,y,1) = -\frac{(x,y)}{2} \mathcal{G}_a(x,y,1)$, a simple integration by parts shows that classical eigenfunctions to (4.1), (4.2) and (4.3) are a "weak eigenfunctions" to (4.1), (4.2) and (4.3), respectively. Moreover, it is easily seen that smooth "weak eigenfunctions" are classical eigenfunctions.

In the spectral analysis of problems (4.1), (4.2), and (4.3), the following one-dimensional eigenvalue problem (where we set $\psi' = d\psi/dy$ for simplicity)

$$-y^{-a}(y^a \psi')' + \frac{y}{2}\psi' = \sigma\psi, \quad y > 0, \tag{4.42}$$

which, similarly as before, is equivalent to

$$-\frac{1}{y^a G_{a+1}}(y^a G_{a+1}\psi')' = \sigma\psi, \quad y > 0, \tag{4.43}$$

will play an important role (here $G_{a+1} = G_{a+1}(y) = \mathcal{G}_a(0,y,1)$, cfr. with Subsection 4.2.1). This is due to the fact that the "fundamental solution" defines a probability measure that can be written as the product of two marginal measures, which are probabilities on the marginal spaces (cfr. with Section 4.2), and the differential Ornstein-Uhlenbeck operators type, defined in (4.1), (4.2) and (4.3) possess a similar property (that we clarify later). For these reasons, we will devote an entire subsection to the analysis of equation (4.42).

4.4.1 Spectral analysis for equation (4.42)

As anticipated before, we now focus on equation (4.42). More precisely, we will get information about the following eigenvalue problems (corresponding to (4.1), (4.2) and (4.3)):

$$\begin{cases} -y^{-a}(y^a\psi')' + \frac{y}{2}\psi' = \sigma\psi, & \text{for } y > 0, \\ -\partial_y^a \psi = 0 & \text{for } y = 0, \end{cases} \tag{4.44}$$

$$\begin{cases} -y^{-a}(y^a\psi')' + \frac{y}{2}\psi' = \sigma\psi, & \text{for } y > 0, \\ \psi = 0 & \text{for } y = 0, \end{cases} \tag{4.45}$$

and

$$-|y|^{-a}(|y|^a\psi')' + \frac{y}{2}\psi' = \sigma\psi, \quad y \neq 0. \tag{4.46}$$

In the following definition, we introduce the concept of "weak eigenfunction" to (4.44), (4.45), and (4.46), respectively.

Definition 4.11. *Again we proceed in three cases.*
• *A nontrivial function* $\psi \in H^1(\mathbb{R}_+, d\mu_y)$ *is said to be a "weak eigenfunction" to problem* (4.44) *with eigenvalue* $\sigma \in \mathbb{R}$ *if*

$$\int_{\mathbb{R}_+} \psi'\eta' \, d\mu_y = \sigma \int_{\mathbb{R}_+} \psi\eta \, d\mu_y, \quad \textit{for all } \eta \in H^1(\mathbb{R}_+, d\mu_y). \tag{4.47}$$

• *A nontrivial function* $\psi \in H^1_0(\mathbb{R}_+, d\mu_y)$ *is said to be a "weak eigenfunction" to problem* (4.45) *with eigenvalue* $\sigma \in \mathbb{R}$ *if*

$$\int_{\mathbb{R}_+} \psi' \eta' \, d\mu_y = \sigma \int_{\mathbb{R}_+} \psi \eta \, d\mu_y, \quad \text{for all } \eta \in H^1_0(\mathbb{R}_+, d\mu_y). \tag{4.48}$$

• *A nontrivial function* $\psi \in H^1(\mathbb{R}, d\mu_y)$ *is said to be a "weak eigenfunction" to problem* (4.46) *with eigenvalue* $\sigma \in \mathbb{R}$ *if*

$$\int_{\mathbb{R}} \psi' \eta' \, d\mu_y = \sigma \int_{\mathbb{R}} \psi \eta \, d\mu_y, \quad \text{for all } \eta \in H^1(\mathbb{R}, d\mu_y). \tag{4.49}$$

Note that in the previous definition we employ the probability measure $d\mu_y := G_{a+1}(y)dy$ in problems (4.47) and (4.48), while its even extension in (4.49) (cfr. with Section 4.2).

We now carry out a detailed analysis of the spectrum of equation (4.42):

$$-y^{-a} \left(y^a \psi'\right)' + \frac{y}{2} \psi' = \sigma \psi, \quad y > 0.$$

In order to work in the framework as general as possible, we do not impose boundary conditions at $y = 0$ but only the following integrability conditions (which are necessary to have "weak eigenfunctions")

$$\int_0^\infty \psi^2(y) \, y^a e^{-\frac{y^2}{4}} \, dy < +\infty, \qquad \int_0^\infty \left|\psi'(y)\right|^2 \, y^a e^{-\frac{y^2}{4}} \, dy < +\infty. \tag{4.50}$$

Our procedure will naturally distinguish the solutions with Dirichlet and/or Neumann boundary conditions.

So, let us set $\psi(y) = \zeta(y^2/4)$. It is easily seen that the equation for $\zeta = \zeta(r)$, $r = y^2/4$ is:

$$r \frac{d^2 \zeta}{dr^2} + \left(1 + \frac{a-1}{2} - r\right) \frac{d\zeta}{dr} + \sigma \zeta = 0, \quad r > 0, \tag{4.51}$$

which can be seen as a Kummer Confluent Hypergeometric type equation, with

$$b_1 = -\sigma \qquad \text{and} \qquad b_2 = 1 + \frac{a-1}{2} = \frac{1+a}{2},$$

and/or a Laguerre equation with

$$\alpha = \frac{a-1}{2} > -1.$$

A detail report about these topics is given in Appendix 4.6. We know that all solutions are given by

$$\zeta(r) = A_1 M\left(-\sigma, \frac{1+a}{2}, r\right) + A_2 \widetilde{M}\left(-\sigma, \frac{1+a}{2}, r\right), \quad r > 0, \tag{4.52}$$

where $A_1, A_2 \in \mathbb{R}$, and $M(\cdot,\cdot,\cdot)$ and $\widetilde{M}(\cdot,\cdot,\cdot)$ are the Kummer and the Tricomi functions, as explained in the appendix mentioned above. We divide the analysis in three cases:

• *Case* $\sigma \in \mathbb{N}$. As explained in Appendix 4.6, when $\sigma = m \in \mathbb{N} = \{0, 1, \ldots\}$, then to each $\sigma = m$ it corresponds a unique solution (up to multiplicative constants) to (4.51) given by the m^{th} Laguerre polynomial of order $(a-1)/2$:

$$\zeta_m(r) = L_{(\frac{a-1}{2}),m}(r), \quad r > 0, \quad m \in \mathbb{N},$$

and so, for any $\sigma = m \in \mathbb{N}$, we obtain the solutions to (4.42), given by

$$\widetilde{\psi}_m(y) = \widetilde{A}_m \zeta_m\left(\frac{y^2}{4}\right) = \widetilde{A}_m L_{(\frac{a-1}{2}),m}\left(\frac{y^2}{4}\right), \quad y > 0, \tag{4.53}$$

where $\widetilde{A}_m$ is an arbitrary real constant. Note that since $\{L_{(\frac{a-1}{2}),m}\}_{m\in\mathbb{N}}$ is an orthogonal basis for the space

$$L^2(\mathbb{R}_+, d\nu) \qquad \text{where} \qquad d\nu(r) = r^{\frac{a-1}{2}}e^{-r}dr,$$

it is immediate to see that $\{\widetilde{\psi}_m\}_{m\in\mathbb{N}}$ is an orthogonal basis for $L^2(\mathbb{R}_+, d\mu_y)$.
Similarly, one could have used formula (4.52) for the explicit expression of all solutions to (4.51) and note that for $\sigma = m \in \mathbb{N}$, we have

$$M\Big(-m, \frac{1+a}{2}, r\Big) := \widetilde{Q}_m(r) = \sum_{j=0}^{m} \frac{(-m)_j}{(\frac{1+a}{2})_j}\frac{r^j}{j!}, \quad r > 0,$$

while (cfr. with Appendix 4.6)

$$\begin{aligned}\widetilde{M}\Big(-m, \frac{1+a}{2}, r\Big) &= \frac{\pi}{\sin\left(\frac{1+a}{2}\pi\right)}\left[\frac{M(-m, \frac{1+a}{2}, r)}{\Gamma(\frac{1-a}{2}-m)\Gamma(\frac{1+a}{2})} - r^{\frac{1-a}{2}}\frac{M(\frac{1-a}{2}-m, 1+\frac{1-a}{2}, r)}{\Gamma(-m)\Gamma(1+\frac{1-a}{2})}\right]\\ &= \frac{\Gamma(\frac{1-a}{2})}{\Gamma(\frac{1-a}{2}-m)}\widetilde{Q}_m(r), \quad r > 0,\end{aligned}$$

where we have used the well-known properties of the Gamma function

$$\frac{1}{\Gamma(-z)} = 0, \quad z \in \mathbb{N} \qquad \text{and} \qquad \frac{\pi}{\sin(z\pi)} = \Gamma(1-z)\Gamma(z), \quad z \notin \mathbb{Z}.$$

Consequently, it follows an equivalent expression for $\widetilde{\psi}_m = \widetilde{\psi}_m(y)$ given by

$$\widetilde{\psi}_m(y) = \widetilde{A}_m\widetilde{Q}_m\left(\frac{y^2}{4}\right) = \widetilde{A}_m\sum_{j=0}^{m}\frac{(-m)_j}{4^j j!(\frac{1+a}{2})_j}y^{2j}, \quad y > 0,$$

which is equivalent to the expression found in (4.53) in view of (4.65). In what follows we will adopt the notation used in (4.53), but always keeping in mind that

$$L_{(\frac{a-1}{2}),m}\left(\frac{y^2}{4}\right) = \sum_{j=0}^{m}\frac{(-m)_j}{4^j j!(\frac{1+a}{2})_j}y^{2j}, \quad y > 0,$$

up to a multiplicative constant.
• *Case $\sigma \notin \mathbb{N}$ but $\sigma - (1-a)/2 \in \mathbb{N}$.* If $\sigma = (1-a)/2 + m$, $m \in \mathbb{N}$, it follows

$$\begin{aligned}\widetilde{M}\Big(-\sigma, \frac{1+a}{2}, r\Big) &= \frac{\pi}{\sin\left(\frac{1+a}{2}\pi\right)}\left[\frac{M(-\sigma, \frac{1+a}{2}, r)}{\Gamma(-m)\Gamma(\frac{1+a}{2})} - r^{\frac{1-a}{2}}\frac{M(-m, 1+\frac{1-a}{2}, r)}{\Gamma(-\sigma)\Gamma(1+\frac{1-a}{2})}\right]\\ &= -\frac{2\Gamma(\frac{1+a}{2})}{(1-a)\Gamma(-\frac{1-a}{2}-m)}r^{\frac{1-a}{2}}\widetilde{P}_m(r), \quad r > 0,\end{aligned}$$

where we have employed again the properties of the Gamma function and we have defined

$$\widetilde{P}_m(r) = M\Big(-m, 1+\frac{1-a}{2}, r\Big) = \sum_{j=0}^{m}\frac{(-m)_j}{(1+\frac{1-a}{2})_j}\frac{r^j}{j!}, \quad r > 0.$$

On the other hand,

$$M\Big(-\sigma, \frac{1+a}{2}, r\Big) = \sum_{j=0}^{\infty}\frac{(-\sigma)_j}{(\frac{1+a}{2})_j}\cdot\frac{r^j}{j!} \sim \frac{\Gamma(\frac{1+a}{2})}{\Gamma(-\sigma)}e^r r^{-\sigma-\frac{1+a}{2}}, \quad \text{for } r \sim +\infty,$$

since $\sigma \notin \mathbb{N}$ (cfr. with (4.63)). From the above asymptotic expansion for $r \sim +\infty$, it is not difficult to see that the function

$$y \to M\left(-\sigma, \frac{1+a}{2}, \frac{y^2}{4}\right)$$

does not satisfies the first bound in (4.50) and so, we have to take $A_1 = 0$. Consequently, for $\sigma = (1-a)/2 + m$, $m \in \mathbb{N}$, we deduce that

$$\zeta_{a,m}(r) = \widetilde{A}_m\, r^{\frac{1-a}{2}} \widetilde{P}_m(r) = \widetilde{A}_m \sum_{j=0}^{m} \frac{(-m)_j}{j!(1+\frac{1-a}{2})_j} r^{\frac{1-a}{2}+j}, \quad r > 0,$$

and so, coming back to the variable y (we recall that $r = y^2/4$),

$$\widehat{\psi}_m(y) = \widetilde{A}_m\, y^{1-a} \widetilde{P}_m\left(\frac{y^2}{4}\right) = \widetilde{A}_m \sum_{j=0}^{m} \frac{(-m)_j}{4^j j!(1+\frac{1-a}{2})_j} y^{1+2j-a}, \quad y > 0.$$

Exactly as before, we can recall formula (4.64) to deduce $L_{(\frac{1-a}{2}),m}(r) = M(-m, 1+\frac{1-a}{2}, r) = \widetilde{P}_m(r)$ (up to a multiplicative constant), and so

$$\widehat{\psi}_m(y) = \widetilde{A}_m\, y^{1-a} L_{(\frac{1-a}{2}),m}\left(\frac{y^2}{4}\right), \quad y > 0. \tag{4.54}$$

Again, since $\{L_{(\frac{1-a}{2}),m}\}_{m\in\mathbb{N}}$ is an orthogonal basis for the space (note the difference in the measure ν w.r.t. the previous case)

$$L^2(\mathbb{R}_+, d\nu) \qquad \text{where now} \qquad d\nu(r) = r^{\frac{1-a}{2}} e^{-r} dr,$$

it follows again that $\{\widehat{\psi}_m\}_{m\in\mathbb{N}}$ is an orthogonal basis for $L^2(\mathbb{R}_+, d\mu_y)$.

• *Case $\sigma \notin \mathbb{N}$ and $\sigma - (1-a)/2 \notin \mathbb{N}$.* Proceeding as before, we get

$$\zeta(r) = \left(A_1 + \frac{\Gamma(\frac{1-a}{2})}{\Gamma(\frac{1-a}{2}-\sigma)} A_2\right) \sum_{j=0}^{\infty} \frac{(-\sigma)_j}{(\frac{1+a}{2})_j} \frac{r^j}{j!} - \frac{2\Gamma(\frac{1+a}{2})}{(1-a)\Gamma(-\sigma)} A_2\, r^{\frac{1-a}{2}} \sum_{j=0}^{\infty} \frac{(\frac{1-a}{2}-\sigma)_j}{(1+\frac{1-a}{2})_j} \frac{r^j}{j!},$$

and so, since the coefficient in front of the first series as to be zero (for the same reason of the above case), we deduce

$$\begin{aligned}\zeta(r) &= \frac{2\Gamma(\frac{1+a}{2})\Gamma(\frac{1-a}{2}-\sigma)}{(1-a)\Gamma(\frac{1-a}{2})\Gamma(-\sigma)} A_2\, r^{\frac{1-a}{2}} \sum_{j=0}^{\infty} \frac{(\frac{1-a}{2}-\sigma)_j}{(1+\frac{1-a}{2})_j} \frac{r^j}{j!} = \overline{A}_2 r^{\frac{1-a}{2}} M\left(\frac{1-a}{2} - \sigma, 1 + \frac{1-a}{2}, r\right) \\ &\sim \frac{\Gamma(1+\frac{1-a}{2})}{\Gamma(\frac{1-a}{2}-\sigma)} \overline{A}_2\, e^r r^{-1-\sigma}, \quad \text{for } r \sim +\infty.\end{aligned}$$

Exactly as before, the above expansion for $r \sim +\infty$ tells us that $\psi(y) = \zeta(y^2/4)$ does not satisfies the first bound in (4.50) and so we to take $\overline{A_2} = 0$, i.e. $A_2 = A_1 = 0$.

• *Conclusions.* From the analysis carried out before, it follows that the set of eigenvalues for equation (4.42) is given by

$$\{\widetilde{\sigma}_m\}_{m\in\mathbb{N}} \cup \{\widehat{\sigma}_m\}_{m\in\mathbb{N}} \qquad \text{where} \qquad \widetilde{\sigma}_m = m, \qquad \widehat{\sigma}_m = \frac{1-a}{2} + m,$$

and to the eigenvalue $\widetilde{\sigma}_m = m$ it corresponds (up to multiplicative constants) the eigenfunction

$$\widetilde{\psi}_m(y) = \widetilde{A}_m L_{(\frac{a-1}{2}),m}\left(\frac{y^2}{4}\right), \quad y > 0,$$

where $L_{(\frac{a-1}{2}),m}(\cdot)$ it the m^{th} Laguerre polynomial of order $(a-1)/2$, whilst the eigenvalue $\widehat{\sigma}_m = (1-a)/2+m$ possesses (up to multiplicative constants) the eigenfunction

$$\widehat{\psi}_m(y) = y^{1-a} L_{(\frac{1-a}{2}),m}\left(\frac{y^2}{4}\right), \quad y > 0.$$

We point out that the existence of two distinct orthogonal basis of eigenfunctions of $L^2(\mathbb{R}_+, d\mu_y)$ comes from the fact that $\{\widetilde{\psi}_m\}_{m\in\mathbb{N}}$ and $\{\widehat{\psi}_m\}_{m\in\mathbb{N}}$ are solutions to different eigenvalue problems. Indeed, for all $m \in \mathbb{N}$, we have

$$\widetilde{\psi}_m(0) \neq 0 \qquad \text{and} \qquad \partial_y^a \widetilde{\psi}_m = 0,$$

while

$$\widehat{\psi}_m(0) = 0 \qquad \text{and} \qquad \partial_y^a \widehat{\psi}_m \neq 0,$$

which mean that the functions $\widetilde{\psi}_m$ correspond to the eigenfunctions to equation (4.42) with homogeneous Neumann boundary condition at $y = 0$, i.e., to problem (4.44), while $\widehat{\psi}_m$ are the eigenfunctions to equation (4.42) with homogeneous Dirichlet boundary condition at $y = 0$, i.e., to problem (4.45). Note that it is not hard to verify that $\widetilde{\psi}_m$ and $\widehat{\psi}_m$ are "weak eigenfunctions" to (4.44) and (4.45), respectively, i.e. they satisfy (4.47) and (4.48), respectively. This easily follows by an integration by parts and noting that $\widetilde{\psi}_m$ and $\widehat{\psi}_m$ satisfy both bounds in (4.50). We have thus showed the following lemma.

Lemma 4.12. *Fix $-1 < a < 1$. Then the following two assertions hold:*
(i) The eigenvalues and the "weak eigenfunctions" to the homogeneous Neumann problem (4.44) *are*

$$\widetilde{\sigma}_m = m \in \mathbb{N}, \qquad \widetilde{\psi}_m(y) = \widetilde{A}_m L_{(\frac{a-1}{2}),m}\left(\frac{y^2}{4}\right), \quad y > 0,$$

where $\widetilde{A}_m \in \mathbb{R}$ is arbitrary and $L_{(\frac{a-1}{2}),m}(\cdot)$ is the m^{th} Laguerre polynomial of order $(a-1)/2$.
(ii) The eigenvalues and the "weak eigenfunctions" to the homogeneous Dirichlet problem (4.45) *are*

$$\widehat{\sigma}_m = \frac{1-a}{2} + m \in \mathbb{N}, \qquad \widehat{\psi}_m(y) = \widetilde{A}_m y^{1-a} L_{(\frac{1-a}{2}),m}\left(\frac{y^2}{4}\right), \quad y > 0,$$

where $\widetilde{A}_m \in \mathbb{R}$ is arbitrary and $L_{(\frac{1-a}{2}),m}(\cdot)$ is the m^{th} Laguerre polynomial of order $(1-a)/2$. Finally, both sets $\{\widetilde{\psi}_m\}_{m\in\mathbb{N}}$ and $\{\widehat{\psi}_m\}_{m\in\mathbb{N}}$ are orthogonal basis of $L^2(\mathbb{R}_+, d\mu_y)$.

Let us now consider equation (4.46):

$$-|y|^{-a}\left(|y|^a \psi'\right)' + \frac{y}{2}\,\psi' = \sigma\psi, \quad y \neq 0.$$

We begin our analysis by noting that if $\psi = \psi(y)$ is a "weak eigenfunction" to equation (4.46), then $\widetilde{\psi}(y) = \psi(-y)$ is a "weak eigenfunction" too, and so are all their linear combinations. Consequently, it suffices to consider solutions $\psi^+ = \psi^+(y)$ to equation (4.42) (already analyzed before) and then using their "reflections" (w.r.t. $y = 0$) to get all solutions to (4.46) defined for all $y \in \mathbb{R}$.
From the analysis carried out in the proof of Lemma 4.12, we get that the only admissible values for σ are

$$\widetilde{\sigma}_m := m \quad \text{with eigenfunctions} \quad \widetilde{\psi}_m^+(y) = \widetilde{A}_m L_{(\frac{a-1}{2}),m}\left(\frac{y^2}{4}\right), \quad m \in \mathbb{N},$$

defined for all $y > 0$, satisfying $\partial_y^a \psi_m^+ = 0$ and both bounds in (4.50), and

$$\widehat{\sigma}_m := \frac{1-a}{2} + m \quad \text{with eigenfunctions} \quad \widehat{\psi}_m^+(y) = \widetilde{A}_m y^{1-a} L_{(\frac{1-a}{2}),m}\left(\frac{y^2}{4}\right), \quad m \in \mathbb{N},$$

defined for all $y > 0$, satisfying $\widehat{\psi}_m^+(0) = 0$ and both bounds in (4.50). As always $\widetilde{A}_m \in \mathbb{R}$ and $m \in \mathbb{N}$.

We begin by focusing on $\widehat{\psi}_m^+ = \widehat{\psi}_m^+(y)$. Clearly, there are two different ways to build a solution to equation (4.46) with $\sigma = \widehat{\sigma}_m$, through "reflection" methods. An odd or an even extension:

$$\widehat{\psi}_m^o(y) := \begin{cases} \widehat{\psi}_m^+(y) & \text{if } y \geq 0 \\ -\widehat{\psi}_m^+(|y|) & \text{if } y < 0, \end{cases} \qquad \widehat{\psi}_m^e(y) := \begin{cases} \widehat{\psi}_m^+(y) & \text{if } y \geq 0 \\ \widehat{\psi}_m^+(|y|) & \text{if } y < 0. \end{cases} \tag{4.55}$$

Note that both $\widehat{\psi}_m^o$ and $\widehat{\psi}_m^e$ satisfy both bounds in (4.50) (this follows from the analysis done in Subsection 4.4.1). However, it is not difficult to see that the even extension (the second formula in (4.55)) "produces" a candidate eigenfunction $\widehat{\psi}_m^e = \widehat{\psi}_m^e(y)$ which is not a "weak eigenfunction" to problem (4.3), i.e., it does not satisfy (4.41). Indeed, using the equation of $\widehat{\psi}_m^e = \widehat{\psi}_m^e(y)$ it is not hard to show that

$$\int_{\mathbb{R}} (\widehat{\psi}_m^e)'\eta' \, d\mu_y = \widehat{\sigma}_m \int_{\mathbb{R}} \widehat{\psi}_m^e \eta \, d\mu_y - 2(1-a)\eta(0), \quad \text{for all } \eta \in H^1(\mathbb{R}, d\mu_y), \tag{4.56}$$

where we recall that $d\mu_y := |y|^a G_{a+1}(y)dy$. On the other hand, repeating the same procedure for the odd extension $\widehat{\psi}_m^o = \widehat{\psi}_m^o(y)$, we get that (4.56) holds for $\widehat{\psi}_m^o$ without the extra term $2(1-a)\eta(0)$ (it cancels thanks to the oddness of $\widehat{\psi}_m^o$), and so $\widehat{\psi}_m^o = \widehat{\psi}_m^o(y)$ is an eigenfunction associated to the eigenvalue $\widehat{\sigma}_m$.

Repeating the same procedure for $\widetilde{\psi}_m^+ = \widetilde{\psi}_m^+(y)$ it is easily seen that its even reflection is a "weak eigenfunction" corresponding to the eigenvalue $\widetilde{\sigma}_m = m$, while its odd extension cannot be a "weak eigenfunction" since it has a jump discontinuity at $y = 0$.

Consequently, we can conclude that problem (4.46) has eigenvalues and corresponding "weak eigenfunctions" defined by

$$\widetilde{\sigma}_m := m \qquad \widetilde{\psi}_m(y) = \widetilde{A}_m L_{(\frac{a-1}{2}),m}\left(\frac{y^2}{4}\right), \quad m \in \mathbb{N},$$

$$\widehat{\sigma}_m := \frac{1-a}{2} + m \qquad \widehat{\psi}_m(y) = \widetilde{A}_m \, y|y|^{-a} L_{(\frac{1-a}{2}),m}\left(\frac{y^2}{4}\right), \quad m \in \mathbb{N},$$

defined for all $y \in \mathbb{R}$. We stress that, using the parity and the oddness of $\widetilde{\psi}_m = \widetilde{\psi}_m(y)$ and $\widehat{\psi}_m = \widehat{\psi}_m(y)$, respectively, we immediately see that (up to multiplicative constants)

$$\int_{\mathbb{R}} \widetilde{\psi}_m(y)\widetilde{\psi}_n(y)d\mu_y = \int_{\mathbb{R}} \widehat{\psi}_m(y)\widehat{\psi}_n(y)d\mu_y = \delta_{m,n} \quad \text{for all } m, n \in \mathbb{N},$$

while

$$\int_{\mathbb{R}} \widetilde{\psi}_m(y)\widehat{\psi}_n(y)d\mu_y = 0 \quad \text{for all } m, n \in \mathbb{N},$$

so that it follows that the family $\{\psi_m\}_{m\in\mathbb{N}} \cup \{\widehat{\psi}_m\}_{m\in\mathbb{N}}$ is an orthogonal set of $L^2(\mathbb{R}, d\mu_y)$. The fact that the family $\{\psi_m\}_{m\in\mathbb{N}} \cup \{\widehat{\psi}_m\}_{m\in\mathbb{N}}$ is a basis of $L^2(\mathbb{R}, d\mu_y)$ follows since both $\{\widetilde{\psi}_m\}_{m\in\mathbb{N}}$ and $\{\widehat{\psi}_m\}_{m\in\mathbb{N}}$ are orthogonal basis of $L^2(\mathbb{R}_+, d\mu_y)$. We thus have proved the following lemma.

Lemma 4.13. *Fix $-1 < a < 1$. Then the set of eigenvalues of problem* (4.46) *is*

$$\{\widetilde{\sigma}_m\}_{m\in\mathbb{N}} \cup \{\widehat{\sigma}_m\}_{m\in\mathbb{N}},$$

where $\widetilde{\sigma}_m$ and $\widehat{\sigma}_m$ correspond to the eigenvalues of the Neumann problem (4.44) *and of the Dirichlet on* (4.45), *respectively, and are defined in Lemma 4.12. Moreover, the eigenfunctions corresponding to $\widetilde{\sigma}_m$ are*

$$\widetilde{\psi}_m(y) = \widetilde{A}_m L_{(\frac{a-1}{2}),m}\left(\frac{y^2}{4}\right), \quad y \in \mathbb{R},$$

where $\widetilde{A}_m \in \mathbb{R}$ is arbitrary and $L_{(\frac{a-1}{2}),m}(\cdot)$ is the m^{th} Laguerre polynomial of order $(a-1)/2$, while the eigenfunctions corresponding to $\widehat{\sigma}_m$ are

$$\widehat{\psi}_m(y) = \widetilde{A}_m \, y|y|^{-a} L_{(\frac{1-a}{2}),m}\left(\frac{y^2}{4}\right), \quad y \in \mathbb{R},$$

where $\widetilde{A}_m \in \mathbb{R}$ is arbitrary and $L_{(\frac{1-a}{2}),m}(\cdot)$ is the m^{th} Laguerre polynomial of order $(1-a)/2$. Finally, the set $\{\widetilde{\psi}_m\}_{m\in\mathbb{N}} \cup \{\widehat{\psi}_m\}_{m\in\mathbb{N}}$ is an orthogonal basis of $L^2(\mathbb{R}, d\mu_y)$.

4.4.2 End of the proof of Theorem 4.1

We begin by proving part (i) and (ii). As in Section 4.2 and following again the ideas of [109], we look for solutions to problems (4.1) and/or (4.2) in "separate variables" form $V(x,y) = \varphi(x)\psi(y)$, $x \in \mathbb{R}^N$, $y > 0$. So, substituting into (4.1) and/or (4.2), it is not difficult to get

$$\left(-\Delta_x\varphi + \frac{x}{2}\cdot\nabla_x\varphi\right)\psi + \left[-y^{-a}\left(y^a\psi'\right)' + \frac{y}{2}\,\psi'\right]\varphi = \kappa\varphi\psi \quad \text{in } \mathbb{R}^{N+1}_+,$$

where, as always, $\psi' = d\psi/dy$. Hence, we recover the eigenvalue $\kappa \in \mathbb{R}$ as the sum of $\nu \in \mathbb{R}$ and $\sigma \in \mathbb{R}$ eigenvalues to

$$-\Delta_x\varphi + \frac{x}{2}\cdot\nabla_x\varphi = \nu\varphi, \quad x \in \mathbb{R}^N \tag{4.57}$$

and equation (4.42):

$$-y^{-a}\left(y^a\psi'\right)' + \frac{y}{2}\,\psi' = \sigma\psi, \quad y > 0,$$

respectively.

Step1: Analysis of (4.57). Evidently (4.57) is the classical Ornstein-Uhlenbeck eigenvalue problem in the all Euclidean space. It possesses the sequence of eigenvalues

$$\nu_n = \frac{n}{2}, \quad n \in \mathbb{N} = \{0, 1, \ldots\},$$

and a eigenfunction basis composed by the so-called Hermite polynomials

$$H_\alpha(x) = H_{n_1}(x_1)\ldots H_{n_N}(x_N), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N,$$

where $\alpha = (n_1, \ldots, n_N) \in \mathbb{Z}^N_{\geq 0}$ and $H_{n_j}(\cdot)$, $j = 1, \ldots, N$, is the n_j^{th} 1 dimensional Hermite polynomial (cfr. with [25, 200] and Appendix 4.6).

Step2: Analysis of (4.42) *and conclusion.* The complete analysis has bee carried out in Subsection 4.4.1. Consequently, part (i) and (ii) of Theorem 4.1 follow by combining the above mentioned facts about Hermite polynomials and classical Ornstein-Uhlenbeck eigenvalue problem, with the statement of Lemma 4.12. In particular, the fact that $\widetilde{V}_{\alpha,m}(x,y) = H_\alpha(x)L_{(\frac{a-1}{2}),m}(y)$ and $\widehat{V}_{\alpha,m}(x,y) = H_\alpha(x)\,y^{1-a}L_{(\frac{1-a}{2}),m}(y)$ form an orthogonal basis of $L^2(\mathbb{R}^{N+1}_+, d\mu)$ comes from the fact that both $H_\alpha(x)$ and $L_{(\frac{a-1}{2}),m}(y)$ (resp. $y^{1-a}L_{(\frac{1-a}{2}),m}(y)$) are orthogonal basis for $L^2(\mathbb{R}^N, d\mu_x)$ and $L^2(\mathbb{R}_+, d\mu_y)$, respectively, since the measure $d\mu = d\mu(x,y)$ defined on $\mathbb{R}^{N+1}_+$ is obtained the measure product of $d\mu_x$ and $d\mu_y$.

Let us now focus on part (iii). Exactly as before we look for solutions to problem (4.3) with form $V(x,y) = \varphi(x)\psi(y)$, $x \in \mathbb{R}^N$, $y \in \mathbb{R}$ and we obtain that $V = V(x,y)$ is an eigenfunction with eigenvalue $\kappa = \nu + \sigma$ if $\varphi = \varphi(x)$ satisfies (4.57) and $\psi = \psi(y)$ satisfies (4.46):

$$-|y|^{-a}\left(|y|^a\psi'\right)' + \frac{y}{2}\,\psi' = \sigma\psi, \quad y \neq 0.$$

Since the analysis of the above equation has already been done, we deduce part (iii) of Theorem 4.1 exactly as before, but in this case we apply Lemma 4.13 instead of Lemma 4.12. We end the proof by stressing that the family $\{\widetilde{V}_{\alpha,m}(x,y)\}_{(\alpha,m)} \cup \{\widehat{V}_{\alpha,m}(x,y)\}_{(\alpha,m)}$ is an orthogonal basis of $L^2(\mathbb{R}^{N+1}, d\mu)$. This follows exactly as before. □

4.5 Some Gaussian-Poincaré type inequalities and application

In this section we study some applications of the spectral analysis carried out above. In particular, we will show three Gaussian-Poincaré type inequalities that will easily follows from the spectral decomposition of the space L^2_μ through the eigenfunctions to problems (4.1), (4.2) and (4.3). There is wide literature Gaussian-Poincaré inequalities. The ones we get in Lemma 4.14 and Theorem 4.15 are not completely new (cfr. for instance with the works of Talenti and Tomaselli [187] and [191], or even Muckenhoupt [156] and Barthe and Roberto [22]). However, w.r.t. the above quoted papers, we will obtain *sharp* inequalities with optimal constants. This is essentially due to the availability of a orthogonal basis of eigenfunctions constructed in the previous section.
We end the paragraph by stressing that these kind of inequalities have a self interest (as the literature show) and, moreover, they will play an important role in the proof of some Liouville type theorems, essential tools to get L^∞_{loc} convergence of the blow-up sequences. Out of clarity, we proceed with the one dimensional case.

4.5.1 One-dimensional Gaussian Poincaré inequality

In Subsection 4.4.1 we have studied eigenvalue problem (4.46):

$$-|y|^{-a}\left(|y|^a\psi'\right)' + \frac{y}{2}\,\psi' = \sigma\psi, \quad y \neq 0,$$

where the above equation is intended in the "weak" sense (cfr. with Definition 4.11) and we have proved Lemma 4.13 which completely characterizes the spectrum of the above equation. In particular, it provides a basis of eigenfunctions $\{\psi_m\}_{m\in\mathbb{N}} = \{\widetilde{\psi}_m\}_{m\in\mathbb{N}} \cup \{\widehat{\psi}_m\}_{m\in\mathbb{N}}$ for the space $L^2(\mathbb{R}, d\mu_y)$ (as in Lemma 4.13), where, as always,

$$d\mu_y = \frac{1}{2^{1+a}\Gamma(\frac{1+a}{2})}|y|^a e^{-\frac{y^2}{4}}\,dy.$$

Let us highlight a very elementary but crucial fact. We know that $\{\psi_m\}_{m\in\mathbb{N}} \subset H^1(\mathbb{R}, d\mu_y)$ and

$$\int_{\mathbb{R}} \psi_m'\psi'\,d\mu_y = \sigma_m \int_{\mathbb{R}} \psi_m\psi\,d\mu_y, \quad \text{for all } m \in \mathbb{N},$$

and for any $\psi \in H^1(\mathbb{R}, d\mu_y)$, where $\{\sigma_m\}_{m\in\mathbb{N}} = \{\widetilde{\sigma}_m\}_{m\in\mathbb{N}} \cup \{\widehat{\sigma}_m\}_{m\in\mathbb{N}}$ (as in Lemma 4.13). From the above integral characterization, we easily deduce

$$\int_{\mathbb{R}} \psi_m'\psi_n'\,d\mu_y = \sigma_m \int_{\mathbb{R}} \psi_m\psi_n\,d\mu_y = \sigma_m\delta_{m,n}, \quad \text{for all } m,n \in \mathbb{N}, \tag{4.58}$$

up to "normalization", thanks to the $L^2(\mathbb{R}, d\mu_y)$-orthogonality of the eigenfunctions ($\delta_{n,m}$ denotes the Kronecker delta). This means that not only the eigenfunctions $\psi_m = \psi_m(y)$ are orthogonal in $L^2(\mathbb{R}, d\mu_y)$, but also in $H^1(\mathbb{R}, d\mu_y)$. We end this paragraph by stressing that (cfr. with Subsection 4.2.2) the space $L^2(\mathbb{R}, d\mu_y)$ is the closure of the space $C_c^\infty(\mathbb{R})$ w.r.t. to the norm

$$\|\psi\|^2_{L^2_{\mu_y}} := \int_{\mathbb{R}} \psi^2(y)\,d\mu_y,$$

while $H^1(\mathbb{R}, d\mu_y)$ is the closure of the same space but w.r.t. to the norm

$$\|\psi\|^2_{H^1_{\mu_y}} := \int_{\mathbb{R}} \psi^2(y)\,d\mu_y + \int_{\mathbb{R}} (\psi')^2(y)\,d\mu_y.$$

A natural consequence of the spectral analysis is the following Gaussian Poincaré type inequality.

Lemma 4.14. *The following three statements hold:*
(i) For any $\psi \in H^1(\mathbb{R}, d\mu_y)$, it holds

$$\int_{\mathbb{R}} \psi^2 \, d\mu_y - \left(\int_{\mathbb{R}} \psi \, d\mu_y \right)^2 \leq \frac{2}{1-a} \int_{\mathbb{R}} (\psi')^2 \, d\mu_y.$$

Furthermore, the equality is attained if and only if $\psi(y) = A$ or $\psi(y) = Ay|y|^{-a}$, $A \in \mathbb{R}$.
(ii) For any $\psi \in H_0^1(\mathbb{R}_+, d\mu_y)$, it holds

$$\int_{\mathbb{R}_+} \psi^2 \, d\mu_y \leq \frac{2}{1-a} \int_{\mathbb{R}_+} (\psi')^2 \, d\mu_y.$$

Furthermore, the equality is attained if and only if $\psi(y) = Ay^{1-a}$, $A \in \mathbb{R}$.
(iii) For any $\psi \in H^1(\mathbb{R}_+, d\mu_y)$, it holds

$$\int_{\mathbb{R}_+} \psi^2 \, d\mu_y - \left(\int_{\mathbb{R}_+} \psi \, d\mu_y \right)^2 \leq \int_{\mathbb{R}_+} (\psi')^2 \, d\mu_y.$$

Furthermore, the equality is attained if and only if $\psi(y) = A$ or $\psi(y) = A(1 - \frac{1-a}{2} - \frac{y^2}{4})$, $A \in \mathbb{R}$.

Proof. Take $\psi \in H^1(\mathbb{R}, d\mu_y)$ with $\overline{\psi} := \int_{\mathbb{R}} \psi \, d\mu_y$, and consider $\Psi_M := \sum_{m=1}^M c_m \psi_m$ such that

$$\Psi_M \to \psi - \overline{\psi} \quad \text{in } L^2(\mathbb{R}, d\mu_y),$$

where, of course, $c_m := \int_{\mathbb{R}} \psi \psi_m \, d\mu_y$. Note that the sum Ψ_M starts from the first nonconstant eigenfunction, since $\psi - \overline{\psi}$ is orthogonal to the eigenspace generated by the constants. Now, using the definition of "weak eigenfunction" and the orthogonality condition in (4.58), it is not difficult to see that

$$\int_{\mathbb{R}} (\Psi_M')^2 \, d\mu_y = \int_{\mathbb{R}} \Psi_M' \psi' \, d\mu_y \quad \text{for all } M \in \mathbb{N}_0,$$

i.e. Ψ_M' and $\Psi_M' - \psi'$ are orthogonal in $L^2(\mathbb{R}, d\mu_y)$. Consequently, we have

$$0 \leq \int_{\mathbb{R}} (\psi' - \Psi_M')^2 d\mu_y = \int_{\mathbb{R}} (\psi')^2 d\mu_y - 2 \int_{\mathbb{R}} \Psi_M' \psi' d\mu_y + \int_{\mathbb{R}} (\Psi_M')^2 d\mu_y = \int_{\mathbb{R}} (\psi')^2 d\mu_y - \int_{\mathbb{R}} (\Psi_M')^2 d\mu_y,$$

and so $\int_{\mathbb{R}} (\Psi_M')^2 d\mu_y \leq \int_{\mathbb{R}} (\psi')^2 d\mu_y < +\infty$, uniformly in $M \in \mathbb{N}_0$ (note that here we have used the fact that $\psi \in H^1(\mathbb{R}, d\mu_y)$). Finally, from the above bound and employing (4.58) again, we have

$$\begin{aligned}\int_{\mathbb{R}} (\psi')^2 d\mu_y \geq \int_{\mathbb{R}} (\Psi_M')^2 \, d\mu_y &= \sum_{m=1}^M c_m^2 \int_{\mathbb{R}} (\psi_m')^2 \, d\mu_y = \sum_{m=1}^M \sigma_m c_m^2 \int_{\mathbb{R}} \psi_m^2 \, d\mu_y \\ &\geq \underline{\sigma} \sum_{m=1}^M c_m^2 \int_{\mathbb{R}} \psi_m^2 \, d\mu_y = \underline{\sigma} \int_{\mathbb{R}} \Psi_M^2 \, d\mu_y,\end{aligned}$$

where we have set $\underline{\sigma} := \min_{m \in \mathbb{N}_0} \{\sigma_m\} = \widehat{\sigma}_0 = (1-a)/2$. Consequently, passing to the limit as $M \to +\infty$, we deduce

$$\int_{\mathbb{R}} (\psi')^2 \, d\mu_y \geq \int_{\mathbb{R}} (\psi - \overline{\psi})^2 \, d\mu_y = \int_{\mathbb{R}} \psi^2 \, d\mu_y - \left(\int_{\mathbb{R}} \psi \, d\mu_y \right)^2,$$

obtaining the desired inequality. The last part of the statement follows from the fact that $\psi(y) = Ay|y|^{-a}$ is the first non constant eigenfunction to problem (4.46) (with eigenvalue $\widehat{\sigma}_0 = (1-a)/2$).

The proof of (ii) and (iii) is very similar to the one of part (i) and we omit it (note that we just have to use the bases of eigenfunctions $\{\widehat{\psi}_m\}_{m\in\mathbb{N}}$ for (ii) and $\{\widetilde{\psi}_m\}_{m\in\mathbb{N}}$ for (iii), instead of $\{\psi_m\}_{m\in\mathbb{N}}$). We just mention that in (ii) we work with the space $H^1_0(\mathbb{R}, d\mu_y)$ which does not contain constant functions different from the trivial one (this is the reason because we do not need to employ the average of ψ in Gaussian Poincaré inequality), while in (iii) it is easily seen that the first nonconstant eigenfunction $\psi(y) = A(1 - \frac{1-a}{2} - \frac{y^2}{4})$, $A \in \mathbb{R}$ has zero mean (w.r.t. to the measure $d\mu_y$). Note that since we are now working with functions defined for $y > 0$, the normalization constant of the probability measure $d\mu_y$ is different from the one of Lemma 4.14 (cfr. with Subsection 4.2.1). □

4.5.2 (N+1)-dimensional Gaussian Poincaré inequality

For the general case, we proceed as before. We consider problem (4.3):

$$-\mathcal{L}_a V + \frac{(x,y)}{2} \cdot \nabla V = \kappa V \quad \text{in } \mathbb{R}^{N+1}.$$

From Theorem 4.1 we know that it has the set of eigenvalues $\{\kappa_{n,m}\}_{n,m\in\mathbb{N}} = \{\widehat{\kappa}_{n,m}\}_{n,m\in\mathbb{N}} \cup \{\widetilde{\kappa}_{n,m}\}_{n,m\in\mathbb{N}}$ where

$$\widetilde{\kappa}_{n,m} := \frac{n}{2} + m, \qquad \widehat{\kappa}_{n,m} := \frac{n}{2} + m + \frac{1-a}{2} \qquad \text{for all } n, m \in \mathbb{N},$$

while the set of eigenfunctions is $\{V_{\alpha,m}\}_{(\alpha,m)} = \{\widetilde{V}_{\alpha,m}\}_{(\alpha,m)} \cup \{\widehat{V}_{\alpha,m}\}_{(\alpha,m)}$, where the eigenfunctions can be of two types

$$\widetilde{V}_{\alpha,m}(x,y) = H_\alpha(x)\widetilde{\psi}_m(y), \quad \widehat{V}_{\alpha,m}(x,y) = H_\alpha(x)\widehat{\psi}_m(y), \qquad \text{for all } (\alpha, m) \in \mathbb{Z}^N_{\geq 0} \times \mathbb{N},$$

where as before $\widetilde{\psi}_m(y) = L_{(\frac{a-1}{2}),m}(y^2/4)$ and $\widehat{\psi}_m(y) = y|y|^{-a} L_{(\frac{1-a}{2}),m}(y^2/4)$ and $H_\alpha(\cdot)$ is a N-dimensional Hermite polynomial of order $|\alpha|$. We recall that similarly to the 1-dimensional case the set $\{V_{\alpha,m}\}_{(\alpha,m)}$ is an orthogonal basis of $L^2(\mathbb{R}^{N+1}, d\mu)$, where

$$d\mu(x,y) = \frac{1}{2^{1+a}\Gamma(\frac{1+a}{2})(4\pi)^{N/2}} |y|^a e^{-\frac{|x|^2+y^2}{4}} dxdy.$$

Since the sets of eigenvalues and eigenfunctions are countable, we can drop one index and denote by κ_j the j^{th} eigenvalue with corresponding eigenfunctions V_j, $j \in \mathbb{N}$. In this setting, we have that the first nonzero eigenvalue depends on the parameter $-1 < a < 1$:

$$\nu_* := \min_{j\in\mathbb{N}} \{\kappa_j \neq 0\} = \frac{1}{2} \min\{1, 1-a\}.$$

This is the unique remarkable difference w.r.t. the 1-dimensional case. We thus have the following theorem.

Theorem 4.15. *The following three statements hold:*
(i) For any $V \in H^1(\mathbb{R}^{N+1}, d\mu)$, it holds

$$\int_{\mathbb{R}^{N+1}} V^2 \, d\mu - \left(\int_{\mathbb{R}^{N+1}} V \, d\mu \right)^2 \leq P_a \int_{\mathbb{R}^{N+1}} |\nabla V|^2 \, d\mu,$$

where $P_a := 1/\nu_* = 2/\min\{1, 1-a\}$. Furthermore, the equality is attained if and only if $V(x,y) = A$ or, depending on $-1 < a < 1$:

$$V(x,y) = \begin{cases} Ax_j & \text{if } a < 0 \quad \text{for some } j \in \{1, \dots, N\} \\ Ax_j & \text{if } a = 0 \quad \text{for some } j \in \{1, \dots, N+1\} \\ Ay|y|^{-a} & \text{if } a > 0, \end{cases}$$

where $A \in \mathbb{R}$ and we have used the convention $x_{N+1} = y$.
(ii) For any $V \in H^1_0(\mathbb{R}^{N+1}_+, d\mu)$, it holds

$$\int_{\mathbb{R}^{N+1}_+} V^2\, d\mu \leq \frac{2}{1-a} \int_{\mathbb{R}^{N+1}_+} |\nabla V|^2\, d\mu.$$

Furthermore, the equality is attained if and only if $V(x,y) = Ay^{1-a}$, $A \in \mathbb{R}$.
(iii) For any $V \in H^1(\mathbb{R}^{N+1}_+, d\mu)$, it holds

$$\int_{\mathbb{R}^{N+1}_+} V^2\, d\mu - \left(\int_{\mathbb{R}^{N+1}_+} V\, d\mu\right)^2 \leq 2 \int_{\mathbb{R}^{N+1}_+} |\nabla V|^2\, d\mu.$$

Furthermore, the equality is attained if and only if $V(x,y) = A$ or $V(x,y) = Ax_j$, for some $j \in \{1,\dots,N\}$ and $A \in \mathbb{R}$.

Remark. Note that, since the Gaussian-Poincaré constants do not depend on the spacial dimension N, they can be extended to infinite dimensional spaces (cfr. with Beckner [23]).

Proof. We begin by proving part (i), following the ideas of the case $N = 1$. Take $V \in H^1(\mathbb{R}^{N+1}, d\mu)$ with $\overline{V} := \int_{\mathbb{R}^{N+1}} V\, d\mu$, and we approximate it with the sequence $\Psi_J := \sum_{j=1}^J c_j V_j$ such that

$$\Psi_J \to V - \overline{V} \quad \text{in } L^2(\mathbb{R}^{N+1}, d\mu), \qquad c_j := \int_{\mathbb{R}^{N+1}} V_j V\, d\mu.$$

Proceeding as before, we easily find that

$$\int_{\mathbb{R}^{N+1}} |\nabla \Psi_J|^2\, d\mu = \int_{\mathbb{R}^{N+1}} \nabla \Psi_J \nabla V\, d\mu \quad \text{for all } J \in \mathbb{N}_0,$$

and so $\int_{\mathbb{R}^{N+1}} |\nabla \Psi_J|^2\, d\mu \leq \int_{\mathbb{R}^{N+1}} |\nabla V|^2\, d\mu$. Hence,

$$\begin{aligned}\int_{\mathbb{R}^{N+1}} |\nabla V|^2 d\mu \geq \int_{\mathbb{R}^{N+1}} |\nabla \Psi_J|^2\, d\mu &= \sum_{j=1}^J c_j^2 \int_{\mathbb{R}^{N+1}} |\nabla V_j|^2\, d\mu = \sum_{j=1}^J \sigma_j c_j^2 \int_{\mathbb{R}^{N+1}} V_j^2\, d\mu \\ &\geq \nu_* \sum_{j=1}^J c_j^2 \int_{\mathbb{R}^{N+1}} V_j^2\, d\mu = \nu_* \int_{\mathbb{R}^{N+1}} \Psi_J^2\, d\mu,\end{aligned}$$

where now the minimum of the eigenvalues $\nu_* := \min_j\{\kappa_j \neq 0\} = \min\{1, 1-a\}/2$. Passing to the limit as $J \to \infty$ we get the inequality of statement (i).
To prove the second part of the statement, let us firstly fix $-1 < a < 0$. In this case we have $\nu_* = 1/2$ and the corresponding eigenfunctions are $V(x,y) = Ax_j$ for $j \in \{1,\dots,N\}$. We thus conclude by the definition of "weak eigenfunction" and the fact that $V(x,y) = Ax_j$ has mean zero for any $j \in \{1,\dots,N\}$ (w.r.t. the measure $d\mu$). Similarly, if $0 < a < 1$ it turns out that $\nu_* = (1-a)/2$ and the corresponding eigenfunctions are $V(x,y) = Ay|y|^{-a}$ (also in this case they have mean zero). Finally, if $a = 0$ we have $\nu_* = 1/2$ and the eigenfunctions are $V(x,y) = Ax_j$ for $j \in \{1,\dots,N\}$ and $V(x,y) = Ay$ (note that we get back the classical statement, cfr. for instance with [23]). This conclude the proof of part (i).

To prove part (ii) and (iii) we employ the bases of eigenfunctions $\{\widehat{V}_j\}_j = \{\widehat{V}_{(\alpha,m)}\}_{(\alpha,m)}$ for (ii) and $\{\widetilde{V}_j\}_j = \{\widetilde{V}_{(\alpha,m)}\}_{(\alpha,m)}$ for (iii), instead of $\{V_j\}_j = \{V_{(\alpha,m)}\}_{(\alpha,m)}$. Note that w.r.t. the proof of Lemma 4.14, the first eigenvalue to the Neumann problem (4.2) is $\nu_* = 1/2$ with corresponding eigenfunctions $V(x,y) = Ax_j$ for $j \in \{1,\dots,N\}$. □

4.6 Appendix: Hermite and Laguerre polynomials

In this brief appendix, we recall some very well known facts about the spectrum of the eigenvalue problems

$$-\Delta\varphi + \frac{x}{2}\cdot\nabla\varphi = \nu\varphi, \quad x \in \mathbb{R}^N, \quad \nu \in \mathbb{R}, \tag{4.59}$$

and

$$-r\frac{d^2\zeta}{dr} - (1+\alpha-r)\frac{d\zeta}{dr} = \sigma\zeta, \quad r > 0, \quad \mu \in \mathbb{R}, \tag{4.60}$$

known in literature as Ornstein-Uhlenbeck and Laguerre eigenvalue problems, respectively. The basic reference is the classical book of Szegö [186]. For problem (4.59), we will refer also to the more recent work [25, 200].

Review for problem (4.59). We divide the presentation in two cases, depending on the dimension $N = 1$ or $N \geq 2$.

• *Case* $N = 1$. As explained in Section 7 of the recent survey [200], the set of eigenvalues of problem (4.59) is given by the half nonnegative integers

$$\nu_n = \frac{n}{2}, \quad n \in \mathbb{N} = \{0, 1, \ldots\},$$

and the corresponding eigenfunctions are the Hermite polynomials $\{H_n\}_{n\in\mathbb{N}}$, given by the compact formula (see Chapter 5 of [186])

$$\begin{aligned} \widetilde{H}_n(x) &= \frac{(-1)^n}{e^{-x^2}}\frac{d^n}{dx^n}\left(e^{-x^2}\right), \\ H_n(x) &= \widetilde{H}_n\left(\frac{x}{2}\right) \qquad\qquad x \in \mathbb{R}, \quad n \in \mathbb{N}, \end{aligned}$$

or the recursive one

$$\begin{aligned} \widetilde{H}_n(x) &= \left(2x - \frac{d}{dx}\right)\widetilde{H}_{n-1}(x), \\ H_n(x) &= \widetilde{H}_n\left(\frac{x}{2}\right) \qquad\qquad x \in \mathbb{R}, \quad n \in \mathbb{N}_0, \end{aligned}$$

where $\widetilde{H}_0(x) = 1$. For instance:

$$H_0(x) = 1, \quad H_1(x) = x, \quad H_2(x) = x^2 - 2, \quad H_3(x) = x^3 - 6x,$$

and so on, and their generating function is

$$e^{\vartheta(x-\vartheta)} = \sum_{n=0}^{\infty} \vartheta^n \frac{H_n(x)}{n!}, \quad x \in \mathbb{R}.$$

Finally, the family $\{H_n\}_{n\in\mathbb{N}}$ is an orthogonal basis of $L^2(\mathbb{R}, d\mu_x^1)$, where

$$d\mu_x^1 := \frac{1}{\sqrt{4\pi}}e^{-\frac{x^2}{4}}dx,$$

in the sense that if $\varphi = \varphi(x)$ belongs to $L^2(\mathbb{R}, d\mu_x^1)$, i.e., it is Lebesgue measurable and satisfies

$$\int_{\mathbb{R}} \varphi^2(x) d\mu_x^1 < +\infty,$$

then

$$\varphi(\cdot) = \sum_{n=0}^{\infty} \varphi_n H_n(\cdot) \quad \text{in } L^2(\mathbb{R}, d\mu_x^1),$$

where $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ (see Chapter 5 of [186]) and they are orthogonal in the sense that

$$\int_{\mathbb{R}} \varphi_m(x)\varphi_n(x) d\mu_x^1 = 2^n n!\, \delta_{n,m}, \quad n, m \in \mathbb{N}.$$

• *Case* $N \geq 2$. In the multidimensional case, for any multi-index $\alpha = (n_1, \ldots, n_N) \in \mathbb{Z}^N_{\geq 0}$, we set:

$$H_\alpha(x) = H_{n_1}(x_1) \ldots H_{n_N}(x_N), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N, \tag{4.61}$$

where $H_{n_j}(\cdot)$, $j = 1, \ldots, N$, is the n_j^{th} 1-dimensional Hermite polynomial. Then the set $\{H_\alpha\}_{\alpha\in\mathbb{Z}^N_{\geq 0}}$ (see Theorem 11 of [25]) is an orthogonal basis of $L^2(\mathbb{R}^N, d\mu_x)$, where

$$d\mu_x = \frac{1}{(4\pi)^{N/2}} e^{-\frac{|x|^2}{4}}, \quad x \in \mathbb{R}^N.$$

The "decomposition" in (4.61), is due to the "factorization" property of the operator

$$\mathcal{O}_x\varphi := -\Delta_x\varphi + \frac{x}{2}\cdot\nabla_x\varphi.$$

Indeed, if $x = (x_1, x_2) \in \mathbb{R}^N$ and $\varphi(x) = \varphi_1(x_1)\varphi_2(x_2)$, it is straightforward to see that

$$\mathcal{O}_x\varphi = \mathcal{O}_x(\varphi_1\varphi_2) = \varphi_2\, \mathcal{O}_{x_1}\varphi_1 + \varphi_1\, \mathcal{O}_{x_2}\varphi_2,$$

and so, we recover the eigenvalues $\nu_{\mathcal{O}_x}$ as the sum of the eigenvalues $\nu_{\mathcal{O}_{x_1}}$ and $\nu_{\mathcal{O}_{x_2}}$, while the eigenfunctions $\varphi(x)$ as the product of φ_1 and φ_2. From this observation it follows in particular that the eigenvalues of $\mathcal{O}_x$ are the nonnegative half-integers even when the space dimension is higher than one.
Finally, the space generated by the eigenfunctions of "order n":

$$\mathcal{V}_{|\alpha|} := \text{span}\{H_\alpha : |\alpha| := n_1 + \ldots + n_N = n\}$$

is a subspace of $L^2(\mathbb{R}^N, d\mu_x)$ of dimension

$$\dim \mathcal{V}_\alpha = \binom{n + N - 1}{n}.$$

For instance, we have $\mathcal{V}_0 = \mathbb{R}$, $\mathcal{V}_1 = \text{span}\{x_j : j = 1, \ldots, N\}$, and so on.

Review for problem (4.60). For $\alpha > -1$, we consider solutions to (4.60) defined on the half-real line $\mathbb{R}_+ = (0, +\infty)$ or $\overline{\mathbb{R}}_+ = [0, +\infty)$.
Let us firstly note that equation (4.60) is a Kummer confluent hypergeometric equation (cfr. with Chapter 13 of [1]):

$$r\frac{d^2\zeta}{dr^2} + (b_2 - r)\frac{d\zeta}{dr} - b_1\zeta = 0, \quad r > 0, \tag{4.62}$$

with $b_1 = -\sigma$ and $b_2 = 1 + \alpha$. For any $b_1 \in \mathbb{R}$ and $b_2 \neq 0$, equation (4.62) possesses two independent solutions

$$M(b_1, b_2, r) = \sum_{j=0}^{\infty} \frac{(b_1)_j}{(b_2)_j}\frac{r^j}{j!}, \quad r > 0$$

where $(b)_j := b(b+1)\dots(b+j-1)$, $(b)_0 := 1$ and $(\cdot)_j$ denotes the Pochhammer's symbol (i.e. the rising factorial), and

$$\widetilde{M}(b_1,b_2,r) = \frac{\pi}{\sin b_2\pi}\left[\frac{M(b_1,b_2,r)}{\Gamma(1+b_1-b_2)\Gamma(b_2)} - r^{1-b_2}\frac{M(1+b_1-b_2,2-b_2,r)}{\Gamma(b_1)\Gamma(2-b_2)}\right], \quad r>0.$$

The function $M(b_1,b_2,\cdot)$ is called Kummer function (or confluent hypergeometric function) while $\widetilde{M}(b_1,b_2,\cdot)$ is known as Tricomi function (or confluent hypergeometric function of the second kind). It thus follows that all solutions to (4.62) are given by

$$\zeta(r) = A_1 M(b_1,b_2,r) + A_2\widetilde{M}(b_1,b_2,r), \quad r>0,$$

where $A_1, A_2 \in \mathbb{R}$. This two independent solutions have some remarkable properties that are needed in our proofs. We briefly recall the most important ones.

• *Properties of the Kummer function.* First of all, we note that $M(b_1,b_2,r)$ is finite for $r \sim 0^+$, while singular for $r \sim +\infty$, depending on $b_1 \in \mathbb{R}$.

So, if $-b_1 = m \in \mathbb{N} = \{0,1,\dots\}$, then $M(-m,b_2,r)$ is a polynomial in $r>0$:

$$M(-m,b_2,r) = \sum_{j=0}^{m}\frac{(-m)_j}{(b_2)_j}\frac{r^j}{j!}, \quad r>0.$$

If $-b_1 \notin \mathbb{N}$, the function $M(b_1,b_2,\cdot)$ behaves at infinity as follows (cfr. with formula 13.1.4 of [1]):

$$M(b_1,b_2,r) \sim \frac{\Gamma(b_2)}{\Gamma(b_1)}e^r r^{b_1-b_2}, \quad \text{for } r\sim+\infty. \tag{4.63}$$

• *Properties of the Tricomi function.* From the expression of $\widetilde{M}(b_1,b_2\cdot)$, it follows that the Tricomi function can be singular for $r\sim 0^+$. In particular, if $b_2>1$, it holds (cfr. with formulas 13.5.6, 13.5.7, 13.5.8 of [1])

$$\widetilde{M}(b_1,b_2,r) \sim \frac{\Gamma(b_2-1)}{\Gamma(b_1)}r^{1-b_2}, \quad \text{for } r\sim 0^+. \tag{4.64}$$

Now, coming back to equation (4.60):

$$-r\frac{d^2\zeta}{dr} - (1+\alpha-r)\frac{d\zeta}{dr} = \sigma\zeta, \quad r>0, \quad \sigma\in\mathbb{R},$$

it is well-known that for $\sigma = m \in \mathbb{N}$ (i.e. $-b_1\in\mathbb{N}$ with the notation of equation (4.62)), there is a unique solution to (4.60) denoted with

$$\zeta(r) = L_{(\alpha),m}(r), \quad r>0, \quad m\in\mathbb{N}$$

and called m^{th} Laguerre polynomial of order $\alpha > -1$ (we mention that the usual way to refer to Laguerre polynomials is $L_m^{(\alpha)}$ and here we change notation to simplify the writing). It is well-known that

$$L_{(\alpha),m}(r) = \binom{m+\alpha}{m}M(-m,1+\alpha,r), \quad r>0, \quad m\in\mathbb{N}, \tag{4.65}$$

where $M(\cdot,\cdot,\cdot)$ denotes the Kummer function introduced before. As for the Hermite, there are different ways for generating all Laguerre polynomials, like the so called Rodrigues formula (cfr. with Chapter 5 of [186]):

$$L_{(\alpha),m}(r) = \frac{1}{m!\,r^\alpha e^{-r}}\frac{d^m}{dr^m}\left(r^{\alpha+m}e^{-r}\right), \quad r>0, \quad m\in\mathbb{N},$$

and/or

$$L_{(\alpha),m}(r) = \sum_{j=0}^{m} (-1)^j \binom{m+\alpha}{m-j} \frac{r^j}{j!}, \quad r > 0, \quad m \in \mathbb{N},$$

(cfr. with the expression given in terms of Kummer and Tricomi functions). For instance:

$$L_{(\alpha),0}(r) = 1, \quad L_{(\alpha),1}(r) = 1 + \alpha - r, \quad L_{(\alpha),2}(r) = \frac{(1+\alpha)(2+\alpha)}{2} - (2+\alpha)r + \frac{r^2}{2},$$

and so on. Moreover, it will be useful to keep in mind that

$$\frac{dL_{(\alpha),m}}{dr}(r) = -L_{(\alpha+1),m-1}(r) \quad r > 0, m \in \mathbb{N}_0, \tag{4.66}$$

which links the derivative of the m^{th} Laguerre polynomial of order α with the $(m-1)^{th}$ Laguerre polynomial of order $\alpha + 1$. Going on with the parallelism with Hermite polynomials, we have that Laguerre polynomials generating function is

$$\frac{1}{(1-\vartheta)^{1+\alpha}} e^{-\frac{\vartheta}{1-\vartheta} r} = \sum_{m=0}^{\infty} \vartheta^m L_{(\alpha),m}(r), \quad r > 0.$$

Finally, we recall that the family $\{L_{(\alpha),m}(r)\}_{m\in\mathbb{N}}$ is an orthogonal basis of $L^2(\mathbb{R}_+, d\nu)$, where

$$d\nu(r) := r^\alpha e^{-r} dr,$$

in the sense that if $\zeta = \zeta(r)$ belongs to $L^2(\mathbb{R}_+, d\nu)$, i.e., it is Lebesgue measurable and satisfies

$$\int_0^\infty \zeta^2(r) d\nu(r) = \int_0^\infty \zeta^2(r) r^\alpha e^{-r} dr < +\infty,$$

then

$$\zeta(\cdot) = \sum_{m=0}^{\infty} \zeta_{(\alpha),m} L_{(\alpha),m}(\cdot) \quad \text{in } L^2(\mathbb{R}_+, d\nu),$$

where $\{\zeta_{(\alpha),m}\}_{m\in\mathbb{N}} \subset \mathbb{R}$ (see Chapter 5 of [186]). In particular, we have the orthogonality condition

$$\int_0^\infty L_{(\alpha),m}(r) L_{(\alpha),n}(r) d\nu(r) = \Gamma(1+\alpha) \binom{n+\alpha}{n} \delta_{n,m}, \quad n, m \in \mathbb{N}.$$

Chapter 5

Blow-up analysis and nodal set of solutions to equation (1)

This is the last chapter and it contains the most significative results of this second part. We focus on the study of the asymptotic behaviour as $\lambda \to 0^+$ of the "normalized blow-up family"

$$U_\lambda(x,y,t) := \frac{U(\lambda x, \lambda y, \lambda^2 t)}{\sqrt{H(\lambda^2, U)}}, \quad \lambda > 0,$$

where $U = U(x,y,t)$ is a "strong solution" to equation (7), or to problems (8)/(9), and $H(\cdot, U)$ is defined as in Section 4.3. In this analysis the parabolically homogeneous profiles studied in Chapter 4 and the decomposition of the space $L^2(\mathbb{R}^{N+1}, d\mu)$ in orthogonal eigenfunctions play a crucial role. As we will see, this blow-up procedure will be essential in the study of the nodal set of solutions to equation (1), which is one of the most important aims of the entire work.

5.1 Main results

As we have said, in this chapter the blow-up procedure begins and we prove two main "blow-up classification" results. The first one is prove in Section 5.2 by combining the monotonicity properties of the Almgren-Poon quotient and the spectral analysis studied in the previous chapter.

Theorem 5.1. *Let $U = U(x,y,t)$ be a nontrivial "strong solution" to equation* (7). *Then there exist $n_0, m_0 \in \mathbb{N}$ such that the following assertions hold:*

(i) The Almgren-Poon quotient $N(t,U)$ (cfr. with the formula in (4.33)*) satisfies*

$$\lim_{t\to 0^+} N(t,U) = \kappa_{n_0,m_0},$$

where the admissible values for $\kappa_{n,m}$ are

$$\kappa_{n,m} = \widetilde{\kappa}_{n,m} := \frac{n}{2} + m \qquad or \qquad \kappa_{n,m} = \widehat{\kappa}_{n,m} := \frac{n}{2} + m + \frac{1-a}{2},$$

are the eigenvalues of problem (4.3)*, for any $m, n \in \mathbb{N}$.*

(ii) For all $T_* > 0$, we have as $\lambda \to 0^+$

$$\int_0^{T_*} \left\| \lambda^{-2\kappa_{n_0,m_0}} U(\lambda x, \lambda y, \lambda^2 t) - t^{\kappa_{n_0,m_0}} \sum_{(\alpha,m)\in J_0} v_{\alpha,m} \overline{V}_{\alpha,m}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \right\|^2_{H^1(\mathbb{R}^{N+1}, d\mu_t)} dt \to 0,$$

where $v_{\alpha,m}$ are suitable constants, the sum is done over the set of indices

$$J_0 := \{(\alpha, m) \in \mathbb{Z}^N_{\geq 0} \times \mathbb{N} : |\alpha| = n \in \mathbb{N} \text{ and } \kappa_{n,m}, = \kappa_{n_0,m_0}\},$$

and the integration probability measure is defined in (4.15). *Moreover,*

$$\overline{V}_{\alpha,m}(x,y) := \frac{V_{\alpha,m}(x,y)}{\|V_{\alpha,m}\|_{L^2_\mu}}, \qquad \alpha \in \mathbb{Z}^N_{\geq 0},\ m \in \mathbb{N},$$

are the normalized versions of the eigenfunctions $V_{\alpha,m} = V_{\alpha,m}(x,y)$ to problem (4.3) *corresponding to the eigenvalue $\kappa_{n,m}$ and defined in the statement of Theorem 4.1.*

Some comments are now in order. First of all, we stress that the above theorem provides relevant (and in some sense sharp) information about the behaviour of the "normalized" blow-up of solutions to equation (7), in terms of *explicit* eigenfunctions, given by combinations of N-dimensional Hermite polynomials and Laguerre polynomials of different orders. The second important fact is that we obtain a version of the above theorem for the "extensions" of the solutions to the nonlocal equation (1):

$$H^s u = 0 \quad \text{a.e. in } \mathbb{R}^N \times (-T, 0),$$

where $0 < s < 1$ and $T > 0$ are fixed, H is the "Heat Operator" $H := \partial_\tau - \Delta$, and $u \in \mathrm{dom}(H^s)$ (cfr. with what explained in the introduction of Part II). As we have recalled before, the key fact in our approach is that the function $U = U(x,y,t)$ defined in (2) satisfies problem (6) and so:

$$\begin{cases} \partial_\tau U - y^{-a}\nabla\cdot(y^a \nabla U) = 0 & \text{in } \mathbb{R}^{N+1}_+ \times \mathbb{R}, \\ U(x,0,\tau) = u(x,\tau) & \text{in } L^2(\mathbb{R}^{N+1}) \end{cases}$$

with

$$\lim_{y\to 0^+} y^a \partial_y U(x,y,\tau) = \partial^a_y U = 0 \quad \text{in } L^2(\mathbb{R}^N \times (-T,0)).$$

The blow-up classification result for the extension of solutions to the nonlocal equation (1) is the following.

Corollary 5.2. *Let $u \in dom(H^s)$ be a nontrivial solution to equation* (1) *and let $U = U(x,y,\tau)$ be defined as in* (2). *Then there exist $n_0, m_0 \in \mathbb{N}$ such that the following assertions hold:*

(i) The Almgren-Poon quotient $N(t,U)$ (cfr. with the formula in (4.26)*) satisfies*

$$\lim_{t\to 0^+} N(t,U) = \lim_{t\to 0^+} \frac{t\int_{\mathbb{R}^{N+1}_+} |\nabla U|^2(x,y,-t)\,d\mu_t(x,y)}{\int_{\mathbb{R}^{N+1}_+} U^2(x,y,-t)\,d\mu_t(x,y)} = \widetilde{\kappa}_{n_0,m_0},$$

where the admissible values for $\widetilde{\kappa}_{n,m}$ are the eigenvalues to problem (4.1)*:*

$$\widetilde{\kappa}_{n,m} = \frac{n}{2} + m, \quad n, m \in \mathbb{N}.$$

(ii) For all $T_* > 0$, we have as $\lambda \to 0^+$

$$\int_0^{T_*} \left\| \lambda^{-2\widetilde{\kappa}_{n_0,m_0}} U(\lambda x, \lambda y, -\lambda^2 t) - t^{\widetilde{\kappa}_{n_0,m_0}} \sum_{(\alpha,m)\in\widetilde{J}_0} v_{\alpha,m} \overline{V}_{\alpha,m}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \right\|^2_{H^1(\mathbb{R}^{N+1}_+, d\mu_t)} dt \to 0,$$

where $v_{\alpha,m}$ are suitable constants, the sum is done over the set of indices

$$\widetilde{J}_0 := \{(\alpha, m) \in \mathbb{Z}^N_{\geq 0} \times \mathbb{N} : |\alpha| = n \in \mathbb{N} \text{ and } \widetilde{\kappa}_{n,m}, = \widetilde{\kappa}_{n_0,m_0}\},$$

and the integration probability measure is defined in (4.12). *Moreover,*

$$\overline{V}_{\alpha,m}(x,y) := \frac{V_{\alpha,m}(x,y)}{\|V_{\alpha,m}\|_{L^2_\mu}}, \qquad \alpha \in \mathbb{Z}^N_{\geq 0},\ m \in \mathbb{N},$$

are the normalized versions of the eigenfunctions $V_{\alpha,m} = V_{\alpha,m}(x,y)$ *to problem* (4.1) *corresponding to the eigenvalue* $\widetilde{\kappa}_{n,m}$ *and defined in the statement of Theorem 4.1.*

Note that Theorem 5.2 gives a blow-up classification for the *extension* of solutions $u = u(x,t)$ to equation (1). It is natural to ask themselves whether it is possible to deduce a blow-up classification for $u = u(x,t)$, in terms of a suitable fractional Sobolev norm. However, w.r.t. the elliptic setting, in the case of the operator $(\partial_\tau - \Delta)^s$, it is not clear which is the formula linking the H^1 type norm of the extension $U = U(x,y,t)$ and a fractional Sobolev norm of $u = u(x,t)$ (cfr. with formula (3.7) of [54]) and, moreover, if the H^1-Gaussian type norm defined in $\mathbb{R}^{N+1}_+$ (as $d\mu_t = d\mu_t(x,y)$) possess a corresponding fractional version on the trace $\mathbb{R}^N \times \{0\}$.

We complete the blow-up analysis showing that the convergence obtained in Theorem 5.1 is also locally uniform in $\mathbb{R}^{N+1} \times (0,T_*)$. To do that, some Liouville type results that we will show in Section 5.3 turn out to be essential. More precisely, we show the following theorem.

Theorem 5.3. *Let* $U = U(x,y,t)$ *be a "strong solution" to equation* (7) *and take* $n_0, m_0 \in \mathbb{N}$ *such that*

$$\lim_{t\to 0^+} N(t,U) = \kappa_{n_0,m_0},$$

where κ_{n_0,m_0} *are as in Theorem 5.1. Then, for any* $T_* > 0$ *and any compact set* $K \subset \mathbb{R}^{N+1} \times (0,T_*)$*, we have as* $\lambda \to 0^+$

$$\left\| \lambda^{-2\kappa_{n_0,m_0}} U(\lambda x, \lambda y, \lambda^2 t) - t^{\kappa_{n_0,m_0}} \sum_{(\alpha,m)\in J_0} v_{\alpha,m} \overline{V}_{\alpha,m}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \right\|_{L^\infty(K)} \to 0,$$

where $v_{\alpha,m} \in \mathbb{R}$, $\overline{V}_{\alpha,m} = \overline{V}_{\alpha,m}(x,y)$ *and* J_0 *are defined as in the statement of Theorem 5.1.*

As before, we obtain a corresponding version for the extensions of solutions to the nonlocal equation (1).

Corollary 5.4. *Let* $u \in dom(H^s)$ *be a nontrivial solution to equation* (1) *and let* $U = U(x,y,\tau)$ *be defined as in* (2)*. Take* $n_0, m_0 \in \mathbb{N}$ *such that*

$$\lim_{t\to 0^+} N(t,U) = \lim_{t\to 0^+} \frac{t \int_{\mathbb{R}^{N+1}_+} |\nabla U|^2(x,y,-t)\, d\mu_t(x,y)}{\int_{\mathbb{R}^{N+1}_+} U^2(x,y,-t)\, d\mu_t(x,y)} = \widetilde{\kappa}_{n_0,m_0},$$

where $\widetilde{\kappa}_{n_0,m_0}$ *are as in Corollary 5.2. Then, for any* $T_* > 0$ *and any compact set* $K \subset \overline{\mathbb{R}^{N+1}_+} \times (0,T_*)$*, we have as* $\lambda \to 0^+$

$$\left\| \lambda^{-2\widetilde{\kappa}_{n_0,m_0}} U(\lambda x, \lambda y, -\lambda^2 t) - t^{\widetilde{\kappa}_{n_0,m_0}} \sum_{(\alpha,m)\in \widetilde{J}_0} v_{\alpha,m} \overline{V}_{\alpha,m}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \right\|_{L^\infty(K)} \to 0,$$

where $v_{\alpha,m} \in \mathbb{R}$, $\overline{V}_{\alpha,m} = \overline{V}_{\alpha,m}(x,y)$ *and* $\widetilde{J}_0$ *are defined as in the statement of Corollary 5.2.*

Now, to better describe the results concerning the analysis of the nodal set of solutions to equation (1), we need to introduce some notations. So, let $U = U(x,y,t)$ be a "strong solution" to problem (8) and let

$$\Gamma(U) := \{(x,y,t) \in \overline{\mathbb{R}^{N+1}_+} \times (0,T) : U(x,y,t) = 0\} = U^{-1}(\{0\})$$

be its nodal set. For any $p_0 \in \Gamma(U) \cap \Sigma$, we consider the transformation

$$U_{p_0}(x, y, t) := U(x_0 + x, y, t_0 + t), \tag{5.1}$$

and the Almgren-Poon quotient "centered at $p_0 = (X_0, t_0) = (x_0, 0, t_0)$":

$$N(p_0, t, U) := \frac{(t - t_0)I(p_0, t, U)}{H(p_0, t, U)}, \quad t > t_0$$

where, setting $\mathcal{G}_a^{p_0}(x, y, t) := \mathcal{G}_a(x - x_0, y, t - t_0)$, we define

$$H(p_0, t, U) := \int_{\mathbb{R}^{N+1}_+} |y|^a U^2(x, y, t)\mathcal{G}_a^{p_0}(x, y, t)\, dxdy, \quad t > t_0$$

$$I(p_0, t, U) := \int_{\mathbb{R}^{N+1}_+} |y|^a |\nabla U|^2(x, y, t)\mathcal{G}_a^{p_0}(x, y, t)\, dxdy, \quad t > t_0.$$

Recall that

$$\Sigma := \{(x, y, t) \in \overline{\mathbb{R}^{N+1}_+} \times (0, T) : y = 0\} = \mathbb{R}^N \times \{0\} \times (0, T).$$

Now, setting $H(\mathbf{0}, t, U) := H(t, U)$, $I(\mathbf{0}, t, U) := I(t, U)$ (and consequently $N(\mathbf{0}, t, U) := N(t, U)$, cfr. with (4.33)), it immediately seen that

$$\begin{aligned} H(p_0, t, U) &= H(\mathbf{0}, t - t_0, U_{p_0}), \\ I(p_0, t, U) &= I(\mathbf{0}, t - t_0, U_{p_0}), \\ N(p_0, t, U) &= N(\mathbf{0}, t - t_0, U_{p_0}), \quad t > t_0. \end{aligned} \tag{5.2}$$

As a first important observation, we point out that the function

$$p_0 \to N(p_0, t_0^+, U) := \lim_{t \to t_0^+} N(p_0, t, U),$$

is upper semi-continuous on $\Gamma(U) \cap \Sigma$. This easily follows from the fact that $N(\cdot, t_0^+, U)$ is defined as an infimum of a family of continuous functions $t \to N(p_0, t, U)$ (cfr. with Lemma 4.5).
Secondly, since problem (7) is invariant under translations of type (5.1), we deduce that both Corollary 5.2 and Corollary 5.4 still hold true, by replacing the blow-up $U_\lambda(x, y, t) = \lambda^{-2\kappa} U(\lambda x, \lambda y, \lambda^2 t)$ with

$$U_{p_0,\lambda}(x, y, t) := \frac{U_{p_0}(\lambda x, \lambda y, \lambda^2 t)}{\lambda^{2\kappa}} = \frac{U(x_0 + \lambda x, \lambda y, t_0 + \lambda^2 t)}{\lambda^{2\kappa}}, \tag{5.3}$$

and the limit profiles $\Theta = \Theta(x, y, t)$, with

$$\Theta_{p_0}(x, y, t) = t^\kappa \sum_{(\alpha,m)\in J_0} v_{\alpha,m} \overline{V}_{\alpha,m}(x/\sqrt{t}, y/\sqrt{t}),$$

where $\kappa = \kappa(p_0) := \lim_{t \to t_0^+} N(p_0, t, U)$ and $v_{\alpha,m} = v_{\alpha,m}(p_0)$ may depend on p_0. Of course, thanks to (5.2) it must be

$$\kappa = \kappa(p_0) \in \widetilde{\mathcal{K}} := \{\widetilde{\kappa}_{n,m}\}_{n,m\in\mathbb{N}}, \tag{5.4}$$

where, as always, $\widetilde{\kappa}_{n,m} := \frac{n}{2} + m$. More precisely, for such $p_0 \in \Gamma(U) \cap \Sigma$ and for all $T_* > 0$, we have as $\lambda \to 0^+$

$$\int_0^{T_*} \left\| \frac{U(x_0 + \lambda x, \lambda y, t_0 + \lambda^2 t)}{\lambda^{2\kappa}} - \Theta_{p_0}(x, y, t) \right\|^2_{H^1(\mathbb{R}^{N+1}_+, d\mu_t)} dt \to 0. \tag{5.5}$$

Similarly, for any compact set $K \subset \overline{\mathbb{R}^{N+1}_+} \times [0, T_*)$, it holds

$$\left\| \frac{U(x_0 + \lambda x, \lambda y, t_0 + \lambda^2 t)}{\lambda^{2\kappa}} - \Theta_{p_0}(x, y, t) \right\|_{L^\infty(K)} \to 0, \quad \text{as } \lambda \to 0^+.$$

The above observations motivate the following important definitions.

Definition 5.5. *Let $U = U(x, y, t)$ be a "strong solution" to problem* (8)*. For any $\kappa \in \widetilde{\mathcal{K}}$, we define*

$$\Gamma_\kappa(U) := \left\{ p_0 \in \Gamma(U) \cap \Sigma : \lim_{t \to t_0^+} N(p_0, t, U) = \kappa \right\}.$$

In particular, we set

$$\mathcal{R}(U) := \Gamma_{1/2}(U)$$
$$\mathcal{S}(U) := \Gamma(U) \setminus \mathcal{R}(U).$$

Definition 5.6. *Let $U = U(x, y, t)$ be a "strong solution" to problem* (8) *with $p_0 \in \Gamma_\kappa(U)$ and $\kappa \in \widetilde{\mathcal{K}}$. We define the set of all the possible "blow-up limits" of U at p_0 as*

$$\mathfrak{B}_\kappa(U) := \left\{ \Theta_{p_0}(x, y, t) = t^{\kappa_{n_0,m_0}} \sum_{(\alpha,m)\in J_0} v_{\alpha,m} \overline{V}_{\alpha,m}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}} \right) \right\},$$

where $n_0, m_0 \in \mathbb{N}$ are chosen so that $\kappa = \kappa_{n_0,m_0}$ (cfr. with (5.4)*), while J_0, $v_{\alpha,m}$, and $\overline{V}_{\alpha,m} = \overline{V}_{\alpha,m}(\cdot, \cdot)$ are as in Corollary 5.2.*
Furthermore, the "tangent map" of U at p_0 is the unique $\Theta_{p_0} \in \mathfrak{B}_\kappa(U)$ such that for any compact set $K \subset \overline{\mathbb{R}^{N+1}_+} \times [0, \infty)$

$$\frac{U(x_0 + \lambda x, \lambda y, t_0 + \lambda^2 t)}{\lambda^{2\kappa}} \to \Theta_{p_0}(x, y, t) \quad \textit{uniformly on } K,$$

as $\lambda \to 0^+$.

Remark. If $u \in \mathrm{dom}(H^s)$ is a nontrivial solution to (1), $\overline{u}(x, t) = u(x, -t)$ and $\overline{U}(x, y, t) = U(x, y, -t)$, where $U = U(x, y, \tau)$ is the extension of $u = u(x, \tau)$ defined in (2), we will set

$$\begin{aligned} &\Gamma_\kappa(u) := \{(x_0, \tau_0) \in \Gamma(u) : (x_0, -\tau_0) \in \Gamma_\kappa(\overline{u})\} \quad \text{for all } \kappa \in \widetilde{\mathcal{K}} \\ &\mathcal{R}(u) := \Gamma_{1/2}(u) \\ &\mathcal{S}(u) := \Gamma(u) \setminus \mathcal{R}(u). \end{aligned} \tag{5.6}$$

where $\Gamma_\kappa(\overline{u}) := \Gamma_\kappa(\overline{U})$ for all $\kappa \in \widetilde{\mathcal{K}}$ and $\mathcal{R}(\overline{u}) := \mathcal{R}(\overline{U}) = \Gamma_{1/2}(\overline{U})$. We are now ready to state the first main theorem which gives a natural but not trivial dimensional estimate on the nodal set of solutions to (1). Its proof is based the blow-up theorems stated above and a classical theorem called Federer's reduction principle (cfr. with Chen [63]).

Theorem 5.7. *Let $u \in \mathrm{dom}(H^s)$ be a nontrivial solution to* (1)*. Then the Hausdorff dimension of its nodal set satisfies the bound:*

$$\dim_{\mathcal{P}}(\Gamma(u)) \le N + 1,$$

where $\dim_{\mathcal{P}}(E)$ denotes the "parabolic Hausdorff dimension" of a set $E \subset \mathbb{R}^N \times \mathbb{R}$.

Once the bound above is established, we move forward with the analysis of the "regular" set $\mathcal{R}(u) = \Gamma_{1/2}(u)$. In particular, we clarify the relation between "regular points" and the limit of the Almgren-Poon quotient "centered at p_0" of "strong solutions" to problem (8). The main goal is to prove the "regular" points are regular indeed.

Theorem 5.8. *(Regularity of $\mathcal{R}(U)$) The following two assertions hold.*
(i) Let $U = U(x, y, t)$ be a "strong solution" to problem (8) *and let us define*

$$\mathcal{R}(U) := \Gamma_{1/2}(U) = \left\{ p_0 \in \Gamma(U) \cap \Sigma : \lim_{t \to t_0^+} N(p_0, t, U) = \frac{1}{2} \right\}.$$

Then, for any $p_0 = (x_0, 0, t_0) \in \mathcal{R}(U)$ it holds $\nabla_x U(p_0) \neq \mathbf{0}$.

(ii) Let $u \in dom(H^s)$ be a nontrivial solution to (1) *and let $\mathcal{R}(u)$ be defined as in* (5.6)*. Then $\mathcal{R}(u)$ is a locally C^1-manifold of Hausdorff dimension N.*

We are thus left to study the "singular set" $\mathcal{S}(u)$ of $u = u(x,\tau)$ defined in (5.6):

$$\mathcal{S}(u) := \Gamma(u) \setminus \mathcal{R}(u) = \bigcup_{\kappa \in 1+\frac{\mathbb{N}}{2}} \Gamma_\kappa(u) = \bigcup_{\kappa \in 1+\frac{\mathbb{N}}{2}} \left\{ (x_0, 0, -t_0) \in \Gamma(\overline{U}) \cap \Sigma : \lim_{t \to t_0^+} N(p_0, t, \overline{U}) = \kappa \right\},$$

where, as always, $\overline{U}(x,y,t) = U(x,y,-t)$ and $U = U(x,y,\tau)$ is the extension of $u = u(x,\tau)$ defined in (2) (cfr. with Definition 5.5).

This part contains the main novelties w.r.t. the classical case. Note indeed that Theorems 5.7 and 5.8 hold also for solutions to the classical Heat Equation, as explained in the introduction (cfr. with the formulas in (12)-(13)). However, we will see that for solutions to (1) the dimensional estimate on the singular set cannot be valid in this nonlocal framework (cfr. with the second formula in (12) and the last to formulas of (13)). This can be easily guessed by taking $N = 1$ and noting that if $U = U(x,y,t)$ is a "strong solution" to problem (8) with $(0,0) \in \Gamma(U)$ and Almgren-Poon limit $\kappa = 1$, then its "blow up limit" must be a linear combination of the "right" re-scaled eigenfunctions (cfr. with the end of Section 5.2):

$$\begin{aligned}\Theta^{A,B}(x,y,t) &= A\widetilde{\Theta}_{2,0}(x,y,t) + B\widetilde{\Theta}_{0,1}(x,y,t) = A(x^2 - 2t) + B\left[\left(\frac{1+a}{2}\right)t - \frac{y^2}{4}\right] \\ &= Ax^2 + \left[\frac{1+a}{2}B - 2A\right]t - \frac{B}{4}y^2.\end{aligned}$$

In particular, taking $A = 1$, $B = 4/(1+a)$ and $A = 0$, $B = 4/(1+a)$, we obtain

$$\Theta^{1,\frac{4}{1+a}}(x,y,t) = x^2 - \frac{y^2}{1+a}, \qquad \Theta^{0,\frac{4}{1+a}}(x,y,t) = 2t - \frac{y^2}{1+a},$$

with traces on $\mathbb{R}^N \times \mathbb{R}$

$$\vartheta^{1,\frac{4}{1+a}}(x,t) := \Theta^{1,\frac{4}{1+a}}(x,0,t) = x^2, \qquad \vartheta^{0,\frac{4}{1+a}}(x,t) := \Theta^{0,\frac{4}{1+a}}(x,0,t) = 2t,$$

respectively. Consequently, since $\dim_{\mathcal{H}}(\mathcal{S}(\vartheta^{1,\frac{4}{1+a}})) = \dim_{\mathcal{H}}(\mathcal{S}(\vartheta^{0,\frac{4}{1+a}})) = 1 \neq 0$, we understand that the non-locality of our operator plays a central role in the way in which "s-caloric functions" approach their nodal sets. Similarly, recalling that

$$\mathcal{Z}_t(u) := \{x \in \mathbb{R}^N : u(x,t) = 0\},$$

we note that $\dim_{\mathcal{H}}(\mathcal{Z}_0(\vartheta^{0,\frac{4}{1+a}})) = 1 \neq 0$ and so all the relations in (13) seems not hold, too.

Now, the first main result concerns the asymptotic behaviour and the differentiability of a solution to problem (8) near nodal points in $\Gamma_\kappa(U)$. Furthermore, it establishes the continuity of the tangent map $p_0 \to \Theta_{p_0}$, seen as a function from $\Gamma_\kappa(U)$ to $\mathfrak{B}_\kappa(U)$.

Theorem 5.9. *(Continuous dependence of the "blow-up limits") Let $U = U(x,y,t)$ be a nontrivial "strong solution" to problem* (8)*, $p_0 = (X_0, t_0) \in \Gamma_\kappa(U)$, and $\Theta_{p_0} = \Theta_{p_0}(x,y,t) \in \mathfrak{B}_\kappa(U)$ its "tangent map" at p_0 (cfr. with Definition 5.6 and Theorem 5.1). Then the following assertions hold:*

(i) We have as $\|(x,y,t)\|^2 := |x|^2 + y^2 + t \to 0^+$

$$U_{p_0}(x,y,t) = \Theta_{p_0}(x,y,t) + o(\|(x,y,t)\|^{2\kappa}).$$

(ii) The map $p_0 \to \Theta_{p_0}$ from $\Gamma_\kappa(U)$ to $\mathfrak{B}_\kappa(U)$ is continuous.

(iii) For any compact set $K \subset \Gamma_\kappa(U)$ there exists a modulus of continuity $\sigma = \sigma(K)$ with $\sigma(0^+) = 0$, such that as $\|(x,y,t)\| \to 0^+$

$$|U_{p_0}(x,y,t) - \Theta_{p_0}(x,y,t)| \le \sigma(\|(x,y,t)\|)\,\|(x,y,t)\|^{2\kappa},$$

for any $p_0 \in K$.

Finally, to state the last theorem of this treatise, we must give two more definitions inspired by the work of Danielli, Garofalo, Petrosyan and To [71] (cfr. with Definition 12.9 of that paper).

Definition 5.10. *Let $u \in dom(H^s)$ be a nontrivial solution to* (1) *and $(x_0,\tau_0) \in \Gamma_\kappa(u)$. We define the spatial dimension of $\Gamma_\kappa(u)$ at (x_0,τ_0) as*

$$d^\kappa_{(x_0,\tau_0)} := \dim\left\{\xi \in \mathbb{R}^N : \xi \cdot \nabla_x \partial_x^\alpha \partial_t^j \vartheta_{p_0} = 0, \text{ for any } \alpha \in \mathbb{Z}^N_{\ge 0},\ j \in \mathbb{N} \text{ with } |\alpha| + 2j = 2\kappa - 1\right\},$$

where $p_0 = (x_0,-\tau_0)$, $\vartheta_{p_0}(x,t) = \Theta_{p_0}(x,0,t)$ and $\Theta_{p_0} \in \mathfrak{B}_\kappa(\overline{U})$ is the "blow-up limit" of $\overline{U} = \overline{U}(x,y,t)$ at p_0. Finally, for any $d = 0,\dots,N$, we define

$$\Gamma^d_\kappa(u) := \left\{(x_0,\tau_0) \in \Gamma_\kappa(u) : d^\kappa_{(x_0,\tau_0)} = d\right\}.$$

Finally, we introduce the notion of "space-like" and "time-like" manifolds.

Definition 5.11. *(Definition 12.11 of [71]) A $(d+1)$-dimensional manifold $\mathcal{M} \subset \mathbb{R}^N \times \mathbb{R}$ (with $d = 0,\dots,N-1$) is said to be "space-like" of class $C^{1,0}$ if it can be locally represented as a graph of a $C^{1,0}$ function $g : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^{N-d}$*

$$(x_{d+1},\dots,x_N) = g(x_1,\dots,x_d,t),$$

up to rotation of coordinate axis in $\mathbb{R}^N$.

A N-dimensional manifold $\mathcal{M} \subset \mathbb{R}^N \times \mathbb{R}$ is said to be "time-like" of class C^1 if it can be locally represented as a graph of a C^1 function $g : \mathbb{R}^N \to \mathbb{R}$

$$t = g(x_1,\dots,x_N).$$

We can now state our "Structure of the singular set theorem". Its proof is based on the techniques due to Garofalo and Petrosyan [112] (elliptic setting) and Danielli, Garofalo, Petrosyan and To [71] (parabolic setting), based based on a ingenious combination of the Implicit function theorem and a parabolic version of the Whitney's extension theorem, which we recall later.

Theorem 5.12. *(Structure of the singular set) Let $u \in dom(H^s)$ be a nontrivial solution to* (1)*. Then the set $\Gamma^d_\kappa(u)$ is contained in a countable union of $(d+1)$-dimensional "space-like" $C^{1,0}$ manifolds for any $d = 0,\dots,N-1$ while $\Gamma^N_\kappa(u)$ is contained in a countable union of N-dimensional "time-like" C^1 manifolds.*

The presentation of the main results is ended so that we can proceed with their proofs.

5.2 Blow-up analysis I: proof of Theorem 5.1 and Corollary 5.2

In the following paragraphs the blow-up analysis starts. Following some ideas of [99], our approach is based on the monotonicity formulae obtained in Section 4.3 and a priori compactness results in $C^0(t_1,t_2;B)$ type spaces where $0 < t_1 < t_2 < \infty$ and B is a Banach space (see [179] or Lemma 5.13). W.r.t. Section 7 of [19] and [71], where the authors considered average versions of $H(t,U)$ and $I(t,U)$ (cfr. with Section 4.3), our methods are possibly naiver but suffice for our purposes. All the proof relies on the following compactness criterion that we recall for clarity.

Lemma 5.13. *(Simon [179, Corollary 8]) Let $X \subset B \subseteq Y$ be Banach spaces satisfying the following two properties:*

- *X is compactly embedded in Y.*
- *There exist $0 < \theta < 1$ and $C > 0$ such that $\|V\|_B \le C\|V\|_X^{1-\theta}\|V\|_Y^{\theta}$, for all $V \in X \cap Y$.*

Fix $-\infty < t_1 < t_2 < +\infty$ and $1 \le p, r \le \infty$. Let $\mathcal{F}$ be a bounded family in $L^p(t_1, t_2; X)$, with $\partial\mathcal{F}/\partial t$ bounded in $L^r(t_1, t_2; Y)$. Then the following two assertions hold true:

- *If $\theta(1 - 1/r) \le (1 - \theta)/p$ then $\mathcal{F}$ is relatively compact in $L^p(t_1, t_2; B)$, for all $p < p_*$, where $1/p_* = (1 - \theta)/p - \theta(1 - 1/r)$.*
- *If $\theta(1 - 1/r) > (1 - \theta)/p$ then $\mathcal{F}$ is relatively compact in $C^0(t_1, t_2; B)$,*

where we recall that $C^0(t_1, t_2; B) := \{V : [t_1, t_2] \to B$ continuous $: \|V\|_{C^0(t_1,t_2;B)} < \infty\}$, where

$$\|V\|_{C^0(t_1,t_2;B)} := \max_{t\in[t_1,t_2]} \|V(t)\|_B.$$

We will apply the above criterion taking $X = H^1_\mu = H^1(\mathbb{R}^{N+1}, d\mu)$ and $B = Y = L^2_\mu = L^2(\mathbb{R}^{N+1}, d\mu)$ and $p = r = \infty$. To do so, we have to prove the following lemma.

Lemma 5.14. *The space $H^1_\mu = H^1(\mathbb{R}^{N+1}, d\mu)$ is compactly embedded in $L^2_\mu = L^2(\mathbb{R}^{N+1}, d\mu)$.*

Proof. Let $\{U_j\}_j$ be a bounded sequence in H^1_μ. Up to subsequences, we can assume the existence of a function $U \in H^1_\mu$ such that $U_j \rightharpoonup U$ (weakly) in H^1_μ and we must prove that

$$U_j \to U \quad \text{(strongly) in } L^2_\mu,$$

as $j \to +\infty$. Let us set $V_j := U_j - U$. For any $A > 0$, we can write

$$\int_{\mathbb{R}^{N+1}} V_j^2\, d\mu(x,y) = \int_{\mathbb{B}_A} V_j^2\, d\mu(x,y) + \int_{\mathbb{B}_A^c} V_j^2\, d\mu(x,y), \tag{5.7}$$

where, following [19], we have defined

$$\mathbb{B}_A := \{(x,y) \in \mathbb{R}^{N+1} : |x|^2 + y^2 < A^2\} \quad \text{with} \quad \mathbb{B}_A^c = \mathbb{R}^{N+1} \setminus \mathbb{B}_A.$$

Now, since the measures $|y|^a\, dxdy$ and $|y|^a\mathcal{G}_a(x,y,1)\, dxdy$ are equivalent on $\mathbb{B}_A$, we obtain that the first term in the r.h.s. of (5.7) goes to zero as $j \to +\infty$, thanks to the well-known $H^1(\mathbb{B}_A, |y|^a) \hookrightarrow L^2(\mathbb{B}_A, |y|^a)$ compact immersions type (cfr. for instance with [97, 165]). Consequently, we are left to prove that also the second integral in the r.h.s. of (5.7) is converging to zero as $j \to +\infty$.

To do so, we repeat the procedure carried out in [19] (cfr. with Lemma 7.4) with some modifications due to our slightly different framework. Before moving forward, we recall the definition of the "fundamental solution" and how it splits into the product of two Gaussians:

$$\mathcal{G}_a(x,y,t) = G_N(x,t)G_{a+1}(y,t) = \frac{1}{(4\pi t)^{\frac{N}{2}}} e^{-\frac{|x|^2}{4t}} \frac{1}{2^{1+a}\Gamma(\frac{1+a}{2})} \frac{1}{t^{\frac{1+a}{2}}} e^{-\frac{y^2}{4t}},$$

and that the following two log-Sobolev type inequalities hold (cfr. with Lemma 7.7 of [71] or formulas (7.16)-(7.17) of [19]):

$$\log\left(\frac{1}{c_a \int_{|f|>0} \mathcal{G}_a(\cdot,1)}\right) \int_{\mathbb{R}^{N+1}} f^2 \mathcal{G}_a(\cdot,1) \le 2 \int_{\mathbb{R}^{N+1}} |\nabla f|^2 \mathcal{G}_a(\cdot,1), \tag{5.8}$$

for all $f \in H^1(\mathbb{R}^{N+1}, d\mu)$, and

$$\log\left(\frac{1}{\int_{|f|>0} G_N(\cdot,1)}\right)\int_{\mathbb{R}^N} f^2 G_N(\cdot,1) \le 2\int_{\mathbb{R}^N} |\nabla f|^2 G_N(\cdot,1), \tag{5.9}$$

for all $f \in H^1(\mathbb{R}^N, d\mu_x)$. Note that, w.r.t. [71] and/or [19], in (5.8) we have introduced the quantity $c_a := 2^{1+a}\Gamma(\frac{1+a}{2})/\sqrt{4\pi}$ due to the different normalization constant of the "fundamental solution".

Now, fix $\varepsilon > 0$, and take $A > 4$ large enough such that

$$c_a \int_{\mathbb{B}^c_{A/2}} \mathcal{G}_a(x,y,1)\,dxdy \le e^{-1/\varepsilon}, \qquad \int_{B^c_{A/2}} G_N(x,1)\,dx \le e^{-1/\varepsilon}, \tag{5.10}$$

where $B_A := \{x \in \mathbb{R}^N : |x| < A\}$ with $B^c_A = \mathbb{R}^N \setminus B_A$. As explained in [19], (5.8) cannot be directly applied to estimate the integral

$$\int_{\mathbb{B}^c_A} V_j^2\, d\mu(x,y) = \int_{\mathbb{B}^c_A} V_j^2\, |y|^a \mathcal{G}_a(x,y,1)dxdy,$$

due to some regularity issues. So, we write

$$\int_{\mathbb{B}^c_A} |y|^a V_j^2\, \mathcal{G}_a(x,y,1)dxdy = \int_{\widetilde{R}_A} |y|^a V_j^2\, \mathcal{G}_a(x,y,1)dxdy + \int_{R_A} |y|^a V_j^2\, \mathcal{G}_a(x,y,1)dxdy, \tag{5.11}$$

where

$$\widetilde{R}_A := \mathbb{B}^c_A \cap (\mathbb{R}^N \times \{|y| \le A/2\}) \quad \text{and} \quad R_A := \mathbb{B}^c_A \cap (\mathbb{R}^N \times \{|y| > A/2\}).$$

Let us start with estimating the first integral in the r.h.s. of (5.11). For any $(x,y) \in \widetilde{R}_A$, we have both $|x|^2 + y^2 \ge A^2$ and $y^2 \le A^2/4$, so that $|x| \ge (\sqrt{3}/2)A \ge A/2$. Consequently,

$$\int_{\widetilde{R}_A} |y|^a V_j^2(x,y)\, \mathcal{G}_a(x,y,1)dxdy \le \int_{-A/2}^{A/2} |y|^a G_{a+1}(y,1)\left(\int_{\{|x|\ge(\sqrt{3}/2)A\}} V_j^2(x,y) G_N(x,1)\,dx\right)dy.$$

Now, if $\varphi \in C^\infty_c(\mathbb{R}^N)$ is a radially decreasing cut-off function satisfying $\varphi = 1$ in $B_{A/2}$ and $\varphi = 0$ outside B_A with $|\nabla\varphi| \le 1$, we firstly observe that

$$\int_{\{|x|\ge(\sqrt{3}/2)A\}} V_j^2(x,y) G_N(x,1)\,dx \le C\int_{\mathbb{R}^N} V_j^2(x,y)[1-\varphi(x)]^2 G_N(x,1)\,dx,$$

for some constant $C > 0$ depending only on φ and for a.e. $y \in \mathbb{R}$ (the above inequality can be directly verified using the properties of the cut-off function $\varphi = \varphi(x)$). Secondly, observing that the second inequality in (5.10), together with the fact that

$$\int_{|V_j(1-\varphi)|>0} G_N(x,1)\,dx \le \int_{B^c_{A/2}} G_N(x,1)\,dx$$

implies

$$\log\left(\frac{1}{\int_{|V_j(1-\varphi)|>0} G_N(\cdot,1)}\right) \ge \frac{1}{\varepsilon}$$

we deduce (by applying (5.9) with $f = V_j(1-\varphi)$):

$$\int_{\mathbb{R}^N} V_j^2(x,y)[1-\varphi(x)]^2 G_N(x,1)\,dx \le \varepsilon\, C\int_{\mathbb{R}^N}\left[V_j^2 + |\nabla_x V_j|^2\right] G_N(x,1)\,dx \le \varepsilon\, C,$$

where $C > 0$ is constant independent of $j \in \mathbb{N}$ (this follows from the properties of $\varphi = \varphi(x)$ and since $\{V_j\}_j$ is bounded in H^1_μ). Consequently, we obtain

$$\int_{\widetilde{R}_A} |y|^a V_j^2(x,y)\,\mathcal{G}_a(x,y,1)dxdy \le \varepsilon\, C \int_{-A/2}^{A/2} |y|^a G_{a+1}(y,1)\,dy \le \varepsilon\, C,$$

where $C > 0$ is new constant not depending on $j \in \mathbb{N}$. Let us focus the second integral in the r.h.s. of (5.11). We introduce the new cut-off functions

- $\overline{\varphi} \in C_c^\infty(\mathbb{R}^{N+1})$ with $\overline{\varphi} = 1$ in $\mathbb{B}_{A/2}$ and $\overline{\varphi} = 0$ outside $\mathbb{B}_A$ with $|\nabla_{x,y}\overline{\varphi}| \le 1$.
- $\widetilde{\varphi} \in C_c^\infty(\mathbb{R})$ with $\widetilde{\varphi}(y) = 1$ for $|y| \le A/4$ and $\widetilde{\varphi}(y) = 0$ for $|y| \ge A/2$ with $|\widetilde{\varphi}'| \le 1$,

and we immediately see that

$$\int_{R_A} |y|^a V_j^2\,\mathcal{G}_a(x,y,1)dxdy \le \int_{\mathbb{R}^{N+1}} |y|^a V_j^2\,[1-\overline{\varphi}(x,y)]^2[1-\widetilde{\varphi}(y)]^2\,\mathcal{G}_a(x,y,1)dxdy.$$

Now, exactly as in [19] (cfr. with formula (7.24)), we set $f = |y|^{a/2}V_j(1-\overline{\varphi})(1-\widetilde{\varphi})$ and we estimate

$$|\nabla f|^2 \le C\left[\,|y|^a(V_j^2 + |\nabla V_j|^2) + |y|^{a-2}V_j^2(1-\overline{\varphi})^2(1-\widetilde{\varphi})^2\,\right] \le C|y|^a(V_j^2 + |\nabla V_j|^2),$$

where $C > 0$ independent of $j \in \mathbb{N}$. The last inequality follows since for any $|y| \ge A/4$, we have

$$|y|^{a-2}V_j^2(1-\overline{\varphi})^2(1-\widetilde{\varphi})^2 \le |y|^{a-2}V_j^2 \le |y|^a V_j^2,$$

where we have used the properties of the support of $\widetilde{\varphi} = \widetilde{\varphi}(y)$ and the fact the we have chosen $A > 4$ from the beginning. Consequently, applying (5.8) (with $f = |y|^{a/2}V_j(1-\overline{\varphi})(1-\widetilde{\varphi})$), we get

$$\int_{R_A} |y|^a V_j^2\,\mathcal{G}_a(x,y,1)dxdy \le \varepsilon\, C \int_{\mathbb{R}^{N+1}} |y|^a\left[V_j^2 + |\nabla V_j|^2\right]\mathcal{G}_a(x,y,1)\,dxdy \le \varepsilon\, C,$$

where we have used the first inequality in (5.10) and the uniform H^1_μ bound on the V_j's. Summing up, from (5.11) and the above bounds, we have got

$$\int_{\mathbb{B}_A^c} |y|^a V_j^2\,\mathcal{G}_a(x,y,1)dxdy \le \varepsilon\, C,$$

which completes the proof, by the arbitrariness of $\varepsilon > 0$. □

Remark. An almost identical proof allows to show that the space $H^1_\mu = H^1(\mathbb{R}^{N+1}_+, d\mu)$ is compactly embedded in $L^2_\mu = L^2(\mathbb{R}^{N+1}_+, d\mu)$.

We are now ready to prove Theorem 5.1. We anticipate that w.r.t. to the above mentioned papers, we first obtain a convergence result in the space $L^2(t_*, T_*; H^1_\mu)$ (and also in $C^0(t_*, T_*; L^2_\mu)$) for any $0 < t_* < T_* < \infty$, and then, in a second moment, we will improve it obtaining convergence in $L^2(0, T_*; H^1_\mu)$.

Proof of Theorem 5.1. We divide the proof in several steps.

Step1: Basic definitions. For $-1 < a < 1$ and $0 < T, T_* < \infty$ fixed, we consider a nontrivial "strong solution" (cfr. with Definition 4.2) $U = U(x, y, t)$ to (7):

$$\partial_t U + \mathcal{L}_a U = 0 \quad \text{in } \mathbb{R}^{N+1} \times (0, T).$$

We recall that in particular $U = U(x, y, t)$ satisfies (4.17):

$$\int_{\mathbb{R}^{N+1}} \left[\partial_t U + \frac{(x, y)}{2t} \nabla U \right] \eta \, d\mu_t(x, y) = \int_{\mathbb{R}^{N+1}} \nabla U \nabla \eta \, d\mu_t(x, y),$$

for a.e. $0 < t < T$ and all $\eta \in L^2(0, T; H^1_{\mu_t})$. We then define the "normalized blow-up" family

$$U_\lambda(x, y, t) := \frac{U(\lambda x, \lambda y, \lambda^2 t)}{\sqrt{H(\lambda^2, U)}}, \quad \lambda > 0,$$

where $H(\cdot, U)$ is defined at the beginning of Subsection 4.3.2. As a preliminary observation, it is not difficult to see that for any $\lambda > 0$, $U_\lambda = U_\lambda(x, y, t)$ is a "strong solution" to equation

$$\partial_t U_\lambda + \mathcal{L}_a U_\lambda = 0 \quad \text{in } \mathbb{R}^{N+1} \times (0, T/\lambda^2),$$

and (4.17) holds for $U_\lambda = U_\lambda(x, y, t)$, too. Now, we define

$$H(t, U_\lambda) := \int_{\mathbb{R}^{N+1}} |y|^a U_\lambda^2(x, y, t) \mathcal{G}_a(x, y, t) \, dxdy,$$
$$I(t, U_\lambda) := \int_{\mathbb{R}^{N+1}} |y|^a |\nabla U_\lambda|^2(x, y, t) \mathcal{G}_a(x, y, t) \, dxdy,$$

for $t \in (0, T/\lambda^2)$. By scaling we have

$$H(t, U_\lambda) = \frac{H(\lambda^2 t, U)}{\sqrt{H(\lambda^2, U)}}, \qquad I(t, U_\lambda) = \frac{\lambda^2 I(\lambda^2 t, U)}{\sqrt{H(\lambda^2, U)}}$$

and so the frequency function corresponding to U_λ:

$$N(t, U_\lambda) := \frac{tI(t, U_\lambda)}{H(t, U_\lambda)}, \quad \text{satisfies} \quad N(t, U_\lambda) = N(\lambda^2 t, U), \quad 0 < t < T/\lambda^2.$$

Finally, we consider the family

$$\widetilde{U}_\lambda(x, y, t) := U_\lambda\left(\sqrt{t}x, \sqrt{t}y, t \right), \quad 0 < t < T/\lambda^2, \ \lambda > 0,$$

which is a "normalized" and "re-scaled" version of $U_\lambda = U_\lambda(x, y, t)$. Note that by scaling we have

$$\widetilde{U}_\lambda \in L^2(0, T/\lambda^2; H^1_\mu), \qquad t\partial_t \widetilde{U}_\lambda \in L^2(0, T/\lambda^2; L^2_\mu),$$

and moreover, $\widetilde{U}_\lambda = \widetilde{U}_\lambda(x, y, t)$ is a "strong solution" to (4.19):

$$t\partial_t \widetilde{U}_\lambda + \frac{1}{|y|^a \mathcal{G}_a} \nabla \cdot (|y|^a \mathcal{G}_a \nabla \widetilde{U}_\lambda) = 0 \quad \text{in } \mathbb{R}^{N+1} \times (0, T/\lambda^2). \tag{5.12}$$

In particular, it holds

$$t \int_{\mathbb{R}^{N+1}} \partial_t \widetilde{U}_\lambda \, \eta \, d\mu(x, y) = \int_{\mathbb{R}^{N+1}} \nabla \widetilde{U}_\lambda \nabla \eta \, d\mu(x, y), \tag{5.13}$$

for a.e. $0 < t < T/\lambda^2$ and all $\eta \in L^2(0, T/\lambda^2; H^1_\mu)$. Before moving forward, we need to make an important observation. For a.e. $0 < t < T$, the linear functional

$$\eta(t) \to \int_{\mathbb{R}^{N+1}_+} \nabla\widetilde{U}_\lambda(t)\nabla\eta(t)\, d\mu(x,y), \quad \eta \in H^1_\mu$$

is well-defined and continuous on H^1_μ. Consequently, by the Hahn-Banach Theorem it can be continuously extended to a linear functional F^t_λ defined and continuous on the all L^2_μ, with

$$\|F^t_\lambda\|_{(L^2_\mu)^\star} \le \|\widetilde{U}_\lambda(t)\|_{H^1_\mu}, \quad \text{for a.e. } 0 < t < T.$$

On the other hand, from the Riesz Theorem, there exists a function $f^t_\lambda \in L^2_\mu$ such that

$$\langle F^t_\lambda, \eta(t)\rangle = \int_{\mathbb{R}^{N+1}} f^t_\lambda\, \eta(t)\, d\mu(x,y), \quad \eta(t) \in L^2_\mu,$$

and so, matching with (5.13), we get $t\partial_t\widetilde{U}_\lambda(t) = f^t_\lambda$ in L^2_μ for a.e. $0 < t < T$. In particular, since $\|F^t_\lambda\| = \|f^t_\lambda\|$, we obtain

$$\|\partial_t\widetilde{U}_\lambda(t)\|_{L^2_\mu} \le \frac{1}{t}\|\widetilde{U}_\lambda(t)\|_{H^1_\mu} \quad \text{for a.e. } 0 < t < T/\lambda^2. \tag{5.14}$$

Step2: Uniform bounds for $\widetilde{U}_\lambda$. Let us assume $1 \le T_* < \infty$ (the case $0 < T_* < 1$ is easier). Using the scaling properties of $\mathcal{G}_a = \mathcal{G}_a(x,y,t)$, it can be easily seen that

$$\begin{aligned}\int_{\mathbb{R}^{N+1}} |y|^a\widetilde{U}^2_\lambda(x,y,t)\mathcal{G}_a(x,y,1)\, dxdy &= \frac{1}{H(\lambda^2,U)}\int_{\mathbb{R}^{N+1}} |y|^a U^2(x,y,\lambda^2 t)\mathcal{G}_a(x,y,\lambda^2 t)\, dxdy \\ &= \frac{H(\lambda^2 t,U)}{H(\lambda^2,U)}, \quad \lambda > 0,\end{aligned} \tag{5.15}$$

and, furthermore,

$$\begin{aligned}\int_{\mathbb{R}^{N+1}} |y|^a|\nabla\widetilde{U}_\lambda|^2(x,y,t)\mathcal{G}_a(x,y,1)\, dxdy &= \frac{\lambda^2 t}{H(\lambda^2,U)}\int_{\mathbb{R}^{N+1}} |y|^a|\nabla U|^2(x,y,\lambda^2 t)\mathcal{G}_a(x,y,\lambda^2 t)\, dxdy \\ &= \frac{\lambda^2 t I(\lambda^2 t,U)}{H(\lambda^2,U)} = N(\lambda^2 t,U)\frac{H(\lambda^2 t,U)}{H(\lambda^2,U)}, \quad \lambda > 0.\end{aligned}$$

Now, assume both $t \in (0, T_*]$ and $\lambda \in (0, 1/\sqrt{T_*}]$. Using the monotonicity of the functions $t \to H(t,U)$ and $t \to N(t,U)$ (cfr. with the proof of Lemma 4.6) and formula (4.35) of Corollary 4.8, we obtain

$$\frac{H(\lambda^2 t,U)}{H(\lambda^2,U)} \le \frac{H(\lambda^2 T_*,U)}{H(\lambda^2,U)} \le T_*^{2N(\lambda^2 T_*,U)} \le T_*^{2N_1},$$

where we have set $N(\lambda^2 t,U) \le N(1,U) := N_1 < \infty$ and used that $\lambda^2 t \le 1$. Consequently, we deduce

$$\int_{\mathbb{R}^{N+1}} \widetilde{U}^2_\lambda(x,y,t)\, d\mu(x,y) \le T_*^{2N_1}, \qquad \int_{\mathbb{R}^{N+1}} |\nabla\widetilde{U}_\lambda|^2(x,y,t)\, d\mu(x,y) \le N_1 T_*^{2N_1},$$

for a.e. $0 < t \le T_*$ and all $0 < \lambda \le 1/\sqrt{T_*}$. The last two formulas imply that the family

$$\{\widetilde{U}_\lambda\}_{\lambda\in(0,1/\sqrt{T_*}]} \text{ is uniformly bounded in } L^\infty(0,T_*;H^1_\mu), \tag{5.16}$$

where we recall that $H^1_\mu := H^1(\mathbb{R}^{N+1}, d\mu)$. Furthermore, from (5.14), we get that

$$\|\partial_t\widetilde{U}_\lambda(t)\|_{L^2_\mu} \le \frac{N_1 T_*^{2N_1}}{t} \quad \text{for a.e. } 0 < t \le T_*, \text{ and all } 0 < \lambda \le 1/\sqrt{T_*}$$

From this last bound we deduce that for any $0 < t_* < T_*$ the family

$$\{\partial_t \widetilde{U}_\lambda\}_{\lambda\in(0,1]} \text{ is uniformly bounded in } L^\infty(t_*, T_*; L^2_\mu). \tag{5.17}$$

So it turns out that the family $\{\widetilde{U}_\lambda\}_{\lambda\in(0,1]}$ is relatively compact in $C^0\left(t_*, T_*; L^2_\mu\right)$ for all fixed $0 < t_* < T_*$, where $L^2_\mu := L^2(\mathbb{R}^{N+1}, d\mu)$ (cfr. for instance with Corollary 8 of [179]).

Step3: Compactness and properties of the limit. So, for any $\lambda_n \to 0^+$ and any fixed $0 < t_* < 1$, we can extract a sub-sequence $\lambda_{n_j} \to 0^+$ (that we rename $\lambda_j := \lambda_{n_j}$ by convenience) such that

$$\widetilde{U}_{\lambda_j} \to \widetilde{\Theta} \quad \text{in } C^0\left(t_*, T_*; L^2_\mu\right), \quad \text{as } j \to +\infty, \tag{5.18}$$

where $\widetilde{\Theta} \in \bigcap_{t_*\in(0,T_*)} C^0\left(t_*, T_*; L^2_\mu\right)$. Note that *a priori* the sequence λ_j depends on $0 < t_* < T_*$. However, an easy diagonal procedure (that we skip not to weight down our presentation) allows us to eliminate this dependence.

Let us now present some important properties of the limit $\widetilde{\Theta} = \widetilde{\Theta}(x, y, t)$. First of all, since $\|\widetilde{U}_\lambda(\cdot, \cdot, 1)\|_{L^2_\mu} = 1$ for all $\lambda > 0$ (cfr. (5.15) with $t = 1$), it follows

$$\|\widetilde{\Theta}(\cdot, \cdot, 1)\|_{L^2_\mu} = 1,$$

too and so the limit is not trivial (this crucial property follows since we have normalized the blow-up family $U_\lambda = U_\lambda(x, y, t)$). Moreover, (5.16) and (5.17) allow us to take the subsequence $\lambda_j \to 0^+$ such that

$$\widetilde{U}_{\lambda_j} \rightharpoonup \widetilde{\Theta} \quad \text{weakly in } L^2(t_*, T_*; H^1_\mu),$$
$$\partial_t\widetilde{U}_{\lambda_j} \rightharpoonup \partial_t\widetilde{\Theta} \quad \text{weakly in } L^2(t_*, T_*; L^2_\mu),$$

for all $0 < t_* < T_*$. This follows from the fact that $L^2(t_*, T_*; H^1_\mu) \subset L^\infty(t_*, T_*; H^1_\mu)$ and $L^2(t_*, T_*; L^2_\mu) \subset L^\infty(t_*, T_*; L^2_\mu)$, and from the reflexivity of $L^2(t_*, T_*; H^1_\mu)$ and $L^2(t_*, T_*; L^2_\mu)$. Consequently, thanks to the previous two convergence properties and the arbitrariness of $0 < t_* < T_*$, it satisfies

$$t\int_{\mathbb{R}^{N+1}} \partial_t\widetilde{U}\eta(t)\, d\mu(x, y) = \int_{\mathbb{R}^{N+1}} \nabla\widetilde{U}\nabla\eta(t)\, d\mu(x, y),$$

for a.e. $t_* < t < T_*$ and for all $\eta(t) \in H^1_\mu$ (cfr. with (5.13)). Now subtracting the equations of $\widetilde{U}_{\lambda_j}$ and $\widetilde{U}$, respectively, and testing with $\eta(t) = (\widetilde{U}_{\lambda_j} - \widetilde{\Theta})(t) \in H^1_\mu$ we get

$$t\left\langle \partial_t(\widetilde{U}_{\lambda_j} - \widetilde{\Theta})(t), (\widetilde{U}_{\lambda_j} - \widetilde{\Theta})(t)\right\rangle_{L^2_\mu} = \int_{\mathbb{R}^{N+1}} |\nabla(\widetilde{U}_{\lambda_j} - \widetilde{\Theta})(t)|^2\, d\mu(x, y), \quad \text{for a.e. } t_* \le t \le T_*,$$

and so, integrating between t_* and T_*, we easily find

$$\begin{aligned}\int_{t_*}^{T_*} &\|(\widetilde{U}_{\lambda_j} - \widetilde{\Theta})(t)\|^2_{L^2_\mu} + \|\nabla(\widetilde{U}_{\lambda_j} - \widetilde{\Theta})(t)\|^2_{L^2_\mu}\, dt \\ &= \frac{1}{2}\left[\|(\widetilde{U}_{\lambda_j} - \widetilde{\Theta})(t_*)\|^2_{L^2_\mu} + \|(\widetilde{U}_{\lambda_j} - \widetilde{\Theta})(T_*)\|^2_{L^2_\mu}\right] \to 0, \quad \text{as } j \to +\infty,\end{aligned}$$

thanks to (5.18). Note that we have used that $2(\partial_t(\widetilde{U}_{\lambda_j} - \widetilde{\Theta}), \widetilde{U}_{\lambda_j} - \widetilde{\Theta})_{L^2_\mu} = \partial_t(\|\widetilde{U}_{\lambda_j} - \widetilde{\Theta}\|^2_{L^2_\mu})$ and integrated by parts (w.r.t. t). This implies

$$\widetilde{U}_{\lambda_j} \to \widetilde{\Theta} \quad \text{in } L^2\left(t_*, T_*; H^1_\mu\right), \quad \text{as } j \to +\infty. \tag{5.19}$$

Step4: Properties of the re-scaling of $\widetilde{U}_\lambda$. Consequently, scaling back to $U_\lambda = U_\lambda(x,y,t)$, for any $0 < t_* < T_*$, it holds

$$\lim_{j\to+\infty}\int_{t_*}^{T_*}\left\|U_{\lambda_j}(\cdot,\cdot,t)-\Theta(\cdot,\cdot,t)\right\|^2_{H^1(\mathbb{R}^{N+1},d\mu_t)}\,dt \to 0, \quad \text{as } j\to+\infty, \tag{5.20}$$

$$\lim_{j\to+\infty}\sup_{t\in[t_*,T_*]}\left\|U_{\lambda_j}(\cdot,\cdot,t)-\Theta(\cdot,\cdot,t)\right\|^2_{L^2(\mathbb{R}^{N+1},d\mu_t)} \to 0, \quad \text{as } j\to+\infty, \tag{5.21}$$

where we recall that

$$U_\lambda(x,y,t) := \widetilde{U}_\lambda\left(\frac{x}{\sqrt{t}},\frac{y}{\sqrt{t}},t\right) \qquad \text{and} \qquad \Theta(x,y,t) := \widetilde{\Theta}\left(\frac{x}{\sqrt{t}},\frac{y}{\sqrt{t}},t\right).$$

Note that from weak convergence we obtain also

$$N(t,U_{\lambda_j}) := \frac{tI(t,U_{\lambda_j})}{H(t,U_{\lambda_j})} = \frac{t\int_{\mathbb{R}^{N+1}}|y|^a|\nabla U_{\lambda_j}|^2(x,y,t)\mathcal{G}_a(x,y,t)\,dxdy}{\int_{\mathbb{R}^{N+1}}|y|^a U^2_{\lambda_j}(x,y,t)\mathcal{G}_a(x,y,t)\,dxdy} \to N(t,\Theta) = \frac{tI(t,\Theta)}{H(t,\Theta)},$$

for a.e. $t_* < t < T_*$, as $j\to+\infty$, where

$$H(t,\Theta) := \int_{\mathbb{R}^{N+1}}|y|^a\Theta^2(x,y,t)\mathcal{G}_a(x,y,t)\,dxdy,$$
$$I(t,\Theta) := \int_{\mathbb{R}^{N+1}}|y|^a|\nabla\Theta|^2(x,y,t)\mathcal{G}_a(x,y,t)\,dxdy.$$

Moreover, by scaling, it turns out that $\Theta = \Theta(x,y,t)$ satisfies (4.17) in $\mathbb{R}^{N+1}\times(t_*,T_*)$:

$$\int_{\mathbb{R}^{N+1}}\left[\partial_t\Theta + \frac{(x,y)}{2t}\cdot\nabla\Theta\right]\eta\,d\mu_t(x,y) = \int_{\mathbb{R}^{N+1}}\nabla\Theta\cdot\nabla\eta\,d\mu_t(x,y), \tag{5.22}$$

for a.e. $t_* < t < T_*$ and all $\eta(t)\in H^1_{\mu_t}$. Consequently, the function $t\to N(t,\Theta)$ is well-defined and non-decreasing in (t_*,T_*) (recall that $H(t,\Theta)>0$ for all $t_*<t<T_*$, in view of Corollary 4.9, since $\Theta\not\equiv 0$).

Step4: The limit is a re-scaled eigenfunction. Now, since $N(t,U_{\lambda_j}) = N(\lambda_j^2 t,U)$, we can fix $t_*<t<T_*$ and take the limit as $j\to+\infty$ to obtain

$$N(t,U_{\lambda_j}) \to \kappa := \lim_{t\to 0}N(t,U), \quad \text{as } j\to+\infty,$$

for a.e. $t_*<t<T_*$. Consequently,

$$N(t,\Theta)\equiv\kappa, \quad \text{for a.e. } t_*<t<T_*, \tag{5.23}$$

and so, as in the proof of Lemma 4.6, the re-scaled version $\widetilde{\Theta}(x,y,t) = \Theta(\sqrt{t}x,\sqrt{t}y,t)$ must be a "weak" eigenfunction to the Ornstein-Uhlenbeck eigenvalue problem type (4.3), in the sense that the identity

$$\int_{\mathbb{R}^{N+1}}\nabla\widetilde{\Theta}\cdot\nabla\eta(t)\,d\mu(x,y) = \kappa\int_{\mathbb{R}^{N+1}}\widetilde{\Theta}\,\eta(t)\,d\mu(x,y), \tag{5.24}$$

is satisfied for a.e. $t_*<t<T_*$ and all $\eta(t)\in H^1_\mu$. From the above identity, it thus follows that κ is an eigenvalue of the Ornstein-Uhlenbeck operator $\mathcal{O}_a$ (cfr. with Section 4.4) and we complete the proof of part (i).

Note that, from Lemma 4.5 (cfr. with *Step4*) we know that the function $t \to t^{-\kappa}\widetilde{\Theta}(x, y, t)$ is constant in time (cfr. with (4.31)) and so, if

$$\frac{\widetilde{\Theta}(x, y, t)}{t^\kappa} = V(x, y),$$

where of course $V = V(x, y)$ still satisfies (5.24), we deduce

$$\Theta(x, y, t) = t^\kappa V\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right),$$

for a.e. $(x, y, t) \in \mathbb{R}^{N+1} \times (t_*, T_*)$, obtaining a precise expression for the blow-up limits in terms of linear combinations of "re-scaled" eigenfunctions to problem (4.3).

Step5: Uniqueness of the blow-up. We complete the proof of the theorem by showing that (5.20) and (5.21) do not depend on the subsequences λ_n and λ_{n_j}. We begin by showing that

$$\lim_{t\to 0^+} t^{-2\kappa} H(t, U) > 0.$$

Note that from Corollary 4.7 we have that the above limit exists finite and it is nonnegative. Note that it gives an improvement of formula (4.36) for small times, in the sense that

$$H(t, U) \sim Ct^{2\kappa} \quad \text{as } t \sim 0^+, \tag{5.25}$$

for some constant $C > 0$. This will be important later. Now, assume by contradiction $t^{-2\kappa}H(t, U) \to 0$ as $t \to 0^+$ and consider the family of eigenfunctions

$$\left\{V_{\alpha,m} = V_{\alpha,m}(x, y) : \alpha \in \mathbb{Z}^N_{\geq 0} \text{ with } |\alpha| = n,\ n \in \mathbb{N}, m \in \mathbb{N}\right\},$$

found in Theorem 4.1. We thus obtain an orthonormal basis of L^2_μ defining

$$\overline{V}_{\alpha,m}(x, y) = \frac{V_{\alpha,m}(x, y)}{\|V_{\alpha,m}\|_{L^2_\mu}}, \qquad \alpha \in \mathbb{Z}^N_{\geq 0} \text{ with } |\alpha| = n,\ n \in \mathbb{N}, m \in \mathbb{N}.$$

Now, by scaling, we have that the "not-normalized blow-up" of $U = U(x, y, 1)$:

$$U^\lambda(x, y, 1) := \sqrt{H(\lambda^2, U)}\, U_\lambda(x, y, 1) = U(\lambda x, \lambda y, \lambda^2) \tag{5.26}$$

belongs to H^1_μ for a.e. $0 < \lambda \leq 1/\sqrt{T_*}$ and so we can write

$$U^\lambda(x, y, 1) = \sum_{\alpha,m} u_{\alpha,m}(\lambda)\overline{V}_{\alpha,m}(x, y) \quad \text{in } L^2_\mu,$$

where the coefficients are given by

$$u_{\alpha,m}(\lambda) = \int_{\mathbb{R}^{N+1}} U^\lambda(x, y, 1)\overline{V}_{\alpha,m}(x, y)\, d\mu(x, y).$$

Note that by the orthonormality of the eigenfunctions $\overline{V}_{\alpha,m} = \overline{V}_{\alpha,m}(x, y)$ and the usual scaling properties, it easy to see that

$$\begin{aligned} H(\lambda^2, U) &= \int_{\mathbb{R}^{N+1}} \left[U^\lambda(x, y, 1)\right]^2 d\mu(x, y) \\ &= \int_{\mathbb{R}^{N+1}} \left[\sum_{\alpha,m} u_{\alpha,m}(\lambda)\overline{V}_{\alpha,m}(x, y)\right]^2 d\mu(x, y) \geq u^2_{\alpha,m}(\lambda), \end{aligned}$$

for any $\alpha \in \mathbb{Z}^N_{\geq 0}$, $m \in \mathbb{N}$, so that the assumption $t^{-2\kappa}H(t,U) \to 0$ as $t \to 0^+$ implies

$$\lambda^{-2\kappa}u_{\alpha,m}(\lambda) \to 0 \quad \text{as } \lambda \to 0^+, \tag{5.27}$$

where we have used the fact that $H(\lambda^2, U) = H(1, U^\lambda)$. Now, from the regularity assumptions on U^λ, we have that

$$\frac{d}{d\lambda}U^\lambda(x,y,1) = \sum_{\alpha,m} u'_{\alpha,m}(\lambda)\overline{V}_{\alpha,m}(x,y) \quad \text{in } L^2_\mu. \tag{5.28}$$

On the other hand,

$$\begin{aligned}\frac{d}{d\lambda}U^\lambda(x,y,1) &= (x,y)\cdot\nabla U(\lambda x,\lambda y,\lambda^2) + 2\lambda\partial_t U(\lambda x,\lambda y,\lambda^2)\\ &= \frac{2}{\lambda}\left[\frac{(x,y)}{2}\nabla U^\lambda(x,y,1) + \partial_t U^\lambda(x,y,1)\right] \quad \text{for a.e. } (x,y)\in\mathbb{R}^{N+1},\end{aligned}$$

and so, testing with $|y|^a\eta(x,y)\mathcal{G}_a(x,y,1)$ we obtain

$$\int_{\mathbb{R}^{N+1}} \frac{dU^\lambda}{d\lambda}\eta\, d\mu(x,y) = \frac{2}{\lambda}\int_{\mathbb{R}^{N+1}} \nabla U^\lambda \nabla\eta\, d\mu(x,y),$$

for $\eta \in H^1_\mu$. Moreover, since $\overline{V}_{\alpha,m} = \overline{V}_{\alpha,m}(x,y)$ are solutions to (4.41) with $\kappa = \kappa_{n,m}$ (defined in the statement of Theorem 4.1), we deduce

$$\begin{aligned}\frac{2}{\lambda}\int_{\mathbb{R}^{N+1}}\nabla U^\lambda\nabla\eta\, d\mu &= \frac{2}{\lambda}\sum_{\alpha,m} u_{\alpha,m}(\lambda)\int_{\mathbb{R}^{N+1}}\nabla\overline{V}_{\alpha,m}\nabla\eta\, d\mu\\ &= \frac{2}{\lambda}\sum_{\alpha,m}\kappa_{n,m}u_{\alpha,m}(\lambda)\int_{\mathbb{R}^{N+1}}\overline{V}_{\alpha,m}\eta\, d\mu,\end{aligned}$$

and so, matching with (5.28) and using the orthogonality of $\overline{V}_{\alpha,m} = \overline{V}_{\alpha,m}(x,y)$, we get

$$u'_{\alpha,m}(\lambda) = \frac{2\kappa_{n,m}}{\lambda}u_{\alpha,m}(\lambda) \qquad \text{for all } \alpha\in\mathbb{Z}^N_{\geq 0} \text{ with } |\alpha| = n,\ n\in\mathbb{N}, m\in\mathbb{N}.$$

Thus, integrating between $0 < \underline{\lambda} < \lambda \leq 1$, we find

$$\lambda^{-2\kappa_{n,m}}u_{\alpha,m}(\lambda) = \underline{\lambda}^{-2\kappa_{n,m}}u_{\alpha,m}(\underline{\lambda}), \tag{5.29}$$

and so, letting $\underline{\lambda} \to 0^+$ and using (5.27) (recall that $\kappa = \kappa_{n,m}$ for some $n,m\in\mathbb{N}$ as showed in *Step4*), it follows $u_{\alpha,m}(\lambda) = 0$ for all $0 < \lambda \leq 1$ which implies $U^\lambda \equiv 0$, getting the desired contradiction.

Now, let us assume (5.25) to be valid and proceed with the second part of the proof. In *Step4* we have showed that if $U = U(x,y,t)$ is a "strong solution" to equation (7), then for any $0 < t_* < T_* < \infty$ and for any sequence $\lambda_n \to 0^+$, there exists a subsequence $\lambda_j = \lambda_{n_j}$ such that (cfr. with (5.20) and (5.21))

$$\lim_{j\to+\infty}\int_{t_*}^1 \left\|U_{\lambda_j}(\cdot,\cdot,t) - \Theta(\cdot,\cdot,t)\right\|^2_{H^1(\mathbb{R}^{N+1},d\mu_t)} dt \to 0, \quad \text{as } j\to+\infty,$$

$$\lim_{j\to+\infty}\sup_{t\in[t_*,T_*]}\left\|U_{\lambda_j}(\cdot,\cdot,t) - \Theta(\cdot,\cdot,t)\right\|^2_{L^2(\mathbb{R}^{N+1},d\mu_t)} \to 0, \quad \text{as } j\to+\infty,$$

where

$$U_\lambda(x,y,t) := \frac{U(\lambda x,\lambda y,\lambda^2 t)}{\sqrt{H(\lambda^2,U)}}, \qquad \Theta(x,y,t) := t^\kappa\, V\left(\frac{x}{\sqrt{t}},\frac{y}{\sqrt{t}}\right),$$

and $V = V(x,y)$ is a "weak" eigenfunction to (4.3):

$$-\mathcal{L}_a V + \frac{(x,y)}{2}\cdot\nabla V = \kappa V \quad \text{in } \mathbb{R}^{N+1}.$$

Moreover, we know that $\kappa = \kappa_{n,m}$ for some $n, m \in \mathbb{N}$ where $\kappa_{n,m}$ are defined in the statement of Theorem 4.1.
Now, if $n_0, m_0 \in \mathbb{N}$ are such that $\kappa = \kappa_{n_0,m_0}$ and $\mathcal{V}_{\alpha_0,m_0}$ is the associated eigenspace, and $\overline{V}_{\alpha,m} = \overline{V}_{\alpha,m}(x,y)$ are the normalized eigenfunctions, we have that

$$V(x,y) = \sum_{(\alpha,m)\in J_0} v_{\alpha,m}\overline{V}_{\alpha,m}(x,y) \quad \text{in } L^2_\mu,$$

where $J_0 := \left\{(\alpha,m)\in\mathbb{Z}^N_{\geq 0}\times\mathbb{N} : |\alpha| = n\in\mathbb{N} \text{ and } \kappa_{n,m} = \kappa_{n_0,m_0}\right\}$. Consequently, in view of (5.25), we can re-write the above convergence properties as

$$\int_{t_*}^{T_*}\left\|\lambda_j^{-2\kappa}U(\lambda_j x,\lambda_j y,\lambda_j^2 t) - t^\kappa\sum_{(\alpha,m)\in J_0} v_{\alpha,m}\overline{V}_{\alpha,m}\left(\frac{x}{\sqrt{t}},\frac{y}{\sqrt{t}}\right)\right\|^2_{H^1(\mathbb{R}^{N+1},d\mu_t)} dt \to 0, \tag{5.30}$$

$$\sup_{t\in[t_*,T_*]}\left\|\lambda_j^{-2\kappa}U(\lambda_j x,\lambda_j y,\lambda_j^2 t) - t^\kappa\sum_{(\alpha,m)\in J_0} v_{\alpha,m}\overline{V}_{\alpha,m}\left(\frac{x}{\sqrt{t}},\frac{y}{\sqrt{t}}\right)\right\|^2_{L^2(\mathbb{R}^{N+1},d\mu_t)} \to 0, \tag{5.31}$$

as $j\to+\infty$. In particular, taking $t = 1$ (recall that we are assuming $T_*\geq 1$ for simplicity), we deduce

$$\lambda_j^{-2\kappa}U(\lambda_j x,\lambda_j y,\lambda_j^2) \to \sum_{(\alpha,m)\in J_0} v_{\alpha,m}\overline{V}_{\alpha,m}(x,y) \quad \text{in } L^2_\mu. \tag{5.32}$$

So, in other to prove that (5.30) and (5.31) hold for any subsequence $\lambda_{n_j}\to 0^+$, or, equivalently, hold for $\lambda\to 0^+$, we are left to show that the coefficients v_{α,m_0} are independent from both λ_n and its subsequence λ_{n_j}. Thus, proceeding as in the first part of this step, we expand

$$U(\lambda x,\lambda y,\lambda^2) = \sum_{\alpha,m} u_{\alpha,m}(\lambda)\overline{V}_{\alpha,m}(x,y),$$

in series of normalized eigenfunctions. So, multiplying (5.32) by $|y|^a\overline{V}_{\alpha,m}(x,y)\mathcal{G}_a(x,y,1)$ and integrating on $\mathbb{R}^{N+1}$, we get

$$\lambda_j^{-2\kappa}u_{\alpha,m}(\lambda_j)\to v_{\alpha,m} \quad \text{as } j\to+\infty,$$

where we have used again the orthogonality of the eigenfunctions $\overline{V}_{\alpha,m} = \overline{V}_{\alpha,m}(x,y)$. On the other hand, exactly as before (cfr. with (5.29)), it holds

$$\lambda^{-2\kappa_{n,m}}u_{\alpha,m}(\lambda) = \underline{\lambda}^{-2\kappa_{n,m}}u_{\alpha,m}(\underline{\lambda}),$$

for all $0<\underline{\lambda}<\lambda\leq 1/\sqrt{T_*}$ which means that the function

$$\lambda\to\lambda^{-2\kappa}u_{\alpha,m}(\lambda) = \lambda^{-2\kappa}\int_{\mathbb{R}^{N+1}} U(\lambda x,\lambda y,\lambda^2)\overline{V}_{\alpha,m}(x,y)\,d\mu(x,y),$$

is constantly equal to $v_{\alpha,m}$ for all $0<\lambda<1/\sqrt{T_*}$. It follows in particular that $v_{\alpha,m}$ depends neither on λ_n nor on λ_{n_j}. This shows the uniqueness of the blow-up limit.

Step6: Improved convergence. Now, for any $0 < t < T_*$ and $0 < \lambda < 1/\sqrt{T_*}$, we consider $U_\lambda(x, y, t) = \lambda^{-2\kappa_{n_0,m_0}} U(\lambda x, \lambda y, \lambda^2 t)$ and

$$\Theta(x, y, t) = t^{\kappa_{n_0,m_0}} V\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) = t^{\kappa_{n_0,m_0}} \sum_{(\alpha,m)\in J_0} v_{\alpha,m} \overline{V}_{\alpha,m}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right),$$

for some $v_{\alpha,m} \in \mathbb{R}$, and $n_0, m_0 \in \mathbb{N}$ are suitably chosen (cfr. with part (i)). So far, we have proved that for all $0 < t_* < T_*$, it holds (cfr. with (5.19))

$$\lim_{\lambda\to 0} \int_{t_*}^{T_*} \|U_\lambda - \Theta\|^2_{L^2_{\mu_t}} + t\,\|\nabla U_\lambda - \nabla\Theta\|^2_{L^2_{\mu_t}}\, dt = 0,$$

and we want to get

$$\lim_{\lambda\to 0} \int_{0}^{T_*} \|U_\lambda - \Theta\|^2_{L^2_{\mu_t}} + t\,\|\nabla U_\lambda - \nabla\Theta\|^2_{L^2_{\mu_t}}\, dt = 0. \tag{5.33}$$

So, we assume by contradiction that there exists $\varepsilon > 0$ and a subsequence $\lambda_j \to 0$ (as $j \to +\infty$), such that

$$\int_{0}^{T_*} \left\|U_{\lambda_j} - \Theta\right\|^2_{L^2_{\mu_t}} + t\left\|\nabla U_{\lambda_j} - \nabla\Theta\right\|^2_{L^2_{\mu_t}} dt \geq \varepsilon, \tag{5.34}$$

for all large $j \in \mathbb{N}$. Now, since $S_{\lambda_j} := U_{\lambda_j} - \Theta$ still satisfies the assumptions of Lemma 4.6, we known that (cfr. with *Step1* of the above quoted lemma):

$$2\int_{t_1}^{t_2} I(t, S_{\lambda_j})\, dt = H(t_2, S_{\lambda_j}) - H(t_1, S_{\lambda_j}),$$

for any choice of times $0 < t_1 < t_2 < T_*$ and so, taking $t_2 = t_*$ and $t_1 \to 0$, we deduce

$$\int_0^{t_*}\int_{\mathbb{R}^{N+1}} |\nabla S_{\lambda_j}|^2\, d\mu_t(x, y)dt \leq \frac{1}{2}\int_{\mathbb{R}^{N+1}} S^2_{\lambda_j}(x, y, t_*)\, d\mu^{t_*}(x, y) = \frac{1}{2}H(t_*, S_{\lambda_j}).$$

Furthermore, setting $\nu_* = \min\{1, 1-a\}/2$ and applying the Gaussian-Poincaré inequality (cfr. Theorem (4.15), part (i)), we get

$$\begin{aligned}\int_0^{t_*}\int_{\mathbb{R}^{N+1}} S^2_{\lambda_j}\, d\mu_t(x, y)dt &\leq \frac{1}{\nu_*}\int_0^{t_*} t\int_{\mathbb{R}^{N+1}} |\nabla S_{\lambda_j}|^2\, d\mu_t(x, y)dt \\ &\leq \frac{t_*}{\nu_*}\int_0^{t_*}\int_{\mathbb{R}^{N+1}} |\nabla S_{\lambda_j}|^2\, d\mu_t(x, y)dt \leq \frac{t_*}{2\nu_*}H(t_*, S_{\lambda_j}),\end{aligned}$$

so that it follows

$$\int_0^{t_*} \left\|U_{\lambda_j} - \Theta\right\|^2_{L^2_{\mu_t}} + t\left\|\nabla U_{\lambda_j} - \nabla\Theta\right\|^2_{L^2_{\mu_t}} dt \leq t_*\left(\frac{1+\nu_*}{2\nu_*}\right)H(t_*, S_{\lambda_j}).$$

Note that, in view of (5.18) we can assume $H(t_*, S_{\lambda_j}) = \|U_{\lambda_j} - \Theta\|_{L^2(\mathbb{R}^{N+1}, d\mu^{t_*})} \leq C$ for all large $j \in \mathbb{N}$, where $C > 0$ is constant not depending on λ_j. It thus follows

$$\int_0^{t_*} \left\|U_{\lambda_j} - \Theta\right\|^2_{L^2_{\mu_t}} + t\left\|\nabla U_{\lambda_j} - \nabla\Theta\right\|^2_{L^2_{\mu_t}} dt \leq Ct_*,$$

for all fixed $0 < t_* < T_*$, where $C > 0$ is a new constant not depending on λ_j. Consequently, (5.34) begins

$$\begin{aligned}\int_{t_*}^{T_*} \left\|U_{\lambda_j} - \Theta\right\|^2_{L^2_{\mu_t}} + t\left\|\nabla U_{\lambda_j} - \nabla\Theta\right\|^2_{L^2_{\mu_t}} dt &\geq \varepsilon - \int_0^{t_*} \left\|U_{\lambda_j} - \Theta\right\|^2_{L^2_{\mu_t}} + t\left\|\nabla U_{\lambda_j} - \nabla\Theta\right\|^2_{L^2_{\mu_t}} dt \\ &\geq \varepsilon - Ct_* > \frac{\varepsilon}{2},\end{aligned}$$

if $0 < t_* < \varepsilon/(2C)$, and we obtain the desired contradiction since the l.h.s. must converge to 0 as $\lambda_j \to 0$. On the other hand, we have proved that

$$\lim_{\lambda\to 0} \sup_{t\in[t_*,T_*]} \|U_\lambda - \Theta\|^2_{L^2_{\mu_t}} = \lim_{\lambda\to 0} \sup_{t\in[t_*,T_*]} H(t, U_\lambda - \Theta) = 0.$$

However, since the function $t \to H(t, U_\lambda - \Theta)$ is non-decreasing, it follows $\sup_{t\in[t_*,T_*]} H(t, U_\lambda - \Theta) = \sup_{t\in(0,T_*]} H(t, U_\lambda - \Theta)$ and so

$$\lim_{\lambda\to 0} \|U_\lambda - \Theta\|^2_{C^0(0,T;L^2_{\mu_t})} := \lim_{\lambda\to 0} \sup_{t\in(0,T_*]} \|U_\lambda - \Theta\|^2_{L^2_{\mu_t}} \tag{5.35}$$

The proof is now completed. □

Remark. We point out that formula (5.25) implies that

$$ct^\kappa \le H(t, U) \le Ct^\kappa \quad \text{for } t \sim 0^+,$$

for some $0 < c < C$. These are fundamental relations and will be crucial in the proof of Lemma 5.20.

Proof of Corollary 5.2. Let $u \in \mathrm{dom}(H^s)$ be a solution to (1):

$$H^s u = 0 \quad \text{a.e. in } \mathbb{R}^N \times (-T, 0),$$

where $0 < s < 1$ and $T > 0$ are fixed and H is the "Heat Operator" $H := \partial_\tau - \Delta$. Then if $a := 2s - 1$ and $U = U(x, y, t)$ is defined as in (2), it follows by Section 5 and 8 of [19]) that $\overline{U}(x, y, t) := U(x, y, -t)$ is a "strong solution" to problem (8):

$$\begin{cases} \partial_t \overline{U} + y^{-a}\nabla\cdot(y^a\nabla\overline{U}) = 0 & \text{in } \mathbb{R}^{N+1}_+ \times (0, T) \\ -\partial^a_y \overline{U} = 0 & \text{in } \mathbb{R}^N \times \{0\} \times (0, T), \end{cases}$$

satisfying by regularity (cfr. with Corollary 1.3 of [183] and Theorem 5.1 of [19]):

$$\overline{U}(x, 0, t) = U(x, 0, -t) = u(x, -t) \quad \text{for all } (x, t) \in \mathbb{R}^N \times (0, T).$$

Keeping in mind this fact, we claim that the following assertions hold true (cfr. with the statement of Theorem 5.1). There exist $n_0, m_0 \in \mathbb{N}$ such that

(i) The Almgren-Poon quotient $N(t, \overline{U})$ (cfr. with the formula in (4.26)) satisfies

$$\lim_{t\to 0^+} N(t, \overline{U}) = \widetilde{\kappa}_{n_0,m_0},$$

where

$$\widetilde{\kappa}_{n,m} := \frac{n}{2} + m \qquad m, n \in \mathbb{N}$$

are the eigenvalues of problem (4.1) (cfr. with the statement of Theorem 4.1, part (i)).

(ii) For all $T_* > 0$, we have as $\lambda \to 0^+$

$$\int_0^{T_*} \left\| \lambda^{-2\widetilde{\kappa}_{n_0,m_0}} \overline{U}(\lambda x, \lambda y, \lambda^2 t) - t^{\widetilde{\kappa}_{n_0,m_0}} \sum_{(\alpha,m)\in\widetilde{J}_0} v_{\alpha,m} \overline{V}_{\alpha,m}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) \right\|^2_{H^1(\mathbb{R}^{N+1}_+, d\mu_t)} dt \to 0, \tag{5.36}$$

where $v_{\alpha,m}$ are suitable constants, the sum is done over the set of indices

$$\widetilde{J}_0 := \{(\alpha, m) \in \mathbb{Z}^N_{\ge 0} \times \mathbb{N} : |\alpha| = n \in \mathbb{N} \text{ and } \widetilde{\kappa}_{n,m}, = \widetilde{\kappa}_{n_0,m_0}\},$$

and the integration probability measure is given in the statement of Corollary 5.2. Similarly, for $\overline{V}_{\alpha,m} = \overline{V}_{\alpha,m}(x,y)$ which are the normalized versions of the eigenfunctions $\widetilde{V}_{\alpha,m} = \widetilde{V}_{\alpha,m}(x,y)$ to problem (4.1) corresponding to the eigenvalue $\widetilde{\kappa}_{n,m}$ and defined by:

$$\widetilde{V}_{\alpha,m}(x,y) = H_\alpha(x) L_{(\frac{a-1}{2}),m}(y^2/4),$$

where $H_\alpha(\cdot)$ is a N-dimensional Hermite polynomial of order $|\alpha|$, while $L_{(\frac{a-1}{2}),m}(\cdot)$ is the m^{th} Laguerre polynomial of order $(a-1)/2$ (cfr. with the statement of Theorem 4.1, part (i)).

The proof of the above claims is immediate since it almost coincides with the one of Theorem 5.1 proved above. The main ingredients are the monotonicity of the Almgren-Poon quotient and the spectral properties of parabolically homogeneous solutions studied in Section 4.4. Both these aspect have been investigated in Subsection 4.3.1 and Section 4.4 for "strong solutions" to problem (8). It is thus straightforward to repeat the above proof by replacing the "quantities" related to "strong solutions" to equation (7) with the corresponding quantities related to "strong solutions" to problem (8). We just mention that the main difference w.r.t. the previous case is that the only admissible eigenvalues are the half-integers with corresponding eigenfunctions according to part (i) of Theorem 4.1. Finally, it is immediately seen that the statement of Corollary 5.2 follows by using the definition of $\overline{U}(x,y,t) = U(x,y,-t)$ in claims (i) and (ii). □

Examples. Out of clarity, we complete the section with some concrete examples of "blow-up profiles". Let us put ourselves in the easiest case when the spacial dimension is $N = 1$ and denote by

$$\widetilde{\Theta}_{\alpha,m}(x,y,t) = t^{\widetilde{\kappa}_{n,m}} V_{\alpha,m}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) = t^{\widetilde{\kappa}_{n,m}} H_n\left(\frac{x}{\sqrt{t}}\right) L_{(\frac{a-1}{2}),m}\left(\frac{y^2}{4t}\right),$$

the "blow-up profile" corresponding to $\widetilde{\kappa}_{n,m} = \frac{n}{2} + m$ (i.e. to the eigenfunction $V_{\alpha,m} = V_{\alpha,m}(x,y)$). Then we have:

$$\begin{aligned}
&\widetilde{\kappa}_{0,0} = 0 && \widetilde{\Theta}_{0,0}(x,y,t) = 1\\
&\widetilde{\kappa}_{1,0} = \frac{1}{2} && \widetilde{\Theta}_{1,0}(x,y,t) = x\\
&\widetilde{\kappa}_{2,0} = 1 && \widetilde{\Theta}_{2,0}(x,y,t) = x^2 - 2t\\
&\widetilde{\kappa}_{0,1} = 1 && \widetilde{\Theta}_{0,1}(x,y,t) = \left(\frac{1+a}{2}\right)t - \frac{y^2}{4}\\
&\widetilde{\kappa}_{3,0} = \frac{3}{2} && \widetilde{\Theta}_{3,0}(x,y,t) = x(x^2 - 6t)\\
&\widetilde{\kappa}_{1,1} = \frac{3}{2} && \widetilde{\Theta}_{1,1}(x,y,t) = x\left[\left(\frac{1+a}{2}\right)t - \frac{y^2}{4}\right]\\
&\widetilde{\kappa}_{4,0} = 2 && \widetilde{\Theta}_{4,0}(x,y,t) = x^4 - 12x^2t + 12t^2\\
&\widetilde{\kappa}_{2,1} = 2 && \widetilde{\Theta}_{2,1}(x,y,t) = (x^2 - 2t)\left[\left(\frac{1+a}{2}\right)t - \frac{y^2}{4}\right]\\
&\widetilde{\kappa}_{0,2} = 2 && \widetilde{\Theta}_{0,2}(x,y,t) = \frac{1}{8}\left[(1+a)(3+a)t^2 - (3+a)y^2t + \frac{y^4}{4}\right].
\end{aligned}$$

On the other hand, since the "blow-up profiles" corresponding to $\widehat{\kappa}_{n,m} = \frac{n}{2} + m + \frac{1-a}{2} = \widetilde{\kappa}_{n,m} + \frac{1-a}{2}$ satisfy:

$$\begin{aligned}
\widehat{\Theta}_{\alpha,m}(x,y,t) &= t^{\widehat{\kappa}_{n,m}} V_{\alpha,m}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right) = t^{\widehat{\kappa}_{n,m}} H_n\left(\frac{x}{\sqrt{t}}\right) \frac{y}{\sqrt{t}} \left|\frac{y}{\sqrt{t}}\right|^{-a} L_{(\frac{1-a}{2}),m}\left(\frac{y^2}{4t}\right)\\
&= t^{\widetilde{\kappa}_{n,m}} y|y|^{-a} H_n\left(\frac{x}{\sqrt{t}}\right) L_{(\frac{1-a}{2}),m}\left(\frac{y^2}{4t}\right),
\end{aligned}$$

we easily see that

$$\begin{aligned}
\widehat{\kappa}_{0,0} &= \frac{1-a}{2} & \widehat{\Theta}_{0,0}(x,y,t) &= y|y|^{-a}\\
\widehat{\kappa}_{1,0} &= \frac{2-a}{2} & \widehat{\Theta}_{1,0}(x,y,t) &= xy|y|^{-a}\\
\widehat{\kappa}_{2,0} &= \frac{3-a}{2} & \widehat{\Theta}_{2,0}(x,y,t) &= y|y|^{-a}(x^2-2t)\\
\widehat{\kappa}_{0,1} &= \frac{3-a}{2} & \widehat{\Theta}_{0,1}(x,y,t) &= y|y|^{-a}\left[\left(\frac{3-a}{2}\right)t-\frac{y^2}{4}\right]\\
\widehat{\kappa}_{3,0} &= \frac{4-a}{2} & \widehat{\Theta}_{3,0}(x,y,t) &= x(x^2-6t)y|y|^{-a}\\
\widehat{\kappa}_{1,1} &= \frac{4-a}{2} & \widehat{\Theta}_{1,1}(x,y,t) &= xy|y|^{-a}\left[\left(\frac{3-a}{2}\right)t-\frac{y^2}{4}\right]\\
\widehat{\kappa}_{4,0} &= \frac{5-a}{2} & \widehat{\Theta}_{4,0}(x,y,t) &= y|y|^{-a}(x^4-12x^2t+12t^2)\\
\widehat{\kappa}_{2,1} &= \frac{5-a}{2} & \widehat{\Theta}_{2,1}(x,y,t) &= y|y|^{-a}(x^2-2t)\left[\left(\frac{3-a}{2}\right)t-\frac{y^2}{4}\right]\\
\widehat{\kappa}_{0,2} &= \frac{5-a}{2} & \widehat{\Theta}_{0,2}(x,y,t) &= \frac{1}{8}y|y|^{-a}\left[(3-a)(5-a)t^2-(5-a)y^2t+\frac{y^4}{4}\right].
\end{aligned}$$

5.3 Liouville type theorems

In this section we present some Liouville type theorems. The first result will be obtained as an easy application of the monotonicity formulae proved in Lemma 4.5 and Lemma 4.6, the Gaussian Poincaré inequalities proved in Section 4.5, and Theorem 5.1, while the second one requires a different monotonicity formula of Alt-Caffarelli-Friedman type, that we prove in Subsection 5.3.1. Again this new monotonicity formula easily follows from the Gaussian-Poincaré inequality (cfr. with Theorem 4.15), combined with Theorem 5.1.

5.3.1 Alt-Caffarelli-Friedman monotonicity formula

In this short subsection, we prove a "mono-species" Alt-Caffarelli-Friedman ([8]) monotonicity formula (cfr. for instance with [14, Lemma 5.4], [49, Theorem 1.1.4], [53, Theorem 12.11], [72, Section 2]), which turns out to be an easy consequence of the Gaussian-Poincaré type inequalities proved in Theorem 4.15. As always we set

$$\nu_* = \frac{1}{2}\min\{1, 1-a\},$$

and we prove the following lemma.

Lemma 5.15. *(Alt-Caffarelli-Friedman monotonicity formula) The following three statements hold:*
(i) Let $U = U(x,y,t)$ be a "strong solution" to equation (7). Then the function

$$t \to J(t,U) := \frac{1}{t^{2\nu_*}}\int_0^t \int_{\mathbb{R}^{N+1}} |\nabla U|^2(x,y,\tau)\, d\mu^\tau(x,y) d\tau \tag{5.37}$$

is nondecreasing for all $0 < t < T$ and it is constant if and only if $U(x,y,t) = A$ or, depending on $-1 < a < 1$:

$$U(x,y,t) = \begin{cases} Ax_j & \text{if } a \le 0 \quad \text{for some } j \in \{1,\dots,N+1\} \\ Ay|y|^{-a} & \text{if } a \ge 0, \end{cases}$$

where $A \in \mathbb{R}$ and we have used the convention $x_{N+1} = y$.

(ii) Let $U = U(x, y, t)$ be a "strong solution" to problem (8). *Then the function*

$$t \to J(t, U) := \frac{1}{t} \int_0^t \int_{\mathbb{R}^{N+1}_+} |\nabla U|^2(x, y, \tau)\, d\mu^\tau(x, y) d\tau \tag{5.38}$$

is nondecreasing for all $0 < t < T$ and it is constant if and only if $U(x, y, t) = A$ or $U(x, y, t) = Ax_j$ for some $j \in \{1, \ldots, N\}$ and $A \in \mathbb{R}$.

(iii) Let $U = U(x, y, t)$ be a "strong solution" to problem (9). *Then the function*

$$t \to J(t, U) := \frac{1}{t^{1-a}} \int_0^t \int_{\mathbb{R}^{N+1}_+} |\nabla U|^2(x, y, \tau)\, d\mu^\tau(x, y) d\tau \tag{5.39}$$

is nondecreasing for all $0 < t < T$ and it is constant if and only if $U(x, y, t) = Ay^{1-a}$ and $A \in \mathbb{R}$.

Proof. We begin by proving assertion (i). First of all, we have

$$J'(t) = -\frac{2\nu_*}{t^{2\nu_*+1}} \int_0^t \int_{\mathbb{R}^{N+1}} |\nabla U|^2\, d\mu^\tau(x, y) d\tau + \frac{1}{t^{2\nu_*}} \int_{\mathbb{R}^{N+1}} |\nabla U|^2\, d\mu^\tau(x, y), \tag{5.40}$$

for all $0 < t < T$. Furthermore, testing the equation with $\eta = U$, we get

$$\begin{aligned}
\int_0^t \int_{\mathbb{R}^{N+1}} |\nabla U|^2\, d\mu^\tau(x, y) d\tau &= \int_0^t \int_{\mathbb{R}^{N+1}} |y|^a \nabla U \cdot \nabla U\, \mathcal{G}_a(x, y, \tau)\, dx dy d\tau \\
&= \int_0^t \int_{\mathbb{R}^{N+1}} |y|^a U \partial_\tau U \mathcal{G}_a\, dx dy d\tau - \int_0^t \int_{\mathbb{R}^{N+1}} |y|^a U \nabla U \cdot \nabla \mathcal{G}_a\, dx dy d\tau \\
&= \frac{1}{2} \int_0^t \int_{\mathbb{R}^{N+1}} |y|^a \partial_\tau \left(U^2\right) \mathcal{G}_a\, dx dy d\tau + \frac{1}{2} \int_0^t \int_{\mathbb{R}^{N+1}} |y|^a U^2 \partial_\tau \mathcal{G}_a\, dx dy d\tau \\
&= \frac{1}{2} \int_{\mathbb{R}^{N+1}} \int_0^t |y|^a \partial_\tau \left(U^2 \mathcal{G}_a\right)\, dx dy d\tau \\
&\le \frac{1}{2} \int_{\mathbb{R}^{N+1}} U^2(x, y, t)\, d\mu_t(x, y)
\end{aligned}$$

where we have used the equations of $U = U(x, y, t)$ and $\mathcal{G}_a = \mathcal{G}_a(x, y, t)$, and the identity

$$2 \int_{\mathbb{R}^{N+1}} |y|^a U \nabla U \cdot \nabla \mathcal{G}_a\, dx dy = - \int_{\mathbb{R}^{N+1}} U^2 \nabla \cdot (|y|^a \nabla \mathcal{G}_a)\, dx dy,$$

for a.e. $0 < \tau < T$. Consequently, combining (5.40) and the above estimate, it suffices to prove

$$\frac{1}{t^{2\nu_*+1}} \left\{ t \int_{\mathbb{R}^{N+1}} |\nabla U|^2\, d\mu_t(x, y) - \nu_* \int_{\mathbb{R}^{N+1}} U^2\, d\mu_t(x, y) \right\} \ge 0,$$

which can be easily re-written as

$$\frac{t \int_{\mathbb{R}^{N+1}} |\nabla U|^2\, d\mu_t(x, y)}{\int_{\mathbb{R}^{N+1}} U^2\, d\mu_t(x, y)} \ge \nu_*, \tag{5.41}$$

and, passing to the re-normalized variables $\widetilde{U}(x, y, t) = U(x/\sqrt{t}, y/\sqrt{t}, t)$, as

$$\frac{\int_{\mathbb{R}^{N+1}} |\nabla \widetilde{U}|^2\, d\mu(x, y)}{\int_{\mathbb{R}^{N+1}} \widetilde{U}^2\, d\mu(x, y)} \ge \nu_*, \tag{5.42}$$

which turns out to be valid by the definition of $\nu_* = \min\{1, 1-a\}/2$ and the Gaussian-Poincaré inequality proved in Theorem 4.15, part (i) (note that we can always assume that $\widetilde{U}(\cdot,\cdot,t)$ has zero mean for all $t > 0$, as in the proof of Lemma 5.16).

To complete the proof, we firstly fix $-1 < a \leq 0$ (so that $\nu_* = 1/2$) and note that Theorem 4.15 implies that the equality is attained in (5.42) if and only if

$$\widetilde{U}(x,y,t) = A(t)x_j \qquad \text{i.e.} \qquad U(x,y,t) = \sqrt{t}A(t)\,x_j,$$

for some $j \in \{1,\ldots,N+1\}$ and function $A = A(t)$ (recall the re-labeling convention $x_{N+1} = y$). Since such $U = U(x,y,t)$ must be a solution to (7) and it satisfies $\nabla\cdot(|y|^a\nabla U) = 0$, we deduce that $A(t) = 1/\sqrt{t}$ (up to multiplicative constants), completing the proof in the case $-1 < a \leq 0$. In the case $0 \leq a < 1$ ($\nu_* = (1-a)/2$), we proceed in the same way, noting that the equality is attained in (5.42) if and only if

$$\widetilde{U}(x,y,t) = A(t)\,y|y|^{-a} \qquad \text{i.e.} \qquad U(x,y,t) = t^{\frac{1-a}{2}}A(t)\,y|y|^{-a}.$$

Consequently, taking $A(t) = t^{\frac{a-1}{2}}$ (up to multiplicative constants), we conclude the proof of the case $0 \leq a < 1$, too.

The proofs of part (ii) and (iii) are very similar to the one just presented and we skip them. We just mention that it is enough to repeat the above procedure using the integral formulation for "strong solutions" to (8) and (9), and the "right" Gaussian-Poincaré inequality (cfr. with part (ii) and (iii) of Theorem 4.15). □

Remark. Let $J = J(t,U)$ be defined as in (5.37). Then, from the fact that

$$\int_0^t\int_{\mathbb{R}^{N+1}}|\nabla U|^2\,d\mu^\tau(x,y)d\tau \leq \frac{1}{2}\int_{\mathbb{R}^{N+1}}U^2(x,y,t)\,d\mu_t(x,y) = \frac{1}{2}H(t,U), \tag{5.43}$$

for all $t > 0$, it easily follows that $J = J(t,U)$ is in fact well-defined and positive (except for constant functions), for all $t > 0$. Note that, in view of the proof of Theorem 5.1 (cfr. with formula (5.25)), we have $H(t,U) \sim Ct^{2\nu_*}$ for $t \sim 0^+$ which gives us

$$\int_0^t\int_{\mathbb{R}^{N+1}}|\nabla U|^2\,d\mu^\tau(x,y)d\tau \leq Ct^{2\nu_*}, \quad \text{for } t \sim 0^+, \tag{5.44}$$

and a suitable constant $C > 0$. Moreover, the converse inequality in (5.44) holds too, for any nonconstant solutions $U = U(x,y,t)$ (that, as always, we can assume to have zero mean w.r.t. $d\mu_t = d\mu_t(x,y)$, for all $t > 0$). Indeed, from (5.41), we get

$$\begin{aligned}\frac{1}{\nu_*}J(t,U) &\geq \frac{1}{t^{2\nu_*}}\int_0^t\frac{1}{\tau}\int_{\mathbb{R}^{N+1}}U^2(x,y,\tau)\,d\mu^\tau(x,y)d\tau = \frac{1}{t^{2\nu_*}}\int_0^t\frac{H(\tau,U)}{\tau}\,d\tau\\ &\geq \frac{1}{t^{2\nu_*+1}}\int_0^t H(\tau,U)\,d\tau \geq \frac{C}{t^{2\nu_*+1}}\int_0^t\tau^{2\nu_*}\,d\tau \geq C > 0, \quad \text{for } t \sim 0^+,\end{aligned}$$

where we have used formula (5.25) again. Consequently, we have

$$\int_0^t\int_{\mathbb{R}^{N+1}}|\nabla U|^2\,d\mu^\tau(x,y)d\tau \sim Ct^{2\nu_*}, \quad \text{for } t \sim 0^+,$$

and as an immediate consequence, we obtain $\lim_{t\to 0^+}J(t,U) > 0$ when $U = U(x,y,t)$ is nonconstant. The very same observations hold if we replace the functional (5.37) with (5.38) and/or (5.39), with simple modifications.

5.3.2 Liouville type theorems

Let us now pass to the proof of some Liouville type results. In this context, it is important to stress that we focus on the *global* quantitative behaviour of "strong solutions", instead of the *local* one (this is easily seen, for instance, in the interest of the asymptotic behaviour of the quantity $H(t,U)$ as $t \to \infty$, instead of $t \to 0^+$). In what follows we employ the following definition of "parabolic distance":

$$d^2(x,y,t) := |x|^2 + |y|^2 + t, \qquad \text{for all } (x,y,t) \in \mathbb{R}^{N+1} \times (0,\infty).$$

Note that it is not a distance in the usual sense, since the triangular inequality does not hold, but it is invariant under parabolic dilatations, in the sense that $d(x,y,t) = \sqrt{t}\, d(x/\sqrt{t}, y/\sqrt{t}, 1)$.

Lemma 5.16. *The following three statements hold.*

(i) Let $U = U(x,y,t)$ be a "strong solution" to equation (7) *in $\mathbb{R}^{N+1} \times (0,\infty)$ (i.e. $T = \infty$) and assume it satisfies the bound*

$$U^2(x,y,t) \le C[1 + d^{2\nu}(x,y,t)], \tag{5.45}$$

for some $C > 0$ and some exponent $\nu > 0$ satisfying

$$0 < \nu < \min\{1, 1-a\}.$$

Then U is constant in $\mathbb{R}^{N+1} \times (0,\infty)$.

(ii) Let $U = U(x,y,t)$ be a "strong solution" to problem (8) *in $\mathbb{R}^{N+1}_+ \times (0,\infty)$ (i.e. $T = \infty$) and assume it satisfies the bound in* (5.45) *for some $C > 0$ and some exponent $\nu > 0$ satisfying*

$$0 < \nu < 1.$$

Then U is constant in $\mathbb{R}^{N+1}_+ \times (0,\infty)$.

(iii) Let $U = U(x,y,t)$ be a "strong solution" to problem (9) *in $\mathbb{R}^{N+1}_+ \times (0,\infty)$ (i.e. $T = \infty$) and assume it satisfies the bound in* (5.45) *for some $C > 0$ and some exponent $\nu > 0$ satisfying*

$$0 < \nu < 1 - a.$$

Then U is identically zero in $\mathbb{R}^{N+1}_+ \times (0,\infty)$.

Proof. We begin by proving part (i). First of all, we note that for any $\nu > 0$, if $U = U(x,y,t)$ satisfies the point-wise bound (5.45), it holds

$$\begin{aligned} H(t,U) &= \int_{\mathbb{R}^{N+1}} |y|^a U^2(x,y,t) \mathcal{G}_a(x,y,t)\, dxdy \\ &\le C \int_{\mathbb{R}^{N+1}} |y|^a \left[(|x|^2 + |y|^2 + t)^\nu + 1\right] \mathcal{G}_a(x,y,t)\, dxdy \\ &= C + t^\nu \int_{\mathbb{R}^{N+1}} |y|^a (|x|^2 + |y|^2 + 1)^\nu \mathcal{G}_a(x,y,1)\, dxdy \le Ct^\nu, \end{aligned} \tag{5.46}$$

for a suitable constant $C > 0$ and $t > 0$ large. Now, set as always $\kappa := \lim_{t\to 0^+} N(t,U)$ and assume by contradiction that $U = U(x,y,t)$ is non constant.

Let us start with the case $\kappa > 0$. In this case, it must be

$$\kappa \ge \frac{1}{2}\min\{1, 1-a\} := \nu_*, \tag{5.47}$$

since $\kappa > 0$ has to be an eigenvalue of problem (4.3) (cfr. with Theorem 5.1) and $1/2$ and $(1-a)/2$ are the first nontrivial eigenvalue depending on $-1 < a < 0$ or $0 < a < 1$, respectively (cfr. with Theorem 4.1). Now, in *Step5* of Theorem 5.1, we have shown that

$$\lim_{t\to 0^+} t^{-2\kappa} H(t,U) := C > 0,$$

for some positive constant $C > 0$. Moreover, proceeding as in Corollary 4.8, it holds

$$\frac{H'(t,U)}{H(t,U)} \geq \frac{2\kappa}{t}, \quad \text{for all } t > 0,$$

and so, integrating between $0 < t_0 < t$ and t, we get

$$H(t,U) \geq t_0^{-2\kappa} H(t_0,U) t^{2\kappa}, \quad \text{for all } 0 < t_0 \leq t.$$

Consequently, taking the limit as $t_0 \to 0^+$ we finally obtain $H(t,U) \geq Ct^{2\kappa}$ for all $t > 0$ and so, thanks to (5.47), we deduce

$$H(t,U) \geq Ct^{\min\{1,1-a\}} = Ct^{2\nu_*},$$

for $t > 0$ large enough, which is in contradiction with the bound from above in 5.46, unless $U = U(x,y,t)$ is constant.

Now, assume that $\kappa = 0$. In this case, the bound in (5.46) is still true, but instead of the last bound from below, we just get $H(t,U) \geq C$ for all $t > 0$, which is not enough to immediately deduce the desired contradiction. So, we pass to the re-scaling $\widetilde{U}(x,y,t) = U(\sqrt{t}x, \sqrt{t}y, t)$ and we recall that $\widetilde{U}(t) \in H^1(\mathbb{R}^{N+1}, d\mu)$ for all $t > 0$. Consequently, we can apply the Gaussian Poincaré inequality of Theorem 4.15, part (i), to deduce

$$\int_{\mathbb{R}^{N+1}} \widetilde{U}^2(t)\, d\mu - \left(\int_{\mathbb{R}^{N+1}} \widetilde{U}(t)\, d\mu \right)^2 \leq P_a \int_{\mathbb{R}^{N+1}} |\nabla \widetilde{U}|^2(t)\, d\mu, \quad \text{for all } t > 0, \tag{5.48}$$

where $P_a := 1/\nu_* = 2/\min\{1, 1-a\}$. Note that we can assume $\int_{\mathbb{R}^{N+1}} \widetilde{U}(t)\, d\mu = 0$ for all $t > 0$. Indeed, we have already seen that the function

$$t \to \int_{\mathbb{R}^{N+1}} \widetilde{U}(t)\, d\mu$$

is constant (cfr. with (4.22)) and, since $\widetilde{U} = \widetilde{U}(x,y,t)$ satisfies a linear equation, we can replace $\widetilde{U}$ by

$$\widetilde{U}(x,y,t) - \int_{\mathbb{R}^{N+1}} \widetilde{U}(t)\, d\mu,$$

to get a "strong solution" to equation (7) with zero mean for all $t > 0$. Consequently, (5.48) can be easily re-written as

$$\nu_* = \frac{1}{2}\min\{1, 1-a\} \leq N(1, \widetilde{U}),$$

and, since the l.h.s. is strictly positive, while the r.h.s. is converging to zero as $t \to 0^+$, we get a contradiction unless $\widetilde{U} \equiv 0$ and we conclude the proof of part (i) (note that we get $\widetilde{U} \equiv 0$ since we have assumed that $\widetilde{U}$ as zero mean for any $t > 0$)

For what concerns part (ii) and (iii), it is easily seen that the proof of part (i) works also in these different settings with straightforward modifications. The most significative is that we have to employ the "right" Gaussian Poincaré inequality proved in Theorem 4.15, depending on the problem (Neuman or Dirichlet) satisfied by $U = U(x,y,t)$. □

Lemma 5.17. *The following three statements hold.*

(i) Let $U = U(x,y,t)$ be a "strong solution" to equation (7) *in $\mathbb{R}^{N+1} \times (0,\infty)$ (i.e. $T = \infty$) and assume it satisfies the bound*

$$|\nabla U(x,y,t)|^2 \leq C\, d^{2\nu}(x,y,t), \tag{5.49}$$

for some $C > 0$ and some exponent $\nu \in \mathbb{R}$ satisfying

$$\nu < \min\{0, -a\}.$$

Then U is constant in $\mathbb{R}^{N+1} \times (0, \infty)$.

(ii) Let $U = U(x, y, t)$ be a "strong solution" to problem (8) *in $\mathbb{R}^{N+1}_+ \times (0, \infty)$ (i.e. $T = \infty$) and assume it satisfies the bound in* (5.49) *for some $C > 0$ and some exponent $\nu \in \mathbb{R}$ satisfying*

$$\nu < 0.$$

Then U is constant in $\mathbb{R}^{N+1}_+ \times (0, \infty)$.

(iii) Let $U = U(x, y, t)$ be a "strong solution" to problem (9) *in $\mathbb{R}^{N+1}_+ \times (0, \infty)$ (i.e. $T = \infty$) and assume it satisfies the bound in* (5.49) *for some $C > 0$ and some exponent $\nu \in \mathbb{R}$ satisfying*

$$\nu < -a.$$

Then U is identically zero in $\mathbb{R}^{N+1}_+ \times (0, \infty)$.

Proof. As before, we prove part (i), while (ii) and (iii) follow similarly. W.r.t. to the above proof, the present one is based on the monotonicity of the functional $t \to J(t, U)$ defined in (5.37) instead of the Almgren-Poon quotient $t \to N(t, U)$.

So, if (5.49) holds true, then

$$\begin{aligned} J(t, U) &= \frac{1}{t^{2\nu_*}} \int_0^t \int_{\mathbb{R}^{N+1}} |\nabla U|^2(x, y, \tau)\, d\mu^\tau(x, y) d\tau \\ &\leq \frac{C}{t^{2\nu_*}} \int_0^t \int_{\mathbb{R}^{N+1}} (|x|^2 + |y|^2 + \tau)^\nu\, d\mu^\tau(x, y) d\tau \\ &\leq \frac{C}{t^{2\nu_*}} \int_0^t \tau^\nu d\tau = \frac{C}{t^{2\nu_* - \nu - 1}} \to 0, \end{aligned}$$

as $t \to +\infty$, thanks to the assumption on $\nu \in \mathbb{R}$. From Lemma 5.15 it thus follows that $J(t, U) = 0$ for any $t > 0$, and so, since the measure $d\mu_t = d\mu_t(x, y)$ is nonnegative, it follows $|\nabla U| = 0$ a.e. in $\mathbb{R}^{N+1} \times (0, \infty)$, i.e. $U = U(x, y, t)$ is constant. □

Remark. Note that the above two Liouville type results have a quite different nature. In the first one, the "critical growth condition" is imposed one the function $U = U(x, y, t)$, whilst, in the second, a sort of decaying property of the norm of the gradient is required. Finally, note that a independent proof could have been proposed by using the Gaussian-Poincaré inequality (Theorem 4.15) and repeating the proof of Lemma 5.16.

5.4 Blow-up analysis II: proof of Theorem 5.3 and Corollary 5.4

In this section we go forward with study of the asymptotic behaviour of the normalized blow-up sequence studied in Theorem 5.1 by showing Theorem 5.3. To do so, we adapt a very ingenious technique firstly employed in [189, 190] to study the elliptic framework, to the parabolic one. The method is based on a blow-up-of-the-blow-up sequence and on the construction of two auxiliary sequences combined with the application of Liouville type results proved in Lemma 5.16.

Proof of Theorem 5.3. Let $U = U(x, y, t)$ be a "strong solution" to equation (7). As in Section 5.2, we consider the blow-up sequence

$$U_\lambda(x, y, t) = \frac{U(\lambda x, \lambda y, \lambda^2 t)}{\lambda^{2\kappa}}, \qquad \kappa := \lim_{t \to 0^+} N(t, U),$$

so that $U_\lambda = U_\lambda(x,y,t)$ satisfies the same equation in $\mathbb{R}^{N+1}\times(0,T/\lambda^2)$, $\lambda>0$. Moreover, we us fix a time $T_*>0$ and a compact set $K\subset\mathbb{R}^{N+1}\times(0,T_*)$ (note that since $\lambda\to 0$, we can assume also $T_*<T/\lambda^2$). The first step of the proof consists in showing that the family $\{U_\lambda\}_\lambda$ is uniformly bounded in the Hölder space $C^{2\nu,\nu}(K)$ for any $0<\nu<\nu_*=\min\{1,1-a\}/2$, i.e.,

$$\|U_\lambda\|_{L^\infty(K)}+[U_\lambda]_{C^{2\nu,\nu}(K)}\le C,$$

for some constant $C>0$ independent of $\lambda>0$, where the symbol $[\,\cdot\,]_{C^{2\nu,\nu}(K)}$ denotes the Hölder semi-norm. Note that it suffices to prove the above bound with $K=\mathbb{Q}_{1/2}:=\mathbb{B}_{1/2}\times(3/4,1)$ and deduce the general case through standard covering and/or scaling procedures.

If $\mathbb{Q}_1:=\mathbb{B}_1\times(0,1)$, a uniform bound for $\|U_\lambda\|_{L^\infty(\mathbb{Q}_1)}$ can be obtained through the Harnack inequality proved by Chiarenza and Serapioni in Theorem 2.1 of [65] (note that from L^2-L^∞ estimates we already know that U_λ is locally bounded, cfr. with Lemma 2.1 of [65] and/or formula (5.7) of [19]). Indeed, applying that theorem to the family of solutions $U_\lambda=U_\lambda(x,y,t)$ we easily deduce that

$$\sup_{\mathbb{Q}_1}|U_\lambda|\le C\inf_{\overline{\mathbb{Q}}_1}|U_\lambda|,\quad \lambda>0,$$

for some constant $C>0$ independent of $\lambda>0$, where $\overline{\mathbb{Q}}_1:=\mathbb{B}_1\times(6,7)$. Consequently, if by contradiction, there exists a sequence $\lambda_j\to 0$ such that $\|U_{\lambda_j}\|_{L^\infty(\mathbb{Q}_1)}\to+\infty$ as $j\to+\infty$, we obtain that

$$U_{\lambda_j}(x,y,t)\to+\infty\qquad\text{for all }(x,y,t)\in\underline{\mathbb{Q}}_1,$$

for any subset $\underline{\mathbb{Q}}_1\subset\overline{\mathbb{Q}}_1$, obtaining a contradiction with the $L^2(0,T_*,H^1_{\mu_t})$-convergence type of Theorem 5.1 toward locally uniformly bounded profiles.

For what concerns the Hölder semi-norm, we assume the uniform bound of $\|U_\lambda\|_{L^\infty(\mathbb{Q}_1)}$, and we show that for any $0<\nu<\nu_*=\min\{1,1-a\}/2$, there exists a constant $C>0$ (independent of $\lambda>0$), such that

$$[U_\lambda]_{C^{2\nu,\nu}(\mathbb{Q}_{1/2})}:=\sup_{\mathbb{Q}_{1/2}}\frac{|U_\lambda(X_1,t_1)-U_\lambda(X_2,t_2)|}{(|X_1-X_2|^2+|t_1-t_2|)^\nu}\le C,\tag{5.50}$$

where we have set by convenience $X:=(x,y)\in\mathbb{R}^{N+1}$ and $|X|=\sqrt{|x|^2+|y|^2}$. More precisely, for any $0<\nu<\nu_*:=\min\{1,1-a\}/2$, we will show

$$[\eta U_\lambda]_{C^{2\nu,\nu}}(\mathbb{Q}_1)\le C,\tag{5.51}$$

where $\eta=\eta(X,t)$ is a smooth function satisfying

$$\begin{cases}\eta(X,t)=1 & \text{for }(X,t)\in\mathbb{Q}_{1/2}\\ 0<\eta(X,t)\le 1 & \text{for }(X,t)\in\mathbb{Q}_1\setminus\mathbb{Q}_{1/2}\\ \eta(X,t)=0 & \text{for }(X,t)\in\partial\mathbb{Q}_1,\end{cases}$$

where $\partial\mathbb{Q}_1:=[\partial\mathbb{B}_1\times(0,1)]\cup[\mathbb{B}_1\times\{0\}]\cup[\mathbb{B}_1\times\{1\}]$. From the definition of $\eta=\eta(X,t)$, (5.50) easily follows from (5.51).

Step1: Reductio ad absurdum and first definitions. Assume by contradiction that (5.51) does not hold, i.e. there exists $0<\nu<\nu_*$ and a sequence $\lambda_j\to 0^+$ as $j\to+\infty$ such that

$$[\eta U_{\lambda_j}]_{C^{2\nu,\nu}}(\mathbb{Q}_1)=\sup_{\mathbb{Q}_1}\frac{|\eta(X_1,t_1)U_{\lambda_j}(X_1,t_1)-\eta(X_2,t_2)U_{\lambda_j}(X_2,t_2)|}{(|X_1-X_2|^2+|t_1-t_2|)^\nu}:=L_j\to+\infty,$$

as $j\to+\infty$. We may assume that for any $j\in\mathbb{N}$, L_j is achieved by $(X_{1,j},t_{1,j}),(X_{2,j},t_{2,j})\in\overline{\mathbb{Q}}_1$, i.e.

$$L_j:=\frac{|\eta(X_{1,j},t_{1,j})U_{\lambda_j}(X_{1,j},t_{1,j})-\eta(X_{2,j},t_{2,j})U_{\lambda_j}(X_{2,j},t_{2,j})|}{r_j^{2\nu}},\tag{5.52}$$

for any index $j \in \mathbb{N}$, where, for future simplicity, we have defined

$$r_j := (|X_{1,j} - X_{2,j}|^2 + |t_{1,j} - t_{2,j}|)^{\frac{1}{2}}, \quad j \in \mathbb{N}.$$

From (5.52), we immediately deduce the bound

$$L_j \le \frac{\|U_{\lambda_j}\|_{L^\infty(\mathbb{Q}_1)}}{r_j^{2\nu}} |\eta(X_{1,j}, t_{1,j}) - \eta(X_{2,j}, t_{2,j})|,$$

and so, since we know that U_{λ_j} is uniformly bounded in $\mathbb{Q}_1$, we get that $r_j \to 0$ as $j \to +\infty$. Furthermore, from the same bound on L_j and the smoothness of $\eta = \eta(X,t)$, it is easily seen that

$$\frac{\mathrm{dist}((X_{1,j}, t_{1,j}), \partial\mathbb{Q}_1)}{r_j} + \frac{\mathrm{dist}((X_{2,j}, t_{2,j}), \partial\mathbb{Q}_1)}{r_j} \ge \frac{L_j r_j^{2\nu-1}}{L\|U_{\lambda_j}\|_{L^\infty(\mathbb{Q}_1)}},$$

where $L > 0$ is taken such that $|\eta(X_1, t_1) - \eta(X_2, t_2)| \le L(|X_1 - X_2|^2 + |t_1 - t_2|)^{\frac{1}{2}}$ and, since $0 < 2\nu < 1$, we get

$$\frac{\mathrm{dist}((X_{1,j}, t_{1,j}), \partial\mathbb{Q}_1)}{r_j} + \frac{\mathrm{dist}((X_{2,j}, t_{2,j}), \partial\mathbb{Q}_1)}{r_j} \to +\infty, \tag{5.53}$$

as $j \to +\infty$.

Step2: Auxiliary sequences. Following the ideas of [189, Section 6] and [190, Section 4], the remaining part of the proof is based on the analysis of two different sequences:

$$\begin{aligned} W_j(X,t) &:= \eta(\widehat{X}_j, \widehat{t}_j) \frac{U_{\lambda_j}(\widehat{X}_j + r_j X, \widehat{t}_j + r_j^2 t)}{L_j r_j^{2\nu}}, \\ \overline{W}_j(X,t) &:= \frac{(\eta U_{\lambda_j})(\widehat{X}_j + r_j X, \widehat{t}_j + r_j^2 t)}{L_j r_j^{2\nu}}, \end{aligned} \tag{5.54}$$

for any sequence of points $\widehat{P}_j = (\widehat{X}_j, \widehat{t}_j) \in \mathbb{Q}_1$ and

$$(X,t) \in \mathbb{Q}^j = (\mathbb{Q}_1 - \widehat{P}_j)/r_j = \mathbb{B}_{1/r_j}(\widehat{X}_j) \times (-\widehat{t}_j/r_j^2, (1-\widehat{t}_j)/r_j^2), \quad j \in \mathbb{N}.$$

The definitions of the new sequences in (5.54) are motivated by two crucial facts. The first one, is that the Hölder semi-norm of order $0 < \nu < \nu_*$ of $\overline{W}_j = \overline{W}_j(X,t)$ is bounded independently of $j \in \mathbb{N}$:

$$[\overline{W}_j]_{C^{2\nu,\nu}(\mathbb{Q}^j)} = 1 \qquad \text{for all } j \in \mathbb{N}.$$

This easily follows by the definition of $\overline{W}_j = \overline{W}_j(X,t)$, L_j and r_j. On the other hand, the first sequence satisfies the notable equation

$$\partial_t W_j + \mathcal{L}_a^j W_j = 0 \quad \text{in } \mathbb{R}^{N+1} \times (-\widehat{t}_j/r_j^2, (1-\widehat{t}_j)/r_j^2), \tag{5.55}$$

where $\mathcal{L}_a^j W = |\widehat{y}_j r_j^{-1} + y|^{-a} \nabla \cdot (|\widehat{y}_j r_j^{-1} + y|^a \nabla W_j)$ and $\widehat{X}_j = (\widehat{x}_j, \widehat{y}_j)$, for all $j \in \mathbb{N}$, with $W_j(0,0) = \overline{W}_j(0,0)$. Another important feature of these two sequences is that they are asymptotically equivalent on compact sets of $\mathbb{R}^{N+1} \times \mathbb{R}$. Indeed, if $K \subset \mathbb{R}^{N+1} \times \mathbb{R}$ is compact and $(X,t) \in K$, we have

$$\begin{aligned} |W_j(X,t) - \overline{W}_j(X,t)| &\le \frac{\|\eta U_{\lambda_j}\|_{L^\infty(\mathbb{Q}_1)}}{r_j^{2\nu} L_j} |\eta(\widehat{X}_j + r_j X, \widehat{t}_j + r_j^2 t) - \eta(\widehat{X}_j, \widehat{t}_j)| \\ &\le L \frac{\|\eta U_{\lambda_j}\|_{L^\infty(\mathbb{Q}_1)}}{r_j^{2\nu-1} L_j} (|X|^2 + t)^{\frac{1}{2}} \to 0, \end{aligned}$$

as $j \to +\infty$ (recall that $r_j \to 0$, $L_j \to +\infty$, and $2v < 1$). In particular, it follows

$$\|W_j - \overline{W}_j\|_{L^\infty(K\cap \mathbb{Q}^j)} \to 0 \quad \text{as } j \to +\infty. \tag{5.56}$$

Furthermore, from the bound above and recalling that $\overline{W}_j = \overline{W}_j(X,t)$ is ν-Hölder continuous with $\overline{W}_j(0,0) = W_j(0,0)$, it is easily seen that

$$\begin{aligned} |W_j(X,t) - W_j(0,0)| &\le |W_j(X,t) - \overline{W}_j(X,t)| + |\overline{W}_j(X,t) - \overline{W}_j(0,0)| \\ &\le C\left(\frac{1}{r_j^{2\nu-1}L_j}(|X|^2+t)^{\frac{1}{2}} + (|X|^2+t)^{\nu}\right), \end{aligned}$$

from which we deduce the existence of a constant $C > 0$ depending on the compact set $K \subset \mathbb{R}^{N+1}\times\mathbb{R}$ such that

$$\sup_{(X,t)\in K\cap \mathbb{Q}^j} |W_j(X,t) - W_j(0,0)| \le C. \tag{5.57}$$

These last two properties will be crucial in the next step.

Step3: Asymptotic behaviour of $(X_{1,j}, t_{1,j})$ *and* $(X_{2,j}, t_{2,j})$. We now show that the sequences $(X_{1,j}, t_{1,j})$ and $(X_{2,j}, t_{2,j})$ approach the characteristic manifold $\Sigma = \{(x,y) \in \mathbb{R}^{N+1} : y = 0\}$ as $j \to +\infty$. More precisely, we prove the existence of a constant $C > 0$ such that

$$\frac{\operatorname{dist}((X_{1,j},t_{1,j}), \mathbb{Q}_1\cap\Sigma)}{r_j} + \frac{\operatorname{dist}((X_{2,j},t_{2,j}), \mathbb{Q}_1\cap\Sigma)}{r_j} \le C, \tag{5.58}$$

for $j \in \mathbb{N}$ large enough. Arguing by contradiction, we assume that

$$\frac{\operatorname{dist}((X_{1,j},t_{1,j}), \mathbb{Q}_1\cap\Sigma)}{r_j} + \frac{\operatorname{dist}((X_{2,j},t_{2,j}), \mathbb{Q}_1\cap\Sigma)}{r_j} \to +\infty \quad \text{as } j \to +\infty.$$

Let us take $(\widehat{X}_j, \widehat{t}_j) = (X_{1,j}, t_{1,j})$ in the definition of $W_j = W_j(X,t)$ and $\overline{W}_j = \overline{W}_j(X,t)$, so that, thanks to (5.53),

$$\mathbb{Q}^j \to \mathbb{R}^{N+1}\times\mathbb{R} \qquad \text{or} \qquad \mathbb{Q}^j \to \mathbb{R}^{N+1}\times(0,\infty) \quad \text{as } j \to +\infty.$$

In what follows, we will assume $\mathbb{Q}^j \to \mathbb{R}^{N+1}\times\mathbb{R}$ since the other case can be treated similarly. Now, consider the new sequences

$$\begin{aligned} Z_j(X,t) &:= W_j(X,t) - W_j(0,0), \\ \overline{Z}_j(X,t) &:= \overline{W}_j(X,t) - \overline{W}_j(0,0), \end{aligned} \tag{5.59}$$

and let $j \in \mathbb{N}$ be large enough such that $K \subset \mathbb{Q}^j$, where $K \subset \mathbb{R}^{N+1}\times\mathbb{R}$ is a fixed compact set. Since the sequence $\{\overline{Z}_j\}_{j\in\mathbb{N}}$ is uniformly bounded in K with uniformly bounded ν-Hölder semi-norm, we can apply Ascoli-Arzelà Theorem to deduce the existence of a continuous function $Z \in C(K)$, uniform limit of $\overline{Z}_j = \overline{Z}_j(X,t)$ as $j \to +\infty$ (we stress that the uniform L^∞ bound for $\{\overline{Z}_j\}_{j\in\mathbb{N}}$ follows from having subtracted the value $\overline{W}_j(0,0) = W_j(0,0)$ to $\overline{W}_j = \overline{W}_j(X,t)$ and using their Hölder continuity). Note that, using a standard covering argument and the arbitrariness of K, we get that $\overline{Z}_j \to Z$ uniformly on compact sets of $\mathbb{R}^{N+1}\times\mathbb{R}$ and, by (5.56) and (5.57), we get $Z_j \to Z$ uniformly on compact sets of $\mathbb{R}^{N+1}\times\mathbb{R}$, too. Furthermore, from the definition of $\overline{Z}_j = \overline{Z}_j(X,t)$

$$|\overline{Z}_j(X_1,t_1) - \overline{Z}_j(X_2,t_2)| = |\overline{W}_j(X_1,t_1) - \overline{W}_j(X_2,t_2)| \le (|X_1-X_2|^2 + |t_1-t_2|)^{\nu},$$

for any choice $(X_1,t_1), (X_2,t_2) \in \mathbb{R}^{N+1}\times\mathbb{R}$ and $j \in \mathbb{N}$ large enough. Consequently, taking $j \to +\infty$ in the above inequality, we obtain $Z \in C^{2\nu,\nu}(\mathbb{R}^{N+1}\times\mathbb{R})$, for any $0 < \nu < \nu_* = \min\{1, 1-a\}/2$. In particular, it satisfies

$$|Z(x,y,t)| \le C(1 + (|X|^2+|t|)^{\nu}] \le C(1 + d^{2\nu}(x,y,t)) \tag{5.60}$$

for any $(x, y, t) \in \mathbb{R}^{N+1} \times \mathbb{R}_+$, $0 < 2\nu < \min\{1, 1-a\}$, and some $C > 0$.
Now, multiplying the equation of $Z_j = Z_j(X,t)$ by $|\widehat{y_j} r_j^{-1} + y|^a \varphi$ (note that Z_j satisfies the same equation of W_j), where $\varphi = \varphi(X,t)$ is any test function belonging to $C_0^\infty(\mathbb{R}^{N+1} \times \mathbb{R})$, and integrating by parts both in space and time, we easily see that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{N+1}} \left[|1 + r_j \widehat{y_j}^{-1} y|^a \partial_t \varphi - \nabla \cdot \left(|1 + r_j \widehat{y_j}^{-1} y|^a \nabla \varphi \right) \right] Z_j \, dxdydt = 0.$$

Hence, exploiting the fact that $|1 + r_j \widehat{y_j}^{-1} y|^a \to 1$ on any compact set of $\mathbb{R}^{N+1} \times \mathbb{R}$ and using the uniform convergence of the sequence $Z_j = Z_j(X,t)$, we pass to the limit as $j \to +\infty$ in the above relation to conclude

$$\int_{\mathbb{R}} \int_{\mathbb{R}^{N+1}} \left(\partial_t \varphi - \Delta_{x,y} \varphi \right) Z \, dxdydt = 0,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^{N+1} \times \mathbb{R})$. In particular, this implies that $Z = Z(X,t)$ is a "backward caloric function" in $\mathbb{R}^{N+1} \times \mathbb{R}_+$, in the "very weak sense". Consequently, from the classical regularity theory of the Heat Equation, the bound in (5.60), and the Liouville type theorem proved in Lemma 5.16 (part (i) with $a = 0$, see also the classical Hirschman's paper [125]), we immediately deduce that $Z = Z(X,t)$ must be constant. Now, since we have chosen $(\widehat{X_j}, \widehat{t_j}) = (X_{1,j}, t_{1,j})$, it follows

$$\frac{(X_{2,j}, t_{2,j}) - (\widehat{X_j}, \widehat{t_j})}{r_j} = \frac{(X_{2,j}, t_{2,j}) - (X_{1,j}, t_{1,j})}{(|X_{1,j} - X_{2,j}|^2 + |t_{1,j} - t_{2,j}|)^{\frac{1}{2}}} \to (X_2, t_2),$$

for some $(X_2, t_2) \in \mathbb{Q}_1$, up to subsequences, and so, using the uniform convergence and the uniform Hölder bound on $\overline{W}_j = \overline{W}_j(X,t)$, we get

$$\begin{aligned} 1 &= \left| \overline{W}_j \left(\frac{X_{2,j} - \widehat{X_j}}{r_j}, \frac{t_{2,j} - \widehat{t_j}}{r_j^2} \right) - \overline{W}_j \left(\frac{X_{1,j} - \widehat{X_j}}{r_j}, \frac{t_{1,j} - \widehat{t_j}}{r_j^2} \right) \right| \\ &= \left| \overline{Z}_j \left(\frac{X_{2,j} - \widehat{X_j}}{r_j}, \frac{t_{2,j} - \widehat{t_j}}{r_j^2} \right) - \overline{Z}_j \left(\frac{X_{1,j} - \widehat{X_j}}{r_j}, \frac{t_{1,j} - \widehat{t_j}}{r_j^2} \right) \right| \\ &= \left| \overline{Z}_j \left(\frac{X_{2,j} - \widehat{X_j}}{r_j}, \frac{t_{2,j} - \widehat{t_j}}{r_j^2} \right) - \overline{Z}_j(0,0) \right| \to |Z(X_2, t_2) - Z(0,0)| = 1, \end{aligned} \tag{5.61}$$

as $j \to +\infty$, i.e. $Z = Z(x,t)$ is nonconstant, in contradiction with the Liouville type theorem. This conclude the proof of (5.58).

Step4: Final part of the proof of (5.50). In view of (5.58) and the arbitrariness of $\widehat{P_j} = (\widehat{X_j}, \widehat{t_j}) \in \mathbb{Q}_1$ in the definition of $W_j = W_j(X,t)$ and $\overline{W}_j = \overline{W}_j(X,t)$ (cfr. with (5.54)), we can choose $\widehat{P_j} = (X_{1,j}, t_{1,j}) = (x_{1,j}, 0, t_{1,j}) \in \Sigma$, for all $j \in \mathbb{N}$. Consequently, it follows that $W_j = W_j(X,t)$ is a "strong solution" to the equation (instead of (5.55)):

$$\partial_t W_j + \mathcal{L}_a W_j = 0 \quad \text{in } \mathbb{R}^{N+1} \times (-\widehat{t_j}/r_j^2, (1 - \widehat{t_j})/r_j^2) \tag{5.62}$$

and, repeating the procedure followed in *Step3* together with the $L^2(0, T_*; H^1_{\mu_t})$ type convergence proved in Theorem 5.1, we deduce that the limit function $Z = Z(X,t)$ (of the sequences in (5.59)) satisfies

$$\int_{\mathbb{R}^{N+1}} \left[\partial_t Z + \frac{(x,y)}{2t} \cdot \nabla Z \right] \eta \, d\mu_t(x,y) = \int_{\mathbb{R}^{N+1}} \nabla Z \cdot \nabla \eta \, d\mu_t(x,y),$$

for a.e. $0 < t < T$ and all $\eta \in L^2_{loc}(0, T; H^1_{\mu_t})$, together with the global bound (5.60). It thus follows that $Z = Z(X, t)$ must be constant (it suffices to apply Lemma 5.16 again) while (5.61) holds also in this case. This gives the final contradiction.

Step5: Uniqueness of the blow-up limit. Having shown (5.50), we can assume that there exists a subsequence $\lambda_j \to 0^+$ and a locally bounded continuous function $\overline{\Theta} = \overline{\Theta}(x, y, t)$ such that for any $T_* > 0$ and compact set $K \subset \mathbb{R}^{N+1} \times (0, T_*)$, it holds

$$\|U_{\lambda_j} - \overline{\Theta}\|_{L^\infty(K)} \to 0,$$

as $j \to +\infty$. On the other hand, setting as always

$$\Theta(x, y, t) = t^{\kappa_{n_0,m_0}} \sum_{(\alpha,m)\in J_0} v_{\alpha,m} \overline{V}_{\alpha,m}\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right),$$

where $v_{\alpha,m} \in \mathbb{R}$, $\overline{V}_{\alpha,m} = \overline{V}_{\alpha,m}(x, y)$ and J_0 are defined as in the statement of Theorem 5.1, and $n_0, m_0 \in \mathbb{N}$ are chosen such that $\kappa = \kappa_{n_0,m_0}$, we have

$$\|U_\lambda - \Theta\|_{L^2(0,T_*;H^1_{\mu_t})} \to 0,$$

as $\lambda \to 0^+$. Now, we note that

$$\begin{aligned}\int_K |\Theta - \overline{\Theta}|^2 d\mu_t(x, y)dt &\leq \|U_{\lambda_j} - \overline{\Theta}\|^2_{L^\infty(K)} \int_0^{T_*} \int_{\mathbb{R}^{N+1}} d\mu_t(x, y)dt + \|U_{\lambda_j} - \Theta\|^2_{L^2(0,T_*;L^2_{\mu_t})} \\ &= T_*\|U_{\lambda_j} - \overline{\Theta}\|^2_{L^\infty(K)} + \|U_{\lambda_j} - \Theta\|^2_{L^2(0,T_*;L^2_{\mu_t})} \to 0,\end{aligned}$$

as $j \to +\infty$, from which we deduce $\Theta = \overline{\Theta}$ on any compact set $K \subset \mathbb{R}^{N+1} \times (0, T_*)$ (note that the continuity of the limit profiles is used) and conclude the proof of the theorem. □

Important remark. We end the section by pointing out an easy but interesting consequence of the above theorem. If $U = U(x, y, t)$ is a locally bounded "strong solution" to equation (7) (let us say $\|U\|_{L^\infty(Q)} \leq C$), then $U = U(x, y, t)$ is uniformly bounded in some Hölder space

$$[U]_{C^{2\nu,\nu}(Q_{1/2})} \leq C \qquad \text{for all } 0 < \nu < \nu_* := \frac{1}{2}\min\{1, 1-a\}, \tag{5.63}$$

and some constant $C > 0$ not depending on $U = U(x, y, t)$. This fact can be easily obtained by repeating the above proof, replacing U_λ by U and it is very significative since it is the first *quantitative* Hölder estimate of solutions to (7). We stress that Hölder continuity of solutions to (7) was already proved by Chiarenza and Serapioni in [65] and by Banerjee and Garofalo in [19], but in those cases the "interval of Hölderianity" was not studied. Note that the above bound is almost optimal, in the sense that there are "strong solutions" to (7) satisfying (5.63) with $\nu = \nu_*$. A simple example for the case $0 < a < 1$ is given by the function $U(x, y, t) = y|y|^{-a}$, while, for $-1 < a \leq 0$, $U(x, y, t) = x$.

Proof of Corollary 5.4. The proof of this corollary is almost identical to the above one. Defining $\overline{U}(x, y, t) = U(x, y, -t)$, where $U = U(x, u, \tau)$ is the extension of $u \in \mathrm{dom}(H^s)$ solution to (1) (cfr. with the proof of Corollary 5.2), we have that the even extension of $\overline{U} = \overline{U}(x, y, t)$ w.r.t. the variable y is a "strong solution" to (7). We can thus repeat the proof of Theorem 5.3 by applying Corollary 5.2 instead of Theorem 5.1. □

As an easy consequence of Theorem 5.3/Corollary 5.4, we obtain a "strong unique continuation" property for solutions to equation (1).

"Strong unique continuation" properties for solutions to elliptic and parabolic equations have a long story and have been intensively studied in both the elliptic and parabolic framework, for both local and nonlocal equations. In the elliptic case we quote the works of Garofalo and Lin [110, 111] for the local framework, and Fall and Felli [98] for the nonlocal one (see also Rüland [173, 174]). For what concerns the parabolic framework, the first result was given by Poon in [167] (see also more recent results in [92, 93] and [209]) for the local case. For the nonlocal setting, a "strong unique continuation" theorem have been proved by Banerjee and Garofalo in [19] for equation (1) with potential. Even though their result is stronger and more general, we have decided to present the proof since it easily follows from Theorem 5.3/Corollary 5.4.

Corollary 5.18. *(Banerjee and Garofalo, Theorem 1.2. of [19]) Let $u \in dom(H^s)$ be a solution to equation* (1) *and assume it vanishes of infinite order at* $(0,0) \in Q_r := B_r \times (-r^2, 0]$*, i.e.,*

$$\sup_{Q_r} |u| = O(r^{2n}) \quad \text{as } r \to 0, \tag{5.64}$$

for all $n > 0$. Then $u \equiv 0$ in $\mathbb{R}^N \times (-T, 0)$.

Proof. Let $u \in \mathrm{dom}(H^s)$ be a solution to (1), satisfying the assumption in (5.64) and $U = U(x, y, -t)$ its extension defined as in (2). Assume by contradiction $u \not\equiv 0$ and define

$$u_\lambda(x,-t) = \lambda^{-2\kappa} u(\lambda x, -\lambda^2 t), \qquad U_\lambda(x,y,-t) = \lambda^{-2\kappa} U(\lambda x, \lambda y, -\lambda^2 t) \quad \lambda > 0,$$

for all $0 < t < T$, where, as always

$$\kappa := \lim_{t \to 0^+} N(t, \overline{U}),$$

with the convention $\overline{U}(x, y, t) = U(x, y, -t)$. From the regularity results of [185] (see also [19]) and the definition of the extension, we have that

$$u_\lambda(x,-t) = U_\lambda(x, 0, -t) \quad \text{in } \mathbb{R}^N \times (0, T).$$

On the other hand, for any fixed $(x, -t) \in Q_r$ with $r > 0$ small enough, we get by (5.64):

$$|u_\lambda(x,-t)| = \lambda^{-2\kappa} |u(\lambda x, -\lambda^2 t)| \le C \lambda^{2(n-\kappa)} r^{2n} \to 0 \quad \text{as } \lambda \to 0^+,$$

for all $n > \kappa$. However, from Corollary 5.4, we know that

$$U_\lambda(x,y,-t) \to t^\kappa \sum_{(\alpha,m) \in \widetilde{J}_0} \frac{v_{\alpha,m}}{\|V_{\alpha,m}\|_{L^2_\mu}} H_\alpha\left(\frac{x}{\sqrt{t}}\right) L_{(\frac{a-1}{2}),m}\left(\frac{y^2}{4t}\right) \quad \text{as } \lambda \to 0^+,$$

uniformly on $\mathbb{Q}_r^+ := \overline{\mathbb{B}_r^+} \times (-r^2, 0] = \{(x,y) \in \mathbb{B}_r,\ y \ge 0\} \times (-r^2, 0]$. In particular, using the continuity (up to $y = 0$) of U_λ and the fact that $L_{(\frac{a-1}{2}),m}(0) = A_m \neq 0$ (cfr. with Section 4.6), we get

$$u_\lambda(x,-t) \to t^\kappa \sum_{(\alpha,m) \in \widetilde{J}_0} \frac{v_{\alpha,m} A_m}{\|V_{\alpha,m}\|_{L^2_\mu}} H_\alpha\left(\frac{x}{\sqrt{t}}\right) \quad \text{as } \lambda \to 0^+,$$

uniformly on Q_r which, since $H_\alpha \not\equiv 0$ in Q_r, gives the desired contradiction. □

Corollary 5.19. *Let $U = U(x, y, t)$ be a "strong solution" to equation* (7) *and assume it vanishes of infinite order at* $(\mathbf{0}, 0) \in \mathbb{Q}_r := \mathbb{B}_r \times [0, r^2)$*, i.e.,*

$$\sup_{\mathbb{Q}_r} |U| = O(r^{2n}) \quad \text{as } r \to 0$$

for all $n > 0$. Then $U \equiv 0$ in $\mathbb{R}^{N+1} \times (0, T)$.

Proof. The proof is very similar to the one above and we skip it. In this case, it suffices to work with "strong solutions" to (7) and applying Theorem 5.3 instead of Corollary 5.4. □

5.5 Nodal properties of solutions to equation (1)

In this final part of the work, we employ the theory developed in the previous sections to get information about the Hausdorff dimension, regularity and structure of the nodal set of solutions to equation (1). As we have mentioned in the introduction, the decisive tools turn out to be the blow-up classification of Section 5.2 and Section 5.4 and classical theorems as Federer's Reduction Principle (cfr. with Theorem 5.25) and Whitney's Extension Theorem (cfr. with Theorem 5.26), together with some "easier" technical results.

We begin with two preliminary results. Firstly, we prove that $\Gamma_\kappa(U)$ is F_σ. Secondly, we will study the decaying properties of solutions to problem (8) near their nodal points of order $\kappa \in \widetilde{\mathcal{K}}$. Both of them will turn out to be crucial in the proof of Theorem 5.12.

Lemma 5.20. *Let $U = U(x, y, t)$ be a nontrivial "strong solution" to problem* (8). *Then:*
(i) The set $\mathcal{R}(U)$ is relatively open in $\Gamma(U)$.
(ii) For any $\kappa = 1, \frac{3}{2}, 2, \ldots$, the set $\Gamma_\kappa(U)$ is a union of countably many closed sets.

Proof. Part (i) directly follows from the upper semi-continuity of the map

$$p_0 \to N(p_0, t_0^+, U) := \lim_{t \to t_0^+} N(p_0, t, U),$$

and Corollary 5.2, which gives us

$$\mathcal{R}(U) = \left\{ p_0 \in \Gamma(U) \cap \Sigma : \lim_{t \to t_0^+} N(p_0, t, U) = \frac{1}{2} \right\} = \left\{ p_0 \in \Gamma(U) \cap \Sigma : N(p_0, t_0^+, U) < 1 \right\}.$$

For what concerns part (ii) we note that thanks to formula (5.25), we have

$$\Gamma_\kappa(U) = \bigcup_{j=1}^{\infty} E_j \qquad E_j := \left\{ p_0 \in \Gamma_\kappa(U) : \frac{1}{j} t^{2\kappa} \le H(p_0, t, U) < j t^{2\kappa}, \text{ as } t \sim 0 \right\}, \tag{5.65}$$

for any $1 \le j \in \mathbb{N}$. Consequently, it is enough to show that all the E_j's are closed sets. We show that for any fixed $1 \le j \in \mathbb{N}$ and $p_0 \in \overline{E_j}$, it holds $p_0 \in E_j$. It is instantly seen that for such p_0, the inequalities in (5.65) hold by continuity, while, since the function $p_0 \to N(p_0, t_0^+, U)$ (cfr. with (5.2)) is upper semi-continuous, we get that $N(p_0, t_0^+, U) \ge \kappa$. Finally, if by contradiction $N(p_0, t_0^+, U) = \overline{\kappa} > \kappa$, combining formula (5.25) with $\kappa = \overline{\kappa}$ and (5.65), we easily get

$$\frac{1}{j} t^{2\kappa} \le H(p_0, t, U) < j t^{2\overline{\kappa}}, \text{ as } t \sim 0,$$

which is contradiction. Consequently, $p_0 \in \Gamma_\kappa(U)$ and so $p_0 \in E_j$. □

Lemma 5.21. *("Regularity") Let $U = U(x, y, t)$ be a nontrivial "strong solution" to problem* (8), *with $p_0 = (X_0, t_0) \in \Gamma_\kappa(U)$. Then there exists a constant $C > 0$, such that*

$$\sup_{\overline{\mathbb{B}_r^+(X_0)} \times (t_0, t_0 + 2r^2)} |U| \le C r^{2\kappa}, \quad \text{as } r \to 0.$$

Proof. Again we take $(X_0, t_0) = (\mathbf{0}, 0)$ and we begin with the following observation. If $|x|^2 + y^2 < r^2$ and $6r^2 < t < 7r^2$, for some $r > 0$, we easily see that

$$7t^{-\frac{N+a+1}{2}} \geq r^{-(N+a+1)}, \qquad e^{-\frac{|x|^2+y^2}{4t}} \geq e^{-\frac{1}{24}},$$

so that

$$\int_{6r^2}^{7r^2} \int_{\mathbb{B}_r^+} d\mu_t(x,y)dt \geq Cr^{-(N+a+1)} \int_{6r^2}^{7r^2} \left(\int_{\mathbb{B}_r^+} |y|^a \, dxdy \right) dt \geq Cr^2,$$

for some $C > 0$. Now, using Theorem 2.1 of [65] again (or formula (5.12) of [19]) and the above estimate, we get

$$\begin{aligned}
\left(\sup_{\overline{\mathbb{B}_r^+} \times (0,2r^2)} |U| \right)^2 &\leq C \left(\inf_{\mathbb{B}_r^+ \times (6r^2, 7r^2)} |U| \right)^2 \leq C \frac{\int_{6r^2}^{7r^2} \int_{\mathbb{B}_r^+} U^2 \, d\mu_t(x,y)dt}{\int_{6r^2}^{7r^2} \int_{\mathbb{B}_r^+} d\mu_t(x,y)dt} \\
&\leq \frac{C}{r^2} \int_{6r^2}^{7r^2} \int_{\mathbb{B}_r^+} U^2 \, d\mu_t(x,y)dt \leq \frac{C}{r^2} \int_{6r^2}^{7r^2} \int_{\mathbb{R}_+^{N+1}} U^2 \, d\mu_t(x,y)dt.
\end{aligned}$$

Consequently, in view of (5.25), we deduce for $r \sim 0$:

$$\int_{6r^2}^{7r^2} \int_{\mathbb{R}_+^{N+1}} U^2 \, d\mu_t(x,y)dt = \int_{6r^2}^{7r^2} H(t,U) \, dt \leq C \int_{6r^2}^{7r^2} t^{2\kappa} \, dt \leq Cr^{4\kappa+2},$$

which, matching with the above chain of inequalities, it follows

$$\sup_{\overline{\mathbb{B}_r^+} \times (0,2r^2)} |U| \leq Cr^{2\kappa}, \quad \text{as } r \to 0,$$

i.e., the thesis. □

5.5.1 Proof of Theorem 5.7 and Theorem 5.8

As the title of the section subsection suggests, we now pass to the problem of estimating the Hausdorff dimension of the nodal set of solutions to (1) and, as explained above, we will make use of the so called Federer's Reduction Principle (cfr. for instance with Simon [178, Appendix A], or Lin [143, Section 2]). Its parabolic version of it is less employed in literature and so, out of completeness, we review it in the next paragraphs. We follow the notable work of Chen [63, Section 8], simplifying its quite general setting to our more specific framework. We begin with the definition of Hausdorff and *parabolic* Hausdorff dimension.

Definition 5.22. *(Parabolic Hausdorff dimension, [63, Definition 8.1]) For any $E \subset \mathbb{R}^N \times \mathbb{R}$, any real number $d \geq 0$, and $0 < \delta \leq \infty$, we define*

$$\mathcal{P}_\delta^d(E) := \inf \left\{ \sum_{j=1}^\infty r_j^d \ : \ E \subset \bigcup_{j=1}^\infty Q_{r_j}(x_j, t_j) \ \text{with} \ 0 < r_j < \delta \right\},$$

where, as always

$$Q_r(x,t) := \left\{ (x', t') \in \mathbb{R}^N \times \mathbb{R} : |x - x'| < R, |t - t'| < R^2 \right\}.$$

Then we define the "d-dimensional cylindrical Hausdorff measure" by

$$\mathcal{P}^d(E) := \lim_{\delta \to 0^+} \mathcal{P}_\delta^d(E) = \sup_{\delta > 0} \mathcal{P}_\delta^d(E).$$

We call "parabolic Hausdorff dimension" of E, the number

$$\dim_{\mathcal{P}}(E) := \inf \left\{ d \geq 0 \ : \ \mathcal{P}^d(E) = 0 \right\} = \sup \left\{ d \geq 0 \ : \ \mathcal{P}^d(E) = +\infty \right\}.$$

Before moving forward, we recall that the above definition is just a parabolic version of the more classical "spherical Hausdorff" measure (Hausdorff dimension, resp.). Indeed, if $E \subset \mathbb{R}^N$, $d \geq 0$, and $0 < \delta \leq \infty$, we define

$$\mathcal{H}^d_\delta(E) := \inf \left\{ \sum_{j=1}^{\infty} r_j^d \ : \ E \subset \bigcup_{j=1}^{\infty} B_{r_j}(x_j) \ \text{ with } \ 0 < r_j < \delta \right\},$$

where now $B_r(x)$ is the ball of radius $r > 0$, centered at x. Consequently, the "d-dimensional spherical Hausdorff measure" can be defined as

$$\mathcal{H}^d(E) := \lim_{\delta \to 0^+} \mathcal{H}^d_\delta(E) = \sup_{\delta > 0} \mathcal{H}^d_\delta(E).$$

The "Hausdorff dimension" of E is the number

$$\dim_{\mathcal{H}}(E) := \inf \left\{ d \geq 0 \ : \ \mathcal{H}^d(E) = 0 \right\} = \sup \left\{ d \geq 0 \ : \ \mathcal{H}^d(E) = +\infty \right\}.$$

Let us start with the following lemma.

Lemma 5.23. *([63, Lemma 8.2]) The following two assertions hold:*

(i) For any linear subspace $E \subset \mathbb{R}^N$ and any $-\infty \leq a < b \leq +\infty$, it holds

$$\dim_{\mathcal{P}}(E \times (a, b)) = \dim_{\mathcal{H}}(E) + 2.$$

In particular, $\dim_{\mathcal{P}}(\mathbb{R}^N \times \mathbb{R}) = N + 2$.

(ii) For any set $E \subset \mathbb{R}^N \times \mathbb{R}$, any point $p_0 = (x_0, t_0) \in \mathbb{R}^N \times \mathbb{R}$, and $\lambda > 0$, define

$$E_{p_0,\lambda} = \frac{E - p_0}{\lambda} := \left\{ (x, t) \in \mathbb{R}^N \times \mathbb{R} : (x_0 + \lambda x, t_0 + \lambda^2 t) \in E \right\}.$$

Then

$$\mathbb{P}^d_\delta(E_{p_0,\lambda}) = \lambda^{-d} \mathbb{P}^d_\delta(E),$$

for any $d \geq 0$ and $\delta > 0$.

The above lemma clarifies the relation between the more natural Hausdorff dimension and the little bit more ambiguous parabolic Hausdorff dimension (for which $\dim_{\mathcal{P}}(\mathbb{R}^N \times \mathbb{R}) = N + 2$!). This (maybe strange) fact is actually quite natural if the natural parabolic scaling is taken into account.

Definition 5.24. *(Locally asymptotically self-similar family, cfr. with [63, Definition 8.3])* *Let $\mathcal{F} \subset L^\infty_{loc}(\mathbb{R}^N \times \mathbb{R})$ be a family of functions and consider a map*

$$\mathcal{S} : \mathcal{F} \to \mathcal{C} := \{C \subset \mathbb{R}^N \times \mathbb{R} : C \textit{ is closed}\}.$$

Moreover, define the "blow-up family"

$$u_{p_0,\lambda,\varrho}(x, t) = \frac{u(x_0 + \lambda x, t_0 + \lambda^2 t)}{\varrho},$$

for any $p_0 = (x_0, t_0) \in \mathbb{R}^N \times \mathbb{R}$ and $\lambda, \varrho > 0$. We say that the pair $(\mathcal{F}, \mathcal{S})$ is a "locally asymptotically self-similar family" if it satisfies the properties (A1), (A2) and (A3) below.

(A1) (Closure under re-scaling, translation and normalization) For any $Q_\lambda(x_0, t_0) \subset Q_1(0, 0)$, $\varrho > 0$ and $u \in \mathcal{F}$, it holds

$$u_{p_0,\lambda,\varrho} \in \mathcal{F}.$$

(A2) (Convergence of the normalized "blow-up sequence") For any $p_0 \in Q_1(0,0)$, $u \in \mathcal{F}$ and $\lambda_j \to 0^+$, there exist a number $\kappa \in \mathbb{R}$ and a function $\vartheta_{p_0} \in \mathcal{F}$ parabolically κ-homogeneous such that as $j \to +\infty$, up to pass to a subsequence of λ_j, it holds

$$u_{p_0,\lambda_j,\lambda_j^{2\kappa}} \to \vartheta_{p_0} \quad \textit{locally uniformly in } \mathbb{R}^N \times \mathbb{R}.$$

Moreover, if $u_{p_0,\lambda_j,\lambda_j^{2\kappa}} \to \vartheta_{p_0}$ and $u_{p_0,\lambda_j,\lambda_j^{2k}} \to \theta_{p_0}$, then $\kappa = k$ and $\vartheta = \theta$.

(A3) (Singular Set assumptions) The map $\mathcal{S} : \mathcal{F} \to C$ satisfies the following properties:
(i) For any $Q_\lambda(x_0,t_0) \subset Q_1(0,0)$, $\varrho > 0$ and $u \in \mathcal{F}$, it holds

$$\mathcal{S}(u_{p_0,\lambda,\varrho}) = (\mathcal{S}(u))_{p_0,\lambda}\,.$$

(ii) For any $p_0 \in Q_1(0,0)$, $u, \vartheta_{p_0} \in \mathcal{F}$, $\kappa \in \mathbb{R}$ and $\lambda_j \to 0^+$ such that $u_{p_0,\lambda_j,\lambda_j^{2\kappa}} \to \vartheta_{p_0}$ uniformly on compact sets of $\mathbb{R}^N \times \mathbb{R}$, the following "continuity property" holds: for any $\varepsilon > 0$, there exists $j_\varepsilon > 0$ such that

$$\mathcal{S}(u_{p_0,\lambda_j,\lambda_j^{2\kappa}}) \cap Q_1(0,0) \subseteq \{p \in \mathbb{R}^N \times \mathbb{R} : \mathrm{dist}(p, \mathcal{S}(\vartheta_{p_0})) < \varepsilon\}, \quad \textit{for all } j \geq j_\varepsilon.$$

(iii) If $u \in \mathcal{F}$ and $\kappa \in \mathbb{R}$ are such that $u_{p_0,\lambda,\lambda^{2\kappa}} = u$ for all $(x_0,t_0) \in \mathbb{R}^N \times \mathbb{R}$ and $\lambda > 0$, then $\mathcal{S}(u) = \emptyset$.

Remark. Some comments are now in order. In Definition 8.3 of [63], the author present a much more general definition that we have decided to adapt to our specific context. Note in particular that we have replaced the heavy notation $g(x_0,t_0;\lambda,1/\varrho)u$ with the simpler one $u_{p_0,\lambda,\varrho}$. Moreover, we have eliminated the assumptions $(A1.1)$, $(A1.2)$, and $(A3.2)$ (of [63]) since they are immediate in our case. Moreover, the second part of $(A1.3)$ and $(A1.4)$ (of [63]) corresponds to (i) and (iii) of $(A3)$, respectively, and finally, $(A2)$, $(A3.1)$ and $(A3.3)$ (of [63]), corresponds to our assumption $(A2)$.

We are now ready to state Federer's reduction principle.

Theorem 5.25. *(Federer's Reduction Principle, Chen [63, Theorem 8.5])*
Let $(\mathcal{F},\mathcal{S})$ be a "locally asymptotically self-similar family" and assume there exists at least one $u \in \mathcal{F}$ such that $\mathcal{S}(u) \cap Q_1(0,0) \neq \emptyset$. Then:

(i) There exists an integer $0 \leq d \leq N+1$ such that for all $u \in \mathcal{F}$ it holds

$$\dim_{\mathcal{P}} [\mathcal{S}(u) \cap Q_1(0,0)] \leq d.$$

(ii) There exist $u \in \mathcal{F}$, $\kappa \in \mathbb{R}$ and a linear subspace of $\mathbb{R}^N \times \mathbb{R}$ such that

$$E_{(0,0),\lambda} = E, \quad \textit{for any } \lambda > 0, \qquad \mathcal{S}(u) = E, \qquad \dim_{\mathcal{P}} E = d,$$

and

$$u_{p_0,\lambda,\lambda^{2\kappa}} = u \quad \textit{for any } \lambda > 0 \;\; \textit{and} \;\; p_0 \in E.$$

Proof of Theorem 5.7. The statement is obtained as by product of Federer's theorem and the blow-up classification of Sections 5.2 and 5.4. So, if $u \in \mathrm{dom}(H^s)$ is a nontrivial solution to (1) and $\overline{u}(x,t) = u(x,-t)$, we define

$$\mathcal{F} := \{\overline{u} \in \mathrm{dom}(H^s) : u \text{ is a non trivial solution to equation (1)}\} \subset L^\infty_{loc}(\mathbb{R}^N \times \mathbb{R}).$$

Note that the family $\mathcal{F}$ satisfies assumption (A_1) of Definition 5.24 thanks to the properties of the operator $(\partial_\tau - \Delta)^s$ such as linearity. Now, let $U = U(x,y,\tau)$ the extension of $u = u(x,\tau)$ defined as in (2) so that $\overline{U}(x,y,t) = U(x,y,-t)$ satisfies problem (8). From Corollary 5.4 we have

$$\overline{U}_{p_0,\lambda_j} \to \Theta_{p_0} \quad \text{locally uniformly in } \overline{\mathbb{R}^{N+1}_+} \times \mathbb{R}_+,$$

as $j \to +\infty$, where $\Theta_{p_0} \in \mathfrak{B}_\kappa(U)$ is the unique (parabolically κ-homogeneous) "tangent map" of U at $p_0 \in \Gamma(U) \cap \Sigma = \Gamma(u)$ (see Definitions 5.6 and 5.6) and $\lambda_j \to 0$. Consequently,

$$\overline{u}_{p_0,\lambda_j} \to \vartheta_{p_0} \quad \text{locally uniformly in } \mathbb{R}^N \times \mathbb{R}_+,$$

as $j \to +\infty$, where $\vartheta_{p_0}(x,t) := \Theta_{p_0}(x,0,t)$ and $\overline{u}_{p_0,\lambda_j}(x,t) = \overline{U}_{p_0,\lambda_j}(x,0,t)$. Since Θ_{p_0} is parabolically κ-homogeneous for some $\kappa \geq 0$, we deduce that assumption (A_2) Definition 5.24 is satisfied too, by taking

$$\overline{u}_{p_0,\lambda_j,\varrho_j}(x,t) = \overline{u}_{p_0,\lambda_j}(x,t) \qquad (\text{with } \varrho_j := \lambda_j^{2\kappa}).$$

Now, let us consider the map

$$\mathcal{S} : \overline{u} \to \mathcal{S}(\overline{u}) := \Gamma(\overline{u}) \in \mathcal{C},$$

since $\Gamma(\overline{U}) \cap Q_1 \cap \Sigma$ is closed by the continuity of $\overline{U} = \overline{U}(x,y,t)$. Moreover, by the uniform convergence of $\overline{U}_{p_0,\lambda}$ towards their "tangent maps" Θ_{p_0} (cfr. with Theorem 5.3), we easily see that also the assumption (A_3) (of Definition 5.24) is satisfied and so, $(\mathcal{F},\mathcal{S})$ is a "locally asymptotically self-similar family".
Hence, in view of Federer's Reduction Principle, we conclude the proof since $d \leq N+1$. □

We end this subsection by proving Theorem 5.8, which focuses on the analysis of the "regular" points of the nodal set $\mathcal{R}(u)$ of solutions to equation (1).

Proof of Theorem 5.8. From the blow up classification of Corollary 5.4 and the spectral Theorem 4.1, we know that as $\lambda \to 0$

$$U_{p_0,\lambda} = \frac{U(x_0 + \lambda x, \lambda y, t_0 + \lambda^2 t)}{\lambda} \to \Theta_{p_0}(x,y,t) = \sum_{i=1}^{N} v_j x_j$$

uniformly on compact sets of $\mathbb{R}^{N+1}_+ \times (0,T)$, where $v_j \neq 0$ at least for some $j = 1,\dots,N$. Consequently, if $(e_j)_{i=1}^{N+1}$ is the standard basis of $\mathbb{R}^{N+1}$ and $v_j \neq 0$, it holds

$$\partial_{x_i} U(p_0) = \lim_{\lambda \to 0^+} \frac{U(p_0 + \lambda e_i) - U(p_0)}{\lambda} = v_i,$$

thanks to the above convergence and that $U(p_0) = 0$ (here δ_{ij} is the Kronecker delta). It thus follows $\nabla_x U(p_0) \neq \mathbf{0}$ (note that however $\partial_t U(p_0) = 0$). Part (i) is thus proved.

For what concerns part (ii), it is enough to apply the Implicit function theorem at point $p_0 \in \mathcal{R}(U)$ and use part (i). □

5.5.2 Proof of Theorem 5.9 and Theorem 5.12

We begin with the proof of Theorem 5.9. We point that its last part will be crucial in the application of Whitney's theorem, in the proof of Theorem 5.12.

Proof. Part (i) easily follows from Theorem 5.3. Indeed, assuming for simplicity $p_0 = (\mathbf{0},0) \in \Gamma_\kappa(U)$ and taking $K = \mathbb{Q}_1^+ = \mathbb{B}_1^+ \times (0,1)$, we have as $\lambda \to 0^+$

$$\begin{aligned} o(1) &= \sup_{(x,y,t)\in\mathbb{Q}_1^+} |U_\lambda(x,y,t) - \Theta(x,y,t)| = \sup_{(x,y,t)\in\mathbb{Q}_1^+} \left| \frac{U(\lambda x, \lambda y, \lambda^2 t)}{\lambda^{2\kappa}} - \frac{\Theta(\lambda x, \lambda y, \lambda^2 t)}{\lambda^{2\kappa}} \right| \\ &= \lambda^{-2\kappa} \sup_{(x,y,t)\in\mathbb{Q}_\lambda^+} |U(x,y,t) - \Theta(x,y,t)|, \end{aligned}$$

where $\mathbb{Q}_\lambda^+ := \mathbb{B}_\lambda^+ \times (0,\lambda^2)$.

To prove part (ii) we follow the ideas of Theorem 14.4 of [71]. As in Remark 12.7 of that paper, we point out that since $\mathfrak{P}_\kappa(U)$ is a finite dimensional space composed by parabolically κ-homogeneous polynomials, the continuity of the function $p_0 \to \Theta_{p_0}$ can be verified w.r.t. any norm on $\mathfrak{P}_\kappa(U)$. We proceed in two separate steps.

Step1: Monneau's and Weiss' type monotonicity formulae. Taking the norm $C^0(0,1;L^2_{\mu_t})$ (cfr. with Lemma 5.13) and repeating the above scaling procedure using the $C^0(0,1;L^2_{\mu_t})$ convergence (instead of the uniform one), we have that for any $p_0 \in \Gamma_\kappa(U)$ (cfr. with (5.35) with $T_* = 1$):

$$\frac{1}{\lambda^{4\kappa}} \max_{t\in[0,\lambda^2]} \|U_{p_0}(t) - \Theta_{p_0}(t)\|^2_{L^2_{\mu_t}} = \frac{1}{\lambda^{4\kappa}} \max_{t\in[0,\lambda^2]} H(t, U_{p_0} - \Theta_{p_0}) \to 0 \quad \text{as } \lambda \to 0^+.$$

On the other hand, we easily see that

$$\frac{1}{\lambda^{4\kappa}} \max_{t\in[0,\lambda^2]} H(t, U_{p_0} - \Theta_{p_0}) = \frac{H(\lambda^2, U_{p_0} - \Theta_{p_0})}{\lambda^{4\kappa}}, \qquad \lambda > 0,$$

thanks to the monotonicity of the function $t \to H(t, U_{p_0} - \Theta_{p_0})$ (cfr. with Section 4.3). Now, we claim that the function

$$t \to M^\kappa_{U_{p_0},\Theta_{p_0}}(t) := \frac{H(t, U_{p_0} - \Theta_{p_0})}{t^{2\kappa}}, \tag{5.66}$$

is monotone non-decreasing (note that this does not follows immediately from (4.36) since we are replacing U_{p_0} with $U_{p_0} - \Theta_{p_0}$). The map $t \to M^\kappa_{U_{p_0},\Theta_{p_0}}(t)$ defined in (5.66) is a Monneau type function (cfr. with [154, 155]) and its derivative can be easily computed as follows

$$\begin{aligned}
\frac{d}{dt} M^\kappa_{U_{p_0},\Theta_{p_0}}(t) &= -2\kappa t^{-2\kappa-1} H(t, U^*_{p_0}) + 2t^{-2\kappa} I(t, U^*_{p_0}) \\
&= \frac{2}{t^{2\kappa+1}} \left[tI(t, U^*_{p_0}) - \kappa H(t, U^*_{p_0}) \right] = \frac{2}{t} M^\kappa_{U_{p_0},\Theta_{p_0}}(t) \left[N(t, U^*_{p_0}) - \kappa \right] \\
&:= \frac{2}{t} W^\kappa_{U_{p_0},\Theta_{p_0}}(t),
\end{aligned}$$

where we have set $U^*_{p_0} := U_{p_0} - \Theta_{p_0}$ and used the fact that $H'(t, U^*_{p_0}) = 2I(t, U^*_{p_0})$. The map $t \to W^\kappa_{U_{p_0},\Theta_{p_0}}(t)$ is known in literature as Weiss type function (cfr. with [206, 207]).
We point out that in Theorem 13.4 of [71] the authors introduced Monneau's and Weiss' type functions which are a sort of averaged versions of ours. In this framework, these "averaged" versions are not needed.

Now, as the reader can easily see, if $U = U(x, y, t)$ has Almgren-Poon limit κ at p_0 and tangent map $\Theta_{p_0} \in \mathfrak{P}_\kappa(U)$, we have that the Weiss type function

$$t \to W^\kappa_{U_{p_0},0}(t) := M^\kappa_{U_{p_0},0}(t) \left[N(t, U_{p_0}) - \kappa \right] = \frac{H(t, U_{p_0})}{t^{2\kappa}} \left[N(t, U_{p_0}) - \kappa \right],$$

is monotone non-decreasing (since it is the product of two non-decreasing functions, cfr. with (4.36) and Lemma 4.5), nonnegative and $W^\kappa_{\Theta_{p_0},0} \equiv 0$ (cfr. with Lemma 4.5 again and use the homogeneity of Θ_{p_0}). Consequently, if we show that

$$W^\kappa_{U_{p_0},\Theta_{p_0}}(t) = W^\kappa_{U_{p_0},0}(t) \quad \text{for all } 0 < t < 1,$$

we deduce that the Monneau's function $t \to W^\kappa_{U_{p_0},\Theta_{p_0}}(t)$ is monotone non-decreasing, as claimed. So, we have

$$
\begin{aligned}
t^{2\kappa} W^{\kappa}_{U_{p_0},0}(t) &= t^{2\kappa}\left[W^{\kappa}_{U_{p_0},0}(t) + W^{\kappa}_{\Theta_{p_0},0}(t)\right] \\
&= \left\{t\left[I(t,U_{p_0}) + I(t,\Theta_{p_0})\pm\right] - \kappa\left[H(t,U_{p_0}) + H(t,\Theta_{p_0})\right]\right\} \\
&= \left[tI(t,U_{p_0}-\Theta_{p_0}) - \kappa H(t,U_{p_0}-\Theta_{p_0})\right] + 2\left[t\int_{\mathbb{R}^{N+1}_+}\nabla\Theta\cdot\nabla U\,d\mu^t - \kappa\int_{\mathbb{R}^{N+1}_+}\Theta U\,d\mu^t\right] \\
&= t^{2\kappa} W^{\kappa}_{U_{p_0},\Theta_{p_0}}(t) + 2\left[\int_{\mathbb{R}^{N+1}_+}\nabla\widetilde{\Theta}\cdot\nabla\widetilde{U}\,d\mu - \kappa\int_{\mathbb{R}^{N+1}_+}\widetilde{\Theta}\widetilde{U}\,d\mu\right] \\
&= t^{2\kappa} W^{\kappa}_{U_{p_0},\Theta_{p_0}}(t) \quad \text{for all } 0 < t < 1,
\end{aligned}
$$

as desired. Note that we have passed to the re-scaled versions $\widetilde{U}(x,y,t) = U(\sqrt{t}x, \sqrt{t}y, t)$ and $\widetilde{\Theta}(x,y,t) = \Theta(\sqrt{t}x, \sqrt{t}y, t)$, and used the definition of weak eigenfunction as in Definition 4.10 (recall that we can test with $\widetilde{U}$ since $\widetilde{U}(t) \in H^1_\mu$ for any $0 < t < 1$). We can thus conclude that the function

$$
\lambda \to \frac{1}{\lambda^{4\kappa}} \max_{t\in[0,\lambda^2]} H(t, U_{p_0} - \Theta_{p_0}) = \frac{H(\lambda^2, U_{p_0} - \Theta_{p_0})}{\lambda^{4\kappa}} = M^{\kappa}_{U_{p_0},\Theta_{p_0}}(\lambda^2),
$$

is monotone non-decreasing for $0 < \lambda < 1$.

Step2: End of the proof of part (ii). For $\varepsilon > 0$ fixed, we take $\lambda_\varepsilon > 0$ such that

$$
M^{\kappa}_{U_{p_0},\Theta_{p_0}}(\lambda_\varepsilon) = \frac{H(\lambda_\varepsilon^2, U_{p_0} - \Theta_{p_0})}{\lambda_\varepsilon^{4\kappa}} < \frac{\varepsilon}{2}, \qquad M^{\kappa}_{U_{p_0},U_{p_1}}(\lambda_\varepsilon) = \frac{H(\lambda_\varepsilon^2, U_{p_0} - U_{p_1})}{\lambda_\varepsilon^{4\kappa}} < \frac{\varepsilon}{2},
$$

where $p_1 \in \Gamma_\kappa(U)$ satisfies $|p_0 - p_1| < \delta_\varepsilon$ and $\delta_\varepsilon > 0$ is small enough depending on $\varepsilon > 0$ and p_0 (here we have simply used the $C^0(0,1;L^2_{\mu^t})$ continuous dependence U_{p_0} on $p_0 \in \Gamma(U)$). Consequently,

$$
H^{\kappa}_{U_{p_1},\Theta_{p_0}}(\lambda_\varepsilon) = \frac{H(\lambda_\varepsilon^2, U_{p_1} - \Theta_{p_0})}{\lambda_\varepsilon^{4\kappa}} \le \frac{H(\lambda_\varepsilon^2, U_{p_1} - U_{p_0})}{\lambda_\varepsilon^{4\kappa}} + \frac{H(\lambda_\varepsilon^2, U_{p_0} - \Theta_{p_0})}{\lambda_\varepsilon^{4\kappa}} < \varepsilon,
$$

where, similar to [71], we have used the continuity of the map $\Gamma(U) \ni p \to U_p$ w.r.t. the norm $C^0(0,1;L^2_{\mu_t})$. Thus, from the monotonicity of $\lambda \to H^{\kappa}_{U_{p_0},\Theta_{p_0}}(\lambda)$ and the fact the both $p_0, p_1 \in \Gamma_\kappa(U)$, it follows

$$
\max_{t\in[0,1]} \|U_{p_1,\lambda}(t) - \Theta_{p_0}(t)\|^2_{L^2_{\mu_t}} \le H^{\kappa}_{U_{p_1},\Theta_{p_0}}(\lambda_\varepsilon) < \varepsilon, \quad \text{for all } 0 < \lambda < \lambda_\varepsilon,
$$

and so, taking the limit as $\lambda \to 0^+$, we obtain

$$
\max_{t\in[0,1]} \|\Theta_{p_1}(t) - \Theta_{p_0}(t)\|^2_{L^2_{\mu_t}} < \varepsilon,
$$

which completes the proof of part (ii).

Finally, to prove part (iii) we combine what showed in the above part and the Harnack inequality proved by Chiarenza and Serapioni in [65]. So, we fix a compact set $K \subset \Gamma_\kappa(U)$, $\varepsilon > 0$ and $p_0 \in K$ and we take $\lambda_\varepsilon, \delta_\varepsilon > 0$ such that

$$
\|U_{p,\lambda} - \Theta_p\|^2_{C^0(0,1;L^2_{\mu_t})} < \varepsilon \quad \text{for all } 0 < \lambda < \lambda_\varepsilon,
$$

for any $p \in K$ satisfying $|p_0 - p| \le \delta_\varepsilon$ (this immediately follows from part (ii)). Now, covering K with a finite union of cylinders $Q_{\delta_\varepsilon}(p_i)$ with $p_i \in K$, we deduce the existence of a constant $C > 0$ such that

$$
\|U_{p,\lambda} - \Theta_p\|^2_{L^2(0,1;L^2_{\mu_t})} \le \|U_{p,\lambda} - \Theta_p\|^2_{C^0(0,1;L^2_{\mu_t})} < C\varepsilon \quad \text{for all } 0 < \lambda < \lambda_\varepsilon,
$$

for all $p \in K$, since $C^0(0,1;L^2_{\mu_t}) \subset L^2(0,1;L^2_{\mu_t})$. Consequently, to end the proof, it is enough to show that

$$\sup_{\overline{\mathbb{B}_1^+}\times(0,2)} |U_{p,\lambda} - \Theta_p| \leq C\|U_{p,\lambda} - \Theta_p\|_{L^2(0,1;L^2_{\mu_t})}, \tag{5.67}$$

for some constant $C > 0$. Indeed, using the homogeneity of $\Theta_p = \Theta_p(x, y, t)$, we get

$$\sup_{\overline{\mathbb{B}_\lambda^+}\times(0,2\lambda^2)} |U_p - \Theta_p| \leq C\varepsilon\lambda^{2\kappa},$$

which implies the thesis, thanks to the arbitrariness of $\varepsilon > 0$. So, we prove (5.67) by miming the proof of Lemma 5.21. We set $W = U_p - \Theta_p$ and we begin with the following observation. If $|x|^2 + y^2 < \lambda^2$ and $6\lambda^2 < t < 7\lambda^2$, for some $\lambda > 0$, we easily see that

$$7t^{-\frac{N+a+1}{2}} \geq \lambda^{-(N+a+1)}, \qquad e^{-\frac{|x|^2+y^2}{4t}} \geq e^{-\frac{1}{24}},$$

so that

$$\int_{6\lambda^2}^{7\lambda^2} \int_{\mathbb{B}_\lambda^+} d\mu_t(x,y)dt \geq C\lambda^{-(N+a+1)} \int_{6\lambda^2}^{7\lambda^2} \left(\int_{\mathbb{B}_\lambda^+} |y|^a\, dxdy \right) dt \geq C\lambda^2,$$

for some $C > 0$. Now, using Theorem 2.1 of [65] again (or formula (5.12) of [19]) and the above estimate, we get

$$\begin{aligned}
\left(\sup_{\overline{\mathbb{B}_\lambda^+}\times(0,2\lambda^2)} |W| \right)^2 &\leq C\left(\inf_{\mathbb{B}_\lambda^+\times(6\lambda^2,7\lambda^2)} |W| \right)^2 \leq C\frac{\int_{6\lambda^2}^{7\lambda^2}\int_{\mathbb{B}_\lambda^+} W^2\, d\mu_t(x,y)dt}{\int_{6\lambda^2}^{7\lambda^2}\int_{\mathbb{B}_\lambda^+} d\mu_t(x,y)dt} \\
&\leq \frac{C}{\lambda^2}\int_{6\lambda^2}^{7\lambda^2}\int_{\mathbb{B}_\lambda^+} W^2\, d\mu_t(x,y)dt \leq \frac{C}{\lambda^2}\int_{6\lambda^2}^{7\lambda^2}\int_{\mathbb{R}_+^{N+1}} W^2\, d\mu_t(x,y)dt \\
&\leq C\int_0^1\int_{\mathbb{R}_+^{N+1}} W_\lambda^2\, d\mu_t(x,y)dt,
\end{aligned}$$

where $C > 0$ is a new constant. This complete the proof of (5.67) and part (iii). □

We are almost ready to show our "Structure of the singular set theorem". Its proof is based on the techniques due to Garofalo and Petrosyan [112] (elliptic setting) and Danielli, Garofalo, Petrosyan and To [71] (parabolic setting). They are based on a ingenious combination of the Implicit function theorem and a parabolic version of the Whitney's extension theorem, which we recall for completeness.

Theorem 5.26. *(Parabolic Whitney's extension, [71, Theorem B.1]) Let E be a compact subset of $\mathbb{R}^N \times \mathbb{R}$, $f : E \to \mathbb{R}$ a continuous function, and $\{f_{\alpha,j}\}_{|\alpha|+2j\leq 2m}$ with $f_{0,0} = f$ and $m \in \mathbb{N}$, $\alpha \in \mathbb{Z}^N_{\geq 0}$ a multi-index. Assume that there exist a family of moduli of continuity $\{\omega_{\alpha,j}\}_{|\alpha|+2j\leq 2m}$ such that*

$$f_{\alpha,j}(x,t) = \sum_{|\beta|+2i\leq 2m-|\alpha|-2j} \frac{f_{\alpha+\beta,j+i}(x_0,t_0)}{\beta!i!}(x-x_0)^\beta(t-t_0)^i + R_{\alpha,j}(x,t;x_0,t_0)$$

and

$$|R_{\alpha,j}(x,t;x_0,t_0)| \leq \omega_{\alpha,j}(\|(x-x_0,t-t_0)\|)\|(x-x_0,t-t_0)\|^{2m-|\alpha|-2j}.$$

Then there exists a function $F \in C^{2m,m}(\mathbb{R}^N \times \mathbb{R})$ such that $F = f$ on E and $\partial_x^\alpha\partial_t^j F = f_{\alpha,j}$ on E, for $|\alpha| + 2j \leq 2m$.

Together with the above result, we will use the following lemma, which explains the importance of having introduced the notion of "time-like" manifold (cfr. with Definition 5.11).

Lemma 5.27. *("Time-like" singular points) Let $u \in dom(H^s)$ be a nontrivial solution to* (1)*, $(x_0, \tau_0) \in \Gamma_\kappa^N(u)$. Then*

$$\vartheta_{p_0}(x,t) = Ct^{\lfloor\kappa\rfloor},$$

for some nonzero $C \in \mathbb{R}$, where the symbol $\lfloor\cdot\rfloor$ denotes the floor function and, as always, $\vartheta_{p_0}(x,t) = \Theta_{p_0}(x,0,t)$ and $\Theta_{p_0} \in \mathfrak{B}_\kappa(\overline{U})$ is the "blow-up limit" of $\overline{U} = \overline{U}(x,y,t)$ at $(x_0, 0, -\tau_0)$.

Proof. Let us assume for simplicity $(x_0, \tau_0) = (0,0)$ and set $\vartheta := \vartheta_{p_0}$. From the homogeneity of $\vartheta = \vartheta(x,t)$, we can write

$$\vartheta(x,t) = \sum_{|\alpha|+2j=2\kappa} \frac{a_{\alpha,j}}{\alpha!j!} x^\alpha t^j, \qquad \text{with } a_{\alpha,j} = \partial_x^\alpha \partial_t^j \vartheta(\mathbf{0},0). \tag{5.68}$$

On the other hand, the fact that $(0,0) \in \Gamma_\kappa^N(u)$ means that

$$\partial_{x_i}\partial_x^\alpha \partial_t^j \vartheta(0,0) = 0,$$

for any multi-index $\alpha \in \mathbb{Z}_{\geq 0}^N$ and $j = 0, \dots, \lfloor\kappa\rfloor$ such that $|\alpha| + 2j = 2\kappa - 1$, and all $i = 1, \dots, N$. Consequently, it follows

$$\partial_x^\beta \partial_t^j \vartheta(0,0) = 0,$$

for any multi-index $\beta \in \mathbb{Z}_{\geq 0}^N$ and $j = 0, \dots, \lfloor\kappa\rfloor$ such that $|\beta| + 2j = 2\kappa$ and $|\beta| = |\alpha| + 1$, and so, the unique nonzero coefficient in the sum (5.68) turns out to be $a_{0,\lfloor\kappa\rfloor}$ and proof is completed. □

Proof of Theorem 5.12. Let $u \in \text{dom}(H^s)$ be a nontrivial solution to (1) and define $\overline{u}(x,t) := u(x,-t)$ (with extension $\overline{U} = \overline{U}(x,y,t)$). Following the ideas of [71, Theorem 12.12], we divide the proof in two steps.

Step1: Parabolic Whitney's extension. Let $p_0 = (x_0, t_0) \in \Gamma_\kappa(\overline{u})$ (cfr. with (5.6)) and let $\vartheta_{p_0}^\kappa(x,t) := \Theta_{p_0}(x,0,t)$, where, as always, $\Theta_{p_0} = \Theta_{p_0}(x,y,t) \in \mathfrak{B}_\kappa(\overline{U})$ is the "blow-up limit" of $\overline{U} = \overline{U}(x,y,t)$ at $p_0 = (x_0, 0, t_0)$. Since, ϑ_{p_0} is a parabolically κ-homogeneous polynomial of degree 2κ, we can write it in the form

$$\vartheta_{p_0}(x,t) = \sum_{|\alpha|+2j=2\kappa} \frac{a_{\alpha,j}(x_0,t_0)}{\alpha!j!} x^\alpha t^j, \tag{5.69}$$

where $\alpha \in \mathbb{Z}_{\geq 0}^N$, $j = 0, 1, \ldots \leq \kappa$, and the coefficient functions $p_0 \to a_{\alpha,j}(p_0)$ are continuous on $\Gamma_\kappa(\overline{u})$, thanks to part (ii) of Theorem 5.9. Now, we define

$$f_{\alpha,j}(x,t) := \begin{cases} 0 & \text{if } |\alpha| + 2j < 2\kappa \\ a_{\alpha,j}(x,t) & \text{if } |\alpha| + 2j = 2\kappa, \end{cases}$$

for any $\alpha \in \mathbb{Z}_{\geq 0}^N$ and $j = 0, 1, \ldots \leq \kappa$. We proceed by proving the following claim.

CLAIM: Let $K = E_j$ for some $j \in \mathbb{N}$, where $E_j = \cap_{n>\kappa} E_j^n$ and E_j^n are defined in (5.65). Then for any $(x_0, t_0), (x,t) \in K$,

$$f_{\alpha,j}(x,t) = \sum_{|\beta|+2i \leq 2\kappa-|\alpha|-2j} \frac{f_{\alpha+\beta,j+i}(x_0,t_0)}{\beta!i!}(x-x_0)^\beta (t-t_0)^j + R_{\alpha,j}(x,t;x_0,t_0), \tag{5.70}$$

with

$$|R_{\alpha,j}(x,t;x_0,t_0)| \leq \sigma_{\alpha,j}(\|(x-x_0, t-t_0)\|)\|(x-x_0,t-t_0)\|^{2\kappa-|\alpha|-2j}, \tag{5.71}$$

where $\sigma_{\alpha,j}$ are suitable modulus of continuity (depending on K).

CLAIM's proof. The case $|\alpha| + 2j = 2\kappa$ follows by the continuity of the functions $p_0 \to a_{\alpha,j}(p_0)$ on $\Gamma_\kappa(\overline{u})$, by taking

$$R_{\alpha,j}(x,t;x_0,t_0) = a_{\alpha,j}(x,t) - a_{\alpha,j}(x_0,t_0).$$

The case $0 \le |\alpha| + 2j < 2\kappa$ is harder. According to Taylor expansion theory, we define

$$\begin{aligned} R_{\alpha,j}(x,t;x_0,t_0) &:= -\sum_{\substack{(\beta,i)\ge(\alpha,j):\\ |\gamma|+2i=2\kappa}} \frac{a_{\beta,i}(x_0,t_0)}{(\beta-\alpha)!(i-j)!}(x-x_0)^{\beta-\alpha}(t-t_0)^{i-j} \\ &= -\partial_x^\alpha \partial_t^j \vartheta_{p_0}(x-x_0,t-t_0), \qquad t \ge t_0. \end{aligned}$$

Now, assume by contradiction, there is no modulus of continuity $\sigma_{\alpha,j} = \sigma_{\alpha,j}(\cdot)$ such that (5.71) is satisfied, i.e., there are sequences $p_l := (x_l,t_l), p_{0l} := (x_{0l},t_{0l}) \in K$,

$$\delta_l := \|(x_l - x_{0l}, t_l - t_{0l})\| \to 0 \quad \text{as } l \to +\infty,$$

such that

$$\begin{aligned} |R_{\alpha,j}(x,t;x_0,t_0)| &= \left| \sum_{\substack{(\beta,i)\ge(\alpha,j):\\ |\gamma|+2i=2\kappa}} \frac{a_{\beta,i}(x_0,t_0)}{(\beta-\alpha)!(i-j)!}(x-x_0)^{\beta-\alpha}(t-t_0)^{i-j} \right| \\ &\ge \underline{\sigma}\|(x_l - x_{0l}, t_l - t_{0l})\|^{2\kappa-|\alpha|-2j}, \qquad \text{for all } l \ge 0. \end{aligned} \tag{5.72}$$

Defining the families

$$\overline{u}_{p_{0l},\delta_l}(x,t) = \frac{\overline{u}_{p_{0l}}(\delta_l x, \delta_l^2 t)}{\delta_l^{2\kappa}}, \qquad (\xi_l,\theta_l) := \left(\frac{x_l - x_{0l}}{\delta_l}, \frac{t_l - t_{0l}}{\delta_l^2}\right),$$

we may assume (up to subsequences) $(x_{0l},t_{0l}) \to (x_0,t_0) \in K$ and $(\xi_{0l},\vartheta_{0l}) \to (\xi_0,\theta_0) \in \overline{B}_1 \times (-1,1)$, and so, from Corollary 5.4, it follows

$$\|\overline{u}_{p_{0l},\delta_l} - \vartheta_{p_0}\|_{L^\infty(Q_R)} \le \|\overline{u}_{p_{0l},\delta_l} - \overline{u}_{p_0,\delta_l}\|_{L^\infty(Q_R)} + \|\overline{u}_{p_0,\delta_l} - \vartheta_{p_0}\|_{L^\infty(Q_R)} \to 0,$$

as $l \to \infty$, for any $Q_R = B_R \times (0,R^2)$. The same holds true for the sequence

$$\overline{u}_{p_l,\delta_l}(x,t) = \frac{\overline{u}_{p_l}(\delta_l x, \delta_l^2 t)}{\delta_l^{2\kappa}},$$

so that

$$\begin{aligned} \overline{u}_{p_{0l},\delta_l} &\to \vartheta_{p_0} \quad \text{in } L^\infty(Q_R) \\ \overline{u}_{p_l,\delta_l} &\to \vartheta_{p_0} \quad \text{in } L^\infty(Q_R), \end{aligned} \tag{5.73}$$

and, consequently,

$$\begin{aligned} &\|\overline{u}_{p_{0l},\delta_l}(\cdot+\xi_l,\cdot+\theta_l) - \overline{u}_{p_l,\delta_l}(\cdot,\cdot)\|_{L^\infty(B_R\times(-R^2,R^2))} \to 0 \\ &\|\overline{u}_{p_l,\delta_l}(\cdot,\cdot) - \overline{u}_{p_{0l},\delta_l}(\cdot-\xi_l,\cdot-\theta_l)\|_{L^\infty(B_R\times(-R^2,R^2))} \to 0, \end{aligned} \tag{5.74}$$

for any $R > 0$. Now, as in [71], we proceed by splitting the remaining part of the proof in two cases:

(i) There are infinitely many indexes $l \in \mathbb{N}$ such $\theta_l \ge 0$.

(ii) There are infinitely many indexes $l \in \mathbb{N}$ such $\theta_l \le 0$.

Let us start with case (i). After passing to a subsequence, we can assume $\theta_l \ge 0$ for any $l \in \mathbb{N}$. So, since $\theta_l \ge 0$, for any $(x,t) \in Q_1$ we have $(x - \xi_l, t_l - \theta_l) \in Q_2 := B_2 \times (-4,4)$ and, furthermore,

$$\begin{aligned} \|\vartheta_{p_0}(\cdot,\cdot) - \vartheta_{p_0}(\cdot-\xi_0,\cdot-\theta_0)\|_{L^\infty(Q_1)} &\le \|\vartheta_{p_0} - \overline{u}_{p_l,\delta_l}\|_{L^\infty(Q_1)} \\ &\quad + \|\overline{u}_{p_l,\delta_l}(\cdot,\cdot) - \overline{u}_{p_{0l},\delta_l}(\cdot-\xi_0,\cdot-\theta_0)\|_{L^\infty(Q_1)} \\ &\quad + \|\overline{u}_{p_{0l},\delta_l}(\cdot-\xi_0,\cdot-\theta_0) - \vartheta_{p_0}(\cdot-\xi_0,\cdot-\theta_0)\|_{L^\infty(Q_1)} \to 0, \end{aligned}$$

as $l \to +\infty$, thanks to (5.73) and (5.74). Consequently, using the real analyticity of the polynomial $\vartheta_{p_0} = \vartheta_{p_0}(x,t)$, it follows

$$\vartheta_{p_0}(x+\xi_0, t+\theta_0) = \vartheta_{p_0}(x,t) \quad \text{for all } (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

and so

$$\partial_x^\alpha \partial_t^j \vartheta_{p_0}(\xi_0,\theta_0) = \partial_x^\alpha \partial_t^j \vartheta_{p_0}(0,0) = 0 \quad \text{for all } |\alpha| + 2j < 2\kappa.$$

On the other hand, diving both sides of (5.72) by $\delta_l^{2\kappa-|\alpha|-2j}$ ad taking the limit as $l \to +\infty$, we obtain

$$|\partial_x^\alpha \partial_t^j \vartheta_{p_0}(\xi_0,\theta_0)| = \left| \sum_{\substack{(\beta,i)\ge(\alpha,j):\\ |\gamma|+2i=2\kappa}} \frac{a_{\beta,i}(x_0,t_0)}{(\beta-\alpha)!(i-j)!} \xi_0^{\beta-\alpha} \theta_0^{i-j} \right| \ge \underline{\sigma} > 0,$$

in contradiction with the computation above. The proof of case (ii) is almost identical to the previous one and we skip it (we just have to using the first convergence in (5.74) instead of the second one).

We have thus verified the assumptions of Whitney's Theorem. Consequently, we deduce the existence of a function $F \in C^{2\kappa,\lfloor\kappa\rfloor}(\mathbb{R}^N \times \mathbb{R})$ such that

$$\partial_x^\alpha \partial_t^j F = f_{\alpha,j} \quad \text{in } K,$$

for all $|\alpha| + 2j \le 2\kappa$, where $\lfloor\cdot\rfloor$ denotes the floor function.

Step2: Implicit function theorem. Let $(x_0,t_0) \in \Gamma_\kappa^d \cap K$ and $d = 0,\dots,N$. As in [71, Theorem 12.12] we consider two subcases.

Let us begin by assuming $d = 0,\dots,N-1$. For these choices of the dimension, we have that there are multi-indexes α_i and nonnegative integers j_i with $|\alpha_i| + j_i = 2\kappa - 1$ such that

$$v_i = \nabla_x \partial_x^{\alpha_i} \partial_t^{j_i} \vartheta_{p_0} = \nabla_x \partial_x^{\alpha_i} \partial_t^{j_i} F(x_0,t_0) \quad \text{for } i = 1,\dots,N-d,$$

are linearly independent vectors. At the same time, in view of Lemma 5.21, we have

$$\Gamma_\kappa^d(u) \cap K \subset \bigcap_{i=1}^{N-d} \Big\{(x,t) \in \mathbb{R}^N \times \mathbb{R} : \partial_x^{\alpha_i} \partial_t^{j_i} F(x,t) = 0\Big\},$$

and so, using the linear independence of the v_i's and the Implicit function theorem, we immediately conclude that $\Gamma_\kappa^d(u) \cap K$ is contained in a $(d+1)$-dimensional "space-like" $C^{1,0}$ manifold (cfr. with Definition 5.11). Finally, recalling that we have chosen $K = E_j$ (for some arbitrary $j \in \mathbb{N}$) and since $\Gamma_\kappa(u) = \cup_{j\in\mathbb{N}} E_j$ (cfr. with Lemma 5.20), the proof in the case $d = 0,\dots,N-1$ is ended.

Assume now $d = N$ and $(x_0,t_0) \in \Gamma_\kappa^N \cap K$. Hence from Lemma 5.27, we deduce

$$\partial_t^{\lfloor\kappa\rfloor} F(x_0,t_0) = \partial_t^{\lfloor\kappa\rfloor} \vartheta_{p_0} \neq 0,$$

while

$$\Gamma_\kappa^N \cap K \subset \Big\{(x,t) \in \mathbb{R}^N \times \mathbb{R} : \partial_t^{\lfloor\kappa\rfloor-1} F(x,t) = 0\Big\},$$

and so, for the Implicit function theorem we get that $\Gamma_\kappa^N \cap K$ is contained in a N-dimensional "time-like" C^1 manifold. The proof is then completed. □

5.6 Extensions, comments and open problems

We end the paper discussing some possible extensions and open problems.

On the nodal set of solutions to equation (7). Different from the rest of the work, in Section 5.5 we have focused on solutions to equation (1). It is thus natural to ask themselves which are the nodal properties of "strong solutions" to (7) near/on the set Σ (recall that far away from Σ the dimensional estimates on the regular and singular set still hold). This problem seems to be nontrivial due to the "odd" component of solutions. However, we can easily adapt the proof of Proposition 5.8 to prove the following result.

Proposition 5.28. *Let $U = U(x, y, t)$ be a smooth "strong solution" to equation* (7). *The following two assertions hold.*

(i) Let us define

$$\mathcal{R}(U) := \Gamma_{\nu_*}(U) = \left\{ p_0 \in \Gamma(U) \cap \Sigma : \lim_{t \to t_0^+} N(p_0, t, U) = \nu_* = \frac{1}{2}\min\{1, 1-a\} \right\}.$$

Then, for any $p_0 = (x_0, 0, t_0) \in \mathcal{R}(U)$ it holds

$$\nabla^a_{x,y} U(p_0) := \left(\nabla_x U, \lim_{y \to 0} |y|^a \partial_y U \right)(p_0) \neq (\mathbf{0}, 0),$$

where by definition

$$\lim_{y \to 0} |y|^a \partial_y U(p_0) := (1-a) \lim_{y \to 0} \frac{U(x_0, y, t_0) - U(x_0, 0, t_0)}{y|y|^{-a}}.$$

(ii) $\mathcal{R}(U)$ is a locally C^1-manifold of Hausdorff dimension $N + 1$.

Proof. Let us start from part (i). If $-1 < a < 0$ we have $\nu_* = 1/2$ and the proof of Proposition 5.8 applies here. On the other hand, if $0 < a < 1$, we have $\nu_* = (1-a)/2$ and as $\lambda \to 0$

$$U_{p_0,\lambda} = \frac{U(x_0 + \lambda x, \lambda y, t_0 + \lambda^2 t)}{\lambda^{1-a}} \to \Theta_{p_0}(x, y, t) = y|y|^{-a},$$

uniformly on compact sets of $\mathbb{R}^{N+1} \times (0, T)$ (and up to a nonzero multiplicative constant). So, as before,

$$\lim_{y \to 0} |y|^a \partial_y U(x_0, y, t_0) = \pm(1-a) \lim_{\lambda \to 0^+} \frac{U(x_0, \pm\lambda, t_0)}{\lambda^{1-a}} = 1 - a \neq 0,$$

and we conclude the proof of the case $0 < a < 1$, too.

To prove part (ii), it is enough to apply the Implicit function theorem once the transformation $z = y|y|^{-a}$ (which changes $|y|^a \partial_y U$ to $\partial_z U$) is performed. □

Nonlocal equations with potentials. A very important remark is that our results also apply to solutions to equation (1) with *potential*, namely:

$$H^s u = V(x, t) u \quad \text{in } \mathbb{R}^{N+1}, \tag{5.75}$$

with potential $V = V(x, t)$ satisfying suitable assumptions as in formula (1.2) of [19]. This fact can be easily seen combining the monotonicity formula proved in Section 6 of [19] and their blow-up procedure. The main point is that the blow-up sequences of the extensions of solutions to (5.75) converge to solutions to problem (8), in which there is not potential. Consequently, we can apply the blow-up classification of Corollary 5.2 and Corollary 5.4, together with the results on the stratification, structure and regularity of the nodal set.

Other extensions. In [185], the study of solutions $u = u(x,t)$ to

$$H^s u = 0 \quad \text{in } \mathbb{R}^{N+1}$$

(or generalizations of it), where as always $H^s = (\partial_\tau - \Delta)^s$, relies on the well-known semigroups formula (see [183, 184])

$$H^s f(x) = \frac{1}{\Gamma(-s)} \int_0^{+\infty} \left(e^{-\tau H} f(x) - f(x)\right) \frac{d\tau}{\tau^{1+s}},$$

for $0 < s < 1$, functions $f \in \mathcal{S}(\mathbb{R}^N)$ and positive real operators H, and on the fact the above formula holds true for operators $H \in \mathbb{C}$, with $\mathrm{Re}(H) > 0$ (this comes from the fact that the function $z \to z^s$ can be analytically continued to $\mathrm{Re}(z) > 0$). As pointed out by the authors of [185] (see Remarks 2.3, 4.1 and 5.2) most of the results proved in [185] can be adapted to more general operators in the form

$$H^s = (\partial_\tau - L)^s,$$

where $Lu(x) = \nabla\cdot(A(x)\nabla u)$ is an elliptic operator in divergence form on a bounded domain $\Omega \subset \mathbb{R}^N$ with suitable boundary conditions, or $L = \Delta_M$, where Δ_M is the Laplace-Beltrami operator on a Riemannian manifold, or $L = \Delta_h$ is the "discrete Laplacian" (see [66]), and so on. It is thus natural conjecture that it is possible to adapt our methods to prove new Almgren-Poon monotonicity formulas to study blow-up classifications and nodal properties of more general fractional operators.

Miming [99], another possible extension is the study of equation (1) with Hardy type potentials and nonlinear sources:

$$H^s u + \frac{a(x/|x|)}{|x|^{2s}} u + f(x,t,u) = 0 \quad \text{in } \mathbb{R}^N \times (-T, 0),$$

with suitable assumptions on $a = a(x/|x|)$ and $f(x,t,u)$. As mentioned above, the classical case $s = 1$ was treated in [99] where the authors studied the blow-up sequences near the singular point $x = 0$ with the decisive use of Gaussian-Hardy type inequalities. When $0 < s < 1$, the analysis of this kind of equations is motivated by the so-called "nonlocal" Hardy type inequality (cfr. with formula (2.1) of [105]):

$$\int_{\mathbb{R}^N} |x|^{-2s} |v(x)|^2 \, dx \le C_{s,N}^{-1} \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{v}(\xi)|^2 d\xi, \qquad v \in C_c^\infty(\mathbb{R}^N), \tag{5.76}$$

valid for any $0 < 2s < N$, where $\hat{v} = \hat{v}(\xi)$ is the Fourier transform of $v = v(x)$ and

$$C_{s,N} := 2^{2s} \frac{\Gamma^2((N+2s)/4)}{\Gamma^2((N-2s)/4)}.$$

Inequality (5.76) was proved independently by different authors (see for instance [24, 123, 211]), but the corresponding Gaussian version (cfr. with Lemma 2.1 of [99]) seems not to be known. It is thus left to us to understand the role played by a "nonlocal" Gaussian-Hardy inequality in the blow-up analysis of solutions to nonlocal parabolic equations with Hardy potentials.

Reaction-diffusion systems with strong competition. Reaction-diffusion systems with strong competitions have been widely studied in these last years. For the elliptic setting we quote the works of Caffarelli et al. [8, 50, 53] and Terracini et al. [70, 67, 68, 69, 162, 188] (local framework) and [51, 52, 189, 190, 203] (nonlocal framework). The parabolic case is less studied and to our knowledge there is literature for the local setting only (see the papers of Dancer et al. [72, 73, 74, 205]). It is thus left to study the parabolic nonlocal version of the above mentioned papers, namely

$$(\partial_t - \Delta)^s u_i = f_i(t, x, u_i) - \beta u_i^p \sum_{j \neq i} a_{ij} u_j^q \quad \text{for } i = 1, \ldots, M$$

in the limit case of $\beta \to +\infty$, for suitable choices of $p, q > 0$, functions $f_i(\cdot)$ and interaction coefficients $a_{ij} = a_{ji}$ ($0 < s < 1$ is fixed from the beginning). The main interest in the systems above is that (at least for the elliptic setting) it can be proved the existence of *segregated* solutions (in the regime $\beta \to +\infty$), i.e., solutions of the type

$$\mathbf{u} = (u_1, \dots, u_M) \quad \text{with} \quad u_i \cdot u_j = 0 \text{ for } i \neq j.$$

Consequently, if the same property is established for the nonlocal parabolic setting, the study of both the optimal Hölder regularity of segregated solutions and the qualitative/quantitative properties of the *free boundary* (i.e. the set in which $u_i = 0$ for any $i = 1, \dots, M$) seems to be very interesting and challenging open problems.

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