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Semiclassical measures and asymptotic distribution of eigenvalues for quantum KAM systems

Tesis presentada por Víctor Arnaiz Solórzano
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Director: Fabricio Macià Lang

A mis padres: Isabel y Roberto

Resumen

Desvarío laborioso y empobrecedor el de componer vastos libros;
el de explayar en quinientas páginas una idea cuya perfecta exposición
oral cabe en pocos minutos. Mejor procedimiento es simular que
esos libros ya existen y ofrecer un resumen, un comentario.

J.L. Borges. *Ficciones.*

Esta tesis aborda el estudio de la dinámica de la ecuación de Schrödinger en el régimen *semiclásico*, es decir, cuando la longitud de onda de las soluciones es comparable con una escala de tamaño $\hbar > 0$ con respecto a la *métrica* con la que se mide. El parámetro $\hbar$ en la literatura a veces se identifica con la constante de Planck normalizada. El *principio de correspondencia* establece que el comportamiento asintótico cuando $\hbar \to 0^+$ de estas soluciones altamente oscilantes se rige por la dinámica clásica subyacente. El estudio riguroso de este fenómeno recibe el nombre de *análisis semiclásico* y se ha desarrollado ampliamente durante las últimas tres décadas, abarcando numerosos problemas de ecuaciones en derivadas parciales lineales y no lineales.

Motivado por los resultados previos de Fabricio Macià y Gabriel Rivière sobre la dinámica de la ecuación de Schrödinger asociada a pequeñas perturbaciones de sistemas completamente integrables cuyo flujo es periódico, como la esfera con la métrica canónica o, más generalmente, las *variedades de Zoll*, este trabajo estudia el problema análogo para perturbaciones de hamiltonianos completamente integrables con flujo no necesariamente periódico, como el sistema de d osciladores armónicos con frecuencias independientes o, más ampliamente, sistemas tipo KAM.

La tesis se divide en cuatro partes que siguen un hilo conductor: el estudio de la *distribución de Wigner*, que describe la concentración o dispersión de la función de onda en el espacio de fases (espacio de posiciones y momentos), asociada a soluciones de la ecuación de Schrödinger en distintas situaciones y regímenes. Los puntos de acumulación de sucesiones de distribuciones de Wigner cuando $\hbar \to 0^+$ reciben el nombre de *medidas semiclásicas*.

En la primera parte de la tesis se obtienen resultados sobre las propiedades de propagación e invarianza de las medidas semiclásicas dependientes del tiempo, es decir, asociadas a las soluciones de la ecuación de propagación de Schrödinger. Asimismo, se muestran aplicaciones de estos resultados para las soluciones de la ecuación de Schrödinger estacionaria. En concreto, se prueba que una pequeña pertubación del oscilador armónico puede destruir los conjuntos minimales (toros invariantes) sobre los que las sucesiones de autofunciones pueden concentrarse si existen resonancias

entre las frecuencias del oscilador. Sin embargo, si el vector de frecuencias es *diofántico*, esto es, los cocientes entre sus componentes se aproximan "mal" por números racionales, se prueba que los toros invariantes maximales asociados son más estables y pueden ser conjuntos de acumulación de la energía de sucesiones de soluciones de la ecuación de Schrödinger dependiente del tiempo para rangos de tiempo polinomialmente largos.

En la segunda parte de la tesis se estudia la distribución asintótica de los autovalores del operador asociado a perturbaciones no autoadjuntas del oscilador armónico. Este problema está relacionado con el estudio del decaimiento de la energía para soluciones de la ecuación de ondas amortiguada. Los resultados obtenidos muestran la influencia de la perturbación en la franja del plano complejo donde los autovalores pueden concentrarse y la escala a la que se produce dicha concentración. Con hipótesis de analiticidad se prueba que los autovalores no pueden acumularse cerca de la recta real, es decir, existe un *gap* espectral. En el caso diferenciable, la estimación es más débil, pero permite obtener una cota sobre la norma de la resolvente del operador.

La tercera parte se ocupa del estudio de las medidas semiclásicas asociadas a perturbaciones de campos vectoriales diofánticos sobre el toro. Se demuestra que para un conjunto cantoriano de frecuencias, el espectro puntual del operador es estable. Para estas frecuencias se caracterizan los puntos de acumulación de sucesiones de autofunciones o *límites cuánticos* del operador perturbado. Este resultado puede verse como una versión semiclásica del teorema KAM clásico sobre perturbaciones de campos vectoriales sobre el toro.

Finalmente, la cuarta y última parte de esta memoria estudia el problema de *renormalización* desde el punto de vista semiclásico. Dada una perturbación acotada de un hamiltoniano lineal con frecuencias diofánticas sobre el toro, se obtiene la existencia de un operador integrable (que solo depende de las coordenadas acción) tal que sumado al operador perturbado lo "renormaliza" dando lugar a un operador integrable y unitariamente equivalente al operador sin perturbar. Como consecuencia, se obtiene que los conjuntos de límites cuánticos y medidas semiclásicas de sucesiones de autofunciones para el operador renormalizado coinciden con aquellos del operador no perturbado.

Abstract

This thesis adresses the study of the Schrödinger dynamics in the semiclassical regime, that is, when the wave length of the solutions is comparable with a scale of size $h > 0$ with respect to the metric size. This parameter h is sometimes identified with the normalized Planck constant. The correspondence principle states that the asymptotic behavior of these solutions as h tends to zero is governed by the underlying classical dynamics. The rigorous study of this phenomenon is called semiclassical analysis and has been widely developed during the last three decades, covering numerous problems of linear and nonlinear PDE. Motivated by the previous results of Fabricio Macià and Gabriel Rivière on the dynamics of the Schrödinger equation associated with small perturbations of completely integrable systems whose flow is periodic, such as the sphere with the canonical metric or, more generally, the Zoll manifolds, this work studies the analogous problem for perturbations of completely integrable Hamiltonians with not necessarily periodic flow, such as the system of d harmonic oscillators with independent frequencies or, more generally, KAM type systems. The thesis is divided in four parts that follow a common thread: the study of Wigner's distribution, which describes the concentration or dispersion of the wave function in the phase space (space of positions and momenta), associated with solutions of the Schrödinger equation in different situations and regimes. The accumulation points of sequences of Wigner distributions as h tends to zero are called semiclassical measures. In the first part of the thesis some results are obtained on the propagation and invariance properties of time-dependent semiclassical measures, that is, associated with the solutions of the time-dependent Schrödinger equation. Also, applications of these results are shown for the solutions of the stationary Schrödinger equation. In particular, it is proved that a small perturbation of the harmonic oscillator can destroy the minimal sets (invariant tori) on which sequences of eigenfunctions can be concentrated if resonances exist between the oscillator frequencies. However, if the vector of frequencies is Diophantine, that is, the quotients between its components are "badly" approximated by rational numbers, it is proved that the associated maximal invariant tori are more stable and can be accumulation sets of the sequence of Wigner distributions for solutions of the time-dependent Schrödinger equation for polynomially long time ranges. In the second part of the thesis the asymptotic distribution of the eigenvalues of the operator associated with non-selfadjoint perturbations of the harmonic oscillator is studied. This problem is related to the study of the decay of energy for solutions of the damped wave equation. The results obtained show the influence of the perturbation on the stripe of the

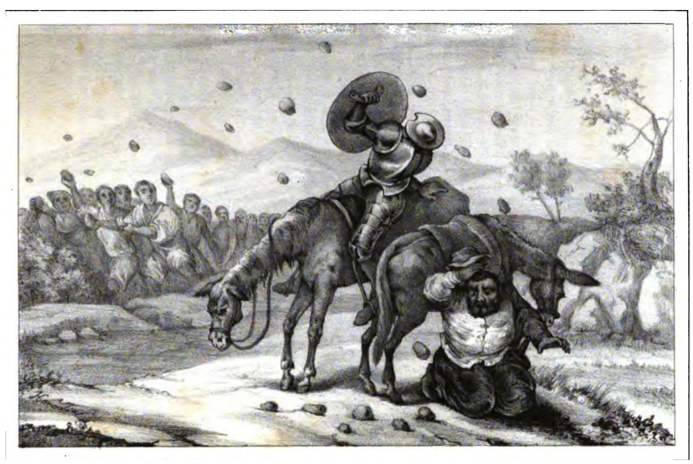
complex plane where the eigenvalues can be concentrated and the scale at which this concentration occurs. With analytical hypothesis, it is proved that the eigenvalues can not accumulate close to the real line, that is, there is a spectral gap. In the smooth case, the estimate is weaker, but it allows to obtain a bound on the resolvent norm of the non-selfadjoint operator. The third part deals with the study of semi-classical measures associated with perturbations of Diophantine vector fields on the torus. It is proved that for a Cantorian set of frequencies, the point spectrum of the operator is stable. For these frequencies, the accumulation points of sequences of L^2-densities of eigenfunctions, or quantum limits, of the perturbed operator are characterized. This result can be seen as a semiclassical version of the classic KAM theorem on perturbations of vector fields on the torus. Finally, the fourth and last part of this report studies the renormalization problem from the semi-classical point of view. Given a bounded perturbation of a linear Hamiltonian with Diophantine frequencies on the torus, we obtain the existence of an integrable operator (which only depends on the action coordinates) that, added to the system, renormalize it becoming integrable and unitary equivalent to the non-perturbed operator. As a consequence, we obtain that the sets of quantum limits and semi-classical measures of sequences of eigenfunctions for the renormalized operator coincide with those of the unperturbed operator.

Gratitud

De gente bien nacida es agradecer los beneficios que reciben, y uno de los pecados que más a Dios ofende es la ingratitud.

M. de Cervantes. *El Ingenioso Hidalgo Don Quijote de la Mancha.*

Estas palabras dirigió Don Quijote a una cadena de desdichados galeotes, tras liberarlos del destino que la justicia del Rey les había impuesto en pago de sus cuantiosos delitos. Lejos de agradecer el generoso gesto del caballero, los prófugos, tras negarse a encomendar la cadena que los tuvo presos a la señora Dulcinea del Toboso, por requerimiento de Don Quijote, acabaron arremetiendo contra éste, su caballo Rocinante, Sancho y su sufrido asno con una lluvia de piedras que los dejó malparados y mohínos, incrédulos del mal que les hacían aquellos a quienes tanto bien habían procurado.



No queriendo caer en semejante bajeza, pretendo recordar a las personas que me han liberado, en mayor o menor medida, del peso que uno acarrea al realizar una tesis doctoral, y a agradecer las dádivas que he recibido.

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Admito que he recibido mucho más de lo que puede ser restituído de mis padres, que me han dado barcaje y criterio para dirigirme en esta vida, de mi incondicional hermano Luis y de mi querida Vanessa.

Víctor Arnaiz Solórzano,

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Contents

Chapter 1

Introduction and main results

> Respetable público... No, respetable público no, público solamente, y no es que el autor no considere al público respetable, todo lo contrario, sino que detrás de esta palabra hay como un delicado temblor de miedo y una especie de súplica para que el auditorio sea generoso con la mímica de los actores y el artificio del ingenio. El poeta no pide benevolencia, sino atención, una vez que ha saltado hace mucho tiempo la barra espinosa de miedo que los autores tienen a la sala.
>
> F.G. Lorca. *La zapatera prodigiosa.*

1.1. Motivation

The results in this thesis pertain to the field known as Semiclassical Analysis, which is succintly described by M. Zworski in the preface to [122] as:

> Semiclassical analysis provides PDE techniques based on the *classical-quantum* (particle-wave) correspondence. These techniques include such well-known tools as geometric optics and the Wentzel-Kramers-Brillouin (WKB) approximation. Examples of problems studied in this subject are high-energy eigenvalue asymptotics or effective dynamics for solutions of evolution equations. From the mathematical point of view, semiclassical analysis is a branch of *microlocal analysis* which, broadly speaking, applies *harmonic analysis* and *symplectic geometry* to the study of linear and non-linear PDE.

The quantum-classical correspondence principle states that the laws of quantum mechanics, valid at atomic scales, should tend to their classical (Newtonian) counterparts in the high-frequency limit. Let us make this statement a bit more precise in a specific example.

One of the most fundamental models in quantum mechanics is the Schrödinger equation, which in its simplest form is:

$$\begin{cases} i\partial_t u\left(t,x\right)+\dfrac{1}{2}\Delta_x u\left(t,x\right)-W(x)u(t,x)=0, & (t,x)\in\mathbb{R}\times\mathbb{R}^d, \\ u|_{t=0}=u^0, \quad \|u^0\|_{L^2(\mathbb{R}^d)}=1. \end{cases} \tag{1.1}$$

The Schrödinger equation (1.1) is a mathematical model for the propagation of a free quantum particle (an electron, for instance) in $\mathbb{R}^d$. If u is a solution to (1.1) then for every measurable set $U\subset\mathbb{R}^d$ and every $t\in\mathbb{R}$, the quantity

$$\int_U \left|u\left(t,x\right)\right|^2 dx \tag{1.2}$$

is the probability for the particle that was at $t=0$ at the state u^0, to be in the region U at time t[1].

In this setting, the correspondence principle roughly states that, if the characteristic length scale over which the potential W varies significantly is much larger than the characteristic wave length of a solution u to (1.1) then the probability density $\left|u\left(t,\cdot\right)\right|^2$, which is called the *position density*, should follow a propagation law based on classical mechanics. Classical or Newtonian mechanics refers here to the Hamiltonian flow ϕ_t^H corresponding to the *classical Hamiltonian*

$$H(x,\xi):=\frac{1}{2}|\xi|^2+W(x), \quad (x,\xi)\in\mathbb{R}^d\times\mathbb{R}^d.$$

Recall that $\phi_t^H(x_0,\xi_0)=(x(t),\xi(t))$, where

$$\dot{x}=\partial_\xi H(x,\xi), \quad \dot{\xi}=-\partial_x H(x,\xi), \quad (x(0),\xi(0))=(x_0,\xi_0).$$

In order to formulate a precise mathematical statement, let us suppose that we have normalized the problem in order to have that the characteristic wave length of the solution under consideration is equal to one. The potential varies at a macroscopic scale much larger than the wave length; suppose this scale is of order $1/\hbar$ with $\hbar>0$ small. Therefore, if the microscopic variable for the position is x, the potential can be written in those variables as $W\left(\hbar x\right)$. The corresponding Schrödinger equation is:

$$i\partial_t u\left(t,x\right)+\frac{1}{2}\Delta_x u\left(t,x\right)-W\left(\hbar x\right)u\left(t,x\right)=0.$$

If a change to macroscopic variables is performed:

$$t\longmapsto T=\hbar t, \quad x\longmapsto X=\hbar x, \quad u_\hbar\left(T,X\right)=\frac{1}{\hbar^{d/2}}u\left(\frac{T}{\hbar},\frac{X}{\hbar}\right),$$

[1] The fact that $|u(t,\cdot)|^2$ is a probability density for every $t\in\mathbb{R}$ follows from the fact that the L^2-norm is conserved by the evolution equation (1.1).

then the *semiclassical* Schrödinger equation is obtained:

$$i\hbar\partial_T u_\hbar (T,X) + \frac{\hbar^2}{2}\Delta_X u_\hbar (T,X) - W(X)\, u_\hbar (T,X) = 0. \tag{1.3}$$

One expects that in the limit $\hbar \to 0^+$ the position density $|u_\hbar (T,\cdot)|^2$ can be described in terms of ϕ_T^H[2]. This can be readily verified when $W = 0$. Let $(x_0, \xi_0) \in \mathbb{R}^d \times \mathbb{R}^d$ and consider the sequence of initial data:

$$u_\hbar^0 (x) = \frac{1}{(\pi\hbar)^{d/4}} e^{-\frac{|x-x_0|^2}{2\hbar}} e^{i\frac{\xi_0}{\hbar}\cdot x}. \tag{1.4}$$

This type of sequence is usually known as a *wave-packet* or a *coherent state* centered at (x_0, ξ_0). As $\hbar \to 0^+$ the sequence $(u_\hbar^0)$ concentrates near x_0 and oscillates rapidly in the direction of ξ_0. It is straightforward to check that:

$$\left|u_\hbar^0 (x)\right|^2 = \frac{1}{(\pi h)^{d/2}} e^{-\frac{|x-x_0|^2}{\hbar}} \rightharpoonup^\star \delta_{x_0}(x), \quad \text{as } \hbar \to 0^+,$$

where δ_{x_0} is the Dirac mass centered at x_0 (the convergence takes place on the space of Radon measures equipped with the weak-$\star$ topology).

An explicit computation involving the Fourier transform gives that $u_\hbar$, the corresponding solutions to (1.3) issued from these initial data with $W = 0$ satisfy:

$$\begin{aligned} |u_\hbar(t,\cdot)(x)|^2 &= |e^{i\hbar t\Delta_x/2} u_\hbar^0(x)|^2 \\ &= \frac{1}{(\pi\hbar\,(1+t^2))^{d/2}} e^{-\frac{|x-x_0-t\xi_0|^2}{\hbar(1+t^2)}} \rightharpoonup^\star \delta_{x_0+t\xi_0}(x), \quad \text{as } \hbar \to 0^+, \end{aligned} \tag{1.5}$$

Therefore, in the limit $\hbar \to 0^+$, the position densities converge to the deterministic probability density concentrated on a particle that propagates according to the Hamiltonian flow ϕ_t^H, which in the case $W = 0$ is simply:

$$(x, \xi) \longmapsto (x + t\xi, \xi).$$

When W does not vanish identically one can no longer apply directly the Fourier transform and perform an explicit computation. However, an analogous result holds: this is known as Egorov's theorem (see Section 2.5) and is one of the fundamental results in Semiclassical Analysis. In our context it gives the following result.

Theorem 1.1 (Semiclassical limit)**.** *Let* $(u_\hbar^0)$ *be the sequence defined in* (1.4)*. Then the corresponding solutions* $(u_\hbar)$ *to* (1.3) *satisfy:*

$$|u_\hbar(t,\cdot)|^2 \rightharpoonup^\star \delta_{x(t)}, \quad \hbar \to 0^+, \tag{1.6}$$

where $x(t)$ *is the projection on the* x*-variable of* $\phi_t^H(x_0, \xi_0)$*.*

[2] Be aware of the fact that the parameter $\hbar$ should not be identified to Planck's constant; this notation for the characteristic frequency may be unfortunate, but we maintain it as it is widely used in the literature.

The convergence in (1.6) is locally uniform in $t \in \mathbb{R}$. Due to the dispersive nature of equation (1.3) one cannot expect that (1.6) holds uniformly in time: for fixed $\hbar$ and as t increases, the wave-packet $e^{it/\hbar(\hbar^2\Delta/2-W)}u_\hbar^0$ will become less and less concentrated around $x(t)$. The study of the simultaneous limits $\hbar \to 0$ and $t \to \infty$ is a notoriously difficult problem. In the most general framework, it is known [32, 18, 53, 25] that (1.6) holds uniformly for

$$|t| \leq T_{\mathrm{E}}(\hbar) := \Gamma^{-1} \log(1/\hbar), \tag{1.7}$$

where $\Gamma > 0$ is a dynamical constant related to the Lyapunov exponents of the Hamiltonian flow on the energy level $H^{-1}(H(x_0, \xi_0))$. This upper bound $T_{\mathrm{E}}(\hbar)$, known as the *Ehrenfest time*, has been shown to be optimal for some one-dimensional systems, see [33, 72].

Understanding the validity of the semiclassical limit for values of t beyond the Ehrenfest time is a very difficult question, although some results have been proved for specific geometries and initial data [25, 39, 95, 107]. However, the analysis becomes more tractable if one performs a time average. The problem we will be interested in consists in averaging the probability position densities $|u_\hbar(t, \cdot)|^2$ over time intervals of size comparable with $\tau_\hbar$, where

$$\tau_\hbar \to \infty \quad \text{as } \hbar \to 0^+.$$

A simple change of variables shows that this amounts to study the accumulation points of the family of measures $\nu_\hbar$ defined on $\mathbb{R}_t \times \mathbb{R}_x^d$ obtained by scaling in time the position densities:

$$\nu_\hbar(dt, dx) := |u_\hbar(\tau_\hbar t, x)|^2 dx\, dt.$$

We will present next a brief account on known results regarding this and related problems.

Quantum limits

The type of problems we are interested in can be formulated in the context of a Riemannian manifold. This generalization is convenient, since the corresponding classical dynamical system is the *geodesic flow* of the manifold. Geodesic flows constitute a widely studied class of dynamical systems, for which dynamical hypotheses can be formulated in geometric terms (curvature, for instance), see [94] among many references.

Let (M, g) be a compact Riemannian manifold; we denote by dx the measure induced by the Riemannian volume. Use this measure to define $L^2(M)$. From now on, Δ_g will denote the corresponding Laplace-Beltrami operator, which is selfadjoint on $L^2(M)$. We will consider real-valued potentials W defined on M and, for the sake of simplicity, we will assume that they are smooth functions. Consider now the *semiclassical Schrödinger operator*:

$$\widehat{H}_\hbar := -\frac{\hbar^2}{2}\Delta_g + W(x), \tag{1.8}$$

which is selfadjoint over $L^2(M)$.

The classical counterpart of $\widehat{H}_\hbar$ is the Hamiltonian $H \in \mathcal{C}^\infty(T^*M)$ defined by:

$$H(x,\xi) := \frac{1}{2}\langle \xi, \xi\rangle_{g(x)} + W(x), \quad (x,\xi) \in T^*M.$$

Above, $\sqrt{\langle \xi, \xi\rangle_{g(x)}}$ stands for the Riemannian norm defined on covectors. Using the canonical symplectic form in T^*M one can define the Hamiltonian vector field X_H, that is given locally by:

$$X_H(x,\xi) = \partial_\xi H(x,\xi)\cdot \partial_x - \partial_x H(x,\xi)\cdot\partial_\xi.$$

We denote by ϕ_t^H the flow of X_H; this is a complete flow, since the level sets $H^{-1}(E)$ are compact. Note that when $W = 0$, ϕ_t^H is nothing but the geodesic flow on T^*M.

We will again consider solutions of the Schrödinger equation:

$$i\hbar\partial_t\, u_\hbar(t,x) = \widehat{H}_\hbar\, u_\hbar(t,x), \quad u_\hbar|_{t=0} = u_\hbar^0 \in L^2(M). \tag{1.9}$$

The unitary propagator $e^{-i\frac{t}{\hbar}\widehat{H}_\hbar}$ associated to (1.3) will be referred to as the *semiclassical Schrödinger flow.*

Let $(u_\hbar^0)$ be a sequence with $\|u_\hbar^0\|_{L^2(M)} = 1$ and let $(\tau_\hbar)$ be a sequence of real numbers that tends to infinity. We assume that moreover, the sequence is $\hbar$-oscillating, meaning that:

$$\lim_{R\to\infty}\limsup_{\hbar\to 0}\left\|\mathbf{1}_{[0,R]}(-\hbar^2\Delta_g)u_\hbar^0\right\|_{L^2(M)} = 1. \tag{1.10}$$

Here, $\mathbf{1}_{[0,R]}$ stands for the characteristic function of the interval $[0,R]$ and $\mathbf{1}_{[0,R]}(-\hbar^2\Delta_g)$ is defined using the functional calculus of selfadjoint operators. It is possible to show that, modulo the extraction of a subsequence, there exist a t-measurable family of probability measures ν_t defined on M such that:

$$\lim_{\hbar\to 0^+}\int_{\mathbb{R}\times M}\varphi(t,x)|e^{-i\frac{\tau_\hbar t}{\hbar}\widehat{H}_\hbar}u_\hbar^0(x)|^2 dx\, dt = \int_{\mathbb{R}\times M}\varphi(t,x)\nu_t(dx)dt, \quad \forall\varphi\in\mathcal{C}_c(\mathbb{R}\times M). \tag{1.11}$$

This follows from the compactness of M. We will denote by $\mathcal{N}(\widehat{H}_\hbar,\tau_\hbar)$ the set of all measures obtained in this way, as $(u_\hbar^0)$ varies among all $\hbar$-oscillating, normalized sequences in $L^2(M)$.

Problem 1. Characterize the set $\mathcal{N}(\widehat{H}_\hbar,\tau_\hbar)$; that is find all probability measures ν_t that can be obtained as an accumulation points in the sense of (1.11) for some sequence $(u_\hbar^0)$ in in $L^2(M)$ that is $\hbar$-oscillating and normalized.

Note that, since $\widehat{H}_\hbar$ is selfadjoint and has compact resolvent, its spectrum is discrete and unbounded, and every solution to (1.3) can be expressed as a superposition of periodic oscillations:

$$u_\hbar(t,\cdot) = \sum_{\lambda_\hbar\in\mathrm{Sp}(\widehat{H}_\hbar)} e^{-i\frac{t}{\hbar}\widehat{H}_\hbar}\,\Pi_{\lambda_\hbar}u_\hbar^0, \tag{1.12}$$

where $\Pi_{\lambda_\hbar}$ is the projection in $L^2(M)$ onto the eigenspace associated to $\lambda_\hbar$. The position densities of each term in this sum are invariant under time-scaling: If $\Psi_\hbar := \Pi_{\lambda_\hbar} u_\hbar^0$, then

$$\widehat{H}_\hbar \Psi_\hbar = \lambda_\hbar \Psi_\hbar \Rightarrow |e^{\frac{i\tau_\hbar t}{\hbar}\widehat{H}_\hbar}\Psi_\hbar|^2 = |\Psi_\hbar|^2;$$

and therefore, every normalized sequence $(\Psi_\hbar)$ of eigenfunctions of $\widehat{H}_\hbar$ with eigenvalues $\lambda_\hbar$ lying in a bounded set of $\mathbb{R}$ satisfies that any accumulation point of $(|\Psi_\hbar|^2)$ is in $\mathcal{N}(\widehat{H}_\hbar, \tau_\hbar)$ for any sequence of time-scales $(\tau_\hbar)$. Let us denote by $\mathcal{N}(\widehat{H}_\hbar)$ the set of all those accumulation points; with this notation:

$$\mathcal{N}(\widehat{H}_\hbar) \subseteq \mathcal{N}(\widehat{H}_\hbar, \tau_\hbar), \quad \forall(\tau_\hbar), \ \lim_{\hbar\to 0^+} \tau_\hbar = \infty. \tag{1.13}$$

Measures in $\mathcal{N}(\widehat{H}_\hbar)$ are called *quantum limits*; by extension, we will refer to elements of $\mathcal{N}(\widehat{H}_\hbar, \tau_\hbar)$ as *time-dependent quantum limits.* A notoriously difficult problem is:

Problem 2. Identify all probability measures in M that are quantum limits for a given Schrödinger operator $\widehat{H}_\hbar$; in other words, characterize the set $\mathcal{N}(\widehat{H}_\hbar)$.

Of course, a solution to Problem 1 for some time-scale $\tau = (\tau_\hbar)$ automatically gives information on Problem 2, because of (1.13). Problem 2 has received a lot of attention in the last fifty years; the systematic study of Problem 1 is more recent. References [8, 83] provide a survey of these and related problems.

In both cases, the answer to these questions involves global properties of the dynamics of the classical Hamiltonian flow ϕ_t^H. The cases that have been more studied are:

1. *Chaotic dynamics.* More precisely, ϕ_t^H is ergodic, (non-uniformly) hyperbolic, or has the Anosov property.

2. *Regular dynamics.* The flow ϕ_t^H is completely integrable in the Liouville sense, or has a certain (relatively large number) of Poisson-commuting first integrals.

For background on these concepts we refer to [91, 54].

Chaotic Dynamics

As stressed in [8], the fact that the Hamiltonian has well-understood chaotic properties would in principle lead to expect that the the corresponding Schrödinger flow has good dispersive properties. This motivates some very strong conjectures on the answer to Problems 1 and 2, such as the quantum unique ergodicity conjecture (QUE) which we partly describe below. On the other hand, these same chaotic properties make it difficult to approximate the Schrödinger dynamics by the classical dynamics: the quantum-classical correspondence is only valid up to the Ehrenfest time, and this leaves little hope to use it to prove those conjectures. From now on, we will assume $W = 0$, so that $\widehat{H}_\hbar = -\hbar^2\Delta_g$ and ϕ_t^H is the geodesic flow.

We first state a version of the Snirelman theorem (see [110] for the original work of Snirelman, Zelditch [118] for the case of compact hyperbolic surfaces, Colin de Verdière [30] in the case of eigenfunctions of the Laplacian for more general chaotic systems, Helffer, Martinez and Robert [56] for semiclassical pseudodifferential operators, and Zelditch [120] in the case of C^* dynamical systems). Suppose that the geodesic flow is ergodic (with respect to the Liouville measure). Then for every $\varepsilon > 0$ let $(\Psi_{\hbar,j})$ be an orthonormal basis of the span of the eigenspaces of $\widehat{H}_\hbar$ associated to eigenvalues in $[1-\varepsilon, 1+\varepsilon]$. Then there exist a subset $\Lambda(\hbar) \subset \mathrm{Sp}(\widehat{H}_\hbar) \cap [1-\varepsilon, 1+\varepsilon]$ such that $\Lambda(\hbar)$ has density one:

$$\lim_{\hbar\to 0^+} \frac{\#\Lambda(\hbar)}{\#\mathrm{Sp}(\widehat{H}_\hbar) \cap [1-\varepsilon, 1+\varepsilon]} = 1,$$

and,

$$\lim_{\hbar\to 0^+,\, \lambda_{\hbar,k}\in\Lambda(\hbar)} \int_M \phi(x)|\Psi_{\hbar,k}(x)|^2 dx = \int_M \phi(x)dx, \quad \forall \phi \in \mathcal{C}(M).$$

The result says that a typical sequence of eigenfunctions becomes equidistributed (in fact, the original statement of Snirelman's theorem expresses the stronger fact that equidistribution takes place both in the "x-variable" and in the "ξ-variable"). At this level of generality, it is not well understood if the whole sequence of eigenfunctions converges, or if there can be exceptional subsequences with a different limiting behavior (that is, if we can take $\Lambda(\hbar) = \mathrm{Sp}(\widehat{H}_\hbar) \cap [1-\varepsilon, 1+\varepsilon]$ or not). In other words, one wonders whether or not

$$\mathcal{N}(\widehat{H}_\hbar) = \{dx\}. \tag{1.14}$$

There are manifolds (or Euclidean domains) with ergodic geodesic flows, but with exceptional subsequences of eigenfunctions [55], but these examples have only been found very recently, and the proof is not constructive (the exceptional subsequences, whose existence is proved, are not exhibited explicitly). Negatively curved manifolds have ergodic geodesic flows, but actually the understanding of the chaotic properties of the flow is so good that one could hope to go beyond the Snirelman theorem. It may seem surprising that the question is still widely open, even in the case of manifolds of constant negative curvature (where the local geometry is completely explicit). The QUE conjecture that (1.14) holds for eigenfunctions of the Laplacian on a negatively curved compact manifold. It was stated by Rudnick and Sarnak [105, 103]. So far, the only complete result is due to Lindenstrauss [24, 76], who proved the conjecture in the case when M is an *arithmetic congruence surface*, and the eigenfunctions (Ψ_n) are common eigenfunctions of Δ_g and of the *Hecke operators*. There are partial results, due to Anantharaman [2]; Anantharaman and Nonnenmacher [10]; and Rivière [101], which hold in great generality, on any compact negatively curved manifold that show that concentration on sets of low Haussdorff dimension is not possible (a closed geodesic, for instance). This type of results have been generalized to the time dependent equation, and in particular can be applied to give some insight on the characterization of $\mathcal{N}(\widehat{H}_\hbar, \hbar^{-1})$, see [11]. Recently, Dyatlov and Jin [36] have shown that, in the case of surfaces of constant negative curvature, elements in $\mathcal{N}(\widehat{H}_\hbar)$ must charge every open set $U \subset M$.

Regular Dynamics

When the geodesic flow is completely integrable, in the sense of Liouville, the situation is very different. The Arnold-Liouville theorem shows that the classical phase space is foliated by families of tori or cylinders that are invariant by the Hamiltonian flow. Moreover, the classical Hamiltonian flow can be conjugated, by symplectic diffeomorphisms, to a flow on a cylinder that is of the form:

$$\phi_t^G : (x, \xi) \longmapsto (x + tdG(\xi), \xi), \quad (x, \xi) \in (\mathbb{T}^r \times \mathbb{R}^{d-r})_x \times \mathbb{R}^d_\xi, \quad 0 < r \leq d,$$

for some Hamiltonian $G \in \mathcal{C}^\infty(\mathbb{R}^d)$ that only depends on the actions. We again refer to [91] for a more precise statement of this result. Manifolds with this property include non-negative constant curvature manifolds, compact-rank-one symmetric spaces, surfaces of revolution, Zoll manifolds, harmonic oscillators, the hydrogen atom, etc.

One expects in this situation to have a wider variety of quantum limits, since the dispersive effects exhibited by the Schrödinger flow are weaker than in the chaotic case. It turns out that this intuition is partially true. Again, the results we describe next assume that $W = 0$; we thus focus on the case $\widehat{H}_\hbar = -\hbar^2\Delta$ and ϕ_t^H is the geodesic flow.

In the case of the the sphere $\mathbb{S}^d$ endowed with its canonical metric, Jakobson and Zelditch [67] proved that:

$$\mathcal{N}(\widehat{H}_\hbar) = \overline{\mathrm{Conv}\{\delta_\gamma \, : \, \gamma \text{ is a geodesic in } \mathbb{S}^d\}}. \tag{1.15}$$

Above, δ_γ stands for the uniform probability measure on the closed curve γ. This result can be proved using an explicit construction involving spherical harmonics (see for instance [83]). Property (1.15) also holds in manifolds of constant positive curvature [17] or compact-rank-one symmetric spaces [79]. One can also show that in all these cases $\mathcal{N}(\widehat{H}_\hbar, \tau_\hbar) = \mathcal{N}(\widehat{H}_\hbar)$ for every time-scale $\tau_\hbar$.

A natural question in this setting is that of understanding whether or not the same holds on a Zoll manifold (that is, a manifold all whose geodesics are closed [21]). In [80] it is shown that in this case:

$$\mathcal{N}(\widehat{H}_\hbar, \tau_\hbar) = \overline{\mathrm{Conv}\{\delta_\gamma \, : \, \gamma \text{ is a geodesic in } M\}}, \qquad \text{provided that } \tau_\hbar = o(\hbar^{-2}).$$

However, very recently Macià and Rivière [84] have shown the exitence of Zoll surfaces such that (1.15) fails. The examples in [84] are Zoll surfaces of revolution; it turns out that there exist an open set of geodesics such that $\delta_\gamma \notin \mathcal{N}(\widehat{H}_\hbar)$ for γ in an open set in the space of geodesics.

On the torus $\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d$, the behavior of quantum limits is very different. Bourgain proved that $\mathcal{N}(\widehat{H}_\hbar) \subset L^1(\mathbb{T}^d)$; and in particular that quantum limits cannot concentrate on closed curves, as was the case on the sphere (this result was reported in [66]). In that same reference, Jakobson proved that for $d = 2$ the density of any quantum limit is a trigonometric polynomial, whose frequencies satisfy a certain Pell equation. In higher dimensions, one can only prove certain regularity properties of the densities, involving decay of its Fourier coefficients. These results are based on arithmetic consideration (distribution of lattice points on spheres) and results on integer points on elliptic curves. Moreover, Jaffard [65] proved that any quantum limit charges any open set:

$$f\,dx \in \mathcal{N}(\widehat{H}_\hbar) \quad \Longrightarrow \int_U f\,dx > 0,\ \forall U \subseteq \mathbb{T}^d \text{ open}. \tag{1.16}$$

This last result also holds for time-dependent quantum limits in $\mathcal{N}(\widehat{H}_\hbar, \hbar^{-1})$. Again, this result is based on the explicit form of solutions in terms of Fourier series, combined with results on Kahane's theory of non-harmonic Fourier series.

One could wonder if there is a proof of these results based only on the dynamical properties of the geodesic flow and that could encompass both results [66, 65]. This is the case, the result was obtained by Macià [81, 82], Anantharaman and Macià [9]. Their proof is based on microlocal methods adapted to the dynamics of the completely integrable geodesic flow and does not make use of the explicit form of the solutions in terms of Fourier series. In fact, they prove a stronger result that allows to deal with (non-semiclassical) perturbations of order one and time-dependent quantum limits in $\mathcal{N}(\widehat{H}_\hbar, \hbar^{-1})$ (see also [22] for results in the non-perturbed case); it is also possible to obtain more precise results on the regularity of the densities [1]. It turns out that this strategy of proof is rather robust, and can be extended to more general completely integrable Hamiltonian flows [4], at least in regions where global action-angle coordinate exist. It also allows to deal with domains in the Euclidean space. Birkhoff's conjecture state that the only such domains that have integrable generalized geodesic (billiard) flow are disks and ellipses. Recently, Anantharaman, Léautaud and Macià proved [6, 7] that the set of time-dependent quantum limits in $\mathcal{N}(\widehat{H}_\hbar, \hbar^{-1})$ on the Euclidean unit disk $\mathbb{D}$ is of the form:

$$\mathcal{N}(\widehat{H}_\hbar, \hbar^{-1}) \subseteq \{\alpha f\,dx + (1-\alpha)\delta_{\partial\mathbb{D}} \,:\, \alpha \in [0,1],\ \|f\|_{L^1(\mathbb{D})} = 1\}.$$

The presence of the singular term $\delta_{\partial\mathbb{D}}$ is due to the fact that action-angle coordinates become degenerate at the boundary of the disk. in fact, it is easy to produce solutions such that their quantum limit is $\delta_{\partial\mathbb{D}}$, the so called *whispering-gallery modes*. In addition, it is possible to show that these results are stable under perturbation and that the densities of quantum limits charge every open set $U \subseteq \mathbb{D}$.

Weak perturbations

We conclude this motivation section presenting the class of systems that we will study more closely in this thesis. The motivation comes from the rather simple observation that non-semiclassical problems can be written and studied in semiclassical terms. For instance, if u solves the non-semiclassical Schrödinger equation:

$$i\partial_t u(t,x) = \left(-\frac{\Delta_g}{2} + W(x)\right) u(t,x), \quad (t,x) \in \mathbb{R}\times M, \quad u|_{t=0} = u^0, \tag{1.17}$$

then $v_\hbar(t,\cdot) := u(t/\hbar,\cdot)$ solves:

$$i\hbar\partial_t\, v_\hbar(t,x) = \left(-\frac{\hbar^2\Delta_g}{2} + \hbar^2 W(x)\right) v_\hbar(t,x), \quad (t,x)\in\mathbb{R}\times M, \quad v_\hbar|_{t=0} = u^0. \tag{1.18}$$

Analogously, if Ψ solves the eigenvalue problem:

$$\left(-\frac{\Delta_g}{2} + W(x)\right)\Psi(x) = \lambda\Psi(x), \quad x\in M, \tag{1.19}$$

for $\lambda > 0$ big, then it is also a solution to the semiclassical problem:

$$\left(-\frac{\hbar^2\Delta_g}{2} + \hbar^2 W(x)\right)\Psi(x) = \Psi(x), \quad x\in M, \quad \text{with } \hbar := \lambda^{-1/2} \text{ small.} \tag{1.20}$$

Both cases involve the *perturbed semiclassical operator* $\widehat{H}_\hbar := -\hbar^2\Delta_g + \hbar^2 W$. Note that, in contrast to the operator defined in (1.8), the potential is multiplied by the coefficient $\hbar^2$ that tends to zero as $\hbar\to 0^+$. It is therefore, a purely *quantum perturbation* that vanishes in the semiclassical limit. From this point of view, it makes sense to consider more general perturbed operators of the form:

$$\widehat{P}_\hbar := \widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar, \tag{1.21}$$

where $\varepsilon_\hbar \to 0$ as $\hbar\to 0^+$, and $\widehat{V}_\hbar$ is a uniformly bounded family of operators on $L^2(M)$. One can define in a similar way the sets of quantum limits $\mathcal{N}(\widehat{P}_\hbar)$ and $\mathcal{N}(\widehat{P}_\hbar,\tau_\hbar)$. This regime can be viewed as an intermediate regime between the KAM. setting (which corresponds to ε small *but fixed*), and the unperturbed regime $\varepsilon = 0$.

The series of works [79, 80, 81, 9, 4, 7, 6] already mentioned fit in this setting in the particular case $\varepsilon_\hbar = \hbar^2$. It is natural then to ask if those results still hold under the presence of stronger perturbations; this is not the case, as was proved by Macià and Rivière in a series of works [84, 85, 86]. On the sphere, they showed that one no longer can concentrate in any geodesic; only the ones that are critical points of the Radon transform of the symbol of $\widehat{V}_\hbar$. On the torus, the situation is the opposite, the absolute continuous character of quantum limits of the Laplace Beltrami operator is lost generically if one adds a potential, and singularities appear generically. Again, this is related to the Radon transform of the perturbation. The proofs of these results are based on quantum versions of the averaging method in classical mechanics, that go back to Weinstein [115].

In this thesis we will deepen in this subject from several angles. In Chapter 3, we will consider the perturbation problem for the quantum harmonic oscillator; Chapter 4 addresses the case of non-selfadjoint perturbations, a subject that is closely related to the decay rates for the damped wave equation. Chapters 5 and 6 consider KAM problems from the semiclassical point of view.

SOME NOTATIONS

Before starting with the presentation of the main results of this thesis, we emphasize here some notations that will appear along the text. Let $(x_\hbar)$ and $(y_\hbar)$ be two sequences of positive real numbers. We will write

$$x_\hbar \lesssim y_\hbar$$

if there exists some universal constant $C > 0$ such that $x_\hbar \leq C y_\hbar$ for every $\hbar \in (0, 1]$. We will also write

$$x_\hbar \ll y_\hbar$$

if $\lim_{\hbar \to 0^+} x_\hbar / y_\hbar = 0$. Finally, we will say that $x_\hbar \sim y_\hbar$ if $\lim_{\hbar \to 0^+} x_\hbar / y_\hbar \to 1$.

1.2. SEMICLASSICAL MEASURES FOR PERTURBED HARMONIC OSCILLATORS

This initial part of the thesis is joint work with Fabricio Macià, and it is the content of the preprint [13]. We study the dynamics of the semiclassical Schrödinger equation associated to small perturbations of the quantum harmonic oscillator.

In order to justify the convenience of the semiclassical point of view adopted along this work, we start by introducing the harmonic oscillator without the semiclassical parameter $\hbar$ and then we will redefine it with $\hbar$ playing the role of a scaling parameter. To this aim, we first consider $\widehat{H}$ to be the quantum harmonic oscillator defined on $L^2(\mathbb{R}^d)$ by[3]

$$\widehat{H} := \frac{1}{2} \sum_{j=1}^{d} \omega_j (\partial_{x_j}^2 + x_j^2), \quad x \in \mathbb{R}^d. \tag{1.22}$$

The spectrum of $\widehat{H}$ in $L^2(\mathbb{R}^d)$ is given by the unbounded discrete set

$$\mathrm{Sp}_{L^2(\mathbb{R}^d)} \big(\widehat{H}\big) = \left\{ \lambda_k = \sum_{j=1}^{d} \left(k_j + \frac{1}{2} \right) \omega_j, \quad k = (k_1, \ldots, k_d) \in \mathbb{N}^d, \quad \omega_j > 0 \right\}.$$

Let (Ψ_k) be a sequence of normalized eigenfunctions of $\widehat{H}$ with eigenvalues $(\lambda_k)_{k \in \mathbb{N}^d}$, we aim at understanding the accumulation of mass of sequences of densities $|\Psi_k(x)|^2$ as $|k| \to \infty$. Unfortunately[4], one can verify that for any of these sequences,

$$|\Psi_k(x)|^2 \rightharpoonup^{\star} 0, \quad \text{as } |k| \to \infty.$$

[3]One can show that $\widehat{H}$ is the diagonal form of the operator $\widehat{H}_Q := \frac{1}{2}\big(-\Delta_x + x \cdot Qx\big)$, where Q is a positive-definite symmetric real matrix of size $d \times d$ with eigenvalues $\{\omega_1^2, \ldots, \omega_d^2\}$. Hence it is a particular example of Hamiltonian of the form $-\frac{\Delta_x}{2} + W$ with $W = \frac{1}{2} x \cdot Qx$.

[4]Compare with the case $\widehat{H} = -\frac{1}{2}\Delta_g$ on a compact Riemannian manifold M. In this case, the accumulation points of L^2-mass sequences of eigenfunctions $(|\Psi_n|^2)$ as $\lambda_n \to +\infty$ are probability measures on M.

As will be highlighted below, what underlines this phenomenon is that the L^2-mass of the sequence disperses on regions of diameter growing like $\sqrt{\lambda_k}$ as $|k| \to \infty$. Therefore, if we want to study this distribution of mass, it is very convenient to rescale the problem, introducing a semiclassical parameter $\hbar > 0$ so that the eigenmodes are rescaled into

$$\Psi_{k,\hbar}(x) := S_\hbar \Psi_k(x) = \frac{1}{\hbar^{d/4}} \Psi_k\left(\frac{x}{\sqrt{\hbar}} \right), \quad k \in \mathbb{N}^d.$$

This can be addressed considering the semiclassical harmonic oscillator $\widehat{H}_\hbar$ defined by

$$\widehat{H}_\hbar := \hbar\, S_\hbar \widehat{H} S_\hbar^* = \frac{1}{2} \sum_{j=1}^d \omega_j \big(-\hbar^2 \partial_{x_j}^2 + x_j^2 \big), \quad \omega_j > 0. \tag{1.23}$$

Note that the spectrum of $\widehat{H}_\hbar$ is just the spectrum of $\widehat{H}$ multiplied by $\hbar$:

$$\mathrm{Sp}_{L^2(\mathbb{R}^d)} \big(\widehat{H}_\hbar \big) = \{ \lambda_{k,\hbar} = \hbar \lambda_k, \quad k \in \mathbb{N}^d \}.$$

Moreover, for every sequence $(k^\hbar) \subset \mathbb{N}^d$ such that

$$\lambda_\hbar := \lambda_{k^\hbar,\hbar} \to 1, \quad \text{as } \hbar \to 0^+,$$

and for every sequence $(\Psi_{k^\hbar,\hbar})$ of normalized eigenfunctions of $\widehat{H}_\hbar$ with eigenvalues $(\lambda_\hbar)$, there exists a probability measure $\nu \in \mathcal{P}(\mathbb{R}^d)$, which we will call quantum limit, such that, modulo the extraction of a subsequence,

$$|\Psi_{k^\hbar,\hbar}|^2 \rightharpoonup^\star \nu, \quad \text{as } \hbar \to 0^+.$$

From the semiclassical point of view, $\widehat{H}_\hbar = \mathrm{Op}_\hbar(H)$ is the semiclassical Weyl quantization (see Section 2.2) of the symbol H given by the classical harmonic oscillator:

$$H(x,\xi) = \frac{1}{2} \sum_{j=1}^d \omega_j \big(\xi_j^2 + x_j^2 \big), \quad (x,\xi) \in \mathbb{R}^{2d}, \tag{1.24}$$

whose induced Hamiltonian flow will be denoted by ϕ_t^H.

From now on we fix $\widehat{H}_\hbar$ to be defined by (1.23). Let $\varepsilon = (\varepsilon_\hbar) \subset \mathbb{R}_+$ be a sequence of positive real numbers satisfying $\varepsilon_\hbar \to 0^+$ as $\hbar \to 0^+$, we consider a semiclassical perturbation of $\widehat{H}_\hbar$ of the form

$$\widehat{P}_\hbar := \widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar, \tag{1.25}$$

where $\widehat{V}_\hbar$ is the semiclassical Weyl quantization of a symbol $V \in \mathcal{C}^\infty(\mathbb{R}^{2d};\mathbb{R})$ which is bounded together with all its derivatives (i.e. it belongs to the class $S^0(\mathbb{R}^{2d})$, see (2.20) in Section 2.3). By the Calderón-Vaillancourt Theorem (see Lemma 2.5), $\widehat{V}_\hbar$ is a bounded operator on $L^2(\mathbb{R}^d)$.

Note that this operator is of the form (1.21) introduced in the previous section. We aim at understanding the long-time dynamics of the Schrödinger equation

$$\big(i\hbar\,\partial_t + \widehat{P}_\hbar\big)v_\hbar(t,x) = 0, \quad v_\hbar(0,x) = u_\hbar \in L^2(\mathbb{R}^d), \tag{1.26}$$

as well as the asymptotic distribution of energy on the phase space of solutions of the stationary problem

$$\widehat{P}_\hbar\,\Psi_\hbar = \lambda_\hbar\,\Psi_\hbar, \quad \|\Psi_\hbar\|_{L^2(\mathbb{R}^d)} = 1, \tag{1.27}$$

as $\hbar \to 0^+$. In order to study the asymptotic behavior of solutions of (1.26), we start by considering sequences of initial data $(u_\hbar)$ on $L^2(\mathbb{R}^d)$ satisfying $\|u_\hbar\|_{L^2} = 1$. The distribution of energy of each function u_h on the phase space $T^*\mathbb{R}^d = \mathbb{R}^{2d}$ can be described in terms of its related Wigner distribution [117] (see Section 2.6). We recall that the Wigner distribution $W^\hbar_{u_\hbar} \in \mathcal{D}'(\mathbb{R}^{2d})$ of $u_\hbar$ is defined by the map

$$\mathcal{C}^\infty_c(\mathbb{R}^{2d}) \ni a \longmapsto \big\langle u_\hbar, \mathrm{Op}_\hbar(a)u_\hbar\big\rangle_{L^2(\mathbb{R}^d)} =: W^\hbar_{u_\hbar}(a),$$

where $\mathrm{Op}_\hbar(a)$ denotes the semiclassical Weyl quantization of the symbol a and we use the following convention for the scalar product on $L^2(\mathbb{R}^d)$:

$$\langle f, g\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx.$$

After possibly extracting a subsequence, there exists a positive Radon measure $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ such that

$$W^\hbar_{u_\hbar} \rightharpoonup^\star \mu_0, \quad \text{as } \hbar \to 0^+,$$

where the convergence takes place in the sense of distributions. The measure μ_0 is called the semiclassical measure associated to the (sub)sequence $(u_\hbar)$ and it satisfies

$$0 \le \int_{\mathbb{R}^{2d}} \mu_0(dx, d\xi) \le 1. \tag{1.28}$$

One can show, for instance using linear combinations of wave-packets, that every positive Radon measure $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ satisfying (1.28) can be obtained as the semiclassical measure of a normalized sequence $(u_\hbar)$. We will restrict our attention to those sequences $(u_\hbar)$ with related semiclassical measure $\mu_0 \in \mathcal{P}(H^{-1}(1))$ i.e., a probability measure on the level set $H^{-1}(1)$. This holds if the sequence $(u_\hbar)$ satisfies the following hypothesis of $\hbar$-oscillation associated to the harmonic oscillator[5]:

$$\lim_{\delta\to 0^+}\lim_{\hbar\to 0^+} \|\mathbf{1}_{[1-\delta,1+\delta]}(\widehat{H}_\hbar)u_\hbar\|_{L^2(\mathbb{R}^d)} = 1, \tag{1.29}$$

[5]Compare with condition (1.10).

where $\mathbf{1}_{[1-\delta,1+\delta]}$ stands for the characteristic function of the interval $[1-\delta,1+\delta]$ and $\mathbf{1}_{[1-\delta,1+\delta]}(\widehat{H}_\hbar)$ is defined using the functional calculus for selfadjoint operators. That is, if

$$u_\hbar = \sum_{\lambda_\hbar \in \mathrm{Sp}(\widehat{H}_\hbar)} \Pi_{\lambda_\hbar} u_\hbar,$$

then

$$\mathbf{1}_{[1-\delta,1+\delta]}(\widehat{H}_\hbar) u_\hbar = \sum_{\lambda_\hbar \in [1-\delta,1+\delta]} \Pi_{\lambda_\hbar} u_\hbar.$$

We denote by $v_\hbar(t)$ the sequence of solutions of (1.26) with initial data $(u_\hbar)$. By Stone's Theorem (see Lemma 2.7), $v_\hbar(t)$ is given by the unitary transformation

$$v_\hbar(t) = e^{-\frac{it}{\hbar}\widehat{P}_\hbar} u_\hbar, \quad t \in \mathbb{R}. \tag{1.30}$$

The correspondence principle [45], [51], [77] (see Section 2.6.2) stablishes that if μ_0 is the semiclassical measure associated to the sequence $(u_\hbar)$ then, for every $t \in \mathbb{R}$, there exists a unique semiclassical measure $\mu(t)$ for the sequence $(v_\hbar(t))$ and it satifies

$$\mu(t) = (\phi_t^H)_* \mu_0,$$

where $(\phi_t^H)_*$ is the push-forward of the classical Hamiltonian flow ϕ_t^H generated by H. Note that the perturbation $\varepsilon_\hbar \widehat{V}_\hbar$ does not influence the semiclassical measure $\mu(t)$ at this regime of time. The situation changes if instead of considering the Wigner distribution at fixed time t, we introduce a time scale $\tau := (\tau_\hbar)$ such that

$$\tau_\hbar \to \infty, \quad \text{as } \hbar \to 0^+,$$

and we look at the Wigner distributions associated to the sequence $(v_\hbar(t\tau_\hbar))$:

$$W_\hbar^{\tau,\varepsilon}(t)(a) := \big\langle v_\hbar(t\tau_\hbar), \mathrm{Op}_\hbar(a) v_\hbar(t\tau_\hbar) \big\rangle_{L^2(\mathbb{R}^d)}, \quad a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d}), \quad t \in \mathbb{R}. \tag{1.31}$$

As we have already mention in the motivation, we can not expect to have any limit object for fixed t if the time scale $(\tau_\hbar)$ is larger than the Ehrenfest time (see Bambusi et. al. [18]). However, as first done by Macià in [80], we can consider the Wigner distributions $W_\hbar^{\tau,\varepsilon}$ as elements of the space $L^\infty(\mathbb{R}, \mathcal{D}'(\mathbb{R}^{2d}))$. Modulo extracting a subsequence, one obtain the existence of a measure $\mu \in L^\infty(\mathbb{R}, \mathcal{M}_+(\mathbb{R}^{2d}))$ such that, for every $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ and for every φ in $L^1(\mathbb{R})$,

$$\lim_{\hbar \to 0^+} \int_{\mathbb{R}} \varphi(t) W_\hbar^{\tau,\varepsilon}(t)(a) dt = \int_{\mathbb{R}} \int_{\mathbb{R}^{2d}} \varphi(t) a(x,\xi) \mu(t, dx, d\xi) dt.$$

Moreover, the measure $\mu(t,\cdot)$ obtained is invariant by the flow ϕ_t^H generated by the Hamiltonian H (see Macià [80] and Section 2.6.4). The aim of this work is to study the extra invariance or propagation properties satisfied by $\mu(t,\cdot)$ as the time scale τ grows up with respect to the perturbation scale ε. We denote by $\mathcal{M}(\widehat{P}_\hbar, \tau_\hbar)$ the set of accumulation points of sequences $(W_\hbar^{\tau,\varepsilon})$ as $(u_\hbar)$ varies among normalized $\hbar$-oscillating sequences in the sense of (1.29). As we did in Section 1.1, we can also define the set $\mathcal{N}(\widehat{P}_\hbar, \tau_\hbar)$ of measures $\nu \in L^\infty(\mathbb{R}; \mathcal{P}(\mathbb{R}^d))$ such that

$$\lim_{\hbar \to 0^+} \int_{\mathbb{R}\times\mathbb{R}^d} \varphi(t,x) |e^{-i\frac{\tau_\hbar t}{\hbar}\widehat{P}_\hbar} u_\hbar(x)|^2 dx\, dt = \int_{\mathbb{R}\times\mathbb{R}^d} \varphi(t,x)\nu(t,dx)dt, \quad \forall \varphi \in \mathcal{C}_c(\mathbb{R}\times\mathbb{R}^d).$$

By construction, $\nu \in \mathcal{N}(\widehat{P}_\hbar, \tau_\hbar)$ if and only if there exists if $\mu \in \mathcal{M}(\widehat{P}_\hbar, \tau_\hbar)$ (obtained from the same sequence of initial data) such that, for almost every $t \in \mathbb{R}$,

$$\nu(t,x) = \int_{\mathbb{R}^d} \mu(t,x,d\xi).$$

Analogously, we define $\mathcal{M}(\widehat{P}_\hbar)$ to be the set of semiclassical measures associated to sequences $(\Psi_\hbar)$ of normalized eigenfunctions of $\widehat{P}_\hbar$ with eigenvalues $\lambda_\hbar \to 1$ and $\mathcal{N}(\widehat{P}_\hbar)$ to be the set of measures ν obtained as weak limits

$$|\Psi_\hbar|^2 \rightharpoonup^\star \nu, \quad \text{as } \hbar \to 0^+.$$

Again, $\nu \in \mathcal{M}(\widehat{P}_\hbar)$ if and only if there exists $\mu \in \mathcal{M}(\widehat{P}_\hbar)$ (obtained form the same sequence of eigenfunctions) such that

$$\nu(x) = \int_{\mathbb{R}^d} \mu(x,d\xi).$$

But in principle, and this is crucial to obtain propagation laws and invariance properties, the sets $\mathcal{M}(\widehat{P}_\hbar)$ and $\mathcal{M}(\widehat{P}_\hbar, \tau_\hbar)$ contain more information than $\mathcal{N}(\widehat{P}_\hbar)$ and $\mathcal{N}(\widehat{P}_\hbar, \tau_\hbar)$, since they describe the distribution of the sequence in the phase space and not only in the position variable

1.2.1. Long time dynamics for the Schrödinger equation

We next explain the propagation laws and flow invariances of elements of $\mathcal{M}(\widehat{P}_\hbar, \tau_\hbar)$. Given the vector of frequencies $\omega := (\omega_1, \dots, \omega_d)$ of the harmonic oscillator H, we consider the submodule

$$\Lambda_\omega := \{k \in \mathbb{Z}^d \, : \, k\cdot\omega = 0\}. \tag{1.32}$$

The nontriviality of this set implies that the vector of frequencies is not irrational. As we will see below, a major role in our study will be played by the average of the symbol V along the orbits of the flow ϕ_t^H. The average $\langle a \rangle$ of a symbol $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ by the flow ϕ_t^H is defined by

$$\langle a \rangle (x,\xi) := \lim_{T\to\infty} \frac{1}{T}\int_0^T a \circ \phi_t^H(x,\xi) dt. \tag{1.33}$$

This limit is well defined and the convergence takes place in the $\mathcal{C}^\infty(\mathbb{R}^{2d})$ topology[6].

[6]Recall that $a_n \to a$ in the $\mathcal{C}^\infty(\mathbb{R}^{2d})$ topology if for every compact set K and every $k \in \mathbb{N}$, $\|a_n - a\|_{\mathcal{C}^k(K)} \to 0$ as $n \to \infty$.

Recall also that the harmonic oscillator H defines a completely integrable Hamiltonian system on $\mathbb{R}^{2d}$ (see Section 3.1). Indeed, a maximal set of linearly independent integrals $\{H_1, \ldots, H_d\}$ is given by the set of decoupled harmonic oscillators

$$H_j(x,\xi) = \frac{1}{2}\big(\xi_j^2 + x_j^2\big), \quad j \in \{1, \ldots, d\}.$$

Note that $\{H_j, H_k\} = 0$, and hence these integrals are in involution. In particular, we can write H as a function of $H_1, \ldots, H_d$. Precisely,

$$H = \mathcal{L}_\omega(H_1, \ldots, H_d),$$

where $\mathcal{L}_\omega : \mathbb{R}^d_+ \to \mathbb{R}$ is the linear form defined by $\mathcal{L}_\omega(E) = \omega \cdot E$. Observe also that, for every energy-tuple $E = (E_1, \ldots, E_d) \in \mathcal{L}_\omega^{-1}(1)$, the torus

$$\mathbb{T}_E := H_1^{-1}(E_1) \cap \cdots \cap H_d^{-1}(E_d) \subset H^{-1}(1) \tag{1.34}$$

is invariant by the Hamiltonian flow ϕ_t^H. In general $\mathbb{T}_E$ is not a minimal invariant set, since it can be foliated by Kronecker invariant tori of smaller dimension. We now introduce the transformation

$$\Phi_\tau^H := \phi_{t_d}^{H_d} \circ \cdots \circ \phi_{t_1}^{H_1}, \quad \tau = (t_1, \ldots, t_d) \in \mathbb{R}^d, \tag{1.35}$$

and note that $\tau \mapsto \Phi_\tau^H$ is $2\pi\mathbb{Z}^d$-periodic, hence we can view it as a function defined on the torus $\mathbb{T}^d := \mathbb{R}^d / 2\pi\mathbb{Z}^d$. We consider also the Kronecker torus $\mathbb{T}_\omega$ defined by

$$\mathbb{T}_\omega := \Lambda_\omega^\perp / (2\pi\mathbb{Z}^d \cap \Lambda_\omega^\perp) \subset \mathbb{T}^d,$$

where $\Lambda_\omega^\perp$ denotes the linear space orthogonal to Λ_ω. This torus stands for the minimal invariant set of angle-coordinates where the orbits of ϕ_t^H are dense. The dimension of $\mathbb{T}_\omega$ is $d_\omega = d - \operatorname{rk} \Lambda_\omega$. This allows us to decompose any function $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ in a Fourier series as follows:

$$a(x,\xi) = \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} a_k(x,\xi), \quad a_k(x,\xi) := \int_{\mathbb{T}^d} a \circ \Phi_\tau^H(x,\xi) e^{-ik\cdot\tau} d\tau,$$

note that $a_k \circ \Phi_\tau^H = a_k\, e^{ik\cdot\tau}$, and hence (see Section 3.1), write the average $\langle a \rangle$ as

$$\langle a \rangle(x,\xi) = \frac{1}{(2\pi)^d} \sum_{k \in \Lambda_\omega} a_k(x,\xi) = \int_{\mathbb{T}_\omega} a \circ \Phi_\tau^H(x,\xi) \mathfrak{h}_\omega(d\tau), \tag{1.36}$$

where $\mathfrak{h}_\omega$ denotes the Haar measure on the torus $\mathbb{T}_\omega$ (i.e. the uniform probability measure on $\mathbb{T}_\omega$). We next define the following equivalence relation on $H^{-1}(1)$ to obtain the reduction by the action of the Hamiltonian flow ϕ_t^H: we say that two points $z, z' \in H^{-1}(1)$ satisfy $z \sim_\omega z'$ if they share the same minimal invariant set by ϕ_t^H, i.e, $\mathcal{O}^H(z) = \mathcal{O}^H(z')$, where

$$\mathcal{O}^H(z) := \{\Phi_\tau^H(z) \,:\, \tau \in \mathbb{T}_\omega\}.$$

For any $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$, we denote by $\langle a \rangle_\omega$ the reduction of the average $\langle a \rangle$ on $H^{-1}(1)/\sim_\omega$. That is, for every $\rho \in H^{-1}(1)/\sim_\omega$, denoting $\pi_\omega : H^{-1}(1) \to H^{-1}(1)/\sim_\omega$ the projection,

$$\langle a \rangle_\omega(\rho) = \langle a \rangle(z), \quad \forall z \in H^{-1}(1), \quad \pi_\omega(z) = \rho.$$

Another important issue to take into account is the *Diophantine* nature of the vector of frequencies ω. It will also play an important role in our study, since it is related to the speed of convergence of

$$\frac{1}{T}\int_0^T a \circ \phi_t^H dt$$

to the average $\langle a \rangle$ as $T \to \infty$ and, as we shall see, this is crucial when dealing with the classic problem of *small denominators* in KAM theory.

Definition 1.1. *A vector $\omega \in \mathbb{R}^d_+$ is called partially Diophantine if*

$$|\omega \cdot k| \geq \frac{\varsigma}{|k|^{\gamma-1}}, \quad \forall k \in \mathbb{Z}^d \setminus \Lambda_\omega, \tag{1.37}$$

for some fixed constants $\varsigma > 0$ and $\gamma > d$.

Remark 1.1. Here the vector ω is not required to be rationally independent. If $\operatorname{rk}\Lambda_\omega = 0$ (when the components of ω are rationally independent), then the condition (1.37) means simply that ω is Diophantine in the usual way. However, in the particular case when $\operatorname{rk}\Lambda_\omega = d-1$, one has that ω is always partially Diophantine, since it is of the form αk_0, with $\alpha > 0$ and $k_0 \in \mathbb{N}^d$. Indeed,

$$|\omega \cdot k| = \alpha|k_0 \cdot k| \geq \alpha > 0, \quad \forall k \in \mathbb{Z}^d \setminus \Lambda_\omega.$$

This case corresponds to the periodic harmonic oscillator. It is well known that the set of Diophantine vectors has full Lebesgue measure (see for instance [34]). Hence the set of partially Diophantine vectors has also full Lebesgue measure, since they contain the set of Diophantine vectors.

We next state our first result:

Theorem 1.2. *Let $\mu \in \mathcal{M}(\widehat{P}_\hbar, \tau_\hbar)$ and denote by μ_0 the semiclassical measure associated to the sequence of initial data used to generate μ. Then the following holds:*

(i) If $\tau_\hbar \varepsilon_\hbar \to 0^+$ then μ is constant with respect to t and, for every $a \in \mathcal{C}_c(\mathbb{R}^{2d})$ and every $t \in \mathbb{R}$:

$$\mu(t)(a) = \mu_0(\langle a \rangle).$$

(ii) If $\tau_\hbar \varepsilon_\hbar \to 1$ then μ is continuous with respect to t and, for every $a \in \mathcal{C}_c(\mathbb{R}^{2d})$ and every $t \in \mathbb{R}$:

$$\mu(t)(a) = \mu_0(\langle a \rangle \circ \phi_t^{\langle V \rangle}),$$

where $\phi_t^{\langle V \rangle}$ denotes de Hamiltonian flow generated by $\langle V \rangle$.

(iii) If $\tau_\hbar \varepsilon_\hbar \to \infty$ then μ has an additional invariance property. For almost every $t \in \mathbb{R}$ and every $s \in \mathbb{R}$:

$$(\phi_s^{\langle V \rangle})_* \mu(t) = \mu(t).$$

Remark 1.2. Note that the flows ϕ_t^H and $\phi_s^{\langle V \rangle}$ commute (i.e. $\{H, \langle V \rangle\} = 0$). Hence $\phi_s^{\langle V \rangle}$ preserves the energy level $H^{-1}(1)$. Moreover, the action of $\phi_s^{\langle V \rangle}$ on $H^{-1}(1)$ is determined only by the values of $\langle V \rangle_\omega$ but not on the values of $\langle V \rangle$ transversally to $H^{-1}(1)$.

Remark 1.3. We emphasize that in the statement of Theorem 1.2 there is not restriction on the size of the sequence $\varepsilon = (\varepsilon_\hbar)$.

Remark 1.4. In the case $\operatorname{rk} \Lambda_\omega = 0$, points *(ii)* and *(iii)* of Theorem 1.2 are empty since for every $z \in H^{-1}(1)$, $X_{\langle V \rangle} \in T_z \mathcal{O}^H(z)$, where $X_{\langle V \rangle}$ denotes the Hamiltonian vector field generated by $\langle V \rangle$. Indeed, in this case $\langle V \rangle = \mathcal{I}_{\langle V \rangle}(H_1, \ldots, H_d)$, i.e. the average is taken over all of $\mathbb{T}_E$:

$$\mathcal{I}_{\langle V \rangle}(E_1, \ldots, E_d) = \langle V \rangle(x, \xi), \quad \text{for all } (x, \xi) \in \mathbb{T}_E, \quad E = (E_1, \ldots, E_d) \in \mathbb{R}_+^d.$$

Theorem 1.5 below will cover this case.

Observe that if $\tau_\hbar \varepsilon_\hbar \to 0^+$, the first point of Theorem 1.2 implies that

$$\mathcal{M}(\widehat{P}_\hbar, \tau_\hbar) = \mathcal{M}(H),$$

where $\mathcal{M}(H)$ denotes de set of probability measures supported on $H^{-1}(1)$ that are invariant by the flow ϕ_t^H (compare with Theorem 1.7 below). This is just the result of Macià [80] adapted to the case of the harmonic oscillator. On the other hand, if $\tau_\hbar \varepsilon_\hbar \to +\infty$, Theorem 1.2 implies in particular that, if the critical set $\mathcal{C}(V)$ defined by

$$\mathcal{C}(V) := \{ z \in H^{-1}(1) \, : \, X_{\langle V \rangle}\big|_z \in T_z \mathcal{O}^H(z) \},$$

satisfies $\mathcal{C}(V) \neq H^{-1}(1)$ then there exist infinetely many invariant tori $\mathcal{O}^H(z)$ such that, for almost every $t \in \mathbb{R}$,

$$\mu(t)\big(\mathcal{O}^H(z)\big) = 0.$$

In particular,

$$\mathcal{M}(\widehat{P}_\hbar, \tau_\hbar) \neq \mathcal{M}(H),$$

since we exclude the delta measures not supported on critical orbits. This results in an adaptation of the methods of Macià and Rivière [84, 85] for the harmonic oscillator. Otherwise, if the critical set satisfies $\mathcal{C}(V) = H^{-1}(1)$, that is when $\langle V \rangle_\omega$ is a constant function, we can say something more, provided that the vector ω is partially Diophantine.

Theorem 1.3. *Assume that ω is partially Diophantine. Suppose also that $\langle V \rangle_\omega$ is identically constant. Let $\mu \in \mathcal{M}(\widehat{P}_\hbar, \tau_\hbar)$, denote by μ_0 the semiclassical measure associated to the sequence of initial data used to generate μ. Let $V^\eth$ be the function defined by*

$$V^\eth := \frac{1}{2}\{V^\sharp, V\}, \tag{1.38}$$

where $V^\sharp$ is given by

$$V^\sharp(x,\xi) = \frac{1}{(2\pi)^d} \sum_{k\in\mathbb{Z}^d\setminus\Lambda_\omega} \frac{V_k(x,\xi)}{ik\cdot\omega}. \tag{1.39}$$

Then the following holds:

(i) If $\tau_\hbar\,\varepsilon_\hbar^2 \to 0^+$ then μ is continuous with respect to t and, for every $a \in \mathcal{C}_c(\mathbb{R}^{2d})$ and every $t\in\mathbb{R}$:

$$\mu(t)(a) = \mu_0(\langle a\rangle).$$

(ii) If $\tau_\hbar\,\varepsilon_\hbar^2 \to 1$ then μ is continuous with respect to t and, for every $a \in \mathcal{C}_c(\mathbb{R}^{2d})$ and every $t\in\mathbb{R}$:

$$\mu(t)(a) = \mu_0(\langle a\rangle \circ \phi_t^{\langle V^\eth\rangle}).$$

(iii) If $\tau_\hbar\,\varepsilon_\hbar^2 \to \infty$ then μ has an additional invariance property. For almost every $t\in\mathbb{R}$ and every $s\in\mathbb{R}$:

$$(\phi_s^{\langle V^\eth\rangle})_*\mu(t) = \mu(t).$$

Remark 1.5. In the periodic case $\omega = (1,\dots,1)$, the function $\langle V^\eth\rangle$ has the simpler form

$$\langle V^\eth\rangle = \frac{1}{4\pi}\int_0^{2\pi}\int_0^t \{V\circ\phi_s^H, V\circ\phi_t^H\}ds\,dt. \tag{1.40}$$

Remark 1.6. It is not difficult to find examples of perturbations $V\in S^0(\mathbb{R}^{2d})$ for which $\langle V\rangle_\omega$ is constant but $\langle V^\eth\rangle_\omega$ is not, see Example 3.1.

To prove Theorem 1.3, we will conjugate the operator $\widehat{P}_\hbar$ by some suitable unitary operator so that the perturbation $\varepsilon_\hbar\widehat{V}_\hbar$ is averaged by the quantum flow $e^{i\frac{t}{\hbar}\widehat{H}_\hbar}$ up to order $\varepsilon_\hbar^N$ for arbitrary large N. Let $T>0$ and let $\widehat{A}_\hbar := \mathrm{Op}_\hbar(a)$ with $a\in S^0(\mathbb{R}^{2d})$, we define its quantum average $\langle\widehat{A}_\hbar\rangle_T$ at time T by

$$\langle\widehat{A}_\hbar\rangle_T := \frac{1}{T}\int_0^T e^{-i\frac{t}{\hbar}\widehat{H}_\hbar}\,\widehat{A}_\hbar\,e^{i\frac{t}{\hbar}\widehat{H}_\hbar}dt. \tag{1.41}$$

The following is consequence of Egorov's theorem (Lemma 2.8), which is exact since H is a polynomial of degree two, the Calderón-Vaillancourt theorem (Lemma 2.5) and the fact that the partial average $\frac{1}{T}\int_0^T a\circ\phi_t^H dt$ converges to $\langle a\rangle$ as $T\to\infty$ in the $\mathcal{C}^\infty(\mathbb{R}^{2d})$ topology:

Proposition 1.1. *The limit*

$$\langle\widehat{A}_\hbar\rangle := \lim_{T\to\infty}\langle\widehat{A}_\hbar\rangle_T \tag{1.42}$$

is well defined in the strong operator $\mathcal{L}(L^2)$-norm, and it satisfies

$$\langle\widehat{A}_\hbar\rangle = \mathrm{Op}_\hbar(\langle a\rangle).$$

We will show in Section 3.2 that if ω is partially Diophantine then, for every $N \in \mathbb{N}$, there exists a sequence of unitary operators $(U_{N,\hbar})$ on $L^2(\mathbb{R}^d)$ such that

$$\widehat{P}_\hbar^N = U_{N,\hbar}^*\big(\widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar\big)U_{N,\hbar} = \widehat{H}_\hbar + \varepsilon_\hbar \langle \widehat{V}_\hbar\rangle + \sum_{j=2}^{N} \varepsilon_\hbar^j \langle \widehat{R}_{j,\hbar}\rangle + O(\varepsilon_\hbar^{N+1}),$$

where $\widehat{R}_{1,\hbar} = \widehat{V}_\hbar$, and $\widehat{R}_{j,\hbar}$ are L^2-bounded pseudodifferential operators. Let $R_j(\hbar)$ be the symbol of $\widehat{R}_{j,\hbar}$, which can be expanded as

$$R_j(\hbar) \sim \sum_{k=0}^{\infty} r_{j,k}\, \hbar^k,$$

we have, in particular, $\langle V^{\eth}\rangle = \langle r_{2,0}\rangle$. Moreover, it could be possible that $\langle V\rangle_\omega$ and $\langle V^{\eth}\rangle_\omega$ where constant, but there was some first element $r_{j,k}$ in the series such that $\langle r_{j,k}\rangle_\omega$ was not identically constant (see Example 3.2 for a particular case). The following result deals with this situation:

Theorem 1.4. *Assume that $\varepsilon_\hbar = \hbar^\alpha$ for some $\alpha > 0$. If there exists a function $L = L(V)$ given by the sum of all terms $r_{j,k}$ in the series such that*

$$\langle L\rangle_\omega = \sum_{\delta_\hbar = \varepsilon_\hbar^j \hbar^k} \langle r_{j,k}\rangle_\omega \tag{1.43}$$

is not constant, and such that the order $\delta_\hbar$ is maximal with respect to this condition, then the three alternatives of Theorem 1.2 hold replacing the critical scale $\tau_\hbar \sim 1/\varepsilon_\hbar$ by $\tau_\hbar \sim 1/(\varepsilon_\hbar^j \hbar^k)$ and the symbol $\langle V\rangle$ by $\langle L\rangle$.

Remark 1.7. The assumption $\varepsilon_\hbar = \hbar^\alpha$ prevents patological situations. For instance, in principle it could be possible that $\langle R_1(\hbar)\rangle_\omega$ was not identically constant, but $\langle r_{1,k}\rangle_\omega \equiv 0$ for all $k \geq 0$. Then, if $\varepsilon_\hbar \ll \hbar^k$ for all k, the order of $\langle R_1(\hbar)\rangle$ would be larger than the one of $\langle R_2(\hbar)\rangle$, and we could not find $\langle L\rangle$. Another patological situation could be that $\varepsilon_\hbar^j \gg \hbar$ and $\langle r_{j,0}\rangle_\omega \equiv 0$ for all $j \geq 1$, but $\langle r_{1,1}\rangle_\omega$ was not identically constant. Again, we could not find $\langle L\rangle$.

Recall that, in the case $\operatorname{rk}\Lambda_\omega = 0$, for every $L \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ one has $\langle L\rangle = \mathcal{I}_{\langle L\rangle}(H_1, \dots, H_d)$. Thus points *(ii)* and *(iii)* of Theorems 1.3 and 1.4 are empty. Our result in this case reads:

Theorem 1.5. *Suppose that ω is Diophantine (in particular, $\operatorname{rk}\Lambda_\omega = 0$). Let τ be such that there exists an integer $N \geq 1$ verifying*

$$\tau_\hbar \varepsilon_\hbar^N \to 0, \quad \text{as } \hbar \to 0^+.$$

Then, μ is continuous in the t variable and, for every $a \in \mathcal{C}_c(\mathbb{R}^{2d})$ and every $t \in \mathbb{R}$:

$$\mu(t)(a) = \mu_0(\langle a\rangle),$$

hence $\mathcal{M}(\widehat{P}_\hbar, \tau_\hbar) = \mathcal{M}(H)$. *More precisely, in this case:*

$$\int_{\mathbb{R}^{2d}} \langle a\rangle(x,\xi)\mu_0(dx,d\xi) = \int_{\mathcal{L}_\omega^{-1}(1)} \mathcal{I}_{\langle a\rangle}(E)\, \mathscr{H}_*\mu_0(dE),$$

where the measure $\mathscr{H}_*\mu_0$ *is given by the disintegration of the Liouville measure on the Lagrangian tori* $\mathbb{T}_E$*: for every* $f \in \mathcal{C}_c(\mathbb{R}^d_+)$,

$$\int_{\mathbb{R}^d_+} f(E)\, \mathscr{H}_*\mu_0(dE) := \int_{\mathbb{R}^{2d}} f\big(H_1(x,\xi),\dots,H_d(x,\xi)\big)\mu_0(dx,d\xi).$$

Remark 1.8. The Diophantine assumption on ω is only necessary when $\tau_\hbar\varepsilon_\hbar \to \infty$. Otherwise, it is sufficient to assume that $\operatorname{rk}\Lambda_\omega = 0$.

1.2.2. Some improvements in 2D

In this section we assume $d = 2$ and we consider the periodic harmonic oscillator, $\omega = (1,1)$. In [50], Guillemin, Uribe and Wang proved the following. Given $\varepsilon_\hbar = \hbar^2$ fixed. Let $I \subset \mathbb{R}$ denote an open interval in the image of H. For any $\lambda \in I$, let

$$\mathbb{S}^2_\lambda = H^{-1}(\lambda)/\mathbb{S}^1$$

be the reduced space by the free $\mathbb{S}^1$ action generated by H. Denote by

$$\langle V\rangle_\lambda : \mathbb{S}^2_\lambda \to \mathbb{R}$$

the reduction of $\langle V\rangle$ at $\mathbb{S}^2_\lambda$. If, for all $\lambda \in I$, $\langle V\rangle_\lambda$ is a perfect Morse function, that is, it has only two critical points, a maximun and a minimun then, for every $N \in \mathbb{N}$, there exists a Fourier integral operator $\mathcal{F}_\hbar$ that conjugates $\widehat{H}_\hbar + \hbar^2\widehat{V}_\hbar$ into the normal form

$$\widehat{P}^N_\hbar = \widehat{H}_\hbar + \hbar^2 G_2\big(\operatorname{Op}_\hbar(H_1), \operatorname{Op}_\hbar(H_2)\big) + \dots + \hbar^N G_N\big(\operatorname{Op}_\hbar(H_1), \operatorname{Op}_\hbar(H_2)\big) + \hbar^2\widehat{R}_\hbar + O(\hbar^{N+1}),$$

where G_j is a two-variable smooth function for $j = 2,3,\dots$, and $\widehat{R}_\hbar$ is a pseudodifferential operator whose *microsupport*[7] is disjoint from $\mathcal{V} := H^{-1}(I)$. Moreover, the Fourier integral operator $\mathcal{F}_\hbar$ quantizes a symplectic transformation $\kappa_\hbar = \kappa + O(\hbar)$ that provides the normal form at classic level. This means that, given $a \in \mathcal{C}^\infty_c(\mathcal{V})$,

$$\mathcal{F}_\hbar \operatorname{Op}_\hbar(a)\mathcal{F}^*_\hbar = \operatorname{Op}_h(\kappa^* a) + O(\hbar), \tag{1.44}$$

where $\kappa_\hbar = \kappa + O(\hbar) : \mathbb{R}^{2d} \to \mathbb{R}^{2d}$ is symplectomorphism such that $(H,V)\circ\kappa = (H, G_2(H_1,H_2))$ on $\mathcal{V}$ and

$$P_N = \big(H + \hbar^2 V\big)\circ\kappa_\hbar = H + \hbar^2 G_2(H_1,H_2) + \ldots + \hbar^N G_N(H_1,H_2) + O(\hbar^{N+1}),$$

on $\mathcal{V}$. In this case, we have:

[7] We say that $(x,\xi) \in \mathbb{R}^{2d}$ does not belong to the microsupport of $\widehat{R}_\hbar$ if its symbol $r(x,\xi,\hbar)$ vanishes to infinite order in $\hbar$ in an open neighborhood of (x,ξ).

Theorem 1.6. *Assume $d = 2$, $\omega = (1,1)$, and $\langle V\rangle_\lambda$ is a perfect Morse function for all $\lambda \in I \ni 1$. Let κ and $\mathcal{F}$ be the transformations satisfying* (1.44)*. Let τ be a time scale such that $\tau_\hbar \hbar^2 \to \infty$ and assume there exists an integer $N \geq 3$ satisfying*

$$\tau_\hbar \hbar^N \to 0^+, \quad \text{as } \hbar \to 0.$$

Then, μ does not depend on the t variable and, for every $a \in \mathcal{C}_c(\mathcal{V})$ and every $t \in \mathbb{R}$:

$$\mu(t)(a) = \mu_0\big(\mathcal{A}_{(H,V)}(a)\big),$$

where the double average $\mathcal{A}_{(H,V)}(a)$ is defined by

$$\mathcal{A}_{(H,V)}(a) := \lim_{T\to\infty} \frac{1}{T} \int_0^T \langle a\rangle \circ \phi_t^{\langle V\rangle} dt.$$

Remark 1.9. If $a \in \mathcal{C}_c^\infty(\mathcal{V})$ then $\mathcal{A}_{(H,V)}(a) \in \mathcal{C}_c^\infty(\mathcal{V})$. More precisely, as we shall show in the proof, we have the explicit formula $\mathcal{A}_{(H,V)}(a) = (\kappa^*)^{-1}\mathcal{A}_{(H_1,H_2)}(\kappa^* a)$, where

$$\mathcal{A}_{(H_1,H_2)}(a) := \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} a \circ \phi_{t_1}^{H_1} \circ \phi_{t_2}^{H_2}\, dt_1\, dt_2.$$

1.2.3. Weak limits of sequences of eigenfunctions

We next state some applications of our previous results in the study of the semiclassical measures associated to sequences of solutions for the stationary problem

$$\widehat{P}_\hbar\, \Psi_\hbar = \lambda_\hbar\, \Psi_\hbar, \quad \|\Psi_\hbar\|_{L^2(\mathbb{R}^d)} = 1,$$

with $\lambda_\hbar \to 1$ as $\hbar \to 0^+$. To fix ideas, we consider the set $\mathcal{M}(\widehat{P}_\hbar)$ of probability measures obtained as semiclassical measures associated to sequences of normalized eigenfunctions for $\widehat{P}_\hbar$ with eigenvalues satisfying $\lambda_\hbar \to 1$. In particular, if $\varepsilon = (\varepsilon_\hbar)$ vanishes identically, we denote the corresponding set of measures by $\mathcal{M}(\widehat{H}_\hbar)$. One always has

$$\mathcal{M}(\widehat{P}_\hbar) \subset \mathcal{M}(H), \tag{1.45}$$

where recall that $\mathcal{M}(H)$ is the set of probability measures supported on $H^{-1}(1)$ that are invariant by the flow ϕ_t^H, and the inclusion may be strict, even if $\varepsilon_\hbar \equiv 0$, as we will show below. We aim at understanding the influence of the perturbation $\varepsilon_\hbar \widehat{V}_\hbar$ on the concentration properties of the elements of $\mathcal{M}(\widehat{P}_\hbar)$. Before looking at the perturbed operator $\widehat{P}_\hbar = \widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar$, we first explain the situation for the nonperturbed harmonic oscillator $\widehat{H}_\hbar$, obtaining a complete characterization of $\mathcal{M}(\widehat{H}_\hbar)$. Roughly speaking, the basic idea is the following: the more multiplicity that each eigenvalue of the sequence has, the more concentration that the related sequence of eigenfunctions can reach. Observe that the spectrum of $\widehat{H}_\hbar$ is given by

$$\mathrm{Sp}_{L^2(\mathbb{R}^d)}\left(\widehat{H}_\hbar\right) = \left\{ \lambda_{k,\hbar} = \sum_{j=1}^{d} \hbar \left(k_j + \frac{1}{2} \right) \omega_j, \quad k = (k_1, \ldots, k_d) \in \mathbb{N}^d \right\},$$

hence the multiplicity of each eigenvalue $\lambda_{k,\hbar}$ depends on the arithmetic relations between components of the vector of frequencies $\omega = (\omega_1, \ldots, \omega_d)$. Considering the quotient set[8] $\mathbb{N}^d/\Lambda_\omega$, we observe that, for every $[k] \in \mathbb{N}^d/\Lambda_\omega$, if $k, k' \in [k]$ then $\lambda_{k,\hbar} = \lambda_{k',\hbar}$ (exact degeneracy of the eigenvalue). Thus, if we choose a sequence $(k^\hbar, \hbar) \subset \mathbb{N}^d \times (0, 1]$ such that $\lambda_{k^\hbar,\hbar} \to 1$ as $\hbar \to 0^+$, then the sequence $k^\hbar$ accumulates on the quotient set $\mathcal{L}_\omega^{-1}(1)/\,[\Lambda_\omega]$, where $[\Lambda_\omega]$ is the linear span of Λ_ω in $\mathbb{R}^d$. For every $[E] \in \mathcal{L}_\omega^{-1}(1)/[\Lambda_\omega]$, we define $\mathcal{M}_{[E]}(H)$ to be the set of ϕ_t^H-invariant probability measures supported on

$$\bigcup_{E \in [E]} \mathbb{T}_E \subset H^{-1}(1),$$

where recall that $\mathbb{T}_E$ is defined by (1.34) for every $E \in \mathcal{L}_\omega^{-1}(1)$. In particular, $\mathcal{M}_{[E]}(H) \subset \mathcal{M}(H)$ for every $[E] \in \mathcal{L}_\omega^{-1}(1)/[\Lambda_\omega]$. The following is a generalization of the result of Ojeda-Valencia and Villegas-Blas [93, Prop. 5] for the non-periodic harmonic oscillator:

Theorem 1.7. *For the nonperturbed harmonic oscillator $\widehat{H}_\hbar$:*

$$\mathcal{M}(H) \supset \mathcal{M}(\widehat{H}_\hbar) = \bigcup_{[E] \in \mathcal{L}_\omega^{-1}(1)/[\Lambda_\omega]} \mathcal{M}_{[E]}(H).$$

Remark 1.10. If $\mathrm{rk}\,\Lambda_\omega = 0$ then, for every $E \in \mathcal{L}_\omega^{-1}(1)$, $[E] = \{E\}$, that is,

$$\mathcal{L}_\omega^{-1}(1)/[\Lambda_\omega] \simeq \mathcal{L}_\omega^{-1}(1).$$

In this case,

$$\mathcal{M}(\widehat{H}_\hbar) = \bigcup_{E \in \mathcal{L}_\omega^{-1}(1)} \{\mathfrak{h}_E\},$$

where $\mathfrak{h}_E$ is the Haar measure on the torus $\mathbb{T}_E$. On the other hand, if $\mathrm{rk}\,\Lambda_\omega = d - 1$ (periodic case) our result reduces to the one of [93, Prop. 5]:

$$\mathcal{M}(\widehat{H}_\hbar) = \mathcal{M}(H).$$

Note that, in this case, $\mathcal{L}_\omega^{-1}(1)/[\Lambda_\omega] = \{\mathcal{L}_\omega^{-1}(1)\}$ and, for every $E \in \mathcal{L}_\omega^{-1}(1)$, $\mathcal{M}_{[E]}(H) = \mathcal{M}(H)$.

In the perturbed case, the following result is consequence of Theorems 1.2, 1.3 and 1.4:

[8] The quotient set $A/B \subset \mathcal{P}(A)$ is defined as follows: $[a] \in A/B$ if for all $a, a' \in [a]$, $a - a' \in B$.

Theorem 1.8. *Let $z \in H^{-1}(1)$ such that*

$$X_{\langle L\rangle}\big|_z \notin T_z\mathcal{O}^H(z),$$

where $\mathcal{O}^H(z)$ is the closure of the orbit of ϕ_t^H issued from the point z and $\langle L\rangle$ is given by $\langle V\rangle$, (resp. $\langle V^{\eth}\rangle$ or $\sum\langle r_{j,k}\rangle$) in the hypothesis of Theorem 1.2 (resp. the hypothesis of Theorems 1.3 or 1.4). In particular,

$$\mathcal{M}(H) \supset \mathcal{M}(\widehat{P}_\hbar) \neq \bigcup_{[E]\in\mathcal{L}_\omega^{-1}(1)/[\Lambda_\omega]} \mathcal{M}_{[E]}(H).$$

1.3. DISTRIBUTION OF EIGENVALUES FOR NON-SELFADJOINT HARMONIC OSCILLATORS

This part of the work is joint work with Gabriel Rivière and it is the content of the preprint [14]. The motivation comes from the study of the damped wave equation on Riemannian manifolds. Let (M, g) be a compact Riemannian manifold, the damped wave equation is given by the following partial differential equation:

$$\begin{cases} \left(\partial_t^2 - \Delta_g + a(x)\partial_t\right) u(t,x) = 0, \\ \left(u(0,\cdot), \partial_t u(0,\cdot)\right) = (u_0, u_1) \in H^1(M) \times L^2(M), \end{cases} \tag{1.46}$$

where Δ_g is the Laplace-Beltrami operator (see Section 2.7) and $a \in \mathcal{C}^\infty(M; \mathbb{R}_+)$ is called damping term. One can check that the energy

$$E(u,t) := \frac{1}{2}\Big(\|\nabla_g u(\cdot,t)\|^2_{L^2(M)} + \|\partial_t u(\cdot,t)\|^2_{L^2(M)}\Big)$$

of every solution u to (1.46) tends to zero as $t \to \infty$. Rauch and Taylor gave in [100] a necessary and sufficient condition for the existence of a uniform decay rate, that is, of a function $f(t) \to 0$ as $t \to \infty$ such that:

$$E(u,t) \leq f(t)E(u,0), \quad \forall (u_0,u_1) \in H^1(M) \times L^2(M). \tag{1.47}$$

In fact, (1.47) holds if and only if $\{a > 0\}$ satisfies the so-called Geometric Control Condition:

$$\exists T, c > 0, \quad \inf_{z_0 \in S^*M} \int_0^T a(\phi_s(z_0))ds \geq c, \tag{1.48}$$

where ϕ_s denotes the geodesic flow on M issued from z_0. The decay rate must necessarily be of exponential type: $f(t) \leq Me^{-\beta t}$ for t big enough.

Studying the large time behavior of solutions of (1.46), Lebeau in [74] obtained a formula for the exponential decay rate of energy in terms of the distribution of eigenvalues $\lambda = \alpha + i\beta \in \mathbb{C}$ of the non-selfadjoint problem

$$\big(- \Delta_g + i\lambda a(x)\big) v(x) = \lambda^2 v(x), \quad v \in L^2(M),$$

and of the average of $a(x)$ along the geodesics of (M, g). Motivated by this work, Sjöstrand in [109] showed that the eigenfrequencies lie in a strip of the complex plane which can be completely determined in terms of the average of the damping term along the geodesic flow [109] (see also [100, 74] for earlier related results). Showing these results for the damped wave equation turns out to be equivalent to obtaining analogous results for the following nonselfadjoint semiclassical problem:

$$\big(- \hbar^2\Delta_g + i\hbar a(x)\big) v_\hbar(x) = \lambda_\hbar v_\hbar, \quad \lambda_\hbar = \alpha_\hbar + i\hbar\beta_\hbar \in \mathbb{C}, \quad v_\hbar \in L^2(M),$$

which has since then be the object of several works. More precisely, it was investigated how these generalized eigenvalues are asymptotically distributed inside the strip determined by Sjöstrand and how the dynamics of the underlying classical Hamiltonian influences this asymptotic distribution. Mostly two questions have been considered in the literature. First, one can ask about the precise distribution of eigenvalues inside the strip and this question was addressed both in the completely integrable framework [57, 58, 59, 64, 60, 61, 62, 63] and in the chaotic one [3]. Second, it is natural to ask how eigenfrequencies can accumulate at the boundary of the strip and also if one can get resolvent estimates near the boundary of the strip. Again, this question has been explored both in the integrable case [16, 5, 26] and in the chaotic one [28, 106, 92, 29, 102, 68].

We consider the second question in the case of completely integrable systems. More precisely, we aim at describing the influence of the subprincipal symbol of the selfajoint part of our semi-classical operators on the asymptotic distribution of eigenvalues but also on resolvent estimates. For the sake of simplicity, we will restrict ourselves to the case of non-selfadjoint perturbations of semiclassical harmonic oscillators on $\mathbb{R}^d$ but it is most likely that the methods presented here can be adapted to deal with more general semiclassical operators associated with completely integrable systems. In [16, Th. 2.3], Asch and Lebeau showed how a selfadjoint pertubation of the principal symbol of the damped wave operator can create a spectral gap inside the spectrum in the high frequency limit. We will also explain how this result can be extended in our context[9]. Recall that a major ingredient in the proof of [16] and in the works of Hitrik-Sjöstrand [58, 59, 64, 60, 61, 62, 63] is the analyticity of the involved operators.

One of the novelty of our work compared with the references above is that we will also explore what can be said when we only suppose that the operators are smooth and how this is influenced by the subprincipal symbols of the selfadjoint part as in [16]. This will be achieved by building on the dynamical construction used for studying semiclassical Wigner measures of semiclassical harmonic oscillators introduced in [13], see Section 1.2 above, and see also [84, 85] in the case of Zoll manifolds.

[9]Observe that, compared with [16], our operators are not necessarily associated with a periodic flow.

We now describe the spectral framework we are interested in. We fix $\omega = (\omega_1, \dots, \omega_d)$ to be an element of $\mathbb{R}^d_+$, and we set $\widehat{H}_\hbar$ to be the semiclassical harmonic oscillator given by (1.23). We will study the spectral properties of nonselfadjoint perturbations of $\widehat{H}_\hbar$. We now fix two smooth functions A and V in $\mathcal{C}^\infty(\mathbb{R}^{2d}, \mathbb{R})$ all of whose derivatives are bounded. We set the Weyl quantization of these smooth symbols:

$$\widehat{A}_\hbar := \mathrm{Op}_\hbar(A), \quad \text{and} \quad \widehat{V}_\hbar := \mathrm{Op}_\hbar(V)$$

(see Section 2.2). These are selfadjoint operators which are bounded on $L^2(\mathbb{R}^d)$ thanks to the Calderón-Vaillancourt Theorem (see Lemma 2.5). We are aiming at describing the asymptotic properties of the following nonselfadjoint operator in the semiclassical limit $\hbar \to 0$:

$$\widehat{P}_\hbar := \widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar + i\hbar \widehat{A}_\hbar, \tag{1.49}$$

where $\varepsilon_\hbar \to 0$ as $\hbar \to 0$. More precisely, we focus on sequences of (pseudo-)eigenvalues

$$\lambda_\hbar = \alpha_\hbar + i\hbar\beta_\hbar$$

such that there exist $\beta \in \mathbb{R}$ and $(v_\hbar)_{\hbar\to 0^+}$ in $L^2(\mathbb{R}^d)$ so that

$$(\alpha_\hbar, \beta_\hbar) \to (1, \beta), \quad \text{as } \hbar \to 0, \quad \text{and} \quad \widehat{P}_\hbar v_\hbar = \lambda_\hbar v_\hbar + r_\hbar, \quad \|v_\hbar\|_{L^2} = 1. \tag{1.50}$$

Here $r_\hbar$ should be understood as a small remainder term which will be typically of order $o(\hbar)$. This allows us to deal with the case of quasimodes which is important to get resolvent estimates. Recall from [109, Th. 5.2] that true eigenvalues exist[10] and that, counted with their algebraic multiplicity, they verify a Weyl asymptotics as $\hbar \to 0$. It follows from the works of Rauch-Taylor [100], Lebeau [74] and Sjöstrand [109, Lemma 2.1] that:

Proposition 1.2. *Let $(\lambda_\hbar = \alpha_\hbar + i\hbar\beta_\hbar)_{\hbar\to 0^+}$ be a sequence of (cuasi-)eigenvalues verifying* (1.50) *with $\beta_\hbar \to \beta$ and $r_\hbar = o(\hbar)$. Then, one has*

$$\beta \in \left[\min_{z\in H^{-1}(1)} \langle A\rangle(z), \max_{z\in H^{-1}(1)} \langle A\rangle(z) \right]. \tag{1.51}$$

We include a proof of this proposition in Section 4.2. Note that one always has

$$\min_{z\in H^{-1}(1)} A(z) \le A_- := \min_{z\in H^{-1}(1)} \langle A\rangle(z) \le A_+ := \max_{z\in H^{-1}(1)} \langle A\rangle(z) \le \max_{z\in H^{-1}(1)} A(z),$$

where the inequalities may be strict. In some particular important cases the damping function A satisfies the so called *geometric-control* condition:

[10]This article deals with the case of non-selfadjoint perturbations of the Laplacian, which can be faithfully transferred to our case.

(GC) For every $z \in H^{-1}(1)$ there exists $T \in \mathbb{R}$ such that $A \circ \phi_T^H(z) > 0$.

As a consequence of Proposition 1.2, if $A \geq 0$ and it satisfies the condition **(GC)**, then $\beta > 0$. In particular, there exists a spectral gap in the spetrum of $\widehat{P}_\hbar$ near the real axis.

Let us now explain our main results which show how the selfadjoint term $\widehat{V}_\hbar$ influences the way that the eigenvalues may accumulate on the boundary of the interval given by Proposition 1.2. Recall that, if we come back to the damped wave equation, studying such a problem is related to the way that the energy of the waves decay (see e.g. [29]). The main assumption on the functions A and V will be the following weaker geometric-control condition:

(WGC) For every $z \in H^{-1}(1) \cap \langle A \rangle^{-1}(0)$ there exists $T \in \mathbb{R}$ such that $\langle A \rangle \circ \phi_T^{\langle V \rangle}(z) > 0$, where $\phi_t^{\langle V \rangle}$ is the Hamiltonian flow generated by $\langle V \rangle$.

Note that this condition implies that undamped trajectories are moved to damped ones through the flow $\phi_t^{\langle V \rangle}$. As we shall see, ensuring this dynamical property depends on the Diophantine properties of ω. Recall that, to each ω, one can associate the submodule

$$\Lambda_\omega := \{k \in \mathbb{Z}^d \, : \, \omega \cdot k = 0\}. \tag{1.52}$$

When the rank of Λ_ω is equal to 0, our geometric control condition can only be satisfied if $\langle A \rangle > 0$, since $\operatorname{rk} \Lambda_\omega = 0$ implies that each average is constant on each invariant Lagrangian torus, so it is a function of the actions. Hence, two averages Poisson-commute: $\{\langle V \rangle, \langle A \rangle\} = 0$. A typical case in which our dynamical condition holds is when $H^{-1}(1) \cap \langle A \rangle^{-1}(0)$ consists in a disjoint union of a finite number of minimal ϕ_t^H-invariant tori $(\mathcal{O}^H(z_k))_{k=1,\dots,N}$. In this case, condition **(WGC)** is equivalent to say that the Hamiltonian vector field $X_{\langle V \rangle}$ satisfies

$$\forall 1 \leq k \leq N, \quad \forall z \in \mathcal{O}^H(z_k), \quad \frac{d}{dt}\left(\phi_t^{\langle V \rangle}(z)\right)\Big|_{t=0} \notin \mathcal{O}^H(z_k).$$

1.3.1. The smooth case

First, we will assume an extra condition on the vector of frequencies ω of the harmonic oscillator H. We recall that $\omega := (\omega_1, \dots, \omega_d) \in \mathbb{R}_+^d$ is *partially Diophantine* if it satisfies (1.37). To keep an example in mind, note that the vector $\omega = (1, \dots, 1)$ is obviously partially Diophantine.

In the smooth case, our main result reads as follows:

Theorem 1.9. *Let ω be partially Diophantine. Suppose that $\langle A \rangle \geq 0$ on $H^{-1}(1)$, and that A and V satisfies the geometric-control condition* **(WGC)**. *Assume also that*

$$\varepsilon_\hbar \gg \hbar^2.$$

Then, for every sequence $(\lambda_\hbar = \alpha_\hbar + i\hbar\beta_\hbar)_{\hbar \to 0^+}$ of (quasi-)eigenvalues verifying (1.50) *with reminder term $r_\hbar$ satisfying $\|r_\hbar\| = o(\hbar\varepsilon_\hbar)$,*

$$\beta_\hbar \gg \varepsilon_\hbar.$$

This result shows that, under a certain geometric control condition, eigenvalues cannot accumulate too fast on the real axis as $\hbar \to 0$. We emphasize that, compared with the analytic case treated in [16], our result apply a priori to quasimodes. Hence, it also yields the following resolvent estimate in the smooth case. For every $R > 0$, and every $\hbar > 0$, there exists some constant $C_R > 0$ such that, for $\hbar > 0$ small enough,

$$\frac{\Im \lambda}{\hbar} \leq R\varepsilon_\hbar \Rightarrow \left\| \left(\widehat{P}_\hbar - \lambda \right)^{-1} \right\|_{\mathcal{L}(L^2)} \leq \frac{C_R}{\hbar \varepsilon_\hbar}, \tag{1.53}$$

which is usefull regarding energy decay estimates and asymptotic expansion of the corresponding semigroup (see e.g. [29]).

When $\langle A \rangle > 0$ everywhere on $H^{-1}(1)$, our Theorem is exactly the result of Rauch-Taylor [100] and Lebeau [74] adapted to the case of the harmonic oscillator. If $A \geq 0$ then the Diophantine property on the vector ω is not required, and the proof of Theorem 1.9 can be simplified a bit (see [14]). Our proof will crucially use the Fefferman-Phong inequality (hence the Weyl quantization) and this allows us to reach perturbations of size $\varepsilon_\hbar \gg \hbar^2$. If we had dealt with more general completely integrable systems (e.g. on compact manifolds), we would have probably been able to use the Garding inequality and it would have lead us to the stronger restriction $\varepsilon_\hbar \gg \hbar$.

1.3.2. The analytic case

We now consider the case when the functions A and V enjoy some analyticity properties. This will be achieved by following a method introduced by Asch and Lebeau in the case of the damped wave equation on the 2-sphere [16]. We will explain how to adapt this strategy in the framework of harmonic oscillators which are not necessarly periodic. Hence, the upcoming results should be viewed as an extension of Asch-Lebeau's construction to semiclassical harmonic oscillators and as an illustration on what can be gained via analyticity compared with the purely dynamical approach used to prove Theorem 1.9. Yet, we emphasize that the argument presented here only holds for true eigenmodes, i.e. $r_\hbar = 0$ in (1.50). In particular, it does not seem to yield any resolvent estimate like (1.53) which is crucial to deduce some results on the semigroup generated by $\widehat{P}_\hbar$.

Definition 1.2. *Let $s > 0$. We say that $a \in L^1(\mathbb{R}^{2d})$ belongs to the space $\mathcal{A}_s$ if*

$$\|a\|_s := \int_{\mathbb{R}^{2d}} |\widehat{a}(w)| e^{s|w|} \, dw < \infty,$$

where $\widehat{a}$ denotes the Fourier transform of a and $|w|$ the Euclidean norm on $\mathbb{R}^{2d}$. Let also $\rho > 0$, we introduce the space $\mathcal{A}_{\rho,s}$ of functions $a \in L^1(\mathbb{R}^{2d})$ such that

$$\|a\|_{\rho,s} := \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}^d} \|a_k\|_s \, e^{\rho |k|} < \infty, \tag{1.54}$$

where

$$a_k(z) = \int_{\mathbb{T}^d} a \circ \Phi_\tau^H(z) e^{-ik\cdot\tau}\, d\tau, \quad k \in \mathbb{Z}^d.$$

Remark 1.11. By the Calderón-Vaillancourt theorem (Lemma 4.4), one has

$$\forall\, a \in \mathcal{A}_s, \quad \| \operatorname{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} \leq C_{d,s}\|a\|_s. \tag{1.55}$$

Our next result reads:

Theorem 1.10. *Assume that ω is partially Diophantine and that $\langle A\rangle \geq 0$. Suppose also that A and V satisfy the geometric-control condidition* **(WGC)** *and belong to the space $\mathcal{A}_{\rho,s}$ for some fixed $\rho, s > 0$. If $\varepsilon_\hbar \geq \hbar$ then there exists $\varepsilon := \varepsilon(A,V) > 0$ such that, for any solution to* (1.50) *with $r_\hbar = 0$,*

$$\beta \geq \varepsilon. \tag{1.56}$$

Moreover, if $\varepsilon_\hbar \gg \hbar$ then, for all $T > 0$ such that

$$T < \frac{\sigma^2}{2\|\langle V\rangle\|_s}, \tag{1.57}$$

where $\sigma < \min\{\rho, s/3\}$, the following holds:

$$\beta \in \left[\min_{z\in H^{-1}(1)} \frac{1}{T}\int_0^T \langle A\rangle \circ \phi_t^{\langle V\rangle}(z), \max_{z\in H^{-1}(1)} \frac{1}{T}\int_0^T \langle A\rangle \circ \phi_t^{\langle V\rangle}(z)\right]. \tag{1.58}$$

Remark 1.12. The analytic assumption of Theorem 1.10 and the condition **(WGC)** imply that the set $H^{-1}(1) \cap \langle A\rangle^{-1}(0)$ consists of a disjoint union of a finite number of minimal invariant tori $(\mathcal{O}^H(z_k))_{k=1,\dots,N}$. Then, (1.58) gives an explicit lower bound of $\beta > 0$.

This Theorem shows that eigenvalues of the non-selfadjoint operator $\widehat{P}_\hbar$ cannot accumulate on the boundary of the strip given by Proposition 1.2. This means that there is a spectral gap. As was alredy mentionned, this result extends Asch-Lebeau's one [16, Th. 2.3] to our context. Again, we emphasize that, compared with Theorem 1.9, it only deals with the case of true eigenvalues and that it does not seem that a resolvent estimate can be easily deduced from the proof below.

1.4. Quantum limits for KAM families of vector fields on the torus

In this part of the thesis, we focus on the study of *quantum limits* for some families of vector fields on the torus. It will appear together with the results of the next section in an article by the author [12].

Let $\mathbb{T}^d := \mathbb{R}^d/(2\pi\mathbb{Z}^d)$ be the flat torus equipped with the flat metric, we consider the $\hbar$-homogeneous operator

$$\widehat{P}_{\omega,\hbar} := \omega \cdot \hbar D_x + v(x;\omega) \cdot \hbar D_x - \frac{i\hbar}{2} \operatorname{Div} v(x;\omega),$$

where $\omega \in \mathbb{R}^d$, $v \in \mathcal{C}^\infty(\mathbb{T}^d \times \mathbb{R}^d; \mathbb{R}^d)$ is a vector field depending on the parameter ω, and we use the notation

$$D_x = (D_{x_1}, \ldots, D_{x_d}), \quad D_{x_j} := -i\partial_{x_j}.$$

This operator generates the transport along the vector field $X_v(\omega) := \omega + v(x;\omega)$, meaning that the solution to the Schrödinger equation

$$\big(i\hbar\partial_t + \widehat{P}_{\omega,\hbar}\big) u_\hbar(t,x) = 0; \quad u_\hbar(0,x) = u_\hbar^0(x) \in L^2(\mathbb{T}^d)$$

is given by

$$u_\hbar(t,x) = u_\hbar^0(\phi_t^{X_v(\omega)}(x))\sqrt{|d\phi_t^{X_v(\omega)}(x)|},$$

where $\phi_t^{X_v(\omega)}$ is the flow generated by the vector field $X_v(\omega)$, and the operator $\widehat{P}_{\omega,\hbar}$ is selfadjoint thanks to the component $-i\hbar \operatorname{Div} v/2$.

Note that the operator

$$\widehat{L}_{\omega,\hbar} := \omega \cdot \hbar D_x \tag{1.59}$$

on $L^2(\mathbb{T}^d)$ is not elliptic and hence its point-spectrum, given by

$$\mathrm{Sp}^p_{L^2(\mathbb{T}^d)}\big(\widehat{L}_{\omega,\hbar}\big) = \{\hbar\,\omega \cdot k \,:\, k \in \mathbb{Z}^d\},$$

is highly unstable under perturbations, in the sense that the point spectrum could be transformed into continuous spectrum by the perturbation. However, we will use classical KAM theory to show that under certain conditions on the perturbation v, the spectrum of $\widehat{P}_{\omega,\hbar}$ is stable for a Cantor set of frequencies ω, modulo a translation in the vector ω. As was shown by Wenyi and Chi in [116], this KAM stability is equivalent to the hypoellipticity of the operator $\widehat{P}_{\omega,\hbar}$.

The operator $\widehat{P}_{\omega,\hbar}$ is the Weyl quantization of the linear Hamiltonian

$$P_\omega(x,\xi) = \mathcal{L}_\omega(\xi) + v(x;\omega)\cdot\xi,$$

where

$$\mathcal{L}_\omega(\xi) := \omega \cdot \xi.$$

In [89], Moser introduced a new approach to the study of quasiperiodic motions by considering the frequencies of the Kronecker tori as independent parameters. We refer to the work of Pöschel [98] for a brief introduction to the subject. If $\Omega \subset \mathbb{R}^d$ is a compact Cantor set of frequencies satisfying some Diophantine condition (see condition (1.60) below) and the perturbation v is sufficiently small in some suitable norm, then there exists a close-to-the-identity change of coordinates

$$\varphi : \Omega \to \mathbb{R}^d$$

so that the related set of Hamiltonians $P_{\varphi(\omega)}$ can be canonically conjugated (frequency by frequency) into the constant linear hamiltonian on $T^*\mathbb{T}^d$ with frequency ω. More precisely, for every $\omega \in \Omega$ there exists a canonical transformation $\Theta_\omega : T^*\mathbb{T}^d \to T^*\mathbb{T}^d$ so that

$$\Theta_\omega^* P_{\varphi(\omega)}(x, \xi) = \mathcal{L}_\omega(\xi).$$

In particular, the hamiltonian $P_{\varphi(\omega)}$ is completely integrable for every $\omega \in \Omega$.

We focus on the study of the high-energy structure of the eigenfunctions of $\widehat{P}_{\omega,\hbar}$. Precisely, we will study the set of *quantum limits* of the system, that is, the weak-$\star$ accumulation points of sequences of L^2-densities of eigenfunctions. We next recall the precise definition of quantum limit in this setting:

Definition 1.3. *We say that a probability measure $\nu \in \mathcal{P}(P_\omega^{-1}(1))$ is a quantum limit of $\widehat{P}_{\omega,\hbar}$ if there exist a sequence $\lambda_\hbar$ of eigenvalues for $\widehat{P}_{\omega,\hbar}$ such that $\lambda_\hbar \to 1$ as $\hbar \to 0$, and a related sequence of L^2-normalized eigenfunctions $(\Psi_\hbar)$ satisfying:*

$$|\Psi_\hbar(x)|^2 dx \rightharpoonup^\star \nu, \quad \text{as } \hbar \to 0,$$

where the convergence takes place in the weak-$\star$ topology for Radon measures. We will denote by $\mathcal{N}(\widehat{P}_{\omega,\hbar})$ the set of quantum limits of $\widehat{P}_{\omega,\hbar}$.

As we have already mention in the motivation, several previous works deal with the study of quantum limits in the completely-integrable and the KAM settings. Zelditch in [119] and [121] studied the high energy distribution of eigenfunctions of the Laplace-Beltrami operator in some completely integrable systems. Other related works are those of Toth [112], Jakobson and Zelditch [67], Toth and Zelditch [113], Anantharaman, Fermanian-Kammerer and Macià [4], and Macià and Rivière [84, 85]. The case of the Laplacian ($+W$) on the flat torus has deserved special attention. The works of Bourgain [22], Jakobson [66], Anantharaman and Macià [9], and Bourgain, Burq and Zworski [23] deal with this case. Quantum limits in this setting are shown to be absolutely continuous. If the classical system is close to completely integrable, in the sense that the classical KAM theorem applies, then the persistence of invariant tori at classic level is expected to imply an analogous result at quantum level. Most of the works dealing with this case are based on the construction of *quasimodes*, or approximate eigenfunctions, studying the asymptotic properties of oscillation and concentration of these quasimodes around the classical invariant tori, but do not conclude analogous results for the quantum limits associated to the true eigenfunctions of the system. The foundations of this study of quasimodes for KAM systems can be found in Lazutkin [73]. Construction of quasimodes with exponentially small error terms is given by Popov [96] and [97]. In a recent work, Gomes [46] applies this result to discard *quantum ergodicity* for these systems.

Perturbations of $\widehat{L}_{\omega,\hbar}$ have been studied by Graffi and Paul [48]. They showed that some particular analytic bounded perturbations of $\widehat{L}_{\omega,\hbar}$ with Diophantine assumptions on ω remain integrable. Actually, they prove the convergence of the quantum normal form providing an exact quantization formula for these systems (see Section 1.5 below).

We consider $\widehat{P}_{\omega,\hbar}$ with frequencies ω in a small neighborhood of a Cantor set of Diophantine vectors $\Omega \subset \mathbb{R}^d$ satisfying:

$$|k \cdot \omega| \geq \frac{\varsigma}{|k|^{\gamma-1}}, \quad k \in \mathbb{Z}^d \setminus \{0\}, \tag{1.60}$$

for some constants $\varsigma > 0$ and $\gamma > d$. For any $\rho > 0$, let Ω_ρ be the complex neighborhood of Ω given by

$$\Omega_\rho := \{z \in \mathbb{C}^d \,:\, \operatorname{dist}(z, \Omega) < \rho\},$$

and, given $s > 0$, we consider also the complex neighborhood of the d-torus

$$D_s := \{z \in \mathbb{T}^d + i\mathbb{R}^d \,:\, |\Im z| < s\}.$$

We introduce the following family of linear symbols on $T^*\mathbb{T}^d$:

Definition 1.4. *A function $V \in \mathcal{C}^\omega(T^*\mathbb{T}^d \times \Omega_\rho)$ belongs to the space of linear symbols $\mathscr{L}_{s,\rho}$ if*

$$V(x, \xi; \omega) = \xi \cdot v(x; \omega) = \sum_{k \in \mathbb{Z}^d} \xi \cdot \widehat{v}(k; \omega) e_k(x), \tag{1.61}$$

for some $v \in \mathcal{C}^\omega(D_s \times \Omega_\rho; \mathbb{C}^d)$, where $\widehat{v}(k;\omega) \in \mathbb{C}^d$ is the k^{th}-Fourier coefficient of v:

$$\widehat{v}(k;\omega) := \big\langle v(\cdot;\omega), e_k \big\rangle_{L^2(\mathbb{T}^d)}, \quad e_k(x) := \frac{e^{ik\cdot x}}{(2\pi)^{d/2}}, \quad k \in \mathbb{Z}^d,$$

and

$$|V|_{s,\rho} := \sup_{\omega \in \Omega_\rho} \sum_{k \in \mathbb{Z}^d} |\widehat{v}(k;\omega)| e^{|k|s} < \infty. \tag{1.62}$$

The space $\big(\mathscr{L}_{s,\rho}, |\cdot|_{s,\rho}\big)$ is a Banach space. We denote $\mathscr{L}_s \subset \mathscr{L}_{s,\rho}$ the subspace of symbols that do not depend on $\omega \in \Omega_\rho$.

Let $s, \rho > 0$, and let $V \in \mathscr{L}_{s,\rho}$ be real analytic. We consider the family of operators given by

$$\widehat{P}_{\omega,\hbar} := \widehat{L}_{\omega,\hbar} + \operatorname{Op}_\hbar\big(V(\cdot;\omega)\big), \tag{1.63}$$

where

$$\operatorname{Op}_\hbar(V) := v \cdot \hbar D_x - \frac{i\hbar}{2} \operatorname{Div} v$$

is the semiclassical Weyl quantization of V (see Section 2.7.1).

Our first result of this part is the following:

Theorem 1.11. *Let $s, \rho > 0$ and $V \in \mathscr{L}_{s,\rho}$ be real analytic and assume*

$$|V|_{s,\rho} \le \varepsilon, \tag{1.64}$$

where ε is a small positive constant depending only on s, ρ, γ and ς. Then there exists a real change of frequencies $\varphi : \Omega \to \Omega_\rho$ such that the point-spectrum of $\widehat{P}_{\varphi(\omega),\hbar}$ is

$$\mathrm{Sp}^p_{L^2(\mathbb{T}^d)}\left(\widehat{P}_{\varphi(\omega),\hbar}\right) = \{\hbar\,\omega \cdot k \,:\, k \in \mathbb{Z}^d\},$$

and, for every $\omega \in \Omega$, there exists a diffeomorphism $\theta_\omega : \mathbb{T}^d \to \mathbb{T}^d$ of the torus homotopic to the identity so that

$$\mathcal{N}(\widehat{P}_{\varphi(\omega),\hbar}) = \left\{ \frac{1}{(2\pi)^d} (\theta_\omega)_* dx \right\}.$$

Moreover,

$$\sup_{\omega \in \Omega} |\varphi(\omega) - \omega| \le C_1 |V|_{s,\rho}, \quad \sup_{x \in \mathbb{T}^d} |\theta_\omega(x) - x| \le C_2 |V|_{s,\rho},$$

where C_1 and C_2 are positive constants depending only on s, ρ, γ and ς,

To prove Theorem 1.11, we will use a classical KAM theorem about perturbations of analytic vector fields on the torus. We will recall a work of Pöschel [99] which uses a new idea of Rüssmann [104] that simplifies the KAM-iteration argument. On the other hand, we will use Egorov's theorem to establish the classic-quantum duality and obtain our result in terms of the quantum system. The approach is similar to that of Bambusi et. al. in [19], in which they obtain reducibility for a class of perturbations of the quantum harmonic oscillator.

If we consider semiclassical perturbations of the form

$$\widehat{P}^{\varepsilon}_{\omega,\hbar} := \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \,\mathrm{Op}_\hbar\left(V(\cdot;\omega)\right),$$

with $\varepsilon_\hbar \to 0$ as $\hbar \to 0$, then the following holds:

Corollary 1.1. *Let $\rho, s > 0$ and let $V \in \mathscr{L}_{s,\rho}$ be real analytic. Then there exists a sequence of real maps $\varphi_\hbar : \Omega \to \Omega_\rho$ satisfying*

$$\lim_{\hbar \to 0} \sup_{\omega \in \Omega} |\varphi_\hbar(\omega) - \omega| = 0,$$

so that $\widehat{P}^{\varepsilon}_{\varphi_\hbar(\omega),\hbar}$ has pure point spectrum and

$$\mathcal{N}(\widehat{P}^{\varepsilon}_{\varphi_\hbar(\omega),\hbar}) = \left\{ \frac{1}{(2\pi)^d} dx \right\}.$$

The proof of Theorem 1.11 is divided in two parts. First we prove that the family $\widehat{P}_{\varphi(\omega),\hbar}$ is unitarily equivalent to $\widehat{L}_{\omega,\hbar}$. This shows the stability of the spectrum along this family. The following holds:

Theorem 1.12. *Let $s, \rho > 0$ and $V \in \mathscr{L}_{s,\rho}$ be real analytic verifying* (1.64). *Then, there exist a real change of coordinates $\varphi : \Omega \to \Omega_\rho$ satisfying*

$$\sup_{\omega \in \Omega} |\varphi(\omega) - \omega| \leq C_1 |V|_{s,\rho},$$

and a family of unitary operators $\Omega \ni \omega \longmapsto \mathcal{U}_\omega$ on $L^2(\mathbb{T}^d)$ such that

$$\mathcal{U}_\omega^* \widehat{P}_{\varphi(\omega),\hbar} \mathcal{U}_\omega = \widehat{L}_{\omega,\hbar}. \tag{1.65}$$

Remark 1.13. In particular, if $V = 0$ then $\varphi = \mathrm{Id}$ and $\mathcal{U}_\omega = \mathrm{Id}$.

The second part of the proof of Theorem 1.11 will follow by applying Egorov's theorem, comparing the quantum limits of $\widehat{P}_{\varphi(\omega),\hbar}$ with those of $\widehat{L}_{\omega,\hbar}$. We will also study the sequences of Wigner distributions associated to the eigenfunctions of $\widehat{P}_{\varphi(\omega),\hbar}$ and its weak-$\star$ limits: the semiclassical measures of the system (see Section 2.6).

If μ is the semiclassical measure associated to a sequence of eigenfunctions $(e_\hbar)$ of $\widehat{L}_{\omega,\hbar}$ with related eigenvalues $\lambda_\hbar \to 1$, then μ is in fact a positive Radon measure on the level-set $\mathbb{T}^d \times \mathcal{L}_\omega^{-1}(1)$. Indeed, by Lemma 2.4 below, for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$,

$$0 = \left\langle \mathrm{Op}_\hbar(a)(\widehat{L}_{\omega,\hbar} - \lambda_\hbar) e_\hbar, e_\hbar \right\rangle_{L^2(\mathbb{T}^d)} = \left\langle \mathrm{Op}_\hbar(a(\mathcal{L}_\omega - \lambda_\hbar)) e_\hbar, e_\hbar \right\rangle_{L^2(\mathbb{T}^d)} + O(\hbar),$$

and hence

$$\int_{T^*\mathbb{T}^d} a(x,\xi)\big(\mathcal{L}_\omega(\xi) - 1\big)\mu(dx, d\xi) = 0.$$

If moreover the measure μ turns out to be a probability measure, then its projection onto the position space

$$\nu(x) = \int_{\mathcal{L}_\omega^{-1}(1)} \mu(x, d\xi)$$

is the quantum limit of the sequence. But since $\mathcal{L}_\omega^{-1}(1)$ is not compact, there exist some sequences of eigenfunctions that oscillate too fast so that the only weak-$\star$ accumulation point of the related sequence of Wigner measures is the zero measure. However, there can exist quantum limits also for that high-oscillating sequences although they can not be obtained as projections of the related semiclassical measures. As we will see, this is not an important difficulty in our case, since we can apply Egorov's theorem directly to the projection of the Wigner distributions onto the position space, by testing against functions that only depends on x before passing to the weak-$\star$ limit.

Recall that the Hamiltonian flow generated by $\mathcal{L}_\omega$ is given by

$$\phi_t^{\mathcal{L}_\omega}(x,\xi) = (x + t\omega, \xi), \quad (x,\xi) \in T^*\mathbb{T}^d.$$

If ω is nonresonant (its components are rationally independent over the rationals), then the flow $\phi_t^{\mathcal{L}_\omega}$ is uniquely ergodic on each torus $\mathbb{T}^d \times \{\xi\}$, meaning that the unique probability measure that is invariant by $\phi_t^{L_\omega}$ and is supported on $\mathbb{T}^d \times \{\xi\}$ is the Haar measure $\mathfrak{h}_{\mathbb{T}^d\times\{\xi\}}$ of the torus $\mathbb{T}^d \times \{\xi\}$. This translates into a quantum result. Let $\mathcal{M}_\omega$ be the set defined by

$$\mathcal{M}_\omega := \bigcup_{\xi\in\mathcal{L}_\omega^{-1}(1)} \left\{\mathfrak{h}_{\mathbb{T}^d\times\{\xi\}}\right\}, \tag{1.66}$$

the following standard result holds:

Proposition 1.3. *Let $\omega \in \mathbb{R}^d$ be nonresonant, meaning that $\operatorname{rk}\Lambda_\omega = 0$. Let $\mathcal{M}\big(\widehat{L}_{\omega,\hbar}\big)$ be the set of non-vanishing semiclassical measures of sequences of eigenfunctions for $\widehat{L}_{\omega,\hbar}$ with eigenvalues $\lambda_\hbar \to 1$. Then*

$$\mathcal{M}\big(\widehat{L}_{\omega,\hbar}\big) = \mathcal{M}_\omega.$$

Moreover,

$$\mathcal{N}(\widehat{L}_{\omega,\hbar}) = \left\{\frac{1}{(2\pi)^d}dx\right\}.$$

For the sake of completeness, we will include a proof of Proposition 1.3 in Section 5.4. With respect to the perturbed operator $\widehat{P}_{\omega,\hbar}$, our result reads:

Theorem 1.13. *Let $s,\rho > 0$ and $V \in \mathscr{L}_{s,\rho}$ be real analytic verifying (1.64). Let $\mathcal{M}\big(\widehat{P}_{\varphi(\omega),\hbar}\big)$ be the set of probability measures obtained as semiclassical measures of sequences of normalized eigenfunctions $(\Psi_\hbar)$ of the Hamiltonian $\widehat{P}_{\varphi(\omega),\hbar}$ with associated sequence of eigenvalues satisfying $\lambda_\hbar \to 1$ as $\hbar \to 0$. Then, there exists a symplectomorphism $\Theta_\omega : T^*\mathbb{T}^d \to T^*\mathbb{T}^d$ such that*

$$\mathcal{M}\big(\widehat{P}_{\varphi(\omega),\hbar}\big) = (\Theta_\omega)_*\,\mathcal{M}_\omega := \bigcup_{\xi\in\mathcal{L}_\omega^{-1}(1)} \left\{(\Theta_\omega)_*\mathfrak{h}_{\mathbb{T}^d\times\{\xi\}}\right\},$$

where $(\Theta_\omega)_*$ is the pushforward of Θ_ω. In particular, if $V = 0$, then $\Theta_\omega = \mathrm{Id}$. Moreover, there exists a diffeomorphism $\theta_\omega : \mathbb{T}^d \to \mathbb{T}^d$ homotopic to the identity such that

$$\Theta_\omega(x,\xi) = \big(\theta_\omega(x), [(\partial_x\theta_\omega(x))^T]^{-1}\xi\big),$$

and it satisfies

$$\sup_{\omega\in\Omega}\sup_{x\in\mathbb{T}^d} |\theta_\omega(x) - x| \leq C_2 |V|_{s,\rho}.$$

Remark 1.14. The unitary operator $\mathcal{U}_\omega$ obtained in Theorem 1.12 and the diffeomorphism θ_ω are related by:

$$\mathcal{U}_\omega\psi(x) = \psi(\theta_\omega(x))\sqrt{|d\theta_\omega(x)|}.$$

1.5. RENORMALIZATION OF SEMICLASSICAL KAM OPERATORS

Another interesting problem of KAM theory which can be studied from the semiclassical point of view is the *renormalization problem.* Let $\mathbb{T}^d$ be the flat torus, as in the previous section, we consider the linear symbol $\mathcal{L}_\omega : \mathbb{R}^d \to \mathbb{R}$ defined by

$$\mathcal{L}_\omega(\xi) = \omega \cdot \xi, \tag{1.67}$$

where the vector of frequencies ω satisfies the Diophantine condition (1.60). The renormalization problem in the classical framework [42], [44] asks if, given a small analytic perturbation V of the linear Hamiltonian $\mathcal{L}_\omega$, with $V = V(x, \xi; \varepsilon)$ defined on $\mathbb{T}^d \times \mathbb{R}^d \times [0, \varepsilon_0]$ for some $\varepsilon_0 > 0$ sufficiently small, there exists another function $R = R(\xi; \varepsilon)$ on $\mathbb{R}^d \times [0, \varepsilon_0]$, called *counterterm* in the literature, such that the renormalized Hamiltonian

$$Q(x, \xi; \varepsilon) = \mathcal{L}_\omega(\xi) + V(x, \xi; \varepsilon) - R(\xi; \varepsilon)$$

is integrable and canonically conjugate to the unperturbed hamiltonian. This was conjectured by Gallavotti in [42] and first proven by Eliasson in [38]. This result can be regarded as a control theory theorem. Despite the fact that small perturbations of $\mathcal{L}_\omega$ could generate even ergodic behavior (see Katok [70]), this shows that modifying in a suitable way the completely integrable part of the Hamiltonian, the system remains stable. Renormalization techniques have been studied by several authors in the context of quantum field theory, as well as its connection with KAM theory [27, 40, 42, 43, 71, 108].

Our goal is to prove a semiclassical version of the renormalization problem. We consider the semiclassical Weyl quantization of $\mathcal{L}_\omega$:

$$\widehat{L}_{\omega,\hbar} := \mathrm{Op}_\hbar(\mathcal{L}_\omega) = \omega \cdot \hbar D_x. \tag{1.68}$$

Let $(\varepsilon_\hbar)_\hbar$ be a semiclassical scaling such that

$$\varepsilon_\hbar \leq \hbar, \tag{1.69}$$

and let $V \in \mathcal{C}^\omega(T^*\mathbb{T}^d; \mathbb{R})$ be a bounded real analytic function. We aim at performing a quantum KAM iteration procedure to construct a counterterm $R_\hbar = R_\hbar(V) \in \mathcal{C}^\omega(\mathbb{R}^d)$, uniformly bounded in $\hbar \in (0, 1]$, so that the quantum Hamiltonian

$$\widehat{Q}_\hbar := \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \, \mathrm{Op}_\hbar(V - R_\hbar) \tag{1.70}$$

is unitarily equivalent to the unperturbed operator $\widehat{L}_{\omega,\hbar}$. This will show that the spectrum of the operator $\widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \, \mathrm{Op}_\hbar(V)$ can be stabilized by adding the counterterm $\varepsilon_\hbar \, \mathrm{Op}_\hbar(R_\hbar)$ to the system. Moreover, we will study the set of quantum limits of $\widehat{Q}_\hbar$ and the set of *semiclassical* *measures* associated to sequences of eigenfunctions for the operator $\widehat{Q}_\hbar$. We will show that these sets coincide with those of the unperturbed operator $\widehat{L}_{\omega,\hbar}$.

In a related work, Graffi and Paul [48] showed that the perturbed operator

$$\widehat{P}_\hbar = \widehat{L}_{\omega,\hbar} + \mathrm{Op}_\hbar(V_\omega)$$

can be conjugated to a convergent quantum normal form for a specific class of bounded analytic perturbations of the form

$$V_\omega(x,\xi) = V(x, \omega\cdot\xi), \quad (x,\xi)\in T^*\mathbb{T}^d, \tag{1.71}$$

(see Gallavotti [42] for a discussion of this condition). As a consequence, it could be possible to show that the set of semiclassical measures is stable under perturbations of this type, without necessity of renormalization. Despite the fact that we need to assume that $\delta_\hbar \leq \hbar$, we consider more general perturbations than the ones of (1.71). The main difference in our approach is the substitution of the particular dependence on $\omega\cdot\xi$ of V, which is stable under the conjugacies employed by Graffi and Paul in their work, by the addition of the renormalization function $R_\hbar$.

We emphasize that, compared to [37], [38] and [44], our work is not based on the study of the convergence of Lindstedt series, and we do not know how to adapt their approach to this problem. Alternatively, we will use an algorithm similar to that of Govin et al. [47] to construct a normal form, obtaining the counterterm $R_\hbar$ step by step. We expect that condition (1.69) is not sharp. One should be able to manage perturbations of order $O(1)$.

We will deal with semiclassical perturbations $\mathrm{Op}_\hbar(V)$ whose symbol V belongs to a suitable Banach space of bounded analytic functions. Similarly as we did in Section 1.3.2, we consider the following spaces of analytic functions (compare with those spaces of Definition 1.2 in the Euclidean case).

Definition 1.5. *Given $s>0$, we define the Banach space $\mathcal{A}_s(\mathbb{R}^d)$ of functions $f\in\mathcal{C}^\omega(\mathbb{R}^d;\mathbb{R})$ such that*

$$|f|_s := \int_{\mathbb{R}^d} |\widehat{f}(\eta)|\, e^{|\eta|s}\, d\eta < \infty,$$

where $\widehat{f}$ denotes the Fourier transform of f. We introduce also the Banach space $\mathcal{A}_s(T^*\mathbb{T}^d)$ of analytic functions $g\in\mathcal{C}^\omega(T^*\mathbb{T}^d;\mathbb{R})$ such that

$$\|g\|_s := \sum_{k\in\mathbb{Z}^d} |\widehat{g}(k,\cdot)|_s\, e^{|k|s} < \infty,$$

where

$$\widehat{g}(k,\xi) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g(x,\xi) e^{-ix\cdot k}\, dx, \quad k\in\mathbb{Z}^d.$$

By the Calderón-Vaillancourt Theorem (see Lemma 6.1), the semiclassical Weyl quantization $\mathrm{Op}_\hbar(a)$ of a symbol $a \in \mathcal{A}_s(T^*\mathbb{T}^d)$ satisfies

$$\|\mathrm{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} \leq C_{d,s}\|a\|_s,$$

for all $\hbar \in (0,1]$.

We next proceed to state our result about the phase-space distribution of energy of sequences of eigenfunctions of the operator $\widehat{Q}_\hbar$. The following holds:

Theorem 1.14. *Let $\omega \in \mathbb{R}^d$ be a strongly non resonant frequency satisfying (1.60), and let V be a real valued function that belongs to $\mathcal{A}_s(T^*\mathbb{T}^d)$ for some fixed $s > 0$. Assume that*

$$\|V\|_s \leq \varepsilon, \tag{1.72}$$

where $\varepsilon > 0$ is a small constant that depends only on s, ϱ and γ. Let $(\varepsilon_\hbar)$ be a sequence of positive real numbers satisfying $\varepsilon_\hbar \leq \hbar$. Then, there exists a sequence of integrable[11] counterterms $R_\hbar = R_\hbar(V) \in \mathcal{A}_{s/2}(\mathbb{R}^d)$ such that $\|R_\hbar\|_{s/2} \lesssim \|V\|_s$, uniformly in $\hbar \in (0,1]$, and

$$\mathrm{Sp}^p_{L^2(\mathbb{T}^d)}\left(\widehat{Q}_\hbar\right) = \mathrm{Sp}^p_{L^2(\mathbb{T}^d)}\left(\widehat{L}_{\omega,\hbar}\right) = \{\hbar\,\omega\cdot k \,:\, k \in \mathbb{Z}^d\}.$$

Moreover, denoting by $\mathcal{M}\big(\widehat{Q}_\hbar\big)$ the set of probability measures obtained as semiclassical measures of sequences of normalized eigenfunctions of the Hamiltonian $\widehat{Q}_\hbar$ with eigenvalues verifying $\lambda_\hbar \to 1$ as $\hbar \to 0$,

$$\mathcal{M}\big(\widehat{Q}_\hbar\big) = \mathcal{M}\big(\widehat{L}_{\omega,\hbar}\big) = \mathcal{M}_\omega,$$

and the set of quantum limits of $\widehat{Q}_\hbar$ is precisely

$$\mathcal{N}(\widehat{Q}_\hbar) = \left\{\frac{1}{(2\pi)^d}dx\right\}.$$

Remark 1.15. In the case $\varepsilon_\hbar \ll \hbar$, condition (1.72) can be removed.

[11]That is, $R_\hbar$ is a function only of the action variable $\xi \in \mathbb{R}^d$.

Chapter 2

Basics of semiclassical analysis

Del pensamiento querría yo ayudarme,
si él me obedeciese a mi contento.

D. Hurtado de Mendoza. *Canciones.*

One of the goals of this thesis is understanding the process of quantization of classical dynamical systems, and how classical dynamics affect the asymptotic properties of the quantum system. The subject of *semiclassical analysis* starts from the so called *correspondence principle* between classical and quantum dynamics. Roughly speaking, the high energy behavior of a quantum system is expected to be governed by the dynamics devoloped by its classical counterpart. By "high energy" we mean a scale in the frequencies of size $1/\hbar$ compared to the classical or macroscopic scale, where $\hbar > 0$ denotes a small parameter called semiclassical parameter, which sometimes is identified with the normalized Planck's constant. Asymptotically as the semiclassical parameter $\hbar$ tends to zero, the classical dynamics emerge and influence the properties of the quantum system.

In this chapter we introduce some notions and tools of semiclassical analysis and recall some standard results in the field. This is a brief presentation of the basics that we will use and extend along the work. For an extensive treatment of the subject, we refer to the books of Dimassi and Sjöstrand [35], Martinez [87], and Zworski [122].

2.1. The problem of quantization

In semiclassical analysis we deal with quantum systems that come from classical Hamiltonian systems. A set of classical particles moving according to the action of certain conservative forces is described in the *phase space*, namely the space of possible positions and momenta of each particle. We assume that the set of possible positions of the system defines a differentiable manifold M of dimension d. The phase space is then T^*M, its cotangent bundle. We denote by (x, ξ) the local variables on T^*M standing for position and momentum respectively.

A classical observable $a : T^*M \to \mathbb{R}$ is a smooth function that recovers concrete information from the system. For example, the functions

$$(x, \xi) \mapsto x_j, \quad (x, \xi) \mapsto \xi_j,$$

for $j \in \{1, \dots, d\}$, are called respectively the position and momentum observables.

We shall denote by $H : T^*M \to \mathbb{R}$ the observable associated to the total energy of the system, called the Hamiltonian. For instance, if (M, g) is a Riemannian manifold then the Hamiltonian

$$H(x, \xi) := \frac{1}{2}\langle \xi, \xi \rangle_{g(x)} + W(x), \quad W \in \mathcal{C}^\infty(M; \mathbb{R}), \tag{2.1}$$

given by the sum of the kinetic energy and the potential energy, is a paradigmatical example of Hamiltonian, where

$$\langle \xi, \xi \rangle_{g(x)} = \sum_{j,k=1}^{d} g^{jk}(x)\, \xi_j \xi_k, \qquad (g^{jk}) = (g_{jk})^{-1},$$

is the inner product defined on the fiber $T^*_x M$ by the metric g given in local coordinates by the matrix (g_{jk}), and W is the potential.

Given a point of the phase space, one can determine the past and future evolution of the system according to the Hamilton equations. More precisely, from the Hamiltonian H one can defines a vector field X_H on $T(T^*M)$, called *Hamiltonian vector field*, via the identity

$$X_H(a) = \{H, a\}, \quad a \in \mathcal{C}^\infty(T^*M),$$

where $\{\cdot, \cdot\}$ denotes the Poisson bracket, which is given in local coordinates by

$$\{H, a\}(x, \xi) := \sum_{j=1}^{d} \big(\partial_{\xi_j} H\, \partial_{x_j} a - \partial_{x_j} H\, \partial_{\xi_j} a\big)(x, \xi).$$

If the Hamiltonian H is good enough, the vector field X_H generates a global-in-time flow ϕ_t^H on T^*M satisfying

$$\partial_t\, \phi_t^H(x, \xi) = X_H\big|_{\phi_t^H(x,\xi)}, \quad (x, \xi) \in T^*M, \quad t \in \mathbb{R}. \tag{2.2}$$

For example, if H is bounded together with all its derivatives (hence $H \in S^0(T^*M)$, see definition (2.20) below) or, in the case when H is given by the total energy (2.1), (M, g) is geodesically complete and V is bounded from below, then the hamiltonian vector field X_H generates a global-in-time flow ϕ_t^H. Thus the evolution of any observable $a \in \mathcal{C}^\infty(T^*M)$ is determined by the differential equation:

$$\partial_t\, a \circ \phi_t^H(x, \xi) = \{H, a\} \circ \phi_t^H(x, \xi), \quad (x, \xi) \in T^*M, \quad t \in \mathbb{R}.$$

From this, we observe that any observable $a \in \mathcal{C}^\infty(T^*M)$ that commutes with H, in the sense that

$$\{H, a\} = 0,$$

is a conserved quantity by the flow ϕ_t^H. In particular, the total energy H is a conserved quantity.

To finish this elementary review on classical hamiltonian dynamics, we set the commutator relations of the position and momentum observables:

$$\{x_j, x_k\} = 0, \quad \{\xi_j, \xi_k\} = 0, \quad \{\xi_j, x_j\} = \delta_{jk}. \tag{2.3}$$

Now, with any Hamiltonian system we want to associate a quantum system satisfying some fundamental axioms due to Heisenberg, Schrödinger, Dirac, von Neumann and others [41], [88], known as the postulates of Quantum Mechanics. We start by defining the set of possible quantum states associated to the classic states $(x, \xi) \in T^*M$. This set can be considered to be the projective space of $L^2(M)$, that is:

$$\big(L^2(M) \setminus \{0\}\big) / \sim, \quad \psi \sim \psi' \quad \text{if } \psi = \rho\, \psi', \quad \rho \in \mathbb{C} \setminus \{0\}.$$

The problem of quantization consists in constructing a map

$$\mathcal{C}^\infty(M; \mathbb{R}) \ni a \mapsto \mathrm{Op}_\hbar(a), \quad \hbar > 0,$$

from the set of classic observables to the space of (non bounded) linear operators on $L^2(M)$. As we will see, the semiclassical parameter $\hbar$ will localize the observable in a suitable energy level in T^*M. For the sake of simplicity, we introduce the process of quantization in the case $M = \mathbb{R}^d$, and later on we will explain how to generalize it to the case of general smooth manifolds.

We first define the quantum observables associated to the position and momentum observables. Given a state $\psi \in L^2(M)$ of a quantum system with $\|\psi\|_{L^2} = 1$, we define the position observable $\widehat{Q}_j$ and the momentum observable $\widehat{P}_j$ acting on ψ by

$$\widehat{Q}_j\, \psi(x) := x_j\, \psi(x) \tag{2.4}$$

$$\widehat{P}_j\, \psi(x) := -i\hbar\, \partial_{x_j} \psi(x). \tag{2.5}$$

The meaning of the position observable $\widehat{Q}_j$ comes from the interpretation of the wave function of a system as a probability wave. Precisely, the probability of finding the system with wave function ψ in a Borel set $B \subset \mathbb{R}^d$ is given by

$$\big\langle \mathbf{1}_{\{x \in B\}} \psi, \psi \big\rangle_{L^2(\mathbb{R}^d)} = \int_B |\psi(x)|^2 dx,$$

where $\mathbf{1}_{\{x \in B\}}$ denotes the indicator function of the set B. For any Borel set $B \subset \mathbb{R}$, if we denote $\Pi_j(B)$ the operator of multiplication by $\mathbf{1}_{\{x_j \in B\}}$, then the probability of finding x_j in the borelian set $B \subset \mathbb{R}$ is $\langle \Pi_j(B)\psi, \psi\rangle_{L^2}$. The projection valued measure Π_j can be associated to the selfadjoint operator $\widehat{Q}_j$ via the spectral theorem [41]. Precisely, we have

$$\widehat{Q}_j = \int_{\mathbb{R}} \lambda \, d\Pi_j(\lambda), \quad \Pi_j(B) = \mathbf{1}_{\{x_j \in B\}}(\widehat{Q}_j).$$

Hence the expected value of the operator $\widehat{Q}_j$ is

$$\big\langle \widehat{Q}_j \, \psi, \psi \big\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}} \lambda \langle d\Pi_j(\lambda)\psi, \psi\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d} x_j |\psi(x)|^2 dx.$$

On the other hand, the interpretation of the momentum operator $\widehat{P}_j$ is based on the Fourier decomposition of the wave function. Defining the semiclassical Fourier transform $\mathscr{F}_\hbar$ by

$$\mathscr{F}_\hbar \psi(\xi) := \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} \psi(x) e^{-\frac{i}{\hbar} x\cdot\xi} dx, \quad \xi \in \mathbb{R}^d,$$

we have, by the inverse Fourier formula,

$$\psi(x) = \frac{1}{(2\pi\hbar)^{d/2}} \int_{\mathbb{R}^d} \mathscr{F}_\hbar \psi(\xi) e^{\frac{i}{\hbar} x\cdot\xi} d\xi.$$

This means that any wave function ψ can be viewed as a superposition of plain waves $e^{\frac{i}{\hbar} x\cdot\xi}/(2\pi\hbar)^{d/2}$, each of them oscillating with frequency $\xi/\hbar$. The "momentum" associated to each plain wave of this form is identified with ξ via the De Broglie relation [87], [88]. Hence, for any Borel set $B \subset \mathbb{R}^d$, the probability of having momentum in the set B is given by

$$\big\langle \mathbf{1}_{\{\xi \in B\}} \mathscr{F}_\hbar \psi, \mathscr{F}_\hbar \psi \big\rangle_{L^2(\mathbb{R}^d)} = \int_B |\mathscr{F}_\hbar(\xi)|^2 d\xi.$$

We obtain that

$$\widehat{P}_j = \mathscr{F}_\hbar^{-1} \, \widehat{Q}_j \, \mathscr{F}_\hbar.$$

One can check the following commutator relations for the position and momentum observables:

$$[\widehat{Q}_j, \widehat{Q}_k] = 0, \quad [\widehat{P}_j, \widehat{P}_k] = 0, \quad [\widehat{P}_k, \widehat{Q}_j] = -i\hbar \, \delta_{jk} \, \mathrm{Id}\,. \tag{2.6}$$

Some desirable conditions for the map $a \mapsto \mathrm{Op}_\hbar(a)$ are to associate the classical observables x_j and ξ_j to the quantum observables $\widehat{Q}_j$ and $\widehat{P}_j$ respectively, and the constant observables c with the quantum operators $c\,\mathrm{Id}$. This preserves the basic commutator relations (2.3) and (2.6) and allows to define the Heisenberg group establishing the parallelism between the generators of the algebras of classic and quantum observables, [41]. Moreover, one would like the algebra of the classic observables itself to be conserved, that is:

$$\mathrm{Op}_\hbar(a+\rho\, b) = \mathrm{Op}_\hbar(a) + \rho\; \mathrm{Op}_\hbar(b), \quad \rho \in \mathbb{C}, \tag{2.7}$$

$$\mathrm{Op}_\hbar(a)\,\mathrm{Op}_\hbar(a) = \mathrm{Op}_\hbar(a^2), \tag{2.8}$$

$$[\mathrm{Op}_\hbar(a), \mathrm{Op}_\hbar(b)] = -i\hbar\,\mathrm{Op}_\hbar(\{a,b\}). \tag{2.9}$$

Unfortunately, some easy examples working with $a = x_1$ and $b = \xi_1$, see [41], show that one can not expect to obtain such a procedure of quantization satisfying all these conditions. However, in the semiclassical regime $\hbar \to 0$, as we will see in the following sections, one can construct a procedure of quantization so that the conditions above are all satisfied modulo small error terms in $\hbar$.

Commutator relations (2.3) and (2.6) are also the basis of the *Heisenberg uncertainty principle*. The following holds:

Lemma 2.1 ([122, Thm. 3.9], Uncertainty principle)**.** *We have*

$$\big\langle \widehat{Q}_j\,\psi, \psi \big\rangle_{L^2(\mathbb{R}^d)} \big\langle \widehat{P}_j\,\psi, \psi \big\rangle_{L^2(\mathbb{R}^d)} \geq \frac{\hbar^2}{4}\|\psi\|^4_{L^2(\mathbb{R}^d)}. \tag{2.10}$$

This shows that the wave function can not be arbitrarily localized simultaneously in position x and momentum ξ. As we will see later, in the semiclassical limit $\hbar \to 0$ the uncertainty principle can be neutralized by adjusting properly the scales of concentration and oscillation of the wave function (see Remark 2.4 of Section 2.6.1).

2.2. Weyl's quantization

There are several ways to associate an operator with a given observable $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$. In this framework, the observable a is often called the *symbol* of the operator. Along this thesis we will work mostly with the Weyl quantization, which has good properties in several settings.

Definition 2.1. *Let $a \in \mathscr{S}(\mathbb{R}^{2d})$ be a symbol in the Schwartz class, we define its Weyl quantization $\mathrm{Op}_\hbar(a)$ by*

$$\mathrm{Op}_\hbar(a)\,\psi(x) := \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^{2d}} a\left(\frac{x+y}{2}, \xi\right)\psi(y) e^{\frac{i}{\hbar}(x-y)\cdot\xi}\,dy\,d\xi, \quad \psi \in \mathscr{S}(\mathbb{R}^d). \tag{2.11}$$

This definition can be extended for more general symbols. See [122, Chp. 4] for a discussion on the most common symbol classes.

Examples.

1. If $a(x,\xi) = \xi^\alpha$, with $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, then
$$\mathrm{Op}_\hbar(a) = (\hbar D)^\alpha, \quad D := -i\nabla_x.$$

2. If $H(x,\xi) = \dfrac{1}{2}|\xi|^2 + W(x)$, with $W \in \mathcal{C}^\infty(\mathbb{R}^d)$, then
$$\mathrm{Op}_\hbar(H) = -\frac{\hbar^2}{2}\Delta + W(x).$$
This means that the quantization of the total energy is the semiclassical Schrödinger operator.

3. If $H(x,\xi) = X(x)\cdot\xi$, with $X \in \mathcal{C}^\infty(\mathbb{R}^d;\mathbb{R}^d)$, then
$$\mathrm{Op}_\hbar(H) = X(x)\cdot\hbar D + \frac{\hbar}{2i}\,\mathrm{Div}(X).$$

We hereafter refer to any operator of the form $\mathrm{Op}_\hbar(a)$ as a *semiclassical pseudodifferential operator*. One of the most important properties of the Weyl quantization is that the formal adjoint of $\mathrm{Op}_\hbar(a)$ is
$$\mathrm{Op}_\hbar(a)^* = \mathrm{Op}_\hbar(\overline{a}). \tag{2.12}$$
In particular, if a is real, then $\mathrm{Op}_\hbar(a)$ is formally selfadjoint. We will later show that, for a very general class of symbols a, the operator $\mathrm{Op}_\hbar(a)$ is bounded on $L^2(\mathbb{R}^d)$, in which case $\mathrm{Op}_\hbar(a)$ is selfadjoint provided that a is real.

2.3. Symbolic calculus

In this section we study the algebra of Weyl's pseudodifferential operators. First, it is convinient to introduce the Schrödinger representation on $L^2(\mathbb{R}^d)$, see [41]. It provides the basic commutator relations between Weyl's operators.

Definition 2.2. *The semiclassical Schrödinger representation $U_\hbar$ on $L^2(\mathbb{R}^d)$ is defined by the unitary group*
$$U_\hbar(w) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d), \quad w = (y,\eta) \in \mathbb{R}^{2d},$$
acting as
$$U_\hbar(w)\psi(x) := e^{\frac{i}{\hbar}y\cdot x + \frac{i}{2\hbar}y\cdot\eta}\,\psi(x+\eta). \tag{2.13}$$

Defining, for any $w = (y, \eta) \in \mathbb{R}^{2d}$, the linear form L_ω given by

$$L_w(z) = w \cdot z = y \cdot x + \eta \cdot \xi, \quad z = (x, \xi) \in \mathbb{R}^{2d},$$

one can check from (2.13) the following commutator relations:

$$U_\hbar(w)^* = U_\hbar(-w), \quad U_\hbar(w)U_\hbar(w') = e^{\frac{i}{2\hbar}\{L_w, L_{w'}\}} U_\hbar(w + w'), \quad w, w' \in \mathbb{R}^{2d}. \tag{2.14}$$

Moreover, the unitary operator $U_\hbar(w)$ is itself the Weyl quantization of a very simple symbol, given by the complex exponential of the linear form L_w. Precisely, the following holds:

Lemma 2.2 ([122, Thm. 4.7])**.** *For every* $w \in \mathbb{R}^{2d}$*:*

$$\operatorname{Op}_\hbar(e^{\frac{i}{\hbar} L_w}) = U_\hbar(w). \tag{2.15}$$

This lemma allows to write the Weyl quantization of any symbol as a Fourier decomposition. Indeed, we can write

$$a(z) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} \widehat{a}_\hbar(w) e^{\frac{i}{\hbar} L_w(z)} dw, \quad z = (x, \xi) \in \mathbb{R}^{2d}, \tag{2.16}$$

where the semiclassical Fourier transform $\widehat{a}_\hbar$ is given by

$$\widehat{a}_\hbar(w) := \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} a(z) e^{-\frac{i}{\hbar} L_w(z)} dz, \quad w \in \mathbb{R}^{2d}.$$

Then, using (2.16) and Lemma 2.2, the semiclassical Weyl's quantization of a symbol $a \in \mathscr{S}(\mathbb{R}^{2d})$ reads

$$\operatorname{Op}_\hbar(a)\psi(x) = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} \widehat{a}_\hbar(w) \operatorname{Op}_\hbar\left(e^{\frac{i}{\hbar} L_w(z)}\right)\psi(x) dw = \frac{1}{(2\pi\hbar)^d} \int_{\mathbb{R}^{2d}} \widehat{a}_\hbar(w) U_\hbar(w) \psi(x) dw.$$

This expression is very usefull to get the laws of composition of Weyl's operators. Using the commutator relations (2.14), we observe that the composition of two Weyl's pseudodifferential operators is again a Weyl's pseudodifferential operator. Indeed,

$$\operatorname{Op}_\hbar(a) \operatorname{Op}_\hbar(b) = \operatorname{Op}_\hbar(a \sharp_\hbar b),$$

where the new symbol $a\sharp_\hbar b$ is given by the Moyal product:

$$a \sharp_\hbar b(z) = \frac{1}{(2\pi\hbar)^{2d}} \int_{\mathbb{R}^{4d}} \widehat{a}_\hbar(w) \widehat{b}_\hbar(w') e^{\frac{i}{2\hbar}\{L_w, L_{w'}\}} e^{\frac{i}{\hbar} L_{w+w'}(z)} dw' \, dw.$$

Note also that, denoting $\zeta := (w, w') \in \mathbb{R}^{4d}$, and defining

$$A(\zeta) := \frac{1}{2}\{L_w, L_{w'}\},$$

we have

$$a\sharp_\hbar b(x,\xi) = \left(\mathscr{F}_\hbar^{-1}\, e^{\frac{i}{\hbar}A(\zeta)}\, \mathscr{F}_\hbar\, c\right)\Big|_{\substack{x=y\\ \xi=\eta}},$$

where

$$c(x,y,\xi,\eta) = a(x,\xi)b(y,\eta),$$

and $\mathscr{F}_\hbar : \mathscr{S}(\mathbb{R}^{4d}) \to \mathscr{S}(\mathbb{R}^{4d})$ is the semiclassical Fourier transform on $\mathbb{R}^{4d}$. Using Taylor's theorem to expand the exponential $e^{\frac{i}{\hbar}A(\zeta)}$ as

$$e^{\frac{i}{\hbar}A(\zeta)} = \sum_{k=0}^{N} \frac{1}{k!}\left(\frac{iA(\zeta)}{\hbar}\right)^k + \frac{1}{N!}\left(\frac{iA(\zeta)}{\hbar}\right)^N \int_0^1 (1-t)^N e^{\frac{it}{\hbar}A(\zeta)}dt,$$

we obtain the following asymptotic expansions for the Moyal product $a\sharp_\hbar b$, which show that, modulo small $\hbar$-errors, the algebra of classical observables is preserved, at least in the Schwartz class:

Lemma 2.3 ([122, Thm. 4.12])**.** *We have, for $N = 0,1,\ldots$,*

$$a\sharp_\hbar b(x,\xi) = \sum_{k=0}^{N} \frac{i^k\hbar^k}{k!} A(D)^k\big(a(x,\xi)b(y,\eta)\big)\Big|_{\substack{y=x\\ \eta=\xi}} + O_{\mathscr{S}}(\hbar^{N+1}), \quad \hbar \to 0, \tag{2.17}$$

where

$$A(D) := \frac{1}{2}\big(D_\xi \cdot D_y - D_x \cdot D_\eta\big).$$

In particular,

$$a\sharp_\hbar b = ab + \frac{\hbar}{2i}\{a,b\} + O_{\mathscr{S}}(\hbar^2), \tag{2.18}$$

and

$$[a,b]_\hbar := a\sharp_\hbar b - b\sharp_\hbar a = \frac{\hbar}{i}\{a,b\} + O_{\mathscr{S}}(\hbar^3). \tag{2.19}$$

Remark 2.1. The notation $\varphi = O_{\mathscr{S}}(\hbar^N)$ means that for all multiindices α, β,

$$|\varphi|_{\alpha,\beta} := \sup_{z\in\mathbb{R}^{2d}} |z^\alpha \partial_\beta\, \varphi(z)| \le C_{\alpha,\beta}\hbar^N.$$

This symbolic calculs can be extended to more general classes of symbols. Let us define, for $m \in \mathbb{Z}$, the family

$$S^m(\mathbb{R}^{2d}) := \left\{a \in \mathcal{C}^\infty(\mathbb{R}^{2d}) \;:\; \|\partial_z^\alpha a\|_{L^\infty(\mathbb{R}^{2d})} \le C_\alpha(1+|z|^2)^{m/2}, \quad \alpha \in \mathbb{N}^{2d}\right\}. \tag{2.20}$$

The following holds:

Lemma 2.4 ([122, Thm. 4.18, Thm 9.5])**.** *If $a \in S^{m_1}(\mathbb{R}^{2d})$ and $b \in S^{m_2}(\mathbb{R}^{2d})$, then*

$$a\sharp_\hbar b \in S^{m_1+m_2}(\mathbb{R}^{2d}),$$

and

$$\mathrm{Op}_\hbar(a)\,\mathrm{Op}_\hbar(b) = \mathrm{Op}_\hbar(a\sharp_\hbar b)$$

as operators mapping $\mathscr{S}(\mathbb{R}^d)$ to $\mathscr{S}(\mathbb{R}^d)$. Moreover,

$$a\sharp_\hbar b = ab + \frac{\hbar}{2i}\{a,b\} + O_{S^{m_1+m_2}}(\hbar^2),$$

and

$$[a,b]_\hbar = \frac{\hbar}{i}\{a,b\} + O_{S^{m_1+m_2}}(\hbar^3).$$

2.4. Operators on L^2

Since we will work mostly on the Hilbert space $L^2(\mathbb{R}^d)$, (or more generally $L^2(M)$), where the Schrödinger formalism is formulated naturally, it is important to state how pseudodifferential operators, with symbols in a suitable class, act on this space.

Lemma 2.5 ([122, Thm. 4.23], Calderón-Vaillancourt theorem)**.** *If the symbol a belongs to $S^0(\mathbb{R}^{2d})$, then*

$$\mathrm{Op}_\hbar(a) : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$$

is bounded, with

$$\|\mathrm{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} \leq C_d \sum_{|\alpha|\leq N_d} \hbar^{|\alpha|/2} \sup_{z\in\mathbb{R}^{2d}} |\partial^\alpha a(z)|,$$

where the constants $C_d > 0$ and $N_d \in \mathbb{N}$ depend only on the dimension d, and $\|\cdot\|_{\mathcal{L}(L^2)}$ denotes the strong operator norm. In particular,

$$\|\mathrm{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} \leq C_d\|a\|_{L^\infty(\mathbb{R}^{2d})} + O(\hbar^{1/2}).$$

Remark 2.2. The bound $O(\hbar^{1/2})$ can be improved up to $O(\hbar)$, see [122, Thm. 13.13].

If the symbol a is non negative, then one can also get a lower bound for the operator $\mathrm{Op}_\hbar(a)$. The following lemma will be a key ingredient in the proof of Theorem 1.9:

Lemma 2.6 ([122, Thm. 4.32], Sharp Garding inequality, Fefferman-Phong inequality)**.** *Assume $a \in S^0(\mathbb{R}^{2d})$ and*

$$a \geq 0, \quad \text{on} \quad \mathbb{R}^{2d}.$$

Then there exist constants $C > 0$ and $\hbar_0 > 0$ such that

$$\left\langle \mathrm{Op}_\hbar(a)\psi, \psi \right\rangle_{L^2(\mathbb{R}^d)} \geq -C\hbar^2\|\psi\|^2_{L^2(\mathbb{R}^d)},$$

for all $0 < \hbar \leq \hbar_0$ and $\psi \in L^2(\mathbb{R}^d)$.

2.5. Egorov's theorem

In this section we recall a fundamental result relating classical and quantum dynamics, known in the mathematical literature as Egorov's theorem (see [25], [122, Chp. 11]).

We assume that, comparing with Section 2.1, we have a smooth family of real Hamiltonians $H(t;x,\xi)$ on $\mathbb{R}^{2d}$, with $t\in[0,T]$. It defines a flow ϕ_t^H according to the differential equation

$$\begin{cases} \partial_t\,\phi_t^H = \left(\phi_t^H\right)_* X_H, \quad (0\le t\le T), \\ \quad \phi_0^H = \mathrm{Id}\,. \end{cases} \tag{2.21}$$

The quantum analog of equation (2.21) is the following operator equation:

$$\begin{cases} \hbar D_t\,\mathcal{U}_\hbar(t)+\mathcal{U}_\hbar(t)\,\mathrm{Op}_\hbar(H) = 0, \quad (0\le t\le T), \\ \qquad\qquad\qquad\qquad \mathcal{U}_\hbar(0) = \mathrm{Id}\,. \end{cases} \tag{2.22}$$

The existence of a unitary operator $\mathcal{U}_\hbar$ solving (2.22) is not always true, even if the solution of (2.21) exists. If the Hamiltonian H does not depend on time, then $\mathcal{U}_\hbar$ exists provided that $\mathrm{Op}_\hbar(H)$ is selfadjoint on $L^2(\mathbb{R}^d)$. In this case, we have:

Lemma 2.7 ([122, Thm. C.13], Stone's Theorem)**.** *Suppose* $\mathrm{Op}_\hbar(H)$ *is a (possibly unbounded) selfadjoint operator on* $L^2(\mathbb{R}^d)$*. Then*

$$\mathcal{U}_\hbar(t) := e^{-\frac{it}{\hbar}\,\mathrm{Op}_\hbar(H)}, \quad t\in\mathbb{R},$$

defines a strongly continuous unitary group satisfying:

1. $\mathcal{U}_\hbar(t)\,\mathcal{U}_\hbar(s)=\mathcal{U}_\hbar(t+s)$*, and* $\mathcal{U}_\hbar(t)^*=\mathcal{U}_\hbar(-t)$*.*
2. $\lim_{t\to 0}\|\mathcal{U}_\hbar(t)\,\psi-\psi\|_{L^2(\mathbb{R}^d)}=0$*, for every* $\psi\in L^2(\mathbb{R}^d)$*.*
3. $\hbar D_t\,\mathcal{U}_\hbar(t)+\mathcal{U}_\hbar(t)\,\mathrm{Op}_\hbar(H)=0$*, and* $\mathcal{U}_\hbar(0)=\mathrm{Id}$*.*

On the other hand, if the Hamiltonian H depends on the time variable, the selfadjointness of $\mathrm{Op}_\hbar(H(t,\cdot))$ for every $0\le t\le T$ is not sufficient to ensure the existence of $\mathcal{U}_\hbar(t)$ solving (2.22). However, if the family $H(t,\cdot)$ belongs smoothly to the class $S^0(\mathbb{R}^{2d})$, then the unitary operator $\mathcal{U}_\hbar(t)$ solving (2.22) exists for every $t\in\mathbb{R}$, see [122, Thm. 10.1]. The same holds if $H(t,\cdot)$ belongs to $S^k(\mathbb{R}^{2d})$ uniformly in t and, moreover, it satisfies:

$$\sup_{t\in[0,T]}\sup_{z\in\mathbb{R}^{2d}}|\partial_t^n\partial_z^\alpha H(t,z)|\le C_{n,\alpha}(1+|z|^2)^{k/2}, \quad n\in\mathbb{N}, \quad \alpha\in\mathbb{N}^{2d},$$

and $H(t,\cdot)$ is uniformly elliptic, meaning that

$$|H(t,z)| \geq \frac{(1+|z|^2)^{k/2}}{C} - C,$$

for a time-independent constant C, see [122, Thm. 10.3].

The Egorov's theorem relates the propagator of equation (2.22) with this of equation (2.21) at symbolic level, modulo a small error term in $\hbar$:

Lemma 2.8 ([25], Egorov's Theorem)**.** *Let $a \in \mathscr{S}(\mathbb{R}^{2d})$. Then, for $0 \leq t \leq T$, we have*

$$\mathcal{U}_\hbar(-t)\operatorname{Op}_\hbar(a)\,\mathcal{U}_\hbar(t) = \operatorname{Op}_\hbar\left(a \circ \phi_t^H\right) + R_\hbar(t), \tag{2.23}$$

where $\|R_\hbar(t)\|_{\mathcal{L}(L^2)} \leq \rho(|t|)\hbar^2$ for some continuous function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ with $\rho(0) = 0$. In the particular case when $H(t,x,\xi)$ is a polynomial in (x,ξ) of degree at most two, then $R_\hbar \equiv 0$.

Conjugation in the left hand side of (2.23) follows the *Heisenberg picture* of quantum mechanics. The assertion is that the evolution of $\operatorname{Op}_\hbar(a)$ by the quantum unitary propagator $\mathcal{U}_\hbar(t)$ is well approximated up to time T by the classical flow.

2.6. Semiclassical measures

In this section we introduce one of the main objects of study in this thesis: the notion of *semiclassical* or *Wigner measure*. We consider a sequence of wave functions $(\psi_\hbar)$ as $\hbar \to 0$. With slightly abuse of notation, we use the same notation for the index of the sequence $(\psi_\hbar)$ and for the semiclassical parameter. What we mean is that, with each index of the sequence $(\psi_\hbar)$, we associate an element of a sequence of parameters $(\hbar)$ such that $\hbar \to 0$. Since we will have to take subsequences frequently, it would make the reading difficult to highlight the indices every time and, anyway, this notation will remain consistent.

One way to understand the asymptotic properties of a given sequence of wave functions $(\psi_\hbar)$ as $\hbar \to 0^+$ is to look at its associate sequence of position densities $|\psi_\hbar(x)|^2$ and at its related sequence of momentum densities $|\mathscr{F}_\hbar\psi_\hbar(\xi)|^2$. If the sequence $(\psi_\hbar)$ is normalized in $L^2(\mathbb{R}^d)$, then, modulo extracting a subsequence, there exist positive Radon measures $\nu_1 \in \mathcal{M}(\mathbb{R}^d_x)$ and $\nu_2 \in \mathcal{M}(\mathbb{R}^d_\xi)$ such that

$$|\psi_\hbar(x)|^2 \rightharpoonup^\star \nu_1, \qquad |\mathscr{F}_\hbar\psi_\hbar(\xi)|^2 \rightharpoonup^\star \nu_2,$$

where the convergence takes place in the weak-$\star$ topology for Radon measures. However, since the physics behind the quantum and classic mechanics occur on the phase space $T^*\mathbb{R}^d$ rather than on the position space $\mathbb{R}^d_x$ or the momentum space $\mathbb{R}^d_\xi$, we need a way of studying the sequence $(\psi_\hbar)$ directly on the phase space to catch all the relevant information involved in the asymptotic propagation of the wave function.

In 1932, Wigner [117] introduced the later on called Wigner quasiprobability distribution, trying to adapt the statatistical-mechanic formalism to the quantum theory. The Wigner distribution is defined by the map:

$$W_\psi^\hbar : \mathcal{C}_c^\infty(\mathbb{R}^{2d}) \ni a \longmapsto \left\langle \operatorname{Op}_\hbar(a)\psi, \psi \right\rangle_{L^2(\mathbb{R}^d)} \in \mathbb{C}.$$

For each classical observable $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$, the Wigner distribution localizes the wave function on the support of a simultaneously in position and momentum. One can check that $W_\psi^\hbar$ has the following expression:

$$W_\psi^\hbar(x,\xi) = \int_{\mathbb{R}^d} \psi\left(x - \frac{\hbar}{2}v\right) \overline{\psi\left(x + \frac{\hbar}{2}v\right)} \, e^{i\xi\cdot v} \frac{dv}{(2\pi)^d}.$$

In particular,

$$\int_{\mathbb{R}^d} W_\psi^\hbar(x,\xi)\, d\xi = |\psi(x)|^2, \quad \int_{\mathbb{R}^d} W_\psi^\hbar(x,\xi)\, dx = |\mathscr{F}_\hbar\psi(\xi)|^2,$$

hence, $W_\psi^\hbar$ contains more information that $|\psi(x)|^2$ and $|\mathscr{F}_\hbar\psi(\xi)|^2$ separately.

By Calderón-Vaillancourt theorem (Lemma 2.5), given a L^2-normalized sequence $(\psi_\hbar)$, we have the estimate

$$|W_{\psi_\hbar}^\hbar(a)| \leq C\|a\|_{L^\infty(\mathbb{R}^{2d})} + O(\hbar).$$

Thus $(W_{\psi_\hbar}^\hbar)$ is a bounded sequence in the space of distributions $\mathcal{D}'(\mathbb{R}^{2d})$, and then the sequence $(W_{\psi_\hbar}^\hbar)$ has at least one accumulation point $\mu \in \mathcal{D}'(\mathbb{R}^{2d})$ (with respect to the inductive limit weak-$\star$ topology in $\mathcal{D}'(\mathbb{R}^{2d})$). Moreover, using the density of $\mathcal{C}_c^\infty(\mathbb{R}^{2d})$ in $\mathcal{C}_c(\mathbb{R}^{2d})$ and a diagonal extraction argument together with the Riesz Representation Theorem (see [122, Thm. 5.2]), one can show that μ is actually a complex Radon measure on $\mathbb{R}^{2d}$. Furthermore, the distribution μ turns out to be real and nonnegative, hence it is a positive Radon measure $\mu \in \mathcal{M}(\mathbb{R}^{2d})$. Indeed, by the sharp Garding inequality (Lemma 2.6), for every $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ such that $a \geq 0$, one has

$$|W_{\psi_\hbar}^\hbar(a)| \geq -C\hbar^2,$$

then, taking limit $\hbar \to 0$, this implies

$$\int_{\mathbb{R}^{2d}} a(x,\xi)\mu(dx,d\xi) \geq 0.$$

The measure μ is called the semiclassical or Wigner measure of the (sub)sequence $(\psi_\hbar)$. The notion of semiclassical measure was introduced by Gérard [51, 52], Lions and Paul [78], and Tartar [111].

2.6.1. Properties and examples

We next clarify some aspects of the above definition and provide some examples to illustrate the main properties of semiclassical measures. Let $(\psi_\hbar)$ be a L^2-normalized sequence of wave functions and suppose that:

$$|\psi_\hbar(x)|^2 \rightharpoonup^\star \nu_1, \quad \text{as } \hbar \to 0,$$
$$|\mathscr{F}_\hbar \psi_\hbar(x)|^2 \rightharpoonup^\star \nu_2, \quad \text{as } \hbar \to 0,$$

for some Radon measures $\nu_1, \nu_2 \in \mathcal{M}(\mathbb{R}^d)$. Suppose moreover that $\mu \in \mathcal{M}(\mathbb{R}^{2d})$ is the semiclassical measure of the sequence $(\psi_\hbar)$. Then the following holds for every $a \in \mathcal{C}_c(\mathbb{R}^d)$:

$$\int_{\mathbb{R}^{2d}} a(x)\mu(dx, d\xi) \leq \int_{\mathbb{R}^d} a(x)\nu_1(dx), \tag{2.24}$$

$$\int_{\mathbb{R}^{2d}} a(\xi)\mu(dx, d\xi) \leq \int_{\mathbb{R}^d} a(\xi)\nu_2(d\xi). \tag{2.25}$$

In general, equality in (2.24) and (2.25) may not hold. This is due to the non-compactness of $T^*\mathbb{R}^d = \mathbb{R}^d_x \times \mathbb{R}^d_\xi$, which allows some loss of mass of $W^\hbar_{\psi_\hbar}$ at infinity as $|x| \to \infty$ or $|\xi| \to \infty$. The following restrictions prevent this loss of mass to occur.

Definition 2.3. *We define the following dual properties:*

1. *We say that the sequence $(\psi_\hbar)$ is compact at infinity provided that*

$$\limsup_{\hbar \to 0} \int_{|x| \geq R} |\psi_\hbar(x)|^2 dx \to 0, \quad \text{as } R \to \infty. \tag{2.26}$$

2. *We say that the sequence $(\psi_\hbar)$ is $\hbar$-oscillating if*

$$\limsup_{\hbar \to 0} \int_{|\xi| \geq R} |\mathscr{F}_\hbar \psi_\hbar(\xi)|^2 d\xi \to 0, \quad \text{as } R \to \infty. \tag{2.27}$$

We next state the main properties of semiclassical measures:

Lemma 2.9. *Let $(\psi_\hbar)$ be a L^2-normalized sequence. The following hold:*

1. *$(\psi_\hbar)$ is $\hbar$-oscillating if and only if*

$$\nu_1(x) = \int_{\mathbb{R}^d} \mu(x, d\xi).$$

2. *$(\psi_\hbar)$ is compact at infinity if and only if*

$$\nu_2(\xi) = \int_{\mathbb{R}^d} \mu(dx, \xi).$$

3. *Let $(\psi_\hbar)$ be $\hbar$-oscillating and compact at infinity. If $\psi_\hbar \to \psi$ strongly in $L^2(\mathbb{R}^d)$ then*

$$\mu(x, \xi) = |\psi(x)|^2 \delta_0(\xi).$$

4. *If $\psi_\hbar \rightharpoonup \psi$ in $L^2(\mathbb{R}^d)$ then*

$$\mu(x, \xi) \geq |\psi(x)|^2 \delta_0(\xi).$$

Remark 2.3. Points *3* and *4* show that μ can be viewed as a measure of the defect of compacity of the sequence $(\psi_\hbar)$. In fact, semiclassical measures are often called *microlocal defect measures*, see Gérard [52] and Macià [83].

Examples.

1. *Oscillating sequence.* Let

$$\psi_\hbar(x) := \psi(x) e^{\frac{i}{\varepsilon_\hbar} x \cdot \xi_0},$$

where $\varepsilon_\hbar \to 0$ as $\hbar \to 0$, $\xi_0 \in \mathbb{R}^d$ and $\psi \in L^2(\mathbb{R}^d)$ with $\|\psi\|_{L^2(\mathbb{R}^d)} = 1$. Then

$$\mu(x, \xi) = \begin{cases} |\psi(x)|^2 \, \delta_0(\xi) & \text{if } \hbar \ll \varepsilon_\hbar, \\ |\psi(x)|^2 \, \delta_{\xi_0}(\xi) & \text{if } \hbar = \varepsilon_\hbar, \\ 0 & \text{if } \hbar \gg \varepsilon_\hbar. \end{cases}$$

2. *Concentrating sequence.* Let

$$\psi_\hbar(x) := \frac{1}{\varepsilon_\hbar^{d/2}} \psi\left(\frac{x - x_0}{\varepsilon_\hbar}\right),$$

where $x_0 \in \mathbb{R}^d$. Then

$$\mu(x, \xi) = \begin{cases} \|\psi\|^2_{L^2(\mathbb{R}^d)} \, \delta_{x_0}(x) \, \delta_0(\xi) & \text{if } \hbar \ll \varepsilon_\hbar, \\ \delta_{x_0}(x) |\mathscr{F}_1 \psi(\xi)|^2 & \text{if } \hbar = \varepsilon_\hbar, \\ 0 & \text{if } \hbar \gg \varepsilon_\hbar. \end{cases}$$

3. *Wave packet or Coherent state.* Let

$$\psi_\hbar(x) := \frac{1}{\varepsilon_\hbar^{d/2}} \psi\left(\frac{x - x_0}{\varepsilon_\hbar}\right) e^{\frac{i}{\hbar} x \cdot \xi_0},$$

with $\varepsilon_\hbar \gg \hbar$. Then

$$\mu(x, \xi) = \|\psi\|^2_{L^2(\mathbb{R}^d)} \delta_{x_0}(x) \delta_{\xi_0}(\xi).$$

Remark 2.4. Despite the fact that the uncertainty principle (Lemma 2.1) prevents each function $\psi_\hbar$ to concentrate simultaneusly in both position and momentum variables, point *3* shows that, adjusting the scale of concentration as $\varepsilon_\hbar \gg \hbar$, it is possible to localize in the point (x_0, ξ_0) as $\hbar \to 0$. As we will see in the next section, this is the germ of the *correspondence principle*.

2.6.2. The Correspondence Principle

Semiclassical measures allow to formalize in a rigorous manner the *correspondence principle* between classical and quantum dynamics. Consider the semiclassical Schrödinger operator given by

$$\widehat{H}_\hbar := \mathrm{Op}_\hbar(H) = -\frac{1}{2}\hbar^2\Delta + W(x),$$

where

$$H(x,\xi) = \frac{1}{2}|\xi|^2 + W(x)$$

is the classical Hamiltonian consisting of the sum of the kinetic and the potential energy. We assume that $W \in \mathcal{C}^\infty(\mathbb{R}^d)$ is Lipschitz continuous and bounded from below. Then the Hamilton equation (2.21) is globally well posed, hence the hamiltonian flow

$$\phi_t^H : \mathbb{R}^{2d} \to \mathbb{R}^{2d}, \quad t \in \mathbb{R}$$

is globally defined. Analogously, if $V \in L^\infty_{loc}(\mathbb{R}^d)$ is essentially bounded from below, then the operator $\widehat{H}_\hbar$ is selfadjoint on $L^2(\mathbb{R}^d)$ and hence, by the Stone's theorem (Lemma 2.7), it generates a unitary group $e^{-\frac{it}{\hbar}\widehat{H}_\hbar}$ solving the quantum equation (2.22). Therefore, the semiclassical Schrödinger equation

$$\big(i\hbar\,\partial_t + \widehat{H}_\hbar\big)\psi_\hbar(t,x) = 0, \quad \psi_\hbar(0,x) = \psi_\hbar^0(x) \in L^2(\mathbb{R}^d) \tag{2.28}$$

has a unique and globally defined solution $\psi(t,x)$ given by

$$\psi_\hbar(t,x) = e^{-\frac{it}{\hbar}\widehat{H}_\hbar}\,\psi_\hbar^0(x),$$

that satisfies $\|\psi_\hbar(t,x)\|_{L^2(\mathbb{R}^d)} = \|\psi_\hbar^0(x)\|_{L^2(\mathbb{R}^d)}$.

Using semiclassical measures and Egorov's theorem (Lemma 2.8), one can formalize the correspondence principle in the following way:

Theorem 2.1 ([51, 78])**.** *Let $(\psi_\hbar^0)$ be a normalized sequence of initial data for the equation* (2.28)*. Assume that $\mu_0 \in \mathcal{M}(\mathbb{R}^{2d})$ is the semiclassical measure of the sequence $(\psi_\hbar)$. Then, for every $t \in \mathbb{R}$, there exists a semiclassical measure $\mu(t,\cdot) \in \mathcal{M}(\mathbb{R}^{2d})$ for the sequence $\psi_\hbar(t,\cdot)$ of solutions of* (2.28) *with data $(\psi_\hbar^0)$, and it satisfies:*

$$\int_{\mathbb{R}^{2d}} a(x,\xi)\mu(t,dx,d\xi) = \int_{\mathbb{R}^{2d}} a\circ\phi_t^H(x,\xi)\mu_0(dx,d\xi), \quad a \in \mathcal{C}_c(\mathbb{R}^{2d}). \tag{2.29}$$

Note that identity (2.29) is equivalent to the fact that $\mu(t,\cdot)$ is obtained as the push-forward of μ_0 along ϕ_t^H; this can be written as:

$$\mu(t,\cdot) = \left(\phi_t^H\right)_* \mu_0, \quad t \in \mathbb{R}.$$

Corollary 2.1. *Suppose the hypothesis of Theorem 2.1 hold. Then, for every $t \in \mathbb{R}$ and $a \in \mathcal{C}_c(\mathbb{R}^d)$, the following holds:*

$$\lim_{\hbar\to 0} \int_{\mathbb{R}^d} a(x)|\psi_\hbar(t,x)|^2 dx = \int_{\mathbb{R}^{2d}} a \circ \pi_x \circ \phi_t^H(x,\xi)\mu_0(dx,d\xi),$$

where $\pi_x : \mathbb{R}_x^d \times \mathbb{R}_\xi^d \to \mathbb{R}_x^d$ is the canonical projection.

2.6.3. Semiclassical measures and the Schrödinger equation

We next state some standard results concerning the semiclassical measures obtained from sequences of approximate solutions of the Schrödinger equation. Let us consider, for the sake of simplicity (we could consider more general elliptic selfadjoint operators), the semiclassical harmonic oscillator $\widehat{H}_\hbar$ defined on $L^2(\mathbb{R}^d)$ by

$$\widehat{H}_\hbar := \frac{1}{2}\sum_{j=1}^d \omega_j(-\hbar^2\partial_{x_j}^2 + x_j^2), \quad \omega_j > 0.$$

The following hold:

Lemma 2.10 ([122, Thm. 5.3])**.** *Consider a sequence $(v_\hbar)$ normalized in $L^2(\mathbb{R}^d)$ such that*

$$\left(\widehat{H}_\hbar - \lambda_\hbar\right)v_\hbar = o(1), \quad \lambda_\hbar \to 1, \quad \text{as } \hbar \to 0^+.$$

Then

$$\operatorname{supp} \mu \subset H^{-1}(1)$$

for any semiclassical measure μ associated with the sequence $(v_\hbar)$.

Lemma 2.11 ([122, Thm. 5.4])**.** *Consider a sequence $(v_\hbar)$ normalized in $L^2(\mathbb{R}^d)$ such that*

$$\left(\widehat{H}_\hbar - \lambda_\hbar\right)v_\hbar = o(\hbar), \quad \lambda_\hbar \to 1, \quad \text{as } \hbar \to 0^+.$$

Then

$$\int_{\mathbb{R}^{2d}} \{H,a\}(x,\xi)\mu(dx,d\xi) = 0$$

for all $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ and any semiclassical measure μ associated with the sequence $(v_\hbar)$.

Remark 2.5. This shows that the semiclassical measure μ is invariant by the hamiltonian ϕ_t^H flow generated by H. In other words,

$$\left(\phi_t^H\right)_* \mu = \mu, \quad \forall t \in \mathbb{R}.$$

2.6.4. TIME-DEPENDENT SEMICLASSICAL MEASURES

In this section we recall some facts about time-dependent semiclassical measures instroduced by Macià in [80] adapted to the case of perturbed harmonic oscillators.

We cosider the Hamiltonian $\widehat{P}_\hbar = \widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar$ introduced in Section 1.2. Let $(u_\hbar)$ be a normalized and $\hbar$-oscillating sequence in $L^2(\mathbb{R}^d)$. For a given semiclassical scale $\tau = (\tau_\hbar)$ such that

$$\tau_\hbar \to \infty, \quad \text{as } \hbar \to 0^+,$$

we denote the Wigner distribution by

$$W_\hbar^{\tau,\varepsilon}(a) := \big\langle v_\hbar(t\tau_\hbar), \mathrm{Op}_\hbar(a) v_\hbar(t\tau_\hbar) \big\rangle_{L^2}, \quad a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d}),$$

where $v_\hbar(t)$ is the soultion of the Schrödinger equation

$$\big(i\hbar\,\partial_t + \widehat{P}_\hbar\big) v_\hbar(t,x) = 0, \quad v_\hbar(0,x) = u_\hbar(x).$$

Using the Calderón-Vaillancourt theorem, we deduce

$$\left| \int_{\mathbb{R}} W_\hbar^{\tau,\varepsilon}(a(t))dt \right| \lesssim \sum_{|\alpha| \le K_d} \hbar^{|\alpha|/2} \int_{\mathbb{R}} \| \partial_{x,\xi}^\alpha a(t,\cdot) \|_{L^\infty(\mathbb{R}^{2d})} dt, \quad a \in \mathcal{C}_c^\infty(\mathbb{R}_t \times \mathbb{R}_{x,\xi}^{2d}).$$

Hence the sequence $(W_\hbar^{\tau,\varepsilon})$ is relatively compact in $\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}^{2d}_{x,\xi})$. In particular, for any accumulation point μ of this sequence and every $a \in \mathcal{C}_c^\infty(\mathbb{R}_t \times \mathbb{R}^{2d}_{x,\xi})$, one has

$$\left| \int_{\mathbb{R}_t \times \mathbb{R}^{2d}_{x,\xi}} a(t,x,\xi) \mu(dt,dx,d\xi) \right| \lesssim \int_{\mathbb{R}} \|a(t,\cdot)\|_{L^\infty(\mathbb{R}^{2d})} dt.$$

Thus, μ can be extended to a continuous linear form on $L^1(\mathbb{R}, \mathcal{C}_c(\mathbb{R}^{2d}))$. Therefore, the limit distribution $t \mapsto \mu(t,\cdot)$ belongs to $L^\infty(\mathbb{R}, \mathcal{M}_{\mathbb{C}}(\mathbb{R}^{2d}))$, where $\mathcal{M}_{\mathbb{C}}(\mathbb{R}^{2d}))$ denotes the set of finite complex measures on $\mathbb{R}^{2d}$. For any converging subsequence in $\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}^{2d}_{x,\xi})$ we note that the following also holds: for every $\varphi \in L^1(\mathbb{R})$ and for every $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$,

$$\lim_{\hbar \to 0^+} \int_{\mathbb{R}} \varphi(t) W_\hbar^{\tau,\varepsilon}(a(t,\cdot))dt = \int_{\mathbb{R}} \int_{\mathbb{R}^{2d}} \varphi(t) a(x,\xi) \mu(t,dx,d\xi) dt.$$

Finally, by the Garding inequality, the limit distribution is in fact a real and positive measure for a. e. $t \in \mathbb{R}$. Using Egorov's theorem, one can also verify that for a. e. $t \in \mathbb{R}$, $\mu(t,\cdot)$ is invariant by the Hamiltonian flow ϕ_t^H.

2.7. Pseudodifferential operators on manifolds

In this section we extend the process of quantization to differentiable manifolds. To this aim, it is very convenient to use symbols in the Kohn-Nirenberg classes. For every $m \in \mathbb{Z}$, we set

$$\mathbf{S}^m := \{a \in \mathcal{C}^\infty(\mathbb{R}^{2d}) \,:\, \|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty(\mathbb{R}^{2d})} \leq C_{\alpha,\beta}(1+|\xi|^2)^{(m-|\beta|)/2}, \quad \alpha, \beta \in \mathbb{N}^d\}. \tag{2.30}$$

This family is very well behaved under changes of coordinates. Precisely, the following holds:

Lemma 2.12 ([122, Thm 9.4])**.** *Given a diffeomorphism $\gamma : \mathbb{R}^d \to \mathbb{R}^d$ such that*

$$|\partial^\alpha \gamma(x)|, |\partial^\alpha \gamma^{-1}(x)| \leq C_\alpha, \quad \alpha \in \mathbb{N}^d.$$

Then, for each symbol $a \in \mathbf{S}^m(\mathbb{R}^{2d})$, its pull-back

$$\gamma^* a(x,\xi) := a\big(\gamma(x), [\partial_x \gamma(x)^{-1}]^T \xi\big)$$

also belongs to $\mathbf{S}^m(\mathbb{R}^{2d})$.

Let M be a manifold of dimension d, and let T^*M be its cotangent bundle. Assume we have an atlas $\mathscr{A} = (U_j, \gamma_j)_{j\in J}$ of homeomorphisms between open sets

$$\gamma_j : V_j \to U_j, \quad U_j \subset M, \quad V_j \subset \mathbb{R}^d,$$

satisfying the usual compatibility conditions.

Definition 2.4. *We say that $a \in \mathbf{S}^m(T^*M)$ if $a \in \mathcal{C}^\infty(T^*M)$ and, for every chart $(U_j, \gamma_j) \in \mathscr{A}$, the function*

$$\gamma_j^* a : T^*V_j \to \mathbb{C}$$

belongs to $\mathbf{S}^m(V_j \times \mathbb{R}^d)$ under the identification

$$T^*V_j \simeq V_j \times \mathbb{R}^d \subset \mathbb{R}^{2d}.$$

By Lemma 2.12, this definition does not depend on the choice of the atlas $\mathscr{A}$. We now consider a locally finite partition of unity $\{\chi_j\}_{j\in J}$ associated to the atlas $\mathscr{A}$, that is

$$\sum_{j\in J} \chi_j(x) \equiv 1, \quad \chi_j \in \mathcal{C}_c^\infty(U_j).$$

Given symbol $a \in \mathbf{S}^m(T^*M)$, we define its Weyl's quantization by

$$\mathrm{Op}_\hbar(a)\psi(x) = \sum_{j\in J} \big(\widehat{A}_{j,\hbar}\, \chi_j \psi\big)(x), \tag{2.31}$$

where
$$\widehat{A}_{j,\hbar} = (\gamma_j^{-1})^* \operatorname{Op}_\hbar \big(\gamma_j^*(\chi_j a)\big)\gamma_j^*.$$
We denote $\operatorname{Op}_\hbar(a) \in \Psi^m(M)$ if $a \in \mathbf{S}^m(T^*M)$. Obviously, this definition is by no means intrinsic, it depends on the atlas $\mathscr{A}$ as well as on the partition of unity. However, one can define the symbol of an operator $A \in \Psi^m(M)$ in the equivalent class

$$\mathbf{S}^m(T^*M)\big/\hbar\,\mathbf{S}^{m-1}(T^*M),$$

see [122, Thm 14.1]. We will not be more precise at this point because along this thesis we will focus on the cases $M = \mathbb{R}^d$ and $M = \mathbb{T}^d$ (the flat torus). This last case deserves special atention, so we will dedicate the following section to deal with it.

If (M, g) is a Riemannian manifold, one can consider the quantization of the Hamiltonian (2.1). In this case, there exists an intrisic operator $\widehat{H}_\hbar$ given by

$$\widehat{H}_\hbar := -\hbar^2\Delta_g + V(x),$$

that quantizes the symbol H , where Δ_g is the Laplace-Beltrami operator $\Delta_g := \operatorname{Div}_g(\nabla_g \cdot)$. Here the gradient and the divergence are taken with respect to the Riemannian metric g. In a local chart:
$$\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^{d} \partial_{x_j}\big(\sqrt{\det g}\, g^{jk}\partial_{x_k}\big).$$

In terms of pseudodifferential operators:

$$-\hbar^2\Delta_g = \operatorname{Op}_\hbar(H) + i\hbar \operatorname{Op}_\hbar(r) + \hbar^2 \operatorname{Op}_\hbar(m),$$

where $m \in \mathcal{C}^\infty(M)$ is a function of x alone, that only depends on the derivatives up to order two of the metric g, and the function r is given in local coordinates by

$$r(x,\xi) = \frac{1}{\sqrt{\det g(x)}} \sum_{j,k=1}^{d} g^{jk}(x)\partial_{x_j}\sqrt{\det g(x)}\,\xi_k.$$

This shows that
$$-\hbar^2\Delta_g = \operatorname{Op}_\hbar(H), \quad \operatorname{mod}\, \hbar\,\Psi^1(M).$$

2.7.1. Weyl's quantization on the torus

In this section we assume $M = \mathbb{T}^d$, the flat torus defined by

$$\mathbb{T}^d := \mathbb{R}^d/2\pi\mathbb{Z}^d,$$

equipped with the standard flat metric. Its cotangent bundle $T^*\mathbb{T}^d$ is identified with the product space $T^*\mathbb{T}^d \simeq \mathbb{T}^d \times \mathbb{R}^d$. There are several ways to define the Weyl quantization on $T^*\mathbb{T}^d$. Of course, one is to use the general definition for manifolds introduced in the previous section. However, the particular structure of the torus allows to choose a canonical way of quantization by taking advantage of the definition of $\mathbb{T}^d$ as a quotient space of $\mathbb{R}^d$. The key observation is to note that the operator $\mathrm{Op}_\hbar(a)$ defined in the usual Euclidean case, with symbol $a \in \mathcal{C}^\infty(T^*\mathbb{T}^d)$ seen as a $(2\pi\mathbb{Z}^d)$-periodic function in the x variable, preserves the space of periodic distributions $\mathcal{D}'(\mathbb{T}^d)$, namely the space of distributions $u \in \mathcal{D}'(\mathbb{R}^d)$ satisfying

$$\langle u, \phi \rangle = \langle u, \phi(\cdot + 2\pi k) \rangle, \quad k \in \mathbb{Z}^d, \quad \phi \in \mathcal{C}_c^\infty(\mathbb{R}^d).$$

This approach allows to justify all the symbolic calculus and main theorems obtained in the Euclidean case also for the torus case. See [9] and [122, Section 5.3.1] for further details.

On the other hand, from the point of view of the intuition of the dynamics on the "quantum" phase space of the torus, it is maybe more illustrative to motivate the Weyl quantization from the definition of position and momentum observables, as we did in the Euclidean case. The main difference lies in the definition of the momentum observable $\widehat{P}_j$, which in this case has discrete spectrum, comparing with the Euclidean momentum.

We define the position and momentum observables on $L^2(\mathbb{T}^d)$ by

$$\widehat{Q}_j \, \psi(x) := x_j \, \psi(x), \tag{2.32}$$

$$\widehat{P}_j \, \psi(x) := -i\hbar \, \partial_{x_j} \psi(x). \tag{2.33}$$

The Fourier transform of $\psi \in L^2(\mathbb{T}^d)$ is defined by

$$\widehat{\psi}(k) := \big\langle \psi, e_k \big\rangle_{L^2(\mathbb{T}^d)}, \quad e_k(x) := \frac{1}{(2\pi)^{d/2}} e^{ik \cdot x}, \quad k \in \mathbb{Z}^d,$$

where we use the usual convention for the scalar product

$$\big\langle \psi, \varphi \big\rangle_{L^2(\mathbb{T}^d)} := \int_{\mathbb{T}^d} \psi(x) \overline{\varphi(x)} dx.$$

Then we can decompose the wave function ψ as

$$\psi(x) = \sum_{k \in \mathbb{Z}^d} \widehat{\psi}(k) e_k(x).$$

Note that the spectrum of $\widehat{P}_j$ is the discrete subset of $\mathbb{R}$ given by

$$\mathrm{Sp}_{L^2(\mathbb{T}^d)} \big(\widehat{P}_j \big) = \{ \hbar k_j \, : \, k \in \mathbb{Z}^d \}, \quad j \in \{1, \dots, d\}.$$

In this sense the momentum is quantized. On the other hand, the operators $\widehat{Q}_j$ and $\widehat{P}_j$ still satisfy the commutator identities (2.6).

Similarly, given $a \in \mathcal{C}^\infty(T^*\mathbb{T}^d)$, we define

$$\widehat{a}(k,\xi) := \big\langle a(\cdot,\xi), e_k \big\rangle_{L^2(\mathbb{T}^d)}, \quad k \in \mathbb{Z}^d,$$

and then

$$a(x,\xi) = \sum_{k\in\mathbb{Z}^d} \widehat{a}(k,\xi) e_k(x), \quad (x,\xi) \in T^*\mathbb{T}^d.$$

We also regard the Fourier decomposition of a in both $(x,\xi) \in T^*\mathbb{T}^d$. It is convinient to write it under a Lebesgue-Stieltjes integral. We consider the product measure on $\mathscr{Z}^d := \mathbb{Z}^d \times \mathbb{R}^d$ defined by:

$$\kappa(l,\eta) = \mathcal{K}_{\mathbb{Z}^d}(l) \otimes \mathcal{L}_{\mathbb{R}^d}(\eta), \quad (l,\eta) \in \mathscr{Z}^d, \tag{2.34}$$

where $\mathcal{L}_{\mathbb{R}^d}$ denotes the Lebesgue measure on $\mathbb{R}^d$, and

$$\mathcal{K}_{\mathbb{Z}^d}(l) := \sum_{k\in\mathbb{Z}^d} \delta(l-k), \quad l \in \mathbb{Z}^d.$$

For any Schwartz function $a \in \mathscr{S}(T^*\mathbb{T}^d)$, we define:

$$\big(\mathscr{F} a\big)(w) := \frac{1}{(2\pi)^{2d}} \int_{T^*\mathbb{T}^d} a(z) e^{-iz\cdot w}\, dz, \quad z = (x,\xi) \in T^*\mathbb{T}^d, \tag{2.35}$$

where $w = (k,\eta) \in \mathscr{Z}^d$. Then:

$$a(z) = \int_{\mathscr{Z}^d} \big(\mathscr{F} a\big)(w) e^{iL_w(z)}\, \kappa(dw), \tag{2.36}$$

where

$$L_w(z) := w \cdot z, \quad z \in T^*\mathbb{T}^d.$$

Analogously as we did in the Euclidean case, we introduce the semiclassical Schrödinger representation on $L^2(\mathbb{T}^d)$:

Definition 2.5. *The semiclassical Schrödinger representation $U_\hbar$ on $L^2(\mathbb{T}^d)$ is defined by the unitary group*

$$U_\hbar(w) : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$$

acting as

$$U_\hbar(w)\psi(x) := e^{ik\cdot x + \frac{i}{2} k\cdot\hbar\eta}\, \psi(x+\hbar\eta), \tag{2.37}$$

for every $w = (k,\eta) \in \mathscr{Z}^d$ and every $\hbar \in (0,1]$. Note the different scaling comparing with (2.13).

It is easy to check from (2.37) that the semiclassical Schrödinger representation satisfies the following conmutator relations:

$$U_\hbar(w)^* = U_\hbar(-w), \quad U_\hbar(w)U_\hbar(w') = e^{\frac{i\hbar}{2}\{L_w, L_{w'}\}} U_\hbar(w+w'), \quad w, w' \in \mathcal{Z}^d, \tag{2.38}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket (compare with (2.14)). We use the Schrödinger representation to define directly the Weyl quantization via the following:

Definition 2.6 (Semiclassical Weyl's quantization)**.** *Let $a \in \mathscr{S}(T^*\mathbb{T}^d)$, we define the semiclassical Weyl quantization* $\mathrm{Op}_\hbar(a)$ *acting on* $\psi \in L^2(\mathbb{T}^d)$ *by*

$$\mathrm{Op}_\hbar(a)\psi(x) := \int_{\mathcal{Z}^d} \big(\mathscr{F}a\big)(w)\, U_\hbar(w)\psi(x)\, \kappa(dw).$$

The composition of two pseudodifferential operators on the torus can be obtained by using Definition 2.6 and commutator formula (2.38). In terms of the Moyal product:

$$\mathrm{Op}_\hbar(a)\, \mathrm{Op}_\hbar(b) = \mathrm{Op}_\hbar(a \sharp_\hbar b),$$

where the Moyal product $\sharp_\hbar$ is given by

$$a\sharp_\hbar b(z) = \int_{\mathcal{Z}^{2d}} \big(\mathscr{F}a\big)(w')\big(\mathscr{F}b\big)(w-w') e^{\frac{i\hbar}{2}\{L_{w'}, L_{w-w'}\}} e^{iL_w(z)} \kappa(dw')\, \kappa(dw). \tag{2.39}$$

This can also be written as

$$a\sharp_\hbar b(x,\xi) = \sum_{k,l \in \mathbb{Z}^d} \widehat{a}\left(k, \xi + \frac{\hbar(l-k)}{2}\right) \widehat{b}\left(l-k, \xi - \frac{\hbar k}{2}\right) e_l(x). \tag{2.40}$$

Regarding the action of the Weyl quantization in terms of Fourier decomposition, the following is an inmediately consequence of Definition 2.6:

Lemma 2.13. *For every $a \in \mathscr{S}(T^*\mathbb{T}^d)$ and $\psi \in L^2(\mathbb{T}^d)$:*

$$\mathrm{Op}_\hbar(a)\psi(x) = \sum_{j,k \in \mathbb{Z}^d} \widehat{a}\left(j-k, \frac{\hbar(j+k)}{2}\right) \widehat{\psi}(k) e_j(x). \tag{2.41}$$

The notions of Wigner distribution and semiclassical measures can be extended to the case of manifolds. Note that, in the particular case of the torus, the Wigner distribution of a wave function $\psi \in L^2(\mathbb{T}^d)$ can be written using (2.41) as

$$W_\psi^\hbar(a) = \sum_{j,k \in \mathbb{Z}^d} \widehat{a}\left(j-k, \frac{\hbar(j+k)}{2}\right) \psi(k)\overline{\psi(-j)}, \quad a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d). \tag{2.42}$$

Finally, the Weyl quantization defined above can be extended for more general symbols, in the same way we explained for the Euclidean case. We define, for $m \in \mathbb{Z}$:

$$S^m(T^*\mathbb{T}^d) := \big\{ a \in \mathcal{C}^\infty(T^*\mathbb{T}^d) \,:\, \|\partial_\xi^\beta a\|_{L^\infty(\mathbb{R}^{2d})} \leq C_\beta (1 + |\xi|^2)^{m/2}, \quad \beta \in \mathbb{N}^d \big\}. \tag{2.43}$$

We conclude this chapter with a version of Calderón-Vaillancourt theorem on the torus:

Lemma 2.14 ([48, Prop 3.5])**.** *Let $a \in S^0(T^*\mathbb{T}^d)$. Then*

$$\| \operatorname{Op}_\hbar(a) \|_{\mathcal{L}(L^2)} \leq C_d \sum_{|\alpha| \leq N_d} \|\partial_x^\alpha a\|_{L^\infty(T^*\mathbb{T}^d)}.$$

In particular, only derivatives in x are requiered to estimate the strong operator norm of $\operatorname{Op}_\hbar(a)$*, compared with Lemma 2.5.*

Chapter 3

Semiclassical measures for perturbed harmonic oscillators

Había sin embargo horas tristes, como todo
el mundo tiene, en que uno creía no haber
logrado lo más mínimo, y le parecía que solo
los procesos destinados desde el principio a
un feliz resultado terminaban bien.

F. Kafka. *El Proceso.*

In this chapter we focus on the study of the semiclassical measures of solutions of the Schrödinger equations (1.26) and (1.27) generated by the perturbed harmonic oscillator:

$$\widehat{P}_\hbar := \widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar,$$

where $\widehat{H}_\hbar$ is given by (1.23), the operator $\widehat{V}_\hbar = \mathrm{Op}_\hbar(V)$ has symbol $V \in S^0(\mathbb{R}^{2d})$, and $\varepsilon_\hbar \to 0$ as $\hbar \to 0^+$. Section 3.1 is devoted to introduce some basic facts about the classic dynamical system associated to the harmonic oscillator. In particular, in Section 3.1.1 we will show how to solve the cohomological equations appearing in the process of averaging. In Section 3.2 we explain the averaging method in the selfadjoint case, and we use it to obtain a normal form via conjugating $\widehat{P}_\hbar$ by a suitable unitary operator up to order $N \geq 1$. In Section 3.3 we focus on the time-dependent Schrödinger equation, proving Theorems 1.2, 1.3, 1.4 and 1.5. The particular case of the periodic harmonic oscillator in 2D is treated in Section 3.3.1, where we prove Theorem 1.6. Finally, in Section 3.4 we study the semiclassical measures associated to sequences of solutions of the stationary problem, giving the proof of Theorems 1.7 and 1.8.

3.1. THE CLASSICAL HARMONIC OSCILLATOR

We start by recalling the basic properties of the dynamical system associated to the Hamiltonian

$$H(x,\xi) = \frac{1}{2}\sum_{j=1}^{d} \omega_j\big(\xi_j^2 + x_j^2\big), \quad \omega = (\omega_1, \dots, \omega_d) \in \mathbb{R}_+^d, \quad (x,\xi) \in \mathbb{R}^{2d}.$$

The Hamilton equations corresponding to H are given by

$$\begin{cases} \dot{x}_j = \omega_j \xi_j, \\ \dot{\xi}_j = -\omega_j x_j, \quad j = 1, \dots, d. \end{cases} \tag{3.1}$$

Hence we can write the solution of this system as a superposition of d-independent commuting flows as follows:

$$\big(x(t), \xi(t)\big) = \phi_t^H(x,\xi) := \phi_{\omega_d t}^{H_d} \circ \cdots \circ \phi_{\omega_1 t}^{H_1}(x,\xi), \quad (x,\xi) \in \mathbb{R}^{2d}, \quad t \in \mathbb{R},$$

where

$$H_j(x,\xi) = \frac{1}{2}(\xi_j^2 + x_j^2), \quad j \in \{1, \dots, d\},$$

and $\phi_t^{H_j}(x,\xi)$ denotes the associated Hamiltonian flow. In other words, the solution of (3.1) can be written in terms of the unitary block matrices

$$\begin{pmatrix} x_j(t) \\ \xi_j(t) \end{pmatrix} = \begin{pmatrix} \cos(\omega_j t) & \sin(\omega_j t) \\ -\sin(\omega_j t) & \cos(\omega_j t) \end{pmatrix} \begin{pmatrix} x_j \\ \xi_j \end{pmatrix}, \quad j = 1, \dots, d. \tag{3.2}$$

Observe that each flow $\phi_t^{H_j}$ is periodic with period $T_j = 2\pi$.

For any function $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$, it is clear that $a \circ \phi_t^H = a \circ \Phi_{t\omega}^H$, where recall that

$$\Phi_\tau^H = \phi_{t_1}^{H_1} \circ \cdots \circ \phi_{t_d}^{H_d}, \quad \tau = (t_1, \dots, t_d) \in \mathbb{T}^d,$$

and then we can write its average $\langle a \rangle$ by the flow ϕ_t^H as

$$\langle a \rangle(x,\xi) := \lim_{T\to\infty} \frac{1}{T}\int_0^T a \circ \phi_t^H(x,\xi)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T a \circ \Phi_{t\omega}^H(x,\xi)dt. \tag{3.3}$$

This limit is well defined and it takes place in the $\mathcal{C}^\infty(\mathbb{R}^{2d})$ topology.

The energy hypersurface $H^{-1}(E) \subset \mathbb{R}^{2d}$ is compact for every $E \geq 0$ and, due to the complete integrability of the system, each of these hypersurfaces is foliated by Kronecker tori that are invariant by the flow ϕ_t^H. Considering the submodule

$$\Lambda_\omega := \{k \in \mathbb{Z}^d \,:\, k \cdot \omega = 0\} \tag{3.4}$$

we can define the minimal torus contained in the space of angles $\tau \in \mathbb{T}^d$:

$$\mathbb{T}_\omega := \Lambda_\omega^\perp/(2\pi\mathbb{Z}^d \cap \Lambda_\omega^\perp),$$

where $\Lambda_\omega^\perp$ denotes the linear space orthogonal to Λ_ω. The dimension of $\mathbb{T}_\omega$ is $d_\omega = d - \operatorname{rk}\Lambda_\omega$. Kronecker's theorem states that the family of probability measures on $\mathbb{T}^d$ defined by

$$\frac{1}{T}\int_0^T \delta_{t\omega}\, dt$$

converges (in the weak-$\star$ topology) to the normalized Haar measure $\mathfrak{h}_\omega$ on the subtorus $\mathbb{T}_\omega \subset \mathbb{T}^d$. Moreover, the family of functions $\frac{1}{T}\int_0^T a \circ \phi_t^H dt$ converges to $\langle a \rangle$ in the $\mathcal{C}^\infty(\mathbb{R}^{2d})$ topology, and

$$\langle a \rangle(x,\xi) = \int_{\mathbb{T}_\omega} a \circ \Phi_\tau^H(x,\xi)\mathfrak{h}_\omega(d\tau), \tag{3.5}$$

and in particular, if $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ then $\langle a \rangle \in \mathcal{C}^\infty(\mathbb{R}^{2d})$. Observe that $1 \leq d_\omega \leq d$. In the case $d_\omega = 1$ and $\omega = \omega_1(1,\dots,1)$, the flow ϕ_t^H is $2\pi/\omega_1$-periodic. On the other hand, if $d_\omega = d$, then, for every $a \in \mathcal{C}^\infty(\mathbb{R}^{2d})$, there exists $\mathcal{I}_{\langle a \rangle} \in \mathcal{C}^\infty(\mathbb{R}^d)$ such that

$$\langle a \rangle(z) = \mathcal{I}_{\langle a \rangle}(H_1(z),\dots,H_d(z)).$$

In particular, for every a and b in $\mathcal{C}^\infty(\mathbb{R}^{2d})$, one has $\{\langle a \rangle, \langle b \rangle\} = 0$ whenever $d_\omega = d$.

3.1.1. Cohomological equations

In the process of averaging, we will deal with cohomological equations [34, Sec. 2.5] as

$$\{H, f\} = g, \tag{3.6}$$

where $g \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ is a smooth function such that $\langle g \rangle = 0$. We look for a function $f \in \mathcal{C}^\infty(\mathbb{R}^{2d})$ solving (3.6) and preserving as much as possible the smooth properties of g. For any $f \in \mathcal{C}^\infty(\mathbb{R}^{2d})$, we can write $f \circ \Phi_\tau^H$ as a Fourier series:

$$f \circ \Phi_\tau^H(x,\xi) = \sum_{k\in\mathbb{Z}^d} f_k(x,\xi)\frac{e^{ik\cdot\tau}}{(2\pi)^d}, \quad f_k(x,\xi) := \int_{\mathbb{T}^d} f \circ \Phi_\tau^H(x,\xi)e^{-ik\cdot\tau}d\tau. \tag{3.7}$$

Note that $f_k \circ \Phi_\tau^H(x,\xi) = f_k(x,\xi)e^{ik\cdot\tau}$. This combined with (3.3) gives:

$$\langle f \rangle(x,\xi) = \frac{1}{(2\pi)^d}\sum_{k\in\Lambda_\omega} f_k(x,\xi) = \int_{\mathbb{T}_\omega} f \circ \Phi_\tau^H(x,\xi)\mathfrak{h}_\omega(d\tau). \tag{3.8}$$

Observe that if f is a solution of (3.6), then so is $f+\lambda\langle f\rangle$ for any $\lambda\in\mathbb{R}$, since $\{H,\langle f\rangle\}=0$. Thus we can try to solve the equation for $\langle f\rangle=0$ fixed, imposing

$$f(x,\xi)=\frac{1}{(2\pi)^{2d}}\sum_{k\in\mathbb{Z}^d\setminus\Lambda_\omega} f_k(x,\xi).$$

Writing down

$$\{H,f\}=\frac{d}{dt}\left(f\circ\Phi^H_{t\omega}\right)|_{t=0}=\frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d\setminus\Lambda_\omega} ik\cdot\omega\, f_k=\frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d\setminus\Lambda_\omega} g_k,$$

we obtain that the solution of (3.6) is given (at least formally) by

$$f(x,\xi)=\frac{1}{(2\pi)^d}\sum_{k\in\mathbb{Z}^d\setminus\Lambda_\omega}\frac{1}{ik\cdot\omega}\, g_k(x,\xi). \tag{3.9}$$

It is not difficult to see that, unless we impose some quantitive restriction on how fast $|k\cdot\omega|^{-1}$ can grow, the solutions given formally by (3.9) may fail to be even distributions (see for instance [34, Ex. 2.16.]). But if ω is partially Diophantine, and $g\in\mathcal{C}^\infty(\mathbb{R}^{2d})$ is such that $\langle g\rangle=0$, then (3.9) defines a smooth solution $f\in\mathcal{C}^\infty(\mathbb{R}^{2d})$ of (3.6).

Finally, the following lemma gives a simpler formula for the solutions of (3.6) in the periodic case. We assume for symplicity $\omega=(1,\dots,1)$.

Lemma 3.1. *If $\omega=(1,\dots,1)$, then the solution of the cohomological equation* (3.6) *is given by the explicit formula*

$$f=\frac{-1}{2\pi}\int_0^{2\pi}\int_0^t g\circ\phi^H_s\,ds\,dt, \tag{3.10}$$

provided that $\langle f\rangle=\langle g\rangle=0$.

Proof. From the identity

$$\frac{d}{dt}(g\circ\phi^H_t)=\{H,g\}\circ\phi^H_t,$$

we have that

$$g\circ\phi^H_t-g=\{H,\int_0^t g\circ\phi^H_s ds\}.$$

Then,

$$\langle g\rangle-g=\frac{1}{2\pi}\int_0^{2\pi}(g\circ\phi^H_t-g)\,dt=\frac{1}{2\pi}\int_0^{2\pi}\{H,\int_0^t g\circ\phi^H_s ds\}\,dt,$$

and therefore the solution f of (3.6) is given by

$$f=-\frac{1}{2\pi}\int_0^{2\pi}\int_0^t g\circ\phi^H_s\,ds\,dt, \tag{3.11}$$

provided that $\langle g\rangle=\langle f\rangle=0$. ☐

3.2. THE AVERAGING METHOD

The averaging method is a technique arising in classical and quantum mechanics (see Moser [90], Weinstein [115], Guillemin [49], Arnold [15], Uribe [114], Colin de Verdière [31]), that appears in the analysis of perturbed completely integrable systems. Roughly speaking, it consists in averaging the perturbation along the orbits of the completely integrable system to a given order. This section is devoted to prove the following:

Proposition 3.1. *If ω is partially Diophantine then, for every $N \in \mathbb{N}$, there exists a sequence of unitary operators $(U_{N,\hbar})$ on $L^2(\mathbb{R}^d)$ such that*

$$\widehat{P}_\hbar^N := U_{N,\hbar}^*\big(\widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar\big)U_{N,\hbar} = \widehat{H}_\hbar + \sum_{j=1}^{N} \varepsilon_\hbar^j \langle \widehat{R}_{j,\hbar}\rangle + O_{\mathcal{L}(L^2)}(\varepsilon_\hbar^{N+1}), \tag{3.12}$$

where $\widehat{R}_{1,\hbar} = \widehat{V}_\hbar$, and $\widehat{R}_{j,\hbar}$ are L^2-bounded pseudodifferential operators that do not depend on N.

Moreover,

$$\big\|U_{N,\hbar}\operatorname{Op}_\hbar(a)U_{N,\hbar}^* - \operatorname{Op}_\hbar(a)\big\|_{\mathcal{L}(L^2)} = O_{\mathcal{L}(L^2)}(\varepsilon_\hbar), \quad \textit{for all } a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d}). \tag{3.13}$$

Proof. We fix $N \geq 1$ arbitrary. Consider $F_1 \in S^0(\mathbb{R}^{2d})$ to be chosen later, and denote its Weyl quantization by $\widehat{F}_{1,\hbar} = \operatorname{Op}_\hbar(F_1)$. We define the following unitary operator:

$$\mathcal{U}_1(t) := \exp\left[-\frac{it\varepsilon_\hbar}{\hbar}\widehat{F}_{1,\hbar}\right] = \sum_{j=0}^{\infty}\frac{1}{j!}\left(-\frac{it\varepsilon_\hbar}{\hbar}\widehat{F}_{1,\hbar}\right)^j, \quad t \in [0,1], \tag{3.14}$$

where the series converges in the $\mathcal{L}(L^2)$-norm provided that $\widehat{F}_{1,\hbar}$ is a bounded operator on $L^2(\mathbb{R}^d)$. We denote $\mathcal{U}_1 = \mathcal{U}_1(1)$ and conjugate $\widehat{P}_\hbar = \widehat{P}_\hbar^0 := \widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar$ by $\mathcal{U}_1$, obtaining:

$$\begin{aligned}\widehat{P}_\hbar^1 := \mathcal{U}_1^*\,\widehat{P}_\hbar^0\,\mathcal{U}_1 &= \widehat{H}_\hbar + \varepsilon_\hbar\widehat{V}_\hbar + \sum_{j=1}^{N}\frac{\varepsilon_\hbar^j}{j!}\left(\frac{i}{\hbar}\right)^j \operatorname{Ad}^j_{\widehat{F}_{1,\hbar}}(\widehat{H}_\hbar)\\ &+ \sum_{j=1}^{N-1}\frac{\varepsilon_\hbar^{j+1}}{j!}\left(\frac{i}{\hbar}\right)^j \operatorname{Ad}^j_{\widehat{F}_{1,\hbar}}(\widehat{V}_\hbar) + \varepsilon_\hbar^{N+1}\widehat{T}_\hbar,\end{aligned}$$

where $\operatorname{Ad}_P^j(Q) := [P, \operatorname{Ad}_P^{j-1}(Q)]$, $\operatorname{Ad}_P^0(Q) = Q$, and the Taylor reminder $\widehat{T}_\hbar$ is given by

$$\begin{aligned}\widehat{T}_\hbar &= \int_0^1 \frac{(1-t)^N}{N!}\left(\frac{i}{\hbar}\right)^{N+1}\mathcal{U}_1(t)^*\operatorname{Ad}^{N+1}_{\widehat{F}_{1,\hbar}}(\widehat{H}_\hbar)\mathcal{U}_1(t)dt\\ &+ \int_0^1 \frac{(1-t)^{N-1}}{(N-1)!}\left(\frac{i}{\hbar}\right)^{N}\mathcal{U}_1(t)^*\operatorname{Ad}^{N}_{\widehat{F}_{1,\hbar}}(\widehat{V}_\hbar)\mathcal{U}_1(t)dt.\end{aligned}$$

We set F_1 to be the solution of the cohomological equation

$$\frac{i}{\hbar}[\widehat{F}_{1,\hbar}, \widehat{H}_\hbar] = \langle \widehat{V}_\hbar \rangle - \widehat{V}_\hbar, \tag{3.15}$$

where the quantum average $\langle \widehat{V}_\hbar \rangle$ (recall Proposition 1.1) is given by

$$\langle \widehat{V}_\hbar \rangle = \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{-i\frac{t}{\hbar}\widehat{H}_\hbar}\, \widehat{V}_\hbar\, e^{i\frac{t}{\hbar}\widehat{H}_\hbar} dt = \mathrm{Op}_\hbar(\langle V \rangle).$$

Note that the commutator in the left is equal to $\mathrm{Op}_\hbar(\{F_1, H\})$, since H is a polynomial of degree two, hence equation 3.15 at symbol level is just

$$\{H, F_1\} = V - \langle V \rangle, \tag{3.16}$$

and we can find the solution $F_1 \in S^0(\mathbb{R}^{2d})$ with $\langle F_1 \rangle = 0$ provided that ω is partially Diophantine (see Section 3.1.1). Precisely,

$$F_1 = V^\sharp = \frac{1}{(2\pi)^d} \sum_{k\in\mathbb{Z}^d\setminus\Lambda_\omega} \frac{1}{ik\cdot\omega} V_k, \quad V_k := \int_{\mathbb{T}^d} V \circ \Phi_\tau^H e^{-ik\cdot\tau} d\tau. \tag{3.17}$$

In the periodic case, $\omega = (1, \dots, 1)$, F_1 has the simpler expression

$$F_1 = -\frac{1}{2\pi} \int_0^{2\pi} \int_0^t (V - \langle V \rangle) \circ \phi_s^H \, ds\, dt, \tag{3.18}$$

given by Lemma 3.1. Thus

$$\widehat{P}_\hbar^1 = \mathcal{U}_1^* \, \widehat{P}_\hbar^0 \mathcal{U}_1 = \widehat{H}_\hbar + \varepsilon_\hbar \langle \widehat{V}_\hbar \rangle + \sum_{j=2}^N \varepsilon_\hbar^j \widehat{R}_{j,\hbar}^1 + \varepsilon_\hbar^{N+1} \widehat{T}_\hbar. \tag{3.19}$$

The remainder terms $\widehat{R}_{j,\hbar}^1$ in (3.19) can be computed explicitely:

$$\widehat{R}_{j,\hbar}^1 = \frac{1}{j!} \left(\frac{i}{\hbar}\right)^{j-1} \mathrm{Ad}_{\widehat{F}_{1,\hbar}}^{j-1} \left(\langle \widehat{V}_\hbar \rangle + (j-1)\widehat{V}_\hbar \right), \quad j = 2, \dots, N. \tag{3.20}$$

Using equation (3.15), one can also deduce the following formula for $\widehat{T}_\hbar$:

$$\widehat{T}_\hbar = \int_0^1 \frac{(1-t)^{N-1}}{N!} \left(\frac{i}{\hbar}\right)^N \mathcal{U}_1(t)^* \, \mathrm{Ad}_{\widehat{F}_{1,\hbar}}^N \Big((1-t)\langle \widehat{V}_\hbar \rangle + (N-1-t)\widehat{V}_\hbar \Big) \mathcal{U}_1(t) dt.$$

Moreover, by the symbolic calculus for Weyl pseudodifferential operators, the Calderón-Vaillancourt theorem and the fact that $\mathcal{U}_1(t)$ is unitary, we have $\widehat{T}_\hbar = O_{\mathcal{L}(L^2)}(1)$.

We next proceed to show the induction step. Assume that we have constructed the Hamiltonian (3.12) until order $1 \le k-1 \le N-1$. More precisely, suppose there exist $k-1$ unitary operators

$$\mathcal{U}_j := \exp\left[-\frac{i\varepsilon_\hbar^j}{\hbar}\widehat{F}_{j,\hbar}\right], \quad j = 1, \ldots, k-1,$$

such that

$$\widehat{P}_\hbar^{k-1} := \mathcal{U}_{k-1}^* \cdots \mathcal{U}_1^* \, \widehat{P}_\hbar^0 \, \mathcal{U}_1 \cdots \mathcal{U}_{k-1} = \widehat{H}_\hbar + \sum_{j=1}^{k-1} \varepsilon_\hbar^j \langle \widehat{R}_{j,\hbar}^{k-1} \rangle + \sum_{j=k}^{N} \varepsilon_\hbar^j \widehat{R}_{j,\hbar}^{k-1} + O(\varepsilon_\hbar^{N+1}).$$

We next set a unitary operator

$$\mathcal{U}_k := \exp\left[-\frac{i\varepsilon_\hbar^k}{\hbar}\widehat{F}_{k,\hbar}\right]$$

so that

$$\widehat{P}_\hbar^k := \mathcal{U}_k^* \, \widehat{P}_\hbar^{k-1} \, \mathcal{U}_k = \widehat{H}_\hbar + \sum_{j=1}^{k} \varepsilon_\hbar^j \langle \widehat{R}_{j,\hbar}^{k} \rangle + \sum_{j=k+1}^{N} \varepsilon_\hbar^j \widehat{R}_{j,\hbar}^{k} + O(\varepsilon_\hbar^{N+1}).$$

First, expanding the left hand side, we have:

$$\begin{aligned}
\widehat{P}_\hbar^k = {} & \widehat{H}_\hbar + \sum_{j=1}^{k-1} \varepsilon_\hbar^j \langle \widehat{R}_{j,\hbar}^{k-1} \rangle + \sum_{j=k}^{N} \varepsilon_\hbar^j \widehat{R}_{j,\hbar}^{k-1} \\
& + \sum_{\substack{1 \le l \\ lk \le N}} \varepsilon_\hbar^{lk} \frac{1}{l!} \operatorname{Ad}_{\widehat{F}_{k,\hbar}}^l (\widehat{H}_\hbar) \\
& + \sum_{j=1}^{k-1} \sum_{\substack{1 \le l \\ lk+j \le N}} \varepsilon_\hbar^{lk+j} \frac{1}{l!} \left(\frac{i}{\hbar}\right)^l \operatorname{Ad}_{\widehat{F}_{k,\hbar}}^l (\langle \widehat{R}_{j,\hbar}^{k-1} \rangle) \\
& + \sum_{j=k}^{N} \sum_{\substack{1 \le l \\ lk+j \le N}} \varepsilon_\hbar^{lk+j} \frac{1}{l!} \left(\frac{i}{\hbar}\right)^l \operatorname{Ad}_{\widehat{F}_{k,\hbar}}^l (\widehat{R}_{j,\hbar}^{k-1}) + O(\varepsilon_\hbar^{N+1}).
\end{aligned}$$

The following conditions are sufficient to prove the induction step:

$$\widehat{R}_{j,\hbar}^k = \widehat{R}_{j,\hbar}^{k-1}, \quad \text{for } 1 \le j \le k; \tag{3.21}$$

$$\frac{i}{\hbar}[\widehat{F}_{k,\hbar}, \widehat{H}_\hbar] = \langle \widehat{R}_{k,\hbar}^k \rangle - \widehat{R}_{k,\hbar}^k. \tag{3.22}$$

Note that the cohomological equation (3.22) can be solved using Section 3.1.1 due to the fact that the commutator in the left is exact since H is a polynomial of degree two and then the equation reduces to its classical counterpart for the whole symbol $R^k_{k,\hbar}$. Moreover, for every $k+1 \leq j \leq N$, the remainder terms $\widehat{R}^k_{j,\hbar}$ are given by:

(a) If $j = k + j'$, for $1 \leq j' \leq k-1$:

$$\widehat{R}^k_{j,\hbar} = \widehat{R}^{k-1}_{j,\hbar} + \frac{i}{\hbar}[\widehat{F}_{k,\hbar}, \langle \widehat{R}^{k-1}_{j',\hbar} \rangle]. \tag{3.23}$$

(b) If $j = 2k$:

$$\widehat{R}^k_{j,\hbar} = \widehat{R}^{k-1}_{j,\hbar} + \frac{i}{2\hbar}[\widehat{F}_{k,\hbar}, \langle \widehat{R}^{k-1}_{k,\hbar} \rangle + \widehat{R}^{k-1}_{k,\hbar}].$$

(c) If $j = lk + j'$, for $l \geq 2$ and $1 \leq j' \leq k-1$:

$$\begin{aligned}\widehat{R}^k_{j,\hbar} = \widehat{R}^{k-1}_{j,\hbar} &+ \frac{1}{l!}\left(\frac{i}{\hbar}\right)^l \operatorname{Ad}^l_{\widehat{F}_{k,\hbar}}(\langle \widehat{R}^{k-1}_{j',\hbar} \rangle) \\ &+ \sum_{m=1}^{l-1} \frac{1}{m!}\left(\frac{i}{\hbar}\right)^m \operatorname{Ad}^m_{\widehat{F}_{k,\hbar}}(\widehat{R}^{k-1}_{(l-m)k+j',\hbar}).\end{aligned}$$

(d) If $j = lk$, for $l \geq 3$:

$$\begin{aligned}\widehat{R}^k_{j,\hbar} = \widehat{R}^{k-1}_{j,\hbar} &+ \frac{1}{l!}\left(\frac{i}{\hbar}\right)^{l-1} \operatorname{Ad}^{l-1}_{\widehat{F}_{k,\hbar}}(\langle \widehat{R}^{k-1}_{k,\hbar} \rangle + (l-1)\widehat{R}^{k-1}_{k,\hbar}) \\ &+ \sum_{m=1}^{l-2} \frac{1}{m!}\left(\frac{i}{\hbar}\right)^m \operatorname{Ad}^m_{\widehat{F}_{k,\hbar}}(\widehat{R}^{k-1}_{(l-m)k,\hbar}).\end{aligned}$$

Therefore, the unitary operator $\widehat{U}_{N,\hbar} := \mathcal{U}_1 \cdots \mathcal{U}_N$ satisfies (3.12) with

$$\widehat{R}_{j,\hbar} := \widehat{R}^N_{j,\hbar}, \quad 1 \leq j \leq N.$$

Finally, (3.13) holds since

$$\widehat{U}_{N,\hbar} \operatorname{Op}_\hbar(a) \widehat{U}^*_{N,\hbar} = \operatorname{Op}_\hbar(a) - \frac{i\varepsilon_\hbar}{\hbar}[\widehat{F}_{1,\hbar}, \operatorname{Op}_\hbar(a)] + O_{\mathcal{L}(L^2)}(\varepsilon_\hbar^2),$$

and $[\widehat{F}_{1,\hbar}, \operatorname{Op}_\hbar(a)] = O_{\mathcal{L}(L^2)}(\hbar)$ for all $a \in \mathcal{C}^\infty_c(\mathbb{R}^{2d})$. $\square$

Remark 3.1. We have the following explicit formulas. The remainder term $\widehat{R}_{2,\hbar}$ is given by

$$\widehat{R}_{2,\hbar} = \frac{i}{2\hbar}[\widehat{F}_{1,\hbar}, \langle \widehat{V}_\hbar \rangle + \widehat{V}_\hbar], \tag{3.24}$$

and its principal symbol is, see (3.17),

$$r_{2,0} = \frac{1}{2}\{V^\sharp, \langle V \rangle + V\}. \tag{3.25}$$

3.3. Transport and invariance

In this section we prove Theorems 1.2, 1.3 and 1.4.

Proof of Theorem 1.2. Given $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$, denote $\widehat{A}_\hbar := \mathrm{Op}_\hbar(a)$. Let $\langle \widehat{A}_\hbar \rangle$ be the quantum average of $\widehat{A}_\hbar$ given by (1.42), the Wigner distribution $W_\hbar^{\tau,\varepsilon}(t)$ satisfies the differential equation

$$\frac{d}{dt} W_\hbar^{\tau,\varepsilon}(t)(\langle a \rangle) = \frac{i\tau_\hbar}{\hbar} \left\langle [\widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar, \langle \widehat{A}_\hbar \rangle] v_\hbar(t\tau_\hbar), v_\hbar(t\tau_\hbar) \right\rangle_{L^2}, \quad t \in \mathbb{R}. \tag{3.26}$$

Since

$$[\widehat{H}_\hbar, \langle \widehat{A}_\hbar \rangle] = \frac{\hbar}{i} \mathrm{Op}_\hbar(\{H, \langle a \rangle\}) = 0,$$

we can use the commutator rule for Weyl pseudodifferential operators to obtain

$$\frac{i\tau_\hbar}{\hbar}[\widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar, \langle \widehat{A}_\hbar \rangle] = \tau_\hbar \varepsilon_\hbar \mathrm{Op}_\hbar(\{V, \langle a \rangle\}) + O(\tau_\hbar \varepsilon_\hbar \hbar^2). \tag{3.27}$$

If $\tau_\hbar \varepsilon_\hbar \to 0$ then, after integrating both sides of (3.26) on the interval $t \in [0, t_0]$ and taking limits as $\hbar \to 0^+$, the following holds for every $t_0 \in \mathbb{R}$:

$$\mu(t_0)(\langle a \rangle) = \mu_0(\langle a \rangle).$$

Since $\mu(t_0)$ is invariant by the flow ϕ_t^H, and hence $\mu(t)(a) = \mu(t)(\langle a \rangle)$ for all t, this concludes the proof of *(i)* of Theorem 1.2.

We next prove part *(ii)*. Recall that $\{H, \langle a \rangle\}$. Integrating (3.26) on $t \in [0, t_0]$, letting $\hbar \to 0^+$, and using the pseudodifferential calculus, give that, for every $t_0 \in \mathbb{R}$:

$$\mu(t_0)(\langle a \rangle) - \mu_0(\langle a \rangle) = \int_0^{t_0} \mu(t)(\{V, \langle a \rangle\}) dt. \tag{3.28}$$

Moreover, since $\mu(t)$ is invariant by the flow ϕ_t^H we have, for every $t \in \mathbb{R}$,

$$\mu(t)(\{V, \langle a \rangle\}) = \mu(t)(\{\langle V \rangle, \langle a \rangle\}).$$

Using this and (3.28) with $a \circ \phi_{-t}^{\langle V \rangle}$ instead of a, and noting that

$$\langle a \circ \phi_{-t}^{\langle V \rangle} \rangle = \langle a \rangle \circ \phi_{-t}^{\langle V \rangle},$$

provided that $\{H, \langle V \rangle\} = 0$, we obtain

$$\frac{d}{dt}\Big(\mu(t)\big(\langle a \circ \phi_{-t}^{\langle V \rangle}\rangle\big)\Big) = \mu(t)\left(\{\langle V \rangle, \langle a \circ \phi_{-t}^{\langle V \rangle}\rangle\}\right) - \mu(t)\left(\{\langle V \rangle, \langle a \rangle\} \circ \phi_{-t}^{\langle V \rangle}\right) = 0.$$

Therefore,

$$\mu(t)(a) = \mu(t)(\langle a \rangle) = \mu_0(\langle a \rangle \circ \phi_t^{\langle V \rangle}), \quad \forall t \in \mathbb{R}, \quad \forall a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d}).$$

Since the space $\mathcal{C}_c^\infty(\mathbb{R}^{2d})$ is dense in $\mathcal{C}_c(\mathbb{R}^{2d})$, this completes the proof of point *(ii)*.

For the large regime $\tau_\hbar \varepsilon_\hbar \to +\infty$ we use integration by parts in (3.26) to show that, for every $\varphi \in \mathcal{C}_c^1(\mathbb{R})$,

$$\int_{\mathbb{R}} \varphi'(t) \left\langle \langle \widehat{A}_\hbar \rangle v_\hbar(t\tau_\hbar), v_\hbar(t\tau_\hbar) \right\rangle_{L^2} dt = -\frac{i\tau_\hbar}{\hbar} \int_{\mathbb{R}} \varphi(t) \left\langle [\widehat{P}_\hbar, \langle \widehat{A}_\hbar \rangle] v_\hbar(t\tau_\hbar), v_\hbar(t\tau_\hbar) \right\rangle_{L^2} dt.$$

By (3.27), we obtain

$$\int_{\mathbb{R}} \varphi(t) \, \langle \mathrm{Op}_\hbar(\{V, \langle a \rangle\}) v_\hbar(t\tau_\hbar), v_\hbar(t\tau_\hbar) \rangle \, dt = O\big((\tau_\hbar \varepsilon_\hbar)^{-1}\big).$$

Taking the limit $\hbar \to 0^+$ and using once more that $\mu(t)$ is invariant by ϕ_t^H show that

$$\int_{\mathbb{R}} \varphi(t) \mu(t)(\{V, \langle a \rangle\}) dt = \int_{\mathbb{R}} \varphi(t) \mu(t)(\{\langle V \rangle, \langle a \rangle\}) dt = 0.$$

By density of $\mathcal{C}_c^1(\mathbb{R})$ in $L^1(\mathbb{R})$, this concludes the proof of part *(iii)* and hence of Theorem 1.2. □

We next show a particular example of perturbation $V \in S^0(\mathbb{R}^{2d})$ such that $\langle V \rangle_\omega$ is constant but $\langle V^\eth \rangle_\omega$ is not:

Example 3.1. Assume $d = 2$ and $\omega = (1,1)$ for simplicity. We will use action-angle coordinates $(\tau, E) \in \mathbb{T}^2 \times \mathbb{R}_+^2$ defined by

$$(x, \xi) = \Phi_\tau^H(\sqrt{2E}, 0), \quad (x, \xi) \in \mathbb{R}^4, \tag{3.29}$$

where we use the notation $\sqrt{2E} = (\sqrt{2E_1}, \sqrt{2E_2})$. By (1.36), we can take V such that $V_k(x, \xi) \equiv 0$ for every $k \in \Lambda_{(1,1)}$. This ensures that $\langle V \rangle \equiv 0$. On the other hand, we have

$$V^\eth = \frac{1}{2}\{V^\sharp, V\} = \frac{1}{(2\pi)^4} \sum_{j,k \in \mathbb{Z}^2 \setminus \Lambda_{(1,1)}} \frac{1}{2i(k_1 + k_2)} \{V_k, V_j\} = \frac{1}{(2\pi)^2} \sum_{l \in \mathbb{Z}^2} V_l^\eth, \tag{3.30}$$

where

$$V_l^{\eth} = \frac{1}{(2\pi)^2} \sum_{j,l-j \in \mathbb{Z}^2 \setminus \Lambda_{(1,1)}} \frac{1}{2i(l_1 - j_1 + l_2 - j_2)} \{V_{l-j}, V_j\}, \quad l \in \mathbb{Z}^2. \tag{3.31}$$

Thus

$$\langle V^{\eth} \rangle = \frac{1}{(2\pi)^2} \sum_{l \in \Lambda_{(1,1)}} V_l^{\eth}, \tag{3.32}$$

and we still have a lot of freedom to choose the coefficients V_j for $j \notin \Lambda_{(1,1)}$. Indeed, let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^2_+)$ be not identically zero in $\mathcal{L}_\omega^{-1}(1)$, and assume that the support of χ is contained in the open set

$$\Omega_\varepsilon := \{E = (E_1, E_2) \in \mathbb{R}^2_+ \, : \, E_1 \cdot E_2 > \varepsilon\}, \quad \varepsilon > 0.$$

We define, for every $r \in \mathbb{R}^2_+$,

$$V_j(\sqrt{2E}, 0) := \begin{cases} \chi(E), & \text{if } j \in \{(1,0), (-1,0)\}; \\ 0, & \text{otherwise.} \end{cases}$$

Note that, since the change to action-angle coordinates (3.29) is a canonical transformation,

$$\{V_{(\pm 1,0)} \circ \Phi_\tau^H(\sqrt{2E}, 0), V_{(\mp 1,0)} \circ \Phi_\tau^H(\sqrt{2E}, 0)\} = \{\chi(E) e^{\pm i\tau_1}, \chi(r) e^{\mp i \tau_1}\} = \mp 2i \chi(E) \cdot \partial_{E_1} \chi(E),$$

and hence

$$\begin{aligned} V^{\eth}_{(0,0)}(\sqrt{2E}, 0) &= \frac{1}{2i} \{\chi(E) e^{i\tau_1}, \chi(r) e^{-i\tau_1}\} + \frac{1}{-2i} \{\chi(E) e^{-i\tau_1}, \chi(E) e^{i\tau_1}\} \\ &= -2\chi(E) \cdot \partial_{E_1} \chi(E). \end{aligned}$$

Therefore, we have $\langle V \rangle \equiv 0$ and

$$\langle V^{\eth} \rangle \circ \Phi_\tau^H(\sqrt{2E}, 0) = V^{\eth}_{(0,0)}(\sqrt{2E}, 0) = -2\chi(E) \cdot \partial_{E_1} \chi(E),$$

which satisfies that $\langle V^{\eth} \rangle_\omega$ is not identically constant provided that the gradient $\partial_E(\chi \cdot \partial_{E_1} \chi)(E)$ is not proportional to $\omega = (1,1)$ for some $E \in \mathcal{L}_\omega^{-1}(1)$ (recall that, in action-angle coordinates, $H \circ \Phi_\tau^H(\sqrt{2E}, 0) = \omega \cdot E$, hence $X_H = (\omega, 0)$).

Proof of Theorems 1.3 and 1.4. Let $N \geq 1$, by Lemma 3.1, there exists a sequence of unitary operators $(U_{N,\hbar})$ on $L^2(\mathbb{R}^d)$ such that

$$\widehat{P}_\hbar^N = U_{N,\hbar}^* \big(\widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar \big) U_{N,\hbar} = \widehat{H}_\hbar + \varepsilon_\hbar \langle \widehat{V}_\hbar \rangle + \sum_{j=2}^N \varepsilon_\hbar^j \langle \widehat{R}_{j,\hbar} \rangle + O(\varepsilon_\hbar^{N+1}).$$

Let $R_j(\hbar)$ be the full symbol of $\widehat{R}_{j,\hbar}$ (recall Lemma 2.4) expanded as

$$R_j(\hbar) \sim \sum_{k=0}^{\infty} r_{j,k}\,\hbar^k.$$

The symbol $L = L(V)$ is obtained as the sum of all terms $r_{j,k}$ in the series such that

$$\langle L\rangle_\omega = \sum_{\delta_\hbar=\varepsilon_\hbar^j\hbar^k} \langle r_{j,k}\rangle_\omega \tag{3.33}$$

is not constant, and such that the order $\delta_\hbar$ is maximal with respect to this condition. In the hypothesis of Theorem 1.3, by Remark 3.1,

$$L(V) = r_{2,0} = \frac{1}{2}\{V^\sharp, \langle V\rangle + V\}.$$

Note, in particular, that

$$\langle r_{2,0}\rangle = \langle V^\eth\rangle = \frac{1}{2}\langle\{V^\sharp, V\}\rangle.$$

Using (3.13), we have

$$\begin{aligned}
W_\hbar^{\tau,\varepsilon}(t)(a) &= \left\langle v_\hbar(t\tau_\hbar), U_{N,\hbar}\operatorname{Op}_\hbar(a)U_{N,\hbar}^* v_\hbar(t\tau_\hbar)\right\rangle_{L^2} + O(\varepsilon_\hbar)\\
&= \left\langle U_{N,\hbar}^* v_\hbar(t\tau_\hbar), \operatorname{Op}_\hbar(a)U_{N,\hbar}^*\, v_\hbar(t\tau_\hbar)\right\rangle_{L^2} + O(\varepsilon_\hbar)\\
&= \widetilde{W}_\hbar^{\tau,\varepsilon}(t)(a) + O(\varepsilon_\hbar),
\end{aligned}$$

where $\widetilde{W}_\hbar^{\tau,\varepsilon}$ is the Wigner distribution associated to the Schrödinger equation

$$i\hbar\partial_t\, v_\hbar'(t) = \widehat{P}_\hbar^N\, v_\hbar'(t), \quad v_\hbar'|_{t=0} = u_\hbar' = U_{N,\hbar}^* u_\hbar.$$

Taking limits as $\hbar \to 0^+$ it follows that both distributions $W_\hbar^{\tau,\varepsilon}$ and $\widetilde{W}_\hbar^{\tau,\varepsilon}$ converge to the same semiclassical measure μ.

Now, take $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$, and denote $\widehat{A}_\hbar := \operatorname{Op}_\hbar(a)$. The following Wigner equation holds:

$$\frac{d}{dt}\widetilde{W}_\hbar^{\tau,\varepsilon}(t)(\langle a\rangle) = \frac{i\tau_\hbar}{\hbar}\left\langle [\widehat{P}_\hbar^N, \langle \widehat{A}_\hbar\rangle] v_\hbar'(t\tau_\hbar), v_\hbar'(t\tau_\hbar)\right\rangle_{L^2}. \tag{3.34}$$

Using the pseudodifferential calculus and (3.33), we obtain:

$$\frac{i\tau_\hbar}{\hbar}[\widehat{P}_\hbar^N, \langle\widehat{A}_\hbar\rangle] = \tau_\hbar\varepsilon_\hbar^j\hbar^k \operatorname{Op}_\hbar(\{\langle L\rangle, \langle a\rangle\}) + o(\tau_\hbar\varepsilon_\hbar^j\hbar^k). \tag{3.35}$$

From this point, the proof of points *(i), (ii) and (iii)* follows the same lines of the proof of Theorem 1.2, replacing the critical scale $\tau_\hbar \sim 1/\varepsilon_\hbar$ by $\tau_\hbar \sim 1/(\varepsilon_\hbar^j \hbar^k)$.

It remains to prove the assertion of Remark 1.5. Note that

$$\begin{aligned} V^\sharp \circ \phi_t^H - V^\sharp &= \int_0^t \frac{d}{ds}(V \circ \phi_s^H) ds = \int_0^t \{H, V^\sharp\} \circ \phi_s^H ds = \int_0^t (V - \langle V \rangle) \circ \phi_s^H ds \\ &= -t\langle V \rangle + \int_0^t V \circ \phi_s^H ds. \end{aligned}$$

Thus,

$$\begin{aligned} \langle V^\eth \rangle &= \frac{1}{2} \langle \{V^\sharp, V\} \rangle \\ &= \frac{1}{4\pi} \int_0^{2\pi} \{V^\sharp \circ \phi_t^H, V \circ \phi_t^H\} dt \\ &= \frac{1}{4\pi} \int_0^{2\pi} \{V^\sharp - t\langle V \rangle + \int_0^t V \circ \phi_s^H ds, V \circ \phi_t^H\} dt \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_0^t \{V \circ \phi_s^H, V \circ \phi_t^H\} \, ds dt. \end{aligned}$$

This concludes the proof. □

We next show an example of perturbation $V \in S^0(\mathbb{R}^{2d})$ such that $\langle V \rangle_\omega$ and $\langle V^\eth \rangle_\omega \equiv 0$ but $\langle L \rangle_\omega$ is not identically constant:

Example 3.2. Assume that $\varepsilon_\hbar \gg \hbar^{1/2}$, hence the first term of largest order after $r_{2,0}$ is $r_{3,0}$, which has order $\varepsilon_\hbar^3$. By (3.20), (3.21) and (3.23), we have

$$r_{3,0} = \frac{1}{3!} \{V^\sharp, \{V^\sharp, \langle V \rangle + 2V\}\} + \{F_{2,0}, \langle V \rangle\},$$

where $F_{2,0}$ solves the cohomological equation $\{H, F_{2,0}\} = r_{2,0} - \langle r_{2,0} \rangle$ and $\langle F_{2,0} \rangle = 0$. Thus

$$\langle r_{3,0} \rangle = \frac{1}{3!} \langle \{V^\sharp, \{V^\sharp, \langle V \rangle + 2V\}\} \rangle.$$

Let $\chi \in \mathcal{C}_c^\infty(\mathbb{R}_+^2)$ as in Remark 3.1, we set, for every $E \in \mathbb{R}_+^2$,

$$V_j(\sqrt{2E}, 0) := \begin{cases} \chi(E), & \text{if } j \in J := \{(1,0), (2,0), (-3,0)\}; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $\langle V \rangle = \langle V^{\eth} \rangle \equiv 0$. Then

$$\langle r_{3,0} \rangle = \frac{1}{3} \langle \{V^\sharp, \{V^\sharp, V\}\} \rangle.$$

On the other hand, we define the set

$$J \uplus J := \{j + k \,:\, j, k \in J \,,\, j \neq k\} = \{(3,0), (-2,0), (-1,0)\},$$

and notice that, for every $l \in J \uplus J$, there exists a unique par $j^{(l)}, k^{(l)} \in J$ such that $j^{(l)} + k^{(l)} = l$. Therefore, by (3.30) and (3.31),

$$\begin{aligned} \{V^\sharp, V\}_l \circ \Phi_\tau^H(\sqrt{2E}, 0) &= \frac{1}{(2\pi)^2} \left(\frac{1}{2ij_1^{(l)}} - \frac{1}{2ik_1^{(l)}} \right) \{\chi(E) e^{ij_1^{(l)} \tau_1}, \chi(E) e^{ik_1^{(l)} \tau_1}\} \\ &= \frac{C_l}{(2\pi)^2} \chi(E) \cdot \partial_{E_1} \chi(E) e^{il\cdot\tau}, \end{aligned}$$

for every $l \in J \uplus J = \{(3,0), (-2,0), (-1,0)\}$, where

$$C_l = \frac{(j_1^{(l)} - k_1^{(l)})^2}{2j_1^{(l)} k_1^{(l)}},$$

while $\{V^\sharp, V\}_l = 0$ for $l \in \mathbb{Z}^2 \setminus (J \uplus J)$. In particular, this and (3.32) show that $\langle V^{\eth} \rangle \equiv 0$. Finally, since

$$\begin{aligned} \{V^\sharp, \{V^\sharp, V\}\} \circ \Phi_\tau^H(\sqrt{2E}, 0) &= \frac{1}{(2\pi)^4} \sum_{l,k \in \mathbb{Z}^d \setminus \Lambda_{(1,1)}} \frac{1}{i(k_1 + k_2)} \{V_k, \{V^\sharp, V\}_l\} \circ \Phi_\tau^H(\sqrt{2E}, 0) \\ &= \frac{1}{(2\pi)^6} \sum_{k \in J} \sum_{l \in J \uplus J} \frac{C_l}{ik_1} \{\chi(E) e^{ik_1 \tau_1}, \chi(E) \cdot \partial_{E_1} \chi(E) e^{il_1 \tau_1}\}, \end{aligned}$$

we obtain

$$\begin{aligned} \langle r_{3,0} \rangle \circ \Phi_\tau^H(\sqrt{2E}, 0) &= \frac{1}{3} \langle \{V^\sharp, \{V^\sharp, V\}\} \rangle \circ \Phi_\tau^H(\sqrt{2E}, 0) \\ &= \frac{\chi(E)}{3(2\pi)^6} \big(2\partial_{E_1} \chi(E)^2 + \chi(E) \partial_{E_1}^2 \chi(E)\big) \sum_{l \in J \uplus J} C_l \\ &= \frac{-3\chi(E)}{(2\pi)^6} \big(2\partial_{E_1} \chi(E)^2 + \chi(E) \partial_{E_1}^2 \chi(E)\big). \end{aligned}$$

Then $L = r_{3,0}$ satisfies that $\langle L \rangle_\omega$ is not identically constant provided that the gradient vector field $\partial_E \big(2\chi \cdot (\partial_{E_1} \chi)^2 + \chi^2 \cdot \partial_{E_1}^2 \chi\big)(E)$ is not proportional to $\omega = (1,1)$ for some $E \in \mathcal{L}_\omega^{-1}(1)$.

Proof of Theorem 1.5. First, we apply Lemma 3.1 to conjugate the Hamiltonian $\widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar$ into $\widehat{P}_\hbar^N$ given by (3.12). Recall, from Section 3.1, that the average of a symbol a is given by

$$\langle a \rangle(x,\xi) = \int_{\mathbb{T}_\omega} a \circ \Phi_\tau^H \, \mathfrak{h}_\omega(d\tau).$$

Since $\operatorname{rk} \Lambda_\omega = 0$, it follows that $\Lambda_\omega = \{0\}$ and then $\mathbb{T}_\omega = \mathbb{T}^d$. We can rewrite the average of a as

$$\langle a \rangle(x,\xi) = \mathcal{I}_{\langle a \rangle}(H_1, \dots, H_d)(x,\xi),$$

where $\mathcal{I}_{\langle a \rangle} \in \mathcal{C}^\infty(\mathcal{L}_\omega^{-1}(1))$ is defined by

$$\mathcal{I}_{\langle a \rangle}(E) = \langle a \rangle(x,\xi), \quad \forall (x,\xi) \in \mathbb{T}_E.$$

By the Whitney extension theorem, there exists an extension of $\mathcal{I}_{\langle a \rangle}$ to a smooth function $\mathcal{I}_{\langle a \rangle} \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then, using the functional calculus for semiclassical pseudodifferential operators (see for instance [35, Chp. 8]), the following holds:

$$\langle \widehat{A}_\hbar \rangle = \operatorname{Op}_\hbar(\langle a \rangle) = \mathcal{I}_{\langle a \rangle}(\operatorname{Op}_\hbar(H_1), \dots, \operatorname{Op}_\hbar(H_d)) + O(\hbar).$$

Then

$$\Big\langle \langle \widehat{A}_\hbar \rangle v_\hbar, v_\hbar \Big\rangle_{L^2} = \big\langle \mathcal{I}_{\langle a \rangle}(\operatorname{Op}_\hbar(H_1), \dots, \operatorname{Op}_\hbar(H_d)) \, v_\hbar, v_h \big\rangle_{L^2} + O(\hbar).$$

On the other hand,

$$[\langle \widehat{R}_{j,\hbar} \rangle, \operatorname{Op}_\hbar(H_k)] = \frac{\hbar}{i} \operatorname{Op}_\hbar(\{\langle R_j(\hbar) \rangle, H_k\}) = 0, \quad k = 1, \dots, d.$$

Therefore, the following identity holds:

$$[\langle \widehat{R}_{j,\hbar} \rangle, \mathcal{I}_{\langle a \rangle}(\operatorname{Op}_\hbar(H_1), \dots, \operatorname{Op}_\hbar(H_d))] = 0.$$

In view of this, we have that

$$\frac{i\tau_\hbar}{\hbar}[\widehat{P}_\hbar^N, \mathcal{I}_{\langle a \rangle}(\operatorname{Op}_\hbar(H_1), \dots, \operatorname{Op}_\hbar(H_d))] = O(\tau_\hbar \varepsilon_\hbar^N).$$

Since $\tau_\hbar \varepsilon_\hbar^N \to 0$ as $\hbar \to 0$, from the Wigner equation (3.34) with $\mathcal{I}_{\langle a \rangle}(\operatorname{Op}_\hbar(H_1), \dots, \operatorname{Op}_\hbar(H_d))$ instead of $\langle \widehat{A}_\hbar \rangle$, by the same argument used to prove part *(i)* of Theorem 1.2, we deduce that in this regime

$$\mu(t)(a) = \mu_0(\langle a \rangle),$$

for all t and all $a \in \mathcal{C}_c(\mathbb{R}^{2d})$, which concludes the proof of the first part of Theorem 1.5.

For the second part we reason as follows. The disintegration Theorem gives, for $\mathscr{H}_*\mu_0$-a.e. $E \in \mathcal{L}_\omega^{-1}(1)$, the existence of a family of probability measures $\mu_E(x,\xi)$ supported on $\mathbb{T}_E$ such that

$$\int_{\mathbb{R}^{2d}} \langle a\rangle(x,\xi)\mu_0(dx,d\xi) = \int_{\mathcal{L}_\omega^{-1}(1)} \int_{\mathbb{T}_E} \langle a\rangle(x,\xi)\mu_E(dx,d\xi)\mathscr{H}_*\mu_0(E). \tag{3.36}$$

Since

$$\langle a\rangle(x,\xi) = \mathcal{I}_{\langle a\rangle}(H_1,\dots,H_d)(x,\xi),$$

we have

$$\begin{aligned}
\int_{\mathbb{R}^{2d}} a(x,\xi)\mu_0(dx,d\xi) &= \int_{\mathcal{L}_\omega^{-1}(1)} \int_{\mathbb{T}_E} \langle a\rangle(x,\xi)\mu_E(dx,d\xi)\, \mathscr{H}_*\mu_0(dE) \\
&= \int_{\mathcal{L}_\omega^{-1}(1)} \int_{\mathbb{T}_E} \mathcal{I}_{\langle a\rangle}(H_1,\dots,H_d)(x,\xi)\mu_E(dx,d\xi)\, \mathscr{H}_*\mu_0(dE) \\
&= \int_{\mathcal{L}_\omega^{-1}(1)} \mathcal{I}_{\langle a\rangle}(E) \int_{\mathbb{T}_E} \mu_E(dx,d\xi)\, \mathscr{H}_*\mu_0(dE) \\
&= \int_{\mathcal{L}_\omega^{-1}(1)} \mathcal{I}_{\langle a\rangle}(E)\, \mathscr{H}_*\mu_0(dE),
\end{aligned}$$

since the μ_E are probability measures. □

3.3.1. THE 2D CASE

In this section we prove Theorem 1.6.

Proof of Theorem 1.6. According to [50, Sect. 3.], there exists a symplectomorphism $\kappa : \mathcal{V} \to \mathcal{V}$ such that:

$$(H, \langle V\rangle) \circ \kappa = (H, G_2(H_1, H_2)). \tag{3.37}$$

This implies that the following diagram is commutative for every $\lambda \in I$:

$$\begin{array}{ccc}
H^{-1}(\lambda) & \xrightarrow{\ \kappa\ } & H^{-1}(\lambda) \\
{\scriptstyle \pi_\lambda}\big\downarrow & & \big\downarrow{\scriptstyle \pi_\lambda} \\
\mathbb{S}^2_\lambda & \xrightarrow{\ \kappa_\lambda\ } & \mathbb{S}^2_\lambda
\end{array}$$

and then

$$\langle V\rangle_\lambda \circ \kappa_\lambda = G_2(H_1, H_2)_\lambda. \tag{3.38}$$

Since $\langle V\rangle_\lambda$ is a perfect Morse function, so is $G_2(H_1,H_2)_\lambda$, and then, for every $\lambda\in I$, the only two orbits of the flow ϕ_t^H contained in $H^{-1}(\lambda)$ that are invariant by the flow $\phi_t^{G_2(H_1,H_2)}$ are the two orbits:

$$\gamma_j(\lambda)=H_j^{-1}(0)\cap H^{-1}(\lambda),\quad j=1,2.$$

These orbits correspond to the two critical values of $\langle V\rangle_\lambda$. As a consequence, since the Hamiltonian vector fields $X_{G_2(H_1,H_2)}$ and X_H satisfy

$$X_{G_2(H_1,H_2)}=\nabla G_2\cdot(X_{H_1}+X_{H_2}),$$
$$X_H=\omega\cdot(X_{H_1}+X_{H_2}),$$

then $\nabla G_2|_{(E_1,E_2)}$ is not proportional to $\omega=(1,1)$ for every $E\in\{E\in\mathcal{L}_\omega^{-1}(\lambda)\ :\ E_1\cdot E_2\neq 0\}$. Then:

$$\lim_{T\to\infty}\int_0^T\langle a\rangle\circ\phi_t^{G_2(H_1,H_2)}dt=\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}a\circ\Phi_\tau^H d\tau=\mathcal{A}_{(H_1,H_2)}(a),\quad a\in\mathcal{C}_c^\infty(\mathcal{V}). \tag{3.39}$$

Now we proceed as in the proofs of Theorems 1.3 and 1.4. Let $\widetilde{W}_\hbar^{\tau,\varepsilon}(t)$ be the Wigner distribution acting as

$$\widetilde{W}_\hbar^{\tau,\varepsilon}(t)(a):=\big\langle\mathcal{F}_\hbar^*v_\hbar(t\tau_\hbar),\mathrm{Op}_\hbar(a)\mathcal{F}_\hbar^*v_\hbar(t\tau_\hbar)\big\rangle_{L^2}.$$

By (1.44), $\widetilde{W}_\hbar^{\tau,\varepsilon}(t)$ converges weakly, modulo a subsequence, to the semiclassical measure $\kappa_*\mu(t)$. Using the Wigner equation

$$\frac{1}{\tau_\hbar}\frac{d}{dt}\widetilde{W}_\hbar^{\tau,\varepsilon}(t)(a)=\frac{i}{\hbar}\big\langle[\widehat{P}_\hbar^N,\mathrm{Op}_\hbar(a)]\mathcal{F}_\hbar^*v_\hbar(t\tau_\hbar),\mathcal{F}_\hbar^*v_\hbar(t\tau_\hbar)\big\rangle_{L^2},$$

and taking limits as $\hbar\to 0^+$, we obtain that $\kappa_*\mu$ is invariant by the flows generated by H and $G_2(H_1,H_2)$ in the regime $\tau_\hbar\hbar^2\to\infty$. Thus

$$\mu(t)(a)=\mu(t)\big((\kappa^*)^{-1}\mathcal{A}_{(H_1,H_2)}(\kappa^*a)\big),$$

for all $t\in\mathbb{R}$ and all $a\in\mathcal{C}_c^\infty(\mathcal{V})$. By (3.38), one can show that

$$\mathcal{A}_{(H,V)}(a)=(\kappa^*)^{-1}\mathcal{A}_{(H_1,H_2)}(\kappa^*a),$$

for every $a\in\mathcal{C}_c^\infty(\mathcal{V})$. Finally, if there exists $N\geq 3$ such that $\tau_\hbar\hbar^N\to 0$, we observe, by Whitney's extension theorem and the functional calculus for pseudodifferential operators, that, for all $a\in\mathcal{C}_c^\infty(\mathcal{V})$, there exists $\mathcal{I}_{\langle a\rangle}\in\mathcal{C}_c^\infty(\mathbb{R}^d)$ such that

$$\mathrm{Op}_\hbar(\mathcal{A}_{H_1,H_2}(a))=\mathcal{I}_{\langle a\rangle}(\mathrm{Op}_\hbar(H_1),\mathrm{Op}_\hbar(H_2))+O(\hbar).$$

Then, since

$$[\mathcal{I}_{\langle a\rangle}(\mathrm{Op}_\hbar(H_1),\mathrm{Op}_\hbar(H_2)),G_k(\mathrm{Op}_\hbar(H_1),\mathrm{Op}_\hbar(H_2))]=0,\quad k=2,\ldots,$$

we have that
$$\frac{i\tau_\hbar}{\hbar}[\widehat{P}_\hbar^N, \mathcal{I}_{\langle a\rangle}(\mathrm{Op}_\hbar(H_1), \mathrm{Op}_\hbar(H_2))] = O(\tau_\hbar \hbar^N).$$
Combining this with the Wigner equation
$$\frac{d}{dt}\widetilde{W}_\hbar^{\tau,\varepsilon}(t) = \frac{i\tau_\hbar}{\hbar}\langle[\widehat{P}_\hbar^N, \mathcal{I}_{\langle a\rangle}(\mathrm{Op}_\hbar(H_1), \mathrm{Op}_\hbar(H_2))]\mathcal{F}_\hbar^* v_\hbar, \mathcal{F}_\hbar^* v_\hbar\rangle + O(\hbar),$$
and taking limit $\hbar \to 0^+$, we obtain:
$$\mu(t)(\kappa^* a) = \mu_0(\kappa^* \mathcal{A}_{(H_1,H_2)}(a)), \quad \forall a \in \mathcal{C}_c^\infty(\mathcal{V}),$$
i.e.,
$$\mu(t)(a) = \mu_0(\mathcal{A}_{(H,V)}(a)), \quad \forall a \in \mathcal{C}_c^\infty(\mathcal{V}).$$
□

3.4. WEAK LIMITS OF SEQUENCES OF EIGENFUNCTIONS

This section is devoted to prove Theorems 1.7 and 1.8.

Proof of Theorem 1.7. We recall that the spectrum of $\widehat{H}_\hbar$ is given by
$$\lambda_{k,\hbar} = \sum_{j=1}^d \hbar\left(k_j + \frac{1}{2}\right)\omega_j, \quad k \in \mathbb{N}^d.$$
For any index $k \in \mathbb{N}^d$, we define the finite set $[k] \in \mathbb{N}^d/\Lambda_\omega$ of indeces that provide the same eigenvalue $\lambda_{k,\hbar}$:
$$[k] := \{m \in \mathbb{N}^d \,:\, m - k \in \Lambda_\omega\}.$$
The multiplicity of the eigenvalue $\lambda_{k,\hbar}$ is precisely the cardinal $N_k \in \mathbb{N}$ of $[k]$. On the other hand, the associated eigenstates are given by suitable linear combinations of the semiclassical Hermite functions $(\psi_{k,\hbar})_{k\in\mathbb{N}^d}$. Each Hermite function is defined by:
$$\psi_{k,\hbar}(x) = \frac{1}{(\pi\hbar)^{d/4}}\prod_{j=1}^d \mathfrak{H}_{k_j}\left(\frac{x_j}{\sqrt{\hbar}}\right)e^{-\frac{x_j^2}{2\hbar}}, \quad k = (k_1, \dots, k_d) \in \mathbb{N}^d, \tag{3.40}$$
where $\mathfrak{H}_{k_j}$ are the Hermite polynomials. Hence, any normalized eigenfunction $\varphi_{k,\hbar}$ of $\widehat{H}_\hbar$ with associated eigenvalue $\lambda_{k,\hbar}$ has the form
$$\varphi_{k,\hbar} = \sum_{m\in[k]} \sigma_{m,\hbar}\,\psi_{m,\hbar}, \quad \sum_{m\in[k]} |\sigma_{m,\hbar}|^2 = 1. \tag{3.41}$$

To facilitate the calculations, we will exploit the Bargmann space representation of the harmonic oscillator (see for instance Berezin and Shubin [20]). We consider the Hilbert space of holomorphic functions

$$\mathcal{H}_\hbar := L^2_{\text{hol}}\left(\mathbb{C}^d, e^{-\frac{|z|^2}{2\hbar}}\frac{dz\,d\overline{z}}{(2\pi\hbar)^{d/2}}\right).$$

The Bargmann transform $\mathcal{B}_\hbar : L^2(\mathbb{R}^d) \to \mathcal{H}_\hbar$ is the isomorphism defined by the following integral operator:

$$\mathcal{B}_\hbar\, u(z) := \frac{1}{(\pi\hbar)^{d/4}}\int_{\mathbb{R}^d}\exp\left[-\frac{1}{2\hbar}(|z|^2+|x|^2-2\sqrt{2}z\cdot x)\right]u(x)dx.$$

Under the Bargmann transform, the eigenfunctions of the harmonic oscillator have a particular convenient form:

$$\mathcal{B}_\hbar\,\psi_{k,\hbar}(z) = \frac{z^k}{\big((2\hbar)^{|k|}k!\big)^{1/2}}, \quad k\in\mathbb{N}^d, \quad z^k = z_1^{k_1}\cdots z_d^{k_d},$$

while the harmonic oscillator $\widehat{H}_\hbar$ itself is conjugated into

$$\mathcal{B}_\hbar\,\widehat{H}_\hbar\,\mathcal{B}_\hbar^{-1} = \hbar\sum_{j=1}^d \omega_j\left(z_j\frac{\partial}{\partial z_j}+\frac{1}{2}\right).$$

Moreover, the Bargmann transform $\mathcal{B}_\hbar$ intertwins anti-Wick operators with Toeplitz operators. Identifying $\mathbb{C}^d$ with $\mathbb{R}^{2d}$ via $z = x + i\xi$, the following holds:

$$\mathcal{B}_\hbar\,\mathrm{Op}_\hbar^{\mathrm{AW}}(a)\,\mathcal{B}_\hbar^{-1} = T_\hbar(a),$$

where the anti-Wick quantization of a is defined by

$$\mathrm{Op}_\hbar^{\mathrm{AW}}(a) := \mathrm{Op}_\hbar(e^{\hbar\Delta/4}a),$$

and the Toeplitz operator $T_\hbar(a) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar$ is given by

$$T_\hbar(a) = \Pi_\hbar M(a),$$

where $M(a)$ defines the multiplication operator on $L^2\big(\mathbb{C}^d,\, e^{-\frac{|z|^2}{2\hbar}}dz\,d\overline{z}\big)$, and

$$\Pi_\hbar : L^2\left(\mathbb{C}^d, e^{-\frac{|z|^2}{2\hbar}}\frac{dz\,d\overline{z}}{(2\pi\hbar)^{d/2}}\right) \to \mathcal{H}_\hbar$$

is the orthogonal projection onto the holomorphic subspace. The Anti-Wick quantization and the Weyl quantization are equivalent in the semiclassical limit. Indeed, one can show that

$$\mathrm{Op}_\hbar^{\mathrm{AW}}(a) = \mathrm{Op}_\hbar(a) + O_{\mathcal{L}(L^2)}(\hbar), \quad a\in\mathcal{C}_c^\infty(\mathbb{R}^{2d}). \tag{3.42}$$

The point of introducing the Bargmann representation is to calculate more easily the semiclassical measures associated to sequences of eigenfunctions of the harmonic oscillator. The following identity holds:

$$\left\langle \psi_{m,\hbar}, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\psi_{m',\hbar}\right\rangle_{L^2} = \frac{1}{C_{\hbar,m,m'}}\int_{\mathbb{C}^d} z^m a(z)\overline{z}^{m'} e^{-\frac{|z|^2}{2\hbar}} dz\, d\overline{z}, \tag{3.43}$$

where

$$C_{\hbar,m,m'} = \pi^d (2\hbar)^{d+\frac{|m|+|m'|}{2}} (m!)^{\frac{1}{2}} (m'!)^{\frac{1}{2}}. \tag{3.44}$$

Lemma 3.2 (Concentration on a minimal set)**.** *Let $z_0 \in H^{-1}(1)$, then*

$$\mathfrak{h}_{\mathcal{O}^H(z_0)} \in \mathcal{M}(\widehat{H}_\hbar),$$

where $\mathfrak{h}_{\mathcal{O}^H(z_0)}$ is the Haar measure on the minimal set

$$\mathcal{O}^H(z_0) = \{\Phi_\tau^H(z_0) \,:\, \tau \in \mathbb{T}_\omega\}. \tag{3.45}$$

Proof. We consider a sequence $(k^\hbar, \hbar)$ such that

$$\hbar k^\hbar \to E^0 := (H_1(z_0), \dots, H_d(z_0))\,, \quad \text{as } \hbar \to 0.$$

In particular, $\lambda_{k^\hbar,\hbar} \to H(z_0) = 1$. Let us consider the set $\mathcal{K}_\hbar$ of indices with same energy as $k^\hbar$ and at distance smaller or iqual than $\hbar^{-1/2}$:

$$\mathcal{K}_\hbar := [k^\hbar] \cap B_{k^\hbar}(\hbar^{-1/2}), \quad B_{k^\hbar}(\hbar^{-1/2}) := \{m \in \mathbb{N}^d : |m - k^\hbar| \leq \hbar^{-1/2}\}, \tag{3.46}$$

and denote by $N_\hbar \in \mathbb{N}$ the cardinal of $\mathcal{K}_\hbar$. We consider the particular sequence of eigenfunctions $(\varphi_\hbar)$ given by

$$\varphi_\hbar := \frac{1}{N_\hbar^{1/2}} \sum_{m \in \mathcal{K}_\hbar} \psi_{m,\hbar}.$$

By (3.42), the associated Wigner distributions satisfy:

$$\left\langle \varphi_\hbar, \mathrm{Op}_\hbar(a)\varphi_\hbar\right\rangle_{L^2} = \left\langle \varphi_\hbar, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\varphi_\hbar\right\rangle_{L^2} + O(\hbar).$$

On the other hand:

$$\begin{aligned}\left\langle \varphi_\hbar, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\varphi_\hbar\right\rangle_{L^2} &= \frac{1}{N_\hbar}\sum_{m,m' \in \mathcal{K}_\hbar} \left\langle \psi_{m,\hbar}, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\psi_{m',\hbar}\right\rangle_{L^2} \\ &= \frac{1}{N_\hbar}\sum_{m \in \mathcal{K}_\hbar}\sum_{l \in \mathcal{K}_\hbar - \{m\}} \left\langle \psi_{m,\hbar}, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\psi_{m+l,\hbar}\right\rangle_{L^2}.\end{aligned}$$

By (3.43),

$$\langle \psi_{m,\hbar}, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\psi_{m+l,\hbar}\rangle_{L^2} = \frac{1}{C_{\hbar,m,m+l}}\int_{\mathbb{C}^d} z^m a(z)\overline{z}^{m+l} e^{-\frac{|z|^2}{2\hbar}} dz\, d\overline{z}.$$

We will require the following intermediate lemma:

Lemma 3.3. *Let $m \in \mathbb{N}^d$ and $l \in \mathbb{N}^d - \{m\}$. Define*

$$z_{m,l,\hbar} := \left(\sqrt{\hbar(2m_1 + l_1 + 1)}, \dots, \sqrt{\hbar(2m_d + l_d + 1)}, 0, \dots, 0\right) \in \mathbb{R}^{2d}.$$

Then

$$\frac{1}{C_{\hbar,m,m+l}} \int_{\mathbb{C}^d} z^m a(z) \overline{z}^{m+l} e^{-\frac{|z|^2}{2\hbar}} dz\, d\overline{z} = \frac{\Lambda(m,l)}{(2\pi)^d} \int_{\mathbb{T}^d} a \circ \Phi^H_\tau(z_{m,l,\hbar}) e^{il\cdot\tau} d\tau + O\left(\frac{\hbar^{1/2}}{1+|l|^{d+1}}\right),$$

as $\hbar \to 0$, where

$$\Lambda(m,l) := \frac{1}{[m!(m+l)!]^{\frac{1}{2}}} \prod_{j=1}^d \Gamma\left(\frac{2m_j + l_j + 2}{2}\right),$$

Γ denotes the Gamma function, and the constant in the $O(\cdot)$ depends only on $a \in \mathcal{C}^\infty_c(\mathbb{R}^{2d})$ and not on $m, l \in \mathbb{Z}^d$.

Remark 3.2. Note that

$$\Lambda(m,l) \to 1, \quad \text{as } |m|, |m+l| \to \infty.$$

Proof. Taking polar coordinates

$$z = \Phi^H_\tau(r, 0), \quad \tau \in \mathbb{T}^d, \quad r = (r_1, \dots, r_d) \in \mathbb{R}^d_+,$$

we have:

$$\int_{\mathbb{C}^d} z^m a(z) \overline{z}^{m+l} e^{-\frac{|z|^2}{2\hbar}} dz\, d\overline{z} = \int_{\mathbb{R}^d_+} \int_{\mathbb{T}^d} a \circ \Phi^H_\tau(r, 0) e^{il\cdot\tau} \prod_{j=1}^d r_j^{2m_j + l_j + 1} e^{-\frac{r_j^2}{2\hbar}} dr d\tau.$$

Now, we perform the following change of variables, shifting the center to $z_{m,l,k}$ and zooming by $1/(2\hbar)^{1/2}$:

$$r_j = \sqrt{2\hbar} s_j + \sqrt{\hbar(2m_j + l_j + 1)}, \quad s_j \in \left[-\sqrt{\frac{2m_j + l_j + 1}{2}}, \infty\right), \quad j = 1, \dots, d.$$

We want to show that

$$\frac{(2\hbar)^{\frac{d}{2}}}{C_{\hbar,m,m+l}} \prod_{j=1}^d r_j^{2m_j + l_j + 1} e^{-\frac{r_j^2}{2\hbar}} \leq C_d e^{-\frac{|s|^2}{2}}, \tag{3.47}$$

for some constant $C_d > 0$ depending only on the dimension d. Indeed, by the following inequality

$$(\sqrt{2}s + \sqrt{B})^B e^{-\frac{(\sqrt{2}s + \sqrt{B})^2}{2}} \leq e^{-\frac{s^2}{2}} \left(\frac{B}{e}\right)^{\frac{B}{2}}, \quad s \geq -\left(\frac{B}{2}\right)^{\frac{1}{2}}, \quad B \geq 0,$$

we have

$$r_j^{2m_j+l_j+1}e^{-\frac{r_j^2}{2\hbar}} \leq e^{-\frac{s_j^2}{2}}e^{-\frac{2m_j+l_j+1}{2}}\big(\hbar(2m_j+l_j+1)\big)^{\frac{2m_j+l_j+1}{2}}.$$

Using Stirling's formula

$$n! \geq \sqrt{2\pi}n^{n+\frac{1}{2}}e^{-n}, \quad n \geq 1,$$

this can be bounded by

$$e^{-\frac{s_j^2}{2}}e^{-\frac{2m_j+l_j+1}{2}}\big(\hbar(2m_j+l_j+1)\big)^{\frac{2m_j+l_j+1}{2}} \leq e^{-\frac{s_j^2}{2}}\left(\frac{\hbar^{2m_j+l_j+1}(2m_j+l_j+1)!}{\sqrt{2\pi}(2m_j+l_j+1)^{\frac{1}{2}}}\right)^{\frac{1}{2}}.$$

Recall, from (3.44), that

$$C_{\hbar,m,m+l} = \pi^d(2\hbar)^{d+\frac{|m|+|m+l|}{2}}\big[m!(m+l)!\big]^{\frac{1}{2}}.$$

Then, using the following standard property of the Gamma function[1]:

$$\Gamma(2x) \lesssim x^{\frac{1}{2}}\Gamma(x)^2 2^{2x-1} \lesssim x^{\frac{1}{2}}\Gamma(x-\alpha)\Gamma(x+\alpha)2^{2x-1}, \quad x \geq 0,$$

with

$$x = \frac{2m_j+l_j+2}{2}, \quad \alpha = \frac{l_j}{2},$$

we conclude (3.47).

On the other hand, denoting

$$\widetilde{a}(r,\tau) := a \circ \Phi_\tau^H(r,0), \quad (r,\tau) \in \mathbb{R}_+^d \times \mathbb{T}^d,$$

and using Taylor's theorem, we can expand

$$a \circ \Phi_\tau^H(z_{m,l,\hbar} + \sqrt{2\hbar}s, 0) = a \circ \Phi_\tau^H(z_{m,l,\hbar}) + \sqrt{2\hbar}s \cdot \int_0^1 \partial_r \widetilde{a}\big(z_{m,l,\hbar} + t\sqrt{2\hbar}s, \tau\big)dt.$$

Since $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$, its Fourier coefficients decay faster than any polynomial. In particular,

$$\|\partial_r \widetilde{a}_{-l}\|_{L^\infty(\mathbb{R}_+^d)} \leq \frac{C}{1+|l|^{d+1}}, \quad l \in \mathbb{Z}^d,$$

where, recalling (3.7),

$$\widetilde{a}_{-l}(r) := a_{-l}(r,0) = \int_{\mathbb{T}^d} a \circ \Phi_\tau^H(r,0)e^{il\cdot\tau}d\tau, \quad r \in \mathbb{R}_+^d.$$

[1]The notation $\lesssim$ means that the inequality holds modulo multiplication by some universal constant.

Therefore, since

$$\int_0^\infty r^{2m_j+l_j+1} e^{-\frac{r_j^2}{2\hbar}} dr_j = \frac{1}{2}\Gamma\left(\frac{2m_j+l_j+2}{2}\right)(2\hbar)^{\frac{2m_j+l_j+2}{2}},$$

we obtain

$$\left|\frac{1}{C_{\hbar,m,m+l}}\int_{\mathbb{C}^d} z^m a(z)\overline{z}^{m+l} e^{-\frac{|z|^2}{2\hbar}} dz\, d\overline{z} - \frac{\Lambda(m,l)}{(2\pi)^d}\int_{\mathbb{T}^d} a\circ\Phi_\tau^H(z_{m,l,\hbar}) e^{il\cdot\tau} d\tau\right|$$
$$\leq C_d \hbar^{\frac{1}{2}} \|\partial_r \widetilde{a}_{-l}\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d_+} |s| e^{-\frac{|s|^2}{2}} ds = O\left(\frac{\hbar^{1/2}}{1+|l|^{d+1}}\right).$$

□

We now proceed to finish the proof of Lemma 3.2. For every $m \in \mathcal{K}_\hbar$,

$$\frac{1}{C_{\hbar,m,m+l}}\int_{\mathbb{C}^d} z^m a(z)\overline{z}^{m+l} e^{-\frac{|z|^2}{\hbar}} dz\, d\overline{z} = \frac{\Lambda(m,l)}{(2\pi)^d}\int_{\mathbb{T}^d} a\circ\Phi_\tau^H(z_{m,l,\hbar}) e^{il\cdot\tau} d\tau + O\left(\frac{\hbar^{1/2}}{1+|l|^{d+1}}\right),$$

hence

$$\left\langle \varphi_\hbar, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\varphi_\hbar \right\rangle_{L^2} = \sum_{m\in\mathcal{K}_\hbar}\sum_{l\in\mathcal{K}_\hbar-\{m\}} \frac{\Lambda(m,l)}{(2\pi)^d N_\hbar}\int_{\mathbb{T}^d} a\circ\Phi_\tau^H(z_{m,l,\hbar}) e^{il\cdot\tau} d\tau + O\left(\hbar^{1/2}\right).$$

By definition (3.46), for every $m \in \mathcal{K}_\hbar$,

$$\hbar|k^\hbar - m| = O(\hbar^{1/2}), \quad \text{as } \hbar \to 0.$$

Then, setting

$$z_0^* := \left(\sqrt{2H_1(z_0)}, \dots, \sqrt{2H_d(z_0)}, 0, \dots, 0\right) \in H^{-1}(1),$$

we have

$$\sup_{m\in\mathcal{K}_\hbar}\ \sup_{l\in\mathcal{K}_\hbar-\{m\}} |z_{m,l,\hbar} - z_0^*| = o(1), \quad \text{as } \hbar \to 0.$$

Using Remark 3.2 and definition (3.46), we observe that

$$\sup_{m\in\mathcal{K}_\hbar}\ \sup_{l\in\mathcal{K}_\hbar-\{m\}} \Big(\Lambda(m,l) - 1\Big) = o(1), \quad \text{as } \hbar \to 0.$$

Then, since

$$\bigcup_{m\in\mathcal{K}_\hbar} \big(\mathcal{K}_\hbar - \{m\}\big) = \big(\mathcal{K}_\hbar - \mathcal{K}_\hbar\big) \to \Lambda_\omega, \quad \text{as } \hbar \to 0^+,$$

where $\mathcal{K}_\hbar - \mathcal{K}_\hbar = \{m - m' \, : m, m' \in \mathcal{K}_\hbar\}$, and the limit is taken in the sense of sets,

$$\lim_{\hbar \to 0} \left\langle \varphi_\hbar, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\varphi_\hbar \right\rangle_{L^2} = \frac{1}{(2\pi)^d} \sum_{l \in \Lambda_\omega} a_l(z_0^*).$$

By the Poisson summation formula:

$$\frac{1}{(2\pi)^d} \sum_{l \in \Lambda_\omega} e^{il\cdot\tau} = \mathfrak{h}_\omega,$$

which is just (1.36), we obtain

$$\lim_{\hbar \to 0} \left\langle \varphi_\hbar, \mathrm{Op}_\hbar(a)\varphi_\hbar \right\rangle_{L^2} = \frac{1}{(2\pi)^d} \sum_{l \in \Lambda_\omega} a_l(z_0^*) = \int_{\mathbb{T}_\omega} a \circ \Phi_\tau^H(z_0^*)\mathfrak{h}_\omega(d\tau).$$

Finally, we define

$$\widehat{\mathscr{H}_\hbar} := \big(\mathrm{Op}_\hbar(H_1), \ldots, \mathrm{Op}_\hbar(H_d)\big)$$

and set $\tau_0 \in \mathbb{T}^d$ such that $\Phi_{\tau_0}^H(z_0^*) = z_0$. By Egorov's theorem,

$$e^{-\frac{i}{\hbar}\tau_0\cdot\widehat{\mathscr{H}_\hbar}} \mathrm{Op}_\hbar(a) e^{\frac{i}{\hbar}\tau_0\cdot\widehat{\mathscr{H}_\hbar}} = \mathrm{Op}_\hbar(a \circ \Phi_{\tau_0}^H), \quad a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d}). \tag{3.48}$$

Thus, the sequence of rotations along the torus $\mathbb{T}_E$:

$$\widetilde{\varphi}_\hbar := e^{\frac{i}{\hbar}\tau_0\cdot\widehat{\mathscr{H}_\hbar}}\varphi_\hbar$$

is also a sequence of eigenfunctions with same related sequence of eigenvalues, and it satisfies

$$\left\langle \widetilde{\varphi}_\hbar, \mathrm{Op}_\hbar(a)\widetilde{\varphi}_\hbar \right\rangle_{L^2} \to \int_{\mathbb{T}_\omega} a \circ \Phi_\tau^H(z_0)\mathfrak{h}_\omega(d\tau).$$

In other words,

$$\mathfrak{h}_{\mathcal{O}^H(z_0)} \in \mathcal{M}(\widehat{H}_\hbar),$$

as we wanted to prove. □

Lemma 3.4 (Closed convex hull of minimal-set measures)**.** *The following holds:*

$$\mathcal{M}(\widehat{H}_\hbar) \supset \bigcup_{[E] \in \mathcal{L}_\omega^{-1}(1)/[\Lambda_\omega]} \mathcal{M}_{[E]}(H).$$

Proof. For any $z \in H^{-1}(1)$, denote

$$E(z) := \big(H_1(z), \dots, H_d(z)\big).$$

First, we show that given two points $z_0, z_1 \in H^{-1}(1)$ such that

$$E(z_0) - E(z_1) \in [\Lambda_\omega], \quad \mathcal{O}^H(z_0) \cap \mathcal{O}^H(z_1) = \emptyset, \tag{3.49}$$

we can find, for any $0 \leq \delta \leq 1$, a normalized sequence of eigenfunctions $(\varphi_\hbar^\delta)$ with associated semiclassical measure given by

$$\mu^\delta = \delta \mathfrak{h}_{\mathcal{O}^H(z_0)} + (1-\delta)\mathfrak{h}_{\mathcal{O}^H(z_1)}. \tag{3.50}$$

Indeed, using the same construction of the proof of Lemma 3.2, we can find two sequences $(k_0^\hbar, \hbar)$, $(k_1^\hbar, \hbar)$ such that $k_0^\hbar - k_1^\hbar \in \Lambda_\omega$ (so they have the same energy) for all $\hbar$,

$$\hbar k_\iota^\hbar \to E^\iota := \big(H_1(z_\iota), \dots, H_d(z_\iota)\big), \quad \text{as } \hbar \to 0,$$

and two sequences of eigenfunctions $(\varphi_\hbar^0)$, $(\varphi_\hbar^1)$ such that

$$W_{\varphi_\hbar^\iota} \rightharpoonup \mathfrak{h}_{\mathcal{O}(z_\iota^*)}, \quad \iota = 0, 1,$$

where

$$z_\iota^* := \left(\sqrt{2H_1(z_\iota)}, \dots, \sqrt{2H_d(z_\iota)}, 0, \dots, 0\right) \in H^{-1}(1).$$

Then, setting

$$\widetilde{\varphi}_\hbar^\iota := e^{\frac{i}{\hbar}\tau_\iota \cdot \widehat{\mathscr{H}_\hbar}} \varphi_\hbar^\iota,$$

where $z_\iota = \Phi_{\tau_\iota}^H(z_\iota^*)$, we obtain (3.50) for $\delta \in \{0, 1\}$. Since $\mathcal{O}^H(z_0) \cap \mathcal{O}^H(z_1) = \emptyset$, we can use ([51], Prop. 3.3) to obtain, for the sequence of eigenfunctions given by

$$\widetilde{\varphi}_\hbar^\delta := \widetilde{\varphi}_\hbar^0 \sqrt{\delta} + \widetilde{\varphi}_\hbar^1 \sqrt{1-\delta},$$

that

$$W_{\widetilde{\varphi}_\hbar^\delta} \rightharpoonup \mu^\delta.$$

Now, by Krein-Milman Theorem, if $\mu \in \mathcal{M}([E])$, then it can be obtained as a limit of the form

$$\mu = \lim_{N \to \infty} \sum_{j=1}^N c_j(N) \mathfrak{h}_{\mathcal{O}^H(z_j)}, \tag{3.51}$$

where the limit is considered in the weak-$\star$ sense, $0 \leq c_j(N) \leq 1$, $\sum_{j=1}^N c_j(N) = 1$, $z_j \in \mathbb{R}^{2d}$, and

$$E(z_j) - E(z_{j'}) \in [\Lambda_\omega], \quad \mathcal{O}^H(z_j) \cap \mathcal{O}^H(z_{j'}) = \emptyset, \quad \text{for all } j \neq j'.$$

For each z_j, we assume that there exists a sequence of eigenfunctions $(\varphi_\hbar^j)$ with same sequence of eigenvalues such that its semiclassical measure is $\lambda_{\mathcal{O}^H(z_j)}$. Then take

$$\vartheta_\hbar^N := \sum_{j=1}^N \sqrt{c_j(N)}\varphi_\hbar^j,$$

which is also an eigenfunction (despite it is not estrictly normalized, asymptotically it is). Since

$$\lim_{\hbar\to 0}\langle \vartheta_\hbar^N, \mathrm{Op}_\hbar(a)\vartheta_\hbar^N\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} a(x,\xi)\left(\sum_{j=1}^N c_j(N)\mathfrak{h}_{\mathcal{O}^H(z_j)}\right)(dx,d\xi)$$

for all N, and using (3.51), we can extract a diagonal subsequence $(\hbar_n)$ such that

$$\lim_{n\to\infty}\langle \vartheta_{\hbar_n}^{N_n}, \mathrm{Op}_{\hbar_n}(a)\vartheta_{\hbar_n}^{N_n}\rangle_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} a(x,\xi)\mu(dx,d\xi).$$

The result holds. □

It remains to show that

$$\mathcal{M}(\widehat{H}_\hbar) \subset \bigcup_{[E]\in\mathcal{L}_\omega^{-1}(1)/[\Lambda_\omega]} \mathcal{M}_{[E]}(H).$$

We will use the following intermediate lemma:

Lemma 3.5. *Given a sequence $(k^\hbar,\hbar)$ such that*

$$k^\hbar\hbar \to E^0 = \left(E_1^0,\dots,E_d^0\right) \in \mathcal{L}_\omega^{-1}(1).$$

Let $(m^\hbar,\hbar)$ be any other sequence with $m^\hbar \in [k^\hbar]$, and let $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ such that

$$\operatorname{supp} a \cap \bigcup_{E\in[E^0]} \mathbb{T}_E = \emptyset, \tag{3.52}$$

and $[E^0]$ is the equivalent class of E^0 in $\mathcal{L}_\omega^{-1}(1)/[\Lambda_\omega]$. Then the following holds for every index $l \in [k^\hbar] - \{m^\hbar\}$:

$$\left\langle \psi_{m^\hbar,\hbar}, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\psi_{m^\hbar+l,\hbar}\right\rangle_{L^2} = O\left(\frac{\hbar^{1/2}}{1+|l|^{d+1}}\right),$$

where the $O(\cdot)$ depends only on a[2].

[2]From (3.52) we actually expect $O(\hbar^\infty)$.

Proof. Observe that

$$\lim_{\hbar\to 0} \operatorname{dist}\big([E^0], \{\hbar m^\hbar : m^\hbar \in [k^\hbar]\}\big) = 0. \tag{3.53}$$

Then, by Lemma 3.3,

$$\left|\langle \psi_{m,\hbar}, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\psi_{m+l,\hbar}\rangle_{L^2}\right| \leq \frac{\Lambda(m^\hbar, l)}{(2\pi)^d}|a_{-l}(z_{m^\hbar,l,\hbar})| + O\left(\frac{\hbar^{1/2}}{1+|l|^{d+1}}\right).$$

The result follows by using the hypothesis (3.52) for $\hbar$ sufficiently small, provided that

$$\lim_{\hbar\to 0} \sup_{l\in[k^\hbar]-\{m^\hbar\}} \operatorname{dist}\left(\bigcup_{E\in[E^0]} \mathbb{T}_E, z_{m^\hbar,l,\hbar}\right) = 0.$$

□

Finally, let $\mu \in \mathcal{M}(\widehat{H}_\hbar)$ be the semiclassical measure of some sequence $\varphi_{k^\hbar,\hbar}$ with eigenvalues $\lambda_{k^\hbar,\hbar} \to 1$. By compacity of the set $\mathcal{L}_\omega^{-1}(1)$, there exists $E^0 \in \mathcal{L}_\omega^{-1}(1)$ and a subsequence $(k^\hbar, \hbar)$ such that

$$\hbar k^\hbar \to E^0, \quad \text{as } \hbar \to 0.$$

By (3.41), any associated eigenfunction has the form

$$\varphi_\hbar = \sum_{m\in[k^\hbar]} \sigma_{m,\hbar}\psi_{m,\hbar},$$

and then, using (3.42), we can write

$$\begin{aligned}\big\langle \varphi_\hbar, \mathrm{Op}_\hbar(a)\varphi_\hbar\big\rangle_{L^2} &= \sum_{m,m'\in[k^\hbar]} \sigma_{m,\hbar}\overline{\sigma}_{m',\hbar}\big\langle \psi_{m,\hbar}, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\psi_{m',\hbar}\big\rangle_{L^2} + O(\hbar)\\ &= \sum_{m\in[k^\hbar]}\sum_{l\in[k^\hbar]-\{m\}} \sigma_{m,\hbar}\overline{\sigma}_{m+l,\hbar}\big\langle \psi_{m,\hbar}, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\psi_{m+l,\hbar}\big\rangle_{L^2} + O(\hbar),\end{aligned}$$

Since

$$\operatorname{supp}(a) \cap \bigcup_{E\in[E^0]} \mathbb{T}_E = \emptyset,$$

by Lemma 3.5, for every $m \in [k^\hbar]$ and $l \in [k^\hbar] - \{m\}$:

$$\langle \psi_{m,\hbar}, \mathrm{Op}_\hbar^{\mathrm{AW}}(a)\psi_{m+l,\hbar}\rangle_{L^2} = O\left(\frac{\hbar^{1/2}}{1+|l|^{d+1}}\right).$$

Thus, using Cauchy-Schwartz inequality,

$$\sum_{m\in[k^\hbar]}\sum_{l\in[k^\hbar]-\{m\}}\left|\sigma_{m,\hbar}\overline{\sigma}_{m+l,\hbar}\left\langle\psi_{m,\hbar},\mathrm{Op}_\hbar^{\mathrm{AW}}(a)\psi_{m+l,\hbar}\right\rangle_{L^2}\right|$$

$$\leq\sum_{m\in[k^\hbar]}\sum_{l\in[k^\hbar]-\{m\}}|\sigma_m|^2\times O\left(\frac{\hbar^{1/2}}{1+|l|^{d+1}}\right)=O(\hbar^{1/2}).$$

Therefore, any weak accumulation point of the sequence $(W^\hbar_{\varphi_\hbar})$ is supported on the set $\bigcup_{E\in[E^0]}\mathbb{T}_E$. In other words:

$$\mu\in\mathcal{M}_{[E]}(H),$$

for some $[E]\in\mathcal{L}_\omega^{-1}(1)/[\Lambda_\omega]$. □

Proof of Theorem 1.8. Assume there exists a sequence $(\Psi_\hbar)$ of eigenfunctions of $\widehat{H}_\hbar+\varepsilon_\hbar\widehat{V}_\hbar$ with associated semiclassical measure μ such that

$$\mu(\mathcal{O}^H(z))>0. \tag{3.54}$$

For every time scale $\tau=(\tau_\hbar)$, the sequence of Wigner distributions associated to the functions

$$\varphi_\hbar^\tau(t):=e^{-\frac{it\tau_\hbar}{\hbar}(\widehat{H}_\hbar+\varepsilon_\hbar\widehat{V}_\hbar)}\Psi_\hbar=e^{-\frac{it\tau_\hbar}{\hbar}\lambda_\hbar}\Psi_\hbar$$

has the same weak limit μ. Then μ is invariant by the flow $\phi_t^{\langle L\rangle}$. This and the assumption

$$X_{\langle L\rangle}\big|_z\notin T_z\mathcal{O}^H(z)$$

clearly contradict (3.54). □

Chapter 4

Distribution of eigenvalues for non-selfadjoint harmonic oscillators

Querido Rubén, los versos debieran publicarse con todo su proceso, desde lo que usted llama monstruo hasta la manera definitiva. Tendrían entonces un valor como las pruebas de aguafuerte.

R. M. del Valle-Inclán. *Luces de Bohemia. El Marqués de Bradomín a Rubén Darío.*

In this chapter we focus on the study of the asymptotic distribution of eigenvalues for the non-selfadjoint semiclassical operator (1.49):

$$\widehat{P}_\hbar = \widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar + i\hbar \widehat{A}_\hbar,$$

where $\widehat{H}_\hbar$ is given by (1.23),

$$\widehat{V}_\hbar = \mathrm{Op}_\hbar(V), \quad \widehat{A}_\hbar = \mathrm{Op}_\hbar(A), \quad V, A \in S^0(\mathbb{R}^{2d}),$$

and $\varepsilon_\hbar \to 0$ as $\hbar \to 0$. We prove Theorems 1.9 and 1.10. In Section 4.1 we explain the averaging method for non-selfadjoint operators following the works of Sjöstrand [109] and Hitrik [57], and we use it to obtain a normal form via conjugating $\widehat{P}_\hbar$ by a suitable Fourier integral operator. In Section 4.2 we study the properties of the semiclassical measures associated to sequences of quasimodes for $\widehat{P}_\hbar$ and, from these properties, we prove Theorem 1.9. Section 4.3 is devoted to develop some tools of analytic symbolic calculus. Finally, in Section 4.4 we give the proof of Theorem 1.10, which is based on a second conjugation of $\widehat{P}_\hbar$ by another Fourier integral operator with analytic symbol.

4.1. The averaging method in the non-selfadjoint case

We first recall how to perform a semiclassical averaging method in the context of nonselfadjoint operators. This consists in averaging both the operators $\widehat{V}_\hbar$ and $\widehat{A}_\hbar$ by the quantum flow generated by $\widehat{H}_\hbar$ via conjugation by a suitable Fourier integral operator. Given $a \in S^0(\mathbb{R}^{2d})$, we recall that the quantum average $\langle \mathrm{Op}_\hbar(a) \rangle$ of the operator $\mathrm{Op}_\hbar(a)$ was given by (1.41) and Proposition 1.1:

$$\langle \mathrm{Op}_\hbar(a) \rangle := \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{i\frac{t}{\hbar}\widehat{H}_\hbar} \mathrm{Op}_\hbar(a) e^{-i\frac{t}{\hbar}\widehat{H}_\hbar}\, dt. \tag{4.1}$$

Moreover, by Egorov's theorem (see Lemma 2.8), which is exact in this case since H is polynomial of order two:

$$\langle \mathrm{Op}_\hbar(a) \rangle = \mathrm{Op}_\hbar(\langle a \rangle).$$

We also require the following nonselfadjoint version of Egorov's theorem:

Lemma 4.1 (Non-selfadjoint Egorov's theorem)**.** *Let $\mathcal{G}_\hbar(t)$ be a family of Fourier integral operators of the form*

$$\mathcal{G}_\hbar(t) := e^{\frac{it}{\hbar}(\widehat{G}_{1,\hbar} - i\hbar \widehat{G}_{2,\hbar})}, \quad t \in \mathbb{R},$$

where $\widehat{G}_{j,\hbar} = \mathrm{Op}_\hbar(G_j)$ for $G_j \in S^0(\mathbb{R}^{2d})$ and $j = 1,2$. Then, for every $t \in \mathbb{R}$ and every $a \in S^0(\mathbb{R}^{2d})$, the following holds:

$$\mathcal{G}_\hbar(t)\, \mathrm{Op}^w_\hbar(a) \mathcal{G}_\hbar(-t) = \mathrm{Op}^w_\hbar(a \circ \phi^{G_1}_t) + O_t(\hbar).$$

Proof. By [69, Thm. III.1.3.], the family $\mathcal{G}_\hbar(t)$ defines a strongly continuous semigroup on $L^2(\mathbb{R}^d)$ such that

$$\|\mathcal{G}_\hbar(t)\|_{\mathcal{L}(L^2)} \le e^{|t| \|\widehat{G}_{2,\hbar}\|_{\mathcal{L}(L^2)}}. \tag{4.2}$$

Let $t \ge 0$. For every $r \in [0,t]$, we define

$$a_r := a \circ \phi^{G_1}_{t-r}.$$

By the product rule:

$$\frac{d}{dr}\big(\mathcal{G}_\hbar(r)\, \mathrm{Op}_\hbar(a_r) \mathcal{G}_\hbar(-r)\big)$$
$$= \mathcal{G}_\hbar(r) \left(\frac{i}{\hbar}[\widehat{G}_{1,\hbar}, \mathrm{Op}_\hbar(a_r)] + [\widehat{G}_{2,\hbar}, \mathrm{Op}_\hbar(a_r)] + \mathrm{Op}_\hbar(\partial_r a_r) \right) \mathcal{G}_\hbar(-r).$$

Using the symbolic calculus for Weyl pseudodifferential operators, we have

$$\frac{i}{\hbar}[\widehat{G}_{j,\hbar}, \mathrm{Op}_\hbar(a_r)] = \mathrm{Op}_\hbar(\{G_j, a_r\}) + O(\hbar^2), \quad j = 1,2.$$

Moreover:

$$\partial_r a_r = -\{G_1, a_r\}.$$

These facts and (4.2) give:

$$\mathcal{G}_\hbar(t)\operatorname{Op}_\hbar(a)\mathcal{G}_\hbar(-t) - \operatorname{Op}_\hbar(a\circ\phi_t^{G_1}) = \int_0^t \frac{d}{dr}\big(\mathcal{G}_\hbar(r)\operatorname{Op}_\hbar(a_r)\mathcal{G}_\hbar(-r)\big)dr = O_t(\hbar).$$

□

The goal of this section is to prove the following:

Lemma 4.2. *There exists a Fourier integral operator $\mathcal{F}_\hbar$ such that*

$$\widehat{P}_\hbar^\dagger := \mathcal{F}_\hbar \widehat{P}_\hbar \mathcal{F}_\hbar^{-1} = \widehat{H}_\hbar + \varepsilon_\hbar\langle\widehat{V}_\hbar\rangle + i\hbar\langle\widehat{A}_\hbar\rangle + \widehat{R}_\hbar, \tag{4.3}$$

where the reminder term $\widehat{R}_\hbar = \operatorname{Op}_\hbar(R)$ satisfies:

$$\operatorname{Op}_\hbar(\Re R) = O_{\mathcal{L}(L^2)}(\varepsilon_\hbar^2 + \hbar^2), \quad \operatorname{Op}_\hbar(\Im R) = O_{\mathcal{L}(L^2)}(\varepsilon_\hbar\hbar). \tag{4.4}$$

Proof. We define

$$\widehat{F}_\hbar = \operatorname{Op}_\hbar(F) := \operatorname{Op}_\hbar(\varepsilon_\hbar F_1 + i\hbar F_2),$$

where F_1 and F_2 are two real valued and smooth functions on $\mathbb{R}^{2d}$ that will be determined later on. We make the assumption that all the derivatives of F_1 and F_2 are bounded. For every t in $[0,1]$, we set

$$\mathcal{F}_\hbar(t) = e^{\frac{i}{\hbar}t\widehat{F}_\hbar}.$$

We shall denote $\mathcal{F}_\hbar = \mathcal{F}_\hbar(1)$ and we will study the properties of the conjugated operator

$$\widehat{P}_\hbar^\dagger := \mathcal{F}_\hbar \widehat{P}_\hbar \mathcal{F}_\hbar^{-1},$$

for appropriate choices of F_1 and F_2. Precisely, we define the symbols F_1 and F_2 as the solutions of the following cohomological equations (see Section 3.1.1):

$$\{H, F_1\} = V - \langle V\rangle, \tag{4.5}$$

$$\{H, F_2\} = A - \langle A\rangle. \tag{4.6}$$

Observe that F_j are real valued for $j = 1, 2$. Using Taylor's theorem we can write the conjugated operator $\widehat{P}_\hbar^\dagger$ as

$$\begin{aligned}\widehat{P}_\hbar^\dagger = \mathcal{F}_\hbar \widehat{P}_\hbar \mathcal{F}_\hbar^{-1} &= \widehat{H}_\hbar + \varepsilon_\hbar\widehat{V}_\hbar + i\hbar\widehat{A}_\hbar + \frac{i}{\hbar}[\widehat{F}_\hbar, \widehat{H}_\hbar] \\ &+ \frac{i}{\hbar}\int_0^1 \mathcal{F}_\hbar(t)[\widehat{F}_\hbar, \varepsilon_\hbar\widehat{V}_\hbar + i\hbar\widehat{A}_\hbar]\mathcal{F}_\hbar(t)^{-1}dt \\ &+ \left(\frac{i}{\hbar}\right)^2\int_0^1 (1-t)\mathcal{F}_\hbar(t)[\widehat{F}_\hbar, [\widehat{F}_\hbar, \widehat{H}_\hbar]]\mathcal{F}_\hbar(t)^{-1}dt.\end{aligned}$$

Recall that, by the symbolic calculus for Weyl pseudodifferential operators,

$$\frac{i}{\hbar}[\widehat{F}_{j,\hbar}, \widehat{H}_\hbar] = \mathrm{Op}_\hbar(\{F_j, H\}), \quad j = 1, 2.$$

Since F_1 and F_2 solve cohomological equations (4.5) and (4.6), we obtain

$$\widehat{P}_\hbar^\dagger = \widehat{H}_\hbar + \varepsilon_\hbar \langle \widehat{V}_\hbar \rangle + i\hbar \langle \widehat{A}_\hbar \rangle + \widehat{R}_\hbar,$$

where

$$\widehat{R}_\hbar = \mathrm{Op}_\hbar(R) = \frac{i}{\hbar} \int_0^1 \mathcal{F}_\hbar(t)[\widehat{F}_\hbar, \widehat{K}_\hbar(t)]\mathcal{F}_\hbar(t)^{-1} dt, \tag{4.7}$$

and

$$\widehat{K}_\hbar(t) = t(\varepsilon_\hbar \widehat{V}_\hbar + i\hbar \widehat{A}_\hbar) + (1-t)(\varepsilon_\hbar \langle \widehat{V}_\hbar \rangle + i\hbar \langle \widehat{A}_\hbar \rangle), \quad t \in [0,1].$$

The symbol of $\widehat{K}_\hbar(t)$ is given by

$$K(t) = t(\varepsilon_\hbar V + i\hbar A) + (1-t)(\varepsilon_\hbar \langle V \rangle + i\hbar \langle A \rangle),$$

thus

$$\mathrm{Op}_\hbar \big(\Re K(t) \big) = O_{\mathcal{L}(L^2)}(\varepsilon_\hbar), \quad \mathrm{Op}_\hbar \big(\Im K(t) \big) = O_{\mathcal{L}(L^2)}(\hbar).$$

Using the pseudodifferential calculus one more time, one can show that

$$\mathrm{Op}_\hbar \big(\Re [F, K(t)]_\hbar \big) = O_{\mathcal{L}(L^2)}(\varepsilon_\hbar^2 + \hbar^2),$$
$$\mathrm{Op}_\hbar \big(\Im [F, K(t)]_\hbar \big) = O_{\mathcal{L}(L^2)}(\varepsilon_\hbar \hbar),$$

where recall that $[A, B]_\hbar$ is the symbol of the commutator $[\mathrm{Op}_\hbar(A), \mathrm{Op}_\hbar(B)]$. Finally, observe that Lemma 4.1 implies in particular that conjugation by $\mathcal{F}_\hbar(t)$ preserves the order of the real and imaginary parts of the principal symbol. Then we obtain that the reminder term $\widehat{R}_\hbar$ satisfies

$$\mathrm{Op}_\hbar(\Re R) = O_{\mathcal{L}(L^2)}(\varepsilon_\hbar^2 + \hbar^2), \quad \mathrm{Op}_\hbar(\Im R) = O_{\mathcal{L}(L^2)}(\varepsilon_\hbar \hbar). \tag{4.8}$$

□

4.2. Study of semiclassical measures

Note that, after conjugation by $\mathcal{F}_\hbar$, the eigenvalue equation (1.50) is transformed into:

$$\widehat{P}_\hbar^\dagger \, v_\hbar^\dagger = \lambda_\hbar \, v_\hbar^\dagger + r_\hbar^\dagger, \quad \| v_\hbar^\dagger \|_{L^2} = 1, \tag{4.9}$$

where $\lambda_\hbar = \alpha_\hbar + i\hbar\beta_\hbar$, $(\alpha_\hbar, \beta_\hbar) \to (1, \beta)$ as $\hbar \to 0^+$, and

$$v_\hbar^\dagger = \frac{\mathcal{F}_\hbar v_\hbar}{\|\mathcal{F}_\hbar v_\hbar\|_{L^2(\mathbb{R}^d)}}, \quad r_\hbar^\dagger = \frac{\mathcal{F}_\hbar r_\hbar}{\|\mathcal{F}_\hbar v_\hbar\|_{L^2(\mathbb{R}^d)}} = o(\hbar\varepsilon_\hbar).$$

Let $(v_\hbar^\dagger)$ be a sequence satisfying (4.9). We consider the Wigner distribution $W^\hbar_{v_\hbar^\dagger} \in \mathcal{D}'(\mathbb{R}^{2d})$ (see Section 2.6.3) associated to $v_\hbar^\dagger$:

$$W^\hbar_{v_\hbar^\dagger} : \mathcal{C}_c^\infty(\mathbb{R}^{2d}) \ni a \longmapsto W^\hbar_{v_\hbar^\dagger}(a) := \left\langle \operatorname{Op}_\hbar^w(a) v_\hbar^\dagger, v_\hbar^\dagger \right\rangle_{L^2(\mathbb{R}^d)}.$$

By Lemma 2.10, modulo extracting a subsequence, there exists a probability measure $\mu^\dagger \in \mathcal{P}(H^{-1}(1))$ such that

$$W^\hbar_{v_\hbar^\dagger} \rightharpoonup \mu^\dagger.$$

Proof of Proposition 1.2. From the identity (4.9), we have

$$\left\langle \widehat{P}_\hbar^\dagger v_\hbar^\dagger, v_\hbar^\dagger \right\rangle_{L^2} = \lambda_\hbar \|v_\hbar^\dagger\|_{L^2}^2 + \left\langle r_\hbar^\dagger, v_\hbar^\dagger \right\rangle_{L^2}.$$

Taking imaginary parts, using $\|v_\hbar^\dagger\|_{L^2} = 1$ and $r_\hbar^\dagger = o(\hbar\varepsilon_\hbar)$, we obtain:

$$\left\langle \langle \widehat{A}_\hbar \rangle v_\hbar^\dagger, v_\hbar^\dagger \right\rangle_{L^2} = \beta_\hbar + o(\varepsilon_\hbar).$$

Recall that $\langle \widehat{A}_\hbar \rangle = \operatorname{Op}_\hbar(\langle A \rangle)$ and hence, modulo the extraction of a subsequence,

$$\lim_{\hbar \to 0^+} \left\langle \langle \widehat{A}_\hbar \rangle v_\hbar^\dagger, v_\hbar^\dagger \right\rangle_{L^2} = \int_{H^{-1}(1)} \langle A \rangle(z) \mu^\dagger(dz).$$

Therefore:

$$\beta = \int_{H^{-1}(1)} \langle A \rangle(z) \mu^\dagger(dz),$$

and the result holds. □

Lemma 4.3. *Let $\mu^\dagger$ be a semiclassical measure associated to the sequence $(v_\hbar^\dagger)$. Then*

$$\operatorname{supp} \mu^\dagger \subset \left\{ z \in H^{-1}(1) \,:\, \beta = \langle A \rangle(z) \right\}. \tag{4.10}$$

Proof. Denote, for any two operators P and Q:

$$\lfloor P, Q \rfloor := PQ + QP,$$

the anticommutator. Using the symbolic calculus for Weyl pseudodifferential operators, we have, for every $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$,

$$\left\langle [\widehat{H}_\hbar + \varepsilon_\hbar \langle \widehat{V}_\hbar \rangle, \operatorname{Op}_\hbar(a)] v_\hbar^\dagger, v_\hbar^\dagger \right\rangle_{L^2(\mathbb{R}^d)} = \frac{\hbar}{i} \left\langle \operatorname{Op}_\hbar(\{H, a\}) v_\hbar^\dagger, v_\hbar^\dagger \right\rangle_{L^2(\mathbb{R}^d)} + O(\varepsilon_\hbar \hbar).$$

On the other hand, using identity (4.9), we also have

$$\begin{aligned}\big\langle [\widehat{H}_\hbar + \varepsilon_\hbar \langle \widehat{V}_\hbar\rangle, \mathrm{Op}_\hbar(a)] v_\hbar^\dagger, v_\hbar^\dagger\big\rangle_{L^2(\mathbb{R}^d)} \\ &= i\hbar \big\langle \lfloor (\langle \widehat{A}_\hbar\rangle - \beta_\hbar), \mathrm{Op}_\hbar(a)\rfloor v_\hbar^\dagger, v_\hbar^\dagger\big\rangle_{L^2(\mathbb{R}^d)} + O(\varepsilon_\hbar \hbar) \\ &= i\hbar \big\langle \mathrm{Op}_\hbar\big(2a(\langle A\rangle - \beta_\hbar)\big) v_\hbar^\dagger, v_\hbar^\dagger\big\rangle_{L^2(\mathbb{R}^d)} + O(\varepsilon_\hbar \hbar).\end{aligned}$$

Then, taking limit $\hbar \to 0^+$, the following equation holds:

$$\int_{\mathbb{R}^{2d}} \{H, a\}\mu^\dagger(dz) = -\int_{\mathbb{R}^{2d}} 2a\big(\langle A\rangle(z) - \beta\big)\mu^\dagger(dz).$$

This is equivalent to

$$\int_{\mathbb{R}^{2d}} a(z)\mu^\dagger(dz) = \int_{\mathbb{R}^{2d}} a\circ\phi_t^H(z)\, e^{2t(\langle A\rangle(z)-\beta)}\mu^\dagger(dz), \tag{4.11}$$

for every $t \in \mathbb{R}$. Moreover, for every $a \in \mathcal{C}_c^\infty(\mathbb{R}^{2d})$ such that $a = \langle a\rangle$, identity (4.11) implies:

$$0 = \int_{\mathbb{R}^{2d}} \langle a\rangle(z)\big(1 - e^{2t(\langle A\rangle(z)-\beta)}\big)\mu^\dagger(dz), \quad \forall t \in \mathbb{R}. \tag{4.12}$$

If $z_0 \notin \{z \in H^{-1}(1) \,:\, \beta = \langle A\rangle(z)\}$, then we can fix a to be a smooth function that does not vanish in a small neighborhood of z_0. As a consequence, $\langle a\rangle$ does not vanish in a small neigborhood of of the Kronecker tori issued from z_0. Then $\mu^\dagger(\langle a\rangle) = 0$. This implies that z_0 does not belong to the support of $\mu^\dagger$ and concludes the proof.

□

Proof of Theorem 1.9. Let us now reproduce the same argument but suppose that $a = \langle a\rangle$, implying in particular that $\{H, \langle a\rangle\} = 0$. From this, since $\Re R = O(\varepsilon_\hbar^2 + \hbar^2)$, we get

$$\begin{aligned}\big\langle [\widehat{H}_\hbar + \varepsilon_\hbar\langle \widehat{V}_\hbar\rangle + \mathrm{Op}_\hbar(\Re R), \mathrm{Op}_\hbar(\langle a\rangle)] v_\hbar^\dagger, v_\hbar^\dagger\big\rangle_{L^2(\mathbb{R}^d)} &= \frac{\hbar\varepsilon_\hbar}{i}\big\langle \mathrm{Op}_\hbar(\{\langle V\rangle, \langle a\rangle\}) v_\hbar^\dagger, v_\hbar^\dagger\big\rangle_{L^2(\mathbb{R}^d)} \\ &\quad + O(\varepsilon_\hbar^2\hbar + \varepsilon_\hbar\hbar^2 + \hbar^3).\end{aligned}$$

As before, one still has

$$\begin{aligned}\big\langle [\widehat{H}_\hbar + \varepsilon_\hbar\langle \widehat{V}_\hbar\rangle + \mathrm{Op}_\hbar(\Re R), \mathrm{Op}_\hbar(\langle a\rangle)] v_\hbar^\dagger, v_\hbar^\dagger\big\rangle_{L^2(\mathbb{R}^d)} \\ &= 2i\hbar\big\langle \mathrm{Op}_\hbar(\langle a\rangle(\langle A\rangle - \beta_\hbar + \Im R\,\hbar^{-1})) v_\hbar^\dagger, v_\hbar^\dagger\big\rangle_{L^2(\mathbb{R}^d)} + O(\|r_\hbar^\dagger\|) + O(\varepsilon_\hbar\hbar^2).\end{aligned}$$

Hence,

$$\left\langle \operatorname{Op}_\hbar\left(\left(2(\langle A\rangle - \beta_\hbar + \Im R\,\hbar^{-1}) + \varepsilon_\hbar X_{\langle V\rangle}\right)\langle a\rangle\right) v_\hbar^\dagger, v_\hbar^\dagger\right\rangle_{L^2(\mathbb{R}^d)} = O(\|r_\hbar^\dagger\|\hbar^{-1}) + O(\varepsilon_\hbar^2) + O(\varepsilon_\hbar\hbar),$$

where $X_{\langle V\rangle}$ is the Hamiltonian vector field of $\langle V\rangle$. Suppose now that $\langle A\rangle \geq 0$ and $\langle a\rangle \geq 0$. Using the Fefferman-Phong inequality (Lemma 2.6) and the assumption $\|r_\hbar^\dagger\| = o(\varepsilon_\hbar\hbar)$, one gets that

$$2\beta_\hbar\left\langle \operatorname{Op}_\hbar(\langle a\rangle)\, v_\hbar^\dagger, v_\hbar^\dagger\right\rangle_{L^2(\mathbb{R}^d)} - 2\left\langle \operatorname{Op}_\hbar\left(\Im R\,\hbar^{-1}\langle a\rangle\right) v_\hbar^\dagger, v_\hbar^\dagger\right\rangle_{L^2(\mathbb{R}^d)}$$
$$\geq \varepsilon_\hbar\left\langle \operatorname{Op}_\hbar\left(X_{\langle V\rangle}\langle a\rangle\right) v_\hbar^\dagger, v_\hbar^\dagger\right\rangle_{L^2(\mathbb{R}^d)} + O(\hbar^2) + o(\varepsilon_\hbar).$$

Now, we would like to show that $\beta_\hbar/\varepsilon_\hbar \to +\infty$. To that end, we proceed by contradiction and suppose that, up to an extraction, one has $2\frac{\beta_\hbar}{\varepsilon_\hbar} \to c_0 \in \mathbb{R}_+$ (in particular $\beta = 0$). Recalling that R was defined by (4.7), we can use the Weyl pseudodifferential calculus (see Section 2.3) and Theorem 4.1 to show that there exists a symbol $R_0 \in S^0(\mathbb{R}^{2d})$ which does not depend on $\hbar$ such that

$$\Im R\,\hbar^{-1} = \varepsilon_\hbar R_0 + o(\varepsilon_\hbar), \quad \text{as } \hbar \to 0.$$

Finally, using that $\varepsilon_\hbar \gg \hbar^2$ and Lemma 4.3 (note that if $\beta = 0$ then $\operatorname{supp}\mu^\dagger \subset H^{-1}(1) \cap \langle A\rangle^{-1}(0)$), one obtains the existence of some constant $C \geq 0$ which does not depend on a such that, after letting $\hbar \to 0^+$,

$$(c_0 + C)\mu^\dagger(\langle a\rangle) \geq \mu^\dagger\left(X_{\langle V\rangle}\langle a\rangle\right).$$

This implies that

$$\frac{d}{dt}\left(e^{-(c_0+C)t}\int_{\mathbb{R}^{2d}}\langle a\rangle \circ \phi_t^{\langle V\rangle} d\mu^\dagger\right) \leq 0, \quad \forall t \in \mathbb{R},$$

where $\phi_t^{\langle V\rangle}$ is the flow generated by $X_{\langle V\rangle}$. Hence, for every $t \geq 0$,

$$\int_{\mathbb{R}^{2d}}\langle a\rangle \circ \phi_t^{\langle V\rangle}(z)\mu^\dagger(dz) \leq e^{(c_0+C)t}\int_{\mathbb{R}^{2d}}\langle a\rangle(z)\mu^\dagger(dz). \tag{4.13}$$

By condition **(WGC)** and compacity of the set $H^{-1}(1) \cap \langle A\rangle^{-1}(0)$, there exist some $T > 0$ such that

$$\int_0^T \langle A\rangle \circ \phi_t^{\langle V\rangle}(z)dt \geq \varepsilon_0 > 0, \quad \forall z \in H^{-1}(1) \cap \langle A\rangle^{-1}(0).$$

Let U be a small neighborhood of $H^{-1}(1)$. Taking $a \in \mathcal{C}_c(U)$ such that

$$\langle a\rangle(z) = \langle A\rangle(z), \quad \forall z \in H^{-1}(1),$$

we obtain, by (4.13) and Lemma 4.3:

$$\varepsilon_0 \leq \int_0^T \int_{\mathbb{R}^{2d}}\langle a\rangle \circ \phi_t^{\langle V\rangle}(z)\mu^\dagger(dz)dt \leq 0,$$

which yields the expected contradiction and concludes the proof of the Theorem. □

4.3. Symbolic calculus in the spaces $\mathcal{A}_s$

In order to prove Theorem 1.10, we start by introducing some basic lemmas about the spaces $\mathcal{A}_s$. First of all, we prove the following version of Calderón-Vaillancourt theorem:

Lemma 4.4 (Calderón-Vaillancourt Theorem). *Let $s > 0$. For any $a \in \mathcal{A}_s(T^*\mathbb{T}^d)$, the following holds:*

$$\| \operatorname{Op}_\hbar(a)\|_{\mathcal{L}(L^2(\mathbb{T}^d))} \leq C_{d,s}\|a\|_s, \tag{4.14}$$

for all $\hbar \in (0,1]$.

Proof. By the standard Calderón-Vaillancourt theorem (Lemma 2.5), for any $a \in \mathcal{C}^\infty(T^*\mathbb{T}^d)$,

$$\| \operatorname{Op}_\hbar(a)\|_{\mathcal{L}(L^2(\mathbb{T}^d))} \leq C_d \sum_{|\alpha|\leq K_d} \sup_{z\in\mathbb{R}^{2d}} |\partial_z^\alpha a(z)|.$$

Then, using

$$\sup_{t\geq 0} t^m e^{-ts} = \left(\frac{m}{se}\right)^m, \quad m > 0, \tag{4.15}$$

we obtain

$$\sup_{z\in\mathbb{R}^{2d}} |\partial_z^\alpha a(z)| \leq \int_{\mathbb{R}^{2d}} |w^\alpha||\widehat{a}(w)|dw \leq \left(\frac{|\alpha|}{se}\right)^{|\alpha|} \int_{\mathbb{R}^{2d}} |\widehat{a}(w)|e^{|w|s}dw = C_{\alpha,s}\|a\|_s,$$

where $\widehat{a}$ denotes the Fourier transform of a in $\mathbb{R}^{2d}$. □

Let $a, b \in \mathcal{A}_s$, the operator given by the composition $\operatorname{Op}_\hbar(a)\operatorname{Op}_\hbar(b)$ is another pseudodifferential operator with symbol c given by the Moyal product $c = a\sharp_\hbar b$, which can be written by the following integral formula (see Section 2.3):

$$c(z) = a\sharp_\hbar b(z) = \frac{1}{(2\pi)^{4d}} \int_{\mathbb{R}^{4d}} \widehat{a}(w')\widehat{b}(w-w')e^{\frac{i\hbar}{2}\{L_{w'},L_{w-w'}\}}e^{iw\cdot z}dw\,dw', \tag{4.16}$$

Recall the notation for the commutator

$$[a,b]_\hbar := a\sharp_\hbar b - b\sharp_\hbar a.$$

Given $a, F \in \mathcal{A}_s$, the following conjugation formula holds formally:

$$e^{\frac{t}{\hbar}\operatorname{Op}_\hbar(F)}\operatorname{Op}_\hbar(a)e^{-\frac{t}{\hbar}\operatorname{Op}_\hbar(F)} = \operatorname{Op}_\hbar(\Psi_{\hbar,t}^F a),$$

where

$$\Psi_{\hbar,t}^F a := \sum_{j=0}^\infty \frac{1}{j!}\left(\frac{it}{\hbar}\right)^j \operatorname{Ad}_F^{\sharp_\hbar,j}(a), \quad t \in \mathbb{R}, \tag{4.17}$$

and

$$\operatorname{Ad}_F^{\sharp_\hbar,j}(a) = [F, \operatorname{Ad}_F^{\sharp_\hbar,j-1}(a)]_\hbar, \quad \operatorname{Ad}_F^{\sharp_\hbar,0}(a) = a.$$

The main goal of this section is to prove the following sharper version of the Egorov's theorem:

Lemma 4.5 (Analytic Egorov's Theorem)**.** *Let $0 < \sigma < s/2$. Consider the family of Fourier integral operators $\{\mathcal{G}_\hbar(t) : t \in \mathbb{R}\}$ defined by*

$$\mathcal{G}_\hbar(t) := e^{\frac{it}{\hbar}(\widehat{G}_{1,\hbar} - i\hbar \widehat{G}_{2,\hbar})},$$

where $\widehat{G}_{j,\hbar} = \mathrm{Op}_\hbar(G_j)$ and $G_j \in \mathcal{A}_s$ are real valued for $j = 1, 2$. Assume

$$\Gamma := \frac{|t|\big(\|G_1\|_s + \hbar\|G_2\|_s\big)}{\sigma^2} \leq \frac{1}{2}.$$

Then, for every $a \in \mathcal{A}_s$:

1. $\Psi_{\hbar,t}^{G_1 - i\hbar G_2} a \in \mathcal{A}_{s-\sigma}$.
2. $a \circ \phi_t^{G_1} \in \mathcal{A}_{s-\sigma}$.
3. $\|\Psi_{\hbar,t}^{G_1 - i\hbar G_2} a - a \circ \phi_t^{G_1}\|_{s-2\sigma} = O_t(\hbar)$.

Before proceeding to the proof, we need some preliminary results.

Lemma 4.6. *For every $a, b \in \mathcal{A}_s$,*

$$\|ab\|_s \leq \|a\|_s \|b\|_s.$$

Proof. The proof makes use of the definition of $\|\cdot\|_s$ and the Young's convolution inequality:

$$\begin{aligned}
\|ab\|_s &= \int_{\mathbb{R}^{2d}} |\widehat{ab}(w)| e^{s|w|} dw \\
&= \int_{\mathbb{R}^{2d}} \left| \int_{\mathbb{R}^{2d}} \widehat{a}(w - w')\widehat{b}(w') dw' \right| e^{s|w|} dw \\
&\leq \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{2d}} |\widehat{a}(w - w')| e^{s|w-w'|} |\widehat{b}(w')| e^{s|w'|} dw'\, dw \\
&\leq \|a\|_s \|b\|_s.
\end{aligned}$$

□

Lemma 4.7. *Let $a, b \in \mathcal{A}_s$. Then, for every $0 < \sigma_1 + \sigma_2 < s$, $[a, b]_\hbar \in \mathcal{A}_{s-\sigma_1-\sigma_2}$ and:*

$$\|[a, b]_\hbar\|_{s-\sigma_1-\sigma_2} \leq \frac{2\hbar}{e^2 \sigma_1(\sigma_1 + \sigma_2)} \|a\|_s \|b\|_{s-\sigma_2}.$$

Proof. By (4.16), we have

$$[a,b]_\hbar(z) = 2i\int_{\mathbb{R}^{4d}} \widehat{a}(w')\widehat{b}(w-w')\sin\left(\frac{\hbar}{2}\{L_{w'},L_{w-w'}\}\right)\frac{e^{iw\cdot z}}{(2\pi)^{4d}}dw'\,dw.$$

Then, using that

$$|\{L_{w'},L_{w-w'}\}| \le 2|w'||w-w'|, \tag{4.18}$$

we obtain:

$$\begin{aligned}
&\|[a,b]_\hbar\|_{s-\sigma_1-\sigma_2}\\
&\quad\le \frac{2\hbar}{(2\pi)^{4d}}\int_{\mathbb{R}^{4d}} |\widehat{a}(w')||w'||\widehat{b}(w-w')||w-w'|e^{(s-\sigma_1-\sigma_2)(|w-w'|+|w'|)}dw'\,dw\\
&\quad\le \frac{2\hbar}{(2\pi)^{4d}}\big(\sup_{r\ge 0} re^{-\sigma_1 r}\big)\big(\sup_{r\ge 0} re^{-(\sigma_1+\sigma_2)r}\big)\|a\|_s\|b\|_{s-\sigma_2}\\
&\quad\le \frac{2\hbar}{e^2\sigma_1(\sigma_1+\sigma_2)}\|a\|_s\|\|_{s-\sigma_2}.
\end{aligned}$$

□

Lemma 4.8. *Let $a,b\in\mathcal{A}_s$ and $0<\sigma<s$. Then there exists $C_\sigma>0$ such that*

$$\|\frac{i}{\hbar}[a,b]_\hbar - \{a,b\}\|_{s-\sigma} \le C_\sigma\,\hbar^2\|a\|_s\|b\|_s. \tag{4.19}$$

Proof. First write:

$$\begin{aligned}
&[a,b]_\hbar(z) + i\hbar\{a,b\}(z)\\
&\quad= 2i\int_{\mathbb{R}^{4d}} \widehat{a}(w')\widehat{b}(w-w')\left(\sin\left(\frac{\hbar}{2}\{L_{w'},L_{w-w'}\}\right)-\frac{\hbar}{2}\{L_{w'},L_{w-w'}\}\right)\frac{e^{iw\cdot z}}{(2\pi)^{4d}}dw'\,dw.
\end{aligned}$$

Using that

$$\sin(x) = x - \frac{x^2}{2}\int_0^1 \sin(tx)(1-t)dt,\quad x\in\mathbb{R},$$

and (4.18), we obtain:

$$\begin{aligned}
&\|[a,b]_\hbar + i\hbar\{a,b\}\|_{s-\sigma}\\
&\quad\le \frac{4\hbar^3}{(2\pi)^{4d}}\int_{\mathbb{R}^{4d}} |\widehat{a}(w')||w'|^3|\widehat{b}(w-w')||w-w'|^3e^{(s-\sigma)(|w-w'|+|w'|)}dw'\,dw\\
&\quad\le C_\sigma\,\hbar^3\|a\|_s\|b\|_{s-\sigma_2}.
\end{aligned}$$

□

Lemma 4.9. *Assume $a, F \in \mathcal{A}_s$. Let $0 < \sigma < s$ and $t \in \mathbb{R}$ such that*

$$\Gamma = \frac{|t|\|F\|_s}{\sigma^2} \le \frac{1}{2},$$

then

$$\left\|\Psi^F_{\hbar,t} a - a\right\|_{s-\sigma} \le \Gamma \|a\|_s.$$

Proof. By definition (4.17), we have

$$\|\Psi^F_{\hbar,t} a - a\|_{s-\sigma} \le \sum_{j=1}^{\infty} \frac{1}{j!}\left(\frac{|t|}{\hbar}\right)^j \|\operatorname{Ad}_F^{\sharp_\hbar, j}(a)\|_{s-\sigma}.$$

Now, for every $j \ge 1$, we use Lemma 4.7 j-times to obtain

$$\|\operatorname{Ad}_F^{\sharp_\hbar, j}(a)\|_{s-\sigma} \le \frac{2^j \hbar^j j^{2j}}{e^{2j}\sigma^{2j} j!} \|a\|_s \|F\|_s^j.$$

By the Stirling formula,

$$\frac{j^j}{e^{j-1} j!} \le 1, \quad j \ge 1,$$

we finally obtain

$$\|\Psi^F_{\hbar,t} a - a\|_{s-\sigma} \le \sum_{j=1}^{\infty} \frac{|t|^j \|F\|_s^j}{e^2 \sigma^{2j}} \|a\|_s \le \Gamma \|a\|_s.$$

□

Corollary 4.1. *Let $a, F \in \mathcal{A}_s$. Assume F is real valued. Let $0 < \sigma < s$ and $t \in \mathbb{R}$ such that*

$$\Gamma := \frac{|t|\|F\|_s}{\sigma^2} \le \frac{1}{2},$$

then

$$\|a \circ \phi_t^F - a\|_{s-\sigma} \le \Gamma \|a\|_s.$$

Proof. The proof use the same estrategy as the one of Lemma 4.9. Note that Lemma 4.7 also holds for $\hbar\{a, b\}$ instead of $[a, b]_\hbar$. □

Proof of Lemma 4.5. Observe that point *(1)* is direct consequence of Lemma 4.9, while point *(2)* is just Corollary 4.1. Let $t \ge 0$. For every $r \in [0, t]$, we define

$$a_r := a \circ \phi_{t-r}^{G_1}.$$

By the Leibniz rule:

$$\frac{d}{dr}\big(\Psi_{\hbar,r}^{G_1-i\hbar G_2}a_r\big) = \Psi_{\hbar,r}^{G_1-i\hbar G_2}\left(\frac{i}{\hbar}[G_1,a_r]_\hbar + [G_2,a_r]_\hbar + \partial_r a_r\right).$$

Moreover,

$$\partial_r a_r = -\{G_1, a_r\}.$$

Using Lemma 4.8, we have

$$\|\frac{i}{\hbar}[G_j,a_r]_\hbar - \{G_j,a_r\}\|_{s-\sigma} = O(\hbar^2), \quad j=1,2.$$

Finally, using these facts and Lemma 4.9 we conclude that:

$$\|\Psi_{\hbar,t}^{G_1-i\hbar G_2}a - a\circ\phi_t^{G_1}\|_{s-2\sigma} \le \int_0^t \|\frac{d}{dr}\big(\Psi_{\hbar,r}^{G_1-i\hbar G_2}a_r\big)\|_{s-2\sigma}dr = O_t(\hbar).$$

□

To conclude the section, we prove the following:

Lemma 4.10. *If $a \in \mathcal{A}_s$ then $\langle a\rangle \in \mathcal{A}_s$ and $\|\langle a\rangle\|_s \le \|a\|_s$.*

Proof. By (3.5), we can write the Fourier transform of $\langle a\rangle$ as

$$\widehat{\langle a\rangle}(w) = \int_{\mathbb{T}_\omega} \widehat{a\circ\Phi_\tau^H}(w)\mathfrak{h}_\omega(d\tau).$$

Moreover, since $\widehat{a\circ\Phi_\tau^H}(w) = \widehat{a}\circ\Phi_\tau^H(w)$ thanks to unitary matrices (3.2), we have that $\widehat{\langle a\rangle} = \langle\widehat{a}\rangle$. Thus, using unitary matrices (3.2) one more time:

$$\begin{aligned}\|\langle a\rangle\|_s &= \int_{\mathbb{R}^{2d}} |\langle\widehat{a}\rangle(w)|e^{s|w|}dw \\ &\le \int_{\mathbb{T}_\omega}\int_{\mathbb{R}^{2d}} |\widehat{a}\circ\Phi_\tau^H(w)|e^{s|w|}dw\,\mathfrak{h}_\omega(d\tau) \\ &= \int_{\mathbb{R}^{2d}} |\widehat{a}(w)|e^{s|w|}dw = \|a\|_s.\end{aligned}$$

□

4.4. Existence of spectral gap in the analytic case

This section is devoted to prove Theorem 1.10. We recall from Section 4.1 that the non-selfadjoint operator $\widehat{P}_\hbar = \widehat{H}_\hbar + \varepsilon_\hbar \widehat{V}_\hbar + i\hbar \widehat{A}_\hbar$ was conjugated by a Fourier integral operator $\mathcal{F}_\hbar$ to a new operator $\widehat{P}_\hbar^\dagger$ given by

$$\widehat{P}_\hbar^\dagger = \widehat{H}_\hbar + \varepsilon_\hbar \langle \widehat{V}_\hbar \rangle + i\hbar \langle \widehat{A}_\hbar \rangle + \widehat{R}_\hbar.$$

In the analytic case, we can obtain the following analytic estimates on the remainder $\widehat{R}_\hbar$ (recall that in this case we assumed that $\varepsilon_\hbar \geq \hbar$):

Proposition 4.1. *If $A, V \in \mathcal{A}_{\rho,s}$, then for every $\sigma < \min\{\rho, s/2\}$, the symbol R of the remainder $\widehat{R}_\hbar = \mathrm{Op}_\hbar(R)$ satisfies*

$$\|\Re R\|_{s-2\sigma} = O(\varepsilon_\hbar^2), \quad \|\Im R\|_{s-2\sigma} = O(\varepsilon_\hbar \hbar).$$

Proof. First we estimate the analytic norms of the solutions of the cohomological equations (4.5) and (4.6). By (1.37), for every $\sigma < \rho$ the following holds:

$$\|F_1\|_s \leq \|F_1\|_{\rho-\sigma,s} \leq \frac{C\nu^\nu}{(\sigma e)^\nu} \|V\|_{\rho,s},$$

$$\|F_2\|_s \leq \|F_2\|_{\rho-\sigma,s} \leq \frac{C\nu^\nu}{(\sigma e)^\nu} \|A\|_{\rho,s}.$$

On the other hand, recalling that the symbol of $\widehat{K}_\hbar(t)$ is given by

$$K(t) = t(\varepsilon_\hbar V + i\hbar A) + (1-t)(\varepsilon_\hbar \langle V \rangle + i\hbar \langle A \rangle),$$

we use Lemma 4.10 to obtain

$$\|\Re K(t)\|_s = O(\varepsilon_\hbar), \quad \|\Im K(t)\|_s = O(\hbar).$$

From this and Lemma 4.7, one can show that:

$$\|\Re [F, K(t)]_\hbar\|_{s-\sigma} = O(\varepsilon_\hbar^2),$$

$$\|\Im [F, K(t)]_\hbar\|_{s-\sigma} = O(\varepsilon_\hbar \hbar).$$

Finally, for $\hbar$ sufficiently small, the condition

$$\Gamma_\hbar = \frac{\hbar}{\sigma^2} \|F\|_{s-\sigma} \leq \frac{1}{2}$$

holds. Then we use Lemma 4.5 to conclude that

$$\|\Re R\|_{s-2\sigma} = O(\varepsilon_\hbar^2), \quad \|\Im R\|_{s-2\sigma} = O(\varepsilon_\hbar \hbar).$$

□

Lemma 4.11. *Assume $\varepsilon_\hbar = \hbar$ and $\sigma < \min\{\rho, s/3\}$. Let $F_3 \in \mathcal{A}_{s-2\sigma}$, define the Fourier integral operator*

$$\widetilde{\mathcal{F}}_\hbar := e^{\frac{\delta}{\hbar}\operatorname{Op}_\hbar(\langle F_3\rangle)}, \quad \delta > 0.$$

Then there exists $\delta_0 = \delta_0(s, \sigma, F_3) > 0$ such that, for every $0 < \delta \leq \delta_0$,

$$\widehat{P}_\hbar^{\dagger\dagger} := \widetilde{\mathcal{F}}_\hbar \widehat{P}_\hbar^\dagger \widetilde{\mathcal{F}}_\hbar^{-1} = \widehat{H}_\hbar + \hbar\langle\widehat{V}_\hbar\rangle + i\hbar\langle\widehat{A}_\hbar\rangle - i\delta\hbar\operatorname{Op}_\hbar(\{\langle F_3\rangle, \langle V\rangle\}) + \widehat{R}'_\hbar,$$

where the remainder term $\widehat{R}'_\hbar = \operatorname{Op}_\hbar(R')$ satisfies

$$\|R'\|_{s-3\sigma} = O(\delta^2\hbar). \tag{4.20}$$

Proof. Recall that, by (4.17), for every $a \in \mathcal{A}_{s-2\sigma}$ we have

$$\widetilde{\mathcal{F}}_\hbar \operatorname{Op}_\hbar(a)\widetilde{\mathcal{F}}_\hbar^{-1} = \operatorname{Op}_\hbar\left(\Psi_{\hbar,\delta/\hbar}^{-i\langle F_3\rangle} a\right),$$

where the symbol $\Psi_{\hbar,\delta/\hbar}^{-i\langle F_3\rangle} a$ can be expanded as

$$\begin{aligned}\Psi_{\hbar,\delta/\hbar}^{-i\langle F_3\rangle} a &= \sum_{j=0}^\infty \frac{1}{j!}\left(\frac{\delta}{\hbar}\right)^j \operatorname{Ad}_{\langle F_3\rangle}^{\sharp_\hbar, j}(a) \\ &= a + \frac{\delta}{\hbar}[\langle F_3\rangle, a]_\hbar + \left(\frac{\delta}{\hbar}\right)^2 \int_0^1 (1-t)\Psi_{\hbar,t\delta/\hbar}^{-i\langle F_3\rangle}\big([\langle F_3\rangle, [\langle F_3\rangle, a]_\hbar]_\hbar\big)dt.\end{aligned}$$

Using Lemmas 4.7 and 4.9, we obtain the existence of some $\delta_0 = \delta_0(s, \sigma, F_3) > 0$ so that the following estimate holds for every $0 < \delta \leq \delta_0$:

$$\left\|\Psi_{\hbar,\delta/\hbar}^{-i\langle F_3\rangle} a - a - \frac{\delta}{\hbar}[\langle F_3\rangle, a]_\hbar\right\|_{s-3\sigma} \leq C_\sigma \delta^2 \|a\|_{s-2\sigma}. \tag{4.21}$$

We use this estimate for $a = \hbar\langle V\rangle$, $a = i\hbar\langle A\rangle$ and $a = R$ given by Proposition 4.1. Moreover, by Lemma 4.8,

$$\frac{\delta}{\hbar}\operatorname{Op}_\hbar\left([\langle F_3\rangle, \hbar\langle V\rangle]_\hbar\right) = -i\delta\hbar\operatorname{Op}_\hbar\left(\{\langle F_3\rangle, \langle V\rangle\}\right) + O(\delta\hbar^3).$$

On the other hand,

$$[\operatorname{Op}_\hbar(\langle F_3\rangle), \widehat{H}_\hbar] = \frac{\hbar}{i}\operatorname{Op}_\hbar(\{\langle F_3\rangle, H\}) = 0,$$

hence $\widetilde{\mathcal{F}}_\hbar \widehat{H}_\hbar \widetilde{\mathcal{F}}_\hbar^{-1} = \widehat{H}_\hbar$. The result then holds. ◻

Lemma 4.12. *Assume $\varepsilon_\hbar \gg \hbar$ and $\sigma < \min\{\rho, s/3\}$. Let $F_3 \in \mathcal{A}_{s-2\sigma}$, define the Fourier integral operator*

$$\widetilde{\mathcal{F}}_\hbar := e^{\frac{1}{\varepsilon_\hbar}\operatorname{Op}_\hbar(\langle F_3\rangle)}.$$

Then

$$\widehat{P}_\hbar^{\dagger\dagger} := \widetilde{\mathcal{F}}_\hbar \widehat{P}_\hbar^{\dagger} \widetilde{\mathcal{F}}_\hbar^{-1} = \widehat{H}_\hbar + \varepsilon_\hbar \langle \widehat{V}_\hbar\rangle + i\hbar\langle \widehat{A}_\hbar\rangle - i\hbar \operatorname{Op}_\hbar(\{\langle F_3\rangle, \langle V\rangle\}) + \widehat{R}'_\hbar,$$

where the remainder term $\widehat{R}'_\hbar = \operatorname{Op}_\hbar(R')$ satisfies

$$\|R'\|_{s-3\sigma} = o(\hbar). \tag{4.22}$$

Proof. By (4.17), for every $a \in \mathcal{A}_{s-2\sigma}$, we have

$$\widetilde{\mathcal{F}}_\hbar \operatorname{Op}_\hbar(a)\widetilde{\mathcal{F}}_\hbar^{-1} = \operatorname{Op}_\hbar\left(\Psi^{-i\langle F_3\rangle}_{\hbar,1/\varepsilon_\hbar} a\right),$$

where now the symbol $\Psi^{-i\langle F_3\rangle}_{\hbar,1/\varepsilon_\hbar} a$ can be expanded as

$$\begin{aligned}\Psi^{-i\langle F_3\rangle}_{\hbar,1/\varepsilon_\hbar} a &= \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{1}{\varepsilon_\hbar}\right)^j \operatorname{Ad}^{\sharp_\hbar, j}_{\langle F_3\rangle}(a)\\ &= a + \frac{\delta}{\varepsilon_\hbar}[\langle F_3\rangle, a]_\hbar + \left(\frac{1}{\varepsilon_\hbar}\right)^2 \int_0^1 (1-t)\Psi^{-i\langle F_3\rangle}_{\hbar,t/\varepsilon_\hbar}\big([\langle F_3\rangle,[\langle F_3\rangle,a]_\hbar]_\hbar\big)dt.\end{aligned}$$

Using Lemmas 4.7 and4.9 we obtain the following estimate for $\hbar$ sufficiently small:

$$\left\|\Psi^{-i\langle F_3\rangle}_{\hbar,1/\varepsilon_\hbar} a - a - \frac{1}{\varepsilon_\hbar}[\langle F_3\rangle, a]_\hbar\right\|_{s-3\sigma} = O\left(\frac{\hbar^2}{\varepsilon_\hbar^2}\right)\cdot \|a\|_{s-2\sigma}. \tag{4.23}$$

We use this estimate for $a = \varepsilon_\hbar\langle V\rangle$, $a = i\hbar\langle A\rangle$ and $a = R$ given by Proposition 4.1. Moreover, by Lemma 4.8,

$$\frac{1}{\varepsilon_\hbar}\operatorname{Op}_\hbar\left([\langle F_3\rangle, \varepsilon_\hbar\langle V\rangle]_\hbar\right) = -i\hbar\operatorname{Op}_\hbar\left(\{\langle F_3\rangle, \langle V\rangle\}\right) + O(\hbar^3).$$

Finally, the result holds after observing that

$$[\operatorname{Op}_\hbar(\langle F_3\rangle), \widehat{H}_\hbar] = \frac{\hbar}{i}\operatorname{Op}_\hbar(\{\langle F_3\rangle, H\}) = 0,$$

and hence $\widetilde{\mathcal{F}}_\hbar \widehat{H}_\hbar \widetilde{\mathcal{F}}_\hbar^{-1} = \widehat{H}_\hbar$. □

Proof of Theorem 1.10. Assume first that $\varepsilon_\hbar = \hbar$. Set

$$v_\hbar^{\dagger\dagger} := \frac{\widetilde{\mathcal{F}}_\hbar \mathcal{F}_\hbar v_\hbar}{\|\widetilde{\mathcal{F}}_\hbar \mathcal{F}_\hbar v_\hbar\|_{L^2}}.$$

By the identity

$$\big\langle \widehat{P}_\hbar^{\dagger\dagger} v_\hbar^{\dagger\dagger}, v_\hbar^{\dagger\dagger} \big\rangle_{L^2} = \lambda_\hbar \| v_\hbar^{\dagger\dagger}\|_{L^2}^2,$$

we can use the Lemma 4.11 and take imaginary parts to get

$$\big\langle \operatorname{Op}_\hbar(\langle A\rangle - \delta\{\langle F_3\rangle, \langle V\rangle\}) v_\hbar^{\dagger\dagger}, v_\hbar^{\dagger\dagger} \big\rangle_{L^2} = \beta_\hbar + O(\delta^2).$$

Recall that $-\{\langle F_3\rangle, \langle V\rangle\} = X_{\langle V\rangle}\langle F_3\rangle$. Thus, modulo extracting a subsequence, there exists a probability measure $\mu^{\dagger\dagger} \in \mathcal{P}(H^{-1}(1))$ such that

$$\lim_{\hbar\to 0} \big\langle \operatorname{Op}_\hbar(\langle A\rangle + \delta X_{\langle V\rangle}\langle F_3\rangle) v_\hbar^{\dagger\dagger}, v_\hbar^{\dagger\dagger} \big\rangle_{L^2} = \int_{H^{-1}(1)} \big(\langle A\rangle(z) + \delta X_{\langle V\rangle}\langle F_3\rangle\big)\mu^{\dagger\dagger}(dz).$$

Then

$$\beta = \int_{H^{-1}(1)} \big(\langle A\rangle(z) + \delta X_{\langle V\rangle}\langle F_3\rangle\big)\mu^{\dagger\dagger}(dz) + O(\delta^2). \tag{4.24}$$

Observe that, since $\langle A\rangle$ and $\langle V\rangle$ are analytic, condition **(WGC)** implies that, for every point $z \in H^{-1}(1) \cap \langle A\rangle^{-1}(0)$, there exists $T > 0$[1] such that

$$\int_0^T \langle A\rangle \circ \phi_t^{\langle V\rangle}(z) dt > 0.$$

Now define

$$\langle F_3\rangle(z) = \frac{1}{T}\int_0^T \int_0^t \langle A\rangle \circ \phi_s^{\langle V\rangle}(z)\, ds\, dt, \quad z \in \mathbb{R}^{2d}. \tag{4.25}$$

By Lemma 4.5, for T sufficiently small, $\langle F_3\rangle \in \mathcal{A}_{s-\sigma}$. Moreover,

$$X_{\langle V\rangle}\langle F_3\rangle = \frac{1}{T}\int_0^T \langle A\rangle \circ \phi_t^{\langle V\rangle}(z) dt > 0, \quad \forall z \in H^{-1}(1) \cap \langle A\rangle^{-1}(0).$$

Finally, substituting $\langle F_3\rangle$ defined by (4.25) in (4.24) and taking $\delta > 0$ sufficiently small, we obtain

$$\beta \geq \min_{z\in H^{-1}(1)} \left\{ (1-\delta)\langle A\rangle(z) + \frac{\delta}{T}\int_0^T \langle A\rangle \circ \phi_t^{\langle V\rangle}(z) \right\} + O(\delta^2) = \varepsilon(A, V) > 0.$$

It remains to show (1.58) provided that $\varepsilon_\hbar \gg \hbar$. Let $T > 0$ satisfying

$$T < \frac{\sigma^2}{2\|\langle V\rangle\|_s}, \tag{4.26}$$

[1] Note that this T can be taken as small as necessary since the function $t \mapsto \langle A\rangle \circ \phi_t^{\langle V\rangle}(z)$ is analytic and then it can not be flat at $t = 0$.

where $\sigma < \min\{\rho, s/3\}$. Note that the function $\langle F_3\rangle$ solves the cohomological equation

$$\{\langle V\rangle, \langle F_3\rangle\} = \mathcal{A}_T^{\langle V\rangle}(A) - \langle A\rangle, \quad \mathcal{A}_T^{\langle V\rangle}(A) := \frac{1}{T}\int_0^T \langle A\rangle \circ \phi_t^{\langle V\rangle} dt, \tag{4.27}$$

and, moreover, by Corollary 4.1 and condition (4.26), it satisfies $\|\langle F_4\rangle\|_{s-\sigma} \leq 2T\|\langle A\rangle\|_s$. We now consider the Fourier integral operator

$$\widetilde{\mathcal{F}}_\hbar := e^{\frac{1}{\varepsilon_\hbar}\operatorname{Op}_\hbar(\langle F_3\rangle)},$$

which verifies, in view of cohomological equation (4.27), that

$$\begin{aligned}\widehat{P}_\hbar^{\dagger\dagger} = \widetilde{\mathcal{F}}_\hbar \widehat{P}_\hbar^{\dagger} \widetilde{\mathcal{F}}_\hbar^{-1} &= \widehat{H}_\hbar + \varepsilon_\hbar \langle \widehat{V}_\hbar\rangle + i\hbar\langle \widehat{A}_\hbar\rangle - i\hbar \operatorname{Op}_\hbar(\{\langle F_4\rangle, \langle V\rangle\}) + \widehat{R}_\hbar' \\ &= \widehat{H}_\hbar + \varepsilon_\hbar \langle \widehat{V}_\hbar\rangle + i\hbar \operatorname{Op}_\hbar(\mathcal{A}_T^{\langle V\rangle}(A)) + \widehat{R}_\hbar'.\end{aligned}$$

By Lemma 4.12, the remainder term $\widehat{R}_\hbar' = \operatorname{Op}_\hbar(R')$ satisfies

$$\|R'\|_{s-3\sigma} = o(\hbar)$$

provided that $\varepsilon_\hbar \gg \hbar$. Finally, as we did before, we can set

$$v_\hbar^{\dagger\dagger} := \frac{\widetilde{\mathcal{F}}_\hbar \mathcal{F}_\hbar v_\hbar}{\|\widetilde{\mathcal{F}}_\hbar \mathcal{F}_\hbar v_\hbar\|_{L^2}}.$$

By the identity

$$\big\langle \widehat{P}_\hbar^{\dagger\dagger} v_\hbar^{\dagger\dagger}, v_\hbar^{\dagger\dagger}\big\rangle_{L^2} = \lambda_\hbar \|v_\hbar^{\dagger\dagger}\|_{L^2}^2,$$

we can take imaginary parts to get

$$\big\langle \operatorname{Op}_\hbar(\mathcal{A}_T^{\langle V\rangle}(A)) v_\hbar^{\dagger\dagger}, v_\hbar^{\dagger\dagger}\big\rangle_{L^2} = \beta_\hbar + o(1).$$

Then, modulo the extraction a subsequence, there exists a probability measure $\mu^{\dagger\dagger} \in \mathcal{P}(H^{-1}(1))$ such that

$$\beta = \int_{H^{-1}(1)} \mathcal{A}_T^{\langle V\rangle}(A)(z)\mu^{\dagger\dagger}(dz). \tag{4.28}$$

The result then holds.

□

Chapter 5

Quantum limits for KAM families of vector fields on the torus

—Siempre está usted descubriendo mediterráneos, amigo Mairena.

—Es el destino ineluctable de todos los navegantes, amigo Tortolez.

A. Machado. *Juan de Mairena.*

This chapter is devoted to the study of the asymptotic properties of solutions of the eigenvalue problem

$$\widehat{P}_{\omega,\hbar}\,\Psi_\hbar = \lambda_\hbar\,\Psi_\hbar, \quad \|\Psi_\hbar\|_{L^2(\mathbb{T}^d)} = 1, \tag{5.1}$$

where the semiclassical operator $\widehat{P}_{\omega,\hbar}$ is given by

$$\widehat{P}_{\omega,\hbar} := \omega\cdot\hbar D_x + v(x;\omega)\cdot\hbar D_x - \frac{i\hbar}{2}\operatorname{Div} v(x;\omega),$$

with $\omega \in \mathbb{R}^d$ belonging to some neighborhood of a Cantor set of Diophantine frequencies, and $v \in \mathcal{C}^\omega(\mathbb{T}^d\times\mathbb{R}^d;\mathbb{R}^d)$. In Section 5.1 we state Egorov's theorem in the particular case of linear Hamiltonians. In Section 5.2 we recall a classical KAM theorem due to Moser [89] on small perturbations of Diophantine vector fields on the torus. For the sake of simplicity, we will assume analyticity of the involved vector fields but it is most likely that our results remain valid with more general regularity assumptions. In the proof of the classical KAM theorem we will follow the work of Pöschel [99] that simplifies the KAM iterative argument. In Sections 5.3 and 5.4 we prove the main results of this chapter concerning the phase-space accumulation of mass of sequences of eigenfunctions of $\widehat{P}_{\omega,\hbar}$ as $\hbar \to 0$.

5.1. Egorov's theorem for linear Hamiltonians

In this section we state Egorov's theorem in the particular case of Hamiltonians with linear symbols (see Section 2.5 for the general statement).

Given $V(x,\xi) = \xi\cdot v(x)$ and $W(x,\xi) = \xi\cdot w(x)$ with $v, w \in \mathcal{C}^\infty(\mathbb{T}^d)$, as a consequence of (2.40), we have:

$$[V,W]_\hbar(x,\xi) = \frac{\hbar}{i}\{V,W\}(x,\xi) = \frac{\hbar\,\xi}{i}\cdot\big([\partial_x w(x)]v(x) - [\partial_x v(x)]w(x)\big), \tag{5.2}$$

where $\{\cdot,\cdot\}$ stands for the Poisson bracket. Let

$$\mathscr{F}_T := \{F(t,x,\xi) = \xi\cdot f(t,x),\ f\in\mathcal{C}^\infty([0,T]\times\mathbb{T}^d)\}$$

be a smooth family of hamiltonians, we consider the classical system of Hamilton equations

$$\begin{cases} \dot{x}(t) &= f(t,x(t)), \\ \dot{\xi}(t) &= -[\partial_x f(t,x(t))]\,\xi(t), \quad 0\le t\le T. \end{cases} \tag{5.3}$$

The solution of (5.3) for initial data $(x,\xi)\in T^*\mathbb{T}^d$ is given by the symplectic lift of the diffeomorphism ϕ_t^f:

$$\Phi_t^F(x,\xi) = \big(\phi_t^f(x), [(\partial_x\phi_t^f(x))^T]^{-1}\xi\big), \tag{5.4}$$

where ϕ_t^f is the flow on $\mathbb{T}^d$ solving the first equation of (5.3) with $\phi_0^f(x) = x$.

Reciprocally, let $\{\phi_t : \mathbb{T}^d\to\mathbb{T}^d\ :\ t\in[0,T]\}$ be a smooth family of diffeomorphisms of the torus, then, denoting

$$f(t,x) = \frac{d}{dt}\,\phi_t(x),$$

we can define the smooth family of linear hamiltonians

$$\mathscr{F}_T := \{F(t,x,\xi) = \xi\cdot f(t,x),\ f\in\mathcal{C}^\infty([0,T]\times\mathbb{T}^d)\},$$

with related flow Φ_t^F given by (5.4).

As for the quantum counterpart, given $F(x,\xi) = \xi\cdot f(x)$ with $f\in\mathcal{C}^\infty(\mathbb{T}^d)$, the operator $\mathrm{Op}_\hbar(F)$ is essentially selfadjoint on $H^1(\mathbb{T}^d)$. Then, by Stone's Theorem,

$$\Big\{U_\hbar^F(t) := e^{-\frac{i}{\hbar}t\,\mathrm{Op}_\hbar(F)}\ :\ t\in\mathbb{R}\Big\}$$

defines a family of unitary operators on $L^2(\mathbb{T}^d)$. To be precise, the propagator is given by the unitary transfer operator associated with the diffeomorphism ϕ_t^f:

$$U_\hbar^F(t)\,u(x) = u\big(\phi_t^f(x)\big)\sqrt{|\det d\phi_t^f(x)|}. \tag{5.5}$$

More generally, let

$$\mathscr{F}_T := \{F(t,x,\xi) = \xi\cdot f(t,x),\ f\in\mathcal{C}^\infty([0,T]\times\mathbb{T}^d)\}$$

be a smooth family of linear hamiltonians, then the operator equation

$$\begin{cases} \hbar D_t U_\hbar^F(t) + U_\hbar^F(t)\,\mathrm{Op}_\hbar(F) = 0 \\ \qquad\qquad U_\hbar^F\big|_{t=0} = I, \quad 0 \le t \le T, \end{cases} \tag{5.6}$$

has a unique solution of unitary operators $\{U_\hbar^F(t)\}_{0\le t\le T}$ on $L^2(\mathbb{T}^d)$ given by (5.5), where ϕ_t^f in this case denotes the flow associated to the time-dependent vector field $f = f(t,x)$.

Lemma 5.1 (Egorov's theorem for linear Hamiltonians)**.** *Let*

$$\mathscr{F}_T := \{F(t,x,\xi) = \xi \cdot f(t,x),\ f \in \mathcal{C}^\infty([0,T]\times\mathbb{T}^d)\}$$

be a smooth family of linear hamiltonians and let $V(x,\xi) := \xi \cdot v(x)$ with $v \in \mathcal{C}^\infty(\mathbb{T}^d)$. Then

$$U_\hbar^F(-t)\,\mathrm{Op}_\hbar(V)U_\hbar^F(t) = \mathrm{Op}_\hbar(V \circ \Phi_t^F), \quad 0 \le t \le T, \tag{5.7}$$

where $\Phi_t^F : T^*\mathbb{T}^d \to T^*\mathbb{T}^d$ is the classical flow generated by the Hamiltonian $F = F(t,\cdot)$.

Remark 5.1. Notice that this is an exact Egorov's theorem.

Proof. The identiy is clearly true for $t = 0$. The left-hand-side of (5.7) satisfies the Heisenberg-von Neumann equation

$$\frac{d}{dt}\Big(U_\hbar^F(-t)\,\mathrm{Op}_h(V)U_\hbar^F(t)\Big) = \frac{i}{\hbar}\big[\,\mathrm{Op}_\hbar(F), U_\hbar^F(-t)\,\mathrm{Op}_\hbar(V)U_\hbar^F(t)\big].$$

On the other hand, the right-hand-side of (5.7) satisfies the equation

$$\frac{d}{dt}\,\mathrm{Op}_\hbar(V \circ \Phi_t^F) = \mathrm{Op}_h(\{F, V \circ \Phi_t^F\}).$$

Then we have

$$\begin{aligned} &\frac{d}{ds}\big(U_\hbar^F(-s)\,\mathrm{Op}_\hbar(V \circ \Phi_{t-s}^F)U_\hbar^F(s)\big) \\ &\quad = U_\hbar^F(-s)\left(\frac{i}{\hbar}\left[\mathrm{Op}_\hbar(F), \mathrm{Op}_\hbar(V \circ \Phi_{t-s}^F)\right] - \mathrm{Op}_\hbar(\{F, V \circ \Phi_{t-s}^F\})\right)U_\hbar^F(s), \end{aligned}$$

and thus

$$\begin{aligned} &U_\hbar^F(-t)\,\mathrm{Op}_\hbar(V)U_\hbar^F(t) - \mathrm{Op}_\hbar(V \circ \Phi_t^F) \\ &\quad = \int_0^t U_\hbar^F(-s)\left(\frac{i}{\hbar}\left[\mathrm{Op}_\hbar(F), \mathrm{Op}_\hbar(V \circ \Phi_{t-s}^F)\right] - \mathrm{Op}_\hbar(\{F, V \circ \Phi_{t-s}^F\})\right)U_\hbar^F(s)ds. \end{aligned}$$

We observe that $V \circ \Phi_t^F$ is a linear symbol of the form:

$$V \circ \Phi_t^F(x,\xi) = v \circ \phi_t^f(x) \cdot [(\partial_x \phi_t^f(x))^T]^{-1}\xi,$$

where $\phi_t^f : \mathbb{T}^d \to \mathbb{T}^d$ is the diffeomorphism of $\mathbb{T}^d$ given by the solution of the evolution equation

$$\begin{cases} \dot{x}(t) = f(t, x(t)), \\ x(0) = x. \end{cases}$$

Using the exact formula for the commutator of two linear symbols, we obtain

$$\frac{i}{\hbar}\left[\mathrm{Op}_\hbar(F), \mathrm{Op}_\hbar(V \circ \Phi_{t-s}^F)\right] = \mathrm{Op}_\hbar(\{F, V \circ \Phi_{t-s}^F\}). \tag{5.8}$$

□

We will also use the following version of Egorov's theorem:

Lemma 5.2. *Let*

$$\mathscr{F}_T := \{F(t,x,\xi) = \xi \cdot f(t,x),\ f \in \mathcal{C}^\infty([0,T] \times \mathbb{T}^d)\}$$

be a smooth family of linear hamiltonians. Then, for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$:

$$U_\hbar^F(-t)\,\mathrm{Op}_\hbar(a) U_\hbar^F(t) = \mathrm{Op}_\hbar(a \circ \Phi_t^F) + O_T(\hbar^2), \quad 0 \le t \le T, \tag{5.9}$$

where the O_T is taken in the $L^2 \to L^2$ strong operator norm.

The proof is standard and follows the same scheme as the one given in the proof of Lemma 5.1. The commutator appearing instead of (5.8) is not exact in this case, but it can be bounded using the commutator rule for pseudodifferential calculus and the Calderón-Vaillancourt theorem. The error term $\hbar^2$ is genuine of the Weyl quantization. We omit the details here and refer the reader to [25].

5.2. A CLASSICAL KAM THEOREM

In this section we recall the result of Pöschel [99]. We use the Diophantine property (1.60) for the sake of simplicity, but the more general *Rüssmann condition* considered in [99] would be valid aswell.

Theorem 5.1 ([99])**.** *Let $\Omega \subset \mathbb{R}^d$ be a compact set of strongly nonresonant frequencies, that is, $\omega \in \Omega$ satisfies* (1.60)*. Let $s, \rho > 0$ and $V \in \mathscr{L}_{s,\rho}$ such that*

$$|V|_{s,\rho} = \varepsilon < \frac{\rho}{16} \le \frac{\varsigma}{32\lambda^\gamma}, \tag{5.10}$$

where λ is so large that

$$r := 8\gamma \left(\frac{1 + \log \lambda}{\lambda} \right) < \frac{s}{2}. \tag{5.11}$$

Then there exists a real map $\varphi : \Omega \to \Omega_\rho$, and for every $\omega \in \Omega$ a real analytic diffeomorphism θ_ω of the d-torus such that, denoting

$$\Theta_\omega(x, \xi) = \big(\theta_\omega(x), [(\partial_x \theta_\omega(x))^T]^{-1} \xi\big),$$

the following holds:

$$\big(\mathcal{L}_{\varphi(\omega)} + V(\cdot; \varphi(\omega))\big) \circ \Theta_\omega = \mathcal{L}_\omega. \tag{5.12}$$

Moreover,

$$\sup_{\omega \in \Omega} |\varphi(\omega) - \omega| \le 7\varepsilon, \quad \sup_{\omega \in \Omega} \sup_{x \in \mathbb{T}^d} |\theta_\omega(x) - x| \le r\, \varsigma^{-1} \lambda^\gamma \varepsilon. \tag{5.13}$$

5.2.1. Symbolic calculus in the spaces $\mathscr{L}_s$

We first prove the following two technical lemmas.

Lemma 5.3. *Let $V \in \mathscr{L}_s$*[1] *and $W \in \mathscr{L}_{s'}$. Then, for $0 < r < \min\{s, s'\}$,*

$$|\{V, W\}|_r \le \frac{1}{e} \left(\frac{1}{s - r} + \frac{1}{s' - r} \right) |V|_s |W|_{s'}.$$

Proof. By definition,

$$V(x, \xi) = \xi \cdot v(x) = \sum_{k \in \mathbb{Z}^d} \xi \cdot \widehat{v}(k) e_k(x),$$

$$W(x, \xi) = \xi \cdot w(x) = \sum_{k \in \mathbb{Z}^d} \xi \cdot \widehat{w}(k) e_k(x).$$

We have

$$[\partial_x v(x)] w(x) = \sum_{k,l \in \mathbb{Z}^d} \big(ik \cdot \widehat{w}(l)\big) \widehat{v}(k) e_{k+l}(x) = \sum_{k,l \in \mathbb{Z}^d} \big(ik \cdot \widehat{w}(l - k)\big) \widehat{v}(k) e_l(x),$$

$$[\partial_x w(x)] v(x) = \sum_{k,l \in \mathbb{Z}^d} \big(il \cdot \widehat{v}(k)\big) \widehat{w}(l) e_{k+l}(x) = \sum_{k,l \in \mathbb{Z}^d} \big(i(l - k) \cdot \widehat{v}(k)\big) \widehat{w}(l - k) e_l(x).$$

[1] That is $V(x, \xi) = v(x) \cdot \xi$. We emply lower case letters to denote the vector depending on x.

Then, using the second equality of (5.2) and the Young's convolution inequality,

$$\begin{aligned}
|\{V,W\}|_r &\le \sum_{k,l\in\mathbb{Z}^d} \big(|k|+|l-k|\big)|\widehat{v}(k)||\widehat{w}(l-k)|e^{|l|r} \\
&\le \sum_{k,l\in\mathbb{Z}^d} \big(|k|+|l-k|\big)|\widehat{v}(k)|e^{|k|r}|\widehat{w}(l-k)|e^{|l-k|r} \\
&\le \left(\sup_{t\ge 0} te^{-(s-r)t} + \sup_{t\ge 0} te^{-(s'-r)t}\right) \sum_{k,l\in\mathbb{Z}^d} |\widehat{v}(k)|e^{|k|s}|\widehat{w}(l-k)|e^{|l-k|s'} \\
&\le \frac{1}{e}\left(\frac{1}{s-r}+\frac{1}{s'-r}\right)\left(\sum_{k\in\mathbb{Z}^d}|\widehat{v}(k)|e^{|k|s}\right)\left(\sum_{l\in\mathbb{Z}^d}|\widehat{w}(l)|e^{|l|s'}\right) \\
&\le \frac{1}{e}\left(\frac{1}{s-r}+\frac{1}{s'-r}\right)|V|_s|W|_{s'}.
\end{aligned}$$

□

Lemma 5.4. *Let $F \in \mathscr{L}_{s+\lambda\sigma}$ with $0<\sigma<s$ and $\lambda>0$. If*

$$\beta := \frac{|F|_{s+\lambda\sigma}}{\sigma} \le \frac{1}{2},$$

then

$$|V\circ\Phi_t^F|_{s-\sigma} \le (1+\beta t)e^{1/\lambda}|V|_s, \quad 0\le t\le 1.$$

Proof. The proof follows by estimating the Lie series expansion of $V\circ\Phi_t^F$. Formally, we have

$$V\circ\Phi_t^F = \sum_{n=0}^{\infty}\frac{t^n}{n!}\operatorname{Ad}_F^n(V),$$

where $\operatorname{Ad}_F^n(V) = \{F, \operatorname{Ad}_F^{n-1}(V)\}$ for $n\ge 1$ and $\operatorname{Ad}_F^0(V) = V$. By the preceding lemma, for every $n\ge 1$,

$$\begin{aligned}
|\operatorname{Ad}_F^n(V)|_{s-\sigma} &= |\{F,\operatorname{Ad}_F^{n-1}(V)\}|_{s-\sigma} \\
&\le \left(\frac{1}{e\sigma}+\frac{1}{e\lambda\sigma}\right)|\operatorname{Ad}_F^{n-1}(V)|_{s-\frac{n-1}{n}\sigma}\|F\|_{s+\lambda\sigma} \\
&= \frac{n}{e\sigma}\left(1+\frac{1}{\lambda n}\right)|\operatorname{Ad}_F^{n-1}(V)|_{s-\frac{n-1}{n}\sigma}\|F\|_{s+\lambda\sigma}.
\end{aligned}$$

Applying this step n times, we obtain

$$|\operatorname{Ad}_F^n(V)|_{s-\sigma} \leq \left(\frac{n}{e\sigma}\right)^n e^{1/\lambda}\|V\|_s\|F\|_{s+\lambda\sigma}^n.$$

Summing up and replacing $\sigma^{-1}|F|_{s+\lambda\sigma}$ by β we get

$$|V \circ \Phi_t^F|_{s-\sigma} \leq e^{1/\lambda}|V|_s \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{n\beta t}{e}\right)^n.$$

With the Stirling estimate

$$n! \geq \frac{n^n}{e^{n-1}}, \quad n \geq 1,$$

and $0 \leq \beta t \leq 1/2$, we conclude that

$$\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{n\beta t}{e}\right)^n \leq 1 + \sum_{n\geq 1} \frac{(\beta t)^n}{e} \leq 1 + \beta t.$$

□

5.2.2. Outline of the proof

We first recall the general structure of the iterative KAM argument. We aim at finding suitable sequences of positive real numbers $(s_j)_{j\geq 1}$, $(\rho_j)_{j\geq 1}$ so that

$$s = s_0 > s_1 > \cdots \to s - 2r > 0,$$
$$\rho = \rho_0 > \rho_1 > \rho_2 > \cdots \to 0,$$

for some $0 < r < s/2$, a sequence of real analytic maps $\varphi_N : \Omega_{\rho_N} \to \Omega_\rho$, and a sequence of real analytic transformations $\theta_N : D_{s_N} \times \Omega_{\rho_N} \to D_s$ such that, denoting

$$\Theta_N(x, \xi; \omega) := (\theta_N(x;\omega), [(\partial_x \theta_N(x;\omega))^T]^{-1}\xi),$$

the following holds:

$$\Big(\mathcal{L}_{\varphi_N(\omega)} + V(\cdot;\varphi_N(\omega))\Big) \circ \Theta_N(\cdot;\omega) = \mathcal{L}_\omega + V_N(\cdot;\omega). \tag{5.14}$$

Moreover, the remainder term $V_N \in \mathscr{L}_{s_N,\rho_N}$ satisfies

$$|V_N|_{s_N,\rho_N} \leq q^N |V|_{s,\rho} \tag{5.15}$$

for some universal constant $0 < q < 1$, and the transformation (φ_N, θ_N) satisfies the following uniform estimates:

$$\sup_{\omega\in\Omega_{\rho_N}} |\varphi_N(\omega) - \omega| \leq 7\varepsilon, \quad \sup_{\omega\in\Omega_{\rho_N}} \sup_{z\in D_{s_N}} |\theta_N(z;\omega) - z| \leq r\varsigma^{-1}\lambda^\gamma\varepsilon. \tag{5.16}$$

The proof concludes by applying Weierstrass and Montel theorems to obtain, modulo a subsequence, a real map

$$\varphi = \lim_{N\to\infty} \varphi_N,$$

and a real analytic diffeomorphism of the torus $\mathbb{T}^d$,

$$\theta_\omega = \lim_{N\to\infty} \theta_N(\cdot;\omega),$$

verifying (5.12) and (5.13).

5.2.3. Step Lemma

The key ingredient in the proof of Theorem 5.1 is the following step lemma, which will be iterated successively.

Lemma 5.5. *Let $0 < \sigma < s/2$, and $\lambda \geq 1$. Set $\alpha = 1 - e^{-\lambda\sigma}$, and let $V \in \mathscr{L}_{s,\rho}$ be such that*

$$|V|_{s,\rho} = \varepsilon < \min\left\{\frac{\rho}{2\alpha}, \frac{\varsigma}{4\lambda^\gamma}\right\}, \quad \rho \leq \frac{\varsigma}{2\lambda^\gamma}. \tag{5.17}$$

Then there exist two real analytic functions $\psi : \Omega_{\rho-2\varepsilon} \to \Omega_\rho$ and $\phi : D_{s-2\sigma} \times \Omega_{\rho-2\varepsilon} \to D_s$ such that, denoting

$$\Phi(x,\xi;\omega) = (\phi(x;\omega), [(\partial_x\phi(x;\omega))^T]^{-1}\xi),$$

the following holds:

$$\Big(\mathcal{L}_{\psi(\omega)} + V(\cdot;\psi(\omega))\Big) \circ \Phi(\cdot;\omega) = \mathcal{L}_\omega + V^+(\cdot;\omega), \tag{5.18}$$

where

$$|V^+|_{s-2\sigma,\rho-2\varepsilon} \leq q\varepsilon,$$

and

$$q = (1 - \alpha + \alpha^2\beta)(1+\beta)e^\alpha, \quad \beta := 2\varsigma^{-1}\lambda^\gamma\varepsilon.$$

Moreover,

$$\sup_{\omega\in\Omega_{\rho-2\varepsilon}} |\psi(\omega) - \omega| \leq \alpha\varepsilon, \quad \sup_{z\in D_{s-2\sigma}} \sup_{\omega\in\Omega_{\rho-2\varepsilon}} |\phi(z;\omega) - z| \leq \beta\sigma.$$

Proof. The idea of the proof is to split V into an infrared part V_1 and a (mostly) ultraviolet part V_2 and obtain Φ as the time-1-map of a flow Φ_t^F, with F solving the cohomological equation

$$\{\mathcal{L}_\omega, F\} = V_1 - \langle V_1 \rangle, \tag{5.19}$$

where

$$\langle V_1 \rangle(\xi;\omega) := \xi \cdot \widehat{v}(0;\omega).$$

Using Taylor's theorem, one can write

$$(\mathcal{L}_\omega + V) \circ \Phi_1^F = \mathcal{L}_\omega + \{F, \mathcal{L}_\omega\} + \int_0^1 (1-t)\{F, \{F, \mathcal{L}_\omega\}\} \circ \Phi_t^F dt$$

$$+ V_1 + \int_0^1 \{F, V_1\} \circ \Phi_t^F dt + V_2 \circ \Phi_1^F.$$

Substituting the cohomological equation (5.19), this can be simplified as

$$(\mathcal{L}_\omega + V) \circ \Phi_1^F = \mathcal{L}_\omega + \langle V_1 \rangle + R,$$

where

$$R = \int_0^1 \{F, W_t\} dt + V_2 \circ \Phi_1^F,$$

and

$$W_t = tV_1 + (1-t)\langle V_1 \rangle, \quad t \in [0,1].$$

The transformation ψ will be obtained as the inverse, defined in a suitable subdomain of Ω_ρ, of the map

$$\omega \longmapsto \omega + \widehat{v}(0;\omega).$$

This yields

$$\Big(\mathcal{L}_{\psi(\omega)} + V(\cdot;\psi(\omega))\Big) \circ \Phi_1^{F(\cdot;\psi(\omega))}(\cdot;\psi(\omega)) = \mathcal{L}_\omega + R(\cdot;\psi(\omega)).$$

Finally, defining

$$V^+(\cdot;\omega) := R(\cdot, \psi(\omega)), \quad \Phi(\cdot;\omega) := \Phi_1^{F(\cdot;\psi(\omega))}(\cdot;\psi(\omega)),$$

we obtain (5.18).

Now we proceed to the heart of the proof. Define

$$V_2(x,\xi;\omega) := \sum_{|k|\geq\lambda} \xi \cdot \widehat{v}(k;\omega) e_k(x) + (1-\alpha) \sum_{|k|<\lambda} \xi \cdot \widehat{v}(k;\omega) e^{|k|\sigma} e_k(x).$$

Since $e^{-\lambda\sigma} = 1 - \alpha$,

$$|V_2|_{s-\sigma,\rho} \leq (1-\alpha)|V|_{s,\rho} = (1-\alpha)\varepsilon.$$

On the other hand, the remainder term V_1 is a trigonometric polynomial

$$V_1(x,\xi;\omega) = \sum_{0\le|k|<\lambda} \xi\cdot\widehat{v}_1(k;\omega)e_k(x), \quad \widehat{v}_1(k,\omega) := (1-(1-\alpha)e^{|k|\sigma})\widehat{v}(k;\omega),$$

which can be bounded in a stronger norm $|\cdot|_{s+\widetilde{\sigma},\rho}$, with $\widetilde{\sigma} = \sigma(1-\alpha)/\alpha$:

$$|V_1|_{s+\widetilde{\sigma},\rho} \le \sup_{0\le t\le\lambda}(1-(1-\alpha)e^{t\sigma})e^{t\widetilde{\sigma}} \sum_{|k|<\lambda}\sup_{\omega\in\Omega_\rho}|\widehat{v}(k;\omega)|e^{|k|s} \le \alpha\varepsilon.$$

The cohomological equation (5.19) is solved by

$$F(x,\xi;\omega) = \xi\cdot f(x;\omega) = \sum_{0<|k|<\lambda}\xi\cdot\frac{\widehat{v}_1(k;\omega)}{ik\cdot\omega}\,e_k(x).$$

For any $\omega\in\Omega_\rho$ we can choose $\omega_0\in\Omega$ with

$$|\omega-\omega_0| < \rho \le \frac{\varsigma}{2\lambda^\gamma},$$

and hence, in view of $\Lambda(\lambda) = \lambda\Delta(\lambda)$,

$$|k\cdot(\omega-\omega_0)| \le |k||\omega-\omega_0| \le \lambda\rho \le \frac{\lambda\varsigma}{2\lambda^\gamma} = \frac{\varsigma}{2\lambda^{\gamma-1}}.$$

On the other hand, as ω_0 satisfies (1.60), we have

$$|k\cdot\omega| \ge |k\cdot\omega_0| - |k\cdot(\omega-\omega_0)| \ge \frac{\varsigma}{\lambda^{\gamma-1}} - \frac{\varsigma}{2\lambda^{\gamma-1}} = \frac{\varsigma}{2\lambda^{\gamma-1}}.$$

Then, using $\alpha = 1-e^{-\lambda\sigma} \le \lambda\sigma$ and the definition $\beta := 2\varsigma^{-1}\lambda^\gamma\varepsilon$, we get

$$|F|_{s+\widetilde{\sigma},\rho} \le 2\varsigma^{-1}\lambda^{\gamma-1}|V_1|_{s+\widetilde{\sigma},\rho} \le 2\varsigma^{-1}\lambda^{\gamma-1}\alpha\varepsilon \le \beta\sigma. \tag{5.20}$$

In particular, by (5.17), $|F|_{s+\widetilde{\sigma},\rho}\le\sigma$, so the function $f(\cdot;\omega)$ generates a flow

$$\phi_t^f(\cdot;\omega) : D_{s-2\sigma} \to D_{s-\sigma}$$

such that

$$\sup_{z\in D_{s-2\sigma}}|\phi_t^f(z;\omega)-z| \le \sup_{z\in D_{s-2\sigma}}\int_0^t |f(\phi_u^f(z;\omega);\omega)|du \le \beta\sigma,$$

for all $\omega\in\Omega_\rho$ and all $t\in[0,1]$. We define $\phi := \phi_1^f$.

To estimate V^+, observe that

$$|W_t|_{s+\widetilde{\sigma},\rho} \le t|V_1|_{s+\widetilde{\sigma},\rho} + (1-t)|\langle V_1\rangle|_{s+\widetilde{\sigma},\rho} \le \alpha\varepsilon,$$

and

$$(s+\widetilde{\sigma})-(s-\sigma)=\frac{1-\alpha}{\alpha}\sigma+\sigma=\frac{\sigma}{\alpha}.$$

Lemma 5.3 thus implies

$$|\{F,W_t\}|_{s-\sigma,\rho}\leq\frac{\alpha}{\sigma}|W_t|_{s+\widetilde{\sigma},\rho}|F|_{s+\widetilde{\sigma},\rho}\leq\lambda^\gamma 2\varsigma^{-1}\alpha^2\varepsilon^2=\alpha^2\beta\varepsilon.$$

Since F satisfies (5.20), we can apply Lemma 5.4 to obtain

$$\int_0^1|\{F,W_t\}\circ\Phi_t^F|_{s-2\sigma}dt\leq(1+\beta)e^\alpha|\{F,W_t\}|_{s-\sigma}\leq\alpha^2\beta(1+\beta)e^\alpha\varepsilon.$$

Analogously,

$$|V_2\circ\Phi_t^F|_{s-2\sigma}\leq(1+\beta)e^\alpha|V_2|_{s-\sigma}\leq(1-\alpha)(1+\beta)e^\alpha\varepsilon.$$

Both estimates together yield the stated estimate of V^+.

It remains to prove the existence of a map $\psi:\Omega_{\rho-2\varepsilon}\to\Omega_\rho$ such that

$$\psi(\omega+\widehat{v}(0;\omega))=\omega.$$

It follows from the following lemma:

Lemma 5.6. *Assume $f:\Omega_\rho\to\mathbb{C}^d$ is analytic and*

$$\sup_{\omega\in\Omega_\rho}|f(\omega)-\omega|\leq\varepsilon<\frac{\rho}{2}.$$

Then f has an analytic inverse $\psi:\Omega_{\rho-2\varepsilon}\to\Omega_\rho$, and

$$\sup_{\omega\in\Omega_{\rho-2\varepsilon}}|\psi(\omega)-\omega|\leq\varepsilon.$$

Proof. By the Cauchy's inequality, for any $0<\rho'<\rho-2\varepsilon$,

$$\sup_{\omega\in\Omega_{\rho'+\varepsilon}}|Df(\omega)-\omega|\leq\frac{\sup_{\omega\in\Omega_\rho}|f(\omega)-\omega|}{\rho-(\rho'+\varepsilon)}\leq\frac{\varepsilon}{\rho-(\rho'+\varepsilon)}<1.$$

Therefore, the operator

$$T:\psi\mapsto\operatorname{Id}-(f-\operatorname{Id})\circ\psi$$

defines a contraction on the space of analytic maps

$$\mathcal{A}_{\rho,\rho',\varepsilon}:=\{\psi:\Omega_{\rho'}\to\Omega_\rho\,,\quad\sup_{\omega\in\Omega_{\rho'}}|\psi(\omega)-\omega|\leq\varepsilon\}.$$

Its unique fixed point ψ is the analytic inverse of f on $\Omega_{\rho'}$. Letting $\rho'\to\rho-2\varepsilon$ we obtain the claim. Note that we can take the limit since, for all $\rho'<\rho-2\varepsilon$,

$$\psi:\Omega_{\rho'}\to\Omega_\rho,$$

hence we can define ψ in the open set $\Omega_{\rho-2\varepsilon}$. □

Finally, since

$$\sup_{\omega\in\Omega_\rho} |\widehat{v}(0;\omega)| \leq |V_1|_{s,\rho} \leq \alpha\varepsilon < \frac{\rho}{2},$$

Lemma 5.6 implies the existence of $\psi : \Omega_{\rho-2\varepsilon} \to \Omega_\rho$ such that

$$\psi(\omega + \widehat{v}(0;\omega)) = \omega,$$

and

$$\sup_{\omega\in\Omega_{\rho-2\varepsilon}} |\psi(\omega) - \omega| \leq \alpha\varepsilon.$$

This concludes the proof of the step lemma. □

5.2.4. ITERATION

We explain now how to iterate the Step Lemma. First observe that one can fix $0 < \alpha < 1$ and $0 < \beta \leq 1/2$ so that

$$q = (1-\alpha+\alpha^2\beta)(1+\beta)e^{\alpha} < 1.$$

One possible choice of the constants is $\alpha = 1/2$ and $\beta = 1/16$. This provides

$$q \approx \frac{9}{10}.$$

Now assume $\varepsilon_0, s_0, \rho_0, \lambda_0 > 0$ satisfy the following initial condition:

$$|V|_{s_0,\rho_0} = \varepsilon_0 < \min\left\{\frac{1-q}{2\alpha}\rho_0, \frac{\varsigma\beta}{2\lambda_0^\gamma}\right\}, \quad \rho_0 \leq \frac{\varsigma}{2\lambda_0^\gamma}, \tag{5.21}$$

Define geometric sequences

$$\varepsilon_N = \varepsilon_0 q^N, \quad \rho_N = \rho_0 q^N, \quad \lambda_N = \lambda_0 q^{-N/\gamma}.$$

and define also σ_N and s_N through

$$1-\alpha = e^{-\lambda_N\sigma_N}, \quad s_{N+1} = s_N - 2\sigma_N.$$

With these sequences, one can apply the Step Lemma repeatedly. Indeed,

$$\varepsilon_N\lambda_N^\gamma = \varepsilon_0\lambda_0^\gamma < \beta \leq \frac{1}{2}, \quad \rho_N\lambda_N^\gamma = \rho_0\lambda_0^\gamma < \frac{\varsigma}{2},$$

and

$$\varepsilon_N < \frac{1-q}{2}\rho_N, \quad \frac{\rho_N - 2\varepsilon_N}{\rho_{N+1}} = \frac{\rho_0 - 2\varepsilon_0}{q\rho_0} \geq 1.$$

Then we obtain sequences $\psi_N : \Omega_{\rho_N} \to \Omega_{\rho_{N-1}}$ and $\phi_N : D_{s_N} \times \Omega_{\rho_N} \to D_{s_{N-1}}$ such that, defining:

$$\begin{aligned}
\varphi_0 &:= \mathrm{Id} : \Omega_{\rho_0} \to \Omega_{\rho_0}, \\
\varphi_N &:= \psi_1 \circ \cdots \circ \psi_N : \Omega_{\rho_N} \to \Omega_{\rho_0}, \\
\varphi_{j,N} &:= \varphi_j^{-1} \circ \varphi_N : \Omega_{\rho_N} \to \Omega_{\rho_j}, \quad j = 0, \dots, N, \\
\theta_N &:= \phi_1(\cdot; \varphi_{1,N}(\cdot)) \circ \cdots \circ \phi_N(\cdot; \varphi_{N,N}(\cdot)) : \Omega_{\rho_N} \times D_{s_N} \to D_{s_0}, \\
\Theta_N &:= \big(\theta_N, [(d_x\theta_N)^T]^{-1}\big),
\end{aligned}$$

identity (5.14) and estimate (5.15) hold. Note that

$$\phi_j(\cdot; \varphi_{j,N}(\omega)) = \phi_t^{f_{j,N}}|_{t=1}, \quad j = 1, \dots, N, \tag{5.22}$$

where $f_{j,N} = f_j(\cdot; \varphi_{j,N}(\omega))$ and the f_j are obtained at each application of the step lemma. Moreover,

$$\begin{aligned}
\sup_{\omega \in \Omega_{\rho_N}} |\varphi_N(\omega) - \omega| &\le \sum_{j=0}^{N-1} \sup_{\omega \in \Omega_{\rho_N}} |\varphi_{j,N}(\omega) - \varphi_{j+1,N}(\omega)| \\
&\le \sum_{j=0}^{N-1} \sup_{\omega \in \Omega_{\rho_j}} |\psi_j(\omega) - \omega| \\
&\le \alpha\varepsilon_0 \sum_{j=0}^{N-1} q^j \le \frac{\alpha\varepsilon_0}{1-q},
\end{aligned}$$

and

$$\begin{aligned}
\sup_{z \in D_{s_N}} \sup_{\omega \in \Omega_{\rho_N}} |\theta_N(z;\omega) - z| &\le \sum_{j=0}^{N-1} \sup_{\omega \in \Omega_{\rho_j}} \sup_{z \in D_{s_j}} |\phi_j(z;\omega) - z| \\
&\le \beta \sum_{j=0}^{N-1} \sigma_j.
\end{aligned}$$

It remains to prove that, for λ_0 sufficiently large, the sequence (s_j) converges to a positive number. Indeed,

$$\sum_{N=1}^{\infty} \frac{1}{\lambda_N} \le \frac{1}{\lambda_0} \int_0^\infty q^{u/\gamma} du = \frac{1}{\lambda_0} \frac{\gamma}{\log q^{-1}}.$$

Then, requiring $\lambda_0^\gamma \geq q^{-1}$, we obtain

$$\sum_{N=0}^{\infty} \frac{1}{\lambda_N} \leq \frac{1}{\log q^{-1}} \frac{\gamma(1+\log \lambda_0)}{\lambda_0}.$$

From this, it follows that

$$r := \sum_{N=0}^{\infty} \sigma_N = \sum_{N \geq 0} \frac{\log(1-\alpha)^{-1}}{\lambda_N} \leq \frac{\log(1-\alpha)}{\log q^{-1}} \cdot \frac{\gamma(1+\log \lambda_0)}{\lambda_0}.$$

Hence, by choosing λ_0 sufficiently large, we can achieve that $r < s/2$, and thus

$$s_N \to s - 2r > 0.$$

The choice of the constants $\alpha = 1/2$ and $\beta = 1/16$ provides

$$\frac{\log(1-\alpha)}{\log q} \leq 8.$$

Thus, the hypothesis of Theorem 5.1 are sufficient to obtain (5.21) and thus to initialize the Step Lemma.

5.2.5. Isotopic deformation of the diffeomorphism θ_ω

In this section we prove the following:

Proposition 5.1. *There exists a smooth isotopy $\mathcal{H}_\omega : [0,1] \times \mathbb{T}^d \to \mathbb{T}^d$ so that*

$$\mathcal{H}_\omega(0,\cdot) = \mathrm{Id}, \quad \mathcal{H}_\omega(1,\cdot) = \theta_\omega.$$

Proof. The diffeomorphism $\theta_\omega : \mathbb{T}^d \to \mathbb{T}^d$ is given by

$$\theta_\omega = \lim_{N\to\infty} \theta_N(\cdot;\omega) = \lim_{N\to\infty} \phi_t^{f_{1,N}} \circ \cdots \circ \phi_t^{f_{N,N}}\big|_{t=1},$$

for some real analytic functions $f_{j,N} : D_{s-2r} \to \mathbb{C}$. With the notation of (5.22),

$$f_{j,N}(z) = f_j(z; \varphi_{j,N}(\omega)), \quad j = 1, \ldots, N.$$

We recover the time dependence to define:

$$H_N(t;z) = \phi_t^{f_{1,N}} \circ \cdots \circ \phi_t^{f_{N,N}}(z), \quad t \in [0,1].$$

The sequence (H_N) is uniformly bounded in $z \in D_{s-2r}$. Moreover, it is uniformly bounded and equicontinuous in $t \in [0,1]$. Indeed, the following holds:

Lemma 5.7. *Let $0 < 2r < s' < s$. Then, for every $N \geq 1$:*

$$\sup_{t\in[0,1]} \sup_{z\in D_{s-2r}} |H_N(t,z) - z| \leq r,$$

$$\sup_{t\in[0,1]} \sup_{z\in D_{s'-2r}} \left|\frac{d}{dt} H_N(t;z)\right| \leq re^{\Gamma r};$$

where

$$\Gamma := \frac{1}{e(s-s')}.$$

Proof. For every $1 \leq j \leq N$,

$$\left.\begin{aligned} \frac{d}{dt}\phi_t^{f_{j,N}}(z) = f_{j,N}(\phi_t^{f_{j,N}}(z)) \\ \phi_0^{f_{j,N}}(z) = z \end{aligned}\right\}. \tag{5.23}$$

Denoting $F_{j,N}(x,\xi) := \xi \cdot f_{j,N}(x)$, by (5.20) and Section 5.2.4,

$$\sup_{z\in D_{s-r}} |f_{j,N}(z)| \leq |F_{j,N}|_{s-r} \leq \sigma_{j-1}, \tag{5.24}$$

where the sequence (σ_j) satisfies

$$r = \sum_{j=0}^{\infty} \sigma_j < \frac{s}{2}.$$

By (5.23) and (5.24),

$$\sup_{z\in D_{s-2r}} |\phi_t^{f_{j,N}}(z) - z| \leq \sup_{z\in D_{s-2r}} \int_0^t |f_{j,N}(\phi_u^{f_{j,N}}(z))| du \leq \sigma_{j-1},$$

and thus

$$\sup_{z\in D_{s-2r}} |H_N(t,z) - z| \leq \sum_{j=0}^{N-1} \sigma_j \leq r.$$

On the other hand,

$$\frac{d}{dt}\partial_z\phi_t^{f_{j,N}}(z) = \left[\partial_z f_{j,N}\left(\phi_t^{f_{j,N}}(z)\right)\right]\partial_z\phi_t^{f_{j,N}}(z).$$

Since $\partial_z \phi_0^{f_{j,N}}(z) = \mathrm{Id}$ and, in view of (5.24), for every $2r < s' < s$,

$$\begin{aligned}\sup_{z\in D_{s'-r}} |\partial_z f_{j,N}(z)| &\le |\partial_z F_{j,N}|_{s'-r} \\ &\le \sum_{k\in\mathbb{Z}^d} |k||\widehat{f}_{j,N}(k)|e^{|k|(s'-r)} \\ &\le \sup_{u\ge 0} u\, e^{-u(s-s')}|F_{j,N}|_{s-r} \\ &\le \frac{\sigma_{j-1}}{e(s-s')} = \Gamma\sigma_{j-1},\end{aligned}$$

we can use the Gronwall inequality to obtain

$$\sup_{z\in D_{s'-r}} |\partial_z \phi_t^{f_{j,N}}(z)| \le e^{\Gamma\sigma_{j-1}}.$$

Therefore, using the chain rule

$$\frac{d}{dt}(\phi_t^f \circ g(t,z)) = f\big(\phi_t^f \circ g(t,z)\big) + \big[\partial_z \phi_t^f(g(t,z))\big]\frac{d}{dt}g(t,z)$$

successively, we get

$$\sup_{t\in[0,1]} \sup_{z\in D_{s'-2r}} \left|\frac{d}{dt}H_N(t,z)\right| \le \sigma_0 + e^{\Gamma\sigma_0}\sigma_1 + \cdots + e^{\Gamma(\sigma_0+\cdots+\sigma_{N-2})}\sigma_{N-1} \le re^{\Gamma r}.$$

□

Then by Arzela-Ascoli and Montel theorems, modulo extracting a subsequence, there exists a limit $H_\omega(t,z)$ which is analytic in the variable $z \in D_{s'-2r}$ and continuous in the variable $t \in [0,1]$. Moreover, $H_\omega(1,\cdot) = \theta_\omega$ and $H_\omega(0,\cdot) = \mathrm{Id}$.

Finally, by the Whitney approximation theorem [75, Thm. 6.29], there exists a smooth homotopy

$$\mathcal{H}_\omega : [0,1] \times \mathbb{T}^d \to \mathbb{T}^d$$

with $\mathcal{H}_\omega(0,\cdot) = \mathrm{Id}$ and $\mathcal{H}_\omega(1,\cdot) = \theta_\omega$. This concludes the proof of Proposition 5.1.

□

5.3. CONSTRUCTION OF THE UNITARY OPERATOR $\mathcal{U}_\omega$

Proof of Theorem 1.12. By Proposition 5.1, there exists a smooth homotopy $\mathcal{H}_\omega : [0,1]\times\mathbb{T}^d \to \mathbb{T}^d$ such that

$$\mathcal{H}_\omega(0,x) = x, \quad \mathcal{H}_\omega(1,x) = \theta_\omega(x).$$

We define the vector field

$$f(t,x) := \frac{d}{dt}\mathcal{H}_\omega(t,x),$$

and the associated smooth family of hamiltonians

$$\mathscr{F}_T = \{F(t;x,\xi) = f(t,x)\cdot\xi, \quad t\in[0,1]\}. \tag{5.25}$$

We construct the operator $\mathcal{U}_\omega$ as the solution at time $t=1$ of the operator equation

$$\begin{cases} \hbar D_t U_\hbar^F(t) + U_\hbar^F(t)\,\mathrm{Op}_\hbar(F) = 0 \\ \qquad\qquad\qquad\qquad U_\hbar^F\big|_{t=0} = I, \quad 0\le t\le 1, \end{cases} \tag{5.26}$$

for the family (5.25). Finally, the exact Egorov's theorem given by Lemma 5.1 implies

$$\mathcal{U}_\omega^*\,\widehat{P}_{\varphi(\omega),\hbar}\mathcal{U}_\omega = \mathrm{Op}_\hbar\left((\mathcal{L}_{\varphi(\omega)} + V(\cdot;\varphi(\omega)))\circ\Phi_1^F\right),$$

and, since

$$\Theta_\omega(x,\xi) := \Phi_1^F(x,\xi) = \left(\theta_\omega(x), (\partial_x\theta_\omega(x)^T)^{-1}\xi\right),$$

we conclude using Theorem 5.1:

$$\mathcal{U}_\omega^*\,\widehat{P}_{\varphi(\omega),\hbar}\mathcal{U}_\omega = \mathrm{Op}_\hbar\left((\mathcal{L}_{\varphi(\omega)} + V(\cdot;\varphi(\omega)))\circ\Theta_\omega\right) = \widehat{L}_{\omega,\hbar}.$$

□

5.4. SEMICLASSICAL MEASURES AND QUANTUM LIMITS

In this section we prove Theorems 1.11 and 1.13. First we give the proof of Proposition 1.3:

Proof of Proposition 1.3. We recall that the point-spectrum of $\widehat{L}_{\omega,\hbar}$ is given by

$$\mathrm{Sp}^p_{L^2(\mathbb{T}^d)}\left(\widehat{L}_{\omega,\hbar}\right) = \{\lambda_{k,\hbar} = \hbar\,\omega\cdot k \,:\, k\in\mathbb{Z}^d\}.$$

Each eigenvalue has multiplicity equal to 1 due to the nonresonant condition on ω. The associated eigenfunction is just

$$e_k(x) = \frac{e^{ik\cdot x}}{(2\pi)^{d/2}}.$$

By a direct calculation using identity (2.42) for the Wigner distribution on the torus, for every test function $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$, the following holds:

$$W_{e_k}^\hbar(a) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} a(x,\hbar k)\,dx, \quad k \in \mathbb{Z}^d.$$

Equivalently, $W_{e_k}^\hbar = \mathfrak{h}_{\mathbb{T}^d\times\{\hbar k\}}$. Given a sequence

$$\lambda_{k_j,\hbar_j} = \hbar_j\,\omega\cdot k_j \to 1, \quad \text{as } \hbar_j \to 0, \tag{5.27}$$

the only possible accumulation points of the sequence $(\hbar_j k_j)$ are precisely those points $\xi \in \mathcal{L}_\omega^{-1}(1)$, and then the only possible accumulation points of subsequences of measures $(W_{e_{k_j}})$ are $\mathfrak{h}_{\mathbb{T}^d\times\{\xi\}}$ for some $\xi \in \mathcal{L}_\omega^{-1}(1)$. Reciprocally, any point $\xi \in \mathcal{L}_\omega^{-1}(1)$ can be obtained as the limit of a sequence $(\hbar_j k_j)$ satisfying (5.27), and hence any measure $\mathfrak{h}_{\mathbb{T}^d\times\{\xi\}}$ is the semiclassical measure associated to a sequence of eigenfunctions. In other words, $\mu \in \mathcal{M}\big(\widehat{L}_{\omega,\hbar}\big)$ if an only if $\mu = \mathfrak{h}_{\mathbb{T}^d\times\{\xi\}}$ for some point $\xi \in \mathcal{L}_\omega^{-1}(1)$.

The second assertion is trivial since

$$|e_k(x)|^2 = \frac{1}{(2\pi)^d}, \quad k \in \mathbb{Z}^d.$$

□

Proof of Theorems 1.11 and 1.13. Since

$$\widehat{L}_{\omega,\hbar} = \mathcal{U}_\omega^*\,\widehat{P}_{\varphi(\omega),\hbar}\,\mathcal{U}_\omega,$$

where $\mathcal{U}_\omega$ is unitary on $L^2(\mathbb{T}^d)$, the spectrum of $\widehat{P}_{\varphi(\omega),\hbar}$ is the same as the spectrum of $\widehat{L}_{\omega,\hbar}$, and the eigenfunctions are precisely

$$\Psi_k = \mathcal{U}_\omega e_k, \quad k \in \mathbb{Z}^d.$$

Then, applying Lemma 5.2,

$$W_{\Psi_k}^\hbar(a) = W_{e_k}^\hbar(a\circ\Theta_\omega) + O(\hbar^2), \quad a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d),$$

and similarly, using (5.5),

$$\int_{\mathbb{T}^d} b(x)|\Psi_k(x)|^2 dx = \int_{\mathbb{T}^d} b\circ\theta_\omega(x)|e_k(x)|^2 dx, \quad b \in \mathcal{C}^\infty(\mathbb{T}^d), \tag{5.28}$$

and then the proofs of Theorems 1.11 and 1.13 reduce to the proof of Proposition 1.3. □

Proof of Corollary 1.1. We can fix constants $0 < \hbar_0 \leq 1$, $s_0 = s$, $0 < \rho_0 \leq \rho$ and r_0 such that, for every $0 < \hbar \leq \hbar_0$, conditions (5.10) and (5.11) hold for $\varepsilon_\hbar V$ instead of V. Then, applying Theorem 5.1, there exist $\varphi_\hbar : \Omega \to \Omega_\rho$ and $\theta_{\omega,\hbar} : \mathbb{T}^d \to \mathbb{T}^d$ satisfying

$$\big(\mathcal{L}_{\varphi_\hbar(\omega)} + \varepsilon_\hbar V(\cdot;\varphi(\omega))\big) \circ \Theta_{\omega,\hbar} = \mathcal{L}_\omega,$$

where $\Theta_{\omega,\hbar} = \big(\theta_{\omega,\hbar}, [(d\theta_{\omega,\hbar})^T]^{-1}\big)$, and

$$\sup_{\omega\in\Omega} |\varphi_\hbar(\omega) - \omega| \leq C_1\varepsilon_\hbar, \quad \sup_{\omega\in\Omega}\sup_{x\in\mathbb{T}^d} |\theta_{\omega,\hbar}(x) - x| \leq C_2\varepsilon_\hbar.$$

Thus we can apply Theorem 1.12 to obtain a sequence of unitary operators $\mathcal{U}^\varepsilon_{\omega,\hbar}$ conjugating $\widehat{P}^\varepsilon_{\varphi_\hbar(\omega),\hbar}$ into $\widehat{L}_{\omega,\hbar}$. Using the same argument as in the proof of Theorem 1.11, we apply (5) to conclude

$$\begin{aligned}\int_{\mathbb{T}^d} b(x)|\Psi_{k,\hbar}(x)|^2 dx &= \int_{\mathbb{T}^d} b\circ\theta_{\omega,\hbar}(x)|e_k(x)|^2 dx \\ &= \int_{\mathbb{T}^d} b(x)|e_k(x)|^2 dx + O(\varepsilon_\hbar),\end{aligned}$$

for every $b \in \mathcal{C}^\infty(\mathbb{T}^d)$. The result then holds by Proposition 1.3. □

Chapter 6

Renormalization of semiclassical KAM operators

> Cet univers désormais sans maître ne lui paraît ni stérile ni futile. Chacun des grains de cette pierre, chaque éclat minéral de cette montagne plein de nuit, à lui seul, forme un monde. La lutte elle-même vers les sommets suffit à remplir un coeur d'homme. Il faut imaginer Sishyphe heureux.
>
> A. Camus. *Le mythe de Sishyphe.*

This chapter is devoted to study the renormalization problem in the semiclassical framework. Given $V \in \mathcal{A}_s(T^*\mathbb{T}^d)$, we aim at finding an integrable counterterm $R_\hbar = R_\hbar(V) \in \mathcal{A}_{s/2}(\mathbb{R}^d)$ that only depends on ξ so that the operator

$$\widehat{Q}_\hbar = \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \operatorname{Op}_\hbar(V - R_\hbar)$$

is unitarily equivalent to the unperturbed operator $\widehat{L}_{\omega,\hbar}$. In Section 6.1 we will construct a normal form for $\widehat{Q}_\hbar$ which will allow us to obtain $R_\hbar$ step by step. Precisely, we will use an algorithm similar to that of Govin et. al. in [47] for the finite dimesional case. Our proof of the convergence will use standard ideas of classic KAM theory adapted to the Weyl pseudodifferential calculus for analytic symbols. In Section 6.2 we will obtain the characterization of the set of quantum limits and semiclassical measures for $\widehat{Q}_\hbar$. The key ingredient will be a precise estimate of the remainder terms appearing in the analytic symbolic calculus.

6.1. KAM iterative algorithm

In this section we explain the iterative argument we will use to prove Theorem 1.14. First of all, we redefine the spaces of analytic symbols we will work with all along this chapter:

Definition 6.1. *Given $s > 0$, we define the Banach space $\mathcal{A}_s(\mathbb{R}^d)$ of functions $f \in \mathcal{C}^\omega(\mathbb{R}^d;\mathbb{R})$ such that*

$$|f|_s := \int_{\mathbb{R}^d} |\widehat{f}(\eta)|\, e^{|\eta| s}\, d\eta < \infty,$$

where $\widehat{f}$ denotes the Fourier transform of f. We introduce also the Banach space $\mathcal{A}_s(T^*\mathbb{T}^d)$ of analytic functions $g \in \mathcal{C}^\omega(T^*\mathbb{T}^d;\mathbb{R})$ such that

$$\|g\|_s := \sum_{k\in\mathbb{Z}^d} |\widehat{g}(k,\cdot)|_s\, e^{|k| s} < \infty,$$

where

$$\widehat{g}(k,\xi) := \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g(x,\xi) e^{-ix\cdot k}\, dx, \quad k \in \mathbb{Z}^d.$$

The main part of the proof of Theorem 1.14 is based on the following quantum version of the renormalization problem:

Theorem 6.1. *Let $\omega \in \mathbb{R}^d$ be a strongly non resonant frequency satisfying (1.60), and let V be a real valued function that belongs to $\mathcal{A}_s(T^*\mathbb{T}^d)$ for some fixed $s > 0$. Assume that $\varepsilon_\hbar \le \hbar$, and*

$$\|V\|_s \le \frac{\varsigma}{64}\left(\frac{\sqrt{s}}{2(\gamma-1)}\right)^{2(\gamma-1)}. \tag{6.1}$$

Then there exist unitary operators $\mathcal{U}_\hbar : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$, and counterterms $R_\hbar \in \mathcal{A}_{s/2}(\mathbb{R}^d)$ such that

$$\mathcal{U}_\hbar \left(\widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \operatorname{Op}_\hbar(V - R_\hbar)\right)\mathcal{U}_\hbar^* = \widehat{L}_{\omega,\hbar}. \tag{6.2}$$

Moreover,

$$|R_\hbar|_{s/2} \le 2\|V\|_s, \quad \hbar \in (0,1].$$

Remark 6.1. If $\varepsilon_\hbar \ll \hbar$ then condition (6.1) can be removed.

6.1.1. Strategy

We will start from the full renormalized operator $\widehat{Q}_\hbar$ with $\widehat{R}_\hbar$ as unknown and then we will construct $\mathcal{U}_\hbar$ and $\widehat{R}_\hbar$ by an iterative algorithm. We will find the renormalization function $R_\hbar$ as an infinite sum of the form

$$R_\hbar := \sum_{j=1}^\infty R_{j,\hbar},$$

where each $R_{j,\hbar}$ will be determine at each step of the iteration and the sum will be proven to converge in $\mathcal{A}_{s/2}(\mathbb{R}^d)$. We initially set $V_1 := V$, and consider

$$\widehat{Q}_{1,\hbar} := \widehat{Q}_\hbar = \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar\left(\operatorname{Op}_\hbar(V_1) - \sum_{j=1}^\infty \operatorname{Op}_\hbar(R_{j,\hbar})\right). \tag{6.3}$$

As in the previous normal forms constructed so far, the goal is to average the term V_1 by the quantum flow generated by $\widehat{L}_{\omega,\hbar}$ and estimate the remainder terms. Given $a \in \mathcal{C}^\infty(T^*\mathbb{T}^d)$ with bounded derivatives, we define the average of its semiclassical Weyl quantization $\operatorname{Op}_\hbar(a)$ by

$$\langle \operatorname{Op}_\hbar(a) \rangle := \lim_{T\to\infty} \frac{1}{T} \int_0^T e^{\frac{it}{\hbar}\widehat{L}_{\omega,\hbar}} \operatorname{Op}_\hbar(a)\, e^{-\frac{it}{\hbar}\widehat{L}_{\omega,\hbar}}\, dt. \tag{6.4}$$

The limit is well defined in the strong $\mathcal{L}(L^2)$-norm for operators, since

$$\frac{1}{T}\int_0^T a \circ \phi_t^{\mathcal{L}_\omega}(x,\xi) dt = \frac{1}{T}\int_0^T a(x + t\omega, \xi) dt$$

converges to $\langle a \rangle$ in the $\mathcal{C}^\infty(T^*\mathbb{T}^d)$ topology, where

$$\langle a \rangle(\xi) := \lim_{T\to\infty} \frac{1}{T}\int_0^T a \circ \phi_t^{\mathcal{L}_\omega}(x,\xi)\, dt = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} a(x,\xi) dx = \frac{1}{(2\pi)^{d/2}} \widehat{a}(0,\xi). \tag{6.5}$$

By Egorov's theorem, which is exact in this case since $\mathcal{L}_\omega$ is a polynomial of degree one, we have

$$\langle \operatorname{Op}_\hbar(a) \rangle = \operatorname{Op}_\hbar(\langle a \rangle).$$

In the first step of the iteration, we set $R_{1,\hbar} := \langle V_1 \rangle$ and consider a unitary operator of the form

$$U_{1,\hbar}(t) := e^{\frac{it\varepsilon_\hbar}{\hbar}\operatorname{Op}_\hbar(F_1)} = \sum_{j=0}^\infty \frac{1}{j!}\left(\frac{it\varepsilon_\hbar}{\hbar}\right)^j \operatorname{Op}_\hbar(F_1)^j, \quad t \in [0,1],$$

where $\operatorname{Op}_\hbar(F_1)$ will be chosen to solve the cohomological equation

$$\frac{i}{\hbar}[\widehat{L}_{\omega,\hbar}, \operatorname{Op}_\hbar(F_1)] = \operatorname{Op}_\hbar(V_1 - R_1), \quad \langle V_1 - R_1 \rangle = 0. \tag{6.6}$$

We will show in Section 6.1.2 how to solve this cohomological equation. Moreover, the Diophantine condition (1.60) on ω will allow us to bound the solution F_1 in a suitable space $\mathcal{A}_{s-\sigma}(T^*\mathbb{T}^d)$ provided that $V_1 \in \mathcal{A}_s(T^*\mathbb{T}^d)$. We denote $U_{1,\hbar} = U_{1,\hbar}(1)$ and define $\widehat{Q}_{2,\hbar} := U_{1,\hbar}\,\widehat{Q}_{1,\hbar}\,U_{1,\hbar}^*$. Using Taylor's theorem we expand this expression as

$$\begin{aligned}
\widehat{Q}_{2,\hbar} &= \widehat{L}_{\omega,\hbar} + \frac{i\varepsilon_\hbar}{\hbar}[\operatorname{Op}_\hbar(F_1), \widehat{L}_{\omega,\hbar}] + \varepsilon_\hbar \operatorname{Op}_\hbar(V_1 - R_1) \\
&\quad + \left(\frac{i\varepsilon_\hbar}{\hbar}\right)^2 \int_0^1 (1-t) U_{1,\hbar}(t)[\operatorname{Op}_\hbar(F_1), [\operatorname{Op}_\hbar(F_1), \widehat{L}_{\omega,\hbar}]] U_{1,\hbar}(t)^* dt \\
&\quad + \frac{i\varepsilon_\hbar^2}{\hbar}\int_0^1 U_{1,\hbar}(t)[\operatorname{Op}_\hbar(F_1), \operatorname{Op}_\hbar(V_1 - R_1)] U_{1,\hbar}(t)^* dt \\
&\quad - \varepsilon_\hbar \sum_{j=2}^\infty U_{1,\hbar} \operatorname{Op}_\hbar(R_{j,\hbar}) U_{1,\hbar}^*.
\end{aligned}$$

Using this and the cohomological equation (6.6),

$$\widehat{Q}_{2,\hbar} = \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \left(\mathrm{Op}_\hbar(V_{2,\hbar}) - \sum_{j=2}^{\infty} U_{1,\hbar} \, \mathrm{Op}_\hbar(R_{j,\hbar}) U_{1,\hbar}^* \right),$$

where

$$\mathrm{Op}_\hbar(V_{2,\hbar}) = \frac{i\varepsilon_\hbar}{\hbar} \int_0^1 t U_{1,\hbar}(t)[\mathrm{Op}_\hbar(F_1), \mathrm{Op}_\hbar(V_1 - R_1)] U_{1,\hbar}(t)^* dt. \tag{6.7}$$

This concludes the first step of the iteration.

Now we proceed to explain the induction step. Assume we have constructed unitary operators $U_{1,\hbar}, \dots, U_{n-1,\hbar}$ and counterterms $R_{1,\hbar}, \dots, R_{n-1,\hbar}$ so that

$$\widehat{Q}_{n,\hbar} = U_{n-1,\hbar} \cdots U_{1,\hbar} \, \widehat{Q}_{1,\hbar} \, U_{1,\hbar}^* \cdots U_{n-1,\hbar}^* = \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \left(\mathrm{Op}_\hbar(V_{n,\hbar}) - \sum_{j=n}^{\infty} \widehat{E}_{n,j,\hbar} \right),$$

where

$$\widehat{E}_{n,j,\hbar} := U_{n-1,\hbar} \cdots U_{1,\hbar} \, \mathrm{Op}_\hbar(R_{j,\hbar}) U_{1,\hbar}^* \cdots U_{n-1,\hbar}^*.$$

At symbol level, $\widehat{E}_{n,j,\hbar} = \mathrm{Op}_\hbar(E_{n,j,\hbar})$, where

$$E_{n,j,\hbar} = \Psi_{\hbar,1}^{\varepsilon_\hbar F_{n-1}} \circ \cdots \circ \Psi_{\hbar,1}^{\varepsilon_\hbar F_1} R_{j,\hbar}, \quad j \geq n.$$

We will find $R_{n,\hbar}$ to be the unique solution of the following equation (see Lemma 6.5 below):

$$\langle E_{n,n,\hbar} \rangle = \langle \Psi_{\hbar,1}^{\varepsilon_\hbar F_{n-1}} \circ \cdots \circ \Psi_{\hbar,1}^{\varepsilon_\hbar F_1} R_{n,\hbar} \rangle = \langle V_{n,\hbar} \rangle. \tag{6.8}$$

We next consider the unitary operator

$$U_{n,\hbar}(t) := e^{\frac{it\varepsilon_\hbar}{\hbar} \mathrm{Op}_\hbar(F_{n,\hbar})} = \sum_{j=0}^{\infty} \frac{1}{j!} \left(\frac{it\varepsilon_\hbar}{\hbar} \right)^j \mathrm{Op}_\hbar(F_{n,\hbar})^j, \quad t \in [-1, 1],$$

where $\mathrm{Op}_\hbar(F_{n,\hbar})$ solves the cohomological equation

$$\frac{i}{\hbar} [\widehat{L}_{\omega,\hbar}, \mathrm{Op}_\hbar(F_{n,\hbar})] = \mathrm{Op}_\hbar(V_{n,\hbar} - E_{n,n,\hbar}), \quad \langle V_{n,\hbar} - E_{n,n,\hbar} \rangle = 0. \tag{6.9}$$

As in the first step, we denote $U_{n,\hbar} := U_{n,\hbar}(1)$. Defining $\widehat{Q}_{n+1,\hbar} := U_{n,\hbar}\,\widehat{Q}_{n,\hbar}\,U^*_{n,\hbar}$, we use Taylor's theorem to expand

$$\begin{aligned}\widehat{Q}_{n+1,\hbar} &= \widehat{L}_{\omega,\hbar} + \frac{i\varepsilon_\hbar}{\hbar}[\mathrm{Op}_\hbar(F_{n,\hbar}), \widehat{L}_{\omega,\hbar}] + \varepsilon_\hbar \,\mathrm{Op}_\hbar(V_{n,\hbar} - E_{n,n,\hbar}) \\ &\quad + \left(\frac{i\varepsilon_\hbar}{\hbar}\right)^2 \int_0^1 (1-t) U_{n,\hbar}(t)[\mathrm{Op}_\hbar(F_{n,\hbar}), [\mathrm{Op}_\hbar(F_{n,\hbar}), \widehat{L}_{\omega,\hbar}]] U_{n,\hbar}(t)^* dt \\ &\quad + \frac{i\varepsilon_\hbar^2}{\hbar} \int_0^1 U_{n,\hbar}(t)[\mathrm{Op}_\hbar(F_{n,\hbar}), \mathrm{Op}_\hbar(V_{n,\hbar} - E_{n,n,\hbar})] U_{n,\hbar}(t)^* dt \\ &\quad - \varepsilon_\hbar \sum_{j=n+1}^{\infty} U_{n,\hbar}\,\mathrm{Op}_\hbar(E_{n,j,\hbar}) U^*_{n,\hbar}.\end{aligned}$$

With this and the cohomological equation (6.9), we obtain

$$\widehat{Q}_{n+1,\hbar} = \widehat{L}_{\omega,\hbar} + \varepsilon_\hbar \left(\mathrm{Op}_\hbar(V_{n+1,\hbar}) - \sum_{j=n+1}^{\infty} \mathrm{Op}_\hbar(E_{n+1,j,\hbar}) \right),$$

where

$$\mathrm{Op}_\hbar(V_{n+1,\hbar}) = \frac{i\varepsilon_\hbar}{\hbar} \int_0^1 t U_{n,\hbar}(t)[\mathrm{Op}_\hbar(F_n), \mathrm{Op}_\hbar(V_{n,\hbar} - E_{n,n,\hbar})] U_{n,\hbar}(t)^* dt, \tag{6.10}$$

and

$$\widehat{E}_{n+1,j,\hbar} := U_{n,\hbar}\,\mathrm{Op}_\hbar(E_{n,j,\hbar}) U^*_{n,\hbar},$$

or, equivalently at symbol level, $\widehat{E}_{n+1,j,\hbar} = \mathrm{Op}_\hbar(E_{n+1,j,\hbar})$ with

$$E_{n+1,j,\hbar} = \Psi^{\varepsilon_\hbar F_n}_{\hbar,1} \circ \cdots \circ \Psi^{\varepsilon_\hbar F_1}_{\hbar,1} R_{j,\hbar}, \quad j \geq n+1.$$

This iteration procedure will converge provided that $V \in \mathcal{A}_s(T^*\mathbb{T}^d)$ is sufficiently small. Precisely, we will obtain a unitary operator $\mathcal{U}_\hbar$ as the limit, in the strong operator norm,

$$\mathcal{U}_\hbar := \lim_{n\to\infty} U_{n,\hbar} \cdots U_{1,\hbar},$$

so that $\mathcal{U}_\hbar\,\widehat{Q}_\hbar\,\mathcal{U}^*_\hbar = \widehat{L}_{\omega,\hbar}$.

6.1.2. Tools of analytic symbolic calculus on the torus

In order to complete the technical parts of the proof of Theorem 1.14, some analytic symbolic calculus like that introduced in Section 4.3 is required.

Lemma 6.1 (Calderón-Vaillancourt theorem revisited). *Let $s > 0$. For any $a \in \mathcal{A}_s(T^*\mathbb{T}^d)$, the following holds:*

$$\| \operatorname{Op}_\hbar(a)\|_{\mathcal{L}(L^2(\mathbb{T}^d))} \le C_{d,s}\|a\|_s, \tag{6.11}$$

for all $\hbar \in (0,1]$.

Proof. The proof is completely analogous to that of Lemma 4.4, but using Lemma 2.14 instead of Lemma 2.5. □

Lemma 6.2. *Assume $a, F \in \mathcal{A}_s(T^*\mathbb{T}^d)$. Let $0 < \sigma < s$. If*

$$\beta = \frac{2|t|\|F\|_s}{\sigma^2} \le 1/2,$$

then

$$\|\Psi^F_{\hbar,t}(a) - a\|_{s-\sigma} \le \beta\|a\|_s, \quad |t| \le 1.$$

Proof. The proof can be faithfully transferred from that of Lemma 4.9. □

Note that, in order to bound $\Psi^F_{\hbar,t}(a)$, some loss of analyticity has been required. On the other hand, if one wants to avoid this loss of analyticity, one can use the following weaker lemma:

Lemma 6.3. *Assume $a, F \in \mathcal{A}_s(T^*\mathbb{T}^d)$. Let $\varepsilon_\hbar \le \hbar$. If*

$$\beta = 2|t|\|F\|_s \le 1/2,$$

then

$$\|\Psi^{\varepsilon_\hbar F}_{\hbar,t}(a) - a\|_s \le \beta\|a\|_s, \quad |t| \le 1.$$

Proof. Since

$$[F,a]_\hbar(z) = 2\int_{\mathbb{R}^{4d}} \widehat{F}(w')\,\widehat{a}(w-w')\sin\left(\frac{\hbar}{2}\{L_{w'},L_{w-w'}\}\right) e^{iL_w(z)}\,\kappa(dw')\kappa(dw),$$

we have the trivial bound

$$\|[F,a]_\hbar\|_s \le 2\|F\|_s\|a\|_s. \tag{6.12}$$

Then

$$\|\Psi^{\varepsilon_\hbar F}_{\hbar,t}(a) - a\|_s \le \sum_{j=1}^\infty \frac{1}{j!}\left(\frac{t}{\hbar}\right)^j \|\operatorname{Ad}^{\sharp_\hbar,j}_{\varepsilon_\hbar F}(a)\|_s \le \sum_{j=1}^\infty \frac{2^j\|F\|_s^j\|a\|_s}{j!} \le \beta\|a\|_s.$$

□

Lemma 6.4. *Let $A \in \mathcal{A}_s(T^*\mathbb{T}^d)$. Then, the cohomological equation*

$$\frac{i}{\hbar}[\widehat{L}_{\omega,\hbar}, \mathrm{Op}_\hbar(F)] = \mathrm{Op}_\hbar(V - \langle V \rangle). \tag{6.13}$$

has a unique solution $F \in \mathcal{A}_{s-\sigma}(T^*\mathbb{T}^d)$ for every $0 < \sigma \leq s$ such that

$$\|F\|_{s-\sigma} \leq \varsigma^{-1} \left(\frac{\gamma - 1}{e\sigma} \right)^{\gamma - 1} \|V\|_s.$$

Proof. Write

$$A(x,\xi) = \sum_{k \in \mathbb{Z}^d} \widehat{A}(k,\xi) e_k(x).$$

Using the properties of the symbolic calculus for the Weyl quantization, equation (6.13) at symbol level is just

$$\{L_\omega, F\} = V - \langle V \rangle, \tag{6.14}$$

Recall also that the average of V is given by

$$\langle V \rangle(\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} V(x,\xi) dx = \frac{1}{(2\pi)^{d/2}} \widehat{V}(0,\xi).$$

On the other hand,

$$\{L_\omega, F\}(x,\xi) = \sum_{k \in \mathbb{Z}^d} i\omega \cdot k \, \widehat{F}(k,\xi) e_k(x),$$

and then we obtain the following expression for the solution of (6.14):

$$F(x,\xi) = \sum_{k \in \mathbb{Z}^d} \frac{\widehat{V}(k,\xi)}{i\omega \cdot k} e_k(x). \tag{6.15}$$

Finally, by Diophantine condition (1.60) and the following estimate,

$$\sup_{t \geq 0} t^{\gamma-1} e^{-t\sigma} = \left(\frac{\gamma - 1}{e\sigma} \right)^{\gamma-1},$$

we conclude that

$$\|F\|_{s-\sigma} \leq \varsigma^{-1} \left(\frac{\gamma - 1}{e\sigma} \right)^{\gamma - 1} \|V\|_s.$$

□

Lemma 6.5. *Assume $\varepsilon_\hbar \le \hbar$. Let $\langle V\rangle \in \mathcal{A}_s(\mathbb{R}^d)$ and $F_1, \dots, F_n \in \mathcal{A}_s(T^*\mathbb{T}^d)$ such that*

$$2\|F_j\|_s \le \beta\, \alpha^{j-1}, \quad j \in \{1, \dots n\},$$

where $\alpha, \beta > 0$ satisfy

$$\lambda := e^{\frac{\beta}{1-\alpha}} - 1 < 1.$$

Then, there exists $R \in \mathcal{A}_s(\mathbb{R}^d)$ so that

$$\langle \Psi_{\hbar,1}^{\varepsilon_\hbar F_n} \circ \cdots \circ \Psi_{\hbar,1}^{\varepsilon_\hbar F_1} R\rangle = \langle V\rangle,$$

and

$$\|R\|_s \le \frac{1}{1-\lambda}\|\langle V\rangle\|_s, \quad \|\Psi_{\hbar,1}^{\varepsilon_\hbar F_n} \circ \cdots \circ \Psi_{\hbar,1}^{\varepsilon_\hbar F_1} R\|_s \le \frac{1+\lambda}{1-\lambda}\|\langle V\rangle\|_s.$$

Proof. Define the map $T : \mathcal{A}_s(\mathbb{R}^d) \to \mathcal{A}_s(\mathbb{R}^d)$:

$$T(R) := \langle \Psi_{\hbar,1}^{\varepsilon_\hbar F_n} \circ \cdots \circ \Psi_{\hbar,1}^{\varepsilon_\hbar F_1} R\rangle.$$

By Lemma 6.3, we have

$$\|T(R) - R\|_s \le \left[\prod_{j=1}^{n}(1+\beta\alpha^{j-1}) - 1\right]\|R\|_s \le \left(e^{\frac{\beta}{1-\alpha}} - 1\right)\|R\|_s = \lambda\|R\|_s.$$

Then, there exists an inverse map $T^{-1} : \mathcal{A}_s \to \mathcal{A}_s$ defined by Neumann series, and

$$\|T^{-1}\|_{\mathcal{A}_s \to \mathcal{A}_s} \le \frac{1}{1-\lambda}.$$

Finally, applying Lemma 6.3 one more time, we obtain:

$$\|\Psi_{\hbar,1}^{\varepsilon_\hbar F_n} \circ \cdots \circ \Psi_{\hbar,1}^{\varepsilon_\hbar F_1} R\|_s \le \frac{1+\lambda}{1-\lambda}\|\langle V\rangle\|_s.$$

This concludes the proof of the Lemma. □

6.1.3. Convergence

We next show that the algorithm sketched in Section 6.1.1 converges provided that $V \in \mathcal{A}_s(T^*\mathbb{T}^d)$ is sufficiently small. This will allow us to prove Theorem 6.1:

Proof of Theorem 6.1. We start by fixing the following universal constants:

$$\alpha := \frac{1}{4}, \quad \beta := \frac{1}{16}, \quad \lambda := e^{\frac{\beta}{1-\alpha}} - 1. \tag{6.16}$$

Now set

$$s_1 := s, \quad \sigma_1 := \frac{s}{2e(\gamma-1)}\alpha^{\frac{1}{2(\gamma-1)}}.$$

By Lemma 6.4 and hypothesis (6.1),

$$\|F_1\|_{s_1-\sigma_1} \leq \varsigma^{-1}\left(\frac{\gamma-1}{e\sigma_1}\right)^{\gamma-1}\|V_1\|_{s_1} \leq \frac{\beta}{2}.$$

Then, using (6.7), which at symbol level reads

$$V_{2,\hbar} = \frac{i\varepsilon_\hbar}{\hbar}\int_0^1 t\Psi^{\varepsilon_\hbar F_1}_{\hbar,t}\big([F_1, V_1 - R_1]_\hbar\big)dt,$$

the trivial bound (6.12), and Lemma 6.3,

$$\|V_2\|_{s_1-\sigma_1} \leq \beta(1+\beta)\|V_1\|_{s_1} \leq \alpha\|V_1\|_{s_1}.$$

Moreover,

$$\|R_1\|_{s_1} = \|\langle V_1\rangle\|_{s_1} \leq \|V_1\|_{s_1}.$$

This shows the first step of the induction. Now define sequences

$$\sigma_{n+1} := \sigma_n\alpha^{\frac{1}{2(\gamma-1)}}, \quad s_{n+1} := s_n - \sigma_n, \quad n \geq 1,$$

and assume the following induction hypothesis: for every $n \geq 2$ and $1 \leq j \leq n-1$,

$$\|F_j\|_{s_j} \leq \frac{\beta\alpha^{\frac{j-1}{2}}}{2}, \quad \|R_{j,\hbar}\|_{s_j} \leq \frac{\alpha^{j-1}}{1-\lambda}\|V_1\|_{s_1}, \tag{6.17}$$

and

$$\|V_{n,\hbar}\|_{s_n} \leq \alpha^{n-1}\|V_1\|_{s_1}. \tag{6.18}$$

We next prove the induction step. First, by Lemma 6.4 and hypothesis (6.1):

$$\|F_{n,\hbar}\|_{s_n-\sigma_n} \leq \varsigma^{-1}\left(\frac{\gamma-1}{e\sigma_n}\right)^{\gamma-1}\|V_n\|_{s_n} \leq \varsigma^{-1}\left(\frac{\gamma-1}{e\sigma_1}\right)^{\gamma-1}\alpha^{\frac{n-1}{2}}\|V_1\|_{s_1} \leq \frac{\beta\alpha^{\frac{n-1}{2}}}{2}.$$

Using Lemma 6.5, we also have

$$\|R_{n,\hbar}\|_{s_n} \leq \frac{1}{1-\lambda}\|V_{n,\hbar}\|_{s_n} \leq \frac{\alpha^{n-1}}{1-\lambda}\|V_1\|_{s_1}.$$

Note that, with the choice of the constants (6.16):

$$\beta(1+\beta)\left(1+\frac{1+\lambda}{1-\lambda}\right) \leq \alpha.$$

Then, recalling (6.10), which at symbol level reads

$$V_{n+1,\hbar} = \frac{i\varepsilon_\hbar}{\hbar} \int_0^1 t\Psi^{\varepsilon_\hbar F_n}_{\hbar,t}\big([F_n, V_{n,\hbar} - E_{n,n,\hbar}]_\hbar\big)dt,$$

we can apply the trivial bound (6.12) and Lemmas 6.3 and 6.5 to obtain:

$$\begin{aligned}\|V_{n+1,\hbar}\|_{s_n-\sigma_n} &\leq \beta(1+\beta)\left(1+\frac{1+\lambda}{1-\lambda}\right)\|V_{n,\hbar}\|_{s_n} \\ &\leq \alpha\|V_{n,\hbar}\|_{s_n} \leq \alpha^n\|V_1\|_{s_1}.\end{aligned}$$

This finishes the induction step. Note that, with our choices of the constants, we have

$$\sum_{n=1}^{\infty}\sigma_n = \sigma_1\sum_{j=0}^{\infty}\left(\frac{1}{2}\right)^{\frac{j}{\gamma-1}} \leq \frac{s}{2e(\gamma-1)}\frac{1}{\log 2^{\frac{1}{\gamma-1}}} \leq \frac{s}{2e\log 2} \leq \frac{s}{2}.$$

Moreover,

$$\|R_\hbar\|_{s/2} \leq \sum_{j=1}^{\infty}\|R_{j,\hbar}\|_{s_j} \leq \left(\frac{1}{1-\lambda}\sum_{j=0}^{\infty}\alpha^j\right)\|V_1\|_{s_1} \leq 2\|V_1\|_{s_1}.$$

It remains to show that there exists a unitary operator $\mathcal{U}_\hbar$ so that

$$\mathcal{U}_\hbar := \lim_{n\to\infty} U_{n,\hbar}\cdots U_{1,\hbar}.$$

For every $1 \leq n$, we set the unitary operator $\mathcal{U}_{n,\hbar}$ by

$$\mathcal{U}_{n,\hbar} := U_{n,\hbar}\cdots U_{1,\hbar}.$$

We have, for $p \geq 1$:

$$\mathcal{U}_{n+p,\hbar} - \mathcal{U}_{n,\hbar} = \mathcal{U}_{n,\hbar}\,\mathscr{R}_\hbar(n,p),$$

where

$$\mathscr{R}_h(n,p) := e^{\frac{i\varepsilon_\hbar}{\hbar}\widehat{F}_{n+1,\hbar}}\cdots e^{\frac{i\varepsilon_\hbar}{\hbar}\widehat{F}_{n+p,\hbar}} - I, \quad \widehat{F}_{j,\hbar} := \mathrm{Op}_\hbar(F_j).$$

By Taylor's theorem, we can write

$$e^{\frac{i\varepsilon_\hbar}{\hbar}\widehat{F}_{j,\hbar}} = I + \widehat{\beta}_{j,\hbar}, \quad \widehat{\beta}_{j,\hbar} := \frac{i\varepsilon_\hbar}{\hbar}\widehat{F}_{j,\hbar}\int_0^1 e^{\frac{it\varepsilon_\hbar}{\hbar}\widehat{F}_{j,\hbar}}\,dt.$$

Moreover, Lemma 6.1 allows us to bound the $\mathcal{L}(L^2)$ norm of $\widehat{\beta}_{j,\hbar}$ by:

$$\|\widehat{\beta}_{j,\hbar}\|_{\mathcal{L}(L^2)} \leq \frac{C_{d,s}\beta\alpha^{\frac{j-1}{2}}}{2}.$$

Then

$$\|\mathscr{R}_\hbar(n,p)\|_{\mathcal{L}(L^2)} \leq -1 + \prod_{j=1}^{p}\left(1 + \|\widehat{\beta}_{n+j,\hbar}\|_{\mathcal{L}(L^2)}\right) \leq -1 + \exp\left[\frac{C_{d,s}\beta\alpha^{\frac{n-1}{2}}}{2(1-\alpha^{1/2})}\right].$$

Finally, taking the limit $n \to \infty$, we obtain that the sequence $\{\mathcal{U}_{n,\hbar}\}_{n\geq 1}$ is a Cauchy sequence in the operator norm, and then the result holds. □

6.2. Description of Semiclassical measures

Finally, we shall prove Theorem 1.14. We will require the following two lemmas:

Lemma 6.6. *For every $a \in \mathcal{A}_s(T^*\mathbb{T}^d)$,*

$$\|\mathcal{U}_\hbar^* \operatorname{Op}_\hbar(a)\mathcal{U}_\hbar - \operatorname{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} = O(\varepsilon_\hbar).$$

Proof. For $n \geq 1$, define:

$$\rho_n = \left(\frac{1}{2}\right)^{\frac{n-1}{3}}\rho_1, \quad \rho_1 := \frac{s}{10}.$$

Note that

$$\sum_{n=1}^{\infty}\rho_n \leq \frac{s}{2}$$

By (6.17), we have

$$\|F_n\|_{s_n} \leq C_s\rho_n^3,$$

where the constant C_s depends only on s. Hence, for $\hbar$ sufficiently small, the following holds for every $n \geq 1$:

$$\frac{2\|\varepsilon_\hbar F_n\|_{s_n}}{\rho_n^2} \leq C_s\rho_n\,\varepsilon_\hbar \leq \frac{1}{2}.$$

Using Lemma 6.2; for every $a \in \mathcal{A}_s(T^*\mathbb{T}^d)$, we have

$$\|\Psi_{1,\hbar}^{\varepsilon_\hbar F_n}a - a\|_{s_n-\rho_n} \leq C_s\rho_n\,\varepsilon_\hbar. \tag{6.19}$$

Finally, since $\mathcal{U}_\hbar = \lim_{n\to\infty} U_{n,\hbar}\cdots U_{1,\hbar}$, that every operator $U_{n,\hbar}$ is unitary on $L^2(\mathbb{T}^d)$, Lemma 6.1 and (6.19):

$$\|\mathcal{U}_\hbar^* \operatorname{Op}_\hbar(a)\mathcal{U}_\hbar - \operatorname{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} \leq C_s\sum_{n=1}^{\infty}\|\Psi_{1,\hbar}^{\varepsilon_\hbar F_n}a - a\|_{s_n-\rho_n} \leq C_s\varepsilon_\hbar\sum_{n=1}^{\infty}\rho_n \leq C_s\varepsilon_\hbar.$$

□

Lemma 6.7. *For every* $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$,

$$\|\mathcal{U}_\hbar^* \operatorname{Op}_\hbar(a)\mathcal{U}_\hbar - \operatorname{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} = o(1).$$

Proof. Let $\varepsilon > 0$ and $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$. Assume that there exists $a^\dagger \in \mathcal{A}_s(T^*\mathbb{T}^d)$ such that

$$\|a - a^\dagger\|_{L^\infty(T^*\mathbb{T}^d)} \leq \varepsilon.$$

Then, by the triangular inequality and Lemma 2.14 (recall that $\varepsilon_\hbar \leq \hbar$):

$$\begin{aligned} &\|\mathcal{U}_\hbar^* \operatorname{Op}_\hbar(a)\mathcal{U}_\hbar - \operatorname{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} \\ &\qquad \leq \|\mathcal{U}_\hbar^* \operatorname{Op}_\hbar(a - a^\dagger)\mathcal{U}_\hbar\|_{\mathcal{L}(L^2)} + \|\mathcal{U}_\hbar^* \operatorname{Op}_\hbar(a^\dagger)\mathcal{U}_\hbar - \operatorname{Op}_\hbar(a^\dagger)\|_{\mathcal{L}(L^2)} + \|\operatorname{Op}_\hbar(a - a^\dagger)\|_{\mathcal{L}(L^2)} \\ &\qquad \leq C_d\|a - a^\dagger\|_{L^\infty(T^*\mathbb{T}^d)} + O(\hbar), \end{aligned}$$

and hence

$$\limsup_{\hbar\to 0} \|\mathcal{U}_\hbar^* \operatorname{Op}_\hbar(a)\mathcal{U}_\hbar - \operatorname{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} \leq C_d\varepsilon.$$

Since the choice $\varepsilon > 0$ was arbitrarily, we conclude that

$$\lim_{\hbar\to 0} \|\mathcal{U}_\hbar^* \operatorname{Op}_\hbar(a)\mathcal{U}_\hbar - \operatorname{Op}_\hbar(a)\|_{\mathcal{L}(L^2)} = 0.$$

It remains to show that, for all $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$, there exists $a^\dagger \in \mathcal{A}_s(T^*\mathbb{T}^d)$ such that

$$\|a - a^\dagger\|_{L^\infty(T^*\mathbb{T}^d)} \leq \varepsilon.$$

Write

$$a(z) = \int_{\mathcal{Z}^d} \widehat{a}(w)e^{iz\cdot w}\kappa(dw), \quad z = (x,\xi) \in T^*\mathbb{T}^d.$$

For $R \geq 1$, we define $a_R \in \mathcal{A}_s(T^*\mathbb{T}^d)$ by

$$\widehat{a}_R(w) = \widehat{a}(w)e^{-\frac{|w|^2}{R}}.$$

It satisfies

$$\|a_R - a\|_{L^\infty(T^*\mathbb{T}^d)} \leq \int_{\mathcal{Z}^d} \big|\widehat{a}(w)\big|\big|e^{-\frac{|w|^2}{R}} - 1\big|\kappa(dw) \to 0, \quad \text{as } R \to \infty.$$

Then it is sufficient to take $a^\dagger = a_R$ for R sufficiently large. □

Proof of Theorem 1.14. We know, by Proposition 1.3:

$$\mathcal{M}\big(\widehat{L}_{\omega,\hbar}\big) = \mathcal{M}_\omega. \tag{6.20}$$

On the other hand, Theorem 6.1 implies that the set of normalized eigenfunctions of $\widehat{Q}_\hbar$ is given precisely by the orthonormal basis of $L^2(\mathbb{T}^d)$ given by

$$\{\Psi_{k,\hbar} = \mathcal{U}_\hbar e_k \,:\, k \in \mathbb{Z}^d\},$$

Using Lemma 6.7, we obtain, for every $a \in \mathcal{C}_c^\infty(T^*\mathbb{T}^d)$,

$$W^\hbar_{\Psi_{k,\hbar}}(a) = W^\hbar_{e_k}(a) + o(1), \quad k \in \mathbb{Z}^d.$$

Hence the proof of the Theorem follows by (6.20).

□

Conclusiones

A continuación se resumen las aportaciones más importantes de esta tesis:

Teoremas 1.2, 1.3 y 1.4. Estos resultados muestran la inestablidad de las medidas semiclásicas dependientes del tiempo $\mathcal{M}(\widehat{H}_\hbar, \tau_\hbar)$ asociadas al operador sin perturbar $\widehat{H}_\hbar$, con presencia de resonancias en el vector ω, bajo perturbaciones semiclásicas de la forma $\varepsilon_\hbar \widehat{V}_\hbar$. En particular, se muestra que, en el régimen $\varepsilon_\hbar \tau_\hbar \to +\infty$, existen toros invariantes por el flujo clásico ϕ_t^H sobre el que los elementos de $\mathcal{M}(\widehat{P}_\hbar, \tau_\hbar)$ no pueden estar soportados. Dicha inestabilidad se produce sobre aquellos toros invariantes no invariantes por el flujo $\phi_t^{\langle V \rangle}$. Las mejoras sucesivas de los Teoremas 1.3 y 1.4 hacen balance de la generalidad con la que se da esta inestabilidad y el régimen crítico $\varepsilon_\hbar \tau_\hbar$ a la que se produce la ruptura de los toros invariantes bajo hipótesis dinámicas sobre el símbolo V de la perturbación.

Teorema 1.5. Este resultado aborda el caso en el que el vector de frecuencias es diofántico. Se muestra que los toros lagrangianos maximales invariantes por ϕ_t^H pueden ser conjuntos de acumulación de las sucesiones de Wigner de soluciones dependientes del tiempo para rangos $\tau_\hbar$ polinomialmente largos sobre el tamaño $\varepsilon_\hbar$ de la perturbación.

Teorema 1.6. En el caso dos dimensional con $\omega = (1,1)$ se obtienen mejoras significativas sobre la caracterización de los elementos de $\mathcal{M}(\widehat{P}_\hbar, \tau_\hbar)$ para rangos de tiempo $\tau_\hbar$ de orden polinomial con respecto al tamaño de la perturbación $\varepsilon_\hbar \widehat{V}_\hbar$. En particular, se muestra que los conjuntos minimales sobre los que las sucesiones de Wigner pueden concentrarse son 2-toros invariantes por los flujos ϕ_t^H y $\phi_t^{\langle V \rangle}$.

Teorema 1.7. Se obtiene una caracterización completa del conjunto $\mathcal{M}(\widehat{H}_\hbar)$ para cualquier vector de frecuencias $\omega \in \mathbb{R}_+^d$.

Teorema 1.8. Como aplicación de los Teoremas 1.2, 1.3 y 1.4 se obtiene que, genéricamente, los toros minimales invariantes por el flujo ϕ_t^H, cuando existen resonancias entre las componentes del vector ω, no pueden ser conjuntos de acumulación para sucesiones de Wigner asociadas a autofunciones del operador perturbado $\widehat{P}_\hbar$.

Teorema 1.9. En el caso diferenciable, se da una estimación para la distribución de los autovalores del operador no autoadjunto $\widehat{P}_\hbar$ cerca de la recta real. Este resultado también se verifica para cuasi-autovalores, lo permite obtener una cota sobre la resolvente del operador.

Teorema 1.10. En el caso analítico, se mejora el resultado anterior probando que los autovalores del operador perturbado no pueden concentrarse cerca de la recta real, es decir, existe un *gap* espectral. Sin embargo, no se optiene una nueva cota sobre la resolvente del operador.

Teoremas 1.11 y 1.13. Estos resultados constituyen un análogo semiclásico del teorema de Moser sobre perturbaciones de campos vectoriales constantes sobre el toro. En particular, se muestra la estabilidad de autovalores bajo perturbaciones para un conjunto cantoriano de frecuencias diofánticas y se caracterizan los conjuntos de medidas semiclásicas y límites cuánticos asociados a sucesiones de autofunciones del operador perturbado.

Teorema 1.14. Este resultado es una versión semiclásica del teorema clásico de renormalización. Dada una perturbación acotada de tamaño $\varepsilon_\hbar \leq \hbar$ del operador $\widehat{L}_{\omega,\hbar}$ sobre el toro con hipótesis diofánticas sobre ω, se obtiene la existencia de un operador integrable $\widehat{R}_\hbar$, cuyo símbolo solo depende de las coordenadas acción, tal que sumado al operador perturbado lo hace unitariamente equivalente al operador sin perturbar. Como consecuencia se caracteriza el conjunto de medidas semiclásicas y límites cuánticos del operador renormalizado.

Conclusions

We next state the main contributions of this report:

Theorems 1.2, 1.3 and 1.4. These results show the instability of the set $\mathcal{M}(\widehat{H}_\hbar, \tau_\hbar)$ of time-dependent semiclassical measures associated to the unperturbed operator $\widehat{H}_\hbar$, with resonancies between the components of the vector ω, under semiclassical perturbations of the form $\varepsilon_\hbar \widehat{V}_\hbar$. In particular, this shows that, in the regime $\varepsilon_\hbar \tau_\hbar \to \infty$, there exist invariant tori for the classical flow ϕ_t^H on which the elements of $\mathcal{M}(\widehat{P}_\hbar, \tau_\hbar)$ can not be supported. This instability appears in those tori which are not invariant by the flow $\phi_t^{\langle V \rangle}$. The successive improvements of Theorems 1.3 and 1.4 deal with the generality of this instability under geometrical hypothesis on the symbol V of the perturbation.

Theorem 1.5. This result adresses the case when the vector ω is Diophantine. It is shown that the maximal Lagrangian tori which are invariant by ϕ_t^H can become accumulation sets for sequences of Wigner distributions associated to time-dependent solutions for ranges $\tau_\hbar$ polynomially large with respect to the size $\varepsilon_\hbar$ of the perturbation.

Theorem 1.6. In the 2D case and $\omega = (1, 1)$ we obtain improvements on the characterization of $\mathcal{M}(\widehat{P}_\hbar, \tau_\hbar)$ for ranges of time $\tau_\hbar$ of polynomial order with respect to the size $\varepsilon_\hbar$ of the perturbation. In particular, it is shown that the minimal sets on which sequences of Wigner sequences can concentrate are 2-tori which are invariant by both the flows ϕ_t^H and $\phi_t^{\langle V \rangle}$.

Theorem 1.7. A complete characterization of the set $\mathcal{M}(\widehat{H}_\hbar)$ is obtained for any vector of frequencies $\omega \in \mathbb{R}_+^d$.

Theorem 1.8. As a consequence of Theorems 1.2, 1.3 and 1.4, it is shown that, generecally, a minimal invariant torus by the flow ϕ_t^H, when resonences between the components of the vector ω exist, can not be the accumulation set for a Wigner sequence assocaited to the eigenfunctions of the perturbed operator $\widehat{P}_\hbar$.

Theorem 1.9. In the smooth case, we show an estimate for the distribution of eigenvalues for the non-selfadjoint operator $\widehat{P}_\hbar$ near the real axis. This results also holds for quasi-eigenvalues, which allows us to obtain a bound on the resolvent of $\widehat{P}_\hbar$.

Theorem 1.10. In the analytic case, we improve the pervious result showing that the eigenvalues of $\widehat{P}_\hbar$ can not concentrate near the real axis, that is, there exists a spectral gap. However, we do not obtain a new bound on the resolvent of the operator.

Theorems 1.11 and 1.13. These two results give a semiclassical analog to the theorem of Moser about perturbations of vector fields on the torus. In particular, a stability result on the point-spectrum under perturbations for a Cantor set of Diophantine frequencies is shown, and the sets of semiclassical measures and quantum limits are characterized.

Theorem 1.14. This result is a semiclassical version of the classical problem of renormalization. Given a bounded perturbation of size $\varepsilon_\hbar \leq \hbar$ of the linear operator $\widehat{L}_{\omega,\hbar}$ on the torus, with Diophantine assumptions on ω, we obtain the existence of an integrable operator $\widehat{R}_\hbar$, whose symbol only depends on the action variables, such that, added to the perturbed operator, renormalizes it making it unitarily equivalent to the unperturbed operator. As a consequence, the sets of semiclassical measures and quantum limits are characterized for the renormalized operator.

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