Aspects of F-Theory-engineered Quantum Field Theories

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Ai miei maestri.

ABSTRACT

In this thesis we discuss three examples of quantum field theories engineered from the IIB superstring theory and F-Theory.

Firstly we consider a model of $SU(5)$ GUT in F-Theory, with E_7 enhancement. Yukawa couplings for the two heaviest families of MSSM are computed, as well as one CKM entry. Realistic masses for the fermions can be obtained by considering certain values of the parameters entering in the model.

Secondly, we discuss the phenomenon of supersymmetry enhancement in QFTs, in which a theory with 4 supercharges flows in the IR to a theory with 8 supercharges. New systematic scans are performed in order to find more theories showing this peculiar feature. Furthermore, an explanation of this SUSY enhancement is given by geometrically engineering the QFT of a D3 probing a F-Theory singularity corresponding to a T-brane background.

Thirdly, we consider mixed branches of 3d $\mathcal{N} = 4$ QFTs. We devise a way to compute the Hilbert Series of a generic mixed branch of a particular set of theories of this kind, called $T[SU(N)]$. In order to understand how the usual formulae for a Coulomb Branch Hilbert series get modified in the mixed branch case, it was crucial to engineer the $T[SU(N)]$ in IIB superstring theory, by the use of an Hanany-Witten setup.

RESUMEN

En esta tesis damos de tres ejemplos de teorias cuánticas de campos que pueden ser realizadas en la teoría de supercuerdas IIB, y teoría F.

Para empezar, consideramos un modelo modelo de Gran Unificación (GUT) de grupo gauge $SU(5)$ en teoría F, con E_7 enhancement. Se calculan los acoplos de Yukawa para las dos familias más pesadas del MSSM, como también una entrada de la matriz CKM. Masas realistas para los fermiones pueden ser encontradas fijando algunos valores de los parámentros que entran en el modelo.

A continuación, consideramos el fenómeno de incremento de supersimetría en Teorías Cuánticas de Campos (QFTs), en que una teoría con 4 supercargas fluye en el IR a una teoría con 8 supercargas. Hacemos nuevas búsquedas sistematicas para encontrar más teorías que tienen esta propriedad particular. Además, damos una explicación de este fenómeno a través del geometrical engineering de la QFT con una D3 que explora una singularidad en teoría F que corresponde a una configuración gauge no-Abeliana conocida como T-brana.

Por ultimo, consideramos ramas mixtas de algunas teorías cuánticas de campos 3d $\mathcal{N} = 4$. Damos una manera para calcular la Serie de Hilbert de una cualquier rama mixta para un conjunto dado de teorias, llamadas $T[SU(N)]$. Para entender como se modifica la fórmula de la Serie de Hilbert del Coulomb Branch en el caso de un mixed branch, es crucial pensar a la teoría $T[SU(N)]$ en el contexto de la teoría de supercuerdas IIB, a través una construcción de Hanany-Witten.

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CONTENTS

CHAPTER 1

GENERAL INTRODUCTION

At the time of writing, it is believed that four fundamental interactions exist. Ordering them from the strongest to the weakest at our energy scales, they are the strong nuclear force, electromagnetism, the weak nuclear force, and gravity.

The theory of General Relativity, introduced more than a century ago, describes gravity at the classical and macroscopic level. Typically General Relativity is used to discuss galaxies, planetary systems, or the evolution of the universe itself: those are all scenarios in which gravity dominates compared to the other interactions. The success of this theory is enormous, passing all experimental tests to date and also recently finding a very non-trivial experimental confirmation by the discovery of gravitational waves. On the other hand, the other three fundamental forces are described at the microscopic level by Quantum Mechanics, and in particular Quantum Field Theory. The so called Standard Model of Particle Physics is also extremely successful in reproducing experimental evidences, up to date. It is typically employed to study particles scatterings and decays, in a regime in which gravity is by all means negligible. We see therefore that we have two set of laws for describing Nature at the fundamental level, and we use one or the other depending on the specific problem we want to address.

But what happens if an object is small and heavy at the same time? Since it is small we should use Quantum Field Theory, and since it is heavy we should use General Relativity. We don't know what to do then. Using at the same time both formalisms is sadly impossible. The main obstacle in putting together General Relativity and Quantum Field Theory is that General

Relativity is not renormalizable. In particular, this implies that General Relativity has to be thought as an effective field theory, valid only at energy sufficiently low compared to a natural cutoff known as the Planck's scale. The way in which gravity behaves at the quantum level is not yet fully understood.

Fortunately, there is active research in this direction and numerous candidate theories of quantum gravity have been proposed over the years. Out of all of them, the most developed and studied one is String Theory.

Originally introduced as a model for the description of the strong interactions - model afterwards put aside in favor of the modern QCD approach - the basic idea of String Theory is that particles are actually small vibrating strings, which we perceive to be pointlike just because we don't resolve them with enough energy. This extremely simple idea has deep consequences, as it was shown that among the modes of vibration of the string there is a spin 2 excitation which can be interpreted as a graviton: the mediator of gravity. String Theory is also consistent with quantum mechanics, namely it is possible to quantize String Theory, making it *de facto* a consistent theory of quantum gravity.

Some initial rapid development culminated with the formulation of five consistent (and apparently different) Superstring Theories. They are named type I, type IIA, type IIB, heterotic-E and heterotic-O. Mathematical consistency of those superstring theories puts several constraints, as for example the fact that spacetime dimension is fixed to be 10, and in order to develop a suitable four dimensional limit of the theory which reproduces known particle physics phenomenology we need to perform a *compactification*. Our understanding of those superstring theories in the perturbative regime is fairly good. The same cannot be said for the non-perturbative regime.

It was later noticed, during the '90s, that all these versions of superstring theory are connected by a intricate web of dualities, often relating a weakly coupled theory led limit of version with the strongly couped limit of another. New dynamical objects such as the D-branes were discovered, and played a major role in explaining the existence of string dualities. A unified picture started to emerge, with a 11d M-theory sitting at some central point in the duality web. Extensive use of these dualities has allowed us to perform numerous computations before unaccessible, therefore allowing us to explore at least part of the non-perturbative regime of the theory. More importantly the existence of dualities has explained us that the 5 versions discussed before have to be thought as somehow equivalent: String Theory is a unique and well defined framework for both phenomenological and formal studies.

By now String Theory is 50 years old. It is fair to say that the old dream, the one of finding a complete theory of quantum gravity able to recover both particle physics phenomenology and General Relativity in some suitable limits, is still unfulfilled. However, tremendous progress have been done over the years in this direction. The derivative of our understanding of String Phenomenology with respect to time has always been positive. Furthermore the understanding of String Theory has shed light also on different other topics in theoretical physics such as Quantum Field Theory itself, or even pure mathematics.

Two important examples of the connection between String Theory and Quantum Field Theory are the following. First of all, it is possible to think about string theory, in the perturbative regime, as some specific 2d superconformal field theory living in the worldsheet. Therefore in some sense perturbative String Theory *is* a Quantum Field Theory. The second connection is the famous AdS/CFT correspondence, for which a theory of quantum gravity on a AdS space is holographically dual to a conformal quantum field theory living on the boundary of AdS. This fact can be turned around and properties of quantum gravity theory (in AdS) can be directly defined by properties of the boundary CFT. Remarkably, we can use Quantum Gravity to understand more QFTs, and we can also use QFTs to understand more Quantum Gravity.

This thesis will be entirely based on the interplay between those two fascinating research topics. Let us now set up the plan of the thesis.

Plan of the thesis

Along the years of the PhD studies, the author of this thesis worked on different projects culminating in the papers [1–5]. This thesis will discuss only a subset of those, namely [1, 3, 5]. The reason for this choice is that these three papers all address the common fact of discussing Quantum Field Theories which can be engineered from String Theory: this is the specific topic we would like to present here.

The thesis is divided in three parts (roughly corresponding to the three papers mentioned above) which can be read independently, although they share numerous common features. The *trait d'union* will be IIB superstring theory, often in a non-perturbative regime. Furthermore, we will always work on local models, where gravity is decoupled because we either take infinitely extended extra dimensions, or we work in a local patch of them, or we only focus on brane worldvolume theories. To all effects, in this thesis we will use a theory of quantum gravity in order to decouple gravity, not care about it anymore, and just study Quantum Field Theories. In particular we will show three concrete examples in which IIB superstring theory and F-Theory can be used to understand QFTs both from the phenomenological point of view or a more formal point of view.

In chapter 2 we study $SU(5)$ Grand Unified Theory, realized within the formalism of F-Theory. In the first half of this chapter we review basic features of GUTs and we give a short survey of the basic aspects of F-Theory. In the second half of this chapter we introduce an explicit F-Theory model realizing $SU(5)$ GUT in a way that at the Yukawa point there is E_7 enhancement. This model allows to compute the Yukawa couplings for the two heaviest families of MSSM, as well as one angle of the CKM matrix.

In chapter 3 we discuss the recently discovered phenomenon of supersymmetry enhancement in quantum field theory. In this context, 4d $\mathcal{N} = 1$ QFTs are found to flow in the IR to 4d $\mathcal{N} = 2$ QFTs. In the first part of this chapter we review some basic aspects of extended supersymmetric and superconformal field theories in 4 dimensions. We also review one of the standard procedures to engineer the aforementioned flows. In the second part of this chapter we discuss some new scans that we performed in order to look for other $\mathcal{N} = 1$ theories which show the same enhancement phenomenon. In this latter part we also discuss how to engineer such QFTs and the RG flow in the framework of F-Theory, giving a way to understand geometrically why some $\mathcal{N} = 1$ theories enjoy these peculiar features.

In chapter 4 we discuss Moduli Spaces of 3d $\mathcal{N} = 4$ supersymmetric gauge theories, considering in particular mixed branches. In the first part of this chapter we review some basic features about extended supersymmetry in three dimensions, and in particular we review how to engineer a subset of these theories in the context of type IIB superstring theory. We further review the Hilbert Series, a counting function useful to investigate properties of moduli spaces of 3d QFTs. In the second part of this chapter we focus on a special class of 3d theories, called $T[SU(N)]$. After a brief introduction, we discuss how to compute the Hilbert Series of a mixed branch of such theories. The IIB realization of $T[SU(N)]$ will be crucial for this task.

Technical details and complementary material are added in different appendices. In the appendix 7.1 we briefly explain the Dynkin label notation we will use in most of the thesis, to denote irreducible representation of complex semisimple Lie algebras. In the appendix 7.2 we discuss few properties of the $\mathfrak{e}_7$ Lie algebra. In appendices 7.3 and 7.4 we collect a series of details on the computation of the wavefunctions for the A and B models considered in chapter 2. In the appendices 7.5 and 7.6 we discuss some extra non-trivial checks of the "Restriction rule" which is the main result of chapter 4. Finally in the appendix 7.7 we discuss basic properties of nilpotent orbits in complex semisimple Lie Algebras, which are used in chapter 3.

CHAPTER 2

YUKAWA COUPLINGS

2.1 The Standard Model and particles' masses.

One of the most astonishing results of 20th century science was the formulation of the Standard Model (SM) of particle physics. Such model describes quantitatively the microscopic dynamics of all the fundamental particles known at the time of writing this thesis, and the precise way such particles can interact by three out of the four fundamental forces. The Standard Model of particle physics is a quantum field theory based on the Lie group[1]

$$G = SU(3) \times SU(2)_L \times U(1)_Y \tag{2.1}$$

Where the $SU(3)$ sector describes the strong nuclear force while the $SU(2) \times U(1)$ describes the electroweak force. The matter content of the theory consists in three generations (families)

[1] A true fact somewhat overlooked in the literature is that *we are not quite sure about which is the right SM gauge group.* [6] Call $G = SU(3) \times SU(2)_L \times U(1)_Y$. Of course we know by experiments that the Lie algebra of the Standard Model must be $\mathfrak{g} = \mathfrak{su}(3) \oplus \mathfrak{su}(2)_L \oplus \mathfrak{u}(1)_Y$, but it is actually very difficult to determine experimentally the exact Lie group. In particular for every Lie group G, the Lie algebra of G/Γ is isomorphic to the Lie algebra of G if Γ is a discrete subgroup of the center of G. In this case of the Standard Model the same Lie Algebra $\mathfrak{g}$ can thus come from both G and G/Γ where Γ can be $\mathbb{Z}_2$, $\mathbb{Z}_3$ or $\mathbb{Z}_2 \times \mathbb{Z}_3 \simeq \mathbb{Z}_6$. Such a difference in the global properties of the group has no effect in any n-point function of local operators. However, the spectrum of Wilson and 't Hooft lines of the theory will be significantly different in the four cases outlined above and the theory will ultimately differ. In the following, if not explicitly stated otherwise, we will stick with the standard choice of taking Γ to be the trivial group.

of quarks and leptons. We will take all fermions to be complex 2-component left-chirality Weyl spinors. Apart from the fermions and the gauge bosons we will also have a complex scalar H, called the Higgs field. The representations of the gauge group in which matter fields transform are given in table [2.1].

Field	$SU(3)$	$SU(2)$	$U(1)_Y$
$Q_L^i = (U^i, D^i)_L$	$[1,0]$	$[1]$	$1/6$
U_R^i	$[0,1]$	$[0]$	$-2/3$
D_R^i	$[0,1]$	$[0]$	$1/3$
$L^i = (\nu^i, E_L^i)$	$[0,0]$	$[1]$	$-1/2$
E_R^i	$[0,0]$	$[0]$	1
$H = (H^0, H^-)$	$[0,0]$	$[1]$	$-1/2$

Table 2.1: A table enconding the SM matter fields and their representations

The electroweak symmetry $SU(2) \times U(1)$ is broken spontaneously to the electromagnetic $U(1)$ by the vacuum expectation value of the Higgs field. Indeed the Higgs field has a scalar potential given by

$$V = -\mu^2 |H|^2 + \lambda |H|^4 \tag{2.2}$$

which gets minimized by

$$v := \langle |H| \rangle^2 = \frac{\mu^2}{2\lambda} \tag{2.3}$$

Three generators of the gauge group get broken in the vacuum therefore generating three Goldstone bosons which are immediately paired up with three different linear combination of the four gauge bosons of $SU(2)_L \times U(1)_Y$. Such Goldstone bosons provide a third polarization state to such linar combinations of the gauge bosons therefore making them massive. As a result we end up with massive vector bosons $W^\pm$ and Z^0 and a massless photon γ.

The Higgs mechanism also provides mass terms to quarks and leptons. In the Standard Model Lagrangian we have the following terms

$$\mathcal{L}_{SM} \supset \mathcal{L}_{Yuk} = Y_U^{ij} \bar{Q}_L^i U_R^j H^* + Y_D^{ij} \bar{Q}_L^i D_R^j H + Y_L^{ij} \bar{L}^i E_R^j H + h.c. \tag{2.4}$$

and we see that after the Higgs field condensates to its vev (2.3) we will have three mass matrices respectively for the up-type quarks, the down-type quarks and the leptons. Here the main point to stress out is that such mass matrices do not only depend on the Higgs vev, but also on the Yukawa couplings. Then also their eigenvalues, which correspond to quark and lepton masses, will depend explicitly on the Yukawa couplings.

Within the Standard Model there is no theoretical explaination nor mechanism which fixes the Yukawa couplings to the specific values that we know they have. They are to all effects some free parameters which need to be specified by hand when writing the SM lagrangian. The presence in a QFT of some arbitrary parameters which need to be specified by hand is clearly unsatisfactory. In this particular case, we are saying that basically the Higgs mechanism explains *why* fermions have a mass, but does not explain *which* specific mass they have. If we simply stay within the Standard Model, there is no explaination then for why an electron weights 0.511MeV/c^2 and not, who knows, seven times this value.

One would expect that such numerical values for fermion masses could be predicted by using some theory which extends the Standard Model. In order to do so, some mechanism to compute the Yukawa couplings should be provided. In the following we will discuss one instance in which it is possible to partially solve this problem by using strongly coupled IIB Superstring Theory (or better F-Theory) in order to build a quantum field theory which resembles SM. The string theoretical construction will then give an explaination of numerical value of the Yukawa coupling of at least some families, as well as the value of some of one entry of the CKM matrix.

Before discussing this construction, we will first need to review some basic features of Grand Unification Theories, and F-Theory.

2.2 Grand Unified Theories

A recurring theme in the history of physics has been the idea of *Unification*. The Standard Model of particle Physics, very briefly reviewed in section (2.1), is up to date the end result of a unification process which has its roots back in the 17th century, shortly after the beginning of Modern Science. Along the path that lead us to such an astonishing result we have witnessed many unifications of different kinds. First Newton unified "the earth and the sky", showing that the laws that regulate planetary motion and celestial dynamics are the same that regulate the motion of a free falling object near the surface of the earth, or the motion of a ball rolling down an inclined plane: in all such circustances it is gravity the force who plays the main role. Later Maxwell showed that phenomena of electricity and magnetism, which appear distinct at a naive first sight, are indeed unified in what we call today electromagnetism: a single force responsible for both these classes of phenomena.

Another great unification, yet of a different type, is the one linking classical mechanics to thermodynamics, via the framework of statistical mechanics. A complex macroscopic system, such a gas, is governed by only a few parameters such as volume, temperature, pression, energy, entropy, etc. Via statistical mechanics it was explained how such parameters emerge from some suitably taken average over the the microscopic degrees of freedom of the system, therefore unifying mechanics and thermodynamics.

During the 20th century two other fundamental forces were discovered: the strong nuclear force and the weak nuclear force. Summing to gravity and electromagnetism, this accounts for the four fundamental interactions. Another more modern unification is the one which happened in the second half of last century: the one present in the Standard Model of particle physics. The electromagnetic force and the weak nuclear force are unified in the electroweak force. The reason for which we see electromagnetism and weak nuclear force as distinct is explained by the phenomenom of spontaneuos symmetry breaking by the Higgs vev, as briefly reviewed in section (2.1).

It is natural then to wonder if such a path of unification can be pursued more, and if there will be maybe an unification of the strong nuclear force (corresponding to $SU(3)$ symmetry in SM) with the electroweak force (corresponding to $SU(2)$). Any theory for which such unification exist will be called Grand Unified Theory, or in short GUT.[2]

2.2.1 Gauge coupling unification

The first conjecture of the existence of a GUT was made in 1974 by Georgi and Glashow [7] from noticing that the gauge coupling constant of the three simple group factors of the SM gauge group seem to unify at some scale of approximately $10^{15} GeV$. Indeed, gauge couplings run due to quantum effects. Quantitatively, the running is given by

$$\frac{1}{\alpha_i(\mu)} = \frac{1}{\alpha_i(M_W^2)} + \frac{b_i}{4\pi} \log \frac{M_W^2}{\mu^2} \tag{2.5}$$

where M_W is the electroweak scale and μ is th energy scale at which we are considering the couplings. The one loop beta function coefficient for a $SU(N)$ gauge theory is given by the NSVZ formula [8]:

$$b = -\frac{11}{3}N + \frac{2}{3}T(\mathcal{R}_f)n_f + \frac{1}{3}T(\mathcal{R}_s)n_s \tag{2.6}$$

[2]Actually, in this thesis we will use the word GUT in a slightly broader sense. We will consider as a GUT any theory which gauge group contains the SM group, and which has some spontaneous symmetry breaking to it. We don't require the GUT group to be simple. For example, we will consider the Trinification model to be a GUT despite the fact that strong and electroweak forces are not unified.

where n_f is the number of left handed Weyl spinors charged in the representation $\mathcal{R}_f$ of $SU(N)$, n_s is the number of complex scalars charged in the representation $\mathcal{R}_s$ of $SU(N)$, and $T(\mathcal{R})$ is the quadratic casimir. For a $U(1)$ gauge theory there is a similar result for the one loop beta function coefficient. In the end, we can compute that for the three gauge coupling costants of SM we have $(b_1, b_2, b_3) = \left(\frac{41}{10}, -\frac{19}{6}, -7\right)$. By plotting the evolution of the coupling costants, one can see that they almost converge to a single point, however perfect unification is not achieved. By doing the same computation in the context of the MSSM, much better unification is achieved.

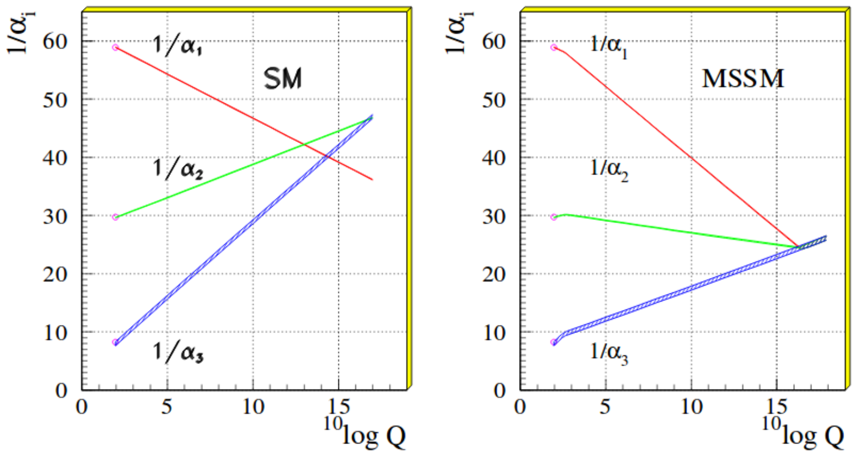


Figure 2.1: Plot of gauge coupling unification in SM compared with MSSM, taken from [10]. The superpartners are assumed to contribute only above the scale of 1 TeV. We see that unification is achieved better in the supersymmetric scenario.

We are therefore lead to take seriously the possibility that a GUT theory exists in Nature. The following question is how to make some explicit models of GUTS. We have at least two requiremets: the GUT gauge group must contain the SM gauge group as a proper subgroup, and also the GUT Lie algebra must admit complex representations in order to allow for a chiral spectrum[3]. In the following we will review the most famous and studied GUT model.

2.2.2 Georgi-Glashow $SU(5)$ GUT

In this section we recall briefly one of the most studied GUT models: the Georgi-Glashow GUT, first introduced in [7]. This is a quantum field theory based on the gauge group $SU(5)$ which is

[3]This simple remark excludes, for example, any kind of GUT model building based on the Lie groups E_7, E_8, as they only have real representations. [9]

spontaneously broken to the SM gauge group by a second Higgs mechanism. The vector field of the $SU(5)$ theory sits in the adjoint representation. In order to see which fields will this lead to after GUT symmetry breaking, we need to look at the branching rules for the decomposition of the adjoint of the Lie algebra $\mathfrak{su}(5)$ into the Lie algebra of SM. Such branching rules give[4]

$$\begin{aligned} \mathfrak{su}(5) &\to \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \\ [1,0,0,1] &\mapsto [1,1;0]_0 \oplus [0,0;2]_0 \oplus [0,0;0]_0 \oplus \\ &\oplus [1,0;1]_{-\frac{5}{6}} \oplus [0,1;1]_{\frac{5}{6}} \end{aligned} \tag{2.7}$$

We clearly see that we recover all SM gauge bosons (irrepses grouped in the first line) together with some extra bosons charged in the fundamental (resp antifundamental) representation of $SU(3)$ called X bosons (resp Y bosons). The Higgs mechanism breaking the GUT down to SM will give these latter bosons a mass of the order of GUT scale. As we will discuss later, those are mediators for proton decay.

In order to recover the matter fields of Standard Model, we will also need to consider some matter fields for the $SU(5)$ theory. In particular we see that a full family of SM fermions can be fit exactly in the sum of two irreducible representations of $\mathfrak{su}(5)$ GUT, namely the $[0,0,0,1]$ and the $[0,1,0,0]$. In this cases the branching rules read

$$\begin{aligned} \mathfrak{su}(5) &\to \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \\ [0,0,0,1] &\mapsto [0,1;0]_{\frac{1}{3}} \oplus [0,0;1]_{-\frac{1}{2}} \\ [0,1,0,0] &\mapsto [0,1;0]_{-\frac{2}{3}} \oplus [1,0;1]_{\frac{1}{6}} \oplus [0,0;0]_1 \end{aligned} \tag{2.8}$$

Comparing this with table (2.1), we see that we recover all the matter fields. The only field of the Standard Model which so far we have not recovered is the Higgs field. In order to do that, we will need to add a scalar field in the $SU(5)$ theory, charged in the $[1,0,0,0]$ representation of $\mathfrak{su}(5)$. Indeed the branching rules in this case will be

$$\begin{aligned} \mathfrak{su}(5) &\to \mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1) \\ [1,0,0,0] &\mapsto [1,0;0]_{\frac{1}{3}} \oplus [0,0;1]_{-\frac{1}{2}} \end{aligned} \tag{2.9}$$

We see that apart from the $SU(2)$ doublet, now we have also some triplet of scalar fields, sometimes called “colored Higgs”. As we will discuss later, they are mediators for proton decay process, which can be extremely dangerous in model building.

[4] Througout this thesis, when we will talk about irreducible representation of complex semisimple Lie Algebras, we will mostly use a Dynkin label notation. See appendix 7.1 for more details

As a last comment, we mention what happens in the supersymmetric case. All matter fields discussed up to now will be the fermionic component of chirals superfields, and all the vector bosons discussed now will be the vector components of vector superfields. Therefore the Susy $SU(5)$ GUT has one vector multiplet in the $[1,0,0,1]$ and chiral multiplets in the $[0,0,0,1]$ and $[0,1,0,0]$. Regarding the Higgs field, it is well known that in MSSM we will need to introduce two chiral multiplet doublets for the Higgses[5], therefore in case we consider the supersymmetric version of the $SU(5)$ GUT, we will need to add a chiral multiplet in the $[1,0,0,0]$ for the usual SM higgs and higgsino, plus also a chiral multiplet in the $[0,0,0,1]$ in order to get the second higgs and the second higgsino.

2.2.3 Anomaly cancellation.

Anomaly cancellation in Standard Model poses some extremely non-trivial constraints on the possible representation of matter fields. It is actually known that local gauge anomaly cancellation is so strong that it almost[6] fixes all the weak hypercharges of all the fields. [11] The fact that mathematical consistency of the Model is able to single out (almost) exactly the right hypercharges (and therefore electric charges) we see in Nature is remarkable.

On the other hand, in GUTS[7] the local gauge anomaly cancellation simplifies enormously. The reason is that we typically don't have $U(1)$ factors of the GUT gauge group G, and $U(1)$ factors are the most likely to give non-trivial triangle diagrams. In the case in which the GUT gauge group is simple, the situation is even better as now the only dangerous diagram can only be the one with three G currents. Having a less stringent anomaly cancellation condition is good for model building, as allows much more possibilities to chose the matter content of the theory. On the other end, one could argue that it is also bad from the point of view that the theory is now somewhat more arbitrary: matter content is no loger fixed by some mathematical

[5]Essentially for two conceptually unrelated reasons: in order to cancel Witten anomaly [12] (as with just one higgsino we would have now a odd number of fermions in the $\mathfrak{su}(2)$ fundamental representation), and in order to be able to write the supersymmetrization of the Yukawa couplings, as now the superpotential needs to be an holomorphic function so the idea is to replace H^* in (2.4) with a different scalar (which will then be the lowest component of a new chiral superfield).

[6]One could take the SM field content and give unspecified hypercharges q_i to all the field of table (2.1), and then impose anomaly cancellation in order to try to solve for the q_i. The system of homogeneus equations coming from the non-trivial triangle diagrams and also gauge-gravity anomaly cancellation will admit only two solutions: one gives the right SM hypercharges, the other gives the so called *bizzarre solution* where all fields have vanishing hypercharge apart from two of them.

[7]And especially in those with simple gauge group.

consistency condition.

As an example of the fact that anomaly cancellation simplifies in GUTs we can consider $SU(5)$ GUT with only fields in the $[1,0,0,0]$ or the $[0,1,0,0]$ or their complex conjugated representations (namely $[0,0,0,1]$ and $[0,0,1,0]$)

Let $n(\mathcal{R})$ be the number of fields that we have in the representation $\mathcal{R}$. Cancellation of the cubic $SU(5)$ anomaly will then be simply the statement that

$$n([0,1,0,0]) - n([0,0,1,0]) = n([1,0,0,0]) - n([0,0,0,1]) \tag{2.10}$$

To conclude, we mention two interesting facts. The first is that Witten anomaly[8] [12] is automatically absent in basically all interesting GUT models, since the gauge group is usually chosen to not have $Sp(n)$ factors. The second fact is that also the mixed-gauge-gravity anomaly is automatically vanishing as the gauge group is chosen not to have $U(1)$ factors.

2.2.4 GUTs' first prediction: Proton decay

Proton decay is one of the most crucial predictions of many Grand Unification Theories. In Standard Model the proton is stable essentially because baryon and lepton number are global symmetries. However, in some GUTs like the $SU(5)$ model and the $SO(10)$ model discussed above (together with their supersymmetric versions) this is no longer the case. As a result the proton is no longer stable, and may decay.

Indeed if we consider for example $SU(5)$ GUT, proton decay can be mediated by the X and Y bosons, as well as the Higgs triplet, for the decay channel $p \to \pi^0 e^+$ with the π^0 later decading to two photons. The predicted lifetime of the proton can be computed, and the result is $\tau_p \sim 10^{29}$ years. Such a prediction is falsified by the Super Kamiokande experiment, which puts a lower bound on the proton lifetime of 10^{33} years [13–17]. This effectively rules out the minimal $SU(5)$ GUT, as well as many other non-supersymmetric GUTs. However, considering the supersymmetric version GUTs, the predicted proton lifetime increases by some orders of magnitude, and the Super Kamiokande bound can still be satisfied.

It must be said that the predicted proton lifetime is very model dependent, and the choice of different GUT models will give different results. In the following table, we summarize the predictions for different famous GUTs.

An important point to discuss is the different role played by X, Y and the Higgs triplet in mediating proton decay. Both these fields will induce some higer dimensional operators

[8]We will give more details about Witten anomaly in section 3.1 of this thesis

GUT	Predicted p lifetime
$SU(5)$	$\sim 10^{31}$ years
$SO(10)$	$\sim 10^{35}$ years
Flipped Susy $SU(5)$	$\sim 10^{36}$ years
Susy $SU(5)$	$\sim 10^{34}$ years
Susy $SO(10)$	$\sim 10^{35}$ years

Table 2.2: A table summarizing experimental bounds on proton lifetime, in different GUT models.

proportional to the value of their mass. The mass of the X and Y bosons is of the order of the GUT scale, while the mass of the Higgs triplet may or may not be of this value. Now the GUT scale is essentially fixed to be of order $10^{15} - 10^{16} GeV$ by the argument of gauge coupling unification, so there is no way to tune the contribution of X and Y to the proton decay process. Instead, the mass of the Higgs triplet can be tuned to be sufficiently high such that its contribution to the proton decay process is small. To illustrate this, consider supersymmetric $SU(5)$ GUT. The masses of the higgs doublet and higgs triplet are determined by the Higgs sector superpotential which reads

$$W = M H_5 H_{\bar{5}} + \lambda_5 H_{\bar{5}} \Phi_{24} h_{\bar{5}} \tag{2.11}$$

Once the field Φ_{24} takes a vev $< \Phi_{24} >= v \; \mathrm{diag}(1, 1, 1, -\frac{3}{2}, -\frac{3}{2})$ and breaks to the SM gauge group, the doublet and triplet gets some mass by the adjoint higgs mechanism. Such masses are

$$m_{H_2} = \left(M - \frac{3}{2} \lambda_5 v \right), \qquad\qquad m_{H_3} = (M + \lambda_5 v). \tag{2.12}$$

so by some fine-tuning of λ_5 and M one can achieve to have the triplets much more massive than the doublets which should be massless. When such a thing is done it is said that we solved the doublet-triplet splitting problem. Such a result is not spoiled by RG flow down to the electroweak scale, but still is some fine-tuning done by hand and ad hoc, at this level. In the following we will see how realizing $SU(5)$ GUT in String Theory will sometimes automatically solve this problem by explaining how such fine tuning is realized.

2.2.5 GUTs' second prediction: 't Hoft- Polyakov monopoles

Another crucial prediction of GUTs, apart from proton decay, is the existence of 't Hoft Polyakov monopoles [18, 19], formed during the symmetry breaking transition from a GUT to SM, back in the early times of the history of the Universe. Such monopoles are topological solitons of codimension 3 in a QFT, and are classified by $\pi_2(G/H)$ where $\pi_n(X)$ is the n-th homotopy group of a topological space X, G is the GUT group and and H is the unbroken group after symmetry breaking. For a good review of monopoles see [20].

Homotopy groups of the type $\pi_n(G/H)$ can often be computed easily, provided the the homotopy groups of G and H are known. This can be done by employing exact sequences. Explaining this method in full generality and with adequate proofs of the mathematical results is out of the scope of this thesis. For that we refer the reader to a standard textbook in homological algebra. Let un instead explain how this method works by giving a clear example, which can be easily generalized to most cases. Let us now prove that $SU(5)$ GUTs predicts the existence of monopoles. We need to show that $\pi_2(X)$ is nontrivial, where we defined $X = G/H$ with $G = SU(5)$ and $H = SU(3) \times SU(2) \times U(1)$ is the gauge group of the Standard Model.

Now, it can be proven that a quotient of topological spaces G/H defines a short exact sequence given by

$$0 \to H \to G \to G/H \to 0 \tag{2.13}$$

By passing to homotopy, such short exact sequence induces a long exact sequence of the form

$$... \to \pi_n(H) \to \pi_n(G) \to \pi_n(G/H) \to \pi_{n-1}(H) \to ... \to \pi_0(G) \to \pi_0(G/H) \to 0 \tag{2.14}$$

Supposing that we know the homotopy of G and H separately, we can look at the long exact sequence and often "solve" for the π_n of G/H. In the case of monopoles for the $SU(5)$ GUT we are interested in the part of the sequence which reads

$$... \to \pi_2(SU(5)) \to \pi_2(X) \to \pi_1(SU(3) \times SU(2) \times U(1)) \to \pi_1(SU(5)) \to ... \tag{2.15}$$

By plugging back in the sequence the following facts that

1. $\pi_1(SU(n)) \simeq 0$
2. $\pi_2(SU(5)) \simeq 0$
3. $\pi_1(SU(3 \times SU(2) \times U(1)) = \pi_1(SU(3) \times \pi_1(SU(2) \times \pi_1(U(1)) \simeq \pi_1(U(1)) \simeq \mathbb{Z}$

we get to the following sequence

$$... \to 0 \to \pi_2(X) \to \mathbb{Z} \to 0 \to ... \tag{2.16}$$

from which it is clear to see that the map betwenn $\pi_2(X)$ and $\mathbb{Z}$ must be a isomorphism, proving therefore that the $SU(5)$ GUT predicts the existence of magnetic monopoles labelled by an integer $z \in \mathbb{Z}$, the magnetic charge. A similar computation can be done for all the othe GUTs.[9]

2.3 A brief introduction to F-Theory

In the following part of this chapter we will discuss how to realize $SU(5)$ GUT models within the context of F-Theory, and how to compute the Yukawa couplings. In the next chapter, on the other hand, we will discuss about how F-Theory can be used to engineer many QFTs, and in particular to partially understand the phenomenom of Supersymmetry Enhanchign RG flows. Since F-Thory plays a central role in this thesis, it is due now to briefly recall the basics of such framework, firstly introduced in 1996 in the seminal paper [21].

The main idea of F-Theory is simple: in IIB superstring theory, the axiodilaton $\tau = C_0 + ie^{\phi}$ can be regarded as the complex structure of an auxiliary torus fibered on top of the usual 10-dimensional spacetime. Locations at which this fibration become singular will then correspond to the position of some 7-brane stacks, as we will review. In such a way, one is able to completely translate the information of the 7-brane dynamics into geometry of the elliptic fibration. Geometrical tools avaiable to date are then sufficiently well developed to allow us to understand even non-perturbative regimes or exotic 7-brane stacks (for example those carrying a E-type gauge group on their worldvolume), opening therefore a huge windows of possibilities both in phenomenological applications and in more formal studies of QFTs.

While the natural entry-point for F-Theory is IIB supersting theory with varying axiodilaton profile, it was soon realized that F-Theory is naturally linked by dualities also to M-Theory and to Heterotic $E_8 \times E_8$ theory. In this section we will discuss these three standard approaches to the subject.

We refer to the reviews [22, 23] for a much more complete treatement of the subject.

[9]Note that this argument does not change if we take the standard model group to be $SU(3) \times SU(2) \times U(1)/\Gamma$ with Γ a discrete subgroup of $\mathbb{Z}_6$, basically because it is possible to prove that $\pi_2(G) = \pi_2(\Gamma)$ for any discrete group Γ.

2.3.1 F-Theory from IIB superstring

Let us start with IIB superstring theory and consider the low energy effective SUGRA action. The bosonic fields present will be the 10-dimensional graviton g_{MN}, the 2-form B-field B_{MN}, the scalar field dilaton ϕ, and a set of p-forms C_p with $p = 0, 2, 4$ coming from the RR sector[10]. It is customary to put toghether C_0 and ϕ in a single complex scalar field called the *axiodilaton* and denoted by τ.

$$\tau = C_0 + ie^{-\phi} \tag{2.17}$$

It is possible to show that the effective action for IIB supergravity is invariant under a $SL(2, \mathbb{R})$ acting on the fields in the following way

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \tag{2.18}$$

with $a, b, c, d \in \mathbb{R} \mid ad - bc = 1$. Both the metric and the field strnght $\tilde{F}_5 = dC_4 - \frac{1}{2}C_2 \wedge H_2 + \frac{1}{2}B_2 \wedge F_3$ will be invariant under this action. Once quantum effects are taken into account such symmetry group is reduced to a discrete subgroup Γ of $SL(2, \mathbb{R})$, because of charge quantization. It is customary[11] to take $\Gamma = SL(2, \mathbb{Z})$.

$SL(2, \mathbb{Z})$ has two generators, which we will call S and T, and they are given by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.19}$$

satisfying the relations $S^2 = (ST)^3 = -1$. The action of S on τ in particular will give $\tau \to -\dfrac{1}{\tau}$ therefore inverting the string coupling $g_s = e^{-\phi}$. For this reason this is called the Strong-Weak duality of IIB superstring theory. Naively one could then expect that everytime we have a nonperturbative regime in which the string coupling is large, we could use this S-transformation to go to a different duality frame in which the string coupling is small and perturbation theory is avaiable. Such logic is wrong in general.

In most cases in which the axiodilaton is not constant but its vev varies along (at least part) of the 10-d spacetime, we cannot use such S-duality in the naive way. To explain this consider some IIB SUGRA solution for which τ undergoes $SL(2, \mathbb{Z})$ monodromies once encircling some

[10]In some democratic formulation of the IIB action it is customary to allow also for extra C_p forms with $p = 6, 8$ which will be related to the original C_ps by hodge dualities, often imposed at the level of the EOMs by lagrange multipliers in the action. We will not use this approach in the following.

[11]Recent developings seem to point out that this is not true in case O3-planes and O1-planes are considered. See for example the appendix of [24].

loci. Such solutions exist and are perfectly well-defined. In these cases we should consider τ not to be a simple function of spacetime coordinates but rather a section of some $SL(2,\mathbb{Z})$ bundle, trasforming nontrivially under changes of patches in the base space of the bundle. This latter point of view implies immediately that in general g_s is not a function but a section of a bundle, and therefore the whole concept of perturbation theory at weak g_s is not well posed in general. The whole notion of perturbation series would need to trasform nontrivially under such changes of g_s, making therefore the concept not well-defined in general.

It is very easy to show explicitly some SUGRA solution of this type. For example, consider a stack of D7 branes. Call z the complex coordinate parametrizing the transverse $\mathbb{C}$ to the D7s, and z_0 the value of z at which the D7 is located. The equation of motion for τ can be solved, giving

$$\tau = \tau_0 + \frac{1}{2\pi i}\log(z - z_0) + ... \tag{2.20}$$

This shows immediately that under encircling z_0, τ will suffer a $SL(2,\mathbb{Z})$ monodromy. In this case, the monodromy can be easily computed and is $\tau \to \tau + 1$. We see that this monodromy is not "dangerous" as it only affects the real part of τ by shifting C_0. One could still go to a weakly coupled duality frame and use perturbation techniques, in this example.

Let us then consider more complicated SUGRA solutions: we will find that indeed there are cases in which the $SL(2,\mathbb{Z})$ monodromies also affect g_s. As an example, consider a stack of (p,q) 7-branes, with p and q coprime. The monodromy matrix for this case can be computed to be

$$T_{(p,q)} = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix} \tag{2.21}$$

therefore affecting the imaginary part of τ as well.

The explaination of how to treat such backgrounds was given in [21] where the complex structure τ was considered to be the complex structure of an auxiliary torus fibered over the 10-d spacetime. The Kahler modulus of this torus is physically meaningless. Such construction automatically encodes the $SL(2,\mathbb{Z})$ monodromies, even in a generic case. Also, from (2.20) we see that $\tau \to i\infty$ when we approach to the location of the D7–brane. This means that the auxiliary torus $\mathbb{T}^2$ will be singular at such point.

From a study of the type of singularities of the fibration we can then read off the location of 7–branes, as well as computing the gauge group they carry on their worldvolume and much more informations. In the following we explain how to do so.

A comment about F-Theory and Seiberg-Witten theory.

The idea of encoding the axiodilaton as the complex structure of an auxiliary torus is amazingly similar to the idea of Seiberg-Witten solution of 4d $\mathcal{N} = 2$ dynamics[12]. Indeed, consider the Seiberg-Witten solution for the effective action on the Coulomb Branch of pure $4d\,\mathcal{N} = 2$ $SU(2)$ gauge theory [33, 34]. The Seiberg-Witten curve is a torus with complex structure given by the complexified YM coupling, automatically encoding $SL(2, \mathbb{Z})$ electromagnetic duality, and it is fibered over the Coulomb Branch. This is perfectly analog with the auxiliary torus of F-Theory which has a complex structure given by the complexified IIB coupling, encoding automatically the $SL(2, \mathbb{Z})$ dualities of IIB superstring theory, and being fibered over the 10d target space of the IIB superstring. The idea, in both contexts, is essentially the same.

Such analogy is not just formal. In some contexts it is exactly possible to identify the two curves. For example consider a single $D3$-brane probing the locus of the singularity of the elliptic fiber, and study the low energy dynamics on the Coulomb branch of the worldvolume theory of the $D3$. Its Seiberg-Witten curve will be given *exactly* by the elliptic fiber of F-Theory. We will use this factin the next chapter.

2.3.2 F-Theory from M-Theory

It is well known that M-Theory on a circle S^1_1 is dual to type IIA superstring theory. Parameters in both sides are identified as

$$R = g_s l_s, \quad l_p^3 = g_s l_s^3 \tag{2.22}$$

where R is the radius of the S^1_1, g_s is the string coupling, l_s the string length and l_p the 11-d Planck length. Now by taking a T-duality of IIA on a different circle S^1_2 we can get to type IIB theory. Therefore we see that M-theory is dual to IIB superstring theory whenever there is a torus $\mathbb{T}^2 = S^2_1 \times S^1_2$ in the background geometry. The two extra dimensions of F-Theory can be seen to arise by taking now the limit in which the torus shrinks to zero volume, but keeping the complex structure τ at a finite value. This can be generalized to the case in which the M-Theory background is some elliptically fibred manifold Y. Following the duality, complex structure τ of the elliptic fiber will be equal to the axiodilaton of IIB.

Therefore we can always use the aforementioned duality as a working definition of F-Theory backgrounds.

[12]We will discuss more Seiberg-Witten theory in the next chapter.

2.3.3 F-Theory from Heterotic

We will not use this duality in the following, but for completeness it is useful to discuss it as well. It is possible to find F-Theory duals of both versions of the Heterotic superstring, but here we will focus mainly in the $E_8 \times E_8$ case[13]. The claim here is that Heterotic string theory compactified on $\mathbb{T}^2$ is dual to F-Theory compactified on $K3$. This was observed originally in [21] by looking at properties of the moduli space in both sides, and comparing.

As a very preliminary remark, notice that number of preserved supercharges in both sides is 16, as F-Theory starts with 32 and K3 compactifications preserve a half of them, while on the dual side Heterotic starts with 16 and a $\mathbb{T}^2$ compactification preserves all of them.

Looking more in detail in the moduli space, on the F-Theory side the elliptic K3 is a $\mathbb{T}^2$ fibered over a $\mathbb{CP}_1$. The Kähler modulus of $\mathbb{CP}_1$ (its size) is mapped to the coupling costant g_S in the Heterotic side. In the Heterotic side we have also other moduli: the complex structure and Kähler moduli of the compactification $\mathbb{T}^2$ as well as the wilson lines moduli on the various cycles of the $\mathbb{T}^2$. It is well known [26] that such moduli will be local coordinates parametrizing the moduli space

$$\mathcal{M} = \Lambda_{2,18} \backslash O(2,18,\mathbb{R})/(O(2,\mathbb{R}) \times O(18,\mathbb{R})) \tag{2.23}$$

where here $\Lambda_{2,18}$ is a discrete group whose action takes into account T-duality.

The complete matching of such moduli space with the one in the F-Theory side is still an open question. A partial result is understood in the so called stable degeneration limit for the K3 manifold. The idea is taking the K3 to be composed of two "half K3s" (dP_9 surfaces). The intersection locus of those half K3 is a $\mathbb{T}^2$ which is identified with the compactification torus in the Heterotic side. Each half K3 will contain a local E_8 singularity which will be deformed to a smaller ADE singularity by a generic choice of the complex structure fields. This defines a set of curves intersecting the $\mathbb{T}^2$ at points. Such points will be in one to one correspondence with the Wilson line moduli in the Heterotic side.

It has to be mentioned that some partial result is also known away of the stable degeneration limit, but only in the special case in which the $E_8 \times E_8$ group is unbroken, or at most broken to $E_7 \times E_8$.

[13] Heterotic-E is dual to Heterotic-O when compactified on a S^1 with suitable Wilson lines turned on. It is therefore intuitive that if one of the Heterotics has a F-Theory dual, also the othe should. See for example [25] for F-Theory duals of Heterotic-O.

As a final remark, this duality can be also extended to lower dimension, by fibering the elliptic K3 (resp the $\mathbb{T}^2$) on some suitable base. The statement becomes that F-Theory on a CY $n+1$-fold which is an elliptic $K3$ fibration over a complex $n-1$ dimensional base B is dual to Heterotic theory on an elliptically fibered CY $n-fold$ with base B, for $n=1,2,3,4$ at least.[14]

2.3.4 Elliptic curves, fibrations, and their singularities

Let us define $\mathbb{P}^2_{231}$ to be the weighted projective space with weights $2,3,1$. Homogeneous coordinates in such space are $(x,y,z) \sim (\lambda^2 x, \lambda^3 y, \lambda z)$. An elliptic curve can be descibed algebraically by a (complex) codimension one locus in $\mathbb{P}^2_{231}$ which can always be defined as the zero locus of the equation

$$y^2 = x^3 + fxz^4 + gz^6 \tag{2.24}$$

where f,g are complex parameters. Such a hypersurface is called a *Weierstrass model* for the elliptic curve. It is customary to work in the patch in which $z=1$. The axiodilaton τ is identified with the complex structure of such elliptic curve, and is related to the Weierstrass model by

$$j(\tau) = 1728\frac{f^3}{\Delta} \tag{2.25}$$

where $j(\cdot)$ is the Jacobi-j function which is a $SL(2,\mathbb{Z})$ modular function of weight zero, and $\Delta = 4f^3 + 27g^2$ is the discriminant of the polynomial $x^3 + fx + g$. The discriminant Δ will vanish when two or more roots of the polynomial $\Delta = 4f^3 + 27g^2$ coincide. In this case we see that the elliptic curve degenerates, with one ot its cycles pinching. From equation (2.25) we see that the axiodilaton diverges in this case, therefore signaling the presence of a 7-brane. The idea is then that we should consider the elliptic curve (2.24) to be fibered over some spacetime base B, and seven branes will be located at places (in transverse space) in which (2.24) becomes singular. Therefore we need now to take (2.24) and fiber it over some base space.

In order to build an elliptically fibered manifold Y with base B, it is sufficient to[15] promote

[14]The status of compactifying F-Theory on a 6-fold is very interesting and yet unclear. The reason is that the elliptic fiber in F-theory is not really a (psuedo)Riemmanian manifold as the rest of spacetime is, as the Kähler modulus is physically meaningless. Therefore it is unclear what a F-theory compactification to 0d will look like, or if it physically makes sense in the first place. On the other hand, it certainly makes sense to compactify the Heterotic Theory on an elliptically fibered 5-fold and finding some matrix model, so one could maybe try to study (or define) the F-Theory compactification to 0d by declaring that the Heterotic/F-Theory duality also holds in this case, and working in the dual side. To the best our knowledge, this was never attempted so far.

[15]This is true only in the case in which the elliptic fibration admits a golbally well-defined section. While cases of elliptic fibrations without any sections have been extensively studied and are relevant in F-Theory (for

f and g in (2.24) to section of appropriate line bundles. Let us now demand Y to be a CY manifold, and postpone the explanation of such choice. In this case, we get to the conditions

$$f \in \Gamma(K_B^{-4}), \quad g \in \Gamma(K_B^{-6}) \tag{2.26}$$

where now K_X denotes the anticanonical bundle of a given algebraic manifold X, $\Gamma(\cdot)$ is the set of sections of a bundle, and exponents mean exterior power of line bundles. Notice that this implies that $\Delta \in \Gamma(K_B^{-12})$. This last fact has a deep physical meaning: it implies that the homology class of the linear combination of holomorphic cycles wrapped by the 7 branes must be equal to the Poincaré dual of $12c_1(B)$. This statement is the F-theoretical analog of the cancellation of the tadpoles in type IIB. Let us comment more on this point: suppose we have a IIB system with D7 branes wrapping some holomorphic divisors D_i in the compactification manifold. The tadpole cancellation condition demand the existence of orientifold planes wrapping other divisors, in a way that the constraint

$$\sum_i N_i[D_i] + N_i^\star[D_i^\star] = 4[D_{O7}] \tag{2.27}$$

is satisfied. This type of conditions corresponds in F-Theory to the contidion

$$\sum_i [D_i] = 12P_{dual}[c_1(B)] \tag{2.28}$$

as we discussed above. Therefore we see that in F-Theory the tadpole cancellation for 7-branes is ensured by the positive curvature of the base of the fibration B. This in turn implies by the adjunction formula that $c_1(Y) = 0$ and therefore Y is CY. This explains why we demanded to have the total space Y of the elliptic fibration to be CY when finding the line bundles in which f and g transform. Imposing the CY condition is equivalent to the physical request of cancelling tadpoles for the D-branes. As a last comment on this, notice that in F-theory there are no orientifolds nor 7-branes: only geometry of the elliptic fibration.

Having discussed elliptic fibrations, we now move to discussing their singularities, as we have seen that singularities encode the position of 7-branes in transverse space. Singularities of an elliptic fibration of complex codimension one were completely classified by Kodaira. We report in table 2.3.4 the result of such classification.

From the type of singularity we can deduce the type of gauge group present on the worldvolume of the 7-brane stack. A detailed treatment of how the gauge group can be read off from the

example to realize $U(1)$ symmetries or discrete symmetries) in this thesis we will never consider such case and always assume that a global section exists. We refer the reader to [27, 28] (and references cited by those papers) for more details.

Fibre type	ord(f)	ord(g)	ord(Δ)	Singularity
I_0	≥ 0	≥ 0	0	Smooth
I_n	0	0	n	A_{n-1}
II	≥ 1	1	2	smooth
III	1	≥ 2	3	A_1
IV	≥ 2	2	4	A_2
I_0^*	≥ 2	≥ 3	6	D_4
I_n^*	2	3	$n+6$	D_{n+4}
IV^*	≥ 3	4	8	E_6
III^*	3	≥ 5	9	E_7
II^*	≥ 4	5	10	E_8
non minimal	≥ 4	≥ 6	≥ 12	non canonical

Table 2.3: Kodaira's classification of singular elliptic fibers. Here ord($\cdot$) denotes the order of vanishing at the singularity locus.

singularity is out of the scope of this thesis. Anyways, the logic can be summarized as follows: consider such singularity in the dual M-Theory picture. Resolve the singularity by blowing up several times the singular points. For the resolution to be consistent, the resolution $\mathbb{CP}_1$ such created will intersect like the affine Dynkin diagram of some complex semisimple Lie algebra $\mathfrak{g}$ of ADE type. W bosons will then be realized by $M2$ branes wrapping such resolution $\mathbb{CP}_1$s and later mapped back to F-Theory by the duality. The mass of such W-bosons will be given by the Kähler modulus of the resolution $\mathbb{CP}_1$ (their size). The Cartans, on the other hand, will be related to the the compactification of the C_3-flux on such $\mathbb{CP}_1$s. Going now to the singular limit by "unresolving" the singularity ammounts to shrink all the resolution $\mathbb{CP}_1$s to zero size. The W bosons become masseless and we "un-do" the Higgs mechanism in field theory, giving therefore symmetry enhancement. Another way to see this is that from the point of view of the gauge theory living in the worldvolume of the 7-branes, resolving the singularity means going to the Coulomb Branch of the moduli space, therefore generically breaking the gauge group to its maximal torus and giving mass to the W bosons. For more details, see a standard F-Theory review like [23].

Up to now we have discussed the case of complex codimension one, which corresponds to a F-Theory compactification to 6d. We want now to discuss what happens in higher complex codimension. As a first thing we mention that now the list of possible gauge groups is enlarged. All complex semisimple Lie algebras can be realized, including non-simply laced cases like the B and C series, or also G_2 and F_4. The reason is that is is always possible to construct a non-simply laced lie algebra by quotienting a simply laced Lie algebra by outer automorphisms[16] and such oter automorphism are physically present in the construction only in codimention 2 or above.

A second important feature which can appear in complex codimension 2 is that divisors generically intersect at curves. So we can have the case of two divisors D_1, D_2 on which the elliptic fiber is singular, and which carry gauge algebra respectively $\mathfrak{g}_1$ and $\mathfrak{g}_2$ intersecting at a complex curve. We know from IIB theory that when two D7 branes intersect there is localized matter at the intesection locus, given by strings stretching from one D7 to the other. We are therefore lead to generalize this to the F-Theory context. It was shown in [29] that this is indeed the case: at the intersection locus the singularity type correspond to some $\mathfrak{g}$ which has both $\mathfrak{g}_2$ and $\mathfrak{g}_2$ as proper subalgebras. From the branching rules for $\mathfrak{g} \to \mathfrak{g}_1 \oplus \mathfrak{g}_2$ we see that the adjoint representation of $\mathfrak{g}$ decomposes as

$$\begin{aligned} \mathfrak{g} &\to \mathfrak{g}_1 \oplus \mathfrak{g}_2 \\ \text{Adj} &\to (Adj_{\mathfrak{g}_1}, 1) \oplus (1, Adj_{\mathfrak{g}_2}) \ \oplus_i (R_{1,i}, R_{2,i}) \end{aligned} \tag{2.29}$$

We will call such a curve a *matter curve*, for obvious reasons.

Let us now consider what can happen in complex codimension 3. Now different matter curves can intesect at points. A case in which three matter curves intersect at a point will lead to a cubic interaction between the fields living in the different matter curves, leading therefore to a Yukawa coupling in the 4d effective theory. This way of realizing the Yukawa coupling is the crucial point of this chapter of the thesis. In the following we will describe an explicit F-Theory GUT where we carried on the computation of the Yukawa couplings for the two heaviest families, as well as an entry of the CKM matrix.

2.4 Yukawas and exceptional groups in F-theory GUTs

The standard scheme of F-theory GUT models [140–143] (see [179, 180] for reviews) requires a Calabi-Yau fourfold elliptically fibered over a three-fold base B, such that the fiber degenerates over a four-cycle $S_{\text{GUT}} \subset B$. At such locus the Dynkin diagram of the fiber singularity cor-

[16] A procedure often called *folding a Dynkin diagram*

responds to the Lie group $G_{\rm GUT}$, except at subloci like complex curves $\Sigma \subset S_{\rm GUT}$ and their intersections where the fiber exhibits a higher singularity type. A quite powerful feature that arises out of this geometric picture is that of localization of GUT degrees of freedom. Indeed, one finds that the 4d gauge bosons that generate the gauge group have an internal profile localized at the four-cycle $S_{\rm GUT}$, and that the curves Σ further localize 4d chiral matter charged under $G_{\rm GUT}$. This statement remains true when one adds a four-form flux G_4 threading $S_{\rm GUT}$ which specifies the 4d chiral matter content of the model and, if chosen appropriately, breaks $G_{\rm GUT}$ to the subgroup $SU(3) \times SU(2) \times U(1)_Y$ and implements a double-triple splitting mechanism.[17]

This feature of localization is easier to detect with an alternative description of the degrees of freedom localized at $S_{\rm GUT}$, which uses a 8d action related to the 7-branes wrapping $S_{\rm GUT}$ and those intersecting them at matter curves. Such action is defined on a four-cycle S and in terms of a non-Abelian symmetry group G that contains $G_{\rm GUT}$ and all the enhanced symmetry groups at the matter curves and their intersections. Under this description the 4d effective theory that corresponds to the GUT sector of the compactification can be obtained upon dimensional reduction of the 8d action. In particular the computation of the Yukawa couplings is encoded in terms of the superpotential

$$W \,=\, m_*^4 \int_S {\rm tr}\, \left(F \wedge \Phi \right) \tag{2.30}$$

and the D-term

$$D \,=\, \int_S \omega \wedge F + \frac{1}{2} [\Phi, \Phi^\dagger] \, . \tag{2.31}$$

Here m_* is the F-theory characteristic scale, $F = dA - iA \wedge A$ is the field strength of the 7-branes gauge boson A, and Φ is the so-called Higgs field: a (2,0)-form on the four-cycle S describing the 7-branes transverse geometrical deformations. Both A and Φ transform in the adjoint of the initial gauge group G, which is nevertheless broken to a subgroup due to their non-trivial profile. On the one hand the profile $\langle \Phi \rangle$ is such that it only commutes with the generators of $G_{\rm GUT}$ in the bulk of $S_{\rm GUT}$, while on top of the matter curves of $S_{\rm GUT}$ it also commutes with further roots of G. On the other hand the profile $\langle A \rangle$ is such that it further breaks $G_{\rm GUT}$ to the MSSM gauge group through a component along the hypercharge generator. These profiles are not arbitrary but need to solve the equations of motion that arise from minimization of (2.30) and (2.31). Similarly, given the background for Φ and A one can compute the zero mode equations for their fluctuations via the two functionals W and D, and then plug them into (2.30) to obtain the Yukawa coupling through a triple wavefunction overlap.

[17] In certain models the presence of this flux may also generate 4d chiral matter that is not localized at any matter curve. We will however not consider this possibility here.

With this alternative description one can extract several key features regarding the computation of Yukawa couplings in F-theory GUTs:

- If we consider $G_{\rm GUT} = SU(5)$ (as we will do in the following) up-type Yukawas $\mathbf{10} \times \mathbf{10} \times \mathbf{5}$ will arise from (2.30) only if G contains the exceptional group E_6.
- The holomorphic piece of the Yukawas does not depend on the worldvolume flux profile $\langle F \rangle$, but only on the geometry around the intersection of the corresponding matter curves [149]. Therefore one can compute holomorphic Yukawas by specifying $\langle \Phi \rangle$ on a neighborhood $U_p \subset S$ of the matter curves intersection point p.
- The flux $\langle F \rangle$ localizes the internal zero mode wavefunctions at particular regions of the matter curves. If the MSSM fields are sufficiently peaked within a patch $U \subset S$ one can compute their physical Yukawas by knowing $\langle \Phi \rangle$ and $\langle F \rangle$ in this patch and replacing $S \to U$ in (2.30) and (2.31) [171].
- For $G = E_7$ or E_8 all the Yukawa couplings for charged MSSM fermions can be described from a single patch U, a scheme favored by the empirical values of the CKM matrix [144].
- One can engineer GUT models where the Yukawa matrices are of rank one by imposing a topological condition on the matter curves [154]. However, this is only compatible with well-defined zero mode wavefunctions if one considers a non-Abelian background profile for Φ [151, 154], dubbed T-brane background.

All these results point to a very suggestive setup for an F-theory GUT model, in which all the Yukawa couplings of the MSSM charged fermions are originated from a patch U where $G = E_7$ or E_8. The Yukawa matrices are of rank one, and therefore a mass hierarchy is generated between the third and the first two families of quarks and leptons, which at this point are massless.

Following [152], one may now take into account the non-perturbative effects originated at a different four-cycle $S_{\rm np} \subset B$, that modify the 7-brane superpotential to

$$W \, = \, m_*^4 \int_S \mathrm{tr}\, (F \wedge \Phi) + \epsilon \, \frac{\theta_0}{2} \mathrm{tr}\, (F \wedge F) \tag{2.32}$$

with ϵ measuring the strength of the non-perturbative effect and θ_0 a holomorphic function that depends on the embedding of the four-cycle $S_{\rm np}$.[18] This deformed superpotential will generically

[18]Namely $\theta_0 \, = \, (4\pi^2 m_*)^{-1} [\log h/h_0]_S$, with h a divisor function such that $S_{\rm np} = \{h = 0\}$, and $h_0 = \int_S h$. The full superpotential contains additional terms of the form $\theta_k \mathrm{STr}(\Phi_{xy}^k F^2)$, $k \geq 2$. These additional terms are suppressed by additional terms of m_* and therefore will be suppressed compared to the contribution coming from θ_0. We refer to [152, 156, 157] for further details.

increase the rank of the Yukawa matrices from one to three, as has been shown by the explicit analysis of several cases of interest [156–159]. One can summarize the results of this approach as follows:

- The hierarchy of fermion masses between different families can already be seen at the level of holomorphic Yukawas and in terms of the parameter ϵ. Typically one either obtains a hierarchy of the form $(\mathcal{O}(1), \mathcal{O}(\epsilon), \mathcal{O}(\epsilon^2))$ or $(\mathcal{O}(1), \mathcal{O}(\epsilon^2), \mathcal{O}(\epsilon^2))$, depending on the structure of matter curves around their intersection point.

- From these two possibilities only the pattern $(\mathcal{O}(1), \mathcal{O}(\epsilon), \mathcal{O}(\epsilon^2))$ allows to reproduce a realistic mass spectrum for charged fermions, with a typical value of $\epsilon \sim 10^{-4}$. The precise fit with empiric data depends of the worldvolume flux densities threading the curves around their intersection points, and on the mass running from the TeV to the compactification scale. The latter usually selects $\tan \beta \sim 20 - 50$.

- The departure from the GUT mass relations is mainly due to the dependence of the physical Yukawas on the hypercharge flux, whose effect is different for each family. Obtaining an appreciable effect entails an hypercharge flux density which is non-negligible in units of m_*.

- Fermion masses are complicated functions of the flux densities that arise from the components of $\langle F \rangle$, and which in this local approach are treated as parameters. Nevertheless, mass ratios display a much simpler dependence in only a few of these parameters.

- If the patch U contains both intersection points $p_{\rm up}$ and $p_{\rm down}$ where up and down-like Yukawas are respectively generated one can also compute the CKM matrix of quark mixing angles in this local approach. The observed mixing between the third and second families constrains $p_{\rm up}$ and $p_{\rm down}$ to be very close to each other compared to the size of $S_{\rm GUT}$, pointing to a symmetry group G which is either E_7 or E_8.

This last point was analyzed quantitatively in [159] for the case of a $SU(5)$ model with symmetry group E_8. Such class of local models have been highlighted in [144] as a tantalizing possibility to generate all fermion masses (including the neutrino sector) from a local patch of the compactification. It was then seen in [159] that such proposal could be made compatible with the above scheme to generate hierarchical Yukawas via non-perturbative effects, at least for the sector of fermions charged under the MSSM gauge group. However a realistic fermion mass spectrum would not happen automatically, but only for certain choices of matter curves/T-brane backgrounds.

In the following we would like to see if this scheme for generating flavor hierarchies can also be applied to models with E_7 enhancement. Notice that in principle a patch of E_7 enhancement does not describe how the masses for the neutrino sector of a F-theory GUT model may be generated [144]. Nevertheless, from the viewpoint of the approach in [156–159] which only deals with Yukawas for the MSSM charged fermions, these models are as equally compelling as the ones with E_8 enhancement. Moreover these models are simpler in the sense that they admit fewer matter curve embeddings than the E_8 case. In fact, in the next section we will see that imposing rank-one Yukawa matrices at tree-level selects a unique model of E_7 enhancement, which due to its simplicity will be analyzed in great detail in the subsequent sections.

2.4.1 $SU(5)$ models with E_7 enhancement

As mentioned in the introduction, one of the interesting features of $SU(5)$ models with E_7 enhancement is that they are rather universal, in the sense that there are very few ways to embed $SU(5)$ into E_7 in an F-theory construction. In fact, we will see that in this context there is essentially only one possibility to generate a hierarchical pattern of Yukawa couplings by means of the mechanism proposed in [152]. Recall that in this scheme one needs to consider an exceptional group E_n Higgsed by a T-brane background such that the resulting pattern of matter curves can embed the full chiral content of the MSSM. The T-brane profile should also be such that only one family of quarks and leptons develops non-trivial Yukawa couplings from the tree-level superpotential [154]. Finally, by including non-perturbative effects the remaining families will also develop Yukawa couplings, creating a hierarchy of masses between families. If this hierarchy is of the form $(1, \epsilon, \epsilon^2)$, with ϵ a small number that measures the strength of the non-perturbative effect, then one obtains Yukawa matrices at the GUT scales that are suitable to reproduce experimental masses for charged fermions [156–159].

As in [159] one may classify the different embeddings of $SU(5)_{\rm GUT}$ into E_n by looking at the pattern of matter curves of the local models, which is in turn specified in terms of the Higgs background $\langle\Phi\rangle$. Such background takes values in the algebra $\mathfrak{g}_\perp$ defined such that $\mathfrak{g}_{\rm GUT} \oplus \mathfrak{g}_\perp$ is a maximal subalgebra of $\mathfrak{g}_p = {\rm Lie}(G_p)$. In our case $\mathfrak{g}_p = \mathfrak{e}_7$ and $\mathfrak{g}_\perp = \mathfrak{su}_3 \oplus \mathfrak{u}_1$, so the maximal decomposition of the adjoint representation reads

$$\begin{aligned} \mathfrak{e}_7 \quad &\supset \quad \mathfrak{su}_5^{\rm GUT} \oplus \mathfrak{su}_3 \oplus \mathfrak{u}_1 \\ \mathbf{133} \quad &\to \quad (\mathbf{24},\mathbf{1})_0 \oplus (\mathbf{1},\mathbf{8})_0 \oplus (\mathbf{1},\mathbf{1})_0 \oplus (\mathbf{10},\overline{\mathbf{3}})_{-1} \oplus (\mathbf{5},\overline{\mathbf{3}})_2 \oplus (\mathbf{5},\overline{\mathbf{1}})_{-3} \oplus c.c. \end{aligned} \tag{2.33}$$

By construction $\langle\Phi\rangle$ commutes with $\mathfrak{su}_5^{\rm GUT}$, but it acts non-trivially on the representations $\mathcal{R}$ of $\mathfrak{g}_\perp = \mathfrak{su}_3 \otimes \mathfrak{su}_1$ that appear as $(\mathcal{R}_{\rm GUT}, \mathcal{R})$ in (2.33). This action can be expressed in terms of a matrix $\Phi_\mathcal{R}$ such that $[\langle\Phi\rangle, \mathcal{R}] = \Phi_\mathcal{R}\mathcal{R}$. Then at the locus where $\det \Phi_\mathcal{R} = 0$ there will be a matter curve hosting zero modes in the representation $\mathcal{R}_{\rm GUT}$ of $\mathfrak{su}_5^{\rm GUT}$.

One may now classify different profiles for $\langle\Phi\rangle$ in terms of the block diagonal structure of the matrices $\Phi_\mathcal{R}$, which we assume reconstructible in the sense of [154]. Because $\mathfrak{g}_\perp$ factorizes as $\mathfrak{su}_3 \otimes \mathfrak{u}_1$, we may directly focus on their block diagonal structure within $\mathfrak{su}_3$. In order to discuss the block diagonal structure of the Higgs field it is convenient to choose $\mathcal{R} = \mathbf{3}$, the fundamental representation of $SU(3)$ as the action of the Higgs field on any other representation may be constructed by taking suitable tensor products of the fundamental representation. With this choice the three different possibilities we have are

i) $\Phi_\mathbf{3}$ is diagonal

ii) $\Phi_\mathbf{3}$ has a $2+1$ block structure

iii) $\Phi_\mathbf{3}$ has a single block

Out of these three options the first one represents a $\langle\Phi\rangle$ taking values in the Cartan subalgebra of $\mathfrak{e}_7$, and so it does not correspond to a T-brane background. Option *iii)* was analyzed in [155], obtaining that up-type Yukawa couplings identically vanish. Hence, we are left with a splitting of the form *ii)* as the only possibility to obtain realistic hierarchical pattern of Yukawa couplings.

Reconstructible models with the split $2+1$ can be characterized with a profile for $\langle\Phi\rangle$ lying in the subalgebra $\mathfrak{su}_2 \oplus \mathfrak{u}_1 \subset \mathfrak{su}_3 \subset \mathfrak{g}_\perp$. Hence in order to read the spectrum of matter curves one may adapt the above branching rules for the adjoint of $\mathfrak{e}_7$ to the non-maximal decomposition $\mathfrak{su}_5^{\rm GUT} \oplus \mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1$. We obtain[19]

$$\begin{aligned} \mathfrak{e}_7 \quad &\supset \quad \mathfrak{su}_5^{\rm GUT} \oplus \mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1 \\ \mathbf{133} \quad &\to \quad (\mathbf{24},\mathbf{1})_{0,0} \oplus (\mathbf{1},\mathbf{3})_{0,0} \oplus 2(\mathbf{1},\mathbf{1})_{0,0} \oplus ((\mathbf{1},\mathbf{2})_{-2,1} \oplus c.c.) \\ &\quad\ \oplus (\mathbf{10},\mathbf{2})_{1,0} \oplus (\mathbf{10},\mathbf{1})_{-1,1} \oplus (\mathbf{5},\mathbf{2})_{0,-1} \oplus (\mathbf{5},\mathbf{1})_{-2,0} \oplus (\mathbf{5},\mathbf{1})_{1,1} \oplus c.c. \end{aligned} \tag{2.34}$$

[19] In writing the decomposition of the $\mathfrak{e}_7$ Lie algebra under $\mathfrak{su}_5^{\rm GUT} \oplus \mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1$ we choose two particular combinations of the generators of $\mathfrak{u}_1 \oplus \mathfrak{u}_1$.

and so we have two different kinds of $\mathbf{10}$ matter curves and three kinds of $\mathbf{5}$ matter curves. In order to have a rank one up-type Yukawa matrix we need to identify the matter curve $\mathbf{10}_M$ with $(\mathbf{10},\mathbf{2})_{1,0}$. Hence the curve containing the Higgs up is fixed to be $(\mathbf{5},\mathbf{1})_{-2,0}$, or otherwise the Yukawa coupling $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ cannot be generated. Finally, the remaining two $\mathbf{5}$-curves must host the family representations $\bar{\mathbf{5}}_M$ and down Higgs representation $\bar{\mathbf{5}}_D$, respectively.

To summarize, we find that in order to obtain a hierarchical pattern of Yukawa couplings we only have two possible ways to identify the matter curves with the representations of $SU(5)_{\mathrm{GUT}}$. Namely those are:

1. **Model A**

$$\begin{aligned} (\mathbf{10},\mathbf{2})_{1,0} &= \mathbf{10}_M \\ (\mathbf{5},\mathbf{1})_{-2,0} &= \mathbf{5}_U \\ (\bar{\mathbf{5}},\mathbf{2})_{0,1} &= \bar{\mathbf{5}}_M \\ (\bar{\mathbf{5}},\mathbf{1})_{-1,-1} &= \bar{\mathbf{5}}_D \end{aligned} \tag{2.35}$$

2. **Model B**

$$\begin{aligned} (\mathbf{10},\mathbf{2})_{1,0} &= \mathbf{10}_M \\ (\mathbf{5},\mathbf{1})_{-2,0} &= \mathbf{5}_U \\ (\bar{\mathbf{5}},\mathbf{1})_{-1,-1} &= \bar{\mathbf{5}}_M \\ (\bar{\mathbf{5}},\mathbf{2})_{0,1} &= \bar{\mathbf{5}}_D \end{aligned} \tag{2.36}$$

The Yukawa couplings for both of these scenarios have been computed. Even if we obtain a favorable hierarchical pattern $(1, \epsilon, \epsilon^2)$ for both of them, we advance that only Model A will reveal itself physically viable. Therefore we will focus only on this first case, deferring many computational details regarding Model B to Appendix **??**.

2.4.2 Yukawa hierarchies in the E_7 model

Let us now consider in more detail the two models with a local E_7 enhancement highlighted in the previous section. Since the difference between them amounts to how matter fields are distributed among matter curves, it is possible to give a description of the local background for the Higgs field Φ and the gauge connection A that applies to both models at the same time.

Indeed, such local models with E_7 enhancement is specified by choosing a Higgs field Φ and a gauge connection A valued in the algebra $\mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1$. To preserve supersymmetry in the low-energy 4d theory it is necessary to choose background fields satisfying the following supersymmetry equations

$$\bar{\partial}_A \Phi = 0 \tag{2.37a}$$

$$F^{(0,2)} = 0 \tag{2.37b}$$

$$\omega \wedge F + \frac{1}{2}[\Phi, \Phi^\dagger] = 0\,. \tag{2.37c}$$

The first two equations ensure the vanishing of the F-terms and may be obtained by varying the superpotential (2.30) while the third equation ensures the vanishing of the D-term (2.31).[20] A common strategy to find a solution of the previous set of equation is to exploit the fact that the F-term equations are invariant under complexified gauge transformations. In particular this gives the possibility of fixing a particular gauge, usually called *holomorphic gauge*, where the gauge connection satisfies $A^{(0,1)} = 0$. In this gauge the F-term equations greatly simplify and any choice of holomorphic Higgs field is a solution. While this solution is not a physical one (in the sense that the gauge connection is not real and the D-term equations are not satisfied) it still gives insight on the structure of matter curves and the rank of the Yukawa matrix. To reach a physically sensible solution of the equations of motion we may perform a complexified gauge transformation that brings the gauge fields in a real gauge that also satisfies the D-term equations. This is a rather cumbersome task in models with T-branes but nevertheless it is a necessary step to extract the physical values of the Yukawa couplings.

With this approach in mind we will start introducing the background value of the Higgs field in holomorphic gauge discussing moreover the structure of the various matter curves. After this we will consider the passage to a real gauge and impose the D-term equations. This will force the introduction of some fluxes which are non-primitive and leaves open the possibility to add primitive fluxes. We will discuss the addition of these fluxes that allow for a chiral spectrum in the 4d theory and the breaking of the GUT group down to the MSSM gauge group. We close this section with a direct computation of the Yukawa matrices for both models introduced in the previous section.

[20] When we consider the corrected superpotential (2.32) these F-terms equations will be modified, shifting the background values for Φ and A [157, 158]. This shift will be taken into account when computing the zero mode wavefunctions in section 2.4.5.

2.4.3 Higgs background

Holomorphic gauge

The first element that enters in the definition of our local model is the vacuum expectation value of the Higgs field $\langle\Phi\rangle = \langle\Phi_{xy}\rangle\, dx \wedge dy$ which constitutes the primary source of breaking the symmetry group E_7 down to $SU(5)_{GUT}$. Our choice in holomorphic gauge is the following one

$$\langle\Phi_{xy}\rangle = m\left(E^+ + mxE^-\right) + \mu_1^2\,(ax-y)\,Q_1 + \left[\mu_2^2\,(bx-y)+\kappa\right] Q_2 \tag{2.38}$$

where Q_i and $E^\pm$ are generators of the Lie algebra of E_7 whose definition (along with other details involving the E_7 Lie algebra) are given in appendix 7.2. In the definition of the Higgs background we introduced the complex constants m, $\mu_{1,2}$ and κ with dimension of mass and $a, b \in \mathbb{C}$ which are dimensionless parameters. The constant κ has a particular rôle in the sense that it controls the separation of the points where the Yukawa couplings for the up and the down-type quarks are generated, as we will now see.

This background for the Higgs field takes values in the subalgebra $\mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1$ orthogonal to $\mathfrak{su}_5^{\mathrm{GUT}}$ in $\mathfrak{e}_7$. As discussed in the previous section there are two possible assignments of matter fields that give rank one Yukawa couplings at tree-level. Here we recall the two possible assignments by specifying their charges under $SU(2) \times U(1) \times U(1)$

- **Model A**

$$\mathbf{10}_M : \mathbf{2}_{1,0}\,, \quad \mathbf{5}_U : \mathbf{1}_{-2,0}\,, \quad \bar{\mathbf{5}}_M : \mathbf{2}_{0,1}\,, \quad \bar{\mathbf{5}}_D : \mathbf{1}_{-1,-1}\,, \tag{2.39}$$

- **Model B**

$$\mathbf{10}_M : \mathbf{2}_{1,0}\,, \quad \mathbf{5}_U : \mathbf{1}_{-2,0}\,, \quad \bar{\mathbf{5}}_M : \mathbf{1}_{-1,-1}\,, \quad \bar{\mathbf{5}}_D : \mathbf{2}_{0,1}\,. \tag{2.40}$$

These assignments specify how the Higgs field background (2.38) enters the zero mode equation for each matter fields, and therefore the curves at which they are localized. As in [159] we define $\Phi|_{\mathcal{R}_{\mathrm{GUT}}}$ as the action of $\langle\Phi_{xy}\rangle$ on the $\mathfrak{g}^\perp$ part of $(\mathcal{R}_{\mathrm{GUT}}, \mathcal{R}) \subset \mathbf{133}$. For the model A we obtain

$$\begin{aligned}
\Phi|_{\mathbf{10}_M} &= \begin{pmatrix} \mu_1^2(ax-y) & m \\ m^2x & \mu_1^2(ax-y) \end{pmatrix}, \quad \Phi|_{\mathbf{5}_U} = -2\mu_1^2(ax-y)\,, \\
\Phi|_{\bar{\mathbf{5}}_M} &= \begin{pmatrix} \mu_2^2(bx-y)+\kappa & m \\ m^2x & \mu_2^2(bx-y)+\kappa \end{pmatrix}, \quad \Phi|_{\bar{\mathbf{5}}_D} = -\mu_1^2(ax-y) - \mu_2^2(bx-y) - \kappa\,.
\end{aligned} \tag{2.41}$$

where the action in the model B may be easily obtained by simply interchanging the actions on $\bar{\mathbf{5}}_M$ and $\bar{\mathbf{5}}_D$. The location of the matter curve hosting the representation $\mathcal{R}_{\rm GUT}$ is then found by computing $\det\Phi|_{\mathcal{R}_{\rm GUT}} = 0$. For the model A the explicit location is

$$\begin{aligned} &\Sigma_{\mathbf{10}_M} : \mu_1^4(ax-y)^2 - m^3 x = 0\,, \quad \Sigma_{\bar{\mathbf{5}}_M} : \left[\mu_2^2(bx-y)+\kappa\right]^2 - m^3 x = 0\,, \\ &\Sigma_{\mathbf{5}_U} : \mu_1^2(ax-y) = 0\,, \quad \Sigma_{\bar{\mathbf{5}}_D} : \mu_1^2(ax-y) + \mu_2^2(bx-y) + \kappa = 0\,. \end{aligned} \tag{2.42}$$

This expression for the matter curves allows to compare the present model with the model of E_8 enhancement considered in [159], see eq.(4.8) therein. In particular we see that the present model is more general, and that we recover the same matter curves as in [159] if we set $a = b$.[21] However, as we will see below considering $a \neq b$ will be crucial in order to implement the doublet-triplet splitting mechanism of [142] and it will also greatly increase the region of parameters for which we can reproduce the empirical masses and mixing for charged MSSM fermions.

Finally, Yukawa couplings for the matter fields are generated at the intersection of these matter curves. In particular the Yukawa coupling $\mathbf{10}_M \times \mathbf{10}_M \times \mathbf{5}_U$ of the up-type quarks is generated at the point where the curves $\Sigma_{\mathbf{10}_M}$ and $\Sigma_{\mathbf{5}_U}$ meet whereas the Yukawa coupling $\mathbf{10}_M \times \bar{\mathbf{5}}_M \times \bar{\mathbf{5}}_D$ of the leptons and down-type quarks is generated where the curves $\Sigma_{\mathbf{10}_M}$, $\Sigma_{\bar{\mathbf{5}}_M}$ and $\Sigma_{\bar{\mathbf{5}}_D}$ meet. These two points are

$$\begin{aligned} Y_U &: \Sigma_{\mathbf{10}_M} \cap \Sigma_{\mathbf{5}_U} = \{x = y = 0\} = p_{\rm up}\,, \\ Y_{D/L} &: \Sigma_{\mathbf{10}_M} \cap \Sigma_{\bar{\mathbf{5}}_D} \cap \Sigma_{\bar{\mathbf{5}}_M} = \{x = x_0,\, y = y_0\} = p_{\rm down}\,, \end{aligned} \tag{2.44}$$

where

$$x_0 = \frac{\kappa^2\mu_1^4}{m^3(\mu_1^2+\mu_2^2)^2} + \mathcal{O}(\kappa^3)\,, \quad y_0 = \frac{\kappa}{\mu_1^2+\mu_2^2}\left(1 + \frac{\kappa\mu_1^4(a\mu_1^2+b\mu_2^2)}{m^3(\mu_1^2+\mu_2^2)^2}\right) + \mathcal{O}(\kappa^3)\,. \tag{2.45}$$

This shows that the two Yukawa points do not necessarily coincide and that the parameter κ controls the separation between them. Setting $\kappa = 0$ both couplings are generated at the same point while the separation of the two points increases with κ.

Real gauge

The background fields described so far are in holomorphic gauge and therefore to achieve a physical solution, namely one in which the gauge fields are real and the D-term equations are

[21] The precise dictionary connecting the two models would be

$$\begin{aligned} -(1+2d)\mu^2 &\to \mu_2^2 \\ -2d\kappa &\to \kappa \end{aligned} \tag{2.43}$$

where the parameters in the left hand side of (2.43) are the ones appearing in [159] and the parameters in the right hand side are the ones of this paper.

satisfied, it is necessary to go to a real gauge. This may be attained by simply performing a gauge transformation defined by an element g of the complexified gauge group $G_{\mathbb{C}}$ so that the D-term equations simply become a set of differential equations for g. More explicitly the effect of this gauge transformation on the background fields is the following one

$$\Phi_{xy} \to g\,\Phi_{xy}\,g^{-1}\,, \quad A_{0,1} \to A_{0,1} + ig\,\bar{\partial} g^{-1}\,, \tag{2.46}$$

where in our case we take $g \in SU(2)_{\mathbb{C}}$. Following [154] we take the following Ansatz for g

$$g = \exp\left[\frac{1}{2} f P\right]\,, \tag{2.47}$$

where $P = [E^+, E^-]$. After this gauge transformation the background fields become

$$\Phi_{xy} = m\left(e^f E^+ + mxe^{-f} E^-\right) + \mu_1^2\,(ax-y)\,Q_1 + \left[\mu_2^2\,(bx-y) + \kappa\right] Q_2\,, \tag{2.48a}$$

$$A_{0,1} = -\frac{i}{2}\bar{\partial} f P\,. \tag{2.48b}$$

Plugging this Ansatz into the D-term equations one obtains a differential equation for f. More precisely, by taking the following expression for the Kähler form

$$\omega = \frac{i}{2}(dx \wedge d\bar{x} + dy \wedge d\bar{y})\,, \tag{2.49}$$

the D-term equations become

$$(\partial_x \bar{\partial}_{\bar{x}} + \partial_y \bar{\partial}_{\bar{y}}) f = m^2 (e^{2f} - m^2 |x|^2 e^{-2f})\,. \tag{2.50}$$

As in [158, 159] we take f to depend only on $r = (x\bar{x})^{\frac{1}{2}}$. Defining $s = \frac{8}{3}(mr)^{\frac{3}{2}}$ and h as

$$e^{2f} = mre^{2h}\,, \tag{2.51}$$

eq.(2.50) becomes

$$\left(\frac{d^2}{ds^2} + \frac{1}{s}\frac{d}{ds}\right) h = \frac{1}{2}\sinh(2h)\,. \tag{2.52}$$

This is a particular instance of the Painlevé III equation and its solution valid over the entire complex plane may be found in [181]. Since our model is defined only in a local patch of S_{GUT} it suffices to expand the solution around the origin and retain only the lowest order terms in r

$$f(r) = \log c + c^2 m^2 r^2 + m^4 r^4 \left(\frac{c^4}{2} - \frac{1}{4c^2}\right) + \ldots\,. \tag{2.53}$$

The constant c in this equation may be fixed if we ask for a solution regular for all values of r, and the explicit value is

$$c = 3^{\frac{1}{3}} \frac{\Gamma\left[\frac{2}{3}\right]}{\Gamma\left[\frac{1}{3}\right]} \sim 0.73\,. \tag{2.54}$$

However since we are only interested in a local solution we shall not restrict to this value in the following and leave c as a free parameter controlling the strength of the non-primitive flux at the origin.

2.4.4 Primitive fluxes

While the background fields specified in the previous section are a consistent solution to the equations of motion it is still possible to consider a more general supersymmetric background for the gauge field strength F. In particular one may add an extra flux besides (2.48b) that is primitive and commutes with Φ. The most general choice of gauge flux that satisfies these constraints and does not break $SU(5)_{GUT}$ is

$$F_Q = i(dx \wedge d\bar{x} - dy \wedge d\bar{y})\,[M_1 Q_1 + M_2 Q_2] + i(dx \wedge d\bar{y} + dy \wedge d\bar{x})\,[N_1 Q_1 + N_2 Q_2]\ . \tag{2.55}$$

This flux has the main effect of inducing 4d chirality in the matter field spectrum because modes of opposite chirality will feel it differently. We will discuss more in detail how the presence of fluxes selects a preferred 4d chirality later in this section.

Finally, an important ingredient missing so far is a mechanism to achieve the breaking of $SU(5)_{GUT}$ down to the SM gauge group. We choose to employ the standard mechanism for GUT breaking in F-theory [142, 143] and add a flux along the hypercharge generator. We assume that the integrals for the hypercharge flux are such that no mass term is generated for the hypercharge gauge boson, a condition that can only be checked in a global realization of our model. In our local approach we may choose the following parametrization for this flux

$$F_Y = i\left[\tilde{N}_Y(dy \wedge d\bar{y} - dx \wedge d\bar{x}) + N_Y(dx \wedge d\bar{y} + dy \wedge d\bar{x})\right] Q_Y\ , \tag{2.56}$$

where we defined the hypercharge generator as

$$Q_Y = \frac{1}{3}\,(H_1 + H_2 + H_3) - \frac{1}{2}(H_4 + H_5)\,. \tag{2.57}$$

To summarize, the total primitive flux present in our model is

$$F_p = iQ_R(dy \wedge d\bar{y} - dx \wedge d\bar{x}) + iQ_S(dy \wedge d\bar{x} + dx \wedge d\bar{y})\,, \tag{2.58}$$

where we defined the generators

$$Q_R = -M_1 Q_1 - M_2 Q_2 + \tilde{N}_Y Q_Y\ , \quad Q_S = N_1 Q_1 + N_2 Q_2 + N_Y Q_Y\ . \tag{2.59}$$

These fluxes will enter explicitly in the equations of motion for the physical zero modes and because of this they will enter directly in the expression of the Yukawa couplings. Just like in [156–159] the holomorphic Yukawa couplings that enter in the superpotential are not affected by the fluxes. However, the physical Yukawa couplings will depend on them after imposing correct normalization of the kinetic terms for the matter fields. We have chosen to summarize how the primitive flux is felt by the various MSSM fields for the case of the model A in Table 2.4 specifying the two combinations q_R and q_S that will be relevant for the computation in the following sections.

MSSM	Sector	$SU(2)\times U(1)\times U(1)$	G_{MSSM}	q_R	q_S
Q	$\mathbf{10}_M$	$\mathbf{2}_{1,0}$	$(\mathbf{3},\mathbf{2})_{-\frac{1}{6}}$	$-\frac{1}{6}\tilde{N}_Y - M_1$	$-\frac{1}{6}N_Y + N_1$
U	$\mathbf{10}_M$	$\mathbf{2}_{1,0}$	$(\bar{\mathbf{3}},\mathbf{1})_{\frac{2}{3}}$	$\frac{2}{3}\tilde{N}_Y - M_1$	$\frac{2}{3}N_Y + N_1$
E	$\mathbf{10}_M$	$\mathbf{2}_{1,0}$	$(\mathbf{1},\mathbf{1})_{-1}$	$-\tilde{N}_Y - M_1$	$-N_Y + N_1$
D	$\bar{\mathbf{5}}_M$	$\mathbf{2}_{0,1}$	$(\bar{\mathbf{3}},\mathbf{1})_{-\frac{1}{3}}$	$-\frac{1}{3}\tilde{N}_Y - M_2$	$-\frac{1}{3}N_Y + N_2$
L	$\bar{\mathbf{5}}_M$	$\mathbf{2}_{0,1}$	$(\mathbf{1},\mathbf{2})_{\frac{1}{2}}$	$\frac{1}{2}\tilde{N}_Y - M_2$	$\frac{1}{2}N_Y + N_2$
H_u	$\mathbf{5}_U$	$\mathbf{1}_{-2,0}$	$(\mathbf{1},\mathbf{2})_{-\frac{1}{2}}$	$-\frac{1}{2}\tilde{N}_Y + 2M_1$	$-\frac{1}{2}N_Y - 2N_1$
H_d	$\bar{\mathbf{5}}_D$	$\mathbf{1}_{-1,-1}$	$(\mathbf{1},\mathbf{2})_{\frac{1}{2}}$	$\frac{1}{2}\tilde{N}_Y + M_1 + M_2$	$\frac{1}{2}N_Y - N_1 - N_2$

Table 2.4: Different sectors and charges for the E_7 model of this section. Here q_R and q_S are the E_7 operators (2.59) evaluated at each different sector. All the multiplets in the table have the same chirality.

Local chirality of matter fields

One of the most important consequences of the addition of gauge fluxes on the worldvolume of 7-branes is the generation of a chiral spectrum in the 4d effective theory. It is possible to compute the net chiral spectrum of the modes localized on a matter curve Σ as an index [141]

$$\chi\left(\Sigma, \mathcal{L}\otimes K_\Sigma^{\frac{1}{2}}\right) = \int_\Sigma c_1(\mathcal{L})\,, \tag{2.60}$$

where $\mathcal{L}$ is a line bundle on Σ whose first Chern class is equal to the magnetic flux threading the matter curve. Therefore a suitable choice of fluxes can give the correct chiral spectrum in the 4d theory. Moreover, since part of the flux triggers the breaking of the GUT group, fields in different representations of the SM group that are in the same representation of the GUT group may have a different chiral spectrum in 4d. This kind of mechanism allows for a simple implementation of doublet-triplet splitting in F-theory GUTs by imposing the absence of massless Higgs triplets in the 4d theory.

Notice that in our local setup we are not able to compute explicitly the chiral index for the various matter representations because this would require to specify the geometry around a patch containing S_{GUT} and in particular the matter curves Σ. It is however still possible to discuss chirality in our local model by employing the concept of local chirality. This notion introduced in [171] amounts to compute a chiral index for those wavefunctions which are localized around the Yukawa point. To gain a better understanding of how local chirality is formulated it is useful to consider models of magnetized D9-branes which are T-dual to our setting, as in [156]. In order to do so we identify the gauge connection $A_{\bar{z}}$ with Φ where we called z the direction transverse to the 7-branes. All fields do not depend on z and therefore $F_{x\bar{z}} = D_x\Phi$ and $F_{y\bar{z}} = D_y\Phi$ and so on. To formulate local chirality we need the expression of the index of the Dirac operator which for a representation $\mathcal{R}$ is

$$\text{index}_{\mathcal{R}} \not{D} = \frac{1}{48(2\pi)^2} \int \left(\text{tr}_{\mathcal{R}} F \wedge F \wedge F - \frac{1}{8} \text{tr}_{\mathcal{R}} F \wedge \text{tr}\, R \wedge R \right) . \tag{2.61}$$

Asking for the existence of a chiral mode in the representation $\mathcal{R}$ amounts to the condition $\mathcal{I}_{\mathcal{R}} < 0$ where $\mathcal{I}_{\mathcal{R}}$ is the integrand in (2.61). Note that since $\mathcal{I}_{\mathcal{R}} = -\mathcal{I}_{\overline{\mathcal{R}}}$ the spectrum in the 4d theory will be chiral. Taking a local patch where we can approximate our configuration by constant fluxes and vanishing curvature we find

$$\begin{aligned} \mathcal{I}_{\mathcal{R}} \equiv \frac{i}{6} \text{tr}_{\mathcal{R}} \, (F \wedge F \wedge F)_{x\bar{x}y\bar{y}z\bar{z}} = i \, \text{tr}_{\mathcal{R}} \big(& F_{x\bar{x}} \{F_{y\bar{y}}, F_{z\bar{z}}\} + F_{x\bar{z}} \{F_{y\bar{x}}, F_{z\bar{y}}\} + \\ & F_{x\bar{y}} \{F_{y\bar{z}}, F_{z\bar{x}}\} - \{F_{x\bar{x}}, F_{y\bar{z}}\} F_{z\bar{y}} - \{F_{x\bar{y}}, F_{y\bar{x}}\} F_{z\bar{z}} - \{F_{x\bar{z}}, F_{y\bar{y}}\} F_{z\bar{x}} \big). \end{aligned} \tag{2.62}$$

Then, evaluating this expression for the various sectors of our model we obtain[22]

$$\mathcal{I}_{\mathbf{10,2}} = -2m^4 c^4 q_R^{(\mathbf{10,2})} \tag{2.63}$$

$$\mathcal{I}_{\bar{\mathbf{5}},\mathbf{2}} = -2m^4 c^4 q_R^{(\bar{\mathbf{5}},\mathbf{2})} \tag{2.64}$$

$$\mathcal{I}_{\mathbf{5,1}} = -4\mu_1^4 [q_R^{(\mathbf{5,1})} (|a|^2 - 1) + 2\text{Re}[a] q_S^{(\mathbf{5,1})}] \tag{2.65}$$

$$\mathcal{I}_{\bar{\mathbf{5}},\mathbf{1}} = -\left\{ q_R^{(\bar{\mathbf{5}},\mathbf{1})} [|a\mu_1^2 + b\mu_2^2|^2 - |\mu_1^2 + \mu_2^2|^2] + 2q_S^{(\bar{\mathbf{5}},\mathbf{1})} \text{Re}[(a\mu_1^2 + b\mu_2^2)(\mu_1^2 + \mu_2^2)] \right\} . \tag{2.66}$$

The conditions that we need to impose in order to obtain the correct chiral spectrum in 4d are the following ones

$$\begin{aligned} &\mathcal{I}_{\mathcal{R}} < 0 \,, \qquad \mathcal{R} = Q, U, E, D, L, H_u, H_d \,, \\ &\mathcal{I}_{\mathcal{R}} = 0 \,, \qquad \mathcal{R} = T_u, T_d \,. \end{aligned} \tag{2.67}$$

We spell out the explicit form of these conditions for our models in Appendix 7.3, where we also write the explicit form of the equations (2.67) and discuss the existence of solutions to

[22] In writing $\mathcal{I}_{\mathbf{10,2}}$ and $\mathcal{I}_{\bar{\mathbf{5}},\mathbf{2}}$ we have neglected some terms involving μ_1 and μ_2. We chose to do so because as we will discuss later we shall restrict to the case $\mu_1, \mu_2 \ll m$ implying that these additional terms will give negligible contributions to the local chiral index.

the system. As shown in there for the particular case of $a = b = 1$ considered in [159] the previous system does not admit solutions, and therefore it is not possible (at least in terms of local chirality) to obtain the MSSM chiral spectrum without Higgs triplets. Therefore we are led to consider models where $a \neq b$ and so, compared to the analysis in [159] our configurations have one further parameter $(a-b)$. As we will see in section 2.4.11 imposing that this parameter is non-vanishing will allow to fit the empiric data for fermion masses in a much wider region of parameter space.

2.4.5 Residue formula for Yukawa couplings

Knowing the distribution of matter fields on each matter curve it is possible to compute the holomorphic Yukawa couplings by simply performing a dimensional reduction of the 7-brane superpotential

$$W = m_*^4 \int_S \mathrm{Tr}\,(\Phi \wedge F) + \frac{\epsilon}{2}\theta_0 \mathrm{Tr}\,(F \wedge F)\ . \tag{2.68}$$

As discussed in section 2.4, θ_0 is a holomorphic section of a line bundle on S and ϵ is a parameter that measures the strength of the non-perturbative effect. It is important to note that this additional term in the superpotential will affect the supersymmetry equations for the background that we discussed in the previous sections. This implies that the background values of Φ and F will be deformed and have $\mathcal{O}(\epsilon)$ corrections. As shown in [157] this does not affect the computation of holomorphic Yukawa couplings and so we may safely ignore this background shift in the discussion below.

To obtain the zero mode equations of motion we separate the 7-brane fields into background and fluctuations around the background

$$\Phi = \langle\Phi\rangle + \varphi\,, \quad A = \langle A\rangle + a\,, \tag{2.69}$$

and retain only the terms linear in the fluctuations in the supersymmetry equations obtained from the superpotential in (2.68). The resulting zero mode equations are

$$\begin{aligned} &\bar\partial_{\langle A\rangle} a = 0\,, \\ &\bar\partial_{\langle A\rangle}\varphi = i[a, \langle\Phi\rangle] - \epsilon\partial\theta_0 \wedge (\partial_{\langle A\rangle} a + \bar\partial_{\langle A\rangle} a^\dagger)\,. \end{aligned} \tag{2.70}$$

where we have taken into account the shift in the value of $\langle\Phi\rangle$ as compared to (2.38) due to non-perturbative corrections, see [157, 158] for details. The same procedure applied to the D-term equation will yield an additional equation for the zero modes but we shall neglect it in this section, for we are only interested in the holomorphic part of the Yukawa couplings. The solution of the system (2.70) is

$$\begin{aligned} a &= \bar{\partial}_{\langle A\rangle}\xi\,, \\ \varphi &= h - i[\langle\Phi\rangle,\xi] + \epsilon\partial\theta_0\wedge(a^\dagger - \partial_{\langle A\rangle}\xi)\,, \end{aligned} \tag{2.71}$$

where ξ is a section of $\Omega^{(0,0)}(S)\otimes \mathrm{ad}(E_7)$ and h is a holomorphic section of $\Omega^{(2,0)}(S)\otimes \mathrm{ad}(E_7)$. The presence of terms involving $a^\dagger$ in (2.71) is a bit puzzling at first sight because it seems that non-holomorphic terms may be present in the 4d superpotential. However when performing the dimensional reduction of the 7-brane superpotential these terms will appear only in total derivatives and will therefore be absent in the 4d superpotential [157]. Indeed, plugging the solutions (2.71) into the superpotential and evaluating cubic terms in the fluctuations one finds that the Yukawa couplings read [149, 154, 157, 158]

$$Y = -i\frac{m_*^4}{3}\int_S \mathrm{Tr}(h\wedge\bar{\partial}_{\langle A\rangle}\xi\wedge\bar{\partial}_{\langle A\rangle}\xi)\,. \tag{2.72}$$

Finally, it is interesting to note that the computation of Yukawa couplings can be translated in a simple residue computation, as first noticed in [149] and generalized in [157, 158] for the setup at hand. The final expression reads

$$Y = m_*^4\pi^2 f_{abc}\,\mathrm{Res}_p\left[\eta^a\eta^b h_{xy}\right] = m_*^4\pi^2 f_{abc}\int_{\mathcal{C}}\eta^a\eta^b h_{xy}dx\wedge dy \tag{2.73}$$

with $\mathcal{C}$ a cycle in $\mathbb{C}^2$ which can be continuously contracted to a product of unit circles surrounding the Yukawa point p without encountering singularities in the integrand. Also the function η is defined as

$$\eta = -i\Phi^{-1}\left[h_{xy} + i\epsilon\partial_x\theta_0\partial_y\left(\Phi^{-1}h_{xy}\right) - i\epsilon\partial_y\theta_0\partial_x\left(\Phi^{-1}h_{xy}\right)\right]\,. \tag{2.74}$$

2.4.6 Holomorphic Yukawa couplings for the E_7 model

We now have all the necessary ingredients to perform the computation of the Yukawa couplings in both E_7 models. Here we will only report the results for the model A deferring the results for the model B to Appendix **??**. We focus our attention on the matter curves including the fields charged under the MSSM gauge group and therefore to the two Yukawa matrices for the couplings $\mathbf{10}_M\times\mathbf{10}_M\times\mathbf{5}_U$ and $\mathbf{10}_M\times\bar{\mathbf{5}}_M\times\bar{\mathbf{5}}_D$. The functions h_{xy} for the different fields are

$$\begin{aligned} h_{\mathbf{10}_M} &= \gamma_{10,i}\,m_*^{3-i}(ax-y)^{3-i} & h_{\bar{\mathbf{5}}_M} &= \gamma_{5,i}\,m_*^{3-i}(a(x-x_0)-(y-y_0))^{3-i} \\ h_{\mathbf{5}_U} &= \gamma_U & h_{\bar{\mathbf{5}}_D} &= \gamma_D, \end{aligned} \tag{2.75}$$

where (x_0, y_0) corresponds to the coordinates (2.45) of the down-type Yukawa point $p_{\rm down}$, while recall that $p_{\rm up}$ is located at the origin. Finally, the constants $\gamma_{10,i}, \gamma_{5,i}, \gamma_U, \gamma_D$ are normalization factors to be computed in the next section and $i = 1, 2, 3$ is a family index. With this form one can compute the functions η in (2.74) which in turn are needed to compute the holomorphic couplings via the residue formula (2.73). We relegate the expressions for such η's to Appendix 7.4 and turn to discuss the Yukawa matrices that result from them.

Below we display the Yukawa matrix for the up-type quarks up to first order in the expansion parameter ϵ. For the Yukawa matrix of down quarks and leptons we find an explicit dependence on κ, the parameter controlling the separation between the two Yukawa points. Since the dimensionless combination $\tilde{\kappa} = \kappa/m_*$ will turn out to be very small we chose to retain only the first two orders in $\tilde{\kappa}$ in the Yukawa matrix (dropping also terms of order $\mathcal{O}(\epsilon\tilde{\kappa})$ which are extremely suppressed). Moreover for the Yukawa matrix of down quarks and leptons we also perform an expansion on the parameter $(a-b)$ which we will eventually find to be small as well. Our computations in section 2.4.11 will however be based on the full $(a-b)$ dependence of $Y_{D/L}$, which can be found in appendix 7.4.3.

$$Y_U = \frac{\pi^2\, \gamma_U\, \gamma_{10,3}^2}{2\rho_m\rho_\mu} \begin{pmatrix} 0 & 0 & \tilde{\epsilon}\frac{\gamma_{10,1}}{2\rho_\mu\gamma_{10,3}} \\ 0 & \tilde{\epsilon}\frac{\gamma_{10,2}^2}{2\rho_\mu\gamma_{10,3}^2} & 0 \\ \tilde{\epsilon}\frac{\gamma_{10,1}}{2\rho_\mu\gamma_{10,3}} & 0 & 1 \end{pmatrix} + \mathcal{O}(\epsilon^2)\,, \tag{2.76}$$

$$Y_{D/L} = Y_{D/L}^{(0)} + (a-b)\, Y_{D/L}^{(1)} + \mathcal{O}((a-b)^2) \tag{2.77}$$

where

$$Y_{D/L}^{(0)} = -\frac{\pi^2\gamma_{5,3}\gamma_{10,3}\gamma_D}{(d+1)\rho_\mu\rho_m} \begin{pmatrix} 0 & \tilde{\kappa}\tilde{\epsilon}\frac{2\gamma_{5,2}\gamma_{10,1}}{(d+1)^2\rho_\mu^2\gamma_{5,3}\gamma_{10,3}} & \frac{\gamma_{10,1}}{(d+1)\rho_\mu\gamma_{10,3}}\left(\frac{2\tilde{\kappa}^2}{(d+1)^2\rho_\mu} - \tilde{\epsilon}\right) \\ \tilde{\kappa}\tilde{\epsilon}\frac{\gamma_{5,1}\gamma_{10,2}}{(d+1)^2\rho_\mu^2\gamma_{5,3}\gamma_{10,3}} & -\tilde{\epsilon}\frac{\gamma_{5,2}\gamma_{10,2}}{(d+1)\rho_\mu\gamma_{5,3}\gamma_{10,3}} & -\tilde{\kappa}\frac{\gamma_{10,2}}{(d+1)^2\rho_\mu\gamma_{10,3}} \\ \tilde{\epsilon}\frac{\gamma_{5,1}}{(d+1)\rho_\mu\gamma_{5,3}} & 0 & 1 \end{pmatrix} \tag{2.78}$$

$$Y_{D/L}^{(1)} = -\frac{\pi^2}{(d+1)^3} \begin{pmatrix} 0 & y^{(12)} & y^{(13)} \\ y^{(21)} & y^{(22)} & y^{(23)} \\ y^{(31)} & y^{(32)} & y^{(33)} \end{pmatrix} \tag{2.79}$$

with the entries given by

$$y^{(12)} = -\epsilon\tilde{\kappa}\,\frac{(d-1)\gamma_{5,2}\gamma_{10,1}\gamma_D\theta_y}{(d+1)\rho_\mu^3\rho_m} \tag{2.80}$$

$$y^{(13)} = \frac{(d-1)\,\gamma_{5,3}\gamma_{10,1}\gamma_D}{2\rho_\mu^2\rho_m}\left[\epsilon\,\theta_y - \tilde{\epsilon}\,\tilde{\kappa}\frac{4d(5d-1)\rho_\mu}{(d^2-1)\,\rho_m^{3/2}}\right] \tag{2.81}$$

$$y^{(21)} = -\epsilon\tilde{\kappa}\,\frac{(d-1)\gamma_{5,1}\gamma_{10,2}\gamma_D\theta_y}{2(d+1)\rho_\mu^3\rho_m} \tag{2.82}$$

$$y^{(22)} = \frac{(d-1)\,\gamma_{5,2}\gamma_{10,2}\gamma_D}{2\rho_\mu^2\rho_m}\left[\epsilon\,\theta_y + \tilde{\epsilon}\tilde{\kappa}\,\frac{18d\rho_\mu}{(d^2-1)\,\rho_m^{3/2}}\right] \tag{2.83}$$

$$y^{(23)} = \frac{3d^2\gamma_{5,3}\gamma_{10,2}\gamma_D}{\rho_m^{5/2}}\left[\tilde{\epsilon} + \tilde{\kappa}^2\frac{2}{d(1+d)\rho_\mu}\right] \tag{2.84}$$

$$y^{(31)} = \frac{(d-1)\gamma_{5,1}\gamma_{10,3}\gamma_D}{2\rho_\mu^2\rho_m}\left[\epsilon\,\theta_y + \tilde{\kappa}\tilde{\epsilon}\frac{4(d-2)\rho_\mu}{(d^2-1)\,\rho_m^{3/2}}\right] \tag{2.85}$$

$$y^{(32)} = -\tilde{\epsilon}\,\frac{3d\gamma_{5,2}\gamma_{10,3}\gamma_D}{\rho_m^{5/2}} \tag{2.86}$$

$$y^{(33)} = -\tilde{\kappa}\,\frac{2d\gamma_{5,3}\gamma_{10,3}\gamma_D}{\rho_m^{5/2}} \tag{2.87}$$

and where we have defined the following quantities

$$d = \frac{\mu_2^2}{\mu_1^2}\,, \quad \rho_\mu = \frac{\mu_1^2}{m_*^2}\,, \quad \rho_m = \frac{m^2}{m_*^2}\,, \quad \tilde{\kappa} = \frac{\kappa}{m_*}\,, \quad \tilde{\epsilon} = \epsilon(\theta_x + a\theta_y)\,. \tag{2.88}$$

2.4.7 Normalization factors and physical Yukawas

So far we have been performing the computation of the Yukawa couplings merely at the holomorphic level, i.e. we have performed the computation of the four dimensional superpotential for the zero modes. To complete the computation and obtain results comparable with measured data it is necessary to compute the kinetic terms of the zero modes and take them to a basis where they are canonically normalized. To compute the kinetic terms it is necessary first to go in a real gauge and solve the zero mode equations in there, which has the effect to induce a dependence on the local flux densities in the kinetic terms.

In this section we will solve the wavefunctions in a real gauge and use this result to obtain the various normalization factors. In the sectors affected by the T-brane background we will not be able to find an analytical solution. However like in [158, 159] we will be able to find an approximate solution in some regions of the parameter space of our local model. We will first compute the wavefunctions that correspond to the tree-level superpotential and show that no kinetic mixing is present at the level of approximation that we are working. We will then include the non-perturbative corrections and argue that the result does not change.

To summarize, in this section we will compute the normalization factors for the chiral wavefunctions of the E_7 model. At tree-level and $\mathcal{O}(\epsilon)$ they correspond to kinetic terms with a diagonal structure, a result that is not changed by non-perturbative effects. This implies that we may compute the final result for the physical Yukawa couplings by employing the holomorphic result discussed in the previous section together with the normalization factors that we are going to derive below.

2.4.8 Perturbative wavefunctions

Without non-perturbative corrections the equations of motion for the zero modes can be obtained from (2.37) expanding the fields as $\Phi = \langle\Phi\rangle + \varphi$ and $A = \langle A\rangle + a$ and retaining only the terms linear in φ and a. The resulting equations are

$$\bar{\partial}_{\langle A\rangle} a = 0\,, \tag{2.89a}$$

$$\bar{\partial}_{\langle A\rangle} \varphi = i[a, \langle\Phi\rangle]\,, \tag{2.89b}$$

$$\omega \wedge \partial_{\langle A\rangle} a = \frac{1}{2}[\langle\bar{\Phi}\rangle, \varphi]\,. \tag{2.89c}$$

In (2.89) we choose $\langle\Phi\rangle$ and $\langle A\rangle$ to be background fields in a real gauge. These equations may be solved by using techniques already employed in [156–159]. To keep the discussion contained we will simply quote the results in this section deferring more details regarding the computation to Appendix 7.4.

Henceforth we are going to use the following notation for the zero modes

$$\vec{\varphi}_\rho = \begin{pmatrix} a^s_{\bar{x}} \\ a^s_{\bar{y}} \\ \varphi^s_{xy} \end{pmatrix} E_{\rho,s} \tag{2.90}$$

where $E_{\rho,s}$ denotes the particular set of roots, labeled by s, corresponding to a given sector ρ. In our models we have that for the sectors unaffected by the T-brane background s takes a single value whereas in the other sectors we have that s takes two values.

Sectors not affected by T-brane

In the sectors not affected by the T-brane background the solution may be computed analytically. In the models we consider we have two sectors that fall in this class and transform as $(\mathbf{5},\mathbf{1})_{-2,0}$ and $(\mathbf{5},\mathbf{1})_{1,1}$ under $SU(5)\times SU(2)\times U(1)\times U(1)$. We recall here that in both models $\mathbf{5}_U := (\mathbf{5},\mathbf{1})_{-2,0}$, while $\bar{\mathbf{5}}_D := (\bar{\mathbf{5}},\mathbf{1})_{-1,-1}$ in the model A and $\bar{\mathbf{5}}_M := (\bar{\mathbf{5}},\mathbf{1})_{-1,-1}$ in the model B.

The solution for both sectors is the following one

$$\vec{\varphi} = \begin{pmatrix} -\frac{i\zeta}{2\tilde{\mu}_a} \\ \frac{i(\zeta-\lambda)}{2\tilde{\mu}_b} \\ 1 \end{pmatrix} \chi(x,y) \tag{2.91}$$

where

$$\chi(x,y) = e^{\frac{q_R}{2}(x\bar{x}-y\bar{y})-q_S\mathrm{Re}(x\bar{y})+(\tilde{\mu}_a x+\tilde{\mu}_b y)(\zeta_1\bar{x}-\zeta_2\bar{y})}\, f(\zeta_2 x+\zeta_1 y) \tag{2.92}$$

and where we have defined

$$\zeta = \frac{\tilde{\mu}_a\,(4\tilde{\mu}_a\tilde{\mu}_b+\lambda q_S)}{\tilde{\mu}_a q_S+\tilde{\mu}_b\,(\lambda+q_R)}\,, \quad \zeta_1 = \frac{\zeta}{\tilde{\mu}_a}\,, \quad \zeta_2 = \frac{\zeta-\lambda}{\tilde{\mu}_b}\,, \tag{2.93}$$

and λ is the lowest solution to the cubic equation (7.35). The parameters $\tilde{\mu}_a$ and $\tilde{\mu}_b$ are directly related to the ones describing the background Higgs fields and for both sectors are given by

	$(\mathbf{5},\mathbf{1})_{-2,0}$	$(\bar{\mathbf{5}},\mathbf{1})_{-1,-1}$
$\tilde{\mu}_a$	$a\mu_1^2$	$\frac{1}{2}(a\mu_1^2+b\mu_2^2)$
$\tilde{\mu}_b$	$-\mu_1^2$	$-\frac{1}{2}(\mu_1^2+\mu_2^2)$

Finally, the function $f(\zeta_2 x+\zeta_1 y)$ is a holomorphic function which can be approximated by a constant if the sector we consider contains an MSSM Higgs. In the remaining case, namely the identification $\bar{\mathbf{5}}_M := (\bar{\mathbf{5}},\mathbf{1})_{-1,-1}$ for the model B we can choose

$$f^i_{\bar{\mathbf{5}}_M}(x,y) = m_*^{3-i}(\zeta_2(x-x_0)+\zeta_1(y-y_0))^{3-i}\,, \tag{2.94}$$

where $i=1,2,3$ is a family index.

Sectors affected by T-brane

In the two sectors affected by the T-brane background the equations of motion become more complicated. Here the fields involved in the solution are doublets of $SU(2)$ in the decomposition of E_7 as $SU(5)\times SU(2)\times U(1)\times U(1)$ and therefore we are going to write the solution as

$$\vec{\varphi} = \begin{pmatrix} a^+_{\bar{x}} \\ a^+_{\bar{y}} \\ \varphi^+_{xy} \end{pmatrix} E^+ + \begin{pmatrix} a^-_{\bar{x}} \\ a^-_{\bar{y}} \\ \varphi^-_{xy} \end{pmatrix} E^- = \vec{\varphi}_+ E^+ + \vec{\varphi}_- E^-\,, \tag{2.95}$$

where we denote with a $+$ the upper component of the $SU(2)$ doublet and with a $-$ the lower one. The equations for the zero modes are generally difficult to solve analytically. Nevertheless as discussed in appendix 7.4 in the limit $\mu_1,\mu_2,\kappa \ll m$ it is possible to find approximate solutions. In both models the solution for the $\mathbf{10}_M$ sector in real gauge is

$$\vec{\varphi}^{\,i}_{10} = \gamma^i_{10}\begin{pmatrix} \frac{i\lambda_{10}}{m^2} \\ -i\frac{\lambda_{10}\zeta_{10}}{m^2} \\ 0 \end{pmatrix} e^{f/2}\chi^i_{10}E^+ + \gamma^i_{10}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-f/2}\chi^i_{10}E^- \tag{2.96}$$

with λ_{10} the negative solution to the cubic equation (7.51) and $\zeta_{10} = -q_S/(\lambda_{10} - q_R)$. Finally the wavefunctions χ^i_{10} are

$$\chi^i_{10} = e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+\lambda_{10}x(\bar{x}-\zeta_{10}\bar{y})} g^i_{10}(y+\zeta_{10}x)\,, \tag{2.97}$$

where g^i_{10} are holomorphic functions of the variable $y + \zeta_{10}x$ and $i = 1, 2, 3$ is a family index. As in [157, 158] we choose these holomorphic functions in the following way

$$g^i_{10}(y+\zeta_{10}x) = m_*^{3-i}(y+\zeta_{10}x)^{3-i}\,. \tag{2.98}$$

The other sector affected by the T-brane background is the $(\bar{\mathbf{5}}, \mathbf{2})_{0,1}$. In the model A we identify it with the $\bar{\mathbf{5}}_M$ sector and the solution is

$$\vec{\varphi}^{\,i}_5 = \gamma^i_5 \begin{pmatrix} \frac{i\lambda_5}{m^2} \\ -i\frac{\lambda_5\zeta_5}{m^2} \\ 0 \end{pmatrix} e^{i\tilde{\psi}+f/2}\chi^i_5(x, y-\nu/a)E^+ + \gamma^i_5 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{i\tilde{\psi}-f/2}\chi^i_5(x, y-\nu/a)E^- \tag{2.99}$$

with $\tilde{\psi}$ defined in (7.55) and $\nu = \kappa/\mu_2^2$. Also, λ_5 is defined as the lowest solution to (7.51) and $\zeta_5 = -q_S/(\lambda_5 - q_R)$. Finally the wavefunctions χ^i_5 are

$$\chi^i_5(x,y) = e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+\lambda_5 x(\bar{x}-\zeta_5\bar{y})} g^i_5(y+\zeta_5 x)\,, \tag{2.100}$$

where g^i_5 are holomorphic functions of $y + \zeta_5 x$ and $i = 1, 2, 3$ is a family index. Analogously, the family functions are

$$g^i_5(y+\zeta_5 x) = m_*^{3-i}(y+\zeta_5 x)^{3-i}\,. \tag{2.101}$$

In the model B where we identify the $(\bar{\mathbf{5}}, \mathbf{2})_{0,1}$ sector with the $\bar{\mathbf{5}}_D$ we find exactly the same solution and the only difference involves the function $g_5(y + \zeta_5 x)$ which in this sector is taken to be constant.

2.4.9 Normalization factors

With the information regarding the perturbative wavefunctions we can compute the normalization factors for the various sectors. The factor appearing in front of the kinetic terms for the various fields is

$$K^{ij}_\rho \;=\; \langle \vec{\varphi}^{\,i}_\rho | \vec{\varphi}^{\,j}_\rho \rangle \;=\; m_*^4 \int_S \mathrm{tr}\,(\vec{\varphi}^{\,i\,\dagger}_\rho \cdot \vec{\varphi}^{\,j}_\rho)\,\mathrm{dvol}_S \tag{2.102}$$

which follows from direct dimensional reduction.

For both models we find that $K^{ij}_\rho = 0$ for $i \neq j$ and no kinetic mixing is present. Therefore the choice of the normalization factors $|\gamma^i_\rho|^2 = (K^{ii}_\rho)^{-1}$ is sufficient to ensure canonically normalized kinetic terms. The computation of these factors is similar to the one performed in [157,158] and will not be repeated it here. For the model A we find

$$|\gamma_{U/D}|^2 = -\frac{4^2}{\pi^2 m_*^4}\frac{(2\mathrm{Re}[\zeta_1\tilde{\mu}_a]+q_R)\,(2\mathrm{Re}[\zeta_2\tilde{\mu}_b]+q_R)+|\zeta_2\tilde{\mu}_a-\zeta_1^*\tilde{\mu}_b^*+q_S|^2}{\zeta_1^2+\zeta_2^2+4} \tag{2.103a}$$

$$|\gamma_{10,j}|^2 = -\frac{c}{m_*^2\pi^2(3-j)!}\frac{1}{\frac{1}{2\mathrm{Re}[\lambda_{10}]+q_R(1+|\zeta_{10}|^2)-|m|^2c^2}+\frac{c^2|\lambda_{10}|^2}{|m|^4}\frac{1}{2\mathrm{Re}[\lambda_{10}]+q_R(1+|\zeta_{10}|^2)+|m|^2c^2}}\left(\frac{q_R}{m_*^2}\right)^{4-j} \tag{2.103b}$$

$$|\gamma_{5,j}|^2 = -\frac{c}{m_*^2\pi^2(3-j)!}\frac{1}{\frac{1}{2\mathrm{Re}[\lambda_5]+q_R(1+|\zeta_5|^2)-|m|^2c^2}+\frac{c^2|\lambda_5|^2}{|m|^4}\frac{1}{2\mathrm{Re}[\lambda_5]+q_R(1+|\zeta_5|^2)+|m|^2c^2}}\left(\frac{q_R}{m_*^2}\right)^{4-j}. \tag{2.103c}$$

We display the normalization factors for the model B in appendix **??**.

Note that the parameters λ and ζ that appear in the various normalization factors depend on the local flux densities and in particular on the flux hypercharge. This implies that the normalization factors of the MSSM multiplets sitting in the same GUT multiplet will be different. As pointed out in [156–158] this is a key feature to obtain realistic mass ratios, as we will see in section 2.4.11.

2.4.10 Non-perturbative corrections to the wavefunctions

So far we have been discussing the kinetic terms of the matter fields neglecting non-perturbative corrections. However, as we are computing Yukawa couplings up to first order in the parameter ϵ, one should consider the expression for the kinetic terms at the same level of approximation. We will discuss now how these effects enter in the computation of the kinetic terms and show that for our models no relevant correction is produced. This implies that the result obtained above may be used in the computation of physical Yukawa matrices.

The F-term equations of motion corrected at $\mathcal{O}(\epsilon)$ are

$$\begin{aligned} &\bar{\partial}_{\langle A\rangle}a = 0\,, \\ &\bar{\partial}_{\langle A\rangle}\varphi = i[a,\langle\Phi\rangle] - \epsilon\partial\theta_0\wedge(\partial_{\langle A\rangle}a+\bar{\partial}_{\langle A\rangle}a^\dagger)\,, \end{aligned} \tag{2.104}$$

which need to be supplemented with the D-term equation (2.89c) which is not affected by non-perturbative corrections [156]. In this section we shall simply show the final result and discuss the impact of non-perturbative corrections on the kinetic terms, deferring the details of the computation to Appendix 7.4.

Sectors not affected by T-brane

In both sectors not affected by the T-brane background the corrections take the same form

$$\overrightarrow{\varphi} = \gamma \begin{pmatrix} -\frac{i\zeta}{2\tilde{\mu}_a} \\ \frac{i(\zeta-\lambda)}{2\tilde{\mu}_b} \\ 1 \end{pmatrix} e^{\frac{q_R}{2}(x\bar{x}-y\bar{y})-q_S\mathrm{Re}(x\bar{y})+(\mu_a x+\mu_b y)(\zeta_1\bar{x}-\zeta_2\bar{y})} \left[f(\zeta_2 x+\zeta_1 y)+\epsilon B(\zeta_2 x+\zeta_1 y)+\epsilon\Upsilon\right] . \tag{2.105}$$

The function Υ that controls the $\mathcal{O}(\epsilon)$ correction is

$$\begin{aligned}\Upsilon = &\frac{1}{4}(\zeta_1\bar{x}-\zeta_2\bar{y})^2(\theta_y\mu_a-\theta_x\mu_b)f(\zeta_2 x+\zeta_1 y)+\frac{1}{2}(\zeta_1\bar{x}-\zeta_2\bar{y})(\zeta_2\theta_y-\zeta_1\theta_x)f'(\zeta_2 x+\zeta_1 y)+\\ &+\left[\frac{\delta_1}{2}(\zeta_1 x-\zeta_2 y)^2+\delta_2(\zeta_1 x-\zeta_2 y)(\zeta_2 x+\zeta_1 y)\right]f(\zeta_2 x+\zeta_1 y)\,,\end{aligned} \tag{2.106}$$

where

$$\begin{aligned}\delta_1 &= \frac{1}{(\zeta_1^2+\zeta_2^2)^2}\left[\bar{\theta}_x(q_S\zeta_1-q_R\zeta_2)+\bar{\theta}_y(q_R\zeta_1+q_S\zeta_2)\right] ,\\ \delta_2 &= \frac{1}{(\zeta_1^2+\zeta_2^2)^2}\left[\bar{\theta}_x(q_R\zeta_1+q_S\zeta_2)-\bar{\theta}_y(q_S\zeta_1-q_R\zeta_2)\right] .\end{aligned} \tag{2.107}$$

The holomorphic function $B(\zeta_2 x+\zeta_1 y)$ in (2.105) is not determined by the equations of motion, and it may be fixed by asking for regularity of the function ξ that appears in (2.71). We shall nevertheless not discuss this point here since it does not affect the result for the kinetic terms.

Having the correction it is now possible to discuss the effect of the correction on the kinetic term. We can use the fact that the integrand has to be invariant under the symmetry $(x,y)\to e^{i\alpha}(x,y)$ to check whether the corrections actually contribute to the kinetic terms. In the cases when the sector hosts a Higgs field (this happens in the model A for both the $\mathbf{5}_U$ and the $\bar{\mathbf{5}}_D$ and in the model B for the $\mathbf{5}_U$) no correction is generated. In the remaining case, namely the $\bar{\mathbf{5}}_M$ in the model B, there are non-diagonal terms in the kinetic terms inducing a mixing between the first and the third families of down quarks and leptons. This however will not affect the computation of Yukawa couplings, because this $\mathcal{O}(\epsilon)$ correction in the kinetic terms will only induce a $\mathcal{O}(\epsilon^2)$ correction in the Yukawa matrix. See [157] for a more detailed discussion of this point in a similar context.

Sectors affected by T-brane

As shown in Appendix 7.4 the structure of the solution for both sectors charged under the T-brane background is

$$\vec{\varphi}_{10^+} = \begin{pmatrix} \bullet \\ \bullet \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ 0 \\ \bullet \end{pmatrix} + \mathcal{O}(\epsilon^2) \qquad \vec{\varphi}_{10^-} = \begin{pmatrix} 0 \\ 0 \\ \bullet \end{pmatrix} + \epsilon \begin{pmatrix} \bullet \\ \bullet \\ 0 \end{pmatrix} + \mathcal{O}(\epsilon^2). \tag{2.108}$$

The peculiar structure of the $\mathcal{O}(\epsilon)$ corrections allows us to demonstrate that these corrections will not affect the kinetic terms without needing to write explicitly their form. Indeed it follows from (2.102) that the corrections vanish because at $\mathcal{O}(\epsilon)$ the correction is proportional to[23] $\vec{\varphi}^{(0)}_{10^+} \cdot \vec{\varphi}^{(1)}_{10^-}$ and $\vec{\varphi}^{(0)}_{10^-} \cdot \vec{\varphi}^{(1)}_{10^+}$ and these scalar products are both zero. This structure is similar to the one observed in [158], to which we refer the reader for a more detailed discussion of this point.

2.4.11 Fitting fermion masses and mixing angles

Gathering the results of the last two sections one may write the final expression for the physical Yukawa matrices at the GUT scale. In particular, for the model A one obtains the matrices (2.76) and (2.77) with the normalization factors given by (2.103). As noted above the value of the normalization factors varies for MSSM field with different hypercharge even if they sit inside the same GUT multiplet, something that we will indicate by adding a superscript to distinguish between them.

Based on these result in this section we explore whether it is possible to find some regions in the parameter space of our models where we may reproduce the realistic values for fermion masses and mixings. Our calculations are performed at the GUT scale which is usually taken around 10^{16} GeV and therefore to compare the values for the fermion masses it is necessary to follow the values of the fermion masses along the renormalization group flow. We show in table 2.5 the extrapolation of the fermion masses up to the unification scale taken from [182] in the context of the MSSM. Since in the MSSM two Higgs fields are present the values depend on an additional parameter $\tan\beta$ which controls how the observed vev of the Higgs is distributed between the H_u and the H_d Higgs fields of the MSSM. More specifically $\langle H_u \rangle = V\cos\beta$ and $\langle H_d \rangle = V\sin\beta$ where V is the measured value of the Higgs field and is given by $V \approx 174$ GeV.

[23]Here the superscript (0) denotes the tree-level term and (1) the $\mathcal{O}(\epsilon)$ correction.

We shall now discuss the comparison between these extrapolated data and the values for the Yukawa couplings that we obtain in our local E_7 models.

$\tan\beta$	10	38	50
m_u/m_c	$2.7 \pm 0.6 \times 10^{-3}$	$2.7 \pm 0.6 \times 10^{-3}$	$2.7 \pm 0.6 \times 10^{-3}$
m_c/m_t	$2.5 \pm 0.2 \times 10^{-3}$	$2.4 \pm 0.2 \times 10^{-3}$	$2.3 \pm 0.2 \times 10^{-3}$
m_d/m_s	$5.1 \pm 0.7 \times 10^{-2}$	$5.1 \pm 0.7 \times 10^{-2}$	$5.1 \pm 0.7 \times 10^{-2}$
m_s/m_b	$1.9 \pm 0.2 \times 10^{-2}$	$1.7 \pm 0.2 \times 10^{-2}$	$1.6 \pm 0.2 \times 10^{-2}$
m_e/m_μ	$4.8 \pm 0.2 \times 10^{-3}$	$4.8 \pm 0.2 \times 10^{-3}$	$4.8 \pm 0.2 \times 10^{-3}$
m_μ/m_τ	$5.9 \pm 0.2 \times 10^{-2}$	$5.4 \pm 0.2 \times 10^{-2}$	$5.0 \pm 0.2 \times 10^{-2}$
Y_τ	0.070 ± 0.003	0.32 ± 0.02	0.51 ± 0.04
Y_b	0.051 ± 0.002	0.23 ± 0.01	0.37 ± 0.02
Y_t	0.48 ± 0.02	0.49 ± 0.02	0.51 ± 0.04

Table 2.5: Running mass ratios of quarks and leptons at the unification scale from ref. [182].

2.4.12 Fermion masses

Knowing the Yukawa matrices we can easily extract the values of the fermion masses which depend on the eigenvalues of the matrices. From the Yukawa matrices in (2.76) and (2.77) we see that the eigenvalues are

$$\begin{aligned}
&Y_t = \gamma_U\, \gamma^Q_{10,3}\gamma^U_{10,3}\, Y^U_{33} && Y_c = \epsilon\, \gamma_U\, \gamma^Q_{10,2}\gamma^U_{10,2}\, Y^U_{22} \\
&Y_b = \gamma_D \left(\gamma^Q_{10,3}\gamma^D_{5,3}Y^{D/L}_{33} + \epsilon\, \gamma^Q_{10,2}\gamma^D_{5,2}\delta\right) && Y_s = \epsilon\, \gamma_D \left(\gamma^Q_{10,2}\gamma^D_{5,2}Y^{D/L}_{22} - \gamma^Q_{10,2}\gamma^D_{5,2}\delta\right) \\
&Y_\tau = \gamma_D \left(\gamma^E_{10,3}\gamma^L_{5,3}Y^{D/L}_{33} + \epsilon\, \gamma^E_{10,2}\gamma^L_{5,2}\delta\right) && Y_\mu = \epsilon\, \gamma_D \left(\gamma^E_{10,2}\gamma^L_{5,2}Y^{D/L}_{22} - \gamma^E_{10,2}\gamma^L_{5,2}\delta\right)
\end{aligned} \tag{2.109}$$

while for the first family we have that

$$Y_u,\ Y_d,\ Y_e\ \sim\ \mathcal{O}(\epsilon^2) \tag{2.110}$$

Here the normalization factors are those given in the previous section, and we have defined

$$\delta = -\tilde{\kappa}\, \frac{\pi^2 d(a-b)\left[\theta_y(a(d-2)-b(4d+1)) - 3(d+1)\theta_x\right]}{(d+1)^5\rho_\mu\, \rho_m^{5/2}} \tag{2.111}$$

Therefore we see that when $a \neq b$ the eigenvalues of the down-type Yukawa matrix are different from the diagonal entries of the matrix. However we note that this correction will be of order $\mathcal{O}(\tilde{\kappa})$, which at the end of this section will be fixed to be $10^{-5} - 10^{-6}$ by fixing the value of the quark mixing angles. In this sense we can neglect δ as compared to the contribution coming from the diagonal entries of the down-type Yukawa matrix, as well as any $\tilde{\kappa}$ dependence on these entries. After this it is easy to see manifestly the $(\mathcal{O}(1), \mathcal{O}(\epsilon), \mathcal{O}(\epsilon^2))$ hierarchy between the three families of quarks and leptons. Because the explicit expression for the eigenvalues of the lightest family cannot be computed at the level of approximation that we are working, we turn to discuss the masses for the two heavier families.

Masses for the second family

The strategy that we choose to follow to see if it is possible to fit all fermions masses is to look first at the mass ratios between the second and third families, which do not depend on $\tan\beta$. More specifically we will start by considering the following mass ratios

$$\frac{m_\mu/m_\tau}{m_s/m_b}\,, \qquad \frac{m_c/m_t}{m_s/m_b}\,, \tag{2.112}$$

which, in addition to being independent of $\tan\beta$ do not depend on the parameter ϵ which measures the strength of the non-perturbative effects. From the data in table 2.5 and the discussion in [157, 158] we aim to reproduce the following values

$$\frac{m_\mu/m_\tau}{m_s/m_b} = 3.3 \pm 1\,, \qquad \frac{m_c/m_t}{m_s/m_b} = 0.13 \pm 0.03\,. \tag{2.113}$$

To complete the discussion of the masses of the second family we can look at an additional mass ratio, namely m_c/m_t. Being able to correctly fix this quantity and (2.113) allows us to obtain correct mass values of masses for the second family of quarks and leptons when the masses of the third family are fitted later on.

Let us now discuss the behavior of these particular ratios of masses in the two models we have been discussing so far. We will see already at this level that the model B does not allow for good values of these ratio of masses.

Model A We can compute the aforementioned ratios for the case of the model A and the result is

$$\frac{m_c}{m_t} = \left|\frac{\tilde{\epsilon}}{2\rho_\mu}\right| \sqrt{q_R^Q\, q_R^U} = \left|\frac{\tilde{\epsilon}\,\tilde{N}_Y}{2\rho_\mu}\right| \sqrt{\left(x - \frac{1}{6}\right)\left(x + \frac{2}{3}\right)} \tag{2.114a}$$

$$\frac{m_s}{m_b} = \frac{Y_s}{Y_b} \sqrt{q_R^Q\, q_R^D} \simeq \frac{\left|\tilde{N}_Y\right| [(d+1)\theta_x + (a+bd)\theta_y]}{(d+1)^2 \rho_\mu} \sqrt{\left(x - \frac{1}{6}\right)\left(y - \frac{1}{3}\right)} \tag{2.114b}$$

$$\frac{m_\mu}{m_\tau} = \frac{Y_\mu}{Y_\tau} \sqrt{q_R^Q\, q_R^D} \simeq \frac{\left|\tilde{N}_Y\right| [(d+1)\theta_x + (a+bd)\theta_y]}{(d+1)^2 \rho_\mu} \sqrt{(x-1)\left(y + \frac{1}{2}\right)} \tag{2.114c}$$

where we defined

$$x = -\frac{M_1}{\tilde{N}_Y}\,, \quad y = -\frac{M_2}{\tilde{N}_Y}\,, \quad d = \frac{\mu_2^2}{\mu_1^2}\,. \tag{2.115}$$

In writing the final expression for m_s/m_b and m_μ/m_τ we neglected the δ shifts appearing in the expressions for the eigenvalues of the down quark and lepton Yukawa matrix as well as the $\mathcal{O}(\epsilon)$ correction appearing in $Y_{33}^{D/L}$. The reason behind this choice is that these contributions are much smaller when compared to the other terms and therefore will not affect the final results. Once these contributions are neglected the expressions for the ratio of masses become much simpler and depend on a smaller subset of parameters giving therefore more analytical control. Using (2.114) we can compute the ratio of masses (2.112) and the results are

$$\frac{m_\mu/m_\tau}{m_s/m_b} = \sqrt{\frac{(x-1)\left(y - \frac{1}{2}\right)}{\left(x - \frac{1}{6}\right)\left(y - \frac{1}{3}\right)}}\,, \tag{2.116}$$

$$\frac{m_c/m_t}{m_s/m_b} = \frac{(d+1)^2\sqrt{2+3x}(a\theta_y + \theta_x)}{2\sqrt{3y-1}\,[(d+1)\theta_x + (a+bd)\theta_y]}\,, \tag{2.117}$$

Chirality conditions place some constraints in the allowed regions for x and y, and in particular we find that for $\tilde{N}_Y < 0$ we need $x < -2/3$ and $y < 1/2$ and for $\tilde{N}_Y > 0$ we need $x > 1$ and $y > 1/3$. Between the two possibilities we find that it is simpler to fit the empirical data by choosing $\tilde{N}_Y > 0$. Moreover it seems reasonable to take $\theta_x \sim \theta_y$ which implies that both ratios of masses will depend only on three parameters, namely x, y and $\hat{d}$ where

$$\hat{d} = \frac{(d+1)^2(a+1)}{a+1+d(b+1)}\,. \tag{2.118}$$

We show in figure 2.2 of the x and y parameter space where we find values for the ratios of masses in agreement with the empirical ones. The remaining mass ratio m_c/m_t has also a nice analytical expression in terms of the parameters of our local model

$$\frac{m_c}{m_t} = \frac{\sqrt{\left(x - \frac{1}{6}\right)\left(\frac{2}{3} + x\right)}|\tilde{N}_Y|}{2\mu_1^2}\,\tilde{\epsilon}\,. \tag{2.119}$$

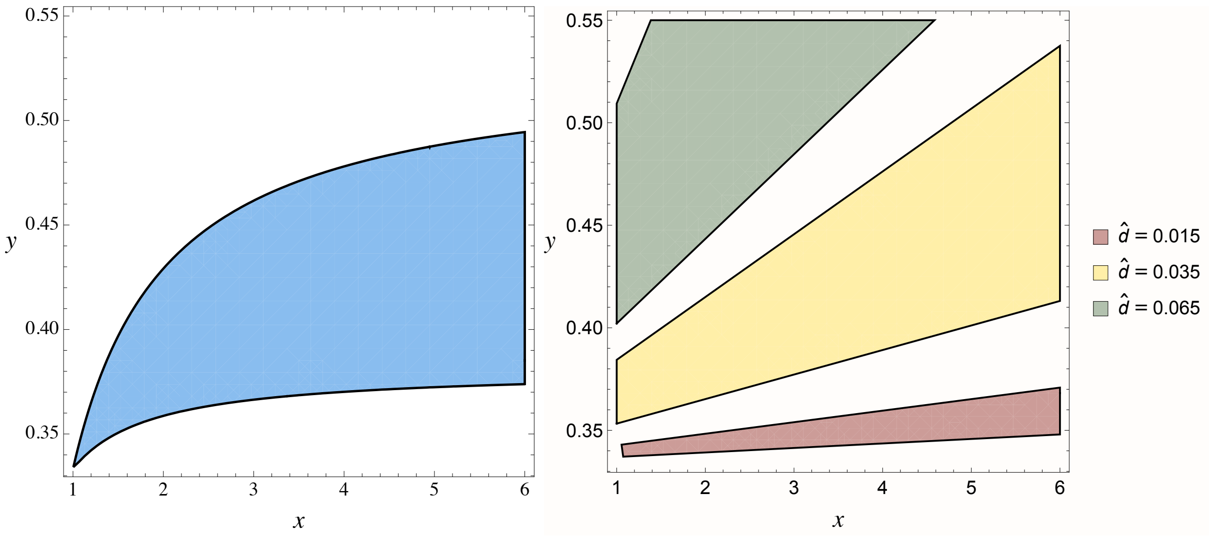


Figure 2.2: On the left the region in the $x - y$ plane for the ratio of masses (2.116) compatible with the realistic value in (2.113). On the right the region in the $x - y$ plane for the ratio of masses (2.117) compatible with (2.113), for different values of $\hat{d}$.

In figure 2.3 we show in which region of the x and $\tilde{\epsilon}\,|\tilde{N}_Y|/\mu_1^2$ parameter space we are able to find good values for this last ratio of masses.

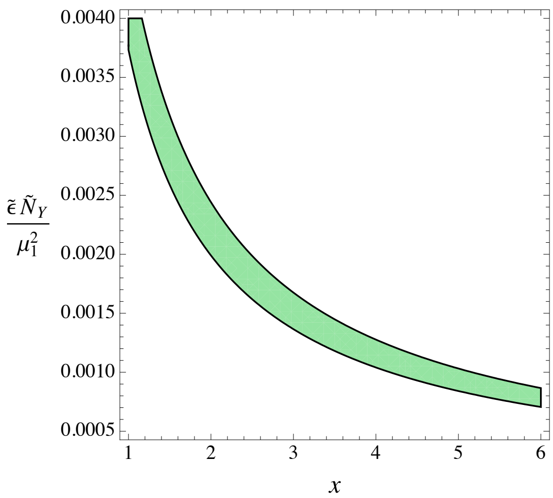


Figure 2.3: Region in the plane $x - \tilde{\epsilon}\tilde{N}_Y/\mu_1^2$ for the ratio (2.119) to be compatible with the range of values in table 2.5.

Model B Contrary to model A, for the model B one does not find simplified expressions for the fermion mass ratios. We have explored numerically different regions in parameter space trying to reproduce the value in (2.113) for the ratio of ratios $(m_\mu/m_\tau)/(m_s/m_b)$ without success. In fact, in figure 2.4 shows how trying to achieve a realistic value for this pushes us to a region of the parameter space in which $\tilde{N}_Y$ is negative, which is in conflict with a condition necessary for a realistic chiral spectrum in this model. It would be interesting to have a more intuitive understanding of why this model fails to reproduce the empiric data as compared to case of the model A.

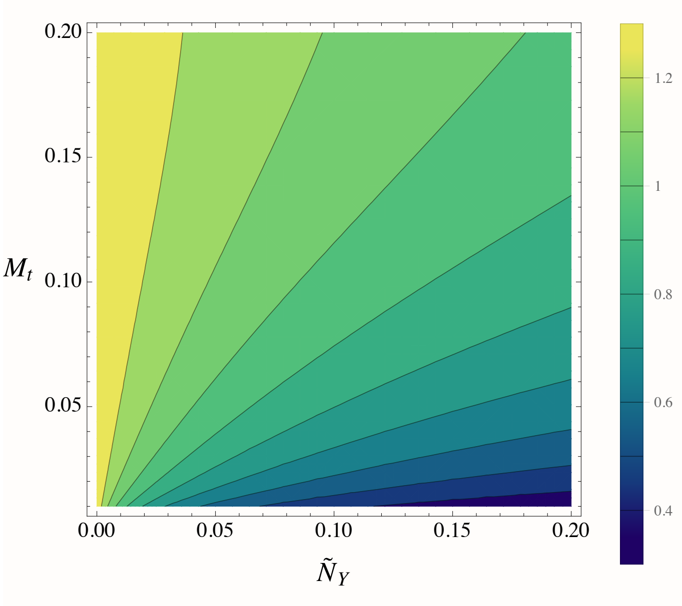


Figure 2.4: Value of the ratio of ratios $(m_\mu/m_\tau)/(m_s/m_b)$ in the model B in the $\tilde{N}_Y - M_t$ plane, where are taking $M_t = -M_1 - \tilde{N}_Y > 0$ and $\tilde{N}_Y > 0$ as dictated by eq.(7.16).

Yukawa couplings for the third family

Given that in the case of the model A we have been able to find regions where the mass ratios between the second and third families are compatible with the MSSM, all we need to fix now are the masses for the fermions in the third family. We start by looking at the ratio between the mass of the τ-lepton and the b-quark. Such ratio can be expressed in terms of normalization factors only

$$\frac{Y_\tau}{Y_b} = \frac{\gamma^E_{10,3}\gamma^L_{5,3}}{\gamma^Q_{10,3}\gamma^D_{5,3}}\,, \tag{2.120}$$

but in terms of the model parameters it acquires a rather complicated form, so it is quite hard to describe analytically the region of parameter space that is compatible with the expected value

$$\frac{Y_\tau}{Y_b} = 1.37 \pm 0.1 \pm 0.2\,. \tag{2.121}$$

We have therefore performed a numerical scan over the values of the local flux densities which are compatible with the conditions for chirality and doublet-triplet splitting, and with the fermion mass ratios just discussed. More precisely we have chosen the following point in parameter space[24]

$$\begin{aligned}(\rho_m, \rho_\mu, d, c, a, b, \epsilon\,\theta_x, \epsilon\,\theta_y) &= (0.23, 2.5\times 10^{-3}, -0.9, 0.25, -0.4, -0.6, 10^{-4}, 10^{-4})\\ (M_1, M_2, N_1, N_2, \tilde{N}_Y, N_Y) &= (-0.17, -0.0136, -0.14, 0.008, 0.034, 0.1953)\end{aligned} \tag{2.122}$$

and scanned over the allowed values for x and $\tilde{N}_Y$ that do not spoil the constraints above. We show our results in figure 2.5, which displays a rather large region of these parameters.

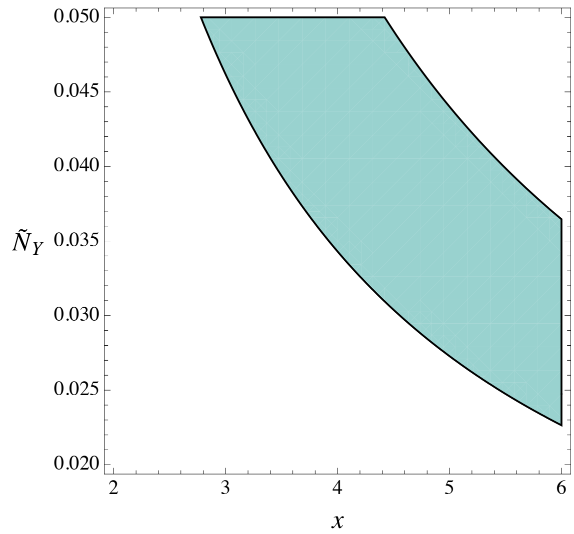


Figure 2.5: Region in the $x - \tilde{N}_Y$ plane with a ratio Y_τ/Y_b compatible with table 2.5.

Finally we may wish to see whether all constraints for chirality, doublet-triplet splitting and realistic fermion mass ratios may be solved simultaneously. We find that this is true for large regions of the parameter space. To illustrate this fact, in figure 2.6 we plot regions in the $m-\tilde{N}_Y$ parameter space where all constraints are fulfilled for different values of c. By inspecting the plot we see that regions fulfilling all constraints exist for different values of c which are of the same order as (2.54).

In these regions we can look at the typical value of the b-quark Yukawa to estimate the value of $\tan\beta$ that we typically obtain from our scan. We show in figure 2.7 the possible values of Y_b and by comparison with the content of the table 2.5 we obtain an approximated value of $\tan\beta \simeq 10 - 20$.

[24]We normalize all local flux densities in units of m_{st}^2 where the string scale m_{st} is related to the typical F-theory scale m_* by $m_{st}^4 = (2\pi)^3 g_s m_*^4$. In all the computation done in this section we take $g_s \sim \mathcal{O}(1)$.

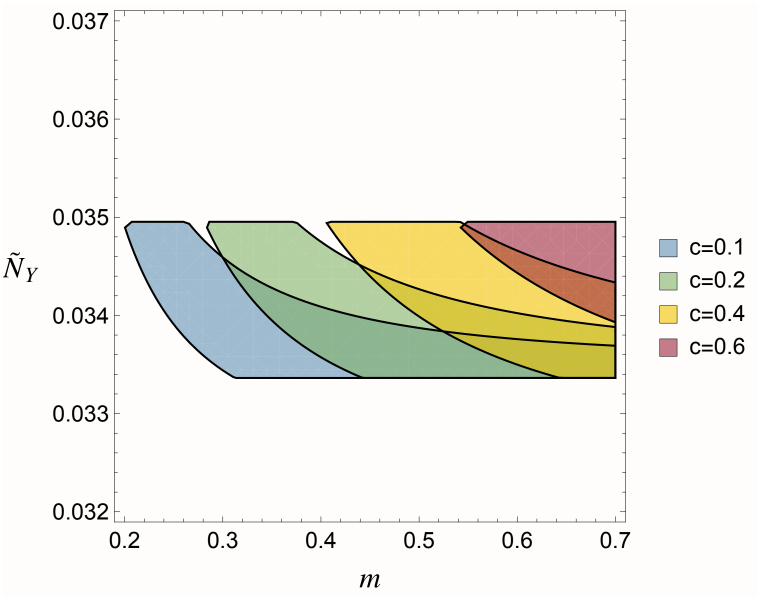


Figure 2.6: Regions in the m-$\tilde{N}_Y$ plane where all constraints are fulfilled for different values of c.

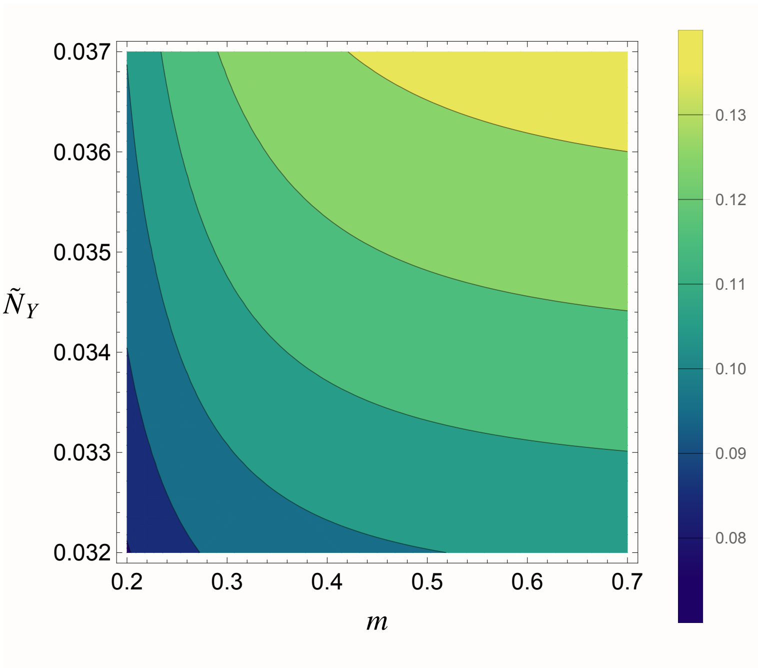


Figure 2.7: Value of Y_b in the $m - \tilde{N}_Y$ plane with the other parameters fixed.

Comparison with previous scans

While the Yukawa couplings just discussed arise from the E_7 model built in section 2.4.2, they are in fact more general, in the sense that they also correspond to certain models with E_8 enhancement. In particular, as mentioned below eq.(2.42) the matter curve content containing the MSSM chiral fields is identical to the one of the E_8 model explored in [159].[25] More precisely, we have that we recover the matter curves and the Yukawas of such E_8 model if in the parameters that describe the Higgs vev in (2.38) we fix $a = b = 1$. As mentioned in section 2.4.4, such particular choice of parameters prevents to implement the doublet-triplet splitting mechanism that by means of the hypercharge flux threading the matter curves $\mathbf{5}_U$ and $\mathbf{5}_D$ [142].

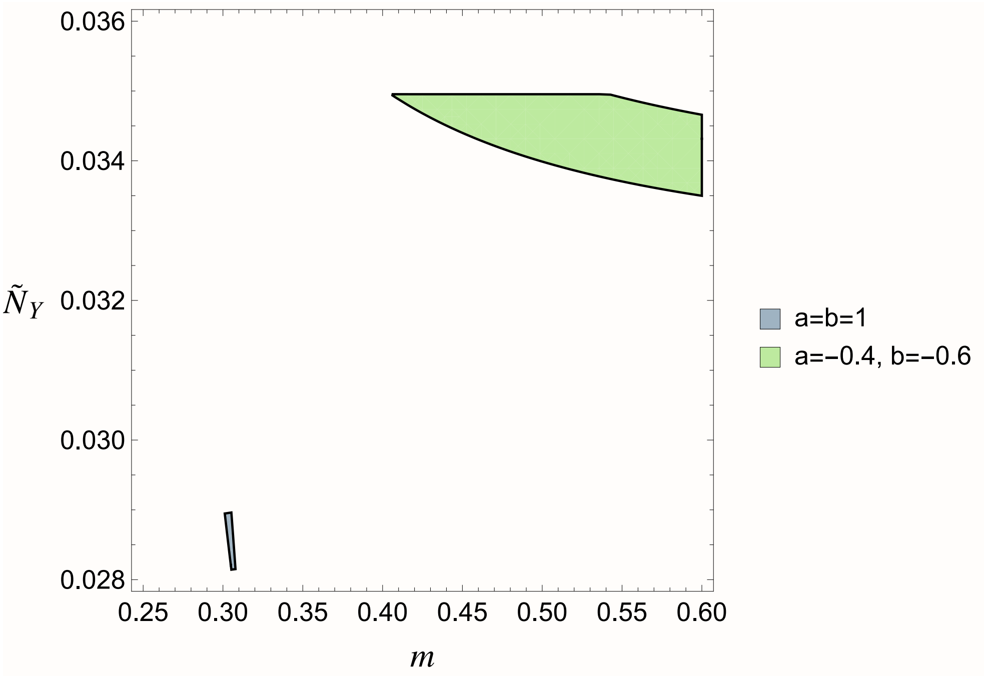


Figure 2.8: Comparison between the current scan and the one in [159], which considers $a = b = 1$.

We find quite intriguing that, by opening this new directions in parameter spaces that allow for doublet-triplet splitting, we are also able to fit the fermion masses much more easily than in previous attempts. We have illustrated such effect by means of figure 2.8, where we plot the allowed regions for realistic fermion masses in the case of the current scan and the one performed for the model in [159], which assumes $a = b = 1$. For illustrative purposes we have again used the plane $\tilde{N}_Y - m$ of parameters, but the fact that the region where realistic fermion masses are reproduced is wider in this case than the one in [159] is true for any direction in parameter space. Finally, a further advantage of exploring this new region of parameters is that the worldvolume flux densities are now much lower than in previous cases (c.f. eq.(2.122) as compared to eq.(6.12) in [159]). Being in the regime of diluted fluxes is quite important to

[25] In such model the T-brane structure of $\langle\Phi\rangle$ is more complicated, but however the matter sectors containing the MSSM chiral content are unaffected by such extra structure. Therefore one can directly apply the computation of Yukawa couplings performed in this paper to such local E_8 model.

construct 7-brane local models where α' corrections are negligible, and therefore the 7-brane action of [141, 142] can be used reliably. In particular, for models where the flux densities are larger than m_* one may worry that the D-term (2.31) receives non-trivial corrections that could modify the computation of wavefunctions in the real gauge.

2.4.13 Quark mixing angles

An additional piece of information that we may extract from the Yukawa matrices involves the quark mixing angles, which are conventionally encoded in the CKM matrix. The definition of the CKM matrix involves a pair of unitary matrices V_U and V_D which diagonalize the product $YY^\dagger$ of the quark Yukawa matrices. More specifically we have that

$$M_U = V_U Y_U Y_U^\dagger V_U^\dagger \tag{2.123a}$$

$$M_D = V_D Y_D Y_D^\dagger V_D^\dagger \tag{2.123b}$$

with M_U and M_D diagonal. Using this we may define the CKM matrix as

$$V_{CKM} = V_U V_D^\dagger \,. \tag{2.124}$$

We can directly apply this definition to the Yukawa matrices of our model, which are accurate up to $\mathcal{O}(\epsilon^2)$ corrections. In general the result is quite complicated, but again we find that we recover the CKM structure of [159] when we set $a = b$. To compare to the results therein we expand our more general CKM matrix in the new parameter $\xi \equiv a - b$. From the above analysis we know that for realistic fermion mass values $|\xi| \sim 0.1$, and so this expansion will quickly converge.

Explicitly we find

$$
\hat{V}_U = \begin{pmatrix} 1 & 0 & -\frac{\tilde{\epsilon}\gamma^Q_{10,1}}{2\rho_\mu \gamma^Q_{10,3}} \\ 0 & 1 & 0 \\ \frac{\tilde{\epsilon}^* \gamma^Q_{10,1}}{2\rho^*_\mu \gamma^Q_{10,3}} & 0 & 1 \end{pmatrix} \tag{2.125a}
$$

$$
\hat{V}_D = \hat{V}_D^{(0)} + \xi \hat{V}_D^{(1)} + \mathcal{O}(\xi^2) \tag{2.125b}
$$

$$
\hat{V}_D^{(0)} = \begin{pmatrix} 1 & -\frac{i\tilde{\epsilon}\mathrm{Im}[(d+1)\tilde{\kappa}^*\rho_\mu]}{(d+1)|d+1|^2|\rho_\mu|^2\rho_\mu}\frac{\gamma^Q_{10,1}\gamma^Q_{10,2}}{(\gamma^Q_{10,3})^2} & \frac{(d+1)\tilde{\epsilon}\rho_\mu - 2\tilde{\kappa}^2}{(d+1)^2\rho_\mu^2}\frac{\gamma^Q_{10,1}}{\gamma^Q_{10,3}} \\ -\frac{\tilde{\epsilon}^*\tilde{\kappa}^*}{(d^*+1)^2\rho^{*2}_\mu}\frac{\gamma^Q_{10,1}\gamma^D_{10,2}}{(\gamma^Q_{10,3})^2} & 1 - \frac{|\tilde{\kappa}|^2}{2|d+1|^2|\rho_\mu|^2}\frac{(\gamma^Q_{10,2})^2}{(\gamma^Q_{10,3})^2} & \frac{\tilde{\kappa}}{(d+1)\rho_\mu}\frac{\gamma^Q_{10,2}}{\gamma^Q_{10,3}} \\ -\frac{(d^*+1)\tilde{\epsilon}^*\rho^*_\mu - 2\tilde{\kappa}^{*2}}{2(d^*+1)^2\rho^{*\,2}_\mu}\frac{\gamma^Q_{10,1}}{\gamma^D_{10,3}} & -\frac{\tilde{\kappa}^*}{(d^*+1)\rho^*_\mu}\frac{\gamma^Q_{10,2}}{\gamma^Q_{10,3}} & 1 - \frac{|\tilde{\kappa}|^2}{2|d+1|^2|\rho_\mu|^2}\frac{(\gamma^Q_{10,2})^2}{(\gamma^Q_{10,3})^2} \end{pmatrix} \tag{2.125c}
$$

$$
\hat{V}_D^{(1)} = \begin{pmatrix} 0 & \frac{(d-1)\epsilon\tilde{\kappa}^*\theta_y}{2|d+1|^2(d+1)|\rho_\mu|^2}\frac{\gamma^Q_{10,1}\gamma^Q_{10,2}}{(\gamma^Q_{10,3})^2} & \frac{20d^2\tilde{\kappa}\rho_\mu\tilde{\epsilon} - \epsilon\theta_y\left(d^2-1\right)\rho_m^{3/2}}{2(d+1)^3\rho_\mu\rho_m^{3/2}}\frac{\gamma^Q_{10,1}}{\gamma^Q_{10,3}} \\ 0 & \frac{3d^2\tilde{\epsilon}\tilde{\kappa}^*\rho_\mu}{(d+1)|d+1|^2\rho_m^{3/2}\rho^*_\mu}\frac{(\gamma^Q_{10,2})^2}{(\gamma^Q_{10,3})^2} & -\frac{d\left(3d(d+1)\tilde{\epsilon}\rho_\mu + 4\kappa^2\right)}{(d+1)^3\rho_m^{3/2}}\frac{\gamma^Q_{10,2}}{\gamma^Q_{10,3}} \\ -\frac{20d^{*2}\tilde{\kappa}^*\tilde{\epsilon}^*\rho^*_\mu + \epsilon\,\theta_y\rho_m^{*3/2}(1-d^{*2})}{2(d^*+1)^3\rho^*_\mu\,\rho_m^{*3/2}}\frac{\gamma^Q_{10,1}}{\gamma^D_{10,3}} & \frac{d^*\left(3d^*(d^*+1)\tilde{\epsilon}^*\rho^*_\mu + 4\bar{\kappa}^2\right)}{(d^*+1)^3\rho_m^{*3/2}}\frac{\gamma^Q_{10,2}}{\gamma^Q_{10,3}} & \frac{3\tilde{\kappa}d^{*2}\tilde{\epsilon}^*\rho^*_\mu}{|d+1|^2\left(\bar{d}+1\right)\rho_\mu\rho_m^{*3/2}}\frac{(\gamma^Q_{10,2})^2}{(\gamma^Q_{10,3})^2} \end{pmatrix}
$$

$$(2.125d)$$

and one can check that $\hat{V}_D^{(0)}$ is similar to the rotation matrix found in [159], with a strong dependence on the parameter $\tilde{\kappa}$. As in there, one can estimate the effect of $\mathcal{O}(\epsilon^2)$ corrections to the Yukawa matrices and kinetic terms by means of some unknown rotation matrices of the form

$$
V_U = R_U \hat{V}_U\,, \qquad V_D = R_D \hat{V}_D\,, \tag{2.126}
$$

where

$$
R_{U,D} = \begin{pmatrix} 1 & \epsilon^2\alpha_{U,D} & 0 \\ -\epsilon^2\alpha_{U,D} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.127}
$$

and $\alpha_{U,D}$ are some unknown coefficients. The effect of these rotations is to modify the elements of the CKM matrix involving the first family of quarks but will not affect the mixing between the top and bottom quarks. Such mixing can be measured in terms of the V_{tb} entry of the CKM matrix which at the level of approximation we are working is given by

$$
V_{tb} = 1 - \frac{|\tilde{\kappa}|^2 q_{R,Q}}{2|d+1|^2|\rho_\mu|^2}\left[1 + \xi\,\frac{6\tilde{\epsilon}^* d^{*2}\rho_\mu^{*2}}{\left(\bar{d}+1\right)\rho_m^{*3/2}\tilde{\kappa}^*}\right], \tag{2.128}
$$

where the factor multiplying ξ takes the value 4×10^{-3} when we substitute the typical values (2.122), and so it corresponds to a negligible correction to the case $\xi = 0$ considered in [159].

The experimental value for this entry of the CKM matrix is

$$|V_{tb}|_{exp} \simeq 0.9991 \tag{2.129}$$

and so it can be reproduced by taking $\tilde{\kappa} \sim 10^{-5} - 10^{-6}$. Notice that this value is quite different from the one obtained in [159], but this mismatch is only due to the different parametrisation of the distance between the Yukawa points $p_{\rm up}$ and $p_{\rm down}$ taken in that reference and in the present one (c.f. footnote 21). The physical quantity is the distance between these two points in units of the typical scale of $S_{\rm GUT}$, which we can estimate by looking at the V_{tb} entry of the CKM matrix. In fact we have the following relation[26]

$$\sqrt{1-|V_{tb}|} \simeq \frac{|\tilde{\kappa}|\sqrt{q_{R,Q}}}{\sqrt{2}|\rho_\mu||d+1|} \propto m_* \left|\frac{a+bd}{d+1}x_0 - y_0\right| \tag{2.130}$$

where (x_0, y_0) are the coordinates of the down Yukawa point, see (2.45). This implies that the separation of the two points directly controls the mixing between the second and third family. In the case $a = b$ (where we recover the CKM matrix of [159]) this separation is measured along the coordinate $ax - y$ which is precisely the complex coordinate entering in the matter wavefunctions, see (2.75). In this case the whole effect of mixing is due to a mismatch in the wavefunctions bases between the two points as in [147]. It would be however interesting to have an intuitive picture for the general case $a \neq b$. In any case, as in [159] using the relation between the measured V_{tb} entry of the CKM matrix and the relative distance between the two Yukawa points we can directly estimate the latter and see that it is of the order of $10^{-2}V_{GUT}^{1/4}$. Hence we can see explicitly that the distance between the two points is rather small when compared to the typical size of S_{GUT} as claimed in [145].

[26] Here we again discard the $\mathcal{O}(\xi)$ term in the expression for V_{tb} as it is negligible.

CHAPTER 3

SUPERSYMMETRY ENHANCEMENT

3.1 Generalities of $4d$ $\mathcal{N} = 2$ QFTs

In this section we briefly review some of the properties of 4d $\mathcal{N} = 2$ QFTs which will be relevant for the following. The treatment will be schematic. For a more complete treatment see for example [30] for the supersymmetric case.

Supersymmetry algebra

The Poincaré group in four dimensions is $\mathbb{R}^4 \rtimes SO(1,3)$. In the following we will always assume a Minkowski metric with signature $(-,+,+,+)$. The Clifford algebra in four dimension is defined by

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{3.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric. The 4d $\mathcal{N} = 2$ supersymmetry algebra[1] can be written in terms of two Weyl spinors Q^I_α, $I = 1, 2$ and reads:

$$\begin{aligned} \{Q^I_\alpha, \bar{Q}^J_{\dot\beta}\} &= 2\sigma_\mu P^\mu \delta^{IJ} \\ \{Q^I_\alpha, Q^J_\beta\} &= \epsilon_{\alpha\beta} Z^{IJ} \end{aligned} \tag{3.2}$$

where here $\alpha, \dot\alpha$ are indices of the $SU(2) \times SU(2)$ Lorenz group and $I, J = 1, 2$ are indices labeling the different spinors of supercharges. There is a $SU(2)_R \times U(1)_r$ R-symmetry rotating

[1] There are of course other commutators involving bosonic generators P_μ, $M_{\mu\nu}$ which we don't report for simplicity.

the supercharges. Z^{IJ} is an antisymmetric matrix of central charges: it can be shown that it commutes with all the generators of the SUSY algebra.

Multiplets and Lagrangians

We will now introduce the irreducible representations of the 4d $\mathcal{N}=2$ supersymmetric algebra, and the most general renormalizable 4d $\mathcal{N}=2$ lagrangian. We will however write everything in terms of a lagrangian which is only manifestly supersymmetric under a specific $\mathcal{N}=1$ subalgebra. The discussion is completely standard, see for example [31] [30] for references.

Let us first define $\mathcal{N}=1$ chiral and vector superfields. Start by defining the superspace covariant derivative to be

$$\begin{aligned} D_\alpha &= \partial_\alpha + i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu \\ \bar D_{\dot\alpha} &= \bar\partial_{\dot\alpha} + i\theta^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu \end{aligned} \tag{3.3}$$

A chiral (resp antichiral) superfield Φ (resp Ψ) is defined to be a superfield such that $\bar D_{\dot\alpha}\Phi = 0$ (resp $D_\alpha\Psi = 0$). Notice that if Φ is a chiral superfield, then $\bar\Phi$ is an antichiral superfield. A chiral superfield Φ contains the on shell degrees of freedom of a complex scalar and a Weyl fermion.

A vector superfield V is defined by the condition $V = V^\dagger$. It contains the on-shell degrees of freedom of a vector field and a Weyl fermion. V is typically in the adjoint representation of the gauge group. From the vector field we can build the filed strength chiral superfield, which for a generic non-abelian gauge group G takes the form

$$W_\alpha = -\frac{1}{4}\bar D\bar D(e^{-V}D_\alpha e^V) \tag{3.4}$$

W_α contains the field strength $F_{\mu\nu}$ and the gaugino.

With chirals and vector fields one can write generic $\mathcal{N}=1$ Lagrangians. For example, the most general[2] renormalizable and local $\mathcal{N}=1$ lagrangian involving one chiral superfield X and a vector field V is of the form

$$\mathcal{L} = \frac{\operatorname{Im}\tau}{4\pi}\int d^4\theta Tr(Xe^V\bar X) + \int d^2\theta\ W(X) + \int d^2\theta\ \frac{-i}{8\pi}W_\alpha W^\alpha + hc \tag{3.5}$$

Where W is a holomorphic polynomial of the chiral superfield X of degree 3 or less, in order to ensure renormalizability of the action.

[2] We could also add Fayet-Iliopulos parameters if G contains some $U(1)$ factors. We choose to not discuss them for simplicity, referring the reader to a standard textbook like [32]

Our main interest in this chapter are actually $\mathcal{N}=2$ theories, so let us review now $\mathcal{N}=2$ supermultiplets. The $\mathcal{N}=2$ hypermultiplet H consists of a $\mathcal{N}=1$ chiral multiplet and a $\mathcal{N}=1$ antichiral multiplet in the same representation $\mathcal{R}$ of the gauge group, namely $H=(Q,\tilde{Q}^\dagger)$. It is customary to trade the antichiral multiplet $\tilde{Q}^\dagger$ with it's complex conjugated chiral multiplet $\tilde{Q}=(\tilde{Q}^\dagger)^\dagger$ which is now in the conjugated representation $\bar{\mathcal{R}}$ of the gauge group, so people often say that the $\mathcal{N}=2$ hypermultiplet H consists of two $\mathcal{N}=1$ chiral multiplets in conjugated representations $\mathcal{R}$ and $\bar{\mathcal{R}}$ of the gauge group, namely $H=(Q,\tilde{Q})$

The $\mathcal{N}=2$ vector multiplet consists of a $\mathcal{N}=1$ vector multiplet together with a chiral multiplet. The Coleman-Mandula theorem implies, among other things, that gauge and global symmetries commute, therefore forcing this chiral field to be in the same representation of the $\mathcal{N}=1$ vector: the adjoint.

The most generic renormalizable lagrangian with $\mathcal{N}=2$ supersymmetry is essentially fixed by the amount of supersymmetry that we have in this case. In particular, the superpotential is fixed[3]. The lagrangian will take the form

$$\mathcal{L}=\mathcal{L}_{vec}+\mathcal{L}_{hyp} \tag{3.6}$$

where $\mathcal{L}_{vec}$ is seen as the vector multiplet lagrangian and $\mathcal{L}_{hyp}$ the hypermultiplet lagrangian. The first contribution is given by

$$\mathcal{L}_{vec}=\frac{\text{Im }\tau}{4\pi}\int d^4\theta Tr(\Phi^\dagger e^V\Phi)+\int d^2\ \theta\frac{-i}{8\pi}TrW_\alpha W^\alpha+cc \tag{3.7}$$

where the first integral contains the kinetic terms for the adjoint chiral Φ and the latter is the kinetic term for the $\mathcal{N}=1$ vector. The second contribution reads

$$\mathcal{L}_{hyp}=\int d^4\theta\ (Q^{\dagger i}e^VQ_i+\tilde{Q}^{\dagger i}e^{-V}Q_i)+\int d^2\theta\tilde{Q}^i\Phi Q_i+\int d^2\theta\ m^i_j\tilde{Q}^jQ_i+cc \tag{3.8}$$

where the first integral contains the kinetic terms of the hypers $H_i=(Q_i,\tilde{Q}_i)$, the second term contains the superpotential interaction between the chirals in the hyper and the adjoint chiral Φ, and the last term contains a mass term for the chirals in the hyper. In particular by comparing the second and the last term we see that the mass terms enters the lagrangian in the same way Φ does. It is therefore possible to treat is as a background multiplet.

[3]This is what will allow us in the next chapter to say that a quiver diagram is a way to encode (almost) the full information about a particular lagrangian in a theory with 8 supercharges

Effective lagrangians in the Coulomb Branch

The moduli space of vacua of a generic theory with 8 supercharges is a complex algebraic variety. Two relevant subvarieties of the moduli space are called "Coulomb Branch" and "Higgs branch". We will discuss more about those in the third chapter, in the context of $3d\ \mathcal{N}=4$ theories. For the moment, let us say that the Coulomb branch is parametrized by vevs of the scalars in the vector multiplet, and the Higgs branch by vevs of gauge invariant combinations of the scalars in the Hypermultiplets.

Indeed, let us consider the F-term and D-term equations for the lagrangian given in (3.6). We find

$$\begin{cases} \frac{1}{g^2}[\Phi^\dagger,\Phi] + (Q_i Q^{\dagger i} - \tilde{Q}_i^\dagger \tilde{Q}^i)\Big|_{tr} = 0 \\ Q_i \tilde{Q}^i\Big|_{tr} = 0 \\ \Phi Q_i + m_i^j Q_j = 0 \\ \tilde{Q}^i \Phi + m_j^i \tilde{Q}^j = 0 \end{cases} \tag{3.9}$$

where for a $N \times N$ matrix $X|_{tr}$ is defined to be

$$X|_{tr} := X - \frac{1}{N} tr(X) \tag{3.10}$$

We can see that the system of equations (3.9) is solved easily in the case in which all $Q_i, \tilde{Q}_i$ take zero expectation value and Φ takes non zero value but such that $[\Phi^\dagger,\Phi] = 0$. It is also solved in the case in which the mass terms vanish, Φ takes zero vev and $\tilde{Q}_i \tilde{Q}^i\Big|_{tr} = 0$. The first solution identifies vacua in the *Coulomb Branch* (CB), and the second identifies vacua in the *Higgs Branch* (HB).

For the moment, let us focus on the Coulomb branch. In a generic CB vacuum, the gauge group G is broken by the vev of Φ to its maximal torus $U(1)^r$. Then a natural question to ask is which will be the low energy effective action in such vacuum. We know that it will be a theory with just r abelian vector multiplets, let us call them A_i and call a_i the scalar component of those multiplets. In principle the effective lagrangian will be in general non-renormalizable and it will not take the exact form given by (3.6). However, supersymmetry still constraints the lagrangian to be of the following form

$$\mathcal{L}_{eff} = \frac{1}{8\pi}\int d^4\theta K(\bar{a}_i, a_j) + \int d^2\theta\ \frac{-i}{8\pi}\tau^{ij}(a) W_{\alpha,i} W_j^\alpha + cc \tag{3.11}$$

We see now that both the Kähler potential and the gauge kinetic term will be explicitly dependent on the fields a_i. Furthermore it holds that $K(\bar{a}, a)$ and $\tau(a)$ are related. In particular, there exist a locally[4] holomorphic function $\mathcal{F}(a)$ called *prepotential* such that

$$\begin{aligned} \tau^{ij}(a) &= \frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} \\ K(\bar{a}, a) &= i \left(\overline{\frac{\partial F}{\partial a_i}} a_i - \bar{a}_i \frac{\partial F}{\partial a_i} \right) \end{aligned} \tag{3.12}$$

Therefore the idea is that the prepotential completely fixes the effective field theory dynamics for any CB vacuum, in the IR.

Anomalies

In 4d $\mathcal{N} = 2$ theories there are no local gauge anomalies as fermions will always appear in non-chiral representations. Fermions in the vector multiplet come in the adjoint representation which is always a real representation. Fermions in the hypermultiplet come in a sum of a representation $\mathcal{R}$ and its conjugate $\bar{\mathcal{R}}$ and fermions in the half-hypermultiplets are in a pseudoreal representation. Therefore local gauge anomalies are automatically canceled.

There is however a global gauge anomaly which has to be taken into account: Witten anomaly. [12] Consider a theory with $SU(2)$ gauge group and a weyl fermion in the fundamental representation. The path integral for this theory will give

$$Z[A_\mu, \psi_{\alpha,i}, \bar{\psi}_{\dot{\alpha},i}] = \int [DA_\mu][D\psi_{\alpha,i}][D\bar{\psi}_{\dot{\alpha},i}] e^{-\int \bar{\psi}\sigma^\mu D_\mu \psi} \tag{3.13}$$

and suppose we first perform the integral on the fermion. After this integration we will have

$$Z[A_\mu] = \int [D\psi_{\alpha,i}][D\bar{\psi}_{\dot{\alpha},i}] e^{-\int \bar{\psi}\sigma^\mu D_\mu \psi} \tag{3.14}$$

which we still have to integrate over A_μ. In order for this A_μ integration to be consistent, $Z[A_\mu]$ must be gauge invariant. Namely, consider the gauge transformed field $A'_\mu = g^{-1} A_\mu g + g^{-1}\partial_\mu g$, we need to have that $Z[A'_\mu] = Z[A_\mu]$ for any $g : \mathbb{R}^4 \to SU(2)$. We can characterize such maps by looking at the way the gauge transformation g behaves on the sphere S_4 obtained by adding one point to $\mathbb{R}^4$. Call g_{S_4} such restriction of g. We will have $g_{S_4} : S_4 \to SU(2)$. Maps of this type are classified by the fourth homotopy group: we will have

$$\pi_4(SU(2)) = \mathbb{Z}_2 \tag{3.15}$$

so we see that there are two distinct classes of gauge transformations. Let us consider one gauge transformation belonging to the class corresponding to the non-identity element of $\mathbb{Z}_2$.

[4] We will see later that $\mathcal{F}(a)$ is not globally a function, but a section of a $SL(2, \mathbb{Z}$ bundle

With a computation, originally performed in [12] is possible to show that under such gauge transformation the integration measure for the fermions is not invariant. Namely

$$[D\psi_{\alpha,i}][D\bar{\psi}_{\dot{\alpha},i}] \mapsto -[D\psi_{\alpha,i}][D\bar{\psi}_{\dot{\alpha},i}] \tag{3.16}$$

therefore making the path integral inconsistent.

This does not happen only for $SU(2)$ with one Weyl fermion in the fundamental representation. It can be shown that $\pi_4(G) = \mathbb{Z}_2$ for $G = Sp(n)$ and it is trivial otherwise. So in general $Sp(n)$ groups might suffer this anomaly, while other groups are safe. Since in 4d Witten anomaly is always valued in $\mathbb{Z}_2$, full hypermultiplets are automatically anomaly free, whatever the gauge representation they are in. The problem therefore might only occur with a odd number of half-hypermultiplets. We will not consider half-hypermultiplets in this thesis, so Witten anomaly will be always automatically satisfied.

3.2 The Seiberg-Witten curve.

It is strongly believed that every 4d $\mathcal{N} = 2$ supersymmetric theory admits an auxiliary object called *Seiberg Witten curve*[5] [33] [34]. This is a complex curve fibered over the Coulomb branch of the Moduli space, which encodes informations about the low energy effective action in a generic CB vacuum. In particular, the curve carries all the information necessary to compute the exact prepotential for the effective theory (3.11). Of course, a full and detailed review of the subject is by far out of the scope of this PhD thesis. However we will now review very briefly such construction as we will need to use Seiberg-Witten curves in the following. We will basically define the curve in one example and explain how to extract the prepotential of the IR theory from it, without explaining why the curve has a specific form. For a more detailed explanation see [30].

Pure $SU(2)$ theory.

Let us consider as an example the case of pure $SU(2)$ gauge theory.

The lagrangian will be given by

$$\mathcal{L} = \frac{Im\tau}{4\pi}\int d^4\theta Tr\Phi^\dagger e^V\Phi + \int d^2\theta\frac{-i}{8\pi}\tau TrW_\alpha W^\alpha + hc \tag{3.17}$$

In a generic vacuum of the Coulomb branch the gauge group will be broken to $U(1)$ by the vev of the scalar field φ in the vector multiplet. The effective IR lagrangian will be of the form given

[5]However, examples in which the specific curve for a given theory is not known to date exist.

in (3.11). A gauge invariant way to label the CB vacuum is given by

$$u = \langle Tr\varphi^2 \rangle = a^2 + ... \tag{3.18}$$

where the dots denote quantum corrections. In the following, we will define also

$$a_D = \frac{\partial \mathcal{F}}{\partial a} \tag{3.19}$$

where $\mathcal{F}$ is the prepotential of the IR theory.

Let us now consider an auxiliary ambient space $\mathbb{C}^2$ parametrized with coordinates z, x. Consider now the following complex 1-dimensional curve in such ambient space

$$\Lambda^2 z + \frac{\Lambda^2}{z} = x^2 - f \tag{3.20}$$

where f is a fixed complex number and Λ is the strong coupling scale for the IR theory. Now do the following: for every point u of the Coulomb branch, fiber on top of it a curve like (3.20), namely promote f to be a function of u. We choose $f(u) = u$ giving

$$\sigma: \quad \Lambda^2 z + \frac{\Lambda^2}{z} = x^2 - u \tag{3.21}$$

This fibration is the so called *Seiberg-Witten curve*. Together with the fibration, it is defined a differential which in this case will be

$$\lambda = x\frac{dz}{z} \tag{3.22}$$

which is the so called *Seiberg-Witten differential.*

It can be seen that the (3.21), thought as a function $x(z)$ has 4 square root branch points at $z = 0, z = \infty, z = z_{\pm}$. We can therefore take two branch cuts, one going from $z = 0$ to $z = z_-$ and one going from $z = z_+$ to $z = \infty$. This curve will therefore be a double sheeted cover of the z plane. We will define two cycles A and B as it can be seen from figure (3.1).

We then declare that the a and a_D coordinates will be given by integrals of the SW differential on those cycles, namely:

$$a = \frac{1}{2\pi i}\oint_A \lambda, \quad a_D = \frac{1}{2\pi i}\oint_B \lambda, \tag{3.23}$$

From this the prepotential (and therefore the Kähler potential and the gauge coupling) can be computed. For example, the gauge coupling constant for the IR theory in a generic CB vacuum will then be given by

$$\tau(a) = \frac{\partial a_D}{\partial a} \tag{3.24}$$

We will omit an explanation of why the curve takes exactly this form, for this theory. In general, for a given $\mathcal{N} = 2$ theory $\mathcal{T}$ it will exist a curve and a differential encoding the low

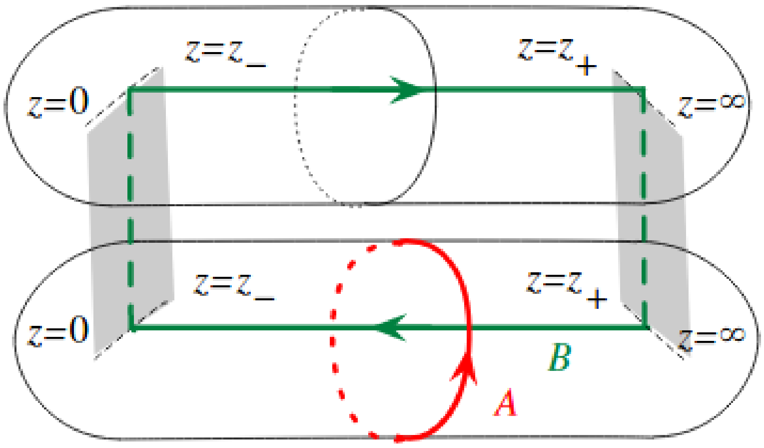


Figure 3.1: A picture, taken from [30], of the SW curve for the pure $SU(2)$ theory.

energy dynamics. The example we gave is for a rank 1 theory. In particular, in a case in which the theory has rank r (namely the CB has complex dimension r) there will be $2r$ cycles and the variables a^i and a^i_D will be given by integrals of the SW differential on those cycles. This fact forces a relation between the genus g of the curve and the rank of the theory: they will always be equal. We will use this fact in the following.

3.3 Generalities of 4d $\mathcal{N} = 2$ SCFTs.

In this section we briefly review some of the properties of 4d $\mathcal{N} = 2$ SCFTs which will be relevant for the following. The treatment will be schematic. In particular, we will refrain to give a full discussion of superconformal representation theory, as it would take us too far apart from the topic we really want to discuss, which is supersymmetry enhancement. We refer the reader to [35] for such discussion.

Let us start by considering the beta function. The key point for us will be that due to the $\mathcal{N} = 2$ non-renormalization theorems [36], the beta function of a 4d $\mathcal{N} = 2$ QFT is one loop exact, and does not receive any non-perturbative correction. The one-loop beta function coefficient for a generic (also non-supersymmetric) QFT is given by the NSVZ formula to be[6]

$$\mu \frac{d}{d\mu} g = -\frac{g^3}{(4\pi)^2} \left[\frac{11}{3} C(\text{adj}) - \frac{2}{3} C(\mathcal{R}_f) - \frac{1}{3} C(\mathcal{R}_s) \right] \tag{3.25}$$

[6]We saw this formula in the special case of $SU(N)$ in the first chapter.

Here $C(\rho)$ is the Dynkin index of a representation ρ defined by

$$tr\rho\left(T^a\right)\rho\left(T^b\right)=C(\rho)\delta^{ab} \tag{3.26}$$

where $\rho(T^a)$ is the matrix representation of the generator T^a. Furthermore in (3.25) we assume that all fermions are left-handed Weyl spinors and $\mathcal{R}_f$ denotes the representation of the fermions and $\mathcal{R}_s$ the representation of the complex scalars.

In case we consider a $\mathcal{N}=2$ such formula simplifies to

$$\mu\frac{d}{d\mu}\frac{8\pi^2}{g^2}=2C(\text{adj})-C(\mathcal{R}_h) \tag{3.27}$$

where now $C(\mathcal{R}_h)$ is the representation of the chiral multiplets inside the hypermultiplet. It is therefore possible to suitably choose the matter content of the theory to make the beta function vanish. If this is the case, the theory is conformally invariant[7]. The easiest example of a superconformal field theory of this type is $SU(2)$ with $N_f=4$ flavors. In this section we will now review basic properties of superconformal field theories.

Central Charges

Consider a generic conformal field theory in 4d. The standard way to define the conformal central charges a and c is by using the OPE of the stress energy tensor [38, 39, 61]. They can also be defined as the coefficients in the conformal anomaly, namely

$$\langle T^\mu_\mu\rangle=\frac{c}{16\pi^2}(\text{Weyl})^2-\frac{a}{16\pi^2}(\text{Euler}) \tag{3.28}$$

where

$$\begin{aligned}(\text{Weyl})^2&=R^2_{\mu\nu\rho\sigma}-2R^2_{\mu\nu}+\frac{1}{3}R^2\\(\text{Euler})&=R^2_{\mu\nu\rho\sigma}-4R^2_{\mu\nu}+R^2\end{aligned} \tag{3.29}$$

are the square of the Weyl tensor and the Euler tensor of the background metric.

If the theory is $\mathcal{N}=2$ supersymmetric, it is possible to relate the central charges a and c to the R-charges of the operators of the theory. If the theory is lagrangian, the central charges will satisfy

$$\begin{aligned}2a-c&=\frac{1}{4}\sum_i\left(2[u_i]-1\right)\\c-a&=\frac{n_h-n_v}{24}\end{aligned} \tag{3.30}$$

[7]Being more precise, the vanishing of the beta function allows us to only say that the theory is scale invariant, and this in 4d generically does not imply invariance under the full conformal group. Some counterexamples are known. In the following, we will assume that all $\mathcal{N}=2$ supersymmetric theories with vanishing beta function are automatically conformally invariant.

where in the first equation we sum the conformal dimensions of all the CB operators. In the second equation n_h (resp n_v) is the number of hypermultiplets (resp vector multiplets). We notice in particular that $n_h - n_v$ is the quaternionic dimension of the Higgs branch of the theory.

If the theory is non lagrangian, computing a and c is harder. There is a formula from topological field theory [40] which gives

$$\begin{aligned} a &= \frac{R(A)}{4} + \frac{R(B)}{6} + \frac{5}{24}r + \frac{h}{24} \\ c &= \frac{R(B)}{3} + \frac{r}{6} + \frac{h}{12} \end{aligned} \tag{3.31}$$

where we denoted with r (resp h) the number of free vector multiplets (resp hypermultiplets) in a generic CB vacuum. Here $R(A) = \sum_i ([u_i] - 1)$ and $R(B)$ can be computed from the discriminant of the Seiberg-Witten curve. We will refrain to give a full expression for $R(B)$ and refer the reader to [40].

Even in the strongly coupled case it will hold again that $c - a$ is related to the Higgs Branch dimension. Then this often opens up a third way to compute $c - a$: it is also related to the dimension of the Coulomb branch of the $3d$ mirror theory, when such mirror is known[8]. This happens because the Higgs branch is invariant under dimensional reduction, and $3d$ mirror symmetry swaps the Coulomb branch and the Higgs branch. This indirect method for computing $c - a$ is particularly useful in case the 4d theory is non-lagrangian, as usually 3d mirrors of non lagrangian theories are lagrangian and it is therefore trivial to compute the Coulomb branch dimension of them.

A crucial point to be discussed now is that a and c are rational in every $\mathcal{N} = 2$ SCFT. This is essentially because a and c are related to the conformal dimensions of CB operators in the theory, or other quantities like $R(B)$ which can nevertheless be read from the Seiberg-Witten curve. Since the Seiberg-Witten curve is a polynomial, this will imply that a and c are rational.[9] The same fact does not hold with $SCFT$s with smaller amount of supersymmetry, essentially because the SW curve will not exist for such theories.[10]

As a last comment, useful for the following, we mention that for a generic $\mathcal{N} = 1$ superconformal theory the central charges a and c can be related to the $R_{\mathcal{N}=1}$ $U(1)$ R-symmetry

[8]We will discuss more about $3d$ Mirro Symmetry in the following chapter

[9]We realize this is not a sharp proof as details on the form of $R(B)$ are not spelled out and also we assume (quite naturally) that every $\mathcal{N} = 2$ SCFT has a SW curve.

[10]To be more precise, there is a notion of $\mathcal{N} = 1$ curve [41] but as far as the author knows, such notion is irrelevant to compute $\mathcal{N} = 1$ superconformal central charges.

by

$$
\begin{aligned}
a &= \frac{3}{32}\left(3trR^3_{\mathcal{N}=1} - trR_{\mathcal{N}=1}\right) \\
c &= \frac{1}{32}\left(9trR^3_{\mathcal{N}=1} - 5trR_{\mathcal{N}=1}\right)
\end{aligned}
\tag{3.32}
$$

where the traces are performed over all the Weyl fermions of the theory.

Lagrangian SCFTs

As mentioned above, the easiest example of a $\mathcal{N}=2$ superconformal field theory is $SU(2)$ with $N_f = 4$ flavors. This can be generalized to $SU(N)$ with $N_f = 2N$ flavors: the beta function (3.27) will vanish because in this case $C(\text{adj}) = N$ and $C(\mathcal{R}_h) = 2 \cdot 2N \cdot \frac{1}{2}$. This can be further generalized to quiver gauge theories[11] with M $SU(N_i)$ nodes, such that every gauge node $i = 1, ...M$ sees a number $N_{fi} = 2N_i$ attached to it. It can be show that the only possible quivers of this type take the form of Dynkin diagrams or affine Dynkin diagrams, with possibly extra fundamental flavors attached at some node. For two examples which we will use in the following, see figures (3.2) and (3.3). Theories of this class are called *superconformal quivers*.

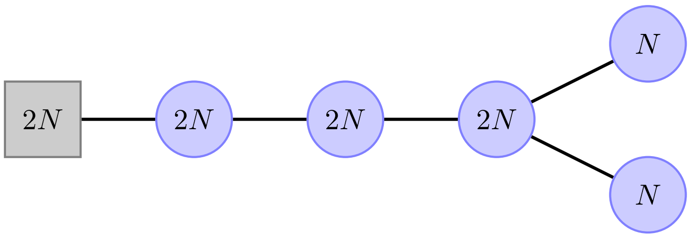


Figure 3.2: A D_5-shaped superconformal quiver

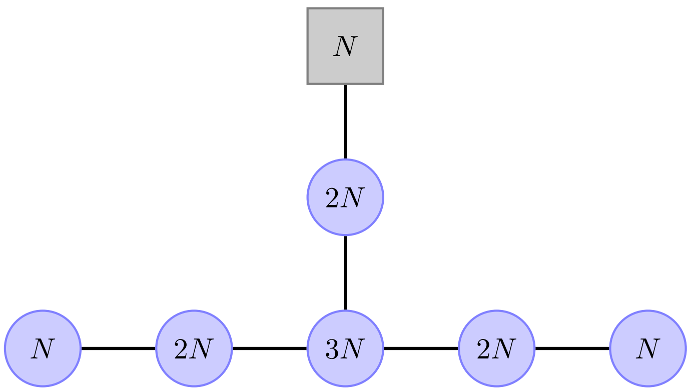


Figure 3.3: A E_6-shaped superconformal quiver

[11]We refer the reader to chapter 4, where a more comprehensive discussion of quiver gauge theories is given.

Class-S and Argyres-Douglas SCFTs.

Only an arguably small subset of 4d $\mathcal{N}=2$ SCFTs are lagrangian.[12]

A very large class of generically non-lagrangian 4d $\mathcal{N}=2$ SCFTs is given by the so called *class-S*. [42]. A full discussion of class-S theories is out of the scope of this thesis. However, due to the prominent and central role of these theories in the landscape of 4d SCFTs, we will now review brief aspects of them. In few words, such theories can be obtained by considering a 6d $(2,0)$ superconformal field theory of ADE type compactified on a Riemann surface with punctures.

In principle the Riemann surface will have no covariantly constant spinors (unless $g=1$) and therefore such compactification would break completely supersymmetry in 4d. Therefore one does a partial topological twist, very much analog to that discussed in the case of F-theory, in the first chapter. This Riemann surface is usually called the *Gaiotto curve*, and the Seiberg-Witten curve of the theory will be given by a N-sheeted cover of the Gaiotto curve. Here N is related to the rank of the ADE-type we chose for the $6d$ $(2,0)$ theory.

Having fixed the 6d theory and the Riemann surface, the punctures can be of different types, called *regular* and *irregular*. For each type, there is a finite list of possible punctures. The full classification of all the regular punctures is performed in the series of papers *"Tinkertoys"* by Distler, Chacaltana, Trimm and collaborators [44–50]. The classification of irregular punctures is performed in the series of papers by Dan Xie and collaborators [51–53]. Punctures always encode flavor symmetry factors for the class-S theory, and well defined and simple rules to read out such flavor factor from the kind of punctures appearing in the Riemann surface are known.

The subset of class-S theories which we will mostly focus in this thesis is given by (generalized) Argyres-Douglas theories. We define a 4d $\mathcal{N}=2$ SCFT to be of (generalized) Argyres-Douglas type if at least one Coulomb branch operator has a fractional non-integer conformal dimension.

Historically the first appearance of Argyres-Douglas theories was in [66]. Here the IR theory at a special point in the CB of pure $SU(3)$ was considered. At such point there are mutually-non local massless dyons, so it is impossible to choose an electromagnetic duality frame in which all the massless particles carry only magnetic charge. This implies that a $\mathcal{N}=2$ lagrangian formulation for the theory does not exist. Let us call this theory H_0. The same theory H_0 can be found by considering a special point in the CB of $SU(2)$ theory with $N_f=1$. The Seiberg

[12]Here by the name *non-lagrangian theory* we do not restrict ourself to saying that a lagrangian for such QFT cannot exist, but we mean that either the lagrangian cannot not exist or that no one has discovered it at the time or writing.

Witten curve and differential for the H_0 theory is given by

$$x^2 = z^2 + mz + u \qquad \lambda = xdz \tag{3.33}$$

Here u is the CB parameter and m is the coupling constant for u. Since integrals of the SW differential will give a and a_D, we must always require that the SW differential has dimension one. Therefore we have

$$[x] + [z] = 1 \tag{3.34}$$

This condition, among with the condition that all the terms in the SW curve are homogeneous with the same degree, is enough to solve for the scaling dimensions of u and m. We find

$$[u] = \frac{6}{5}, \qquad [m] = \frac{4}{5} \tag{3.35}$$

We see that the CB operator u has fractional non-integer scaling dimension, so this theory H_1 is an Argyres-Douglas theory.

After this first example, many other Argyres-Douglas theories where discovered. For example going to special point in the CB of $SU(2)$ with $N_f = 2$ (resp $N_f = 3$) one can find the AD theory H_2 (resp H_3). A much greater generalization comes from considering a special point of the CB of a theory with generic gauge group G and fundamental matter [67,68]. In case G is of ADE type, the Argyres-Douglas theories found can be named as follows: for the ones found at a special point of the CB of $SU(N+1)$ are usually called (A_1, A_N), the ones coming from $SO(2n)$ are called (A_1, D_N) and the ones coming from E_n are called (A_1, E_n). Cecotti-Neitze-Vafa further enlarged this class, by providing a geometrical engineering of it in the context of IIB superstring theory [69]. They considered IIB on a background given by an isolated hypersurface singularity of the form

$$f_G(x_1, x_2) + f_{G'}(x_3, x_4) = 0 \tag{3.36}$$

where $f_G(x, y)$ is a polynomial of the following type

$$\begin{aligned} f_{A_n}(x, y) &= x^2 + y^{n+1} \\ f_{D_n}(x, y) &= x^2 y + y^{n-1} \\ f_{E_6}(x, y) &= x^3 + y^4 \\ f_{E_7}(x, y) &= x^3 + xy^3 \\ f_{E_8}(x, y) &= x^3 + y^5 \end{aligned} \tag{3.37}$$

The theory such engineered will be called AD of (G, G') type. This notation with two ADE labels is also related to the fact that the BPS quiver of the theory (G, G') has a shape given by the disjoint union of the Dynkin diagram of G and of G'. It was also understood that those (G, G') theories can be realized in class-S by taking the $(2,0)$ theory to be compactified on a Riemann sphere with a single suitably chosen irregular singularity.

A point that will be of interest for us is that since all these theories are of AD type, as discussed before it is believed to be impossible to find a $\mathcal{N}=2$ Lagrangian formulation for them. We will now discuss a recently discovered way to overcome this problem with some amazing solution: it turns out that it is possible to find $\mathcal{N}=1$ lagrangians for some UV theory such that at the IR fixed point of the RG flow supersymmetry gets enhanced to $\mathcal{N}=2$, and the IR theory is exactly an Argyres-Douglas theory. We will discuss this in the next sections.

3.4 Maruyoshi-Song flows.

In the past couple of years phenomena of supersymmetry enhancement have been studied. The idea is that certain $\mathcal{N}=1$ supersymmetric theories will have an RG flow leading to a IR fixed point in which a $\mathcal{N}=2$ supersymmetry is restored. There are two ways in which such examples were discovered. The first way is explained in [54, 55] and uses heavily Argyres-Seiberg duality in order to explain the enhancement. The second technique, exploited by Maruyoshi and Song in [56] (and subsequently by Agarwal and Sciarappa in [57, 58]) is more direct: it consists in starting with a $\mathcal{N}=2$ SCFT, turning on a $\mathcal{N}=1$ preserving deformation of a certain type, and following the RG flow until deep IR. We will only briefly discuss this second approach in this thesis. For more details, we refer the reader to the original papers.

The deformation

Let us review how the Maruyoshi-Song procedure goes in detail.

1. Start with a $\mathcal{N}=2$ SCFT in the UV which we call theory $\mathcal{T}$, with a non-abelian flavor symmetry group F. $\mathcal{T}$ can be either lagrangian or non-lagrangian.

2. Add to the theory a $\mathcal{N}=1$ chiral multiplet M charged in the adjoint representation of F.

3. Couple M to the theory $\mathcal{T}$ by turning on a superpotential term of the form $W = TrM\mu$ where μ is the moment-map operator of F. Notice that in case the theory is lagrangian $W = Mq\tilde{q}$ so M will behave like a mass term for the $\mathcal{N}=2$ hypers, although it is a dynamical (non background) field.

4. Give a vacuum expectation value to M, along a nilpotent orbit $\mathcal{O}$ of the complexified Lie algebra $\mathfrak{f}$ of F. Namely

$$\langle M \rangle = \rho(\sigma^+) \tag{3.38}$$

where $\rho : \mathfrak{su}(2) \to \mathfrak{f}$ is the Jacobson-Morozov embedding which is associated with the nilpotent orbit $\mathcal{O}$ of $\mathfrak{f}$. For more details about nilpotent orbits, see appendix [APP] and for a more complete treatment [Collingwood].

5. The procedure done so far has broken explicitly $\mathcal{N} = 2$ down to $\mathcal{N} = 1$. Call the $\mathcal{N} = 1$ theory such obtained $\mathcal{T}^{UV}[\mathcal{O}]$. The deformation we turned on and also triggers a new RG flow from $\mathcal{T}^{UV}[\mathcal{O}]$ down to some IR fixed point $\mathcal{T}^{IR}[\mathcal{O}]$.

6. It happens that depending on the choice of the original $\mathcal{T}$ and the specific nilpotent orbit $\mathcal{O}$, the IR theory $\mathcal{T}^{IR}[\mathcal{O}]$ can be a $\mathcal{N} = 2$ theory, therefore showing a phenomenon of supersymmetry enhancement.

The a-maximization check

In order to see whether the $\mathcal{T}^{IR}[\mathcal{O}]$ is really $\mathcal{N} = 2$ or not, many checks can be performed. The easiest one consists in computing its central charges a and c, and looking if they are rational or not. As discussed before, if a and c are not rational this implies that the theory is $\mathcal{N} = 1$, while if they are both rational the theory *could be* $\mathcal{N} = 2$. Remarkably, in many cases in which a and c are rational they also agree with the central charges of some known $\mathcal{N} = 2$ theories, therefore giving good evidence that the enhancement is real. Let us now discuss how to compute a and c of the IR theory $\mathcal{T}^{IR}[\mathcal{O}]$.

In the UV, the R-symmetry of the theory $\mathcal{T}$ is $SU(2)_r \times U(1)_r$. Let us call I_3 the Cartan of $SU(2)_R$ and r the generator of $U(1)_r$. Let us now define

$$J_+ = 2I_3, \quad J_- = r \tag{3.39}$$

We further define

$$R_0 = \frac{1}{2}(J_+ + J_-), \quad \mathcal{F} = \frac{1}{2}(J_+ - J_-) \tag{3.40}$$

After the introduction of the chiral multiplet M and after giving it a vev, the theory $\mathcal{T}^{UV}[\mathcal{O}]$ will be only $\mathcal{N} = 1$ supersymmetric, with a R-symmetry $R^{UV}_{\mathcal{N}=1}$ given by

$$R^{UV}_{\mathcal{N}=1} = R_0 + \frac{1}{3}\mathcal{F} \tag{3.41}$$

Here R_0 is a residual $U(1)$ from the broken $\mathcal{N}=2$ R-symmetry, and $\mathcal{F}$ is a global $U(1)$ symmetry of the $\mathcal{N}=1$ theory. We see that the actual $\mathcal{N}=1$ R-symmetry $R^{UV}_{\mathcal{N}=1}$ is given by a mixing of the two $U(1)$s. Crucially, along the RG flow the $\mathcal{N}=1$ R-symmetry will change, as the mixing between those $U(1)$s changes. We can then parametrize the $\mathcal{N}=1$ R-symmetry as

$$R_{\mathcal{N}=1} = R_0 + \epsilon\mathcal{F} \tag{3.42}$$

where ϵ is called the *mixing parameter*. At the IR fixed point we will have a new $\mathcal{N}=1$ R-symmmetry given by

$$R^{IR}_{\mathcal{N}=1} = R_0 + \epsilon^*\mathcal{F} \tag{3.43}$$

where ϵ^* is some specific number to be found.

It is known that ϵ^* will maximize the central charge a, now seen as a function of ϵ by plugging (3.43) into equation (3.32). This is called the *a-maximization technique* [70], and it will prove very useful to find the central charge a of $\mathcal{T}^{IR}[\mathcal{O}]$.

Let us consider this procedure in more details, in our case. The central charges of the undeformed UV theory $\mathcal{T}$ can be written in terms of the R-symmetry anomaly coefficients [61] by

$$\begin{aligned}
trJ_+ = trJ_+^3 &= 0\\
trJ_- = trJ_-^3 &= 48(a_\mathcal{T} - c_\mathcal{T})\\
trJ_+^2 J_- &= 8(2a_\mathcal{T} - c_\mathcal{T})\\
trJ_+ J_-^2 &= 0\\
trJ_- T^a T^a &= -\frac{k_F}{2}
\end{aligned} \tag{3.44}$$

where T^a are generators of the flavor symmetry and K_F is the flavor central charge. After the introduction of the $\mathcal{N}=1$ deformation, these relations will change because of two different contributions

1. The introduction of new Weyl fermions coming from the chiral multiplet M that we added forces us to recompute the anomaly coefficients. This is done by looking at what are the J_+ and J_- charges of all the fermions in M, after M has taken the vev, and then adding this contribution to (3.44). When M takes a vev along a nilpotent orbit $\mathcal{O}$ of $\mathfrak{f}$, this vev breaks the adjoint representation of $\mathfrak{f}$ into a direct sum of irreducible representations of the Jacobson-Morozov $\mathfrak{su}(2)$. Let j be the spin of such irrepses. Only components of M with minimal value of the third component of the spin will remain coupled to the theory after

M takes a vev: namely, we will have only $M_{j,-j}$. Each of this Weyl fermion has charges $(J_+, J_-) = (-1, 1+2j)$.

2. The introduction of the superpotential term $W = TrM\mu$ induces a shift in J_-, essentially to keep the superpotential with R-charges $(J_+, J_-) = (2, 2)$. The shift will be

$$J_+ \to J_+, \quad J_- \to J_- - 2\rho(\sigma_3) \tag{3.45}$$

This induces a shift in trJ_-^3 of the form

$$trJ_-^3 \to trJ_-^3 - 6k_F I_\rho \tag{3.46}$$

where I_ρ is the embedding index of the nilpotent orbit $\mathcal{O}$.

After performing a-maximization, it will sometime happen that some operators among the $M_{j,-j}$ or the coulomb branch operators u_i of the theory $\mathcal{T}^{UV}[\mathcal{O}]$ will get a conformal dimension such that they violate the conformal unitarity bound. whenever this happens the interpretation is that such operators became free at some point along the RG flow, and will decouple. The strategy is therefore to eliminate by hand their contributions to the trial central charge $a(\epsilon)$ and then re-compute a-maximization. This procedure needs to be iterated as long as no operators are seen anymore to violate the unitarity bound.

One fully worked example.

Consider the Argyres Douglas theory H_1 briefly discussed in a previous section[13] We will use this theory as the UV starting point $\mathcal{T}^{UV}$ for a Maruyoshi-Song flow. This is a rank 1 superconformal field theory, namely there is one single CB operator u, with dimension $\Delta(u) = \frac{4}{3}$. The central charges are

$$a = \frac{11}{24}, \quad c = \frac{1}{2}, \quad k_F = \frac{8}{3} \tag{3.47}$$

[13] En passant we notice that the same theory is called sometimes $(A_1, A_3) \simeq (A_1, D_3)$ in the context of generalized (G, G') Argyres-Douglas theories, or also $AD_{N_f=2}(SU(2))$ to emphasize that this theory can be found by going to a special point in the CB of $SU(2)$ with $N_f = 2$ and then performing a specific scaling limit of the SW curve.

and the flavor symmetry group is $F = SU(2)$. The anomaly coefficients for the undeformed theory can be computed from equation (3.44) and read

$$
\begin{aligned}
trJ_+ = trJ_+^3 &= 0 \\
trJ_- = trJ_-^3 &= -2 \\
trJ_+^2 J_- &= \frac{10}{3} \\
trJ_+ J_-^2 &= 0
\end{aligned}
\tag{3.48}
$$

We will now proceed to turn on the deformation. $\mathfrak{su}(2)$ has two nilpotent orbits, labeled by partitions of the number 2: the maximal orbit is labeled by the partition [2], and the trivial orbit is labeled by the partition $[1, 1]$. Let us choose the maximal orbit. It can be computed that under the Jacobson-Morozov embedding associated to this orbit, the adjoint of $\mathfrak{su}(2)$ will split as

$$
\text{Adj} \to V_1 \tag{3.49}
$$

where with V_1 we denote the $\mathfrak{su}(2)$ irrep of spin 1. Therefore we have a single extra chiral field $M_{1,-1}$, the fermion in this multiplet has $(J_+, J_-) = (-1, 3)$. Furthermore the embedding index for this orbit is $I_{[2]} = 1$. The anomaly coefficient will thus be modified as follows

$$
\begin{aligned}
trJ_+ = trJ_+^3 &= -1 \\
trJ_- &= 1 \\
trJ_-^3 &= 9 \\
trJ_+^2 J_- &= \frac{19}{3} \\
trJ_+ J_-^2 &= -9
\end{aligned}
\tag{3.50}
$$

The trial central charge will therefore read

$$
\begin{aligned}
a(\epsilon) &= \frac{3}{32}\left(3trR^3_{\mathcal{N}=1} - trR_{\mathcal{N}=1}\right) = \\
&= \frac{3}{32}\left[3tr\left(\frac{1}{2}(J_+ + J_-) + \frac{\epsilon}{2}(J_+ - J_-)\right)^3 - tr\left(\frac{1}{2}(J_+ + J_-) + \frac{\epsilon}{2}(J_+ - J_-)\right)\right] = \\
&= \frac{3}{32}\left[3tr\left(\frac{1+\epsilon}{2}J_+ \frac{1-\epsilon}{2}J_-\right)^3 - tr\left(\frac{1+\epsilon}{2}J_+ \frac{1-\epsilon}{2}J_-\right)\right] = \\
&= \frac{3}{32}\left[3\left(\frac{1+\epsilon}{2}\right)^3 trJ_+^3 + 9\left(\frac{1+\epsilon}{2}\right)^2\left(\frac{1-\epsilon}{2}\right)trJ_+^2J_- + 9\left(\frac{1+\epsilon}{2}\right)\left(\frac{1-\epsilon}{2}\right)^2 trJ_+J_-^2 + \right. \\
&\left. + 3\left(\frac{1-\epsilon}{2}\right)^3 trJ_-^3 - \left(\frac{1+\epsilon}{2}\right)trJ_+ - \left(\frac{1-\epsilon}{2}\right)trJ_-\right] = \\
&= \frac{3}{32}\left[-3\left(\frac{1+\epsilon}{2}\right)^3 + 9\frac{19}{3}\left(\frac{1+\epsilon}{2}\right)^2\left(\frac{1-\epsilon}{2}\right) - 81\left(\frac{1+\epsilon}{2}\right)\left(\frac{1-\epsilon}{2}\right)^2 + \right. \\
&\left. + 27\left(\frac{1-\epsilon}{2}\right)^3 + \left(\frac{1+\epsilon}{2}\right) - \left(\frac{1-\epsilon}{2}\right)\right]
\end{aligned}
\tag{3.51}
$$

By computing now $\dfrac{da(\epsilon)}{d\epsilon}$ and setting it to zero we find two extrema, namely

$$
\epsilon_1 = \frac{1}{21}\left(4 - \sqrt{65}\right), \quad \epsilon_2 = \frac{1}{21}\left(4 + \sqrt{65}\right) \tag{3.52}
$$

By a second derivative test we can easily see that ϵ_1 is a local minimum, and ϵ_2 is a local maximum. We are therefore tempted to say that ϵ_2 is the right mixing parameter for the R-symmetry in the IR. To see if this is consistent, we need to compute the dimensions of the operators in the IR and see if any of them has hit the unitarity bound (or not) along the RG flow.

We see that the original CB operator will violate the unitarity bound, while the $M_{1,-1}$ multiplet remains coupled. In more details, a $\mathcal{N} = 2$ CB multiplet can be decomposed into representations of the $\mathcal{N} = 1$ superconformal algebra[14] as

$$
\mathcal{E}_{r,(0,0)} \to \mathcal{B}_{\frac{1-\epsilon}{2}r,(0,0)} \oplus \mathcal{B}_{\frac{1-\epsilon}{2}r+2\epsilon,(0,0)} \oplus \mathcal{B}_{\frac{1-\epsilon}{2}r+\epsilon,(0,\frac{1}{2})} \oplus \mathcal{B}_{\frac{1-\epsilon}{2}r+\epsilon,(0,-\frac{1}{2})} \tag{3.53}
$$

for the original CB operator u we have that $r = 2\Delta(u) = \dfrac{8}{3}$ so in particular the IR R-symmetry charge of the operator $\mathcal{B}_{\frac{1-\epsilon}{2}r,(0,0)}$ will be $\dfrac{1-\epsilon_2}{2}\dfrac{8}{3} \sim 0,57$. Now the IR dimension for this operator

[14]We did not cover representation theory of the superconformal algebra in this thesis. For that we refer the reader to the appendices of [62]. In few words, here $\mathcal{E}_{r(j_1,j_2)}$ is a $\mathcal{N} = 2$ chiral multiplet with $U(1)_r$ charge r and quantum numbers (j_1, j_2) for the $S(2) \times SU(2)$ Lorentz group. Analogously $\mathcal{B}_{\mathcal{R}_{\mathcal{N}=1}(j_1,j_2)}$ is a $\mathcal{N} = 1$ short multiplet with $U(1)_{\mathcal{N}=1}$ charge r and quantum numbers (j_1, j_2) for the $S(2) \times SU(2)$ Lorentz group

is $\Delta^{IR}\left(\mathcal{B}_{\frac{1-\epsilon_2}{2}r,(0,0)}\right)=\frac{3}{2}\frac{1-\epsilon_2}{2}r\sim 0,85$ which is less then one. A similar check can be performed for $M_{1,-1}$ and we see that in the IR is will become coupled.

We therefore need to repeat the a-maximization with the contribution of the CB multiplet u removed. We define then a new trial central charge

$$a_{\text{new}}(\epsilon)=\frac{3}{32}\left(3trR^3_{\mathcal{N}=1}-trR_{\mathcal{N}=1}\right)-\frac{3}{32}\left(3trR^3_{\mathcal{N}=1,\text{removed}}-trR_{\mathcal{N}=1,\text{removed}}\right) \tag{3.54}$$

where the first term is the old trial central charge and in the second term the traces are done only over the fermions in the removed multiplets. In this case, only over the fermions in the removed CB multiplet.

We then again compute $\frac{da_{\text{new}}(\epsilon)}{d\epsilon}=0$ to find the maximum, and we find in this second case that

$$\epsilon^*=\frac{9}{15} \tag{3.55}$$

By checking again the IR dimension of $M_{1,-1}$ to see if it has decoupled or not, we find that it is above the unitarity bound. Therefore $M_{1,-1}$ remains coupled and the procedure terminates here. We find that the IR central charges are given by

$$a(\epsilon^*)=\frac{43}{120},\quad c(\epsilon^*)=\frac{11}{30} \tag{3.56}$$

and furthermore the dimension of the $M_{1,-1}$ operator is $\frac{6}{5}$.

Remarkably, we have found rational central charges in the IR. This is good evidence of supersymmetry enhancement to $\mathcal{N}=2$, as explained in the previous section. Even more remarkable, we see that the central charges so found and the dimension of the operator $M_{1,-1}$ perfectly match with the central charges and the dimension of the CB operator of the Argyres Douglas theory[15] H_0

To conclude this example, we briefly say what will change in a different case. Changing the original theory $\mathcal{T}$ will of course give different UV superconformal central charges to start with. Changing the orbit will change the decomposition of the adjoint of the flavor symmetry, generically giving more than one field $M_{j,-j}$ and also a different embedding index I_ρ. Apart from this, the computation in *any* case will be essentially analog to the one performed in detail in this section.

[15] En passant we notice that the same theory is called sometimes (A_1, A_2) in the context of generalized (G, G') Argyres-Douglas theories, or also $AD_{N_f=1}(SU(2))$ to emphasize that this theory can be found by going to a special point in the CB of $SU(2)$ with $N_f=1$ and then performing a specific scaling limit of the SW curve.

Another comment is the fact that this example was reported in detail just because it is the simplest. It is however possible to find cases in which the undeformed UV theory $\mathcal{N} = 2$ is lagrangian. Those are the most interesting cases as the UV lagrangian can be used to compute quantities which are invariant under the RG flow like the superconformal index, and therefore learn more about the non-lagrangian IR theory.

Therefore it is interesting to perform systematic scans by changing $\mathcal{T}$ and $\mathcal{O}$, in order to see whether new cases of supersymmetry enhancement are found. This is what we will do in the following.

3.5 New scans: looking for MS flows.

In the original papers [56–58] by Maruyoshi, Song, Agarwal and Sciarappa many scans over $\mathcal{T}$ and $\mathcal{O}$ are performed. We refer the reader to these papers to see the cases there discussed. We now report on the new scans we did which are not explicitly considered in the literature to this date. In order to perform the scans in an efficient way, a Mathematica script was created, which automatically performs the computation explained in the example of the previous section. The code for this script is reported in the appendices.

Minahan-Nemeschansky E_7 theory

We started from the Minahan-Nemeschansky E_7 theory [60]. We recall that such theory is a rank 1 superconformal field theory with flavor group $F = E_7$ and central charges given by

$$a = \frac{59}{24}, \quad c = \frac{19}{6}, \quad k_F = 8 \tag{3.57}$$

The unique CB operator has dimension $\Delta(u) = 4$. We checked all the possible deformations corresponding to all the 45 nilpotent orbits of $\mathfrak{e}_7$. We list all those orbits in appendix 7.7, together with their properties and the decomposition of the adjoint representation under the Jacobson-Morozov embedding. We find that there is enhancement only in the case of the maximal orbit (labeled by Bala-Carter label E_7) and the orbit labeled by Bala-Carter label E_6. In the first case the deformed theory flows in the IR to the H_0 AD theory, and in the second case it flows to the H_1 theory. The latter case is a new result: as far as we know, so far nowhere in the literature is stated that the E_7 theory deformed with some non-maximal orbit, flows to H_1.

Minahan-Nemeschansky E_8 theory

We started from the Minahan-Nemeschansky E_7 theory [60]. We recall that such theory is a rank 1 superconformal field theory with flavor group $F = E_8$ and central charges given by

$$a = \frac{95}{24}, \quad c = \frac{31}{6}, \quad k_F = 12 \tag{3.58}$$

The unique CB operator has dimension $\Delta(u) = 6$. We checked all the possible deformations corresponding to all the 70 nilpotent orbits of $\mathfrak{e}_8$. We list all those orbits in the appendix 7.7, together with their properties and the decomposition of the adjoint representation under the Jacobson-Morozov embedding. We find that there is enhancement only in the case of the maximal orbit (labeled by Bala-Carter label E_8) and the deformed theory flows to the H_0 theory. This was the particular case found in the original papers by Maruyoshi, Song and collaborators. Therefore by our analysis we show that this is the unique case in which the enhacement happens.

D-type and E-type shaped quivers

We considered superconformal quivers of D-type and E-type, analog to those drawn in figures (3.2) and (3.3). In particular we considered all quivers of shape E_6, E_7, E_8, D_4, D_5, D_6 with nodes of ranks smaller or equal than 5 and we scanned over all nilpotent orbits for each case. For this class of theories supersymmetry enhancement has never been found. This leads us to conjecture that in general superconformal quivers of this type will not show supersymmetry enhancement. This is somewhat interesting because superconformal *linear* quivers (both with SU gauge nodes and also alternating $SO - Sp$ gauge nodes) will show supersymmetry enhancement, as found in [58], at least always for the maximal orbit of the flavor symmetry. Up to date there is no complete understanding about why the D and E class of examples behave differently. One conjecture could be that superconformal quivers of D and E type clearly do not belong to class-S, and for some unknown reason supersymmetry enhancement driven by this kind of deformation can only exist if the UV theory $\mathcal{T}$ belongs to class-S. We leave this direction for further investigation.

3.6 A geometric picture for the enhancement.

In this section we want to discuss a geometrical way to understand the phenomenon of supersymmetry enhancement. The strategy will be engineer the UV $\mathcal{N} = 2$ theory and also the $\mathcal{N} = 1$ nilpotent deformation in the context of F-Theory.

In particular, consider a setup in which F-Theory is compactified on a local elliptically fibered and singular $K3$ surface. We further place a $D3$ brane probing such singularity and we investigate on the worldvolume theory living on the D3 brane, which will be the $\mathcal{N} = 2$ supersymmetric theory we will use as a starting point for the Maruyoshi-Song flow. A crucial point is that in this context the Seiberg-Witten curve of the field theory is identified with the fiber of the elliptic fibration, however typically in the field theory description the SW curve and the SW differential are written in a different coordinate base, and a change of variable is needed to recast the SW curve in Weierstrass form. Complex codimension one singularities of elliptic fibrations are completely classified by Kodaira, and are given in table 2.3.4. In this context, we restrict to the subset of singularities reported in table (3.6), which will lead to a series of field theories of rank 1, living in the worldvolume of the D3 probe. In (3.6) table we list the Weierstrass model for the singularity, as well as the gauge group living on the locus in which the fibration is singular, which from the probe point of view will be perceived as a flavor symmetry group.

Singularity	Curve	Flavor group
II^*	$u^2 = v^3 + v(M_2 z^3 + M_8 z^2 + M_{14} z + M_{20}) + (z^5 + M_{12} z^3 + M_{18} z^2 + M_{24} z + M_{30})$	E_8
III^*	$u^2 = v^3 + v(z^3 + M_8 z + M_{12}) + (M_2 z^4 + M_6 z^3 + M_{10} z^2 + M_{14} z + M_{18})$	E_7
IV^*	$u^2 = v^3 + v(M_2 z^2 + M_5 z + M_8) + (z^4 + M_6 z^2 + M_9 z + M_{12})$	E_6
I_0^*	$u^2 = v^3 + v(\tau z^2 + M_2 z + M_4) + (z^3 + \tilde{M}_4 z + M_6)$	$SO(8)$
IV	$u^2 = v^3 + v(M_{1/2} z + M_2) + (z^2 + M_3)$	$SU(3)$
III	$u^2 = v^3 + vz + (M_{2/3} v + M_2)$	$SU(2)$
II	$u^2 = v^3 + vM_{4/5} + z$	no

Table 3.1: Maximally deformed singularities

Here z is the complex coordinate on the base of the elliptic fibration, which from the probe point of view is the vev of the unique Coulomb branch operator. u and v are complex coordinates in the fiber of the elliptic fibration, and M_i are casimir invariants corresponding to versal deformations [73] of the Weierstrass model. When all the casimir invariants are turned off, this correspond to a non-deformed singularity. In IIB language, this case correspond to some stack of seven branes. When the casimirs are turned on, they will give versal deformations of the singularity, which in IIB language can be thought of stacks of T-branes of 7-branes.

The idea will be that fixing the UV theory $\mathcal{T}$ with flavor F and the corresponding Weierstrass model, there is a one to one correspondence between a given nilpotent obit of $\mathfrak{f}$ and the set of casimir operators turned on in the deformed model. So we see how the deformation of the 4d field theory is translated in the deformation of the geometry of the elliptically fibered K3.

The RG flow is interpreted as a local zoom in the neighborhood of the singular point. This is physically very intuitive, as the D3 is a probe in some background, and when the probe has very little energy it cannot resolve global aspects of the singularity, but only local ones.

Remarkably, we will see that in a local neighborhood of the singularity, some terms in the deformed Weierstrass model will drop because they are subleading compared to other ones. The end result will be that we will find the undeformed Weierstrass model for *another* singularity, which is the one corresponding to the IR $\mathcal{N} = 2$ theory, in the cases in which the enhancement is present.

In figure (3.4) we show all the different flows with SUSY enhancement that exist among rank 1 superconformal field theories which can be engineered in F-Theory. All those flows can be understood geometrically by the procedure outlined above. We will now discuss two of the easiest examples. The full study is included in [5].

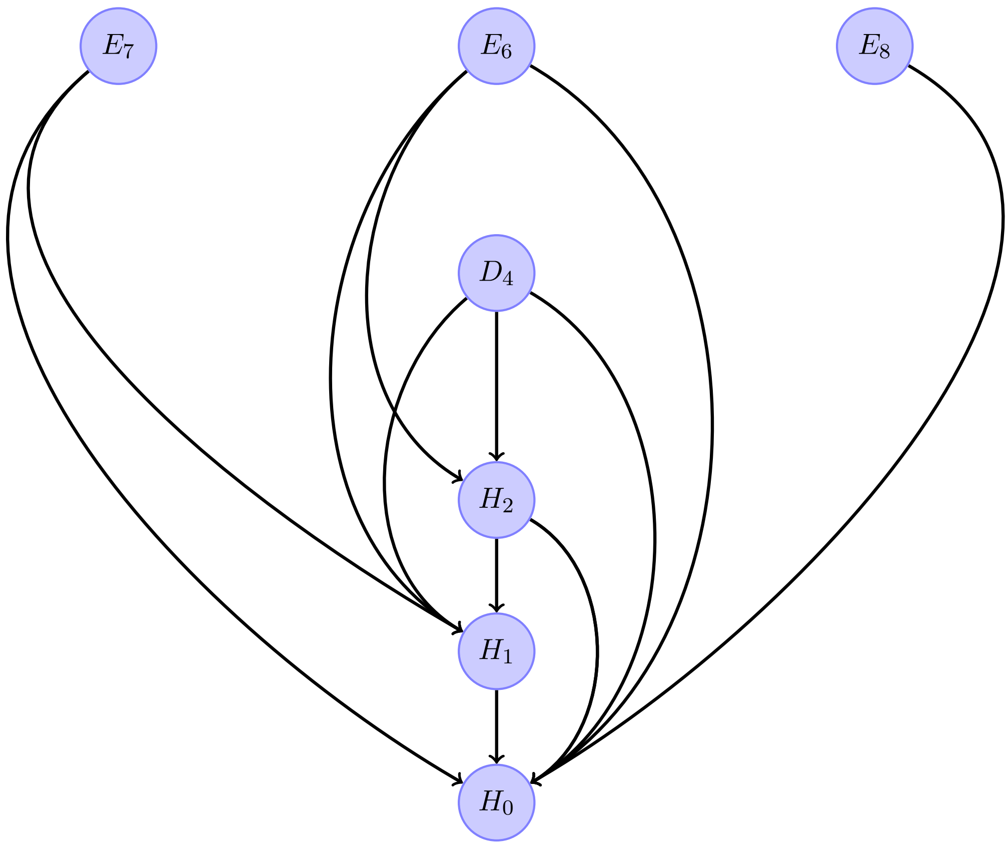


Figure 3.4: A picture of the enhancing flows for rank 1 theories

Flow starting from H_2

We will now give an example in which we can see supersymmetry enhancement from the F-Theory setup. Consider the AD theory H_2, realized by one $D3$ brane probing the Kodaira singularity H_2. The Weierstrass model for the fibration is

$$u^2 = v^3 + z^2 \tag{3.59}$$

The flavor symmetry of this theory is $SU(3)$, as can be seen from Kodaira's table. The nilpotent orbits of the complexified Lie algebra $\mathfrak{sl}(3, \mathbb{C})$ of the flavor symmetry are in one to one correspondence with partitions of the number [3]. Namely we will have three orbits $\mathcal{O}_{[3]}, \mathcal{O}_{[2,1]}, \mathcal{O}_{[1,1,1]}$.

Maximal orbit

The maximal orbit of $\mathfrak{sl}(3, \mathbb{C})$ corresponds to the partition [3]. The Jacobson-Morozov triple will be given by

$$X = \begin{pmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \end{pmatrix}, \qquad H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \tag{3.60}$$

The adjoint representation splits as $V_1 \oplus V_2$. So we will have one lowest weight of spin 1 and one lowest weight of spin 2. Let us identify all of them explicitly. Call $E_{ij} = e_i \otimes e_j$ the matrix which has entry 1 in position (i, j) and entries 0 everywhere else. Then a generic matrix of $\mathfrak{sl}(3, \mathbb{C})$ will be parametrized as

$$A = \sum_{i=1}^{8} a_i A_i \tag{3.61}$$

where we take the generators to be defined as

$$A_1 = E_{11} - E_{33} \tag{3.62a}$$

$$A_2 = E_{22} - E_{33} \tag{3.62b}$$

$$A_3 = E_{12} \tag{3.62c}$$

$$A_4 = E_{13} \tag{3.62d}$$

$$A_5 = E_{21} \tag{3.62e}$$

$$A_6 = E_{23} \tag{3.62f}$$

$$A_7 = E_{31} \tag{3.62g}$$

$$A_8 = E_{32} \tag{3.62h}$$

$$\tag{3.62i}$$

By computing all the commutators of the form

$$[H, A_i] = 2b_i A_i \tag{3.63}$$

we can find all the weights b_i that the eight linear independent matrices A_i have under H. The result we find is summarized in table (3.6).

State A_i	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
Weight b_i	0	0	1	2	-1	1	-2	-1

Table 3.2: Weights under H for the states corresponding to the maximal orbit of $\mathfrak{sl}_3$

We see that there is only one state of weight $b_7 = -2$ so A_7 will be the lowest weight of the irreps V_2. Indeed we can correctly check that $[Y, A_7] = 0$ Now we need to identify the lowest weight of the V_1 irreps. We see that we have two states, A_5 and A_8 with weight $b_5 = b_8 = -1$. By computing $[Y, A_5]$ and $[Y, A_8]$ we see that neither of them gets annihilated. Therefore we can conclude that one linear combination of them is the lowest weight of V_1 and another linear combination will be the state with weight -1 in the V_2 representation. It is easy to check that the lowest weight of V_1 will be $A_5 + A_8$.

Let us write explicitly the lowest weight we find. Call $M_{j,-j}$ the lowest weight of V_j. We then have

$$M_{2,-2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad M_{1,-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{3.64}$$

According to [72] those are the only components remaining after we give a nilpotent vev along this orbit. Therefore it is sufficient to choose a vev for M of the form

$$M = \begin{pmatrix} 0 & m\sqrt{2} & 0 \\ y_1 & 0 & m\sqrt{2} \\ y_2 & y_1 & 0 \end{pmatrix} \tag{3.65}$$

The characteristic polynomial reads

$$P(z) = 2m^2 y_2 + 2\sqrt{2} m y_1 z - z^3 \tag{3.66}$$

So we see that taking only the leading order in m gives $2m^2y_2$ which corresponds to the casimir M_2. The modified Weierstrass model, corresponding to a T-brane background, will be then given by

$$u^2 = v^3 + z^2 + m^3 y_2 \tag{3.67}$$

Now in a local neighborhood of the singularity, the z^2 term will drop as it is subleading compared to the m^3y_2 term, and we end up with

$$u^2 = v^3 + y_2 \tag{3.68}$$

which we recognize as the Weierstrass model for H_0 theory, where now the role of the new CB operator is being played by y_2.

A similar procedure can be performed for the subregular orbit. In that case, we find that the H_2 theory deformed by $\mathcal{O}_{[2,1]}$ flows to H_1. We will omit details of this latter straightforward computation, and refer the reader to [5].

CHAPTER 4

MIXED BRANCHES OF 3D $\mathcal{N} = 4$ QFTS

4.1 3d field theories with 8 supercharges: generalities.

In this section we review some of the properties of 3d $\mathcal{N} = 4$ QFTs, often putting an emphasis on those features which will differ from the analog 4d $\mathcal{N} = 2$ case. The treatment will be schematic. For a more complete treatment see for example [123].

Supersymmetry algebra

The Poincaré group in three dimension is $\mathbb{R}^3 \rtimes SO(1,2)$. In the following we will always assume a Minkowski metric with signature $(-,+,+)$. The Clifford algebra in three dimension is defined by

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu} \tag{4.1}$$

where $\eta_{\mu\nu}$ is Minkowski metric. For the choice of gamma matrices we will follow the convention of [65], namely

$$\gamma_1 = i\sigma_2, \quad \gamma_2 = \sigma_3, \quad \gamma_3 = \sigma_1 \tag{4.2}$$

where the three σ_i are the usual Pauli matrices. The $\mathcal{N} = 4$ susy algebra can be written in term of four real supercharges as

$$\{Q^i{}_\alpha, Q^j{}_\beta\} = 2\delta^{ij}\gamma^\mu_{\alpha\beta}P_\mu + 2i\epsilon_{\alpha\beta}Z^{ij} \tag{4.3}$$

where Z^{ij} is a real antisymmetric matrix of central charges. We can immediately see that there is an $SO(4)$ automorphism of this algebra rotating the four supercharges. It is useful to think of this R-symmetry as

$$SO(4) \simeq SU(2)_R \times SU(2)_L \tag{4.4}$$

Multiplets and Lagrangians

We will now introduce the irreducible representations of the 3d $\mathcal{N} = 4$ supersymmetric algebra, and the most general renormalizable 3d $\mathcal{N} = 4$ lagrangian. We will however write everything in terms of a lagrangian which is only manifestly supersymmetric under a specific $\mathcal{N} = 2$ subalgebra. The discussion is almost completely analog to the familiar case of 4d $\mathcal{N} = 2$ lagrangians written in terms of 4d $\mathcal{N} = 1$ chiral and vector superfields that we discussed in the previous chapter.

To start it is useful to first recall irreducible representations of the 3d $\mathcal{N} = 2$ algebra. Define the superspace covariant derivatives as

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial \theta_\alpha} - i(\gamma^\mu \bar{\theta})_\alpha \partial_\mu \\ \bar{D}_\alpha &= -\frac{\partial}{\partial \bar{\theta}_\alpha} + i(\gamma^\mu \theta)_\alpha \partial_\mu \end{aligned} \tag{4.5}$$

A chiral superfield Φ (resp antichiral superfield $\bar{\Phi}$) is defined as spin zero superfield such that $\bar{D}_\alpha \Phi = 0$ (resp $D_\alpha \Phi = 0$).

A vector superfield is defined as a spin zero superfield V such that $V = V^\dagger$. When vector fields are used as gauge fields, they will be always valued in the adjoint representation of the gauge Lie algebra. Its bosonic components (in Wess-Zumino gauge) contain a scalar field σ and a vector field A^a_μ.

The field strength multiplet is defined as

$$W_\alpha = \frac{1}{4} \bar{D} \bar{D} e^{-V} D_\alpha V \tag{4.6}$$

This superfield contains the field strength tensor $F^a_{\mu\nu}$ as well as the gaugino.

In three dimension there is another superfield which contains $F^a_{\mu\nu}$, namely the linear superfield Σ. This is defined by the three conditions

$$\Sigma = \Sigma^\dagger, \quad \Sigma = \bar{D}^\alpha D_\alpha V, \quad D^\alpha D_\alpha V = \bar{D}^\alpha \bar{D}_\alpha V = 0 \tag{4.7}$$

This completes the list of the basic 3d $\mathcal{N} = 2$ superfields we will need in this thesis. Let us focus now on 3d $\mathcal{N} = 4$ superfields. The 3d $\mathcal{N} = 4$ vector multiplet is given by the set of a 3d $\mathcal{N}=2$ vector superfield V and a 3d $\mathcal{N}=2$ chiral superfield Φ. Since Φ is in the same multiplet of V, it must be also in the same gauge representation. So Φ is adjoint valued. The 3d $\mathcal{N}=4$ hypermultiplet in a gauge representation $\mathcal{R}$ consists of two chiral multiplets Φ and $\tilde{\Phi}$ in conjugated gauge representations $\mathcal{R}$ and $\bar{\mathcal{R}}$.

We will move now in writing the generic form for 3d $\mathcal{N}=4$ lagrangians. We point out immediately that we will omit completely the discussion about Chern-Simons terms in the lagrangians, are we will never need to include them in this thesis. We will also always give all expression in terms of the superfields. It is obviously possible to express everything in terms of components, exactly as in the standard 4d $\mathcal{N}=1$ case, but we will refrain from do that explicitly.

The SUSY actions for $\mathcal{N}=4$ theories will be given by

$$\begin{aligned} S = S_{\text{kin, chirals}} + S_{\text{YM}} + S_{\text{real mass}} + S_{\text{real FI}} + \\ + S_{\text{kin},\Phi} + S_{\text{superpotential}} + S_{\text{complex FI}} + S_{\text{complex mass}} \end{aligned} \tag{4.8}$$

where each term will be defined in the following. Here terms in the first line will be present regardless if the theory is $\mathcal{N}=2$ or $\mathcal{N}=4$. Terms in the second line will be explicitly related to $\mathcal{N}=4$ theory. For example, as we will see the superpotential in $\mathcal{N}=4$ theories has a fixed form, and Fayet-Ilioupulos parameters (resp mass parameters) must come in triplets which can be regarded as formed by one real FI (resp mass) and a complex FI (resp mass).

Let us look in more details at all the terms. They are given by

$$S_{\text{kin, chirals}} = -\int d^3x d^2\theta d^2\bar{\theta} \sum_i \left(\Phi_i^\dagger e^V \Phi_i\right) \tag{4.9}$$

where here the sum is extended over all the chiral superfields Φ_i of the theory which belong to hypermultiplets. In particular the sum does not include the adjoint chiral in the $\mathcal{N}=4$ vector multiplet. The following term S_{YM} is the kinetic term for the vector superfield. It shows novelty compared to the 4d $\mathcal{N}=1$ formalism: there are two equivalent ways of writing it, depending on our choice to use linear multiplets or field strength multiplets.

$$\begin{aligned} S_{\text{YM}} &= \frac{1}{g^2}\int d^3x d^2\theta d^2\bar{\theta}\ \frac{1}{4} Tr\left(\Sigma^2\right) = \\ &= \frac{1}{g^2}\int d^3x d^2\theta\ Tr W_\alpha^2 + h.c. \end{aligned} \tag{4.10}$$

The following term $S_{\text{real mass}}$ can be thought as a mass term for the chiral fields. We can add such term by considering the mass to be the scalar field of a background vector superfield. Therefore we will have

$$S_{\text{real mass}} = -\int d^3x d^2\theta d^2\bar{\theta}\ \sum_i \left(\Phi_i^\dagger e^{\theta\bar{\theta} m} \Phi_i\right) \tag{4.11}$$

The following term $S_{\text{real FI}}$ is a real Fayet-Ilipoulos term, completely analog to the one entering in usual 4d $\mathcal{N}=1$ actions. It is only possible to add one of such terms for every $U(1)$ factor of the gauge group.

$$S_{\text{real FI}} = \int d^3x d^2\theta d^2\bar{\theta}\ \xi V \tag{4.12}$$

The following term $S_{\text{kin},\Phi}$ is a kinetic term for the chiral Φ_{adj} in the adjoint of the $\mathcal{N}=4$ vector multiplet. We chose not to include it in (4.9) as its normalization is different. It will be

$$S_{\text{kin},\Phi} = -\frac{1}{g^2}\int d^3x d^2\theta d^2\bar{\theta}\ \Phi^\dagger_{adj} e^V \Phi_{\text{adj}} \tag{4.13}$$

The following term $S_{\text{superpotential}}$ is a superpotential term which is requested by $\mathcal{N}=4$ supersymmetry.

$$S_{\text{superpotential}} = -i\sqrt{2}\int d^3x d^2\theta \sum_i \left(\tilde{\Phi}_i \Phi_{\text{adj}} \Phi_i\right) + h.c. \tag{4.14}$$

Finally we will have a look at $S_{\text{complex FI}}$ and $S_{\text{complex mass}}$. The complex FI parameter can be thought as the scalar in a background chiral multiplet inside of a hypermultiplet. On the other hand a complex mass can be thought as the scalar in a background chiral multiplet inside of the vector multiplet. Therefore both of them enter in the lagrangian in a way similar to equation (4.14), namely

$$S_{\text{complex FI}} = \xi_{\mathbb{C}} \int d^3x d^2\theta \Phi_{\text{adj}} + h.c. \tag{4.15}$$

$$S_{\text{complex mass}} = \int d^3x d^2\theta \sum_i \left(\tilde{\Phi}_i m_{\mathbb{C}} \Phi_i\right) + h.c. \tag{4.16}$$

This concludes the description about $\mathcal{N}=4$ lagrangians.

Dualization of the photon

A crucial difference between 2+1 field theories and 3+1 field theories is that in three dimensions vector fields are dual to scalar fields. This does not hold in 4d.

A first insight that in three dimensions the vector field can be dual to the scalar is given by the observation that the on-shell degrees of freedom (polarizations) of a vector in d dimensions are $d-2$. Thus in 3d the scalar and the vector have the same number of degrees of freedom. Let us look at the dualization in more details.

Consider a free vector A_μ in three dimensions and call $F_{\mu\nu}$ it's field strength. The duality to a free scalar is given by the relation

$$F_{\mu\nu} = \epsilon_{\mu\nu\rho}\partial^\rho \gamma \tag{4.17}$$

It is possible to show this dualization explicitly from the path-integral.

The statement of above also implies the following: $F = dA$ is a two form. By Bianchi identity we then have

$$d * F = 0 \tag{4.18}$$

therefore implying that the 1-form current $*F$ is conserved. By Noether theorem we can then argue that the theory has a extra $U(1)_J$ symmetry, which is not manifest at the level of the lagrangian. Such $U(1)_J$ is called *topological symmetry* and acts on γ by shifts, therefore γ is a periodic scalar. This dualization extends to a full vector multiplet: it is dual to a chiral multiplet whose lowest component is

$$\Phi|_{\theta=0} = \sigma + i\gamma \tag{4.19}$$

Anomalies

In even dimensions there can be local gauge anomalies. The path integral for a QFT with charged chiral fermions can be anomalous due to the failure of the integration measure to be invariant under a chiral rotation of the fermions. No such anomalies are present in 2+1 dimensions due to the non-existence of chirality in odd dimensions.

However, a similar phenomenon can in principle happen when we consider the variation of the one loop determinant of a charged fermion under a large gauge transformation. It has been shown that such determinant is not invariant but its anomalous transformation can be cured by the suitable addition of a Chern-Simons term in the theory. Such anomaly is called "parity anomaly". It is automatically canceled in theories with 8 supercharges as charged matter comes in hypermultiplet which contain chirals in conjugate representations. Therefore, we will not care about this anomaly in this thesis.

Solitons: the case of the monopoles

In three dimensions there are codimension three solitons. We will call them *monopoles*[1] following the general nomenclature rule for which codimension 4 solitons are instantons, codimension 3 are monopoles, codimension 2 are vortices and codimension 1 are domain walls.

It is important to stress right now that in half of the literature another naming convention is used, and these codimension 3 solitons in 3d are instead called instantons. The reason for this is that such solitons share some properties with 4d 't Hooft-Polyakov monopoles [18, 19]

[1] Not to be confused with the monopole operators which we will introduce later.

(being both codimension 3 objects) but also share some features with 4d instantons (being both maximal codimension objects). We will explain this fact in more details.

As explained in section 2.2.5 monopoles in a QFT with spontaneous symmetry breaking are classified by $\pi_2(G/H)$, the second homotopy group of the broken part of the gauge group. Here G is the gauge group before spontaneous symmetry breaking, and H is the gauge group after it. As we will review later in more depth, the moduli space of $4d$ $\mathcal{N}=4$ theories has one branch called Coulomb Branch, where the gauge group G is broken to its maximal torus $U(1)^r$. Therefore there will be monopoles if $\pi_2(G/U(1)^r)$ is nonvanishing. By using the fact that $\pi_2(G)=0$ for any simple Lie group G and the exact sequence technique explained in section 2.2.5 it is easy to prove that

$$\pi_2(G/U(1)^r) \simeq \mathbb{Z}^r \tag{4.20}$$

so there will always be monopoles when the theory is in a Coulomb branch vacuum. These 3d monopole solutions are exactly identical to the 't Hooft-Polyakov solutions in 4d for the same G broken to $U(1)^r$, where a dimensional reduction is performed by ignoring the time coordinate of the 4d solution.

On the other hand, being a maximal codimension soliton, those monopole solutions are also finite energy solutions of the Euclidean equations of motion of the QFT in the Coulomb branch (analog to 4d instantons). In the path integral, they will be weighted semiclassically

$$e^{-\sigma\cdot\beta_i/g^2} \tag{4.21}$$

where σ is the scalar in the adjoint of the vector multiplet and β_i are roots of the Lie algebra of the gauge group. In the case of theories with 8 supercharges such monopole contributions to the path integral will correct the Coulomb branch metric, but not change the dimension of the Coulomb branch. The same statement does not hold true in cases with less supersymmetry, in which monopoles generate a superpotential which typically lifts some directions of the moduli space, decreasing its dimension.

Monopole operators

A very important class of operators which are present in $4d$ $\mathcal{N}=4$ supersymmetric gauge theories are *Monopole operators*. Such operators are disorder operators, analog to twist operators in 2d CFTs or 't Hooft lines in 4d QFTs.

A bare monopole operator $V_m(x)$ is defined by a boundary condition in the Euclidean path integral, by requiring that the set of gauge connections onto which the path integral is performed will be restricted to a set of connections having a Dirac monopole's singularity (specified by an embedding $U(1) \mapsto G$) at the insertion point x. Namely

$$A_{\pm} \sim \frac{m}{2}(\pm 1 - \cos\theta) d\varphi, \tag{4.22}$$

where spherical coordinates (r, θ, φ) are used and $A_{\pm}$ is the gauge connection on the northern (respectively southern) hemisphere of a sphere S^2 surrounding the insertion point x. Here m is the magnetic charge of the monopole operator, which takes values in the weight lattice of the Langlands (GNO) dual group ${}^L G$ [133], and satisfies a Dirac quantization condition [134]

$$\exp(2\pi i m) = 1_G. \tag{4.23}$$

In order for a monopole operator to be BPS, the scalar in the vector multiplet must also have a singularity corresponding to (4.22) such that

$$d\sigma = \star F \tag{4.24}$$

Notice that this scalar σ can be any of the three real scalars present in the vector multiplet (before dualization of the photon). This amounts of saying that we are picking a specific $\mathcal{N} = 2$ subalgebra at this stage. From now on, we will only focus on BPS monopole operators, often dropping the "BPS" term. A (BPS) bare monopole operator carries a magnetic charge m, defined as the flux of the gauge field through a sphere surrounding the insertion point of the monopole operator. It also has a conformal dimension, determined by its IR R-charge. Such conformal dimension of a bare monopole operator is given in terms of the magnetic charge by the following dimension formula [96, 103, 107–109][2]

$$\Delta(m) = -\sum_{\alpha \in \Delta^+} |\alpha(m)| + \frac{1}{2} \sum_i \sum_{\rho_i} |\rho_i(m)|, \tag{4.25}$$

where α are the positive roots of the gauge algebra, and ρ_i are the weights of the matter representations.

Solitons: the case of the vortices

In three dimensions there are also codimension two solitons. We will call them Abrikosov-Nielsen-Olesen *vortices* following the general nomenclature rule for which codimension 4 solitons

[2]The dimension formula for a monopole operator will be valid when the UV $U(1)_R$ symmetry is equal to the IR R-symmetry. Those theories are called "good" or "ugly" in [107]. The reason for which the dimension formula does not always work is that in the case of "bad" theories it will give as an output some conformal dimension which explicitly violates the unitarity bound.

are instantons, codimension 3 are monopoles, codimension 2 are vortices and codimension 1 are domain walls. Such vortex solutions are present in Higgs branch vacua. In some cases these vortices configurations are BPS. As an example of BPS vortices, consider a $\mathcal{N}=2$ $U(1)$ theory with N_f fundamental hypers with a non-zero real FI parameter $\xi_{\mathbb{R}}$. There are vacua in which only one flavor gets expectation value and in such vacua there are vortex configurations of the form

$$\begin{aligned} \varphi &= \sqrt{\xi_{\mathbb{R}}}e^{\pm i\theta} \\ A_\theta &= \pm\frac{1}{r} \end{aligned} \tag{4.26}$$

The central charge Z (and therefore the vortex mass) is equal to the $\pm\xi_{\mathbb{R}}$. When the FI $\xi_{\mathbb{R}}$ is set to be zero, the vacuum in which only one flavor gets expectation value will be the origin of the Coulomb branch, and we see that there are massless vortices there.

4.2 Quiver gauge theories

A quiver is a special type of oriented graph, which can be defined formally by the four objects $\{V, A, s, e\}$. Here V and A are sets, and $s : A \to V$ and $e : A \to V$ are maps. V is called the *set of vertices* (or nodes) of the quiver, and A is called the *set of arrows*. Every arrow $a \in A$ must start in a node and end in a node (which may also be the same node it started from), therefore the two maps s and e will be defined such that for every arrow $a \in A$, $s(a)$ is the node it started from and $e(a)$ is the note in which it ends. Pictorially it is possible to draw the quiver, for example see figure 4.1.

There are many uses of quivers in supersymmetric field theory and string theory, as an example, the BPS spectrum of many $4d$ $\mathcal{N}=2$ QFTs can be encoded in a BPS quiver [74–79]. Another use of the quivers is in the context of the study of singularities in toric varieties and D-branes probing such singularities [80–83].

Here, we will mainly use quiver graphs to denote in a simple way the matter content of a lagrangian QFT. Whenever this is possible, we will say that we are dealing with a Quiver Gauge Theory. Let us review now the easy dictionary to *read off* the matter content from a quiver. There are two versions of such dictionary, depending if we are working with a theory with 4 or 8 supercharges.

Consider first the case of 4 supercharges. Every round node of the quiver represents a factor of the gauge group with a therefore associated vector multiplet, and every arrow $a \in A$

represents a chiral superfield charged in the fundamental representation of the $s(a)$ group, and the antifundamental representation of the $e(a)$ group. With this dictionary one can read off very easily the matter content, just by looking at the quiver. Square nodes correspond to global symmetries (flavor). As an example, consider the quiver drown in the figure (4.1), which correspond to $U(2)$ with 4 flavors. Such quiver corresponds to the matter assignment explicitly written in table (4.1).

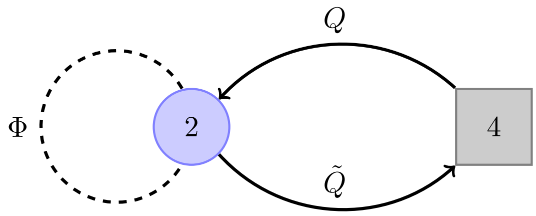


Figure 4.1: The quiver graph for the $U(2)$ gauge theory with 4 flavors, written in a "4-supercharges notation". $Q, \tilde{Q}, \Phi$ are 3d $\mathcal{N} = 2$ chiral multiplets.

	$U(2)_g$		
	$U(1)_g$	$SU(2)_g$	$SU(4)_f$
Q	w_1	$[1]_{w_2}$	$[0,0,1]_{x_1,x_2,x_3}$
$\tilde{Q}$	w_1^{-1}	$[1]_{w_2}$	$[1,0,0]_{x_1,x_2,x_3}$
Φ	1	$[2]_{w_2}$	1

Table 4.1: The charge assignments for the theory of figure 4.1.

The case in which the theory has 8 supercharges is the following. Every node of the quiver still represents a factor of the gauge group. However, the vector multiplet in a theory with 8 supercharges will include also the chiral in the adjoint representation. Each hypermultiplet will now contain a pair of chiral multiplets in conjugated representations of the gauge group. Therefore for every arrow in the quiver there must be an arrow in the opposite direction. It is customary to write a single unoriented line instead of the two arrows. Furthermore, for every node gauge node there must be an arrow starting and ending on it, corresponding to the chiral in the adjoint. It is customary not to draw this arrow and implicitly remember that it must be there. As an example, consider again the the quiver drown in the figure (4.1). Such quiver is actually $\mathcal{N} = 4$ and therefore can be written in $\mathcal{N} = 4$ notation, corresponding to the quiver drawn in figure (4.2) and exactly the same matter assignment of table (4.1).

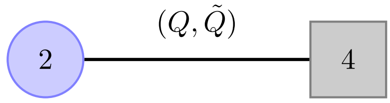


Figure 4.2: The quiver graph for the $U(2)$ gauge theory with 4 flavors written in a "8-supercharges notation". $(Q, \tilde{Q})$ is a 3d $\mathcal{N} = 4$ hypermuliplet.

In order to fully determine the lagrangian of a supersymmetric theory it is not enough to simply know the matter content of the theory: one should also declare which is the Kähler potential and the superpotential. In this context it is common practice to take a canonical Kähler potential.

Regarding the superpotential, if the theory has only 4 supercharges then the superpotential can be chosen at will.[3] However, if the theory has 8 supercharges (or more) then the superpotential will be essentially fixed by the matter assignment. Therefore in the case of 8 supercharges we can say that the quiver itself essentially fixes completely the lagrangian of the theory.

4.3 A look at the moduli space

For a generic field theory, the moduli space $\mathcal{M}$ is defined as a set of gauge-inequivalent vacua. Each point of $\mathcal{M}$ is labeled by vacuum expectation values (vevs) of a set of scalar fields of the theory, which therefore give coordinates on $\mathcal{M}$. Determining which kind of variety $\mathcal{M}$ is (i.e. dimension, topology, possible singularities, metric tensor on top of it, etc) is therefore the first question one should address when studying a quantum field theory, and one of the most basic: understanding its vacuum structure. For non-supersymmetric field theories the moduli space typically consists in a variety of dimension zero: just a set of points.[4] The reason for this is that massless scalars are not protected in the quantum theory: quantum corrections will generically give them a mass. In the case of supersymmetric field theories, instead, it can happen that massless scalar fields don't have any potential, therefore their vacuum expectation value can be chosen at will, and the moduli space is not zero-dimensional. For this reason, from now on we will restrict to the case of Supersymmetric field theories. Let us now review a fruitful strategy to study a moduli space, given some supersymmetric QFT as an input.

As a starting point, it is easy to check that if X, Y are chiral superfields, and therefore

[3] Provided that some obvious constraints are satisfied. For example, it should have R-charge 2.

[4] The moduli space of Standard Model, for example, is just one point.

by definition $D_\alpha X = D_\alpha Y = 0$, then also $X + Y$ and XY are chiral superfields. The set of chiral fields has then a natural structure of a ring which we call *chiral ring* $\mathcal{R}$. Such chiral ring is believed to be isomorphic to the ring $\mathcal{O}$ of holomorphic functions defined over $\mathcal{M}$. In particular, we can associate a holomorphic function on the moduli space $\mathcal{M}$ to every element in $\mathcal{R}$. Now, if the ring of holomorphic functions on an unknown algebraic variety $\mathcal{M}$ is known, one can reconstruct and define $\mathcal{M}$ via usual techniques in algebraic geometry. Namely, $\mathcal{M}$ will be defined as a scheme locally isomorphic to the spectrum of the ring $\mathcal{R}$, with Zariski's topology.

While in principle this strategy will work, it is in general hard to explicitly determine and identify all the elements of the chiral ring (or equivalently all the holomorphic functions on the moduli space), so one settles down to a more modest approach of simply counting chiral operators, grading them by their charges under all the symmetries that the theory under study enjoys. This is a well defined problem which is in general much simpler than computing the full chiral ring exactly. Nevertheless, many informations about the geometry of the moduli space can be extracted by this graded counting. We will later define and perform this counting in a number of examples. For the moment, let us specialize to the case of moduli spaces of theories with 3d $\mathcal{N} = 4$ supersymmetry, and review some of their features.

In a $3d$ $\mathcal{N} = 4$ supersymmetric theory, it is known that the geometry of $\mathcal{M}$ will be locally a product of a Higgs branch factor $\mathcal{H}$ and a Coulomb branch factor $\mathcal{C}$. Both branches of the moduli space are (possibly singular) HyperKähler varieties. We will later review the structure and properties of such branches.

Higgs Branch

The (full) Higgs branch of the moduli space of a $3d$ $\mathcal{N} = 4$ supersymmetric gauge theory is characterized by giving nonzero vev to the scalars in the hypermultiplets. On a generic vacuum of the Higgs branch the gauge group is completely broken by the Higgs mechanism. Unlike the Coulomb branch, the Higgs branch is protected against quantum corrections, and therefore its exact geometry can be studied in the classical theory. The reason for which the Higgs branch metric is classically exact is that the complexified coupling constant $\tau = \frac{4\pi i}{g^2} + \frac{\theta}{2\pi}$ can be thought as the lowest component of a background vector multiplet, therefore not affecting the Higgs Branch. In the Higgs branch, the relevant operators of the chiral ring are gauge invariant operators composed of hypermultiplets, subject to F-term conditions.

Coulomb Branch

The (full) Coulomb branch $\mathcal{C}$ of a $3d$ $\mathcal{N}=4$ supersymmetric gauge theory is characterized by giving nonzero vev to the triplet of scalars in the vector multiplets, and also by the vev of the dual photons. On a generic vacuum of the Coulomb branch the gauge group G is broken to the maximal torus $U(1)^r$ where r is the rank of the gauge group, and all the W-bosons and charged matter fields get massive. The geometry of $\mathcal{C}$ is a HyperKähler variety of the quaternion dimension equal to the rank r of G. Unlike the Higgs branch, the Coulomb branch receives quantum corrections. The relevant operators in the chiral ring of the full Coulomb branch are dressed monopole operators. Indeed, after having given a vev to a monopole operator it is still possible to turn on a vev for a complex scalar in the adjoint representation of the vector multiplet, without spoiling the BPS condition of the monopole [104].

4.4 Hilbert Series for Moduli Spaces of 3d $\mathcal{N}=4$ Theories

In this section, we will explain the Hilbert Series technique for counting operators in the chiral ring of three-dimensional $\mathcal{N}=4$ supersymmetric gauge theories. This method was first developed in [128,129] for the full Higgs branch and in [96] for the full Coulomb branch. This method was successfully tested in different contexts and already produced some interesting applications, for example the computation of the moduli spaces of instantons in [113,115,116,130–132]. Here we recall the minimal notions needed in the following, and we refer to the literature for more details.

The Hilbert series $HS(t)$ is the main tool used for this counting purpose. It is a generating function that keeps track, in a systematic way, of all the operators of the chiral ring. In more details, the coefficient a_n in the Taylor expansion

$$HS(t) = \sum_n a_n t^n, \tag{4.27}$$

will be equal to the number of chiral operators having charge n under the symmetry which is weighted by a fugacity t. This can be refined to the case in which one wishes to grade the chiral operators by more than one symmetry. For example, suppose that the chiral operators are charged under N global symmetries. For each one of them, we choose $x_i,\ i=1,\cdots,N$ as a grading parameter. Then the Hilbert series will be given by

$$HS(t,x_i) = \sum_{k_1}\sum_{k_2}\cdots\sum_{k_N} a_{k_1,k_2,\cdots k_n} \prod_{i=1}^{N} x_i^{k_i}, \tag{4.28}$$

and the interpretation is that $a_{k_1,k_2,\cdots k_N}$ is the number of chiral operators having respectively charges $k_1, k_2, \cdots k_N$ under the N symmetries.

The algorithmic procedure to compute the Hilbert series from the data defining $d = 3$ $\mathcal{N} = 4$ gauge theories varies, depending on the fact that we want to compute the Hilbert series for the chiral ring of the Higgs branch factor $\mathcal{H}$ or the Coulomb branch factor $\mathcal{C}$. Therefore we will split the discussion in two.

4.4.1 Higgs branch moduli space

As written before, relevant operators in the Higgs branch are gauge invariants combination of the scalars in the hypermultiplets, subject to F-term relations.

One can count them by the following three step-procedure if a lagrangian of the theory is known [128, 129]:

1. Generating all the possible symmetric products of the scalars in the chiral multiplets.

   To do this one computes
   $$\mathrm{PE}\left[\sum_{i=1}^{2N_h} \mathrm{char}_{R_i}(w)\mathrm{char}_{R'_i}(x)\tilde{t}\right], \tag{4.29}$$
   where w (resp. x) is a collective notation for all the $\mathrm{rk}(G_g)$ (resp. $\mathrm{rk}(G_f)$) fugacities of the gauge (resp. flavor) group. Also $\mathrm{char}_{R_i}(w)$ (resp. $\mathrm{char}_{R'_i}(x)$) is the character of the gauge (resp. flavor) representation R_i (resp. R'_i) of the i-th chiral multiplet X_i, $\tilde{t}$ is defined as $\tilde{t} = t^{\frac{1}{2}}$ where t is again a fugacity counting the conformal dimension of some operators. Note that a scalar in 3d has dimension $\frac{1}{2}$. We introduced $\tilde{t}$ to avoid the appearance of fractional powers in the expressions. The sum is done over all the set of the $\mathcal{N} = 2$ chiral multiplets belonging to $\mathcal{N} = 4$ hypermultiplets. N_h is the number of hypermultiplets and 2 in front of N_h appears since a $\mathcal{N} = 4$ hypermultiplet is made of two $\mathcal{N} = 2$ chiral multiplets. Here PE is the Plethysitic exponential, a generating function for symmetrizations, defined for any function $f(x_1, \cdots, x_n)$ such that $f(0, \cdots 0) = 0$ as
   $$\mathrm{PE}[f(x_1, \cdots, x_n)] := \exp\left(\sum_{k=1}^{\infty} \frac{f(x_1^k, \cdots x_n^k)}{k}\right). \tag{4.30}$$

2. The F-term prefactor.

   In this second step, one has to take into account the fact that the symmetric products of scalars generated in the step above are not independent, but subject to a number N_r of relations arising from the fact that the F-term conditions need to be satisfied by every vacuum of the Higgs branch. To enforce this fact in the counting procedure, one has to multiply equation (4.29) by a factor

$$\mathrm{Pfc}(w,\tilde{t}) := \mathrm{PE}\left[\sum_{i=1}^{N_r} \mathrm{char}_{R_i''}(w)\tilde{t}^{d_i}\right]^{-1}, \tag{4.31}$$

where $\mathrm{char}_{R_i''}(w)$ is the character of the gauge representation R_i'' of the i-th relation, and d_i is its degree in the conformal dimension: typically $d_i = 2$. The F-term relations will not usually depend on flavor fugacities, due to the fact that the superpotential (in terms of the $\mathcal{N} = 2$ notation) involves a trace on the flavor indices, and this trace always appears also in the F-term equations. One might think that the variation under a hypermultiplet may give rise to an F-term equation that has flavor indices. However, the F-term condition is automatically satisfied since we do not turn on Coulomb branch moduli. Characters and degrees of the classical relations can be extracted easily from the superpotential, we will show detailed examples of how to do this in section 7.6.

3. The Molien-Weyl projection, (see e.g. [137]).

   In order to count only the gauge invariant operators, and not all of the symmetric products, we need to project all the representations that the PE generates onto the gauge singlets. This is done by integrating the gauge fugacities over the whole gauge group. This indeed works since from representation theory it holds that $\int d\mu_G \; \mathrm{char}_{R_i}(w)\mathrm{char}_{\bar{R}_j}(w) = \delta_{ij}$. This implies that only gauge singlets give non-zero contribution after the integration. Therefore, integrating the result of step 1 and 2 over the full gauge group will discard all the gauge-variant operators, and keep only the gauge invariant ones.

   In conclusion the Hilbert series of a Higgs branch is given by[5]

$$HS(\tilde{t},z) = \int_G d\mu_G \; \mathrm{Pfc}(w,\tilde{t}) \prod_i PE\left[\mathrm{char}_{R_i}(w)\mathrm{char}_{R_i'}(z)\tilde{t}\right], \tag{4.32}$$

   where μ_G is the Haar measure of G, defined for any Lie group as (see e.g. [138])

$$\int_G d\mu_G = \frac{1}{(2\pi i)^r}\oint_{|w|_1=1}\cdots\oint_{|w|_r=1}\frac{dw_1}{w_1}\cdots\frac{dw_r}{w_r}\prod_{\alpha\in\Delta^+}\left(1-\prod_{k=1}^{r} w_k^{\alpha_k}\right), \tag{4.33}$$

   where Δ^+ is the set of positive roots of the Lie algebra of G.

[5] Here we have used the well known property of the Plethystic exponential that $\mathrm{PE}[f(t)+g(t)] = \mathrm{PE}[f(t)]\mathrm{PE}[g(t)]$, for any $f(t)$ and $g(t)$ such that $f(0) = g(0) = 0$.

4.4.2 Coulomb branch moduli space

As written before, relevant operators in the Coulomb branch are monopole operators dressed by the adjoint scalar in the vector multiplet. In order to count those operators grading them by their conformal dimension, it is crucial that there is exactly one bare BPS monopole operator for every magnetic charge m [103]. However, there are still different ways in which it can be dressed. Given this, the Hilbert series is defined as

$$HS(t) = \sum_{m \in \Gamma(^L G)/\mathcal{W}_{^L G}} t^{\Delta(m)} P_G(m,t) \tag{4.34}$$

where t is a fugacity keeping track of the conformal dimension of the monopoles. The magnetic charge m runs over all the lattice points of a Weyl chamber, i.e. over the weight lattice $\Gamma(^L G)$ of the Langlands (GNO) dual group of the gauge group modded out by the action of the Weyl group $\mathcal{W}_{^L G}$ [135] . Now, $P_G(m,t)$ is a correction factor taking care of the different dressings.

In details, the factor $P_G(m,t)$ is included due to the following reason. When the vev of a bare monopole operator is turned on in the background, the gauge group is generically broken to a subgroup $H_m \subset G$, defined as the subgroup of G which commutes with the magnetic flux with the magnetic charge m. Then one can consistently turn on a vev for a complex scalar in the adjoint representation of this residual gauge group H_m, without spoiling the BPS conditions for the monopole. $P_G(t,m)$ counts the gauge invariant operators of the residual group H_m. The explicit expression is given by

$$P_G(t,m) = \prod_{i=1}^{r} \frac{1}{1 - t^{d_i(m)}}, \tag{4.35}$$

where r is the rank of H_m and $d_i(m)$ are all the degrees of the r Casimir operators of H_m. As a reference, the degrees of the Casimir operators are given in table 4.2.

In the case in which the gauge group G consists of a product $G = \prod_i G_i$ of factors, and some of them are not simply connected, one can further refine this counting by including fugacities z_i which keep track of charges under the 3d topological $U(1)_J^n$ symmetry. The topological $U(1)_J$ symmetry is a symmetry which induces in the semiclassical picture the shift of the dual photon [136]. The Hilbert series with this latter fugacities included, called now *Refined Hilbert Series*, is then given by

$$HS(t) = \sum_{m \in \Gamma(^L G)/\mathcal{W}_{^L G}} t^{\Delta(m)} \prod_{i=1}^{n} z_i^{J_i(m)} P_G(m,t), \tag{4.36}$$

where $J_i(m)$ represents the charge of the monopole operator under the i-th $U(1)_J$ topological symmetry, where here $i = 1, \cdots n$, and n is the number of non-simply connected factors of G.

Simple Lie Algebra $\mathfrak{g}$	Degrees
$a_l, \quad l \geq 1$	$2, 3, \cdots, l+1$
$b_l, \quad l \geq 2$	$2, 4, \cdots, 2l$
$c_l, \quad l \geq 3$	$2, 4, \cdots, 2l$
$d_l, \quad l \geq 4$	$2, 4, \cdots, 2l-2, l$
e_6	$2, 5, 6, 8, 9, 12$
e_7	$2, 6, 8, 10, 12, 14, 18$
e_8	$2, 8, 12, 14, 18, 20, 24, 30$
f_4	$2, 6, 8, 12$
g_2	$2, 6$

Table 4.2: Degrees of the Casimir invariants of the simple Lie algebras.

4.5 The Hanany-Witten cartoon

It is possible to engineer many $3d\,\mathcal{N}=4$ field theories via some string theory construction. Here we will review one possibility, namely the Hanany-Witten cartoon.

Consider type IIB superstring theory. A flat Dp brane is a 1/2 BPS solitonic state of the theory, and carries on its worldvolume a maximally supersymmetric gauge theory in $p+1$ dimensions, namely Super Yang-Mills with 16 supercharges. It is possible to consider a system of branes suspended between other branes, to reduce the total amount of supersymmetry in the worldvolume theory, provided that one choses the type of D-branes and the dimensions they span in a suitable way. In this section we review how this is achieved.

The type IIB brane system yielding 3d $\mathcal{N}=4$ supersymmetric gauge theories was first analyzed in [94], with more details spelled out in [84–87] The configuration of the branes in the ten-dimensional spacetime is shown in table 4.3. In this configuration, some D3-branes are suspended between NS5-branes, and are of finite length in the x^6-direction. Therefore, the worldvolume theory on the D3-branes is effectively a 3d $\mathcal{N}=4$ theory after the dimensional reduction along the x_6-direction. The rotational symmetry in the (x_3, x_4, x_5)-plane and in the (x_7, x_8, x_9)-plane gives the $SO(3) \times SO(3)$ R-symmetry of the 3d $\mathcal{N}=4$ supersymmetric theory.

This brane configuration can be explicitly drawn in pictures like (4.3), which are called *Hanany-Witten cartoons.*

•	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
D3	-	-	-	x	x	x	-	x	x	x
D5	-	-	-	-	-	-	x	x	x	x
NS5	-	-	-	x	x	x	x	-	-	-

Table 4.3: The brane system realizing a 3d $\mathcal{N} = 4$ theory. In this table "x" means that the brane is pointlike in that direction, while "-" means that it is extended in that direction.

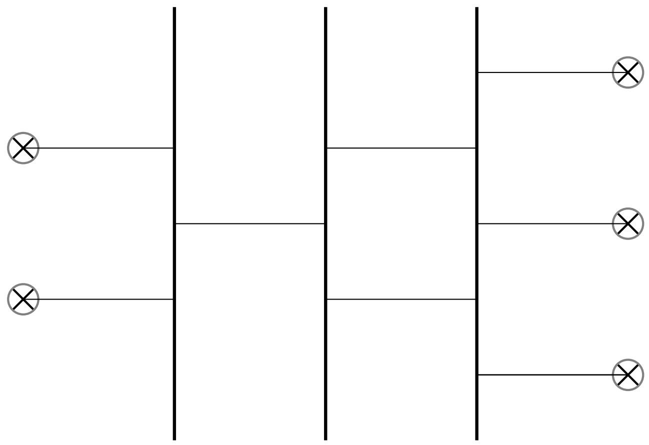


Figure 4.3: An example of Hanany-Witten cartoon. Here the horizontal direction is x_6, the vertical direction in x_7, x_8, x_9, and the "out of the page direction" is x_3, x_4, x_5. Horizontal lines are D3 branes, vertical lines are NS5 branes, and crossed circles are D5 branes. The directions x_0, x_1, x_2 are suppressed since they are spanned by all the branes.

Let us call *cells* the zones delimitated by NS5s and let us numerate the cells by a number $i = 1, 2, 3, ...$ starting from the leftmost one. Let us also denote

$$\begin{aligned} \overrightarrow{m} &= (x_7, x_8, x_9) \\ \overrightarrow{w} &= (x_3, x_4, x_5) \end{aligned} \tag{4.37}$$

The identification of the 3d fields ad parameters from the picture is the following:

1. For every cell there will be a $U(N_i)$ gauge group where N_i is the number of D3 branes in the i-th cell.

2. The vector multiplets of such $U(N_i)$ is given by strings starting from a D3 brane in a given cell and ending on a generic D3 brane (including the same one) in the same cell.

3. There are bifundamental hypermultiplets $(Q, \tilde{Q})_i$ charged under the fundamental representation of $U(N_i)$ and the antifundamental representation of $U(N_{i+1})$. They are identified by strings starting from one D3 in a given cell and ending on a D3 in the next cell.

4. Fundamental flavors are identified with strings starting from a D3 in a given cell and ending on a D3 which ends on a D5 brane.

5. The triplet of mass parameter for the fundamental hypers corresponds to the relative position of the D5s and the D3s in the $\vec{m}$ direction:

$$\vec{m} = \vec{m}_{D5} - \vec{m}_{D3} \tag{4.38}$$

   Notice that this is very intuitive, increasing this relative distance the strings corresponding to fundamental flavors will be more and more elongated.

6. The triplet of FI parameters for the $U(N_i)$ gauge node is given by the relative position of the NS5s in the i-th cell

$$\vec{\xi}_i = \vec{w}_i - \vec{w}_{i+1} \tag{4.39}$$

7. The gauge coupling of the $U(N)$ factor related to a given cell inside two NS5s at positions t_1 and t_2 in x_6 will be

$$\frac{1}{g^2} = |t_1 - t_2| \tag{4.40}$$

By using this rules is easy, for example, to understand that the HW cartoon of figure (4.3) corresponds to the quiver gauge theory of figure (4.4)

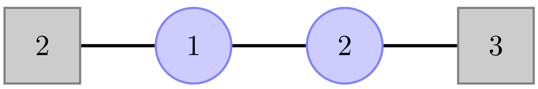


Figure 4.4: The quiver graph for the theory with HW cartoon drawn in figure (4.3)

At this point we need to comment about one important restriction in drawing HW cartoons: the *S-rule* [107]. Namely, in order for this construction to engineer a theory with 8 supercharges, there will be a constraint on the number of D3 branes which can stretch between 5-branes of different type. Namely, take a NS5 brane and a D5 brane. If there is more than one D3 brane between them, then the whole configuration is no longer supersymmetric. This property will be crucial in the following, then we explain how the HW cartoon is able to encode also the structure of the mixed branch of the Moduli space of the 3d QFT under investigation.

Finally, although we will never need it in this thesis, it is worth to mention that a modification of this brane picture can be used to study $\mathcal{N}=2$ and $\mathcal{N}=3$ 3d theories of Chern-Simons Maxwell type, by replacing some of the NS5 branes with $(1,k)$ branes. A further generalization including generic (p,q) five-branes is also studied. [88]

4.6 3d mirror symmetry

There is a strong-strong IR duality in 3d called *3d mirror symmetry* and originally proposed in [92]. Mirror symmetry is the statement that for every 3d $\mathcal{N}=4$ QFT A there exist a 3d $\mathcal{N}=4$ QFT B such that

1. The duality exchanges Coulomb and Higgs branch.
2. The duality exchanges $SU(2)_L$ with $SU(2)_R$.
3. Mass and FI parameters are exchanged.

The name *Mirror Symmetry* in this context comes from the fact that if theory A is realized from type II string theory compactification on $CY_3 \times S^1$, then theory B will be realized by type II string theory compactification on $CY_3' \times S'^1$ where S^1 and S'^1 have inverse radii and CY_3 and CY_3' are mirror dual in the Calabi-Yau sense.

Mirror symmetry can be extensively checked by many different ways, for example by computing the partition functions of two theories which are conjectured to be dual and then matching them. Other checks can be done by matching the superconformal indices or by matching operators in the dual sides. In particular Hilbert series technique we explained before has proven very practical for checking mirror symmetry.

3d Mirror symmetry from the brane picture

In terms of the Hanany-Witten cartoon discussed in the previous section, mirror symmetry is S-duality of IIB superstring theory. Under such duality $D3$ branes are left invariant while $NS5$-branes become $D5$ branes and vice versa. This, combined with Hanany-Witten transitions, makes extremely easy to find the mirror dual of a given linear quiver gauge theory. Let us explain this with an easy illustrative example: suppose we consider the quiver gauge theory of figure (4.5), which is simply $U(1)$ with $N_f = 3$. We wish now to find its mirror dual.

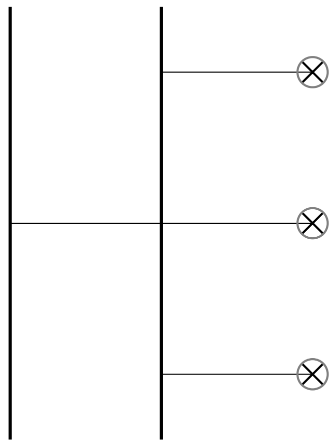


Figure 4.5: The brane picture for $U(1)$ with $N_f = 3$.

As a first thing we bring the D5 branes inside the cell. By an Hanany-Witten transition, the D3 branes stretched between the NS5 and the D5 will thus disappear. We also place the D5s on top of the D3 brane. We then get to the situation explained in (4.6).

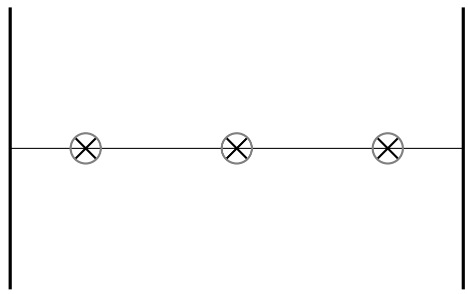


Figure 4.6: The brane picture for $U(1)$ with $N_f = 3$ after the Hanany-Witten transition.

Now we perform S duality of IIB superstring theory. In the new duality frame D3 branes are left invariant while D5s become NS5s and NS5s become D5s. The Hanany Witten cartoon in the S-dual frame will be the one given in figure (4.7).

We see that the 3d mirror theory for $U(1)$ with $N_f = 3$ is a quiver gauge theory with two $U(1)$ gauge nodes and a $U(1)$ flavor node attached to both of them, as depicted in figure (4.8).

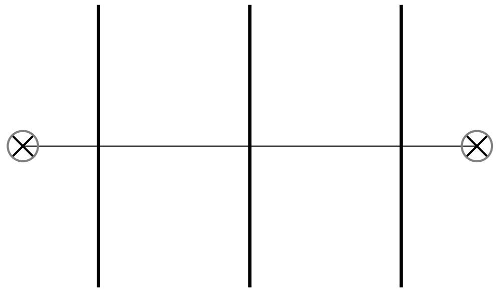


Figure 4.7: The brane picture for the 3s mirror dual of $U(1)$ with $N_f = 3$.

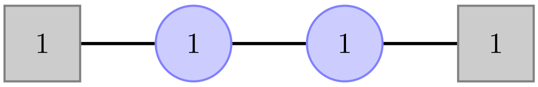


Figure 4.8: The quiver graph for the mirror dual of $U(1)$ with $N_f = 3$.

4.7 $T[SU(N)]$ theory and its relation to class-S.

In the following we will be interested in studying the full moduli space of a particular theory called $T[SU(N)]$. We will now define this theory and motivate why it is an interesting theory to be studied, due to its connection to class-S theories in 4d. The $T[SU(N)]$ theory can be defined by the linear quiver of figure 4.9.

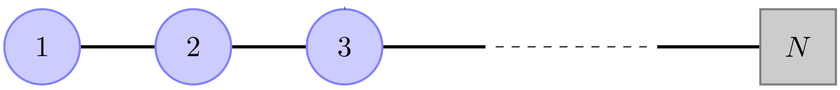


Figure 4.9: The quiver graph for the $T[SU(N)]$ theory.

Here each circle node with a number k, $k = 1, \cdots N-1$ denotes a factor $U(k)$ of the gauge group, and a line between two gauge nodes stand for one hypermultiplet in the bi-fundamental representation of the two gauge groups. The rightmost node with a number N denotes a $SU(N)$ flavor group. In other words, there are N hypermultiplets in the fundamental representation of $U(N-1)$ gauge group. As explained in the previous section, this quiver alone is enough to completely specify the lagrangian of $T[SU(N)]$.

We will now review what is the relation between $T[SU(N)]$ and class-S theories that was mentioned before. By taking three copies of the $T[SU(N)]$ gauge theory and gauging together the three $SU(N)$ flavor symmetries by the introduction of a $SU(N)$ vector multiplet one can realize a star shaped quiver. An example for $N = 3$ is given in figure (4.10).

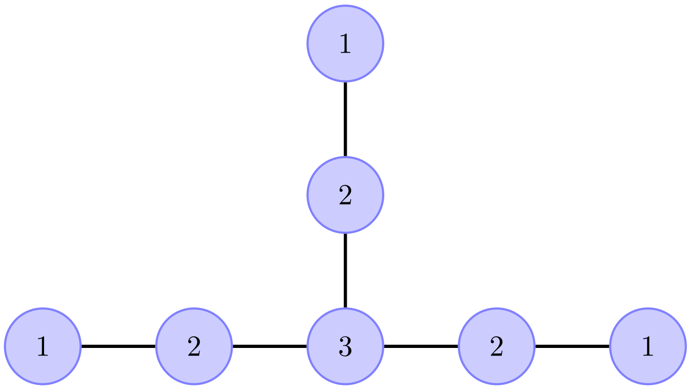


Figure 4.10: Star shaped quiver obtained from gauging three copies of $T[SU(3)]$ theories

Let us call this star shaped quiver gauge theory $\tilde{T}_{3\mathrm{d},N}$. Now this theory is conjectured to have a 3d mirror dual $T_{3\mathrm{d},N}$ which is a non-lagrangian theory [89]. The claim is that $T_{3\mathrm{d},N}$ is the dimensional reduction of Gaiotto's famous T_N theory, the building-block of class-S theories. [42] Many checks of this claims have been performed. Due to the central role of T_N in the landscape of class-S theories, it is therefore interesting to understand its moduli space as fully as possible. In particular is then interesting to study the moduli space of $T[SU(N)]$ as this can shed light on the T_N case by the above chain of gauging, mirror symmetry and dimensional reduction, while still being a very simple and tractable lagrangian field theory[6].

4.8 Mixed Branches of the $T[SU(N)]$ Theory

So far we have focused on a full Coulomb branch and a full Higgs branch of 3d $\mathcal{N} = 4$ theories. In general, 3d $\mathcal{N} = 4$ theories have many mixed branches where we can turn on vevs for scalars both in vector multiplets and hypermultiplets. For example, at some special locus of a full Coulomb branch, we may turn on vevs for scalars in hypermultiplets and there open up some directions in a Higgs branch. Then, the full moduli space of a generic three-dimensional $\mathcal{N} = 4$ theory in fact has the structure

$$\bigcup_{\alpha} \mathcal{C}_\alpha \times \mathcal{H}_\alpha \tag{4.41}$$

where α labels the different mixed branches, $\mathcal{C}_\alpha$ is the Coulomb branch factor and $\mathcal{H}_\alpha$ is the Higgs branch factor. Both $\mathcal{C}_\alpha$ and $\mathcal{H}_\alpha$ are Hyperkähler varieties, where $\mathcal{C}_\alpha$ is parametrized

[6]En passant we comment that the $N = 3$ star shaped quiver in figure 4.10 is the 3d mirror for the T_3 theory, which is just another name for the famous E_6 Minahan-Nemeschansky theory we studied also in chapter 2.

by the vev of scalars in the vector multiplets and the dual photon and $\mathcal{H}_\alpha$ by scalars in the hypermultiplets. The union in equation (4.41) is clearly not a disjoint union, as in general different mixed branches intersect with one another. With this notation, a full Coulomb branch is $\mathcal{C} \times \{0\}$ and a full Higgs branch is $\{0\} \times \mathcal{H}$. Those two full branches intersect at a single point, where typically the theory is a superconformal field theory.

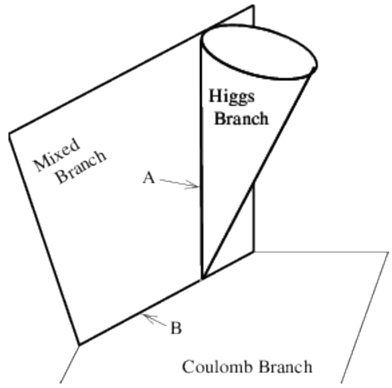


Figure 4.11: A schematic picture for a mixed branch of the Moduli Space.

In this section, we are mainly interested in the mixed branches of the $T[SU(N)]$ theory. In order to visualize the mixed branch structure of the $T[SU(N)]$ theory, it is useful to engineer it with a brane construction in type IIB superstring theory and we will heavily make use of it.

Also the $T[SU(N)]$ theory can be realized by using a brane system in type IIB string theory, which arises as the S-dual of a half-BPS boundary condition of a 4d $\mathcal{N} = 4$ super Yang-Mills theory [107]. The brane configuration which yields the 3d $T[SU(N)]$ theory is given in figure 4.12. We have k D3-branes between the k-th and the $(k+1)$-th NS5-brane[7] for $k = 1, \cdots, N-1$, and N D3-branes are attached only to the last NS5-brane. At the end of each rightmost D3-brane, we may put one D5-brane. The introduction of the D5-branes will be useful for reading off the Higgs branch. The k D3-branes between NS5-branes give rise to a gauge group $U(k)$, and we call them "color D3-branes". On the other hand, the N D3-branes attached to the rightmost NS5-brane realize the $SU(N)$ flavor symmetry, and we call them "flavor D3-branes".

While the N D5-branes in the brane configuration for the $T[SU(N)]$ theory yield the perturbative $SU(N)$ flavor symmetry, the N NS5-branes in fact realize non-perturbative $SU(N)$ global symmetry [107]. From the quiver description of the $T[SU(N)]$ theory, we know that at least we have the $U(1)_J^{N-1}$ topological global symmetry. The $U(1)_J^{N-1}$ topological global symmetry is in fact enhanced to $SU(N)$ by the effect of monopole operators. Moreover, the $T[SU(N)]$ theory

[7]The order is counted from left to right. Namely, the leftmost NS5-branes is the first NS5-brane.

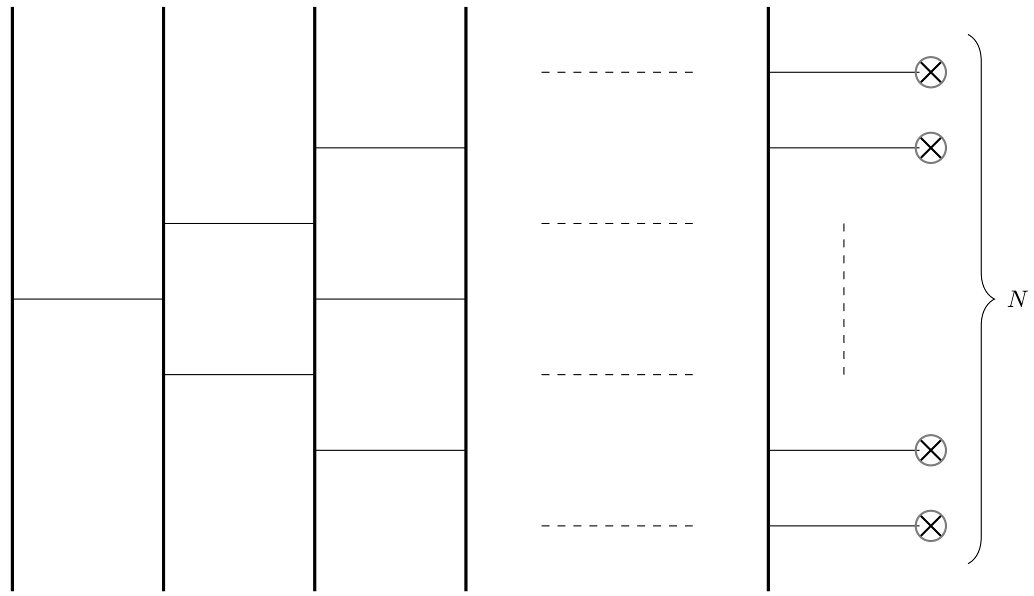


Figure 4.12: The brane picture for the $T[SU(N)]$ theory. In this picture the directions x_0, x_1, x_2 are suppressed, since they are shared among all the branes. The horizontal axis is x_6, the vertical axis corresponds to directions x_7, x_8, x_9 on which the NS5-branes are stretched, and the "out of the page axis" corresponds to x_3, x_4, x_5, on which the flavor D5-branes are stretched. Hence, horizontal lines and vertical lines represent D3-branes and NS5-branes respectively. D5-branes are denoted by $\otimes$.

is self-mirror and the full Coulomb branch moduli space is isomorphic to the full Higgs branch moduli space.

One nice feature about the brane picture is that the Coulomb branch moduli space, the Higgs branch moduli space and all the mixed branches can be pictorially understood from brane motions. The D3-branes suspended between NS5-branes can move along the NS5-branes. These degrees of freedom correspond to the Coulomb branch moduli of the 3d gauge theory[8]. When we tune the positions of the color D3-branes in the (x_7, x_8, x_9)-directions, the flavor D3-branes may be fractionated between D5-branes and can move between the D5-branes in the (x_3, x_4, x_5)-directions. These latter degrees of freedom correspond to the moduli parameterizing the Higgs branch. In particular, when all the positions of the color D3-branes are tuned to zero, the full Higgs branch opens up. Due to this construction, the non-perturbative $SU(N)$ global symmetry

[8]One the other hand, the positions of the flavor D3-branes in the (x_7, x_8, x_9)-directions are related to the mass parameters of the fundamental hypermultiplets.

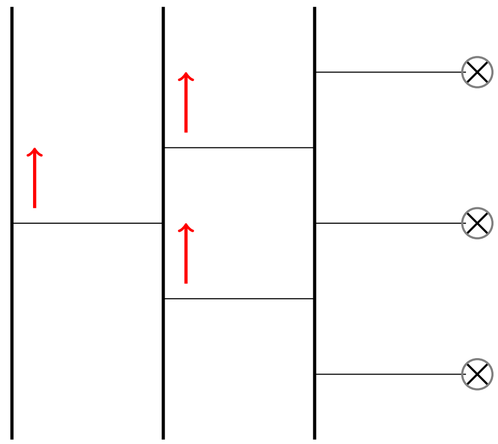


Figure 4.13: The brane picture for the full Coulomb branch of $T[SU(3)]$.

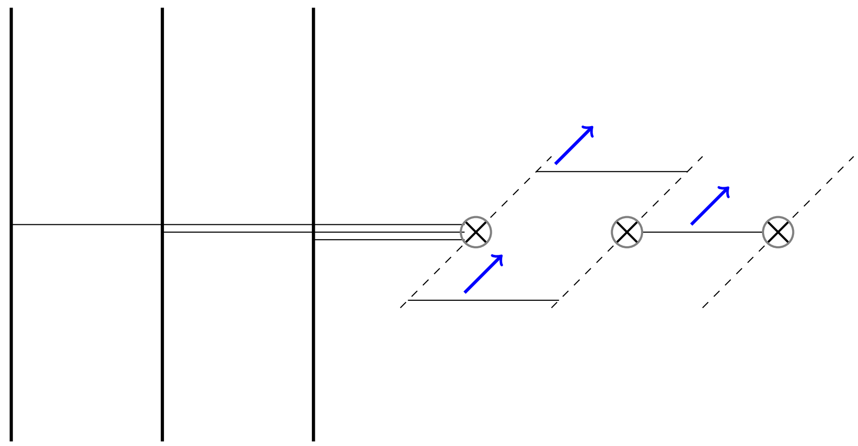


Figure 4.14: The brane picture for the full Higgs branch of $T[SU(3)]$.

is associated to the Coulomb branch and the perturbative $SU(N)$ flavor symmetry is associated to the Higgs branch. The full Coulomb branch and the full Higgs branch of the $T[SU(3)]$ theory is shown in figure 4.13 and 4.14 respectively.

A mixed branch of the $T[SU(N)]$ theory may arise when only a part of the positions of the color D3-branes are tuned. At some subloci of the full Coulomb branch moduli space, a Higgs branch opens up. In fact, the subloci where a Higgs branch opens up are given by nilpotent orbits of $\mathfrak{su}(N)$, and can be classified by a Young diagram with N boxes or equivalently a partition of the integer N [107,124,126]. The correspondence goes as follows. A partition $\rho = [a_1, a_2, \cdots, a_n]$

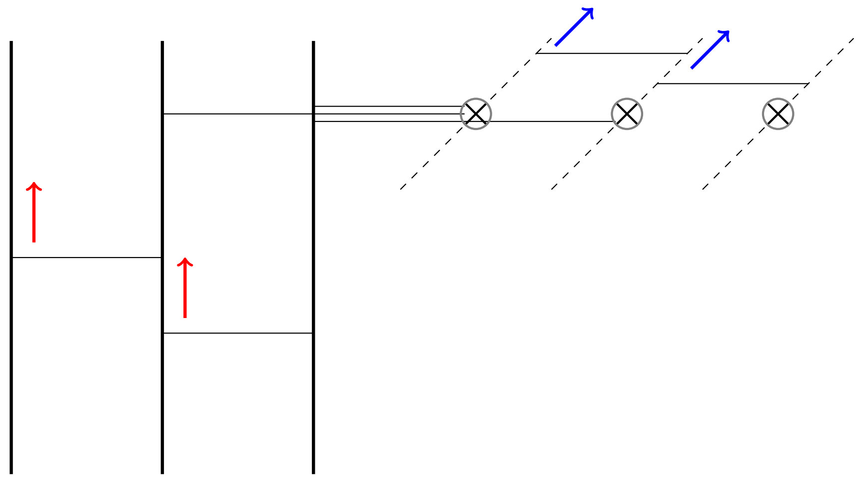


Figure 4.15: The brane picture for the mixed branch $\rho = [2, 1]$.

with $a_1 \geq a_2 \geq \cdots \geq a_n$ and $\sum_{i=1}^{n} a_i = N$[9] means that a_i flavor D3-branes are put together on one D5-brane for each $i = 1, \cdots, n$. Note that this restriction does not only fix the positions of flavor D3-branes but also fix the positions of color D3-branes. This is due to the s-rule which states that only one D3-brane can be suspended between an NS5-brane and a D5-brane in order to preserve the supersymmetry [94]. Therefore, when some flavor D3-branes are put on one D5-brane, some of the flavor D3-branes should connect to some color D3-branes so that the configuration does not break the s-rule. In this way, the Young diagram classification can tune the Coulomb branch moduli.

When some of the positions of the color D3-branes are fixed, some of the flavor D3-branes may be fractionated between D5-branes and hence a mixed branch of the $T[SU(N)]$ theory can be realized. Note that in order to realize the maximal Higgs branch of a mixed branch, one also needs to tune the mass parameters of the remaining fundamental hypermultiplets, An example of the mixed branch corresponding to the partition $\rho = [2, 1]$ of the $T[SU(3)]$ theory is shown in figure 4.15.

Since for the $T[SU(N)]$ theory the mixed branch structure may be completely specified by the partition ρ with N boxes [107, 124, 126], the full moduli space is given by

$$\bigcup_{\rho} \mathcal{C}_\rho \times \mathcal{H}_\rho, \tag{4.42}$$

[9] In terms of a Young diagram, $\rho = [a_1, a_2, \cdots, a_n]$ means that the Young diagram has a_i boxes for the i-th column for $i = 1, \cdots, n$. Due to this correspondence, we will use a partition and the corresponding Young diagram interchangeably and write the Young diagram associated to a partition ρ as Y_ρ.

where ρ is all the possible partitions of the integer N. In particular, $\rho = [1, 1, \cdots, 1]$ gives $\mathcal{C} \times \{0\}$ with the maximal Coulomb branch $\mathcal{C}$, and $\rho = [N]$ gives $\{0\} \times \mathcal{H}$ with the maximal Higgs branch $\mathcal{H}$. The dimension of the Coulomb branch moduli space $\mathcal{C}_\rho$ can be computed from the associated partition [107, 124, 126],

$$\dim_{\mathbb{H}}(\mathcal{C}_\rho) = \frac{1}{2}\left(N^2 - \sum_{i=1}^{n} a_i^2\right), \tag{4.43}$$

where $a_i, i = 1, \cdots, n$ is the entry of the partition $\rho = [a_1, \cdots, a_n]$. For example, one can check

$$\dim_{\mathbb{H}}(\mathcal{C}_{[2,1]}) = \frac{1}{2}\left(3^2 - (2^2 + 1^2)\right) = 2, \tag{4.44}$$

which agrees with the number of color D3-branes that are not frozen in figure 4.15.

In fact, the mirror symmetry of the 3d $\mathcal{N} = 4$ theory implies [107, 124, 126]

$$\mathcal{H}_\rho \simeq \mathcal{C}_{\rho^D}, \tag{4.45}$$

where ρ^D is the dual partition to ρ, which is associated to the transpose of the Young diagram Y_ρ. This property can be inferred from the brane configuration. In terms of the brane configuration, the mirror symmetry is realized by the S-duality in type IIB string theory [94], which exchanges NS5-branes with D5-branes but keep D3-branes unchanged. Since the $T[SU(N)]$ theory is self-mirror, a Higgs branch $\mathcal{H}_\rho$ in a mixed branch specified by ρ of the $T[SU(N)]$ theory is mapped to a Coulomb branch $\mathcal{C}_{\rho'}$ in a different mixed branch specified by a different partition ρ' of the $T[SU(N)]$ theory . The partition ρ' should be related to the number of flavor D3-branes put on one D5-brane in the mirror picture. Hence, in the original theory, ρ' should be related to the number of D3-branes put on one NS5-brane. Suppose ρ is given by $[a_1, \cdots, a_n]$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ and $\sum_{i=1}^n a_i = N$. This means that for example n D3-branes end on the rightmost NS5-brane. In general, if the number of a_i satisfying $a_i \geq k$ is b_k, then there are b_k D3-branes ending on the $(N - k + 1)$-th NS5-brane. Therefore, we find that ρ' is given by the partition $[b_1, \cdots, b_{n'}]$ where b_k is the number of a_i satisfying $a_i \geq k$ for $i = 1, \cdots, n$. Then it is possible to see that the partition ρ' defined in this way is nothing but the dual partition ρ^D, yielding the claim (4.45).

Due to this feature, one can write the full moduli space (4.42) as

$$\bigcup_\rho \mathcal{C}_\rho \times \mathcal{C}_{\rho^D}, \tag{4.46}$$

or

$$\bigcup_\rho \mathcal{H}_{\rho^D} \times \mathcal{H}_\rho. \tag{4.47}$$

The relation (4.45) also implies that the dimension of the Higgs branch $\mathcal{H}_\rho$ of the mixed branch specified by ρ may be given by

$$\dim_{\mathbb{H}}(\mathcal{H}_\rho) = \frac{1}{2}\left(N^2 - \sum_{i=1}^{n'} b_i^2\right), \tag{4.48}$$

where $b_i, i = 1, \cdots, n'$ is the entry of the partition $\rho' = [b_1, \cdots, b_{n'}]$ which is dual to ρ. For example regarding the Higgs branch factor $\mathcal{H}_{[2,1]}$ the dimension can be counted by using the dual partition which is the same as $[2,1]$. Then the dimension of $\mathcal{H}_{[2,1]}$ is again 2 from (4.44), which agrees with the number of mobile D3-branes suspended between D5-branes in figure 4.15.

4.8.1 Hilbert series for the Coulomb branch factor

It is possible to compute the Hilbert series for the Coulomb branch factor $\mathcal{C}_\rho$ in a mixed branch specified by ρ by utilizing the method described in section 4.4. Since the mixed branch is locally given by a product of the Coulomb branch factor $\mathcal{C}_\rho$ and the Higgs branch factor $\mathcal{H}_\rho$, the value of the vevs parameterizing $\mathcal{H}_\rho$ does not affect the Coulomb branch part $\mathcal{C}_\rho$. Hence, in particular we can consider infinitely large vevs for the scalars parameterizing $\mathcal{H}_\rho$. In terms of the brane picture, we send the pieces of D3-branes between D5-branes to infinity. At low energies at the infinitely large vev of the Higgs branch $\mathcal{H}_\rho$, one obtains a different 3d $\mathcal{N} = 4$ theory which we call $T^\rho[SU(N)]$ theory. An example of the brane picture realizing the $T^{[2,1]}[SU(3)]$ theory is shown in figure 4.16.

Since the Coulomb branch moduli space of the $T^\rho[SU(N)]$ theory should be the same as $\mathcal{C}_\rho$, one can consider the Hilbert series of the Coulomb branch for the $T^\rho[SU(N)]$ theory. The Hilbert series can be calculated by going to the gauge theory description of the $T^\rho[SU(N)]$ theory [111,112]. Although it is non-trivial to read off the gauge theory content from the original brane picture with several D3-branes on top of one D5-brane, one can move the D5-brane to the left until no D3-branes are attached to the D5-brane. The annihilation of D3-branes is due to the Hanany-Witten transitions. Then the D5-brane gives a hypermultiplet in the fundamental representation under the gauge group given by color D3-branes in the cell where the D5-brane is located. Once we obtain the gauge theory description of the $T^\rho[SU(N)]$ theory, we can use the method described in section 4.4 to compute the Hilbert series of the Coulomb branch of the $T^\rho[SU(N)]$ theory, which should coincide with the Hilbert series for $\mathcal{C}_\rho$.

The brane picture of the $T^{[2,1]}[SU(3)]$ case after the Hanany-Witten transitions is given in figure 4.17. To read off the gauge theory content we moved the two D5-branes in figure 4.16 to the left and obtain another brane configuration in figure 4.17. From the brane configuration in

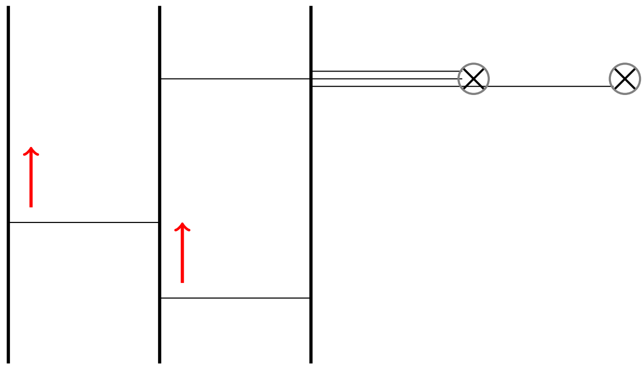


Figure 4.16: The brane picture of the $T^{[2,1]}[SU(3)]$ theory.

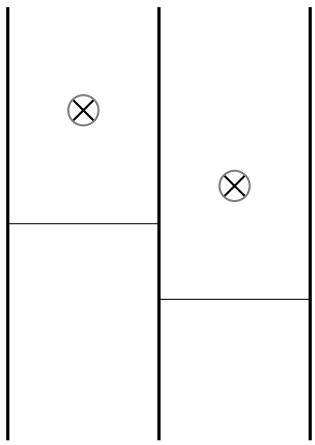


Figure 4.17: The brane picture for the IR theory $T^{[2,1]}[SU(3)]$ after Hanany-Witten transitions compared with the one in figure 4.16.

figure 4.17 the gauge theory description can be inferred as

$$[1] - U(1) - U(1) - [1]. \tag{4.49}$$

Here $[1]-$ or $-[1]$ is one hypermultiplet charged under the $U(1)$ to which the line is connected. The other line between the two $U(1)$'s denotes a hypermultiplet in the bi-fundamental representation under the gauge group $U(1) \times U(1)$. Similarly, we will use a notation where $[n]-$ implies n hypermultiplets in the fundamental representation of the gauge group to which the line is connected and a line between two gauge groups means a hypermultiplet in the bi-fundamental representation of the two gauge groups.

In general, the $T^{\rho}[SU(N)]$ theory where $\rho = [a_1, \cdots, a_n]$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ and

$\sum_{i=1}^{n} a_i = N$ is given by the following linear quiver theory,

$$\begin{array}{ccccc} [\#\,(a_i = N-1)] & & [\#\,(a_i = N-k)] & & [\#\,(a_i = 1)] \\ | & & | & & | \\ U\,(1-N_1) & - \cdots - & U\,(k-N_k) & - \cdots - & U\,(N-1-N_{N-1}), \end{array} \tag{4.50}$$

with

$$N_k = \sum_{i=1}^{n} (a_i - (N-k))\, H\,(a_i - (N-k))\,, \quad k = 1, \cdots, N-1, \tag{4.51}$$

where $H(x)$ is the Heaviside step function with the convention $H(0) = 0$ and $\#(a_i = l)$ is the number of a_i which is equal to l for $i = 1, \cdots, n$.

In the next section, we will describe a different technique, namely the restriction prescription, to compute the Hilbert series of the Coulomb branch factor $\mathcal{C}_\rho$. The method in fact directly uses the brane picture realizing the mixed branch specified by a partition ρ and does not use the IR gauge theory of $T^\rho[SU(N)]$.

4.8.2 Hilbert series for the Higgs branch factor

It is also possible to calculate the Hilbert series of the Higgs branch factor $\mathcal{H}_\rho$ of a mixed branch specified by a partition ρ by utilizing the method for computing the full Higgs branch described in section 4.4. We can again make use of the locally product structure of the mixed branch. Namely, the Higgs branch factor $\mathcal{H}_\rho$ is independent of the value of the Coulomb branch moduli of $\mathcal{C}_\rho$. In particular, we can take infinitely large vevs for the Coulomb branch moduli. In terms of the brane picture, we send the non-fixed positions of the color D3-branes to infinity. At low energies at the infinitely large vev of the Coulomb branch moduli, one obtains a different theory which we call $\tilde{T}^\rho[SU(N)]$ theory. The resulting brane configuration of the the $\tilde{T}^\rho[SU(N)]$ theory is the one at the origin of the Coulomb branch of the $\tilde{T}^\rho[SU(N)]$ theory. By moving to a generic point of the Coulomb branch moduli space, one can read off the gauge theory content of the $\tilde{T}^\rho[SU(N)]$ theory. After knowing the gauge theory description, one can apply the technique for computing the Hilbert series of the Higgs branch introduced in section 4.4 to the gauge theory corresponding to the $\tilde{T}^\rho[SU(N)]$ theory. The full Higgs branch of the $\tilde{T}^\rho[SU(N)]$ theory should be the same as the Higgs branch factor $\mathcal{H}_\rho$ of the mixed branch. Similarly, both the Hilbert series should be the same.

For example, as for the Higgs branch of the mixed branch specified by the partition $[2,1]$ of the $T[SU(3)]$ theory, decoupling the Coulomb branch moduli yields the $U(1)$ gauge theory with 3 flavors as in figure 4.18. Therefore, the Higgs branch factor $\mathcal{H}_{[2,1]}$ is isomorphic to the full Higgs branch of the $U(1)$ gauge theory with 3 flavors.

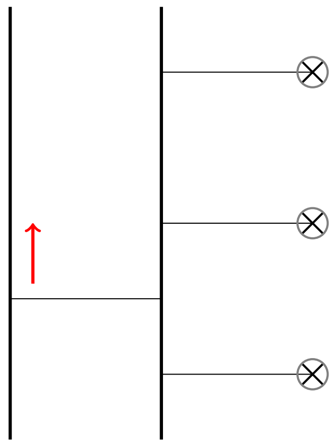


Figure 4.18: The brane picture for the IR theory of $\tilde{T}^{[2,1]}[SU(3)]$ obtained by decoupling all the unfrozen Coulomb branch moduli of the UV theory.

In general, the IR theory at the infinitely large vev for the Coulomb branch part of the mixed branch specified by $\rho = [a_1, \cdots, a_n]$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ and $\sum_{i=1}^{n} a_i = N$ is give by the following linear quiver theory,

$$U(N_1) - U(N_2) - \cdots - U(N_{N-1}) - [N], \tag{4.52}$$

where $N_k, k = 1, \cdots, N-1$ is given by (4.51). When N_k is zero then we remove the gauge node as well as the line attached to it.

4.8.3 The Restriction Rule for the Hilbert Series

In this section we develop the main result of this article. We conjecture that the Hilbert series for the Coulomb branch part of a mixed branch can be obtained from the Hilbert series of the full Coulomb branch, by performing a specific restriction of the latter. We also explain how this restriction rule is easily understood in terms of the type IIB brane picture.

The restriction rule

In section 4.8.1, we described a way to compute the Hilbert series of the Coulomb branch factor in the mixed branch specified by a partition ρ. For that, we made use of the gauge theory description of the $T^{\rho}[SU(N)]$ obtained after certain Hanany-Witten transitions of the corresponding brane diagram. However, we argue that we are able to compute the Hilbert series of the Coulomb branch factor without going to the gauge theory description but directly from the brane configuration realizing the mixed branch $\mathcal{M}_\rho$.

Due to the boundary condition (4.22) at the insertion point of a monopole operator, the BPS condition implies that the real scalar σ in the $\mathcal{N}=2$ vector multiplet inside the $\mathcal{N}=4$ vector multiplet satisfies [104]

$$\sigma \sim \frac{m}{2r} \tag{4.53}$$

where m is the magnetic charge and r is the radial coordinate. On the other hand, vevs of the scalars in the vector multiplet are related to color D3-brane positions. We can therefore relate, in the brane picture, color D3-brane positions with the magnetic charges of monopole operators.

At a point in the Coulomb branch factor of a mixed branch, we tune the positions of some of the color D3-branes so that they coincide with the positions of flavor D3-branes ending on D5-brane. Since the positions of the color D3-branes are the Coulomb branch moduli and the positions of the flavor D3-branes are the mass parameters for fundamental hypermultiplets, the tuning implies that the Coulomb branch moduli are equal to the mass parameters. In order to obtain a mixed branch, we turn off the mass parameters and all the flavor D3-branes are aligned along one line. Then, it is possible to set the values of the masses to zero without loss of generality. This in turn means that the value of the frozen positions of the color D3-branes or equivalently the corresponding Coulomb branch moduli are zero. Then the BPS condition (4.53) means that the corresponding magnetic charges also have to be zero.

Hence, the restriction of the positions of the color D3-branes given by the partition ρ can be translated into the condition that the corresponding magnetic charges are zero. Then, when one computes the Hilbert series of the Coulomb branch factor $\mathcal{C}_\rho$, one can simply insert the condition that some magnetic charges are zero into the Hilbert series for the full Coulomb branch. And the restriction of the magnetic charges can be read off from which color D3-branes are frozen. Physically, the restriction truncates the magnetic charges to a subset corresponding to BPS monopole operators that arise in the Coulomb branch factor.

In more detail, our conjecture of the Hilbert series of a Coulomb branch moduli space of a mixed branch specified by ρ is

$$HS_\rho(t,z_i) = \sum_{m_1|_{R_\rho}} \sum_{(m_{21}\geq m_{22})|_{R_\rho}} \cdots \sum_{(m_{N1}\geq m_{N2}\cdots\geq m_{N-1N-1})|_{R_\rho}} t^{\Delta(m)} \prod_{i=1}^{N-1} z_i^{\sum_j m_{ij}} \left(\prod_{k=1}^{N-1} P_{U(k)}(m,t)\right)\Bigg|_{R_\rho}, \tag{4.54}$$

where the summations are modified in a way prescribed by a restriction map R_ρ associated to the frozen color D3-branes. $z_i, i = 1, \cdots, N-1$ are fugacities for the non-perturbative $SU(N)$ topological symmetry associated to the Coulomb branch moduli. We will now define this map, and explain how it is determined by the partition ρ.

Let us label the cells between adjacent NS5-branes of the brane diagram as 1, 2, $\cdots$, N, starting from the leftmost cell. From the brane picture, the restriction map R_ρ associated to a partition of the type $\rho = [a_1, \cdots a_n]$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ and $\sum_{i=1}^{n} a_i = N$ can be read off as follows:

- The restriction on the magnetic charges.

  From the quiver theory in (4.50), the total number of color D3-branes which are frozen in the k-th cell is given by (4.51), namely

  $$N_k = \sum_{i=1}^{n} \left(a_i - (N-k)\right) H\left(a_i - (N-k)\right). \tag{4.55}$$

  Hence, N_k magnetic charges among the k magnetic charges of $U(k)$ are set to zero in the summations of (4.54).

  N_k is always smaller than k except for the case where there is no Coulomb branch moduli. Then we have several ways to choose N_k magnetic charges which we set to zero among the k magnetic charges in the k-th cell. The rule is that we consider all the possible choices which are compatible with the condition for the magnetic charges to remain in the same Weyl chamber $\Gamma(^LG)/\mathcal{W}_{^LG}$.

- The change of the factor $P_{U(k)}$

  The factor $P_{U(k)}$ should be composed of non-frozen Coulomb branch moduli. Therefore, in the k-th cell, the factor $P_{U(k)}$ is replaced with $P_{U(k-N_k)}$ with the N_k defined in (4.55).

In this way, we propose that the restriction rule gives the Hilbert series for the Coulomb branch part $\mathcal{C}_\rho$ of the mixed branch $\mathcal{M}_\rho$. Furthermore, by using the product structure of the mixed branch of the $T[SU(N)]$ theory (4.46),

$$\mathcal{M}_\rho = \mathcal{C}_\rho \times \mathcal{C}_{\rho^D}, \tag{4.56}$$

the Hilbert series for the mixed branch $\mathcal{M}_\rho$ can be written by

$$HS_{\mathcal{M}_\rho}(t, z_i, y_j) = HS_\rho(t, z_i) \times HS_{\rho^D}\left(t, \prod_{j=1}^{N-1} x_j^{M_{ij}}\right), \tag{4.57}$$

where $x_j, j = 1, \cdots, N-1$ are the fugacities associated to the perturbative $SU(N)$ flavor symmetry. M_{ij} is an element of a matrix yielding a linear combination of the Cartan generators of the flavor symmetry group, depending on the definition of the fugacities.

In the Hilbert series computation for the Coulomb branch, we will use the fugacities $z_i, i = 1, \cdots, N-1$ associated to the Cartan generators H^z_i which give charges for the simple roots of the $SU(N)$[10] as

$$H^z_i|e_j - e_{j+1}\rangle = \delta_{ij}|e_j - e_{j+1}\rangle, \tag{4.58}$$

for $i, j = 1, \cdots, N-1$. On the other hand, for the Hilbert series computation for the Higgs branch, we will use the fugacities $x_i, i = 1, \cdots, N-1$ associated to the Cartan generators H^x_i which give charges to the simple roots of the $su(N)$ Lie algebra as

$$H^x_i|e_j - e_{j+1}\rangle = C^{su(N)}_{ij}|e_j - e_{j+1}\rangle, \tag{4.59}$$

for $i, j = 1, \cdots .N-1$ where $C^{su(N)}_{ij}$ is an element of the Cartan matrix of the $su(N)$ Lie algebra. Due to these choices of the Cartan generators,, the matrix M_{ij} is in fact the Cartan matrix $C^{su(N)}_{ij}$ in the later computation which we will perform.

Although we focus on mixed branches of the $T[SU(N)]$ theory, the restriction rule will be applicable to the computation of mixed branches of more general 3d $\mathcal{N} = 4$ gauge theories which have the type IIB brane construction without orientifolds.

The similar restriction has been made use of for computing the Hilbert series of 3d $\mathcal{N} = 2$ gauge theories [118–120]. In that case, the restriction of the magnetic charges or the corresponding Coulomb branch moduli occurs due to the generation of non-perturbative superpotentials which lift a part of the Coulomb branch moduli. In the current case, the restriction of the Coulomb branch arises since we consider a sublocus of the full Coulomb branch of the $T[SU(N)]$ theory where a Higgs branch opens up. Furthermore, the restriction of the magnetic charges can be understood from the frozen D3-branes in the brane picture.

The restriction rule with an example

The algorithmic rule defined above is quite straightforward to apply, however it can seem involved at first. Hence let us give now an explicit example of how the rule should be applied to determine the frozen magnetic charges, in a nontrivial case of the partition $[3, 2]$. In this case $N = 5$ and $n = 2$. Then,

[10]The simple roots of the $su(N)$ Lie algebra can be expressed as $e_i - e_{i+1}, i = 1, \cdots, N-1$ where $e_i, i = 1, \cdots, N$ are orthonormal bases in $\mathbb{R}^N$.

- For $a_1 = 3$, the restriction appears from the 3rd cell since $a_1 - (5-k) > 0$ when $k \geq 3$. Then,

    1. For $k = 3$, in the 3rd cell we set to zero $a_1 - (5-k) = 1$ magnetic charge.
    2. For $k = 4$, in the 4th cell we set to zero $a_1 - (5-k) = 2$ magnetic charges.

- For $a_2 = 2$, the restriction appears from the 4th cell since $a_2 - (5-k) > 0$ when $k \geq 4$. Then,

    1. For $k = 4$, in the 4th cell we set to zero $a_2 - (5-k) = 1$ magnetic charge.

Therefore, in this case, we see that a total of $2+1=3$ magnetic charges must be put to zero in the 4th cell, and only 1 magnetic charge should be put to zero in the 3rd cell. This information can be also understood in a clear way from the brane picture of the $[3,2]$ branch, as shown in the figure 4.19, where one color D3-brane is fixed in the 3rd cell and three color D3-branes are frozen in the 4th cell.

Now, in the 4th cell we have 4 magnetic charges in total. Let's call the $m_{41}, m_{42}, m_{43}, m_{44}$ and they are subject to be in the the same Weyl chamber of the weight space of $U(4)$, therefore they satisfy

$$m_{41} \geq m_{42} \geq m_{43} \geq m_{44}. \tag{4.60}$$

Among them we should choose three to vanish and the rule is that we must take into account all the possible ways. By looking at the Weyl chamber condition (4.60), we see that there are only two ways. We can have

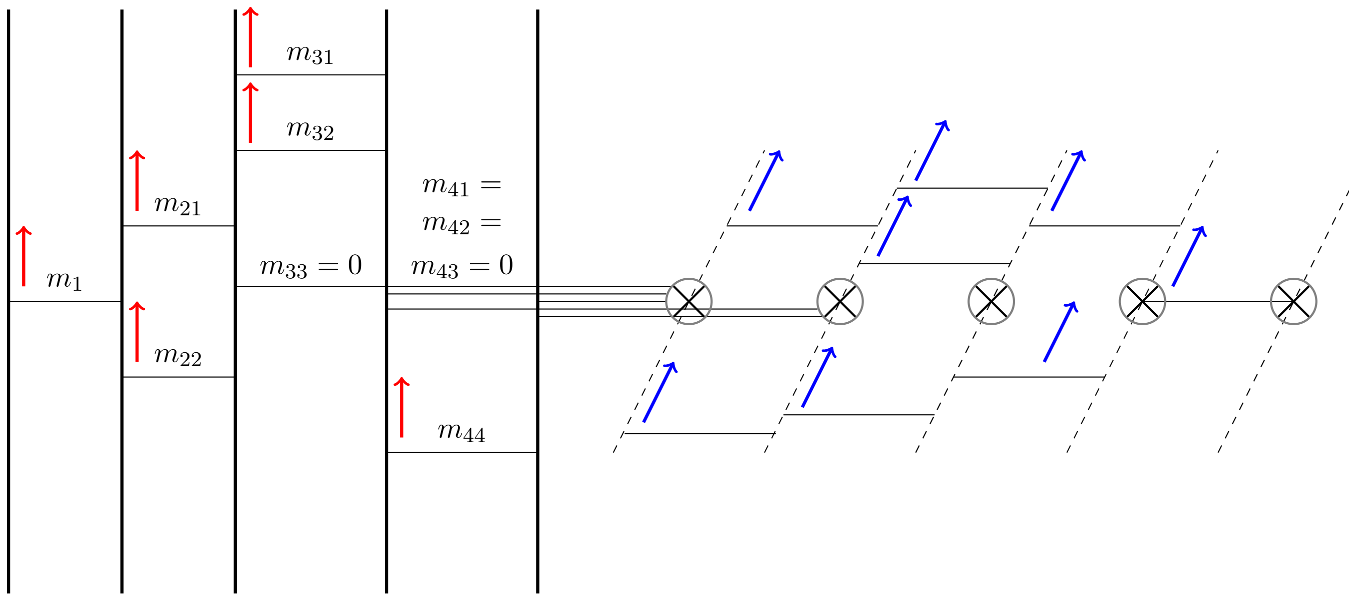
1. $0 = m_{41} = m_{42} = m_{43} \geq m_{44}$,
2. $m_{41} \geq m_{42} = m_{43} = m_{44} = 0$.

A similar reasoning works also for the magnetic charge that should be set to zero in the 3rd cell. In the 3rd cell there are three magnetic charges m_{31}, m_{32}, m_{33} for $U(3)$ satisfying

$$m_{31} \geq m_{32} \geq m_{33}, \tag{4.61}$$

and we see that in this case we have three ways to put one of the magnetic charges to zero,

1. $0 = m_{31} \geq m_{32} \geq m_{33}$,
2. $m_{31} \geq m_{32} = 0 \geq m_{33}$,



1	2	3	4	$= k.$	Number of the cell
0	0	1	3	$= N_k.$	Number of frozen branes
1	2	2	1	$= k - N_k.$	Number of mobile branes.

Figure 4.19: The $[3, 2]$ example, and the different numbers of frozen branes in every cell.

3. $m_{31} \geq m_{32} \geq m_{33} = 0$.

Therefore, in this example, we see there are in total 3×2 different sets of magnetic charges that need to be put to zero, and therefore the Hilbert series of the full Coulomb branch will split in six different sub-sums, depending on the way in which the non-zero charges are chosen. In the restriction of the Hilbert series, one has to take into account all of these conditions and sum over all of them. However, to avoid oversumming, if some value for the magnetic charge is repeated, it should be counted only once. For example, we see that $m_{31} = m_{32} = m_{33} = m_{34} = 0$ is repeated both in the first and the second way for the 4th cell.

For the practical computation of the restriction of the magnetic charges, we can divide the possibilities of setting which magnetic charges to zero into disjoint sets. This will crucially avoid the overcounting problem outlined above. Let us consider a gauge node $U(k)$ with the magnetic charges satisfying the Weyl chamber condition

$$m_1 \geq m_2 \geq \cdots \geq m_k. \tag{4.62}$$

In a Coulomb branch part of a mixed branch, the rule says that N_k of the magnetic charges are zero. Then, there are $k - N_k + 1$ possibilities of which N_k magnetic charges are zero. For $i = 0, \cdots, k - N_k$, we can consider the following set

$$m_1 \geq \cdots \geq m_{i-1} \geq m_i > 0 \geq m_{i+N_k+1} \geq m_{i+N_k+2} \geq \cdots \geq m_k. \tag{4.63}$$

with

$$m_{i+1} = m_{i+2} = \cdots = m_{i+N_k} = 0. \tag{4.64}$$

These sets are all disjoint between each other for all $i = 0, \cdots, k - N_k$ and in fact the sum of the sets exhausts all the elements in the summation after the restriction. Hence, in the practical calculation one can use the disjoint sets (4.63) to sum up all the possibilities of the restriction of the magnetic charges.

CHAPTER 5

GENERAL CONCLUSIONS

In the first part of the thesis we have analyzed the structure of the Yukawa matrices of quarks and leptons in the context of F-theory $SU(5)$ GUTs. The generation of all the Yukawa matrices in the same local patch of S_{GUT} requires a local enhancement of the gauge group to either E_7 or E_8 and we considered the possible models that may be embedded in the former. We have seen that among the set of possible models only one shows a promising structure for the Yukawa matrices. Since these Yukawas are essentially the same ones as found in the context of local E_8 models in [159], our results point towards some sort of universal structure for realistic Yukawas in the context of the proposal made in [152]. All these models require the presence of a non-commuting Higgs background to generate a large mass for the the third family of quarks and leptons and the deformation of the 7-brane superpotential due to non-perturbative effects to generate a mass for the first two families, creating a flavor hierarchy in agreement with experimental measurements.

The details concerning the 7-brane background and fluxes for the local model were discussed in section 2.4.2. In particular, a sufficiently rich set of fluxes was considered in order to obtain a realistic local 4d chiral spectrum, break the $SU(5)$ gauge group down to $SU(3)\times SU(2)\times U(1)_Y$ and implement double-triplet splitting as in [142]. The last feature is an improvement with a previous attempt to obtain a realistic spectrum out of a local patch, and involves considering a more general background compared to the one in [159]. Following similar techniques as those applied in [156–159] we have computed the holomorphic Yukawa couplings of this model, taking into account non-perturbative effects, and shown that they exhibit an appropriate hierarchical structure. Finally, we were able to obtain the physical Yukawas by means of computing the kinetic terms for the MSSM chiral fields and imposing their canonical normalization.

These technical results have led to the analysis of section 2.4.11, where the phenomenological possibilities for two slightly different models with the same structure for the matter curves are studied in detail. In one of the models we find wide regions in the parameter space where the values of the Yukawa couplings are compatible with measured values (model A), while in the other we do not find any compatible region (model B). As in [159] we also find that the empirical values for the mixing $|V_{tb}|$ imply a very small distance among the up and down Yukawa points, therefore supporting the initial hypothesis of E_7 enhancement. The other entries of the CKM matrix are more difficult to analyze, as they heavily depend on non-holomorphic data related to the lightest family of quark and leptons over which we have poor control in this ultra-local approach. In particular, as argued in [159] one would expect that curvature effects within S_{GUT} could play an important rôle in their evaluation, which following the general arguments in [145] could give a rationale for the size of the Cabibbo angle.

The results obtained here show how $SU(5)$ F-theory GUTs possess an interesting and potentially viable flavor structure when the proposal of [152] is implemented in realistic local models. To reach a more precise understanding of this flavor structure it would be necessary to go beyond the leading ϵ contributions considered so far. This would allow to compute the mass of the first generation of fermions as well as additional entries in the CKM matrix. Moreover, in addition to the non-perturbative effects that have been considered in this paper, other effects recently studied in [183] may have an impact in the structure of the Yukawa matrices. It would be therefore desirable to develop in more detail the computation of the couplings generated by such effects and see if, whenever present, they are comparable, dominant or subdominant with respect to the ones considered here. Finally it would be important to see whether they can give rise to novel features in the flavor sector of F-theory GUTs. Another missing ingredient in this construction is the realization of the local models considered in this paper in a fully-fledged F-theory compactification. Extension to global models would be necessary to check the possibility of having the correct chiral spectrum in 4d and also the viability of hypercharge flux GUT breaking. Moreover, it may be that parameters that look independent from a local viewpoint are not such globally, something that is crucial for interpreting our results in the context of the landscape of F-theory vacua.

Finally, while we have gained a good insight over the structure of couplings of quarks and charged leptons, neutrinos and the MSSM Higgs fields remain elusive. A natural mechanism to generate mass terms for them would be via the coupling to singlets which would eventually get a vev. The presence of singlets which are not localized on the GUT divisor makes the computation more involved as the methods employed so far would not be sufficient. It would be therefore desirable to develop techniques for these kind of computations as these missing terms play an important rôle for flavor physics and electroweak symmetry breaking.

In the second part of the thesis we have discussed the phenomenon of supersymmetry enhancement in quantum field theories. We have first reviewed basic features of supersymmetric and superconformal field theories with 8 supercharges in 4d. Then we have reviewed the standard technique of [55–57] to deform a $\mathcal{N} = 2$ in such a way that supersymmetry is broken to $\mathcal{N} = 1$. This involves adding an extra $\mathcal{N} = 1$ chiral multiplet M coupled to the moment-map operator, and giving it a vev along some specific nilpotent orbit of the flavor symmetry algebra. We then reviewed the technique of a-maximization, useful to correctly identify the $U(1)$ R-symmetry in the IR. We gave a specific, completely worked-out example of this procedure, showing as in [56] the flow that exists starting from a deformation of the Argyres-Douglas H_1 theory in the UV, and ending to the Argyres-Douglas H_0 theory.

We later moved to present new results, so far no absent in the literature. By using the a-maximization technique we performed several scans, either by changing the original undeformed $\mathcal{N} = 2$ theory in the UV, or by turning on deformations corresponding to nilpotent orbits which have not been considered yet. We find a new case of supersymmetry enhancement, for non-minimal deformations of Minahan-Nemeschansky E_7 theory. We also prove by exhaustion of possibilites that the only orbit which give enhancement for the Minahan-Nemeschansky E_8 theory is the maximal orbit. We then considered numerous cases in which the UV starting point is a superconformal quiver gauge theory, with quiver type given by the affine Dynkin diagram of D-type and $E-$ type, and SU gauge nodes. We sistematically see no enhancement in any of these cases. The reason for this failure is not understood to date, and we leave it for further studies. One idea could be the fact that all these superconformal quivers cannot be realized by the class-S construction, and we could maybe conjecture that supersymmetry enhancement only can exist within class S.

We further move to discuss another new result, which is the geometric interpretation of supersymmetry enhancement. We engineered the 4d QFT as the worldvolume theory of a D3 brane probing a singularity in F-Theory. The singularity corresponds to a T-brane of 7-branes, in the dual IIB picture[1], and it is this T-brane which realizes the nilpotent vev for the chiral multiplet M which is added in the field theory description. The RG-flow is interpreted as a local zoom into a small neighborhood of the singularity. This is interpreted physically by saying that in the IR the D3 probe does not have enough energy to resolve global aspects of the singularity, and only resolve local aspects. We explicitly show how the Weierstrass model corresponding to the deformed UV theory recovers the Weierstrass model corresponding to the IR theory, in this limit. We wrote down some specific fully worked-out cases, and we refer the reader to [5] for a complete systematic and geometric understanding of susy enhancement for of all the rank 1 cases.

In the third part of the thesis we have first reviewed basic features of theories with 8 supercharges in 3 spacetime dimensions. Then we determined the restriction rule for computing the Hilbert series for the Coulomb branch part of a mixed branch of the 3d $\mathcal{N}=4$ $T[SU(N)]$ theory from the Hilbert series of the full Coulomb branch. In particular, the brane realization of the mixed branch precisely gives an explicit way to truncate the magnetic charges as well as to reduce the classical dressing factor. We confirmed this method by comparing the result obtained from the restriction with the result obtained from the technique of going first to the IR gauge theory.

We also computed the Hilbert series of the Higgs branch part of a mixed branch of the 3d $T[SU(N)]$ theory in two ways. One way is to use the technique of the Molien-Weyl projection discussed in 4.4.1. In order to use this method, we consider an IR gauge theory by decoupling the Coulomb branch moduli. In this way we were able to compute the Hilbert series of the Higgs branch part. The other method consists in utilizing 3d mirror symmetry and the restriction rule for computing the Coulomb branch part of a mixed branch. Intriguingly, this completely different computation exactly gives the same result including flavor fugacities. This provides a non-trivial check of the restriction rule as well as the mirror symmetry of the 3d $\mathcal{N}=4$ theories.

By taking the product of the Hilbert series of the two branches, we are able to compute the Hilbert series of any mixed branch of the $T[SU(N)]$ theory. The restriction rule indeed gives a systematic way to obtain the series from the product of the Hilbert series of the two full branches.

Although our computation determines the Hilbert series of a mixed branch of the 3d $T[SU(N)]$ theory from the restriction rule, it is interesting to consider the Hilbert series of the full moduli space of the 3d $T[SU(N)]$ theory. In fact, the restriction procedure seems to suggest a natural

[1] Whenever the IIB limit exists.

way to obtain it. The basic structure of the Hilbert series of the full Coulomb branch moduli space of the $T[SU(N)]$ theory is that it is given by a sum of a set of magnetic charges $\{\overrightarrow{m}\}$ as[2]

$$HS\,(t) = \sum_{\{\overrightarrow{m}\}} f\,(\{\overrightarrow{m}\}, t)\,. \tag{5.1}$$

The restriction rule says that among the possible summation of $\{\overrightarrow{m}\}$, there are special sub-summations where a Higgs branch opens up. For example, when $\overrightarrow{m} = \overrightarrow{0}$, which implies the origin of the Coulomb branch moduli space, we have the full Higgs branch which shares the origin. There is a natural guess to implement the intersection to the Hilbert series. Namely from the summation (5.1), we remove $\overrightarrow{m} = \overrightarrow{0}$, and add the Hilbert series of the full Higgs branch which we denote by $HS_{\mathcal{H}_{[N]}}(t)$,

$$HS\,(t) = \sum_{\{\overrightarrow{m}\}\backslash\{\overrightarrow{0}\}} f\,(\{\overrightarrow{m}\}, t) + HS_{\mathcal{H}_{[N]}}(t). \tag{5.2}$$

This guess will also lead to a way of incorporating another mixed branch further. For example, there is a mixed branch specified by a partition $[N-1,1]$. The restriction rule says that for computing the Hilbert series of the Coulomb branch $\mathcal{C}_{[N-1,1]}$, we sum over a subset of $\{\overrightarrow{m}\}$ and we denote the subset by $\{\overrightarrow{m}\}|_{R_{[N,1]}}$, which also includes the origin. The $R_{[N-1,1]}$ is the restriction map introduced in section 4.8.3. The Hilbert series of the Coulomb branch part can be written by $HS_{\mathcal{C}_{[N-1,1]}}(t) = \sum_{\{\overrightarrow{m}\}|_{R_{[N,1]}}} f(\{\overrightarrow{m}\}, t)|_{R_{[N-1,1]}}$. Along the sublocus, a Higgs branch $\mathcal{H}_{[N,1]}$ opens up and we denote the Hilbert series for $\mathcal{H}_{[N-1,1]}$ by $HS_{\mathcal{H}_{[N-1,1]}}(t)$. Then the Hilbert series with the mixed branch might be

$$\begin{aligned} HS\,(t) \;\; &= \sum_{\{\overrightarrow{m}\}\backslash\{\overrightarrow{m}\}|_{R_{[N-1,1]}}} f\,(\{\overrightarrow{m}\}, t) \\ &\quad + \left(\sum_{\{\overrightarrow{m}\}|_{R_{[N-1,1]}}\backslash\{\overrightarrow{m}\}|_{R_{[N]}}} f\,(\{\overrightarrow{m}\}, t)\,|_{R_{[N-1,1]}} \right) \times HS_{\mathcal{H}_{[N-1,1]}}(t) \\ &\quad + HS_{\mathcal{H}_{[N]}}(t), \end{aligned} \tag{5.3}$$

where $\{\overrightarrow{m}\}|_{R_{[N]}} = \overrightarrow{0}$. Therefore, the restriction rule yields a natural guess for computing the Hilbert series of the full moduli space by removing some magnetic charges corresponding to a sublocus and adding the Hilbert series of the mixed branch which stems from the sublocus. The repetition of the procedures would give a systematic way to compute the Hilbert series of the full moduli space of the 3d $\mathcal{N} = 4$ $T[SU(N)]$ theory although the combinatorics of dividing the

[2]We suppress the flavor fugacities for simplicity.

summation will be more complicated. At least, we checked the above procedure is consistent with the Hilbert series of a variety made from two $\mathbb{C}^n$-planes glued at a point. A similar gluing was first discussed in [139]. It would be certainly interesting to prove that this guess is correct, and we leave it for future work.

We hope the result obtained in this chapter could be useful for future studies on the mixed branches of the moduli space of more general $3d\,\mathcal{N}=4$ supersymmetric theories. One interesting direction could be including $O3^{\pm}$-planes to the brane picture, and therefore to determine the restriction rule for computing the Hilbert series of mixed branches of the $T[SO(N)]$ and $T[Sp(N)]$ theories constructed in [107].

CHAPTER 6

CONCLUSIONES GENERALES

En este último capítulo escibimos las conclusiones de la tesis en castellano, en una versión mucho más esquemática y reducida comparada con las conclusiones dadas en inglés, en la sección (5).

En la primera parte de la tesis hemos analizado la estructura de las matrices de Yukawa de los quarks y los leptones en el contexto de una teoría de Gran Unificación con grupo gauge $SU(5)$ en teoría F. La generación de todas las matrices de Yukawa en el mismo patch local de S_{GUT} requiere un enhancement local del grupo gauge a E_7 o a E_8; hemos considerado los posibles modelos que pueden ser embebidos en E_7. Hemos visto que, en el conjunto de los modelos posibles, solo uno tiene una buena estructura para las matrices de Yukawa. Estos Yukawas son básicamente del mismo tipo que los que se encuentran en el contexto de modelos locales E_8 en [159]. Por lo tanto, nuestro resultado indica hacia una estructura universal para Yukawas realistas en el contexto de la propuesta hecha en [152]. Todos estos modelos necesitan la presencia de un Higgs background no-comutativo que genere una masa suficientemente grande para la tercera familia de quarks y leptones. También necesita efectos no-perturbativos que deformen el superpotencial de la 7-brana y así generar una masa para las primeras dos familias. Esto crea una jerarquía de sabor que encaja con las medidas experimentales.

En la segunda parte de la tesis hemos hablado del fenómeno de incremento de supersimetría en teorías cuánticas de campos. Para empezar, hemos explicado algunos hechos básicos de la dinámica de teorías supersimétricas y superconformes con 8 supercargas en 4d. Luego hemos explicado la técnica estándar usada para deformar una teoría $\mathcal{N} = 2$ de una manera tal que la supersimetría sea rota a $\mathcal{N}=1$. Esto se hace añadiendo un multiplete quiral M acoplado con el operador de moment-map, y dándole un valor de expectación en el vacío que sea en la misma dirección que una especifica órbita nilpotente del álgebra de sabor. Luego, hemos explicado la técnica de a-maximization, que es muy útil para identificar de manera correcta la simetría R $U(1)$ en el IR. Damos un ejemplo especifico, con todos los detalles, de cómo esto se aplica al estudio del incremento de supersimetría. En particular, como en [56] miramos al caso del flujo de una teoría que en el UV es una deformación de la teoría de Argyres-Douglas H_1, y en el IR es la teoría de Argyres-Douglas H_0.

Luego hemos presentado los resultados nuevos, que hasta ahora no han aparecido en la literatura. Usando a-maximization hemos hecho numerosas búsquedas para encontrar nuevos casos de incremento de supersimetría. Hemos intentado cambiar la teoría original en el UV, y incluso usar deformaciones que corresponden a órbitas nilpotentes nuevas. Encontramos casos nuevos de incremento de supersimetría, por ejemplo el caso de la teoría de Minahan-Nemeschansky E_7 deformada con la órbita sub-regular. También conseguimos demostrar que la teoría de Minahan-Nemeschansky E_8 solo tiene una órbita que da incremento de supersimetría: la órbita maximal. Luego hemos considerados muchos casos en que la teoría UV está dada por un quiver superconforme, con tal quiver dado por el diagrama de Dynkin affín de tipo D o E, y nodos SU. De manera sistemática, nunca encontramos incremento de supersimetría en estos casos. La razón por esta falta de éxito aún no está clara, y la dejamos para estudios en el futuro. Una idea podría ser que todos estos quivers superconformes no se pueden realizar en Clase-S, entonces podríamos conjeturar que el fenómeno de incremento de supersimetría existe, por alguna razón aún no comprendida, solamente para teorías de clase-S.

Luego hablamos de otro resultado nuevo, que es la interpretación geométrica del incremento de supersimetría.

Hacemos el engineering de la QFT en 4d a través de una D3-brana que explora una singularidad en teoría F. La singularidad corresponde a una T-brana de 7-branas, en el contexto de la teoría IIB dual[1], y es esta T-brana la que realiza el multiplete quiral nilpotente M, en la descripción de teoría de campos. El flujo RG es interpretado de manera geométrica como un zoom local en un entorno pequeño de la singularidad. Esto se interpreta físicamente diciendo que en el IR la D3 no tiene bastante energía para explorar aspectos globales de la singularidad, y solo vee aspectos locales. Demostramos explícitamente cómo el modelo de Weierstrass correspondiente a la teoría deformada en el UV se reduce al modelo de Weierstras correspondiente

[1]Cuando el limite IIB existe.

a la teoría en el IR, bajo este zoom local. Damos algunos ejemplos específicos en los que esta cuenta es desarrollada en detalle. Esta técnica se puede aplicar a cualquier caso de teorías de rango 1, cómo se explica en [5].

En la tercera parte de la tesis hemos hecho un review de algunas propiedades básicas de las teorías con 8 supercargas en 3 dimensiones espaciales. Luego hemos determinado una restriction rule para calcular la serie de Hilbert de la parte de Coulomb de una rama mixta de la teoría 3d $\mathcal{N} = 4$ $T[SU(N)]$, a partir de la serie de Hilbert de la rama de Coulomb. En más detalle, la realización de las ramas mixtas a través de un sistema de branas nos da una manera explícita para truncar la suma en las cargas magnéticas tanto cómo cortar el dressing factor clásico. Hemos confirmado este método comparando el resultado obtenido desde la restrición, con el resultado obtenido desde la técnica de ir antes a la teoría en el IR. También hemos calculado la serie de Hilbert de la parte de Higgs de una rama mixta de la teoria 3d $T[SU(N)]$ de dos maneras. Una manera consiste en usar la proyección de Molien-Weyl, explicada en 4.4.1. La otra manera consiste en utilizar 3d mirror symmetry y la regla de restricción para una parte de Coulomb de la rama mixta correspondiente. Estas dos maneras completamente distintas para hacer la cuenta producen exactamente el mismo resultado para la serie de Hilbert, incluyendo también fugacidades de sabór. Esto es una confirmación no-trivial de la regla de restrición que hemos propuesto.

CHAPTER 7

APPENDICES.

7.1 Dynkin Label notation

In this appendix we fix the convention we will use in most of the thesis (apart from section 2.4) in order to denote irreducible representation of complex semisimple Lie Algebras.

Fix a a Lie algebra $\mathfrak{g}$ of rank r. We will denote any irrep $\mathcal{R}$ of $\mathfrak{g}$ by the Dynkin Labels, which are a set of r natural numbers. Namely, $\mathcal{R} = [n_1, n_2, ..., n_r]$. As an example, consider $\mathfrak{su}(3)$. Then the trivial representation will be denoted by $[0, 0]$, the fundamental by $[1, 0]$, the antifundamental by $[0, 1]$, the adjoint by $[1, 1]$, the second rank symmetric product of the fundamental by $[2, 0]$ and so on.

Such Dynkin Labels are defined as follows. For any given irreducible representation $\mathcal{R}$ there will exist a unique highest weight state λ. It is defined by the fact that it is annihilated by the action of any positive root on it. Since they weight space is a vector space, we can expand w on the base of the weight space, which is given by the fundamental weights. The Dynkin labels are then the coefficient of such expansion.

The advantage of using this notation and not other notations as the one with dimensions, or (colored) Young Tableauxes is double. As a first thing, the Dynkin Label notation holds also for irrepses of exceptional Lie algebras, while the notation with (colored) Young Tableauxes does not[1]. Second, in chapter 4 we will often consider series of the form

$$\sum_{i=0}^{\infty} \chi(\mathcal{R}_i) t^i \tag{7.1}$$

where $\chi(\mathcal{R}_i)$ is the character of some representation $\mathcal{R}_i$. It is much more convenient to write these Series with the Dynkin label notation, as the representations appearing will often be such that for different values of the index i, their Dynkin Label will be related in a easy way.

There are different ways for connecting the Dynkin Label notation with the notation based on the dimension of the representation. For example, we recall now the most direct one: computing the dimension of an irrep from its Dynkin labels. This is the Peter-Weyl dimension formula, a special case of the more general Peter-Weyl character formula. Consider a representation $\mathcal{R}_\lambda$ with highest weight λ. Let Δ^+ to be the set of positive roots and define $\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha$ half of the sum of the positve roots. Then, the dimension of $\mathcal{R}_\lambda$ will be given by

$$\dim \mathcal{R}_\lambda = \frac{\prod_{\alpha\in\Delta^+}(\lambda+\rho,\alpha)}{\prod_{\alpha\in\Delta^+}(\rho,\alpha)} \tag{7.2}$$

For example, consider the case of $\mathfrak{su}(3)$. Then by evaluating (7.2), a generic irrep $[m,n]$ will have a dimension given by

$$\dim\,[m,n] = \frac{1}{2}\,(m+1)\,(n+1)\,(m+n+2) \tag{7.3}$$

It is now trivial to check that, for example $[1,1]$ is the only irrep of dimension 8, therefore being the adjoint.

7.2 E_7 machinery

The Lie algebra of E_7 has 133 generators Q_α. We will always work in the Weyl-Cartan basis, where such generators are split in the 7 generators of the Cartan subalgebra $H_i, i = 1...7$ and 126 roots E_ρ. In this basis he commutation rules among Cartan and roots are the following

$$[H_i, E_\rho] = \rho_i E_\rho \tag{7.4}$$

From (7.4) we see that each root E_ρ is uniquely associated with the vector ρ of its charges under the Cartan subalgebra, and so one may identify E_ρ with ρ.

[1] As far as the author knows.

In this notation, the roots of $\mathfrak{e}_7$ take the following form:

$$\left(\underline{\pm 1, \pm 1, 0, 0, 0, 0}, 0\right) \tag{7.5}$$

$$2\left(\pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1, \pm\sqrt{2}\right) \tag{7.6}$$

$$\left(0, 0, 0, 0, 0, 0, \pm\sqrt{2}\right) \tag{7.7}$$

where in (7.6) we consider only charge vectors in which an even number of +1 appear.

In order to choose the vev of the Higgs field, we need to decompose $E_7 \to SU(5) \times SU(2) \times U(1)^2$. The dimension 133 adjoint representation of $\mathfrak{e}_7$ decomposes as follows:

$$\begin{aligned} \mathfrak{e}_7 \quad &\supset \quad \mathfrak{su}_5^{\mathrm{GUT}} \oplus \mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1 \\ \mathbf{133} \quad &\to \quad (\mathbf{24}, \mathbf{1})_{0,0} \oplus (\mathbf{1}, \mathbf{3})_{0,0} \oplus 2(\mathbf{1}, \mathbf{1})_{0,0} \oplus ((\mathbf{1}, \mathbf{2})_{-2,1} \oplus c.c.) \\ &\quad \oplus (\mathbf{10}, \mathbf{2})_{1,0} \oplus (\mathbf{10}, \mathbf{1})_{-1,1} \oplus (\mathbf{5}, \mathbf{2})_{0,-1} \oplus (\mathbf{5}, \mathbf{1})_{-2,0} \oplus (\mathbf{5}, \mathbf{1})_{1,1} \oplus c.c. \end{aligned} \tag{7.8}$$

Let us look for a subset of $\mathfrak{su}(2)$ roots among the roots of $\mathfrak{e}_7$. Notice that among the roots given in (7.5) we can identify two of them which add to zero. The generators associated to these roots will be the raising and lowering operators for the $\mathfrak{su}(2)$ subalgebra. We choose

$$E^+ := E_{\frac{1}{2}(1,1,1,1,1,1,\sqrt{2})} \tag{7.9a}$$

$$E^- := E_{-\frac{1}{2}(1,1,1,1,1,1,\sqrt{2})} \tag{7.9b}$$

From the commutation rules for the root operators we have

$$[E_\alpha, E_\beta] = \alpha \cdot \overrightarrow{H} \tag{7.10}$$

in the case that $\alpha + \beta = 0$. In the case of $E^\pm$ their commutator is given by

$$P := [E^+, E^-] = \frac{1}{2}(H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + \sqrt{2}H_7) \tag{7.11}$$

so that $\{E^+, E^-, P\}$ generates a $\mathfrak{su}(2)$ subalgebra of $\mathfrak{e}_7$.

$$[E^+, E^+] = 0 \tag{7.12a}$$

$$[E^-, E^-] = 0 \tag{7.12b}$$

$$[E^+, E^-] = P \tag{7.12c}$$

In the main text we use two particular linear combination of Cartan generators Q_1 and Q_2, that generate the two Abelian factors in $\mathfrak{su}_5^{\mathrm{GUT}} \oplus \mathfrak{su}_2 \oplus \mathfrak{u}_1 \oplus \mathfrak{u}_1$. These are

$$\begin{aligned} Q_1 &= -\frac{1}{2}\left(H_1 + H_2 + H_3 + H_4 + H_5 - H_6 - 2\sqrt{2}H_7\right) \\ Q_2 &= -\frac{1}{2}\left(2H_6 - \sqrt{2}H_7\right) \end{aligned}$$

With this assignment for the roots of the $SU(2)$ subgroup and the generators of the two $U(1)$s, we can also identify how all the other roots of E_7 split into representations of $SU(5) \times SU(2) \times U(1) \times U(1)$. We find

E_7 generator	$SU(5) \times SU(2)$	Q_1, Q_2 charges
$(\underline{+1,-1,0,0,0},0,0) \oplus H_1, H_2, H_3, H_4$	$(\mathbf{24},\mathbf{1})$	(0,0)
Q_1, Q_2 cartans	$2(\mathbf{1},\mathbf{1})$	(0,0)
ρ_+, ρ_-, P	$(\mathbf{1},\mathbf{3})$	(0,0)
$(0,0,0,0,0,0,\sqrt{2})$ and $\frac{1}{2}(-1,-1,-1,-1,-1,-1,\sqrt{2})$	$(\mathbf{1},\mathbf{2})$	(-2,1)
$\frac{1}{2}(1,1,1,1,1,1,-\sqrt{2})$ and $(0,0,0,0,0,0,-\sqrt{2})$	$(\mathbf{1},\mathbf{2})$	(2,-1)
$(\underline{1,0,0,0,0},1,0)$ and $\frac{1}{2}(\underline{1,-1,-1,-1,-1},1,-\sqrt{2})$	$(\mathbf{5},\mathbf{2})$	(0,-1)
$\frac{1}{2}(\underline{1,-1,-1,-1,-1},1,\sqrt{2})$	$(\mathbf{5},\mathbf{1})$	(-2,0)
$(\underline{1,0,0,0,0},-1,0)$	$(\mathbf{5},\mathbf{1})$	(1,1)
$(\underline{1,1,0,0,0},0,0)$ and $\frac{1}{2}(\underline{1,1,-1,-1,-1},-1,-\sqrt{2})$	$(\mathbf{10},\mathbf{2})$	(1,0)
$\frac{1}{2}(\underline{-1,-1,-1,1,1},-1,\sqrt{2})$	$(\mathbf{10},\mathbf{1})$	(-1,1)

Table 7.1: Roots of E_7 and their charges under the subgroup $SU(5) \times SU(2) \times U(1)^2$.

7.3 Local chirality and doublet-triplet splitting

In this appendix we provide further details on the computations regarding the local chirality for the models A and B, as follows from the discussion of section 2.4.4.

7.3.1 Model A

Using the explicit form of q_R and q_S that may be found in table 2.4 we can write explicitly the chirality conditions (2.67) for all the various sectors appearing in the model. These are

$$q_Y\tilde{N}_Y - M_1 > 0\,, \qquad q_Y = -\frac{1}{6}\,,\frac{2}{3}\,,-1 \tag{7.13a}$$

$$q_Y\tilde{N}_Y - M_2 > 0\,, \qquad q_Y = \frac{1}{2}\,,-\frac{1}{3} \tag{7.13b}$$

$$\left(|a|^2-1\right)\left(2M_1+\frac{\tilde{N}_Y}{3}\right)+2\,\mathrm{Re}[a]\left(\frac{N_Y}{3}-2N_1\right)=0 \tag{7.13c}$$

$$\left(|a|^2-1\right)\left(2M_1-\frac{\tilde{N}_Y}{2}\right)-2\,\mathrm{Re}[a]\left(2N_1+\frac{N_Y}{2}\right)>0 \tag{7.13d}$$

$$-\left(N_1+N_2+\frac{N_Y}{3}\right)\mathrm{Re}[(\mu_1^2+\mu_2^2)(a\mu_1^2+b\mu_2^2)]+\frac{1}{2}\left(M_1+M_2-\frac{\tilde{N}_Y}{3}\right)\hat{\mu}^4=0 \tag{7.13e}$$

$$-\left(N_1+N_2-\frac{N_Y}{2}\right)\mathrm{Re}[(\mu_1^2+\mu_2^2)(a\mu_1^2+b\mu_2^2)]+\frac{1}{2}\left(M_1+M_2+\frac{\tilde{N}_Y}{2}\right)\hat{\mu}^4>0 \tag{7.13f}$$

where we defined $\hat{\mu}^4 = |a\,\mu_1^2+b\,\mu_2^2|^2-|\mu_1^2+\mu_2^2|^2$. From (7.13a) and (7.13b) we find two possible branches according to the sign of $\tilde{N}_Y$

$$\tilde{N}_Y\leq 0 \quad\rightarrow\quad \begin{cases} M_1<\frac{2}{3}\tilde{N}_Y \\ M_2<\frac{1}{2}\tilde{N}_Y \end{cases}, \quad \tilde{N}_Y>0 \quad\rightarrow\quad \begin{cases} M_1<-\tilde{N}_Y \\ M_2<-\frac{1}{3}\tilde{N}_Y \end{cases}. \tag{7.14}$$

For each of these branches the whole system (7.13) has solution and therefore it is possible to obtain (at least locally) the correct chiral spectrum of the MSSM. This is in contrast to what happened in [159] where a solution was not possible. We can easily understand why this occurs by merely taking $a=b=1$ which was the particular case considered in [159]. If we impose (7.13c) and (7.13e) with $a=b=1$ then (7.13d) and (7.13f) reduce to $-N_1>0$ and $N_1>0$ respectively and therefore the system does not allow for solutions.

7.3.2 Model B

For the model B we find a similar set of equations that are necessary to obtain the correct 4d chiral spectrum

$$q_Y \tilde{N}_Y - M_1 > 0\,, \qquad q_Y = -\frac{1}{6}\,, \frac{2}{3}\,, -1 \tag{7.15a}$$

$$\frac{1}{2}\tilde{N}_Y - M_2 > 0\,, \tag{7.15b}$$

$$\frac{1}{3}\tilde{N}_Y + M_2 = 0\,, \tag{7.15c}$$

$$(|a|^2 - 1)\left(2M_1 + \frac{\tilde{N}_Y}{3}\right) + 2\mathrm{Re}[a]\left(\frac{N_Y}{3} - 2N_1\right) = 0 \tag{7.15d}$$

$$(|a|^2 - 1)\left(2M_1 - \frac{\tilde{N}_Y}{2}\right) - 2\mathrm{Re}[a]\left(2N_1 + \frac{N_Y}{2}\right) > 0 \tag{7.15e}$$

$$-\left(N_1 + N_2 + \frac{N_Y}{3}\right)\mathrm{Re}[(\mu_1^2 + \mu_2^2)(a\mu_1^2 + b\mu_2^2)] + \frac{1}{2}\left(M_1 + M_2 - \frac{\tilde{N}_Y}{3}\right)\hat{\mu}^4 > 0 \tag{7.15f}$$

$$-\left(N_1 + N_2 - \frac{N_Y}{2}\right)\mathrm{Re}[(\mu_1^2 + \mu_2^2)(a\mu_1^2 + b\mu_2^2)] + \frac{1}{2}\left(M_1 + M_2 + \frac{\tilde{N}_Y}{2}\right)\hat{\mu}^4 > 0 \tag{7.15g}$$

In this case the sign of $\tilde{N}_Y$ is fixed and there is only one branch of solutions to (7.15a)-(7.15c), namely

$$\begin{cases} M_1 < -\tilde{N}_Y \\ \tilde{N}_Y > 0 \\ M_2 = -\frac{\tilde{N}_Y}{3} \end{cases} . \tag{7.16}$$

Here the whole system (7.15) admits solutions even at the point $a = b = 1$ and therefore it is also possible to obtain the correct chiral spectrum in 4d.

7.4 Zero mode wavefunctions

In this appendix we present details of the computation of the zero modes wavefunctions in holomorphic and real gauge. We start by collecting some data necessary for the computation of the Yukawa couplings in holomorphic gauge and then discuss how to obtain a solution of the full system of F-term and D-term equations in real gauge for all sectors appearing in the model.

7.4.1 Wavefunctions in holomorphic gauge and Yukawa couplings

Once the non-perturbative corrections are taken into account the Yukawa matrix has the following general form

$$Y = m_*^4 \pi^2 f_{abc}\, \mathrm{Res}_p\left[\eta^a \eta^b h_{xy}\right]\,, \tag{7.17}$$

where (at first order in ϵ) η is given by

$$\eta = -i\Phi^{-1}\left[h_{xy} + i\epsilon\partial_x\theta_0\partial_y\left(\Phi^{-1}h_{xy}\right) - i\epsilon\partial_y\theta_0\partial_x\left(\Phi^{-1}h_{xy}\right)\right]. \tag{7.18}$$

The explicit form of η is different for each sector because of the different form of Φ and h_{xy}. Here we give the explicit expression for h and η for the various sectors, taking θ_0 as

$$\theta_0 = i(x\theta_x + y\theta_y). \tag{7.19}$$

The result for each sector is

$(\mathbf{10}, \mathbf{2})_{1,0}$

$$h^i_{\mathbf{10}}/\gamma_{\mathbf{10}} = m_*^{3-i}(ax-y)^{3-i}, \tag{7.20}$$

$$\begin{aligned} i\eta^i_{\mathbf{10}}/\gamma^i_{\mathbf{10}} &= \frac{m_*^{3-i}}{\det\Phi_{\mathbf{10}}}\begin{pmatrix} -m(ax-y)^{3-i} \\ \mu_1^2(ax-y)^{4-i} \end{pmatrix} \\ &+ \epsilon\,\frac{m_*^{3-i}}{(\det\Phi_{\mathbf{10}})^3}\begin{pmatrix} -m(ax-y)^{3-i} \\ \mu_1^2(ax-y)^{4-i} \end{pmatrix}\left[m^3\theta_y - 2\mu_1^4(ax-y)\,(a\theta_y+\theta_x)\right] \\ &+ \epsilon\,\frac{m_*^{3-i}\,(a\theta_y+\theta_x)}{(\det\Phi_{\mathbf{10}})^2}\begin{pmatrix} -i(2i-7)\mu_1^2\,m(ax-y)^{3-i} \\ i(ax-y)^{2-i}\left((i-4)\mu_1^4(y-ax)^2+(i-3)m^3x\right) \end{pmatrix}. \end{aligned} \tag{7.21}$$

$(\mathbf{5}, \mathbf{1})_{-2,0}$

$$h_{\mathbf{5},\mathbf{1}}/\gamma_{\mathbf{5},\mathbf{1}} = 1 \tag{7.22}$$

$$i\eta_{\mathbf{5},\mathbf{1}}/\gamma_{\mathbf{5},\mathbf{1}} = -\frac{1}{\Phi_{\mathbf{5}_U}} + \epsilon\,\frac{2\mu^2(\theta_x + a\,\theta_y)}{\Phi^3_{\mathbf{5}_U}}. \tag{7.23}$$

$(\bar{\mathbf{5}}, \mathbf{1})_{-1,-1}$

$$h^i_{\bar{\mathbf{5}},\mathbf{1}} = m_*^{3-i}\left(\left(\mu_1^2+\mu_2^2\right)x + \left(a\mu_1^2+b\mu_2^2\right)y\right)^{3-i} \tag{7.24}$$

$$\begin{aligned} i\eta^i_{\bar{\mathbf{5}},\mathbf{1}}/\gamma^i_{\bar{\mathbf{5}},\mathbf{1}} &= -\frac{1}{\Phi_{\bar{\mathbf{5}},\mathbf{1}}}h^i_{\bar{\mathbf{5}},\mathbf{1}} + \epsilon\,\frac{h^i_{\bar{\mathbf{5}},\mathbf{1}}}{\Phi^3_{\bar{\mathbf{5}},\mathbf{1}}}\left(\theta_y\left(a\mu_1^2+b\mu_2^2\right)+\left(\mu_1^2+\mu_2^2\right)\theta_x\right) \\ &+ \epsilon\,\frac{m_*h^{i-1}_{\bar{\mathbf{5}},\mathbf{1}}}{\Phi^2_{\bar{\mathbf{5}},\mathbf{1}}}(3-i)\left[\theta_x\left(a\mu_1^2+b\mu_2^2\right)-\left(\mu_1^2+\mu_2^2\right)\theta_y\right] \end{aligned} \tag{7.25}$$

$(\bar{\mathbf{5}},\mathbf{2})_{0,1}$

$$h^i_{\bar{\mathbf{5}},\mathbf{2}} = m_*^{3-i}(a(x-x_0)-(y-y_0))^{3-i}\,, \tag{7.26}$$

$$\begin{aligned} i\eta^i_{\bar{\mathbf{5}},\mathbf{2}}/\gamma^i_{\bar{\mathbf{5}},\mathbf{2}} &= \frac{m_*^{3-i}}{\det\Phi_{\bar{\mathbf{5}},\mathbf{2}}}\begin{pmatrix} -m\left(a\left(x-x_0\right)-y+y_0\right)^{3-i} \\ \left(a\left(x-x_0\right)-y+y_0\right)^{3-i}\left(\mu_2^2(bx-y)+\kappa\right)\end{pmatrix} \\ &+ \epsilon\,\theta_y\frac{m_* h^{i-1}_{\bar{\mathbf{5}},\mathbf{2}}}{(\det\Phi_{\bar{\mathbf{5}},\mathbf{2}})^2}\begin{pmatrix} \mu_2^2 m\left[ab\left((2i-7)x+x_0\right)-2a(i-3)y+b\left(y-y_0\right)\right] \\ -\mu_2^4(bx-y)\left[ab\left((i-4)x+x_0\right)-a(i-3)y+b\left(y-y_0\right)\right]\end{pmatrix} \\ &+ \epsilon\,\theta_y\frac{m_* h^{i-1}_{\bar{\mathbf{5}},\mathbf{2}}}{(\det\Phi_{\bar{\mathbf{5}},\mathbf{2}})^2}\begin{pmatrix} 2a(i-3)\kappa m \\ \kappa\mu_2^2\left[ab\left((7-2i)x-x_0\right)+b\left(y-y_0\right)\right]-a(i-3)(m^3x-2\kappa\mu_2^2y-\kappa^2)\end{pmatrix} \\ &+ \epsilon\,\theta_x\frac{m_* h^{i-1}_{\bar{\mathbf{5}},\mathbf{2}}}{(\det\Phi_{\bar{\mathbf{5}},\mathbf{2}})^2}\begin{pmatrix} -\mu_2^2 m\left(a\left(x-x_0\right)-2b(i-3)x+2iy-7y+y_0\right) \\ \mu_2^4(bx-y)\left(a\left(x-x_0\right)-b(i-3)x+(i-4)y+y_0\right)-(i-3)m^3x\end{pmatrix} \\ &+ \epsilon\,\theta_x\frac{\kappa m_* h^{i-1}_{\bar{\mathbf{5}},\mathbf{2}}}{(\det\Phi_{\bar{\mathbf{5}},\mathbf{2}})^2}\begin{pmatrix} 2(i-3)m \\ \mu_2^2\left(ax-ax_0-2bix+6bx+2iy-7y+y_0\right)-(i-3)\kappa\end{pmatrix} \\ &+ \epsilon\,\theta_y\frac{h^{i}_{\bar{\mathbf{5}},\mathbf{2}}}{(\det\Phi_{\bar{\mathbf{5}},\mathbf{2}})^3}\begin{pmatrix} -2m\left(\mu_2^2(bx-y)+\kappa\right)\left(-2b\kappa\mu_2^2+2b\mu_2^4(y-bx)+m^3\right) \\ \left(-2b\kappa\mu_2^2+2b\mu_2^4(y-bx)+m^3\right)\left(2\kappa\mu_2^2(bx-y)+\mu_2^4(y-bx)^2+\kappa^2+m^3x\right)\end{pmatrix} \\ &+ \epsilon\,\theta_x\frac{h^{i}_{\bar{\mathbf{5}},\mathbf{2}}}{(\det\Phi_{\bar{\mathbf{5}},\mathbf{2}})^3}\begin{pmatrix} 4\mu_2^2 m\left(\mu_2^2(bx-y)+\kappa\right)^2 \\ -2\mu_2^2\left(\mu_2^2(bx-y)+\kappa\right)\left(2\kappa\mu_2^2(bx-y)+\mu_2^4(y-bx)^2+\kappa^2+m^3x\right)\end{pmatrix}. \end{aligned} \tag{7.27}$$

Note that in the model A the sector $(\bar{\mathbf{5}},\mathbf{2})_{0,1}$ contains three families, so we must take $i=1,2,3$ in the expression above, while the sector $(\bar{\mathbf{5}},\mathbf{1})_{-1,-1}$ contains just one, so there we take $i=3$. In the model B the opposite happens, and so $i=1,2,3$ for the sector $(\bar{\mathbf{5}},\mathbf{1})_{-1,-1}$ and $i=3$ for the $(\bar{\mathbf{5}},\mathbf{2})_{0,1}$ sector.

7.4.2 Wavefunctions in real gauge

When computing the zero mode wavefunctions in real gauge we find that there is a great difference in the computation according to if the sector we are considering is charged or not under the T-brane background. Because of this we shall separate the discussion starting with sectors not affected by the T-brane background.

Sectors not affected by the T-brane background

In these sectors which do not feel the effect of the non-commutativity of the background Higgs field it is possible to solve exactly for the wavefunctions using the techniques already employed in [156, 157]. The F-term and D-term equations may be compactly rewritten as a Dirac-like equation

$$\begin{pmatrix} 0 & D_x & D_y & D_z \\ -D_x & 0 & -D_{\bar z} & D_{\bar y} \\ -D_y & D_{\bar z} & 0 & -D_{\bar x} \\ -D_z & -D_{\bar y} & D_{\bar x} & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \\ \vec{\varphi}_U \\ \\ \end{pmatrix} = 0 \tag{7.28}$$

where we defined the covariant derivatives

$$D_x = \partial_x + \frac{1}{2}(q_R\bar x - q_S\bar y) \qquad D_y = \partial_y - \frac{1}{2}(q_R\bar y + q_S\bar x) \qquad D_z = 2i(\tilde\mu_a^2\bar x - \tilde\mu_b^2\bar y) \tag{7.29}$$

and $D_{\bar m}$ are their conjugate. In writing the covariant derivatives we took the following gauge connection

$$A = \frac{i}{2}Q_R(yd\bar y - \bar y dy - xd\bar x + \bar x dx) + \frac{i}{2}Q_S(xd\bar y - \bar y dx + yd\bar x - \bar x dy) - \frac{i}{2}m^2c^2P(xd\bar x - \bar x dx)\,, \tag{7.30}$$

which gives the flux

$$F = iQ_R(dy\wedge d\bar y - dx\wedge d\bar x) + iQ_S(dx\wedge d\bar y + dy\wedge d\bar x) + im^2c^2P dx\wedge d\bar x\,. \tag{7.31}$$

The effect of the fluxes in every sector is different and reflected in the values of the constants q_R and q_S which appear in the covariant derivatives. The various values of q_R and q_S of the different MSSM fields are listed in table 2.4. We can follow the strategy outlined in [157] to find a solution for the previous system of differential equations and the result is

$$\vec{\varphi} = \begin{pmatrix} -\frac{i\zeta}{2\tilde\mu_a} \\ \frac{i(\zeta-\lambda)}{2\tilde\mu_b} \\ 1 \end{pmatrix} \chi(x,y) \tag{7.32}$$

where

$$\chi(x,y) = e^{\frac{q_R}{2}(x\bar x - y\bar y) - q_S\mathrm{Re}(x\bar y) + (\mu_a x + \mu_b y)(\zeta_1\bar x - \zeta_2\bar y)}\, f(\zeta_2 x + \zeta_1 y) \tag{7.33}$$

we have defined

$$\zeta = \frac{\tilde\mu_a\,(4\tilde\mu_a\tilde\mu_b + \lambda q_S)}{\tilde\mu_a q_S + \tilde\mu_b\,(\lambda + q_R)}\,, \quad \zeta_1 = \frac{\zeta}{\tilde\mu_a}\,, \quad \zeta_2 = \frac{\zeta-\lambda}{\tilde\mu_b}\,, \tag{7.34}$$

and λ is defined as the lowest solution of the cubic equation

$$-\lambda^3 + 4\lambda\mu_a^2 + 4\lambda\mu_b^2 + \lambda q_R^2 - 4\mu_a^2 q_R + 4\mu_b^2 q_R + \lambda q_S^2 + 8\mu_a\mu_b q_S = 0\,. \tag{7.35}$$

This general solution applies to any sector whose matter curve goes through the origin. The effect of a non-zero separation (which affects only the $\bar{\mathbf{5}}_{-1,-1}$ sector) can be easily taken into account by performing a shift in the (x,y) plane

$$x \to x - x_0\,, \quad y \to y - y_0\,. \tag{7.36}$$

However by simply performing the shift in the scalar wavefunction χ we would obtain a solution for a shifted gauge field A. This may be easily remedied by a suitable gauge transformation

$$A(x - x_0, y - y_0) = A(x, y) + d\psi\,, \tag{7.37}$$

with

$$\psi = \frac{i}{2}Q_R(y_0\bar{y} - \bar{y}_0 y - x_0\bar{x} + x_0 x) + \frac{i}{2}Q_S(x_0\bar{y} - \bar{y}_0 x + y_0\bar{x} - \bar{x}_0 y) - \frac{i}{2}m^2c^2P(x_0\bar{x} - \bar{x}_0 x). \tag{7.38}$$

Therefore the general shifted solution may be written as

$$\vec{\varphi} = \begin{pmatrix} -\frac{i\zeta}{2\tilde{\mu}_a} \\ \frac{i(\zeta-\lambda)}{2\tilde{\mu}_b} \\ 1 \end{pmatrix} e^{-i\psi}\chi(x - x_0, y - y_0)\,. \tag{7.39}$$

Sectors affected by the T-brane background

The presence of the the T-brane background greatly affects the sectors charged under it and in particular as we are now going to show it turns out prohibitive to find a simple solution to the zero modes equations of motion. However in particular region in the space of parameters, more precisely when the diagonal terms in the Higgs background are negligible compared to the off-diagonal ones, great simplifications occur in the zero-mode equations and a solution may be easily obtained.

The sectors affected by the T-brane background are the $(\mathbf{10},\mathbf{2})_{1,0}$ and the $(\bar{\mathbf{5}},\mathbf{2})_{0,1}$. Since the difference between the two appears in the diagonal entries of the Higgs background and we are going to assume that these contributions are negligible we shall discuss them at the same time in the following.

The general form of the wavefunctions for the sectors charged under the T-brane is the following one

$$\begin{pmatrix} a_{\bar{x}} \\ a_{\bar{y}} \\ \varphi_{xy} \end{pmatrix} = \vec{\varphi}_{10^+}E_1^+ + \vec{\varphi}_{10^-}E_1^-\,. \tag{7.40}$$

The zero-mode equations take the same form of (7.28) when written in terms of

$$a = \begin{pmatrix} a^+ \\ a^- \end{pmatrix}, \qquad \varphi = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}. \tag{7.41}$$

Following [158] we will start by looking for a general solution of the F-term equations and eventually impose the D-term equations on this solution. While the first step may be done for a general choice of the parameters entering in the Higgs background the latter turns out to be feasible if we restrict to the particular case in which the diagonal terms in the Higgs background are negligible as opposed to the off-diagonal ones.

For sake of notational simplicity we will consider the case in which the primitive fluxes are vanishing and reinstate them at the end of the computation. Then the general solution to the F-terms is

$$a = e^{fP/2}\bar{\partial}\xi \tag{7.42a}$$

$$\varphi = e^{fP/2}\left(h - i\Psi\xi\right) \tag{7.42b}$$

where ξ and h are both doublets whose components we denote as $\xi^{\pm}$ and $h^{\pm}$ and P and Ψ when acting on doublets may be represented as

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \tilde{\mu}^2 F(x,y) & m \\ m^2 x & \tilde{\mu}^2 f(x,y) \end{pmatrix}. \tag{7.43}$$

The explicit form of $\tilde{\mu}^2 F(x,y)$ is different in the two sectors that we are considering in this section but it will be unimportant in the upcoming discussion as we will choose these terms to be negligible.

We may now solve (7.42) for ξ obtaining

$$\xi = i\Psi^{-1}\left(e^{-fP/2}\varphi - h\right), \tag{7.44}$$

and plug this solution in the D-term equations for the zero-modes which therefore become an equation in ξ and h

$$\partial_x\partial_{\bar{x}}\xi + \partial_y\partial_{\bar{y}}\xi + \partial_x f P \partial_{\bar{x}}\xi - i\Lambda^{\dagger}\left(h - i\Psi\xi\right) = 0\,. \tag{7.45}$$

Note that in writing (7.45) we have used that the function f does not depend on $(y, \bar{y})$ and we have defined

$$\Lambda = e^{fP}\Psi e^{-fP} = \begin{pmatrix} \tilde{\mu}^2 F(x,y) & me^{2f} \\ m^2 x e^{-2f} & \tilde{\mu}^2 F(x,y) \end{pmatrix}. \tag{7.46}$$

While (7.45) depends on both ξ and h it is possible to write it as an equation for one single doublet U defined as

$$U = e^{-fP/2}\varphi\,, \qquad \rightarrow \qquad \xi = i\Psi^{-1}(U - h)\,. \tag{7.47}$$

When written in terms of U (7.45) becomes

$$\partial_x\partial_{\bar{x}}U + \partial_y\partial_{\bar{y}}U - (\partial_x\Psi)\Psi^{-1}\partial_{\bar{x}}U + (\partial_y\Psi)\Psi^{-1}\partial_{\bar{y}}U + \partial_x f\Psi P\Psi^{-1}\partial_{\bar{x}}U - \Psi\Lambda^{\dagger}U = 0\,. \qquad (7.48)$$

We have managed therefore to translate the full set of zero-mode equations to a system of partial differential equations for the doublet U. In general this system is coupled and therefore finding a solution turns out to be very involved. However when taking $\tilde{\mu}^2 \ll m^2$ the system decouples and may be easily solved. Since no localised solution of this system for U^+ exists we will set to zero henceforth. Then if we Taylor expand f near the Yukawa point as $f = \log c + c^2m^2x\bar{x}$ we find that $U^- = e^{\lambda x\bar{x}}$ where λ the lowest solution to $c^2\lambda^3 + 4c^4m^2\lambda^2 - m^4\lambda = 0$. Using this the solution to the zero mode equations is simply

$$\vec{\varphi}^{\,j}_{+} = \gamma^j \begin{pmatrix} \frac{i\lambda}{m^2} \\ 0 \\ 0 \end{pmatrix} e^{f/2}\chi^j\,, \qquad \vec{\varphi}^{\,j}_{-} = \gamma^j \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-f/2}\chi^j\,, \qquad (7.49)$$

where $e^{f/2} = \sqrt{c}\,e^{m^2c^2x\bar{x}/2}$ and $\chi^j = e^{\lambda x\bar{x}}\,g_j(y)$, with g_j holomorphic functions of y.

It is easy to generalise the computation when extra primitive fluxes are present. Following a similar procedure we obtain a solution which now is

$$\vec{\varphi}^{\,j}_{+} = \gamma^j \begin{pmatrix} \frac{i\lambda}{m^2} \\ -\frac{i\lambda\zeta}{m^2} \\ 0 \end{pmatrix} e^{f/2}\chi^j \qquad \vec{\varphi}^{\,j}_{-} = \gamma^j \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{-f/2}\chi^j \qquad (7.50)$$

where λ is the lowest (negative) solution to

$$m^4(\lambda - q_R) + \lambda c^2\left(c^2m^2(q_R - \lambda) - \lambda^2 + q_R^2 + q_S^2\right) = 0 \qquad (7.51)$$

and $\zeta = -q_S/(\lambda - q_R)$. The scalar wavefunctions χ are

$$\chi^j = e^{\frac{q_R}{2}(|x|^2-|y|^2)-q_S(x\bar{y}+y\bar{x})+\lambda x(\bar{x}-\zeta\bar{y})}\,g_j(y+\zeta x) \qquad (7.52)$$

where g_j holomorphic functions of $y + \zeta x$, and $j = 1, 2, 3$ label the different zero mode families. The family functions we choose to adopt are

$$g_j = m_*^{3-j}(y + \zeta x)^{3-j}. \qquad (7.53)$$

Note that in neglecting the diagonal terms in the Higgs background we may also discard the effect of the separation of the Yukawa points. If however we consider the case $\kappa, \mu_2 \ll m$ with $\kappa/\mu_2^2 = \nu$ finite we find that the down Yukawa point is located at $(x_0, y_0) = (0, \nu/2)$. We may follow the same strategy as in the previous section and obtain the solution by simply performing a shift and the result is

$$\vec{\varphi}^i = \gamma^i \begin{pmatrix} \frac{i\lambda}{m^2} \\ -i\frac{\lambda\zeta}{m^2} \\ 0 \end{pmatrix} e^{i\tilde{\psi}+f/2}\chi^i(x, y-\nu/2)E^+ + \gamma^i \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} e^{i\tilde{\psi}-f/2}\chi^i(x, y-\nu/2)E^- \tag{7.54}$$

where $\tilde{\psi}$ is

$$\tilde{\psi} = \frac{i}{2}Q_R(\nu\bar{y}/2 - \bar{\nu}y/2) + \frac{i}{2}Q_S(\nu\bar{x}/2 - \bar{\nu}x/2)\,, \tag{7.55}$$

and the definitions of χ, ζ and λ are unchanged.

Non-perturbative corrections - sectors not affected by T-branes

The computation of the first order correction to the wavefunction is similar to the one already considered in [158]. The zero-mode equations are

$$\bar{\partial}_{\langle A\rangle} a = 0\,, \tag{7.56}$$

$$\bar{\partial}_{\langle A\rangle}\varphi + i[\langle\Phi\rangle, a] + \epsilon\,\partial\theta_0 \wedge \partial_{\langle A\rangle} a = 0\,, \tag{7.57}$$

$$\omega \wedge \partial_{\langle A\rangle} a - \frac{1}{2}[\langle\bar{\Phi}\rangle, \varphi] = 0\,. \tag{7.58}$$

We find it possible to solve for the first order correction to the wavefunctions and the result is

$$\vec{\varphi}^{(1)} = \gamma \begin{pmatrix} -\frac{i\zeta}{2\tilde{\mu}_a} \\ \frac{i(\zeta-\lambda)}{2\tilde{\mu}_b} \\ 1 \end{pmatrix} e^{\frac{q_R}{2}(x\bar{x}-y\bar{y}) - q_S\mathrm{Re}(x\bar{y}) + (\tilde{\mu}_a x + \tilde{\mu}_b y)(\zeta_1\bar{x} - \zeta_2\bar{y})}\,\Upsilon\,. \tag{7.59}$$

The function Υ that controls the $\mathcal{O}(\epsilon)$ correction is

$$\begin{aligned}\Upsilon = {}& \frac{1}{4}(\zeta_1\bar{x} - \zeta_2\bar{y})^2(\theta_y\mu_a - \theta_x\mu_b)f(\zeta_2 x + \zeta_1 y) + \frac{1}{2}(\zeta_1\bar{x} - \zeta_2\bar{y})(\zeta_2\theta_y - \zeta_1\theta_x)f'(\zeta_2 x + \zeta_1 y) + \\ & + \left[\frac{\delta_1}{2}(\zeta_1 x - \zeta_2 y)^2 + \delta_2(\zeta_1 x - \zeta_2 y)(\zeta_2 x + \zeta_1 y)\right] f(\zeta_2 x + \zeta_1 y)\,,\end{aligned} \tag{7.60}$$

where

$$\begin{aligned}\delta_1 &= \frac{1}{(\zeta_1^2 + \zeta_2^2)^2}\left[\bar{\theta}_x(q_S\zeta_1 - q_R\zeta_2) + \bar{\theta}_y(q_R\zeta_1 + q_S\zeta_2)\right]\,, \\ \delta_2 &= \frac{1}{(\zeta_1^2 + \zeta_2^2)^2}\left[\bar{\theta}_x(q_R\zeta_1 + q_S\zeta_2) - \bar{\theta}_y(q_S\zeta_1 - q_R\zeta_2)\right]\,.\end{aligned} \tag{7.61}$$

Similarly to the order zero in ϵ case it is possible to obtain the solution when there is a non-zero separation between the Yukawa points by performing a shift in the coordinates and a suitable gauge transformation. Because of this similarity we refrain from displaying the result explicitly.

Non-perturbative corrections - sectors affected by the T-brane background

As already mentioned in the main text the mere structure of the first order correction to the wavefunctions is sufficient to ensure that no corrections at $\mathcal{O}(\epsilon)$ are present in the kinetic terms. Here we shall demonstrate how this structure arises without explicitly computing the first order correction as this is unnecessary to compute the kinetic terms.

We start by consider the case when the primitive fluxes are absent. The solution to the F-term equations at first order in ϵ is

$$a = g\,\bar{\partial}\xi \tag{7.62a}$$

$$\varphi = g\,(h - i\Phi\xi - \epsilon\partial\theta_0 \wedge \partial\xi) \;=\; g\,U\,dx \wedge dy \tag{7.62b}$$

with

$$g = \begin{pmatrix} e^{f/2} & 0 \\ 0 & e^{-f/2} \end{pmatrix}, \tag{7.63}$$

where Φ is different in the two sectors and may be found in (2.41). We expand the doublet U in ϵ

$$U \;=\; U^{(0)} + \epsilon\, U^{(1)} +\; \mathcal{O}(\epsilon^2) \tag{7.64}$$

where $U^{(0)}$ was computed previously

$$U_-^{(0)} \;=\; e^{\lambda x\bar{x}}h(y) \qquad\qquad U_+^{(0)} \;=\; 0. \tag{7.65}$$

Then, one may solve for ξ from (7.62b) as

$$\begin{gathered} \xi \;=\; \xi^{(0)} + i\epsilon\Phi^{-1}\left[U^{(1)} + \partial_x\theta_0\partial_y\xi^{(0)} - \partial_y\theta_0\partial_x\xi^{(0)}\right] + \mathcal{O}(\epsilon^2) \\ \xi^{(0)} \;=\; i\Phi^{-1}(U^{(0)} - h) \end{gathered} \tag{7.66}$$

and then solve for $U^{(1)}$ by plugging in this expression into the D-term for the fluctuations (2.89c). This yields $U_-^{(1)} = 0$, in the limit $\tilde{\mu}^2 \ll m^2$. Thus, we find the following structure

$$\xi_+ \;=\; \xi_+^{(0)} + 0 + \mathcal{O}(\epsilon^2) \qquad\qquad \xi_- \;=\; \epsilon\,\xi_-^{(1)} + \mathcal{O}(\epsilon^2). \tag{7.67}$$

This is actually sufficient to prove that the solution when taking into account the first order correction in ϵ has the following form

$$\vec{\varphi}_+ = \begin{pmatrix} \bullet \\ \bullet \\ 0 \end{pmatrix} + \epsilon \begin{pmatrix} 0 \\ 0 \\ \bullet \end{pmatrix} + \mathcal{O}(\epsilon^2) \qquad \vec{\varphi}_- = \begin{pmatrix} 0 \\ 0 \\ \bullet \end{pmatrix} + \epsilon \begin{pmatrix} \bullet \\ \bullet \\ 0 \end{pmatrix} + \mathcal{O}(\epsilon^2) \tag{7.68}$$

This is sufficient for the argument outlined in the main text regarding the computation of the kinetic terms at order ϵ. Moreover by following a similar procedure it is possible to show that this continues to hold if primitive fluxes are taken into account and the shift form the origin is taken into account.

7.4.3 Holomorphic Yukawa matrix

Let us give the explicit expressions for the down-type Yukawas that arises from the residue formula (2.73), for the case of the model A. Unlike in (2.77) the expression below will be given to all orders in the parameter $a-b$. We obtain that

$$Y_{D/L} = \begin{pmatrix} 0 & Y^{(12)} & Y^{(13)} \\ Y^{(21)} & Y^{(22)} & Y^{(23)} \\ Y^{(31)} & Y^{(32)} & Y^{(33)} \end{pmatrix} + \mathcal{O}(\epsilon^2)\,, \tag{7.69}$$

where

$$Y^{(12)} = \frac{2\pi^2\tilde{\kappa}\epsilon\gamma_{5,2}\gamma_{10,1}\gamma_D\left(\theta_y(a+bd)+(d+1)\theta_x\right)}{(d+1)^4\rho_m\rho_\mu^3}\,, \tag{7.70}$$

$$\begin{aligned} Y^{(13)} = &-\frac{\pi^2\gamma_{5,3}\gamma_{10,1}\gamma_D}{(d+1)^5\rho_m^{5/2}\rho_\mu^5}\Big[\epsilon(d+1)^2\rho_m^{3/2}\rho_\mu^6\left(\theta_y(a+bd)+(d+1)\theta_x\right)+2(d+1)^2\tilde{\kappa}^2\rho_m^{3/2}\rho_\mu^2 \\ &-2d\tilde{\kappa}\epsilon(a-b)\rho_\mu^4\left(\theta_y\left(a\left(2d^2+7d-1\right)+3b(d-1)d\right)+\left(5d^2+4d-1\right)\theta_x\right)\Big]\,, \end{aligned} \tag{7.71}$$

$$Y^{(21)} = \frac{\pi^2\tilde{\kappa}\epsilon\gamma_{5,1}\gamma_{10,2}\gamma_D\left(\theta_y(a+bd)+(d+1)\theta_x\right)}{(d+1)^4\rho_m\rho_\mu^3}\,, \tag{7.72}$$

$$\begin{aligned} Y^{(22)} = &-\frac{\pi^2\epsilon\gamma_{5,2}\gamma_{10,2}\gamma_D}{(d+1)^5\rho_m^{5/2}\rho_\mu^2}\Big[(d+1)^2\rho_m^{3/2}\left(\theta_y(a+bd)+(d+1)\theta_x\right) \\ &-d(a-b)\rho_\mu\left(\theta_y(-a(d-8)+10bd+b)+9(d+1)\theta_x\right)\Big]\,, \end{aligned} \tag{7.73}$$

$$\begin{aligned} Y^{(23)} = &\frac{\pi^2\gamma_{5,3}\gamma_{10,2}\gamma_D}{(d+1)^6\rho_m^4\rho_\mu^2}\Big[\epsilon\, d^2(d+1)^2\rho_m^{3/2}\rho_\mu^2(a-b)\left(\theta_y(a(d+4)+b(2d-1))+3(d+1)\theta_x\right) \\ &-6\tilde{\kappa}d^2\epsilon(a-b)^2\rho_\mu^3\left(\theta_y(a(d(d+8)+2)+bd(6d+1))+(d+1)(7d+2)\theta_x\right) \\ &+\tilde{\kappa}(d+1)^4\rho_m^3-6d(d+1)^2\tilde{\kappa}^2\rho_m^{3/2}(a-b)\rho_\mu\Big]\,, \end{aligned} \tag{7.74}$$

$$
\begin{aligned}
Y^{31} &= -\frac{\pi^2 \epsilon \gamma_{5,1}\gamma_{10,3}\gamma_D}{(d+1)^5 \rho_m^{5/2}\rho_\mu^2}\Big[(d+1)^2\rho_m^{3/2}\left(\theta_y(a+bd)+(d+1)\theta_x\right) \\
&\quad - 2\tilde{\kappa}(a-b)\rho_\mu\left(\theta_y(a(2d-1)+b((d-3)d-1))+(d-2)(d+1)\theta_x\right)\Big] \,.
\end{aligned} \tag{7.75}
$$

$$
\begin{aligned}
Y^{32} &= -\frac{\pi^2 \epsilon\, d\gamma_{5,2}\gamma_{10,3}\gamma_D\,(a-b)}{(d+1)^6\rho_m^4}\Big[(d+1)^2\rho_m^{3/2}\left(\theta_y(-a(d-2)+4bd+b)+3(d+1)\theta_x\right) \\
&\quad -2\tilde{\kappa}(a-b)\rho_\mu\Big(\theta_y\left(d^2(19b-6a)+7d(a+b)+a\right)+(d+1)(13d+1)\theta_x\Big)\Big]
\end{aligned} \tag{7.76}
$$

$$
\begin{aligned}
Y^{(33)} &= -\frac{\pi^2\gamma_{5,3}\gamma_{10,3}\gamma_D}{(d+1)^7\rho_m^{11/2}\rho_\mu}\Big[(d+1)^6\rho_m^{9/2} - 2d(d+1)^2\tilde{\kappa}\rho_m^{3/2}(a-b)\rho_\mu\Big(6d\tilde{\kappa}(b-a)\rho_\mu+(d+1)^2\rho_m^{3/2}\Big) \\
&\quad - 2\epsilon\, d^2(d+1)^2\rho_m^{3/2}(a-b)^2\rho_\mu^3\left(\theta_y(a(1-(d-3)d)+bd(5d+2))+(d+1)(4d+1)\theta_x\right) \\
&\quad +4d^3\tilde{\kappa}\epsilon(a-b)^3\rho_\mu^4\left(\theta_y(a((19-7d)d+11)+3bd(11d+6))+(d+1)(26d+11)\theta_x\right)\Big] \,.
\end{aligned} \tag{7.77}
$$

7.5 Coulomb branch examples of restriction rule.

In this appendix we will work out explicitly some examples of the general procedure outlined in chapter 4, in order to explain the rather abstract rule that defines the restriction map in terms of the partition ρ and perform some explicit checks that our conjecture holds.

The case of $[2,1]$ of $T[SU(3)]$

To begin, let us think of the easiest possible case. We consider the $T[SU(3)]$ theory defined by the following linear quiver of figure 7.1. The brane picture is given in figure 7.2 where m_1

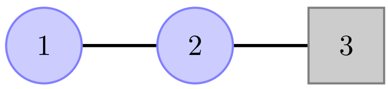


Figure 7.1: The quiver graph for $T[SU(3)]$ theory.

is the magnetic charge for $U(1)$ and m_{21}, m_{22} are the magnetic charges for $U(2)$ which satisfy $m_{21} \geq m_{22}$. The Hilbert series for the full Coulomb branch of is given by the general formula (4.36)

$$
HS(t,z_1,z_2) = \sum_{m_1=-\infty}^{\infty}\sum_{m_{21}\geq m_{22}} t^{\Delta(m_1,m_{21},m_{22})} z_1^{m_1} z_2^{m_{21}+m_{22}} P_{U(1)}(m_1,t)P_{U(2)}(m_{21},m_{22},t), \tag{7.78}
$$

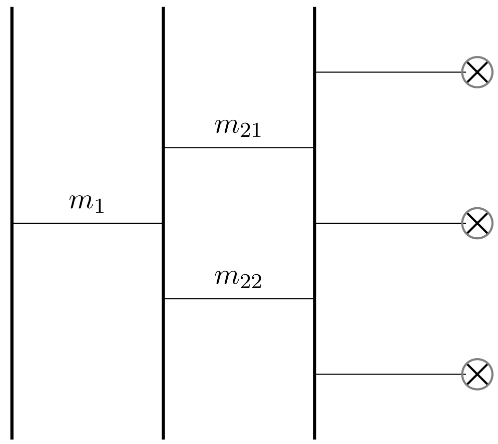


Figure 7.2: The brane picture for the $T[SU(3)]$ theory yielding $U(1) - U(2) - [3]$.

where the dimension formula (4.25) reads

$$\Delta(m_1, m_{21}, m_{22}) = -|m_{21} - m_{22}| + \frac{1}{2}(|m_{21} - m_1| + |m_{22} - m_1| + 3|m_{21}| + 3|m_{22}|), \tag{7.79}$$

and the classical factors are

$$P_{U(1)}(m_1, t) = \frac{1}{1-t}, \tag{7.80}$$

and

$$P_{U(2)}(m_{21}, m_{22}, t) = \begin{cases} \dfrac{1}{(1-t)(1-t^2)}, & \text{for } m_{21} = m_{22}, \\ \dfrac{1}{(1-t)^2}, & \text{for } m_{21} > m_{22}. \end{cases} \tag{7.81}$$

Then we focus on the mixed branch $\rho = [2, 1]$. In order to satisfy the s-rule [94] we must set the position of one of the two D3-branes in the second cell to be exactly equal to one of the mass parameters, and therefore equal to the position of one of the flavor D5-brane. Computationally, this is implemented by setting to zero the magnetic charge associated to the position of that brane. Figure 7.3 shows how the brane system looks for the mixed branch of $\rho = [2, 1]$.

Now we should point out that there are two ways to set one of the two gauge branes to zero: one is $m_{21} = 0 \geq m_{22}$, as shown in figure 7.3 and the other is to set $m_{21} > m_{22} = 0$, as shown in figure 7.4. Both these cases are allowed and we should sum over both of them. With this we mean that the Hilbert series for the Coulomb branch part of mixed branch $\rho = [2, 1]$ will be given by the Hilbert series of the full Coulomb branch (7.78) in which $m_{21} = 0 \geq m_{22}$ plus the Hilbert series of the full Coulomb branch (7.78) in which $m_{21} > m_{22} = 0$. In this addition, we only count the magnetic charge $m_{21} = m_{22} = 0$ once and there is no overcounting.

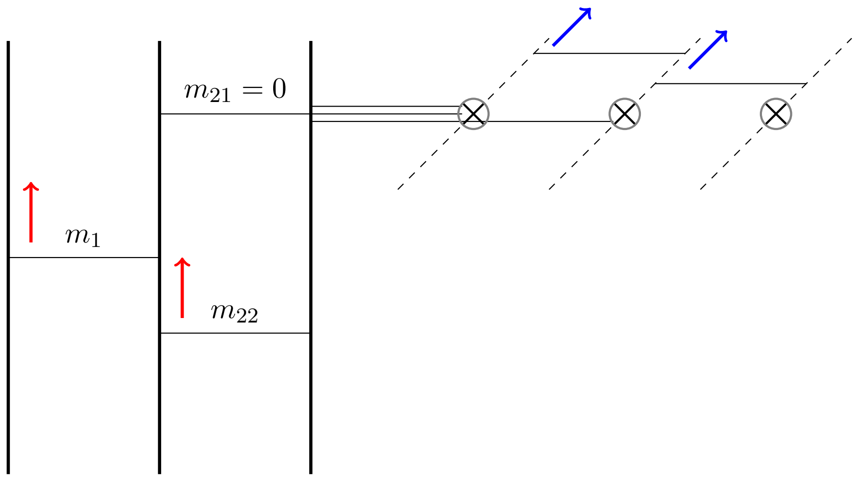


Figure 7.3: The brane picture realizing the mixed branch $\rho = [2, 1]$. In this first case the restriction amounts to take $m_{21} = 0 \geq m_{22}$.

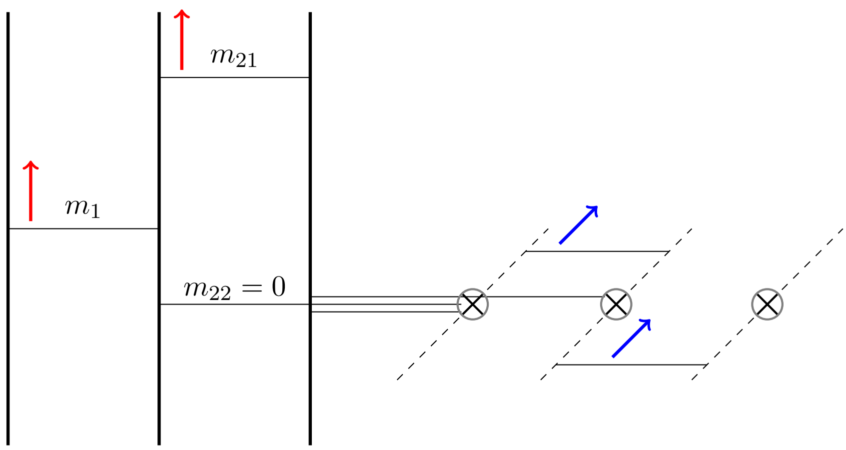


Figure 7.4: Another brane picture for $\rho = [2, 1]$. In this second case the restriction amounts to take $m_{21} > m_{22} = 0$.

By the second rule in section 4.8.3, the map $R_{[2,1]}$ also restricts the classical factors, replacing $P_{U(2)}$ with $P_{U(1)}$. The physical intuition for this fact is that since one of the two branes is frozen to a specific position, the residual gauge group becomes $U(1)$. Therefore, the classical dressing factor will be reduced to $P_{U(1)}$ from $P_{U(2)}$.

The Hilbert series for the Coulomb branch part of the $[2, 1]$ mixed branch is therefore given by

$$
\begin{aligned}
HS(t, z_1, z_2) = & \sum_{m_1=-\infty}^{\infty} \sum_{m_{21}=0, m_{22}\leq 0} t^{\Delta_1(m_1,m_{22})} z_1^{m_1} z_2^{m_2} P_{U(1)}(m_1, t) P_{U(1)}(m_{22}, t) \\
& + \sum_{m_1=-\infty}^{\infty} \sum_{m_{22}=0, m_{21}>0} t^{\Delta_2(m_1,m_{21})} z_1^{m_1} z_2^{m_2} P_{U(1)}(m_1, t) P_{U(1)}(m_{21}, t),
\end{aligned} \tag{7.82}
$$

where $\Delta_1(m_1, m_{22}) = \Delta(m_1, 0, m_{22})$ and $\Delta_2(m_1, m_{21}) = \Delta(m_1, m_{21}, 0)$.

In detail, by performing this truncation of the sum at the ninth order in t, we get[2]

$$
HS(z, t) = \sum_{k=0}^{9} [k, k]_z t^k + \mathcal{O}(t^{10}), \tag{7.84}
$$

where $[n_1, n_2]_z$ is the character of the representation $[n_1, n_2]$ where n_1, n_2 are Dynkin labels of the $su(3)$ Lie algebra[3]. Since the Hilbert series of (7.84) is written by the characters of the $su(3)$ Lie algebra, it implies that the topological symmetry is enhanced to $SU(3)$.

We now want to check that the restriction rule indeed works. We compare the Hilbert series that we obtained by restricting the summation over the magnetic charges, with the Hilbert series of the full Coulomb branch of the $T^{[2,1]}[SU(N)]$ theory. To do so we first go to the IR theory, effectively giving infinite vev to the scalars parameterizing the Higgs branch. The resulting brane configuration after a sequence of Hanany-Witten transitions was already obtained in figure 4.17, yielding the $[1] - U(1) - U(1) - [1]$ linear quiver theory. For this theory the monopole dimension is

$$
\Delta(n_1, n_2) = \frac{1}{2} \left(|n_1| + |n_2 - n_1| + |n_2| \right), \tag{7.85}
$$

where n_1, n_2 are the magnetic charges of the two $U(1)$'s. The Hilbert series of the full Coulomb branch is given by

$$
HS(t, z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} t^{\Delta(n_1,n_2)} z_1^{n_1} z_2^{n_2} P_{U(1)}(n_1, t) P_{U(1)}(n_2, t). \tag{7.86}
$$

---

[2]This computation, writing the HS in terms of characters, was performed explicitly only up to order t^9. However, the obtained result strongly implies that the same structure will continue to higher orders. Therefore we conjecture that the Hilbert series is given by

$$
HS(z, t) = \sum_{k=0}^{\infty} [k, k]_z t^k. \tag{7.83}
$$

.

[3]Here and everywhere else in section **??** and 7.6 we use a Dynkin label notation for the characters of a representation of a Lie algebra. For example, [2] means the character of the adjoint of $\mathfrak{su}(2)$. By the basis of the Cartan generators in (4.58), the character is given by $[2]_z = z + 1 + z^{-1}$. On the other hand, by the basis of the Cartan generators in (4.59), the character is given by $[2]_x = x^2 + 1 + x^{-2}$.

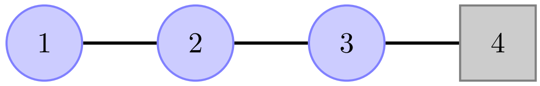


Figure 7.5: The quiver graph for the $T[SU(4)]$ theory.

Performing this computation explicitly gives us

$$HS(z,t) = \sum_{k=0}^{9} [k,k]_z t^k + \mathcal{O}(t^{10}), \tag{7.87}$$

and we see that this exactly matches with the equation (7.84). This matching was checked at order 30 in t.

Other explicit checks

We then exemplify the restriction rules in section 4.8.3 by more non-trivial examples.

The mixed branch $\rho = [2,2]$

In this example we start by considering now the $3d$ $\mathcal{N} = 4$ $T[SU(4)]$ theory given by the quiver diagram depicted in figure 7.5. This theory can be also realized in terms of the brane picture in figure 7.6 where m_1 is the magnetic charge of the $U(1)$, m_{21}, m_{22} are the magnetic charges of the $U(2)$, m_{31}, m_{32}, m_{33} are the magnetic charges of the $U(3)$ and $m_{41}, m_{42}, m_{43}, m_{44}$ are the magnetic charges of the $U(4)$. The monopole dimension formula for the full Coulomb branch reads:

$$\begin{aligned} \Delta\left(\vec{m}\right) = & -|m_{21} - m_{22}| - |m_{31} - m_{32}| - |m_{31} - m_{33}| - |m_{32} - m_{33}| \\ & + \frac{1}{2}\left(|m_{21} - m_{11} + |m_{22} - m_{11}| + |m_{31} - m_{21}| + |m_{31} - m_{22}| + |m_{32} - m_{21}|\right. \\ & \left. + |m_{32} - m_{22}| + |m_{33} - m_{21}| + |m_{33} - m_{22}| + 4|m_{31}| + 4|m_{32}| + 4|m_{33}|\right). \end{aligned} \tag{7.88}$$

with the magnetic charges $\vec{m} = (m_1.m_{21}, m_{22}, m_{31}, m_{32}, m_{33})$ satisfying $m_{21} \geq m_{22}$ and $m_{31} \geq m_{32} \geq m_{33}$.

The Hilbert series for the full Coulomb branch of this theory is

$$\begin{aligned} HS(z,t) := & \sum_{m_1=-\infty}^{\infty} \sum_{m_{21} \geq m_{22}} \sum_{m_{31} \geq m_{32} \geq m_{33}} t^{\Delta(m_1, m_{21}, m_{22}, m_{31}, m_{32}, m_{33})} \\ & \cdot P_{U(1)}(m_1, t) P_{U(2)}(m_{21}, m_{22}, t) P_{U(3)}(m_{31}, m_{32}, m_{33}, t) z_1^{m_1} z_2^{(m_{21}+m_{22})} z_3^{(m_{31}+m_{32}+m_{33})}. \end{aligned} \tag{7.89}$$

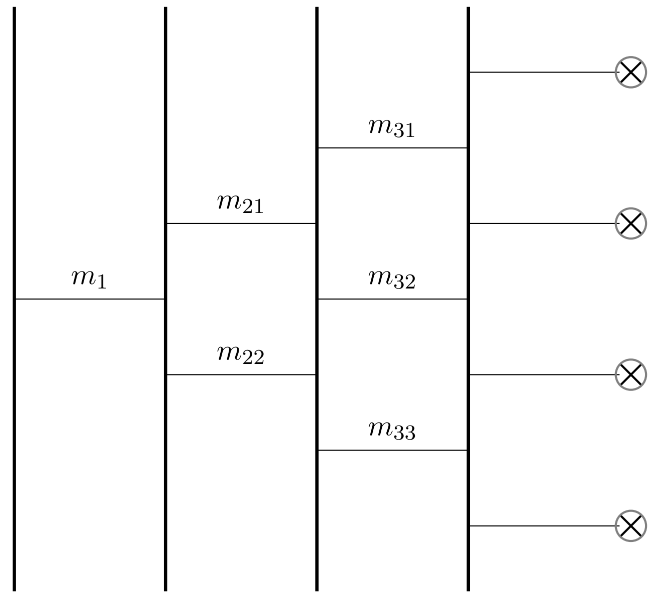


Figure 7.6: The brane picture for the $T[SU(4)]$ theory yielding the linear quiver $U(1) - U(2) - U(3) - [4]$.

Now we are interested in studying the mixed branch given by the partition $\rho = [2, 2]$. The first rule in section 4.8.3 says that we can set two magnetic charges to zero in the 3rd cell. Furthermore, one sees again that there are two different ways to set to zero two magnetic charges in the third cell: one can choose $m_{31} = m_{32} = 0$, as in figure 7.7, or one can choose $m_{32} = m_{33} = 0$ as in figure 7.8.

One can now compute the Hilbert series for the Coulomb branch part of the $\rho = [2, 2]$ mixed branch, by restricting the full summation in the way explained in section 4.8.3. By doing this one finds a series

$$
\begin{aligned}
H(z,t) = 1 &+ [1,0,1]_z\ t + ([2,0,2]_z + [0,2,0]_z)\ t^2 \\
&+ ([3,0,3]_z + [1,2,1]_z)\ t^3 + ([4,0,4]_z + [2,2,2]_z + [0,4,0]_z)\ t^4 \\
&+ ([5,0,5]_z + [3,2,3]_x + [1,4,1]_z)\ t^5 + ([6,0,6]_z + [4,2,4]_z + [2,4,2]_z + [0,6,0]_z)\ t^6 \\
&+ ([7,0,7]_z + [5,2,5]_z + [3,4,3]_z + [1,6,1]_z)\ t^7 \\
&+ ([8,0,8]_z + [6,2,6]_z + [4,4,4]_z + [2,6,2]_z + [0,8,0]_z)\ t^8 \\
&+ ([9,0,9]_z + [7,2,7]_z + [5,4,5]_z + [3,6,3]_z + [1,8,1]_z)\ t^9 + \mathcal{O}(t^{10}).
\end{aligned}
\tag{7.90}
$$

Since the Hilbert series of (7.90) is written by the characters of the $su(4)$ Lie algebra, it implies that the topological symmetry is enhanced to $SU(4)$.

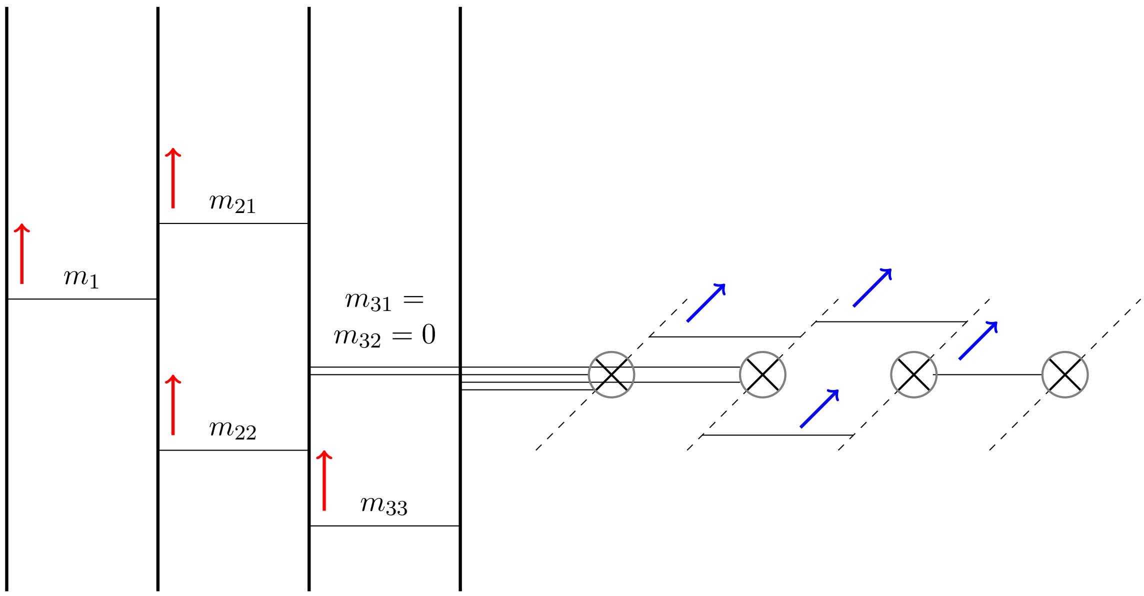


Figure 7.7: The brane picture for the $\rho = [2,2]$ mixed branch. In this subcase, $m_{31} = m_{32} = 0 \geq m_{33}$.

Let us then compare this result with the one obtained from the IR theory after taking the limit where the Higgs branch vevs of all the unfrozen branes become infinite. After performing some Hanany-Witten transitions, the quiver theory at the IR will be given by figure 7.9. For this theory, the dimension formula of the monopole operators is given by

$$\Delta(\overrightarrow{n}) = -|n_{21} - n_{22}| + \frac{1}{2}\left(|n_{21} - n_1| + |n_{22} - n_1| + |n_{21} - n_2| + |n_{22} - n_2| + 2|n_{21}| + 2|n_{22}|\right). \tag{7.91}$$

where $\overrightarrow{n} = (n_1, n_2, n_{21}, n_{22})$, and n_1, n_2 are the magnetic charges of the two $U(1)$'s and n_{21}, n_{22} are the magnetic charges of the $U(2)$.

The Hilbert series for the IR theory is given by

$$HS(z,t) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_{21} \geq n_{22}} t^{\Delta(n_1,n_2,n_{21},n_{22})} z_1^{n_1} z_2^{(n_{21}+n_{22})} z_3^{n_3} P_{U(1)}(n_1,t)\, P_{U(1)}(n_2,t)\, P_{U(2)}(n_{21},n_{22},t) \tag{7.92}$$

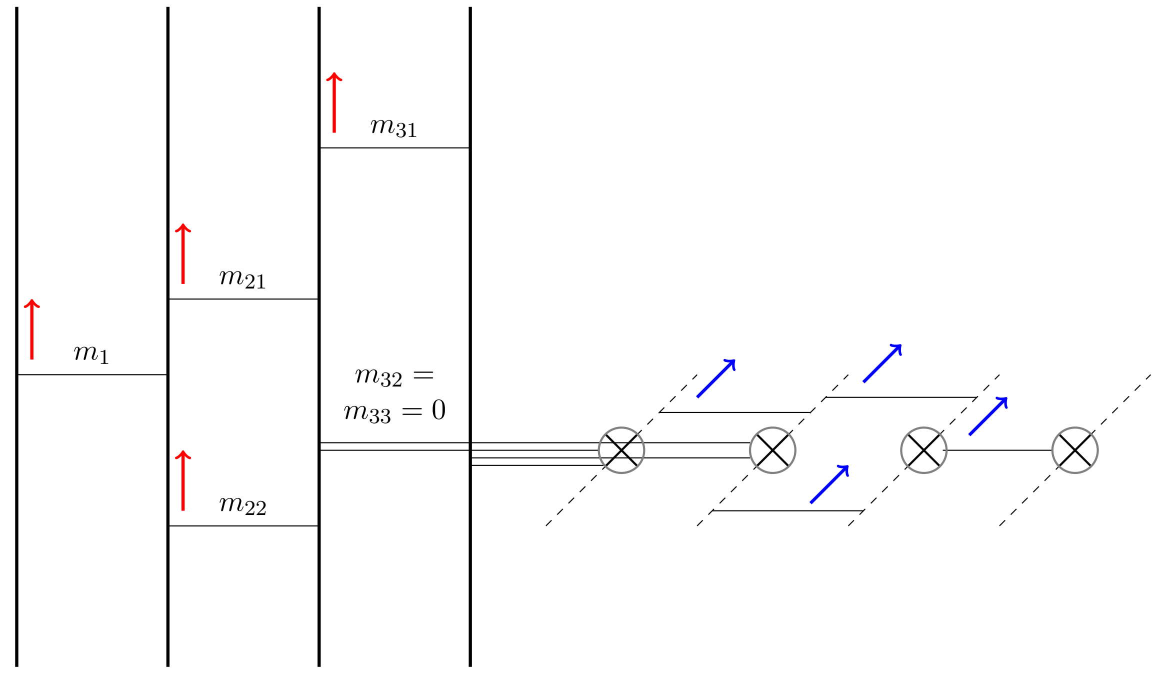


Figure 7.8: The brane picture for the $\rho = [2,2]$ mixed branch. In this subcase, $m_{31} > m_{32} = m_{33} = 0$.

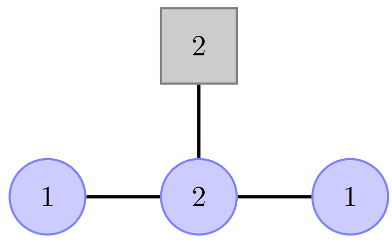


Figure 7.9: The quiver graph for the $T^{[2,2]}[SU(4)]$ theory, obtained by sending to infinity all the unfrozen Higgs branch vevs.

By computing explicitly the Hilbert series in this case we find

$$
\begin{aligned}
H(z,t) = 1 &+ [1,0,1]_z\ t + ([2,0,2]_z + [0,2,0]_z)\ t^2 \\
&+ ([3,0,3]_z + [1,2,1]_z)\ t^3 + ([4,0,4]_z + [2,2,2]_z + [0,4,0]_z)\ t^4 \\
&+ ([5,0,5]_z + [3,2,3]_x + [1,4,1]_z)\ t^5 + ([6,0,6]_z + [4,2,4]_z + [2,4,2]_z + [0,6,0]_z)\ t^6 \\
&+ ([7,0,7]_z + [5,2,5]_z + [3,4,3]_z + [1,6,1]_z)\ t^7 \\
&+ ([8,0,8]_z + [6,2,6]_z + [4,4,4]_z + [2,6,2]_z + [0,8,0]_z)\ t^8 \\
&+ ([9,0,9]_z + [7,2,7]_z + [5,4,5]_z + [3,6,3]_z + [1,8,1]_z)\ t^9 + \mathcal{O}(t^{10}),
\end{aligned} \tag{7.93}
$$

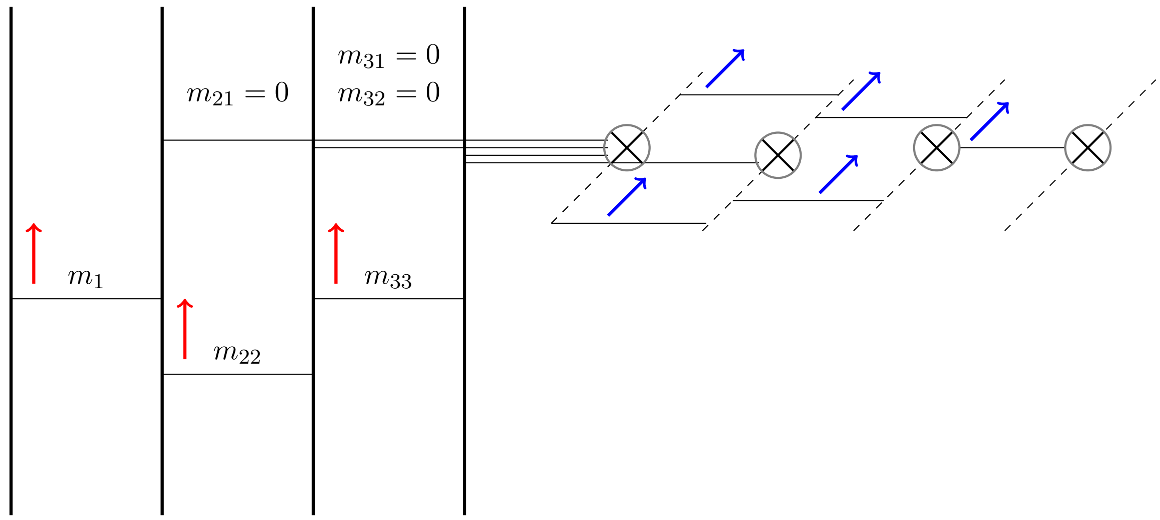


Figure 7.10: The brane picture for the $\rho = [3, 1]$ mixed branch.

which exactly agrees with (7.90).

The mixed branch $\rho = [3, 1]$

Another example is the Coulomb branch moduli part of the mixed branch $\rho = [3, 1]$ of the 3d $T[SU(4)]$ theory. The Hilbert series for the full Coulomb branch of the $T[SU(4)]$ theory is again given by (7.89).

The restriction of the magnetic charges corresponding to $[3, 1]$ is

$$(m_{21} = 0 \text{ or } m_{22} = 0) \text{ and } (m_{31} = m_{32} = 0 \text{ or } m_{32} = m_{33} = 0)\,, \tag{7.94}$$

giving in total 4 possible choices. We have to apply each one of them to equation (7.89) and sum the four resulting sub-sums obtained. After performing such a restriction to the Hilbert series of the full Coulomb branch of the $T[SU(4)]$ theory, as explained in section 4.8.3, we find the following Hilbert series:

$$HS(z,t) = \sum_{k=0}^{9} [k, 0, k]_z t^k + \mathcal{O}(t^{10}). \tag{7.95}$$

We also see the enhancement of the topological symmetry to $SU(4)$ since (7.95) is written by the characters of the $su(4)$ Lie algebra.

On the other hand, the IR theory at the infinitely large Higgs vev is given by the linear quiver of figure 7.11. The dimension formula of this latter IR theory will read

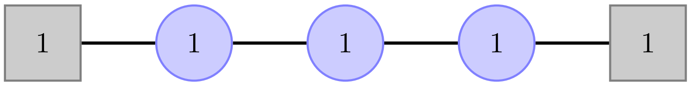


Figure 7.11: The quiver graph for the $T^{[3,1]}[SU(4)]$ theory.

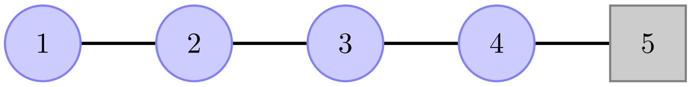


Figure 7.12: The quiver graph for $T[SU(5)]$.

$$\Delta(n_1, n_2, n_3) = \frac{1}{2}\left(|n_1| + |n_2 - n_1| + |n_3 - n_2| + |n_3|\right), \tag{7.96}$$

where n_1, n_2, n_3 are the magnetic charges of the three $U(1)$'s.

The Hilbert series for the IR theory is given by

$$HS(z,t) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{n_3=-\infty}^{\infty} t^{\Delta(n_1,n_2,n_3)} z_1^{n_1} z_2^{n_2} z_3^{n_3} P_{U(1)}(n_1, t)\, P_{U(1)}(n_2, t)\, P_{U(1)}(n_3, t) \tag{7.97}$$

By computing this explicitly we get

$$HS(z,t) = \sum_{k=0}^{9} [k, 0, k]_z t^k + \mathcal{O}(t^{10}), \tag{7.98}$$

which precisely agrees with (7.95). This matching has been checked up to order 12 in t.

The mixed branch $\rho = [3,2]$

As a final example now we consider the mixed branch $\rho = [3,2]$ of the $T[SU(5)]$ theory. The quiver description of the $T[SU(5)]$ is given by figure 7.12. We denote the magnetic charge of the $U(1)$ by m_{11}, the magnetic charges of the $U(2)$ by m_{21}, m_{22}, the magnetic charges of the $U(3)$ by m_{31}, m_{32}, m_{33} and the magnetic charges of the $U(5)$ by $m_{41}, m_{42}, m_{43}, m_{44}$. The dimension formula for the full Coulomb branch of $T[SU(5)]$ is given by

$$\begin{aligned}\Delta(\overrightarrow{m}) = &- |m_{21}-m_{22}| - |m_{31}-m_{32}| - |m_{31}-m_{33}| - |m_{32}-m_{33}| - |m_{41}-m_{42}| \\ &- |m_{41}-m_{43}| - |m_{41}-m_{44}| - |m_{42}-m_{43}| - |m_{42}-m_{44}| - |m_{43}-m_{44}| \\ &+ \frac{1}{2}\left(|m_{21}-m_{11}| + |m_{22}-m_{11}| + |m_{31}-m_{21}| + |m_{31}-m_{22}| + |m_{32}-m_{21}|\right. \\ &+ |m_{32}-m_{22}| + |m_{33}-m_{21}| + |m_{33}-m_{22}| + |m_{41}-m_{31}| + |m_{41}-m_{32}| \\ &+ |m_{41}-m_{33}| + |m_{42}-m_{31}| + |m_{42}-m_{32}| + |m_{42}-m_{33}| + |m_{43}-m_{31}| \\ &+ |m_{43}-m_{32}| + |m_{43}-m_{33}| + |m_{44}-m_{31}| + |m_{44}-m_{32}| + |m_{44}-m_{33}| \\ &\left.+5|m_{41}| + 5|m_{42}| + 5|m_{43}| + 5|m_{44}|\right),\end{aligned} \tag{7.99}$$

where $\overrightarrow{m} = (m_{11}, m_{21}, m_{22}, m_{31}, m_{32}, m_{33}, m_{41}, m_{42}, m_{43}, m_{44})$ satisfying $m_{21} \geq m_{22}$, $m_{31} \geq m_{32} \geq m_{33}$ and $m_{41} \geq m_{42} \geq m_{43} \geq m_{44}$.

The Hilbert Series for the full Coulomb branch of this theory is

$$\begin{aligned}HS(z,t) := &\sum_{m_1=-\infty}^{\infty} \sum_{m_{21}\geq m_{22}} \sum_{m_{31}\geq m_{32}\geq m_{33}} \sum_{m_{41}\geq m_{42}\geq m_{43}\geq m_{44}} t^{\Delta(m_1,m_{21},m_{22},m_{31},m_{32},m_{33},m_{41},m_{42},m_{43},m_{44})} \\ &\cdot P_{U(1)}(m_1,t) P_{U(2)}(m_{21},m_{22},t) P_{U(3)}(m_{31},m_{32},m_{33},t) P_{U(4)}(m_{41},m_{42},m_{43},m_{44},t) \\ &\cdot z_1^{m_1} z_2^{(m_{21}+m_{22})} z_3^{(m_{31}+m_{32}+m_{33})} z_4^{(m_{41}+m_{42}+m_{43}+m_{44})}.\end{aligned} \tag{7.100}$$

By going to the mixed branch we wish to analyze, we have the brane picture in figure 7.13. We see that by using the first rule we have to set to zero 3 magnetic charges of the 4th cell, and 1 magnetic charge of the 3rd cell. Again, there are different ways to do so: in the 4th cell we can have $m_{41} = m_{42} = m_{43} = 0$ or $m_{42} = m_{43} = m_{44} = 0$. For any of these two cases, we have three choices in the 3rd cell, namely $m_{31} = 0$, $m_{32} = 0$ or $m_{33} = 0$. In total, we find six different sub-cases into which equation (7.100) splits, and we must sum over all of them.

By performing the summation over all these subcases we find the following Hilbert series.

$$\begin{aligned}HS(z,t) = &1 + [1,0,0,1]_z t + ([2,0,0,2]_z + [0,1,1,0]_z)\, t^2 \\ &+ ([3,0,0,3]_z + [1,1,1,1]_z)\, t^3 + ([4,0,0,4]_z + [2,1,1,2]_z + [0,2,2,0]_z)\, t^4 \\ &+ ([5,0,0,5]_z + [3,1,1,3]_z + [1,2,2,1]_z)\, t^5 + \\ &+ ([6,0,0,6]_z + [4,1,1,4]_z + [2,2,2,2]_z + [0,3,3,0]_z)\, t^6 + \\ &+ ([7,0,0,7]_z + [5,1,1,5]_z + [3,2,2,3]_z + [1,3,3,1]_z)\, t^7 + \\ &+ ([8,0,0,8]_z + [6,1,1,6]_z + [4,2,2,4]_z + [2,3,3,2]_z + [0,4,4,0]_z)\, t^8 + \\ &+ ([9,0,0,9]_z + [7,1,1,7]_z + [5,2,2,5]_z + [3,3,3,3]_z + [1,4,4,1]_z)\, t^9 + \mathcal{O}(t^{10}).\end{aligned} \tag{7.101}$$

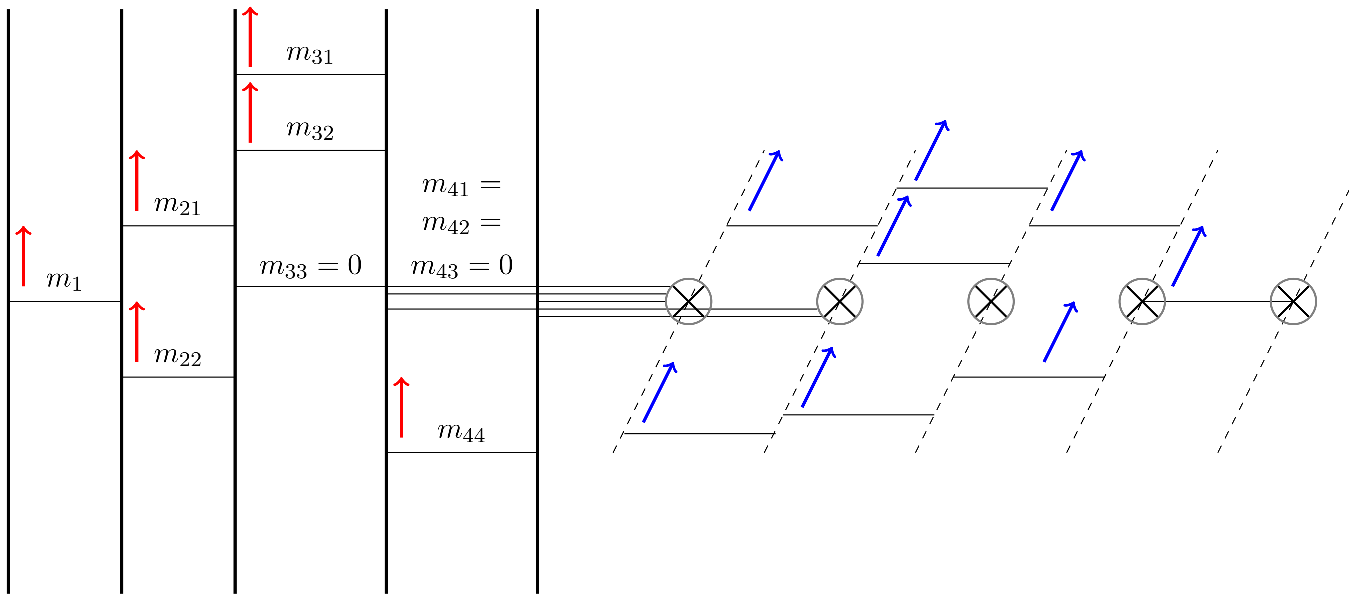


Figure 7.13: The brane picture for the $\rho = [3, 2]$ mixed branch, for the subcase in which $m_{41} = m_{42} = m_{43} = 0 \geq m_{44}$ and $m_{31} \geq m_{32} > m_{33} = 0$.

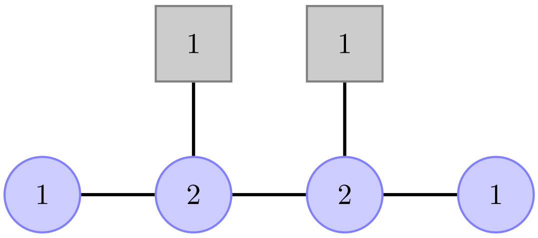


Figure 7.14: The Quiver graph for the $T^{[3,2]}[SU(5)]$ theory.

The Hilbert series is written by the characters of the $su(5)$ Lie algebra, and this implies that the topological symmetry is enhanced to $SU(5)$.

Now we would like to check this result by using the same procedure of the other examples. After going to the IR by giving infinitely large vev to the hypermultiplets and performing some Hanany-Witten transitions, we find the quiver theory of figure 7.14. For this linear quiver theory, the dimension formula is

$$\begin{aligned}\Delta(m_1, m_{21}, m_{22}, n_{21}, n_{22}, n_1) = &-|m_{22} - m_{21}| - |n_{22} - n_{21}| + \frac{1}{2}(|m_{21} - m_1| + |m_{22} - m_1| \\ &+ |n_{21} - m_{21}| + |n_{21} - m_{22}| + |n_{22} - m_{21}| + |n_{22} - m_{22}| \\ &+ |n_1 - n_{21}| + |n_1 - n_{22}| + |m_{21}| + |m_{22}| + |n_{21}| + |n_{22}|).\end{aligned} \tag{7.102}$$

where we assign magnetic charges as follows: m_1 is the magnetic charge for the leftmost $U(1)$, m_{21} and m_{22} (resp. n_{21} and n_{22}) are the magnetic charges for the leftmost (resp. rightmost) $U(2)$ group, and n_1 for the rightmost $U(1)$. The Hilbert series for the IR theory is given by

$$\begin{aligned} HS(z,t) = & \sum_{m_1=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} \sum_{n_{21}\geq n_{22}} \sum_{m_{21}\geq m_{22}} t^{\Delta(m_1,m_{21},m_{22},n_{21},n_{22},n_1)} z_1^{m_1} z_2^{(m_{21}+m_{22})} z_3^{(n_{21}+n_{22})} z_4^{n_1} \\ & \cdot P_{U(1)}(n_1,t)\, P_{U(1)}(n_2,t)\, P_{U(2)}(n_{21},n_{22},t)\, P_{U(2)}(m_{21},m_{22},t) \end{aligned} \tag{7.103}$$

By explicitly computing the refined Hilbert series we get

$$\begin{aligned} HS(z,t) = & 1 + [1,0,0,1]_z t + ([2,0,0,2]_z + [0,1,1,0]_z)\, t^2 \\ & + ([3,0,0,3]_z + [1,1,1,1]_z)\, t^3 + ([4,0,0,4]_z + [2,1,1,2]_z + [0,2,2,0]_z)\, t^4 \\ & + ([5,0,0,5]_z + [3,1,1,3]_z + [1,2,2,1]_z)\, t^5 + \\ & + ([6,0,0,6]_z + [4,1,1,4]_z + [2,2,2,2]_z + [0,3,3,0]_z)\, t^6 + \\ & + ([7,0,0,7]_z + [5,1,1,5]_z + [3,2,2,3]_z + [1,3,3,1]_z)\, t^7 + \\ & + ([8,0,0,8]_z + [6,1,1,6]_z + [4,2,2,4]_z + [2,3,3,2]_z + [0,4,4,0]_z)\, t^8 + \\ & + ([9,0,0,9]_z + [7,1,1,7]_z + [5,2,2,5]_z + [3,3,3,3]_z + [1,4,4,1]_z)\, t^9 + \mathcal{O}(t^{10}). \end{aligned} \tag{7.104}$$

and we see this is in perfect agreement with the Hilbert series found directly by the restriction rule in equation (7.101). This matching has been checked up to order 9 in t.

7.6 Higgs branch examples of the restriction rule

The Hilbert series for the Higgs branch part of any mixed branch of the $T[SU(N)]$ theory can be computed by going to the IR by decoupling all the Coulomb branch moduli which are not frozen, as explained in section 4.8.2, and by using the method described in section 4.4.1. In section 4.8.3, we use yet another way to compute the Hilbert series of the Higgs branch by using the restriction rule as well as the 3d mirror symmetry. In this appendix we will compute explicitly the Hilbert series for the Higgs branch part of all the mixed branches considered in section 7.5 by using the two methods. We will in fact find the complete agreement between the two results which give a nice check for the restriction rule as well as the 3d mirror symmetry.

The mixed branch $\rho = [2,2]$

In this case, we are interested in the Higgs branch part of the $[2,2]$ branch. The brane picture for this mixed branch is given already in figure 7.7. We first compute the Hilbert series of the

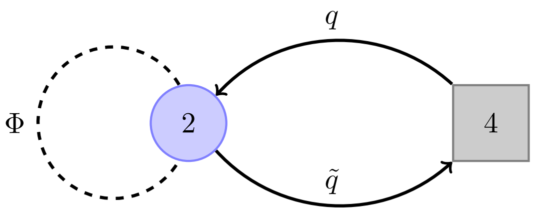


Figure 7.15: The quiver graph for the $U(2)$ gauge theory with 4 flavors. $q, \tilde{q}, \Phi$ are 3d $\mathcal{N} = 2$ chiral multiplets and $q, \tilde{q}$ form a 3d $\mathcal{N} = 4$ hypermultiplet.

	$U(2)_g$		
	$U(1)_g$	$SU(2)_g$	$SU(4)_f$
q	w_1	$[1]_{w_2}$	$[0,0,1]_{x_1,x_2,x_3}$
$\tilde{q}$	w_1^{-1}	$[1]_{w_2}$	$[1,0,0]_{x_1,x_2,x_3}$
Φ	1	$[2]_{w_2}$	1

Table 7.2: The charge assignment of the fields of the $\tilde{T}^{[2,2]}[SU(4)]$ theory, as seen from the quiver in figure 7.15.

Higgs branch factor $\mathcal{H}_{[2,2]}$ by using the method described in 4.8.2. For that we make use of the IR theory of (4.52).

By decoupling all the unfrozen Coulomb branch moduli (i.e. sending to infinity the mobile color D3-branes) we see that we are left only with 2 frozen color D3-branes in the 3rd cell, and 4 flavor D3-branes. The IR theory is therefore given by $U(2)$ with 4 flavors. In figure 7.15 we describe the matter in this IR theory, by using a 3d $\mathcal{N} = 2$ quiver notation, in which one 3d $\mathcal{N} = 4$ hypermultiplet is split in two 3d $\mathcal{N} = 2$ chiral multiplets $(q, \tilde{q})$, and we explicitly write a 3d $\mathcal{N} = 2$ chiral multiplet Φ in the adjoint representation of the $U(2)$, which lies inside a 3d $\mathcal{N} = 4$ vector multiplet. From the quiver, we can read the charge assignment of all the different chiral multiplets, and we then associate fugacities to the gauge and flavor symmetry groups, according to table 7.2. In our notation, if an arrow is pointing toward a group G, then a chiral multiplet associated to the arrow is in the fundamental representation under the symmetry group G.

The quiver in figure 7.15 also has the following superpotential in terms of the 3d $\mathcal{N} = 2$ notation,

$$W = \mathrm{tr}\,(q^i \Phi_{i\bar{j}} \tilde{q}^{\bar{j}}), \tag{7.105}$$

where i (resp. $\bar{j}$) is an index of the fundamental (resp. anti-fundamental) representation of the $U(2)$, and the trace is performed on the flavor indices. By deriving the F-term equations from this superpotential by taking a derivative with respect to $\Phi_{i\bar{j}}$, we notice that on the Higgs branch there is one relation of order 2 in $\tilde{t}$, and carrying both an index i and an index $\bar{j}$. This splits into two independent equations: one in the adjoint and the other in the trivial representation of the $U(2)$. The other F-term conditions are automatically satisfied since the vevs for $\Phi_{i\bar{j}}$ are zero. Out of this information we can derive the F-term prefactor described in section 4.4.1 as

$$\mathrm{Pfc}(w_2, \tilde{t}) = \left(\mathrm{PE}\left[[2]_{w_2}\tilde{t}^2 + \tilde{t}^2\right]\right)^{-1}, \tag{7.106}$$

which we will need to multiply to the integrand of formula of (4.32), as explained in section 4.4.1.

Therefore the Hilbert series is given by

$$HS\left(\tilde{t}, x\right) = \int d\mu_{U(2)} \;\; \mathrm{Pfc}(w_2, \tilde{t}) \cdot \mathrm{PE}\left[w_1[1]_{w_2}[0,0,1]_x\tilde{t} + w_1^{-1}[1]_{w_2}[1,0,0]_x\tilde{t}\right], \tag{7.107}$$

where we recall that the Haar measure for a $U(2)$ gauge group is given by

$$\int d\mu_{U(2)} = \frac{1}{(2\pi i)^2}\oint_{|w_1|=1}\oint_{|w_2|=1}\frac{dw_1}{w_1}\frac{dw_2}{w_2}(1-w_2^2). \tag{7.108}$$

By performing explicitly this residue computation we get the following result,

$$HS\left(x, t\right) = 1 + [1,0,1]_x t + \left([2,0,2]_x + [0,2,0]_x\right)t^2 + \mathcal{O}(t^3), \tag{7.109}$$

where we write the expression in terms of $t = \tilde{t}^2$.

Matching with the dual Coulomb branch part of $[2, 2]$

We then move on to the other method of the computation in section 4.8.3 where we use the restriction rule and the 3d mirror symmetry. The dual partition to $[2, 2]$ is in fact $[2, 2]$. Therefore we can reuse the result of (7.90). By keeping track of the fugacities z_i for the topological symmetry the result was

$$HS(t, z) = 1 + [1,0,1]_z t + \left([2,0,2]_z + [0,2,0]_z\right)t^2 + \mathcal{O}(t^3), \tag{7.110}$$

which exactly coincides with (7.109). This matching has been checked up to order 7 in t. Note that z is related to x by performing a redefinition of the Cartan generators as explained in (4.58) and (4.59). In this case, the relations are

$$z_1 \mapsto x_1^2 x_2^{-1}, \quad z_2 \mapsto x_1^{-1}x_2^2, \tag{7.111}$$

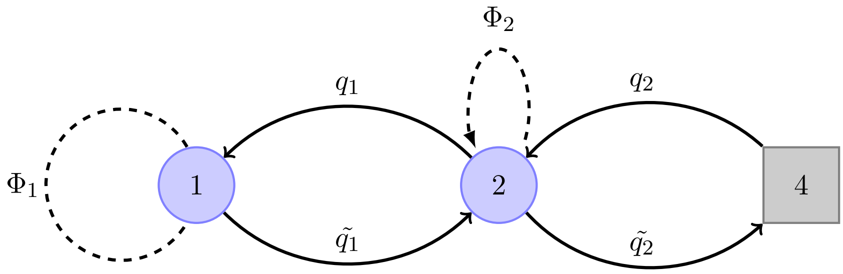


Figure 7.16: The quiver graph for the $\tilde{T}^{[2,2]}[SU(4)]$ theory.

since the Cartan matrix of the $su(3)$ Lie algebra is given by

$$M_{A_2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} . \tag{7.112}$$

The mixed branch $\rho = [3, 1]$

The next example is the Higgs branch part of the mixed branch $[3, 1]$. The brane picture for this mixed branch is given in figure 7.10. We first compute the Hilbert series by the method in section 4.8.2 and hence we send to infinity all the unfrozen color D3-branes. Doing this, we end up with a brane diagram yielding a theory given by the quiver depicted in figure 7.16. Our claim is that the Hilbert series for the full Higgs branch of this theory is the same as the Higgs branch part of the mixed branch of $[3, 1]$. Therefore we compute the Hilbert series, using the techniques in section 4.4.1.

From figure 7.16 we can read what are the matter fields. We assign gauge fugacities w_1 to the $U(1)_1$ factor of the gauge group, and w_2, w_3 to the $U(2)_2$ factor. In particular w_2 will be the fugacity for the overall $U(1)_2 \hookrightarrow U(2)_2$ and w_3 for $SU(2)_2 \hookrightarrow U(2)_2$. We also assign fugacities x_1, x_2, x_3 to the flavor $SU(4)$ group. We summarize the matter fields and their charges in table 7.3.

From the quiver in figure 7.16, we can also read off the superpotential that in this case leads to F-term constraints generating a prefactor

$$\mathrm{Pfc}(w_3, \tilde{t}) = \left(\mathrm{PE}\left[[2]_{w_3}\tilde{t}^2 + 2\tilde{t}^2\right]\right)^{-1} . \tag{7.113}$$

Therefore the Hilbert series is given by

$$\begin{aligned} HS\left(x, \tilde{t}\right) = \int d\mu_{U(1)\times U(2)} \;\; \mathrm{Pfc}(w_3, \tilde{t}) \; \cdot \\ \cdot \, \mathrm{PE}\left[w_1 w_2^{-1}[1]_{w_3}\tilde{t} + w_1^{-1} w_2 [1]_{w_3}\tilde{t} + w_2 [1]_{w_3}[0,0,1]_x \tilde{t} + w_2^{-1}[1]_{w_3}[1,0,0]_x \tilde{t}\right] , \end{aligned} \tag{7.114}$$

	$U(2)_2$			
	$U(1)_1$	$U(1)_2$	$SU(2)_2$	$SU(4)_f$
q_1	w_1^1	w_2^{-1}	$[1]_{w_2}$	1
$\tilde{q}_1$	w_1^{-1}	w_2^1	$[1]_{w_2}$	1
q_2	1	w_2^1	$[1]_{w_2}$	$[0,0,1]_{x_1,x_2,x_3}$
$\tilde{q}_2$	1	w_2^{-1}	$[1]_{w_2}$	$[1,0,0]_{x_1,x_2,x_3}$
Φ_1	1	1	1	1
Φ_2	1	1	$[2]_{w_3}$	1

Table 7.3: The charge assignment of the fields in the $\tilde{T}^{[3,1]}[SU(4)]$ theory, as seen from the quiver in figure 7.16.

where we recall that the Haar measure for a $U(1) \times U(2)$ gauge group is given by

$$\int d\mu_{U(1)\times U(2)} = \frac{1}{(2\pi i)^3} \oint_{|w_1|=1} \oint_{|w_2|=1} \oint_{|w_3|=1} \frac{dw_1}{w_1} \frac{dw_2}{w_2} \frac{dw_3}{w_3} (1 - w_3^2). \tag{7.115}$$

By performing explicitly this residue computation we get to the following result:

$$HS(x,t) = 1 + [1,0,1]_x t + ([2,0,2]_x + [0,2,0]_x + [1,0,1]_x)\, t^2 + \mathcal{O}(t^3), \tag{7.116}$$

where we again used $t = \tilde{t}^2$.

Matching with the dual Coulomb branch part of $[2,1,1]$

We then move on the the mirror computation in section 4.8.3. The dual to the partition $[3,1]$ is $[2,1,1]$. Hence we compute the Hilbert series of $\mathcal{C}_{[2,1,1]}$ by the restriction rule. We do not repeat the process of the computation and quote the result

$$HS(t,z) = 1 + [1,0,1]_z t + ([2,0,2]_z + [0,2,0]_z + [1,0,1]_z)\, t^2 + \mathcal{O}(t^3), \tag{7.117}$$

which completely agrees with (7.116). This matching has been checked up to order 6 in t. Here z is related to x by performing a redefinition of the Cartan generators and it is given by

$$z_1 \mapsto x_1^2 x_2^{-1}, \quad z_2 \mapsto x_1^{-1} x_2^2 x_3^{-1} \quad z_3 \mapsto x_2^{-1} x_3^2, \tag{7.118}$$

from the Cartan matrix of the $su(3)$,

$$M_{A_2} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}. \tag{7.119}$$

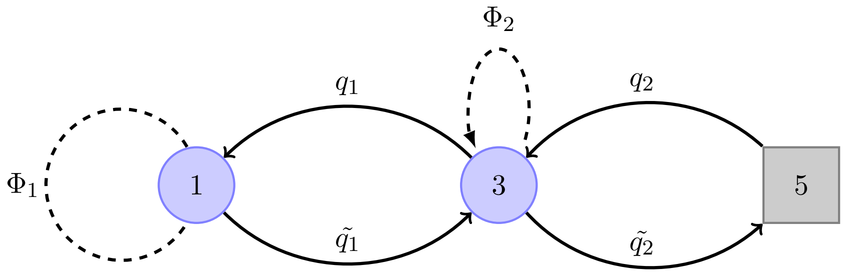


Figure 7.17: The quiver graph for the $\tilde{T}^{[3,2]}[SU(5)]$ theory.

		$U(3)_2$		
	$U(1)_1$	$U(1)_2$	$SU(3)_2$	$SU(5)_f$
q_1	w_1^1	w_2^{-1}	$[1,0]_{w_2,w_3}$	1
$\tilde{q}_1$	w_1^{-1}	w_2^1	$[0,1]_{w_2,w_3}$	1
q_2	1	w_2^1	$[0,1]_{w_2,w_3}$	$[0,0,0,1]_{x_1,x_2,x_3,x_4}$
$\tilde{q}_2$	1	w_2^{-1}	$[1,0]_{w_2,w_3}$	$[1,0,0,0]_{x_1,x_2,x_3,x_4}$
Φ_1	1	1	1	1
Φ_2	1	1	$[1,1]_{w_3,w_4}$	1

Table 7.4: The charge assignment of the fields of the $\tilde{T}^{[3,2]}[SU(5)]$ theory, as seen from the quiver in figure 7.17.

The mixed branch $\rho = [3,2]$

In this last example we are interested in the Higgs branch part of the mixed branch $[3,2]$. For doing the computation in section 4.8.2, we use the quiver diagram of the $\tilde{T}^{[3,2]}[SU(5)]$ theory given in figure 7.17 by using the general result (4.52).

From the quiver, we can read the matter fields and their charges under the global and gauge symmetry groups. This is reported in table 7.4. In particular we assign a fugacity w_1 to the $U(1)$ factor of the gauge group, fugacities w_2, w_3 and w_4 to the $U(3)$ gauge group, and fugacities x_1, x_2, x_3, x_4 to the $SU(5)$ flavor symmetry.

Furthermore, from the quiver we can write down the superpotential, and by writing the F-term equations we see that there is one relation in the adjoint of $SU(3)$ and two relations which are singlets under the gauge group. In particular the prefactor in this example takes the following form

$$\text{Pfc}(w_3, w_4, \tilde{t}) = \text{PE}\left[[1,1]_{w_3,w_4}\tilde{t}^2 + 2\tilde{t}^2\right]^{-1}. \tag{7.120}$$

With this information, we can write the Hilbert series, which in this case is

$$
\begin{aligned}
HS\left(\tilde{t}, x\right) = \int d\mu_{U(1)\times U(2)} \ \ \mathrm{Pfc}(w_3, \tilde{t}) \ \cdot \\
\cdot \ \mathrm{PE}\left[w_1 w_2^{-1}[1,0]_{w_3,w_4}\tilde{t} + w_1^{-1}w_2[0,1]_{w_3,w_4}\tilde{t} + \right. \\
\left. + w_2[0,1]_{w_3,w_4}[0,0,0,1]_x\tilde{t} + w_2^{-1}[1,0]_{w_3,w_4}[1,0,0,0]_x\tilde{t}\right],
\end{aligned}
\tag{7.121}
$$

where we recall that the Haar measure for a $U(1) \times U(3)$ gauge group is given by

$$
\begin{aligned}
\int d\mu_{U(1)\times U(3)} = \frac{1}{(2\pi i)^4} \oint_{|w_1|=1} \oint_{|w_2|=1} \oint_{|w_3|=1} \oint_{|w_4|=1} \frac{dw_1}{w_1}\frac{dw_2}{w_2}\frac{dw_3}{w_3}\frac{dw_4}{w_4} \\
(1 - w_3 w_4)\left(1 - \frac{w_3^2}{w_4}\right)\left(1 - \frac{w_4^2}{w_3}\right).
\end{aligned}
\tag{7.122}
$$

By performing this computation explicitly, and expanding to low order in $t = \tilde{t}^2$ we find

$$
HS(t, x) = 1 + [1,0,0,1]_x t + \left([2,0,0,2]_x + [1,0,0,1]_x + [0,1,1,0]_x\right) t^2 + \mathcal{O}(t^3). \tag{7.123}
$$

Matching with the dual Coulomb branch part of $[2,2,1]$

We then again compare the result (7.123) with the result by using the 3d mirror symmetry and the restriction rule. The dual to the partition $[3,2]$ is $[2,2,1]$. The Hilbert series of the Coulomb branch part $\mathcal{C}_{[2,2,1]}$ is given by

$$
HS(t, z) = 1 + [1,0,0,1]_z t + \left([2,0,0,2]_z + [1,0,0,1]_z + [0,1,1,0]_z\right) t^2 + \mathcal{O}(t^3), \tag{7.124}
$$

which again yields the perfect agreement with (7.123). This matching has been checked up to order 4 in t. The relations between z and x are

$$
z_1 \mapsto x_1^2 x_2^{-1}, \quad z_2 \mapsto x_1^{-1} x_2^2 x_3^{-1}, \quad z_3 \mapsto x_2^{-1} x_3^2 x_4^{-1}, \quad z_4 \mapsto x_3^{-1} x_4^2, \tag{7.125}
$$

since the Cartan matrix of the $su(5)$ Lie algebra is

$$
M_{A_4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}. \tag{7.126}
$$

7.7 Nilpotent Orbits

In this appendix we recall some of the basic features of nilpotent orbits of complex semisimple Lie algebras. A standard reference is [184].

Nilpotent orbits of complex semisimple Lie algebras are completely classified by mathematicians. We will briefly recall here such classification. The key point are the theorems of Jacobson-Morozov and Konstant which ensure that nilpotent orbits of $\mathfrak{g}$ are into one-to-one correspondence with embeddings $\mathfrak{su}(2) \to \mathfrak{g}$. In more details, the theorem of Jacobson-Morozov estabilishes that for every nilpotent element X of a complex semisimple Lie algebra there will exist a full *standard triple* of elements $\{H, X, Y\}$ which are generators of a $\mathfrak{su}(2)$ subalgebra. The theorem of Konstant shows that the inverse map is injective up to conjugation: namely for every standard triple it is possible to single out an unique element (up to conjugation), say X, which will be a nilpotent element of the Lie algebra. Therefore the orbits of nilpotent elements are into one-to-one correspondence with embeddings of $\mathfrak{su}(2)$. The problem is then recasted in the problem of classifying such embeddings. For the case of classical algebras, this was completely solved by Dynkin. For the case of exceptional algebras there exist a more modern classification by Bala and Carter.

7.7.1 A sketch of the classification

1. Nilpotent orbits in $\mathfrak{su}(\mathfrak{n}) \simeq \mathfrak{sl}(\mathfrak{n})$ are classified by partitions of the integer n.For example, $\mathfrak{su}(3)$ has 3 nilpotent orbits given by three different partitions of the number 3: $\mathcal{O}_1 = [3]$, $\mathcal{O}_2 = [2,1]$, $\mathcal{O}_3 = [1,1,1]$

2. Nilpotent orbits in $\mathfrak{so}(2n+1)$ are into one-to-one correspondence with the set of partitions of $2n+1$ in which even parts occurr with even multiplicity. For example in $\mathfrak{so}(7)$ there are seven nilpotent orbits: $\mathcal{O}_1 = [7]$, $\mathcal{O}_2 = [5,1^2]$, $\mathcal{O}_3 = [3,1^4]$, $\mathcal{O}_4 = [3,2^2]$, $\mathcal{O}_5 = [3^2,1]$, $\mathcal{O}_6 = [2^2,1^3]$, $\mathcal{O}_7 = [1^7]$.

3. Nilpotent orbits in $\mathfrak{sp}(2n)$ are into one-to-one correspondence with the set of partitions of $2n$ in which odd parts occurr with odd multiplicity. For example in $\mathfrak{sp}(6)$ there are eight nilpotent orbits: $\mathcal{O}_1 = [6]$, $\mathcal{O}_2 = [4,2]$, $\mathcal{O}_3 = [4,1^2]$, $\mathcal{O}_4 = [3^2]$, $\mathcal{O}_5 = [2^3]$, $\mathcal{O}_6 = [2^2,1^2]$, $\mathcal{O}_7 = [2,1^4]$, $\mathcal{O}_8 = [1^6]$.

4. Nilpotent orbits of $\mathfrak{so}(2n)$ are classified by partitions of the integer $2n$ such that even parts appear an even number of times. Furthermore, if the partition consist only in even parts

each of which appearing an even number of times, then the orbit is called *very even* and there will be a degeneracy. Two different orbits will correspond to the same partition. Therefore it is customary to add a roman numeral to the very even partitions to take into account such degenracy[4]. For example, for the case of $\mathfrak{so}(8)$ we have the following orbits: $\mathcal{O}_1 = [7,1]$, $\mathcal{O}_2 = [5,3]$, $\mathcal{O}_3 = [5,1^3]$, $\mathcal{O}_4 = [4,4]^I$, $\mathcal{O}_5 = [4,4]^{II}$, $\mathcal{O}_6 = [3^2,1^2]$, $\mathcal{O}_7 = [3,2^2,1]$, $\mathcal{O}_8 = [3,1^5]$, $\mathcal{O}_9 = [2^4]^I$, $\mathcal{O}_{10} = [2^4]^{II}$, $\mathcal{O}_{11} = [2^2,1^4]$, $\mathcal{O}_{12} = [1^8]$

5. Nilpotent orbits of exceptional Lie algebras are classified by the Bala-Carter classification. Such classification is quite involved and out of the scope of this thesis. For our purpose, it sufficies to say that nilpotent orbits are classified by a label of the form $G(x_i)$ where G is a Lie algebra, x is a letter which can be a or b, and i is an integer number. Sometimes also labels of the form $G(x_i) + G'(x'_i)$ are considered. For example, see the tables in the following section of this appendix.

7.7.2 Three useful properties

1. All nilpotent orbits are hyperkahler varieties.

2. The set of the nilpotent orbits of a given Lie algebra $\mathfrak{g}$ admits a poset structure. Namely it is possible to define a partial ordering among nilpotent orbits. Given orbits $\mathcal{O}_1$ and $\mathcal{O}_2$ we will say that $\mathcal{O}_1 \leq \mathcal{O}_2$ if the closure of $\mathcal{O}_1$ in $\mathfrak{g}$ is contained in the closure of $\mathcal{O}_2$ in $\mathfrak{g}$. This obviously implies that if $\mathcal{O}_1 \leq \mathcal{O}_2$ then $\dim_{\mathbb{C}}\mathcal{O}_1 \leq \dim_{\mathbb{C}}\mathcal{O}_2$

3. The poset structure has some features which will not depend on the Lie algebra chosen. In particular, for every Lie algebra there will exist one maximal orbit, one subregular orbit, and one minimal orbit such that the Hasse diagram will be given schematically by figure (7.18).

[4] Some author prefer to take into account the degeneracy by referring to the two very even orbits as the *red orbit* and the *blue orbit*.

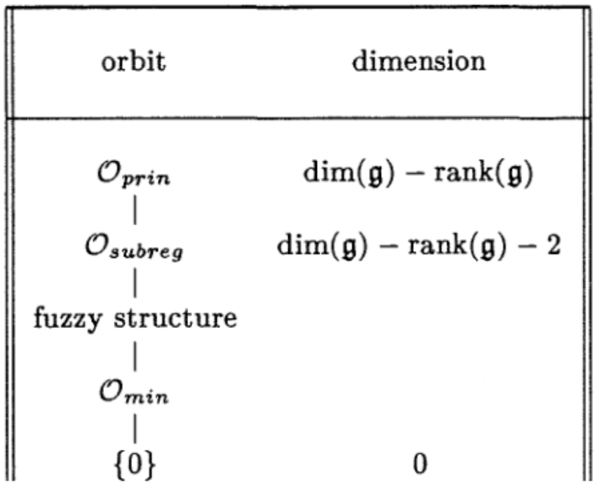


Figure 7.18: Schematic picture of the Hasse diagram for the nilpotent orbits of any complex semisimple Lie algebra $\mathfrak{g}$. This picture is taken from [184]. Here the dimension of the minimal orbit is not spelled out as depends on the choice of $\mathfrak{g}$.

7.7.3 Tables of orbits for the E-series

Here we give a complete list of the nilpotent orbits for the Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$- We list the Bala-Carter label of the orbit, its complex dimension, the branching rules for the decomposition of the adjoint representation in terms of representation of the Jacobson-Morozov $\mathfrak{su}(2)$ and the we say if such nilpotent orbit gives $\mathcal{N} = 2$ enhancement or not. This latter information is the most relevant for us, as the decomposition of the adjoint is what fixes the number (and the R-charges) of the scalar singlets we use to deform the UV theories in chapter3. The notation adopted for these tables is that V_n means the $\mathfrak{su}(2)$ representation with spin n. I.e. $V_n \sim [n+1]$ in Dynkin label notation. Orbits for which we can have supersymmetry enhancement are written in blue.

Orbit $\mathcal{O}$	$\dim_{\mathbb{C}} \bar{\mathcal{O}}$	Decomposition of Adj	Enhacement?
E_6	72	$V_1 \oplus V_4 \oplus V_5 \oplus V_7 \oplus V_8 \oplus V_{11}$	Yes, H_0 theory.
$E_6(a_1)$	70	$V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus 2V_5 \oplus V_7 \oplus V_8$	No
D_5	68	$V_0 \oplus V_1 \oplus 2V_2 \oplus V_3 \oplus V_4 \oplus 3V_5 \oplus V_7$	Yes, H_1 theory.
$E_6(a_3)$	66	$3V_1 \oplus 3V_2 \oplus 2V_3 \oplus 2_4 \oplus 2V_5$	No
$D_5(a_1)$	64	$V_0 \oplus 2V_{\frac{1}{2}} \oplus 2V_1 \oplus V_2 \oplus 2V_{\frac{5}{2}} \oplus 2V_3 \oplus 2V_{\frac{7}{2}} \oplus V_4 \oplus V_5$	No
A_5	64	$3V_0 \oplus V_1 \oplus 2V_{\frac{3}{2}} \oplus V_2 \oplus 2V_{\frac{5}{2}} \oplus V_3 \oplus V_4 \oplus 2V_{\frac{9}{2}} \oplus V_5$	No
$A_4 + A_1$	62	$V_0 \oplus 2V_{\frac{1}{2}} \oplus 2V_1 \oplus 2V_{\frac{3}{2}} \oplus 3V_2 \oplus 2V_{\frac{5}{2}} \oplus V_3 \oplus 2V_{\frac{7}{2}} \oplus V_4$	No
D_4	60	$8V_0 \oplus V_1 \oplus 8V_3 \oplus V_5$	Yes, H_2 theory.
A_4	60	$4V_0 \oplus 5V_1 \oplus 3V_2 \oplus 5V_3 \oplus V_4$	No
$D_4(a_1)$	58	$2V_0 \oplus 9V_1 \oplus 7V_2 \oplus 2V_3$	No
$A_3 + A_1$	56	$4V_0 \oplus 2V_{\frac{1}{2}} \oplus 4V_1 \oplus 6V_{\frac{3}{2}} \oplus 3V_2 \oplus 2V_{\frac{5}{2}} \oplus V_3$	No
$2A_32 + A_1$	54	$3V_0 \oplus 6V_{\frac{1}{2}} \oplus 5V_1 \oplus 4V_{\frac{3}{2}} \oplus 4V_2 \oplus 2V_{\frac{5}{2}}$	No
A_3	52	$11V_0 \oplus V_1 \oplus 8V_{\frac{3}{2}} \oplus 5V_2 \oplus V_3$	No
$A_2 + 2A_1$	50	$4V_0 \oplus 8V_{\frac{1}{2}} \oplus 9V_1 \oplus 4V_{\frac{3}{2}} \oplus 3V_2$	No
$2A_2$	48	$14V_0 \oplus 8V_1 \oplus 8V_2$	No
$A_2 + A_1$	46	$9V_0 \oplus 8V_{\frac{1}{2}} \oplus 8V_1 \oplus 6V_{\frac{3}{2}} \oplus V_2$	No
A_2	42	$16V_0 \oplus 19V_1 \oplus V_2$	No
$3A_1$	40	$11V_0 \oplus 16V_{\frac{1}{2}} \oplus 9V_1 \oplus 2V_{\frac{3}{2}}$	No
$2A_1$	32	$22V_0 \oplus 16V_{\frac{1}{2}} \oplus 8V_1$	No
A_1	22	$35V_0 \oplus 20V_{\frac{1}{2}} \oplus V_1$	No
0	0	$78V_0$	No

Table 7.5: All the nilpotent orbits of $\mathfrak{e}_6$.

Orbit $\mathcal{O}$	$\dim_{\mathbb{C}}\bar{\mathcal{O}}$	Decomposition of Adj	Enhacement?
E_7	126	$V_1 \oplus V_5 \oplus V_7 \oplus V_9 \oplus V_{11} \oplus V_{13} \oplus V_{17}$	Yes, H_0 theory.
$E_7(a_1)$	124	$V_1 \oplus V_3 \oplus 2V_5 \oplus V_7 \oplus V_8 \oplus V_9 \oplus V_{11} \oplus V_{13}$	No
$E_7(a_2)$	122	$2V_1 \oplus V_3 \oplus V_4 \oplus 2V_5 \oplus 2V_7 \oplus V_8 \oplus V_9 \oplus V_{11}$	No
$E_7(a_3)$	120	$2V_1 \oplus V_2 \oplus 2V_3 \oplus V_4 \oplus 4V_5 \oplus 2V_7 \oplus V_8 \oplus V_9$	No
E_6	120	$3V_0 \oplus V_1 \oplus 3V_4 \oplus V_5 \oplus V_7 \oplus 3V_8 \oplus V_{11}$	Yes, H_1 theory.
$E_6(a_1)$	118	$V_0 \oplus V_1 \oplus 3V_2 \oplus V_3 \oplus 3V_4 \oplus 2V_5 \oplus 2V_6 \oplus V_7 \oplus V_8$	No
D_6	118	$3V_0 \oplus V_1 \oplus 2V_{\frac{5}{3}} \oplus V_3 \oplus 2V_{\frac{9}{2}} \oplus 2V_5 \oplus V_7 \oplus 2V_{\frac{15}{2}} \oplus V_9$	No
$E_7(a_4)$	116	$4V_1 \oplus 2V_2 \oplus 3V_3 \oplus 2V_4 \oplus 5V_5 \oplus V_6 \oplus V_7$	No
$D_6(a_1)$	114	$3V_0 \oplus 2V_1 \oplus 2V_{\frac{3}{2}} \oplus 2V_{\frac{5}{2}} \oplus 2V_3 \oplus V_4 \oplus 2V_{\frac{9}{2}} \oplus 2V_5 \oplus 2V_{\frac{11}{2}} \oplus V_7$	No
$D_5 + A_1$	114	$3V_0 \oplus 2V_1 \oplus 2V_{\frac{3}{2}} \oplus 2V_{\frac{5}{2}} \oplus V_3 \oplus 3V_4 \oplus 2V_{\frac{9}{2}} \oplus V_5 \oplus 2V_{\frac{11}{2}} \oplus V_7$	No
A_6	114	$3V_0 \oplus V_1 \oplus 3V_2 \oplus 5V_3 \oplus 3V_4 \oplus V_5 \oplus 3V_6$	No
$E_7(a_5)$	112	$6V_1 \oplus 4V_2 \oplus 5V_3 \oplus 3V_4 \oplus 3V_5$	No
D_5	112	$6V_0 \oplus V_1 \oplus 4V_2 \oplus V_3 \oplus 3V_4 \oplus \oplus 5V_5 \oplus V_7$	No
$E_6(a_3)$	110	$3V_0 \oplus 3V_1 \oplus 7V_2 \oplus 4V_3 \oplus 4V_4 \oplus 2V_5$	No
$D_6(a_2)$	110	$3V_0 \oplus 3V_1 \oplus 4V_{\frac{3}{2}} \oplus V_2 \oplus 2V_{\frac{5}{2}} \oplus 3V_3 \oplus 2V_{\frac{7}{2}} \oplus V_4 \oplus 2V_{\frac{9}{2}} \oplus 2V_5$	No
$D_5(a_1) + A_1$	108	$3V_0 \oplus 7V_1 \oplus 3V_2 \oplus 8V_3 \oplus 3V_4 \oplus V_5$	No
$A_5 + A_1$	108	$3V_0 \oplus 4V_{\frac{1}{2}} \oplus 2V_1 \oplus 2V_{\frac{3}{2}} \oplus 3V_2 \oplus 2V_{\frac{5}{2}} \oplus V_3 \oplus 2V_{\frac{7}{2}} \oplus 3V_4 \oplus 2V_{\frac{9}{2}} \oplus V_5$	No
$(A_5)'$	108	$6V_0 \oplus V_1 \oplus 2V_{\frac{3}{2}} \oplus 3V_2 \oplus 6V_{\frac{5}{2}} \oplus V_3 \oplus 3V_4 \oplus 2V_{\frac{9}{2}} \oplus V_5$	No
$A_4 + A_2$	106	$3V_0 \oplus 6V_1 \oplus 10V_2 \oplus 5V_3 \oplus 3V_4$	No
$D_5(a_1)$	106	$4V_0 \oplus 4V_{\frac{1}{2}} \oplus 4V_1 \oplus V_2 \oplus 4V_{\frac{5}{2}} \oplus 4V_3 \oplus 4V_{\frac{7}{2}} \oplus V_4 \oplus V_5$	No
$A_4 + A_1$	104	$2V_0 \oplus 4V_{\frac{1}{2}} \oplus 4V_1 \oplus 4V_{\frac{3}{2}} \oplus 5V_2 \oplus 4V_{\frac{5}{2}} \oplus 3V_3 \oplus 2V_{\frac{7}{2}} \oplus V_4$	No
$D_4 + A_1$	102	$10V_0 \oplus 4V_{\frac{1}{2}} \oplus 4V_1 \oplus 4V_{\frac{3}{2}} \oplus 5V_2 \oplus 4V_{\frac{5}{2}} \oplus 3V_3 \oplus 2V_{\frac{7}{2}} \oplus V_4$	No
$(A_5)''$	102	$14V_0 \oplus V_1 \oplus 7V_2 \oplus V_3 \oplus 7V_4 \oplus V_5$	No
$A_3 + A_2 + A_1$	100	$3V_0 \oplus 15V_1 \oplus 10V_2 \oplus 15V_3$	No
A_4	100	$9V_0 \oplus 7V_1 \oplus 9V_2 \oplus 7V_3 \oplus V_4$	No
$A_3 + A_2$	98	$4V_0 \oplus 4V_{\frac{1}{2}} \oplus 8V_1 \oplus 8V_{\frac{3}{2}} \oplus 4V_2 \oplus 4V_{\frac{5}{2}} \oplus 3V_3$	No
$D_4(a_1) + A_1$	96	$6V_0 \oplus 4V_{\frac{1}{2}} \oplus 8V_1 \oplus 8V_{\frac{3}{2}} \oplus 5V_2 \oplus 4V_{\frac{5}{2}} \oplus 2V_3$	No
D_4	96	$21V_0 \oplus V_1 \oplus 14V_3 \oplus V_5$	No
$A_3 + 2A_1$	94	$6V_0 \oplus 8V_{\frac{1}{2}} \oplus 7V_1 \oplus 6V_{\frac{3}{2}} \oplus 7V_2 \oplus 4V_{\frac{5}{2}} \oplus V_3$	No
$D_4(a_1)$	94	$9V_0 \oplus 15V_1 \oplus 13V_2 \oplus 2V_3$	No
$(A_3 + A_1)'$	92	$9V_0 \oplus 6V_{\frac{1}{2}} \oplus 6V_1 \oplus 10V_{\frac{3}{2}} \oplus 7V_2 \oplus 2V_{\frac{5}{2}} \oplus V_3$	No
$2A_2 + 2A_1$	90	$6V_0 \oplus 10V_{\frac{1}{2}} \oplus 11V_1 \oplus 8V_{\frac{3}{2}} \oplus 6V_2 \oplus 2V_{\frac{5}{2}}$	No
$(A_3 + A_1)'$	86	$21V_0 \oplus 10V_1 \oplus 15V_2 \oplus V_3$	No
$A_2 + 3A_1$	84	$14V_0 \oplus 28V_1 \oplus 7V_2$	No
$2A_2$	84	$17V_0 \oplus 22V_1 \oplus 10V_2$	No
A_3	84	$24V_0 \oplus V_1 \oplus 16V_{\frac{3}{2}} \oplus 7V_2 \oplus V_3$	No
$A_2 + 2A_1$	82	$9V_0 \oplus 16V_{\frac{1}{2}} \oplus 15V_1 \oplus 8V_{\frac{3}{2}} \oplus 3V_2$	No
$A_2 + A_1$	76	$16V_0 \oplus 16V_{\frac{1}{2}} \oplus 16V_1 \oplus 8V_{\frac{3}{2}} \oplus V_2$	No
$4A_1$	70	$21V_0 \oplus 20V_{\frac{1}{2}} \oplus 16V_1 \oplus 6V_{\frac{3}{2}}$	No
A_2	66	$35V_0 \oplus 31V_1 \oplus V_2$	No
$(3A_1)'$	64	$24V_0 \oplus 28V_{\frac{1}{2}} \oplus 2V_{\frac{3}{2}}$	No
$(3A_1)'$	54	$52V_0 \oplus 27V_1$	No
$2A_1$	52	$39V_0 \oplus 32V_{\frac{1}{2}} \oplus 10V_1$	No
A_1	34	$66V_0 \oplus 32V_{\frac{1}{2}} \oplus V_1$	No
0	0	$133V_0$	No

Table 7.6: All the nilpotent orbits of $\mathfrak{e}_7$.

Orbit $\mathcal{O}$	$\dim_{\mathbb{C}}\bar{\mathcal{O}}$	Decomposition of Adj	Enhacement?
E_8	240	$V_1 \oplus V_7 \oplus V_{11} \oplus V_{13} \oplus V_{17} \oplus V_{19} \oplus V_{23} \oplus V_{29}$	Yes, H_0 theory
$E_8(a_1)$	238	$V_1 \oplus V_5 \oplus V_7 \oplus V_9 \oplus V_{11} \oplus V_{13} \oplus V_{14} \oplus V_{17} \oplus V_{19} \oplus V_{23}$	No
$E_8(a_2)$	236	$V_1 \oplus V_3 \oplus V_5 \oplus V_7 \oplus V_8 \oplus V_9 \oplus 2V_{11} \oplus V_{13} \oplus V_{14} \oplus \oplus_{17} \oplus V_{19}$	No
$E_8(a_3)$	234	$2V_1 \oplus V_4 \oplus 2V_5 \oplus V_7 \oplus V_8 \oplus 2V_9 \oplus V_{11} \oplus 2V_{13} \oplus V_{14} \oplus V_{17}$	No
$E_8(a_4)$	232	$V_1 \oplus V : 2 \oplus V : 3 \oplus V_4 \oplus 2V_5 \oplus 3V_7 \oplus V_8 \oplus 2V_9 \oplus 2V_{11} \oplus V_{13} \oplus V_{14}$	No
E_7	232	$3V_0 \oplus V_1 \oplus 2V_{\frac{9}{2}} \oplus V_5 \oplus V_7 \oplus 2V_{\frac{17}{2}} \oplus V_9 \oplus V_{11} \oplus V_{13} \oplus 2V_{\frac{27}{2}} \oplus V_{17}$	No
$E_8(b_4)$	230	$2V_1 \oplus V_2 \oplus 2V_3 \oplus 2V_5 \oplus V_6 \oplus 2V_7 \oplus 2V_8 \oplus V_9 \oplus V_{10} \oplus 2V_{11} \oplus V_{13}$	No
$E_8(a_5)$	228	$3V_1 \oplus V_2 \oplus V_3 \oplus V_4 \oplus 4V_5 \oplus 2V_6 \oplus 3V_7 \oplus V_8 \oplus V_9 \oplus V_{10} \oplus 2V_{11}$	No
$E_7(a_1)$	228	$3V_0 \oplus V_1 \oplus 2V_{\frac{5}{2}} \oplus V_3 \oplus 2V_5 \oplus 2V_{\frac{11}{2}} \oplus V_7 \oplus$ $\oplus 2V_{\frac{15}{2}} \oplus V_8 \oplus V_9 \oplus 2V_{\frac{21}{2}} \oplus V_{11} \oplus V_{13}$	No
$E_8(b_5)$	226	$4V_1 \oplus V_2 \oplus 2V_3 \oplus 3V_4 \oplus 3V_5 \oplus 3V_7 \oplus 3V_8 \oplus 2V_9 \oplus V_{11}$	No
D_7	226	$3V_0 \oplus V_1 \oplus 2V_{\frac{3}{2}} \oplus V_3 \oplus 2V_{\frac{9}{2}} \oplus V_5 \oplus 2V_{\frac{11}{2}} \oplus$ $\oplus 3V_6 \oplus V_7 \oplus 2V_{\frac{15}{2}} \oplus V_9 \oplus \oplus 2V_{\frac{21}{2}} \oplus V_{11}$	No
$E_8(a_6)$	224	$3V_1 \oplus V_2 \oplus 5V_3 \oplus 3V_4 \oplus 3V_5 \oplus 3V_6 \oplus 3V_7 \oplus V_8 \oplus 2V_9$	No
$E_7(a_2)$	224	$3V_1 \oplus 2V_2 \oplus 2V_{\frac{3}{2}} \oplus V_3 \oplus 2V_{\frac{7}{2}} \oplus V_4 \oplus 2V_{\frac{9}{2}} \oplus 2V_5 \oplus$ $\oplus 2V_7 \oplus 2V_{\frac{15}{2}} \oplus V_8 \oplus 2V_{\frac{17}{2}} \oplus V_9 \oplus V_{11}$	No
$E_6 + A_1$	222	$3V_0 \oplus 4V_{\frac{1}{2}} \oplus 2V_1 \oplus 2V_{\frac{7}{2}} \oplus 3V_4 \oplus 2V_{\frac{9}{2}} \oplus$ $\oplus V_5 \oplus V_7 \oplus 2V_{\frac{15}{2}} \oplus 3V_8 \oplus 2V_{\frac{17}{2}} \oplus V_{11}$	No
$D_7(a_1)$	222	$V_0 \oplus 4V_1 \oplus 2V_2 \oplus 3V_3 \oplus 3V_4 \oplus 6V_5 \oplus V_6 \oplus 3V_7 \oplus 2V_8 \oplus V_9$	No
$E_8(b_6)$	220	$4V_1 \oplus 4V_2 \oplus 5 \oplus_3 \oplus 3V_4 \oplus 6V_5 \oplus 2V_6 \oplus 3V_7 \oplus V_8$	No
$E_7(a_3)$	220	$3V_0 \oplus 2V_{\frac{1}{2}} \oplus 2V_1 \oplus V_2 \oplus 2V_{\frac{5}{2}} \oplus 2V_3 \oplus V_4 \oplus$ $\oplus 4V_{\frac{9}{2}} \oplus 3V_5 \oplus 2V_{\frac{11}{2}} \oplus 2V_7 \oplus 2V_{\frac{15}{2}} \oplus V_8 \oplus V_9$	No
$E_6(a_1) + A_1$	218	$V_0 \oplus 2V_{\frac{1}{2}} \oplus 2V_1 \oplus 2V_{\frac{3}{2}} \oplus 3V_2 \oplus 2V_{\frac{5}{2}} \oplus V_3 \oplus 2V_{\frac{7}{2}} \oplus$ $\oplus 3V_4 \oplus 2V_{\frac{9}{2}} \oplus 2V_5 \oplus 2V_{\frac{11}{2}} \oplus 2V_6 \oplus 2V_{\frac{13}{2}} \oplus V_7 \oplus V_8$	No
A_7	218	$3V_0 \oplus V_1 \oplus 2V_{\frac{3}{2}} \oplus 3V_2 \oplus 2V_{\frac{5}{2}} \oplus V_3 \oplus 4V_{\frac{7}{2}}$ $\oplus 3V_4 \oplus 2V_{\frac{9}{2}} \oplus V_5 \oplus 2V_{\frac{11}{2}} \oplus 3V_6 \oplus V_7 \oplus 2V_{\frac{15}{2}}$	No
$D_7(a_2)$	216	$V_0 \oplus 2V_{\frac{1}{2}} \oplus 2V_1 \oplus 2V_{\frac{3}{2}} \oplus 3V_2 \oplus 2V_{\frac{5}{2}} \oplus 3V_3 \oplus 4V_{\frac{7}{2}} \oplus$ $\oplus 3V_4 \oplus 2V_{\frac{9}{2}} \oplus 2V_5 \oplus 2V_{\frac{11}{2}} \oplus V_6 \oplus 2V_{\frac{13}{2}} \oplus V_7$	No
E_6	216	$14V_0 \oplus V_1 \oplus 7V_4 \oplus V_5 \oplus V_7 \oplus 7V_8 \oplus V_{11}$	No
D_6	216	$10V_0 \oplus V_1 \oplus 4V_{\frac{5}{2}} \oplus V_3 \oplus 4V_{\frac{9}{2}} \oplus 6V_5 \oplus V_7 \oplus 4V_{\frac{15}{2}} \oplus V_9$	No
$D_5 + A_2$	214	$V_0 \oplus 8V_1 \oplus 5V_2 \oplus 5V_3 \oplus 5V_4 \oplus 7V_5 \oplus 2V_6 \oplus V_7$	No
$E_6(a_1)$	214	$8V_0 \oplus V_1 \oplus 7V_2 \oplus V_3 \oplus 7V_4 \oplus 2V_5 \oplus 6V_6 \oplus V_7 \oplus V_8$	No
$E_7(a_4)$	212	$3V_0 \oplus 2V_{\frac{1}{2}} \oplus 4V_1 \oplus 4V_{\frac{3}{2}} \oplus 2V_2 \oplus 2V_{\frac{5}{2}} \oplus 3V_3 \oplus 2V_{\frac{7}{2}} \oplus$ $\oplus 2V_4 \oplus 4V_{\frac{9}{2}} \oplus 4V_5 \oplus 2V \frac{11}{2} \oplus V_6 \oplus V_7$	No
$A_6 + A_1$	212	$3V_0 \oplus 2V_{\frac{1}{2}} \oplus 2V_1 \oplus 2V_{\frac{3}{2}} \oplus 3V_2 \oplus 4V_{\frac{5}{2}} \oplus 5V_3 \oplus$ $4V_{\oplus \frac{7}{2}} \oplus 3V_4 \oplus 2V \frac{9}{2} \oplus V_5 \oplus 2V_{\frac{11}{2}} \oplus 3V_6$	No
$D_6(a_1)$	210	$6V_0 \oplus 6V_1 \oplus 4V_{\frac{3}{2}} \oplus 4V_{\frac{5}{2}} \oplus 2V_3 \oplus 5V_4 \oplus 4V_{\frac{9}{2}} \oplus 2V_5 \oplus 4V_{\frac{11}{2}} \oplus V_7$	No
A_6	210	$6V_0 \oplus 5V_1 \oplus 3V_2 \oplus 13V_3 \oplus 3V_4 \oplus 5V_5 \oplus 3V_6$	No
$E_8(a_7)$	208	$10V_1 \oplus 10V_2 \oplus 10V_3 \oplus 6V_4 \oplus 4V_5$	No
$D_5 + A_1$	208	$6V_0 \oplus 6V_{\frac{1}{2}} \oplus 2V_1 \oplus 2V_{\frac{3}{2}} \oplus 4V_2 \oplus 2V_{\frac{5}{2}} \oplus V_3 \oplus$ $\oplus 2V_{\frac{7}{2}} \oplus 3V_4 \oplus 4V_{\frac{9}{2}} \oplus 5V_5 \oplus 2V_{\frac{11}{2}} \oplus V_7$	No
$E_7(a_5)$	206	$3V_0 \oplus 6V_1 \oplus 6V_{\frac{3}{2}} \oplus 4V_2 \oplus 6V_{\frac{5}{2}} \oplus 5V_3 \oplus 4V_{\frac{7}{2}} \oplus 3V_4 \oplus 2V_{\frac{9}{2}} \oplus 3V_5$	No
$E_6(a_3) + A_1$	204	$3V_0 \oplus 4V_{\frac{1}{2}} \oplus 4V_1 \oplus 4V_{\frac{3}{2}} \oplus 7V_2 \oplus 6V_{\frac{5}{2}} \oplus$ $\oplus 4V_3 \oplus 5V_{\frac{7}{2}} \oplus 4V_4 \oplus 2V_{\frac{9}{2}} \oplus 2V_5$	No
$D_6(a_2)$	204	$6V_0 \oplus 3V_1 \oplus 8V_{\frac{3}{2}} \oplus 5V_2 \oplus 4V_{\frac{5}{2}} \oplus 7V_3 \oplus 4V_{\frac{7}{2}} \oplus V_4 \oplus 4V_{\frac{9}{2}} \oplus 2V_5$	No
$D_5(a_1) + A_2$	202	$3V_0 \oplus 4V_{\frac{1}{2}} \oplus 7V_1 \oplus 4V_{\frac{3}{2}} \oplus 6V_2 \oplus 6V_{\frac{5}{2}} \oplus$ $\oplus 4V_3 \oplus 6V_{\frac{7}{2}} \oplus 3V_4 \oplus 2V_{\frac{9}{2}} \oplus V_5$	No
$A_5 + A_1$	202	$6V_0 \oplus 4V_{\frac{1}{2}} \oplus 2V_1 \oplus 4V_{\frac{3}{2}} \oplus 7V_2 \oplus$ $\oplus 8V_{\frac{5}{2}} \oplus 5V_3 \oplus 2V_{\frac{7}{2}} \oplus 3V_4 \oplus 4V_{\frac{9}{2}} \oplus V_5$	No
$A_4 + A_3$	200	$3V_0 \oplus 4V_{\frac{1}{2}} \oplus 6V_1 \oplus 8V_{\frac{3}{2}} \oplus 6V_2 \oplus 6V_{\frac{5}{2}} \oplus 6V_3 \oplus 4V_{\frac{7}{2}} \oplus 3V_4 \oplus 2V_{\frac{9}{2}}$	No
D_5	200	$21V_0 \oplus V_1 \oplus 8V_2 \oplus V_3 \oplus 7V_4 \oplus 9V_5 \oplus V_7$	No

Table 7.7: All the orbits or $\mathfrak{e}_8$. Part 1.

Orbit $\mathcal{O}$	$\dim_{\mathbb{C}}\mathcal{O}$	Decomposition of Adj	Enhancement?
$E_6(a_3)$	198	$14V_0 \oplus 3V_1 \oplus 15V_2 \oplus 8V_3 \oplus 8V_4 \oplus 2V_5$	No
$D_4 + A_2$	198	$8V_0 \oplus 14V_1 \oplus 7V_2 \oplus 14V_3 \oplus 6V_4 \oplus V_4$	No
$A_4 + A_2 + A_1$	196	$3V_0 \oplus 6V_{\frac{1}{2}} \oplus 7V_1 \oplus 8V_{\frac{3}{2}} \oplus 10V_2 \oplus 6V_{\frac{5}{2}} \oplus 5V_3 \oplus 4V_{\frac{7}{2}} \oplus 3V_4$	No
$D_5(a_1) + A_1$	196	$6V_0 \oplus 10V_{\frac{1}{2}} \oplus 7V_1 \oplus 2V_{\frac{3}{2}} \oplus 3V_2 \oplus 6V_{\frac{5}{2}} \oplus 8V_3 \oplus 6V_{\frac{7}{2}} \oplus 3V_4 \oplus V_5$	No
A_5	196	$17V_0 \oplus V_1 \oplus 2V_{\frac{3}{2}} \oplus 7V_2 \oplus 14V_{\frac{5}{2}} \oplus V_3 \oplus 7V_4 \oplus 2V_{\frac{9}{2}} \oplus V_5$	No
$A_4 + A_2$	194	$6V_0 \oplus 18V_1 \oplus 14V_2 \oplus 13V_3 \oplus 3V_4$	No
$A_4 + 2A_1$	192	$4V_0 \oplus 8V_{\frac{1}{2}} \oplus 9V_1 \oplus 8V_{\frac{3}{2}} \oplus 9V_2 \oplus 8V_{\frac{5}{2}} \oplus 5V_3 \oplus 4V_{\frac{7}{2}} \oplus V_4$	No
$D_5(a_1)$	190	$15V_0 \oplus 8V_{\frac{1}{2}} \oplus 8V_1 \oplus V_2 \oplus 8V_{\frac{5}{2}} \oplus 8V_3 \oplus 8V_{\frac{7}{2}} \oplus V_4 \oplus V_5$	No
$2A_3$	188	$10V_0 \oplus 4V_{\frac{1}{2}} \oplus 6V_1 \oplus 16V_{\frac{3}{2}} \oplus 10V_2 \oplus 4V_{\frac{5}{2}} \oplus 6V_3 \oplus 4V_{\frac{7}{2}}$	No
$A_4 + A_1$	188	$9V_0 \oplus 8V_{\frac{1}{2}} \oplus 8V_1 \oplus 8V_{\frac{3}{2}} \oplus 9V_2 \oplus 8V_{\frac{5}{2}} \oplus 7V_3 \oplus 2V_{\frac{7}{2}} \oplus V_4$	No
$D_4(a_1) + A_2$	184	$8V_0 \oplus 28V_1 \oplus 20V_2 \oplus 8V_3$	No
$D_4 + A_1$	184	$21V_0 \oplus 14V_{\frac{1}{2}} \oplus 2V_1 \oplus 6V_{\frac{5}{2}} \oplus 14V_3 \oplus 6V_{\frac{7}{2}} \oplus V_5$	No
$A_3 + A_2 + A_1$	182	$6V_0 \oplus 10V_{\frac{1}{2}} \oplus 15V_1 \oplus 14V_{\frac{3}{2}} \oplus 10V_2 \oplus 6V_{\frac{5}{2}} \oplus 5V_3$	No
A_4	180	$24V_0 \oplus 11V_1 \oplus 21V_2 \oplus 11V_3 \oplus V_4$	No
$A_3 + A_2$	178	$11V_0 \oplus 8V_{\frac{1}{2}} \oplus 16V_1 \oplus 16V_{\frac{3}{2}} \oplus 8V_2 \oplus 8V_{\frac{5}{2}} \oplus 3V_3$	Nco
$D_4(a_1) + A_1$	176	$9V_0 \oplus 14V_{\frac{1}{2}} \oplus 16V_1 \oplus 12V_{\frac{3}{2}} \oplus 13V_2 \oplus 6V_{\frac{5}{2}} \oplus 2V_3$	No
$A_3 + 2A_1$	172	$13V_0 \oplus 14V_{\frac{1}{2}} \oplus 15V_1 \oplus 16V_{\frac{3}{2}} \oplus 11V_2 \oplus 6V_{\frac{5}{2}} \oplus V_3$	No
$2A_2 + 2A_1$	168	$10V_0 \oplus 20V_{\frac{1}{2}} \oplus 20V_1 \oplus 16V_{\frac{3}{2}} \oplus 10V_2 \oplus 4V_{\frac{5}{2}}$	No
D_4	168	$52V_0 \oplus V_1 \oplus 26V_3 \oplus V_5$	No
$D_4(a_1)$	166	$28V_0 \oplus 27V_1 \oplus 25V_2 \oplus 2V_3$	No
$A_3 + A_1$	164	$24V_0 \oplus 14V_{\frac{1}{2}} \oplus 10V_1 \oplus 18V_{\frac{3}{2}} \oplus 15V_2 \oplus 2V_{\frac{5}{2}} \oplus V_3$	No
$2A_2 + A_1$	162	$17V_0 \oplus 18V_{\frac{1}{2}} \oplus 23V_1 \oplus 16V_{\frac{3}{2}} \oplus 10V_2 \oplus 2V_{\frac{5}{2}}$	No
$2A_2$	156	$28V_1 \oplus 50V_3 \oplus 14V_5$	No
$A_2 + 3A_1$	154	$17V_0 \oplus 28V_{\frac{1}{2}} \oplus 28V_1 \oplus 14V_{\frac{3}{2}} \oplus 7V_2$	No
A_3	148	$55V_0 \oplus V_1 \oplus 32V_{\frac{3}{2}} \oplus 11V_2 \oplus V_3$	No
$A_2 + 2A_1$	146	$24V_0 \oplus 32V_{\frac{1}{2}} \oplus 27V_1 \oplus 16V_{\frac{3}{2}} \oplus 3V_2$	No
$A_2 + A_1$	136	$35V_0 \oplus 32V_{\frac{1}{2}} \oplus 32V_1 \oplus 14V_{\frac{3}{2}} \oplus V_2$	No
$4A_1$	128	$36V_0 \oplus 48V_{\frac{1}{2}} \oplus 28V_1 \oplus 8V_{\frac{3}{2}}$	No
A_2	114	$78V_0 \oplus 55V_1 \oplus V_2$	No
$3A_1$	112	$55V_0 \oplus 52V_{\frac{1}{2}} \oplus 37V_1 \oplus 2V_{\frac{3}{2}}$	No
$2A_1$	92	$78V_0 \oplus 64V_{\frac{1}{2}} \oplus 13V_1$	No
A_1	58	$133V_0 \oplus 56V_{\frac{1}{2}} \oplus V_1$	No
0	0	$248V_0$	No

Table 7.8: All the orbits or $\mathfrak{e}_8$. Part 2.

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