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Recovery of Singularities in Inverse Scattering

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A mi abuelo Aniceto,
de quien más he aprendido

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Resumen y Conclusiones

El objetivo central en scattering inverso es recuperar un potencial $q(x)$, $x \in \mathbb{R}^n$, desconocido, a partir de los datos de scattering u_∞. Sea $k \in (0,\infty)$, y $\theta, \theta' \in \mathbb{S}^{n-1}$. $u_\infty(k,\theta,\theta')$ describe el comportamiento asintótico de las soluciones de scattering $u_s(k,\theta,x)$, $x = |x|\theta'$, que satisfacen

$$\begin{cases} (-\Delta + q - k^2)u = 0 \\ u(x) = e^{ik\theta\cdot x} + u_s(k,\theta,x), \end{cases} \tag{1}$$

junto con la condición de radiación de Sommerfeld.

El problema de recuperar $q(x)$ a partir de $u_\infty(k,\theta,\theta')$ está sobredeterminado, ya que el potencial depende de n parámetros (es una función $\mathbb{R}^n$) y los datos de scattering dependen de $2n-1$. De hecho, la transformada de Fourier de q se puede recuperar puntualmente de los datos totales $u_\infty(k,\theta,\theta')$, y lo mismo ocurre en ciertos casos específicos, cuando los datos dependen solo de $n+1$ parámetros.

Por lo tanto, es natural considerar problemas inversos formalmente bien propuestos, restringiendo el dominio de $u_\infty(k,\theta,\theta')$ para que dependa solo de n parámetros. Desde el punto de vista de las aplicaciones, esto tiene sentido ya que siempre es de interés reducir el número de medidas necesarias. Existen distintas maneras de reducir el número de parámetros. Uno de los más estudiados es el problema de backscattering. En este caso, como el nombre sugiere, solo se consideran las ondas reflejadas por el potencial en la dirección opuesta a la onda incidente (los ecos), esto es, únicamente $u_\infty(k,\theta,-\theta)$ se da por conocido. Éste es el principal problema tratado en esta tesis, pero también estudiaremos el caso de scattering de ángulo fijo, en el cual las ondas se envían desde una dirección fija $\theta = \theta_0$. Además se presentarán algunos resultados relacionados para el caso de datos totales, problema que presenta algunos rasgos en común con los dos anteriores.

En estos tres problemas, una de las maneras usuales de proceder es construir la *aroximación de Born* del potencial, que denotaremos por q_B en el caso de backscattering y q_θ in en el de scattering de ángulo fijo. La aproximación de Born es esencialmente la transformada de Fourier inversa de los datos de scattering reducidos, y es también una función de $\mathbb{R}^n$. En cierto sentido, es la aproximación lineal del problema inverso y es muy utilizada en las aplicaciones. Desde el punto de vista teórico, la transformada de Fourier de q_B se puede entender como una transformada de Fourier no lineal del potencial q. Esto se puede observar en la fórmula

$$\widehat{q_B}(-2k\theta) = \int_{\mathbb{R}^n} e^{ik\theta\cdot y} q(y)(e^{ik\theta\cdot y} + u_s(k,\theta,y))\, dy, \tag{2}$$

donde la no linealidad se debe al hecho de que $u_s(k,\theta,x)$ depende de q a través de (1).

A partir de la fórmula anterior, podemos establecer una relación más inmediata entre la aproximación de Born y el potencial, a través de serie de Born

$$q_B \sim q + \sum_{j=2}^{\infty} Q_j(q).$$

En el problema de scattering de ángulo fijo, existe una relación del todo análoga:

$$q_\theta \sim q + \sum_{j=2}^{\infty} Q_{\theta,j}(q).$$

Utilizamos el símbolo $\sim$ para evitar tratar en este resumen la convergencia de ambas series. $Q_j(q)$ y $Q_{\theta,j}(q)$ son ciertos operadores multilineales que describen la dispersión múltiple de la onda $e^{ik\theta\cdot x}$, y se pueden expresar en términos de la resolvente del laplaciano. Como el nombre sugiere, la aproximación de Born es una buena aproximación para q, en el sentido de que la diferencia $q - q_B$ es pequeña en una norma apropiada para potenciales que satisfacen ciertas condiciones de pequeñez.

Desde el punto de vista matemático, una cuestión importante que no está respondida satisfactoriamente es establecer cuanta información contiene la aproximación de Born sobre el potencial q, y si es posible recuperar completamente q conociendo q_B. Esto es equivalente a preguntarse si la transformada de Fourier no lineal dada por (2) es invertible. Este es un problema de gran complejidad y de hecho, la cuestión de la unicidad solo se ha respondido parcialmente.

Motivado por el uso de la aproximación de Born en las aplicaciones, un posible enfoque para buscar resultados de recuperación parcial del potencial parte de preguntarse qué tipo de información sobre q se puede recuperar directamente de q_B, o de forma muy inmediata. Este es el camino seguido en [36] donde se muestra que la aproximación de Born construida a partir de los datos totales contiene las principales singularidades de q. Desde entonces este tipo de resultados han recibido una gran atención en los tres problemas de scattering aquí comentados.

El objetivo principal de este trabajo es cuantificar la regularidad de $q - q_B$, y comprobar cuando esta diferencia es más regular q, dependiendo de la dimensión n, y de la regularidad a priori del potencial q medida en la escala de Sobolev. Trataremos la misma cuestión en el caso de scattering de ángulo fijo. La mayoría de los resultados de esta tesis se pueden encontrar también en los artículos [27–29]. Se consideran potenciales complejos a lo largo de todo el trabajo, y no se asume ninguna condición de pequeñez sobre los mismos.

Los principales resultados obtenidos se pueden dividir en tres grupos. En primer se estudia la recuperación de singularidades en backscattering para dimensión n, mejorando la mayoría de los resultados previos conocidos hasta ahora (ver teorema 2.3). Además por primera vez se demuestra que existe una condición necesaria que limita las singularidades que se puede obtener de la aproximación de Born (ver teorema 2.2). En segundo lugar, se intentan obtener resultados de recuperación de singularidades que sean óptimos de acuerdo con el teorema 2.2, bien pasando de medir la regularidad de $q - q_B$ de los espacios de Sobolev a la clase de Hölder (teorema 2.7 y corolario 2.8), bien asumiendo que los potenciales son radiales, o satisfacen ciertas condiciones un poco menos restrictivas (teorema 2.9 y corolarios 2.10 y 2.11). Por último, en tercer lugar, se extienden los dos primeros resultados mencionados al caso de scattering de ángulo fijo (ver teoremas 2.13 y 2.14).

Contents

Chapter 1

Introduction

The central problem in inverse scattering for the Schrödinger equation is to recover an unknown potential $q(x)$, $x \in \mathbb{R}^n$, from the scattering data, the so called far field pattern or scattering amplitude u_∞.

Let $k \in (0, \infty)$, and $\theta, \theta' \in \mathbb{S}^{n-1}$. $u_\infty(k, \theta, \theta')$ describes the asymptotic behavior of the scattering solutions $u_s(k, \theta, x)$, $x = |x|\theta'$, which satisfy

$$\begin{cases} (-\Delta + q - k^2)u = 0 \\ u(x) = e^{ik\theta\cdot x} + u_s(k, \theta, x), \end{cases} \tag{1.1}$$

together with a certain condition at infinity (the Sommerfeld radiation condition). Therefore, u_s is just the difference between the generalized eigenvalues of the Hamiltonian $H = -\Delta + q$ and the plane waves, the generalized eigenvalues in the free case. When $\theta \neq \theta'$, $u_\infty(k, \theta, \theta')$ is related to the probability amplitude (in the quantum mechanics sense) that a free particle of energy k^2 incoming from direction θ is scattered by the potential in direction θ'. Indeed, the far field pattern is closely related to the scattering matrix, which has a central role in quantum mechanics.

The problem of recovering $q(x)$ from $u_\infty(k, \theta, \theta')$ is greatly overdetermined since the former depends on n parameters (is a function in $\mathbb{R}^n$) and the latter depends on $2n - 1$. In fact it can be shown that the Fourier transform of q can be recovered pointwise from the whole data $u_\infty(k, \theta, \theta')$, and the same happens in certain specific instances if we restrict the data to depend only on $n + 1$ parameters.

Therefore is natural to consider formally well posed inverse problems by restricting the domain of $u_\infty(k, \theta, \theta')$ to depend only on n parameters. This makes sense from the point of view of applications, where very often is of great interest to reduce the number of measurements as much as possible. There are different natural ways to do this.

One of the most widely studied is the *backscattering problem*. In this case, as the name suggest, only the waves scattered in the opposite direction of the incident wave (the echoes) are taken into account, that is only $u_\infty(k, \theta, -\theta)$ is considered known. This is the main problem studied in this thesis, but we will also treat the *fixed angle scattering problem*, where the waves come instead from a fixed direction $\theta = \theta_0$. Some results will also concern the *full data scattering problem*, that shares some common traits with the previous ones.

In these scattering problems, the usual procedure is to construct the *Born approximation* of the potential, which we denote by q_B in the case of backscattering and q_θ in the case of fixed angle scattering. The Born approximation is essentially the Fourier inverse transform of the restricted scattering data, and it is also an $\mathbb{R}^n$ function as q. In a certain sense, it is a linear approximation to the inverse problem and it is widely used in applications. From the theoretical point of view, the Fourier transform of q_B is a kind of nonlinear Fourier transform of the potential q since we have the formula

$$\widehat{q_B}(-2k\theta) = \int_{\mathbb{R}^n} e^{ik\theta\cdot y} q(y)(e^{ik\theta\cdot y} + u_s(k,\theta,y))\, dy. \tag{1.2}$$

The nonlinearity comes from the fact that $u_s(k,\theta,x)$ also depends on q by (1.1).

From the previous formula, a more direct relation with the potential can be established through the so called *Born series* expansion

$$q_B \sim q + \sum_{j=2}^{\infty} Q_j(q). \tag{1.3}$$

In fixed angle scattering we find a completely analogous relation,

$$q_\theta \sim q + \sum_{j=2}^{\infty} Q_{\theta,j}(q), \tag{1.4}$$

(we use the $\sim$ symbol to avoid claiming anything about convergence yet). $Q_j(q)$ and $Q_{\theta,j}(q)$ are certain multilinear operators describing the multiple dispersion of the wave $e^{ik\theta\cdot x}$ that can be expressed in terms of the resolvent of the Laplacian. We will call the Q_2 operator the double dispersion operator of backscattering and analogously in fixed angle scattering. As the name suggest, the Born approximation is a good approximation for potentials satisfying certain smallness conditions, in the sense that the difference $q - q_B$ is also small in appropriate function spaces.

From a mathematical point of view an important question that is still not completely answered is to establish how much information does the Born approximation contain about the actual potential q, and if it is possible to recover q completely from the knowledge of q_B. This is equivalent to asking if the nonlinear Fourier transform given by (1.2) can be inverted. This is a hard problem, and in fact, the question of uniqueness is only partially answered.

Motivated by the use of the Born approximation in applications, and in search of partial recovery results, another approach is to ask how much and what kind of information about q can be obtained just by looking at q_B, that is, in a very immediate way. In this sense, in [36] it was shown that the Born approximation of full data scattering must contain the leading singularities of q. Since then, this approach has received great amount of attention in all the different scattering problems.

The main objective of this work is to quantify as exactly as possible how much more regular than q can $q - q_B$ be in general, depending on the dimension n, and the a priori regularity of the potential q measured in the Sobolev scale. We will address the same question in fixed angle scattering. Most of the results that we will present in this thesis can also be found in the papers [27–29]. The potentials considered can be complex valued and we don't assume any smallness condition on them.

Chapter 2

Main results

2.1 Historical remarks

Scattering theory studies a huge number of physical phenomena in which particles or waves are scattered by a collection of objects or inhomogeneities in a medium. In our case, the object responsible of the scattering phenomena is $q(x)$, a compactly supported electrostatic potential in $\mathbb{R}^n$. For scattering theory, the most important information is how scattered waves or particles deviate from the free case, loosely speaking the situation in which there are no objects or inhomogeneities disturbing their propagation (for us this clearly will be the case $q = 0$). This is essentially the information contained in the scattering data. Given a certain underlying physical theory (non relativistic quantum mechanics, Maxwell equations, elasticity...), inverse scattering deals with the reconstruction of the parameters which describe the object responsible of the scattering phenomena from the scattering data. As in other inverse problems, some of the main questions are uniqueness, stability and reconstruction.

In mathematical physics, the backscattering problem was extensively studied in [37–41]. From the point of view of the inverse problems, the question of uniqueness is still open. In particular it is not even known if $q_B = 0$ implies that $q = 0$ for potentials in $C_c^\infty(\mathbb{R}^n)$. There are several results concerning uniqueness, but they are related to different definitions of the scattering data. In fact, the backscattering data can be introduced in the context of the stationary Schrödinger equation or in that of the time dependent wave equation. The reader may consult [42] for more details and references. In [51] it is shown, among other uniqueness results, that in $n = 3$ the backscattering data determines the convex hull of the support of a compactly supported potential. Generic uniqueness and uniqueness for small potentials has been obtained in [13–15, 25, 50] for dimensions 2 and 3. Similar results have been obtained in odd dimension $n \geq 3$ in [55] and for even dimension in [56]. Also, in [42] it has been proved for $n = 3$ that two potentials differing in a finite number of spherical harmonics with radial coefficients must be identical if they have the same backscattering data.

In fixed angle scattering, uniqueness for the inverse problem of recovering $q(x)$ from the previous data is also an open question. Generic uniqueness and uniqueness for small potentials has been obtained in [50] for potentials in dimension 3 with certain smoothness conditions. Also, it has been shown in [7] that if the scattering amplitude vanishes for a fixed θ, then q has to be zero.

A different problem that we didn't mention in the introduction is the fixed energy scattering problem. In this case the scattering data $u_\infty(k, \theta, \theta')$ is considered known for a fixed energy $k = k_0$ and for every $\theta, \theta' \in \mathbb{S}^{n-1}$, and hence it is formally well determined only in dimension 2. The Born approximation, as it turns out, is not very useful in this case, and that is one of the main reasons we don't address it in this work. In fact, this problem is strongly related to the Calderon problem, and the main tools used to study it are Calderón-Sylvester-Uhlmann complex exponential solutions. This is because, for compactly supported potentials, knowledge of the scattering amplitude at fixed energy is equivalent to knowing the Dirichlet-Neumann map for the Schrödinger equation, measured on the boundary of a large ball which contains the support of the potential (see, for example [54]).

The interest of studying the reconstruction of singularities from the Born approximation comes from the fact that it is a very simple procedure that yields important qualitative information about the potential. This is of special interest in geophysics, where an algorithm which yields the singularities of the parameters describing the medium of propagation of the waves is denominated a *a migration scheme* (see [10]).

Thee are many works which study the recovery of singularities in backscatering , see [3,8,17,19,27,31,43–45,48], and also in full data scattering, see [2,33–36]. In fixed angle scattering we mention [28, 46]. Also, an analogue of the Born approximation has been introduced to study the recovery of singularities of live loads in Navier elasticity, see [4, 5]. In a slightly different spirit, in [22, 32] they study the problem of reconstructing the corners of the boundary of a penetrable obstacle of polygonal shape from the measurement of a single acoustic scattered wave.

2.2 The Born series expansion

The scattering solutions $u_s(k, \theta, x)$, $k \in (0, \infty)$, $\theta \in \mathbb{S}^{n-1}$ that appeared in the introduction are perturbed solutions of the stationary Schrödinger equation. They satisfy the following equation and conditions

$$\begin{cases} (-\Delta + q - k^2)u = 0 \\ u(x) = e^{ik\theta\cdot x} + u_s(k, \theta, x) \\ \lim_{|x|\to\infty}(\frac{\partial u_s}{\partial r} - iku_s)(x) = o(|x|^{-(n-1)/2}). \end{cases} \tag{2.1}$$

The last line is the outgoing Sommerfeld radiation condition, necessary to have uniqueness of solutions. An important property of the scattering solutions satisfying the outgoing Sommerfeld radiation condition is that if q is compactly supported, then u_s has the following asymptotic behavior when $|x| \to \infty$

$$u_s(k,\theta,x) = C|x|^{-(n-1)/2}k^{(n-3)/2}e^{ik|x|}u_\infty(k,\theta,x/|x|) + o(|x|^{-(n-1)/2}),$$

for a certain function $u_\infty(k,\theta,\theta')$, $k \in (0,\infty)$, $\theta,\theta' \in \mathbb{S}^{n-1}$. As mentioned in the introduction, u_∞ is the so called scattering amplitude or far field pattern that describes the behavior of $u_s(k,\theta,x)$ at infinity. The far field pattern can be expressed directly in terms of the potential,

$$u_\infty(k,\theta,\theta') = \int_{\mathbb{R}^n} e^{-ik\theta'\cdot y}q(y)u(y)\,dy, \tag{2.2}$$

where u is the solution of (2.1), and hence also depends on q, k and θ. This is also a theorem of Rellich.

To construct the scattering solutions u_s we apply to the first line of (2.1) the outgoing resolvent of the Laplacian R_k, an inverse operator of $\Delta + k^2$. Formally, this gives the Lippmann-Schwinger integral equation

$$u_s(k,\theta,x) = R_k(qe^{ik\theta\cdot(\cdot)})(x) + R_k(qu_s(k,\theta,\cdot))(x). \tag{2.3}$$

Since k^2 is in the spectrum of the Laplacian, R_k cannot be a bounded operator in $L^2(\mathbb{R}^n)$, but it is bounded from L^2_δ to $L^2_{-\delta}$ with $\delta > 1/2$. This is the well known limiting absorption principle of [1]. Its Fourier symbol is given by the following limit in the sense of distributions

$$\widehat{R_k(f)}(\xi) = \lim_{\substack{\varepsilon\to 0\\ \varepsilon>0}}(-|\xi|^2 + k^2 + i\varepsilon)^{-1}\widehat{f}(\xi), \tag{2.4}$$

and computing the inverse Fourier transform of the previous expression it can be shown that

$$R_k(f)(x) = i\frac{\pi}{2}k^{n-2}\int_{\mathbb{S}^{n-1}}\widehat{f}(k\omega)e^{ikx\cdot\omega}\,d\sigma(\omega) + P.V.\int_{\mathbb{R}^n} e^{ix\cdot\zeta}\frac{\widehat{f}(\zeta)}{-|\zeta|^2+k^2}\,d\zeta,$$

For a more detailed account of all these facts see [47] or [16] (in the latter the case non-compactly supported real potentials with certain decay in L^∞ is also considered).

The existence of scattering solutions of (2.1) can be reduced then to showing existence for the Lippmann-Schwinger equation (2.3). To do that, certain a priori estimates for R_k are essential. Denote $T_k(h) := R_k(qh)$ and $f = R_k(qe^{ik\theta\cdot(\cdot)})$, then (2.3) can be written as

$$(I - T_k)u_s = f. \tag{2.5}$$

If the potential $q \in L^p$, $p > n/2$ is compactly supported and real, it can be shown that T_k is a compact operator. Then, by Fredholm theorem, using Rellich uniqueness theorem for the Helmoltz equation and unique continuation properties, it follows that for each $k \in (0,\infty)$, $\theta \in \mathbb{S}^{n-1}$ there is a unique u_s solving Lippmann-Schwinger equation (see, for example, [47] or [11]).

If $q \in L^p$ is a complex and compactly supported potential, we cannot use Fredholm theory. In this case, since the norm of T_k decays to zero as $k \to \infty$ in appropriate function spaces, we can use a Neumann series expansion in (2.5) which will be convergent for $k > k_0 \geq 0$, where k_0 depends on the L^p norm of q. See Lemma 6.5 for a detailed proof of these facts. We mention [30] for other examples of complex potentials which enjoy existence of scattering solutions for high energy (see [2] for more references). In both cases the condition $q \in L^p(\mathbb{R}^n)$, $p > n/2$ is required. We remark that by the Sobolev embedding this is satisfied by $q \in W^{\beta,2}(\mathbb{R}^n)$, $\beta \geq 0$, if $\beta > (n-4)/2$.

We can now introduce the Born series expansion. If we insert (2.3) in (2.2), we can expand the Lippmann-Schwinger equation in a Neumann series. Then we obtain the Born series, which relates the scattering amplitude with the Fourier transform of the potential.

$$u_\infty(k,\theta,\theta') = \widehat{q}(k(\theta'-\theta)) + \sum_{j=2}^{\ell} \int_{\mathbb{R}^n} e^{-ik\theta'\cdot y}(qR_k))^{j-1}(q(\cdot)e^{ik\theta\cdot(\cdot)})(y)\,dy + \int_{\mathbb{R}^n} e^{-ik\theta'\cdot y}(qR_k)^{\ell-1}(q(\cdot)u_s(k,\theta,\cdot))(y)\,dy, \tag{2.6}$$

where the last is the error term. Since we are considering complex potentials, $u_\infty(k,\theta,\theta')$ is not defined for $k \leq k_0$ as we have seen. Therefore we also have to ask $k > k_0$ in (2.6).

As we explained in the introduction, the problem of determining q from the knowledge of the scattering amplitude is formally overdetermined in the sense that the data $u_\infty(k,\theta,\theta')$ is described by $2n-1$ variables, while the unknown potential $q(x)$ has only n. And indeed $\widehat{q}(\xi)$ can be recovered pointwise if q is compactly supported and if $u_\infty(k,\theta,\theta')$ is known for every $k \in (0,\infty)$, $\theta' \in \mathbb{S}^{n-1}$ and $\theta \in \mathbb{S}^{n-1} \cap P$, where P is any two dimensional plane passing through the origin (see [47]). Therefore the usual approach is to restrict the number of parameters for which $u_\infty(k,\theta,\theta')$ is considered known. As we have already mentioned, the main cases are the fixed energy scattering problem, the fixed angle scattering problem, and the backscattering problem.

Leaving aside fixed energy scattering, in the following sections we examine more closely the backscattering and fixed angle scattering problems. We will also introduce the full data scattering problem, but first we give the formal definition of the Sobolev spaces that we use in this work.

Let $\langle D\rangle^\alpha$, $\alpha \in \mathbb{R}$ be the Bessel fractional derivative operator given by the Fourier symbol $\langle \xi\rangle^\alpha$ with $\langle x\rangle = (1+|x|^2)^{1/2}$. We consider the weighted Sobolev spaces $W^{\alpha,p}_\delta(\mathbb{R}^n)$, $\delta \in \mathbb{R}$,

$$W^{\alpha,p}_\delta(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \|\langle\cdot\rangle^\delta \langle D\rangle^\alpha f\|_{L^p} < \infty\}.$$

We always use the notation $L^p_\delta(\mathbb{R}^n) := W^{0,p}_\delta(\mathbb{R}^n)$ and $W^{\alpha,p}(\mathbb{R}^n) := W^{\alpha,p}_0(\mathbb{R}^n)$. Also we say that $f \in W^{\alpha,p}_{loc}(\mathbb{R}^n)$ if $\phi f \in W^{\alpha,p}(\mathbb{R}^n)$ for every $\phi \in C^\infty_c(\mathbb{R}^n)$.

2.3 Backscattering

2.3.1 Main results of recovery of singularities

In backscattering we avoid the overdetermination by assuming knowledge only of $u_\infty(k,\theta,-\theta)$ for all $\theta \in \mathbb{S}^{n-1}$ and for all $k > k_0$, if q is complex, or $k > 0$ if q is real. Therefore, this problem is formally well determined. The Born approximation q_B is defined by the identity,

$$\widehat{q_B}(\xi) := u_\infty(k,\theta,-\theta), \quad \text{where} \;\; \xi = -2k\theta. \tag{2.7}$$

Since $u_\infty(k,\theta,-\theta)$ is not defined for $k \le k_0$ in the complex case, from now on we consider that $q_B(x)$ is defined modulo a C^∞ function. We also define the multiple dispersion operators,

$$\widehat{Q_j(q)}(\xi) := \int_{\mathbb{R}^n} e^{ik\theta\cdot y}(qR_k)^{j-1}(q(\cdot)e^{ik\theta\cdot(\cdot)})(y)\,dy, \tag{2.8}$$

and the remainder term

$$\widehat{Q^R_j(q)}(\xi) := \int_{\mathbb{R}^n} e^{ik\theta\cdot y}(qR_k)^{j-1}(q(\cdot)u_s(k,\theta,\cdot))(y)\,dy, \tag{2.9}$$

in both cases with $\xi = -2k\theta$.

Consider a constant $C_0 \ge 1$ and let $0 \le \chi(\xi) \le 1$, $\xi \in \mathbb{R}^n$, be a smooth cut-off function such that

$$\chi(\xi) = 1 \;\text{ if }\; |\xi| > 2C_0 \;\text{ and }\; \chi(\xi) = 0 \;\text{ if }\; |\xi| < C_0. \tag{2.10}$$

We define

$$\widehat{\widetilde{Q}_j(q)}(\xi) := \chi(\xi)\widehat{Q_j(q)}(\xi), \quad \widehat{\widetilde{Q}^R_j(q)}(\xi) := \chi(\xi)\widehat{Q^R_j(q)}(\xi), \tag{2.11}$$

and so, $Q_j(q)$ differs from $\widetilde{Q}_j(q)$ in a smooth function, and the same for the remainder. If we also take $C_0 > 2k_0$ we can write (2.6) as follows

$$\chi(\xi)\widehat{q}_B(\xi) = \chi(\xi)\widehat{q}(\xi) + \sum_{j=2}^{\ell} \widehat{\widetilde{Q}_j(q)}(\xi) + \widehat{\widetilde{Q}^R_\ell(q)}(\xi)$$

for all $\xi \in \mathbb{R}^n$. Then if we take the inverse Fourier transform we can write that, *modulo a C^∞ function*,

$$q_B = q + \sum_{j=2}^{\ell} \widetilde{Q}_\ell(q) + \widetilde{Q}^R_j(q). \tag{2.12}$$

We call the previous series the high frequency Born series of backscattering.

As we mentioned in the introduction, the central problem of this work is to determine with precision which singularities of q can be recovered from q_B or from the scattering data. Essentially we want to determine which is the best $\varepsilon(\beta) > 0$, such that for every $q \in W^{\beta,2}(\mathbb{R}^n)$, $\beta \geq 0$ we have that $q - q_B \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ for all $\alpha < \beta + \varepsilon(\beta)$ (this would be an $\varepsilon(\beta)^-$ derivative gain).

In backscattering there have been a great number of works addressing this problem. For real potentials we mention [31,44] in dimension 2, and [45,48] for dimensions 2 and 3. In [43, 45] it is shown that the derivative gain $\varepsilon(\beta)$ is always at least 1/2 for $n = 2, 3$:

Theorem 2.1 (J. M. Reyes and A. Ruiz)**.** *Let $n = 2, 3$ and $\beta \geq 0$. Assume that $q \in W^{\beta,2}(\mathbb{R}^n)$ is compactly supported and real valued. Then $q - q_B \in W^{\alpha,2}(\mathbb{R}^n)$, modulo a C^∞ function, for all $\alpha < \beta + 1/2$.*

Clearly the condition $q - q_B \in W^{\alpha,2}(\mathbb{R}^n)$ modulo a C^∞ function implies that $q - q_\theta \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$. We will use the same condition in most of the results in this work. Also, in [8], using a certain modification of the Born approximation and of the Q_j operators, they show that it is possible to take $\varepsilon(\beta) = \min(\beta - (n-3)/2, 1)$ for $n \geq 3$ odd and $\beta \geq (n-3)/2$.

Apart from the previous works, which use the Sobolev scale to measure the regularity of $q - q_B$, in [3] they use the Hölder scale. With this approach they are able to obtain for complex potentials and $n = 2$, a whole 1^- derivative gain in the integrability sense. In a different spirit, the recovery of singularities from backscattering data has also been studied in [17,19] without resorting to the notion of the Born approximation. Instead, the authors reconstruct the conormal singularities of q from the scattering data using the time domain approach to scattering.

We now introduce the main results of recovery of singularities in backscattering for general potentials $q \in W^{\beta,2}(\mathbb{R}^n)$.

Theorem 2.2. *Let $n \geq 2$ and $\beta \geq 0$. Assume that $q - q_B \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ for every $q \in W^{\beta,2}(\mathbb{R}^n)$ compactly supported, radial, and real. Then α necessarily satisfies,*

$$\alpha \leq \begin{cases} 2\beta - (n-4)/2, & if \quad (n-4)/2 < \beta < (n-2)/2, \\ \beta + 1, & if \quad (n-2)/2 \leq \beta < \infty, \end{cases} \tag{2.13}$$

Theorem 2.2 is the first result giving upper bounds for the maximum possible regularity that can be obtained from the Born approximation in backscattering. It means that necessarily $\varepsilon(\beta) \leq \min(\beta - (n-4)/2, 1)$. As we shall see, condition (2.13) is a consequence of upper bounds for the regularity of the Q_2 operator given by Theorem 2.5 below.

A remarkable consequence of condition (2.13) is that for $\beta < (n-2)/2$ and $n > 2$ it is not possible to reach the expected gain of one derivative over the regularity of q (see Figure 2.1 for the cases $n = 2, 4$). More surprisingly, if $n \geq 4$, $\varepsilon(\beta)$ vanishes

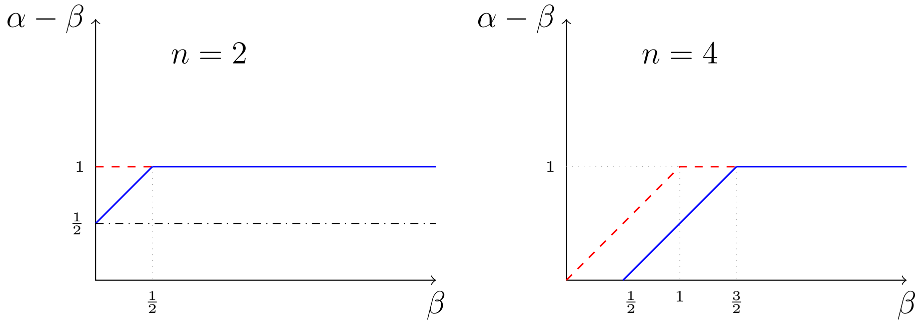

Figure 2.1: The (red) dashed line represents the limitation on the regularity gain given in Theorem 2.2 for $q - q_B$, and the solid (blue) line represents the positive results given in Theorem 2.3. When $n = 2$, the dot dashed line represents the previously known positive results of [45].

when β approaches the value $(n-4)/2$. In practice, the condition $\beta > (n-4)/2$ is imposed to guarantee (by the Sobolev inequality) that $q \in L^p(\mathbb{R}^n)$ for some $p > n/2$. This is required, essentially, to have convergence of the Born series (and more basically, existence of scattering solutions). Therefore it is worth noticing that $\beta \geq (n-4)/2$ is also the necessary condition to obtain a non-negative regularity gain for the double dispersion operator (see Theorem 2.5).

Theorem 2.3 (Recovery of singularities)**.** *Let $n \geq 2$ and $\beta \geq 0$. Assume that $q \in W^{\beta,2}(\mathbb{R}^n)$ is compactly supported. Then $q - q_B \in W^{\alpha,2}(\mathbb{R}^n)$, modulo a C^∞ function, if the following condition also holds*

$$\alpha < \begin{cases} 2\beta - (n-3)/2, & if \quad (n-3)/2 < \beta < (n-1)/2, \\ \beta + 1, & if \quad (n-1)/2 \leq \beta < \infty. \end{cases} \tag{2.14}$$

See Figure 2.1 for a graphic representation of these results for $n = 2, 4$.

As far as we know, these are the first results of recovery of singularities for every dimension $n \geq 2$ in backscattering. Since Theorem 2.2 implies that a one derivative gain is the best possible result, the 1^- derivative gain in (2.14) is optimal except for the limiting case $\alpha = \beta + 1$. A similar result has been obtained in [8, Corollary 4.8] in odd dimension $n \geq 3$ using a certain modified Born approximation. As we shall see briefly, Theorems 2.2 and 2.3 hold identically in fixed angle scattering.

Unfortunately, the previous couple of theorems leave a gap of up to $1/2$ derivative when $\max((n-4)/2, 0) \leq \beta < (n-1)/2$ between the positive and negative results. This gap is only partially closed in dimension 3 in the range $0 \leq \beta \leq 1/2$ thanks to Theorem 2.1. In the next section we will address the problem of closing this gap in more cases.

Theorem 2.3 is a consequence of new estimates for the Q_j operators that we now introduce.

Theorem 2.4. *Let $n \geq 2$ and $j \geq 2$. Assume that $0 \leq \beta \leq \infty$ and that the following condition also holds*

$$\alpha < \begin{cases} \beta + (j-1)(\beta - (n-3)/2), & if \quad (n-3)/2 < \beta < (n-1)/2, \\ \beta + (j-1), & if \quad (n-1)/2 \leq \beta < \infty. \end{cases} \tag{2.15}$$

Then for $q \in W_2^{\beta,2}(\mathbb{R}^n)$ and $j = 2$ we have the estimate

$$\|\widetilde{Q}_2(q)\|_{W^{\alpha,2}} \leq C\|q\|^2_{W_2^{\beta,2}}. \tag{2.16}$$

Otherwise if $j \geq 3$ and $q \in W_4^{\beta,2}(\mathbb{R}^n)$ we have that

$$\|\widetilde{Q}_j(q)\|_{W^{\alpha,2}} \leq C\|q\|^j_{W_4^{\beta,2}}. \tag{2.17}$$

Notice that in this theorem we include weighted Sobolev norms since we consider non compactly supported potentials.

To prove Theorem 2.4 we estimate the Fourier transform of $\widetilde{Q}_j(q)$. One of the most important points in the proof is Proposition 3.2 where we give an explicit formula for $\widehat{Q_j(q)}$ as certain distributions acting on the radial parameters of a spherical operator that we introduce in Section 3.1. The spherical operators basically consist of a series of convolutions of the Fourier transform of q over the so called Ewald spheres. And the radial distributions that act on the spherical operator are, as in (2.18), certain principal values and Dirac deltas. In the case of $j = 2$ the formula is

$$\widehat{Q_2(q)}(\eta) = i\pi S_1(q)(\eta) + P.V. \int_0^\infty \frac{1}{1-r} S_r(q)(\eta)\, dr, \tag{2.18}$$

where S_r is the operator given by

$$S_r(q)(\eta) := \frac{2}{|\eta|(1+r)} \int_{\Gamma_r(\eta)} \widehat{q}(\xi)\widehat{q}(\eta - \xi)\, d\sigma_{r\eta}(\xi),$$

and $\Gamma_r(\eta)$ is the sphere satisfying $\Gamma_r(\eta) = \{\xi \in \mathbb{R}^n : |\xi - \eta/2| = r|\eta|/2\}$, $r \in (0, \infty)$.

The main tools used in the proof of Theorem 2.4 are trace estimates to control the spherical integrals, and a new method based on (2.18) to reduce the estimate of the terms with principal values to estimates of the spherical operator. One advantage of these techniques is that with the same effort we can prove estimates for general dimension $n \geq 2$ and are also useful to study the case of fixed angle scattering. Nonetheless we mention that the estimate of the $\widetilde{Q}_3$ operator for $n = 3$ given in [45] is still the best estimate in the range $0 \leq \beta < 1/4$.

Finally, we give an upper bound for the regularity of the double dispersion operator, which constrains the amount of regularity of q that one can expect to recover from the Born approximation, as stated in Theorem 2.2.

Theorem 2.5. *Let $0 \leq \beta < \infty$ and assume that $Q_2(q) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ for every potential $q \in W^{\beta,2}(\mathbb{R}^n)$ radial, real and compactly supported, then α necessarily satisfies*

$$\alpha \leq \begin{cases} 2\beta - (n-4)/2, & if \quad 0 \leq \beta < (n-2)/2, \\ \beta + 1, & if \quad (n-2)/2 \leq \beta < \infty, \end{cases} \tag{2.19}$$

The proof will be given in Chapter 5.

2.3.2 Optimal estimates for the double dispersion operator

We now address the regularity gap that appears between the results of Theorem 2.4 with $j = 2$ and Theorem 2.5. The final goal is to improve the results of recovery of singularities for $q - q_B$ and, if possible, to also close the gap between Theorem 2.2 and Theorem 2.3.

We follow two different approaches. In the first one we restrict the range to $\beta > (n-2)/2$, and, instead of the Sobolev scale, we use the Hölder scale $\Lambda^\alpha(\mathbb{R}^n)$ to measure the regularity of $Q_2(q)$, as in [3]. In the second one we maintain the Sobolev scale but in exchange we assume the potentials to be radial.

The Hölder spaces $\Lambda^\alpha(\mathbb{R}^n)$, $\alpha \geq 0$ are the Banach spaces given by the norm,

$$\|f\|_{\Lambda^\alpha} = \sum_{|\gamma|<m} \|\partial^\gamma f\|_\infty + \sum_{|\gamma|=m} \sup_{t\neq 0} \frac{|\partial^\gamma f(\cdot) - \partial^\gamma f(\cdot - t)|}{|t|^\sigma},$$

where we are decomposing α in its integer and fractional parts, $\alpha = m + \sigma$ with $m \in \mathbb{N}$ and $\sigma \in [0,1)$.

We now state the following result of [3].

Theorem 2.6 (J. Barceló, D. Faraco, A. Ruiz and A. Vargas)**.** *Let $q \in W^{\beta,2}(\mathbb{R}^2)$ and assume there exists some $p \in (1,2)$ such that*

$$|\cdot|q(\cdot) \in L^p(\mathbb{R}^2),$$

then for any $\beta > (2-p)/p$ and $\alpha < \beta - (2-p)/p$ we have $Q_2(q) \in \Lambda^\alpha(\mathbb{R}^2)$.

As mentioned, we have extended this result to the case $n \geq 3$.

Theorem 2.7. *Let $n \geq 3$ and assume that $q \in W^{\beta,2}_1(\mathbb{R}^n)$ with $\beta > (n-2)/2$. Then*

$$\|Q_2(q)\|_{\Lambda^\alpha} \leq C\|q\|^2_{W^{\beta,2}_1}, \tag{2.20}$$

for all $\alpha < \beta - (n-2)/2$.

Actually we are able to prove a slightly better result than (2.20), an estimate for $\widehat{Q_2(q)}$ in $L^1_\alpha(\mathbb{R}^n)$ (see Proposition 7.1 below). Theorem 2.7 and Theorem 2.6 imply the following result of recovery of singularities (the case $n = 2$ was already derived in [3]).

Theorem 2.8. *Let $n \geq 2$ and assume that $q \in W^{\beta,2}(\mathbb{R}^n)$, $\beta > (n-2)/2$, is compactly supported. Then for any $\alpha < \beta - (n-2)/2$, we have that $q - q_B \in \Lambda^\alpha(\mathbb{R}^n)$ modulo a C^∞ function.*

By the Morrey-Sobolev inequality we have that

$$\|f\|_{\Lambda^{\gamma-(n-2)/2}} \leq C\|f\|_{W^{\gamma+1,2}}, \tag{2.21}$$

if $\gamma > (n-2)/2$. This means that, for $\beta > (n-2)/2$, we can interpret the previous theorem as a 1^- derivative gain in the integrability sense. As a consequence of this, we may consider Theorem 2.8 as nearly optimal since, by (2.13), we know that, when $\beta > (n-2)/2$, the best possible result in the Sobolev scale is a 1 derivative gain. Notice that by (2.21), Theorem 2.3 yields the same result in the Sobolev scale only for $\beta > (n-1)/2$.

For radial potentials we can obtain an optimal result in the Sobolev scale.

Theorem 2.9. *Let $n \geq 2$ and assume that $\beta \geq 0$, and $\beta > (n-4)/2$. Then the following estimate holds*

$$\|\widetilde{Q}_2(q)\|_{W^{\alpha,2}} \leq C\|q\|^2_{W_1^{\beta,2}}, \tag{2.22}$$

for all $\alpha < \beta + \varepsilon(\beta)$ and every $q \in W_1^{\beta,2}(\mathbb{R}^n)$ radial, if and only if

$$\varepsilon(\beta) = \min(\beta - (n-4)/2, 1). \tag{2.23}$$

Notice that this theorem does not imply any result for the limiting case $\alpha = \beta + \varepsilon(\beta)$ which it is still an open question. As a consequence we obtain the following corollary of recovery of singularities.

Corollary 2.10. *Let $n \geq 2$ and let $q \in W^{\beta,2}(\mathbb{R}^n)$, $\beta \geq 0$, be a compactly supported and radial function. Then we have that $q - q_B \in W^{\alpha,2}(\mathbb{R}^n)$ if*

$$\alpha < \begin{cases} \beta + 2(\beta - (n-3)/2), & \text{if} \quad (n-3)/2 < \beta < (n-2)/2, \\ \beta + 1, & \text{if} \quad (n-2)/2 \leq \beta < \infty. \end{cases} \tag{2.24}$$

See Figure 2.2 for a graphic representation of this results. The previous result gives a 1^- derivative gain in the range $\beta > (n-2)/2$ which is the best possible result (except for the limiting case $\alpha = \beta + 1$) by Theorem 2.2. Unfortunately, this is not the case in the range $(n-4)/2 \leq \beta < (n-2)/2$, since, to get an optimal result, the estimates of the other Q_j operators with $j > 2$ should be improved too.

Since the results of recovery of singularities of a potential are non quantitative in nature, to get Corollary 2.10 we don't need necessarily a quantitative estimate like (2.22), it is just enough to show that the norm in the left hand side is finite. Then, instead of asking q to be radial, we can consider potentials which satisfy a weaker assumption. This yields the following corollary.

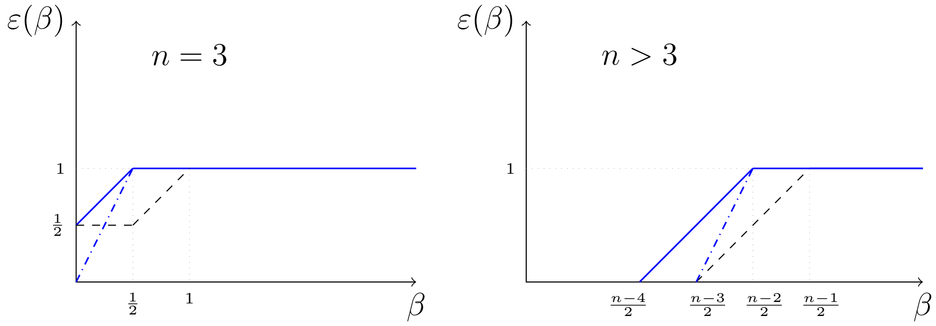

Figure 2.2: The solid (blue) line represents the value of $\varepsilon(\beta)$ for $Q_2(q)$ given by Theorem 2.9. The dashed line represents the results of Theorem 2.4 for $j = 2$, and the results of [45]. The dot dashed (blue) line represents the regularity gain of $q - q_B$ given by (2.24).

Corollary 2.11. *Let $n \geq 2$ and let $q \in W^{\beta,2}(\mathbb{R}^n)$ be compactly supported. Assume also that there exists some $g \in W^{\beta,2}(\mathbb{R}^n)$ radial such that $|\widehat{q}(\xi)| \leq \widehat{g}(\xi)$. Then we have that $q - q_B \in W^{\alpha,2}(\mathbb{R}^n)$ if α and β satisfy* (2.24).

To prove Theorem 2.7 and Theorem 2.9 we need to improve the estimates of the spherical operator S_r used to get Theorem 2.4. For this task, it will be essential to do a finer analysis of the integrals over the spheres $\Gamma_r(\eta)$. In the case of $r = 1$ we use a simple case of Santaló's formula in spheres for which we give a short proof in Section 7.1.2.

2.4 Fixed angle and full data scattering

2.4.1 Fixed angle scattering

In the fixed angle scattering problem one assumes knowledge of $u_\infty(k, \theta, \theta')$ only for a fixed $\theta \in \mathbb{S}^{n-1}$ and its opposite unit vector $-\theta$, and for all $k > k_0$ (or $k > 0$ if the potential is real), and $\theta' \in \mathbb{S}^{n-1}$. Then the problem is formally well determined.

Now, for a fixed θ, the identity $\xi = k(\theta' - \theta)$ is a diffeomorphism from $(0, \infty) \times \mathbb{S}^{n-1}$ to $H_\theta \subset \mathbb{R}^n$, where

$$H_\theta := \{\xi \in \mathbb{R}^n : \xi \cdot \theta < 0\},$$

is an open half space of $\mathbb{R}^n$. For $\xi \in H_\theta$, the inverse of this diffeomorphism, is given by the formulas

$$k(\xi, \theta) := -\frac{|\xi|^2}{\theta \cdot \xi}, \quad \text{and} \quad \theta'(\xi, \theta) := k^{-1}(\xi + k\theta). \tag{2.25}$$

We notice that the condition $k(\xi,\theta) > k_0$ holds if we ask $|\xi| > C_0$ for any constant $C_0 > 2k_0$ since we have always that $2k \geq |\xi|$. Therefore for $|\xi| > C_0$, we can define the Born approximation of fixed angle scattering:

$$\widehat{q_\theta}(\xi) := \begin{cases} u_\infty(k(\xi,\theta),\theta,\theta'(\xi,\theta)), & \text{when} \quad \xi \in H_\theta, \\ u_\infty(k(\xi,-\theta),-\theta,\theta'(\xi,-\theta)), & \text{when} \quad \xi \in H_{-\theta}, \end{cases} \tag{2.26}$$

where we need the data related to the angle $-\theta$ to generate also the opposite half space $H_{-\theta}$, so that we can cover a full measure subset of $\mathbb{R}^n$. This is not necessary for real potentials, since in this case the following reciprocity relation holds

$$u_\infty(k,\theta,\theta') = u_\infty(k,-\theta,-\theta'), \tag{2.27}$$

(see [11]), but we consider directly the general case. Notice that we have avoided to give a definition of $\widehat{q_\theta}(\xi)$ for all $|\xi| < C_0$. As a consequence, from now on it will be understood that the function $q_\theta(x)$ is defined *modulo a C^∞ function* (as in bakscattering, we remind that this subtlety is not necessary when dealing only with real potentials).

We define the multiple dispersion operators and the remainder term of fixed angle scattering, as follows:

$$\widehat{Q_{\theta,j}(q)}(\xi) := B_{\theta,j}(q)(\xi) + B_{-\theta,j}(q)(\xi), \quad \widehat{Q^R_{\theta,j}(q)}(\xi) := B^R_{\theta,j}(q)(\xi) + B^R_{-\theta,j}(q)(\xi), \tag{2.28}$$

where, if $k = k(\xi,\theta)$ and $\theta' = \theta'(\xi,\theta)$ are given by (2.25), we have

$$B_{\theta,j}(q)(\xi) := \begin{cases} \int_{\mathbb{R}^n} e^{-ik\theta'\cdot y}(qR_k)^{j-1}(qe^{ik\theta\cdot(\cdot)})(y)\,dy, & \text{if} \quad \xi \in H_\theta, \\ 0 & \text{if} \quad \xi \notin H_\theta, \end{cases} \tag{2.29}$$

$$B^R_{\theta,j}(q)(\xi) := \begin{cases} \int_{\mathbb{R}^n} e^{-ik\theta'\cdot y}(qR_k)^{j-1}(qu_s(k,\theta,\cdot))(y)\,dy, & \text{if} \quad \xi \in H_\theta, \\ 0 & \text{if} \quad \xi \notin H_\theta, \end{cases} \tag{2.30}$$

Also, we write

$$\widehat{\widetilde{Q}_{\theta,j}(q)}(\xi) := \chi(\xi)\widehat{Q_{\theta,j}(q)}(\xi), \quad \widehat{\widetilde{Q}^R_{\theta,j}(q)}(\xi) := \chi(\xi)\widehat{Q^R_{\theta,j}(q)}(\xi), \tag{2.31}$$

where $\chi(\xi)$ is the cut-off introduced in (2.10). Hence, $\widetilde{Q}_{\theta,j}(q)$ and $Q_{\theta,j}(q)$ differ just in a C^∞ function, and the same for the remainder terms.

Multiplying (2.6) by the cut-off we get

$$\chi(\xi)\widehat{q_\theta}(\xi) = \chi(\xi)\widehat{q}(\xi) + \sum_{j=2}^{\ell} \widehat{\widetilde{Q}_{\theta,j}(q)}(\xi) + \widehat{\widetilde{Q}^R_{\theta,\ell}(q)}(\xi), \tag{2.32}$$

and hence, taking the inverse Fourier transform, we have that

$$q_\theta = q + \sum_{j=2}^{\ell} \widetilde{Q}_{\theta,j}(q) + \widetilde{Q}^R_{\theta,\ell}(q). \tag{2.33}$$

modulo a C^∞ function.

In appearance, the backscattering problem has less complicated expressions and formulas than fixed angle scattering, since in the former, the choice $\theta' = -\theta$ yields $\xi = -2k\theta$, which clearly has a much simpler inverse than (2.25). But, from the point of view of the Sobolev estimates, the fact that in the right hand side of (2.29) θ and θ' are independent is in some cases an advantage. In fact it makes possible the use of the Stein-Thomas restriction theorem for the Fourier transform, which yields in certain instances better regularity estimates for the multiple dispersion operators (see Proposition 6.13, or [46]).

2.4.2 Full data scattering

The full data scattering problem is closely related to the fixed angle scattering problem. As the name suggest in this case one uses all the data u_∞ to construct a Born approximation, by averaging in θ the Born approximation q_θ of fixed angle scattering. One of the advantages of this procedure is to simplify the structure of the multiple dispersion operators by making them radially invariant, as in backscattering. We denote by q_F the Born approximation of fixed angle scattering. Since all the data are used, it is natural to expect that the recovery should be better than in fixed angle scattering or in backscattering. As we shall see in the next section, this appears to be the case for the positive results (see Theorem 2.12), but not for the constraints on the regularity of $q - q_F$ (see 2.13).

To construct the Born approximation we take the average of (2.33) in $\mathbb{S}^{n-1}$. This gives, *modulo a C^∞ function*

$$q = q_F + \sum_{j=2}^{\ell} \widetilde{Q}_{F,j}(q) + \widetilde{Q}^R_{F,\ell}(q), \tag{2.34}$$

where, as before $\widehat{\widetilde{Q}_{F,j}(q)}(\xi) := \chi(\xi)\widehat{Q_{F,j}(q)}(\xi)$, and $\widehat{\widetilde{Q}^R_{F,j}(q)}(\xi) := \chi(\xi)\widehat{Q^R_{F,j}(q)}(\xi)$. Writing explicitly the averages we have

$$q_F(x) = \fint_{\mathbb{S}^{n-1}} q_\theta(x)\, d\sigma(\theta),$$

$$Q_{F,j}(q)(x) = \fint_{\mathbb{S}^{n-1}} Q_{\theta,j}(q)(x)\, d\sigma(\theta), \quad \text{and} \quad Q^R_{F,j}(q)(x) = \fint_{\mathbb{S}^{n-1}} Q^R_{\theta,j}(q)(x)\, d\sigma(\theta).$$

In this work we are not going to need any explicit formula for the $Q_{F,j}$ operators, only to observe that by (2.28) we have

$$\widehat{Q_{F,j}(q)}(\xi) = \frac{2}{|\mathbb{S}^{n-1}|} \int_{\{\theta \in \mathbb{S}^{n-1} : \, \xi\cdot\theta<0\}} B_{\theta,j}(q)(\xi)\, d\sigma(\theta), \tag{2.35}$$

(to see this, notice that the average in θ commutes with the Fourier transform). We also mention that in [33, pp. 708-709] the Fourier symbol of $Q_{F,2}$ as a bilinear operator is computed explicitly,

$$Q_{F,2}(q)(x) = C_n \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i(\xi+\eta)\cdot x} m(\xi,\eta) \frac{\widehat{q}(\xi)}{|\xi|} \frac{\widehat{q}(\eta)}{|\eta|}\, d\xi d\eta,$$

where

$$m(\xi,\eta) = \frac{\xi\cdot\eta}{|\xi||\eta|} + i\left(1 - \left(\frac{\xi\cdot\eta}{|\xi||\eta|}\right)^2\right)^{1/2},$$

and in [2] they use this formula to prove the following result.

Theorem 2.12 (J. Barceló, D. Faraco, A. Ruiz and A. Vargas)**.** *Let $\beta \geq 0$, $1 < r, s < \infty$, and consider $q \in W^{\beta,r}(\mathbb{R}^n)$ compactly supported. Assume that $\beta < \alpha < \beta + 1$ and that*

$$\alpha < \min\left(\beta + 1 + \left(\frac{1}{s} - \frac{1}{r}\right), 2(\beta+1) + \left(\frac{1}{s} - \frac{2}{r}\right)\right).$$

i) Then $\Re\left(Q_{F,2}(q)\right) \in W^{\alpha,s}_{loc}(\mathbb{R}^n)$.

ii) If we also assume for $n > 2$,

$$\frac{2n}{n+1} < r < \frac{2n}{n-1},$$

or for $n = 2$, $6/5 < r < 6$, then $Q_{F,2}(q) \in W^{\alpha,s}_{loc}(\mathbb{R}^n)$.

As a consequence of this theorem, in the same paper they also obtain a result of recovery of singularities (see Theorem 6.15).

2.4.3 Main results

We now introduce the main results of this work concerning fixed angle scattering and full data scattering.

Theorem 2.13. *Let $n \geq 2$ and $\beta \geq 0$. Assume that at least one of the statements $q - q_\theta \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$, or $q - q_F \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ holds for every $q \in W^{\beta,2}(\mathbb{R}^n)$ compactly supported, radial, and real. Then α necessarily satisfies,*

$$\alpha \leq \begin{cases} 2\beta - (n-4)/2, & if \quad (n-4)/2 < \beta < (n-2)/2, \\ \beta + 1, & if \quad (n-2)/2 \leq \beta < \infty. \end{cases}$$

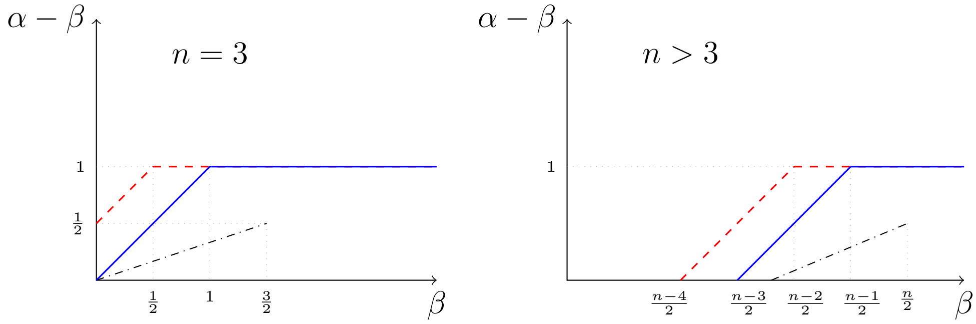

Figure 2.3: The (red) dashed line represents the limitation on the regularity gain given by *Theorem* 2.13 for $q - q_\theta$ and $q - q_F$. The solid (blue) line represents the positive results given in *Theorem* 2.14. The dot dashed line represents the previously known positive results of [46].

As far as we know, this is the first time that necessary conditions are given for the regularity of $q - q_\theta$ and $q - q_F$. As a consequence, Theorem 2.13 and Theorem 2.2, establish that in general it is not possible to have more than one derivative gain in the Sobolev scale in all the three scattering problems considered in this work. It is remarkable that these results are identical in the three cases, since the Born approximations have been constructed in very different ways. Notice that, as we mentioned for backscattering, when β approaches the value $(n-4)/2$, the derivative gain goes to zero (see Figure 2.3). All the comments after Theorem 2.2 also apply to this case.

We also have the following positive results.

Theorem 2.14 (Recovery of singularities)**.** *Let $q \in W^{\beta,2}(\mathbb{R}^n)$ with $0 \le \beta < \infty$ be a compactly supported function. Then $q - q_\theta \in W^{\alpha,2}(\mathbb{R}^n)$, modulo a C^∞ function, if the following condition holds,*

$$\alpha < \begin{cases} 2\beta - (n-3)/2, & if \quad (n-3)/2 < \beta < (n-1)/2, \\ \beta + 1, & if \quad (n-1)/2 \le \beta < \infty. \end{cases}$$

This theorem shows that there is a 1^- derivative gain when $\beta \ge (n-1)/2$, which, except for the limiting case $\alpha = \beta + 1$, is the best possible result by Theorem 2.13. It also improves the results of [46] in the spaces $W^{\alpha,2}(\mathbb{R}^n)$ for every value of β (see Figure 2.3). The key point to prove Theorem 2.14 is to obtain new estimates of the double dispersion operator $Q_{\theta,2}$ that we now introduce. In Theorem 6.15 below we also give a partial improvement of the results of recovery of singularities of full data scattering given in [2].

Theorem 2.15. *Let $n \ge 2$ and $q \in W_2^{\beta,2}(\mathbb{R}^n)$ with $0 \le \beta < \infty$. Then*

$$\|\widetilde{Q}_{\theta,2}(q)\|_{W^{\alpha,2}} \le C\|q\|^2_{W_2^{\beta,2}},$$

if the following conditions also hold,

$$\alpha < \begin{cases} 2\beta - (n-3)/2, & if \quad (n-3)/2 < \beta < (n-1)/2, \\ \beta + 1, & if \quad (n-1)/2 \le \beta < \infty. \end{cases} \tag{2.36}$$

Conversely, if we assume that $Q_{\theta,2}(q) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ *for all* $q \in W^{\beta,2}(\mathbb{R}^n)$ *real, radial and compactly supported, then* α *necessarily satisfies*

$$\alpha \le \min(\beta + 1, 2\beta - (n-4)/2). \tag{2.37}$$

The necessary condition (2.37) also holds for the double dispersion operator $Q_{F,2}$ in full data scattering (see Theorem 5.1 below). It is interesting to notice that this implies that the estimates of $Q_{F,2}$ given in Theorem 2.12 are the best possible for every $0 \le \beta < \infty$ and $r = s = 2$ (except for the limiting case). Hence, in full data scattering the regularity gap between the positive and negative results for the double dispersion operator has been closed for the complete range of β and $n \ge 2$. Unfortunately, this is not translated into optimal results of recovery of singularities, since also the estimates of higher order $Q_{F,j}$ operators should be improved in the range $(n-4)/2 < \beta \le (n-2)/2$.

2.5 Structure of the thesis

Chapter 3 is devoted to prove Theorem 2.4. The main tools are Proposition 3.2 which yields an explicit formula for $\widehat{Q_j(q)}(\eta)$ and Lemma 3.3 which allows us to control the principal value terms. We first study the case $j = 2$ which is the easiest one, and then we apply a similar line of reasoning when $j > 2$. The key result is the estimate of the spherical operator given by Lemma 3.5. This chapter is the most complex part of this work.

In Chapter 4 we estimate the double dispersion operator of fixed angle scattering (we prove the first part of the statement of Theorem 2.15). To do that, we adapt the techniques from the backscattering case introduced in Chapter 3.

In Chapter 5 we prove Theorem 2.5 and Theorem 2.15 by constructing in Proposition 5.4 an explicit family of counterexamples. The main result in this chapter is Theorem 5.1.

Chapter 6 contains the proofs of the main theorems. First, in Lemma 6.1 and Lemma 6.11 we study the remainder term of the Born series. With these lemmas, and the main results proved in previous chapters, we can finally prove Theorem 2.2, Theorem 2.3, and Theorem 2.13. With the same techniques we reduce the proofs of Theorem 2.8 and Corollary 2.10, respectively, to Theorem 2.7 and Theorem 2.9. Also, in Proposition 6.13 we give estimates for the multiple dispersion operators in fixed angle scattering, which allow us to prove Theorem 2.14.

Finally, in Chapter 7 we study more carefully the double dispersion operator in backscattering, in order to prove Theorem 2.7 and Theorem 2.9. Among other things, we will use again Lemma 3.3 to control the principal value term of $Q_2(q)$. The main estimates of the spherical operator are given, respectively, in Lemma 7.5 and Lemma 7.8.

Chapter 3

The multiple dispersion operators in backscattering

As we have explained in the previous chapter, the Q_j operators that appear in the Born series expansion of q can be expressed as a sum of a spherical term and several principal value operators. The usual strategy is to estimate the spherical part and then try to extend this estimate to the other terms. This is generally a very long and technical process that must be repeated case by case if the dimension or the value of j is changed (see [3, 44, 45, 48]). As a consequence, we want understand better the structure of the Q_j operators in order to estimate them by applying recursive arguments. In the first section we will prove a formula that simplifies greatly this task. Later we will study how to reduce the estimate of the principal value terms to the estimate of the spherical operators.

3.1 The structure of the Q_j operators

We begin studying the case of the Q_2 operator, which is simpler. Define the following distributions. Let $f \in C_c^\infty((0,\infty))$, we put

$$d(f) = \int_0^\infty \delta(1-r) f(r)\, dr, \quad \text{and} \quad P(f) = P.V. \int_0^\infty \frac{1}{1-r} f(r)\, dr, \tag{3.1}$$

where δ denotes the Dirac delta distribution, as usual.

Proposition 3.1. *Let $r \in (0,\infty)$ and consider the (modified) Ewald spheres defined by the equation*

$$\Gamma_r(\eta) := \{\xi \in \mathbb{R}^n : |\xi - \eta/2| = r|\eta|/2\}, \tag{3.2}$$

(see Figure 3.1). Then we have that

$$\widehat{Q_2(q)}(\eta) = (i\pi d + P) S_r(q)(\eta), \tag{3.3}$$

where, if we denote by $\sigma_{r\eta}$ the Lebesgue measure of $\Gamma_r(\eta)$,

$$S_r(q)(\eta) := \frac{2}{|\eta|(1+r)} \int_{\Gamma_r(\eta)} \widehat{q}(\xi)\widehat{q}(\eta-\xi)\, d\sigma_{r\eta}(\xi). \tag{3.4}$$

We omit the proof since it is just the case $j = 2$ of Proposition 3.2 below. In the case of $r = 1$ the sphere $\Gamma_1(\eta)$ is just the usual Ewald sphere as introduced in [48].

The case of the Q_j operators is a bit more complicated, since it involves several radial parameters instead of just r. To manage this, let $\ell \geq 1$, and assume we have $\mathbf{r} \in (0,\infty)^\ell$, $\mathbf{r} = (r_1, \dots, r_\ell)$ and $f \in C_c^\infty((0,\infty)^\ell)$. In analogy with the $j = 2$ case, we define the operators,

$$P_i, d_i : C_c^\infty((0,\infty)^\ell) \to C_c^\infty((0,\infty)^{\ell-1}), \quad i = 1, \dots, \ell,$$

following the notation introduced in (3.1),

$$d_i(f)(r_1, \dots, \widehat{r_i}, \dots, r_\ell) := \int_0^\infty \delta(r_i - 1) f(\mathbf{r})\, dr_i,$$
$$P_i(f)(r_1, \dots, \widehat{r_i}, \dots, r_\ell) := P.V. \int_0^\infty \frac{1}{1-r_i} f(\mathbf{r})\, dr_i,$$

where $\widehat{r_i}$ indicates that this coordinate is deleted in the list. Hence, if $\ell = 1$, $d_i(f)$ and $P_i(f)$ are just scalars. Also, if $\mathbf{r} \in (0,\infty)^\ell$ we define the manifold,

$$\Gamma_{\mathbf{r}}(\eta) = \Gamma_{r_1}(\eta) \times \dots \times \Gamma_{r_\ell}(\eta),$$

and we denote by $\sigma_{\mathbf{r}}$ its Lebesgue measure (product of the measures of the spheres $\Gamma_{r_i}(\eta)$),

$$d\sigma_{\mathbf{r}}(\xi_1, \dots, \xi_\ell) = d\sigma_{r_1\eta}(\xi_1) \times \dots \times d\sigma_{r_\ell\eta}(\xi_\ell).$$

Proposition 3.2 ($Q_j(q)$ structure)**.** *Let $n \geq 2$ and $j \geq 2$. Then we have that*

$$\widehat{Q_j(q)}(\eta) = \prod_{i=1}^{j-1} (i\pi d_i + P_i)\, S_{j,\mathbf{r}}(q)(\eta), \tag{3.5}$$

where $\mathbf{r} = (r_1, \dots, r_{j-1})$, and

$$S_{j,\mathbf{r}}(q)(\eta) := \left(\prod_{i=1}^{j-1} \frac{2}{1+r_i} \right) \times \dots$$
$$\frac{1}{|\eta|^{j-1}} \int_{\Gamma_{\mathbf{r}}(\eta)} \widehat{q}(\eta - \xi_1) \left(\prod_{i=1}^{j-2} \widehat{q}(\xi_i - \xi_{i+1}) \right) \widehat{q}(\xi_{j-1})\, d\sigma_{\mathbf{r}}(\xi_1, \dots, \xi_{j-1}). \tag{3.6}$$

Proposition 3.2 implies that the higher order operators Q_j have a similar structure to the Q_2 operator. In fact, when $j = 2$, (3.5) is equivalent to equation (3.3) since with the new notation we have $S_r = S_{2,\mathbf{r}}$ (in this case we have $\mathbf{r} = r$, since there is only one parameter).

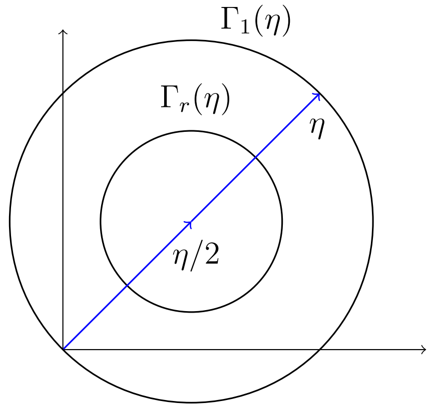

Figure 3.1: The largest sphere is the Ewald sphere $\Gamma_1(\eta)$, and the small one represents the Ewald sphere $\Gamma_r(\eta)$ for some $r < 1$.

Proof of Proposition 3.2. Let $k \in (0, \infty)$, we are going need the identity

$$R_k(f)(x) = i\frac{\pi}{2}k^{n-2}\int_{\mathbb{S}^{n-1}} \widehat{f}(k\omega)e^{ikx\cdot\omega}\,d\sigma(\omega) + P.V.\int_{\mathbb{R}^n} e^{ix\cdot\zeta}\frac{\widehat{f}(\zeta)}{-|\zeta|^2+k^2}\,d\zeta, \quad (3.7)$$

introduced in the previous chapter. We take spherical coordinates in the principal value integral, denoting by $t := |\zeta|$ the radial variable and we use the change of variables $t = rk$ in the radial integral,

$$\begin{aligned} P.V.\int_{\mathbb{R}^n} e^{ix\cdot\zeta}\frac{\widehat{f}(\zeta)}{-|\zeta|^2+k^2}\,d\zeta &= P.V.\int_0^\infty \frac{1}{(k-t)(k+t)}\int_{\mathbb{S}^{n-1}} e^{ix\cdot t\omega}\widehat{f}(t\omega)\,t^{n-1}d\sigma(\omega)\,dt \\ &= P.V.\frac{1}{k}\int_0^\infty \frac{1}{(1-r)(1+r)}\int_{\mathbb{S}^{n-1}} e^{ix\cdot rk\omega}\widehat{f}(rk\omega)\,(rk)^{n-1}d\sigma(\omega)\,dr \\ &= P.V.\frac{1}{k}\int_0^\infty \frac{1}{(1-r)(1+r)}\int_{\Gamma_r(\eta)} e^{i(-\xi-k\theta)\cdot x}\widehat{f}(-\xi-k\theta)\,d\sigma_{r\eta}(\xi)\,dr, \end{aligned}$$

where to obtain the integral over the Ewald sphere in the last line we have used the change of variables $rk\omega = -\xi - k\theta$ in the spherical integral, and that $\Gamma_r(\eta) = \{\xi \in \mathbb{R}^n : |\xi + k\theta| = rk\}$ if $\eta = -2k\theta$ (see (3.2)). Hence, using the analogous change of variables $k\omega = -\xi - k\theta$ in the first integral in (3.7), we finally obtain

$$\begin{aligned} R_k(f)(x) &= i\pi\frac{1}{|\eta|}\int_{\Gamma_1(\eta)} e^{i(-\xi-k\theta)\cdot x}\widehat{f}(-\xi-k\theta)\,d\sigma_\eta(\xi) \\ &\qquad + P.V.\int_0^\infty \frac{2}{|\eta|(1-r)(1+r)}\int_{\Gamma_r(\eta)} e^{i(-\xi-k\theta)\cdot x}\,\widehat{f}(-\xi-k\theta)\,d\sigma_{r\eta}(\xi)\,dr \\ &= (i\pi d + P)\left(\frac{2}{|\eta|(1+r)}\int_{\Gamma_r(\eta)} e^{i(-\xi-k\theta)\cdot x}\,\widehat{f}(-\xi-k\theta)\,d\sigma_{r\eta}(\xi)\right). \end{aligned} \quad (3.8)$$

We recall that by (2.8), we have

$$\widehat{Q_j(q)}(-2k\theta) = \int_{\mathbb{R}^n} e^{ik\theta\cdot y}(qR_k)^{j-1}(q(\cdot)e^{ik\theta\cdot(\cdot)})(y)\, dy. \tag{3.9}$$

Let $m \in \mathbb{N}$, we define

$$f_m(x) := R_k((qR_k)^{m-1}(q(\cdot)e^{ik\theta\cdot(\cdot)}))(x). \tag{3.10}$$

We claim that

$$\begin{aligned} f_m(x) = &\left(\prod_{i=1}^{m}(i\pi d_i + P_i)\right)\left(\prod_{i=1}^{m}\frac{2}{(1+r_i)}\right)\frac{1}{|\eta|^m} \times \dots \\ &\int_{\Gamma_{r_m}(\eta)} \cdots \int_{\Gamma_{r_1}(\eta)} e^{i(-\xi_m - k\theta)\cdot x}\,\widehat{q}(\eta-\xi_1)\left(\prod_{i=1}^{m-1}\widehat{q}(\xi_i-\xi_{i+1})\right) d\sigma_{\mathbf{r}}(\xi_1,\dots,\xi_m). \end{aligned} \tag{3.11}$$

We prove the claim by induction. The case $m = 1$ follows directly from (3.8) using that $\widehat{qe^{ik\theta\cdot(\cdot)}}(\xi) = \widehat{q}(\xi - k\theta)$ and that $\eta = -2k\theta$.

We are going to prove (3.11) for $m + 1$ assuming that it is true for m. On the one hand, by (3.10) and (3.8) we have

$$\begin{aligned} f_{m+1}(x) = R_k(qf_m)(x) = &(i\pi d_{m+1} + P_{m+1})\dots \\ &\left(\frac{2}{(1+r_{m+1})|\eta|}\int_{\Gamma_{r_{m+1}}(\eta)} e^{i(-\xi_{m+1}-k\theta)\cdot x}\,\widehat{(qf_m)}(-\xi_{m+1}-k\theta)\, d\sigma_{r_{m+1}\eta}(\xi_{m+1})\right). \end{aligned} \tag{3.12}$$

On the other hand, by (3.11), changing the order of integration we have

$$\begin{aligned} \widehat{(qf_m)}(\zeta) &= \int_{\mathbb{R}^n} q(y)f_m(y)e^{-i\zeta\cdot y}\, dy \\ &= \left(\prod_{i=1}^{m}(i\pi d_i + P_i)\right)\left(\prod_{i=1}^{m}\frac{2}{1+r_i}\right)\frac{1}{|\eta|^m}\int_{\Gamma_{r_m}(\eta)} \cdots \int_{\Gamma_{r_1}(\eta)} \widehat{q}(\eta-\xi_1) \times \dots \\ &\qquad \left(\prod_{i=1}^{m-1}\widehat{q}(\xi_i-\xi_{i+1})\right)\widehat{q}(k\theta+\zeta+\xi_m)\, d\sigma_{\mathbf{r}}(\xi_1,\dots,\xi_m). \end{aligned} \tag{3.13}$$

Thus, putting $\zeta = -\xi_{m+1} - k\theta$ in the previous equality and using (3.12), we get

$$\begin{aligned} f_{m+1}(x) = &\left(\prod_{i=1}^{m+1}(i\pi d_i + P_i)\right)\left(\prod_{i=1}^{m+1}\frac{2}{(1+r_i)}\right)\frac{1}{|\eta|^{m+1}} \times \dots \\ &\int_{\Gamma_{r_{m+1}}(\eta)} \cdots \int_{\Gamma_{r_1}(\eta)} e^{i(-\xi_{m+1}-k\theta)\cdot x}\,\widehat{q}(\eta-\xi_1)\left(\prod_{i=1}^{m}\widehat{q}(\xi_i-\xi_{i+1})\right) d\sigma_{\mathbf{r}}(\xi_1,\dots,\xi_{m+1}), \end{aligned}$$

which proves the claim.

By (3.9), since $\eta = -2k\theta$, we have that $\widehat{Q_j(q)}(\eta) = \widehat{qf_{j-1}}(-k\theta)$, and hence, in order to obtain (3.5), is enough to put $\zeta = -k\theta$ in (3.13),

$$\widehat{Q_j(q)}(\eta) = \left(\prod_{i=1}^{j-1}(i\pi d_i + P_i)\right)\left(\prod_{i=1}^{j-1}\frac{2}{(1+r_i)}\right)\frac{1}{|\eta|^{j-1}} \times \dots$$
$$\int_{\Gamma_{r_{j-1}}(\eta)} \dots \int_{\Gamma_{r_1}(\eta)} \widehat{q}(\eta - \xi_1)\left(\prod_{i=1}^{j-2}\widehat{q}(\xi_i - \xi_{i+1})\right)\widehat{q}(\xi_{j-1})\, d\sigma_{\mathbf{r}}(\xi_1, \dots, \xi_{j-1}). \quad \square$$

3.2 From the spherical integral to the principal value integral

The formulas (3.3) and (3.5) motivate the following lemma to control the the principal value terms.

Lemma 3.3. *Let $1 \le p < \infty$ and $\alpha \in \mathbb{R}$. Assume that there is a $0 < \delta < 1$, $\tau \in \mathbb{R}$, $\gamma > 0$ and $M > 0$ such that the one parameter family of $L^1_{loc}(\mathbb{R}^n)$ functions $\{F_r\}_{r\in(0,\infty)}$ satisfies*

i) For a.e. $\eta \in \mathbb{R}^n$ fixed, $\partial_r F_r(\eta)$ is a continuous function for all $r \in (1-\delta, 1+\delta)$, and in the same interval satisfies the estimate

$$\|\partial_r F_r\|_{L^p_\tau} \le M. \tag{3.14}$$

ii) For every $r \in (0,\infty)$,

$$\|F_r\|_{L^p_\alpha} \le (1+r)^{-\gamma} M. \tag{3.15}$$

Then we have that

$$\|(i\pi d + P)F_r\|_{L^p_{\alpha'}} \le C_2 M, \tag{3.16}$$

for every $\alpha' < \alpha$ and $C_2 = C_2(\delta, \alpha, \alpha', \tau, p, \gamma)$.

Notice that the value of τ in (3.14) does not have any influence on the value of α' in (3.16).

Proof. By the definition of d in (3.1), we clearly have that $\|d\,(F_r)\|_{L^p_\alpha} \le 2^{-\gamma} M$ follows directly putting $r = 1$ in (3.15). Therefore it remains to estimate in L^p_α the term

$$P(F_r)(\eta) = P.V. \int_0^\infty \frac{1}{1-r} F_r(\eta)\, dr.$$

Set

$$\delta_\eta := \delta\langle\eta\rangle^{-s}, \tag{3.17}$$

for some $s \geq 0$ that will be chosen later. Since for any $a > 0$, $P.V.\int_{|1-r|<a} \frac{dr}{1-r} = 0$, we have that

$$\begin{aligned} P(F_r)(\eta) = \\ &= \int_{|1-r|\leq\delta_\eta} \frac{F_r - F_1}{1-r}(\eta)\, dr + \int_{\delta_\eta<|1-r|<\delta} \frac{F_r(\eta)}{1-r} dr + \int_{\delta\leq|1-r|} \frac{F_r(\eta)}{1-r}\, dr \\ &:= P_A(\eta) + P_B(\eta) + P_C(\eta), \end{aligned} \tag{3.18}$$

where the $P.V.$ is not necessary any more, since we can cancel the singularity in the denominator since $F_r(\eta)$ is C^1 in $(1-\delta, 1+\delta)$ by condition i). Applying Minkowski's integral inequality and estimate (3.15), we obtain that

$$\|P_C\|_{L^p_\alpha} \leq \int_{\delta<|1-r|} \frac{\|F_r\|_{L^p_\alpha}}{|1-r|}\, dr \leq C(\delta,\gamma)M. \tag{3.19}$$

By the fundamental theorem of calculus we have

$$\frac{F_r(\eta) - F_1(\eta)}{1-r} = -\int_0^1 \partial_u F_u(\eta)\big|_{u=u(t)}\, dt,$$

where $u(t) = (r-1)t + 1$. Then the inequality

$$\langle\eta\rangle^{s/2} \leq \delta^{1/2}|1-r|^{-1/2},$$

that holds in the region $|1-r| < \delta_\eta$, yields

$$\begin{aligned} \|P_A\|_{L^p_\alpha} &= \left(\int_{\mathbb{R}^n} \langle\eta\rangle^{p\alpha} \left|\int_{|1-r|<\delta_\eta} \int_0^1 \partial_u F_u(\eta)\big|_{u=u(t)}\, dt\, dr\right|^p d\eta\right)^{1/p} \\ &\leq \delta^{1/2} \left(\int_{\mathbb{R}^n} \langle\eta\rangle^{p(\alpha-s/2)} \left(\int_{|1-r|<\delta} |1-r|^{-1/2} \int_0^1 \left|\partial_u F_u(\eta)\big|_{u=u(t)}\right|\, dt\, dr\right)^p d\eta\right)^{1/p} \\ &\leq \delta^{1/2} \int_{|1-r|<\delta} \int_0^1 |1-r|^{-1/2} \|\partial_u F_u\big|_{u=u(t)}\|_{L^p_{\alpha-s/2}}\, dt\, dr, \end{aligned}$$

where to get the last line we have used Minkowski's inequality.

We have two cases. If in (3.15) and (3.14) we have $\alpha \leq \tau$ we can choose $s = 0$, otherwise, if $\alpha > \tau$, we choose s such that $\alpha - s/2 = \tau$. In both cases by (3.15) we obtain

$$\begin{aligned} \|P_A\|_{L^p_\alpha} &\leq \delta^{1/2} \int_{|1-r|<\delta} \int_0^1 |1-r|^{-1/2} \|\partial_u F_u\big|_{u=u(t)}\|_{L^p_\tau}\, dt\, dr \\ &\leq 4\delta M. \end{aligned} \tag{3.20}$$

To finish we need estimate P_B which is non-zero when $s > 0$, that is when $\alpha > \tau$. We set $N(\eta) = -\log_2(\delta\langle\eta\rangle^{-s})$, and consider the next dyadic decomposition,

$$\begin{aligned} P_B(\eta) :&= \int_B \frac{F_r(\eta)}{1-r}\, dr \\ &= \sum_{0\le j<N(\eta)} \int_{\{2^{-(j+1)}<|1-r|<2^{-j}\}} \chi_{\{|1-r|<\delta_\eta\}}(r) \frac{F_r(\eta)}{1-r}\, dr. \end{aligned}$$

If $j = 0, 1, ..., N(\eta)$, for η fixed, the definition of $N(\eta)$ implies that $2^j \le \langle\eta\rangle^s/\delta$, therefore

$$|P_B(\eta)| \le \sum_{j=0}^{\infty} 2^{j+1} \chi_{(\delta 2^j,\infty)}(\langle\eta\rangle^s) \int_{|1-r|<2^{-j}} |F_r(\eta)|\, dr. \tag{3.21}$$

But observe that in the last line we have an expression of the kind

$$P^\lambda(\eta) := \chi_{(\delta\lambda^{-1},\infty)}(\langle\eta\rangle^s) \int_{|1-r|\le\lambda} |F_r(\eta)|\, dr,$$

with $0 < \lambda \le 1$. Computing its $L^p_{\alpha-\varepsilon}$ norm when $\varepsilon > 0$ and applying Minkowski's integral inequality we obtain

$$\|P^\lambda\|_{L^p_{\alpha-\varepsilon}} \le \lambda^{\varepsilon/s} \int_{\{|1-r|\le\lambda\}} \|F_r\|_{L^p_\alpha}\, dr \le \lambda^{1+\varepsilon/s} M, \tag{3.22}$$

where we have used estimate (3.15), and that in the region where the characteristic function does not vanish we have that $\langle\eta\rangle^{-\varepsilon} \le \delta^{-\varepsilon/s}\lambda^{\varepsilon/s}$. Hence, taking the $L^p_{\alpha'}$ norm of (3.21) and applying estimate (3.22) with $\varepsilon = \alpha - \alpha'$ yields

$$\begin{aligned} \|P_B\|_{L^p_{\alpha'}} \le 2\sum_{j=0}^{\infty} 2^j \|P^{2^{-j}}\|_{L^p_{\alpha'}} &\le 2\delta^{-\varepsilon/s} M \sum_{j=0}^{\infty} 2^{-j\varepsilon/s} \\ &\le C(\delta, \alpha, \alpha', \tau, p) M. \end{aligned} \tag{3.23}$$

Observe that this is the first time we need the strict inequality $\alpha' < \alpha$ in the statement of the theorem. Therefore since $P(F_r) = P_A + P_B + P_C$ we conclude the proof putting together estimates (3.19), (3.20) and (3.23). □

In our case usually the family of functions F_r is given by a multilinear operator over the potentials, as it can be seen in (3.3) where $F_r = S_r(q)$.

3.3 Estimate of the spherical operator of double dispersion

In this section we study in detail the spherical operator S_r associated to the double dispersion operator $\widetilde{Q}_2$ in order to prove Theorem 2.4 for $j = 2$. This section will serve to illustrate the approach that we will follow in the next section to obtain the main estimates of the spherical operators $S_{j,\mathbf{r}}$.

For notational convenience we define the operator

$$\widetilde{S}_r(q)(\eta) := \chi(\eta) S_r(q)(\eta).$$

Then, multiplying both sides of equation (3.3) by the smooth cut-off $\chi(\eta)$ we get

$$\widehat{\widetilde{Q}_2(q)}(\eta) = (i\pi d + P)\widetilde{S}_r(q)(\eta). \tag{3.24}$$

Hence, the main step to estimate the $\widetilde{Q}_2$ operator is to apply Lemma 3.3 to the particular case $F_r = \widetilde{S}_r(q)$. We begin with the necessary estimates for $\widetilde{S}_r(q)$.

Lemma 3.4. *Let $n \geq 2$ and $q \in W_1^{\beta,2}(\mathbb{R}^n)$ with $\beta \geq 0$. Then the estimate*

$$\|\widetilde{S}_r(q)\|_{L^2_\alpha} \leq C(1+r)^{-\gamma} \|q\|^2_{W_1^{\beta,2}},$$

holds when

$$\begin{cases} \alpha \leq \beta + (\beta - (n-3)/2), & if \quad (n-3)/2 < \beta < (n-1)/2, \\ \alpha < \beta + 1, & if \quad (n-1)/2 \leq \beta < \infty, \end{cases} \tag{3.25}$$

for some real number $\gamma > 0$ (possibly depending on β and α).

To simplify later computations we define the operator

$$\widetilde{K}_r(g_1, g_2)(\eta) = \chi(\eta) K_r(g_1, g_2)(\eta), \tag{3.26}$$

where

$$K_r(g_1, g_2)(\eta) := \frac{1}{|\eta|} \int_{\Gamma_r(\eta)} |g_1(\xi)||g_2(\eta - \xi)| \, d\sigma_{r\eta}(\xi). \tag{3.27}$$

Then we have that

$$\left|\widetilde{S}_r(q)(\eta)\right| \leq \frac{2}{1+r} \widetilde{K}_r(\widehat{q}, \widehat{q})(\eta), \tag{3.28}$$

and therefore the proof of Lemma 3.4 is an immediate consequence of the following lemma taking $\gamma = 1 - \lambda$.

Lemma 3.5. *Let $n \geq 2$ and $f_1, f_2 \in W_1^{\beta,2}(\mathbb{R}^n)$ with $\beta \geq 0$. Then the estimate*

$$\|\widetilde{K}_r(\widehat{f}_1, \widehat{f}_2)\|_{L^2_\alpha} \leq C r^\lambda \|f_1\|_{W_1^{\beta,2}} \|f_2\|_{W_1^{\beta,2}}, \tag{3.29}$$

holds when condition (3.25) is also satisfied, for some real number $0 < \lambda < 1$ (possibly depending on β and α).

In the proof we are going to use the following result.

Lemma 3.6. *Let $\mathbb{S}_\rho \subset \mathbb{R}^n$ be any sphere of radius ρ and let $d\sigma_\rho$ be its Lebesgue measure. Then for any $0 < \lambda \leq (n-1)/2$, we have that*

$$\int_{\mathbb{S}_\rho} \frac{1}{|x-y|^{(n-1)-2\lambda}}\, d\sigma_\rho(y) \leq C_\lambda \rho^{2\lambda},$$

for any $x \in \mathbb{R}^n$, and for a constant C_λ that only depends on λ.

This can be proved by direct computation (for a detailed proof see Appendix A.1). Also, We want to highlight the following property of Sobolev norms that we will use frequently in this work.

Remark 3.7. We have that $W^{\beta,2}_\delta \subset W^{\beta',2}_{\delta'}$ if $\beta \geq \beta'$ and $\delta \geq \delta'$. This follows from the equivalence

$$\|\langle\cdot\rangle^\delta \langle D\rangle^\beta f\|_{L^2} \sim \|\langle D\rangle^\beta \langle\cdot\rangle^\delta f\|_{L^2},$$

and Plancherel theorem, see, for example, [21, Definition 30.2.2]. We will also use the inequality

$$\|x_i f\|_{W^{\beta,2}_\delta} \leq C\|f\|_{W^{\beta,2}_{\delta+1}},$$

(this can be proved, for example, for integer values of β and extended by interpolation).

Proof of Lemma 3.5. Since by (2.10) $\chi(\eta) = 0$ for $|\eta| \leq 1$, we have that $|\eta|^{-1} \leq 2\langle\eta\rangle^{-1}$ in the region where χ does not vanish. Then

$$\|\widetilde{K}_r(\widehat{f}_1, \widehat{f}_2)\|^2_{L^2_\alpha} \leq C \int_{\mathbb{R}^n} \langle\eta\rangle^{2\alpha-2} \left(\int_{\Gamma_r(\eta)} |\widehat{f}_1(\xi)||\widehat{f}_2(\eta-\xi)|\, d\sigma_{r\eta}(\xi) \right)^2 d\eta.$$

Now, $\eta = (\eta-\xi)+\xi$, so if we choose any $0 < c < 1/2$ at least one of the conditions $|\xi| > c|\eta|$ and $|\eta-\xi| > c|\eta|$ must hold. But now observe that the change of variables $\xi' = \eta-\xi$ leaves invariant $\Gamma_r(\eta)$ and $\widetilde{K}_r(\widehat{f}_1, \widehat{f}_2)$, except for the fact that interchanges the roles of $\widehat{f}_1$ and $\widehat{f}_2$. Therefore is enough to study only the case of $|\xi| > c|\eta|$ since then the other follows applying the change of variables. We want to estimate

$$I := \int_{\mathbb{R}^n} \langle\eta\rangle^{2\alpha-2} \left(\int_{A_r(\eta)} |\widehat{f}_1(\xi)||\widehat{f}_2(\eta-\xi)|\, d\sigma_{r\eta}(\xi) \right)^2 d\eta,$$

$$\text{where} \quad A_r(\eta) := \{\xi \in \Gamma_r(\eta) : |\xi| > c|\eta|\}.$$

We introduce a real parameter $0 < \lambda \leq (n-1)/2$. Then by Cauchy-Schwarz's inequality we have

$$I \leq C \int_{\mathbb{R}^n} \langle\eta\rangle^{2\alpha-2} \int_{A_r(\eta)} |\widehat{f}_1(\xi)|^2 |\widehat{f}_2(\eta-\xi)|^2 |\eta-\xi|^{n-1-2\lambda}\, d\sigma_{r\eta}(\xi) \times \dots$$

$$\dots \times \int_{\Gamma_r(\eta)} \frac{1}{|\eta-\xi|^{n-1-2\lambda}}\, d\sigma_{r\eta}(\xi)\, d\eta.$$

Then, since $\Gamma_r(\eta)$ has radius $r|\eta|/2$, using Lemma 3.6 to bound the second integral we obtain

$$\begin{aligned} I &\leq Cr^{2\lambda} \int_{\mathbb{R}^n} \langle\eta\rangle^{2\alpha-2}|\eta|^{2\lambda} \int_{A_r(\eta)} |\widehat{f}_1(\xi)|^2 |\widehat{f}_2(\eta-\xi)|^2 \langle\eta-\xi\rangle^{n-1-2\lambda}\, d\sigma_{r\eta}(\xi)\, d\eta \\ &\leq Cr^{2\lambda} \int_{\mathbb{R}^n} \int_{\Gamma_r(\eta)} |\widehat{f}_1(\xi)|^2 \langle\xi\rangle^{2\alpha-2+2\lambda} |\widehat{f}_2(\eta-\xi)|^2 \langle\eta-\xi\rangle^{n-1-2\lambda}\, d\sigma_{r\eta}(\xi)\, d\eta, \end{aligned} \tag{3.30}$$

using also that $\langle\eta\rangle^{2\alpha-2+2\lambda} \leq C\langle\xi\rangle^{2\alpha-2+2\lambda}$, which follows from the fact that $|\eta| \leq c|\xi|$, if we impose the extra condition $\alpha - 1 + \lambda \geq 0$.

We are going to use the trace theorem to bound the $L^2(\Gamma_r(\eta))$ norm given by the second integral in (3.30). The fundamental point is that for spheres, the constant of the trace theorem can be taken to be 1, independently of the radius of the sphere. See Proposition A.1 in the Appendix for an elementary proof of this fact.

$$\begin{aligned} I \leq\ & Cr^{2\lambda} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\widehat{f}_1(\xi)|^2 \langle\xi\rangle^{2\alpha-2+2\lambda} |\widehat{f}_2(\eta-\xi)|^2 \langle\eta-\xi\rangle^{(n-1)-2\lambda} d\xi\, d\eta \\ &+ Cr^{2\lambda} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left|\nabla\left(\widehat{f}_1(\xi)\langle\xi\rangle^{\alpha-1+\lambda}\right)\right|^2 |\widehat{f}_2(\eta-\xi)|^2 \langle\eta-\xi\rangle^{(n-1)-2\lambda} d\xi\, d\eta \\ &+ Cr^{2\lambda} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\widehat{f}_1(\xi)|^2 \langle\xi\rangle^{2\alpha-2+2\lambda} \left|\nabla\left(\widehat{f}_2(\eta-\xi)\langle\eta-\xi\rangle^{(n-1)/2-\lambda}\right)\right|^2 d\xi\, d\eta. \end{aligned}$$

Therefore changing the order of integration and using that by Plancherel theorem we have

$$\int_{\mathbb{R}^n} \left|\nabla(\widehat{f}(\xi)\langle\xi\rangle^t)\right|^2 d\xi \leq C\|f\|^2_{W_1^{t,2}},$$

we obtain

$$I \leq Cr^{2\lambda}\|f_1\|^2_{W_1^{\alpha-1+\lambda,2}}\|f_2\|^2_{W_1^{(n-1)/2-\lambda,2}}.$$

As we have explained before, in the case $|\eta-\xi| > c|\eta|$ we obtain the same estimate but interchanging the roles of f_1 and f_2. Putting both estimates together we get

$$\begin{aligned} &\|\widetilde{K}_r(\widehat{f}_1,\widehat{f}_2)\|_{L^2_\alpha} \\ &\leq Cr^{\lambda}\left(\|f_1\|_{W_1^{\alpha-1+\lambda,2}}\|f_2\|_{W_1^{(n-1)/2-\lambda,2}} + \|f_2\|_{W_1^{\alpha-1+\lambda,2}}\|f_1\|_{W_1^{(n-1)/2-\lambda,2}}\right). \end{aligned}$$

We also add the extra restriction $\lambda < 1$, this is necessary to have a negative value for γ in Lemma 3.4. Now, we also ask λ to satisfy the equality

$$\beta = \alpha - 1 + \lambda. \tag{3.31}$$

Hence, the condition $\alpha - 1 + \lambda \geq 0$ used in the proof implies we must have $\beta \geq 0$. As a consequence of (3.31), equation (3.29) follows directly in the range $\beta \geq (n-1)/2$

(we are using Remark 3.7). But, by the conditions imposed in the proof we have to take into account the restrictions

$$\begin{cases} 0<\lambda<1 \\ 0<\lambda\le\frac{n-1}{2} \end{cases} \iff \begin{cases} \beta<\alpha<\beta+1 \\ \beta+1-\frac{n-1}{2}\le\alpha<\beta+1. \end{cases}$$

We can discard the lower bounds for α using that $\|f\|_{L^2_\alpha} \le \|f\|_{L^2_{\alpha'}}$ always holds if $\alpha \le \alpha'$. Therefore only the restriction $\alpha < \beta + 1$ remains.

Otherwise, if β is in the range $0 \le \beta < (n-1)/2$, estimate (3.29) will follow if we add the extra condition

$$(n-1)/2-\lambda\le\beta. \tag{3.32}$$

Then, since $\lambda < 1$, we must have $\beta > (n-3)/2$ (the other conditions on λ don't add new restrictions). Also (3.31) and (3.32) imply together that $\alpha \le 2\beta - (n-3)/2$, which is a stronger condition than $\alpha < \beta + 1$ since we have $\beta < (n-1)/2$. Hence, we have obtained the ranges of parameters given in the statement. □

Lemma 3.8. *Let $q \in \mathcal{S}'(\mathbb{R}^n)$ such that $\widehat{q}$ is smooth. Then, for every $\eta \neq 0$ fixed, $S_r(q)(\eta)$ is smooth in the r variable. Moreover, we have the following pointwise inequality*

$$|\partial_r S_r(q)(\eta)| \le C K_r(\widehat{q},\widehat{q})(\eta) + C|\eta| \sum_{i=1}^{n} K_r(\widehat{x_i q},\widehat{q}). \tag{3.33}$$

In general, the constant C in the estimate might depend on δ, but this is harmless.

Proof. If $\theta \in \mathbb{S}^{n-1}$, we center the Ewald sphere $\Gamma_r(\eta)$ at the origin with the change $\xi = \eta/2 + r|\eta/2|\theta$,

$$S_r(q)(\eta) = \frac{r^{n-1}|\eta|^{n-2}}{2^{n-2}(1+r)} \int_{\mathbb{S}^{n-1}} \widehat{q}\left(r\frac{|\eta|}{2}\theta+\frac{\eta}{2}\right)\widehat{q}\left(-r\frac{|\eta|}{2}\theta+\frac{\eta}{2}\right)\,d\sigma(\theta). \tag{3.34}$$

Now we can compute derivatives in the r variable. Consider η fixed, then

$$\begin{aligned}
&\partial_r S_r(q)(\eta) = \\
&= \frac{((n-1)r^{n-2}(1+r)-r^{n-1})|\eta|^{n-2}}{2^{n-2}(1+r)^2} \int_{\mathbb{S}^{n-1}} \widehat{q}\left(r\frac{|\eta|}{2}\theta+\frac{\eta}{2}\right)\widehat{q}\left(-r\frac{|\eta|}{2}\theta+\frac{\eta}{2}\right)\,d\sigma(\theta) \\
&+ \frac{r^{n-1}|\eta|^{n-1}}{2^{n-1}(1+r)} \int_{\mathbb{S}^{n-1}} \theta\cdot\nabla\widehat{q}\left(r\frac{|\eta|}{2}\theta+\frac{\eta}{2}\right)\widehat{q}\left(-r\frac{|\eta|}{2}\theta+\frac{\eta}{2}\right)\,d\sigma(\theta) \\
&- \frac{r^{n-1}|\eta|^{n-1}}{2^{n-1}(1+r)} \int_{\mathbb{S}^{n-1}} \widehat{q}\left(r\frac{|\eta|}{2}\theta+\frac{\eta}{2}\right)\theta\cdot\nabla\widehat{q}\left(-r\frac{|\eta|}{2}\theta+\frac{\eta}{2}\right)\,d\sigma(\theta).
\end{aligned}$$

We have passed the derivative inside the integral since we are integrating in finite measure and $\widehat{q}$ is smooth. Observe also that the last two terms are identical (this can be verified with the change $\omega = -\theta$). Hence, if we undo the change to spherical coordinates we get

$$\begin{aligned}\partial_r S_r(q)(\eta) =& \frac{(n-2)r+(n-1)}{r(1+r)^2}\frac{2}{|\eta|}\int_{\Gamma_r(\eta)} \widehat{q}(\xi)\widehat{q}(\eta-\xi)\, d\sigma_{r\eta}(\xi) \\ &+ \frac{2}{(1+r)}\int_{\Gamma_r(\eta)} \frac{(\xi-\eta/2)}{|\xi-\eta/2|}\cdot \nabla\widehat{q}(\xi)\widehat{q}(\eta-\xi)\, d\sigma_{r\eta}(\xi). \end{aligned} \tag{3.35}$$

Therefore by (3.27), if we fix some $0<\delta<1$, for $r\in(1-\delta,1+\delta)$ we obtain

$$|\partial_r S_r(q)(\eta)| \le CK_r(\widehat{q},\widehat{q})(\eta) + C|\eta|K_r(|\nabla\widehat{q}|,\widehat{q})(\eta).$$

The estimate follows then using that

$$K_r(|\nabla\widehat{q}|,\widehat{q}) \le \sum_{i=1}^n K_r(\partial_i\widehat{q},\widehat{q}) = C\sum_{i=1}^n K_r(\widehat{x_i q},\widehat{q}). \qquad \square$$

From Lemmas 3.4 and 3.8 we get the following proposition.

Proposition 3.9. *Let $n\ge 2$ and fix any $0<\delta<1$. Then for every $r\in(1-\delta,1+\delta)$, and $q\in\mathcal{S}(\mathbb{R}^n)$ we have that*

$$\|\partial_r\widetilde{S}_r(q)\|_{L^2_{\alpha-1}} \le C\|q\|^2_{W_2^{\beta,2}},$$

holds when α and $\beta\ge 0$ satisfy condition (3.25).

Observe the appearance of the Sobolev space $W_2^{\beta,2}$ instead of $W_1^{\beta,2}$.

Proof. Multiplying (3.33) by $\chi(\eta)$ we get

$$\|\partial_r\widetilde{S}_r(q)\|_{L^2_{\alpha-1}} \le C\|\widetilde{K}_r(\widehat{q},\widehat{q})\|_{L^2_{\alpha-1}} + C\sum_{i=1}^n \|\widetilde{K}_r(\widehat{x_i q},\widehat{q})\|_{L^2_\alpha}.$$

Notice that we get the L^2_α norm in the last term due to the extra $|\eta|$ factor appearing in (3.33). Then, by Lemma 3.5 we obtain the desired estimate using that

$$\|\widetilde{K}_r(\widehat{x_i q},\widehat{q})\|_{L^2_\alpha} \le C\|x_i q\|_{W_1^{\beta,2}}\|q\|_{W_1^{\beta,2}} \le C\|q\|^2_{W_2^{\beta,2}}. \tag{3.36}$$

The estimate $\|x_i q\|_{W_1^{\beta,2}} \le C\|q\|_{W_2^{\beta,2}}$ can be verified for integer β and extended by interpolation to the general case.

$\square$

By Lemma 3.4 and Proposition 3.9 we can apply Lemma 3.3 to estimate the $\widetilde{Q}_2$ operator, but we leave this for the next section.

3.4 Estimate of the multiple dispersion operators

3.4.1 Proof of Theorem 2.4

We now introduce the $\widetilde{S}_{j,\mathbf{r}}$ spherical operators,

$$\widetilde{S}_{j,\mathbf{r}}(q)(\eta) := \chi(\eta) S_{j,\mathbf{r}}(q)(\eta),$$

as we did for $j = 2$. The following proposition generalizes the results of Lemma 3.4 and Proposition 3.9 for $j \geq 2$. Its proof will be given later on.

Proposition 3.10. *Let $q \in \mathcal{S}(\mathbb{R}^n)$, $n \geq 2$, $j \geq 3$ and $0 < \delta < 1$. Consider all the multi-indices $\mathbf{a} = (a_1, \ldots, a_{j-1})$ with a_i, $1 \leq i \leq j-1$, either 0 or 1. Then the estimate*

$$\|\partial_{\mathbf{r}}^{\mathbf{a}} \widetilde{S}_{j,\mathbf{r}}(q)\|_{L^2_{\alpha-|\mathbf{a}|}} \leq C \left(\prod_{i=1}^{j-1} \frac{1}{(1+r_i)^{\gamma}} \right) \|q\|^j_{W_4^{\beta,2}}, \tag{3.37}$$

holds for $\beta \geq 0$, a certain $\gamma > 0$ (possibly dependent on β), and some constant $C = C(n, j, \alpha, \beta)$, if the following conditions also hold

$$r_i \in (1-\delta, 1+\delta) \text{ if } a_i = 1 \text{ and } r_i \in (0, \infty) \text{ if } a_i = 0, \tag{3.38}$$

$$\begin{cases} \alpha \leq \beta + (j-1)(\beta - (n-3)/2), & if \quad (n-3)/2 < \beta < (n-1)/2, \\ \alpha < \beta + (j-1), & if \quad (n-1)/2 \leq \beta < \infty. \end{cases} \tag{3.39}$$

With this proposition we can prove finally Theorem 2.4, with the help of the following density argument.

Lemma 3.11. *Let $j \geq 2$, $1 < p < \infty$, and $\delta \in \mathbb{R}$. Assume that, for every $q \in C_c^\infty(\mathbb{R}^n)$, the operator $\widetilde{Q}_j$ satisfies an a priori estimate*

$$\|\widetilde{Q}_j(q)\|_{W^{\alpha,2}} \leq C \|q\|^j_{W_\delta^{\beta,p}}. \tag{3.40}$$

Then there is a unique continuous extension $\widetilde{Q}_j : W_\delta^{\beta,p}(\mathbb{R}^n) \longrightarrow W^{\alpha,2}(\mathbb{R}^n)$ of the operator, and it satisfies (3.40) for every $q \in W_\delta^{\beta,p}(\mathbb{R}^n)$.

With this lemma we can extend the estimates for $\widetilde{Q}_j(q)$ without having to give an estimate for the multilinear operator $Q_j(f_1, \ldots, f_j)$ (this operator is defined by putting f_i instead of q in (2.8) following the order of appearance of each q in the formula). The advantage of having $f_i = q$ in most of the estimates in this work is a question of (great) notational simplicity, but it is not an essential restriction in any of them. Lemma 3.11 is a direct consequence of Lemma A.3 in the appendix.

Proof of Theorem 2.4. We begin with the case $j = 2$. By Proposition 3.9 and Lemma 3.4, for each $q \in \mathcal{S}(\mathbb{R}^n)$ we can apply Lemma 3.3 with $F_r = \widetilde{S}_r(q)$, $p = 2$, $\tau = \alpha - 1$ and $M = C\|q\|^2_{W_2^{\beta,2}}$. Therefore by (3.24) this yields the estimate

$$\|\widehat{\widetilde{Q}_2(q)}\|_{L^2_{\alpha'}} \leq C\|q\|^2_{W_2^{\beta,2}}.$$

for $\alpha' < \alpha$ and α in the range (3.25). Then by Plancherel theorem we get the desired estimate for $\widetilde{Q}_2(q)$ in the Sobolev norm, and by Lemma 3.11 we can extend by density these estimates for every $q \in W_2^{\beta,2}(\mathbb{R}^n)$. This is enough to prove estimate (2.16).

Now, let's study the case $j \geq 3$. Consider $f \in \mathcal{S}(\mathbb{R}^n)$. We introduce the following operators,

$$T_{j,1}(r_1, \dots, r_{j-1})(f) := \widetilde{S}_{j,\mathbf{r}}(f), \tag{3.41}$$

$$\begin{aligned} T_{j,k}(r_k, \dots, r_{j-1})(f) &:= (i\pi d_{k-1} + P_{k-1})T_{j,k-1}(r_{k-1}, \dots, r_{j-1})(f) \\ &= \prod_{i=1}^{k-1}(i\pi d_i + P_i)\widetilde{S}_{j,\mathbf{r}}(f), \end{aligned} \tag{3.42}$$

for $2 \leq k \leq j-1$, and

$$\begin{aligned} T_{j,j}(f) := (i\pi d_{j-1} + P_{j-1})T_{j,j-1}(r_{j-1})(f) &= \prod_{i=1}^{j-1}(i\pi d_i + P_i)\widetilde{S}_{j,\mathbf{r}}(f) \\ &= \widehat{\widetilde{Q}_j(f)}, \end{aligned} \tag{3.43}$$

$T_{j,k}(r_k, \dots, r_{j-1})(f)(x)$ is a well defined function, smooth in the variables $r_k, \dots, r_{j-1}$ and x (see Proposition A.4 in the Appendix for more details). As we are going to see, the proof can be reduced to proving the following claim.

Claim 3.12. *Let $1 \leq k \leq j$, and let $\mathbf{a} = (a_1, \dots, a_{j-1})$ with $a_i = 0$ if $1 \leq i \leq k-1$, and $a_i = 0, 1$ if $k \leq i \leq j-1$. Then the estimate*

$$\|\partial_{\mathbf{r}}^{\mathbf{a}} T_{j,k}(r_k, \dots, r_{j-1})(f)\|_{L^2_{\alpha'}} \leq c_k \|f\|^j_{W_4^{\beta,2}}, \tag{3.44}$$

holds for $\alpha' < (\alpha - |\mathbf{a}|)$ if conditions (3.38) and (3.39) are satisfied, with

$$c_k = CC_2^{k-1}\prod_{i=k}^{j-1}\frac{1}{(1+r_i)^\gamma}, \tag{3.45}$$

where C_2 is the constant introduced in Lemma 3.3.

By (3.43), we have that for $q \in \mathcal{S}(\mathbb{R}^n)$, estimate (3.44) with $k = j$, $\mathbf{a} = 0$ and $f = q$ gives

$$\|\widehat{\widetilde{Q}_j(q)}\|_{L^2_{\alpha'}} \leq C\|q\|^j_{W_2^{\beta,2}},$$

for every $\alpha' < \alpha$ and α in the range (3.39). Then, using Plancherel theorem and Lemma 3.11 to extend the resulting estimate for all $q \in W_4^{\beta,2}(\mathbb{R}^n)$, yields estimate (2.17).

We now prove Claim 3.12 by induction in k (observe that j is fixed in the claim). By (3.41), the case $k = 1$ of estimate (3.44) is equivalent to Proposition 3.10. To prove that (3.44) holds true for $2 \leq k \leq j$, in each induction step we are going to use Lemma 3.3 and (3.42).

Let's assume that the claim holds for a certain k, $1 \leq k < j - 1$, then we are going to prove it for $k + 1$. Let $\mathbf{a}' = (a'_1, \dots, a'_{j-1})$ with $a'_i = 0$ if $1 \leq i \leq k$, and $a'_i = 0, 1$ if $k + 1 \leq i \leq j - 1$. We are going to apply Lemma 3.3 with

$$F_{r_k}(x) := \partial_{\mathbf{r}}^{\mathbf{a}'} T_{j,k}(r_k, \dots, r_{j-1})(f)(x).$$

By the induction hypothesis (3.44) with $\mathbf{a} = \mathbf{a}'$, and (3.45) we have

$$\|F_{r_k}\|_{L^2_{\alpha'}} \leq \frac{c_{k+1}}{(1 + r_k)^\gamma} C_2^{-1}\|f\|^j_{W_4^{\beta,2}}, \tag{3.46}$$

for $\alpha' < (\alpha - |\mathbf{a}'|)$ and $r_k \in (0, \infty)$. Moreover, taking now $\mathbf{a}$ with $a_i = a'_i$ for $i \neq k$, and $a_k = 1$, we also get from (3.44) the estimate

$$\|\partial_{r_k} F_{r_k}\|_{L^2_{\alpha'-1}} \leq \frac{c_{k+1}}{(1 + r_k)^\gamma} C_2^{-1}\|f\|^j_{W_4^{\beta,2}}, \tag{3.47}$$

with $\alpha' < (\alpha - |\mathbf{a}'| - 1)$ and $r_k \in (1 - \delta, 1 + \delta)$. Then, for each $f \in \mathcal{S}(\mathbb{R}^n)$ we can apply Lemma 3.3 since condition (3.14) is given by (3.47) and (3.15) by (3.46) with $M = c_{k+1}C_2^{-1}\|f\|^j_{W_4^{\beta,2}}$. Therefore, for $\alpha' < (\alpha - |\mathbf{a}'|)$, we obtain that

$$\begin{aligned}&\|\partial_{\mathbf{r}}^{\mathbf{a}'} T_{j,k+1}(r_k, \dots, r_{j-1})(f)\|_{L^2_{\alpha'}} = \\ &\|(i\pi d_k + P_k)\partial_{\mathbf{r}}^{\mathbf{a}'} T_{j,k}(r_k, \dots, r_{j-1})(f)\|_{L^2_{\alpha'}} = \|(i\pi d_k + P_k)F_{r_k}\|_{L^2_{\alpha'}} \leq c_k\|f\|^j_{W_4^{\beta,2}},\end{aligned}$$

where the first equality is true by Proposition A.4 in the Appendix. This concludes the proof of the claim. □

3.4.2 Proof of Proposition 3.10

We define the operator

$$\begin{aligned}&K_{j,\mathbf{r}}(g_i)(\eta) = K_{j,\mathbf{r}}(g_1, \dots, g_j)(\eta) := \\ &\frac{1}{|\eta|^{j-1}} \int_{\Gamma_{\mathbf{r}}(\eta)} |g_1(\eta - \xi_1)| \left(\prod_{i=1}^{j-2} |g_{i+1}(\xi_i - \xi_{i+1})| \right) |g(\xi_{j-1})| \, d\sigma_{\mathbf{r}}(\xi_1, \dots, \xi_{j-1}),\end{aligned}$$

and $\widetilde{K}_{j,\mathbf{r}}(g_i)(\eta) := \chi(\eta)K_{j,\mathbf{r}}(g_i)(\eta)$. Hence we have that

$$\left|\widetilde{S}_{j,\mathbf{r}}(q)(\eta)\right| \leq \left(\prod_{i=1}^{j-1} \frac{2}{1+r_i}\right) \widetilde{K}_{j,\mathbf{r}}(\widehat{q},\dots,\widehat{q})(\eta). \tag{3.48}$$

The main tool to prove Proposition 3.10 is the following Lemma.

Lemma 3.13. *Let $n \geq 2$ and $j \geq 3$, and consider $f_l \in W_2^{\beta,2}(\mathbb{R}^n)$, $1 \leq l \leq j$ with $\beta \geq 0$. Then the estimate*

$$\|\widetilde{K}_{j,\mathbf{r}}(\widehat{f}_1,\dots,\widehat{f}_j)\|_{L^2_\alpha} \leq C\left(\prod_{i=1}^{j-1}(1+r_i)^\lambda\right)\prod_{l=1}^{j}\|f_l\|_{W_2^{\beta,2}},$$

holds when α is in the range given in (3.39) for some real number $0<\lambda<1$.

Proof. Since

$$\eta = (\eta-\xi_1) + \sum_{i=1}^{j-2}(\xi_i - \xi_{i+1}) + \xi_{j-1},$$

if we fix $c < 1/j$, one of the conditions $|\eta-\xi_1| > c|\eta|$, $|\xi_i - \xi_{i+1}| > c|\eta|$ for some $1 \leq i \leq j-2$, or $|\xi_{j-1}| > c|\eta|$ must hold. Hence, the sets

$$\begin{aligned}
A^1_{\mathbf{r}}(\eta) &= \{(\xi_1,\dots,\xi_{j-1}) \in \Gamma_{\mathbf{r}}(\eta) : \ |\eta-\xi_1| > c|\eta|\}, \\
A^i_{\mathbf{r}}(\eta) &= \{(\xi_1,\dots,\xi_{j-1}) \in \Gamma_{\mathbf{r}}(\eta) : \ |\xi_{i-1}-\xi_i| > c|\eta|\}, \quad i = 2,\dots,j-1, \\
A^j_{\mathbf{r}}(\eta) &= \{(\xi_1,\dots,\xi_{j-1}) \in \Gamma_{\mathbf{r}}(\eta) : \ |\xi_{j-1}| > c|\eta|\},
\end{aligned}$$

satisfy $\Gamma_{\mathbf{r}}(\eta) = \cup_{k=1}^{j} A^k_{\mathbf{r}}(\eta)$. As a consequence

$$\|\widetilde{K}_{j,\mathbf{r}}(\widehat{f}_1,\dots,\widehat{f}_j)\|_{L^2_\alpha} \leq \sum_{k=1}^{j}\|\widetilde{K}^k_{j,\mathbf{r}}(\widehat{f}_1,\dots,\widehat{f}_j)\|_{L^2_\alpha}, \tag{3.49}$$

where $\widetilde{K}^k_{j,\mathbf{r}}$ is defined as $\widetilde{K}_{j,\mathbf{r}}$ but integrating over $A^k_{\mathbf{r}}(\eta)$ instead of $\Gamma_{\mathbf{r}}(\eta)$.

From now on we fix

$$\beta := \alpha - (j-1)(1-\lambda). \tag{3.50}$$

In the region where $\chi(\eta)$ does not vanish, $|\eta| \sim \langle\eta\rangle$, and hence

$$\|\widetilde{K}^k_{j,\mathbf{r}}(\widehat{f}_1,\dots,\widehat{f}_j)\|^2_{L^2_\alpha} \leq$$
$$\int_{\mathbb{R}^n} \langle\eta\rangle^{2\beta}|\eta|^{-2(j-1)\lambda}\left(\int_{A^k_{\mathbf{r}}(\eta)} |\widehat{f}_1(\eta-\xi_1)|\left(\prod_{i=1}^{j-2}|\widehat{f}_{i+1}(\xi_i-\xi_{i+1})|\right)|\widehat{f}_j(\xi_{j-1})|\,d\sigma_{\mathbf{r}}\right)^2 d\eta, \tag{3.51}$$

where λ is a real parameter satisfying $0<\lambda\leq(n-1)/2$, and $d\sigma_{\mathbf{r}} = d\sigma_{\mathbf{r}}(\xi_1,\dots,\xi_{j-1})$.

The analysis is exactly the same for each $\widetilde{K}^k_{j,\mathbf{r}}$ $k = 1,\cdots, j$, so we only show one explicitly, for example, the case $k = j$.

If $\beta \geq 0$ we can use that $\langle\eta\rangle^\beta \leq C\langle\xi_{j-1}\rangle^\beta$ in $A^j_{\mathbf{r}}(\eta)$. Hence multiplying and dividing by $|\eta-\xi_1|^{n-1-2\gamma}\prod_{i=1}^{j-2}|\xi_i-\xi_{i+1}|^{n-1-2\gamma}$ and applying Cauchy-Schwarz inequality, we get the following point-wise estimate for the integrand of (3.51),

$$\begin{aligned}
&\langle\eta\rangle^{2\beta}|\eta|^{-2(j-1)\lambda}\left(\int_{A^j_{\mathbf{r}}(\eta)}|\widehat{f}_1(\eta-\xi_1)|\left(\prod_{i=1}^{j-2}|\widehat{f}_{i+1}(\xi_i-\xi_{i+1})|\right)|\widehat{f}_j(\xi_{j-1})|\,d\sigma_{\mathbf{r}}\right)^2\\
\leq&|\eta|^{-2(j-1)\lambda}\int_{\Gamma_{\mathbf{r}}(\eta)}|\widehat{f}_1(\eta-\xi_1)|^2|\eta-\xi_1|^{n-1-2\lambda}\left(\prod_{i=1}^{j-2}|\widehat{f}_{i+1}(\xi_i-\xi_{i+1})|^2|\xi_i-\xi_{i+1}|^{n-1-2\lambda}\right)\\
&\cdots\times|\widehat{f}_j(\xi_{j-1})|^2\langle\xi_{j-1}\rangle^{2\beta}\,d\sigma_{\mathbf{r}}\int_{\Gamma_{\mathbf{r}}(\eta)}\frac{1}{|\eta-\xi_1|^{n-1-2\gamma}}\prod_{i=1}^{j-2}\frac{1}{|\xi_i-\xi_{i+1}|^{n-1-2\gamma}}\,d\sigma_{\mathbf{r}}. \qquad (3.52)
\end{aligned}$$

Now, by Lemma 3.6 we have that

$$|\eta|^{-2(j-1)\lambda}\int_{\Gamma_{\mathbf{r}}(\eta)}\frac{1}{|\eta-\xi_1|^{n-1-2\gamma}}\prod_{i=1}^{j-2}\frac{1}{|\xi_i-\xi_{i+1}|^{n-1-2\gamma}}\,d\sigma_{\mathbf{r}}\leq C\prod_{i=1}^{j-1}r_i^{2\lambda}, \qquad (3.53)$$

where C is some constant independent of η (to see this, always compute first the integral in the variable ξ_i that only appears in one factor, in this case ξ_{j-1}). Hence, using this in (3.52) and integrating in the η variable we get the estimate

$$\|\widetilde{K}^j_{j,\mathbf{r}}(\widehat{f}_1,\ldots,\widehat{f}_j)\|^2_{L^2_\alpha}\leq C\prod_{i=1}^{j-1}r_i^{2\lambda}\int_{\mathbb{R}^n}\int_{\Gamma_{\mathbf{r}}(\eta)}|F(\xi_1,\ldots,\xi_{j-1},\eta)|^2\,d\sigma_{\mathbf{r}}\,d\eta,$$

where

$$\begin{aligned}
F(\xi_1,\ldots,&\xi_{j-1},\eta):=\widehat{f}_1(\eta-\xi_1)\langle\eta-\xi_1\rangle^{(n-1)/2-\lambda}\times\ldots\\
&\left(\prod_{i=1}^{j-2}\widehat{f}_{i+1}(\xi_i-\xi_{i+1})\langle\xi_i-\xi_{i+1}\rangle^{(n-1)/2-\lambda}\right)\widehat{f}_j(\xi_{j-1})\langle\xi_{j-1}\rangle^\beta.
\end{aligned}$$

Therefore, as in Lemma 3.5, we apply the trace theorem to the integrals in $\Gamma_{r_i}(\eta)$, to obtain

$$\begin{aligned}
\|\widetilde{K}^j_{j,\mathbf{r}}(\widehat{f}_1,\ldots,\widehat{f}_j)\|^2_{L^2_\alpha}\leq C&\left(\prod_{i=1}^{j-1}r_i^{2\lambda}\right)\times\ldots\\
&\sum_{0\leq|\alpha_1|,\ldots,|\alpha_{j-1}|\leq 1}\int_{\mathbb{R}^n}\cdots\int_{\mathbb{R}^n}|\partial^{\alpha_1}_{\xi_1}\ldots\partial^{\alpha_{j-1}}_{\xi_{j-1}}F(\xi_1,\ldots,\xi_{j-1},\eta)|^2\,d\xi_1\ldots d\xi_{j-1}\,d\eta,
\end{aligned} \qquad (3.54)$$

where the α_i are multi-indices related to the corresponding variables ξ_i in $\mathbb{R}^n$, see Lemma A.2 in the Appendix for a more detailed formulation. Now, using the Leibniz rule in (3.54) we can put the derivative operators on the functions $\widehat{f}_i\langle\cdot\rangle^a$. In the worst case we are going to get terms of the kind

$$\partial_{\xi_i}^{\alpha_i}\partial_{\xi_{i+1}}^{\alpha_{i+1}}\left(\widehat{f}_{i+1}(\xi_i-\xi_{i+1})\langle\xi_i-\xi_{i+1}\rangle^a\right),$$

with at most two derivative operators having $|\alpha_i| = |\alpha_{i+1}| = 1$. Therefore, if we integrate each summand first in η and then in $\xi_1, \xi_2, \ldots, \xi_{j-1}$, we obtain

$$\|\widetilde{K}^j_{j,\mathbf{r}}(\widehat{f}_1,\ldots,\widehat{f}_j)\|^2_{L^2_\alpha} \le C\left(\prod_{i=1}^{j-1} r_i^{2\lambda}\right)\|f_j\|^2_{W_2^{\beta,2}}\prod_{l=1}^{j-1}\|f_l\|^2_{W_2^{(n-1)/2-\lambda,2}}. \tag{3.55}$$

Putting together (3.55) and the analogous estimates coming from the analysis of the other cases in (3.49), we obtain

$$\|\widetilde{K}_{j,\mathbf{r}}(\widehat{f}_i)\|_{W_2^{\alpha,2}} \le C\left(\prod_{i=1}^{j-1} r_i^{2\lambda}\right)\sum_{i=1}^{j}\|f_i\|_{W_2^{\beta,2}}\prod_{\substack{1\le l\le j\\ l\ne i}}\|f_l\|_{W_2^{(n-1)/2-\gamma,2}}.$$

We reason as in Lemma 3.5, first we impose the extra condition $\lambda < 1$.

As a consequence of Remark 3.7, equation (3.49) follows directly in the range $\beta \ge (n-1)/2$. The restrictions on λ and (3.50) give us the following restrictions on α,

$$\begin{cases}0<\lambda<1\\ 0<\lambda\le\frac{n-1}{2}\end{cases}\iff\begin{cases}\beta+(j-2)<\alpha<\beta+(j-1)\\ \beta+(j-1)-\frac{n-1}{2}\le\alpha<\beta+(j-1).\end{cases} \tag{3.56}$$

We discard the lower bounds for α as in Lemma 3.5.

Otherwise, if β is in the range $\beta < (n-1)/2$, estimate (3.49) holds if we add the extra condition $(n-1)/2-\lambda \le \beta$. Then, since $\lambda < 1$, we must have $\beta > (n-3)/2$ as in Lemma 3.5. Also, (3.50) together with $(n-1)/2-\lambda\le\beta$ imply that $\alpha \le \beta+(j-1)(\beta-(n-3)/2)$ which is a more restrictive condition than $\alpha < \beta+(j-1)$ since we have $\beta < (n-1)/2$. Hence, we have obtained the ranges of parameters given in the statement. □

Proof of Proposition 3.10. By (3.48), estimate (3.37) follows directly when $\mathbf{a} = 0$. Therefore, we consider the case $\mathbf{a} \ne 0$.

Let $\mathbf{r} = (r_1,\ldots,r_k)$, $1 \le k < \infty$. Doing the same computation we did to obtain $\partial_r S_r(q)$ in (3.35) from (3.34), we have that for a general function $F(\xi_1,\ldots,\xi_k,\eta)$, which is C^1 in the first k variables,

$$\begin{aligned}\partial_{r_i}\left(\frac{1}{1+r_i}\int_{\Gamma_{\mathbf{r}}(\eta)} F(\xi_1,...,\xi_k,\eta)\ d\sigma_{\mathbf{r}}(\xi_1,\dots,\xi_k)\right)\\ =\frac{(n-2)r_i+(n-1)}{r_i(1+r_i)^2}\int_{\Gamma_{\mathbf{r}}(\eta)} F(\xi_1,...,\xi_k,\eta)\ d\sigma_{\mathbf{r}}\\ +|\eta|\frac{1}{1+r_i}\int_{\Gamma_{\mathbf{r}}(\eta)}\theta_i\cdot\nabla_{\xi_i}F(\xi_1,...,\xi_k,\eta)\ d\sigma_{\mathbf{r}},\end{aligned}\tag{3.57}$$

where $\theta_i=\frac{\xi_i-\eta/2}{|\xi_i-\eta/2|}$ is a unitary vector. Observe that the coefficients before the integrals are functions of r_i which are bounded for $r_i\in(1-\delta,1+\delta)$ for any $0<\delta<1$ fixed. Hence if we take a derivative $\partial_{\mathbf{r}}^{\mathbf{a}}$ with $\mathbf{a}=(a_1,\dots,a_k)$ and $a_i=0,1$, we have

$$\begin{aligned}&\left|\partial_{\mathbf{r}}^{\mathbf{a}}\left(\left(\prod_{i=1}^{j-1}\frac{1}{1+r_i}\right)\int_{\Gamma_{\mathbf{r}}(\eta)} F(\xi_1,...,\xi_k,\eta)\,d\sigma_{\mathbf{r}}\right)\right|\\ &\le C\left(\prod_{i=1}^{j-1}\frac{1}{1+r_i}\right)\sum_{\substack{0\le|\alpha_i|\le a_i,\\ 1\le i\le k}}|\eta|^{|\alpha_1|+\cdots+|\alpha_k|}\int_{\Gamma_{\mathbf{r}}(\eta)}\left|\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_k}^{\alpha_k}F(\xi_1,...,\xi_k,\eta)\right|\,d\sigma_{\mathbf{r}},\end{aligned}\tag{3.58}$$

where α_i are multi-indices associated to derivatives in $\mathbb{R}^n$, and we have imposed $r_i\in(1-\delta,1+\delta)$ if $a_i=1$ (to bound the coefficients dependent on r_i as we did before). Notice that $|\alpha_1|+\cdots+|\alpha_k|$ can take all the integer values from 0 to $|\mathbf{a}|$. We are interested in computing $\partial_{\mathbf{r}}^{\mathbf{a}}S_{j,\mathbf{r}}$ so we put $k=j-1$ and

$$F(\xi_1,...,\xi_{j-1},\eta)=\widehat{q}(\eta-\xi_1)\left(\prod_{i=1}^{j-2}\widehat{q}(\xi_i-\xi_{i+1})\right)\widehat{q}(\xi_{j-1}).$$

In this case, each potential is going to be derived at most twice, since in the worst case $\widehat{q}$ is valued on the difference of two variables, ξ_i and ξ_{i+1}. Therefore it suffices to give the following rough estimate,

$$\int_{\Gamma_{\mathbf{r}}(\eta)}\left|\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_{j-1}}^{\alpha_{j-1}}F(\xi_1,...,\xi_{j-1},\eta)\right|\,d\sigma_{\mathbf{r}}\le C\sum_{\substack{0\le|\alpha_i'|\le 2\\ 1\le i\le j-1}}K_{j,\mathbf{r}}(\partial^{\alpha_1'}\widehat{q},\partial^{\alpha_2'}\widehat{q},\dots,\partial^{\alpha_{j-1}'}\widehat{q})(\eta),$$

for some new multi-indices $\alpha_1',\dots,\alpha_{j-1}'$. Hence, by (3.6), putting together (3.58) and the previous estimate, we obtain

$$\begin{aligned}|\partial_{\mathbf{r}}^{\mathbf{a}}S_{j,\mathbf{r}}(q)(\eta)|\le C\left(\prod_{i=1}^{j-1}\frac{1}{1+r_i}\right)\frac{1}{|\eta|^{j-1}}\left(1+|\eta|+\cdots+|\eta|^{|\mathbf{a}|}\right)\times\dots\\ \sum_{\substack{0\le|\alpha_i|\le 2\\ 1\le i\le j-1}}K_{j,\mathbf{r}}(\widehat{qx^{\alpha_1}},\widehat{qx^{\alpha_2}},\dots,\widehat{qx^{\alpha_{j-1}}})(\eta).\end{aligned}$$

Then, since $|\alpha_i| \leq 2$, multiplying the previous inequality by $\chi(\eta)$, (3.37) follows form Lemma 3.13 using that $\|x^{\alpha_i} q\|_{W_2^{\beta,2}} \leq C\|q\|_{W_4^{\beta,2}}$, which can be obtained by the the same reasoning given after (3.36). $\square$

Chapter 4

The double dispersion operator in fixed angle scattering

In this chapter we prove the Sobolev estimates of the double dispersion operator of fixed angle scattering.

Proposition 4.1. *Let $n \geq 2$ and $q \in W_2^{\beta,2}(\mathbb{R}^n)$ with $0 \leq \beta < \infty$. Then*

$$\|\widetilde{Q}_{\theta,2}(q)\|_{W^{\alpha,2}} \leq C\|q\|^2_{W_2^{\beta,2}},$$

if the following conditions also hold,

$$\alpha < \begin{cases} 2\beta - (n-3)/2, & if \quad (n-3)/2 < \beta < (n-1)/2, \\ \beta + 1, & if \quad (n-1)/2 \leq \beta < \infty. \end{cases} \tag{4.1}$$

This is the first part of the statement of Theorem 2.15. Since we can prove an analogous of identity (3.3), it is not surprising that the techniques and the results in this section are similar to the ones of backscattering. The main complication is that, when computing $\widehat{Q_{\theta,2}(q)}(\eta)$, by (2.25), we have that $k = k(\eta,\theta) := -|\eta|^2/(\theta\cdot\eta)$, which implies that k is not comparable with the modulus of $|\eta|$. This means that we need to be more careful when estimating the spherical and principal value operators.

4.1 The double dispersion operator

To prove Proposition 4.1 we provide an explicit formula for the Fourier transform of $Q_{\theta,2}(q)$, as we did in the backscattering problem. Let $\zeta \in \mathbb{R}^n$ and $r \in (0,\infty)$. We remind that in (3.2) we defined the the modified Ewald spheres

$$\Gamma_r(\zeta) := \{\xi \in \mathbb{R}^n : |\xi - \zeta/2| = r|\zeta/2|\}.$$

Since in this chapter we always have $\zeta = -2k\theta$, we change slightly the notation, and denote by σ_{rk} their Lebesgue measure.

Proposition 4.2. *Let $\theta \in \mathbb{S}^{n-1}$, and $\eta \in \mathbb{R}^n$. Then we have that*

$$B_{\theta,2}(q)(\eta) = \chi_{H_\theta}(\eta)\left[i\pi S_{\theta,1}(q)(\eta) + P_\theta(q)(\eta)\right], \tag{4.2}$$

where

$$S_{\theta,r}(q)(\eta) := \frac{1}{k(r+1)} \int_{\Gamma_r(-2k\theta)} \widehat{q}(\xi)\widehat{q}(\eta-\xi)\, d\sigma_{rk}(\xi), \tag{4.3}$$

with $k = k(\eta,\theta)$ given by (2.25), $r \in (0,\infty)$, and

$$P_\theta(q)(\eta) := P.V. \int_0^\infty \frac{1}{1-r} S_{\theta,r}(q)(\eta)\, dr. \tag{4.4}$$

Proof. Inserting the formula for the resolvent (3.7) in (2.29) with $j = 2$, and computing the resulting Fourier transform in the y variable, we get that for $\eta \in H_\theta$, and $k = k(\eta,\theta)$, $\theta' = \theta'(\eta,\theta)$,

$$\begin{aligned}
B_{\theta,2}(q)(\eta) &= i\frac{\pi}{2}k^{n-2}\int_{\mathbb{R}^n} e^{-ik\theta'\cdot y}q(y)\int_{S^{n-1}} \widehat{q}(k\omega - k\theta)e^{iky\cdot\omega}\, d\sigma(\omega)\, dy \\
&\quad + \int_{\mathbb{R}^n} e^{-ik\theta'\cdot y}q(y) P.V.\int_{\mathbb{R}^n} e^{iy\cdot\zeta}\frac{\widehat{q}(\zeta-k\theta)}{-|\zeta|^2+k^2}\, d\zeta\, dy \\
&= i\frac{\pi}{2}k^{n-2}\int_{S^{n-1}} \widehat{q}(k\theta' - k\omega)\widehat{q}(k\omega - k\theta)\, d\sigma(\omega) \\
&\quad + P.V.\int_{\mathbb{R}^n} \widehat{q}(k\theta' - \zeta)\frac{\widehat{q}(\zeta-k\theta)}{-|\zeta|^2+k^2}\, d\zeta.
\end{aligned}$$

If we put $\xi = k(\omega - \theta)$ in the first integral and $\xi = \zeta - k\theta$ in the second, then

$$B_{\theta,2}(q)(\eta) = i\pi S_{\theta,1}(q)(\eta) - P.V.\int_{\mathbb{R}^n} \frac{\widehat{q}(\xi)\widehat{q}(\eta-\xi)}{\xi\cdot(\xi+2k\theta)}\, d\xi,$$

when $\eta \in H_\theta$. Now, notice that

$$-\xi\cdot(\xi+2k\theta) = k^2 - |\xi-(-k\theta)|^2 = (k - |\xi-(-k\theta)|)(k + |\xi-(-k\theta)|).$$

We are going to take spherical coordinates with a radial parameter t around the point $-k\theta$ in the principal value integral, denoting by B_t the ball of center $-k\theta$ and radius t, and by σ_t the Lebesgue measure of the sphere ∂B_t. Hence, if we use the change of variables $t = rk$, $r \in (0,\infty)$, in the radial variable, we obtain

$$\begin{aligned}
&- P.V.\int_{\mathbb{R}^n} \frac{\widehat{q}(\xi)\widehat{q}(\eta-\xi)}{\xi\cdot(\xi+2k\theta)}\, d\xi = P.V.\int_0^\infty \frac{1}{(k-t)\,(k+t)}\int_{\partial B_t} \widehat{q}(\xi)\widehat{q}(\eta-\xi)\, d\sigma_t(\xi)\, dt \\
&= P.V.\int_0^\infty \frac{1}{1-r}\frac{1}{k(1+r)}\int_{\Gamma_r(-2k\theta)} \widehat{q}(\xi)\widehat{q}(\eta-\xi)\, d\sigma_{rk}(\xi)\, dr \\
&= P.V.\int_0^\infty \frac{1}{1-r} S_{\theta,r}(q)(\eta)\, dr.
\end{aligned}$$

□

Notice that using the notation of the previous chapter we can write

$$B_{\theta,2}(q)(\eta) = \chi_{H_\theta}(\eta)(i\pi d + P)S_{\theta,r}(q)(\eta),$$

which is analogous to (3.3), and so, by (2.28) we get

$$\widehat{Q_{\theta,2}(q)}(\eta) = \chi_{H_\theta}(\eta)(i\pi d + P)S_{\theta,r}(q)(\eta) + \chi_{H_{-\theta}}(\eta)(i\pi d + P)S_{-\theta,r}(q)(\eta). \quad (4.5)$$

4.2 Estimate of the spherical operator

We begin in this section with estimates for the spherical operator $S_{r,\theta}$. These estimates are uniform on θ, and will be useful in the following section to bound the operator P_θ given in (4.4). To simplify notation we define

$$\widetilde{S}_{\theta,r}(q)(\eta) := \chi(\eta)S_{\theta,r}(q)(\eta), \quad \text{and} \quad \widetilde{P}_\theta(q)(\eta) := \chi(\eta)P_\theta(q)(\eta), \quad (4.6)$$

as we have done previously in the case of backscattering.

Lemma 4.3. *Let $n \geq 2$ and $q \in W_1^{\beta,2}(\mathbb{R}^n)$ with $\beta \geq 0$. Then if $0 \leq \varepsilon < 1$, the estimate*

$$\|k^\varepsilon \widetilde{S}_{\theta,r}(q)\|_{L^2_\alpha} \leq C(1+r)^{-\gamma}\|q\|^2_{W_1^{\beta,2}}, \quad (4.7)$$

holds when

$$\begin{cases} \alpha \leq \beta + (\beta - (n-3)/2) - \varepsilon, & if \quad (n-3)/2 + \varepsilon < \beta < (n-1)/2, \\ \alpha < \beta + 1 - \varepsilon, & if \quad (n-1)/2 \leq \beta < \infty, \end{cases} \quad (4.8)$$

for some real number $\gamma > 0$ (possibly depending on β).

This lemma is very similar to Lemma 3.4, and in fact the proof follows a similar line of reasoning. The main novelty is the presence of the k^ε factor which is necessary for the estimate of the principal value term $P_\theta(q)$. More precisely, we need to absorb a k^ε factor at some point in the proof of Proposition 4.6. In backscattering, this was strightforward, since, in that case, it was an extra $|\eta|^\varepsilon$ factor, which can be immediately absorbed in the Sobolev norm as a loss of an ε of regularity. This is slightly more complicated in fixed angle scattering, where the radius of the Ewald sphere $\Gamma_r(-2k\theta)$ is proportional to k, which is not equivalent to $|\eta|$.

To simplify later computations we define the operator

$$\widetilde{K}_{\theta,r}(g_1, g_2)(\eta) = \chi(\eta)K_{\theta,r}(g_1, g_2)(\eta),$$

where

$$K_{\theta,r}(g_1, g_2)(\eta) := \frac{1}{k}\int_{\Gamma_r(-2k\theta)} |g_1(\xi)||g_2(\eta - \xi)| \, d\sigma_{rk}(\xi).$$

Then we have that

$$\left|\widetilde{S}_{\theta,r}(q)(\eta)\right| \leq \frac{1}{1+r}\widetilde{K}_{\theta,r}(\widehat{q},\widehat{q})(\eta),$$

and therefore the proof of Lemma 4.3 is an immediate consequence of the following lemma taking $\gamma = 1-\lambda$.

Lemma 4.4. *Let $n \geq 2$ and $f_1, f_2 \in W_1^{\beta,2}(\mathbb{R}^n)$ with $\beta \geq 0$. Then if $0 \leq \varepsilon < 1$, the estimate*

$$\|k^\varepsilon \widetilde{K}_{\theta,r}(\widehat{f}_1,\widehat{f}_2)\|_{L^2_\alpha} \leq C(1+r)^\lambda \|f_1\|_{W_1^{\beta,2}}\|f_2\|_{W_1^{\beta,2}}, \tag{4.9}$$

holds when α is in the range given in (4.8), for some real number $0<\lambda<1$ (possibly depending on β).

Proof. Consider a parameter $\varepsilon < \lambda < 1$, for ε in the statement, and observe that k satisfies $|\eta| \leq 2k$. Since $C_0 \geq 1$ in (2.10), we have that $\chi(\eta) = 0$ for $|\eta| \leq 1$. This means that $|\eta|^{-1} \leq 2\langle\eta\rangle^{-1}$ in the region where χ does not vanish. Since $\lambda < 1$, putting these inequalities together we get $k^{\lambda-1} \leq C\langle\eta\rangle^{\lambda-1}$, and this yields

$$\begin{aligned}
&\|k^\varepsilon \widetilde{K}_{\theta,r}(\widehat{f}_1,\widehat{f}_2)\|^2_{L^2_\alpha} \\
&\leq C\int_{\mathbb{R}^n}\langle\eta\rangle^{2\alpha-2+2\lambda}\left(k^{\varepsilon-\lambda}\int_{\Gamma_r(-2k\theta)}|\widehat{f}_1(\xi)||\widehat{f}_2(\eta-\xi)|\,d\sigma_{rk}(\xi)\right)^2 d\eta.
\end{aligned}$$

Now, we ask λ to also satisfy the relation

$$\beta = \alpha - 1 + \lambda. \tag{4.10}$$

We have $\eta = (\eta-\xi)+\xi$, so if we choose any $0<c<1/2$, for every $\xi \in \Gamma_r(\eta)$ at least one of the conditions $|\xi| > c|\eta|$ and $|\eta-\xi| > c|\eta|$ must hold. Since we have assumed that $\beta \geq 0$, in both cases we are led, respectively, to the estimate

$$\|k^\varepsilon \widetilde{K}_{\theta,r}(\widehat{f}_1,\widehat{f}_2)\|^2_{L^2_\alpha} \leq C(I_1+I_2), \quad \text{where}$$

$$\begin{aligned}
I_1 &:= \int_{\mathbb{R}^n}\left(k^{\varepsilon-\lambda}\int_{\Gamma_r(-2k\theta)}|\widehat{f}_1(\xi)|\langle\xi\rangle^\beta|\widehat{f}_2(\eta-\xi)|\,d\sigma_{rk}(\xi)\right)^2 d\eta,\\
I_2 &:= \int_{\mathbb{R}^n}\left(k^{\varepsilon-\lambda}\int_{\Gamma_r(-2k\theta)}|\widehat{f}_1(\xi)||\widehat{f}_2(\eta-\xi)|\langle\eta-\xi\rangle^\beta\,d\sigma_{rk}(\xi)\right)^2 d\eta,
\end{aligned}$$

We study the case of I_1. Multiplying and dividing by $|\eta-\xi|^{(n-1)/2-\lambda+\varepsilon}$, and applying Cauchy-Schwarz's inequality, since $0 \leq \varepsilon < \lambda$ we have

$$\begin{aligned}
I_1 &\leq C\int_{\mathbb{R}^n}\int_{\Gamma_r(-2k\theta)}|\widehat{f}_1(\xi)|^2\langle\xi\rangle^{2\beta}|\widehat{f}_2(\eta-\xi)|^2|\eta-\xi|^{n-1-2(\lambda-\varepsilon)}\,d\sigma_{rk}(\xi)\times\dots\\
&\qquad\qquad\dots\times\int_{\Gamma_r(-2k\theta)}\frac{k^{-2(\lambda-\varepsilon)}}{|\eta-\xi|^{n-1-2(\lambda-\varepsilon)}}\,d\sigma_{rk}(\xi)\,d\eta\\
&\leq Cr^{2(\lambda-\varepsilon)}\int_{\mathbb{R}^n}\int_{\Gamma_r(-2k\theta)}|\widehat{f}_1(\xi)|^2\langle\xi\rangle^{2\beta}|\widehat{f}_2(\eta-\xi)|^2|\eta-\xi|^{n-1-2(\lambda-\varepsilon)}\,d\sigma_{rk}(\xi)\,d\eta,
\end{aligned} \tag{4.11}$$

where we need to impose the condition $\lambda \leq (n-1)/2+\varepsilon$, to apply Lemma 3.6 and to get the last inequality (recall $\Gamma_r(-2k\theta)$ has radius rk). To simplify the integral over the Ewald sphere we are going to use the trace theorem, as in the proof of Lemma 3.5 (see Proposition A.1). This yields

$$\begin{aligned}\int_{\Gamma_r(-2k\theta)} &|\widehat{f}_1(\xi)|^2\langle\xi\rangle^{2\beta}|\widehat{f}_2(\eta-\xi)|^2\langle\eta-\xi\rangle^{n-1-2(\lambda-\varepsilon)}\,d\sigma_{rk}(\xi)\\ &\leq \int_{\mathbb{R}^n}|\widehat{f}_1(\xi)|^2\langle\xi\rangle^{2\beta}|\widehat{f}_2(\eta-\xi)|^2\langle\eta-\xi\rangle^{(n-1)-2(\lambda-\varepsilon)}\,d\xi\\ &+\int_{\mathbb{R}^n}\left|\nabla_\xi\left(\widehat{f}_1(\xi)\langle\xi\rangle^{\beta}\widehat{f}_2(\eta-\xi)\langle\eta-\xi\rangle^{(n-1)/2-(\lambda-\varepsilon)}\right)\right|^2\,d\xi. \end{aligned}\tag{4.12}$$

Therefore, inserting (4.12) in (4.11) and changing the order of integration we get

$$\begin{aligned} I_1 \leq\ & Cr^{2(\lambda-\varepsilon)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\widehat{f}_1(\xi)|^2\langle\xi\rangle^{2\beta}|\widehat{f}_2(\eta-\xi)|^2\langle\eta-\xi\rangle^{(n-1)-2(\lambda-\varepsilon)}d\xi\,d\eta\\ &+Cr^{2(\lambda-\varepsilon)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\left|\nabla\left(\widehat{f}_1(\xi)\langle\xi\rangle^{\beta}\right)\right|^2|\widehat{f}_2(\eta-\xi)|^2\langle\eta-\xi\rangle^{(n-1)-2(\lambda-\varepsilon)}d\xi\,d\eta\\ &+Cr^{2(\lambda-\varepsilon)}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}|\widehat{f}_1(\xi)|^2\langle\xi\rangle^{2\beta}\left|\nabla\left(\widehat{f}_2(\eta-\xi)\langle\eta-\xi\rangle^{(n-1)/2-(\lambda-\varepsilon)}\right)\right|^2d\xi\,d\eta\\ &\leq Cr^{2(\lambda-\varepsilon)}\|f_1\|^2_{W_1^{\beta,2}}\|f_2\|^2_{W_1^{(n-1)/2-(\lambda-\varepsilon),2}}, \end{aligned}\tag{4.13}$$

since by Plancherel theorem we have

$$\int_{\mathbb{R}^n}\left|\nabla(\widehat{f}(\xi)\langle\xi\rangle^t)\right|^2\,d\xi \leq C\|f\|^2_{W_1^{t,2}}.$$

The estimate of I_2 is nearly identical, the only difference is that we multiply and divide by the weight $|\xi|^{(n-1)/2-(\lambda-\varepsilon)}$ in (4.11), so essentially we recover estimate (4.13) but interchanging the roles of f_1 and f_2. Hence we have that

$$\begin{aligned}&\|k^\varepsilon\widetilde{K}_{\theta,r}(\widehat{f}_1,\widehat{f}_2)\|_{L^2_\alpha}\\ &\leq Cr^{(\lambda-\varepsilon)}\left(\|f_1\|_{W_1^{\beta,2}}\|f_2\|_{W_1^{(n-1)/2-(\lambda-\varepsilon),2}}+\|f_2\|_{W_1^{\beta,2}}\|f_1\|_{W_1^{(n-1)/2-(\lambda-\varepsilon),2}}\right).\end{aligned}$$

Now, as a consequence of (4.10) and that $\lambda > \varepsilon$, equation (4.9) will follow directly in the range $\beta \geq (n-1)/2$. But, by the conditions imposed in the proof we have to take into account the restrictions

$$\begin{cases}\varepsilon<\lambda<1\\ \varepsilon<\lambda\leq\frac{n-1}{2}+\varepsilon\end{cases} \iff \begin{cases}\beta<\alpha<\beta+1-\varepsilon\\ \beta+1-\frac{n-1}{2}-\varepsilon\leq\alpha<\beta+1-\varepsilon.\end{cases}$$

We can discard the lower bounds for α using that $\|f\|_{L^2_\alpha} \leq \|f\|_{L^2_{\alpha'}}$ always holds if $\alpha \leq \alpha'$. Therefore we have only the restriction $\alpha < \beta+1-\varepsilon$.

Otherwise, if β is in the range $0 \leq \beta < (n-1)/2$, estimate (4.9) will follow if we add the extra condition

$$(n-1)/2 - (\lambda - \varepsilon) \leq \beta. \tag{4.14}$$

Then, since $\lambda < 1$, we must have $\beta > (n-3)/2 + \varepsilon$ (the other conditions on λ don't add new restrictions). Also (4.10) and (4.14) imply together that $\alpha \leq 2\beta - (n-3)/2 - \varepsilon$, which is a stronger condition than $\alpha < \beta + 1 - \varepsilon$ since we have $\beta < (n-1)/2$. Hence, we have obtained the ranges of parameters given in the statement. □

In the next section we are going to need the following Lipschitz estimate for $\widetilde{S}_{\theta,r}$ which follows from the previous lemma, to bound the principal value operator $\widetilde{P}_\theta$.

Proposition 4.5. *Let $n \geq 2$ and $q \in \mathcal{S}(\mathbb{R}^n)$. Then for any $0 < \delta < 1$ and $r_1, r_2 \in (1-\delta, 1+\delta)$*

$$\|k^{-1}(\widetilde{S}_{\theta,r_1}(q) - \widetilde{S}_{\theta,r_2}(q))\|_{L^2_\alpha} \leq C|r_1 - r_2| \|q\|^2_{W_2^{\beta,2}}, \tag{4.15}$$

holds when α and β satisfy (4.8) with $\varepsilon = 0$.

In general the constant C in the estimate is going to depend on δ, but this has no special relevance. Observe also that the Sobolev space this time is $W_2^{\beta,2}$ instead of $W_1^{\beta,2}$.

Proof. We center the Ewald spheres in the origin with the change $\xi = rk\omega - k\theta$, where $\omega \in \mathbb{S}^{n-1}$,

$$\begin{aligned} S_{\theta,r}(q)(\eta) &:= \frac{1}{k(1+r)} \int_{\Gamma_r(-2k\theta)} \widehat{q}(\xi)\widehat{q}(\eta - \xi)\, d\sigma_{rk}(\xi) \\ &= \frac{2k^{n-2}r^{n-1}}{(1+r)} \int_{\mathbb{S}^{n-1}} \widehat{q}\,(rk\omega - k\theta)\, \widehat{q}\,(\eta - rk\omega + k\theta)\; d\sigma(\omega). \end{aligned}$$

Now we can compute derivatives in the r variable. Consider η fixed, then

$$\begin{aligned} &\frac{d}{dr} S_{\theta,r}(q)(\eta) = \\ &= \frac{k^{n-2}((n-1)r^{n-2}(1+r) - r^{n-1})}{(1+r)^2} \int_{\mathbb{S}^{n-1}} \widehat{q}\,(rk\omega - k\theta)\, \widehat{q}\,(\eta - rk\omega + k\theta)\; d\sigma(\omega) \\ &+ \frac{k^{n-1}r^{n-1}}{(1+r)} \int_{\mathbb{S}^{n-1}} \omega \cdot \nabla \widehat{q}\,(rk\omega - k\theta)\, \widehat{q}\,(\eta - rk\omega + k\theta)\; d\sigma(\omega) \\ &- \frac{k^{n-1}r^{n-1}}{(1+r)} \int_{\mathbb{S}^{n-1}} \widehat{q}\,(rk\omega - k\theta)\, \omega \cdot \nabla \widehat{q}\,(\eta - rk\omega + k\theta)\; d\sigma(\omega), \end{aligned}$$

(notice that $S_{\theta,r}(q)(\eta)$ is a smooth function in the r variable for every $\eta \neq 0$). Hence, fixing some $0 < \delta < 1$, for $r \in (1-\delta, 1+\delta)$, if we undo the change to spherical coordinates we get

$$\left| \frac{d}{dr} S_{\theta,r}(q)(\eta) \right| \le C K_{\theta,r}(\widehat{q},\widehat{q})(\eta) + Ck K_{\theta,r}(|\nabla \widehat{q}|,\widehat{q})(\eta) + Ck K_{\theta,r}(\widehat{q},|\nabla \widehat{q}|)(\eta), \quad (4.16)$$

taking the absolute values inside the integrals. Notice the k factor multiplying the last terms. Now, if $\eta \neq 0$, by the fundamental theorem of calculus we have

$$\begin{aligned} S_{\theta,r_2}(q)(\eta) - S_{\theta,r_1}(q)(\eta) &= \int_{r_1}^{r_2} \frac{d}{dr} S_{\theta,r}(q)(\eta)\, dr \\ &= (r_2 - r_1) \int_0^1 \left[\frac{d}{dr} S_{\theta,r}(q)(\eta) \right]_{r=r(t)} dt, \end{aligned}$$

where for brevity, $r(t) = (r_2 - r_1)t + r_1$. Then by (4.16) we obtain

$$\begin{aligned} &|S_{\theta,r_2}(q)(\eta) - S_{\theta,r_1}(q)(\eta)| \\ &\quad \le C|r_2 - r_1| \int_0^1 \big(K_{\theta,r(t)}(\widehat{q},\widehat{q})(\eta) + k K_{\theta,r(t)}(|\nabla \widehat{q}|,\widehat{q})(\eta) + k K_{\theta,r(t)}(\widehat{q},|\nabla \widehat{q}|)(\eta) \big)\, dt, \end{aligned}$$

so multiplying by $\chi(\eta)$, and applying Minkowski's integral inequality we have

$$\begin{aligned} &\|k^{-1}(\widetilde{S}_{\theta,r_2}(q) - \widetilde{S}_{\theta,r_2}(q))\|_{L^2_\alpha} \le \\ &C|r_2 - r_1| \int_0^1 \Big(\|k^{-1}\widetilde{K}_{\theta,r(t)}(\widehat{q},\widehat{q})\|_{L^2_\alpha} + \|\widetilde{K}_{\theta,r(t)}(|\nabla \widehat{q}|,\widehat{q})\|_{L^2_\alpha} + \|\widetilde{K}_{\theta,r(t)}(\widehat{q},|\nabla \widehat{q}|)\|_{L^2_\alpha} \Big)\, dt. \end{aligned}$$

Then, since $r(t) \in (1-\delta, 1+\delta)$, we can apply Lemma 4.4 with $\varepsilon = 0$ to estimate the first term (using that $k^{-1} \le 2|\eta|^{-1} \le c$ where χ does not vanish). The others follow similarly, observe that

$$K_{\theta,r}(|\nabla \widehat{q}|,\widehat{q}) \le C \sum_{i=1}^n K_{\theta,r}(\partial_i \widehat{q},\widehat{q}) = C \sum_{i=1}^n K_{\theta,r}(\widehat{x_i q},\widehat{q}),$$

so again we can apply Lemma 4.4 and Remark 3.7 to estimate these terms, which yields

$$\|\widetilde{K}_{\theta,r}(\widehat{x_i q},\widehat{q})\|_{L^2_\alpha} \le C\|x_i q\|_{W_1^{\beta,2}} \|q\|_{W_1^{\beta,2}} \le C\|q\|^2_{W_2^{\beta,2}}. \qquad \square$$

4.3 Estimate of the Principal Value Operator

As a consequence of Lemma 4.3 and Proposition 4.5, we obtain the following estimate for the principal value operator $\widetilde{P}_\theta$ introduced in (4.6). We follow closely the proof of Lemma 3.3.

Proposition 4.6. *Let $n \geq 2$ and $q \in W_2^{\beta,2}(\mathbb{R}^n)$ with $\beta \geq 0$. Then the estimate*

$$\|\widetilde{P}_\theta(q)\|_{L^2_\alpha} \leq C\|q\|^2_{W_2^{\beta,2}},$$

holds when α is in the range (4.1).

Proof. By the density argument of Lemma A.3 we might assume $q \in \mathcal{S}(\mathbb{R}^n)$. Let δ be the same of Proposition 4.5, and set $\delta_k := \delta \min(k^{-2}, 1)$. To simplify notation we define the region

$$B_k := \{r \in (0,\infty) : \delta_k \leq |1-r| \leq \delta\},$$

relevant when $k > 1$. By using that $P.V. \int_{|1-r|<a} \frac{dr}{1-r} = 0$ for any $a > 0$, we have

$$\begin{aligned}\widetilde{P}_\theta(q)(\eta) = \int_{|1-r|<\delta_k} \frac{\widetilde{S}_{\theta,r}(q)(\eta) - \widetilde{S}_{\theta,1}(q)(\eta)}{1-r} dr + \int_{B_k} \frac{\widetilde{S}_{\theta,r}(q)(\eta)}{1-r}\, dr \\ + \int_{\delta<|1-r|} \frac{\widetilde{S}_{\theta,r}(q)(\eta)}{1-r}\, dr \\ := P_{\theta,A}(q)(\eta) + P_{\theta,B}(q)(\eta) + P_{\theta,C}(q)(\eta),\end{aligned} \tag{4.17}$$

where, in the first term on the right, the $P.V.$ is not necessary any more since $q \in \mathcal{S}(\mathbb{R}^n)$ implies that $\widetilde{S}_{\theta,r}(q)(\eta)$ is smooth in the r variable, and hence the singularity in the denominator is cancelled by the numerator.

Applying Minkowski's integral inequality and estimate (4.7) with $\varepsilon = 0$, we obtain

$$\|P_{\theta,C}(q)\|_{L^2_\alpha} \leq \int_{\delta<|1-r|} \frac{\|\widetilde{S}_{\theta,r}(q)\|_{L^2_\alpha}}{|1-r|}\, dr \leq C\|q\|^2_{W_1^{\beta,2}}, \tag{4.18}$$

Using Cauchy-Schwarz's inequality in the r variable and estimate (4.15) we obtain

$$\begin{aligned}\|P_{\theta,A}(q)\|^2_{L^2_\alpha} &= \int_{\mathbb{R}^n} \langle\eta\rangle^{2\alpha} \left| \int_{|1-r|<\delta_k} \frac{\widetilde{S}_{\theta,1}(q)(\eta) - \widetilde{S}_{\theta,r}(q)(\eta)}{1-r}\, dr \right|^2 d\eta \\ &\leq 2\delta \int_{\mathbb{R}^n} \langle\eta\rangle^{2\alpha} \int_{|1-r|<\delta_k} \left(k^{-1} \frac{|\widetilde{S}_{\theta,1}(q)(\eta) - \widetilde{S}_{\theta,r}(q)(\eta)|}{|1-r|} \right)^2 dr\, d\eta \\ &\leq 2\delta \int_{|1-r|<\delta} \frac{\|k^{-1}(\widetilde{S}_{\theta,1}(q)(\eta) - \widetilde{S}_{\theta,r}(q)(\eta))\|^2_{L^2_\alpha}}{|1-r|^2}\, dr \leq C\delta^2 \|q\|^4_{W_2^{\beta,2}}.\end{aligned} \tag{4.19}$$

Now, to estimate $P_{\theta,B}(q)(\eta)$, set $N(k) = -\log_2(\delta k^{-2})$, and consider the next dyadic decomposition,

$$P_{\theta,B}(q)(\eta) = \sum_{0\leq j<N(k)} \int_{\{2^{-(j+1)}<|1-r|<2^{-j}\}} \chi_{B_k}(r) \frac{1}{1-r} \widetilde{S}_{\theta,r}(q)(\eta)\, dr,$$

where χ_{B_k} is the characteristic function of B_k. For η fixed, if $0 \le j < N(k)$, the definition of $N(k)$ implies that $2^j \le k^2/\delta$, therefore

$$|P_{\theta,B}(q)(\eta)| \le \sum_{j=0}^{\infty} 2^{j+1}\chi_{(\delta 2^j,\infty)}(k^2) \int_{|1-r|<2^{-j}} |\widetilde{S}_{\theta,r}(q)(\eta)|\,dr, \tag{4.20}$$

where $\chi_{(\delta 2^j,\infty)}$ is again a characteristic function. But observe that in the last line we have a sublinear operator of the kind

$$P^{\lambda}(q)(\eta) := \chi_{(\delta\lambda^{-1},\infty)}(k^2) \int_{|1-r|\le\lambda} |\widetilde{S}_{\theta,r}(q)(\eta)|\,dr,$$

with $0 < \lambda < 1$. Take $\varepsilon > 0$ small. Computing the L^2_α norm of P^λ and applying Minkowski's integral inequality we obtain

$$\|P^{\lambda}(q)\|_{L^2_\alpha} \le C\lambda^{\varepsilon/2} \int_{|1-r|\le\lambda} \|k^{\varepsilon}\widetilde{S}_{\theta,r}(q)\|_{L^2_\alpha}\,dr \le \lambda^{1+\varepsilon/2} C\|q\|^2_{W_1^{\beta,2}}, \tag{4.21}$$

using estimate (4.7), and that in the region where the characteristic function does not vanish we have that $k^{-\varepsilon} \le C\lambda^{\varepsilon/2}$. Hence, taking the L^2_α norm of (4.20) and applying estimate (4.21),

$$\|P_{\theta,B}(q)\|_{L^2_\alpha} \le 2\sum_{j=0}^{\infty} 2^j \|P^{2^{-j}}(q)\|_{L^2_\alpha} \le C\|q\|^2_{W_1^{\beta,2}} \sum_{j=0}^{\infty} 2^{-j\varepsilon/2}, \tag{4.22}$$

and the dyadic sum converges. This estimate holds when α is in the range given by (4.8), so for every α in the range given by (4.1) is possible to choose ε so that (4.22) holds. Therefore since $\widetilde{P}_\theta = P_{\theta,A} + P_{\theta,B} + P_{\theta,C}$ we conclude the proof putting together estimates (4.18), (4.19) and (4.22). □

Proof of Proposition 4.1. By the definition of $\widetilde{P}_\theta$ and $\widetilde{S}_\theta$, if we multiply (4.5) by the smooth cut-off $\chi(\eta)$ we obtain

$$\widehat{\widetilde{Q}_{\theta,2}(q)}(\eta) = \chi_{H_\theta}(\eta)\left(\widetilde{S}_\theta(q)(\eta) + \widetilde{P}_\theta(q)(\eta)\right) + \chi_{H_{-\theta}}(\eta)\left(\widetilde{S}_{-\theta}(q)(\eta) + \widetilde{P}_{-\theta}(q)(\eta)\right).$$

Then the estimate of the spherical operators follows from *Lemma* 4.3 with $\varepsilon = 0$ and $r = 1$, and the estimate of the principal value operators from Proposition 4.6. □

Chapter 5

Counterexamples

In this chapter we construct a family of real, radial and compactly supported functions g_β to obtain an upper bounds for the regularity gain of the Q_2, $Q_{\theta,2}$ and $Q_{F,2}$ operators. This is the essential step to prove Theorem 2.2 and Theorem 2.13, though we leave the proofs of these theorems for the next chapter.

Theorem 5.1. *For every $0 < \beta < \infty$, if $\alpha_0 := \min(\beta+1, 2\beta-(n-4)/2)$, there is a radial, real and compactly supported function g_β satisfying $g_\beta \in W^{\gamma,2}(\mathbb{R}^n)$ if $\gamma < \beta$, and such that*

i) $Q_2(g_\beta) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ only if $\alpha < \alpha_0$.

ii) $Q_{\theta,2}(g_\beta) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ only if $\alpha < \alpha_0$.

iii) $Q_{F,2}(g_\beta) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ only if $\alpha < \alpha_0$.

The key idea behind the proof of Theorem 5.1 is to study the asymptotic behavior of $|\widehat{Q_2(g_\beta)}(\eta)|$ when $|\eta| \to \infty$. This is greatly simplified by the fact we have the explicit formula (3.3). Now, g_β has a real Fourier transform $\widehat{g_\beta}(\xi)$ by construction, so $\widehat{Q_2(g_\beta)}$ has a real part given by the principal value term in (3.3) and an imaginary part given by $\pi S_1(g_\beta)$. As there is no possible cancellation between the real and imaginary parts, we are going to study only the asymptotic behavior of the spherical integral, which has the advantage of having a positive integrand. We have exactly the same situation when considering $Q_{\theta,2}(g_\beta)$ and $Q_{F,2}(g_\beta)$.

See [9, pp. 20] for an explicit radial counterexample in the case $\beta = (1/2)^-$ and $n = 3$, for the double dispersion operator introduced in that paper.

To simplify notation we write $S(q) := S_1(q)$, $S_\theta(q) := S_{\theta,1}(q)$, $\Gamma(\zeta) := \Gamma_1(\zeta)$ and similarly for analogous cases. The main estimates are given by the following two lemmas.

Lemma 5.2 (Backscattering)**.** *Let $\beta > -n/2$ and assume that $q_\beta \in \mathcal{S}'(\mathbb{R}^n)$ satisfies the following conditions,*

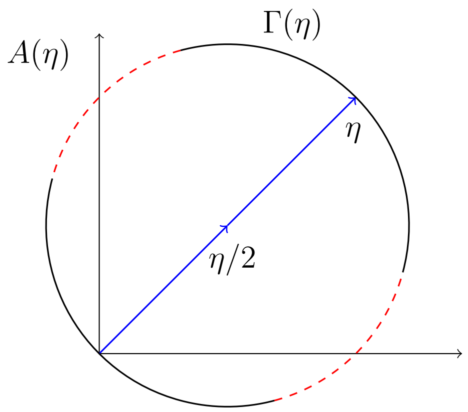

Figure 5.1: The (red) dashed line represent the set $A(\eta) \subset \Gamma(\eta)$.

i) Its Fourier transform $\widehat{q_\beta}(\xi)$ is real and non negative function in all $\mathbb{R}^n$.

ii) There is a constant $c > 0$ such that if $|\xi| > c$, then $\widehat{q_\beta}(\xi) \geq C\langle\xi\rangle^{-n/2-\beta}$.

iii) $\widehat{q_\beta}(\xi)$ is continuous and satisfies $\widehat{q_\beta}(0) > 0$.

Then we have that, if $|\eta| > 4c$,

$$S(q_\beta)(\eta) \geq C \max\left(\langle\eta\rangle^{-\beta-n/2-1}, \langle\eta\rangle^{-2\beta-2}\right).$$

Proof. Since $\widehat{q_\beta}$ is non negative, we have that

$$S(q_\beta)(\eta) \geq \frac{1}{|\eta|}\int_{A(\eta)} \widehat{q_\beta}(\xi)\widehat{q_\beta}(\eta-\xi)\, d\sigma_\eta(\xi), \tag{5.1}$$

where, if we write $\eta = |\eta|\theta$ with θ a unitary vector, $A(\eta) \subset \Gamma(\eta)$ is defined as follows

$$A(\eta) := \{\xi \in \Gamma(\eta) : |(\xi - \eta/2)\cdot\theta| \leq |\eta|/4\}.$$

That is, $A(\eta)$ is a band around the equator orthogonal to η of width proportional to $|\eta|$ (see Figure 5.1). Observe that we have $\xi \in A(\eta)$ if and only if $\eta - \xi \in A(\eta)$, and that in this region $|\xi| \geq |\eta|/4$. Hence, if we consider $|\eta| > 4c$ (where c is given in the statement) and $\xi \in A(\eta)$, we have that $|\xi| > c$ and $|\eta - \xi| > c$, so from (5.1) we get

$$\begin{aligned} S(q_\beta)(\eta) &\geq C\frac{1}{|\eta|}\int_{A(\eta)} \langle\eta-\xi\rangle^{-\beta-n/2}\langle\xi\rangle^{-\beta-n/2}\, d\sigma_\eta(\xi) \\ &\geq C\langle\eta\rangle^{-2\beta-n}|\eta|^{n-2} > C\langle\eta\rangle^{-2\beta-2}, \end{aligned} \tag{5.2}$$

where to get the last line we have used that the measure of $A(\eta)$ is proportional to $|\eta|^{n-1}$, and that $|\xi| \leq |\eta|$ and $|\eta - \xi| \leq |\eta|$ always hold in $\Gamma(\eta)$.

Now, if $\widehat{q_\beta}$ is continuous and $\widehat{q_\beta}(0) > 0$, we can take a ball B_ε around the origin of radius $0 < \varepsilon < c$ such that $\widehat{q_\beta}(\xi)$ is positive in its closure. Also, if $|\eta| > 2c$ and $\xi \in B_\varepsilon \cap \Gamma(\eta)$, we have that $|\eta - \xi| > c$. Using both facts we get

$$\begin{aligned} S(q_\beta)(\eta) &\geq \frac{1}{|\eta|} \int_{B_\varepsilon \cap \Gamma(\eta)} \widehat{q_\beta}(\xi) \widehat{q_\beta}(\eta - \xi)\, d\sigma_\eta(\xi) \\ &\geq C \frac{1}{|\eta|} \int_{B_\varepsilon \cap \Gamma(\eta)} \langle \eta - \xi \rangle^{-\beta - n/2}\, d\sigma_\eta(\xi) \geq C \langle \eta \rangle^{-\beta - n/2 - 1}, \end{aligned} \tag{5.3}$$

where to get the last inequality we have used that $|\eta - \xi| \leq |\eta|$, and that the measure $|B_\varepsilon \cap \Gamma(\eta)|$ is bounded below by a positive constant independent of η (this is because the region $B_\varepsilon \cap \Gamma(\eta)$ approaches, for η large, a flat disc of radius ε). To finish we have just to put together (5.2) and (5.3). □

Similarly, we can prove the following result.

Lemma 5.3 (Fixed angle)**.** *Consider the half cone $D_\theta := \{\eta \in \mathbb{R}^n : \eta \cdot \theta \leq -a|\eta|\}$ for some $0 < a < 1$. Assume also that q satisfies the same conditions stated in Lemma 5.2. Then we have that if $|\eta| > 4c$, there is a constant C independent of η and θ such that*

$$S_\theta(q_\beta)(\eta) \geq C \chi_{D_\theta}(\eta) \max\left(\langle \eta \rangle^{-\beta - n/2 - 1}, \langle \eta \rangle^{-2\beta - 2} \right),$$

where χ_{D_θ} denotes the characteristic function of the cone.

Proof. Observe that if $\eta \cdot \theta \leq -a|\eta|$, by (2.25) we have that $k \sim |\eta|$. Since $\widehat{q_\beta}$ is non negative, we have that

$$S_\theta(q_\beta)(\eta) \geq \frac{1}{2k} \int_{A'(\eta)} \widehat{q_\beta}(\xi) \widehat{q_\beta}(\eta - \xi)\, d\sigma_k(\xi), \tag{5.4}$$

where $A'(\eta) \subset \Gamma(-2k\theta)$ is defined as follows

$$A'(\eta) := \{\xi \in \Gamma(-2k\theta) : |\xi| > c \ \text{ and } \ |\eta - \xi| > c\}.$$

That is, we take the points on the Ewald sphere which are not contained in two balls of radius c centered, respectively, around η and the origin. This implies that the measure of $A'(\eta)$ satisfies $|A'(\eta)| \geq Ck^{n-1}$ for some constant $C > 0$, since $|\eta| > 4c$ implies $k > 2c$. Therefore, by condition *ii*) of q, in (5.4) we get

$$\begin{aligned} S_\theta(q_\beta)(\eta) &\geq C \frac{1}{k} \int_{A'(\eta)} \langle \eta - \xi \rangle^{-\beta - n/2} \langle \xi \rangle^{-\beta - n/2}\, d\sigma_k(\xi) \\ &\geq C \frac{1}{k} \langle \eta \rangle^{-2\beta - n} k^{n-1} \geq C \langle \eta \rangle^{-2\beta - 2}, \end{aligned} \tag{5.5}$$

where we have used that $|\xi| \leq 2k \leq C|\eta|$ and that $|\eta - \xi| \leq |\eta| + 2k \leq C|\eta|$.

Now, if $\widehat{q_\beta}$ is continuous and $\widehat{q_\beta}(0) > 0$, we can take a ball B_ε around the origin of radius $0 < \varepsilon < c$ such that $\widehat{q_\beta}(\xi)$ is positive in its closure. Also, if $|\eta| > 2c$, $\xi \in B_\varepsilon \cap \Gamma(-2k\theta)$ implies $|\eta - \xi| > c$, so

$$\begin{aligned} S_\theta(q_\beta)(\eta) &\geq \frac{1}{k} \int_{B_\varepsilon \cap \Gamma(-2k\theta)} \widehat{q_\beta}(\xi)\widehat{q_\beta}(\eta - \xi)\, d\sigma_k(\xi) \\ &\geq C\frac{1}{k} \int_{B_\varepsilon \cap \Gamma(-2k\theta)} \langle \eta - \xi \rangle^{-\beta - n/2}\, d\sigma_k(\xi) \geq C\langle \eta \rangle^{-\beta - n/2 - 1}, \end{aligned} \tag{5.6}$$

using again that $|\eta - \xi| \leq C|\eta|$ and that the measure $|B_\varepsilon \cap \Gamma(-2k\theta)|$ is bounded below by a positive constant independent of η (the region $B_\varepsilon \cap \Gamma(-2k\theta)$ approaches for η large a flat disc of radius ε, as in the previous lemma). To finish we have just to put together (5.5) and (5.6). □

We now construct the family of functions g_β.

Proposition 5.4. *For every $0 < \beta < \infty$ there is a radial, real and compactly supported function $g_\beta \in W^{\gamma,2}(\mathbb{R}^n)$ for every $\gamma < \beta$, such that $\widehat{g_\beta}$ is non negative in $\mathbb{R}^n$, $\widehat{g_\beta}(0) > 0$, and for some $c > 0$ we have that*

$$\widehat{g_\beta}(\xi) \geq C\langle \xi \rangle^{-n/2-\beta} \quad \text{if} \quad |\xi| > c. \tag{5.7}$$

Proof. We introduce the functions $G_\beta(x)$ given by the relation

$$\widehat{G_\beta}(\xi) := \frac{1}{\langle \xi \rangle^{n/2+\beta}}.$$

These functions are, up to normalizing factors, kernels of Bessel potential operators. We observe that the Fourier transform of a radial and real function in $\mathbb{R}^n$ is also radial and real. As a consequence, the G_β functions satisfy the statement of the proposition except for the condition of compact support.

The regularity properties of the G_β function are determined by its behavior when $|x| \to 0$. Far from the origin $G_\beta(x)$ is smooth with exponential decay (see, for example, chapter V of [52]). This motivates us to choose $g_\beta = \phi G_\beta$ where, ϕ is any $C_c^\infty(\mathbb{R}^n)$ function non-vanishing at the origin. Then clearly we have $g_\beta \in W^{\gamma,2}(\mathbb{R}^n)$ for every $\gamma < \beta$, as desired.

The rest of the properties of g_β follow if we choose ϕ in the following way. Consider again a function $\psi \in C_c^\infty(\mathbb{R}^n)$ radial and real such that $\widehat{\psi}(0) \neq 0$ and put $\phi = \psi * \psi$. Then ϕ is going to be compactly supported, radial, real and non-zero in the origin. Moreover its Fourier transform satisfies $\widehat{\phi}(\xi) = \widehat{\psi}(\xi)^2 \geq 0$ for all $\xi \in \mathbb{R}^n$ and also that $\widehat{\phi}(0) > 0$.

Using this we get that $\widehat{g_\beta}(\xi) = \widehat{\phi} * \widehat{G_\beta}(\xi)$ is non-negative. Also, since $\widehat{\phi}(0) > 0$, there is an $\varepsilon > 0$ such that $\widehat{\phi}(\xi)$ is bounded below in $B_\varepsilon = \{\xi \in \mathbb{R}^n : |\xi| < \varepsilon\}$. This yields (5.7) since we have

$$\widehat{g_\beta}(\xi) = \int_{\mathbb{R}^n} \widehat{G_\beta}(\xi-\zeta)\widehat{\phi}(\zeta)\,d\zeta \geq \int_{B_\varepsilon} \langle \xi-\zeta\rangle^{-n/2-\beta}\widehat{\phi}(\zeta)\,d\zeta$$
$$\geq C\int_{B_\varepsilon} \langle \xi-\zeta\rangle^{-n/2-\beta}\,d\zeta \geq C\langle |\xi|+\varepsilon\rangle^{-n/2-\beta} \geq C\langle \xi\rangle^{-n/2-\beta},$$

for $|\xi| > \varepsilon$. To finish the proof we only have to verify that $\widehat{g_\beta}(0) > 0$. But this is immediate,

$$\widehat{g_\beta}(0) = \int_{\mathbb{R}^n} \widehat{G_\beta}(-\xi)\widehat{\phi}(\xi)\,d\xi > 0,$$

since $\widehat{G_\beta}(\xi) > 0$ and $\widehat{\phi}(\xi) \geq 0$ for every $\xi \in \mathbb{R}^n$. □

We can now prove Theorem 5.1, with the help of the following simple result.

Lemma 5.5. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$ be such that $\widehat{f}$ is a non negative measurable function. Consider an open set E such that for all $x \in E$ and $\lambda > 0$, $\lambda x \in E$ (a conical open set). Assume also that in E, for some $c > 0$, $\gamma \in \mathbb{R}$ and $|\eta| > c$ we have $\widehat{f}(\eta) \geq C\langle\eta\rangle^{-n/2-\gamma}$. Then we have that $f \notin W^{\alpha,2}_{loc}(\mathbb{R}^n)$ for every $\alpha \geq \gamma$.*

Proof. In the proof of Proposition 5.4 we have seen that we can take a function $\phi \in C^\infty_c(\mathbb{R}^n)$ such that $\widehat{\phi}(\xi) \geq 0$ in $\mathbb{R}^n$ and $\widehat{\phi}(0) > 0$. Then we can choose an $0 < \varepsilon < c$ small so that $\widehat{\phi}(\xi)$ is bounded below by a positive constant when $\xi \in B_\varepsilon$. Then, if $E_\varepsilon := \{\eta \in E : (\eta-\xi) \in E, \forall \xi \in B_\varepsilon\}$ and we ask $|\eta| \geq 2c$ and $\eta \in E_\varepsilon$, we obtain that

$$\widehat{\phi f}(\eta) = \int_{\mathbb{R}^n} \widehat{\phi}(\xi)\widehat{f}(\eta-\xi)\,d\xi$$
$$\geq \int_{B_\varepsilon} \widehat{\phi}(\xi)\widehat{f}(\eta-\xi)\,d\xi \geq C\langle\eta\rangle^{-n/2-\gamma}.$$

As a consequence we have that $\phi f \notin W^{\alpha,2}(\mathbb{R}^n)$ for $\alpha \geq \gamma$, which implies that $f \notin W^{\alpha,2}_{loc}(\mathbb{R}^n)$ by definition of the local Sobolev spaces. □

Proof of Theorem 5.1. Let's prove *i*). By Proposition 5.4 the function g_β satisfies all the conditions necessary to apply Lemma 5.2, and hence for $|\eta|$ large we have

$$S(g_\beta)(\eta) \geq C\max\left(\langle\eta\rangle^{-\beta-n/2-1}, \langle\eta\rangle^{-2\beta-2}\right). \tag{5.8}$$

By (3.3), we have that

$$\widehat{Q_2(g_\beta)}(\eta) = P(S_r(g_\beta))(\eta) + i\pi S(g_\beta)(\eta), \tag{5.9}$$

and $\widehat{g}_\beta$ is real, so $P(S_r(g_\beta))$ and $S(g_\beta)$ also are real functions of η. This means that if we assume $Q_2(g_\beta) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$, we must have $\mathcal{F}^{-1}(S(g_\beta)) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$, since there are no possible cancellations between the real and imaginary parts in (5.9). Then, as a consequence of (5.8), applying Lemma 5.5 with $f = \mathcal{F}^{-1}(S(g_\beta))$ and $E = \mathbb{R}^n$ we obtain that α must satisfy simultaneously $\alpha < \beta + 1$ and $\alpha < 2\beta + (n-4)/2$. Hence, we have shown that for every $0 < \beta < \infty$ there is a radial, real and compactly supported function g_β such that $g_\beta \in W^{\gamma,2}(\mathbb{R}^n)$ if and only if $\gamma < \beta$, but we have that $Q_2(g_\beta) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ only if $\alpha < \min(\beta+1, 2\beta-(n-4)/2)$.

We now prove *ii*). In this case, by Lemma 5.3 we have for $|\eta|$ large that

$$S_\theta(g_\beta)(\eta) \geq C\chi_{D_\theta}(\eta) \max\left(\langle\eta\rangle^{-\beta-n/2-1}, \langle\eta\rangle^{-2\beta-2}\right). \tag{5.10}$$

Also, since $D_\theta \subset H_\theta$, by (4.5) we have

$$\chi_{D_\theta}(\eta)\widehat{Q_{\theta,2}(q)}(\eta) = i\pi\chi_{D_\theta}(\eta)S_\theta(q)(\eta) + \chi_{D_\theta}(\eta)P_\theta(q)(\eta).$$

As we mentioned in the case of backscattering, since g_β is real, there are no cancellations possible between $P_\theta(g_\beta)$ and $i\pi S_\theta(g_\beta)$. Hence if we assume that $Q_{\theta,2}(g_\beta) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$, it implies that $\mathcal{F}^{-1}(S_\theta(g_\beta)) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$. As a consequence, by (5.10), applying Lemma 5.5 with $E = D_\theta$ and $f = \mathcal{F}^{-1}(S_\theta(g_\beta))$ we obtain that α must satisfy $\alpha < \beta + 1$ and $\alpha < 2\beta + (n-4)/2$ simultaneously.

To prove *iii*) observe that taking the imaginary part of (2.35), we have

$$\Im(\widehat{Q_{F,2}(g_\beta)})(\eta) = \frac{2\pi}{|\mathbb{S}^{n-1}|}\int_{\{\theta\in S^{n-1}:\, \eta\cdot\theta<0\}} S_\theta(g_\beta)(\eta)\, d\sigma(\theta).$$

Therefore, since the integrand is positive for every θ, if we consider an η fixed satisfying $|\eta| > 4c$, we can restrict the integral to the subset of points $\theta \in \mathbb{S}^{n-1}$ such that $\eta \in D_\theta$. Then we obtain

$$\begin{aligned}\Im(\widehat{Q_{F,2}(g_\beta)})(\eta) &\geq \frac{2\pi}{|\mathbb{S}^{n-1}|}\int_{\{\theta\in S^{n-1}:\ \eta\cdot\theta<-a|\eta|\}} S_\theta(g_\beta)(\eta)\, d\sigma(\theta) \\ &\geq C\max\left(\langle\eta\rangle^{-\beta-n/2-1}, \langle\eta\rangle^{-2\beta-2}\right),\end{aligned}$$

where the last line follows from (5.10). From this estimate, reasoning as in the proof of *i*) and *ii*) we get the desired result. □

Proof of Theorem 2.5. It follows immediately from point *i*) of Theorem 5.1. □

Proof of Theorem 2.15. The first part of the statement follows directly from Proposition 4.1. Condition (2.37) is an immediate consequence of Theorem 5.1. □

In the same spirit of Theorem 5.1 we have the following result which significance will be clear in Chapter 7

Proposition 5.6. *For every $(n-2)/2 < \beta < \infty$, we have that $\widehat{Q_2(g_\beta)} \in L^1_\alpha(\mathbb{R}^n)$ only if $\alpha < \beta - (n-2)/2$.*

Proof. (5.8) implies that $S(g_\beta) \notin L^1_\alpha(\mathbb{R}^n)$ for $\alpha > \beta - (n-2)/2$. Then it follows, reasoning as in the proof of Theorem 5.1, that also $\widehat{Q_2(g_\beta)} \notin L^1_\alpha(\mathbb{R}^n)$. □

Chapter 6

The Born series expansion in Sobolev spaces

In this chapter we are going to study the high frequency Born series in $W^{\alpha,2}(\mathbb{R}^n)$, in backscattering and in fixed angle scattering. The main results are Lemma 6.1 and Lemma 6.11 in which we prove, using the Kenig-Ruiz-Sogge estimates for the resolvent R_k, that the remainder term of the series can be as regular as desired. As a consequence of this result we can finally prove Theorem 2.2, Theorem 2.3 and Theorem 2.13. Apart from this, we apply finer estimates of the resolvent from [46] to control $\widetilde{Q}_j(q)$, and $\widetilde{Q}_{\theta,j}(q)$ in $W^{\alpha,2}(\mathbb{R}^n)$ (see Proposition 6.6 and Proposition 6.13). In the case of backscattering we have already estimated $\widetilde{Q}_j(q)$ in Theorem 2.4, but, in certain instances, the estimates provided in this chapter are better in the low regularity case (that is, when $q \in W^{\beta,2}(\mathbb{R}^n)$ with β near $\max(0,(n-4)/2)$). We will comment more about this later on. In the case of fixed angle scattering, Proposition 6.13 is essential to prove Theorem 2.14, since until now we only have studied the double dispersion operator in in Chapter 4, and we don't have an equivalent result to Theorem 2.4.

The constant C_0 introduced just before (2.10), and used in (2.11) and (2.31) in the definition of the high frequency part of the remainder terms, plays a special role in this chapter. To estimate $\widetilde{Q}_j^R(q)$, we need to choose $C_0 > 2k_0$, where k_0 is large enough, depending on some L^p norm of q with $p > n/2$. This is required since, in order to control the behavior of the scattering solution $u_s(k,\theta,x)$ that appears in the definition of the remainder terms, we need to impose $k \geq k_0$ (see Lemma 6.5). Therefore, from now on we assume in this chapter.

$$C_0 \geq C(n,p)\|q\|_{L^p}^{1/(2-n/p)}, \quad \text{with} \quad p > n/2, \tag{6.1}$$

for an enough large constant $C(n,p) > 0$. From now on we will omit the dependence on n and p in all the constants that appear in this chapter. We recall that C_0 was used in (2.10) to define the cut-off χ.

6.1 The case of backscattering

6.1.1 Proofs of theorems 2.2 and 2.3

Since we have already studied the Q_j operators in Chapter 3, by (2.12), to prove the main results of recovery of singularities, we analyze the behavior of the remainder term $\widetilde{Q}_j^R(q)$. We first state the main result and then we use it to prove Theorem 2.2 and Theorem 2.3. The proof of the estimate of the remainder term will be left for the next section.

Lemma 6.1. *Assume that q is compactly supported, and that $q \in L^p(\mathbb{R}^n)$ for some $p > n/2$. Let $\alpha \in \mathbb{R}$. Then, if we take $C_0 \geq C\|q\|_{L^p}^{1/(2-n/p)}$ with C large enough, we have that $\widetilde{Q}_j^R(q) \in W^{\alpha,2}(\mathbb{R}^n)$ for every $j \geq \frac{n/2+\alpha}{2-n/p}$.*

This is the main lemma necessary to prove, together with the individual estimates for the $\widetilde{Q}_j$ operators, the results of recovery of singularities in backscattering. But we can prove also the following result for the remainder term, which implies that the (tail) of the high frequency Born series converges in $W^{\alpha,2}(\mathbb{R}^n)$.

Proposition 6.2. *Assume that q is compactly supported in B_ρ, the ball of radius ρ, and that $q \in L^p(\mathbb{R}^n)$ for some $p > n/2$. Let $\alpha \in \mathbb{R}$. Then if we take $C_0 \geq C(\rho)\|q\|_{L^p}^{2/(2-n/p)}$ with $C(\rho)$ large enough, and $j \geq \frac{n/2+\alpha}{2-n/p}$, we have that*

$$\lim_{j\to\infty} \|\widetilde{Q}_j^R(q)\|_{W^{\alpha,2}} = 0. \tag{6.2}$$

From Lemma 6.1 we get the following result.

Lemma 6.3. *Assume that the conditions on q and C_0 stated in Lemma 6.1 hold. Assume also that for some $\alpha \geq 0$ and every $j \geq 2$, we know that $\widetilde{Q}_j(q) \in W^{\alpha,2}(\mathbb{R}^n)$. Then we have that $q - q_B \in W^{\alpha,2}(\mathbb{R}^n)$ modulo a C^∞ function.*

Proof. By Lemma 6.1 there is an $\ell \geq 2$ such that $\widetilde{Q}_\ell^R(q) \in W^{\alpha,2}(\mathbb{R}^n)$, and hence the result follows immediately since, modulo a C^∞ function, we have that

$$q - q_B = \sum_{j=2}^{\ell} \widetilde{Q}_j(q) + \widetilde{Q}_\ell^R(q). \tag{6.3}$$

□

Using this lemma we can reduce the proof of Theorem 2.3 to Theorem 2.4, which was proved in Chapter 3.

Proof of Theorem 2.3. By the Sobolev embedding, we have that $q \in L^p(\mathbb{R}^n)$ for some $p > n/2$, since we have that $q \in W^{\beta,2}(\mathbb{R}^n)$ with $\beta \geq 0$, and $\beta > (n-3)/2$. (In fact, as we have mentioned previously, it is enough to have $\beta > (n-4)/2$ instead of the latter condition.) On the other hand, by Theorem 2.4, if $\beta \geq 0$ and $j \geq 2$, we have that $\widetilde{Q}_j(q) \in W^{\alpha,2}(\mathbb{R}^n)$, with

$$\alpha < \begin{cases} 2\beta - (n-3)/2, & if \quad (n-3)/2 < \beta < (n-1)/2, \\ \beta + 1, & if \quad (n-1)/2 \leq \beta < \infty. \end{cases}$$

Therefore we can apply Lemma 6.3 which yields the desired result. □

Similarly we can prove some of the results of recovery of singularities given in Section 2.3.2.

Proof of Corollary 2.10. By Theorem 2.4 and Theorem 2.9, if $\beta \geq 0$, for q radial we have that $\widetilde{Q}_j(q) \in W^{\alpha,2}(\mathbb{R}^n)$ for every $j \geq 2$ with

$$\alpha < \begin{cases} \beta + 2(\beta - (n-3)/2)), & if \quad (n-3)/2 < \beta < (n-1)/2, \\ \beta + 1, & if \quad (n-1)/2 \leq \beta < \infty. \end{cases}$$

On the other hand, as we explained in the previous proof, in this range of β we have that $q \in L^p(\mathbb{R}^n)$ for some $p > n/2$. Hence the result follows applying Lemma 6.3. □

Proof of Theorem 2.8. As in the previous proofs, since $\beta > (n-2)/2$, we have that $q \in L^p$ for some $\beta > n/2$ by the Sobolev embedding. Let $\ell \geq 3$. By (6.3) we have, modulo a C^∞ function, that

$$q - q_B = Q_2(q) + \sum_{j=3}^{\ell} \widetilde{Q}_j(q) + \widetilde{Q}_\ell^R(q). \qquad \square$$

First, by Lemma 6.1 we can choose ℓ such that $\widetilde{Q}_\ell^R(q) \in W^{\beta+1,2}(\mathbb{R}^n)$. Then, since $\beta > (n-2)/2$, by Theorem 2.4 we have that $\sum_{j=3}^{\ell} \widetilde{Q}_j(q) \in W^{\beta+1,2}(\mathbb{R}^n)$ (this is straightforward using (2.15)). Finally, by Theorem 2.7 we have that $Q_2(q) \in \Lambda^\alpha(\mathbb{R}^n)$ for all $\alpha < \beta - (n-2)/2$. Therefore we must also have that $q - q_B \in \Lambda^\alpha(\mathbb{R}^n)$, (modulo a smooth function) since the Morrey-Sobolev inequality (2.21) implies that $W^{\beta+1,2}(\mathbb{R}^n) \subset \Lambda^\alpha(\mathbb{R}^n)$ for $\alpha < \beta - (n-2)/2$.

Also, thanks to Lemma 6.1, the proof of Theorem 2.2 can be reduced to Theorem 2.5, which has been proved in the previous chapter.

Proof of Theorem 2.2. Take $\alpha \geq 0$ and assume that we have that $q - q_B \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ for every compactly supported, real and radial potential $q \in W^{\beta,2}(\mathbb{R}^n)$. We are going to prove that then necessarily $\widetilde{Q}_j(q) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ for all $j \geq 2$.

We denote by $q_B(\lambda)$ the Born approximation of the potential $q(\lambda) = \lambda q$, where $\lambda \in (0,1)$. By the multilinearity of the $\widetilde{Q}_j$ operators, the Born series (2.12) for $q(\lambda)$ becomes

$$\lambda q - q_B(\lambda) = -\sum_{j=2}^{\ell} \lambda^j \widetilde{Q}_j(q) + \widetilde{Q}_\ell^R(\lambda q), \tag{6.4}$$

modulo a C^∞ function (which also depends on λ).

Since by assumption the potential is compactly supported, and satisfies $q \in W^{\beta,2}(\mathbb{R}^n)$ for some $\beta > \max(0,(n-4)/2)$, by the Sobolev inequality we have that $q \in L^p(\mathbb{R}^n)$ for some $p > n/2$. Then, according to Lemma 6.1, we can choose $\ell \geq 2$ such that $\widetilde{Q}_\ell^R(\lambda q) \in W^{\alpha,2}(\mathbb{R}^n)$. Notice also that ℓ does not depend on q. Since by hypothesis $\lambda q - q_B(\lambda) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$, for every $\lambda \in (0,1)$ we have that

$$\sum_{j=2}^{\ell} \lambda^j \widetilde{Q}_j(q) \in W^{\alpha,2}_{loc}(\mathbb{R}^n).$$

But, by choosing $\lambda_i \in (0,1)$ for every $2 \leq i \leq \ell$ such that $\det(\lambda_i^j) \neq 0$ (this is a Vandermonde determinant), we obtain that, for all $2 \leq j \leq \ell$, $\widetilde{Q}_j(q) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$. But, by condition (2.19) of Theorem 2.5, we know that this implies that α must be in the range given by (2.13). □

6.1.2 Estimate of the remainder term

To estimate the remainder term (Lemma 6.1), we need a couple of lemmas. We first introduce some well known estimates of the resolvent R_k. We assume always that $1 < r, r' < \infty$ are Hölder conjugate exponents, that is $\frac{1}{r'} + \frac{1}{r} = 1$.

Lemma 6.4. *Ler r such that either $\frac{2}{n+1} \leq \frac{1}{r'} - \frac{1}{r} \leq \frac{2}{n}$ and $n > 2$ or $\frac{2}{3} \leq \frac{1}{r'} - \frac{1}{r} < 1$ and $n = 2$. Then*

$$\|R_k(f)\|_{L^r} \leq C k^{n\left(\frac{1}{r'}-\frac{1}{r}\right)-2} \|f\|_{L^{r'}}.$$

It follows essentially from [23], with some special care in the case $n = 2$. See [46, Lemma 3.1] for more detailed comments about to the proof of this lemma.

Using Lemma 6.5 we can show the following result of existence and uniqueness of the scattering solutions $u_s(k,\theta,x)$ which satisfy (2.1).

Lemma 6.5. *Let $q \in L^p(\mathbb{R}^n)$ with $p > n/2$ be compactly supported in B_ρ, and let $r = 2p/(p-1)$. Then there is a solution $u_s(k,\theta,x)$ of (2.1) in $L^r(\mathbb{R}^n)$, and it satisfies the estimate*

$$\|u_s(k,\theta,\cdot)\|_{L^r} \leq C(\rho) k^{n/p-2} \|q\|_{L^p}, \tag{6.5}$$

for every $k \geq k_0$, where $k_0 := C\|q\|_{L^p}^{1/(2-n/p)}$, for some large constant C.

Proof. As we explained in Section 2.2, if u_s is solution of (2.1), we have (formally) that

$$u_s = (1 - T_k)^{-1}(R_k(q e^{ik\theta\cdot(\cdot)})), \tag{6.6}$$

where $T_k(f) = R_k(qf)$. As we are going to see now, the previous formula can be made rigorous for k large enough (we cannot use Fredholm theory, since q can be complex valued). By Lemma 6.4 and Hölder inequality, the operator T_k satisfies the estimate

$$\|T_k(f)\|_{L^r} \leq Ck^{n/p-2}\|qf\|_{L^{r'}} \leq Ck^{n/p-2}\|q\|_{L^p}\|f\|_{L^r}, \tag{6.7}$$

since $r' = 2p/(p+1)$ and $\frac{1}{r'} - \frac{1}{r} = \frac{1}{p}$. The condition $\frac{2}{n+1} \leq \frac{1}{r'} - \frac{1}{r} \leq \frac{2}{n}$ necessary to apply Lemma 6.4 implies that we need $n/2 \leq p \leq (n+1)/2$, but we can discard the upper bound using that q is compactly supported (or, alternatively, by interpolation, if the condition $q \in L^2(\mathbb{R}^n)$ is added to the statement). By assumption $n/p - 2 < 0$, and therefore, if we take $k \geq k_0$ where k_0 satisfies $\|T_k\| \leq Ck_0^{n/p-2}\|q\|_{L^p} < 1/2$, we can expand the operator $(1-T_k)^{-1}$ in (6.6) in a Neumann series. Moreover, the operator norm of $(1-T_k)^{-1}$ will be bounded by 2. Using this in (6.6) together with Lemma 6.4, we get

$$\|u_s(k,\theta,\cdot)\|_{L^r} \leq \|R_k(qe^{ik\theta\cdot(\cdot)})\|_{L^r} \leq Ck^{n/p-2}\|q\|_{L^{r'}}.$$

To finish, is enough to use that, by Hölder's inequality,

$$\|q\|_{L^{r'}} \leq C(\rho)\|q\|_{L^p},$$

since q has compact support in B_ρ, and, by definition, $r' \leq p$ always. □

Proofs of Lemma 6.1 and Proposition 6.2. By (2.9) and (2.11), for $\xi = -2k\theta$ we have that

$$\widehat{\widetilde{Q}_j^R(q)}(\xi) = \chi(\xi)\int_{\mathbb{R}^n} e^{ik\theta\cdot y}(qR_k)^{j-1}(q(\cdot)u_s(k,\theta,\cdot))(y)\,dy.$$

Then, since $\chi(\xi)$ vanishes for $|\xi| \leq C_0$,

$$\begin{aligned}\|\widetilde{Q}_j^R(q)\|^2_{W^{\alpha,2}} &\leq \int_{C_0}^{\infty} k^{n-1+2\alpha}\int_{\mathbb{S}^{n-1}}\left|\int_{\mathbb{R}^n} e^{ik\theta\cdot y}(qR_k)^{j-1}(q(\cdot)u_s(k,\theta,\cdot))(y)\,dy\right|^2 d\sigma(\theta)\,dk \\ &\leq \int_{C_0}^{\infty} k^{n-1+2\alpha}\int_{\mathbb{S}^{n-1}}\|(qR_k)^{j-1}(q(\cdot)u_s(k,\theta,\cdot))\|^2_{L^1}\,d\sigma(\theta)\,dk. \end{aligned} \tag{6.8}$$

Since q is compactly supported we might assume without loss of generality that $n/2 < p \leq (n+1)/2$. As in the previous proof, we now take $r = 2p/(p-1)$ and $r' = 2p/(p+1)$. These numbers satisfy $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{r'} - \frac{1}{r} = \frac{1}{p}$. Applying Hölder's inequality, since q has compact support, yields

$$\|(qR_k)^{j-1}(q(\cdot)u_s(k,\theta,\cdot))\|_{L^1} \leq C(\rho)\|(qR_k)^{j-1}(q(\cdot)u_s(k,\theta,\cdot))\|_{L^{r'}}. \tag{6.9}$$

Combining Hölder's inequality and Lemma 6.4 we have

$$\|(qR_k)(f)\|_{L^{r'}} \leq Ck^{n/p-2}\|q\|_{L^p}\|f\|_{L^{r'}}.$$

Using this estimate $j-1$ times in (6.9) and Hölder's inequality, gives

$$\begin{aligned}\|(qR_k)^{j-1}(q(\cdot)u_s(k,\theta,\cdot))\|_{L^1} &\leq C(\rho)k^{(j-1)(n/p-2)}\|q\|^{j-1}_{L^p}\|q(\cdot)u_s(k,\theta,\cdot)\|_{L^{r'}} \\ &\leq C(\rho)k^{(j-1)(n/p-2)}\|q\|^{j}_{L^p}\|u_s(k,\theta,\cdot)\|_{L^r}.\end{aligned}$$

Hence, assuming that $C_0 \geq C\|q\|_{L^p}^{1/(2-n/p)}$, we can use Lemma 6.5 to obtain

$$\|(qR_k)^{j-1}(q(\cdot)u_s(k,\theta,\cdot))\|_{L^1} \leq C(\rho)k^{j(n/p-2)}\|q\|_{L^p}^{j+1}. \tag{6.10}$$

Then, inserting the previous inequality in (6.8) yields

$$\|\widetilde{Q}_j^R(q)\|_{W^{\alpha,2}}^2 \leq C(\rho)\|q\|_{L^p}^{2(j+1)} \int_{C_0}^{\infty} k^{n-1+2\alpha+2j(n/p-2)}\,dk.$$

Since $n/p - 2 < 0$, the previous integral is finite for all $j > \frac{n/2+\alpha}{2-n/p}$, which proves Lemma 6.1. To prove Proposition 6.2 observe that the previous inequality yields

$$\|\widetilde{Q}_j^R(q)\|_{W^{\alpha,2}}^2 \leq C(\rho)\|q\|_{L^p}^{2(j+1)} \frac{C_0^{-2j(2-n/p)+n+2\alpha}}{2j(2-n/p)-n-2\alpha} \leq C(\rho)\|q\|_{L^p}^{2(j+1)} C_0^{-2j(2-n/p)+n+2\alpha},$$

if we consider this time $j > \frac{n/2+\alpha+1/2}{2-n/p}$. Then, the right hand side goes to zero as $j \to \infty$, if take $C_0 \geq C(\rho)\|q\|_{L^p}^{\lambda}$ for any $\lambda > (2-n/p)^{-1}$. For simplicity, in the statement we have chosen $\lambda = 2(2-n/p)^{-1}$. □

6.1.3 Implicit estimates of the multiple dispersion operators

In this section, we give alternative estimates for $Q_j(q)$ which are different from the results given by Theorem 2.4. In general these estimates are worse in terms of regularity for high β, but yield some interesting results in the range $(n-4)/2 < \beta < (n-2)/2$, specially when the dimension is low.

In the proof, we follow the method developed for fixed angle scattering in [46], and for backscattering in [48]. It has also been adapted to the elasticity setting in [4] and [5]. As in the mentioned works, we begin by giving some estimates of the resolvent of the Laplacian, based on different interpolation results between Agmon-Hörmader estimates and the Kenig-Ruiz-Sogge estimates given in Lemma 6.4. We will omit the proof (see, for example, [46], [43] or [47, chapter 5]).

Proposition 6.6. *Let $n \geq 2$, $j \geq 2$ and $\max(0,m) \leq \beta < \infty$, where $m := \frac{n-4}{2} + \frac{2}{n+1}$. If $q \in W^{\beta,2}(\mathbb{R}^n)$ is compactly supported in B_ρ, then*

$$\|\widetilde{Q}_j(q)\|_{W^{\alpha,2}} \leq C^j(\beta,\rho)\frac{C_0^{-(\alpha_j-\alpha)}}{(\alpha_j-\alpha)^{1/2}}\|q\|_{W^{\beta,2}}^j$$

if $\alpha < \alpha_j$, with

$$\alpha_j = \beta + (j-1) - \frac{n}{2} - \frac{(n-1)}{2}(j-2)\max\left(0, \frac{1}{2} - \frac{\beta}{n}\right). \tag{6.11}$$

This proposition improves the regularity gain obtained in [48, Proposition 4.3], for the range $m \le \beta \le n/2$ and in [45] for $\beta \ge n/2$. We have used certain cancellations given by the fractional Laplacian $(-\Delta)^s$ to raise the value of α_j in dimension n (see Section 6.1). It also improves the regularity gain given in [45] for the $\widetilde{Q}_4$ operator with $n = 3$. This would allow us to obtain the results of recovery of singularities in that paper without the very technical proof to estimate $\widetilde{Q}_4(q)$. As a corollary of Proposition 6.6, we can show explicitly that the tail of the high frequency Born series converges absolutely in a certain range of β.

Corollary 6.7. *Let q be as in Proposition 6.6. Then, for every $\alpha \ge 0$ there is an $\ell \ge 2$ such that the series $\sum_{j=\ell}^{\infty} \widetilde{Q}_j(q)$, converges absolutely in $W^{\alpha,2}(\mathbb{R}^n)$ provided we take $C_0 = C\|q\|_{W^{\beta,2}}^{1/\varepsilon}$ for a large constant $C = C(\alpha, \beta, \rho)$, and a certain $\varepsilon = \varepsilon(\beta) > 0$.*

In general the Born series $\sum_{j=2}^{\infty} Q_j(q)$ is not expected to be convergent for low frequencies, without assuming certain smallness conditions on the potential (see, for example, [18, p. 33]). This makes essential the introduction of the high frequency Born series $\sum_{j=2}^{\infty} \widetilde{Q}_j(q)$. Another approach is to modify the definition of the Born series expansion to eliminate the effects of the negative eigenvalues associated to the potentials. This is the approach of [8] where it is shown that, if $(n-3)/2 \le \beta < \infty$, a modified Born series converges in $W^{\alpha,2}(\mathbb{R}^n)$ with α in the same range given in Theorem 2.3.

We define the conjugate resolvent operator

$$R_\theta(q)(x) := e^{-ik\theta\cdot x} R_k \left(e^{ik\theta\cdot(\cdot)} q(\cdot)\right)(x). \tag{6.12}$$

Lemma 6.8. *Let $s \ge 0$ and let r and t be such that $0 \le 1/t - 1/2 \le 1/(n+1)$ and $0 \le 1/2 - 1/r \le 1/(n+1)$. There exist δ, $\delta' > 0$ and C (independent of k) such that*

$$\|R_\theta(q)\|_{W^{s,r}_{-\delta}} \le C k^{-1+(1/t-1/r)(n-1)/2} \|q\|_{W^{s,t}_{\delta'}}.$$

We also need a theorem of Zolesio on the product of functions in the Sobolev spaces (a proof can be found in [20] and for the compactly supported case in [43, pp. 182-183])

Lemma 6.9 (Zolesio)**.** *Let $s_1, s_2, s \ge 0$, $s \le s_1$, $s \le s_2$, and let r, t and p be such that $t < \min(p, r)$ and*

$$s_1 + s_2 - s \ge n\left(\frac{1}{p} + \frac{1}{r} - \frac{1}{t}\right).$$

Then

$$\|qf\|_{W^{s,t}} \le C\|q\|_{W^{s_1,p}}\|f\|_{W^{s_2,r}}.$$

Moreover, if q is compactly supported and $\delta, \delta' \in \mathbb{R}$, then

$$\|qf\|_{W^{s,t}_{-\delta}} \le C(\text{supp } q, \delta, \delta')\|q\|_{W^{s_1,p}}\|f\|_{W^{s_2,r}_{\delta'}}. \tag{6.13}$$

We now introduce the fractional Laplacian. If f is a $\mathcal{S}(\mathbb{R}^n)$ function and $\beta \geq 0$, the fractional Laplacian (see, for example, [12, Section 3]) can be defined by the identity

$$\mathcal{F}\left((-\Delta)^{\beta/2} f\right)(\xi) := |\xi|^\beta \widehat{f}(\xi). \tag{6.14}$$

By definition, $(-\Delta)^{\beta/2}$ is a self-adjoint operator. Also, since its Fourier multiplier is not smooth, $(-\Delta)^{\beta/2}$ cannot be extended by duality to $\mathcal{S}'(\mathbb{R}^n)$, but only to a proper subset of the tempered distributions which satisfy a certain growth restriction (see [49, chapter 2]). Fortunately, this set includes $L^\infty(\mathbb{R}^n)$, so we may write

$$(-\Delta)^{\beta/2} e^{i2k\theta\cdot x} = (2k)^\beta e^{i2k\theta\cdot x}, \tag{6.15}$$

in the sense of distributions.

Lemma 6.10. *Let $f, g \in C_c^\infty(\mathbb{R}^n)$, then if $\beta \geq 0$ we have that*

$$\|(-\Delta)^{\beta/2}(fg)\|_{L^1} \leq C(\beta)\|f\|_{W^{\beta,2}}\|g\|_{W^{\beta,2}}.$$

We leave the proof of this lemma for the end of this section. We can now prove Proposition 6.6.

Proof of Proposition 6.6. Without loss of generality assume $q \in C_c^\infty(B_\rho)$, where B_ρ denotes the ball of radius ρ. In terms of R_θ, defined in (6.12), the expression of Q_j given in (2.8) becomes

$$\widehat{Q_j(q)}(\xi) = \int_{\mathbb{R}^n} e^{i2k\theta\cdot y}(qR_\theta)^{j-1}(q)(y)\, dy,$$

with $\xi = -2k\theta$. In spherical coordinates we can write

$$\|\widetilde{Q}_j(q)\|^2_{W^{\alpha,2}} \leq \int_{C_0}^\infty k^{n-1+2\alpha} \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{R}^n} e^{i2k\theta\cdot y}(qR_\theta)^{j-1}(q)(y)\, dy \right|^2 d\sigma(\theta)\, dk.$$

Since $(qR_\theta)^{j-1}(q) \in C_c^\infty(\mathbb{R}^n)$, we can use in the right hand side that that (6.15) holds in the sense of distributions to obtain

$$\begin{aligned}
&\|\widetilde{Q}_j(q)\|^2_{W^{\alpha,2}} \\
&\leq C(\beta) \int_{C_0}^\infty k^{n-1+2\alpha-2\beta} \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{R}^n} (-\Delta)^{\beta/2}(e^{i2k\theta\cdot y})(qR_\theta)^{j-1}(q)(y)\, dy \right|^2 d\sigma(\theta)\, dk \\
&= C(\beta) \int_{C_0}^\infty k^{n-1+2\alpha-2\beta} \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{R}^n} e^{i2k\theta\cdot y}(-\Delta)^{\beta/2}\left((qR_\theta)^{j-1}(q)\right)(y)\, dy \right|^2 d\sigma(\theta)\, dk \\
&\qquad \leq C(\beta) \int_{C_0}^\infty k^{n-1+2\alpha-2\beta} \int_{\mathbb{S}^{n-1}} \|(-\Delta)^{\beta/2}\left((qR_\theta)^{j-1}(q)\right)\|^2_{L^1(\mathbb{R}^n)}\, d\sigma(\theta)\, dk.
\end{aligned} \tag{6.16}$$

Applying Lemma 6.10 and Remark 3.7 we have

$$\begin{aligned}\|(-\Delta)^{\beta/2}\left((qR_\theta)^{j-1}(q)\right)\|_{L^1} &\le C(\beta)\|\langle\cdot\rangle^{\delta}q\|_{W^{\beta,2}}\|\langle\cdot\rangle^{-\delta}R_\theta((qR_\theta)^{j-2}(q))\|_{W^{\beta,2}}\\ &\le C(\beta,\rho)\|q\|_{W^{\beta,2}}\|R_\theta((qR_\theta)^{j-2}(q))\|_{W^{\beta,2}_{-\delta}},\end{aligned}$$

where we have also use that q is compactly supported. Now, choose δ in the previous equation as in Lemma 6.8. The idea to deal with the norm in the right hand side is to iterate Lemmas 6.8 and 6.9 following the diagram,

$$\begin{array}{ccccccccc} \mathrm{W}^{\beta,t_{j-1}}_{\delta'} & \xrightarrow{\mathrm{R}_\theta} & \mathrm{W}^{\beta,r_{j-1}}_{-\delta} & \xrightarrow{\mathrm{q}\cdot} & \mathrm{W}^{\beta,t_{j-2}}_{\delta'}\dots & \xrightarrow{q\cdot} & W^{\beta,t_1}_{\delta'} & \xrightarrow{\mathrm{R}_\theta} & W^{\beta,r_1}_{-\delta}\\ q & \longrightarrow & R_\theta(q) & \longrightarrow & qR_\theta(q)\dots & \longrightarrow & (qR_\theta)^{j-2}(q) & \longrightarrow & R_\theta(qR_\theta)^{j-2}(q)\end{array}$$

where $r_1 = 2$ and $t_{j-1} = 2$ and r_ℓ and t_ℓ, $\ell = 1, \dots, j-2$ have to satisfy the conditions

$$\begin{array}{lll} 0 \le \dfrac{1}{t_\ell} - \dfrac{1}{2} \le \dfrac{1}{n+1} & \text{and} & 0 \le \dfrac{1}{2} - \dfrac{1}{r_{\ell+1}} \le \dfrac{1}{n+1},\\ t_\ell < 2 & \text{and} & 0 \le \dfrac{1}{2} + \dfrac{1}{r_{\ell+1}} - \dfrac{1}{t_\ell} \le \dfrac{\beta}{n}.\end{array}$$

Hence we obtain

$$\|R_\theta((qR_\theta)^{j-2}(q))\|_{W^{\beta,2}_{-\delta}} \le C^j(\beta,\rho)k^{\gamma_j}\|q\|^{j-1}_{W^{\beta,2}},$$

(in (6.13) the constant depends on the support B_ρ of q) where

$$\begin{aligned}\gamma_j &= -(j-1) + \frac{(n-1)}{2}\sum_{\ell=1}^{j-1}\left(\frac{1}{t_\ell} - \frac{1}{r_\ell}\right)\\ &= -(j-1) + \frac{(n-1)}{2}\sum_{\ell=1}^{j-2}\left(\frac{1}{t_\ell} - \frac{1}{r_{\ell+1}}\right).\end{aligned}$$

Now, for small $\varepsilon > 0$, when $\beta \ge m = (n-4)/2 + 2/(n+1)$ (this restriction comes from the range of t and r in which Lemma 6.8 holds), we can choose r_ℓ and t_ℓ satisfying all the previous conditions and

$$1/t_\ell - 1/r_{\ell+1} = \max(1/2 - \beta/n, \varepsilon),$$

for all $1 \le \ell \le j-2$, and so we obtain

$$\gamma_j = -(j-1) + \frac{(n-1)}{2}(j-2)\max(1/2 - \beta/n, \varepsilon).$$

Putting all the previous estimates together in (6.16) we obtain

$$\|\widetilde{Q}_j(q)\|^2_{W^{\alpha,2}} \leq C^{2j}(\beta,\rho)\,\|q\|^{2j}_{W^{\beta,2}} \int_{C_0}^{\infty} k^{n-1+2\alpha-2\beta+2\gamma_j}\,dk = C^{2j}(\beta,\rho)\frac{C_0^{-2(\alpha_j-\alpha)}}{\alpha_j-\alpha}\|q\|^{2j}_{W^{\beta,2}}, \quad (6.17)$$

with $\alpha < \alpha_j$ and $\alpha_j = \beta + (j-1) - \frac{n}{2} - \frac{(n-1)}{2}(j-2)\max\left(0, \frac{1}{2} - \frac{\beta}{n}\right)$. By density, we can extend estimate (6.17) for $q \in W^{\beta,2}(\mathbb{R}^n)$ compactly supported in B_ρ. This follows from Lemma 3.11, with minor changes to take into account the restriction in the support (or directly by Lemma A.3 with the appropriate spaces X, Y). □

Proof of Corollary 6.7. Choose some $\alpha > 0$. Now, for $\beta \geq m$, α_j grows linearly with j (this can be verified with a tedious but straightforward computation). Then, for any integer $\ell \geq 2$ such that $\alpha_\ell > \alpha$ we have by Proposition 6.6 that

$$\left\|\sum_{j=\ell}^{\infty}\widetilde{Q}_j(q)\right\|_{W^{\alpha,2}} \leq \sum_{j=\ell}^{\infty}\|\widetilde{Q}_j(q)\|_{W^{\alpha,2}} \leq \sum_{j=\ell}^{\infty} C_0^{-(\alpha_j-\alpha)}C^j(\alpha,\beta,\rho)\|q\|^j_{W^{\beta,2}}.$$

Using the linear growth of α_j we can choose some $\varepsilon(\beta) = \varepsilon > 0$ such that for every $j \geq \ell$, $(\alpha_j - \alpha) \geq j\varepsilon$. Therefore we obtain that

$$\left\|\sum_{j=\ell}^{\infty}\widetilde{Q}_j(q)\right\|_{W^{\alpha,2}} \leq \sum_{j=\ell}^{\infty} C_0^{-\varepsilon j}C^j(\alpha,\beta,\rho)\|q\|^j_{W^{\beta,2}},$$

and the right hand side converges taking $C_0 > (C(\alpha,\beta,\rho)\|q\|_{W^{\beta,2}})^{1/\varepsilon}$. □

Proof of Lemma 6.10. Assume first that $0 < \beta < 2$ (the case of $\beta = 0$ is trivial). Then we have the pointwise relation (it can be computed by hand using the principal value formula of the fractional Laplacian see, for example, [6, p. 636])

$$(-\Delta)^{\beta/2}(fg)(x) = f(x)(-\Delta)^{\beta/2}(g)(x) + g(x)(-\Delta)^{\beta/2}(f)(x) + \int_{\mathbb{R}^n}\frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+\beta}}\,dy,$$

where we need $0 < \beta < 2$ so that the singularity in the last integral can be controlled. Since by (6.14) we have that $\|(-\Delta)^{\beta/2}u\|_{L^2} \leq \|u\|_{W^{\beta,2}}$, taking the $L^1(\mathbb{R}^n)$ norm and applying Cauchy-Schwarz inequality to the first two terms we obtain

$$\|(-\Delta)^{\beta/2}(fg)\|_{L^1} \leq 2\|f\|_{W^{\beta,2}}\|g\|_{W^{\beta,2}} + \int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\frac{(f(x)-f(y))(g(x)-g(y))}{|x-y|^{n+\beta}}\,dy\right|\,dx.$$

But the last can be bounded using Cauchy-Schwarz and that

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+\beta}}\,dy\,dx \le C\|f\|^2_{W^{\beta/2,2}},$$

(in fact, the left hand side is an equivalent norm for the homogeneous $\dot{W}^{\beta/2,2}$ when $0<\beta<2$, see [12, Proposition 3.4]).

If now we assume that $\beta \ge 2$, define $k := [\beta/2]$, that is the integer part, and $\tilde{\beta} = \beta - 2k$ so we have now $\tilde{\beta} \in [0,2)$, and

$$(-\Delta)^{\beta/2}(fg) = (-\Delta)^{\tilde{\beta}/2}(-\Delta)^k(fg). \tag{6.18}$$

An integer power of the Laplacian is an homogeneous constant coefficient differential operator of order $2k$, and therefore if $a, b \in \mathbb{N}^n$ we have

$$(-\Delta)^k(fg) = \sum_{|a|+|b|=2k} c_{a,b}\partial^a f \partial^b g,$$

where we are not interested in the particular values of the constants $c_{a,b}$. Then, to bound the $L^1(\mathbb{R}^n)$ norm of (6.18) we can apply the same arguments as before so we obtain

$$\|(-\Delta)^{\beta/2}(fg)\|_{L^1} \le \sum_{|a|+|b|=2k} |c_{a,b}|\|\partial^a f\|_{W^{\tilde{\beta},2}}\|\partial^b g\|_{W^{\tilde{\beta},2}} \le C\|f\|_{W^{\beta,2}}\|g\|_{W^{\beta,2}},$$

using that $\|\partial^a f\|_{W^{\tilde{\beta},2}} \le \|f\|_{W^{\tilde{\beta}+|a|,2}} \le \|f\|_{W^{\beta,2}}$ since $|a| \le 2k$. □

6.2 The case of fixed angle scattering

6.2.1 Proofs of theorems 2.13 and 2.14

As in backscattering, we have the following estimates for the remainder term $Q^R_{\theta,j}(q)$.

Lemma 6.11 (A. Ruiz)**.** *Assume that q is compactly supported, and that $q \in L^p(\mathbb{R}^n)$ for some $p > n/2$. Let $\alpha \in \mathbb{R}$. Then, if we take $C_0 \ge C\|q\|^{1/(2-n/p)}_{L^p}$ with C large enough, we have that $\widetilde{Q}^R_{\theta,j}(q) \in W^{\alpha,2}(\mathbb{R}^n)$ for every $j \ge \frac{n/2+\alpha}{2-n/p}$.*

Lemma 6.11 is a consequence of [46, Proposition 4.5]. We give a different proof without using the restriction theorem for the Fourier transform. This yields a slightly worse result in terms of the regularity of the remainder, but it is enough for our purposes.

Proposition 6.12. *Assume that q is compactly supported in B_ρ, and that $q \in L^p(\mathbb{R}^n)$ for some $p > n/2$. Let $\alpha \in \mathbb{R}$. Then if we take $C_0 \ge C(\rho,\alpha)\|q\|^{2/(2-n/p)}_{L^p}$ with $C(\rho,\alpha)$ large enough, and $j \ge \frac{n/2+\alpha}{2-n/p}$, we have that*

$$\lim_{j\to\infty} \|\widetilde{Q}^R_{\theta,j}(q)\|_{W^{\alpha,2}} = 0. \tag{6.19}$$

Proof of Lemma 6.11 and Proposition 6.12. By (2.28), (2.30), and (2.31), for $\xi \in H_\theta$ we have that $\widehat{\widetilde{Q}^R_{\theta,j}(q)}(\xi) = \chi(\xi)(B^R_{\theta,j}(q)(\xi) + B^R_{-\theta,j}(q)(\xi))$ where

$$B^R_{\theta,j}(q)(\xi) = \int_{\mathbb{R}^n} e^{-ik\theta'\cdot y}(qR_k)^{j-1}(q(\cdot)u_s(k,\theta,\cdot))(y)\,dy, \quad \xi \in H_\theta.$$

Since $\xi = k(\theta' - \theta)$, in the half space H_θ we have the change of variables $d\xi = k^{n-1}|\theta' - \theta|^2 dk\,d\sigma(\theta')$. Using that $\chi(\xi)$ vanishes for $|\xi| \le C_0$, for any $\theta \in \mathbb{S}^{n-1}$ we then have

$$\begin{aligned}
&\|\chi B^R_{\theta,j}(q)\|^2_{L^2_\alpha} \le \int_{H_\theta} |\chi(\xi)B^R_{\theta,j}(q)(\xi)|^2\,d\xi \\
&\le \int_{C_0}^\infty k^{n-1+2\alpha}\int_{\mathbb{S}^{n-1}}\left|\int_{\mathbb{R}^n} e^{ik\theta\cdot y}(qR_k)^{j-1}(q(\cdot)u_s(k,\theta,\cdot))(y)\,dy\right|^2 |\theta-\theta'|^{2+2\alpha}\,d\sigma(\theta')\,dk \\
&\le C(\alpha)\int_{C_0}^\infty k^{n-1+2\alpha}\int_{\mathbb{S}^{n-1}} \|(qR_k)^{j-1}(q(\cdot)u_s(k,\theta,\cdot))\|^2_{L^1}\,d\sigma(\theta')\,dk, \qquad (6.20)
\end{aligned}$$

where we have used that $|\theta - \theta'| \le 2$. Then, inserting (6.10) in (6.20) yields

$$\|\widetilde{Q}^R_{\theta,j}(q)\|^2_{W^{\alpha,2}} \le C(\rho,\alpha)\|q\|^{2(j+1)}_{L^p}\int_{C_0}^\infty k^{n-1+2\alpha+2j(n/p-2)}\,dk.$$

Since $n/p - 2 < 0$, the previous integral is finite for all $j > \frac{n/2+\alpha}{2-n/p}$, which proves Lemma 6.11. Proposition 6.12 follows from the previous estimate, taking C_0 as in the statement. This follows using the same reasoning explained in the proof of Proposition 6.2. □

We now focus on the regularity of the $\widetilde{Q}_{\theta,j}(q)$ terms. The following result is analogous to Proposition 6.6.

Proposition 6.13. *Let $n \ge 2$, $j \ge 2$ and $\max(0,m) \le \beta < \infty$, where $m = \frac{n-4}{2} - \frac{2}{n+1}$. Then if $q \in W^{\beta,2}(\mathbb{R}^n)$ is compactly supported in B_ρ, we have that*

$$\|\widetilde{Q}_{\theta,j}(q)\|_{W^{\alpha,2}} \le C^j(\beta,\rho)\frac{C_0^{-(\alpha_j-\alpha)}}{(\alpha_j-\alpha)^{1/2}}\|q\|^j_{W^{\beta,2}}$$

if $\alpha < \alpha_j$, with

$$\alpha_j = \beta - \frac{1}{2} + (j-1) - (j-1)\frac{(n-1)}{2}\max\left(0, \frac{1}{2} - \frac{\beta}{n}\right). \qquad (6.21)$$

We leave the proof of this proposition for the next section.

This result is originally from [46] (see Theorem 1.1 of that paper), where they consider $q \in W^{\beta,p}(\mathbb{R}^n)$ with $p > 2 - 4/(n+3)$. For simplicity, we reduce to the case

$p = 2$. We have made some modifications to the proof to extend it for $\beta \geq n/2$. As we have mentioned in Section 2.4, the vale of α_j is greater in (6.21) than in the case of backscattering, thanks to the fact that in the proof we can use the restriction theorem of the Fourier transform. This is why in Theorems 2.3 and 2.14 the same regularity gain is obtained even if Theorem 2.15 only yields an estimate for the double dispersion operator, instead of for every j like in Theorem 2.4.

Remark 6.14. By the definition of the $\widetilde{Q}_{F,j}$ operator as an average in θ of $\widetilde{Q}_{\theta,j}$, the previous three propositions hold identically changing $\widetilde{Q}_{\theta,j}$ and $\widetilde{Q}^R_{\theta,j}$, by $\widetilde{Q}_{F,j}$ and $\widetilde{Q}^R_{F,j}$. (This follows from the fact that the estimates of the $\widetilde{Q}_{\theta,j}$ of the resolvent used in the proofs are uniform on θ.)

We can now reduce the proof of Theorem 2.14 and Theorem 2.13 to Theorem 2.15, by using Lemma 6.11.

Proof of Theorem 2.14. Since by assumption $\beta \geq 0$ and $\beta > (n-3)/2$, q satisfies the condition $q \in L^p(\mathbb{R}^n)$ for some $p > n/2$. On the other hand, by Theorem 2.15 and Proposition 6.13, we have that $\widetilde{Q}_{\theta,j}(q) \in W^{\alpha,2}(\mathbb{R}^n)$, for all $j \geq 2$ with

$$\alpha < \begin{cases} 2\beta - (n-3)/2, & if \quad (n-3)/2 < \beta < (n-1)/2, \\ \beta + 1, & if \quad (n-1)/2 \leq \beta < \infty. \end{cases}$$

(For $j = 2$ this is immediate, for $j \geq 3$ it follows after tedious but straightforward computations.) Hence, let us choose α satisfying the previous condition. By (2.33) we have that

$$q_\theta = q + \sum_{j=2}^{\ell} \widetilde{Q}_{\theta,j}(q) + \widetilde{Q}^R_{\theta,\ell}(q). \tag{6.22}$$

Then, by Lemma 6.11 we can choose an ℓ such that $\widetilde{Q}^R_{\theta,\ell}(q) \in W^{\alpha,2}(\mathbb{R}^n)$, which yields the desired result. □

Similarly, from Theorem 2.12 we can deduce the following result of recovery of singularities in full data scattering using Remark 6.14.

Theorem 6.15. *Let $q \in W^{\beta,2}(\mathbb{R}^n)$ with $0 \leq \beta < \infty$ be a compactly supported function. Then $q - q_F \in W^{\alpha,2}(\mathbb{R}^n)$, modulo a C^∞ function, and let $a = (2n-2)^{-1}$ if the following condition holds,*

$$\alpha < \begin{cases} 2\beta - (n-4)/2 - \beta/n, & if \quad (n-3)/2 - 3a < \beta < (n-1)/2 - a, \\ \beta + 1, & if \quad (n-1)/2 - a \leq \beta < \infty. \end{cases}$$

This proposition is basically [2, Theorem 1.1] restricted to the case $p = 2$, but we have extended it to include the range $n/2 < \beta < \infty$. The proof follows directly from Theorem 2.12 and Lemma 6.11, when applied to full data as explained in Remark 6.14. As mentioned in the introduction, this theorem does not yield the optimal result since it would be necessary to improve the estimates of the $\widetilde{Q}_{F,j}(q)$ with $j \geq 3$.

Proof of Theorem 2.13. The main idea is the same as in the proof of *Theorem* 2.2, but for completeness we repeat it in detail. Take $\alpha \geq 0$ and assume that we have that $q - q_\theta \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ for every compactly supported, real and radial potential $q \in W^{\beta,2}(\mathbb{R}^n)$. We are going to prove that then necessarily $\widetilde{Q}_{\theta,j}(q) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ also.

Consider the Born series (6.22) for the potential $q(\lambda) = \lambda q$, where $\lambda \in (0,1)$, and denote by $q_\theta(\lambda)$ its corresponding Born approximation. By the multilinearity of the $\widetilde{Q}_{\theta,j}$ operators we have

$$\lambda q - q_\theta(\lambda) = -\sum_{j=2}^{\ell} \lambda^j \widetilde{Q}_{\theta,j}(q) + \widetilde{Q}^R_{\theta,\ell}(\lambda q), \tag{6.23}$$

modulo a C^∞ function (possibly dependent on λ). By Lemma 6.11, we have that if $\beta \geq (n-4)/2$ (this is necessary to guarantee that $q \in L^p(\mathbb{R}^n)$ for some $p > n/2$), we can take an $\ell \geq 2$, independent of λ, such that $\widetilde{Q}^R_{\theta,\ell}(\lambda q) \in W^{\alpha,2}(\mathbb{R}^n)$. Since by hypothesis also $\lambda q - q_B(\lambda) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$, we have that

$$\sum_{j=2}^{\ell} \lambda^j \widetilde{Q}_{\theta,j}(q) \in W^{\alpha,2}_{loc}(\mathbb{R}^n).$$

But for every $2 \leq i \leq \ell$, we can always choose a $\lambda_i \in (0,1)$ such that $\det(\lambda_i^j) \neq 0$, which implies that $\widetilde{Q}_{\theta,j}(q) \in W^{\alpha,2}_{loc}(\mathbb{R}^n)$ for all $2 \leq j \leq \ell$. But, by Theorem 2.15, we know that this implies that $\alpha \leq \min(2\beta - (n-4)/2, \beta + 1)$.

Using the Remark 6.14 and Theorem 5.1 the case of full data scattering can be proved in the same way. □

6.2.2 Estimate of the multiple dispersion operators

To prove Proposition 6.13 we begin by sating the following lemma from [46], based on the Stein-Thomas restriction theorem.

Lemma 6.16. *Let $\theta' \in \mathbb{S}^{n-1}$. We define the operator*

$$L_k(f)(\theta') = \int_{\mathbb{S}^{n-1}} e^{-ik(\theta'-\theta)\cdot x} f(x)\, dx.$$

Given t such that $0 \leq \frac{1}{t} - \frac{1}{2} \leq \frac{1}{n+1}$, and $\beta \geq 0$, there exists some $\delta = \delta(t)$ such that one has

$$\int_{\mathbb{S}^{n-1}} |L_k(f)(\theta')|^2 |\theta' - \theta|^\beta \, d\sigma(\theta') \leq C k^{(n-1)\left(\frac{1}{t}-\frac{3}{2}\right)-2\beta} \|f\|^2_{W^{\beta,t}_\delta}.$$

Proof of Proposition 6.13. By (2.29) and (6.12) we have that

$$B_{\theta,j}(q)(\xi) = \int_{\mathbb{R}^n} e^{-ik(\theta'-\theta)\cdot y} (qR_\theta)^{j-1}(q)(y)\, dy,$$

where $\xi = k(\theta' - \theta)$ and R_θ was defined in (6.12). In the half space H_θ we have the change of variables $d\xi = k^{n-1}|\theta' - \theta|^2 dk\, d\sigma(\theta')$, and therefore

$$\begin{aligned}
&\|\chi B_{\theta,j}(q)\|^2_{L^2_\alpha} \\
&\leq \int_{C_0}^{\infty} k^{n-1+2\alpha} \int_{\mathbb{S}^{n-1}} \left| \int_{\mathbb{R}^n} e^{-ik(\theta'-\theta)\cdot y} (qR_\theta)^{j-1}(q)(y)\, dy \right|^2 |\theta' - \theta|^{2+2\alpha}\, d\sigma(\theta')\, dk \\
&\leq \int_{C_0}^{\infty} k^{n-1+2\alpha} \int_{\mathbb{S}^{n-1}} \left| L_k\left((qR_\theta)^{j-1}(q)(y)\right)\right|^2 |\theta' - \theta|^{\beta}\, d\sigma(\theta')\, dk,
\end{aligned}$$

where we have used that $|\theta' - \theta| \leq 2$. Then Lemma 6.16 yields

$$\|\chi B_{\theta,j}(q)\|^2_{L^2_\alpha} \leq \int_{C_0}^{\infty} k^{n-1+2\alpha+(n-1)\left(\frac{1}{t_1}-\frac{3}{2}\right)-2\beta} \int_{\mathbb{S}^{n-1}} \|(qR_\theta)^{j-1}(q)\|^2_{W^{\beta,t_1}_\delta}\, d\sigma(\theta')\, dk.$$

Applying Lemma 6.8 and Lemma 6.9 $j-1$ times, as we did in Proposition 6.6, we get

$$\|\chi B_{\theta,j}(q)\|^2_{L^2_\alpha} \leq C^{2j} \|q\|^{2j}_{W^{\beta,2}} \int_{C_0}^{\infty} k^{n-1+2\alpha-2\gamma}\, dk, \tag{6.24}$$

where $C = C(n, \alpha, \beta, \text{supp } q) > 0$,

$$\gamma = \beta - (j-1) - \frac{(n-1)}{2}\left(\frac{1}{t_1} - \sum_{\ell=1}^{j-1}\left(\frac{1}{r_\ell} - \frac{1}{t_{\ell+1}}\right) - \frac{3}{2}\right), \tag{6.25}$$

and t_ℓ, r_ℓ, are parameters that for $\ell = 1, \dots, j-1$ must satisfy the conditions $t_{\ell+1} < 2$, $t_{\ell+1} < r_\ell$ and

$$\begin{aligned}
&0 \leq \frac{1}{t_\ell} - \frac{1}{2} \leq \frac{1}{n+1}, && 0 \leq \frac{1}{t_{j+1}} - \frac{1}{2} \leq \frac{1}{n+1}, \\
&0 \leq \frac{1}{2} - \frac{1}{r_\ell} \leq \frac{1}{n+1}, && 0 \leq \frac{1}{2} + \frac{1}{r_\ell} - \frac{1}{t_{\ell+1}} \leq \frac{\beta}{n}.
\end{aligned} \tag{6.26}$$

The previous inequalities imply that

$$0 < \frac{1}{t_{\ell+1}} - \frac{1}{r_\ell} \leq \frac{2}{n+1},$$

which together with the last condition in (6.26) gives the restriction $\beta \geq \max(0, m)$.

Now, we can always choose $t_1 = 2$, and $t_{\ell+1}$, r_ℓ, $\ell = 1, \dots, j-1$ such that they satisfy all the previous conditions and

$$\frac{1}{t_{\ell+1}} - \frac{1}{r_\ell} = \max\left(\varepsilon, \frac{1}{2} - \frac{\beta}{n}\right),$$

for any $\varepsilon > 0$ small. This choice is slightly different from the one in [46], and it is the only change necessary to extend their results to the range $\beta \geq n/2$.

On the other hand, (6.24) together with (2.28) yield

$$\|\widetilde{Q}_{\theta,j}(q)\|^2_{W^{\alpha,2}} \leq C^{2j}\|q\|^{2j}_{W^{\beta,2}} C_0^{2\alpha-2\gamma+n}, \tag{6.27}$$

if we have that $n-1+2\alpha-2\gamma<-1$. This, together with (6.25) and the previous choice of parameters, implies

$$\alpha < \beta - \frac{1}{2} + (j-1) - (j-1)\frac{(n-1)}{2}\max\left(\varepsilon, \frac{1}{2} - \frac{\beta}{n}\right).$$

Since we can take $\varepsilon > 0$ as small as necessary, this gives the condition $\alpha < \alpha_j$ where α_j satisfies in (6.21). □

As in backscattering, from Proposition 6.13 we get the following corollary that yields absolute convergence for the tail of high frequency Born series.

Corollary 6.17. *Let q be as in Proposition 6.13. Then, for every $\alpha > 0$, there exists an $\ell \geq 2$ such that the series $\sum_{j=\ell}^{\infty} \widetilde{Q}_{\theta,j}(q)$ converges absolutely in $W^{\alpha,2}(\mathbb{R}^n)$ provided we take $C_0 = C\|q\|^{1/\varepsilon}_{W^{\beta,2}}$, for a large constant $C = C(\alpha,\beta,\rho)$ and a certain $\varepsilon = \varepsilon(\beta) > 0$.*

The proof is completely analogous to the proof of Corollary 6.7.

Chapter 7

Optimal estimates for the double dispersion operator in backscattering

In this chapter we return to study the regularity of the Q_2 operator. We first estimate the double dispersion operator in Hölder spaces to prove Theorem 2.7 and then we analyze the radial case to prove Theorem 2.9.

7.1 Hölder estimates

7.1.1 Proof of Theorem 2.7

Proposition 7.1. *Assume $q \in W^{\beta,2}(\mathbb{R}^n)$ where $\beta > (n-2)/2$ and $n \geq 3$. Then we have that*

$$\|\widehat{Q_2(q)}\|_{L^1_\alpha} \leq C\|q\|_{W^{\beta,2}}\|q\|_{W^{(n-2)/2,2}},$$

for all $\alpha < \beta - (n-2)/2$.

This proposition does not include the case $n = 2$, which was treated in [3]. This is due to the fact that certain integrals over the Ewald spheres appear in the proof of Proposition 7.1, which contain the singularity $1/|\xi|$, critical in dimension two. Nonetheless, we think that our line of reasoning could be modified to deal with this difficulty and include the case $n = 2$.

In the introduction we have mentioned that Theorem 2.7 is optimal in the sense that it represents a weaker version of what is expected to be the best possible result in the Sobolev scale. Similarly Proposition 7.1 is also optimal (except possibly for the limiting case $\alpha = \beta - (n-2)/2$), as it can be seen in Proposition 5.6

To prove Theorem 2.7 we also need the following result.

Proposition 7.2. *Let $f \in \mathcal{S}'(\mathbb{R}^n)$. Then we have that*

$$\|f\|_{\Lambda^\alpha} \leq C\|\widehat{f}\|_{L^1_\alpha}.$$

Proof. Let m be the integer part of α, and γ any multi-index such that $|\gamma| \le m$. Then we have that

$$\|\partial^\gamma f\|_{L^\infty} \le C \int_{\mathbb{R}^n} |\xi^\gamma \widehat{f}(\xi)|\, d\xi \le C \int_{\mathbb{R}^n} \langle \xi \rangle^\alpha |\widehat{f}(\xi)|\, d\xi.$$

This means that we can reduce the proof to the case $0 < \alpha < 1$. Expressing $f(x)$ as the inverse Fourier transform of $\widehat{f}(\xi)$ we get

$$\frac{|f(x+t) - f(x)|}{|t|^\alpha} \le C \int_{\mathbb{R}^n} \left| \frac{e^{i\xi\cdot t} - 1}{|t|^\alpha} \right| |\widehat{f}(\xi)|\, d\xi.$$

Then is enough to show that

$$\left| \frac{e^{i\xi\cdot t} - 1}{|t|^\alpha} \right| \le 2|\xi|^\alpha.$$

The previous inequality is immediate for $|\xi| \ge |t|^{-1}$, so we consider $|\xi| \le |t|^{-1}$. In this case we have $|\xi||t| \le 1$ which implies

$$|e^{i\xi\cdot t} - 1| \le 2|\xi||t| \le 2|\xi|^\alpha |t|^\alpha,$$

and this yields the desired result. □

Proof of Theorem 2.7. The desired estimate for $Q_2(q)$ follows immediately from Proposition 7.1 thanks to the previous proposition. □

To prove Proposition 7.1 we begin estimating the spherical operator $S_r(q)$. To do that, we need the following result to change the order of integration in the algebraic submanifold of $\mathbb{R}^n \times \mathbb{R}^n$ defined by the equation $|\xi - \eta/2| = r|\eta/2|$ (recall the definition of $\Gamma_r(\eta)$ given in (3.2)). We leave the proof for Section 7.3.

Lemma 7.3. *Let $f \in C_c^\infty(\mathbb{R}^n)$. Then we have that*

$$\int_{\mathbb{R}^n} \int_{\Gamma_r(\eta)} f(\eta, \xi)\, d\sigma_{r\eta}(\xi)\, d\eta = \int_{\mathbb{R}^n} \int_{N_r(\xi)} f(\eta, \xi) \frac{|\eta|}{|\xi|}\, d\sigma_{r,\xi}(\eta)\, d\xi,$$

where we denote by $\sigma_{r,\xi}$ the restriction of the Lebesgue measure to the hypersurface

$$N_r(\xi) := \{\eta \in \mathbb{R}^n : |\xi - \eta/2| = r|\eta/2|\}. \tag{7.1}$$

If $r \ne 1$, N_r is the sphere of center $\frac{2\xi}{1-r^2}$ and radius $\frac{2|\xi|r}{|1-r^2|}$, otherwise for $r = 1$ it is a hyperplane. We also give the following lemma, a non-homogeneous analogue of Lemma 3.6, which is also proved in the Appendix by direct computation.

Lemma 7.4. *Let $\mathbb{S}_\rho \subset \mathbb{R}^n$ be any sphere of radius ρ and let σ_ρ be its Lebesgue measure. Let $a, b \geq 0$ satisfy $a + b > n-1$ and $a < n-1$, for all $x \in \mathbb{R}^n$ we have that*

$$\int_{S_\rho} \frac{1}{|x-y|^a \langle x-y\rangle^b}\, d\sigma_\rho(y) \leq C_{a,b}, \tag{7.2}$$

where the constant $C_{a,b}$ only depends on the parameters a and b.

Recall that we can control S_r if we estimate first the operator K_r, since by (3.27) we have.

$$|S_r(q)(\eta)| \leq \frac{2}{1+r} K_r(\widehat{q}, \widehat{q})(\eta), \tag{7.3}$$

Lemma 7.5. *Let $n \geq 3$ and $f_1, f_2 \in W^{\beta,2}(\mathbb{R}^n)$ with $\beta > (n-2)/2$. Then the estimate*

$$\|K_r(\widehat{f_1}, \widehat{f_2})\|_{L^1_\alpha} \leq C\|f_1\|_{W^{\beta,2}} \|f_2\|_{W^{(n-2)/2,2}} + C\|f_2\|_{W^{\beta,2}} \|f_1\|_{W^{(n-2)/2,2}}, \tag{7.4}$$

holds when $\alpha < \beta - (n-2)/2$.

Proof. We consider the case $r \neq 1$. The case $r = 1$ can be proved with similar arguments. Nevertheless we provide a different proof in Proposition 7.7, by using a special case of Santaló's formula.

In the first place, observe that the change of variables $\xi' = \eta - \xi$ leaves invariant the Ewald sphere $\Gamma_r(\eta)$, since it changes a point by its antipodal point on the sphere. We define $\Gamma_r^+(\eta) := \{\xi \in \Gamma_r(\eta) : |\xi| \geq |\eta - \xi|\}$, which is exactly a half sphere. Then, using the mentioned change of variables, we can reduce the integrals over $\Gamma_r(\eta)$ to integrals over $\Gamma_r^+(\eta)$,

$$\begin{aligned}\|K_r(\widehat{f_1}, \widehat{f_2})\|_{L^1_\alpha} = \int_{\mathbb{R}^n} \frac{\langle\eta\rangle^\alpha}{|\eta|} \int_{\Gamma_r^+(\eta)} |\widehat{f_1}(\xi)||\widehat{f_2}(\eta-\xi)|\, d\sigma_{r\eta}(\xi)\, d\eta \\ + \int_{\mathbb{R}^n} \frac{\langle\eta\rangle^\alpha}{|\eta|} \int_{\Gamma_r^+(\eta)} |\widehat{f_2}(\xi)||\widehat{f_1}(\eta-\xi)|\, d\sigma_{r\eta}(\xi)\, d\eta.\end{aligned}$$

We are going to estimate only the first term since the estimate of the second follows simply by interchanging the roles of f_1 and f_2. Let's denote it by I_1.

We define the set

$$N_r^+(\xi) := \{\eta \in N_r(\xi) : |\xi| \geq |\eta - \xi|\},$$

and here we have that $|\eta| \leq 2|\xi|$. Now consider $\varepsilon > 0$ and fix $\beta = \alpha + (n-2)/2 + 2\varepsilon$. Changing the order of integration (Lemma 7.3) we obtain

$$\begin{aligned} I_1 &\leq C \int_{\mathbb{R}^n} \frac{1}{|\eta|\langle\eta\rangle^{(n-2)/2+\varepsilon}} \int_{\Gamma_r^+(\eta)} |\widehat{f_1}(\xi)|\langle\xi\rangle^{\beta-\varepsilon} |\widehat{f_2}(\eta-\xi)|\, d\sigma_{r\eta}(\xi)\, d\eta \\ &= C \int_{\mathbb{R}^n} |\widehat{f_1}(\xi)|\langle\xi\rangle^\beta \int_{N_r^+(\xi)} \frac{|\widehat{f_2}(\eta-\xi)|}{|\xi|\langle\xi\rangle^\varepsilon \langle\eta\rangle^{(n-2)/2+\varepsilon}} d\sigma_{r,\xi}(\eta)\, d\xi \\ &\leq C\|f_1\|_{W^{\beta,2}}\, I_2, \end{aligned} \tag{7.5}$$

where to get the last inequality we have applied Cauchy-Schwarz inequality in the ξ variables so that

$$I_2 := \left(\int_{\mathbb{R}^n} \left|\int_{N_r^+(\xi)} \frac{1}{|\xi|\langle\xi\rangle^{\varepsilon}\langle\eta\rangle^{(n-2)/2+\varepsilon}}|\widehat{f}_2(\eta-\xi)|d\sigma_{r,\xi}(\eta)\right|^2 d\xi\right)^{\frac{1}{2}}.$$

Now, let us consider the integral in $N_r^+(\xi)$. Taking into account that $\langle\xi\rangle^{\varepsilon} \geq \langle\xi-\eta\rangle^{\varepsilon}$, we multiply and divide by $|\eta-\xi|^{1/2}\langle\eta-\xi\rangle^{(n-2)/2}$ before using Cauchy-Schwarz inequality in the η variable,

$$\begin{aligned}
&\left(\int_{N_r^+(\xi)} \frac{1}{\langle\xi\rangle^{\varepsilon}\langle\eta\rangle^{(n-2)/2+\varepsilon}}|\widehat{f}_2(\eta-\xi)|d\sigma_{r,\xi}(\eta)\right)^2 \\
&\leq \int_{N_r^+(\xi)} \frac{1}{\langle\eta\rangle^{n-2+2\varepsilon}}|\widehat{f}_2(\eta-\xi)|^2|\eta-\xi|\langle\eta-\xi\rangle^{n-2}d\sigma_{r,\xi}(\eta) \times \dots \\
&\qquad \dots \times \int_{N_r(\xi)} \frac{1}{|\eta-\xi|\langle\eta-\xi\rangle^{n-2+2\varepsilon}}d\sigma_{r,\xi}(\eta). \qquad (7.6)
\end{aligned}$$

But, since $n \geq 3$, we can apply Lemma 7.4 with $a = 1$ and $b = n-2+2\varepsilon$ to get

$$\int_{N_r(\xi)} \frac{1}{|\eta-\xi|\langle\eta-\xi\rangle^{n-2+2\varepsilon}}\, d\sigma_{r,\xi}(\eta) \leq C, \qquad (7.7)$$

where C does not depend in any way on the sphere $N_r(\xi)$. If we put together (7.6) and (7.7), using that $|\eta| \leq 2|\xi|$ and changing again the order of integration we get

$$\begin{aligned}
I_2 &\leq \left(\int_{\mathbb{R}^n}\int_{N_r^+(\xi)} \frac{1}{|\xi|^2\langle\eta\rangle^{n-2+2\varepsilon}}|\widehat{f}_2(\eta-\xi)|^2|\eta-\xi|\langle\eta-\xi\rangle^{n-2}\, d\sigma_{r,\xi}(\eta)\, d\xi\right)^{1/2} \\
&\leq C\left(\int_{\mathbb{R}^n} \frac{1}{|\eta|^2\langle\eta\rangle^{n-2+2\varepsilon}}\int_{\Gamma_r(\eta)} |\widehat{f}_2(\eta-\xi)|^2|\eta-\xi|\langle\eta-\xi\rangle^{n-2}\, d\sigma_{r\eta}(\xi)\, d\eta\right)^{1/2} \\
&\leq C\left(\int_{\mathbb{R}^n} \frac{1}{|\eta|^2\langle\eta\rangle^{n-2+2\varepsilon}}\int_{\Gamma_r(\eta)} |\widehat{f}_2(\xi')|^2|\xi'|\langle\xi'\rangle^{n-2}\, d\sigma_{r\eta}(\xi')\, d\eta\right)^{1/2},
\end{aligned}$$

where we have used in the last line the change of variables $\xi' = \xi - \eta$. Therefore, if we change the order of integration for the last time, returning to (7.5) we finally obtain

$$\begin{aligned}
I_1 &\leq C\|f_1\|_{W^{\beta,2}}\left(\int_{\mathbb{R}^n} |\widehat{f}_2(\xi)|^2\langle\xi\rangle^{n-2}\int_{N_r(\xi)} \frac{1}{|\eta|\langle\eta\rangle^{n-2+2\varepsilon}}\, d\sigma_{r,\xi}(\eta)\, d\xi\right)^{1/2} \\
&\leq C\|f_1\|_{W^{\beta,2}}\|f_2\|_{W^{(n-2)/2,2}},
\end{aligned}$$

where we have applied Lemma 7.4 to the integral in $N_r(\xi)$. Then the previous estimate yields

$$\|K_r(\widehat{f}_1,\widehat{f}_2)\|_{L^1_\alpha} \leq C\left(\|f_1\|_{W^{\beta,2}}\|f_2\|_{W^{(n-2)/2,2}} + \|f_2\|_{W^{\beta,2}}\|f_2\|_{W^{(n-2)/2,2}}\right)$$

for $\alpha = \beta - (n-2)/2 - 2\varepsilon$. Taking $\varepsilon > 0$ as small as necessary, we recover the statement of the lemma. □

Proof of Proposition 7.1. We begin estimating the spherical operator S_r and its radial derivative in order to apply Lemma 3.3.

By Lemma A.3 we can assume $q \in \mathcal{S}(\mathbb{R}^n)$. Then, by (7.3) and Lemma 7.5, it follows directly that if $r \in (0,\infty)$, $\beta > (n-2)/2$ and $\alpha < \beta - (n-2)/2$ we have

$$\|S_r(q)\|_{L^1_\alpha(\mathbb{R}^n)} \le C\frac{1}{1+r}\|q\|^2_{W^{\beta,2}}. \tag{7.8}$$

On the other hand, by Lemma 3.8, taking the $L^1_{\alpha-1}$ norm of (3.33) we have

$$\|\partial_r S_r(q)\|_{L^1_{\alpha-1}} \le C\|K_r(\widehat{q},\widehat{q})\|_{L^1_\alpha} + C\sum_{i=1}^n \|K_r(\widehat{x_i q},\widehat{q})\|_{L^1_\alpha},$$

if, for any $0<\delta<1$ fixed, $r \in (1+\delta, 1-\delta)$. Then we can apply Lemma 7.5 directly to the first term with $f_1 = q = f_2$, and to the second, with $f_1 = q$ and $f_2(x) = x_i q(x)$, which yields

$$\|\partial_r S_r(q)\|_{L^1_{\alpha-1}} \le C\|q\|^2_{W^{\beta,2}} + C\|x_i q\|_{W^{\beta,2}}\|q\|_{W^{(n-2)/2,2}} \le C\|q\|^2_{W_1^{\beta,2}}. \tag{7.9}$$

To obtain the desired result, by (7.8) and (7.9), we can apply Lemma 3.3 for every $q \in \mathcal{S}(\mathbb{R}^n)$ with $p=1$, $\tau = \alpha - 1$ and $M = C\|q\|^2_{W_1^{\beta,2}}$.

□

7.1.2 Santaló's formula and the spherical term

We now give a proof of the estimate of the spherical term S_r for the special case $r=1$. The main tool is Santaló's formula in spheres, which enables us to adapt the arguments of [3] for dimension $n \ge 3$. In this section we denote by σ the restriction of Lebesgue measure to $\mathbb{S}^{n-1}$, independently of the dimension.

Proposition 7.6 (Santaló's formula)**.** *Let f be a $L^1(\mathbb{S}^{n-1})$ function and $\theta \in \mathbb{S}^{n-1}$. Then if we define*

$$\mathbb{S}^{n-2}_\theta = \{\omega \in \mathbb{S}^{n-1} : \theta\cdot\omega = 0\}, \tag{7.10}$$

we have that

$$\int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-2}_\theta} f(\omega)\,d\sigma(\omega)\,d\sigma(\theta) = |\mathbb{S}^{n-2}|\int_{\mathbb{S}^{n-1}} f(\theta)\,d\sigma(\theta). \tag{7.11}$$

Proof. We define the following positive and bounded functional on $C(\mathbb{S}^{n-1})$,

$$F(g) := \int_{\mathbb{S}^{n-1}}\int_{\mathbb{S}^{n-2}_\theta} g(\omega)\,d\sigma(\omega)\,d\sigma(\theta).$$

This means that by the Riesz representation theorem, there exists a Radon measure μ on $\mathbb{S}^{n-1}$ such that

$$F(g) = \int_{\mathbb{S}^{n-1}} g(\theta)\, d\mu(\theta).$$

But observe that if O is any orthogonal matrix we have that

$$\int_{\mathbb{S}^{n-2}_{\theta}} g(O(\omega))\, d\sigma(\omega) = \int_{\mathbb{S}^{n-2}_{O(\theta)}} g(\omega)\, d\sigma(\omega),$$

which in turn implies, integrating in $\mathbb{S}^{n-1}$ both sides of the previous equation, that F is invariant under rotations. Therefore, the following property must hold in the measure representation of F

$$\int_{\mathbb{S}^{n-1}} g(O(\theta))\, d\mu(\theta) = \int_{\mathbb{S}^{n-1}} g(\theta)\, d\mu(\theta). \tag{7.12}$$

One consequence of this fact is that all balls of the same radius in the sphere must have the same μ-measure, that is, μ is a uniformly distributed measure on $\mathbb{S}^{n-1}$. This is a very rigid property for Radon measures. In fact, all uniformly distributed Radon measures must be equal up to a scalar factor (see [24, Proposition 3.1.5]) which implies that μ must be a multiple of the Lebesgue measure on $\mathbb{S}^{n-1}$. To determine the constant it is enough to compute $F(1)$. □

Since $r = 1$ always in this section, to simplify notation we will drop the subindex 1, that is, we write $S(q) := S_1(q)$, $\Gamma(\eta) := \Gamma_1(\eta)$, $N(\xi) := N_1(\xi)$ and analogously for similar cases.

Proposition 7.7. *Let $n \geq 3$ and assume that $q \in W^{\beta,2}(\mathbb{R}^n)$ with $\beta > (n-2)/2$. Then we have that*

$$\|S(q)\|_{L^1_\alpha} \leq C \|q\|_{W^{\beta,2}} \|q\|_{W^{(n-2)/2,2}},$$

for all $\alpha < \beta - (n-2)/2$.

The proof that we now give yields an identical estimate to (7.4) for $K_1(\widehat{f}_1, \widehat{f}_2)$ but for simplicity we work directly with $S(q)$.

Proof. As in the proof of Lemma 7.5, by the symmetry in ξ and $\eta - \xi$, we have that

$$\|S(q)\|_{L^1_\alpha} \leq C \int_{\mathbb{R}^n} \frac{1}{|\eta|} \int_{\Gamma^+(\eta)} |\widehat{q}(\xi)| < \xi >^{\alpha} |\widehat{q}(\eta - \xi)|\, d\sigma_\eta(\xi)\, d\eta,$$

where we have used that in this region $|\xi| \leq |\eta| \leq 2|\xi|$. Let's change the order of integration using Lemma 7.3,

$$\|S(q)\|_{L^1_\alpha} \leq C \int_{\mathbb{R}^n} |\widehat{q}(\xi)| \frac{< \xi >^{\alpha}}{|\xi|} \int_{N^+(\xi)} |\widehat{q}(\eta - \xi)| d\sigma_\xi(\eta) d\xi.$$

Now, if we change variables in the second integral by fixing $v = \eta - \xi$ we have

$$\|S(q)\|_{L^1_\alpha} \leq C \int_{\mathbb{R}^n} |\widehat{q}(\xi)| \frac{< \xi >^\alpha}{|\xi|} \int_{D(\xi)} |\widehat{q}(v)| d\sigma_\xi(v) d\xi, \tag{7.13}$$

where, since $|\xi - \eta/2| = |\eta|/2 \iff \xi \cdot (\eta - \xi)$, $D(\xi)$ is the disc given by $D(\xi) = \{v \in \mathbb{R}^n : v \cdot \xi = 0, \, |v| \leq |\xi|\}$.

If we write the first integral in (7.13) in spherical coordinates taking $\xi = r\theta$, by Cauchy-Schwarz inequality we obtain

$$\begin{aligned} \|S(q)\|_{L^1_\alpha} &\leq C \int_0^\infty r^{n-2}(1+r^2)^{\alpha/2} \int_{\mathbb{S}^{n-1}} |\widehat{q}(r\theta)| \int_{D(r\theta)} |\widehat{q}(v)| \; d\sigma_{(r\theta)}(v) \, d\sigma(\theta) dr \\ &\leq C \int_0^\infty r^{n-2}(1+r^2)^{\alpha/2} \left(\int_{\mathbb{S}^{n-1}} |\widehat{q}(r\theta)|^2 d\sigma(\theta) \right)^{\frac{1}{2}} G(r) \, dr, \end{aligned} \tag{7.14}$$

where, using the definition of $\mathbb{S}^{n-2}_\theta$ given in (7.10), we have

$$\begin{aligned} G(r) &= \left(\int_{\mathbb{S}^{n-1}} \left(\int_{D(r\theta)} |\widehat{q}(v)| \; d\sigma_{(r\theta)}(v) \right)^2 d\sigma(\theta) \right)^{\frac{1}{2}} \\ &= \left(\int_{\mathbb{S}^{n-1}} \left(\int_{\mathbb{S}^{n-2}_\theta} \int_0^r s^{n-2} \, |\widehat{q}(s\omega)| \, ds \, d\sigma(\omega) \right)^2 d\sigma(\theta) \right)^{\frac{1}{2}}, \end{aligned}$$

Then Hölder's inequality and Minkoswski's integral inequality yield

$$\begin{aligned} G(r) &\leq C \left(\int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-2}_\theta} \left(\int_0^r s^{n-2} \, |\widehat{q}(s\omega)| \, ds \right)^2 d\sigma(\omega) \, d\sigma(\theta) \right)^{\frac{1}{2}} \\ &\leq C \int_0^r s^{n-2} \left(\int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-2}_\theta} |\widehat{q}(s\omega)|^2 \, d\sigma(\omega) \, d\sigma(\theta) \right)^{\frac{1}{2}} ds. \end{aligned}$$

Now, using Santaló's formula (7.11) we have that

$$\int_{\mathbb{S}^{n-1}} \int_{\mathbb{S}^{n-2}_\theta} |\widehat{q}(s\omega)|^2 \, d\sigma(\omega) \, d\sigma(\theta) = |\mathbb{S}^{n-2}| \int_{\mathbb{S}^{n-1}} |\widehat{q}(s\theta)|^2 \, d\sigma(\theta),$$

and hence, multiplying and dividing by $\langle s \rangle^{1/2+\varepsilon}$ and using Cauchy-Schwartz inequality we get

$$G(r) \leq C \left(\int_0^r s^{2(n-2)} \langle s \rangle^{1+2\varepsilon} \int_{\mathbb{S}^{n-1}} |\widehat{q}(s\theta)|^2 \, d\sigma(\theta) \, ds \right)^{\frac{1}{2}} \times \dots$$
$$\dots \times \left(\int_0^\infty \frac{1}{\langle s' \rangle^{1+2\varepsilon}} \, ds' \right)^{\frac{1}{2}}$$
$$\leq C \langle r \rangle^{\varepsilon} \left(\int_0^r s^{n-1} \int_{\mathbb{S}^{n-1}} |\widehat{q}(s\theta)|^2 \langle s \rangle^{n-2} \, d\sigma(\theta) ds \right)^{\frac{1}{2}}$$
$$\leq C \langle r \rangle^{\varepsilon} \|q\|_{W^{(n-2)/2,2}},$$

(to get the second line is where we have used implicitly the condition $n \geq 3$, so that the exponent of s^{n-3} is non-negative). Using the estimate for $G(r)$ in (7.14), and repeating again exactly the same reasoning to bound the resulting integral, we finally obtain

$$\|S(q)\|_{L^1_\alpha} \leq C \|q\|_{W^{(n-2)/2,2}} \int_0^\infty r^{n-2} \left(\int_{\mathbb{S}^{n-1}} |\widehat{q}(r\theta)|^2 \langle r \rangle^{2\alpha+2\varepsilon} \, d\sigma(\theta) \right)^{\frac{1}{2}} dr$$
$$\leq C \|q\|_{W^{(n-2)/2,2}} \left(\int_0^\infty r^{2(n-2)} \langle r \rangle^{1+2\varepsilon} \int_{\mathbb{S}^{n-1}} |\widehat{q}(r\theta)|^2 \langle r \rangle^{2\alpha+2\varepsilon} \, d\sigma(\theta) \, dr \right)^{\frac{1}{2}}$$
$$\leq C \|q\|_{W^{\alpha+(n-2)/2+2\varepsilon,2}} \|q\|_{W^{(n-2)/2,2}},$$

so choosing $\beta = \alpha + (n-2)/2 + 2\varepsilon$ we obtain the desired result. □

7.2 Proof of Theorem 2.9

In this section we assume that q is a radial function. We begin by giving estimates for the spherical operator $\widetilde{S}_r(q)$ and its r-derivative in $W^{\alpha,2}(\mathbb{R}^n)$. As usual, we estimate first $\widetilde{K}_r(\widehat{f}_1, \widehat{f}_2)$.

Lemma 7.8. *Let $n \geq 2$ and $f_1, f_2 \in W^{2,\beta}(\mathbb{R}^n)$, and assume that $|\widehat{f}_2(\xi)|$ is a radial function. Then, if $r \in (0, \infty)$, and $\beta_0 = \min(-1/2, (n-7)/4)$ we have that*

$$\|\widetilde{K}_r(\widehat{f}_1, \widehat{f}_2)\|_{L^2_\alpha(\mathbb{R}^n)} \leq C\,(1+r)^{1-\gamma} \|f_1\|_{W^{\beta,2}} \|f_2\|_{W^{\beta,2}}, \tag{7.15}$$

for some $\gamma > 0$, possibly depending on β, if the following condition holds

$$\begin{cases} \alpha \leq 2\beta - (n-4)/2 & if \quad \beta_0 < \beta < (n-2)/2, \\ \alpha < \beta + 1 & if \quad (n-2)/2 \leq \beta < \infty, \end{cases} \tag{7.16}$$

In the proof we use the following result about integration on spheres.

Let $h : (0, \infty) \to \mathbb{C}$ be a measurable function. Let $x \in \mathbb{R}^n/\{0\}$ and $b > 0$, and consider the functional defined by the expression

$$F_{x,b}(h) := \int_{\mathbb{S}_b(x)} h(|z|)\, d\sigma_b(z),$$

where σ_b is the Lebesgue measure of $\mathbb{S}_b(x) \subset \mathbb{R}^n$, the sphere of radius b and center x.

Proposition 7.9. *We can write*

$$F_{x,b}(h) = \int_0^\infty h(t)\, d\mu_{x,b}(t).$$

where $\mu_{x,b}$ is the absolute continuous measure given by

$$\frac{d\mu_{x,b}}{dt} = 2^{3-n} c_{n-1} \chi_{x,b}(t) |x|^{2-n} bt \left((|x|+b)^2 - t^2\right)^{(n-3)/2} \left(t^2 - (|x|-b)^2\right)^{(n-3)/2}, \tag{7.17}$$

where $\chi_{x,b}$ is the characteristic function of the interval $(||x|-b|\,, |x|+b)$ and $c_n = |\mathbb{S}^{n-1}|$.

This formula is a result of [26], a proof can also be found in [9]. With this proposition we can prove the following lemma.

Lemma 7.10. *Let h as before and $f(x) := h(|x|)$. Then, if $r \neq 1$ we have that*

$$\int_{N_r(\xi)} f(\eta - \xi)\, d\sigma_{r,\xi}(\eta) \leq C \left(\frac{r}{1+r^2}\right)^{(n-1)/2} \int_0^\infty h(t) t^{n-2}\, dt. \tag{7.18}$$

Proof. By (7.1) we know that $N_r(\xi)$ is a sphere of center $\frac{2\xi}{1-r^2}$ and radius $b = \frac{2|\xi| r}{|1-r^2|}$. Since in (7.18) f is evaluated in in $\eta - \xi$, we can apply Proposition 7.9 with $x = \frac{2\xi}{1-r^2} - \xi$,

$$\int_{N_r(\xi)} f(\eta - \xi) d\sigma_{r,\xi}(\eta) = \int_{\mathbb{S}_b(x)} h(|z|) d\sigma_b(z) = \int_0^\infty h(t) d\mu_{x,b}(t).$$

On the other hand, if $t \in (||x|-b|, |x|+b)$ we obtain the inequalities

$$t^2 - (|x|-b)^2 \leq t^2 \quad \text{and} \quad (|x|+b)^2 - t^2 \leq 4|x|b,$$

and from (7.17), since $|x| = |\xi| \frac{1+r^2}{|1-r^2|}$, we get

$$\begin{aligned} \frac{d\mu_{x,b}}{dt} &\leq C|x|^{2-n+(n-3)/2} b^{1+(n-3)/2} t^{n-2} = C\left(\frac{b}{|x|}\right)^{(n-1)/2} t^{n-2} \\ &\leq C\left(\frac{r}{1+r^2}\right)^{(n-1)/2} t^{n-2}, \end{aligned}$$

which gives the desired result. □

Proof of Lemma 7.8. Since $\chi(\eta) = 0$ for $|\eta| \leq 1$, $\langle\eta\rangle \leq 2|\eta|$ in the region where χ does not vanish. Then

$$\|\widetilde{K}_r(\widehat{f}_1, \widehat{f}_2)\|_{L^2_\alpha} \leq C\left(\int_{\mathbb{R}^n} |\eta|^{2\alpha-2}\left(\int_{\Gamma_r^+(\eta)} |\widehat{f}_1(\xi)||\widehat{f}_2(\eta-\xi)|\,d\sigma_{r\eta}(\xi)\right)^2 d\eta\right)^{1/2}$$
$$+ C\left(\int_{\mathbb{R}^n} |\eta|^{2\alpha-2}\left(\int_{\Gamma_r^-(\eta)} |\widehat{f}_1(\xi)||\widehat{f}_2(\eta-\xi)|\,d\sigma_{r\eta}(\xi)\right)^2 d\eta\right)^{1/2} := I_1 + I_2,$$

where $\Gamma_r^-(\eta) := \{\xi \in \Gamma(\eta) : |\xi| < |\eta-\xi|\}$ is the complementary of $\Gamma_r^+(\eta)$ (introduced in Lemma 7.5). We begin with the estimate of I_1.

Consider a parameter $0 < \lambda \leq (n-1)/2$. By Cauchy-Schwarz inequality and Lemma 3.6 we have that

$$I_1^2 \leq C\int_{\mathbb{R}^n} |\eta|^{2\alpha-2}\int_{\Gamma_r^+(\eta)} |\widehat{f}_1(\xi)|^2|\widehat{f}_2(\eta-\xi)|^2|\eta-\xi|^{n-1-2\lambda}\,d\sigma_{r\eta}(\xi) \times \dots$$
$$\dots \times \int_{\Gamma_r(\eta)} \frac{1}{|\eta-\xi|^{n-1-2\lambda}}\,d\sigma_{r\eta}(\xi)\,d\eta$$
$$\leq Cr^{2\lambda}\int_{\mathbb{R}^n} |\eta|^{2\alpha-2+2\lambda}\int_{\Gamma_r^+(\eta)} |\widehat{f}_1(\xi)|^2|\widehat{f}_2(\eta-\xi)|^2|\eta-\xi|^{n-1-2\lambda}\,d\sigma_{r\eta}(\xi)\,d\eta.$$

Then, using that $|\eta| \leq 2|\xi|$ in $\Gamma_r^+(\eta)$ and Lemma 7.3 to change the order of integration, yields

$$I_1^2 \leq Cr^{2\lambda}\int_{\mathbb{R}^n} |\widehat{f}_1(\xi)|^2|\xi|^{2\alpha-2+2\lambda}\int_{N_r(\xi)} |\widehat{f}_2(\eta-\xi)|^2|\eta-\xi|^{n-1-2\lambda}\,d\sigma_{r,\xi}(\eta)\,d\xi, \tag{7.19}$$

(notice that we need $2\alpha - 1 + 2\lambda \geq 0$). From now on we fix λ such that

$$\beta = \alpha - 1 + \lambda, \tag{7.20}$$

and assume that $r \neq 1$, so that N_r is a sphere. Since $|\widehat{f}_2(\xi)|$ is a radial function, we can write that $|\widehat{f}_2(\xi)| = g(|\xi|)$ for an appropriate function g. Then we can apply Lemma 7.10 with $h(t) = g(t)^2t^{n-1-2\lambda}$ to the second integral of (7.19), and this yields

$$I_1^2 \leq Cr^{2\lambda}\left(\frac{r}{1+r^2}\right)^{(n-1)/2}\int_{\mathbb{R}^n} |\widehat{f}_1(\xi)|^2|\xi|^{2\beta}\int_0^\infty g(t)^2t^{n-2-2\lambda}t^{n-1}\,dt\,d\xi$$
$$\leq Cr^{2\lambda}\left(\frac{r}{1+r^2}\right)^{(n-1)/2}\|f_1\|^2_{W^{\beta,2}}\|f_2\|^2_{W^{(n-2)/2-\lambda,2}}.$$

Analogously for I_2, by Cauchy-Schwarz inequality and Lemma 3.6 we have

$$I_2^2 \leq C \int_{\mathbb{R}^n} |\eta|^{2\alpha-2} \int_{\Gamma_r^-(\eta)} |\widehat{f}_1(\xi)|^2 |\xi|^{n-1-2\lambda} |\widehat{f}_2(\eta-\xi)|^2 \, d\sigma_{r\eta}(\xi) \times \dots$$
$$\dots \times \int_{\Gamma_r(\eta)} \frac{1}{|\xi|^{n-1-2\lambda}} \, d\sigma_{r\eta}(\xi) \, d\eta$$
$$\leq Cr^{2\lambda} \int_{\mathbb{R}^n} |\eta|^{2\alpha-2+2\lambda} \int_{\Gamma_r^-(\eta)} |\widehat{f}_1(\xi)|^2 |\xi|^{n-1-2\lambda} |\widehat{f}_2(\eta-\xi)|^2 \, d\sigma_{r\eta}(\xi).$$

Then, changing the order of integration (Lemma 7.3) and using that $|\eta| \leq 2|\eta-\xi|$ in $\Gamma_r^-(\eta)$ gives

$$I_2^2 \leq Cr^{2\lambda} \int_{\mathbb{R}^n} |\widehat{f}_1(\xi)|^2 |\xi|^{n-2-2\lambda} \int_{N_r(\xi)} |\eta-\xi|^{2\alpha-1+2\lambda} |\widehat{f}_2(\eta-\xi)|^2 \, d\sigma_{r,\xi}(\eta) \, d\xi.$$

Therefore assuming again $r \neq 1$ we can apply Lemma 7.10, this time with $h(t) = g(t)^2 t^{2\alpha-1+2\lambda}$ and use (7.20) to get

$$I_2^2 \leq Cr^{2\lambda} \left(\frac{r}{1+r^2} \right)^{(n-1)/2} \int_{\mathbb{R}^n} |\widehat{f}_1(\xi)|^2 |\xi|^{n-2-2\lambda} \int_0^\infty |g(t)|^2 t^{2\beta} t^{n-1} \, dt \, d\xi$$
$$\leq Cr^{2\lambda} \left(\frac{r}{1+r^2} \right)^{(n-1)/2} \|f_1\|^2_{W^{(n-2)/2-\lambda,2}} \|f_2\|^2_{W^{\beta,2}}.$$

We now assume that $r = 1$. By (7.19) we have

$$I_1^2 \leq C \int_{\mathbb{R}^n} |\widehat{f}_1(\xi)|^2 |\xi|^{2\alpha-2+2\lambda} \int_{N_1(\xi)} |\widehat{f}_2(\eta-\xi)|^2 |\eta-\xi|^{n-1-2\lambda} \, d\sigma_\xi(\eta) \, d\xi. \tag{7.21}$$

The only difference with the case $r \neq 1$ is that $N(\xi) = N_1(\xi)$ is now an hyperplane and not a sphere. As in (7.14), in the second integral we introduce the change of variables $v = \eta - \xi$ which translates the $N(\xi)$ to the origin. Then we can take polar coordinates $v = s\theta$ in the resulting hyperplane. With a slight abuse of notation we can write that $\widehat{q}(s\theta) = \widehat{q}(s)$, since $\widehat{q}$ is radial. This yields

$$\int_{N(\xi)} |\widehat{f}_2(\eta-\xi)|^2 |\eta-\xi|^{n-1-2\lambda} \, d\lambda_\xi(\eta) = \int_0^{|\xi|} |\widehat{f}_2(s)|^2 s^{n-1-2\lambda} s^{n-2} \, ds$$
$$= \int_{|v| \leq |\xi|} |\widehat{f}_2(v)|^2 |v|^{n-2-2\lambda} \, dv,$$

and hence

$$I_1^2 \leq C \int_{\mathbb{R}^n} |\widehat{f}_1(\xi)|^2 |\xi|^{2\alpha-2+2\lambda} \, d\xi \int_{\mathbb{R}^n} |\widehat{f}_2(v)|^2 |v|^{n-2-2\lambda} \, dv$$
$$\leq C \|f_1\|_{W^{\beta,2}} \|f_2\|_{W^{(n-2)/2-\lambda,2}}.$$

The estimate of I_2 for $r = 1$ follows analogously. Hence, putting together the estimates of I_1 and I_2 for every $r \in (0,\infty)$, yields

$$\|\widetilde{K}_r(\widehat{f}_1,\widehat{f}_2)\|_{L^2_\alpha} \leq C r^\lambda \left(\frac{r}{1+r^2}\right)^{(n-1)/4} \|f_1\|_{W^{\beta,2}}\|f_2\|_{W^{(n-2)/2-\lambda,2}} + \|f_1\|_{W^{(n-2)/2-\lambda,2}}\|f_2\|_{W^{\beta,2}}. \quad (7.22)$$

Since we want to have the bound $r^\lambda\left(\frac{r}{1+r^2}\right)^{(n-1)/4} \leq (1+r)^{1-\gamma}$ for some $\gamma > 0$, we need to ask $\lambda - (n-1)/4 < 1$ and hence we need $\lambda < (n+3)/4$.

By (7.20), the condition $2\alpha - 1 + 2\lambda \geq 0$ used in the proof implies we must have $\beta \geq -1/2$. Then, equation (7.15) follows directly from (7.22) in the range $\beta \geq (n-2)/2$. But, together with (7.20), the restrictions imposed on λ yield

$$\begin{cases} 0 < \lambda < \frac{n+3}{4} \\ 0 < \lambda \leq \frac{n-1}{2} \end{cases} \iff \begin{cases} \beta + 1 - \frac{n+3}{4} < \alpha < \beta + 1 \\ \beta + 1 - \frac{n-1}{2} \leq \alpha < \beta + 1. \end{cases} \quad (7.23)$$

We can discard the lower bounds for α using that $\|f\|_{L^2_\alpha} \leq \|f\|_{L^2_{\alpha'}}$ always holds if $\alpha \leq \alpha'$. Therefore only the restriction $\alpha < \beta + 1$ remains.

Otherwise, if β is in the range $0 \leq \beta < (n-2)/2$, estimate (7.15) will follow if we add the extra condition

$$(n-2)/2 - \lambda \leq \beta. \quad (7.24)$$

Then, we have that $\beta \geq \min(-1/2,(n-7)/4)$ by the conditions on λ given in the left hand side of (7.23). Also, putting together (7.20) and (7.24) we get $\alpha \leq 2\beta - (n-4)/2$, which is a stronger condition than $\alpha < \beta + 1$ since we are in the range $\beta < (n-2)/2$. Hence, we have obtained the ranges of parameters given in the statement. □

Proof of Theorem 2.9. Let $n \geq 2$ and assume that $q \in \mathcal{S}(\mathbb{R}^n)$ is a radial function. Then (3.28) and Lemma 7.8 yield the estimate

$$\|\widetilde{S}_r(q)\|_{L^2_\alpha} \leq C(1+r)^{-\gamma}\|q\|^2_{W^{\beta,2}}, \quad (7.25)$$

for some $\gamma > 0$, which can depend on β, and α in the range (7.16). Also, multiplying (3.33) by $\chi(\eta)$ and taking the $L^2_{\alpha-1}$ norm we get

$$\|\partial_r \widetilde{S}_r(q)\|_{L^2_{\alpha-1}} \leq C\|\widetilde{K}_r(\widehat{q},\widehat{q})\|_{L^2_{\alpha-1}} + C\sum_{i=1}^n \|\widetilde{K}_r(\widehat{x_i q},\widehat{q})\|_{L^2_\alpha},$$

assuming that $r \in (1+\delta, 1-\delta)$, for some $0 < \delta < 1$ fixed. Then we can apply again Lemma 7.8 to the first term on the right hand side with $f_1 = f_2 = q$, and to the remaining terms with $f_2 = q$ to obtain

$$\|\widetilde{K}_r(\widehat{x_i q},\widehat{q})\|_{L^2_\alpha} \leq C\|x_i q\|_{W^{\beta,2}}\|q\|_{W^{\beta,2}} \leq \|q\|_{W^{\beta,2}_1}\|q\|_{W^{\beta,2}},$$

for α in the range (7.16) (we have used again Remark 3.7). This yields

$$\|\partial_r \widetilde{S}_r(q)\|_{L^2_{\alpha-1}} \leq C\|q\|^2_{W_1^{\beta,2}}, \tag{7.26}$$

for $r \in (1+\delta, 1-\delta)$. This means that by (3.24) we can apply Plancherel theorem and Lemma 3.3 with $F_r(\eta) = \widetilde{S}_r(q)(\eta)$, $p = 2$, $M = C\|q\|^2_{W_1^{\beta,2}}$ and $\tau = \alpha - 1$ to get estimate (2.22) for $\widetilde{Q}_2(q)$. The extension for every $q \in W_1^{\beta,2}(\mathbb{R}^n)$ radial, follows by the usual density argument, Lemma A.3 (this time the dense subset D in the lemma will be the subset of radial functions of the Schwartz class).

On the other hand, the necessary condition for $\varepsilon(\beta)$ is given by Theorem 2.5. □

Proof of Corollary 2.11. By Lemma 6.3 and Theorem 2.4 it is enough to show that

$$\|\widetilde{Q}_2(q)\|_{W^{\alpha,2}} < \infty,$$

for $\alpha < \beta + \varepsilon(\beta)$, and $\varepsilon(\beta)$ given by (2.23). We sketch the main ideas of the proof.

Observe that in Lemma 7.8, we have used only that $|\widehat{f}_2(\xi)|$ is a radial function. By (3.26) and (3.27) and the assumption that $|\widehat{q}| \leq \widehat{g}$, we have

$$\widetilde{K}_r(\widehat{f}_1, \widehat{q}) \leq \widetilde{K}_r(\widehat{f}_1, \widehat{g}).$$

Hence, applying Lemma 7.8 to the right hand side with $f_2 = g$ yields $\|\widetilde{K}_r(\widehat{f}_1, \widehat{q})\|_{L^2_\alpha} < \infty$. This estimate can be used to show, exactly as we did to obtain (7.25) and (7.26), that for some constant M,

$$\|\widetilde{S}_r(q)\|_{L^2_\alpha} < (1+r)^{-\gamma}M, \qquad \|\partial_r \widetilde{S}_r(q)\|_{L^2_{\alpha-1}} < M,$$

for α in the range (7.16), and, respectively, for $r \in (1-\delta, 1+\delta)$ and $r \in (0, \infty)$. The reader may object that to get (7.25) and (7.26) we have assumed $q \in \mathcal{S}(\mathbb{R}^n)$. But this restriction is not necessary, since (3.33) holds if $\widehat{q}$ is smooth. This is certainly satisfied in this case since, by assumption, q has compact support. Also, by Lemma 3.8, we know that the fact that q is smooth implies that,for every $\eta \neq 0$ fixed, $\partial \widetilde{S}_r(q)(\eta)$ is smooth in the r variable.

Therefore, by (3.24), we can apply Lemma 3.3 with $F_r(\eta) = \widetilde{S}_r(q)(\eta)$ to get an estimate for $\widehat{\widetilde{Q}_2(q)}$ in L^2_α. To finish, Plancherel theorem yields $\|\widetilde{Q}_2(q)\|_{W^{\alpha,2}} < \infty$. □

7.3 A Fubini theorem in the Ewald spheres

We now give the proof of Lemma 7.3 used in the estimate of the spherical operator S_r. The case of $r = 1$ is proved in [48]. We prove a more general statement that has been used in [5], for Ewald spheres that depend on two independent parameters instead of one. Let $a, b > 0$, we define

$$\Phi := \{(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n : |\xi - a\eta| = b|\eta|\},$$

$$\Gamma_{a,b}(\eta) := \{\xi \in \mathbb{R}^n : |\xi - a\eta| = b|\eta|\}, \qquad N_{a,b}(\xi) := \{\eta \in \mathbb{R}^n : |\xi - a\eta| = b|\eta|\},$$

and let $\sigma_{a,b,\eta}(\xi)$ and $\sigma_{a,b,\xi}(\eta)$ be, respectively, the restriction of the Lebesgue measure to the last two submanifolds of $\mathbb{R}^n$. In this case $N_{a,b}(\xi)$ is the sphere of center $\frac{a}{(a^2-b^2)}\xi$ and radius $\frac{b}{|a^2-b^2|}|\xi|$.

Lemma 7.11. *Let $f \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ and assume that $a \neq b$. Then we have that*

$$\int_{\mathbb{R}^n}\int_{\Gamma_{a,b}(\eta)} f(\eta,\xi)\, d\sigma_{a,b,\eta}(\xi)\, d\eta = \int_{\mathbb{R}^n}\int_{N_{a,b}(\xi)} f(\eta,\xi)\frac{|\eta|}{|\xi|}\, d\sigma_{a,b,\xi}(\eta)\, d\xi.$$

Lemma 7.3 is just the case $a = 1/2$ and $b = r/2$ of the previous statement.

Proof. The result follows by direct computation using the language of differential forms. We denote the volume form of $\mathbb{R}^n$ in coordinates $(\xi_1, \dots, \xi_2)$ and $(\eta_1, \dots, \eta_2)$ by, respectively, $d\xi = d\xi_1 \wedge \cdots \wedge d\xi_n$ and $d\eta = d\eta_1 \wedge \cdots \wedge d\eta_n$. Also, we denote by ω_η the natural volume n-form of the sphere $\Gamma_r(\eta)$ and by ω_ξ the volume n-form of $N_r(\xi)$. Hence ω_η is associated to the measure $\sigma_{a,b,\eta}$ and ω_ξ to $\sigma_{a,b,\xi}$.

Since $\Gamma_{a,b}(\eta)$ is an hypersurface, ω_η is just the contraction of its (exterior) unit normal vector field $\nu(\xi)$ with the volume form $d\xi$. Similarly, ω_ξ is the contraction with the unit normal field to $N_{a,b}(\xi)$, $\nu(\eta)$, with the volume form $d\eta$. Since both hypersurfaces are spheres, these vector fields can be computed very easily in coordinates

$$\nu(\xi) = \frac{1}{b|\eta|}(\xi - a\eta), \quad \text{and} \quad \nu(\eta) = \frac{|a^2-b^2|}{b|\xi|}\left(\eta - \frac{a}{a^2-b^2}\xi\right).$$

Therefore we can compute the following coordinate expressions,

$$\omega_\eta \wedge d\eta = \frac{1}{b|\eta|}\sum_{i=1}^n (-1)^{i+1}(\xi_i - a\eta_i)d\xi_1 \wedge \cdots \wedge \widehat{d\xi_i} \wedge \cdots \wedge d\xi_n \wedge d\eta,$$

$$\omega_\xi \wedge d\xi = \frac{|a^2-b^2|}{b|\xi|}\sum_{i=1}^n (-1)^{i+1}\left(\eta_i - \frac{a}{a^2-b^2}\xi_i\right) d\eta_1 \wedge \cdots \wedge \widehat{d\eta_i} \wedge \cdots \wedge d\eta_n \wedge d\xi,$$

where the notation $\widehat{d\xi_i}$ means that we are omitting the 1-form $d\xi_i$ in the wedge product. $\omega_\eta \wedge d\eta$ and $\omega_\xi \wedge d\xi$ are volume forms on the $\mathbb{R}^n \times \mathbb{R}^n$ submanifold Φ. To compare them, we want to write both forms in coordinates as similarly as possible . This can be achieved by using the structural relation

$$\sum_{i=1}^n \left(2(a^2-b^2)\left(\eta_i - \frac{a}{a^2-b^2}\xi_i\right) d\eta_i + 2(\xi_i - a\eta_i)d\xi_i\right) = 0,$$

obtained just by taking the exterior differential of the function $|\xi - a\eta|^2 - b^2|\eta|^2$, which is constant on Φ by definition. Assume that we are in the open set given by

$(\xi_1 - a\eta_1) \neq 0$ (we can choose any of the other possible conditions $(\xi_i - a\eta_i) \neq 0$ without difference). Then we can write

$$d\xi_1 = \frac{1}{(a\eta_1 - \xi_1)} \left(\sum_{i=1}^{n} (a^2 - b^2) \left(\eta_i - \frac{a}{a^2 - b^2}\xi_i \right) d\eta_i + \sum_{i=2}^{n} (\xi_i - a\eta_i) d\xi_i \right).$$

Introducing this equation in the coordinate expressions of $\omega_\eta \wedge d\eta$ and $\omega_\xi \wedge d\xi$ most products cancel out, and after some computations we obtain that

$$\begin{aligned}
\omega_\eta \wedge d\eta &= \frac{1}{b|\eta|(a\eta_1 - \xi_1)} \sum_{i=1}^{n} -(\xi_i - a\eta_i)^2 d\xi_2 \wedge \ldots \cdots \wedge d\xi_n \wedge d\eta \\
&= -\frac{b|\eta|}{(a\eta_1 - \xi_1)} d\xi_2 \wedge \ldots \cdots \wedge d\xi_n \wedge d\eta, \\
\omega_\xi \wedge d\xi &= \frac{|a^2 - b^2|(a^2 - b^2)}{b|\xi|(a\eta_1 - \xi_1)} \sum_{i=1}^{n} (-1)^{n^2-1} \left(\eta_i - \frac{a}{a^2 - b^2}\xi_i \right)^2 d\xi_2 \wedge \ldots \cdots \wedge d\xi_n \wedge d\eta \\
&= (-1)^{n^2-1} \frac{|a^2 - b^2|}{a^2 - b^2} \frac{b|\xi|}{(a\eta_1 - \xi_1)} d\xi_2 \wedge \ldots \cdots \wedge d\xi_n \wedge d\eta.
\end{aligned}$$

Comparing both expressions we see that except for the sign, both volume forms on Φ differ by a $|\eta|/|\xi|$ factor. This yields the desired result, returning to the notation with the measures $\sigma_{a,b,\eta}$ and $\sigma_{a,b,\xi}$. □

Appendix A

Some technical results

A.1 Spherical Integrals

The following proposition gives an upper bound for the constant in the trace theorem on spheres.

Proposition A.1. *Let $f \in W^{1,2}(\mathbb{R}^n)$, and let $\mathbb{S}_\rho \subset \mathbb{R}^n$ be the any sphere of radius ρ. Then we have that*

$$\int_{\mathbb{S}_\rho} |f(x)|^2 \, d\sigma_\rho(x) \leq \int_{\mathbb{R}^n} |f(x)|^2 \, dx + \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx. \tag{A.1}$$

Proof. Assume that $\mathbb{S}_\rho$ is centered in the origin. The general case follows by the invariance under translations of the Sobolev norm. Without loss of generality consider a real function $f \in C_c^\infty(\mathbb{R}^n)$. Then using spherical coordinates with $\theta \in \mathbb{S}^{n-1}$ we have that

$$\frac{d}{dr}\left(f(r\theta)^2 r^{n-1}\right) = 2\frac{df}{dr}(r\theta) f(r\theta) r^{n-1} + (n-1) f^2(r\theta) r^{n-2}.$$

Fix $\rho \in (0, \infty)$. If we integrate the previous equation in the r variable, by the fundamental theorem of calculus and the compactness of the support of f we have

$$\begin{aligned} f(\rho\theta)^2 \rho^{n-1} &= -\int_\rho^\infty 2\frac{df}{dr}(r\theta) f(r\theta) r^{n-1} \, dr - (n-1)\int_\rho^\infty f^2(r\theta) r^{n-2} \, dr \\ &\leq 2\int_0^\infty |\nabla f(r\theta)||f(r\theta)| r^{n-1} \, dr, \end{aligned}$$

since the second integral in the first line is negative. Then, integrating both sides in the unit sphere $\mathbb{S}^{n-1}$ we recover the statement of the proposition,

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} f(\rho\theta)^2 \rho^{n-1} d\sigma(\theta) &\leq 2\int_{\mathbb{R}^n} |\nabla f(x)||f(x)| \, dx \\ &\leq \int_{\mathbb{R}^n} |f(x)|^2 \, dx + \int_{\mathbb{R}^n} |\nabla f(x)|^2 \, dx. \end{aligned}$$

□

Lemma A.2. *Let $\xi_1, \ldots, \xi_k \in \mathbb{R}^n$, and $\mathbf{r} \in (0,\infty)^k$. Assume that for every $\eta \in \mathbb{R}^n$ the function $F(\xi_1, \xi_2, ..., \xi_k, \eta)$ is C^1 in the first k variables. Then if $\alpha_1, \ldots, \alpha_k$ are multi indices corresponding to the variables $\xi_1, \ldots, \xi_k$ we have that*

$$\int_{\Gamma_{\mathbf{r}}(\eta)} |F(\xi_1, ..., \xi_k, \eta)|^2 \, d\sigma_{\mathbf{r}}(\xi_1, \ldots, \xi_k)$$
$$\leq C \sum_{0 \leq |\alpha_1|, \ldots, |\alpha_k| \leq 1} \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} |\partial_{\xi_1}^{\alpha_1} \ldots \partial_{\xi_k}^{\alpha_k} F(\xi_1, ..., \xi_k, \eta)|^2 \, d\xi_1 \ldots d\xi_k,$$

where the constant C does not depend on η, or $\mathbf{r}$.

Proof. The general case follows inductively from (A.1). □

We now give the proof of Lemma 3.6, used in the estimate of the spherical operator.

Proof of Lemma 3.6. We can consider only the case of spheres $\mathbb{S}_\rho$ centered on the origin. By homogeneity, if $y = \rho\theta$ we have

$$\int_{\mathbb{S}_\rho} \frac{1}{|x-y|^{(n-1)-2\gamma}} \, d\sigma_\rho(y) = \rho^{2\gamma} \int_{\mathbb{S}^{n-1}} \frac{1}{|x'-\theta|^{(n-1)-2\gamma}} \, d\sigma(\theta),$$

where $x'\rho = x$ and $\mathbb{S}^{n-1}$ is the sphere of radius 1 centered on the origin. Hence we need to bound uniformly on x' the last integral. Now, assume that $x' \neq 0$ and take $\omega \in \mathbb{S}^{n-1}$ such that $\omega = x'/|x'|$. Let $P_\omega = \{x \in \mathbb{R}^n : x \cdot \omega = 0\}$, and let $P(z) := z - (z \cdot \omega)\omega$, be the projection of $z \in \mathbb{R}^n$ on the plane P_ω. Consider the half sphere comprised between this plane and the parallel one that goes trough ω. The Jacobian of the projection P restricted to $\mathbb{S}^{n-1}$ is bounded if we exclude a small band of ε width from it. Let's denote this region by S_ε (the half sphere minus the band). In the first place we have

$$\int_{\mathbb{S}^{n-1}} \frac{1}{|x'-\theta|^{(n-1)-2\gamma}} \, d\sigma(\theta) \leq 2n \int_{S_\varepsilon} \frac{1}{|x'-\theta|^{(n-1)-2\gamma}} \, d\sigma(\theta),$$

this is because in the region S_ε the integrand has larger values than in the rest of the sphere, since we are in the half which is closer to x', and it is possible to cover generously $\mathbb{S}^{n-1}$ with $2n$ pieces like S_ε. But since the Jacobian of P is bounded we can use the change of variables $y = P(\theta)$ to integrate in the corresponding region of the plane. Hence

$$\int_{S_\varepsilon} \frac{1}{|x'-\theta|^{(n-1)-2\gamma}} \, d\sigma(\theta) \leq \int_{S_\varepsilon} \frac{1}{|P(\theta)|^{(n-1)-2\gamma}} \, d\sigma(\theta)$$
$$\leq C \int_{P(S_\varepsilon)} \frac{1}{|y|^{(n-1)-2\gamma}} \, dy \leq C \int_{\mathbb{R}^{n-1} \cap B_1} \frac{1}{|y|^{(n-1)-2\gamma}} \, dy,$$

where we have used that $P(x') = 0$. The last integral is finite and it does not depend in any way on x' and therefore we have finished. □

Proof of Lemma 7.4. Consider $\mathbb{S}_\rho$ centered in the origin. Assume that $x \neq 0$, and take $\omega \in \mathbb{S}_\rho$ such that $\omega = x/|x|$. Let $P_\omega = \{x \in \mathbb{R}^n : x \cdot \omega = 0\}$, and let $P(z) := z - (z \cdot \omega)\omega$, be the projection of $z \in \mathbb{R}^n$ on the plane P_ω. Consider the half sphere comprised between the plane P_ω and the parallel one that goes trough x. The Jacobian of the projection P restricted to $\mathbb{S}_\rho$ is uniformly bounded in ρ if we exclude a small band of $\rho\varepsilon$ width from it. Let's denote this region by $S_{\rho\varepsilon}$ (the half sphere minus the band). We have that

$$\int_{\mathbb{S}_\rho} \frac{1}{|x-y|^a \langle x-y\rangle^b}\, d\sigma_\rho(y) \leq 2n \int_{S_{\rho\varepsilon}} \frac{1}{|x-y|^a \langle x-y\rangle^b}\, d\sigma_\rho(y),$$

since in the region $S_{\rho\varepsilon}$ the integrand has larger values than in the rest of the sphere (we are in the half which is closer to x', and it is possible to cover generously $\mathbb{S}^{n-1}$ with $2n$ pieces like $S_{\rho\varepsilon}$). Then we can use the change of variables $z = P(y)$ to integrate in the corresponding region of the plane. Hence, since the integrand is a decreasing function,

$$\begin{aligned}\int_{S_{\rho\varepsilon}} \frac{1}{|x-y|^a \langle x-y\rangle^b}\, d\sigma_\rho(y) &\leq \int_{S_{\rho\varepsilon}} \frac{1}{|P(y)|^a \langle P(y)\rangle^b}\, d\sigma_\rho(y)\\ &\leq C \int_{P(S_{\rho\varepsilon})} \frac{1}{|z|^a \langle z\rangle^b}\, dz \leq C \int_{\mathbb{R}^{n-1}} \frac{1}{|z|^a \langle z\rangle^b}\, dz < \infty,\end{aligned}$$

where we have used that $P(x) = 0$. □

A.2 Density lemma

Lemma A.3. *Let X, Y be Banach spaces, and let $D \subset Y$ be a dense subspace. Consider an operator $T : D \longrightarrow X$ such that T is the restriction to the diagonal of a multlinear operator of order j. That is, assume that there is some $Q : D\times, \dots, \times D \longrightarrow X$ multilinear such that for every $f \in D$, $T(f) = Q(f, \dots, f)$. Then, if for every $f \in D$*

$$\|T(f)\|_X \leq C\|f\|_Y^k, \tag{A.2}$$

we have that there is a unique continuous extension of T to the whole space Y, and it satisfies the estimate (A.2) for every $f \in Y$.

Proof. Let $\{g_i\}_{i\in\mathbb{N}}$, $g_i \in D$ for every $i \in \mathbb{N}$, be a Cauchy sequence in the Y norm. To prove the proposition it is enough to show that then $\{T(g_i)\}_{i\in\mathbb{N}}$ is also a Cauchy sequence in X, since this implies that there must be a unique continuous extension of T to the whole space Y.

Without loss of generality we can consider Q symmetric, since otherwise we can take its symmetric part:

$$Q_S(f_1, \dots, f_j) := \frac{1}{j!} \sum_\sigma Q(f_{\sigma(1)}, \dots, f_{\sigma(j)}),$$

where the sum is over all the permutations σ of j elements. Therefore using the symmetry and the multilinearity we have

$$T(g_k) - T(g_l) = Q(g_k - g_l, g_k, \dots, g_k) + Q(g_l, g_k - g_l, g_k, \dots, g_k) \\ + Q(g_l, \dots, g_l, g_k - g_l). \quad \text{(A.3)}$$

Now we can use a polarization identity for multilinear operators to express each of the previous terms as combinations of diagonal terms. See [53] for the explicit derivation of the identity:

$$j!Q(f_1, \dots, f_k) = \sum_{m=1}^{j} (-1)^{j-m} \sum_{J,|J|=m} T(S_J),$$

where the inner sum in the right hand side is over all distinct subsets $J \subset \{1, 2, \dots, j\}$ of m elements, and $S_J = \sum_{i \in J} f_i$. Since each term in the last line of (A.3) can be treated in the same way, we illustrate only one case. Let $h > 0$ be a (small) constant that we will choose later. Then the polarization identity can be written in the following way

$$Q(g_l, g_k - g_l, g_k, \dots, g_k) = Q(hg_l, h^{-(j-1)}(g_k - g_l), hg_k, \dots, hg_k) \\ = \frac{1}{j!} \sum_{0<a+b+c\leq j} (-1)^{j-1-(a+b+c)} N(a,b,c) T(ah^{-(j-1)}(g_l - g_k) + h(bg_l + cg_k)),$$

where a, b, c are integers satisfying $0 \leq a, b \leq 1$, since g_l and $g_l - g_k$ appear only once in the term we have chosen, and $0 \leq c \leq j - 2$ since g_k appears $j - 2$ times (the integer coefficient $N(a, b, c)$ is just to account for repetitions).

Since $\{g_i\}$ is a Cauchy sequence, it is bounded, so $\|g_i\|_Y \leq M$ for every $i \in \mathbb{N}$ and some constant $M > 0$. Hence, taking the X norm and using estimate (3.40) we obtain

$$\|Q(g_l, g_k - g_l, g_k, \dots, g_k)\|_X \leq \\ = \frac{1}{j!} \sum_{0<a+b+c\leq j} N(a,b,c) \|T(ah^{-(j-1)}(g_l - g_k) + h(bg_l + cg_k))\|_X \\ \leq C(j) \left(h^{-j(j-1)} \|g_l - g_k\|_Y^j + h^j M^j \right) < \varepsilon / j, \quad \text{(A.4)}$$

for the following choices ($C(j)$ is some constant dependent only on j),

$$h^j < \frac{\varepsilon}{2jC(j)M^j}, \quad \text{and} \quad \|g_l - g_k\|_Y < \frac{\varepsilon}{2jC(j)M^{j-1}}.$$

So, using (A.4) for each term in (A.3) we finally obtain

$$\|T(g_k) - T(g_l)\|_X < \varepsilon,$$

which shows that $\{T(g_i)\}_{i \in \mathbb{N}}$ is a Cauchy sequence in X. □

A.3 Smoothness of $T_{j,k}(f)$ in $\mathcal{S}(\mathbb{R}^n)$

Proposition A.4. *Let $f \in \mathcal{S}(\mathbb{R}^n)$ and $1 \le k \le j$. Then we have that*

$$T_{j,k}(f)(\eta) = \prod_{i=1}^{k-1} (i\pi d_i + P_i) \widetilde{S}_{j,\mathbf{r}}(f)(\eta),$$

is a well defined function in $\mathcal{S}\left((0,\infty)^{j-k} \times \mathbb{R}^n\right)$. Moreover, for $k \le m \le j-1$

$$\prod_{i=1}^{k-1} (i\pi d_i + P_i) \partial_{r_m} \widetilde{S}_{j,\mathbf{r}}(f)(\eta) = \partial_{r_m} \prod_{i=1}^{k-1} (i\pi d_i + P_i) \widetilde{S}_{j,\mathbf{r}}(f)(\eta). \tag{A.5}$$

Proof. We only sketch some of the main computations. Let's verify that $\widetilde{S}_{j\mathbf{r}}(f)$ is a function in $\mathcal{S}\left((0,\infty)^{j-1} \times \mathbb{R}^n\right)$ (that is, the case $k = 0$). First any derivative in the η or $\mathbf{r}$ variables can be computed as in (3.34) (see also (3.57)). The $|\eta|$ factors appearing in the expression of $\widetilde{S}_{j,\mathbf{r}}$ and its derivatives are non-smooth for $\eta = 0$, but this is not a problem since we have the smooth cut-off $\chi(\eta)$ which vanishes in the origin. Essentially the estimate of each Schwartz class seminorm can be reduced to the basic case

$$\begin{aligned}
&\langle \eta \rangle^\gamma \int_{\Gamma_{\mathbf{r}}(\eta)} |f(\eta - \xi_1)| \left(\prod_{i=1}^{j-2} |f(\xi_i - \xi_{i+1})| \right) |f(\xi_{j-1})| \, d\sigma_{\mathbf{r}} \le \frac{C}{|\eta|^{n-1}} \cdots \\
&\int_{\Gamma_{\mathbf{r}}(\eta)} |f(\eta - \xi_1)| \langle \eta - \xi_1 \rangle^{\gamma'} \left(\prod_{i=1}^{j-2} |f(\xi_i - \xi_{i+1})| \langle \xi_i - \xi_{i+1} \rangle^{\gamma'} \right) |f(\xi_{j-1})| \langle \xi_{j-1} \rangle^{\gamma'} d\sigma_{\mathbf{r}} \\
&\le \|f \langle \cdot \rangle^{\gamma'}\|_\infty^j,
\end{aligned}$$

where $\gamma' = \gamma + n - 1$. If instead of a weight $\langle \eta \rangle^\gamma$ we have $\langle \mathbf{r} \rangle^\gamma$, an analogous procedure can be followed using that $r_i = 2|\xi_i - \eta/2|/|\eta| \le C(1 + |\xi_i|)$ for $|\eta| > C_0$, that is, where the cut-off $\chi(\eta)$ does not vanish.

We give the following indications to prove (A.5) and that

$$T_{j,k}(f) \in \mathcal{S}\left((0,\infty)^{j-1-k} \times \mathbb{R}^n\right),$$

for $k > 0$. Let $g \in \mathcal{S}((0,\infty)^k)$ be a function of the variable $\mathbf{r} \in (0,\infty)^k$. Is not very difficult to bound the principal value operators in the Schwartz class since we have the estimate

$$\|P_i(g)\| \le C(\|\partial_{r_i} g\|_\infty + \|\langle r_i \rangle^\varepsilon g\|_\infty),$$

for $\varepsilon > 0$. This implies that $\partial_{r_m} P_i(g) = P_i(\partial_{r_m} g)$ since the limit that defines the derivative ∂_{r_m} is continuous in the norms of the right hand side (this is a consequence of the mean value theorem together with the fact that $g \in S((0,\infty)^k)$ which means that all the derivatives are uniformly bounded). The same reasoning can be applied to control the partial derivatives in η of $T_{j,k}(f)$. □

Bibliography

[1] S. Agmon. Spectral properties of Schrödinger operators and scattering theory. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 2(2):151–218, 1975. URL: http://www.numdam.org/item?id=ASNSP_1975_4_2_2_151_0.

[2] J. A. Barceló, D. Faraco, A. Ruiz, and A. Vargas. Reconstruction of singularities from full scattering data by new estimates of bilinear Fourier multipliers. *Math. Ann.*, 346(3):505–544, 2010. doi:10.1007/s00208-009-0398-5.

[3] J. A. Barceló, D. Faraco, A. Ruiz, and A. Vargas. Reconstruction of discontinuities from backscattering data in two dimensions. *SIAM J. Math. Anal.*, 45(6):3494–3513, 2013. doi:10.1137/120902963.

[4] J. A. Barceló, M. Folch-Gabayet, S. Pérez-Esteva, A. Ruiz, and M. C. Vilela. A Born approximation for live loads in Navier elasticity. *SIAM J. Math. Anal.*, 44(4):2824–2846, 2012. doi:10.1137/110856265.

[5] J. A. Barceló, M. Folch-Gabayet, S. Pérez-Esteva, A. Ruiz, and M. C. Vilela. A Born approximation from backscattering data for live loads in Lamé system. *Rev. Mat. Iberoam.*, 31(4):1423–1458, 2015. doi:10.4171/RMI/875.

[6] B. Barrios, I. Peral, F. Soria, and E. Valdinoci. A Widder's type theorem for the heat equation with nonlocal diffusion. *Arch. Ration. Mech. Anal.*, 213(2):629–650, 2014. doi:10.1007/s00205-014-0733-1.

[7] A. Bayliss, Y. Y. Li, and C. S. Morawetz. Scattering by a potential using hyperbolic methods. *Math. Comp.*, 52(186):321–338, 1989. doi:10.2307/2008470.

[8] I. Beltiţă and A. Melin. Local smoothing for the backscattering transform. *Comm. Partial Differential Equations*, 34(1-3):233–256, 2009. doi:10.1080/03605300902812384.

[9] I. Beltiţă and A. Melin. The quadratic contribution to the backscattering transform in the rotation invariant case. *Inverse Probl. Imaging*, 4(4):599–618, 2010. doi:10.3934/ipi.2010.4.599.

[10] G. Beylkin. Imaging of discontinuities in the inverse scattering problem by inversion of a causal generalized Radon transform. *J. Math. Phys.*, 26(1):99–108, 1985. doi:10.1063/1.526755.

[11] D. Colton and R. Kress. *Inverse acoustic and electromagnetic scattering theory*, volume 93 of *Applied Mathematical Sciences*. Springer, New York, third edition, 2013. doi:10.1007/978-1-4614-4942-3.

[12] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012. doi:10.1016/j.bulsci.2011.12.004.

[13] G. Eskin and J. Ralston. The inverse backscattering problem in three dimensions. *Comm. Math. Phys.*, 124(2):169–215, 1989. URL: http://projecteuclid.org/euclid.cmp/1104179141.

[14] G. Eskin and J. Ralston. Inverse backscattering in two dimensions. *Comm. Math. Phys.*, 138(3):451–486, 1991. URL: http://projecteuclid.org/euclid.cmp/1104203038.

[15] G. Eskin and J. Ralston. Inverse backscattering. *J. Anal. Math.*, 58:177–190, 1992. Festschrift on the occasion of the 70th birthday of Shmuel Agmon. doi:10.1007/BF02790363.

[16] G. Eskin. *Lectures on linear partial differential equations*, volume 123 of *Graduate Studies in Mathematics*. American Mathematical Society, 2011. doi:10.1090/gsm/123.

[17] S. Eswarathasan. Microlocal analysis of scattering data for nested conormal potentials. *J. Funct. Anal.*, 262(5):2100–2141, 2012. doi:10.1016/j.jfa.2011.12.013.

[18] A. Galindo and P. Pascual. *Quantum mechanics. II.* Texts and Monographs in Physics. Springer-Verlag, Berlin, 1991. Translated from the second Spanish edition by L. Alvarez-Gaumé. doi:10.1007/978-3-642-84129-3.

[19] A. Greenleaf and G. Uhlmann. Recovering singularities of a potential from singularities of scattering data. *Comm. Math. Phys.*, 157(3):549–572, 1993. URL: http://projecteuclid.org/euclid.cmp/1104254021.

[20] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 24 of *Monographs and Studies in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1985.

[21] L. Hörmander. *The analysis of linear partial differential operators. IV.* Classics in Mathematics. Springer-Verlag, Berlin, 2009. Fourier integral operators, Reprint of the 1994 edition. doi:10.1007/978-3-642-00136-9.

[22] G. Hu, M. Salo, and E. V. Vesalainen. Shape identification in inverse medium scattering problems with a single far-field pattern, 2015. arXiv:arXiv:1507.07846.

[23] C. E. Kenig, A. Ruiz, and C. D. Sogge. Uniform Sobolev inequalities and unique continuation for second order constant coefficient differential operators. *Duke Math. J.*, 55(2):329–347, 1987. doi:10.1215/S0012-7094-87-05518-9.

[24] S. G. Krantz and H. R. Parks. *Geometric integration theory*. Cornerstones. Birkhäuser Boston, Inc., Boston, MA, 2008. doi:10.1007/978-0-8176-4679-0.

[25] R. P. Lagergren. *The back-scattering problem in three dimensions*. ProQuest LLC, Ann Arbor, MI, 2001. Thesis (Ph.D.)–Lunds Universitet (Sweden). URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:C809013.

[26] M. Leckband. A note on the spherical maximal operator for radial functions. *Proc. Amer. Math. Soc.*, 100(4):635–640, 1987. doi:10.2307/2046697.

[27] C. J. Meroño. Recovery of the singularities of a potential from backscattering data in general dimension. arXiv:1709.00748.

[28] C. J. Meroño. Fixed angle scattering: Recovery of singularities and its limitations. *SIAM J. Math. Anal.*, 50(5):5616–5636, 2018. doi:10.1137/18M1164871.

[29] C. J. Meroño. Optimal estimates for the double dispersion operator in backscattering, 2018. arXiv:1807.08961.

[30] K. Mochizuki. Eigenfunction expansions associated with the Schrödinger operator with a complex potential and the scattering theory. *Publ. Res. Inst. Math. Sci. Ser. A*, 4:419–466, 1968/69. doi:10.2977/prims/1195194884.

[31] P. Ola, L. Päivärinta, and V. Serov. Recovering singularities from backscattering in two dimensions. *Comm. Partial Differential Equations*, 26(3-4):697–715, 2001. doi:10.1081/PDE-100001768.

[32] L. Päivärinta, M. Salo, and E. V. Vesalainen. Strictly convex corners scatter. *Rev. Mat. Iberoam.*, 33(4):1369–1396, 2017. doi:10.4171/RMI/975.

[33] L. Päivärinta and V. Serov. Recovery of singularities of a multidimensional scattering potential. *SIAM J. Math. Anal.*, 29(3):697–711, 1998. doi:10.1137/S0036141096305796.

[34] L. Päivärinta, V. S. Serov, and E. Somersalo. Reconstruction of singularities of a scattering potential in two dimensions. *Adv. in Appl. Math.*, 15(1):97–113, 1994. doi:10.1006/aama.1994.1003.

[35] L. Päivärinta and V. Serov. Recovery of jumps and singularities in the multidimensional Schrödinger operator from limited data. *Inverse Probl. Imaging*, 1(3):525–535, 2007. doi:10.3934/ipi.2007.1.525.

[36] L. Päivärinta and E. Somersalo. Inversion of discontinuities for the Schrödinger equation in three dimensions. *SIAM J. Math. Anal.*, 22(2):480–499, 1991. doi:10.1137/0522031.

[37] R. T. Prosser. Formal solutions of inverse scattering problems. *J. Mathematical Phys.*, 10:1819–1822, 1969. doi:10.1063/1.1664766.

[38] R. T. Prosser. Formal solutions of inverse scattering problems. II. *J. Mathematical Phys.*, 17(10):1775–1779, 1976. doi:10.1063/1.522819.

[39] R. T. Prosser. Formal solutions of inverse scattering problems. III. *J. Math. Phys.*, 21(11):2648–2653, 1980. doi:10.1063/1.524379.

[40] R. T. Prosser. Formal solutions of inverse scattering problems. IV. Error estimates. *J. Math. Phys.*, 23(11):2127–2130, 1982. doi:10.1063/1.525267.

[41] R. T. Prosser. Formal solutions of inverse scattering problems. V. *J. Math. Phys.*, 33(10):3493–3496, 1992. doi:10.1063/1.529898.

[42] Rakesh and G. Uhlmann. Uniqueness for the inverse backscattering problem for angularly controlled potentials. *Inverse Problems*, 30(6):065005, 24, 2014. doi:10.1088/0266-5611/30/6/065005.

[43] J. M. Reyes. Problema inverso de scattering para la ecuación de schr odinger: Reconstrucción parcial del potencial a partir de datos de retrodispersión en 2d y 3d. ph.d. thesis. universidad autónoma de madrid., 2007. URL: http://www.uam.es/gruposinv/inversos/publicaciones/index.html.

[44] J. M. Reyes. Inverse backscattering for the Schrödinger equation in 2D. *Inverse Problems*, 23(2):625–643, 2007. doi:10.1088/0266-5611/23/2/010.

[45] J. M. Reyes and A. Ruiz. Reconstruction of the singularities of a potential from backscattering data in 2D and 3D. *Inverse Probl. Imaging*, 6(2):321–355, 2012. doi:10.3934/ipi.2012.6.321.

[46] A. Ruiz. Recovery of the singularities of a potential from fixed angle scattering data. *Comm. Partial Differential Equations*, 26(9-10):1721–1738, 2001. doi:10.1081/PDE-100107457.

[47] A. Ruiz. Harmonic analysis and inverse problems. notes of the 4th summer school in inverse problems, oulu, finland, 2002. URL: http://www.uam.es/gruposinv/inversos/publicaciones/Inverseproblems.pdf.

[48] A. Ruiz and A. Vargas. Partial recovery of a potential from backscattering data. *Comm. Partial Differential Equations*, 30(1-3):67–96, 2005. doi:10.1081/PDE-200044450.

[49] L. E. Silvestre. *Regularity of the obstacle problem for a fractional power of the Laplace operator*. ProQuest LLC, Ann Arbor, MI, 2005. Thesis (Ph.D.)–The University of Texas at Austin. URL: http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mtx:dissertation&res_dat=xri:pqdiss&rft_dat=xri:pqdiss:3183966.

[50] P. Stefanov. Generic uniqueness for two inverse problems in potential scattering. *Comm. Partial Differential Equations*, 17(1-2):55–68, 1992. doi:10.1080/03605309208820834.

[51] P. D. Stefanov. A uniqueness result for the inverse back-scattering problem. *Inverse Problems*, 6(6):1055–1064, 1990. URL: http://stacks.iop.org/0266-5611/6/1055.

[52] E. M. Stein. *Singular integrals and differentiability properties of functions*. Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[53] E. G. F. Thomas. A polarization identity for multilinear maps. *Indag. Math. (N.S.)*, 25(3):468–474, 2014. With an appendix by Tom H. Koornwinder. doi:10.1016/j.indag.2013.11.003.

[54] G. Uhlmann. Inverse boundary value problems and applications. *Astérisque*, (207):153–211, 1992.

[55] G. Uhlmann. A time-dependent approach to the inverse backscattering problem. *Inverse Problems*, 17(4):703–716, 2001. Special issue to celebrate Pierre Sabatier's 65th birthday (Montpellier, 2000). doi:10.1088/0266-5611/17/4/309.

[56] J.-N. Wang. Inverse backscattering in even dimensions. *Math. Z.*, 239(2):365–379, 2002. doi:10.1007/s002090100301.