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Boundedness of fractional elliptic and parabolic operators on Lebesgue and Hölder spaces. A semigroup approach.

Marta de León Contreras

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Supervised by:

José Luis Torrea Hernández

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A mi familia y a Luis,
porque mi mayor suerte es tenerles.

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Contents

Boundedness of fractional elliptic and parabolic operators on Lebesgue and Hölder spaces. A semigroup approach.

Marta de León-Contreras[1],

The connecting thread of this thesis is the semigroup language, a unifying and general technique to formulate and analyze fundamental properties of fractional operators. We have used this approach to deal with different problems. The first chapter is devoted to the discrete fractional derivatives. We have defined them via semigroups and we have proved that they approximate the continuous fractional derivatives. We have also obtained comparison and maximum principles and regularity results for the fractional powers. These results also allow us to prove the pointwise coincidence of the Marchaud and Grünwald-Letnikov derivatives. On the second chapter we consider Schrödinger operators on $\mathbb{R}^n$ with $n \geq 3$, that is, $\mathcal{L} = -\Delta + V$, where V is a nonnegative potential satisfying a reverse Hölder inequality. We have found the appropriated pointwise definition of Lipschitz (or Hölder) classes in the Schrödinger setting for $0 < \alpha < 2$. Secondly, we have defined, for every $\alpha > 0$, new Lipschitz spaces adapted to $\mathcal{L}$ by means of the heat and Poisson semigroups. We prove that in fact these spaces do coincide with the ones defined pointwise. Moreover, we use these new definitions of Lipschitz spaces via semigroups to get regularity results of fractional powers of Schrödinger operators. On the third chapter we deal with the Hermite operator, a Schrödinger operator for which are known a lot of interesting properties. These properties have allowed us to get better results in this case than for general Schrödinger operators. We have got a complete characterization of Lipschitz spaces adapted to the Hermite operator (also in the parabolic case) and we have obtained regularity results for the Hermite fractional operators in those spaces. Finally, the last chapter is devoted to the study of the classical solvability of the parabolic Bessel differential equation and the boundedness on (mixed) weighted L^p spaces of the associated Riesz transforms.

Acotación de operadores elípticos y parabólicos fraccionarios en espacios Hölder y de Lebesgue. Un enfoque a través de semigrupos.

El hilo conductor de esta tesis es el lenguaje de semigrupos, una técnica general y unificadora para formular y analizar propiedades fundamentales de operadores fraccionarios. Hemos usado este enfoque para tratar con diferentes problemas. El primer capítulo está dedicado a las derivadas fraccionarias discretas. Las hemos definido por medio de semigrupos y hemos visto que aproximan a las derivadas fraccionarias continuas. También hemos obtenido principios de comparación y del máximo y resultados de regularidad para las potencias fraccionarias. Estos resultados nos permiten probar la coincidencia puntual de las derivadas de Marchaud y de Grünwald-Letnikov. En el segundo capítulo consideramos operadores de Schrödinger en $\mathbb{R}^n$ con $n \geq 3$, esto es, $\mathcal{L} = -\Delta + V$, donde V es un potencial no negativo que satisface una desigualdad de Hölder inversa. Hemos encontrado la definición puntual apropiada de los espacios de Lipschitz (o Hölder) en el contexto de Schrödinger para $0 < \alpha < 2$. En segundo lugar, hemos definido para cualquier $\alpha > 0$, nuevos espacios Lipschitz adaptados a $\mathcal{L}$ por medio de los semigrupos del calor y del Poisson. Hemos probado que de hecho estos espacios coinciden con los definidos puntualmente. Además, usamos estas nuevas definiciones de espacios Lipschitz a través de semigrupos para obtener resultados de regularidad para potencias fraccionarias de los operadores de Schrödinger. En el tercer capítulo tratamos con el operador de Hermite, un operador de Schrödinger del que se conocen muchas propiedades interesantes. Estas propiedades nos han permitido obtener mejores resultados en este caso que para operadores de Schrödinger generales. Hemos obtenido una caracterización completa de los espacios Lipschitz adaptados al operador de Hermite (también en el caso parabólico) y hemos obtenido resultados de regularidad para operadores fraccionarios de Hermite en estos espacios. Finalmente, el último capitulo se dedica al estudio de la existencia de soluciones clásicas de la ecuación de Bessel parabólica y la acotación de las transformadas de Riesz en espacios L^p (mixtos) y con pesos.

[1]Departamento de Matemáticas
Universidad Autónoma de Madrid, 28049 Madrid, Spain.
email: marta.leon@uam.es

Introducción

La teoría de semigrupos comenzó a estudiarse en los años cuarenta y algunos de los trabajos pioneros son debidos a E. Hille y K. Yosida, [46, 94]. Ambos autores estudiaron de manera independiente el problema de determinar el operador lineal acotado más general T_t, $t \geq 0$, que satisface

1) $T_{t+s} = T_t T_s$.

2) $T_0 = Id$ (Id denota el operador identidad.)

Además, introdujeron la noción de generador infinitesimal A de T_t, definido por

$$A = \lim_{t \to 0} \frac{T_t - Id}{t} \quad \text{(el límite es entendido en el sentido fuerte).}$$

Dado un espacio de Banach X, un *semigrupo* es una familia de operadores lineales acotados en X, $\{T_t\}_{t \geq 0} \subset L(X, X)$, que satisface 1) y 2). Desde 1948, la teoría analítica de semigrupos y sus aplicaciones han hecho grandes progresos. Pronto Yosida aplicó su teorema de generación a las ecuaciones de difusión en una serie de artículos importantes, véase por ejemplo [95, 96], y Hille, inspirado por el trabajo de Yosida, atacó de nuevo en 1949 el problema de Cauchy con la ayuda de la teoría de semigrupos, [47, 48]. Otra de las personas que pronto comenzó a trabajar en la teoría general de semigrupos fue R.S. Phillips, [71, 72], quien completó muchas de las cosas que Hille había dejado atrás y además amplió la teoría usando teoría de representación para álgebras de semigrupos, métodos de perturbación, clases extendidas de semigrupos, etc. A principios de 1952, Hille colaboró con Phillips en la nueva edición de su libro para hacer una revisión exhaustiva, debido a los nuevos avances en la teoría, véase [49].

Además de los autores mencionados anteriormente, muchos otros han hecho contribuciones importantes para desarrollar la teoría de semigrupos, como A.V Balakrishnan, E. B. Davies, N. Dunford, W. Feller, N. Jacob, T. Kato, A. Pazy, J. T. Schwartz, E. Stein, M. Taibleson, H. F. Trotter, etc. Véase por ejemplo [7, 29, 34, 38, 50, 51, 70, 81, 87, 91]. En esta memoria serán particularmente interesantes para nosotros los trabajos de Taibleson and Stein, [81, 87], y su uso de la teoría de semigrupos en análisis armónico. Taibleson, en su tesis dirigida por Stein probó que los espacios de Lipschitz clásicos pueden caracterizarse por medio de los semigrupos del calor y de Poisson generados por $-\Delta$. Como curiosidad, en [81, 87] los autores no mencionan la palabra "semigrupo". Ellos se refieren a los semigrupos de Poisson y del calor como las integrales de Poisson y de Gauss-Weierstrass de una función.

Sin embargo, Stein también publicó en el mismo año su libro [80], que puede considerarse el trabajo pionero más importante donde la teoría de semigrupos se usa en análisis armónico. Allí, él probó su celebrado “teorema maximal”, esto es, un teorema sobre la acotación en L^p, $1 < p \leq \infty$, de la función maximal de un semigrupo contractivo y autoadjunto en L^2, $\{T_t\}_{t\geq 0}$, $\sup_{t>0} |T_t f(x)|$. Este libro fue bastante citado en los primeros veinte años tras su publicación, pero nada que ver con la repercusión que ha tenido en los últimos veinte años. Desde 1995, muchísimos investigadores han usado las ideas de Stein y han aplicado la teoría de semigrupos en el desarrollo del análisis armónico. Algunos de los ejemplos de los artículos a los que han surgido a raíz de este boom son [20, 21, 32, 35, 36, 37, 55, 67, 68, 82].

En 2009, P. R. Stinga y J. L. Torrea se dieron cuenta de que el lenguaje de semigrupos podía utilizarse también para formular y analizar las propiedades fundamentales de las potencias fraccionarias de operadores. En particular, inspirados por el famoso trabajo de Caffarelli y Silvestre sobre el problema de extensión relacionado con el laplaciano fraccionario, [23], probaron en [84] que las potencias fraccionarias de cualquier operador diferencial de segundo orden, normal y no negativo pueden describirse por medio de un problema de extensión. Además, introdujeron una definición puntual de las potencias fraccionarias positivas de operadores por medio de los semigrupos y su conocimiento sobre núcleos del calor. Las potencias fraccionarias de operadores han sido definidas de muchas maneras en análisis funcional, probabilidad, cálculo fraccionario y teoría potencial, véase [5, 7, 52, 78, 93]. El enfoque que Stinga y Torrea utilizaron fue el siguiente.

Sea L un operador general y consideremos su semigrupo del calor $\left\{e^{-tL}\right\}_{t\geq 0}$. Las potencias fraccionarias de L pueden definirse por

$$L^\sigma f = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tL} f - f\right) \frac{dt}{t^{1+\sigma}}, \quad 0 < \sigma < 1, \tag{0.1}$$

ver [52, 78, 93]. Esta fórmula está motivada por la siguiente identidad numérica,

$$\lambda^\sigma = \frac{1}{\Gamma(-\sigma)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+\sigma}}. \tag{0.2}$$

Stinga y Torrea se dieron cuenta de que cuando un núcleo del calor está disponible para e^{-tL}, esta fórmula abstracta da una expresión puntual para las potencias positivas del operador L.

Esta descripción por semigrupos tiene múltiples ventajas, como explicaremos ahora. Consideremos el operador fraccionario más popular, el laplaciano fraccionario, $(-\Delta)^\sigma f$, $0 < \sigma < 1$. Se define como la función cuya transformada de Fourier es $|\xi|^{2\sigma} \hat{f}(\xi)$. De este modo, para obtener una formula para $(-\Delta)^\sigma f$ podríamos invertir la transformada de Fourier, pero esto implicaría cálculos bastante complejos. Una manera alternativa de caracterizar $(-\Delta)^\sigma$, evitando invertir la transformada de Fourier, es la fórmula con semigrupos

$$(-\Delta)^\sigma f = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{t\Delta} f - f\right) \frac{dt}{t^{1+\sigma}}, \qquad 0 < \sigma < 1.$$

En [84], Stinga y Torrea usaron esta fórmula y sustituyeron $e^{t\Delta} f$ por la convolución del núcleo de Gauss-Weierstrass con la función f y consiguieron, de una forma más directa y sencilla, la expresión puntual de $(-\Delta)^\sigma f(x)$ para funciones suficientemente buenas, con el cálculo de las constantes explícitas.

En cuanto a las potencias negativas, la expresión puntual de la comúnmente llamada "integral fraccionaria" ya se conocía desde los orígenes del cálculo fraccionario. Su definición proviene de la generalización de la integral de orden n. Además, las potencias negativas del operador L pueden describirse desde el punto de vista del lenguaje de semigrupos, partiendo de la identidad numérica

$$\lambda^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t\lambda} \frac{dt}{t^{1-\sigma}}, \qquad \sigma > 0, \tag{0.3}$$

a la siguiente fórmula

$$L^{-\sigma} f = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL} f \, \frac{dt}{t^{1-\sigma}}, \quad \sigma > 0. \tag{0.4}$$

Esta expresión (0.4) es clásica, véase por ejemplo [52, 78, 93].

Recientemente, Torrea y colaboradores, [8, 85], extendieron (0.2) y (0.3) para $\lambda \in \mathbb{C}$: $\Re(\lambda) \geq 0$. Esto permite que las expresiones (0.1) y (0.4) sean válidas para una clase mayor de operadores, para los cuales la transformada de Fourier puede no estar disponible. Así, si el núcleo del calor para el semigrupo e^{-tL} es conocido, las correspondientes fórmulas puntuales se obtienen para funciones suficientemente buenas.

Las expresiones puntuales de las potencias fraccionarias de operadores revelan el carácter *no local* de los mismos, esto es, la dependencia de los valores de la función f en todo el dominio. Esta propiedad implica que los métodos locales de EDPs no se pueden aplicar para estudiar problemas para L^s. Sin embargo, en el espíritu de Caffarelli-Silvestre y Stinga-Torrea, una caracterización de L^s por medio de un problema de extensión puede hacerse, véase [19, 40, 84, 85], y esta caracterización nos puede permitir probar algunos resultados de EDPs, como las desigualdades de Harnack.

Por otra parte, es deseable estudiar **propiedades de regularidad** de operadores fraccionarios tales como $L^{\pm\sigma}$, así como transformadas de Riesz, potenciales de Bessel, multiplicadores de tipo transformada de Laplace, etc, en diferentes espacios funcionales. Estos resultados son interesantes no sólo en análisis, sino también en EDPs, porque implican estimaciones a priori de las soluciones a algunas ecuaciones en derivadas parciales. Probar resultados de regularidad en espacios Lipschitz (también conocidos como Hölder) pueden llevarte a considerar diferencias de la forma $|L^\sigma f(x_1) - L^\sigma f(x_2)|$, que podría convertirse en una tarea tediosa y difícil. Sin embargo, el lenguaje de semigrupos también nos permite caracterizar los espacios Lipschitz adaptados a "laplacianos" por medio de estimaciones de las derivadas de los semigrupos del calor y del Poisson asociados a esos "laplacianos", véase [30, 31, 43, 81, 85, 87]. Esta descripción por semigrupos de espacios Hölder nos permite conseguir resultados de regularidad para algunos operadores fraccionarios de una manera más fácil, rápida y elegante.

A lo largo de esta tesis trataremos con diversos problemas que surgen en análisis y EDPs, como los nombrados anteriormente, y veremos cómo el lenguaje de semigrupos nos ayuda a tratarlos y resolverlos de una manera eficiente.

0.1 Descripción de los resultados

0.1.1 Capítulo 1: Derivadas e integrales fraccionarias discretas.

En este capítulo estudiamos **derivadas e integrales fraccionarias discretas.** En 1911, S. Chapman fue el primero en considerar las "diferencias de orden fraccionario", véase [25]. Para $s > 0$, dado una sucesión a_n él definió

$$\triangle^s a_n = \sum_{m=0}^{\infty} \binom{-s-1+m}{m} a_{n+m}. \tag{0.5}$$

Su motivación fue extender la fórmula obvia para las diferencias fraccionarias de orden natural. Lo primero en lo que estamos interesados en este tema es en probar que la definición de las derivadas fraccionarias discretas usando el enfoque de semigrupos coincide con la de Chapman. Aunque la definición de Chapman sólo se preocupa del futuro, nosotros también consideraremos las derivadas discretas que miran al pasado. Para $f : \mathbb{Z} \to \mathbb{R}$, definimos *las derivadas discretas* "desde la derecha" y "desde la izquierda" como

$$\delta_{\text{right}} f(n) = f(n) - f(n+1) \ \ y \ \ \delta_{\text{left}} f(n) = f(n) - f(n-1).$$

Hemos hallado los semigrupos del calor generados por $-\delta_{\text{right}}$ y $-\delta_{\text{left}}$, $\{e^{-t\delta_{\text{right/left}}}\}_{t\geq 0}$, y usando el enfoque de los semigrupos, véase (0.1) y (0.4), definimos para $0 < \alpha < 1$

$$(\delta_{\text{right}})^{\alpha} f(n) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \frac{e^{-t\delta_{\text{right}}} f(n) - f(n)}{t^{1+\alpha}} dt, \ \ (\delta_{\text{right}})^{-\alpha} f(n) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \frac{e^{-t\delta_{\text{right}}} f(n)}{t^{1-\alpha}} dt,$$

y las fórmulas correspondientes para $(\delta_{\text{left}})^{\alpha}$, $-1 < \alpha < 1$. En la Sección 1.1 probamos que esta definición de $(\delta_{\text{right}})^{\alpha}$ coincide con la fórmula (0.5) dada por Chapman. También probamos **principios del máximo y de comparación** para las derivadas fraccionarias discretas, ver **Teoremas 1.6 y 1.7**, y probamos que las derivadas fraccionarias discretas aproximan a las derivadas fraccionarias continuas.

Para probar dicho teorema de aproximación necesitamos adaptar nuestras definiciones. Consideramos una malla de longitud de paso $h > 0$, esto es, $\mathbb{Z}_h = \{jh : j \in \mathbb{Z}\}$. En este contexto definimos

$$\delta_{\text{right}} u(hn) = \frac{u(hn) - u(h(n+1))}{h} \quad \text{y} \quad \delta_{\text{left}} u(hn) = \frac{u(hn) - u(h(n-1))}{h}, \quad n \in \mathbb{Z}.$$

Dada una función u definida en $\mathbb{R}$, sea $r_h u$ su restricción (o discretización) a $\mathbb{Z}_h$, es decir, $r_h u(j) = u(hj)$ para $j \in \mathbb{Z}$. Hemos probado el siguiente resultado de aproximación (ver **Teorema 1.12**).

(i) Sea $u \in C^{0,\beta}(\mathbb{R})$ *y* $0 < \alpha < \beta \leq 1$. *Entonces,*

$$\|(\delta_{\text{right}})^{\alpha}(r_h u) - r_h((D_{right})^{\alpha} u)\|_{\ell^\infty} \leq C_{\alpha} [u]_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha}.$$

(ii) Sea $u \in C^{1,\beta}(\mathbb{R})$ y $0 < \alpha < \beta \leq 1$. Entonces,

$$\|-\delta_{\text{right}}(\delta_{\text{right}})^{\alpha}(r_h u) - r_h(\frac{d}{dx}(D_{right})^{\alpha}u)\|_{\ell^\infty} \leq C_\alpha [u]_{C^{1,\beta}(\mathbb{R})} h^{\beta-\alpha}.$$

Aquí, los operadores $(D_{right/left})^{\alpha}$ son las derivadas de Marchaud, esto es,

$$(D_{right/left})^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(x \pm t) - f(x)}{t^{1+\alpha}} dt. \qquad (0.6)$$

Las clases $C^{k,\beta}(\mathbb{R})$, $k \in \mathbb{N}_0$, $\beta > 0$, son las clases Hölder usuales en la recta real.

También hemos probado los resultados análogos para δ_{left}. Como consecuencia de nuestro teorema de aproximación, probamos la coincidencia **puntual** de las derivadas de **Marchaud** y de **Grünwald-Letnikov** para funciones Hölder continuas (ver **Teorema 1.14**), donde la derivada de Marchaud viene dada por (0.6) y la de Grünwald-Letnikov por

$$\lim_{h \to +0} \sum_{m=0}^{\infty} \frac{\binom{-\alpha-1+m}{m} f(x \pm mh)}{h^\alpha}, \; x \in \mathbb{R}.$$

Hasta ese momento sólo era conocida la coincidencia en el sentido de $L^p(\mathbb{R})$, $1 \leq p < \infty$, para funciones $f \in L^r(\mathbb{R})$, con r y p independientes, véanse [78, Teoremas 20.2, 20.4].

Además, probamos resultados de regularidad para las derivadas e integrales discretas en las clases Hölder discretas $C_h^{k,\beta}$ así como estudiamos el comportamiento de las funciones maximales y las funciones cuadrado de Littlewood-Paley asociadas a los semigrupos del calor y a las funciones generalizadas de Poisson en los espacios de Lebesgue $\ell^p(\mathbb{Z})$. Para probar las acotaciones relacionadas con la función generalizada de Poisson usamos la *teoría de Calderón-Zygmund vector valuada en espacios de tipo homogéneo*, ver Sección 1.5.

0.1.2 Capítulo 2: Espacios Lipschitz de Schrödinger y resultados de regularidad

En este capítulo consideramos operadores de Schrödinger en $\mathbb{R}^n$ con $n \geq 3$, esto es, $\mathcal{L} = -\Delta + V$, donde V es un potencial no negativo que satisface la desigualdad de Hölder inversa:

$$\left(\frac{1}{|B|}\int_B V(y)^q dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)dy, \text{ con un exponente } q > n/2,$$

para cada bola B. Esta hipótesis sobre el potencial fue introducida por Z. Shen en [79] para obtener estimaciones de la solución fundamental de $\mathcal{L}$ en $\mathbb{R}^n$ y de la comparación con la solución fundamental de $-\Delta$ en $\mathbb{R}^n$.

En el contexto de operadores de Schrödinger, los espacios Lipschitz (también llamados Hölder) se han definido (ver [20, 59]) para $0 < \alpha < 1$ como:

$$\left\{ f : \rho(\cdot)^{-\alpha} f(\cdot) \in L^\infty(\mathbb{R}^n) \text{ y } \sup_{|z|>0} \frac{\|f(\cdot + z) - f(\cdot)\|_\infty}{|z|^\alpha} < \infty \right\},$$

donde $\rho(x) := \sup\left\{r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \le 1\right\}$ es la función radio crítico, que normalmente juega un papel importante en este escenario, véase [21, 59, 79].

Debido a que el potencial V carece de regularidad, no está claro cuál sería el operador que jugaría el mismo papel que las derivadas en el caso de los espacios de Hölder clásicos cuando $\alpha > 1$. Por esta razón, hasta el momento no había ninguna definición de espacios Hölder (o Lipschitz) adaptados a $\mathcal{L}$ para $\alpha > 1$. Nosotros hemos extendido la definición puntual de [59] para $0 < \alpha < 2$ usando el enfoque de Zygmund:

$$C^\alpha_{\mathcal{L}} := \left\{ f :\ \rho(\cdot)^{-\alpha} f(\cdot) \in L^\infty(\mathbb{R}^n) \text{ y } \sup_{|z|>0} \frac{\|f(\cdot + z) + f(\cdot - z) - 2f(\cdot)\|_\infty}{|z|^\alpha} < \infty \right\}. \tag{0.7}$$

La definiciones puntuales de los espacios Lipschitz implican que para probar resultados de regularidad de un operador en estos espacios necesitamos su expresión puntual, pero en muchos casos esta puede ser muy complicada. Por esta razón, la descripción de los espacios Lipschitz mediante el lenguaje de semigrupos es más conveniente.

En el espíritu de Taibleson y Stein, ver [81, 87], introducimos los siguientes espacios, definidos a través de los semigrupos del calor y de Poisson asociados a $\mathcal{L}$, $e^{-t\mathcal{L}}$ y $e^{-y\sqrt{\mathcal{L}}}$, para $\alpha > 0$:

$$\Lambda^W_{\alpha/2} := \left\{ f :\ \rho(\cdot)^{-\alpha} f(\cdot) \in L^\infty(\mathbb{R}^n) \text{ y } \left\|\partial^k_t e^{-t\mathcal{L}} f\right\|_{L^\infty(\mathbb{R}^n)} \le C_\alpha t^{-k+\alpha/2},\ \ k = [\alpha/2]+1, t > 0 \right\}.$$

$$\Lambda^P_\alpha := \left\{ f :\ \int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty \text{ y } \left\|\partial^k_y e^{-y\sqrt{\mathcal{L}}} f\right\|_{L^\infty(\mathbb{R}^n)} \le C_\alpha y^{-k+\alpha},\ \ k = [\alpha]+1, y > 0 \right\}.$$

Nuestro resultado principal en este capítulo es el siguiente (ver **Teorema 2.47**):

Sea $0 < \alpha \le 2 - n/q$ y f una función con $\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty$. Los siguientes enunciados son equivalentes:

$$f \in C^\alpha_{\mathcal{L}}, \quad f \in \Lambda^P_\alpha, \quad f \in \Lambda^W_{\alpha/2}.$$

Además, las normas son equivalentes.

En efecto, primero probamos que para $0 < \alpha \le 2 - n/q$, una función f pertenece a $C^\alpha_{\mathcal{L}}$ si, y sólo si, $f \in \Lambda^W_{\alpha/2}$, véase el **Teorema 2.22**. Sin embargo, si queremos añadir el espacio Λ^P_α a la cadena de equivalencias, necesitamos imponer la condición $\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty$.

La restricción $\alpha \le 2 - \frac{n}{q}$ viene impuesta fuertemente por la desigualdad de Hölder inversa que satisface el potencial V. Además, la prueba de estos resultados se basa en la comparación del semigrupo del calor de los operadores de Schrödinger con el semigrupo del calor clásico, y esta sólo se conoce cuando $q > n/2$, véase [37, 79].

Por otra parte, definimos las potencias $\mathcal{L}^{\pm\beta}$ usando el enfoque de semigrupos, ver (0.1) y (0.4), y obtenemos, entre otros, los siguientes **resultados de regularidad** (**Teoremas 2.58** y **2.60**).

- *Si $0 < \beta < \alpha$ y $f \in \Lambda^W_{\alpha/2}$, entonces $\|\mathcal{L}^{\beta/2} f\|_{\Lambda^W_{\frac{\alpha-\beta}{2}}} \le C\|f\|_{\Lambda^W_{\alpha/2}}$.*

- *Si* $\alpha, \beta > 0$, *entonces* $\|\mathcal{L}^{-\beta/2} f\|_{\Lambda^W_{\frac{\alpha+\beta}{2}}} \leq C\|f\|_{\Lambda^W_{\alpha/2}}$.

Estos resultados son completamente nuevos para $\alpha \geq 1$ pues, como dijimos antes, los espacios Lipschitz en el contexto general de Schrödinger sólo se habían definido para $0 < \alpha < 1$, luego los resultados de regularidad conocidos hasta este momento sólo podían considerar $0 < \alpha < 1$.

Además, hemos mejorado los resultados de [20] sobre la acotación de las transformadas de Riesz asociadas.

Consideremos las transformadas de Riesz de primer orden asociadas a los operadores de Schrödinger definidos por

$$\mathcal{R}_i = \partial_{x_i}(\mathcal{L}^{-1/2}), \text{ y } R_i = \mathcal{L}^{-1/2}(\partial_{x_i}), \; i = 1, \ldots, n.$$

- *Si* $0 < \alpha \leq 1 - n/q$, *entonces* $\|\mathcal{R}_i f\|_{\Lambda^W_{\alpha/2}} \leq C\|f\|_{\Lambda^W_{\alpha/2}}$, $i = 1, \ldots, n$,
- *Si* $1 < \alpha \leq 2 - n/q$, *entonces* $\|R_i f\|_{\Lambda^W_{\alpha/2}} \leq C\|f\|_{\Lambda^W_{\alpha/2}}$, $i = 1, \ldots, n$.

0.1.3 Capítulo 3: Espacios Lipschitz de Hermite parabólicos y elípticos. Resultados de regularidad

El oscilador armónico, $\mathcal{H} = -\Delta + |x|^2$, es un caso particular de operador de Schrödinger donde $V(x) = |x|^2$ satisface la desigualdad de Hölder inversa para todo $q > n/2$ ($n \geq 3$) y la función radio crítico asociada a V es $\rho(x) = \frac{1}{1+|x|}$, que es una función acotada. Esto significa que todos los resultados del Capítulo 2 aplican en este contexto sin las restricciones que dependían de q. Además, conocemos explícitamente los núcleos del calor (y por lo tanto de Poisson) asociados a $\mathcal{H}$ en $\mathbb{R}^n$, para $n \geq 1$. Estos hechos nos permitirán conseguir mejores resultados para $\mathcal{H}$ que los que hemos obtenido para los operadores generales de Schrödinger.

En este capítulo iremos más allá y consideraremos no sólo $\mathcal{H}$ sino también el *operador de Hermite parabólico* en $\mathbb{R}^n$, $n \geq 1$,

$$\mathbb{H} := \partial_t + \mathcal{H} = \partial_t - \Delta_x + |x|^2, \; x \in \mathbb{R}^n, \; t > 0.$$

Introduciremos definiciones de espacios Lipschitz (también llamados Hölder) adaptados a $\mathcal{H}$ y $\mathbb{H}$ por medio de sus semigrupos del calor y de Poisson. Veremos que esos espacios tienen caracterizaciones puntuales que implicarán, en el caso de $\mathcal{H}$, la coincidencia con la definición puntual de espacios Hölder introducida por Stinga y Torrea en [86].

Vamos a presentar nuestros resultados principales de este capítulo. El operador $\mathcal{H}$ puede factorizarse como $\mathcal{H} = \frac{1}{2}\sum_{i=1}^n (A_i A_{-i} + A_{-i} A_i)$, $A_i = \partial_{x_i} + x_i$, $A_{-i} = -\partial_{x_i} + x_i$. Los operadores de primer orden $A_{\pm i}$ juegan el papel, con respecto al operador $\mathcal{H}$, de las derivadas $\pm\partial_{x_i}$ con respecto al laplaciano clásico Δ.

Stinga y Torrea en [86] introducen la siguiente definición puntual.

Para $0 < \alpha < 1$,

$$C^\alpha_{\mathcal{H}}(\mathbb{R}^n) := \{f : (1 + |\cdot|)^\alpha f(\cdot) \in L^\infty(\mathbb{R}^n), \text{ y } \|f(\cdot + z) - f(\cdot)\|_{L^\infty(\mathbb{R}^n)} \leq A|z|^\alpha\}$$

con la norma asociada

$$\|f\|_{C^\alpha_{\mathcal{H}}} = [f]_{M^\alpha} + [f]_{C^\alpha_{\mathcal{H}}},$$

donde $[f]_{M^\alpha} = \|(1+|\cdot|)^\alpha f(\cdot)\|_\infty$ *y* $[f]_{C^\alpha_{\mathcal{H}}} = \sup_{|z|>0} \dfrac{\|f(\cdot+z)-f(\cdot)\|_\infty}{|z|^\alpha}$.

Para $\alpha > 1$ *y no entero,* $f \in C^\alpha_{\mathcal{H}}(\mathbb{R}^n)$*, si existen las derivadas de orden* $[\alpha]$ *y la norma*

$$\|f\|_{C^\alpha_{\mathcal{H}}} := [f]_{M^{\alpha-[\alpha]}} + \sum_{\substack{1\le|i_1|,\dots,|i_m|\le n\\ 1\le m\le[\alpha]}} [A_{i_1}\dots A_{i_m} f]_{M^{\alpha-[\alpha]}} + \sum_{1\le|i_1|,\dots,|i_{[\alpha]}|\le n} [A_{i_1}\dots A_{i_{[\alpha]}} f]_{C^{\alpha-[\alpha]}_{\mathcal{H}}},$$

es finita.

Inspirados por la definición anterior y los espacios Hölder parabólicos de Krylov, $C^{\alpha/2,\alpha}$, $0 < \alpha < 3$, ver Sección 3.1, introducimos los siguientes **espacios Hölder de Hermite parabólicos**:

Supongamos que $f \in L^\infty(\mathbb{R}^{n+1})$.

- *Sea* $0 < \alpha < 1$*. Decimos que* $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$ *si* $f \in C^{\alpha/2,\alpha}$ *y*

$$[f]_{M^\alpha} = \sup_{(t,x)\in\mathbb{R}^{n+1}} (1+|x|)^\alpha |f(t,x)| < \infty,$$

  En este caso, $\|f\|_{C^{\alpha/2,\alpha}_{t,\mathcal{H}}} = [f]_{M^\alpha} + [f]_{C^{\alpha/2,\alpha}_{t,\mathcal{H}}}$.

- *Para* $1 < \alpha < 2$*,* $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$ *si* $A_{\pm i} f \in C^{\alpha/2-1/2,\alpha-1}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$*,* $i = 1,\dots,n$*, y* $f(\cdot,x) \in C^{\alpha/2}(\mathbb{R})$ *uniformemente en* x.

- *Para* $2 < \alpha < 3$ *decimos que una función* $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$*, si las funciones* $A_{\pm i}A_{\pm j} f$*,* $i,j = 1,\dots,n$*, y* $\partial_t f$ *pertenecen a* $C^{\alpha/2-1,\alpha-2}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$.

También introducimos los espacios Hölder de Hermite parabólicos definidos por medio del semigrupo de Poisson, $\mathcal{P}_y f = e^{-y\sqrt{\mathbb{H}}} f$.

Sea $\mathcal{P}_y = e^{-y\sqrt{\mathbb{H}}}$ *y* $\alpha > 0$*, consideramos la clase*

$$\Lambda^{\mathcal{P}}_\alpha := \left\{ f :\ f \in L^\infty(\mathbb{R}^{n+1}) \text{ y } \left\|\partial_y^k \mathcal{P}_y f\right\|_{L^\infty(\mathbb{R}^{n+1})} \le C_k y^{-k+\alpha},\ \ k = [\alpha]+1, y > 0 \right\},$$

cuya norma viene dada por $\|f\|_{\Lambda^{\mathcal{P}}_\alpha} := \|f\|_\infty + C$*, donde* C *es el ínfimo de las constantes positivas* C_k *de arriba.*

Del mismo modo, pueden definirse los espacios análogos en el contexto elíptico: sea $P^{\mathcal{H}}_y = e^{-y\sqrt{\mathcal{H}}}$ el semigrupo de Poisson asociado a $\mathcal{H}$,

$$\Lambda^{P^{\mathcal{H}}}_\alpha := \left\{ f :\ f \in L^\infty(\mathbb{R}^n) \text{ y } \left\|\partial_y^k P^{\mathcal{H}}_y f\right\|_{L^\infty(\mathbb{R}^n)} \le C_k y^{-k+\alpha},\ \text{con}\, k = [\alpha]+1, y > 0 \right\}.$$

El siguiente resultado (**Teorema 3.68**) muestra que los espacios $\Lambda_\alpha^{\mathcal{P}}$ tienen una descripción puntual. Además, la restricción a funciones que dependen sólo de x, produce el resultado para $\Lambda_\alpha^{P^{\mathcal{H}}}$ (**Teorema 3.71**).

1. *Supongamos que $0 < \alpha < 2$. Entonces $f \in \Lambda_\alpha^{\mathcal{P}}$ si, y sólo si, existe una constante $C > 0$ tal que*

$$\|f(\cdot-\tau,\cdot-z)+f(\cdot-\tau,\cdot+z)-2f(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^{n+1})} \leq C(|\tau|^{1/2}+|z|)^\alpha, (\tau,z) \in \mathbb{R}^{n+1}$$

   y $(1+|x|)^\alpha f \in L^\infty(\mathbb{R}^{n+1})$. En este caso, si K denota el ínfimo de las constantes C para el cual la desigualdad de arriba es cierta, entonces $\|u\|_{\Lambda_\alpha^{\mathcal{P}}} := [u]_{M^\alpha} + K$, donde $[f]_{M^\alpha} = \|(1+|\cdot|)^\alpha f(\cdot,\cdot)\|_\infty$.
2. *Supongamos que $\alpha > 2$. Entonces, $f \in \Lambda_\alpha^{\mathcal{P}}$ si, y sólo si, $f \in L^\infty(\mathbb{R}^{n+1})$,*

$$A_{\pm i}A_{\pm j}f \in \Lambda_{\alpha-2}^{\mathcal{P}}, \ \ i,j = 1,\dots,n, \quad y \quad \partial_t f \in \Lambda_{\alpha-2}^{\mathcal{P}}.$$

   En este caso, se tiene la siguiente equivalencia

$$\|f\|_{\Lambda_\alpha^{\mathcal{P}}} \sim \|f\|_\infty + \sum_{i,j=1}^n \left(\|A_{\pm i}A_{\pm j}f\|_{\Lambda_{\alpha-2}^{\mathcal{P}}}\right) + \|\partial_t f\|_{\Lambda_{\alpha-2}^{\mathcal{P}}}.$$

Como consecuencia del resultado anterior, probamos el siguiente (**Teorema 3.69**).

Sea $0 < \alpha < 3$, α no entero. Entonces,

$$C_{t,\mathcal{H}}^{\alpha/2,\alpha} = \Lambda_\alpha^{\mathcal{P}},$$

con equivalencia de normas.

El resultado de arriba tiene su paralelo en el caso del operador de Hermite $\mathcal{H} = -\Delta_x + |x|^2$. En particular, ya que el espacio $C_{\mathcal{H}}^\alpha(\mathbb{R}^n)$ está definido para cada $\alpha > 0$, obtendremos en el contexto elíptico que $C_{\mathcal{H}}^\alpha(\mathbb{R}^n) = \Lambda_\alpha^{P^{\mathcal{H}}}$, para cada $\alpha > 0$, $\alpha \notin \mathbb{N}$, véase el **Teorema 3.72**.

Además, también introduciremos espacios Lipschitz de Hermite definidos a través del semigrupo del calor, pero sólo en el caso elíptico. En el caso parabólico sería análogo.

Sea $e^{-y\mathcal{H}} = W_y^{\mathcal{H}}$ el semigrupo del calor asociado a $\mathcal{H}$. Para $\alpha > 0$,

$$\Lambda_{\alpha/2}^{W^{\mathcal{H}}} =: \left\{ f \in L^\infty(\mathbb{R}^n) : \left\|\partial_y^k W_y^{\mathcal{H}} f\right\|_{L^\infty(\mathbb{R}^n)} \leq C_k y^{-k+\alpha/2}, \ k = [\alpha/2]+1 \right\}.$$

En este caso tenemos el siguiente resultado (**Theorem 3.95**).

Para cada $\alpha > 0$, los espacios $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ y $\Lambda_\alpha^{P^{\mathcal{H}}}$ coinciden en el sentido de espacios normados.

Como consecuencia, usando la coincidencia de $\Lambda_\alpha^{P^{\mathcal{H}}}$ y $C_{\mathcal{H}}^\alpha$, tenemos el siguiente corolario.

Si $\alpha > 0$ no es un entero, $\Lambda_{\alpha/2}^{W^{\mathcal{H}}} = \Lambda_{\alpha}^{P^{\mathcal{H}}} = C_{\mathcal{H}}^{\alpha}$, donde las identidades se entienden en el sentido de espacios normados.

Nuestras definiciones de espacios Lipschitz mediante semigrupos nos permitirán obtener **resultados de regularidad** para operadores fraccionarios relacionados con $\mathcal{H}$ y $\mathbb{H}$ de una forma más directa y elegante. Nótese que la coincidencia de $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ y $\Lambda_{\alpha}^{P^{\mathcal{H}}}$ para cualquier $\alpha > 0$ implica que todos los resultados de regularidad que hemos probado en el capítulo anterior aplican aquí cuando $n \geq 3$. Además, en la Sección 3.4 hemos probado los siguientes resultados en el contexto parabólico y su análogo en el elíptico, para $n \geq 1$.

- *Sea $0 < 2\beta < \alpha$ y $f \in \Lambda_{\alpha}^{\mathcal{P}}$, entonces $\mathbb{H}^{\beta} f \in \Lambda_{\alpha-2\beta}^{\mathcal{P}}$ y*

$$\|\mathbb{H}^{\beta} f\|_{\Lambda_{\alpha-2\beta}^{\mathcal{P}}} \leq C\|f\|_{\Lambda_{\alpha}^{\mathcal{P}}}.$$

- *Sea $\alpha, \beta > 0$.*
  (i) Dado $f \in \Lambda_{\alpha}^{\mathcal{P}}$, entonces $\mathbb{H}^{-\beta} f \in \Lambda_{\alpha+2\beta}^{\mathcal{P}}$ y

$$\|\mathbb{H}^{-\beta} f\|_{\Lambda_{\alpha+2\beta}^{\mathcal{P}}} \leq C\|f\|_{\Lambda_{\alpha}^{\mathcal{P}}}.$$

  (ii) Si $f \in L^{\infty}(\mathbb{R}^{n+1})$, entonces $\mathbb{H}^{-\beta} f \in \Lambda_{2\beta}^{\mathcal{P}}$ y

$$\|\mathbb{H}^{-\beta} f\|_{\Lambda_{2\beta}^{\mathcal{P}}} \leq C\|f\|_{\infty}.$$

En cuanto a las transformadas de Riesz, obtenemos mejores resultados que los que probamos para operadores generales de Schrödinger en el capítulo anterior.

Consideremos las transformadas de Riesz parabólicas de orden $m \geq 1$ definidas por

$$R_{\nu} = (A_{\pm 1}^{\nu_1} A_{\pm 2}^{\nu_2} \dots A_{\pm n}^{\nu_n}) \mathbb{H}^{-m/2} \;\; y \;\; R_m = \partial_t^m \mathbb{H}^{-m}$$

donde $\nu_i \geq 0, i = 1, \dots, n$ y $|\nu| = \nu_1 + \dots + \nu_n = m$. Sea $\alpha > 0$, entonces R_{ν} y R_m están acotadas de $\Lambda_{\alpha}^{\mathcal{P}}$ en sí mismo. Un resultado paralelo se obtiene para los operadores $(A_{\pm 1}^{\nu_1} A_{\pm 2}^{\nu_2} \dots A_{\pm n}^{\nu_n}) \mathcal{H}^{-m/2}$ cuando actúan en los espacios $\Lambda_{\alpha}^{P^{\mathcal{H}}}$.

Además, en la Sección 3.4 también obtenemos la acotación de los potenciales de Bessel y de los multiplicadores de tipo transformada de Laplace, definidos a través de los semigrupos del calor y de Poisson, y obtenemos un principio del máximo.

0.1.4 Capítulo 4: Ecuaciones parabólicas en el contexto de Bessel

En este capítulo consideramos las ecuaciones parabólicas

$$\frac{\partial u(t,x)}{\partial t} = \Delta_{\mu} u(t,x) + f(t,x), \;\; (t,x) \in \mathbb{R} \times (0,\infty) \;\; o \;\; (t,x) \in (0,\infty) \times (0,\infty), \qquad (0.8)$$

donde, para cada $\mu > -1$, Δ_{μ} representa el operador de Bessel definido por $\Delta_{\mu} = \partial_x^2 + (\frac{1}{4} - \mu^2) x^{-2}$.

El operador $-\Delta_\mu$ es positivo, autoadjunto en $L^2((0,\infty))$ y genera el semigrupo $\{e^{t\Delta_\mu}\}_{t>0} = \{W_t^\mu\}_{t>0}$ de operadores en $L^2((0,\infty))$ donde, para cada $t>0$ y $\phi \in L^2((0,\infty))$,

$$W_t^\mu(\phi)(x) = \int_0^\infty W_t^\mu(x,y)\phi(y)dy, \quad x \in (0,\infty). \tag{0.9}$$

Aquí, $W_t^\mu(x,y)$ es el núcleo del calor correspondiente. Si, para cada $t>0$, W_t^μ viene dado como en (0.9), $\{W_t^\mu\}_{t>0}$ también define un semigrupo de operadores en $L^p((0,\infty))$, para cada $1<p<\infty$ cuando $\mu > -1/2$ y para cada $1<p<\infty$ tal que $-\mu - 1/2 < \frac{1}{p} < \mu + 3/2$, cuando $-1 < \mu \leq -1/2$.

Vamos a describir los resultados en $\mathbb{R} \times (0,\infty)$. Nuestro primer resultado principal es sobre la existencia de soluciones clásicas de (0.8).

Supongamos que $f \in L^\infty(\mathbb{R}\times(0,\infty))$ tiene soporte compacto en $\mathbb{R}\times(0,\infty)$. Entonces, para $\mu > -1$, la función $u(t,x)$, $(t,x) \in \mathbb{R}\times(0,\infty)$, dada por

$$u(t,x) = \int_0^\infty \int_0^\infty W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau, \quad (t,x) \in \mathbb{R}\times(0,\infty),$$

está definida por una integral que converge absolutamente para cada $(t,x) \in \mathbb{R}\times(0,\infty)$. Además, si f es también $C^2(\mathbb{R}\times(0,\infty))$, entonces para cada $(t,x) \in \mathbb{R}\times(0,\infty)$, $\frac{\partial u(t,x)}{\partial t} = \Delta_\mu u(t,x) + f(t,x)$, siendo

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \lim_{\epsilon\to 0^+} \int_\epsilon^\infty \int_0^\infty \frac{\partial}{\partial \tau} W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau + f(t,x)\\ &= \lim_{\epsilon\to 0^+} \int_{\Omega_\epsilon(x)} \frac{\partial}{\partial \tau} W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau + Af(t,x), \quad t,x \in (0,\infty),\end{aligned}$$

y

$$\begin{aligned}\frac{\partial^2 u(t,x)}{\partial x^2} &= \lim_{\epsilon\to 0^+} \int_\epsilon^\infty \int_0^\infty \frac{\partial^2}{\partial x^2} W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau\\ &= \lim_{\epsilon\to 0^+} \int_{\Omega_\epsilon(x)} \frac{\partial^2}{\partial x^2} W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau - (1-A)f(t,x)\end{aligned}$$

donde, para cada $\epsilon, x \in (0,\infty)$, $\Omega_\epsilon(x) = \{(\tau,y) \in (0,\infty)^2 : \tau^{1/2} + |x-y| > \epsilon\}$, y $A = \frac{1}{\sqrt{\pi}} \int_0^1 e^{-\frac{w^2}{4}} dw$.

El operador de Bessel puede escribirse como $\Delta_\mu = \delta_\mu^* \delta_\mu$, donde $\delta_\mu = x^{\mu+1/2}\frac{d}{dx}x^{-\mu-1/2}$, y $\delta_\mu^* = x^{-\mu-1/2}\frac{d}{dx}x^{\mu+1/2}$ representa el adjunto formal de δ_μ. Ahora consideramos el operador L_μ definido por

$$(L_\mu f)(t,x) = \int_0^\infty \int_0^\infty W_s^\mu(x,y) f(t-s,y)dyds,$$

siendo f una función compleja medible definida en $\mathbb{R}\times(0,\infty)$, siempre que la última integral converja. L_μ puede verse como $(\partial_t - \Delta_\mu)^{-1}$. Teniendo en cuenta las ideas de Stein ([80]), definimos las transformadas de Riesz asociadas al operador parabólico $\partial_t - \Delta_\mu$ como sigue: para cada $f \in C_c^2(\mathbb{R} \times (0,\infty))$,

$$R_\mu(f) = \delta_{\mu+1}\delta_\mu L_\mu(f) \text{ y } \widetilde{R_\mu}(f) = \partial_t L_\mu(f).$$

De acuerdo con el teorema anterior, si $f \in C_c^2(\mathbb{R}\times(0,\infty))$, las definiciones de arriba de $R_\mu(f)$ y $\widetilde{R_\mu}(f)$ tienen sentido. Además, podemos escribir, para cada $f \in C_c^2(\mathbb{R} \times (0,\infty))$,

$$R_\mu(f)(t,x) = \lim_{\epsilon\to 0^+} \int_{\Omega_\epsilon(t,x)} K_\mu(t,x;\tau,y) f(\tau,y) d\tau dy + f(t,x)\frac{1}{\sqrt{\pi}}\int_1^\infty e^{-s^2/4} ds \tag{0.10}$$

y

$$\widetilde{R_\mu}(f)(t,x) = \lim_{\epsilon\to 0^+} \int_{\Omega_\epsilon(t,x)} \widetilde{K_\mu}(t,x;\tau,y) f(\tau,y) d\tau dy + f(t,x)\frac{1}{\sqrt{\pi}}\int_0^1 e^{-s^2/4} ds, \tag{0.11}$$

con $(t,x) \in \mathbb{R} \times (0,\infty)$, donde

$$K_\mu(t,x;\tau,y) = \delta_{\mu+1}\delta_\mu W^\mu_{t-\tau}(x,y)\chi_{(0,\infty)}(t-\tau), \quad x,y \in (0,\infty), \quad t,\tau \in \mathbb{R},$$
$$\widetilde{K_\mu}(t,x;\tau,y) = -\partial_\tau W^\mu_{t-\tau}(x,y)\chi_{(0,\infty)}(t-\tau), \quad x,y \in (0,\infty), \quad t,\tau \in \mathbb{R},$$

y $\Omega_\epsilon(t,x) = \{(\tau,y) \in (0,\infty) \times (0,\infty) : \max\{|t-\tau|^{1/2}, |x-y|\} > \epsilon\}$, para $\epsilon, x \in (0,\infty)$ y $t \in (0,\infty)$.

Vamos a probar la acotación de R_μ y $\widetilde{R_\mu}$ en espacios L^p (con pesos) y espacios L^p mixtos y con pesos.

Nuestro primer resultado en esta línea, **Teorema 4.122**, concierne los espacios L^p (con pesos). Representamos, para cada $1 \le p < \infty$, por $A_p^*(\mathbb{R} \times (0,\infty))$ la clase de pesos de Muckenhoupt en el espacio de tipo homogéneo $(\mathbb{R} \times (0,\infty), m, d)$, donde d es la distancia parabólica y m la medida de Lebesgue en $\mathbb{R} \times (0,\infty)$.

(1) Si $\mu > -1$, las transformadas de Riesz R_μ y $\widetilde{R_\mu}$ están acotadas de $L^2(\mathbb{R} \times (0,\infty))$ en sí mismo.

(2) Supongamos que $\mu > 1/2$ o $\mu = -1/2$. Las transformadas de Riesz R_μ y $\widetilde{R_\mu}$ pueden extenderse de $L^2(\mathbb{R}\times(0,\infty)) \cap L^p(\mathbb{R}\times(0,\infty),\omega)$ a $L^p(\mathbb{R}\times(0,\infty),\omega)$ como operadores acotados de $L^p(\mathbb{R}\times(0,\infty),\omega)$

- *en $L^p(\mathbb{R} \times (0,\infty),\omega)$, para cada $1 < p < \infty$ y $\omega \in A_p^*(\mathbb{R} \times (0,\infty))$.*
- *en $L^{1,\infty}(\mathbb{R} \times (0,\infty),\omega)$, para $p = 1$ y $\omega \in A_1^*(\mathbb{R} \times (0,\infty))$.*

(3) Si $\mu > -1/2$, las transformadas de Riesz R_μ y $\widetilde{R_\mu}$ pueden extenderse de $L^2(\mathbb{R}\times(0,\infty)) \cap L^p(\mathbb{R}\times(0,\infty))$ a $L^p(\mathbb{R}\times(0,\infty))$ como operadores acotados de $L^p(\mathbb{R}\times(0,\infty))$

- *en $L^p(\mathbb{R} \times (0,\infty))$, para cada $1 < p < \infty$.*
- *en $L^{1,\infty}(\mathbb{R} \times (0,\infty))$, para $p = 1$.*

(4) Si $-1 < \mu \leq -1/2$, entonces la transformada de Riesz $\widetilde{R_\mu}$ puede extenderse de $L^2(\mathbb{R}\times(0,\infty))\cap L^p(\mathbb{R}\times(0,\infty))$ a $L^p(\mathbb{R}\times(0,\infty))$ como un operador acotado de $L^p(\mathbb{R}\times(0,\infty))$ en sí mismo, siempre que $-\mu-1/2 < 1/p < \mu+3/2$ y $1 < p < \infty$.

(5) Si $-1 < \mu \leq -1/2$, entonces la transformada de Riesz R_μ puede extenderse de $L^2(\mathbb{R}\times(0,\infty))\cap L^p(\mathbb{R}\times(0,\infty))$ a $L^p(\mathbb{R}\times(0,\infty))$ como un operador acotado de $L^p(\mathbb{R}\times(0,\infty))$ en sí mismo, siempre que $p > \frac{1}{\mu+3/2}$ y $1 < p < \infty$.

Además, cuando $\mu > -1/2$ en todos estos casos las extensiones de los operadores R_μ y $\widetilde{R_\mu}$ están definidas por (0.10) y (0.11), respectivamente, donde el límite existe en casi todo punto $(t,x)\in\mathbb{R}\times(0,\infty)$ y las igualdades se entienden también en casi todo punto $(t,x)\in\mathbb{R}\times(0,\infty)$.

Además, aplicando la teoría de Calderón-Zygmund vector valuada (véase [76]), establecemos las siguientes desigualdades de norma mixta con pesos para las transformadas de Riesz R_μ y $\widetilde{R_\mu}$ (ver **Teorema 4.127**). Para cada $1 \leq p < \infty$, denotamos las clases de pesos de Muckenhoupt clásicas por $A_p(\Omega)$, donde $\Omega = (0,\infty)$ o $\Omega = \mathbb{R}$.

Supongamos que $\mu > 1/2$ o $\mu = -1/2$. Si $1 < p < \infty$ y $v \in A_p((0,\infty))$, entonces las transformadas de Riesz R_μ y $\widetilde{R_\mu}$ pueden extenderse de $L^2(\mathbb{R}\times(0,\infty))\cap L^q(\mathbb{R},u,L^p((0,\infty),v))$ a $L^q(\mathbb{R},u,L^p((0,\infty),v))$ como operadores acotados de $L^q(\mathbb{R},u,L^p((0,\infty),v))$ en sí mismos, siempre que $1 < q < \infty$ y $u \in A_q(\mathbb{R})$; y, para cada $u \in A_1(\mathbb{R})$, de $L^2(\mathbb{R}\times(0,\infty))\cap L^1(\mathbb{R},u,L^p((0,\infty),v))$ a $L^1(\mathbb{R},u,L^p((0,\infty),v))$ como operadores acotados de $L^1(\mathbb{R},u,L^p((0,\infty),v))$ en $L^{1,\infty}(\mathbb{R},u,L^p((0,\infty),v))$.

Estos resultados pueden verse como desigualdades de tipo Sobolev con pesos y mixtas con pesos para soluciones de las ecuaciones de Bessel parabólicas.

También, consideramos el siguiente problema de Cauchy asociado a (0.8) en $(0,\infty)\times(0,\infty)$:

$$\begin{cases} \partial_t u(t,x) = \Delta_\mu u(t,x) + f(t,x), & (t,x)\in(0,\infty)\times(0,\infty), \\ u(0,x) = g(x), & x\in(0,\infty). \end{cases}$$

De forma similar al caso $\mathbb{R}\times(0,\infty)$, probamos la existencia de soluciones clásicas del problema de Cauchy de arriba (ver **Teorema 4.129**). También, para cada $f \in C_c^2((0,\infty)\times(0,\infty))$ definimos las transformadas de Riesz

$$\mathbf{R}_\mu(f)(t,x) = \lim_{\epsilon\to 0^+}\int_0^{t-\epsilon}\int_0^\infty \delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau, \quad t,x\in(0,\infty),$$

y

$$\widetilde{\mathbf{R}_\mu}(f)(t,x) = \lim_{\epsilon\to 0^+}\int_0^{t-\epsilon}\int_0^\infty \partial_\tau W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau, \quad t,x\in(0,\infty).$$

Probaremos la acotación de $\mathbf{R}_\mu$ y $\widetilde{\mathbf{R}_\mu}$ en espacios L^p (con pesos), véase **Teorema 4.130**, y la acotación de $\widetilde{\mathbf{R}_\mu}$ en espacios L^p mixtos (**Teorema 4.131**).

Introduction

The study of semigroup theory has its origins in the forties and some of the pioneering works are due to E. Hille and K. Yosida, see [46, 94]. They independently studied the problem of determining the most general bounded linear operator valued function T_t, $t \geq 0$, which satisfies

1) $T_{t+s} = T_t T_s$.

2) $T_0 = Id$ (Id denotes the identity operator.)

Moreover, they introduced the notion of infinitesimal generator A of T_t defined by

$$A = \lim_{t \to 0} \frac{T_t - Id}{t} \quad \text{(the limit is taken in the strong sense).}$$

Given X is Banach spaces, a *semigroup* is a family of bounded linear operators on X, $\{T_t\}_{t \geq 0} \subset L(X, X)$, which satisfies 1) and 2).

Since 1948 both the analytical theory of semigroups and its applications have made vigorous progress. Yosida proceeded to apply his generation theorem to the diffusion equations in a series of important papers ([95, 96]) and Hille, inspired by Yosida's work, made in 1949 a new attack on Cauchy's problem with the aid of the semigroup theory, see [47, 48]. Another of the early workers in the general theory of semigroups of linear operators was R.S. Phillips, see for instance [71, 72], who filled in many of the gaps which Hille had left behind and then he went on to broaden the theory by using representation theory for semigroups algebras, perturbation methods, extended classes of semigroups, etc. In early 1952, Hille collaborated with Phillips in the new edition of his book to make an extensive revision due to the new advances in the theory of semigroups, see [49].

Apart from the authors mentioned above, many others have made important contributions to develop the semigroup theory such as A.V Balakrishnan, E. B. Davies, N. Dunford, W. Feller, N. Jacob, T. Kato, A. Pazy, J. T. Schwartz, E. Stein, M. Taibleson, H. F. Trotter, etc. See [7, 29, 34, 38, 50, 51, 70, 81, 87, 91]. Of particular interest for us are the works of Taibleson and Stein and their use of the semigroup theory in harmonic analysis, [81, 87]. Taibleson, in his thesis surpervised by Stein, proved that classical Lipschitz spaces can be characterized by means of the heat and the Poisson semigroups generated by $-\Delta$. As a curiosity, along [81, 87] the authors did not mention the word "semigroup". They referred to the Poisson and heat semigroups as the Poisson and Gauss-Weierstrass integrals of a function. However, Stein also published in the same year his book [80], which can be considered the most important pioneer work where the semigroup theory is used in harmonic analysis. There, he proved the celebrated "maximal theorem", that is, a theorem about the boundedness on L^p, $1 < p \leq \infty$, of the maximal function of a contractive and selfadjoint on L^2 semigroup $\{T_t\}_{t\geq 0}$, $\sup_{t>0} |T_t f(x)|$. This book was quite cited in the first twenty years after publication, but nothing to do with the repercussion it has got in the last twenty years. From 1995, many researchers have used Stein's ideas and have applied the theory of semigroups in the development of harmonic analysis. Some examples of the papers this has given rise are [20, 21, 32, 35, 36, 37, 55, 67, 68, 82].

In 2009, P. R. Stinga and J. L. Torrea realized that the semigroup language can be used to formulate and analyze fundamental properties of fractional powers of operators. In particular, inspired by the celebrated work of Caffarelli and Silvestre about the extension problem related to the fractional Laplacian, see [23], they proved that the fractional powers of any nonnegative normal second order differential operator can be described by means of a extension problem, see [84]. Moreover, they introduced a pointwise definition of positive fractional powers of operators by means of semigroups and their knowledge about heat kernels. Fractional powers of operators have been defined in several ways in functional analysis, probability, fractional calculus and potential theory, see [5, 7, 52, 78, 93]. The approach that Stinga and Torrea used was the following.

Let L be a general operator and consider the semigroup $\left\{e^{-tL}\right\}_{t\geq 0}$. The positive fractional powers of L can be defined as

$$L^\sigma f = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{-tL} f - f\right) \frac{dt}{t^{1+\sigma}}, \quad 0 < \sigma < 1,$$

see [52, 78, 93].. This formula is motivated by the following numerical identity,

$$\lambda^\sigma = \frac{1}{\Gamma(-\sigma)} \int_0^\infty (e^{-t\lambda} - 1) \frac{dt}{t^{1+\sigma}}.$$

Stinga and Torrea realized that, when a heat kernel is available for e^{-tL}, this abstract formula gives a pointwise expression for positive powers of the operator L.

This semigroup description has many advantages, as we will explain now. Consider the most popular fractional operator, the fractional Laplacian, $(-\Delta)^\sigma f$, $0 < \sigma < 1$. It is defined as the function whose Fourier transform is given by $|\xi|^{2\sigma} \hat{f}(\xi)$. Thus, to get a formula for $(-\Delta)^\sigma f$ we could invert the Fourier transform, but this would imply rather involve computations. An alternative approach to characterize $(-\Delta)^\sigma$, which avoids inverting the Fourier transform, is the semigroup formula

$$(-\Delta)^\sigma f = \frac{1}{\Gamma(-\sigma)} \int_0^\infty \left(e^{t\Delta} f - f\right) \frac{dt}{t^{1+\sigma}}, \qquad 0 < \sigma < 1.$$

In [84], Stinga and Torrea used this formula by substituting $e^{t\Delta} f$ by the convolution of the Gauss-Weierstrass kernel with the function f and they got, in a direct and easier way, the pointwise expression of $(-\Delta)^\sigma f(x)$ for good enough functions, with the explicit constants.

Regarding the negative powers, the pointwise expression of the so called "fractional integral" was already known from the originis of fractional calculus. Its definition came from the generalization of the integral of order n. Moreover, the negative powers of the operator L can be described from the semigroup language point of view, which drives the numerical identity

$$\lambda^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t\lambda} \frac{dt}{t^{1-\sigma}}, \qquad \sigma > 0,$$

to the following formula

$$L^{-\sigma} f = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-tL} f \, \frac{dt}{t^{1-\sigma}}, \quad \sigma > 0.$$

This formula (0.4) is classical, see [52, 78, 93].

Recently, Torrea and collaborators, see [8, 85], extended (0.2) and (0.3) for $\lambda \in \mathbb{C}$: $\Re(\lambda) \geq 0$. This allows us to get formulae (0.1) and (0.4) for a bigger class of operators, for which the Fourier transform may not be available. Thus, if the heat kernel for the semigroup e^{-tL} is known, the corresponding pointwise formulas for good enough functions are obtained.

Pointwise formulas of fractional powers of operators reveal the *nonlocal nature* of the operators, that is, the dependance on the values of the function f in all the domain. This property implies that the local PDE methods can not be applied to study problems for L^s. However, in the spirit of Caffarelli-Silvestre and Stinga-Torrea, a characterization of L^s by means of a extension problem can be made, see [19, 40, 84, 85], and this characterization may allow us to prove some PDE results, such as Harnack's inequalities.

On the other hand, it is desirable to study **regularity properties** of fractional operators such as $L^{\pm\sigma}$, as well as Riesz transforms, Bessel potentials, Laplace transform-type multipliers, etc, in different functional spaces. These kind of results are interesting not only in analysis, but also in PDE, because they imply a priori estimates of solutions to some partial differential equations. Proving regularity results in Lipschitz (also called Hölder) spaces may lead you to consider differences of the form $|L^\sigma f(x_1) - L^\sigma f(x_2)|$, which could become a difficult and tedious task. However, the semigroup language also allows us to characterize Lipschitz spaces adapted to "Laplacians" by means of estimates of the derivatives of the heat or the Poisson semigroups associated to those "Laplacians", see [30, 31, 43, 81, 85, 87]. This semigroup description of Hölder spaces allows to get regularity results for some fractional operators in a easier, quicker and more elegant way.

Along this thesis we shall deal with different problems that arise in analysis and PDE, as the ones cited above, and we shall show how the semigroup language can be quite useful to manage and solve them in an efficient way.

0.2 Description of the results

0.2.1 Chapter 1: Discrete fractional derivatives and integrals.

In this chapter we study **discrete fractional derivatives and integrals.** S. Chapman in 1911, see [25], was the first author who considered "differences of fractional order". For $s > 0$, given a sequence a_n he defined

$$\triangle^s a_n = \sum_{m=0}^{\infty} \binom{-s-1+m}{m} a_{n+m}.$$

His motivation was to extend the obvious formula for differences of natural order. Our first interest in this theme is to prove if the definition of discrete fractional derivatives by using the semigroup approach would coincide with the Chapman's one. Although Chapman's definition only cares about the future, we also consider the discrete derivatives which cares about the past. For $f : \mathbb{Z} \to \mathbb{R}$, we define *the discrete derivatives* "from the right" and "from the left" as

$$\delta_{\text{right}} f(n) = f(n) - f(n+1) \text{ and } \delta_{\text{left}} f(n) = f(n) - f(n-1).$$

We got the heat semigroups generated by $-\delta_{\text{right}}$ and $-\delta_{\text{left}}$, $\{e^{-t\delta_{\text{right/left}}}\}_{t\geq 0}$, and by the semigroup aproach, see (0.1) and (0.4), we define for $0 < \alpha < 1$

$$(\delta_{\text{right}})^{\alpha} f(n) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \frac{e^{-t\delta_{\text{right}}} f(n) - f(n)}{t^{1+\alpha}} dt, \quad (\delta_{\text{right}})^{-\alpha} f(n) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} \frac{e^{-t\delta_{\text{right}}} f(n)}{t^{1-\alpha}} dt,$$

and the corresponding formulas for $(\delta_{\text{left}})^{\alpha}$, $-1 < \alpha < 1$. In Section 1.1 we prove that this definition of $(\delta_{\text{right}})^{\alpha}$ coincides with the formula (1.1) given by Chapman. Also, we get **maximum and comparison principles** for the discrete fractional derivatives, see **Theorems 1.6 and 1.7**, and we prove that the discrete fractional derivatives **approximate** the continuous fractional derivatives.

To prove that approximation theorem we need to adapt our definitions. We consider a mesh with step length $h > 0$, that is, $\mathbb{Z}_h = \{jh : j \in \mathbb{Z}\}$. In this setting we define

$$\delta_{\text{right}} u(hn) = \frac{u(hn) - u(h(n+1))}{h} \quad \text{and} \quad \delta_{\text{left}} u(hn) = \frac{u(hn) - u(h(n-1))}{h}, \quad n \in \mathbb{Z}.$$

Given a function u defined on $\mathbb{R}$, let $r_h u$ be its restriction (or discretization) to $\mathbb{Z}_h$, that is, $r_h u(j) = u(hj)$ for $j \in \mathbb{Z}$. We have proved the following approximation result (see **Theorem 1.12**).

(i) Let $u \in C^{0,\beta}(\mathbb{R})$ and $0 < \alpha < \beta \leq 1$. Then

$$\|(\delta_{\text{right}})^{\alpha}(r_h u) - r_h((D_{right})^{\alpha} u)\|_{\ell^\infty} \leq C_{\alpha}[u]_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha}.$$

(ii) Let $u \in C^{1,\beta}(\mathbb{R})$ and $0 < \alpha < \beta \leq 1$. Then

$$\|-\delta_{\text{right}}(\delta_{\text{right}})^{\alpha}(r_h u) - r_h(\frac{d}{dx}(D_{right})^{\alpha} u)\|_{\ell^\infty} \leq C_{\alpha}[u]_{C^{1,\beta}(\mathbb{R})} h^{\beta-\alpha}.$$

Here, the operators $(D_{right/left})^{\alpha}$ are the Marchaud derivatives, see [78], that is,

$$(D_{right/left})^{\alpha} f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} \frac{f(x \pm t) - f(x)}{t^{1+\alpha}} dt.$$

The classes $C^{k,\beta}(\mathbb{R})$, $k \in \mathbb{N}_0$, $\beta > 0$, are the usual Hölder classes on the real line.

Analogous results for δ_{left} are also obtained. As a consequence of our approximation theorem, we prove the **pointwise** coincidence of the **Marchaud and the Grünwald-Letnikov derivatives** for Hölder continuous functions (see **Theorem 1.14**), where the Marchaud derivative is given by (0.6) and Grünwald-Letnikov derivative is defined as

$$\lim_{h\to+0}\sum_{m=0}^{\infty}\frac{\binom{-\alpha-1+m}{m}f(x\pm mh)}{h^{\alpha}},\ x\in\mathbb{R}.$$

Until that moment it only was known the coincidence in the $L^p(\mathbb{R})$ sense, $1\le p<\infty$, for functions $f\in L^r(\mathbb{R})$, with r and p independent, see [78, Theorems 20.2, 20.4].

In addition, we get regularity results for both discrete fractional derivatives and integrals in the discrete Hölder classes $C_h^{k,\beta}$ as well as we study the behaviour of the maximal functions and Littlewood-Paley square functions associated to the heat semigroup and the generalized Poisson function in the Lebesgue spaces $\ell^p(\mathbb{Z})$. To prove the boundedness properties related to the generalized Poisson function we use *vector-valued Calderón-Zygmund Theory in spaces of homogeneous type*, see Section 1.5.

0.2.2 Chapter 2: Schrödinger Lipschitz spaces and regularity results

In this chapter we consider Schrödinger operators in $\mathbb{R}^n$ with $n\ge 3$, that is, $\mathcal{L}=-\Delta+V$, where V is a nonnegative potential satisfying the reverse Hölder inequality:

$$\left(\frac{1}{|B|}\int_B V(y)^q dy\right)^{1/q}\le\frac{C}{|B|}\int_B V(y)dy,\ \text{with an exponent } q>n/2,$$

for every ball B. This hypothesis concerning the potential was introduced by Z. Shen in [79] in order to obtain estimates of the fundamental solution to $\mathcal{L}$ on $\mathbb{R}^n$ and the comparison with the fundamental solution to $-\Delta$ on $\mathbb{R}^n$.

In the Schrödinger setting, the natural Lipschitz (also called Hölder) spaces have been defined (see [20, 59]) for $0<\alpha<1$ as follows:

$$\left\{f:\ \rho(\cdot)^{-\alpha}f(\cdot)\in L^\infty(\mathbb{R}^n)\text{ and }\sup_{|z|>0}\frac{\|f(\cdot+z)-f(\cdot)\|_\infty}{|z|^\alpha}<\infty\right\},$$

where $\rho(x):=\sup\left\{r>0:\ \frac{1}{r^{n-2}}\int_{B(x,r)}V(y)dy\le 1\right\}$ is the critical radius function, which usually plays an important role in this context, see [21, 59, 79].

Since no regularity is assumed for the potential V, it is not clear which one would be the operator that would play the same role than the derivatives in the case of the classical Hölder spaces when $\alpha>1$. For this reason, up to this moment there was not any definition of Hölder (or Lipschitz) spaces adapted to $\mathcal{L}$ for $\alpha>1$.

We have extended the pointwise definition of [59] for $0<\alpha<2$ by using Zygmund's approach:

$$C_{\mathcal{L}}^{\alpha}:=\left\{f:\ \rho(\cdot)^{-\alpha}f(\cdot)\in L^\infty(\mathbb{R}^n)\text{ and }\sup_{|z|>0}\frac{\|f(\cdot+z)+f(\cdot-z)-2f(\cdot)\|_\infty}{|z|^\alpha}<\infty\right\}.$$

Pointwise definitions of Lipschitz spaces imply that to prove regularity results of an operator among these spaces we need its pointwise expression, but in many cases this can be a rather involved formula. This is why the description of Lipschitz spaces through semigroup language is more convenient.

In the spirit of Taibleson and Stein, see [81, 87], we introduce the following spaces, defined via the heat and the Poisson semigroups associated to $\mathcal{L}$, $e^{-t\mathcal{L}}$ and $e^{-y\sqrt{\mathcal{L}}}$, for $\alpha > 0$:

$$\Lambda^W_{\alpha/2} := \left\{ f : \rho(\cdot)^{-\alpha} f(\cdot) \in L^\infty(\mathbb{R}^n) \text{ and } \left\| \partial_t^k e^{-t\mathcal{L}} f \right\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha t^{-k+\alpha/2}, \;\; k = [\alpha/2]+1, t > 0 \right\}.$$

$$\Lambda^P_\alpha := \left\{ f : \int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty \text{ and } \left\| \partial_y^k e^{-y\sqrt{\mathcal{L}}} f \right\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha y^{-k+\alpha}, \;\; k = [\alpha]+1, y > 0 \right\}.$$

Our main result in this chapter is the following (see **Theorem 2.47**):

Let $0 < \alpha \leq 2 - n/q$ and f be a function with $\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty$. The following statements are equivalent:

$$f \in C^\alpha_{\mathcal{L}}, \quad f \in \Lambda^P_\alpha, \quad f \in \Lambda^W_{\alpha/2}.$$

Moreover, the norms are equivalent.

In fact, we first prove that for $0 < \alpha \leq 2 - n/q$, a function f belongs to $C^\alpha_{\mathcal{L}}$ if, and only if, $f \in \Lambda^W_{\alpha/2}$, see **Theorem 2.22**. However, if we want to add the space Λ^P_α to the chain of equivalences, we need to add the condition $\int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}} dx < \infty$.

The restriction $\alpha \leq 2 - \frac{n}{q}$ is strongly imposed by the reverse Hölder condition in the potential V. Moreover, the proofs of these results are based on the comparison between the heat semigroup of Schrödinger operators and the heat semigroup of $-\Delta$, and this is only known whenever $q > n/2$, see [37, 79].

On the other hand, we define the powers $\mathcal{L}^{\pm\beta}$ by using the semigroup approach, see (0.1) and (0.4), and we have got, among others, the following **regularity results** (**Theorems 2.58** and **2.60**).

- *If $0 < \beta < \alpha$ and $f \in \Lambda^W_{\alpha/2}$, then $\|\mathcal{L}^{\beta/2} f\|_{\Lambda^W_{\frac{\alpha-\beta}{2}}} \leq C \|f\|_{\Lambda^W_{\alpha/2}}$.*
- *If $\alpha, \beta > 0$, then $\|\mathcal{L}^{-\beta/2} f\|_{\Lambda^W_{\frac{\alpha+\beta}{2}}} \leq C \|f\|_{\Lambda^W_{\alpha/2}}$.*

These results are completely new for $\alpha \geq 1$ because, as we said before, Lipschitz spaces in the general Schrödinger setting were defined only for $0 < \alpha < 1$, so the regularity results known until this moment could only consider $0 < \alpha < 1$.

Moreover, we have improved the results of [20] about the boundedness of the associated Riesz transforms.

Consider the first order Riesz transforms associated to the Schrödinger operators defined by

$$\mathcal{R}_i = \partial_{x_i}(\mathcal{L}^{-1/2}), \text{ and } R_i = \mathcal{L}^{-1/2}(\partial_{x_i}), \; i = 1, \ldots, n.$$

- *If* $0 < \alpha \le 1 - n/q$, *then* $\|\mathcal{R}_i f\|_{\Lambda^W_{\alpha/2}} \le C\|f\|_{\Lambda^W_{\alpha/2}}$, $i = 1, \dots, n$,
- *If* $1 < \alpha \le 2 - n/q$, *then* $\|R_i f\|_{\Lambda^W_{\alpha/2}} \le C\|f\|_{\Lambda^W_{\alpha/2}}$, $i = 1, \dots, n$.

0.2.3 Chapter 3: Parabolic and elliptic Hermite Lipschitz spaces. Regularity results

The harmonic oscillator, $\mathcal{H} = -\Delta + |x|^2$, is a particular case of Schrödinger operator, where $V(x) = |x|^2$ satisfies the reverse Hölder inequality for every $q > n/2$ ($n \ge 3$) and the critical radius function associated to V is $\rho(x) = \frac{1}{1+|x|}$, which is a bounded function. This means that all the results of Chapter 2 apply in this context without the restrictions depending on q. Moreover, we know explicitly the heat (and therefore the Poisson) kernels associated to $\mathcal{H}$ on $\mathbb{R}^n$, for $n \ge 1$. These facts will allow us to get better results for $\mathcal{H}$ than the ones got for general Schrödinger operators.

In this chapter we will go further and we shall consider not only $\mathcal{H}$ but also the *parabolic Hermite operator* on $\mathbb{R}^n$, $n \ge 1$,

$$\mathbb{H} := \partial_t + \mathcal{H} = \partial_t - \Delta_x + |x|^2,\ x \in \mathbb{R}^n,\ t > 0.$$

We shall introduce definitions of Lipschitz (also called Hölder) spaces adapted to $\mathcal{H}$ and $\mathbb{H}$ by means of their heat and Poisson semigroups. We shall see that these spaces have pointwise characterizations, which will produce in the case of $\mathcal{H}$ the coincidence with the pointwise definition of Hölder spaces introduced by Stinga and Torrea in [86].

Let us present our main results of this chapter. The operator $\mathcal{H}$ can be factorized as $\mathcal{H} = \frac{1}{2}\sum_{i=1}^n (A_i A_{-i} + A_{-i} A_i)$, $A_i = \partial_{x_i} + x_i$, $A_{-i} = -\partial_{x_i} + x_i$. The first order operators $A_{\pm i}$ play the role, with respect to operator $\mathcal{H}$, of the derivatives $\pm\partial_{x_i}$ with respect to the classical Laplacian Δ.

Stinga and Torrea in [86] introduced the following pointwise definition.

For $0 < \alpha < 1$,

$$C^\alpha_{\mathcal{H}}(\mathbb{R}^n) := \{f : (1 + |\cdot|)^\alpha f(\cdot) \in L^\infty(\mathbb{R}^n),\ \textit{and}\ \|f(\cdot + z) - f(\cdot)\|_{L^\infty(\mathbb{R}^n)} \le A|z|^\alpha\}$$

with associated norm

$$\|f\|_{C^\alpha_{\mathcal{H}}} = [f]_{M^\alpha} + [f]_{C^\alpha_{\mathcal{H}}},$$

where $[f]_{M^\alpha} = \|(1 + |\cdot|)^\alpha f(\cdot)\|_\infty$ *and* $[f]_{C^\alpha_{\mathcal{H}}} = \sup\limits_{|z|>0} \dfrac{\|f(\cdot + z) - f(\cdot)\|_\infty}{|z|^\alpha}$.

For $\alpha > 1$ *and not integer,* $f \in C^\alpha_{\mathcal{H}}(\mathbb{R}^n)$, *if there exist the derivatives of order* $[\alpha]$ *and the norm*

$$\|f\|_{C^\alpha_{\mathcal{H}}} := [f]_{M^{\alpha-[\alpha]}} + \sum_{\substack{1\le|i_1|,\dots,|i_m|\le n\\ 1\le m\le[\alpha]}} [A_{i_1} \dots A_{i_m} f]_{M^{\alpha-[\alpha]}} + \sum_{1\le|i_1|,\dots,|i_{[\alpha]}|\le n} [A_{i_1} \dots A_{i_{[\alpha]}} f]_{C^{\alpha-[\alpha]}_{\mathcal{H}}},$$

is finite.

Inspired by the previous definition and Krylov's parabolic Hölder spaces, see Section 3.1, we introduce the following **Parabolic Hermite Hölder spaces**:

Assume that $f \in L^\infty(\mathbb{R}^{n+1})$.

- *Let* $0 < \alpha < 1$. *We say that* $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$ *if* $f \in C^{\alpha/2,\alpha}$ *and*

$$[f]_{M^\alpha} = \sup_{(t,x)\in\mathbb{R}^{n+1}} (1+|x|)^\alpha |f(t,x)| < \infty,$$

  In this case, $\|f\|_{C^{\alpha/2,\alpha}_{t,\mathcal{H}}} = [f]_{M^\alpha} + [f]_{C^{\alpha/2,\alpha}}$.

- *For* $1 < \alpha < 2$, $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$ *if* $A_{\pm i} f \in C^{\alpha/2-1/2,\alpha-1}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$, $i = 1,\dots,n$, *and* $f(\cdot,x) \in C^{\alpha/2}(\mathbb{R})$ *uniformly on* x.
- *For* $2 < \alpha < 3$ *we say that* $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$, *if* $A_{\pm i}A_{\pm j} f$, $i,j = 1,\dots,n$, *and* $\partial_t f$ *belong to* $C^{\alpha/2-1,\alpha-2}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$.

Also, we introduce the Parabolic Hermite Lipschitz spaces defined through the Poisson semigroup, $\mathcal{P}_y f = e^{-y\sqrt{\mathbb{H}}} f$.

Let $\mathcal{P}_y = e^{-y\sqrt{\mathbb{H}}}$ *and* $\alpha > 0$, *we consider the class*

$$\Lambda^{\mathcal{P}}_\alpha := \left\{ f : \ f \in L^\infty(\mathbb{R}^{n+1}) \text{ and } \left\| \partial_y^k \mathcal{P}_y f \right\|_{L^\infty(\mathbb{R}^{n+1})} \le C_k y^{-k+\alpha}, \ \ , k = [\alpha]+1, y > 0 \right\},$$

whose norm is given by $\|f\|_{\Lambda^{\mathcal{P}}_\alpha} := \|f\|_\infty + C$, *where* C *is the infimum of the positive constants* C_k *above.*

Similarly, it can be defined analogous space in the elliptic setting: let $P^{\mathcal{H}}_y = e^{-y\sqrt{\mathcal{H}}}$ be the Poisson semigroup associated to $\mathcal{H}$,

$$\Lambda^{P^{\mathcal{H}}}_\alpha := \left\{ f : \ f \in L^\infty(\mathbb{R}^n) \text{ and } \left\| \partial_y^k P^{\mathcal{H}}_y f \right\|_{L^\infty(\mathbb{R}^n)} \le C_k y^{-k+\alpha}, \ \text{with}\, k = [\alpha]+1, y > 0 \right\}.$$

The following result (**Theorem 3.68**) shows that $\Lambda^{\mathcal{P}}_\alpha$ spaces have a pointwise description. Moreover, the restriction to functions depending only on x, produces the result for $\Lambda^{P^{\mathcal{H}}}_\alpha$ (**Theorem 3.71**).

1. *Suppose that* $0 < \alpha < 2$. *Then* $f \in \Lambda^{\mathcal{P}}_\alpha$ *if and only if there exists a constant* $C > 0$ *such that*

$$\|f(\cdot-\tau,\cdot-z)+f(\cdot-\tau,\cdot+z)-2f(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^{n+1})} \le C(|\tau|^{1/2}+|z|)^\alpha, (\tau,z) \in \mathbb{R}^{n+1}$$

   and $(1+|x|)^\alpha f \in L^\infty(\mathbb{R}^{n+1})$. *In this case, if* K *denotes the infimum of the constants* C *for which the inequality above is true, then* $\|u\|_{\Lambda^{\mathcal{P}}_\alpha} := [u]_{M^\alpha} + K$, *where* $[f]_{M^\alpha} = \|(1+|\cdot|)^\alpha f(\cdot,\cdot)\|_\infty$.

2. *Suppose that* $\alpha > 2$. *Then* $f \in \Lambda_\alpha^{\mathcal{P}}$ *if and only if* $f \in L^\infty(\mathbb{R}^{n+1})$,

$$A_{\pm i}A_{\pm j}f \in \Lambda_{\alpha-2}^{\mathcal{P}}, \quad i,j = 1,\dots,n, \qquad \text{and} \qquad \partial_t f \in \Lambda_{\alpha-2}^{\mathcal{P}}.$$

In this case the following equivalence holds

$$\|f\|_{\Lambda_\alpha^{\mathcal{P}}} \sim \|f\|_\infty + \sum_{i,j=1}^n \left(\|A_{\pm i}A_{\pm j}f\|_{\Lambda_{\alpha-2}^{\mathcal{P}}}\right) + \|\partial_t f\|_{\Lambda_{\alpha-2}^{\mathcal{P}}}.$$

As a consequence of the previous result, we prove the following (see **Theorem 3.69**).

Let $0 < \alpha < 3$, α *not an integer. Then*

$$C_{t,\mathcal{H}}^{\alpha/2,\alpha} = \Lambda_\alpha^{\mathcal{P}},$$

with equivalence of norms.

The above result has the parallel result in the case of Hermite operator $\mathcal{H} = -\Delta_x + |x|^2$. In particular, since the space $C_{\mathcal{H}}^\alpha(\mathbb{R}^n)$ is defined for every $\alpha > 0$, we shall get in the elliptic setting that $C_{\mathcal{H}}^\alpha(\mathbb{R}^n) = \Lambda_\alpha^{P^{\mathcal{H}}}$, for every $\alpha > 0$, $\alpha \notin \mathbb{N}$, see **Theorem 3.72**.

Moreover, we will also introduce Hermite Lipschitz spaces defined through the heat semigroup, but only in the elliptic case. In the parabolic case would be analogous.

Let $e^{-y\mathcal{H}} = W_y^{\mathcal{H}}$ *be the heat semigroup associated to* $\mathcal{H}$. *For* $\alpha > 0$,

$$\Lambda_{\alpha/2}^{W^{\mathcal{H}}} =: \left\{ f \in L^\infty(\mathbb{R}^n) : \left\|\partial_y^k W_y^{\mathcal{H}} f\right\|_{L^\infty(\mathbb{R}^n)} \le C_k y^{-k+\alpha/2},\ k = [\alpha/2]+1 \right\}.$$

In this case we get the following result (see **Theorem 3.95**).

For every $\alpha > 0$, *the spaces* $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ *and* $\Lambda_\alpha^{P^{\mathcal{H}}}$ *do coincide in the sense of normed spaces.*

As a consequence, by using the coincidence of $\Lambda_\alpha^{P^{\mathcal{H}}}$ and $C_{\mathcal{H}}^\alpha$, we have next corollary.

If $\alpha > 0$ *is not an integer,* $\Lambda_{\alpha/2}^{W^{\mathcal{H}}} = \Lambda_\alpha^{P^{\mathcal{H}}} = C_{\mathcal{H}}^\alpha$, *where the identities are understood in the sense of normed spaces.*

Our semigroup definitions of Lipschitz spaces will allow us to get **regularity results** for fractional operators related to $\mathcal{H}$ and $\mathbb{H}$ in a more direct and elegant way. Observe that the coincidence of the spaces $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ and $\Lambda_\alpha^{P^{\mathcal{H}}}$ for every $\alpha > 0$ implies that all the regularity results we have proved in the previous chapter apply here, whenever $n \ge 3$. Moreover, in Section 3.4 we have proved the following results in the parabolic setting and its analogous in the elliptic context, for $n \ge 1$.

- *Let* $0 < 2\beta < \alpha$ *and* $f \in \Lambda_\alpha^{\mathcal{P}}$, *then* $\mathbb{H}^\beta f \in \Lambda_{\alpha-2\beta}^{\mathcal{P}}$ *and*

$$\|\mathbb{H}^\beta f\|_{\Lambda_{\alpha-2\beta}^{\mathcal{P}}} \le C\|f\|_{\Lambda_\alpha^{\mathcal{P}}}.$$

- *Let* $\alpha, \beta > 0$.

  (i) Given $f \in \Lambda_\alpha^{\mathcal{P}}$, *then* $\mathbb{H}^{-\beta} f \in \Lambda_{\alpha+2\beta}^{\mathcal{P}}$ *and*

$$\|\mathbb{H}^{-\beta} f\|_{\Lambda_{\alpha+2\beta}^{\mathcal{P}}} \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}}.$$

  (ii) If $f \in L^\infty(\mathbb{R}^{n+1})$, *then* $\mathbb{H}^{-\beta} f \in \Lambda_{2\beta}^{\mathcal{P}}$ *and*

$$\|\mathbb{H}^{-\beta} f\|_{\Lambda_{2\beta}^{\mathcal{P}}} \leq C\|f\|_\infty.$$

Regarding the Riesz transforms, we obtain better results than the ones we have proved for general Schrödinger operators in the previous chapter.

Consider the Parabolic Hermite Riesz transforms of order $m \geq 1$ *defined by*

$$R_\nu = (A_{\pm 1}^{\nu_1} A_{\pm 2}^{\nu_2} \ldots A_{\pm n}^{\nu_n})\mathbb{H}^{-m/2} \;\; \textit{and} \;\; R_m = \partial_t^m \mathbb{H}^{-m}$$

where $\nu_i \geq 0, i = 1, \ldots, n$ *and* $|\nu| = \nu_1 + \cdots + \nu_n = m$. *Let* $\alpha > 0$, *then* R_ν *and* R_m *are bounded from* $\Lambda_\alpha^{\mathcal{P}}$ *into itself. A parallel result holds for the operators* $(A_{\pm 1}^{\nu_1} A_{\pm 2}^{\nu_2} \ldots A_{\pm n}^{\nu_n})\mathcal{H}^{-m/2}$ *when acting on the spaces* $\Lambda_\alpha^{P^{\mathcal{H}}}$.

Moreover, in Section 3.4 we also obtain the boundedness of Bessel potentials and multipliers of Laplace transform type, defined via the heat and Poisson semigroups, and we obtain a maximum principle.

0.2.4 Chapter 4: Parabolic equations in the Bessel setting

Along this chapter we consider the parabolic equations

$$\frac{\partial u(t,x)}{\partial t} = \Delta_\mu u(t,x) + f(t,x), \;\; (t,x) \in \mathbb{R} \times (0,\infty) \;\; \textit{or} \;\; (t,x) \in (0,\infty) \times (0,\infty),$$

where, for every $\mu > -1$, Δ_μ represents the Bessel operator defined by $\Delta_\mu = \partial_x^2 + (\frac{1}{4} - \mu^2)x^{-2}$.

The operator $-\Delta_\mu$ is positive, selfadjoint on $L^2((0,\infty))$ and generates a semigroup $\{e^{t\Delta_\mu}\}_{t>0} = \{W_t^\mu\}_{t>0}$ of operators in $L^2((0,\infty))$ where, for every $t > 0$ and $\phi \in L^2((0,\infty))$,

$$W_t^\mu(\phi)(x) = \int_0^\infty W_t^\mu(x,y)\phi(y)dy, \;\; x \in (0,\infty).$$

Here, $W_t^\mu(x,y)$ is the corresponding heat kernel. If, for every $t > 0$, W_t^μ is given as in (0.9), $\{W_t^\mu\}_{t>0}$ also defines a semigroup of operators on $L^p((0,\infty))$, for each $1 < p < \infty$ when $\mu > -1/2$ and for each $1 < p < \infty$ such that $-\mu - 1/2 < \frac{1}{p} < \mu + 3/2$, when $-1 < \mu \leq -1/2$.

We will describe the results on $\mathbb{R} \times (0,\infty)$. Our first main result is about the classical solvability of (0.8).

Assume that $f \in L^\infty(\mathbb{R}\times(0,\infty))$ has compact support on $\mathbb{R}\times(0,\infty)$. Then, for $\mu > -1$, the function $u(t,x)$, $(t,x)\in\mathbb{R}\times(0,\infty)$, given by

$$u(t,x) = \int_0^\infty \int_0^\infty W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau, \quad (t,x)\in\mathbb{R}\times(0,\infty),$$

is defined by an absolutely convergent integral, for every $(t,x)\in\mathbb{R}\times(0,\infty)$. Moreover, if f is also in $C^2(\mathbb{R}\times(0,\infty))$, then, for every $(t,x)\in\mathbb{R}\times(0,\infty)$, $\frac{\partial u(t,x)}{\partial t} = \Delta_\mu u(t,x) + f(t,x)$, being

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \lim_{\epsilon\to 0^+}\int_\epsilon^\infty\int_0^\infty \frac{\partial}{\partial\tau}W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau + f(t,x)\\ &= \lim_{\epsilon\to 0^+}\int_{\Omega_\epsilon(x)}\frac{\partial}{\partial\tau}W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau + Af(t,x), \quad t,x\in(0,\infty),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 u(t,x)}{\partial x^2} &= \lim_{\epsilon\to 0^+}\int_\epsilon^\infty\int_0^\infty \frac{\partial^2}{\partial x^2}W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau\\ &= \lim_{\epsilon\to 0^+}\int_{\Omega_\epsilon(x)}\frac{\partial^2}{\partial x^2}W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau - (1-A)f(t,x)\end{aligned}$$

where, for every $\epsilon, x\in(0,\infty)$, $\Omega_\epsilon(x) = \{(\tau,y)\in(0,\infty)^2 : \tau^{1/2}+|x-y|>\epsilon\}$, and $A = \frac{1}{\sqrt{\pi}}\int_0^1 e^{-\frac{w^2}{4}}dw$.

The Bessel operator can be written as $\Delta_\mu = \delta_\mu^*\delta_\mu$, where $\delta_\mu = x^{\mu+1/2}\frac{d}{dx}x^{-\mu-1/2}$, and $\delta_\mu^* = x^{-\mu-1/2}\frac{d}{dx}x^{\mu+1/2}$ represents the formal adjoint of δ_μ. We now consider the operator L_μ defined by

$$(L_\mu f)(t,x) = \int_0^\infty\int_0^\infty W_s^\mu(x,y)f(t-s,y)dyds,$$

being f a measurable complex function defined on $\mathbb{R}\times(0,\infty)$, provided that the last integral exists. L_μ can be seen as $(\partial_t - \Delta_\mu)^{-1}$. Keeping in mind Stein's ideas ([80]), we define Riesz transforms associated with the parabolic operator $\partial_t - \Delta_\mu$ as follows: for every $f\in C_c^2(\mathbb{R}\times(0,\infty))$,

$$R_\mu(f) = \delta_{\mu+1}\delta_\mu L_\mu(f) \text{ and } \widetilde{R_\mu}(f) = \partial_t L_\mu(f).$$

According to the previous theorem, if $f\in C_c^2(\mathbb{R}\times(0,\infty))$, the above definitions of $R_\mu(f)$ and $\widetilde{R_\mu}(f)$ have sense. Moreover, we can write, for every $f\in C_c^2(\mathbb{R}\times(0,\infty))$,

$$R_\mu(f)(t,x) = \lim_{\epsilon\to 0^+}\int_{\Omega_\epsilon(t,x)} K_\mu(t,x;\tau,y)f(\tau,y)d\tau dy + f(t,x)\frac{1}{\sqrt{\pi}}\int_1^\infty e^{-s^2/4}ds$$

and

$$\widetilde{R_\mu}(f)(t,x) = \lim_{\epsilon\to 0^+}\int_{\Omega_\epsilon(t,x)} \widetilde{K_\mu}(t,x;\tau,y)f(\tau,y)d\tau dy + f(t,x)\frac{1}{\sqrt{\pi}}\int_0^1 e^{-s^2/4}ds,$$

with $(t,x) \in \mathbb{R} \times (0,\infty)$, where

$$K_\mu(t,x;\tau,y) = \delta_{\mu+1}\delta_\mu W^\mu_{t-\tau}(x,y)\chi_{(0,\infty)}(t-\tau), \quad x,y \in (0,\infty), \quad t,\tau \in \mathbb{R},$$
$$\widetilde{K_\mu}(t,x;\tau,y) = -\partial_\tau W^\mu_{t-\tau}(x,y)\chi_{(0,\infty)}(t-\tau), \quad x,y \in (0,\infty), \quad t,\tau \in \mathbb{R},$$

and $\Omega_\epsilon(t,x) = \{(\tau,y) \in (0,\infty)\times(0,\infty) : \max\{|t-\tau|^{1/2}, |x-y|\} > \epsilon\}$, for $\epsilon, x \in (0,\infty)$ and $t \in (0,\infty)$.

We prove the boundedness of R_μ and $\widetilde{R_\mu}$ on (weighted) and mixed weighted L^p spaces.

Our first result in this line, see **Theorem 4.122**, concerns (weighted) L^p spaces. We represent, for every $1 \leq p < \infty$, by $A_p^*(\mathbb{R}\times(0,\infty))$ the class of Muckenhoupt weigths in the space of homogeneous type $(\mathbb{R}\times(0,\infty), m, d)$, where d is the parabolic distance and m the Lebesgue measure on $\mathbb{R}\times(0,\infty)$.

(1) If $\mu > -1$, the Riesz transformations R_μ and $\widetilde{R_\mu}$ are bounded from $L^2(\mathbb{R}\times(0,\infty))$ into itself.

(2) Suppose that $\mu > 1/2$ or $\mu = -1/2$. The Riesz transformations R_μ and $\widetilde{R_\mu}$ can be extended from $L^2(\mathbb{R}\times(0,\infty)) \cap L^p(\mathbb{R}\times(0,\infty),\omega)$ to $L^p(\mathbb{R}\times(0,\infty),\omega)$ as bounded operators from $L^p(\mathbb{R}\times(0,\infty),\omega)$

- *into $L^p(\mathbb{R}\times(0,\infty),\omega)$, for every $1 < p < \infty$ and $\omega \in A_p^*(\mathbb{R}\times(0,\infty))$.*
- *into $L^{1,\infty}(\mathbb{R}\times(0,\infty),\omega)$, for $p = 1$ and $\omega \in A_1^*(\mathbb{R}\times(0,\infty))$.*

(3) If $\mu > -1/2$, the Riesz transformations R_μ and $\widetilde{R_\mu}$ can be extended from $L^2(\mathbb{R}\times(0,\infty)) \cap L^p(\mathbb{R}\times(0,\infty))$ to $L^p(\mathbb{R}\times(0,\infty))$ as bounded operators from $L^p(\mathbb{R}\times(0,\infty))$

- *into $L^p(\mathbb{R}\times(0,\infty))$, for every $1 < p < \infty$.*
- *into $L^{1,\infty}(\mathbb{R}\times(0,\infty))$, for $p = 1$.*

(4) If $-1 < \mu \leq -1/2$, then the Riesz transformation $\widetilde{R_\mu}$ can be extended from $L^2(\mathbb{R}\times(0,\infty)) \cap L^p(\mathbb{R}\times(0,\infty))$ to $L^p(\mathbb{R}\times(0,\infty))$ as a bounded operator from $L^p(\mathbb{R}\times(0,\infty))$ into itself, provided that $-\mu - 1/2 < 1/p < \mu + 3/2$ and $1 < p < \infty$.

(5) If $-1 < \mu \leq -1/2$, then the Riesz transformation R_μ can be extended from $L^2(\mathbb{R}\times(0,\infty)) \cap L^p(\mathbb{R}\times(0,\infty))$ to $L^p(\mathbb{R}\times(0,\infty))$ as a bounded operator from $L^p(\mathbb{R}\times(0,\infty))$ into itself, provided that $p > \frac{1}{\mu+3/2}$ and $1 < p < \infty$.

Moreover, when $\mu > -1/2$ in all these cases the extensions of the operators R_μ and $\widetilde{R_\mu}$ are defined by (0.10) *and* (0.11)*, respectively, where the limit exist a.e. $(t,x) \in \mathbb{R}\times(0,\infty)$ and the equalities are understood also in a.e. $(t,x) \in \mathbb{R}\times(0,\infty)$.*

In addition, as an application of vector-valued Calderón-Zygmund theory (see [76]), we establish the following mixed weighted norm inequalities for Riesz transforms R_μ and $\widetilde{R_\mu}$ (see **Theorem 4.127**). For every $1 \leq p < \infty$, we denote the classical classes of Muckenhoupt weights by $A_p(\Omega)$, where $\Omega = (0,\infty)$ or $\Omega = \mathbb{R}$.

Assume that $\mu > 1/2$ or $\mu = -1/2$. If $1 < p < \infty$ and $v \in A_p((0,\infty))$, then the Riesz transforms R_μ and $\widetilde{R_\mu}$ can be extended from $L^2(\mathbb{R} \times (0,\infty)) \cap L^q(\mathbb{R}, u, L^p((0,\infty), v))$ to $L^q(\mathbb{R}, u, L^p((0,\infty), v))$ as bounded operators from $L^q(\mathbb{R}, u, L^p((0,\infty), v))$ into itself, provided that $1 < q < \infty$ and $u \in A_q(\mathbb{R})$; and, for every $u \in A_1(\mathbb{R})$, from $L^2(\mathbb{R} \times (0,\infty)) \cap L^1(\mathbb{R}, u, L^p((0,\infty), v))$ to $L^1(\mathbb{R}, u, L^p((0,\infty), v))$ as bounded operators from $L^1(\mathbb{R}, u, L^p((0,\infty), v))$ into $L^{1,\infty}(\mathbb{R}, u, L^p((0,\infty), v))$.

These results can be viewed as weighted and mixed weighted Sobolev type inequalities for solutions of Bessel parabolic equations.

Also, we consider the following Cauchy problem associated with (0.8) in $(0,\infty) \times (0,\infty)$:

$$\begin{cases} \partial_t u(t,x) = \Delta_\mu u(t,x) + f(t,x), \quad (t,x) \in (0,\infty) \times (0,\infty), \\ u(0,x) = g(x), \quad x \in (0,\infty). \end{cases}$$

Similarly to the case $\mathbb{R} \times (0,\infty)$, we prove the classical solvability of the Cauchy problem above (see **Theorem 4.129**). Also, for every $f \in C_c^2((0,\infty) \times (0,\infty))$ we define the Riesz transformations

$$\mathbf{R}_\mu(f)(t,x) = \lim_{\epsilon \to 0^+} \int_0^{t-\epsilon} \int_0^\infty \delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y) f(t-\tau,y)\,dyd\tau, \quad t,x \in (0,\infty),$$

and

$$\widetilde{\mathbf{R}_\mu}(f)(t,x) = \lim_{\epsilon \to 0^+} \int_0^{t-\epsilon} \int_0^\infty \partial_\tau W_\tau^\mu(x,y) f(t-\tau,y)\,dyd\tau, \quad t,x \in (0,\infty).$$

We shall prove the boundedness of $\mathbf{R}_\mu$ and $\widetilde{\mathbf{R}_\mu}$ on (weighted) L^p spaces (see **Theorem 4.130**) and the boundedness of $\widetilde{\mathbf{R}_\mu}$ on mixed L^p spaces (see **Theorem 4.131**).

Chapter 1

Discrete fractional derivatives and integrals.

This chapter **corresponds to [1] and [2]**.

We shall study discrete fractional derivatives and integrals from the semigroup language approach. In Section 1.2, we shall prove maximum and comparison principles as well as regularity results for the discrete fractional derivatives and integrals on the discrete Hölder spaces. In Section 1.3 we shall get approximation theorems for the discrete to the continuous fractional derivatives. Moreover, we shall see that when the functions are good enough (Hölder continuous), these approximation procedures give a measure of the order of approximation. These results also allow us to prove the coincidence, for Hölder continuous functions, of the Marchaud and Grünwald-Letnikov derivatives in every point and the speed of convergence to the Grünwald-Letnikov derivative. Finally, in Section 1.4 we will describe the discrete fractional derivative as a Neumann-Dirichlet operator defined by a semi-discrete extension problem and some operators related to the Harmonic Analysis associated to the discrete derivative will be also considered, in particular their behavior in the Lebesgue spaces $\ell^p(\mathbb{Z})$.

1.1 Definitions via the semigroup language

As far as we know, the first author who considered "differences of fractional order" was S. Chapman in 1911, see [25]. For $s > 0$, given a sequence a_n he defined

$$\triangle^s a_n = \sum_{m=0}^{\infty} \binom{-s-1+m}{m} a_{n+m}. \tag{1.1}$$

His motivation was to extend the obvious formula for differences of natural order. Also, by using Fourier transform in the integers it can be checked that the above definition produces $(\widehat{\triangle^s a_n})(\theta) = (1-e^{i\theta})^s \hat{a_n}(\theta)$ in coherence with the fact $(\widehat{a_n - a_{n+1}})(\theta) = (1-e^{i\theta})\hat{a_n}(\theta)$.

Observe that Chapman's definition only cares about the future, but we shall also consider the discrete derivatives which cares about the past. For $f : \mathbb{Z} \to \mathbb{R}$, we define *"the discrete derivative from the right"* and *"the discrete derivative from the left"* as the operators given by the formulas

$$\delta_{\text{right}} f(n) = f(n) - f(n+1) \text{ and } \delta_{\text{left}} f(n) = f(n) - f(n-1). \tag{1.2}$$

We shall use semigroup language as an alternative approach to differences of fractional order. Given the function $G_t(n) = e^{-t}\frac{t^n}{n!}$, $n \in \mathbb{N}_0$, we define the operators

$$T_{t,+}f(n) = \sum_{j=0}^{\infty} G_t(j) f(n+j), \text{ and } T_{t,-}f(n) = \sum_{j=0}^{\infty} G_t(j) f(n-j), \qquad t > 0,\, n \in \mathbb{Z}. \tag{1.3}$$

Now we shall see that $T_{t,\pm}f(n)$ are markovian semigroups on $\ell^p(\mathbb{Z})$, $1 \le p \le \infty$, whose infinitesimal generators are $-\delta_{\text{right}}$ and $-\delta_{\text{left}}$, so the function $u(n,t) = T_{t,+}f(n)$ satisfies the Cauchy problem (1.4).

Lemma 1.1. *Let $f \in \ell^\infty(\mathbb{Z})$ and $\{T_{t,\pm}\}_{t\ge 0}$ be the families defined in (1.3), then*

$$\lim_{t\to 0} \frac{T_{t,+}f(n) - f(n)}{t} = -\delta_{\text{right}} f(n) \text{ and } \lim_{t\to 0} \frac{T_{t,-}f(n) - f(n)}{t} = -\delta_{\text{left}} f(n), \quad n \in \mathbb{Z}.$$

Proof. Let $f \in \ell^\infty(\mathbb{Z})$ and $n \in \mathbb{Z}$. Observe that

$$\begin{aligned}\frac{T_{t,+}f(n) - f(n)}{t} &= \frac{1}{t} e^{-t} \sum_{j=0}^{\infty} \frac{t^j}{j!} \left(f(n+j) - f(n)\right) = e^{-t} \sum_{j=1}^{\infty} \frac{t^{j-1}}{j!} \left(f(n+j) - f(n)\right) \\ &= e^{-t}(f(n+1) - f(n)) + e^{-t} \sum_{j=2}^{\infty} \frac{t^{j-1}}{j!} \left(f(n+j) - f(n)\right).\end{aligned}$$

Dominated Convergence gives the first identity in the statement. By a similar argument,

$$\lim_{t\to 0} \frac{T_{t,-}f(n) - f(n)}{t} = \lim_{t\to 0} e^{-t}(f(n-1) - f(n)) + \lim_{t\to 0} e^{-t} \sum_{j=2}^{\infty} \frac{t^{j-1}}{j!} \left(f(n-j) - f(n)\right) = -\delta_{\text{left}} f(n).$$

□

The next Proposition shows that although the semigroups are not self adjoint, they satisfy the rest of the properties of the so called symmetric diffusion semigroups in the sense of E. M. Stein, see [81].

Proposition 1.2. *Let $f \in \ell^p(\mathbb{Z})$ with $1 \le p \le \infty$. The families of operators $\{T_{t,\pm}\}_{t\ge 0}$ satisfy*

(i) $\|T_{t,\pm}f\|_{\ell^p} \le \|f\|_{\ell^p}$ *and* $T_{0,\pm}f = f$.

(ii) $T_{t,\pm}T_{s,\pm}f = T_{t+s,\pm}f$.

(iii) $\lim_{t\to 0} T_{t,\pm}f = f$ *on* $\ell^p(\mathbb{Z})$.

(iv) $T_{t,\pm}f \ge 0$ *if* $f \ge 0$.

(v) $T_{t,\pm}1 = 1$.

(vi) $T^*_{t,\pm} = T_{t,\mp}$ *on* $\ell^2(\mathbb{Z})$.

Proof. $T_{0,\pm}f = f$ by definition. We prove the rest of the results for $T_{t,+}$ (the proof is analogous for $T_{t,-}$). Let $f \in \ell^p(\mathbb{Z})$ for $1 \leq p < \infty$. By Minkowski's inequality

$$\|T_{t,+}f\|_{\ell^p} \leq e^{-t}\sum_{j=0}^{\infty}\frac{t^j}{j!}\left(\sum_{n\in\mathbb{Z}}|f(n+j)|^p\right)^{\frac{1}{p}} = \|f\|_{\ell^p}.$$

For $p = \infty$, is analogous. In order to prove (ii) we use the Newton's binomial,

$$\begin{aligned} T_{t,+}T_{s,+}f(n) &= e^{-(t+s)}\sum_{j=0}^{\infty}\frac{t^j}{j!}\sum_{h=0}^{\infty}\frac{s^h}{h!}f(n+j+h) = e^{-(t+s)}\sum_{j=0}^{\infty}\frac{t^j}{j!}\sum_{u=j}^{\infty}\frac{s^{u-j}}{(u-j)!}f(n+u) \\ &= e^{-(t+s)}\sum_{u=0}^{\infty}\frac{f(n+u)}{u!}\sum_{j=0}^{u}\binom{u}{j}t^j s^{u-j} = T_{t+s,+}f(n). \end{aligned}$$

For (iii) we use that $f(n+j) - f(n) = 0$ for $j = 0$, and Minkowski's inequality to get

$$\begin{aligned} \|T_{t,\pm}f - f\|_{\ell^p} &= \left(\sum_{n\in\mathbb{Z}}\left|e^{-t}\sum_{j=1}^{\infty}\frac{t^j}{j!}(f(n+j)-f(n))\right|^p\right)^{\frac{1}{p}} \\ &\leq e^{-t}\sum_{j=1}^{\infty}\left(\sum_{n\in\mathbb{Z}}\left|\frac{t^j}{j!}(f(n+j)-f(n))\right|^p\right)^{\frac{1}{p}} \\ &\leq 2e^{-t}\|f\|_{\ell^p}\sum_{j=1}^{\infty}\frac{t^j}{j!} \to 0 \end{aligned}$$

as $t \to 0$. For $p = \infty$ is similar. We leave the verification of (iv), (v) and (vi) to the reader. □

Remark 1.3. *By Lemma 1.1 and Proposition 1.2 (i)-(iii), observe that the one-parameter operator families* $\{T_{t,\pm}\}_{t\geq 0}$ *are uniformly bounded* C_0*-semigroups on* $\ell^p(\mathbb{Z})$ *for* $1 \leq p \leq \infty$*, generated by* $-\delta_{\text{right/left}}$*, in the sense of the operator theory, see [6]. Furthermore, it is easy to see that the domain of* $\delta_{\text{right/left}}$ *on* $\ell^p(\mathbb{Z})$ *is the whole space.*

As a consequence of Lemma 1.1 and Proposition 1.2 we get that $u(n,t) = T_{t,+}f(n) = \sum_{j\geq 0} G_t(j)f(n+j)$ satisfies the following semi-discrete transport equation

$$\begin{cases} \partial_t u(n,t) + \delta_{\text{right}}u(n,t) = 0, & n \in \mathbb{Z},\ t \geq 0, \\ u(n,0) = f(n), & n \in \mathbb{Z}. \end{cases} \tag{1.4}$$

Moreover, the function $v(n,t) = \sum_{j=0}^{\infty} G_t(j)f(n-j)$ satisfies the analogous equation for δ_{left}.

However, we write here the proof in a few lines. Observe that

$$\frac{\partial}{\partial t}G_t(0) = -G_t(0), \quad \text{and } \frac{\partial}{\partial t}(G_t(j)) = -G_t(j) + G_t(j-1), \quad j \geq 1.$$

On the other hand, for any $A > 0$, $\sum_j \frac{A^j}{j!} \leq C_A < \infty$, hence we can differentiate term by term the series and we have

$$\begin{aligned}\partial_t T_{t,+}f(n) &= -G_t(0)f(n) + \sum_{j=1}^{\infty}(-G_t(j) + G_t(j-1))f(n+j) \\ &= -\sum_{j\geq 0} G_t(j)f(n+j) + \sum_{j\geq 0} G_t(j)f(n+(j+1)) = -\delta_{\text{right}}T_t f(n).\end{aligned}$$

The proof of the result for δ_{left} is analogous.

Once we have enclosed the fractional differences into the frame of semigroups, we take advantage of the method to highlight some properties and interesting results of these operators. We recall to the reader the following Gamma function formulas for an operator L.

$$L^{\alpha} = \frac{1}{\Gamma(-\alpha)}\int_0^{\infty}(e^{-tL} - 1)\frac{dt}{t^{1+\alpha}} \quad \text{and} \quad L^{-\alpha} = \frac{1}{\Gamma(\alpha)}\int_0^{\infty} e^{-tL}\frac{dt}{t^{1-\alpha}}, \tag{1.5}$$

where e^{-tL} is the associated semigroup, see [8, 81, 84, 93]. In particular, for f good enough we define

$$(\delta_{\text{right}})^{\alpha}f(n) = \frac{1}{\Gamma(-\alpha)}\int_0^{\infty}\frac{T_{t,+}f(n) - f(n)}{t^{1+\alpha}}dt, \ 0 < \alpha < 1,$$

and

$$(\delta_{\text{right}})^{-\alpha}f(n) = \frac{1}{\Gamma(\alpha)}\int_0^{\infty}\frac{T_{t,+}f(n)}{t^{1-\alpha}}dt, \ 0 < \alpha < 1,$$

and the corresponding formula for $(\delta_{\text{left}})^{\alpha}$, $-1 < \alpha < 1$. We shall prove that this definition of $(\delta_{\text{right}})^{\alpha}$ coincides with the definition (1.1) given by Chapman in 1911.

Along this chapter, for $\alpha \in \mathbb{R}$, we denote

$$\Lambda^{\alpha}(m) = \frac{-\alpha(-\alpha+1)\cdots(-\alpha+m-1)}{m!}, \quad m \in \mathbb{N},$$

and $\Lambda^{\alpha}(0) = 1$. Note that if $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$ we have that $\Lambda^{\alpha}(m) = \binom{m-\alpha-1}{m} = (-1)^m\binom{\alpha}{m}$ for $m \in \mathbb{N}_0$. Here we highlight some properties of this kernel. Also, if $-1 < \alpha < 0$, then Λ^{α} is decreasing as a function of n, while if $0 < \alpha < 1$, we have $\sum_{n=0}^{\infty}\Lambda^{\alpha}(n) = 0$, so $\sum_{n=1}^{\infty}\Lambda^{\alpha}(n) = -1$.

Also, the kernel $(\Lambda^{\alpha}(n))_{n\in\mathbb{N}_0}$ could be defined by the generating function, that is,

$$\sum_{n=0}^{\infty}\Lambda^{\alpha}(n)z^n = (1-z)^{\alpha}, \quad |z| < 1,$$

and therefore we have

$$\Lambda^{\alpha+\beta}(n) = \sum_{j=0}^{n} \Lambda^{\alpha}(n-j)\Lambda^{\beta}(j), \quad \alpha, \beta \in \mathbb{R},\, n \in \mathbb{N}_0. \tag{1.6}$$

In the following, we will use the asymptotic behaviour of the sequences $\{\Lambda^{\alpha}(n)\}_{n\in\mathbb{N}_0}$. It is known that for every $\alpha \in \mathbb{R}\setminus\mathbb{N}_0$,

$$\Lambda^{\alpha}(n) = \frac{1}{n^{1+\alpha}\Gamma(-\alpha)}\left(1 + O\left(\frac{1}{n}\right)\right), \quad n \in \mathbb{N}, \tag{1.7}$$

see [98, Vol.I, p.77, (1.18)]. In the case $\alpha \in \mathbb{N}_0$, $\Lambda^{\alpha}(n) = 0$ for $n > \alpha$. To see more properties of $\{\Lambda^{\alpha}(n)\}_{n\in\mathbb{N}_0}$ in a general setting, see [98].

Then, for $0 < \alpha < 1$ and $f \in \ell^p$, $1 \le p \le \infty$,

$$\begin{aligned}(\delta_{\text{right}})^{\alpha} f(n) &= \frac{1}{\Gamma(-\alpha)}\int_0^{\infty} \frac{e^{-t}\sum_{j=0}^{\infty}\frac{t^j}{j!}(f(n+j)-f(n))}{t^{1+\alpha}}dt = \sum_{j=0}^{\infty}(f(n+j)-f(n))\int_0^{\infty}\frac{e^{-t}t^{j-\alpha}}{j!\Gamma(-\alpha)}\frac{dt}{t}\\ &= \sum_{j=0}^{\infty}(f(n+j)-f(n))\frac{\Gamma(-\alpha+j)}{\Gamma(-\alpha)j!} = \sum_{j=0}^{\infty}\Lambda^{\alpha}(j)f(n+j) = \sum_{m=n}^{\infty}\Lambda^{\alpha}(m-n)f(m),\end{aligned}$$

where the interchange of the sum and the integral is justified because of the integral converges absolutely. In the last equality we have used that $\sum_{j=0}^{\infty}\Lambda^{\alpha}(j) = 0$, $0 < \alpha < 1$.

In a similar way we also get, for $0 < \alpha < 1$ and $f \in \ell^p$, $1 \le p \le \infty$,

$$(\delta_{\text{left}})^{\alpha} = \sum_{j=0}^{\infty}\Lambda^{\alpha}(j)f(n-j) = \sum_{m=-\infty}^{n}\Lambda^{\alpha}(n-m)f(m).$$

Remark 1.4. *The above expression of $(\delta_{\text{right/left}})^{\alpha}$, $0 < \alpha < 1$, coincides with the formula of the fractional powers of $\delta_{\text{right/left}}$ as generators of uniformly bounded C_0-semigroups on $\ell^p(\mathbb{Z})$, in the sense of Balakrishnan, see [93, Chapter IX, Section 11].*

In order to assure the convergence of the fractional integral, we need to consider our functions in a particular space. For $0 < \alpha < 1$, we define the space $\ell_{-\alpha,h}$ as follows:

$$\ell_{-\alpha,h} = \left\{u : \mathbb{Z}_h \to \mathbb{R} : \text{ for every } n \in \mathbb{Z},\ \sum_{m=0}^{\infty}\frac{|u(m\pm n)h)|}{(1+m)^{1-\alpha}} < \infty\right\}. \tag{1.8}$$

Hence, by using (1.5), for $0 < \alpha < 1$, and $f \in \ell_{-\alpha,1}$, we have

$$\begin{aligned}(\delta_{\text{right}})^{-\alpha} f(n) &= \frac{1}{\Gamma(\alpha)}\int_0^{\infty}\frac{e^{-t}\sum_{j=0}^{\infty}\frac{t^j}{j!}f(n+j)}{t^{1-\alpha}}dt = \sum_{j=0}^{\infty}f(n+j)\int_0^{\infty}\frac{e^{-t}t^{j+\alpha}}{j!\Gamma(\alpha)}\frac{dt}{t}\\ &= \sum_{j=0}^{\infty}f(n+j)\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)j!} = \sum_{j=0}^{\infty}\Lambda^{-\alpha}(j)f(n+j) = \sum_{m=n}^{\infty}\Lambda^{-\alpha}(m-n)f(m),\end{aligned}$$

where the interchange of the sum and the integral is justified because of the integral converges absolutely. By a similar way we also get

$$(\delta_{\text{left}})^{-\alpha} f(n) = \sum_{j=0}^{\infty} \Lambda^{-\alpha}(j) f(n-j) = \sum_{m=-\infty}^{n} \Lambda^{-\alpha}(n-m) f(m).$$

Remark 1.5 (Probabilistic interpretation of the discrete fractional derivative)**.** *Let u be a function defined on $\mathbb{Z}_h$ such that its progressive difference is zero, which is equivalent to write*

$$u(jh) = u((j+1)h), \quad j \in \mathbb{Z}.$$

This implies that the discrete function u is constant. We can interpret the above identity as the movement of a particle that compulsorily jumps to the adjacent right point on the mesh. If now we suppose that $(\delta_{\text{right}})^{\alpha} u = 0$, $0 < \alpha < 1$, then

$$u(jh) = -\sum_{n=1}^{\infty} \Lambda^{\alpha}(n) u((j+n)h).$$

Since $-\sum_{n=1}^{\infty} \Lambda^{\alpha}(n) = 1$, $0 < \alpha < 1$, the fractional identity can describe the movement of a particle which is able to jump to the right points $j+n$ with probability $-\Lambda^{\alpha}(n)$. It is easy to see that we recover the first situation as $\alpha \to 1^-$. If $\alpha \to 0^+$, the particle tends to be still.

1.2 Properties from a PDE point of view

In this section we shall prove maximum and comparison principles as well as regularity results for the discrete fractional derivatives and integrals on the discrete Hölder spaces.

Observe that the definitions in Section 1.1 can be given for a mesh with step length $h > 0$ instead of the integers mesh with step length 1. In other words, we can work in the field $\mathbb{Z}_h = \{jh : j \in \mathbb{Z}\}$. In this way we define, for $u : \mathbb{Z}_h \to \mathbb{R}$,

$$\delta_{\text{right}} u(hn) = \frac{u(hn) - u(h(n+1))}{h}, \quad \delta_{\text{left}} u(hn) = \frac{u(hn) - u(h(n-1))}{h}, \quad n \in \mathbb{Z},$$

and the associated $G_{t/h}(j)$. Notice that $\{T_{\frac{t}{h},\pm}\}_{t\geq 0}$ are the contraction semigroups on $\ell^p(\mathbb{Z}_h)$, $1 \leq p \leq \infty$, generated by $-\delta_{\text{right}}$ and $-\delta_{\text{left}}$. Then, by the results of the last section, for $0 < \alpha < 1$ we can write

$$(\delta_{\text{right}})^{\alpha} u(nh) = \frac{1}{h^{\alpha}} \sum_{m=n}^{\infty} \Lambda^{\alpha}(m-n) u(mh) \quad \text{and} \quad (\delta_{\text{left}})^{\alpha} u(nh) = \frac{1}{h^{\alpha}} \sum_{j=-\infty}^{n} \Lambda^{\alpha}(n-m) u(mh), \tag{1.9}$$

$$(\delta_{\text{right}})^{-\alpha} u(nh) = h^{\alpha} \sum_{m=n}^{\infty} \Lambda^{-\alpha}(m-n) u(mh), \quad (\delta_{\text{left}})^{-\alpha} u(nh) = h^{\alpha} \sum_{m=-\infty}^{n} \Lambda^{-\alpha}(n-m) u(mh), \tag{1.10}$$

whenever the series converge.

In general, for any $\alpha > 0$, it is defined

$$(\delta_{\text{right}})^{\alpha} u = (\delta_{\text{right}})^{m} (\delta_{\text{right}})^{\alpha - m} u, \quad (\delta_{\text{right}})^{-\alpha} u = (\delta_{\text{right}})^{-m} (\delta_{\text{right}})^{-(\alpha - m)} u,$$

where $m = [\alpha]$. In addition, in our case, by (1.6) we have that formulas (1.9) and (1.10) are valid for every $\alpha > 0$. Also, by (1.6) we have

$$(\delta_{\text{right}})^{-\alpha} (\delta_{\text{right}})^{\alpha} u(nh) = u(nh), \quad n \in \mathbb{Z}, u \in \ell^p(\mathbb{Z}_h).$$

Furthermore, for $\alpha, \beta \in \mathbb{R}$, we have

$$(\delta_{\text{right}})^{\alpha} (\delta_{\text{right}})^{\beta} u(nh) = (\delta_{\text{right}})^{\alpha + \beta} u(nh), \quad n \in \mathbb{Z},$$

for u such that the series involved in the identity converge.

Now we shall prove the maximum and comparison principles for the fractional differences $(\delta_{\text{right}})^{\alpha}$ and the uniqueness of the corresponding Dirichlet problems. We state the results and proofs for δ_{right}, being for δ_{left} completely analogous.

Theorem 1.6. *Let $0 < \alpha < 1$ and $1 \le p \le \infty$.*

(i) Let $u \in \ell^p(\mathbb{Z}_h)$ such that $u(j_0 h) = 0$ for some $j_0 \in \mathbb{Z}$, and $u(jh) \ge 0$ for all $j_0 \le j$. Then $(\delta_{\text{right}})^{\alpha} u(j_0 h) \le 0$. Moreover, $(\delta_{\text{right}})^{\alpha} u(j_0 h) = 0$ if and only if $u(jh) = 0$ for all $j_0 \le j$.

(ii) Let $u, v \in \ell^p(\mathbb{Z}_h)$ such that $u(j_0 h) = v(j_0 h)$ for some $j_0 \in \mathbb{Z}$, and $u(jh) \ge v(jh)$ for all $j_0 \le j$. Then $(\delta_{\text{right}})^{\alpha} u(j_0 h) \le (\delta_{\text{right}})^{\alpha} v(j_0 h)$. Moreover, $(\delta_{\text{right}})^{\alpha} u(j_0 h) = (\delta_{\text{right}})^{\alpha} v(j_0 h)$ if and only if $u(jh) = v(jh)$ for all $j_0 \le j$.

Proof. Part (ii) is a straightforward consequence of (i). For (i) we write $(\delta_{\text{right}})^{\alpha} u(j_0 h) = \frac{1}{h^{\alpha}} \sum_{m \in \mathbb{N}_0} \Lambda^{\alpha}(m) u((j_0 + m)h) = \frac{1}{h^{\alpha}} \sum_{m \in \mathbb{N}} \Lambda^{\alpha}(m) u((j_0 + m)h) \le 0$, since $\Lambda^{\alpha}(m) < 0$ for all $m \in \mathbb{N}$. Moreover, by the same argument, if $(\delta_{\text{right}})^{\alpha} u(j_0 h) = 0$ then $u((j_0 + m)h) = 0$, for all $m \in \mathbb{N}$. □

The following result is a consequence of the above theorem.

Theorem 1.7. *Let $j_0 < j_1 \in \mathbb{Z}$ and $u, v \in \ell^p(\mathbb{Z}_h)$ with $1 \le p \le \infty$.*

(i) Let u be a solution of

$$\begin{cases} (\delta_{\text{right}})^{\alpha} u = f, & \text{in } [j_0, j_1) \\ u = 0, & \text{in } [j_1, \infty). \end{cases}$$

If $f \ge 0$ in $[j_0, j_1)$ then $u \ge 0$ in $[j_0, \infty)$.

(ii) If $(\delta_{\text{right}})^{\alpha} u \le 0$ in $[j_0, j_1)$ and $u \le 0$ in $[j_1, \infty)$, then

$$\sup_{j \ge j_0} u(jh) = \sup_{j \ge j_1} u(jh).$$

(iii) *If* $(\delta_{\text{right}})^\alpha u \geq 0$ *in* $[j_0, j_1)$ *and* $u \geq 0$ *in* $[j_1, \infty)$*, then*

$$\inf_{j \geq j_0} u(jh) = \inf_{j \geq j_1} u(jh).$$

(iv) *If*

$$\begin{cases} (\delta_{\text{right}})^\alpha u \geq (\delta_{\text{right}})^\alpha v, & \text{in } [j_0, j_1) \\ \\ u \geq v, & \text{in } [j_1, \infty), \end{cases}$$

then $u \geq v$ *in* $[j_0, \infty)$*. In particular, we have uniqueness of the Dirichlet problem*

$$\begin{cases} (\delta_{\text{right}})^\alpha u = f, & \text{in } [j_0, j_1) \\ \\ u = g, & \text{in } [j_1, \infty). \end{cases}$$

Proof. We prove part (i) by contradiction. We suppose that there exists $m \in [j_0, j_1)$ such that $u(mh) < 0$ is the global minimum of u in $[j_0, \infty)$. So $u(jh) - u(mh) \geq 0$ for all $j \geq j_0$, then, by the maximum principle, $(\delta_{\text{right}})^\alpha (u - u(mh))(mh) = (\delta_{\text{right}})^\alpha u(mh) \leq 0$. On the one hand, if $(\delta_{\text{right}})^\alpha u(mh) = 0$, then $u(jh) = u(mh) < 0$ for all $j \geq m$, which contradicts that $u(jh) = 0$ for all $j \geq j_1$. On the other hand, if $(\delta_{\text{right}})^\alpha u(mh) < 0$, this contradicts the hypothesis on f.

For part (ii), we use again an argument of contradiction. We suppose that $\sup_{j \geq j_0} u(jh)$ is not attained in $[j_1, \infty)$. Then there exists $m \in [j_0, j_1)$ such that $u(mh)$ is the global maximum of u in $[j_0, \infty)$. So $u(mh) - u(jh) \geq 0$ for all $j \geq j_0$, then, by the maximum principle, $(\delta_{\text{right}})^\alpha (u(mh) - u)(mh) = -(\delta_{\text{right}})^\alpha u(mh) \leq 0$, i.e, $(\delta_{\text{right}})^\alpha u(mh) \geq 0$. If $(\delta_{\text{right}})^\alpha u(mh) > 0$, it contradicts that $(\delta_{\text{right}})^\alpha u \leq 0$ in $[j_0, j_1)$ and if $(\delta_{\text{right}})^\alpha u(mh) = 0$, then, by the maximum principle, $u(jh) = u(hm)$ for all $j \geq m$, so the $\sup_{j \geq j_0} u(jh)$ is attained in $[j_1, \infty)$.

Part (ii) implies part (iii) by taking $-u$. Finally, part (iv) is a consequence of (iii), and the uniqueness is a straightforward consequence. □

To state the regularity results, we need to consider the discrete Hölder spaces. Following the notation in [26], for $l, s \in \mathbb{N}_0$, we denote $\delta^{l,s}_{\text{right,left}} := (\delta_{\text{right}})^l (\delta_{\text{left}})^s$.

Definition 1.8. *([26, Definition 2.1]). Let* $0 < \beta \leq 1$ *and* $k \in \mathbb{N}_0$*. A function* $u : \mathbb{Z}_h \to \mathbb{R}$ *belongs to the discrete Hölder space* $C_h^{k,\beta}$ *if*

$$[\delta^{l,s}_{\text{right,left}} u]_{C_h^{0,\beta}} = \sup_{m \neq j} \frac{|\delta^{l,s}_{\text{right,left}} u(jh) - \delta^{l,s}_{\text{right,left}} u(hm)|}{h^\beta |j - m|^\beta} < \infty$$

for each pair $l, s \in \mathbb{N}_0$ *such that* $l + s = k$*. The norm in the spaces* $C_h^{k,\beta}$ *is given by*

$$\|u\|_{C_h^{k,\beta}} = \max_{l+s \leq k} \sup_{m \in \mathbb{Z}} |\delta^{l,s}_{\text{right,left}} u(mh)| + \max_{l+s=k} [\delta^{l,s}_{\text{right,left}} u]_{C_h^{0,\beta}}.$$

Theorem 1.9 (Discrete Hölder estimates)**.** *Let* $0 < \beta \leq 1$ *and* $0 < \alpha < 1$.

(i) Let $u \in C_h^{0,\beta}$ *and* $\alpha < \beta$*. Then* $(\delta_{\text{right}})^\alpha u \in C_h^{0,\beta-\alpha}$ *and*

$$\|(\delta_{\text{right}})^\alpha u\|_{C_h^{0,\beta-\alpha}} \leq C\|u\|_{C_h^{0,\beta}}.$$

(ii) Let $u \in C_h^{1,\beta}$ *and* $\alpha < \beta$*. Then* $(\delta_{\text{right}})^\alpha u \in C_h^{1,\beta-\alpha}$ *and*

$$\|(\delta_{\text{right}})^\alpha u\|_{C_h^{1,\beta-\alpha}} \leq C\|u\|_{C_h^{1,\beta}}.$$

(iii) Let $u \in C_h^{1,\beta}$ *and* $\alpha > \beta$*. Then* $(\delta_{\text{right}})^\alpha u \in C_h^{0,\beta-\alpha+1}$ *and*

$$\|(\delta_{\text{right}})^\alpha u\|_{C_h^{0,\beta-\alpha+1}} \leq C\|u\|_{C_h^{1,\beta}}.$$

(iv) Let $u \in C_h^{k,\beta}$ *and assume that* $k+\beta-\alpha$ *is not an integer, with* $\alpha < k+\beta$*. Then* $(\delta_{\text{right}})^\alpha u \in C_h^{l,s}$ *where* l *is the integer part of* $k+\beta-\alpha$ *and* $s = k+\beta-\alpha-l$*.*

The positive constants C *are independent of* h *and* u*.*

Proof. Let $j, l \in \mathbb{Z}$. We write

$$|(\delta_{\text{right}})^\alpha u(jh) - (\delta_{\text{right}})^\alpha u(lh)| = \frac{1}{h^\alpha}|I_1 + I_2|,$$

with $I_1 = \sum_{1\leq m\leq |j-l|} (u((j+m)h) - u(jh) - u((l+m)h) + u(lh))\Lambda^\alpha(m)$, and
$I_2 = \sum_{m>|j-l|} (u((j+m)h) - u(jh) - u((l+m)h) + u(lh))\Lambda^\alpha(m)$.

To prove (i), note that

$$|I_1| \leq C_\alpha h^\beta [u]_{C_h^{0,\beta}} \sum_{1\leq m\leq |j-l|} \frac{m^\beta}{m^{\alpha+1}} \leq C_\alpha h^\beta [u]_{C_h^{0,\beta}} |j-l|^{\beta-\alpha},$$

where we have used (1.7). For I_2, observe that

$$|u(jh) - u(lh))|, |u((j+m)h) - u((l+m)h)| \leq h^\beta [u]_{C_h^{0,\beta}} |j-l|^\beta,$$

then

$$|I_2| \leq C_\alpha h^\beta [u]_{C_h^{0,\beta}} |j-l|^\beta \sum_{m>|j-l|} \frac{1}{m^{\alpha+1}} \leq C_\alpha h^\beta [u]_{C_h^{0,\beta}} |j-l|^{\beta-\alpha},$$

using again (1.7).

Part (ii) is a straightforward consequence of (i). By definition, if $u \in C_h^{1,\beta}$, then $\delta_{\text{right}} u \in C_h^{0,\beta}$, and as $\delta_{\text{right,left}}$ commutes with $(\delta_{\text{right}})^\alpha$, by using (i) we get $\delta_{\text{right/left}}(\delta_{\text{right}})^\alpha u \in C_h^{0,\beta-\alpha}$.

For part (iii), suppose without loss of generality that $j > l$. We can write $u((j+m)h) - u(jh) = -h \sum_{p=0}^{m-1} \delta_{\text{right}} u((j+p)h)$. Then

$$|I_1| \le h \sum_{1\le m\le |j-l|} \sum_{p=0}^{m-1} |\delta_{\text{right}} u((j+p)h) - \delta_{\text{right}} u((l+p)h)| \Lambda^{\alpha}(m)$$
$$\le C_\alpha h^{\beta+1} \|u\|_{C_h^{1,\beta}} |j-l|^\beta \sum_{1\le m\le |j-l|} \frac{1}{m^\alpha} \le C_\alpha h^{\beta+1} \|u\|_{C_h^{1,\beta}} |j-l|^{\beta+1-\alpha}.$$

To bound I_2 we write $u((j+m)h) - u((l+m)h) = -h \sum_{p=0}^{|j-l|-1} \delta_{\text{right}} u((l+m+p)h)$, then

$$|I_2| \le h \sum_{m>|j-l|} \sum_{p=0}^{|j-l|-1} |\delta_{\text{right}} u((l+m+p)h) - \delta_{\text{right}} u((l+p)h)| \Lambda^{\alpha}(m)$$
$$\le C_\alpha h^{\beta+1} \|u\|_{C_h^{1,\beta}} |j-l| \sum_{m>|j-l|} \frac{m^\beta}{m^\alpha} \le C_\alpha h^{\beta+1} \|u\|_{C_h^{1,\beta}} |j-l|^{\beta+1-\alpha}.$$

Iterating parts (i), (ii) and (iii) we get (iv). □

Theorem 1.10 (Discrete Schauder estimates)**.** *Let $0 < \beta, \alpha < 1$, and $u \in \ell_{-\alpha,h}$, see* (1.8).

(i) Let $u \in C_h^{0,\beta}$ and suppose that $\alpha + \beta < 1$. Then $(\delta_{\text{right}})^{-\alpha} u \in C_h^{0,\beta+\alpha}$ and

$$\|(\delta_{\text{right}})^{-\alpha} u\|_{C_h^{0,\beta+\alpha}} \le C \|u\|_{C_h^{0,\beta}}.$$

(ii) Let $u \in C_h^{0,\beta}$ and suppose that $\alpha + \beta > 1$. Then $(\delta_{\text{right}})^{-\alpha} u \in C_h^{1,\beta+\alpha-1}$ and

$$\|(\delta_{\text{right}})^{-\alpha} u\|_{C_h^{1,\beta+\alpha-1}} \le C \|u\|_{C_h^{0,\beta}}.$$

(iii) Let $u \in C_h^{k,\beta}$ and assume that $k + \beta + \alpha$ is not an integer. Then $(\delta_{\text{right}})^{-\alpha} u \in C_h^{l,s}$ where l is the integer part of $k + \beta + \alpha$ and $s = k + \beta + \alpha - l$.

(iv) Let $u \in \ell^\infty$. Then $(\delta_{\text{right}})^{-\alpha} u \in C_h^{0,\alpha}$ and

$$\|(\delta_{\text{right}})^{-\alpha} u\|_{C_h^{0,\alpha}} \le C \|u\|_\infty.$$

The positive constants C are independent of h and u.

To prove this theorem we need a previous lemma.

Lemma 1.11. *1. For every $j \in \mathbb{N}_0$, and $\alpha \in \mathbb{R}$, $\Lambda^{-\alpha}(j+1) - \Lambda^{-\alpha}(j) = \Lambda^{-(\alpha-1)}(j+1)$.*

2. For every $n, l \in \mathbb{Z}$, with $n > l$, and $0 < \alpha < 1$,

$$\sum_{m=n}^{\infty} (\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l)) - \sum_{m=l}^{n-1} \Lambda^{-\alpha}(m-l) = 0.$$

Proof. At first we prove (1). Observe that $\Lambda^{-\alpha}(1) - \Lambda^{-\alpha}(0) = \alpha - 1 = \Lambda^{-(\alpha-1)}(1)$. Let $j \in \mathbb{N}$. We have that

$$\begin{aligned}\Lambda^{-\alpha}(j+1) - \Lambda^{-\alpha}(j)) &= \frac{\alpha(\alpha+1)\dots(\alpha+j-1)}{j!}\left(\frac{\alpha+j}{j+1} - 1\right) = \frac{(\alpha+j-1)!}{(\alpha-1)!j!}\left(\frac{\alpha-1}{j+1}\right) \\ &= \frac{\Gamma(j+\alpha)}{(j+1)!\Gamma(\alpha-1)} = \Lambda^{-(\alpha-1)}(j+1).\end{aligned}$$

Now we prove (2). Let $n, l \in \mathbb{Z}$, with $n > l$, and $0 < \alpha < 1$. By using the identity in (1) we obtain

$$\begin{aligned}\sum_{m=n}^{\infty} (\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l)) &= \sum_{m=n}^{\infty} (\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-(n-1)) + \Lambda^{-\alpha}(m-(n-1)) \\ &\quad + \dots + \Lambda^{-\alpha}(m-l-1) - \Lambda^{-\alpha}(m-l)) \\ &= \sum_{m=n}^{\infty} (-\Lambda^{-(\alpha-1)}(m-(n-1)) - \Lambda^{-(\alpha-1)}(m-(n-2)) \\ &\quad - \dots - \Lambda^{-(\alpha-1)}(m-l)).\end{aligned}$$

Again, as $\sum_{m=k}^{\infty} \Lambda^{-(\alpha-1)}(m-k) = 0$, we have that

$$\begin{aligned}&- \sum_{m=n}^{\infty} \Lambda^{-(\alpha-1)}(m-(n-1)) = \Lambda^{-(\alpha-1)}(0) = \Lambda^{-\alpha}(0) \\ &- \sum_{m=n}^{\infty} \Lambda^{-(\alpha-1)}(m-(n-2)) = \Lambda^{-(\alpha-1)}(0) + \Lambda^{-(\alpha-1)}(1) \\ &\vdots \\ &- \sum_{m=n}^{\infty} \Lambda^{-(\alpha-1)}(m-l) = \Lambda^{-(\alpha-1)}(0) + \Lambda^{-(\alpha-1)}(1) + \dots + \Lambda^{-(\alpha-1)}(n-l-1).\end{aligned}$$

Thus, using again identity on (1) we get

$$\begin{aligned}&\sum_{m=n}^{\infty} (\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l)) \\ &= (n-l)\Lambda^{-\alpha}(0) + (n-l-1)\Lambda^{-(\alpha-1)}(1) + \dots + \Lambda^{-(\alpha-1)}(n-l-1) \\ &= \sum_{m=l}^{n-1} \Lambda^{-\alpha}(m-l),\end{aligned}$$

and the result follows. □

Proof of Theorem 1.10.

Let $n, l \in \mathbb{Z}$, we assume $n > l$ without loss of generality. First let $u \in C_h^{0,\beta}$ and $\alpha+\beta<1$. By using Lemma 1.11 (2) we can write

$$\begin{aligned}
h^{-\alpha}[(\delta_{\text{right}})^{-\alpha}u(nh) - (\delta_{\text{right}})^{-\alpha}u(lh)] &= \sum_{m=n}^{\infty} \Lambda^{-\alpha}(m-n)u(mh) - \sum_{m=l}^{\infty} \Lambda^{-\alpha}(m-l)u(mh) \\
&= \sum_{m=n}^{\infty} (\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l))(u(mh)-u(lh)) - \sum_{m=l}^{n-1} \Lambda^{-\alpha}(m-l)(u(mh)-u(lh)) \\
&= I + II.
\end{aligned}$$

On the one hand, by using estimate (1.7) and the hypothesis on u, we get

$$|II| \leq C[u]_{C_h^{0,\beta}} h^\beta \sum_{m=l+1}^{n-1} \frac{|m-l|^\beta}{|m-l|^{1-\alpha}} = C[u]_{C_h^{0,\beta}} h^\beta \sum_{k=1}^{n-1-l} \frac{1}{k^{1-\alpha-\beta}} \leq C[u]_{C_h^{0,\beta}} h^\beta (n-l)^{\alpha+\beta}.$$

Before doing the estimation for I, observe that, as $n > l$, by (1.7) we have that $|\Lambda^{-\alpha}(m-n)| \leq \frac{C}{(m-n)^{1-\alpha}}$ and $|\Lambda^{-\alpha}(m-l)| \leq \frac{C}{(m-n)^{1-\alpha}}$ for $m \geq n+1$. Also, by using Lemma 1.11 (1) and (1.7) we get that

$$\begin{aligned}
&|\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l)| \\
&= |-\Lambda^{-(\alpha-1)}(m-(n-1)) - \Lambda^{-(\alpha-1)}(m-(n-2)) - \cdots - \Lambda^{-(\alpha-1)}(m-l)| \\
&\leq \frac{C|n-l|}{(m-(n-1))^{2-\alpha}} \leq \frac{C|n-l|}{(m-n)^{2-\alpha}} \quad m \geq n+1. \qquad (1.11)
\end{aligned}$$

Hence, by using the comments above, the hypothesis on u and (1.7), we obtain that

$$\begin{aligned}
|I| &\leq C[u]_{C_h^{0,\beta}} h^\beta \left(|n-l|^\beta + \frac{|n-l|^\beta}{(n-l)^{1-\alpha}} + \sum_{m=n+1}^{2n-l} \frac{|m-l|^\beta}{(m-n)^{1-\alpha}} + \sum_{m=2n-l+1}^{\infty} \frac{(n-l)|m-l|^\beta}{(m-n)^{2-\alpha}} \right) \\
&\leq C[u]_{C_h^{0,\beta}} h^\beta \left((|n-l|^{\alpha+\beta} + \sum_{k=1}^{n-l} \frac{k^\beta + (n-l)^\beta}{k^{1-\alpha}} + \sum_{k=n-l+1}^{\infty} \frac{(n-l)(k^\beta + (n-l)^\beta)}{k^{2-\alpha}} \right) \\
&\leq C[u]_{C_h^{0,\beta}} h^\beta (n-l)^{\alpha+\beta}.
\end{aligned}$$

Now suppose that $u \in C_h^{0,\beta}$ with $\alpha+\beta>1$. By the definition of the space $C_h^{1,\alpha+\beta-1}$, we have to prove that $\delta_{\text{right}}((\delta_{\text{right}})^{-\alpha}u) \in C_h^{0,\alpha+\beta-1}$. By using $\delta_{\text{right}}((\delta_{\text{right}})^{-\alpha}u) = (\delta_{\text{right}})^{1-\alpha}u$ and Theorem 1.9, we conclude that $\delta_{\text{right}}((\delta_{\text{right}})^{-\alpha}u) \in C_h^{0,\alpha+\beta-1}$, so the result follows.

We prove statement (iii) for $k=1$. The other cases follow by iteration.

Let $u \in C_h^{1,\beta}$ and $\alpha+\beta<1$. By hypothesis, $\delta_{\text{right}}u$ belongs to $C_h^{0,\beta}$. We want to prove that $\delta_{\text{right}}^{-\alpha}u \in C_h^{1,\alpha+\beta}$, that is, $\delta_{\text{right}}(\delta_{\text{right}}^{-\alpha})u = \delta_{\text{right}}^{-\alpha}(\delta_{\text{right}}u) \in C_h^{0,\alpha+\beta}$, and this is consequence of (i).

Now suppose that $u \in C_h^{1,\beta}$ and $\alpha+\beta>1$. By hypothesis, $\delta_{\text{right}}u \in C_h^{0,\beta}$. We want to prove that $\delta_{\text{right}}^{-\alpha}u \in C_h^{2,\alpha+\beta-1}$, that is, $(\delta_{\text{right}})^2(\delta_{\text{right}}^{-\alpha}u) = \delta_{\text{right}}(\delta_{\text{right}}^{1-\alpha})u \in C_h^{0,\alpha+\beta-1}$. By using (ii), we have that $\delta_{\text{right}}^{-\alpha}(\delta_{\text{right}}u) = \delta_{\text{right}}^{1-\alpha}u \in C_h^{1,\alpha+\beta-1}$, and by the definition of the space $C_h^{1,\alpha+\beta-1}$, we conclude that $\delta_{\text{right}}(\delta_{\text{right}}^{1-\alpha})u \in C_h^{0,\alpha+\beta-1}$.

Finally, assume that $u \in \ell^\infty$. Again, we can write

$$
\begin{aligned}
h^{-\alpha}[(\delta_{\text{right}})^{-\alpha}u(nh) - (\delta_{\text{right}})^{-\alpha}u(lh)] &= \sum_{m=n}^{\infty}(\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l))u(mh) \\
&- \sum_{m=l}^{n-1}\Lambda^{-\alpha}(m-l)u(mh).
\end{aligned}
$$

By using (1.11), we have

$$
\left|\sum_{m=2n-l+1}^{\infty}(\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l))u(mh)\right| \leq \|u\|_\infty \sum_{m=2n-l+1}^{\infty}\frac{n-l}{(m-n)^{2-\alpha}} \leq C\|u\|_\infty(n-l)^\alpha
$$

and by using (1.7), we get that

$$
\begin{aligned}
&\left|\sum_{m=n}^{2n-l}(\Lambda^{-\alpha}(m-n) - \Lambda^{-\alpha}(m-l))u(mh)\right| \\
&\leq C\|u\|_\infty\left(1 + \frac{1}{(n-l)^{1-\alpha}} + \sum_{m=n+1}^{2n-l}(|\Lambda^{-\alpha}(m-n)| + |\Lambda^{-\alpha}(m-l)|)\right) \\
&\leq C\|u\|_\infty\left(1 + \frac{1}{(n-l)^{1-\alpha}} + \sum_{m=n+1}^{2n-l}\frac{1}{(m-n)^{1-\alpha}}\right) \leq C\|u\|_\infty(n-l)^\alpha
\end{aligned}
$$

and

$$
\left|\sum_{m=l}^{n-1}\Lambda^{-\alpha}(m-l)u(mh)\right| \leq C\|u\|_\infty\left(1 + \sum_{m=l+1}^{n-1}\frac{1}{(m-l)^{1-\alpha}}\right) \leq C\|u\|_\infty(n-l)^\alpha.
$$

□

1.3 Approximation of fractional derivatives in the line by discrete fractional derivatives. Marchaud and Grünwald-Letnikov fractional derivatives

In this section we compare the discrete fractional derivatives and the discretized continuous fractional derivatives on Hölder spaces, and we estimate the error of the approximation on $\ell^\infty(\mathbb{Z})$. These results also allow us to prove the coincidence, for Hölder continuous functions, of the Marchaud and Grünwald-Letnikov derivatives in every point and the speed of convergence to the Grünwald-Letnikov derivative.

In [8], the fractional powers of the derivatives from the right and the left are considered, where

$$D_{\text{right}} f(x) = \lim_{t\to 0^+} \frac{f(x) - f(x+t)}{t} \text{ and } D_{\text{left}} f(x) = \lim_{t\to 0^+} \frac{f(x) - f(x-t)}{t}$$

are the continuous derivatives from the right and from the left, respectively. We recall that the fractional derivatives in the line (also called the *Marchaud fractional derivative*) for $0 < \alpha < 1$ are given by

$$(D_{\text{right/left}})^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(x \pm t) - f(x)}{t^{1+\alpha}}\, dt,$$

for sufficiently smooth functions f.

Also, recall that a continuous real function u belongs to the Hölder space $C^{k,\beta}(\mathbb{R})$, $k \in \mathbb{N}_0$, $0 < \beta \le 1$, if $u \in \mathcal{C}^k(\mathbb{R})$ and

$$[u^{(k)}]_{C^{0,\beta}(\mathbb{R})} = \sup_{x\neq y\in\mathbb{R}} \frac{|u^{(k)}(x) - u^{(k)}(y)|}{|x-y|^\beta} < \infty, \tag{1.12}$$

where $u^{(k)}$ denotes the k-th derivative of u. The norm in the spaces $C^{k,\beta}(\mathbb{R})$ is

$$\|u\|_{C^{k,\beta}(\mathbb{R})} = \sum_{l=0}^{k} \|u^{(l)}\|_{L^\infty(\mathbb{R})} + [u^{(k)}]_{C^{0,\beta}(\mathbb{R})}.$$

Given a function u defined on $\mathbb{R}$, we consider its restriction $r_h u$ (or discretization) to $\mathbb{Z}_h$, that is, $r_h u(j) = u(hj)$ for $j \in \mathbb{Z}$. We have the following theorem.

Theorem 1.12. *Let $0 < \beta \le 1$ and $0 < \alpha < 1$.*

(i) Let $u \in C^{0,\beta}(\mathbb{R})$ and $\alpha < \beta$. Then

$$\|(\delta_{\text{right}})^\alpha (r_h u) - r_h((D_{right})^\alpha u)\|_{\ell^\infty} \le C_\alpha [u]_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha}.$$

(ii) Let $u \in C^{1,\beta}(\mathbb{R})$ and $\alpha < \beta$. Then

$$\|-\delta_{\text{right}} (\delta_{\text{right}})^\alpha (r_h u) - r_h(\frac{d}{dx}(D_{right})^\alpha u)\|_{\ell^\infty} \le C_\alpha [u]_{C^{1,\beta}(\mathbb{R})} h^{\beta-\alpha}.$$

Here, the operator $(D_{right})^\alpha$ is the Marchaud derivative, that is,

$$(D_{right})^\alpha f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(x+t) - f(x)}{t^{1+\alpha}} dt. \tag{1.13}$$

There are analogous results when we substitute δ_{left} by δ_{right} and D_{left} by D_{right} respectively. To prove Theorem 1.12 we need a previous lemma.

Lemma 1.13. *Let $0 < \alpha < 1$ and $j \in \mathbb{Z}$. We have*

$$\left| \frac{1}{\Gamma(-\alpha)} \int_{(j+m)h}^{(j+m+1)h} \frac{dt}{(t-jh)^{1+\alpha}} - \frac{\Lambda^\alpha(m)}{h^\alpha} \right| \leq \frac{C_\alpha}{h^\alpha m^{2+\alpha}}, \quad m \in \mathbb{N}, \tag{1.14}$$

and

$$\int_{(j+m)h}^{(j+m+1)h} \frac{dt}{(t-jh)^{1+\alpha}} \leq \frac{C_\alpha}{h^\alpha m^{1+\alpha}}, \quad m \in \mathbb{N}. \tag{1.15}$$

Proof. By performing the change of variable $t - jh = zh$ we have

$$\left| \frac{1}{\Gamma(-\alpha)} \int_{(j+m)h}^{(j+m+1)h} \frac{1}{(t-jh)^{1+\alpha}} \, dt - \frac{\Lambda^\alpha(m)}{h^\alpha} \right| = \left| \frac{1}{h^\alpha \Gamma(-\alpha)} \int_m^{m+1} \frac{dz}{z^{1+\alpha}} - \frac{\Lambda^\alpha(m)}{h^\alpha} \right|$$

$$\leq \left| \frac{1}{h^\alpha \Gamma(-\alpha)} \int_m^{m+1} \left(\frac{1}{z^{1+\alpha}} - \frac{1}{m^{1+\alpha}} \right) dz \right| + \frac{1}{h^\alpha} \left| \frac{1}{\Gamma(-\alpha) m^{1+\alpha}} - \Lambda^\alpha(m) \right|.$$

The Mean Value Theorem implies

$$\left| \int_m^{m+1} \left(\frac{1}{z^{1+\alpha}} - \frac{1}{m^{1+\alpha}} \right) dz \right| \leq C_\alpha \left| \int_m^{m+1} \frac{1}{m^{2+\alpha}} dz \right| \leq \frac{C_\alpha}{m^{2+\alpha}},$$

and

$$\left| \frac{1}{\Gamma(-\alpha) m^{1+\alpha}} - \Lambda^\alpha(m) \right| \leq \frac{C_\alpha}{m^{2+\alpha}}$$

by (1.7). So, (1.14) is proved. For (1.15), we have

$$\int_{(j+m)h}^{(j+m+1)h} \frac{dt}{(t-jh)^{1+\alpha}} \leq \int_{(j+m)h}^{(j+m+1)h} \frac{dt}{(mh)^{1+\alpha}} \leq \frac{C_\alpha}{h^\alpha m^{1+\alpha}}.$$

□

Proof of Theorem 1.12.
We suppose the hypothesis of part (i). Let $j \in \mathbb{Z}$, then

$$\begin{aligned} r_h((D_{\text{right}})^\alpha u)(j) &= \frac{1}{\Gamma(-\alpha)} \sum_{m \in \mathbb{N}_0} \int_{(j+m)h}^{(j+m+1)h} \frac{u(t) - u(jh)}{(t-jh)^{1+\alpha}} \, dt \\ &= \frac{1}{\Gamma(-\alpha)} \Bigg(\sum_{m \in \mathbb{N}_0} \int_{(j+m)h}^{(j+m+1)h} \frac{u(t) - u((j+m)h)}{(t-jh)^{1+\alpha}} \, dt \\ &\quad + \sum_{m \in \mathbb{N}} \int_{(j+m)h}^{(j+m+1)h} \frac{u((j+m)h) - u(jh)}{(t-jh)^{1+\alpha}} \, dt \Bigg) \\ &= \frac{1}{\Gamma(-\alpha)} (I_1 + I_2). \end{aligned}$$

On the one hand,

$$
\begin{aligned}
|I_1| &\le C[u]_{C^{0,\beta}(\mathbb{R})}\sum_{m\in\mathbb{N}_0}\int_{(j+m)h}^{(j+m+1)h}\frac{|t-(j+m)h|^\beta}{(t-jh)^{1+\alpha}}\,dt\\
&\le C_\alpha h^{\beta-\alpha}[u]_{C^{0,\beta}(\mathbb{R})}\left(1+\sum_{m\in\mathbb{N}}\frac{1}{m^{1+\alpha}}\right)=C_\alpha h^{\beta-\alpha}[u]_{C^{0,\beta}(\mathbb{R})},
\end{aligned}
$$

where we have used (1.15). On the other hand we compare I_2 with $(\delta_{\text{right}})^\alpha(r_hu)(j)$. By (1.14),

$$
\begin{aligned}
&\left|\frac{I_2}{\Gamma(-\alpha)}-(\delta_{\text{right}})^\alpha(r_hu)(j)\right|\\
&\le\sum_{m\in\mathbb{N}}|u((j+m)h)-u(jh)|\left|\frac{1}{\Gamma(-\alpha)}\int_{(j+m)h}^{(j+m+1)h}\frac{dt}{(t-jh)^{1+\alpha}}-\frac{\Lambda^\alpha(m)}{h^\alpha}\right|\\
&\le C_\alpha h^{\beta-\alpha}[u]_{C^{0,\beta}(\mathbb{R})}\sum_{m\in\mathbb{N}}\frac{m^\beta}{m^{2+\alpha}}\le C_\alpha h^{\beta-\alpha}[u]_{C^{0,\beta}(\mathbb{R})}.
\end{aligned}
$$

For (ii), observe that δ_{right} commutates with $(\delta_{\text{right}})^\alpha$ and $\frac{d}{dx}$ with $(D_{\text{right}})^\alpha$. Then we write

$$
\begin{aligned}
\|-\delta_{\text{right}}(\delta_{\text{right}})^\alpha(r_hu)-r_h(\frac{d}{dx}(D_{\text{right}})^\alpha u)\|_{\ell^\infty}&\le\|(\delta_{\text{right}})^\alpha(-\delta_{\text{right}})(r_hu)-(\delta_{\text{right}})^\alpha(r_h\frac{d}{dx}u)\|_{\ell^\infty}\\
&\quad+\|(\delta_{\text{right}})^\alpha(r_h\frac{d}{dx}u)-r_h((D_{\text{right}})^\alpha(\frac{d}{dx}u))\|_{\ell^\infty}.
\end{aligned}
$$

We apply the part (i) to the second term. Let $j\in\mathbb{Z}$. For the first one, we apply the Mean Value Theorem and the fact that $\sum_{m=0}^\infty\Lambda^\alpha(m)=0$,

$$
\begin{aligned}
&\left|(\delta_{\text{right}})^\alpha(-\delta_{\text{right}})(r_hu)(j)-(\delta_{\text{right}})^\alpha(r_h\frac{d}{dx}u)(j)\right|=\\
&=\frac{1}{h^\alpha}\left|\sum_{m\in\mathbb{N}}\Lambda^\alpha(m)\Big(\frac{u((j+m+1)h)-u((j+m)h)}{h}-u'((j+m)h)-\frac{u((j+1)h)-u(jh)}{h}+u'(jh)\Big)\right|\\
&=\frac{1}{h^\alpha}\left|\sum_{m\in\mathbb{N}}\Lambda^\alpha(m)\Big(u'(\xi_{j+m})-u'((j+m)h)-u'(\xi_j)+u'(jh)\Big)\right|\\
&\le\frac{C}{h^\alpha}[u']_{C^{0,\beta}(\mathbb{R})}\sum_{m\in\mathbb{N}}|\Lambda^\alpha(m)|h^\beta\le C_\alpha[u']_{C^{0,\beta}(\mathbb{R})}h^{\beta-\alpha},
\end{aligned}
$$

where $\xi_j\in(jh,(j+1)h)$ and $\xi_{j+m}\in((j+m)h,(j+m+1)h)$. □

Pointwise coincidence of Marchaud and the Grünwald-Letnikov fractional derivatives.

Consider the *fractional differences of order* α, with $\alpha > 0$,

$$\Delta^\alpha_{h,\pm} f(x) = \sum_{k=0}^\infty (-1)^k \binom{\alpha}{k} f(x \pm kh) = \sum_{k=0}^\infty \Lambda^\alpha(k) f(x \pm kh),\ x \in \mathbb{R}, h > 0.$$

The *Grünwald-Letnikov derivatives* of a function f are defined by

$$f^\alpha_\pm(x) = \lim_{h\to+0} \frac{\Delta^\alpha_{h,\pm} f(x)}{h^\alpha}, \tag{1.16}$$

see [78, pages 371–373].
The coincidence of the Marchaud and the Grünwald-Letnikov derivatives is known in almost everywhere sense or in $L^p(\mathbb{R}), 1 \le p < \infty$, for $f \in L^r(\mathbb{R})$, with r and p independent, see [78, Theorems 20.2 ,20.4]. As a consequence of our Theorem 1.12 we shall prove that, for Hölder continuous functions, both derivatives coincide pointwise. Moreover, we get the speed of convergence of the limit in (1.16), which is of order $h^{\beta-\alpha}$.

Theorem 1.14. *Let $0 < \alpha < \beta \le 1$ and $f \in C^{0,\beta}(\mathbb{R})$. Then $f^\alpha_\pm(x) = (D_{right/left})^\alpha f(x)$ for every point $x \in \mathbb{R}$. Moreover, there exists a positive constant $C_{\alpha,\beta}$ such that*

$$\left| (D_{right/left})^\alpha f(x) - \frac{\Delta^\alpha_{h,\pm} f(x)}{h^\alpha} \right| \le C_{\alpha,\beta} [f]_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha}, \quad x \in \mathbb{R}.$$

Proof. Given $x \in \mathbb{R}$ and $h > 0$, there exists a $j_0 \in \mathbb{Z}$ such that $j_0 h \le x < j_0 h + h$. Then, we have

$$\begin{aligned}\left| (D_{\text{right}})^\alpha f(x) - \frac{\Delta^\alpha_{h,+}}{h^\alpha} f(x) \right| &\le |(D_{\text{right}})^\alpha f(x) - (D_{\text{right}})^\alpha f(j_0 h)| + \left| (D_{\text{right}})^\alpha f(j_0 h) - \frac{\Delta^\alpha_{h,+}}{h^\alpha} f(j_0 h) \right| \\ &\quad + \left| \frac{\Delta^\alpha_{h,+}}{h^\alpha} f(j_0 h) - \frac{\Delta^\alpha_{h,+}}{h^\alpha} f(x) \right| = I + II + III.\end{aligned}$$

On the one hand,

$$\begin{aligned} I &= \left| \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(x+t) - f(x)}{t^{1+\alpha}} dt - \frac{1}{\Gamma(-\alpha)} \int_0^\infty \frac{f(j_0 h + t) - f(j_0 h)}{t^{1+\alpha}} dt \right| \\ &\le \left| \frac{1}{\Gamma(-\alpha)} \int_0^h \frac{f(x+t) - f(x)}{t^{1+\alpha}} dt \right| + \left| \frac{1}{\Gamma(-\alpha)} \int_0^h \frac{f(j_0 h + t) - f(j_0 h)}{t^{1+\alpha}} dt \right| \\ &+ \left| \frac{1}{\Gamma(-\alpha)} \int_h^\infty \frac{f(x+t) - f(j_0 h + t)}{t^{1+\alpha}} dt \right| + \left| \frac{1}{\Gamma(-\alpha)} \int_h^\infty \frac{f(j_0 h) - f(x)}{t^{1+\alpha}} dt \right| \\ &\le C_\alpha [f]_{C^{0,\beta}(\mathbb{R})} \left(\int_0^h \frac{t^\beta}{t^{1+\alpha}} dt + \int_h^\infty \frac{h^\beta}{t^{1+\alpha}} dt \right) = C_{\alpha,\beta} [f]_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha}. \end{aligned}$$

By using Theorem 1.12 we obtain

$$II \le C_\alpha [f]_{C^{0,\beta}(\mathbb{R})} h^{\beta-\alpha}.$$

On the other hand, as $\sum_{k=0}^{\infty}\left|\binom{\alpha}{k}\right| \le C_\alpha$, we obtain

$$III \le \left|\frac{1}{h^\alpha}\sum_{k=0}^{\infty}(-1)^k\binom{\alpha}{k}(f(j_0h+kh)-f(x+kh))\right| \le \frac{C[f]_{C^{0,\beta}(\mathbb{R})}}{h^\alpha}\sum_{k=0}^{\infty}\left|\binom{\alpha}{k}\right|h^\beta$$
$$\le C_\alpha [f]_{C^{0,\beta}(\mathbb{R})}h^{\beta-\alpha},$$

so the result follows.

The proof is analogous for $(D_{\text{left}})^\alpha$ and $\frac{\Delta^\alpha_{h,-}}{h^\alpha}$. □

1.4 Extension problem

The operators $(\delta_{\text{right/left}})^\alpha$ are non-local, see (1.10), but they can be understood as limits (when $t \to 0$) of a local extension problem on $\mathbb{R}_+ \times \mathbb{Z}$. In fact, by following the ideas of [23, 40, 84] we get the next result.

Theorem 1.15. *Let $f \in \ell^p(\mathbb{Z})$ and $0 < \gamma < 1$. Consider the equation*

$$\partial^2_{zz}U(z,\cdot) + \frac{1-2\gamma}{z}\partial_z U(z,\cdot) - \delta_{\text{right}}U(z,\cdot) = 0, \quad z \in S_{\pi/4}, \tag{1.17}$$

where $S_{\pi/4} = \{z \in \mathbb{C} \,|\, z \neq 0 \text{ and } |\arg z| < \pi/4\}$. The formula

$$U(z,\cdot) = \frac{z^{2\gamma}}{4^\gamma\Gamma(\gamma)}\int_0^\infty e^{-\frac{z^2}{4t}}u(\cdot,t)\frac{dt}{t^{1+\gamma}} = \frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-\frac{z^2}{4t}}v(\cdot,t)\frac{dt}{t^{1-\gamma}} \tag{1.18}$$

solves (1.17) on $\ell^p(\mathbb{Z})$, where $u(\cdot,t) = \sum_{j\ge 0}G_t(j)f(\cdot+j)$ and $v(\cdot,t)$ satisfies (1.4) with initial data $(\delta_{\text{right}})^\gamma f(\cdot)$. Moreover,

$$\lim_{z\to 0}U(z,\cdot) = f(\cdot) \quad \text{and} \quad \frac{1}{2\gamma}\lim_{z\to 0}z^{1-2\gamma}\partial_z U(z,\cdot) = \frac{\Gamma(-\gamma)}{4^\gamma\Gamma(\gamma)}(\delta_{\text{right}})^\gamma f(\cdot),$$

where both limits hold through proper subsectors of $S_{\pi/4}$ in the $\ell^p(\mathbb{Z})$ sense.

A parallel result can be stated for δ_{left}.

Remark 1.16. Extension problem for negative powers. *It is clear that if a function g is good enough, $(\delta_{\text{right}})^{-\gamma}g \in \ell^p(\mathbb{Z})$, and in (3.10) we substitute f by $(\delta_{\text{right}})^{-\gamma}g$ and $(\delta_{\text{right}})^\gamma f$ by g, U solves the same equation with the initial Dirichlet condition $(\delta_{\text{right}})^{-\gamma}g$ and the Neumann condition $\frac{\Gamma(-\gamma)}{4^\gamma\Gamma(\gamma)}g$. See [22, 40, 84].*

Theorem 1.15 is a straightforward consequence of Remark 1.3 and [40, Theorem 2.1 and Remark 2.2, (*ii*)]. The formula (3.10) provides an explicit expression in the case $z \in (0,\infty)$ of the generalized Poisson function associated to $\delta_{\text{right/left}}$, that is,

$$U(t,n) = P^\gamma_{t,\pm}f(n) = \frac{t^{2\gamma}}{4^\gamma\Gamma(\gamma)}\int_0^\infty e^{-\frac{t^2}{4s}}T_{s,\pm}f(n)\frac{ds}{s^{1+\gamma}} = \sum_{j=0}^{\infty}\frac{t^{2\gamma}f(n\pm j)}{4^\gamma\Gamma(\gamma)j!}\int_0^\infty e^{-s-t^2/4s}s^{j-\gamma}\frac{ds}{s} \tag{1.19}$$
$$= \sum_{j=0}^{\infty}\frac{t^{j+\gamma}}{2^{j+\gamma-1}\Gamma(\gamma)j!}K_{j-\gamma}(t)f(n\pm j), \;\; t > 0,$$

where we have used the identity [74, 2.3.16.1, p. 344]. The function K_ν is the *Macdonald's function* (also called *modified Bessel function of the third type*) defined in [56, Section 5.7, p. 108]. By completeness we prove that the previous formula solves (1.17) pointwise for $t \in (0,\infty)$. We shall use the following identities (see [56, Section 5.7]):

$$K_{\nu+1}(t) = \frac{2\nu}{t}K_\nu(t) + K_{\nu-1}(t), \qquad \frac{d}{dt}(t^\nu K_\nu(t)) = -t^\nu K_{\nu-1}(t),\ \forall \nu \in \mathbb{R}.$$

We have

$$\partial_t P^\gamma_{t,+} f(n) = \sum_{j=0}^\infty \frac{f(n+j)}{2^{j+\gamma-1}\Gamma(\gamma)j!}(2\gamma t^{j+\gamma-1}K_{j-\gamma}(t) - t^{\gamma+j}K_{j-\gamma-1}(t)),$$

and

$$\begin{aligned}\partial_{tt}P^\gamma_{t,+}f(n) &= \sum_{j=0}^\infty \frac{f(n+j)t^{\gamma+j}}{2^{j+\gamma-1}\Gamma(\gamma)j!}\left(\frac{2\gamma(2\gamma-1)}{t^2}K_{j-\gamma}(t) - \frac{4\gamma+1}{t}K_{j-\gamma-1}(t) + K_{j-\gamma-2}(t)\right)\\ &= \sum_{j=0}^\infty \frac{f(n+j)t^{\gamma+j}}{2^{j+\gamma-1}\Gamma(\gamma)j!}\left(\left(\frac{2\gamma(2\gamma-1)}{t^2}+1\right)K_{j-\gamma}(t) - \frac{2\gamma+2j-1}{t}K_{j-\gamma-1}(t)\right).\end{aligned}$$

Then, we obtain

$$\begin{aligned}(\partial^2_{tt} + \frac{1-2\gamma}{t}\partial_t)P^\gamma_{t,+}f(n) &= \sum_{j=0}^\infty \frac{f(n+j)t^{\gamma+j}}{2^{j+\gamma-1}\Gamma(\gamma)j!}(K_{j-\gamma} - \frac{2j}{t}K_{j-\gamma-1}(t))\\ &= \delta_{\text{right}}P^\gamma_{t,+}f(n).\end{aligned}$$

The analogous result for $\delta_{\text{left}}P^\gamma_{t,-}$ can be also proved by the same way.

Observe that, when $\gamma = 1/2$, $P^{1/2}_{t,\pm}f$ is precisely the Poisson semigroup associated to $\delta_{\text{right/left}}$.

1.5 Maximal operators. Littlewood-Paley functions

In this last section we shall consider the maximal functions and the Littlewood-Paley square functions associated to the heat semigroup and generalized Poisson function naturally linked to $\delta_{\text{right/left}}$. Our first observation is that both functions have bad behavior in the case of the heat semigroups. In fact we have the following.

Claim 1 *The maximal functions of the heat semigroups defined in (1.3), that is,*

$$T^*_\pm f = \sup_{t\geq 0}|T_{t,\pm}f|,$$

are not bounded from $\ell^p(\mathbb{Z})$ into itself for $1 \leq p \leq 2$.

In fact, let $f(0) = 1$ and $f(j) = 0$ for $j \neq 0$, then $T_{t,-}f(n) = 0$ for $n < 0$ and $T_{t,-}f(n) = e^{-t}\frac{t^n}{n!}$ for $n \geq 0$. The maximum of the function $e^{-t}\frac{t^n}{n!}$ is $e^{-n}\frac{n^n}{n!}$, and, by Stirling's formula, it behaves asymptotically like $\frac{1}{\sqrt{n}}$ as $n \to \infty$.

Claim 2 *The Littlewood-Paley functions of the heat semigroups defined by*

$$\mathfrak{g}_{\pm}(f) = \left(\int_0^\infty |t\partial_t T_{t,\pm} f|^2 \frac{dt}{t} \right)^{1/2},$$

are not bounded from $\ell^2(\mathbb{Z})$ into itself.

We shall consider the Fourier transform $\hat{f}(\theta) = \sum_n f(n)e^{-in\theta}$. Observe that $\widehat{T_{t,+}f}(\theta) = e^{-t(1-e^{i\theta})}\hat{f}(\theta)$, and then it follows

$$\widehat{\partial_t T_{t,+}} f(\theta) = (e^{i\theta} - 1)e^{-t(1-e^{i\theta})}\hat{f}(\theta).$$

Let $f \in \ell^2(\mathbb{Z})$ such that $\int_0^{2\pi} \left(\frac{|\hat{f}(\theta)|}{\theta}\right)^2 d\theta = \infty$. Hence, by Plancherel's and Fubini's Theorems, we have

$$\begin{aligned}
\|\mathfrak{g}_+(f)\|_{\ell^2}^2 &= \sum_{n\in\mathbb{Z}} \int_0^\infty |t\partial_t T_{t,+} f(n)|^2 \frac{dt}{t} = \frac{1}{2\pi} \int_0^\infty \int_{\mathbb{T}} |t\widehat{\partial_t T_{t,+}} f(\theta)|^2 d\theta \frac{dt}{t} \\
&= \frac{1}{2\pi} \int_0^\infty \int_{\mathbb{T}} |t(e^{i\theta} - 1)e^{-t(1-e^{i\theta})}|^2 |\hat{f}(\theta)|^2 d\theta \frac{dt}{t} \\
&= \frac{1}{2\pi} \int_0^{2\pi} |\hat{f}(\theta)|^2 \int_0^\infty t^2 |e^{i\theta} - 1|^2 e^{-2t\Re(1-e^{i\theta})} \frac{dt}{t} d\theta \\
&\geq \frac{1}{2\pi} \int_0^{2\pi} |\hat{f}(\theta)|^2 \int_0^\infty t^2 (\sin\theta)^2 e^{-2t(1-\cos\theta)} \frac{dt}{t} d\theta \\
&\underbrace{=}_{2(1-\cos\theta)t=u} \frac{1}{8\pi} \int_0^{2\pi} \frac{|\hat{f}(\theta)|^2 (\sin\theta)^2}{(1-\cos\theta)^2} \int_0^\infty u e^{-u} du\, d\theta = \frac{1}{8\pi} \int_0^{2\pi} \frac{|\hat{f}(\theta)|^2 (\sin\theta)^2}{(1-\cos\theta)^2} d\theta,
\end{aligned}$$

and this integral does not converge.

However, the behavior of the generalized Poisson function is suitable with the classical results in Harmonic analysis. Consider the maximal function associated to the generalized Poisson function

$$P_{\pm}^{\gamma,*} f = \sup_{t\geq 0} |P_{t,\pm}^{\gamma} f|, \quad 0 < \gamma < 1,$$

and the Littlewood-Paley square functions

$$g_{\pm}^{\gamma}(f) = \left(\int_0^\infty |t\partial_t P_{t,\pm}^{\gamma} f|^2 \frac{dt}{t} \right)^{1/2}, \quad 0 < \gamma < 1.$$

We have the following result.

Theorem 1.17. *Let $1 < p < \infty$. Let S be either the maximal function or the square Littlewood-Paley function associated to the generalized Poisson functions defined in (1.19). Then S is bounded from $\ell^p(\mathbb{Z})$ into itself and from $\ell^1(\mathbb{Z})$ into weak-$\ell^1(\mathbb{Z})$ (For the maximal function p can be ∞).*

To prove Theorem 1.17 for these operators, the tool that we shall use is the *vector-valued Calderón-Zygmund Theory in spaces of homogeneous type*, more specifically in the particular case of the integers with the natural distance $d(n,m) = |n-m|$ and measure $\mu(n) = 1$. Given a Banach space E, we denote by $\ell^p_E(\mathbb{Z})$ or also $\ell^p(\mathbb{Z}, E)$, $1 \le p \le \infty$, the space of E-valued functions f defined on $\mathbb{Z}$ such that $\|f\|_E$ belongs to $\ell^p(\mathbb{Z}, d, \mu)$.

Definition 1.18 (Vector-valued (convolution) Calderón-Zygmund operator on $(\mathbb{Z}, d, \mu)$)**.** *We say that a linear operator $\mathfrak{T}$ on the space $(\mathbb{Z}, d, \mu)$ is a Calderón-Zygmund operator if it satisfies the following conditions.*

(I) There exists $1 \le p_0 \le \infty$ such that $\mathfrak{T}$ is bounded from $\ell^{p_0}(\mathbb{Z})$ into $\ell^{p_0}_E(\mathbb{Z})$.

(II) For bounded functions f with compact support, $\mathfrak{T}f$ can be represented as

$$\mathfrak{T}f(n) = \sum_{j\in\mathbb{Z}} K(j)f(j+n),$$

where $K(j) \in \mathcal{L}(\mathbb{R}, E)$ is the space of bounded linear operator from $\mathbb{R}$ to E, and satisfies

(II.1) $\|K(j)\|_{\mathcal{L}(\mathbb{R},E)} \le \dfrac{C}{|j|}$, *for every $j \neq 0$;*

(II.2) $\|K(j) - K(j_0)\|_{\mathcal{L}(\mathbb{R},E)} \le C\dfrac{|j-j_0|}{|j_0|^2}$, *whenever $|j_0| > 2|j-j_0|$, $\quad j_0 \neq 0$;*

for some constant $C > 0$.

The Calderón–Zygmund theorem says that if $\mathfrak{T}$ is a Calderón–Zygmund operator on $(\mathbb{Z}, d, \mu)$ as above then $\mathfrak{T}$ is bounded from $\ell^p(\mathbb{Z})$ into $\ell^p_E(\mathbb{Z})$, for any $1 < p \le \infty$, and it is also of weak type $(1,1)$. For full details see [60, 75, 76].

Now we ready to prove the Theorem 1.17. We prove only the cases associated to $P^\gamma_{t,+}$. For $P^\gamma_{t,-}$. the proof is analogous.

Proof of Theorem 1.17.

Case 1. The maximal function.

For convenience, we will write $P^\gamma_{t,+}f(n) = \sum_{j=0}^{\infty} P^\gamma_t(j)f(n+j)$, where

$$P^\gamma_t(j) = \frac{t^{2\gamma}}{4^\gamma\Gamma(\gamma)j!}\int_0^\infty e^{-s-t^2/4s}s^{j-\gamma}\frac{ds}{s}\ , j \in \mathbb{N}_0.$$

Consider the vector-valued operator

$$\mathfrak{T}f(n) = \Big\{\sum_{j\in\mathbb{Z}} P^\gamma_t(j)f(n+j)\Big\}_{t\ge0} = \sum_{j\in\mathbb{Z}}\Big\{P^\gamma_t(j)\Big\}_{t\ge0} f(n+j),$$

where we have assumed $P^\gamma_t(j) = 0$ for $j < 0$, $t \ge 0$. The operator $\mathfrak{T}$ satisfies $\mathfrak{T} : \ell^\infty(\mathbb{Z}) \longrightarrow$

$\ell^\infty_{L^\infty(0,\infty)}(\mathbb{Z})$. In fact

$$\|\mathfrak{T}f\|_{\ell^\infty_{L^\infty}(\mathbb{Z})} = \sup_{n\in\mathbb{Z}}\sup_{t\geq 0}|P^\gamma_{t,+}f(n)| \leq C_\gamma\|f\|_\infty \sup_{t\geq 0}\sum_{j=0}^\infty \frac{1}{j!}\int_0^\infty e^{-u}e^{-\frac{t^2}{4u}}u^j\left(\frac{t^2}{u}\right)^\gamma\frac{du}{u}$$
$$= C_\gamma\|f\|_\infty \sup_{t\geq 0}\int_0^\infty e^{-\frac{t^2}{4u}}\left(\frac{t^2}{u}\right)^\gamma\frac{du}{u} \underbrace{=}_{\frac{t^2}{4u}=v} C_\gamma\|f\|_\infty\sup_{t\geq 0}\int_0^\infty e^{-v}(4v)^\gamma\frac{dv}{v} < \infty,$$

where we have applied Fubini's Theorem.

Moreover the kernel $\{P^\gamma_t(j)\}_{t\geq 0}$ satisfies

$$\|P^\gamma_t(j)\|_{L^\infty} = \sup_{t\geq 0}\frac{1}{4^\gamma\Gamma(\gamma)j!}\int_0^\infty e^{-u}e^{-\frac{t^2}{4u}}\left(\frac{t^2}{u}\right)^\gamma u^j\frac{du}{u} \leq \sup_{t\geq 0}\frac{C_\gamma}{j!}\int_0^\infty e^{-u}u^j\frac{du}{u} = C_\gamma\frac{1}{j},$$

for $j > 0$ and $\|P^\gamma_t(j)\|_{L^\infty} = 0 \leq \frac{C_\gamma}{|j|}$ for $j < 0$, where we have used that the function $g(u) = e^{-\frac{t^2}{4u}}\left(\frac{t^2}{u}\right)^\gamma$ reaches its maximum at $u = t^2/4\gamma$.

Regarding (II.2), it is easy to see that it is enough to prove that for each $j\in\mathbb{N}$,

$$\sup_{t\geq 0}|P^\gamma_t(j) - P^\gamma_t(j+1)| \leq \frac{C_\gamma}{j^2}.$$

If $j\geq 1$, then for all $t\geq 0$ we have

$$|P^\gamma_t(j) - P^\gamma_t(j+1)| = \frac{1}{4^\gamma\Gamma(\gamma)j!}\left|\int_0^\infty e^{-u}e^{-\frac{t^2}{4u}}\left(\frac{t^2}{u}\right)^\gamma u^j\left(1-\frac{u}{j+1}\right)\frac{du}{u}\right|$$
$$= \frac{1}{4^\gamma\Gamma(\gamma)(j+1)!}\left|\int_0^\infty \partial_u(e^{-u}u^{j+1})e^{-\frac{t^2}{4u}}\left(\frac{t^2}{u}\right)^\gamma\frac{du}{u}\right|$$
$$= \frac{1}{4^\gamma\Gamma(\gamma)(j+1)!}\left|\int_0^\infty e^{-u}u^{j+1}\partial_u\left(e^{-\frac{t^2}{4u}}\left(\frac{t^2}{u}\right)^\gamma\frac{1}{u}\right)du\right|,$$

where we have used integration by parts. As $\left|\partial_u\left(e^{-\frac{t^2}{4u}}\left(\frac{t^2}{u}\right)^\gamma\frac{1}{u}\right)\right| \leq \frac{C_\gamma}{u^2}$, then

$$|P^\gamma_t(j) - P^\gamma_t(j+1)| \leq \frac{C_\gamma}{(j+1)!}\int_0^\infty e^{-u}u^{j-1}du = \frac{C_\gamma\Gamma(j)}{(j+1)!} = \frac{C_\gamma}{j^2}.$$

Finally since $\|\mathfrak{T}f(n)\|_{L^\infty} = P^{\gamma,*}_+f(n)$ the result follows for the maximal operator, by choosing $p_0 = \infty$, and $E = L^\infty(\mathbb{R}_+)$.

Case 2. Littlewood-Paley functions

Consider the vector-valued operator

$$\mathfrak{T}f(n) = \left\{\sum_{j\in\mathbb{Z}} t\partial_t P^\gamma_t(j)f(n+j)\right\}_{t\geq 0} = \sum_{j\in\mathbb{Z}}\left\{t\partial_t P^\gamma_t(j)\right\}_{t\geq 0}f(n+j).$$

In this case we write

$$P_t^\gamma(j) = \frac{t^{2\gamma}}{4^\gamma \Gamma(\gamma) j!}\int_0^\infty e^{-s-t^2/4s} s^{j-\gamma}\frac{ds}{s} = \frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-r} G_{t^2/4r}(j)\frac{dr}{r^{1-\gamma}}\ ,\ j\in\mathbb{N}_0.$$

We have performed the change of variables $r=\frac{t^2}{4s}$ and the sequence G_t is defined in (1.3). Again we assume $P_t^\gamma(j)=0$ for $j<0$, $t\geq 0$. The operator $\mathfrak{T}$ is bounded from $\ell^2(\mathbb{Z})$ into $\ell^2_{L^2((0,\infty),\frac{dt}{t})}$. In fact

$$\|\mathfrak{T}f\|^2_{\ell^2_{L^2((0,\infty),\frac{dt}{t})}} = \frac{1}{2\pi}\int_0^\infty \int_{\mathbb{T}} |\widehat{t\partial_t P_{t,+}^\gamma f}(\theta)|^2 d\theta \frac{dt}{t} = \frac{1}{2\pi}\int_{\mathbb{T}}\int_0^\infty |\widehat{t\partial_t P_t^\gamma}(\theta)|^2\frac{dt}{t}|\hat{f}(-\theta)|^2 d\theta.$$

We shall see that $\int_0^\infty |\widehat{t\partial_t P_t^\gamma}(\theta)|^2\frac{dt}{t} < C_\gamma$, where $C_\gamma>0$ does not depend on θ. Observe that

$$\widehat{t\partial_t P_t^\gamma}(\theta) = \frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-r}\frac{t^2}{2r}(e^{i\theta}-1)e^{-\frac{t^2}{4r}(1-e^{i\theta})}\frac{dr}{r^{1-\gamma}}.$$

Notice that if $\theta=0,2\pi$, the statement is trivial, so we have to consider three cases: $0<\theta<\frac{\pi}{4}$, $\frac{\pi}{4}\leq\theta\leq\frac{7\pi}{4}$ and $\frac{7\pi}{4}<\theta<2\pi$.

If $0<\theta<\frac{\pi}{4}$, we define $z_0=t(1-e^{i\theta})^{1/2}$, with $\varphi_0=\arg z_0\in(-\pi/4,\pi/4)$. Then,

$$\int_0^\infty |\widehat{t\partial_t P_t^\gamma}(\theta)|^2\frac{dt}{t} \quad = \quad C_\gamma\int_0^\infty \Big|\frac{z_0^2}{2}\int_0^\infty e^{-r}e^{-z_0^2/(4r)}\frac{dr}{r^{2-\gamma}}\Big|^2\frac{dt}{t}.$$

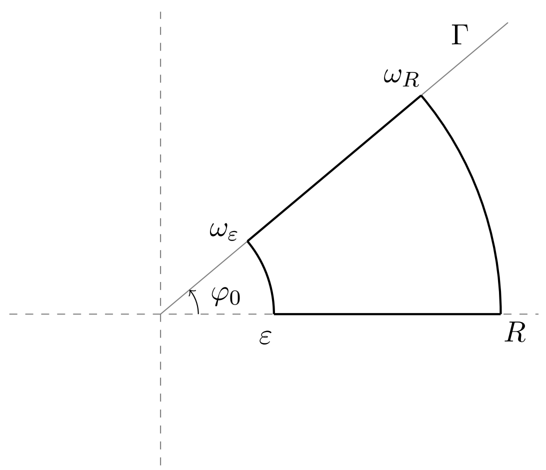
By applying Cauchy's Theorem, for $\varepsilon, R>0$, we get

$$\int_\varepsilon^R e^{-r}e^{-z_0^2/(4r)}\frac{dr}{r^{2-\gamma}} = \Big(\int_{\Gamma_\varepsilon} - \int_{\Gamma_R} + \int_{\omega_\varepsilon}^{w_R}\Big)e^{-\omega}e^{-z_0^2/(4\omega)}\frac{d\omega}{\omega^{2-\gamma}},$$

where

$$\Gamma=\{\omega=sz_0 \mid 0\leq s\leq\infty\},\quad \Gamma_\varepsilon=\{\omega=\varepsilon e^{i\varphi}\ |\varphi\in[0,\varphi_0]\},\quad \Gamma_R=\{\omega=Re^{i\varphi}\ |\varphi\in[0,\varphi_0]\},$$

and $\omega_\varepsilon=\varepsilon z_0$, $\omega_R=Rz_0$, see the next figure.



Notice that

$$\Big|\int_{\Gamma_\varepsilon} e^{-\omega}e^{-z_0^2/(4\omega)}\frac{d\omega}{\omega^{2-\gamma}}\Big| \le \int_{[0,\varphi_0]}|e^{-\varepsilon e^{i\varphi}}e^{-\frac{z_0^2}{4\varepsilon}e^{-i\varphi}}|\frac{d\varphi}{\varepsilon^{1-\gamma}} \le \frac{|\varphi_0|e^{-\frac{|z_0|^2\cos(2\varphi_0)}{4\varepsilon}}}{\varepsilon^{1-\gamma}} \to 0, \text{ as}\,\varepsilon\to 0,$$

since $e^{-\varepsilon\cos\varphi}\le 1$ and it can be checked that $\Re(\frac{z_0^2}{4\varepsilon}e^{-i\varphi}) = \frac{|z_0|^2}{4\varepsilon}\cos(2\varphi_0-\varphi) \ge \frac{|z_0|^2}{4\varepsilon}\cos(2\varphi_0)$.

Similarly, we get

$$\Big|\int_{\Gamma_R} e^{-\omega}e^{-z_0^2/(4\omega)}\frac{d\omega}{\omega^{2-\gamma}}\Big| \le \frac{|\varphi_0|e^{-\frac{|z_0|\cos(2\varphi_0)}{4R}}}{R^{1-\gamma}} \to 0, \text{ as}\,R\to\infty.$$

Hence

$$\frac{z_0^2}{2}\int_0^\infty e^{-r}e^{-z_0^2/(4r)}\frac{dr}{r^{2-\gamma}} = \frac{z_0^2}{2}\int_\Gamma e^{-\omega}e^{-z_0^2/(4\omega)}\frac{d\omega}{\omega^{2-\gamma}} = \frac{z_0^{1+\gamma}}{2}\int_0^\infty e^{-z_0 s}e^{-z_0/(4s)}\frac{ds}{s^{2-\gamma}}$$

and

$$\begin{aligned}\int_0^\infty |\widehat{t\partial_t P_t^\gamma}(\theta)|^2\frac{dt}{t} &= C_\gamma\int_0^\infty \Big|\frac{z_0^{1+\gamma}}{2}\int_0^\infty e^{-z_0 s}e^{-z_0/(4s)}\frac{ds}{s^{2-\gamma}}\Big|^2\frac{dt}{t}\\ &\le C_\gamma\int_0^\infty\left(t^{1+\gamma}\theta^{\frac{1+\gamma}{2}}\int_0^\infty e^{-ct\sqrt{\theta}\cos(\varphi_0)s}e^{-ct\sqrt{\theta}\cos(\varphi_0)/(4s)}\frac{ds}{s^{2-\gamma}}\right)^2\frac{dt}{t}\\ &= C_\gamma\int_0^\infty\left(t^{1+\gamma}\theta^{\frac{1+\gamma}{2}}K_{\gamma-1}(ct\sqrt{\theta}\cos(\varphi_0))\right)^2\frac{dt}{t},\end{aligned}$$

where we have used that $|1-e^{i\theta}|\sim\theta$, for θ being close to 0. In the last identity K_ν denotes the Mcdonald's function, see [74, 2.3.16.1, p. 344]. The following properties can be found in [56, Section 5.7 and Section 5.16]

$$K_{-\nu}(z) = K_\nu(z),\ K_\nu(t)\sim\frac{2^{\nu-1}\Gamma(\nu)}{t^\nu}\ \text{as}\ t\to 0,\ \text{and}\ K_\nu(t)\sim\sqrt{\frac{\pi}{2t}}e^{-t}\ \text{as}\ t\to\infty,\ \nu>0. \tag{1.20}$$

Then

$$\begin{aligned}\int_0^\infty |\widehat{t\partial_t P_t^\gamma}(\theta)|^2\frac{dt}{t} &\le C_\gamma\int_0^\infty\left(t^{1+\gamma}\theta^{\frac{1+\gamma}{2}}K_{1-\gamma}(ct\sqrt{\theta}\cos(\varphi_0))\right)^2\frac{dt}{t}\\ &\le C_\gamma\int_0^{\frac{1}{c\sqrt{\theta}\cos\varphi_0}}\frac{t^{2+2\gamma}\theta^{1+\gamma}}{t^{2-2\gamma}\theta^{1-\gamma}(\cos\varphi_0)^{2-2\gamma}}\frac{dt}{t} + C_\gamma\int_{\frac{1}{c\sqrt{\theta}\cos\varphi_0}}^\infty\frac{t^{2+2\gamma}\theta^{1+\gamma}e^{-2ct\sqrt{\theta}\cos\varphi_0}}{t\sqrt{\theta}\cos\varphi_0}\frac{dt}{t}\\ &\underbrace{=}_{ct\sqrt{\theta}\cos\varphi_0=u} C_\gamma\int_0^1\frac{du}{u^{1-4\gamma}} + C_\gamma\int_1^\infty u^{2\gamma+1}e^{-2u}\frac{du}{u} < \infty.\end{aligned}$$

Observe that in the last identity we have used that $\sqrt{2}/2\le\cos\varphi_0\le 1$.

The case $\frac{7\pi}{4} < \theta < 2\pi$ follows analogously, by using that $|1-e^{i\theta}| = |1-e^{i(\theta-2\pi)}| \sim |\theta-2\pi|$, for θ being closed to 2π.

On the other hand, if $\frac{\pi}{4} \le \theta \le \frac{7\pi}{4}$, then by using again [74, 2.3.16.1, p. 344] we get

$$\begin{aligned}|\widehat{t\partial_t P_t^\gamma}(\theta)| &\le \frac{1}{\Gamma(\gamma)}\int_0^\infty e^{-r}\frac{t^2}{2r}|e^{i\theta}-1|e^{-\frac{t^2}{4r}\Re(1-e^{i\theta})}\frac{dr}{r^{1-\gamma}} \le C_\gamma \int_0^\infty e^{-r}\frac{t^2}{2r}e^{-\frac{t^2}{4r}(1-\cos\theta)}\frac{dr}{r^{1-\gamma}} \\ &\le C_\gamma \int_0^\infty e^{-r}\frac{t^2}{2r}e^{-\frac{t^2}{4r}(1-\sqrt{2}/2)}\frac{dr}{r^{1-\gamma}} \le C_\gamma \frac{t^2}{2}t^{\gamma-1}K_{1-\gamma}\left(t\sqrt{1-\sqrt{2}/2}\right) \\ &= C_\gamma t^{1+\gamma}K_{1-\gamma}(\tilde{c}t).\end{aligned}$$

Hence by estimates (1.20)

$$\begin{aligned}\int_0^\infty |\widehat{t\partial_t P_t^\gamma}(\theta)|^2\frac{dt}{t} &= C_\gamma\int_0^{1/\tilde{c}} \frac{t^{2+2\gamma}}{(\tilde{c}t)^{2-2\gamma}}\frac{dt}{t} + C_\gamma\int_{1/\tilde{c}}^\infty \frac{t^{2+2\gamma}e^{-2\tilde{c}t}}{\tilde{c}t}\frac{dt}{t} \\ &\underbrace{=}_{\tilde{c}t=u} C_\gamma \int_0^1 \frac{du}{u^{1-4\gamma}} + C_\gamma\int_1^\infty u^{2\gamma+1}e^{-2u}\frac{du}{u} < \infty.\end{aligned}$$

Therefore, we have proved that $\|\mathfrak{T}f\|^2_{\ell^2_{L^2((0,\infty),\frac{dt}{t})}} \le C_\gamma\|f\|_2$.

By the representation of the Poisson kernel, (1.19), we can write, for all $n \in \mathbb{Z}$, $t \ge 0$,

$$t\partial_t P^\gamma_{t,\pm}f(n) = \sum_{j=0}^\infty \frac{f(n\pm j)}{2\sqrt{\pi}j!}\int_0^\infty e^{-u}u^{j-\gamma}\left(2\gamma - \frac{t^2}{2u}\right)e^{-\frac{t^2}{4u}}t^{2\gamma}\frac{du}{u} = \sum_{j=0}^\infty t\partial_t P_t^\gamma(j)f(n\pm j).$$

Then, by Minkowski's integral inequality, we get

$$\begin{aligned}\|t\partial_t P_t^\gamma(j)\|_{L^2((0,\infty),\frac{dt}{t})} &= C_\gamma\left(\int_0^\infty \left|\int_0^\infty \left(2\gamma-\frac{t^2}{2u}\right)e^{-u}\frac{u^{j-\gamma}}{j!}e^{-\frac{t^2}{4u}}t^{2\gamma}\frac{du}{u}\right|^2\frac{dt}{t}\right)^{1/2} \\ &\le C_\gamma\int_0^\infty \frac{u^j}{j!}e^{-u}\left(\int_0^\infty \left|2\gamma-\frac{t^2}{2u}\right|^2 e^{-\frac{t^2}{2u}}\left(\frac{t^2}{u}\right)^{2\gamma}\frac{dt}{t}\right)^{1/2}\frac{du}{u} \\ &= C_\gamma\int_0^\infty \frac{u^j}{j!}e^{-u}I^{1/2}\frac{du}{u}, \quad \text{for } j \ge 1.\end{aligned}$$

Observe that

$$\begin{aligned}I &\le C\int_0^{2\sqrt{\gamma u}} 4\gamma^2 e^{-\frac{t^2}{2u}}\left(\frac{t^2}{u}\right)^{2\gamma}\frac{dt}{t} + C\int_{2\sqrt{\gamma u}}^\infty \frac{t^4}{4u^2}e^{-\frac{t^2}{2u}}\left(\frac{t^2}{u}\right)^{2\gamma}\frac{dt}{t} \\ &\underbrace{\le}_{\frac{t}{2\sqrt{\gamma u}}=v} C_\gamma\int_0^1 e^{-2\gamma v^2}v^{4\gamma}\frac{dv}{v} + C_\gamma\int_1^\infty v^{4+4\gamma}e^{-2\gamma v^2}\frac{dv}{v} \le C_\gamma.\end{aligned}$$

Hence, if $j \geq 1$,

$$\|t\partial_t P_t^\gamma(j)\|_{L^2((0,\infty),\frac{dt}{t})} \leq C_\gamma \int_0^\infty \frac{u^j}{j!} e^{-u} \frac{du}{u} = C_\gamma \frac{\Gamma(j)}{\Gamma(j+1)} \leq \frac{C_\gamma}{j},$$

and $\|t\partial_t P_t^\gamma(j)\|_{L^2((0,\infty),\frac{dt}{t})} = 0 \leq \frac{C_\gamma}{|j|}$ for $j < 0$.

Regarding (II.2), if $j \geq 1$,

$$\begin{aligned}
&\|t\partial_t P_t^\gamma(j) - t\partial_t P_t^\gamma(j+1)\|_{L^2((0,\infty),\frac{dt}{t})} \\
&= C_\gamma \left(\int_0^\infty \left| \int_0^\infty \left(2\gamma - \frac{t^2}{2u}\right) \frac{e^{-u}}{j!} u^j \left(1 - \frac{u}{j+1}\right) e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u}\right)^\gamma \frac{du}{u} \right|^2 \frac{dt}{t} \right)^{1/2} \\
&= C_\gamma \left(\int_0^\infty \left| \int_0^\infty \frac{1}{j!} \left(2\gamma - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u}\right)^\gamma \frac{1}{u} \frac{\partial_u(e^{-u}u^{j+1})}{j+1} du \right|^2 \frac{dt}{t} \right)^{1/2} \\
&= C_\gamma \left(\int_0^\infty \left| \int_0^\infty \frac{1}{(j+1)!} e^{-u} u^{j+1} \partial_u \left\{ \left(2\gamma - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u}\right)^\gamma \frac{1}{u} \right\} du \right|^2 \frac{dt}{t} \right)^{1/2}.
\end{aligned}$$

Note that

$$\left| \partial_u \left\{ \left(2\gamma - \frac{t^2}{2u}\right) e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u}\right)^\gamma \frac{1}{u} \right\} \right| \leq C_\gamma \frac{e^{-\frac{t^2}{4u}}}{u^2} \left(\frac{t^2}{u}\right)^\gamma \left[1 + \left(\frac{t^2}{u}\right) + \left(\frac{t^2}{u}\right)^2\right].$$

Hence, by Minkowski's inequality,

$$\begin{aligned}
&\|t\partial_t P_t^\gamma(j) - t\partial_t P_t^\gamma(j+1)\|_{L^2((0,\infty),\frac{dt}{t})} \\
&\leq \frac{C_\gamma}{(j+1)!} \int_0^\infty e^{-u} u^{j-1} \left(\int_0^\infty \left| e^{-\frac{t^2}{4u}} \left(\frac{t^2}{u}\right)^\gamma \left[1 + \left(\frac{t^2}{u}\right) + \left(\frac{t^2}{u}\right)^2\right] \right|^2 \frac{dt}{t} \right)^{1/2} du \\
&\leq C_\gamma \frac{\Gamma(j)}{\Gamma(j+2)} \leq \frac{C_\gamma}{j^2}.
\end{aligned}$$

Finally since $\|\mathfrak{T} f(n)\|_{L^2((0,\infty),\frac{dt}{t})} = g_+^\gamma f(n)$, the result follows for the maximal operator, by choosing $p_0 = 2$, and $E = L^2((0,\infty),\frac{dt}{t})$.

□

Chapter 2

Schrödinger Lipschitz spaces and regularity results

This Chapter **corresponds with [31].**

Classical Lipschitz spaces on $\mathbb{R}^n$, C^α, $\alpha > 0$, are classes of smooth functions that play an important role in function theory, harmonic analysis and partial differential equations. For $0 < \alpha < 1$, they are defined as the set of functions φ such that

$$|\varphi(x+z) - \varphi(x)| \leq C|z|^\alpha \;\; x, z \in \mathbb{R}^n.$$

It could be said that these classes are in between of the continuos functions $\mathcal{C}^0$ and derivable functions with continuous derivative, $\mathcal{C}^1$. However, Zygmund, see [98], argued that for applications in harmonic analysis, the natural limit case when $\alpha \to 1$ corresponds with a space bigger than $\mathcal{C}^1$, the set of continuous functions φ such that $|\varphi(x+z)+\varphi(x-z)-2\varphi(x)| \leq C|z|$, $x, z \in \mathbb{R}^n$. This is why for $\alpha = 1$ this space is also known as the Zygmund space. For $\alpha > 1$, C^α is defined as the class of smooth functions such that their first order derivatives belong to $C^{\alpha-1}$. In the real line, $C^\alpha(\mathbb{R}) = C^{k,\beta}(\mathbb{R})$, where $\alpha = k + \beta$, $k \in \mathbb{N}_0$, $0 < \beta < 1$, see (1.12).

A recurrent object of research is to find some characterizations of these spaces which are more suitable for some applications, like expressions with finite differences, approximation properties, semigroup language, etc. See for instance [53, 81, 87]. The characterizations of bounded Lipschitz functions via the Poisson semigroup, $e^{-y\sqrt{-\Delta}}$, and the Gauss semigroup, $e^{y\Delta}$, are due to Stein and Taibleson, see [81] and [87], and deserve a special mention. The advantage of this approach is that the semigroup language allows to obtain regularity results in these spaces in a more direct way. In particular, it allows to prove the boundedness of some fractional operators, such as fractional Laplacians, fractional integrals, Riesz transforms or Bessel potentials, in a much simpler way than using the classical definition of the spaces.

The works of Taibleson and Stein raise the question of analyzing some Lipschitz spaces adapted to different "Laplacians" and to find pointwise and semigroup estimate characterizations. In the literature sometimes "Lipschitz classes" are also known as "Hölder classes", so we will use both names indistinctly along the thesis. In the case of the Hermite operator $\mathcal{H} = -\Delta + |x|^2$, adapted Hölder classes were defined pointwise in [86]. By using semigroups we have characterized these last classes, also in the parabolic case. We will see this in great detail in Chapter 3.

In this chapter we consider Schrödinger operators in $\mathbb{R}^n$ with $n \geq 3$, that is, $\mathcal{L} = -\Delta + V$, where V is a nonnegative potential satisfying the reverse Hölder inequality:

$$\left(\frac{1}{|B|}\int_B V(y)^q dy\right)^{1/q} \leq \frac{C}{|B|}\int_B V(y)dy, \text{ with an exponent } q > n/2, \tag{2.1}$$

for every ball B.

A particular case is the Hermite operator, where $V(x) = |x|^2$ and satisfies (2.1) for every $q > n/2$. Appropriate Hölder spaces adapted to the operator $\mathcal{L}$ have been analyzed by different authors, but only for $0 < \alpha < 1$. In the paper [20] the authors introduced, for $0 < \alpha < 1$, the space

$$\left\{ f : \rho(\cdot)^{-\alpha} f(\cdot) \in L^\infty(\mathbb{R}^n) \text{ and } \sup_{|z|>0} \frac{\|f(\cdot + z) - f(\cdot)\|_\infty}{|z|^\alpha} < \infty \right\}, \tag{2.2}$$

where $\rho(x)$ is the critical radius associated to the potential V, defined by

$$\rho(x) = \sup\left\{ r > 0 : \frac{1}{r^{n-2}} \int_{B(x,r)} V(y)dy \leq 1 \right\}. \tag{2.3}$$

The content of this chapter is the following: at first, we find the appropriated pointwise definition of Lipschitz classes in the Schrödinger setting for $0 < \alpha < 2$. We shall denote this space by $C^\alpha_\mathcal{L}$. Secondly, we shall define, for every $\alpha > 0$, new Lipschitz spaces adapted to $\mathcal{L}$ by means of the heat semigroup, $e^{-y\mathcal{L}}$, or the Poisson semigroup, $e^{-y\sqrt{\mathcal{L}}}$. We will prove that in fact these spaces coincide with $C^\alpha_\mathcal{L}$ for $0 < \alpha \leq 2 - n/q$. Thirdly, we shall use these new definitions of Lipschitz spaces through semigroups to prove Hölder estimates of negative and positive powers of the operator $\mathcal{L}$, the boundedness of Bessel potentials, Riesz transforms and some multiplier operators associated to $\mathcal{L}$. Due to the semigroup description, the proofs of these estimates will run more smoothly than by using the pointwise definition of the classes. Moreover, the regularity results on these classes defined through heat and Poisson semigroups will be valid for every $\alpha > 0$.

2.1 Schrödinger Lipschitz spaces via the heat semigroup and pointwise characterization.

Motivated by Taibleson and Stein results, we introduce the following definition.

Definition 2.19. *Let* $0 < \alpha < 2$ *and* $\rho(x)$ *the critical radius, see* (2.3). *We shall denote by* $C^\alpha_\mathcal{L}$ *the class of measurable functions such that*

$$M^\mathcal{L}_\alpha[f] := \|\rho(\cdot)^{-\alpha} f(\cdot)\|_\infty < \infty \quad \text{and} \quad N_\alpha[f] := \sup_{|z|>0} \frac{\|f(\cdot + z) + f(\cdot - z) - 2f(\cdot)\|_\infty}{|z|^\alpha} < \infty.$$

We endow this space with the norm

$$\|f\|_{C^\alpha_\mathcal{L}} := M^\mathcal{L}_\alpha[f] + N_\alpha[f].$$

As we said before, in [20] the authors considered the class (2.2). We shall see that our space $C^\alpha_{\mathcal{L}}$ coincides with (2.2) when $0 < \alpha < 1$, see Remark 2.39.

By $W_y = e^{-y\mathcal{L}}$ we will denote the heat semigroup associated to $\mathcal{L}$. From the Feynman-Kac formula, it is well known that

$$W_y(x,z) \le (4\pi y)^{-n/2} e^{-\frac{|x-z|^2}{4y}}.$$

In order to introduce a space of functions for which $W_y f = W_y * f$ and its derivatives will be defined, we need to impose some conditions to the functions. In view of above estimate, we shall say that a function f satisfies a **heat size condition for $\mathcal{L}$** if

$$\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{y}} |f(x)|dx < \infty, \text{ for every } y > 0, \text{ and } \forall \ell \in \mathbb{N}\cup\{0\}, \lim_{y\to\infty} \partial_y^\ell W_y f(x) = 0.$$

When some estimates on the derivatives of the heat semigroup are assumed, the following Theorem shows that this **heat size condition** is equivalent to a controlled growth of the function.

Theorem 2.20. *Let $\alpha > 0$. Let f be a function such that*

$$\left\| \partial_y^k W_y f \right\|_{L^\infty(\mathbb{R}^n)} \le C_\alpha y^{-k+\alpha/2}, \quad \text{with } k = [\alpha/2]+1, y > 0. \tag{2.4}$$

Then, f satisfies a **heat size condition for $\mathcal{L}$** *if and only if $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$.*

We will prove this result on Subsection 2.1.2. Theorem 2.20 leads us to the next definition.

Definition 2.21. *Let $\alpha > 0$. We shall denote by $\Lambda^W_{\alpha/2}$ the set of functions f which satisfy a* **heat size condition for $\mathcal{L}$** *and (2.4). We endow this space with the norm*

$$\|f\|_{\Lambda^W_{\alpha/2}} := S^W_\alpha[f] + M^{\mathcal{L}}_\alpha[f],$$

being $S^W_\alpha[f]$ the infimum of the constants C_α appearing in (2.4).

Now we state the first characterization of the Lipschitz classes by using the derivatives of the heat semigroup. We will prove it on Subsection 2.1.4.

Theorem 2.22. *Let $0 < \alpha \le 2 - \frac{n}{q}$. Then*

$$C^\alpha_{\mathcal{L}} = \Lambda^W_{\alpha/2},$$

with equivalence of norms.

Some observations are in order. The restriction in the range $0 < \alpha \le 2 - \frac{n}{q}$ is due to the reverse Hölder inequality (2.1) that satisfies the potencial V. If the potential V satisfies (2.1) for every $q > n/2$, then we get the result for every $0 < \alpha < 2$. This is the case of the Hermite operator, $\mathcal{H} = -\Delta + |x|^2$, which we will treat in detail on Chapter 3. To prove Theorem 2.22, we compare the spaces $\Lambda^{W}_{\alpha/2}$ with some parallel spaces $\Lambda^{\tilde{W}}_{\alpha/2}$ defined for the classical Laplace operator, see Definition 2.32. We believe that these spaces, more general than the classical Lipschitz spaces, are of independent interest and we study them on subsection 2.1.3. In $\Lambda^{\tilde{W}}_{\alpha/2}$ the functions don't need to be bounded, however a pointwise characterization is also valid as in the classical case, see Theorem 2.37. Once we have this characterization, by using the so called "perturbation formula" for Schrödinger operators, we get a comparison between the classes $\Lambda^{\tilde{W}}_{\alpha/2}$ and $\Lambda^{W}_{\alpha/2}$, see Theorem 2.42. Theorem 2.22 contains as particular cases the results in [20] and [59], when $0 < \alpha < 1$.

2.1.1 Technical results

Let $W_y(x,z)$ be the integral kernel of the semigroup of $e^{-y\mathcal{L}}$ generated by $-\mathcal{L}$. That is, for f satisfying a **heat size condition**

$$e^{-y\mathcal{L}}f(x) = \int_{\mathbb{R}^n} W_y(x,z)f(z)dz, \;\; x \in \mathbb{R}^n.$$

It is known (see [36, 55]) that the integral kernel $W_\zeta(x,y)$ of the extension of $e^{-y\mathcal{L}}$ to the holomorphic semigroup $\{e^{-\zeta\mathcal{L}}\}_{\zeta\in\triangle_{\pi/4}}$ satisfies

$$|W_\zeta(x,z)| \le C_N \frac{e^{-\frac{|x-z|^2}{c\Re\zeta}}}{(\Re\zeta)^{n/2}} \left(1+\frac{\sqrt{\Re\zeta}}{\rho(z)}+\frac{\sqrt{\Re\zeta}}{\rho(x)}\right)^{-N}, \quad x,z\in\mathbb{R}^n, \tag{2.5}$$

for $N > 0$ arbitrary.

Lemma 2.23. *Let $k \ge 1$. There exist constants $c, C_k > 0$ such that, for every $M > 0$,*

$$|\partial_y^k W_y(x,z)| \le C_k \frac{e^{-\frac{|x-z|^2}{cy}}}{y^{k+n/2}} \left(1+\frac{\sqrt{y}}{\rho(z)}+\frac{\sqrt{y}}{\rho(x)}\right)^{-M}.$$

The case $k = 1$ of this Lemma can be found in [35, Formula (2.7)] and [37].

Proof. By Cauchy's integral formula and (2.5) we have

$$|\partial_y^k W_y(x,z)| = k!\Big|\frac{1}{2\pi i}\int_{|\zeta-y|=y/10}\frac{W_\zeta(x,z)}{(\zeta-y)^{k+1}}d\zeta\Big| \le C_k\frac{1}{y^{k+n/2}}\Big(1+\frac{\sqrt{y}}{\rho(x)}+\frac{\sqrt{y}}{\rho(z)}\Big)^{-N}e^{-\frac{|x-z|^2}{cy}}.$$

□

Remark 2.24. *A consequence of the last Lemma is that* $\int_{\mathbb{R}^n}\partial_y^k W_y(x,z)dz \le \frac{C}{y^k}$.

2.1.2 Controlled growth at infinity.

The following Lemma is inspired in [42], we sketch here the proof for completeness.

Lemma 2.25. *Let f be a measurable function such that there exists $y_0 > 0$ for which $\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{y_0}} f(x)dx < \infty$. Then, $\lim_{y\to 0} W_y f(x) = f(x)$, a.e. $x \in \mathbb{R}^n$.*

Proof. Let $|x| \leq A$, $A \in \mathbb{N}$. Given a function f we split

$$f = f\chi_{\{|z|\leq 2A\}} + f\chi_{\{|z|>2A\}} = f_1 + f_2.$$

Observe that, for $|z| > 2A$, $|x - z| \geq \frac{|z|}{2}$. Hence, by using (2.5), we get for $y < y_0/(8c)$,

$$W_y(x,z) \leq C\frac{e^{-\frac{|x-z|^2}{cy}}}{y^{n/2}} \leq C\frac{e^{-\frac{|z|^2}{4cy}}}{y^{n/2}} \leq C\frac{e^{-\frac{A^2}{2cy}}e^{-\frac{|z|^2}{y_0}}}{y^{n/2}}.$$

Hence

$$|W_y f_2(x)| \leq C y^{-n/2} e^{-\frac{A^2}{2cy}} \int_{\mathbb{R}^n} |f(z)| e^{-\frac{|z|^2}{y_0}} dz \to 0, \text{ as } y \to 0.$$

On the other hand, it is known, see [36, Proposition 2.16], that there exists a nonnegative rapidly decaying function w such that

$$|W_y(x,z) - \tilde{W}_y(x-z)| \leq C\Big(\frac{\sqrt{y}}{\rho(x)}\Big)^{2-n/q} w_y(x-z), \text{ for } \sqrt{y} \leq \rho(x), \tag{2.6}$$

where $\tilde{W}_y$ is the Gauss kernel, that is, the kernel of the classical heat semigroup $e^{y\Delta}$. Hence, for $\sqrt{y} \leq \rho(x)$,

$$|W_y f_1(x) - \tilde{W}_y f_1(x)| \leq C\Big(\frac{\sqrt{y}}{\rho(x)}\Big)^{2-n/q} w_y \star f_1(x) \leq C\Big(\frac{\sqrt{y}}{\rho(x)}\Big)^{2-n/q} \|w_y\|_{L^1(\mathbb{R}^n)} \|f_1\|_{L^1(\mathbb{R}^n)}.$$

Therefore, by the standard pointwise convergence for L^1-functions we have

$$\lim_{y\to 0} W_y f_1(x) = f_1(x), \text{ a.e } x \in \mathbb{R}^n.$$

□

Proposition 2.26. *Let $\alpha > 0$, $k = [\alpha/2]+1$ and f be a function satisfying the heat size condition. Then, $\|\partial_y^k W_y f\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha y^{-k+\alpha/2}$ if, and only if, for $m \geq k$, $\|\partial_y^m W_y f\|_{L^\infty(\mathbb{R}^n)} \leq C_m y^{-m+\alpha/2}$. Moreover, for each m, C_m and C_α are comparable.*

Proof. Let $m \geq [\alpha/2] + 1 = k$. By the semigroup property and Remark 2.24 we have

$$\Big|\partial_y^m W_y f(x)\Big| = C\Big|\partial_y^{m-k} W_{y/2}(\partial_u^k W_u f(x)\big|_{u=y/2})\Big| \leq C'_\alpha \frac{1}{y^{m-k}} y^{-k+\alpha/2} = C_m y^{-m+\alpha/2}.$$

For the converse, the fact $|\partial_y^\ell W_y f(x)| \to 0$ as $y \to \infty$, allows us to integrate on y as many times as we need to get $\|\partial_y^k W_y f\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha\, y^{-k+\alpha/2}$.

□

To prove Theorem 2.20, we need some lemmas and Propositions that we present now. The following Lemma can be found in [37, 79].

Lemma 2.27. *There exist constants $C>0$ and $k_0 \geq 1$ such that, for all $x, z \in \mathbb{R}^n$,*

$$C^{-1}\rho(x)\left(1+\frac{|x-z|}{\rho(x)}\right)^{-k_0} \leq \rho(z) \leq C\rho(x)\left(1+\frac{|x-z|}{\rho(x)}\right)^{\frac{k_0}{1+k_0}}.$$

In particular, $\rho(x) \sim \rho(z)$ when $z \in B_r(x)$ and $r \leq C\rho(x)$.

Lemma 2.28. *Let f be a function such that $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$, for some $\alpha>0$. Then, for every $\ell \in \mathbb{N}\cup\{0\}$ and $M>0$, $|\partial_y^\ell W_y f(x)| \leq C_\ell\, M_\alpha^{\mathcal{L}}[f]\, \frac{(\rho(x))^\alpha}{y^\ell}\left(1+\frac{y^{1/2}}{\rho(x)}\right)^{-M}$, $x \in \mathbb{R}^n$, $y>0$.*

Proof. By using (2.5) and Lemma 2.27, for some $\lambda<1$ we have

$$\begin{aligned}
|W_y f(x)| &\leq C_N\, M_\alpha^{\mathcal{L}}[f] \int_{|x-z|<\rho(x)} \frac{e^{-\frac{|x-z|^2}{cy}}\rho(x)^\alpha}{y^{n/2}}\left(1+\frac{y^{1/2}}{\rho(x)}\right)^{-N} dz\\
&+ C_N\, M_\alpha^{\mathcal{L}}[f] \sum_{j=1}^{\infty} \int_{2^{j-1}\rho(x)<|x-z|<2^j\rho(x)} \frac{e^{-\frac{|x-z|^2}{cy}}}{y^{n/2}}\left(1+\frac{y^{1/2}}{\rho(x)}\right)^{-N} e^{-\frac{2^{2(j-1)}\rho(x)^2}{cy}}\rho(x)^\alpha\left(1+\frac{|x-z|}{\rho(x)}\right)^{\lambda\alpha} dz\\
&\leq C_N\, M_\alpha^{\mathcal{L}}[f]\,\rho(x)^\alpha\left(1+\frac{y^{1/2}}{\rho(x)}\right)^{-N} + C_N\, M_\alpha^{\mathcal{L}}[f]\,\rho(x)^\alpha \times\\
&\quad\times \sum_{j=1}^{\infty} 2^{j\alpha(\lambda-2)} \int_{2^{j-1}\rho(x)<|x-z|<2^j\rho(x)} \frac{e^{-\frac{|x-z|^2}{cy}}}{y^{n/2}}\left(\frac{\rho(x)}{y^{1/2}}\right)^{2\alpha}\left(1+\frac{y^{1/2}}{\rho(x)}\right)^{-(N-2\alpha)} e^{-\frac{2^{2(j-1)}\rho(x)^2}{cy}} 2^{2j\alpha} dz\\
&\leq C_N\, M_\alpha^{\mathcal{L}}[f]\,\rho(x)^\alpha\left(1+\frac{y^{1/2}}{\rho(x)}\right)^{-(N-2\alpha)}.
\end{aligned}$$

By choosing $M=N-2\alpha$ we get the result. For the derivatives, we proceed in the same way by using Lemma 2.23. □

The following Proposition is a direct consequence of Lemma 2.28. Moreover, it corresponds with the " if " part of Theorem 2.20.

Proposition 2.29. *Given an operator $\mathcal{L}$, let $\alpha>0$ and f a measurable function. If $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$, then f satisfies a* **heat size condition** *for $\mathcal{L}$.*

Lemma 2.30. *Let $\alpha>0$ and $k=[\alpha/2]+1$. Assume that f satisfies the* **heat size condition** *and (2.4), then for every $j, m \in \mathbb{N}\cup\{0\}$ such that $\frac{m}{2}+j \geq k$, there exists a $C_{m,j}>0$ such that*

$$\left\|\frac{\partial_y^j W_y f}{\rho(\cdot)^m}\right\|_\infty \leq C_m S_\alpha^W[f]\, y^{-(\frac{m}{2}+j)+\alpha/2}.$$

Proof. For $\ell \geq k$, by the semigroup property and Lemma 2.23 we get that

$$\begin{aligned}\left|\frac{\partial_y^\ell W_y f(x)}{\rho(x)^m}\right| &= \left|\frac{C_\ell}{\rho(x)^m}\int_{\mathbb{R}^n}\partial_v^{\ell-k}W_v(x,z)|_{v=y/2}\partial_u^k W_u f(z)|_{u=y/2}dz\right| \\ &\leq \frac{C_\ell\|\partial_u^k W_u f|_{u=y/2}\|_\infty}{\rho(x)^m}\int_{\mathbb{R}^n}\frac{e^{-\frac{|x-z|^2}{cy}}}{y^{n/2+\ell-k}}\left(\frac{\rho(x)}{y^{1/2}}\right)^m dz \\ &\leq C_\ell S_\alpha^W[f]y^{-(\frac{m}{2}+\ell)+\alpha/2}, \quad x\in\mathbb{R}^n.\end{aligned}$$

If $j < k$, since the $y-$derivatives of $W_y f(x)$ tend to zero as $y \to \infty$, we integrate $\ell - j$ times the previous estimate and we get the result.

□

The following Proposition corresponds with the "only if " part of Theorem 2.20.

Proposition 2.31. *Let $\alpha > 0$ and f be a function satisfying the* **heat size condition** *for $\mathcal{L}$ and (2.4). Then $\rho(\cdot)^{-\alpha}f \in L^\infty(\mathbb{R}^n)$.*

Proof. By using Lemma 2.25 we have

$$\begin{aligned}|f(x)| &\leq \sup_{0<y<\rho(x)^2}|W_y f(x)| \\ &\leq \sup_{0<y<\rho(x)^2}|W_y f(x) - W_{\rho(x)^2}f(x)| + |W_{\rho(x)^2}f(x)| \\ &= I + II.\end{aligned}$$

We shall estimate I. Let $k = [\alpha/2]+1$. If α is not even, by Lemma 2.30 with $j = 1$ and $m = 2(k-1)$ we have that

$$\begin{aligned}I &\leq \rho(x)^{2(k-1)}\sup_{0<y<\rho(x)^2}\int_y^{\rho(x)^2}\left|\frac{\partial_z W_z f(x)}{\rho(x)^{2(k-1)}}\right|dz \leq CS_\alpha^W[f]\rho(x)^{2(k-1)}\sup_{0<y<\rho(x)^2}\int_y^{\rho(x)^2}z^{-k+\alpha/2}dz \\ &\leq CS_\alpha^W[f]\rho(x)^{2(k-1)}\sup_{0<y<\rho(x)^2}((\rho(x)^2)^{-(k-1)+\alpha/2} - y^{-(k-1)+\alpha/2}) \leq CS_\alpha^W[f]\rho(x)^\alpha.\end{aligned}$$

When α is even, we write

$$\begin{aligned}I &= \sup_{0<y<\rho(x)^2}\left|\int_y^{\rho(x)^2}\partial_z W_z f(x)dz\right| \\ &= \sup_{0<y<\rho(x)^2}\left|\int_y^{\rho(x)^2}\left(-\int_z^{\rho(x)^2}\partial_u^2 W_u f(x)du + \partial_v W_v f(x)|_{v=\rho(x)^2}\right)dz\right|.\end{aligned}$$

By Lemma 2.30 with $j = 2$ and $m = 2(k-2)$, since $k = \alpha/2 + 1$, we get

$$\begin{aligned}
\Big|\int_y^{\rho(x)^2}\int_z^{\rho(x)^2} \partial_u^2 W_u f(x)\,dudz\Big| &= \rho(x)^{2(k-2)}\left|\int_y^{\rho(x)^2}\int_z^{\rho(x)^2} \frac{\partial_u^2 W_u f(x)}{\rho(x)^{2(k-2)}}dudz\right| \\
&\leq C S_\alpha^W[f]\rho(x)^{\alpha-2}\int_y^{\rho(x)^2}\int_z^{\rho(x)^2} u^{-1}dudz = CS_\alpha^W[f]\rho(x)^{\alpha-2}\int_y^{\rho(x)^2}(\log(\rho(x)^2) - \log z)dz \\
&= CS_\alpha^W[f]\rho(x)^{\alpha-2}\big[\log(\rho(x)^2)(\rho(x)^2 - y) - (\rho(x)^2\log(\rho(x)^2) - \rho(x)^2 - y\log y + y)\big] \\
&= CS_\alpha^W[f]\rho(x)^{\alpha-2}\big[y\log\big(\frac{y}{\rho(x)^2}\big) + \rho(x)^2 - y\big] \leq CS_\alpha^W[f]\rho(x)^\alpha.
\end{aligned}$$

For the second summand of I, Lemma 2.30, with $j = 1$ and $m = 2(k-1)$ applies, so

$$\begin{aligned}
\sup_{0<y<\rho(x)^2}(\rho(x)^2 - y)|\partial_v W_v f(x)|_{v=\rho(x)^2}| &= \sup_{0<y<\rho(x)^2}(\rho(x)^2 - y)\rho(x)^{2(k-1)}\frac{|\partial_v W_v f(x)|_{v=\rho(x)^2}|}{\rho(x)^{2(k-1)}} \\
&\leq CS_\alpha^W[f]\sup_{0<y<\rho(x)^2}(\rho(x)^2 - y)\rho(x)^\alpha(\rho(x)^2)^{-1} \\
&\leq CS_\alpha^W[f]\rho(x)^\alpha.
\end{aligned}$$

Regarding II, by using Lemma 2.30 with $j = 0$ and $m = 2k$ we have

$$II = |W_{\rho(x)^2}f(x)| = \left|\frac{W_{\rho(x)^2}f(x)}{\rho(x)^{2k}}\right|\rho(x)^{2k} \leq CS_\alpha^W[f](\rho(x)^2)^{-k+\alpha/2}\rho(x)^{2k} = CS_\alpha^W[f]\rho(x)^\alpha.$$

□

2.1.3 Some remarks about the classical Lipschitz spaces

In this subsection we define a class of Lipschitz spaces associated to Laplace operator. It will be an auxiliary class for our results about the spaces adapted to the Schrödinger operator. With respect to the classical definitions, see [81], [87], the main and crucial difference is that the functions don't need to be bounded.

Definition 2.32. *Let $\alpha > 0$. We define the spaces $\Lambda_{\alpha/2}^{\tilde{W}}$ as*

$$\Lambda_{\alpha/2}^{\tilde{W}} = \Big\{f:\ (1+|\cdot|)^{-\alpha}f \in L^\infty(\mathbb{R}^n)\ \text{and}\ \left\|\partial_y^k\tilde{W}_y f\right\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha y^{-k+\alpha/2},\ k = [\alpha/2]+1\Big\}.$$

Parallel to the linear spaces $\Lambda_{\alpha/2}^W$, we can endow this class with the norm

$$\|f\|_{\Lambda_{\alpha/2}^{\tilde{W}}} := \tilde{M}_\alpha[f] + \tilde{S}_\alpha[f],$$

with $\tilde{M}_\alpha[f] = \|(1+|\cdot|)^{-\alpha}f(\cdot)\|_\infty$ and $\tilde{S}_\alpha[f]$ being the infimum of the constants C_α appearing above.

Remark 2.33. *Let f be a function such that $\tilde{M}_\alpha[f] < \infty$. Then, for every $\ell \in \mathbb{N} \cup \{0\}$, $\partial_y^\ell \tilde{W}_y f$ is well defined. Observe that*

$$\int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{cy}}}{y^{n/2}} f(z)dz \le \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{cy}}}{y^{n/2}} (1+|z|)^\alpha dz.$$

If $|z| < 2|x|$, the last integral is convergent and bounded by $C(1 + y^{\alpha/2} + |x|^\alpha)$. If $|z| > 2|x|$ then the above integral is less than

$$\int_{\mathbb{R}^n} \frac{e^{-\frac{|z|^2}{cy}}}{y^{n/2}} (1+|z|)^\alpha dz \le C(1 + y^{\alpha/2}).$$

The same arguments can be used for the derivatives $\partial_y^\ell \tilde{W}_y f$, $\ell \in \mathbb{N}$.

Moreover, if $m/2 + \ell \ge [\alpha/2] + 1$, then $\lim_{y\to\infty} \partial_{x_i}^m \partial_y^\ell \tilde{W}_y f(x) = 0$, for every $x \in \mathbb{R}^n$. Indeed, observe that

$$\left|\partial_{x_i}^m \partial_y^\ell \tilde{W}_y f(x)\right| \le C \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{cy}} |f(z)|}{y^{n/2+m/2+\ell}} dz \le C \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{cy}}}{y^{n/2+m/2+\ell}} (1+|z|)^\alpha dz.$$

If $|z| < 2|x|$, the last integral is less than $C(1 + y^{\alpha/2} + |x|^\alpha) y^{-m/2-\ell}$. In the case $|z| > 2|x|$ the integral is less than $C(1 + y^{\alpha/2}) y^{-m/2-\ell}$.

The following Lemma is parallel to Lemma 2.25 and follows the ideas in [42]. We sketch the proof for completeness.

Lemma 2.34. *Let f be a measurable function such that there exists $y_0 > 0$ for which $\int_{\mathbb{R}^n} e^{-\frac{|x|^2}{y_0}} f(x)dx < \infty$. Then, $\lim_{y\to 0} \tilde{W}_y f(x) = f(x)$, a.e. $x \in \mathbb{R}^n$. Moreover, $\tilde{W}_y f(x)$ belongs to $C^\infty((0,\infty) \times \mathbb{R}^n)$.*

Proof. Since

$$\left|\int_{\mathbb{R}^n} \partial_y^\ell \tilde{W}_y(x,y) f(z) dz\right| \le \frac{C}{y^\ell} \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{cy}}}{y^{n/2}} |f(z)| dz,$$

we can derivate $\tilde{W}_y f$ with respect to y.

Let $x_0 \in \mathbb{R}^n$, $\varepsilon < |x_0|/10$. Consider the ball $B_\varepsilon(x_0) := \{|x - x_0| < \varepsilon\} \subset \{|x_0| - \varepsilon < |x| < |x_0| + \varepsilon\}$. Let $x \in B_\varepsilon(x_0)$. Observe that

$$\left|\int_{\mathbb{R}^n} \partial_{x_i} \tilde{W}_y(x-z) f(z) dz\right| \le \frac{C}{y^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} e^{-\frac{|x-z|^2}{cy}} |f(z)| dz, \; y > 0.$$

If $|z| > 2|x_0|$, then $|z| < |x-z| + |x| < |x-z| + |x_0| + \varepsilon < |x-z| + |z|/2 + \varepsilon$. In addition, as $\varepsilon < |x_0|/10 < |z|/20$, we have that

$$\chi_{|z|>2|x_0|} e^{-\frac{|x-z|^2}{cy}} \le \chi_{|z|>2|x_0|} e^{-\frac{(\frac{|z|}{2}-\varepsilon)^2}{cy}} \le C\chi_{|z|>2|x_0|} e^{-\frac{(\frac{|z|}{c})^2}{Cy}}.$$

If $|z| < 2|x_0|$ we have $\chi_{|z|<2|x_0|} e^{-\frac{|x-z|^2}{cy}} \le \chi_{|z|<2|x_0|}$. Hence,

$$e^{-\frac{|x-z|^2}{cy}} |f(z)| \le C\Big(\chi_{|z|>2|x_0|} e^{-\frac{(\frac{|z|}{c})^2}{Cy}} + \chi_{|z|<2|x_0|}\Big)|f(z)|,$$

so we can derivate $\tilde{W}_y f$ with respect to x_i, $i = 1, \dots, n$. □

Proposition 2.35. *Let $\alpha > 0$. A function $f \in \Lambda^{\tilde{W}}_{\alpha/2}$ if, and only if, for all $m \ge [\alpha/2] + 1$, we have $\|\partial_y^m \tilde{W}_y f\|_{L^\infty(\mathbb{R}^n)} \le C_m y^{-m+\alpha/2}$ and $\tilde{M}_\alpha[f] < \infty$.*

The proof of this Proposition is parallel to the proof of Proposition 2.26, we leave the details to the reader.

Lemma 2.36. *Let $\alpha > 0$ and $k = [\alpha/2] + 1$. If $f \in \Lambda^{\tilde{W}}_{\alpha/2}$, then for every $j, m \in \mathbb{N} \cup \{0\}$ such that $\frac{m}{2} + j \ge k$, there exists a $C_{m,j} > 0$ such that*

$$\left\|\partial_{x_i}^m \partial_y^j \tilde{W}_y f\right\|_\infty \le C_{m,j} \tilde{S}_\alpha[f]\, y^{-(m/2+j)+\alpha/2}, \text{ for every } i = 1 \dots, n.$$

Moreover, for each j, m, the constant $C_{m,j}$ is comparable to the constant C_α in Definition 2.32.

Proof. If $j \ge k$, by the semigroup property we get that

$$\begin{aligned}
\left|\partial_{x_i}^m \partial_y^j \tilde{W}_y f(x)\right| &= C \left|\int_{\mathbb{R}^n} \partial_{x_i}^m \partial_v^{j-k} \tilde{W}_v(x-z)|_{v=y/2} \partial_u^k \tilde{W}_u f(z)|_{u=y/2} dz\right| \\
&\le \frac{C_{m,j} \|\partial_u^k \tilde{W}_u f|_{u=y/2}\|_\infty}{y^{\frac{m}{2}+j-k}} \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{cy}}}{y^{n/2}} dz \\
&\le C_{m,j} \tilde{S}_\alpha[f]\, y^{-(\frac{m}{2}+j)+\alpha/2}, \ \ x \in \mathbb{R}^n.
\end{aligned}$$

If $j < k$, by proceeding as before we get that $\left|\partial_{x_i}^m \partial_y^k \tilde{W}_y f(x)\right| \le C\tilde{S}_\alpha[f] y^{-(\frac{m}{2}+k)+\alpha/2}$, $x \in \mathbb{R}^n$, and we get the result by integrating the previous estimate $k-j$ times, since $|\partial_{x_i}^m \partial_y^\ell \tilde{W}_y f(x)| \to 0$ as $y \to \infty$ as far as $\frac{m}{2} + \ell \ge k$, see Remark 2.33.

□

Theorem 2.37. *Let $0 < \alpha < 2$. Then $f \in \Lambda^{\tilde{W}}_{\alpha/2}$ if, and only if*

$$N_\alpha[f] := \sup_{|z|>0} \frac{\|f(\cdot+z) + f(\cdot-z) - 2f(\cdot)\|_\infty}{|z|^\alpha} < \infty \text{ and } \tilde{M}_\alpha[f] = \|(1+|\cdot|)^{-\alpha} f\|_\infty < \infty.$$

Moreover,

$$\|f\|_{\Lambda^{\tilde{W}}_{\alpha/2}} \sim N_\alpha[f] + \tilde{M}_\alpha[f].$$

Proof. Let $x \in \mathbb{R}^n$ and $f \in \Lambda^{\tilde{W}}_{\alpha/2}$. We can write, for every $y > 0$, $z \in \mathbb{R}^n$,

$$\begin{aligned}|f(x+z)+f(x-z)-2f(x)| &\leq |\tilde{W}_y f(x+z)-f(x+z)|+|\tilde{W}_y f(x-z)-f(x-z)|\\ &+2|\tilde{W}_y f(x)-f(x)|+|\tilde{W}_y f(x+z)-\tilde{W}_y f(x)+\tilde{W}_y f(x-z)-\tilde{W}_y f(x)|.\end{aligned}$$

By using Lemma 2.34 we have that

$$|\tilde{W}_y f(x)-f(x)| = \left|\int_0^y \partial_u \tilde{W}_u f(x)du\right| \leq C\tilde{S}_\alpha[f]\int_0^y u^{-1+\alpha/2}du = C\tilde{S}_\alpha[f]y^{\alpha/2}.$$

In a parallel way we handle the two first summands. Regarding the last sumand, by using the chain rule and Lemma 2.36 we have that

$$\begin{aligned}|\tilde{W}_y f(x+z)-\tilde{W}_y f(x)+\tilde{W}_y f(x-z)-\tilde{W}_y f(x)| &= \left|\int_0^1 \partial_\theta(\tilde{W}_y f(x+\theta z)+\tilde{W}_y f(x-\theta z))d\theta\right|\\ &= \left|\int_0^1 (\nabla_u \tilde{W}_y f(x+\theta z)_{|u=x+\theta z}\cdot z - \nabla_v \tilde{W}_y f(x-\theta z)_{|v=x-\theta z}\cdot z)d\theta\right|\\ &= \left|\int_0^1\int_{-1}^1 \partial_\lambda \nabla_u \tilde{W}_y f(x+\lambda\theta z)_{|u=x+\lambda\theta z}\cdot z\, d\lambda d\theta\right|\\ &= \left|\int_0^1\int_{-1}^1 \nabla_u^2 \tilde{W}_y f(x+\lambda\theta z)_{|u=x+\lambda\theta z}\cdot \theta|z|^2\, d\lambda d\theta\right|\\ &\leq C\tilde{S}_\alpha[f]\, y^{-1+\alpha/2}|z|^2,\end{aligned}$$

Thus, by choosing $y = |z|^2$ we get what we wanted.

For the converse, we assume that $N_\alpha[f], \tilde{M}_\alpha[f] < \infty$. Since
$\int_{\mathbb{R}^n} \partial_y \tilde{W}_y(z)f(x+z)dz = \int_{\mathbb{R}^n} \partial_y \tilde{W}_y(-z)f(x-z)dz = \int_{\mathbb{R}^n} \partial_y \tilde{W}_y(z)f(x-z)dz$,
and $\int_{\mathbb{R}^n} \partial_y \tilde{W}_y(z)dz = 0$ we have

$$\begin{aligned}|\partial_y \tilde{W}_y f(x)| &= \left|\frac{1}{2}\int_{\mathbb{R}^n} \partial_y \tilde{W}_y(z)(f(x-z)+f(x+z)-2f(x))dz\right|\\ &\leq CN_\alpha[f]\int_{\mathbb{R}^n} \frac{e^{-\frac{|z|^2}{cy}}|z|^\alpha}{y^{\frac{n}{2}+1}}dz \leq CN_\alpha[f]\, y^{-1+\alpha/2}.\end{aligned}$$

□

The following Proposition shows that in the case $0 < \alpha < 1$ we recover the classical Lipschitz condition.

Proposition 2.38. *Let $0 < \alpha < 1$. If a function $f \in \Lambda^{\tilde{W}}_{\alpha/2}$ then*

$$\sup_{|z|>0} \frac{\|f(\cdot - z) - f(\cdot)\|_\infty}{|z|^\alpha} < \infty.$$

Proof. We assume that $f \in \Lambda_{\alpha/2}^{\tilde{W}}$ with $\|f\|_{\Lambda_{\alpha/2}^{\tilde{W}}} = 1$. Let us take a representative of the function f. We want to show that $|f(x+z) - f(x)| \leq C|z|^\alpha$, $x, z \in \mathbb{R}^n$.

Fix $x \in \mathbb{R}^n$. Assume first that $|x| > 1$. In the case $|z| \geq |x|$, as $\tilde{M}_\alpha[f] < \infty$, we have that $|f(x+z) - f(x)| \leq C(1 + |x| + |z|)^\alpha \leq C|z|^\alpha$. In the case $|z| < |x|$, we choose a nonnegative integer k such that $|x| \leq |2^k z| < 2|x|$. We define

$$g(t) = f(x+t) - f(x), \quad t \in \mathbb{R}^n.$$

By hypothesis and Theorem 2.37,

$$|g(t) - 2g(t/2)| = |f(x+t) + f(x) - 2f(x+t/2)| \leq C|t|^\alpha.$$

Similarly, $|2^{j-1}g(t/2^{j-1}) - 2^j g(t/2^j)| \leq C2^{j-1}\Big(\frac{|t|}{2^{j-1}}\Big)^\alpha$. Therefore, adding up we have

$$|g(t) - 2^k g(t/2^k)| \leq C\sum_{j=1}^{k} 2^{j-1}\Big(\frac{|t|}{2^{j-1}}\Big)^\alpha.$$

Now we choose $t = 2^k z$. We have

$$\begin{aligned} |g(z)| &\leq \frac{|g(2^k z)|}{2^k} + C\frac{1}{2^k}\sum_{j=1}^{k} 2^{j-1}\Big(\frac{2^k|z|}{2^{j-1}}\Big)^\alpha \\ &\leq C|z|^\alpha 2^{k(\alpha-1)} + C2^{k(\alpha-1)}|z|^\alpha \sum_{j=0}^{k} 2^{j(1-\alpha)} \leq C|z|^\alpha. \end{aligned}$$

This implies that $|f(x+z) - f(x)| \leq C|z|^\alpha$.

If $|x| < 1 < |z|$ we can proceed as in the previous case $|x| < |z|$. If $|x| < 1$ and $|z| < 1$, we choose k such that $1 \leq |2^k z| < 2$. We observe that in this case $|g(2^k z)| \leq C$, therefore

$$\begin{aligned} |g(z)| &\leq C\frac{|g(2^k z)|}{2^k} + C\frac{1}{2^k}\sum_{j=0}^{k} 2^{j-1}\Big(\frac{2^k|z|}{2^{j-1}}\Big)^\alpha \\ &\leq C|z| + 2^{k(\alpha-1)}|z|^\alpha \sum_{j=0}^{k} 2^{j(1-\alpha)} \leq C|z|^\alpha. \end{aligned}$$

Observe that we have used in an essencial way that $0 < \alpha < 1$.

□

Remark 2.39. *Observe that Lemma 2.27 for $x = 0$, i.e.*

$$\rho(z) \leq C\rho(0)\left(1 + \frac{|z|}{\rho(0)}\right)^\lambda, \ \textit{for some } 0 < \lambda < 1, \tag{2.7}$$

implies that if $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$, then $(1 + |\cdot|)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$. Therefore, for $0 < \alpha < 1$, Theorem 2.37 and Proposition 2.38 imply that $C^\alpha_{\mathcal{L}}$ coincides with the space (2.2)*, introduced in [20].*

Proposition 2.40. *Let $1 < \alpha < 2$, $f \in \Lambda^{\tilde{W}}_{\alpha/2}$ and assume that for a certain ρ associated to a Schrödinger operator $\mathcal{L}$, we have $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$. Then, for every $i = 1, \dots, n$, $\partial_{x_i} f \in \Lambda^{\tilde{W}}_{\frac{\alpha-1}{2}}$ and $\rho(\cdot)^{-(\alpha-1)}\partial_{x_i} f \in L^\infty(\mathbb{R}^n)$. Moreover,*

$$\|\partial_{x_i} f\|_{\Lambda^{\tilde{W}}_{\frac{\alpha-1}{2}}} \leq C\Big(\tilde{S}_\alpha[f] + M^{\mathcal{L}}_\alpha[f]\Big).$$

Proof. We first prove that $\partial_{x_i} f$ exists. By Lemma 2.36 we have that $\|\partial_y \partial_{x_i} \tilde{W}_y f\|_\infty \leq C\, \tilde{S}_\alpha[f]\, y^{-3/2+\alpha/2} = C\, \tilde{S}_\alpha[f]\, y^{-1+\frac{\alpha-1}{2}}$. For every $x \in \mathbb{R}^n$ we can write

$$\partial_{x_i} \tilde{W}_y f(x) = -\int_y^1 \partial_u \partial_{x_i} \tilde{W}_u f(x)du + \partial_{x_i} \tilde{W}_y f(x)|_{y=1}.$$

Therefore, for every $0 < y_1 < y_2 < 1$ we have

$$\begin{aligned}|\partial_{x_i} \tilde{W}_{y_2} f(x) - \partial_{x_i} \tilde{W}_{y_1} f(x)| &= \left|\int_{y_1}^{y_2} \partial_u \partial_{x_i} \tilde{W}_u f(x)du\right| \\ &\leq C|y_2^{\frac{\alpha-1}{2}} - y_1^{\frac{\alpha-1}{2}}| \leq C|y_2 - y_1|^{\frac{\alpha-1}{2}}.\end{aligned}$$

This means that $\{\partial_{x_i} \tilde{W}_y f\}_{y>0}$ is a Cauchy sequence in the L^∞ norm (as $y \to 0$). In addition, as $\tilde{W}_y f \to f$ as $y \to 0$ we get that $\partial_{x_i} \tilde{W}_y f$ converges uniformly to $\partial_{x_i} f$.

On the other hand, since $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$, integration by parts and (2.7) give $\int_{\mathbb{R}^n} e^{-\frac{|z|^2}{y}} \partial_{z_i} f(z)dz = \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{y}} \frac{2z_i}{y} f(z)dz \leq C \int_{\mathbb{R}^n} \frac{e^{-\frac{|z|^2}{cy}}}{y^{1/2}} |f(z)|dz < \infty$, for every $y > 0$. Moreover, since $\tilde{W}_y f$ is a convolution, by Remark 2.33 we have $|\partial_y \tilde{W}_y(\partial_{x_i} f)(x)| = |\partial_y \partial_{x_i} \tilde{W}_y f(x)| \leq C\tilde{S}_\alpha[f]\, y^{-(3/2)+\alpha/2} = C\, \tilde{S}_\alpha[f]\, y^{-1+(\alpha-1)/2}$.

Let us see the size condition for the derivative. By proceeding as in the proof of Proposition 2.31, we have

$$\begin{aligned}\frac{|\partial_{x_i} f(x)|}{\rho(x)^{\alpha-1}} &\leq \frac{1}{\rho(x)^{\alpha-1}} \sup_{0<y<\rho(x)^2} |\tilde{W}_y(\partial_{x_i} f)(x)| \\ &\leq \frac{1}{\rho(x)^{\alpha-1}} \sup_{0<y<\rho(x)^2} |(\tilde{W}_y(\partial_{x_i} f)(x) - \tilde{W}_{\rho(x)^2}(\partial_{x_i} f)(x)| + \frac{1}{\rho(x)^{\alpha-1}} |\tilde{W}_{\rho(x)^2}(\partial_{x_i} f)(x)| \\ &= I + II.\end{aligned}$$

$$\begin{aligned}I &\leq \frac{1}{\rho(x)^{\alpha-1}} \sup_{0<y<\rho(x)^2} \int_y^{\rho(x)^2} |\partial_z \tilde{W}_z(\partial_{x_i} f)(x)|dz \leq C\frac{\tilde{S}_\alpha[f]}{\rho(x)^{\alpha-1}} \sup_{0<y<\rho(x)^2} \int_y^{\rho(x)^2} z^{-1+\frac{\alpha-1}{2}} dz \\ &\leq C\frac{\tilde{S}_\alpha[f]}{\rho(x)^{\alpha-1}} \sup_{0<y<\rho(x)^2} (\rho(x)^{\alpha-1} - y^{\frac{\alpha-1}{2}}) \leq C\tilde{S}_\alpha[f].\end{aligned}$$

On the other hand, integration by parts and Lemma 2.27 give

$$II = \frac{1}{\rho(x)^{\alpha-1}}\left|\int_{\mathbb{R}^n}\partial_{z_i}\tilde{W}_{\rho(x)^2}(x-z)f(z)dz\right| \le \frac{C\,M_\alpha^{\mathcal{L}}[f]}{\rho(x)^{\alpha-1}}\int_{\mathbb{R}^n}\frac{e^{-\frac{|x-z|^2}{c\rho(x)^2}}}{\rho(x)^{n+1}}\rho(z)^\alpha dz$$

$$\le \frac{C\,M_\alpha^{\mathcal{L}}[f]}{\rho(x)^{\alpha-1}}\Big[\int_{|x-z|<\rho(x)}\frac{e^{-\frac{|x-z|^2}{c\rho(x)^2}}}{\rho(x)^{n+1}}\rho(x)^\alpha dz$$

$$+\sum_{j=1}^{\infty}\int_{2^{j-1}\rho(x)<|x-z|<2^j\rho(x)}\frac{e^{-\frac{|x-z|^2}{c\rho(x)^2}}}{\rho(x)^{n+1}}\rho(x)^\alpha\left(1+\frac{|x-z|}{\rho(x)}\right)^{\lambda\alpha}dz\Big]$$

$$\le C\,M_\alpha^{\mathcal{L}}[f] + \frac{C\,M_\alpha^{\mathcal{L}}[f]}{\rho(x)^{\alpha-1}}\sum_{j=1}^{\infty}2^{-j(1-\lambda)\alpha}\int_{2^{j-1}\rho(x)<|x-z|<2^j\rho(x)}\frac{e^{-\frac{|x-z|^2}{c\rho(x)^2}}e^{-\frac{2^{2j}\rho(x)^2}{c\rho(x)^2}}}{\rho(x)^{n+1}}\rho(x)^\alpha(2^j)^\alpha dz$$

$$\le C\,M_\alpha^{\mathcal{L}}[f].$$

Finally, (2.7) allows us to conclude that $\tilde{M}_{\alpha-1}[\partial_{x_i}f]<\infty$ and hence $\partial_{x_i}f\in\Lambda^{\tilde{W}}_{\frac{\alpha-1}{2}}$.

□

2.1.4 Proof of the pointwise characterization.

The following result will be crucial along this subsection and can be found in [37], [79]. We say that a function ψ defined on $\mathbb{R}^n$ is *rapidly decaying* if, for every $N>0$, there exists a constant C_N such that

$$|\psi(x)|\le C_N(1+|x|)^{-N}.$$

Lemma 2.41. *Let ψ be a rapidly decaying nonnegative function and consider $\psi_y(x)=y^{-n/2}\psi(y^{-1/2}x)$. There exists a constant $C>0$ such that*

$$\int_{\mathbb{R}^n}V(z)\psi_y(x-z)dz\le C\frac{1}{y}\left(\frac{y^{1/2}}{\rho(x)}\right)^{2-\frac{n}{q}},\ \text{whenever } y\le\rho(x)^2.$$

Theorem 2.42. *Let $0<\alpha\le 2-n/q$, and a function f such that $\rho(\cdot)^{-\alpha}f(\cdot)\in L^\infty(\mathbb{R}^n)$. Then, $\|\partial_t\tilde{W}_tf-\partial_tW_tf\|_\infty\le C\,M_\alpha^{\mathcal{L}}[f]\,t^{-1+\alpha/2}$.*

Proof. The existence of the derivatives $\partial_t\tilde{W}_tf(x)$ and $\partial_tW_tf(x)$ follows from Lemma 2.28 and Remark 2.33. We analyze first the case $t\le\rho(x)^2$. As a consequence of Kato-Trotter formula,

$$\tilde{W}_t(x-y)-W_t(x,y)=\int_0^t\int_{\mathbb{R}^n}\tilde{W}_{t-s}(x-z)V(z)W_s(z,y)dzds,$$

see [37], we have the following identity:

$$\begin{aligned}\partial_t(\tilde{W}_tf-W_tf) &= \int_0^{t/2}\frac{\partial}{\partial t}\tilde{W}_{t-s}VW_sfds+\int_{t/2}^t\tilde{W}_{t-s}V\frac{\partial}{\partial s}W_sfds+\tilde{W}_{t/2}VW_{t/2}f\\ &= A+B+E. \qquad (2.8)\end{aligned}$$

On the one hand, we have

$$
\begin{aligned}
A &= \int_0^{t/2} \int_{\mathbb{R}^n} \frac{\partial}{\partial t} \tilde{W}_{t-s}(x-z) V(z) \int_{\mathbb{R}^n} W_s(z,y) f(y)\, dy\, dz\, ds \\
&= \int_0^{t/2} \int_{\mathbb{R}^n} \frac{\partial}{\partial t} \tilde{W}_{t-s}(x-z) V(z) \Big(\int_{|y-z|\le \rho(z)} + \sum_{j=1}^{\infty} \int_{2^{j-1}\rho(z) \le |y-z| \le 2^j \rho(z)} \Big) W_s(z,y) f(y)\, dy\, dz\, ds \\
&= A_0 + \sum_{j=1}^{\infty} A_j.
\end{aligned}
$$

By using (2.5), Lemma 2.27 and the fact $t - s \le t \le \rho(x)^2$, we have

$$
\begin{aligned}
|A_0| &\le C\, M^{\mathcal{L}}_{\alpha}[f] \int_0^{t/2} \frac{1}{t-s} \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{C(t-s)}}}{(t-s)^{n/2}} V(z) \int_{|y-z|\le \rho(z)} \frac{e^{-\frac{|z-y|^2}{cs}}}{s^{n/2}} \rho(z)^{\alpha} dy dz ds \\
&\le C\, M^{\mathcal{L}}_{\alpha}[f] \int_0^{t/2} \frac{1}{t-s} \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{C(t-s)}}}{(t-s)^{n/2}} V(z) \rho(x)^{\alpha} \Big(1 + \frac{|x-z|}{\rho(x)}\Big)^{\lambda\alpha} dz ds \\
&\le C\, M^{\mathcal{L}}_{\alpha}[f] \int_0^{t/2} \frac{1}{t-s} \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{C(t-s)}}}{(t-s)^{n/2}} e^{-\frac{|x-z|^2}{C\rho(x)^2}} V(z) \rho(x)^{\alpha} \Big(1 + \frac{|x-z|}{\rho(x)}\Big)^{\lambda\alpha} dz ds \\
&\le C\, M^{\mathcal{L}}_{\alpha}[f] \rho(x)^{\alpha} \int_0^{t/2} \frac{1}{t-s} \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{C(t-s)}}}{(t-s)^{n/2}} V(z) dz ds \\
&\le C\, M^{\mathcal{L}}_{\alpha}[f]\, \rho(x)^{\alpha} \int_0^{t/2} \frac{1}{(t-s)^2} \Big(\frac{\sqrt{t-s}}{\rho(x)}\Big)^{\alpha} ds \le C\, M^{\mathcal{L}}_{\alpha}[f]\, t^{-1+\alpha/2}.
\end{aligned}
$$

Observe that in the last two lines we have used Lemma 2.41 and the fact that $\alpha \le 2 - \frac{n}{q}$.

Now we shall deal with the summation of A_j. Observe that by using Lemma 2.27 with $N > \lambda\alpha$ and (2.5), we have

$$
\begin{aligned}
&\left| \int_{2^{j-1}\rho(z) \le |y-z| \le 2^j \rho(z)} W_s(z,y) f(y)\, dy \right| \\
&\quad \le C\, M^{\mathcal{L}}_{\alpha}[f] \int_{2^{j-1}\rho(z) \le |y-z| \le 2^j \rho(z)} \frac{e^{-\frac{|z-y|^2}{cs}}}{s^{n/2}} \Big(1 + \frac{\sqrt{s}}{\rho(z)} + \frac{\sqrt{s}}{\rho(y)}\Big)^{-N} \rho(y)^{\alpha} dy \\
&\quad \le C\, M^{\mathcal{L}}_{\alpha}[f] \int_{2^{j-1}\rho(z) \le |y-z| \le 2^j \rho(z)} \frac{e^{-\frac{|z-y|^2}{cs}}}{s^{n/2}} \Big(\frac{\sqrt{s}}{\rho(z)}\Big)^{-N} \rho(z)^{\alpha}\, 2^{j\lambda\alpha} dy \\
&\quad \le C\, M^{\mathcal{L}}_{\alpha}[f] \int_{2^{j-1}\rho(z) \le |y-z| \le 2^j \rho(z)} e^{-\frac{(2^j \rho(z))^2}{cs}} \Big(\frac{2^j \rho(z)}{\sqrt{s}}\Big)^{N} 2^{-jN} \frac{e^{-\frac{|z-y|^2}{cs}}}{s^{n/2}} \rho(z)^{\alpha}\, 2^{j\lambda\alpha} dy \\
&\quad \le C\, M^{\mathcal{L}}_{\alpha}[f]\, \rho(z)^{\alpha} 2^{-j(N-\lambda\alpha)}.
\end{aligned}
$$

The rest of the computation can be finished as in the case of A_0. Now we analyze B.

$$\begin{aligned}
B &= \int_{t/2}^{t}\int_{\mathbb{R}^n}\tilde{W}_{t-s}(x-z)V(z)\int_{\mathbb{R}^n}\frac{\partial}{\partial s}W_s(z,y)f(y)\,dydzds\\
&= \int_{t/2}^{t}\int_{\mathbb{R}^n}\tilde{W}_{t-s}(x-z)V(z)\Big(\int_{|y-z|\le\rho(z)}+\sum_{j=1}^{\infty}\int_{2^{j-1}\rho(z)\le|y-z|\le 2^j\rho(z)}\Big)\frac{\partial}{\partial s}W_s(z,y)f(y)\,dydzds\\
&= B_0+\sum_{j=1}^{\infty}B_j.
\end{aligned}$$

Analogously to A_0 we have

$$\begin{aligned}
|B_0| &\le C\,M_\alpha^{\mathcal{L}}[f]\int_{t/2}^{t}\int_{\mathbb{R}^n}\frac{e^{-\frac{|x-z|^2}{C(t-s)}}}{(t-s)^{n/2}}V(z)\int_{|y-z|\le\rho(z)}\frac{e^{-\frac{|z-y|^2}{cs}}}{s^{n/2+1}}\rho(z)^\alpha dydzds\\
&\le C\,M_\alpha^{\mathcal{L}}[f]\int_{t/2}^{t}\int_{\mathbb{R}^n}\frac{e^{-\frac{|x-z|^2}{C(t-s)}}}{(t-s)^{n/2+1}}V(z)\rho(x)^\alpha\Big(1+\frac{|x-z|}{\rho(x)}\Big)^{\lambda\alpha}dzds.
\end{aligned}$$

We can continue as in the case of A_0. B_j is parallel to the case A_j with the obvious changes. Finally we shall analyze E.

$$\begin{aligned}
E &= C\int_{\mathbb{R}^n}\tilde{W}_{t/2}(x-z)V(z)\int_{\mathbb{R}^n}W_{t/2}(z,y)f(y)\,dydz\\
&= C\int_{\mathbb{R}^n}\tilde{W}_{t/2}(x-z)V(z)\Big(\int_{|y-z|\le\rho(z)}+\sum_{j=1}^{\infty}\int_{2^{j-1}\rho(z)\le|y-z|\le 2^j\rho(z)}\Big)W_{t/2}(z,y)f(y)\,dy\,dz\\
&= E_0+\sum_j E_j.
\end{aligned}$$

Regarding to E_0 we use (2.5), Lemma 2.27 and Lemma 2.41 to get

$$\begin{aligned}
|E_0| &\le C\,M_\alpha^{\mathcal{L}}[f]\int_{\mathbb{R}^n}\frac{e^{-\frac{|x-z|^2}{4t}}}{t^{n/2}}V(z)\int_{|y-z|\le\rho(z)}\frac{e^{-\frac{|z-y|^2}{ct}}}{t^{n/2}}\rho(z)^\alpha dydz\\
&\le C\,M_\alpha^{\mathcal{L}}[f]\int_{\mathbb{R}^n}\frac{e^{-\frac{|x-z|^2}{4t}}}{t^{n/2}}V(z)\rho(x)^\alpha\Big(1+\frac{|x-z|}{\rho(x)}\Big)^{\lambda\alpha}dydz\\
&\le C\,M_\alpha^{\mathcal{L}}[f]\int_{\mathbb{R}^n}\frac{e^{-\frac{|x-z|^2}{8t}}}{t^{n/2}}e^{-\frac{|x-z|^2}{8\rho(x)^2}}V(z)\rho(x)^\alpha\Big(1+\frac{|x-z|}{\rho(x)}\Big)^{\lambda\alpha}dz\\
&\le C\,M_\alpha^{\mathcal{L}}[f]\,\rho(x)^\alpha\int_{\mathbb{R}^n}\frac{e^{-\frac{|x-z|^2}{8t}}}{t^{n/2}}V(z)dz\le C\,M_\alpha^{\mathcal{L}}[f]\,\rho(x)^\alpha\frac{1}{t}\Big(\frac{\sqrt{t}}{\rho(x)}\Big)^\alpha\le Ct^{-1+\alpha/2}.
\end{aligned}$$

E_j, $j=1,2,\dots$, are handled similarly to A_j and B_j with the obvious changes.

Now we consider the case $t \geq \rho(x)^2$. From Lemmas 2.23 and 2.27 we have

$$|\partial_t \tilde{W}_t f(x) - \partial_t W_t f(x)| \leq 2C \int_{\mathbb{R}^n} \frac{1}{t} \frac{e^{-\frac{|x-y|^2}{ct}}}{t^{n/2}} |f(y)| dy \leq C\, M_\alpha^{\mathcal{L}}[f] \int_{\mathbb{R}^n} \frac{1}{t} \frac{e^{-\frac{|x-y|^2}{ct}}}{t^{n/2}} \rho(y)^\alpha dy$$

$$= C\, M_\alpha^{\mathcal{L}}[f]\, \frac{1}{t} \int_{|x-y|\leq \rho(x)} \frac{e^{-\frac{|x-y|^2}{ct}}}{t^{n/2}} \rho(x)^\alpha dy$$

$$+ C\, M_\alpha^{\mathcal{L}}[f]\, \frac{1}{t} \sum_j \int_{2^{j-1}\rho(x)\leq|x-y|\leq 2^j\rho(x)} \frac{e^{-\frac{|x-y|^2}{ct}}}{t^{n/2}} \rho(y)^\alpha dy$$

$$\leq C\, M_\alpha^{\mathcal{L}}[f]\, \frac{1}{t} \rho(x)^\alpha$$

$$+ C \frac{M_\alpha^{\mathcal{L}}[f]}{t} \sum_j \int_{2^{j-1}\rho(x)\leq|x-y|\leq 2^j\rho(x)} \frac{e^{-\frac{|x-y|^2}{ct}}}{t^{n/2}} e^{-\frac{(2^j\rho(x))^2}{ct}} \rho(x)^\alpha (2^j)^{\lambda\alpha} dy$$

$$\leq C\, M_\alpha^{\mathcal{L}}[f]\, t^{-1+\alpha/2}$$

$$+ C\, M_\alpha^{\mathcal{L}}[f]\, \frac{1}{t} t^{\alpha/2} \sum_j 2^{j(\lambda-1)\alpha} \int_{2^{j-1}\rho(x)\leq|x-y|\leq 2^j\rho(x)} \frac{e^{-\frac{|x-y|^2}{ct}}}{t^{n/2}} dy \leq C\, M_\alpha^{\mathcal{L}}[f]\, t^{-1+\alpha/2}.$$

□

As a consequence of the previous results we have the following Theorem.

Theorem 2.43. *For $0 < \alpha \leq 2 - n/q$, a measurable function $f \in \Lambda_{\alpha/2}^W$ if, and only if, $f \in \Lambda_{\alpha/2}^{\tilde{W}}$ and $\rho(\cdot)^{-\alpha} f(\cdot) \in L^\infty(\mathbb{R}^n)$.*

Proof of Theorem 2.22. Theorem 2.43 together with Theorem 2.37 give the proof of Theorem 2.22.

□

2.2 Schrödinger Lipschitz spaces defined through the Poisson semigroup.

There are some important differences when we want to define Lipschitz spaces through the Poisson semigroup. The Poisson semigroup can be defined by the following subordination formula

$$P_y f(x) = e^{-y\sqrt{\mathcal{L}}} f(x) = \frac{y}{2\sqrt{\pi}} \int_0^\infty e^{-\frac{y^2}{4\tau}} e^{-\tau\mathcal{L}} f(x) \frac{d\tau}{\tau^{3/2}}. \tag{2.9}$$

Getting inside the Feynman-Kac estimate of the heat kernel we get that the kernel of the Poisson semigroup, $P_y(x,y)$ satisfies

$$P_y(x,z) \leq C \frac{y}{(|x-z|+y)^{n+1}}.$$

Hence, parallel to the heat semigroup case, we shall say that a function f satisfies a **Poisson size condition for $\mathcal{L}$** if

$$M^P[f] := \int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}}\,dx < \infty. \tag{2.10}$$

On Subsection 2.2.1 we will prove the following.

Theorem 2.44. *Let $\alpha > 0$ and f be a function such that*

$$\left\|\partial_y^k P_y f\right\|_{L^\infty(\mathbb{R}^n)} \le C_\alpha y^{-k+\alpha}, \quad \text{with}\, k = [\alpha]+1, y > 0. \tag{2.11}$$

If f satisfies a **Poisson size condition for $\mathcal{L}$** *then $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$.*

Remark 2.45. *Observe that $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$ implies the* **Poisson size condition for $\mathcal{L}$,** *if $0 < \alpha < 1$ (see (2.7)) or $\rho \in L^\infty(\mathbb{R}^n)$.*

The previous Theorem drives us to the following definition.

Definition 2.46. *Let f be a function that satisfies $M^P[f] < \infty$. Given $\alpha > 0$, we shall say that f belongs to the class Λ_α^P if it satisfies (2.11). The linear space can be endowed with the norm*

$$\|f\|_{\Lambda_\alpha^P} := S_\alpha^P[f] + M_\alpha^{\mathcal{L}}[f], \tag{2.12}$$

where $S_\alpha^P[f]$ is the infimum of the constants C_α appearing in (2.11).

The main result of this section is the following.

Theorem 2.47. *Let f be a function with $M^P[f] < \infty$. For $0 < \alpha \le 2 - n/q$, the following statements are equivalent:*

$$f \in C_{\mathcal{L}}^\alpha, \quad f \in \Lambda_\alpha^P, \quad f \in \Lambda_{\alpha/2}^W.$$

Moreover, the norms are equivalent.

We need to make some remarks about the theorem above.

Since the converse of Theorem 2.44 it is not true in general, we have to assume the hypothesis $M^P[f] < \infty$ in Theorem 2.47. In the case of the Hermite operator, since $\rho(x) = \frac{1}{1+|x|} \in L^\infty(\mathbb{R}^n)$, that hypothesis is not necessary (see Remark 2.45) and the result holds for $0 < \alpha < 2$. A complete characterization of Hermite Hölder spaces by using the Poisson semigroup will be given in Chapter 3.

As we said before, in [59], the authors proved a characterization of the class Λ_α^P in the case $0 < \alpha < 1$ for functions satisfying the integrability condition $\displaystyle\int_{\mathbb{R}^n} \frac{|f(z)|}{(|z|+1)^{n+\alpha+\varepsilon}} dz < \infty$. Our result contains their case; even more, the class of functions for which our results apply is bigger. Moreover, we can extend the characterization beyond 1. We will prove Theorem 2.47 in subsection 2.2.2.

In [20], the authors proved that, in the case $0 < \alpha < 1$, the space $C^{\alpha}_{\mathcal{L}}$ is isometric to the space $BMO^{\alpha}_{\mathcal{L}}$ defined as the set of locally integrable functions such that, for every ball $B = B(x, R), R > 0$

$$\begin{aligned} \int_B |f - f_B| &\leq C|B|^{1+\alpha/n}, \text{ with } f_B = \frac{1}{|B|}\int_B f \\ \text{and } \int_B |f| &\leq C|B|^{1+\alpha/n}, \text{ if } R \geq \rho(x). \end{aligned}$$

Hence, our Theorems 2.22 and 2.47 can be viewed as a sort of Carleson condition characterizations of the space $BMO^{\alpha}_{\mathcal{L}}$. In the case of the Poisson semigroup, a complete Carleson characterization has been given in [59] for a more restricted class of functions.

2.2.1 Some results about the spaces.

Before proving Theorem 2.47 we need to prove some properties about the space Λ^P_α. The following result was proved in [59].

Lemma 2.48. *Given $k \in \mathbb{N}$, for any $N > 0$ there exists a constant $C = C_{N,k}$ such that*

(a) $|P_y(x,z)| \leq C \dfrac{y}{(|x-z|^2 + y^2)^{\frac{n+1}{2}}} \left(1 + \dfrac{(|x-z|^2+y^2)^{1/2}}{\rho(x)} + \dfrac{(|x-z|^2+y^2)^{1/2}}{\rho(z)}\right)^{-N};$

(b) $|\partial_y^k P_y(x,z)| \leq C \dfrac{1}{(|x-z|^2 + y^2)^{\frac{n+k}{2}}} \left(1 + \dfrac{(|x-z|^2+y^2)^{1/2}}{\rho(x)} + \dfrac{(|x-z|^2+y^2)^{1/2}}{\rho(z)}\right)^{-N}.$

As a consequence, we have the following Proposition.

Proposition 2.49. *Let f be a function such that $M^P[f] = \int_{\mathbb{R}^n} \frac{|f(x)|}{(1+|x|)^{n+1}}dx < \infty$. Then, $\lim_{y\to\infty} \partial_y^{\ell} P_y f(x) = 0$, for every $\ell \in \mathbb{N}\cup\{0\}$, $x \in \mathbb{R}^n$, and $\lim_{y\to 0} P_y f(x) = f(x)$, a.e. $x \in \mathbb{R}^n$.*

Proof. The convergence to 0 of the Poisson semigroup and its derivatives follows directly from the previous Lemma. It remains to prove that $\lim_{y\to 0} P_y f(x) = f(x)$, a.e. $x \in \mathbb{R}^n$.

By Lemma 2.48 we have that, for $y < 1$,

$$\begin{aligned} \int_{|x-z|>2|x|} |P_y(x,z)f(z)|dz &\leq \int_{|x-z|>2|x|} \frac{y}{(|x-z|+y)^{n+1}}|f(z)|dz \qquad (2.13) \\ &\leq C\int_{|z|<1} \frac{y}{(2|x|+y)^{n+1}}|f(z)|dz + C\int_{|z|>1} \frac{y}{(\frac{2}{3}|z|+y)^{n+1}}|f(z)|dz \\ &\leq \frac{Cy}{|x|^{n+1}}\int_{|z|<1}|f(z)|dz + Cy\int_{|z|>1}\frac{1}{(|z|+1)^{n+1}}|f(z)|dz \to 0, \text{ as } y \to 0. \end{aligned}$$

To manipulate the other integral, we proceed as in the proof of Lemma 2.25. We compare the Poisson kernel with the kernel of the classical Poisson semigroup, $e^{-y\sqrt{-\Delta}}$, that we will denote by $\tilde{P}_y$.

By using (2.6) we have that

$$\begin{aligned}
&\left|\int_{|x-z|<2|x|} (P_y(x,z) - \tilde{P}_y(x-z))f(z)dz\right| \\
&\le C\int_0^\infty ye^{-\frac{y^2}{4\tau}} \int_{|x-z|<2|x|} |W_\tau(x,z) - \tilde{W}_\tau(x-z)||f(z)|dz\frac{d\tau}{\tau^{3/2}} \\
&\le C\int_0^{\rho(x)^2} ye^{-\frac{y^2}{4\tau}} \int_{|x-z|<2|x|} \left(\frac{\sqrt{\tau}}{\rho(x)}\right)^{2-n/q} w_\tau(x-z)|f(z)|dz\frac{d\tau}{\tau^{3/2}} \\
&\quad + Cy\int_{\rho(x)^2}^\infty \frac{d\tau}{\tau^{3/2}} \int_{|x-z|<2|x|} |f(z)|dz \\
&\le \frac{C}{\rho(x)^\epsilon}\int_0^{\rho(x)^2} \frac{y}{\tau^{1/2}} e^{-\frac{y^2}{4\tau}} (\sqrt{\tau})^\epsilon \frac{d\tau}{\tau} + \frac{Cy}{\rho(x)} \\
&\le \frac{Cy^\epsilon}{\rho(x)^\epsilon} + \frac{Cy}{\rho(x)} \to 0, \text{ as } y \to 0,
\end{aligned}$$

where $0 < \epsilon < 1$.

Finally, by the pointwise convergence of the classical Poisson semigroup to L^1 functions, we deduce the result. □

Parallel to the heat semigroup case, in order to prove Theorem 2.44, we need this previous Lemma.

Lemma 2.50. *Let $\alpha > 0$ and $k = [\alpha]+1$. Assume that $f \in \Lambda_\alpha^P$, then for every $j, m \in \mathbb{N}\cup\{0\}$ such that $m + j \ge k$, there exists a $C_{m,j} > 0$ such that*

$$\left\|\frac{\partial_y^j P_y f}{\rho(\cdot)^m}\right\|_\infty \le C_m S_\alpha^P[f] y^{-(m+j)+\alpha}.$$

Proof. For $\ell > k$, by the semigroup property and Lemma 2.48 we get that

$$\begin{aligned}
\left|\frac{\partial_y^\ell P_y f(x)}{\rho(x)^m}\right| &= \left|\frac{C_\ell}{\rho(x)^m}\int_{\mathbb{R}^n} \partial_v^{\ell-k} P_v(x,z)|_{v=y/2}\partial_u^k P_u f(z)|_{u=y/2}dz\right| \\
&\le \frac{C_\ell \|\partial_u^k P_u f|_{u=y/2}\|_\infty}{\rho(x)^m} \int_{\mathbb{R}^n} \frac{1}{(|x-z|^2+y^2)^{\frac{n+\ell-k}{2}}} \left(\frac{\rho(x)}{y}\right)^m dz \\
&\le C_\ell S_\alpha^P[f] y^{-(m+\ell)+\alpha}, \quad x \in \mathbb{R}^n.
\end{aligned}$$

If $j \le k$, since the $y-$derivatives of $P_y f(x)$ tend to zero as $y \to \infty$, we integrate $\ell - j$ times the previous estimate and we get the result.

□

Proof of Theorem 2.44.

By using Proposition 2.49 we have

$$\begin{aligned} |f(x)| &\leq \sup_{0<y<\rho(x)} |P_y f(x)| \\ &\leq \sup_{0<y<\rho(x)} |P_y f(x) - P_{\rho(x)} f(x)| + |P_{\rho(x)} f(x)| \\ &= I + II. \end{aligned}$$

Let $k = [\alpha] + 1$. By using Lemma 2.50 with $j = 0$ and $m = k$ we have

$$II = |P_{\rho(x)} f(x)| = \left| \frac{P_{\rho(x)} f(x)}{\rho(x)^k} \right| \rho(x)^k \leq C S_\alpha^P [f] (\rho(x))^{-k+\alpha} \rho(x)^k = C\, S_\alpha^P [f] \rho(x)^\alpha.$$

Now we shall estimate I. If α is not integer, by Lemma 2.50 with $j = 1$ and $m = k - 1$ we have that

$$\begin{aligned} I &\leq \rho(x)^{k-1} \sup_{0<y<\rho(x)} \int_y^{\rho(x)} \left| \frac{\partial_z P_z f(x)}{\rho(x)^{k-1}} \right| dz \leq C\, S_\alpha^P [f] \rho(x)^{k-1} \sup_{0<y<\rho(x)} \int_y^{\rho(x)} z^{-k+\alpha} dz \\ &\leq C\, S_\alpha^P [f] \rho(x)^{k-1} \sup_{0<y<\rho(x)} ((\rho(x))^{-(k-1)+\alpha} - y^{-(k-1)+\alpha}) \leq C\, S_\alpha^P [f] \rho(x)^\alpha. \end{aligned}$$

When α is an integer, we write

$$\begin{aligned} I &= \sup_{0<y<\rho(x)} \left| \int_y^{\rho(x)} \partial_z P_z f(x) dz \right| \\ &= \sup_{0<y<\rho(x)} \left| \int_y^{\rho(x)} \left(-\int_z^{\rho(x)} \partial_u^2 P_u f(x) du + \partial_v P_v f(x)|_{v=\rho(x)} \right) dz \right|. \end{aligned}$$

By Lemma 2.50 with $j = 2$ and $m = k - 2$, since $k = \alpha + 1$, we get

$$\begin{aligned} &\Big| \int_y^{\rho(x)} \int_z^{\rho(x)} \partial_u^2 P_u f(x) du dz \Big| = \rho(x)^{k-2} \left| \int_y^{\rho(x)} \int_z^{\rho(x)} \frac{\partial_u^2 P_u f(x)}{\rho(x)^{k-2}} du dz \right| \\ &\leq C\, S_\alpha^P [f] \rho(x)^{\alpha-1} \int_y^{\rho(x)} \int_z^{\rho(x)} u^{-1} du dz = C\, S_\alpha^P [f] \rho(x)^{\alpha-1} \int_y^{\rho(x)} (\log(\rho(x)) - \log z) dz \\ &= C\, S_\alpha^P [f] \rho(x)^{\alpha-1} \big[\log(\rho(x))(\rho(x) - y) - (\rho(x) \log(\rho(x)) - \rho(x) - y \log y + y) \big] \\ &= C\, S_\alpha^P [f] \rho(x)^{\alpha-1} \big[y \log \big(\frac{y}{\rho(x)} \big) + \rho(x) - y \big] \leq C\, S_\alpha^P [f] \rho(x)^\alpha. \end{aligned}$$

For the second summand of I, Lemma 2.50, with $j = 1$ and $m = k - 1$ applies, so

$$\begin{aligned} \sup_{0<y<\rho(x)} (\rho(x) - y) |\partial_v P_v f(x)|_{v=\rho(x)}| &= \sup_{0<y<\rho(x)} (\rho(x) - y) \rho(x)^{k-1} \frac{|\partial_v P_v f(x)|_{v=\rho(x)}|}{\rho(x)^{k-1}} \\ &\leq C\, S_\alpha^P [f] \sup_{0<y<\rho(x)} (\rho(x) - y) \rho(x)^\alpha (\rho(x))^{-1} \leq C\, S_\alpha^P [f] \rho(x)^\alpha. \end{aligned}$$

□

2.2.2 Proof of the pointwise characterization. Equivalence between the spaces defined via the heat and Poisson semigroups.

To prove Theorem 2.47, we need to define an auxiliary class of Lipschitz functions by means of the classical Poisson semigroup, $\tilde{P}_y = e^{-y\sqrt{-\Delta}}$. Again, the crucial difference between this class and the one defined by Stein in [81] is that the functions don't need to be bounded.

We define $\Lambda_\alpha^{\tilde{P}}$ as the collection of functions satisfying $M^P[f] < \infty$ and

$$\left\| \partial_y^k \tilde{P}_y f \right\|_{L^\infty(\mathbb{R}^n)} \leq C_\alpha y^{-k+\alpha}, \quad \text{with}\, k = [\alpha]+1, y > 0. \tag{2.14}$$

We denote by $S_\alpha^{\tilde{P}}[f]$ as the infimum of the constants C_α above.

Remark 2.51. *Observe that the space $\Lambda_\alpha^{\tilde{P}}$ is well defined, because if f is a function such that $M^P[f] < \infty$,then*

(i) $|\partial_{x_i}^m \partial_y^\ell \tilde{P}_y f(x)| \to 0$ *as* $y \to \infty$ *as far as* $m+\ell \geq k \geq 1$*. Indeed,*

$$\begin{aligned} |\partial_{x_i}^m \partial_y^\ell \tilde{P}_y f(x)| &\leq C \int_{|x-z|<|x|} \frac{|f(z)|}{(|x-z|+y)^{n+k}} dz + C \int_{|x-z|>|x|} \frac{|f(z)|}{(|z|+y)^{n+k}} dz \\ &\leq C \frac{1}{y^{n+k}} \int_{|x-z|<|x|} |f(z)| dz + C \int_{|x-z|>|x|} \frac{|f(z)|}{(|z|+y)^{n+k}} dz. \end{aligned}$$

Both summands tend to cero, the second one by dominated convergence.

(ii) $\lim_{y\to 0} \tilde{P}_y f(x) = f(x)$ *a.e.* $x \in \mathbb{R}^n$*. This can be proved as we did in* (2.13) *and by using the a.e. convergence of the classical Poisson semigroup for* L^1 *functions.*

Moreover, we can prove the following results analogously as we did for the heat semigroup.

Proposition 2.52. *Let* $\alpha > 0$, $k = [\alpha]+1$ *and* f *be a function satisfying* $M^P[f] < \infty$*. Then,* $\|\partial_y^k \tilde{P}_y f\|_{L^\infty(\mathbb{R}^n)} \leq C_k y^{-k+\alpha}$ *if, and only if, for* $m \geq k$, $\|\partial_y^m \tilde{P}_y f\|_{L^\infty(\mathbb{R}^n)} \leq C_m y^{-m+\alpha}$.

The following Lemma is parallel to Lemma 2.36. We leave the details of the proof to the interested reader.

Lemma 2.53. *Let* $\alpha > 0$ *and* $k = [\alpha]+1$*. If* $f \in \Lambda_\alpha^{\tilde{P}}$*, then for every* $j, m \in \mathbb{N} \cup \{0\}$ *such that* $m+j \geq k$*, there exists a* $C_{m,j} > 0$ *such that*

$$\left\| \partial_{x_i}^m \partial_y^j \tilde{P}_y f \right\|_\infty \leq C\, S_\alpha^{\tilde{P}}[f] y^{-(m+j)+\alpha}, \text{ for every } i = 1\ldots, n.$$

Theorem 2.54. *Let* $0 < \alpha < 2$*. Then* $f \in \Lambda_\alpha^{\tilde{P}}$*, if and only* $M^P[f] < \infty$ *and*

$$N_\alpha[f] = \sup_{|z|>0} \frac{\|f(\cdot + z) + f(\cdot - z) - 2f(\cdot)\|_\infty}{|z|^\alpha} < \infty.$$

Proof. Let $x \in \mathbb{R}^n$. We can write, for every $y > 0$, $z \in \mathbb{R}^n$,

$$\begin{aligned}|f(x+z)+f(x-z)-2f(x)| &\leq |\tilde{P}_y f(x+z) - f(x+z) + \tilde{P}_y f(x-z) - f(x-z) \\ &\quad + 2(\tilde{P}_y f(x) - f(x))| + |\tilde{P}_y f(x+z) - \tilde{P}_y f(x) + \tilde{P}_y f(x-z) - \tilde{P}_y f(x)| \\ &= A + B.\end{aligned}$$

By using Lemma 2.53 we can proceed as in the proof of Theorem 2.37. We have

$$B = |\tilde{P}_y f(x+z) - \tilde{P}_y f(x) + \tilde{P}_y f(x-z) - \tilde{P}_y f(x)| \leq C\, S_\alpha^{\tilde{P}}[f] y^{-2+\alpha}|z|^2,$$

If $0 < \alpha < 1$, by using Remark 2.51 we have that

$$|\tilde{P}_y f(x) - f(x)| = \left|\int_0^y \partial_u \tilde{P}_u f(x) du\right| \leq C\, S_\alpha^{\tilde{P}}[f] \int_0^y u^{-1+\alpha} du = C\, S_\alpha^{\tilde{P}}[f] y^\alpha,$$

and the same for the other two summands of A.

If $1 < \alpha < 2$, by proceeding as in the proof of Theorem 2.37, by Lemma 2.53 we have that

$$\begin{aligned}A &= \left|\int_0^y (\partial_u \tilde{P}_u f(x+z) + \partial_u \tilde{P}_u f(x-z) - 2\partial_u \tilde{P}_u f(x)) du\right| \\ &= \left|\int_0^y \int_0^1 (\nabla_w \partial_u \tilde{P}_u f(x+\theta z)_{|w=x+\theta z} \cdot z - \nabla_v \partial_u \tilde{P}_u f(x-\theta z)_{|v=x-\theta z} \cdot z) d\theta du\right| \\ &\leq C\, S_\alpha^{\tilde{P}}[f] \int_0^y u^{-2+\alpha} du |z| \leq C\, S_\alpha^{\tilde{P}}[f] y^{-1+\alpha}|z|.\end{aligned}$$

Thus, by choosing $y = |z|$ in each case we get what we wanted.

For $\alpha = 1$, by using that $\partial_u \tilde{P}_u f(x) = -\int_u^y \partial_w^2 \tilde{P}_w f(x) dw + \partial_y \tilde{P}_y f(x)$, we have

$$\begin{aligned}|A| &\leq C \int_0^y \int_u^y w^{-1} dw du + \left|\int_0^y \Big((\partial_y \tilde{P}_y f(x+z) + \partial_y \tilde{P}_y f(x-z) - 2\partial_y \tilde{P}_y f(x))\Big) du\right| \\ &= A_1 + A_2.\end{aligned}$$

Observe that $A_1 \leq Cy$. Regarding A_2, we proceed as in the case $1 < \alpha < 2$ and we have

$$A_2 \leq \left| y \int_0^1 (\nabla_{\tilde{x}} \partial_y \tilde{P}_y f(x+\theta z)_{|\tilde{x}=x+\theta z} \cdot z - \nabla_{\tilde{x}} \partial_y \tilde{P}_y f(x-\theta z)_{|\tilde{z}=x-\theta z} \cdot z) d\theta \right| \leq C\, S_\alpha^{\tilde{P}}[f]\, |z|.$$

When $y = |z|$ we get what we wanted.

For the converse we proceed as in Theorem 2.37.

□

Theorem 2.55. *Let $0 < \alpha \leq 2 - n/q$ and f be a function such that $M^P[f] < \infty$. If $\rho(\cdot)^{-\alpha} f(\cdot) \in L^\infty(\mathbb{R}^n)$, then*

$$\|\partial_y^2 P_y f - \partial_y^2 \tilde{P}_y f\|_\infty \leq C M_\alpha^{\mathcal{L}}[f] y^{-2+\alpha}.$$

Proof. By subordination formula, integration by parts and and Theorem 2.43 we have that

$$
\begin{aligned}
|\partial_y^2 P_y f(x) - \partial_y^2 \tilde{P}_y f(x)| &= \left| \frac{1}{2\sqrt{\pi}} \int_0^\infty \partial_y^2 \left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \right) (W_\tau f - \tilde{W}_\tau f) d\tau \right| \\
&= \left| \frac{1}{2\sqrt{\pi}} \int_0^\infty \partial_\tau \left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \right) (W_\tau f - \tilde{W}_\tau f) d\tau \right| \\
&\le \frac{1}{2\sqrt{\pi}} \int_0^\infty \frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} |\partial_\tau (W_\tau f - \tilde{W}_\tau f)| d\tau \\
&\le C\, M_\alpha^{\mathcal{L}}[f] \int_0^\infty \frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \tau^{-1+\alpha/2} d\tau \\
&\le C\, M_\alpha^{\mathcal{L}}[f] \left(\frac{1}{y^2} \int_0^{y^2} \frac{y^3}{\tau^{3/2}} e^{-\frac{y^2}{4\tau}} \tau^{-1+\alpha/2} d\tau + \int_{y^2}^\infty \tau^{-1+\alpha/2} \frac{d\tau}{\tau} \right) \\
&\le C M_\alpha^{\mathcal{L}}[f] y^{-2+\alpha}.
\end{aligned}
$$

□

A consequence of the previous Theorem is the following.

Theorem 2.56. *Let $0 < \alpha \le 2 - n/q$ and f be a function such that $M^P[f] < \infty$ and $\rho(\cdot)^{-\alpha} f(\cdot) \in L^\infty(\mathbb{R}^n)$. Then, $f \in \Lambda_\alpha^P$ if and only if $f \in \Lambda_\alpha^{\tilde{P}}$.*

The last ingredient to prove Theorem 2.47 is the following result.

Theorem 2.57. *Let $\alpha > 0$ and f a function such that $M^P[f] < \infty$. If $f \in \Lambda_{\alpha/2}^W$, then $f \in \Lambda_\alpha^P$. Moreover, $S_\alpha^P[f] \le C S_\alpha^W[f]$.*

Proof. Let $k = [\alpha/2]+1$ and $f \in \Lambda_{\alpha/2}^W$, then $[\alpha]+1 = [\alpha/2+\alpha/2]+1 \le [\alpha/2]+[\alpha/2]+2 = 2k$. By Lemma 2.50, it is enough to prove that $\|\partial_y^{2k} P_y f\|_\infty \le C y^{-(2k)+\alpha}$.

Since $\partial_y^2 \left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \right) = \partial_\tau \left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \right)$, k-times integration by parts give

$$
\begin{aligned}
|\partial_y^{2k} P_y f(x)| &= \left| \frac{1}{2\sqrt{\pi}} \int_0^\infty \partial_y^{2k} \left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \right) e^{-\tau\mathcal{L}} f(x) d\tau \right| = \left| \frac{1}{2\sqrt{\pi}} \int_0^\infty \partial_\tau^k \left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \right) e^{-\tau\mathcal{L}} f(x) d\tau \right| \\
&= \frac{1}{2\sqrt{\pi}} \left| \int_0^\infty (-1)^k \left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \right) \partial_\tau^k e^{-\tau\mathcal{L}} f(x) d\tau \right| \le C\, S_\alpha^W[f] \int_0^\infty \frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \tau^{-k+\alpha/2} d\tau \\
&\le C\, S_\alpha^W[f] \left(\frac{1}{y^2} \int_0^{y^2} \frac{y^3}{\tau^{3/2}} e^{-\frac{y^2}{4\tau}} \tau^{-k+\alpha/2} d\tau + \int_{y^2}^\infty \tau^{-k+\alpha/2} \frac{d\tau}{\tau} \right) \\
&\le C\, S_\alpha^W[f] y^{-2k+\alpha}.
\end{aligned}
$$

□

Observe that this proof can be used to see that, in general, for $\mathcal{W}_y f$ and $\mathcal{P}_y f$ being the heat and Poisson semigroups related some differential operator, $\|\partial_y^k \mathcal{W}_y f\|_\infty \leq Cy^{-k+\alpha/2}$ implies $\|\partial_y^{2k} \mathcal{P}_y f\|_\infty \leq Cy^{-2k+\alpha}$.

Finally it is easy to see that Theorems 2.37, 2.57, 2.56 and 2.54 have as a consequence that Theorem 2.47 is true.

2.3 Regularity results.

Once we have studied deeply our classes of functions, or aim is to study the regularity of the following operators in the Lipschitz spaces. Their definitions are motivated by the gamma formulas, see [84].

- The *Bessel potential of order* $\beta > 0$,

$$(Id + \mathcal{L})^{-\beta/2} f(x) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty e^{-t} e^{-t\mathcal{L}} f(x) t^{\beta/2} \frac{dt}{t}.$$

- The *fractional integral* of order $\beta > 0$.

$$\mathcal{L}^{-\beta/2} f(x) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty e^{-t\mathcal{L}} f(x) t^{\beta/2} \frac{dt}{t}.$$

- The *fractional "Laplacian"* of order $\beta/2 > 0$

$$\mathcal{L}^{\beta/2} f(x) = \frac{1}{c_\beta} \int_0^\infty (e^{-t\mathcal{L}} - Id)^{[\beta/2]+1} f(x) \frac{dt}{t^{1+\beta/2}}.$$

- The first order Riesz transforms defined by

$$\mathcal{R}_i = \partial_{x_i}(\mathcal{L}^{-1/2}), \text{ and } R_i = \mathcal{L}^{-1/2}(\partial_{x_i}),\ i = 1, \dots, n.$$

We have the following results.

Theorem 2.58. *Let $\alpha, \beta > 0$ and $\mathcal{T}_\beta$ denote the Bessel potential or the fractional integral of order β. Then, $\mathcal{T}_\beta$ satisfies*

(i) $\|\mathcal{T}_\beta f\|_{\Lambda^W_{\frac{\alpha+\beta}{2}}} \leq C\|f\|_{\Lambda^W_{\alpha/2}}$.

(ii) $\|\mathcal{T}_\beta f\|_{\Lambda^W_{\beta/2}} \leq C\|f\|_\infty$.

In the case $\alpha + \beta < 1$ and for the classes $C^\alpha_{\mathcal{L}}$, statement (i) was obtained in [59] and (i) and (ii) were proved in [20].

To prove Theorem 2.58, we need the following Lemma.

Lemma 2.59. *Let $\beta > 0$ and $\mathcal{T}_\beta$ be either the operator $(Id + \mathcal{L})^{-\beta/2}$ or the operator $\mathcal{L}^{-\beta/2}$. If f is a function such that $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$ for some $\alpha > 0$, then $\mathcal{T}_\beta f(x)$ is well-defined and satisfies*

$$M^{\mathcal{L}}_{\alpha+\beta}[\mathcal{T}_\beta f] \leq C M^{\mathcal{L}}_\alpha [f].$$

Moreover if $f \in L^\infty(\mathbb{R}^n)$ then $\mathcal{T}_\beta f(x)$ is well defined and

$$M^{\mathcal{L}}_\beta [\mathcal{T}_\beta f] \leq C \|f\|_\infty.$$

Proof. If $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$ for some $\alpha > 0$, then by Lemma 2.28 we get

$$\begin{aligned} |(Id + \mathcal{L})^{-\beta/2} f(x)| &= \Big| \frac{1}{\Gamma(\beta/2)} \int_0^\infty e^{-t} e^{-t\mathcal{L}} f(x) t^{\beta/2} \frac{dt}{t} \Big| \\ &\leq C\, M^{\mathcal{L}}_\alpha [f] \int_0^{\rho(x)^2} \rho(x)^\alpha t^{\beta/2} \frac{dt}{t} + C\, M^{\mathcal{L}}_\alpha [f] \int_{\rho(x)^2}^\infty \rho(x)^\alpha \left(\frac{\rho(x)^2}{t} \right)^{\beta/2+1} t^{\beta/2} \frac{dt}{t} \\ &= C\, M^{\mathcal{L}}_\alpha [f]\, \rho(x)^{\alpha+\beta},\ x \in \mathbb{R}^n. \end{aligned}$$

The same estimate works for $\mathcal{L}^{-\beta/2} f$. The proof in the second case runs parallel, since Lemma 2.28 has an obvious version for bounded functions.

□

Proof of Theorem 2.58. We prove only (i), estimate (ii) can be proved analogously.

Let $f \in \Lambda^W_{\alpha/2}$. Lemma 2.28 with $\ell = 0$ together with Fubini's Theorem allow us to get $W_y((Id + \mathcal{L})^{-\beta/2} f)(x) = \dfrac{1}{\Gamma(\beta/2)} \displaystyle\int_0^\infty e^{-t} W_y(W_t f)(x) t^{\beta/2} \frac{dt}{t}$. Also observe that by the semigroup property and Lemma 2.30 with $j = 1$ and m such that $[\alpha/2 + \beta/2] + 1 \leq 1 + \frac{m}{2}$, we have

$$\begin{aligned} \int_0^\infty \left| e^{-t} \partial_y W_y(W_t f)(x) \right| t^{\beta/2} \frac{dt}{t} &= \int_0^\infty \left| e^{-t} \partial_w W_w f(x) \big|_{w=y+t} \right| t^{\beta/2} \frac{dt}{t} \\ &\leq C\, S_\alpha [f] \int_0^\infty e^{-t} \rho(x)^m (y+t)^{-(m/2+1)+\alpha/2} t^{\beta/2} \frac{dt}{t}. \end{aligned}$$

The last integral can be bounded by a uniform (in a neighborhood of y) integrable function (of t). This means that we can interchange the derivative with respect to y and the integral with respect to t in the above expression.

Let $\ell = [\alpha/2 + \beta/2] + 1$. By iterating the above arguments and using the hypothesis we have

$$\begin{aligned}|\partial_y^\ell W_y((Id+\mathcal{L})^{-\beta/2}f(x))| &= \left|\frac{1}{\Gamma(\beta/2)}\int_0^\infty e^{-t}\partial_y^\ell W_y(W_tf)(x)t^{\beta/2}\frac{dt}{t}\right| \\ &\le C\,S_\alpha[f]\int_0^\infty e^{-t}(\partial_w^\ell W_w f(x)\Big|_{w=y+t})t^{\beta/2}\frac{dt}{t} \\ &\le C\,S_\alpha[f]\int_0^\infty e^{-t}(y+t)^{-\ell+\alpha/2}t^{\beta/2}\frac{dt}{t} \\ &\overset{\frac{t}{y}=u}{\le} C\,S_\alpha[f]\,y^{\alpha/2+\beta/2-\ell}\int_0^\infty \frac{u^{\beta/2}e^{-yu}}{(1+u)^{\ell-\alpha/2}}\frac{du}{u} \\ &\le C\,S_\alpha[f]\,y^{\alpha/2+\beta/2-\ell}.\end{aligned}$$

When $f\in L^\infty(\mathbb{R}^n)$ we apply Lemma 2.23 and we get for $\ell=[\beta/2]+1$ that $|\partial_y^\ell W_y W_\nu f(x)|\le C\frac{\|f\|_\infty}{y^\ell}$. Then we can proceed as before.

Now by using Lemma 2.59 we end the proof of the theorem. □

Theorem 2.60 (Hölder estimates)**.** *Let $0<\beta<\alpha$ and $f\in\Lambda^W_{\alpha/2}$. Then,*

$$\|\mathcal{L}^{\beta/2}f\|_{\Lambda^W_{\frac{\alpha-\beta}{2}}}\le C\|f\|_{\Lambda^W_{\alpha/2}}.$$

For the classes $C^\alpha_{\mathcal{L}}$, $0<\alpha<1$, the result was obtained in [59].

To prove Theorem 2.60 we need the following lemma.

Lemma 2.61. *Let $0<\beta<\alpha$ and f be a function in the space $\Lambda^W_{\alpha/2}$. Then $\mathcal{L}^{\beta/2}f$ is well defined and*

$$M^{\mathcal{L}}_{\alpha-\beta}[\mathcal{L}^{\beta/2}f]\le C_{\alpha,\beta}\|f\|_{\Lambda^W_{\alpha/2}}.$$

Proof. We can write

$$|\mathcal{L}^{\beta/2}f(x)| = \left|\frac{1}{c_\beta}\left(\int_0^{\rho(x)^2}+\int_{\rho(x)^2}^\infty\right)(e^{-t\mathcal{L}}-Id)^{[\beta/2]+1}f(x)\,\frac{dt}{t^{1+\beta/2}}\right| = |I+II|.$$

As $\rho(\cdot)^{-\alpha}f\in L^\infty(\mathbb{R}^n)$, by Lemma 2.28 we have

$$|II|\le CM^{\mathcal{L}}_\alpha[f]\int_{\rho(x)^2}^\infty \rho(x)^\alpha\frac{dt}{t^{1+\beta/2}} = CM^{\mathcal{L}}_\alpha[f]\rho(x)^{\alpha-\beta}.$$

Now we shall estimate $|I|$. Let $\ell=[\beta/2]+1$, by the semigroup property we have $(e^{-t\mathcal{L}}-Id)^{[\beta/2]+1}f(x) = \int_0^t\underbrace{\dots}_{\ell}\int_0^t \partial_{y_1}\dots\partial_{y_\ell}W_{y_1+\dots+y_\ell}f(x)dy_1\dots dy_\ell$.

If $\beta/2<\alpha/2<\ell$, then $k:=[\alpha/2]+1=\ell$ and

$$|(e^{-t\mathcal{L}}-Id)^\ell f(x)|\le C\,S^W_\alpha[f]\int_0^t\underbrace{\dots}_{\ell}\int_0^t\frac{dy_\ell\dots dy_1}{(y_1+\dots+y_\ell)^{\ell-\alpha/2}}\le C\,S^W_\alpha[f]\,t^{\alpha/2}$$

so $|I| \leq C\, S_\alpha^W[f] \int_0^{\rho(x)^2} t^{\alpha/2} \frac{dt}{t^{1+\beta/2}} = C\, S_\alpha^W[f]\, \rho(x)^{\alpha-\beta}$.

If $\ell < \alpha/2$, then $k > \ell$ and by Lemma 2.30 we get, for $0 < t \leq \rho(x)^2$,

$$\begin{aligned}
|(e^{-t\mathcal{L}} - Id)^\ell f(x)| &= \Big| \int_0^t \underbrace{\cdots}_{\ell} \int_0^t \Big(- \int_{y_1+\cdots+y_\ell}^{\ell(\rho(x))^2} \frac{\partial_u^{\ell+1} W_u f(x)}{(\rho(x)^2)^{k-(\ell+1)}} du (\rho(x)^2)^{k-(\ell+1)} \\
&\qquad + \frac{\partial_\nu^\ell W_\nu f(x)\Big|_{\nu=\ell\rho(x)^2}}{(\rho(x)^2)^{k-\ell}} (\rho(x)^2)^{k-\ell} \Big) dy_1 \dots dy_\ell \Big| \\
&\leq C\, S_\alpha^W[f] (\rho(x)^2)^{k-(\ell+1)} \int_0^t \underbrace{\cdots}_{\ell} \int_0^t \int_{y_1+\cdots+y_\ell}^{\ell(\rho(x))^2} u^{-k+\alpha/2} du dy_1 \dots dy_\ell \\
&\qquad + C\, S_\alpha^W[f] (\ell\rho(x)^2)^{-k+\alpha/2} (\rho(x)^2)^{k-\ell}\, t^\ell.
\end{aligned}$$

Therefore, if α is not even we have, for $0 < t \leq \rho(x)^2$,

$$\begin{aligned}
&|(e^{-t\mathcal{L}} - Id)^\ell f(x)| \\
&\leq C\, S_\alpha^W[f] (\rho(x)^2)^{k-(\ell+1)} \int_0^t \underbrace{\cdots}_{\ell} \int_0^t ((y_1 + \cdots + y_\ell)^{-k+\alpha/2+1} + (\ell\rho(x)^2)^{-k+\alpha/2+1}) dy_1 \dots dy_\ell \\
&\quad + C\, S_\alpha^W[f] (\rho(x)^2)^{\alpha/2-\ell} t^\ell \\
&\leq C\, S_\alpha^W[f] ((\rho(x)^2)^{k-(\ell+1)} t^{-k+\alpha/2+\ell+1} + (\rho(x)^2)^{-\ell+\alpha/2} t^\ell).
\end{aligned}$$

Thus, in this case we get

$$\begin{aligned}
|I| &\leq C\, S_\alpha^W[f] \left((\rho(x)^2)^{k-\ell-1} \int_0^{\rho(x)^2} t^{-k+\alpha/2+\ell-\beta/2} dt + (\rho(x)^2)^{-\ell+\alpha/2} \int_0^{\rho(x)^2} t^{\ell-\beta/2-1} dt \right) \\
&= C\, S_\alpha^W[f] \rho(x)^{\alpha-\beta}.
\end{aligned}$$

If α is even, then $k = \alpha/2 + 1$ and, for $0 < t \leq \rho(x)^2$,

$$\begin{aligned}
&|((e^{-t\mathcal{L}} - Id)^\ell f(x)| \\
&\qquad \leq C\, S_\alpha^W[f] (\rho(x)^2)^{\alpha/2-\ell} \int_0^t \underbrace{\cdots}_{\ell} \int_0^t (\log(\ell(\rho(x))^2) - \log(y_1 + \cdots + y_\ell)) dy_1 \dots dy_\ell \\
&\qquad\quad + C\, S_\alpha^W[f] (\rho(x)^2)^{\alpha/2-\ell} t^\ell.
\end{aligned}$$

In order to solve the last integral we can perform the change of variables $\tilde{y}_1 = y_1, \tilde{y}_2 = y_2, \cdots, \tilde{y}_{\ell-1} = y_{\ell-1}, \tilde{y} = y_1 + \cdots + y_\ell$. Then we proceed as in the proof of Proposition 2.31. Putting together the above computations we get in this case

$$\int_0^{(\rho(x))^2} \frac{(e^{-t\mathcal{L}} - Id)^\ell f(x)}{t^{1+\beta/2}} dt \leq C\, S_\alpha^W[f] (\rho(x))^{\alpha-\beta}.$$

□

Proof of Theorem 2.60.

Let $\ell = [\beta/2] + 1$ and $m = \left[\frac{\alpha-\beta}{2}\right] + 1$. Then, $m + \ell = \left[\frac{\alpha-\beta}{2}\right] + 1 + [\beta/2] + 1 > \alpha/2 - \beta/2 + \beta/2 = \alpha/2$. As $m + \ell \in \mathbb{N}$ we get $m + \ell \geq [\alpha/2] + 1$.

By using the arguments in the proof of Lemma 2.61 we have

$$\begin{aligned}
\left|\partial_y^m W_y(\mathcal{L}^\beta f)(x)\right| &= \left|c_\beta \int_0^\infty \partial_y^m W_y\Big(\underbrace{\int_0^t \dots \int_0^t}_{\ell} \partial_\nu^\ell W_\nu|_{\nu=s_1+\dots+s_\ell} f(x) ds_1 \dots ds_\ell\Big)\frac{dt}{t^{1+\beta/2}}\right| \\
&= \left|c_\beta \int_0^\infty \Big(\underbrace{\int_0^t \dots \int_0^t}_{\ell} \partial_\nu^{m+\ell} W_\nu|_{\nu=y+s_1+\dots+s_\ell} f(x) ds_1 \dots ds_\ell\Big)\frac{dt}{t^{1+\beta/2}}\right| \\
&\leq C_\beta\, S_\alpha^W[f] \int_0^\infty \Big(\underbrace{\int_0^t \dots \int_0^t}_{\ell} (y + s_1 + \dots s_\ell)^{-(m+\ell)+\alpha/2} ds_1 \dots ds_\ell\Big)\frac{dt}{t^{1+\beta/2}} \\
&= C_\beta\, S_\alpha^W[f] \int_0^y (\dots)\frac{dt}{t^{1+\beta/2}} + C_\beta\, S_\alpha[f] \int_y^\infty (\dots)\frac{dt}{t^{1+\beta/2}} = C_\beta\, S_\alpha^W[f]\,(A+B).
\end{aligned}$$

Now we shall estimate A and B.

$$\begin{aligned}
A &= C_\beta y^{-m+\alpha/2} \int_0^y \underbrace{\int_0^{t/y} \dots \int_0^{t/y}}_{\ell} (1 + s_1 + \dots s_\ell)^{-(m+\ell)+\alpha/2} ds_1 \dots ds_\ell \frac{dt}{t^{1+\beta/2}} \\
&\leq C_\beta\, y^{-m+\alpha/2} \int_0^y \left(\frac{t}{y}\right)^\ell \frac{dt}{t^{1+\beta/2}} = C_\beta\, y^{-m+\alpha/2-\ell} \int_0^y \frac{dt}{t^{1+\beta/2-\ell}} = C_\beta\, y^{-m+(\alpha-\beta)/2}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
B &\leq \int_y^\infty \sum_{j=0}^\ell \frac{C_j}{(y+jt)^{m-\alpha/2}} \frac{dt}{t^{1+\beta/2}} = \sum_{j=0}^\ell \int_y^\infty \frac{C_j}{(y+jt)^{m-\alpha/2}} \frac{dt}{t^{1+\beta/2}} \\
&\leq \sum_{j=0}^\ell C_j y^{-m+(\alpha-\beta)/2}.
\end{aligned}$$

The last inequality is obtained by observing that $y \leq y + jt \leq (1+\ell)t$ inside the integrals together with the discussion about the sign of $m - \alpha/2$.

□

Theorem 2.62.

- *For* $1 < \alpha \leq 2 - n/q$, *then* $\|R_i f\|_{\Lambda^W_{\alpha/2}} \leq C\|f\|_{\Lambda^W_{\alpha/2}}$, $i = 1, \dots, n$.
- *For* $0 < \alpha \leq 1 - n/q$, *then* $\|\mathcal{R}_i f\|_{\Lambda^W_{\alpha/2}} \leq C\|f\|_{\Lambda^W_{\alpha/2}}$, $i = 1, \dots, n$.

The results were known in the case $C^\alpha_{\mathcal{L}}$, $0 < \alpha < 1$, with restrictions in α motivated by the reverse Hölder inequality, see [21].

Proof. Let $0 < \alpha \le 1 - n/q$ and $f \in \Lambda^W_{\alpha/2}$. By Theorem 2.58 we have that $\mathcal{L}^{-1/2} f \in \Lambda^W_{\frac{\alpha+1}{2}}$ and by Theorem 2.43 this means that $\mathcal{L}^{-1/2} f \in \Lambda^{\tilde{W}}_{\frac{\alpha+1}{2}}$ and $\rho(\cdot)^{-(\alpha+1)}\mathcal{L}^{-1/2} f \in L^\infty(\mathbb{R}^n)$. Therefore, by Proposition 2.40 we get that $\mathcal{R}_i f = \partial_{x_i}(\mathcal{L}^{-1/2} f) \in \Lambda^{\tilde{W}}_{\frac{\alpha}{2}}$ and $\rho(\cdot)^{-\alpha}\mathcal{R}_i f \in L^\infty(\mathbb{R}^n)$. Thus, Theorem 2.43 gives the first statement of the theorem.

Suppose now $1 < \alpha \le 2 - n/q$ and $f \in \Lambda^W_{\alpha/2}$. By Theorem 2.43 this means that $f \in \Lambda^{\tilde{W}}_{\frac{\alpha}{2}}$ and $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$. Then, Proposition 2.40 gives that $\partial_{x_i} f \in \Lambda^{\tilde{W}}_{\frac{\alpha-1}{2}}$ and $\rho(\cdot)^{-(\alpha-1)}\partial_{x_i} f \in L^\infty(\mathbb{R}^n)$. Again, by Theorem 2.43 this means that $\partial_{x_i} f \in \Lambda^W_{\frac{\alpha-1}{2}}$ and by Theorem 2.58 we get that $R_i f = \mathcal{L}^{-1/2}(\partial_{x_i} f) \in \Lambda^W_{\frac{\alpha}{2}}$.

□

Theorem 2.63. *Let a be a measurable bounded function on $[0,\infty)$ and consider*

$$m(\lambda) = \lambda \int_0^\infty e^{-s\lambda} a(s)ds, \ \lambda > 0.$$

Then, for every $\alpha > 0$, the multiplier operator of the Laplace transform type $m(\mathcal{L})$ is bounded from $\Lambda^W_{\alpha/2}$ into itself.

In the case $0 < \alpha < 1$ the result was obtained in [59] for the classes $C^\alpha_{\mathcal{L}}$.

Proof. Lemmas 2.28 and 2.30 guaranty the integrability of $\partial_s(W_s f(x))$ as a function of s. Then, we can write

$$m(\mathcal{L}f)(x) = \int_0^\infty (-\partial_s(W_s f(x))a(s)ds \le \left(\int_0^{\rho(x)^2} + \int_{\rho(x)^2}^\infty\right) \partial_s(W_s f(x))a(s)ds = I + II.$$

By using Lemma 2.28, we get

$$\begin{aligned}\big|II\big| &\le C\|a\|_\infty M^{\mathcal{L}}_\alpha \rho(x)^\alpha \int_{\rho(x)^2}^\infty \frac{1}{s}\left(1+\frac{s}{\rho(x)^2}\right)^{-M} ds \\ &= C\|a\|_\infty M^{\mathcal{L}}_\alpha[f]\, \rho(x)^\alpha \int_1^\infty \frac{1}{u(1+u)^M} du \le C\|a\|_\infty M^{\mathcal{L}}_\alpha[f]\, \rho(x)^\alpha.\end{aligned}$$

Now we estimate I. Let $k = [\alpha/2] + 1$. If α is not even, by Lemma 2.30 we get

$$\begin{aligned}\big|I\big| &= (\rho(x))^{2(k-1)} \int_0^{\rho(x)^2} \frac{|\partial_s W_s f(x)|}{(\rho(x))^{2(k-1)}} |a(s)|ds \le C\|a\|_\infty S^W_\alpha[f]\, (\rho(x))^{2(k-1)} \int_0^{\rho(x)^2} s^{-k+\alpha/2} ds \\ &= C\|a\|_\infty S^W_\alpha[f]\, \rho(x)^\alpha.\end{aligned}$$

If α is even, by Lemma 2.30 we have

$$
\begin{aligned}
|I| &= \left| \int_0^{\rho(x)^2} \left(\int_s^{\rho(x)^2} \frac{\partial_u^2 W_u f(x)}{(\rho(x))^{2(k-2)}} du (\rho(x))^{2(k-2)} - \frac{\partial_\nu W_\nu f(x)\big|_{\nu=\rho(x)^2}}{(\rho(x))^{2(k-1)}} (\rho(x))^{2(k-1)} \right) a(s) ds \right| \\
&\leq C\|a\|_\infty S_\alpha^W[f] \int_0^{\rho(x)^2} \left(\int_s^{\rho(x)^2} u^{-1} du (\rho(x))^{\alpha-2} + (\rho(x))^{\alpha-2} \right) ds \\
&= C\|a\|_\infty S_\alpha^W[f] \left(\int_0^{\rho(x)^2} (\log(\rho(x)^2) - \log s)(\rho(x))^{\alpha-2} ds + \rho(x)^\alpha \right) \\
&= C\|a\|_\infty S_\alpha^W[f]\rho(x)^\alpha.
\end{aligned}
$$

Up to now, we have shown that $M_\alpha^{\mathcal{L}}[m\mathcal{L}f] \leq \|f\|_{\Lambda_{\alpha/2}^W}$.

Now we want to see that $\|\partial_u^k W_y m(\mathcal{L}f)\|_\infty \leq Cy^{-k+\alpha/2}$. Fubini's Theorem together with Lemmas 2.30 and 2.28 allow us to interchange integral with derivatives and kernels. Then,

$$
\begin{aligned}
|\partial_u^k W_y m(\mathcal{L}f)(x)| &= \left| \int_0^\infty \partial_u^{k+1} W_u f(x)\big|_{u=y+s} a(s) ds \right| \leq C\|a\|_\infty S_\alpha^W[f] \int_0^\infty \frac{ds}{(y+s)^{k+1-\alpha/2}} ds \\
&= C\|a\|_\infty S_\alpha^W[f]\, y^{-(k+1)+\alpha/2} \int_0^\infty \frac{y}{(1+r)^{k+1-\alpha/2}} dr = C\|a\|_\infty S_\alpha^W[f] y^{-k+\alpha/2}.
\end{aligned}
$$

□

Chapter 3

Parabolic and elliptic Hermite Lipschitz spaces. Regularity results

This Chapter **corresponds with [30].**

The harmonic oscillator, $\mathcal{H} = -\Delta + |x|^2$, is a particular case of Schrödinger operator, where $V(x) = |x|^2$ satisfies the reverse Hölder inequality for every $q > n/2$ ($n \geq 3$) and the critical radius function associated to V is $\rho(x) = \frac{1}{1+|x|}$, which is a bounded function. This means that all the results of Chapter 2 apply in this context without the restrictions depending on q. Moreover, we know explicitly the heat (and therefore the Poisson) kernels associated to $\mathcal{H}$ on $\mathbb{R}^n$, for $n \geq 1$. These facts will allow us to get better results for $\mathcal{H}$ than the ones got for general Schrödinger operators.

In this chapter we will go further and we shall consider not only $\mathcal{H}$ but also the *parabolic Hermite operator* on $\mathbb{R}^n$, $n \geq 1$,

$$\mathbb{H} := \partial_t + \mathcal{H} = \partial_t - \Delta_x + |x|^2,\ x \in \mathbb{R}^n,\ t > 0. \tag{3.1}$$

We shall introduce definitions of Lipschitz (also called Hölder) spaces adapted to $\mathcal{H}$ and $\mathbb{H}$ by means of their heat and Poisson semigroups. We shall see that these spaces have pointwise characterizations, which will produce in the case of $\mathcal{H}$ the coincidence with the pointwise definitions of Hölder spaces introduced by Stinga and Torrea in [86]. Our semigroup definition of Hölder spaces will allow us to get regularity results for related fractional operators in a quicker and more elegant way.

The organization of this chapter is the following. In Section 3.1 we shall recall the pointwise definition of Hölder spaces adapted to $\mathcal{H}$ and we will introduce a new pointwise definition of parabolic Hölder spaces adapted to $\mathbb{H}$. In Section 3.2 we will introduce Hölder spaces adapted to $\mathcal{H}$ and $\mathbb{H}$ defined through their Poisson semigroups and we will prove pointwise characterizations and their coincidence with the spaces of Section 3.1. In Section 3.3 we will introduce Hölder spaces adapted to $\mathcal{H}$ defined via the heat semigroup and we will show the coincidence with the spaces defined through the Poisson semigroup (and therefore the coincidence with the spaces of Section 3.1). Finally, in Section 3.4 is devoted to applications. We will prove a maximum principle and regularity results of fractional operators related to $\mathcal{H}$ and $\mathbb{H}$ in the Hölder spaces introduced in Sections 3.2 and 3.3.

3.1 Pointwise description of Hermite Hölder spaces.

The operator $\mathcal{H}$ can be factorized as $\mathcal{H} = \frac{1}{2}\sum_{i=1}^n (A_iA_{-i} + A_{-i}A_i)$, $A_i = \partial_{x_i} + x_i$, $A_{-i} = -\partial_{x_i} + x_i$. The first order operators $A_{\pm i}$ play the role, with respect to operator $\mathcal{H}$, of the derivatives $\pm\partial_{x_i}$ with respect to the classical Laplacian Δ, see [82] and [86]. This fact will be crucial to define Hölder spaces adapted to $\mathcal{H}$.

The following definition was introduced by Stinga and Torrea in [86].

Definition 3.64 (Hermite Hölder spaces)**.** *Let $0<\alpha<1$. We consider the space of functions*

$$C^\alpha_{\mathcal{H}}(\mathbb{R}^n) = \{f : (1+|\cdot|)^\alpha f(\cdot) \in L^\infty(\mathbb{R}^n),\ \text{and}\ \|f(\cdot+z)-f(\cdot)\|_{L^\infty(\mathbb{R}^n)} \le A|z|^\alpha\}$$

with associated norm

$$\|f\|_{C^\alpha_{\mathcal{H}}} = [f]_{M^\alpha} + [f]_{C^\alpha_{\mathcal{H}}},$$

where $[f]_{M^\alpha} = \|(1+|\cdot|)^\alpha f(\cdot)\|_\infty$ *and* $[f]_{C^\alpha_{\mathcal{H}}} = \sup\limits_{|z|>0} \dfrac{\|f(\cdot+z)-f(\cdot)\|_\infty}{|z|^\alpha}$.
For $\alpha>1$ and not integer, we say that $f\in C^\alpha_{\mathcal{H}}(\mathbb{R}^n)$, if there exist the derivatives of order $[\alpha]$ and the norm

$$\|f\|_{C^\alpha_{\mathcal{H}}} := [f]_{M^{\alpha-[\alpha]}} + \sum_{\substack{1\le|i_1|,\dots,|i_m|\le n\\ 1\le m\le[\alpha]}} [A_{i_1}\dots A_{i_m}f]_{M^{\alpha-[\alpha]}} + \sum_{1\le|i_1|,\dots,|i_{[\alpha]}|\le n} [A_{i_1}\dots A_{i_{[\alpha]}}f]_{C^{\alpha-[\alpha]}_{\mathcal{H}}},$$

is finite.

On the other hand, some parabolic Hölder spaces were considered by N. Krylov, see [54]. Namely,

(i) Let $0<\alpha<1$, $C^{\alpha/2,\alpha}$ was defined as the set of bounded functions such that

$$[f]_{C^{\alpha/2,\alpha}} = \sup_{(\tau,z)\neq(0,0)} \frac{\|f(\cdot-\tau,\cdot-z)-f(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^{n+1})}}{(|\tau|^{1/2}+|z|)^\alpha} < \infty.$$

(ii) For $1<\alpha<2$, $f\in C^{\alpha/2,\alpha}$ if $f\in L^\infty(\mathbb{R}^{n+1})$, $\partial_{x_i}f \in C^{\alpha/2-1/2,\alpha-1}$, $i=1,\dots,n$, and $f(\cdot,x)\in C^{\alpha/2}(\mathbb{R})$ uniformly on x.

(iii) Let $0<\alpha<1$, $C^{1+\alpha/2,2+\alpha}$ if $f\in L^\infty(\mathbb{R}^{n+1})$, $\partial_{x_i}\partial_{x_j}f$, $i,j=1,\dots,n$, and $\partial_t f$ belong to $C^{\alpha/2,\alpha}$.

These Krylov's definitions together with Definition 3.64 drive us to consider the following definition.

Definition 3.65 (Parabolic Hermite Hölder spaces)**.**

- *Let $0<\alpha<1$. We say that $f\in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$ if $f\in C^{\alpha/2,\alpha}$ and*

$$[f]_{M^\alpha} = \sup_{(t,x)\in\mathbb{R}^{n+1}} (1+|x|)^\alpha |f(t,x)| < \infty,$$

  In this case, $\|f\|_{C^{\alpha/2,\alpha}_{t,\mathcal{H}}} = [f]_{M^\alpha} + [f]_{C^{\alpha/2,\alpha}}$.

- *For* $1 < \alpha < 2$, $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$ *if* $f \in L^\infty(\mathbb{R}^{n+1})$, $A_{\pm i} f \in C^{\alpha/2-1/2,\alpha-1}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$, $i = 1, \dots, n$, *and* $f(\cdot, x) \in C^{\alpha/2}(\mathbb{R})$ *uniformly on* x.

- *For* $2 < \alpha < 3$ *we say that a function* $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$, *if* $f \in L^\infty(\mathbb{R}^{n+1})$, $A_{\pm i}A_{\pm j} f$, $i, j = 1, \dots, n$, *and* $\partial_t f$ *belong to* $C^{\alpha/2-1,\alpha-2}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$.

In the next result we will show that the functions in $C^{\alpha/2,\alpha}_{t,\mathcal{H}}$, $0 < \alpha < 1$, can be taken to be continuous, so the inequality $|f(t-\tau, x+z) - f(t,x)| \le C(\tau^{1/2} + |z|)^\alpha$ holds for every $x, z \in \mathbb{R}^n$, $t, \tau \in \mathbb{R}$.

Proposition 3.66. *For* $0 < \alpha < 1$, *every* $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$ *can be modified on a set of measure zero so that it becomes continuous.*

Proof. Let $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$, $0 < \alpha < 1$. We will follow the ideas in Stein [81, page 142]. Let $\mathcal{P}_y f = e^{-y\sqrt{\mathbb{H}}} f$ be the Poisson semigroup associated to $\mathbb{H}$, which is well defined for $f \in L^\infty(\mathbb{R}^{n+1})$, see subsection 3.2.1. By the hypothesis on f, Lemma 3.78 (i) and Lemma 3.77 (3) we have

$$
\begin{aligned}
&|\mathcal{P}_y f(t,x) - f(t,x)| \\
&\le \left| \int_{\mathbb{R}^{n+1}} \mathcal{P}_y(\tau, x, z)(f(t-\tau, x-z) - f(t,x)) d\tau dz \right| + \left| f(t,x) \left(\int_{\mathbb{R}^{n+1}} \mathcal{P}_y(\tau, x, z) d\tau dz - 1 \right) \right| \\
&\le [f]_{C^{\alpha/2,\alpha}} \Big(\int_{\mathbb{R}^n} \int_0^\infty \frac{y e^{-\frac{y^2+|z|^2}{c\tau}} (\tau^{1/2} + |z|)^\alpha}{\tau^{\frac{n+3}{2}}} d\tau dz \Big) + \|f\|_\infty \left| e^{-y\sqrt{\mathbb{H}}} 1(t,x) - 1 \right| \le C \|f\|_{C^{\alpha/2,\alpha}} y^\alpha.
\end{aligned}
$$

In particular, we conclude that $\mathcal{P}_y f$ converges uniformly to f as y goes to zero. As $\mathcal{P}_y f$ is continuous, f can be taken to be continuous. □

3.2 Parabolic Hermite Hölder spaces via the Poisson semigroup.

At first, we will introduce the Parabolic Hermite Lipschitz spaces defined through the Poisson semigroup, $\mathcal{P}_y f = e^{-y\sqrt{\mathbb{H}}} f$.

Definition 3.67 (Parabolic Hermite Lipschitz spaces)**.** *Let* $\mathcal{P}_y = e^{-y\sqrt{\mathbb{H}}}$ *and* $\alpha > 0$, *we consider the class*

$$
\Lambda^{\mathcal{P}}_\alpha = \left\{ f : \ f \in L^\infty(\mathbb{R}^{n+1}) \ \text{and} \ \left\| \partial_y^k \mathcal{P}_y f \right\|_{L^\infty(\mathbb{R}^{n+1})} \le C_k y^{-k+\alpha}, \ \text{with}\, k = [\alpha] + 1, y > 0. \right\},
$$

whose norm is given by $\|f\|_{\Lambda^{\mathcal{P}}_\alpha} := \|f\|_\infty + C$, *where* C *is the infimum of the positive constants* C_k *above.*

In this case the condition $f \in L^\infty(\mathbb{R}^{n+1})$ is enough to define our spaces. See Remark 3.73 for the proof in the elliptic case.

Now we state our main results of this section. They will be proved in Subsection 3.3.2.

The following theorem shows that $\Lambda_\alpha^{\mathcal{P}}$ spaces have a pointwise description. Moreover, a restriction to functions depending only on x, produces the natural Definition 3.70 and Theorem 3.71 for the case of Hermite operator in $\mathbb{R}^n$.

Theorem 3.68.

1. *Suppose that* $0 < \alpha < 2$. *Then* $f \in \Lambda_\alpha^{\mathcal{P}}$ *if and only if there exists a constant* $C > 0$ *such that*

$$\|f(\cdot - \tau, \cdot - z) + f(\cdot - \tau, \cdot + z) - 2f(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^{n+1})} \leq C(|\tau|^{1/2} + |z|)^\alpha, (\tau, z) \in \mathbb{R}^{n+1} \quad (3.2)$$

and $(1+|x|)^\alpha f \in L^\infty(\mathbb{R}^{n+1})$. *In this case, if* K *denotes the least constant* C *for which the inequality above is true, then* $\|u\|_{\Lambda_\alpha^{\mathcal{P}}} := [u]_{M^\alpha} + K$, *where* $[f]_{M^\alpha} = \|(1+|\cdot|)^\alpha f(\cdot,\cdot)\|_\infty$.

2. *Suppose that* $\alpha > 2$. *Then* $f \in \Lambda_\alpha^{\mathcal{P}}$ *if and only if* $f \in L^\infty(\mathbb{R}^{n+1})$,

$$A_{\pm i} A_{\pm j} f \in \Lambda_{\alpha-2}^{\mathcal{P}}, \;\; i, j = 1, \dots, n, \qquad \text{and} \qquad \partial_t f \in \Lambda_{\alpha-2}^{\mathcal{P}}.$$

In this case the following equivalence holds

$$\|f\|_{\Lambda_\alpha^{\mathcal{P}}} \sim \|f\|_\infty + \sum_{i,j=1}^n \left(\|A_{\pm i} A_{\pm j} f\|_{\Lambda_{\alpha-2}^{\mathcal{P}}} \right) + \|\partial_t f\|_{\Lambda_{\alpha-2}^{\mathcal{P}}}.$$

As a consequence of the Theorem 3.68 we shall prove the following.

Theorem 3.69. *Let* $0 < \alpha < 3$, α *not an integer. Then*

$$C_{t,\mathcal{H}}^{\alpha/2,\alpha} = \Lambda_\alpha^{\mathcal{P}},$$

with equivalence of norms.

The above results have the following parallel results in the case of Hermite operator $\mathcal{H} = -\Delta_x + |x|^2$.

Definition 3.70 (Hermite Lipschitz spaces)**.** *Let* $P_y^{\mathcal{H}} = e^{-y\sqrt{\mathcal{H}}}$ *and* $\alpha > 0$, *we consider the class*

$$\Lambda_\alpha^{P^{\mathcal{H}}} = \left\{ g : \; g \in L^\infty(\mathbb{R}^n) \text{ and } \left\| \partial_y^k P_y^{\mathcal{H}} g \right\|_{L^\infty(\mathbb{R}^n)} \leq C_k y^{-k+\alpha}, \text{ with } k = [\alpha] + 1, y > 0 \right\},$$

whose norm is given by $\|g\|_{\Lambda_\alpha^{P^{\mathcal{H}}}} := \|g\|_\infty + C$, *where* C *is the infimum of the positive constants* C_k *above.*

Theorem 3.71.

1. *Suppose that* $0 < \alpha < 2$. *Then* $g \in \Lambda_\alpha^{P^{\mathcal{H}}}$ *if and only if* $(1 + |\cdot|)^\alpha g \in L^\infty(\mathbb{R}^n)$ *and there exists a constant* $C > 0$ *such that*

$$\|g(\cdot - z) + g(\cdot + z) - 2g(\cdot)\|_{L^\infty(\mathbb{R}^n)} \le C|z|^\alpha,\ z \in \mathbb{R}^n.$$

In this case, if K *denotes the least constant* C *for which the inequality above is true, then* $\|g\|_{\Lambda_\alpha^{P^{\mathcal{H}}}} := [g]_{M^\alpha} + K$, *where* $[g]_{M^\alpha} = \|(1 + |\cdot|)^\alpha g(\cdot)\|_\infty$.

2. *Suppose that* $\alpha > 1$. *Then* $g \in \Lambda_\alpha^{P^{\mathcal{H}}}$ *if and only if* $g \in L^\infty(\mathbb{R}^n)$ *and*

$$A_{\pm i} g \in \Lambda_{\alpha-1}^{P^{\mathcal{H}}}\ \ i = 1, \dots, n.$$

In this case the following equivalence holds

$$\|g\|_{\Lambda_\alpha^{P^{\mathcal{H}}}} \sim \|g\|_\infty + \sum_{i=1}^n \|A_{\pm i} g\|_{\Lambda_{\alpha-1}^{P^{\mathcal{H}}}}.$$

As a consequence of the Theorem 3.71 we shall prove the following.

Theorem 3.72. *Let* $\alpha > 0$, $\alpha \notin \mathbb{N}$. *Then*

$$C_{\mathcal{H}}^\alpha = \Lambda_\alpha^{P^{\mathcal{H}}},$$

with equivalence of norms.

Remark 3.73. *Notice that, when we defined on Chapter 2 Lipschitz spaces adapted to Schrödinger operators via the Poisson semigroup, we imposed an integrability condition on* f *(the condition* $M^P[f] < \infty$*). We can see that, in the Hermite setting, the Schrödinger Lipschitz spaces* Λ_α^P, *where* $\mathcal{L} = \mathcal{H}$, *are equivalent to* $\Lambda_\alpha^{P^{\mathcal{H}}}$. *Indeed, it is clear that* $f \in L^\infty(\mathbb{R}^n) \implies M^P[f] < \infty$. *On the other hand, if* $f \in \Lambda_\alpha^P$, *where* $\mathcal{L} = \mathcal{H}$, *by using Theorem 2.44 and the fact that* $\rho(\cdot) = \frac{1}{1+|\cdot|}$ *we get that* $(1 + |\cdot|)^\alpha f \in L^\infty(\mathbb{R}^n)$, *so in particular* f *is bounded.*

Along the section we present the computations and the results in such a way that the parabolic case includes as particular case the Hermite case. This will be clarified in the paragraphs called **Elliptic Hermite setting**.

3.2.1 Preliminary considerations and results.

For functions $g \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, the heat semigroup $e^{-\tau\mathcal{H}}$ has the pointwise expression

$$e^{-\tau\mathcal{H}} g(x) = \int_{\mathbb{R}^n} \frac{e^{-\frac{|x-z|^2}{4}\coth\tau} e^{-\frac{|x+z|^2}{4}\tanh\tau}}{(2\pi \sinh 2\tau)^{n/2}} g(z)\, dz,\ \ x \in \mathbb{R}^n,$$

see [82], [88]. On the other hand, the operator ∂_t in (3.1) is taking care of the past, in other words, its heat semigroup is given by $e^{-\tau\partial_t}\varphi(t)=\varphi(t-\tau)$, $t\in\mathbb{R}$, $\tau>0$. Hence, for functions $f\in\mathcal{C}^1_{L^p(\mathbb{R}^n)}(\mathbb{R})$ we have $e^{-\tau\mathbb{H}}f(t,x)=e^{-\tau\mathcal{H}}\Big(e^{-\tau\partial_t}f(t,\cdot)\Big)(x)$, moreover

$$e^{-\tau\mathbb{H}}f(t,x)=e^{-\tau\mathcal{H}}(f(t-\tau,\cdot))(x)=\int_{\mathbb{R}^n}\frac{e^{-\frac{|x-z|^2}{4}\coth\tau}e^{-\frac{|x+z|^2}{4}\tanh\tau}}{(2\pi\sinh 2\tau)^{n/2}}f(t-\tau,z)\,dz. \tag{3.3}$$

The Fourier-Hermite transform of a function $f\in L^1(\mathbb{R}^{n+1})$ can be defined as

$$\mathcal{F}(f)(\rho,\mu)=\int_{\mathbb{R}^{n+1}}f(t,x)e^{-i\rho t}h_\mu(x)dtdx,\ \ \rho\in\mathbb{R},\ \mu\in\mathbb{N}_0^n. \tag{3.4}$$

Where $h_\mu(x)=\prod_{j=1}^n h_{\mu_j}(x_j)$, $\quad x=(x_1,\dots,x_n)\in\mathbb{R}^n$. For $k\in\mathbb{N}$, h_k is the Hermite function defined by

$$h_k(t)=\frac{(-1)^k}{(2^kk!\pi^{1/2})^{1/2}}\,H_k(t)\,e^{-t^2/2},\quad t\in\mathbb{R}.$$

Here H_k denotes the Hermite polynomial of degree k (see [88]). These functions are eigenvectors of the Hermite operator $\mathcal{H}$. In fact $\mathcal{H}h_\mu=(2|\mu|+n)\,h_\mu$. Consequently, for functions $f\in L^1(\mathbb{R}^{n+1})$ we have

$$\mathcal{F}(e^{-\tau\mathbb{H}}f)(\rho,\mu)=e^{-\tau(i\rho+2|\mu|+n)}\mathcal{F}(f)(\rho,\mu),\ \rho\in\mathbb{R},\ \mu\in\mathbb{N}^n. \tag{3.5}$$

By analytic continuation, Bocher's subordination formula can be extended for $z\in\mathbb{C}$ with $\Re z\geq 0$, that is,

$$e^{-t\sqrt{z}}=\frac{y}{2\sqrt{\pi}}\int_0^\infty e^{-y^2/4\tau}e^{-\tau z}\frac{d\tau}{\tau^{3/2}},\ \text{for } t>0 \text{ and } z\in\mathbb{C} \text{ such that } \Re z\geq 0.$$

Hence, for $f\in L^1(\mathbb{R}^{n+1})$ we have

$$e^{-y\sqrt{i\rho+2|\mu|+n}}\mathcal{F}(f)(\rho,\mu)=\frac{y}{2\sqrt{\pi}}\int_0^\infty e^{-y^2/4\tau}e^{-\tau(i\rho+2|\mu|+n)}\mathcal{F}(f)(\rho,\mu)\frac{d\tau}{\tau^{3/2}}.$$

This last expression can be written as

$$\mathcal{F}(e^{-y\sqrt{\mathbb{H}}}f)(\rho,\mu)=\frac{y}{2\sqrt{\pi}}\int_0^\infty e^{-y^2/4\tau}\mathcal{F}(e^{-y\mathbb{H}}f)(\rho,\mu)\frac{d\tau}{\tau^{3/2}}.$$

The Fourier transform defined in (3.4) is an isometry in $L^2(\mathbb{R}^{n+1})$ and in particular we have, in the $L^2(\mathbb{R}^{n+1})$ sense,

$$\mathcal{P}_yf(t,x)=e^{-y\sqrt{\mathbb{H}}}f(t,x)=\frac{y}{2\sqrt{\pi}}\int_0^\infty e^{-y^2/4\tau}e^{-\tau\mathbb{H}}f(t,x)\frac{d\tau}{\tau^{3/2}}. \tag{3.6}$$

For functions f good enough, formulas (3.3) and (3.6) give the following pointwise expression

$$\mathcal{P}_yf(t,x)=\frac{y}{2\sqrt{\pi}}\int_0^\infty\int_{\mathbb{R}^n}e^{-y^2/4\tau}\frac{e^{-\frac{|x-z|^2}{4}\coth\tau}e^{-\frac{|x+z|^2}{4}\tanh\tau}}{(2\pi\sinh 2\tau)^{n/2}}f(t-\tau,z)\,dz\frac{d\tau}{\tau^{3/2}},\ x\in\mathbb{R}^n,\ t\in\mathbb{R}. \tag{3.7}$$

On the other hand,

$$ye^{-y^2/4\tau}\frac{e^{-\frac{|x-z|^2}{4}\coth\tau}e^{-\frac{|x+z|^2}{4}\tanh\tau}}{(2\pi\sinh 2\tau)^{n/2}\tau^{3/2}}\chi_{\{\tau>0\}} \le C\frac{y}{\tau^{1/2}}\frac{e^{-\frac{y^2}{4\tau}}}{\tau}\frac{e^{-\frac{|x-z|^2}{4\tau}}}{\tau^{n/2}}\chi_{\{\tau>0\}} = \Phi_y(\tau, x-z).$$

As Φ_y belongs to $L^1(\mathbb{R}^{n+1})$, the formula (3.7), defining the *Parabolic Poisson Hermite integral,* remains valid for any $f \in L^p(\mathbb{R}^{n+1})$, $1 \le p \le \infty$. Moreover, this integral satisfies a *Parabolic Hermite Laplace equation* as the following Proposition shows.

Proposition 3.74. *Assume $f \in L^\infty(\mathbb{R}^{n+1})$. Then $\mathcal{P}_y f(t,x)$ satisfies the equation*

$$\partial_y^2\mathcal{P}_y f(t,x) - \mathbb{H}\mathcal{P}_y f(t,x) = 0, \ \ (t,x) \in \mathbb{R}^{n+1}. \tag{3.8}$$

Proof. We observe that

$$\begin{aligned}\Big|\partial_y^2\Big(ye^{-y^2/4\tau}&\frac{e^{-\frac{|x-z|^2}{4}\coth\tau}e^{-\frac{|x+z|^2}{4}\tanh\tau}}{(2\pi\sinh 2\tau)^{n/2}\tau^{3/2}}\chi_{\{\tau>0\}}\Big)\Big| \\ &+\Big|\Delta_x\Big(ye^{-y^2/4\tau}\frac{e^{-\frac{|x-z|^2}{4}\coth\tau}e^{-\frac{|x+z|^2}{4}\tanh\tau}}{(2\pi\sinh 2\tau)^{n/2}\tau^{3/2}}\chi_{\{\tau>0\}}\Big)\Big| \\ &+\Big|\partial_\tau\Big(ye^{-y^2/4\tau}\frac{e^{-\frac{|x-z|^2}{4}\coth\tau}e^{-\frac{|x+z|^2}{4}\tanh\tau}}{(2\pi\sinh 2\tau)^{n/2}\tau^{3/2}}\chi_{\{\tau>0\}}\Big)\Big| \\ &\le \frac{C}{\tau}\frac{e^{-\frac{y^2}{4\tau}}}{\tau}\frac{e^{-\frac{|x-z|^2}{4\tau}}}{\tau^{n/2}}\chi_{\{\tau>0\}}.\end{aligned} \tag{3.9}$$

Hence, for $y > 0$ and $|x-z| > 0$, the function $ye^{-y^2/4\tau}\frac{e^{-\frac{|x-z|^2}{4}\coth\tau}e^{-\frac{|x+z|^2}{4}\tanh\tau}}{(2\pi\sinh 2\tau)^{n/2}\tau^{3/2}}\chi_{\{\tau>0\}}$ is smooth in all its variables. In particular, we can write

$$\mathcal{P}_y f(t,x) = \frac{y}{2\sqrt{\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}^n} e^{-y^2/4(t-\tau)}\frac{e^{-\frac{|x-z|^2}{4}\coth(t-\tau)}e^{-\frac{|x+z|^2}{4}\tanh(t-\tau)}}{(2\pi\sinh 2(t-\tau))^{n/2}}f(\tau,z)\,dz\,\chi_{\{t-\tau>0\}}\frac{d\tau}{(t-\tau)^{3/2}}.$$

The above estimates (3.9) also show that we can interchange the derivatives with the integral for $y > 0$ and $|x-z| > 0$. Hence, the result follows since the kernel of the integral defining $\mathcal{P}_y f$ satisfies the equation (3.8). □

Remark 3.75. *The proof of the previous Proposition also shows that for functions $f \in L^\infty(\mathbb{R}^{n+1})$ we can write*

$$\begin{aligned}\mathcal{P}_y f(t,x) &= \int_{\mathbb{R}^{n+1}}\mathcal{P}_y(\tau,x,z)f(t-\tau,x-z)dzd\tau \\ &= \frac{y}{2\sqrt{\pi}}\int_{\mathbb{R}}\int_{\mathbb{R}^n}\frac{e^{-\frac{|z|^2}{4}\coth\tau}e^{-\frac{|2x-z|^2}{4}\tanh\tau}e^{-\frac{y^2}{4\tau}}}{(2\pi\sinh(2\tau))^{n/2}\tau^{3/2}}f(t-\tau,x-z)dz\,\chi_{\{\tau>0\}}\,d\tau.\end{aligned} \tag{3.10}$$

For simplicity in the notation, along the chapter we will denote by $\mathcal{P}_y f$ the Poisson semigroup of the parabolic operator $\mathbb{H}$ and by $\mathcal{P}_y(\tau,x,z)$ the corresponding Poisson integral kernel, because since we always write the dependance on the variables, there is not any confusion.

Elliptic Hermite setting.

Given $g \in L^\infty(\mathbb{R}^n)$, consider the function $f(t,x) = g(x)$, then formula (3.7) becomes

$$\mathcal{P}_y f(t,x) = \frac{y}{2\sqrt{\pi}} \int_0^\infty \int_{\mathbb{R}^n} e^{-y^2/4\tau} \frac{e^{-\frac{|x-z|^2}{4}\coth\tau} e^{-\frac{|x+z|^2}{4}\tanh\tau}}{(2\pi\sinh 2\tau)^{n/2}} g(z)\,dz \frac{d\tau}{\tau^{3/2}} = P_y^{\mathcal{H}} g(x), \quad (3.11)$$

where $P_y^{\mathcal{H}} g(x)$ is the Poisson semigroup associated to the operator $\mathcal{H} = -\Delta_x + |x|^2$. The thoughts developed along this section show that:

- For functions $g \in L^\infty(\mathbb{R}^n)$, $P_y^{\mathcal{H}} g(x)$ satisfies the equation $\partial_y^2 P_y^{\mathcal{H}} g(x) - \mathcal{H} P_y^{\mathcal{H}} g(x) = 0$, $x \in \mathbb{R}^n$.
- Identities (3.10) and (3.11) give that $\int_{\mathbb{R}} \mathcal{P}_y(\tau,x,z)d\tau = P_y^{\mathcal{H}}(x,z)$, for all $x, z \in \mathbb{R}^n$, where $\mathcal{P}_y$ is the Poisson kernel associated to $\mathbb{H}$ and $P_y^{\mathcal{H}}$ is the Poisson kernel associated to the harmonic oscillator, $\mathcal{H}$. Again, to ease notation, we denote by $P_y^{\mathcal{H}} g$ the Poisson semigroup of the operator $\mathcal{H}$ and by $P_y^{\mathcal{H}}(x,z)$ the corresponding Poisson integral kernel.

The following results will be used systematically along this chapter.

Remark 3.76. *Let $\tau > 0$.*

(1) If $\tau < 1$, then $\sinh\tau \sim \tau$, $\cosh\tau \sim C$, $\coth\tau \sim \frac{1}{\tau}$ and $\tanh\tau \sim \tau$.

(2) If $\tau > 1$, then $\sinh\tau \sim e^\tau$, $\cosh\tau \sim e^\tau$, $\coth\tau \sim C$ and $\tanh\tau \sim C$.

(3) Given $n, \ell > 0$ and $\lambda \geq 0$, there exists a positive constant $C_{\ell,n,\lambda}$ such that $\frac{1}{(\coth\tau)^\ell (\sinh\tau)^n} = \frac{(\tanh\tau)^\ell}{(\sinh\tau)^n} \leq C_{\ell,n,\lambda}\tau^{-n+\ell-\lambda}$.

(4) Let $z \geq 0$ and $\alpha \geq 0$ there exists a constant $C_\alpha > 0$ such that $z^\alpha e^{-z} \leq C_\alpha e^{-z/2}$.

As usual, by $A \sim B$ we mean there exist constants C_1, C_2 such that $C_1 A \leq B \leq C_2 A$.

Lemma 3.77. *For each $x \in \mathbb{R}^n$ and $\tau > 0$, we have:*

(1) $e^{-\tau\mathbb{H}}1(t,x) = \dfrac{e^{-\frac{\tanh(2\tau)}{2}|x|^2}}{(\cosh(2\tau))^{n/2}}$.

(2) $|\partial_\tau e^{-\tau\mathbb{H}}1(t,x)| \leq C(\min\{\tau,1\} + |x|^2)$.

(3) Given $0 < \alpha < 1$, there exists $C_\alpha > 0$ such that

$$\left| e^{-y\sqrt{\mathbb{H}}}1(t,x) - 1 \right| \leq C_\alpha (1+|x|)^\alpha y^\alpha. \quad (3.12)$$

Proof. By using formula (3.3) we have

$$
\begin{aligned}
&(2\pi\sinh(2\tau))^{n/2}e^{-\tau\mathbb{H}}1(t,x)\\
&=\int_{\mathbb{R}^n}\exp\Big(-\frac{1}{4}\coth\tau(|x|^2+|z|^2-2xz)\Big)\exp\Big(-\frac{1}{4}\tanh\tau(|x|^2+|z|^2+2xz)\Big)dz\\
&=\exp(-\frac{1}{4}|x|^2(\coth\tau+\tanh\tau))\\
&\times\int_{\mathbb{R}^n}\exp\Big(-\frac{1}{4}\Bigg[\Bigg(\sqrt{(\coth\tau+\tanh\tau)}z-\frac{\coth\tau-\tanh\tau}{\sqrt{(\coth\tau+\tanh\tau)}}x\Bigg)^2-\frac{(\coth\tau-\tanh\tau)^2|x|^2}{(\coth\tau+\tanh\tau)}\Bigg]\Big)dz\\
&=\exp\Big(-\frac{1}{4}|x|^2(\coth\tau+\tanh\tau)\Big)\exp\Big(\frac{1}{4}\frac{(\coth\tau-\tanh\tau)^2}{(\coth\tau+\tanh\tau)}|x|^2\Big)\int_{\mathbb{R}^n}e^{-\frac{u^2}{4}}\frac{du}{(\coth\tau+\tanh\tau)^{n/2}}\\
&=\exp\Big(-\frac{1}{2}|x|^2\tanh(2\tau)\Big)\frac{(\sinh(2\tau))^{n/2}}{(2\cosh(2\tau))^{n/2}}2^n\pi^{n/2}.
\end{aligned}
$$

In the third identity we have done the change of variables $u=\sqrt{(\coth\tau+\tanh\tau)}z-\frac{\coth\tau-\tanh\tau}{\sqrt{(\coth\tau+\tanh\tau)}}x$. This concludes the proof of (1).

By using the estimates of Remark 3.76, it is easy to show that

$$
|\partial_\tau e^{-\tau\mathbb{H}}1(t,x)|\le C\Big(\tanh(2\tau)+(1+\tanh^2(2\tau))|x|^2\Big)e^{-\tau\mathbb{H}}1(t,x)\le C(\min\{\tau,1\}+|x|^2).
$$

For (3), consider first the case $|x|>1$. By the Mean Value Theorem and epigraphs (1), (2) in this Lemma we get

$$
\begin{aligned}
\Big|e^{-y\sqrt{\mathbb{H}}}1(t,x)-1\Big|&=\frac{1}{2\sqrt{\pi}}\Big|\Big(\int_0^{1/|x|^2}+\int_{1/|x|^2}^{\infty}\Big)\frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{1/2}}(e^{-\tau\mathbb{H}}1(t,x)-1)\frac{d\tau}{\tau}\Big|\\
&\le C\int_0^{1/|x|^2}\frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{1/2}}|x|^2\tau\frac{d\tau}{\tau}+C\int_{1/|x|^2}^{\infty}\frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{1/2}}\frac{d\tau}{\tau}\\
&\underbrace{=}_{\frac{y^2}{4\tau}=v}C\Big(|x|^2y^2\int_{\frac{|x|^2y^2}{c}}^{\infty}v^{1/2}e^{-v}\frac{dv}{v^2}+\int_0^{\frac{|x|^2y^2}{4}}v^{1/2}e^{-v}\frac{dv}{v}\Big)\\
&\le\frac{C|x|^2y^2}{(|x|^2y^2)^{1-\alpha/2}}\int_{\frac{|x|^2y^2}{c}}^{\infty}v^{1/2-\alpha/2}e^{-v}\frac{dv}{v}+C|x|^\alpha y^\alpha\int_0^{\frac{|x|^2y^2}{4}}v^{1/2-\alpha/2}e^{-v}\frac{dv}{v}\\
&\le C\Gamma(1/2-\alpha/2)|x|^\alpha y^\alpha.
\end{aligned}
$$

Regarding the case $|x|<1$, by the Mean Value Theorem we get

$$\begin{aligned}\Big|\int_0^1 \frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{1/2}}(e^{-\tau\mathbb{H}}1(t,x)-1)\frac{d\tau}{\tau}\Big| &\le C\int_0^1 \frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{1/2}}(\tau+|x|^2)\tau\frac{d\tau}{\tau}\\ &\le C\int_0^{1/|x|^2} \frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{1/2}}|x|^2\tau\frac{d\tau}{\tau}+\int_0^1 \frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{1/2}}\tau^2\frac{d\tau}{\tau}\\ &\underbrace{\le}_{\frac{y^2}{4\tau}=v} C|x|^2y^2\int_{\frac{|x|^2y^2}{4}}^\infty v^{1/2}e^{-v}\frac{dv}{v^2}+C\int_{\frac{y^2}{4}}^\infty v^{1/2}e^{-v}\left(\frac{y^2}{v}\right)^2\frac{dv}{v}\\ &\le C|x|^\alpha y^\alpha\int_{\frac{|x|^2y^2}{4}}^\infty v^{1/2-\alpha/2}e^{-v}\frac{dv}{v}+Cy^\alpha\int_{\frac{y^2}{4}}^\infty v^{1/2-\alpha/2}e^{-v}\frac{dv}{v}\\ &\le C\Gamma(1/2-\alpha/2)y^\alpha.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Big|\int_1^\infty \frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{1/2}}(e^{-\tau\mathbb{H}}1(t,x)-1)\frac{d\tau}{\tau}\Big| \le C\int_1^\infty \frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{1/2}}\frac{d\tau}{\tau} &\le Cy^\alpha\int_1^\infty \frac{y^{1-\alpha}e^{-\frac{y^2}{4\tau}}}{\tau^{1/2-\alpha/2}}\frac{d\tau}{\tau}\\ &\le C\Gamma(1/2-\alpha/2)y^\alpha.\end{aligned}$$

□

Lemma 3.78. *Let $\mathcal{P}_y(\tau,x,z)$ be the Poisson kernel associated with the parabolic harmonic oscillator, $\mathbb{H}$, given by* (3.10)*. Then,*

(i) There exists a constant C such that for every x,z in $\mathbb{R}^n$ and $\tau>0$,
$\Big|\mathcal{P}_y(\tau,x,z)\Big|\le Cy\,e^{-\frac{y^2+|z|^2}{c\tau}}\tau^{-(\frac{n+3}{2})}$ *and* $\Big|\partial_y^k\mathcal{P}_y(\tau,x,z)\Big|\le C_k\,e^{-\frac{y^2+|z|^2}{c\tau}}\tau^{-(\frac{n+k}{2}+1)}$*, $k\ge1$.*

(ii) Let $\gamma,\nu\ge0, s\ge0$. For each $\ell,k,m\in\mathbb{N}\cup\{0\}$, there exists a constant $C_{\gamma,\nu,k,\ell,m,s}>0$ such that, for every $x\in\mathbb{R}^n$ and $\tau>0$,

$$\int_{\mathbb{R}^{n+1}}|x|^\gamma|z|^\nu|\partial_y^k\partial_{z_i}^m\partial_{x_j}^\ell\mathcal{P}_y(\tau,x,z)|d\tau dz\le\begin{cases}C_{\gamma,\nu,\ell,k,m,s}\,y^{-(k+m-\ell-\nu+\gamma+s)}, & \text{if } s\ge0,\zeta>0,\\ C_{\gamma,\nu,\ell,k,m,s}\,y^{-s}, & \text{if } s>0,\zeta\le0,\end{cases}$$

for $\zeta=k+m-\ell-\nu+\gamma$ and $i=1,\dots,n,\ j=1,\dots,n$.

(iii) Let f such that $|x|^\alpha f\in L^\infty(\mathbb{R}^{n+1})$, $0<\alpha\le1$ and $s\ge0$. There exists a constant $C_{s,\alpha}>0$ such that, for every $x\in\mathbb{R}^n$ and $\tau>0$,

$$\int_{\mathbb{R}^{n+1}}|\partial_{x_i}\partial_y^2\mathcal{P}_y(\tau,x,z)f(t-\tau,x-z)|dzd\tau\le\begin{cases}C_{s,\alpha}y^{-(1-\alpha+s)}, & \text{if } s\ge0,\ \alpha<1.\\ C_{s,\alpha}y^{-s}, & \text{if } s>0,\ \alpha=1.\end{cases}$$

(iv) There exists a constant C such that for every $x\in\mathbb{R}^n$ and $\tau>0$,

$$\int_{\mathbb{R}^{n+1}}|\partial_\tau\mathcal{P}_y(\tau,x,z)|dzd\tau\le Cy^{-2}. \tag{3.13}$$

Proof. Along this proof will use Remark 3.76 and the estimates: $\partial_y^k\Big(\frac{ye^{-\frac{y^2}{4\tau}}}{\tau^{3/2}}\Big) \le C_k\, e^{-\frac{y^2}{8\tau}}\tau^{-(k/2+1)}$, $\Big|\partial_{x_i}^\ell\Big(e^{-\frac{|2x-z|^2\tanh\tau}{4}}\Big)\Big| \le C_\ell e^{-\frac{|2x-z|^2\tanh\tau}{8}}(\tanh\tau)^{\ell/2}$ and $|\partial_{z_i}^m e^{-\frac{|z|^2\coth\tau}{4}}| \le C_m e^{-\frac{|z|^2\coth\tau}{8}}(\coth\tau)^{m/2}$.

Estimate (i) is consequence of Remark 3.76. In order to prove (ii), as

$$|\partial_y^k\partial_{z_i}^m\partial_{x_j}^\ell\mathcal{P}_y(\tau,x,z)| \le \frac{C}{(\sinh(2\tau))^{n/2}}e^{-\frac{y^2}{c\tau}}\tau^{-(k/2+1)}e^{-\frac{|z|^2\coth\tau}{c}}(\coth\tau)^{m/2}e^{-c|x-\frac{z}{2}|^2\tanh\tau}(\tanh\tau)^{\ell/2},$$

again by Remark 3.76, for every $\lambda \ge 0$ we get

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}}|x|^\gamma|z|^\nu|\partial_y^k\partial_{z_i}^m\partial_{x_j}^\ell\mathcal{P}_y(\tau,x,z)|d\tau dz\\
&\le C\int_{\mathbb{R}^n}\int_0^\infty\frac{(|x-\frac{z}{2}|^\gamma+|\frac{z}{2}|^\gamma)|z|^\nu e^{-\frac{y^2}{c\tau}}e^{-\frac{|z|^2\coth\tau}{c}}e^{-c|x-\frac{z}{2}|^2\tanh\tau}}{\tau^{k/2}(\sinh(2\tau))^{n/2}}(\coth\tau)^{m/2}(\tanh\tau)^{\ell/2}\frac{d\tau}{\tau}dz\\
&\le C\int_{\mathbb{R}^n}\int_0^\infty\frac{e^{-\frac{y^2}{c\tau}}e^{-\frac{|z|^2\coth\tau}{c}}}{\tau^{\frac{k+n+m-\ell+\gamma-\nu+\lambda}{2}}}\frac{d\tau}{\tau}dz \le C\int_0^\infty\frac{e^{-\frac{y^2}{c\tau}}}{\tau^{\frac{k+m-\ell+\gamma-\nu+\lambda}{2}}}\frac{d\tau}{\tau}.
\end{aligned}$$

The constant C depends on γ,ν,ℓ,k,m and λ. The result follows by choosing $\lambda = s$ in the case $k+m-\ell+\gamma-\nu > 0$, and $\lambda = -(k+m-\ell+\gamma-\nu)+s$ in the case $k+m-\ell+\gamma-\nu \le 0$.

For (iii), as $|x|^\alpha f \in L^\infty(\mathbb{R}^{n+1})$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^{n+1}}|\partial_{x_i}\partial_y^2\mathcal{P}_y(\tau,x,z)f(t-\tau,x-z)|dzd\tau\\
&\le C\int_{\mathbb{R}^n}\int_0^\infty\frac{e^{-\frac{y^2}{c\tau}}e^{-\frac{|z|^2\coth\tau}{4}}e^{-\frac{|2x-z|^2\tanh\tau}{c}}\tanh\tau|2x-z|^{1-\alpha}|2x-z|^\alpha|f(t-\tau,x-z)|}{\tau(\sinh(2\tau))^{n/2}}\frac{d\tau}{\tau}dz\\
&\le C\int_{\mathbb{R}^n}\int_0^\infty\frac{e^{-\frac{y^2}{c\tau}}e^{-\frac{|z|^2\coth\tau}{4}}(\tanh\tau)^{\frac{1+\alpha}{2}}|x-z|^\alpha|f(t-\tau,x-z)|}{\tau(\sinh(2\tau))^{n/2}}\frac{d\tau}{\tau}dz\\
&+C\int_{\mathbb{R}^n}\int_0^\infty\frac{e^{-\frac{y^2}{c\tau}}e^{-\frac{|z|^2\coth\tau}{4}}(\tanh\tau)^{\frac{1+\alpha}{2}}|z|^\alpha|f(t-\tau,x-z)|}{\tau(\sinh(2\tau))^{n/2}}\frac{d\tau}{\tau}dz\\
&\le C\||x|^\alpha f\|_\infty\int_{\mathbb{R}^{n+1}}\int_0^\infty\frac{e^{-\frac{y^2}{c\tau}}e^{-\frac{|z|^2\coth\tau}{4}}(\tanh\tau)^{\frac{1+\alpha}{2}}}{\tau(\sinh(2\tau))^{n/2}}\frac{d\tau}{\tau}dz\\
&+C\|f\|_\infty\int_{\mathbb{R}^n}\int_0^\infty\frac{e^{-\frac{y^2}{c\tau}}e^{-\frac{|z|^2\coth\tau}{c}}(\tanh\tau)^{\frac{1+\alpha}{2}}}{\tau(\sinh(2\tau))^{n/2}(\coth\tau)^{\alpha/2}}\frac{d\tau}{\tau}dz\\
&\le C(\||x|^\alpha f\|_\infty+\|f\|_\infty)\int_0^\infty\frac{e^{-\frac{y^2}{c\tau}}}{\tau^{\frac{1-\alpha+s}{2}}}\frac{d\tau}{\tau}.
\end{aligned}$$

Regarding (iv), we have

$$\int_{\mathbb{R}^{n+1}} |\partial_\tau \mathcal{P}_y(\tau,x,z)|dzd\tau$$
$$\leq C\int_0^\infty \int_{\mathbb{R}^n} \frac{ye^{-\frac{y^2}{c\tau}}e^{-\frac{|z|^2\coth\tau}{c}}e^{-\frac{|2x-z|^2\tanh\tau}{c}}}{\tau^{3/2}(\sinh 2\tau)^{n/2}}\Big(\frac{1}{\tau}+\frac{\cosh(2\tau)}{\sinh(2\tau)}+\frac{|y|^2}{\tau^2}+\frac{|z|^2}{(\sinh\tau)^2}+\frac{|2x-z|^2}{(\cosh\tau)^2}\Big)dzd\tau$$
$$\leq C\int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-\frac{y^2}{c\tau}}e^{-\frac{|z|^2}{c\tau}}}{\tau^{1+n/2}}dz\frac{1}{\tau}d\tau \leq C\int_0^\infty \frac{e^{-\frac{y^2}{c\tau}}}{\tau}\frac{d\tau}{\tau} \leq \frac{C}{y^2}.$$

□

Remark 3.79. *Observe that the previous Lemma remains valid when we substitute* $\partial_{x_i}^\ell$ *and* $\partial_{x_i}^m$ *by* $\partial_{x_1}^{\delta_1}\dots\partial_{x_n}^{\delta_n}$ *and* $\partial_{z_1}^{\mu_1}\dots\partial_{z_n}^{\mu_n}$*, respectively, where* $\delta_i,\mu_i\in\mathbb{N}_0$ *and* $\delta_1+\dots+\delta_n=\ell$ *and* $\mu_1+\dots+\mu_n=m$.

Remark 3.80. *Observe that for bounded functions f, Lemma 3.78 (ii) assures that* $\left\|\partial_y^k\mathcal{P}_yf\right\|_{L^\infty(\mathbb{R}^{n+1})}\leq C\|f\|_\infty y^{-k}$. *Therefore, we can assume in Definition 3.67 that* $y<1$.

Lemma 3.81. *Let* $f\in L^\infty(\mathbb{R}^{n+1})$, $\alpha>0$, *and k,l integers bigger than α. Then, for $y>0$, the following conditions are equivalent:*

(a) $\left\|\partial_y^k\mathcal{P}_yf\right\|_{L^\infty(\mathbb{R}^{n+1})}\leq A_ky^{-k+\alpha}$

(b) $\left\|\partial_y^l\mathcal{P}_yf\right\|_{L^\infty(\mathbb{R}^{n+1})}\leq A_ly^{-l+\alpha}$,

where A_k and A_l are positive constants with $A_k\sim A_l$.

Proof. We only do the proof for $l=k+1$. For the other cases the proof is analogous. Let $l=k+1$. Suppose that f satisfies (a). By using the semigroup property and Lemma 3.78 (ii) we have, for every $(t,x)\in\mathbb{R}^{n+1}$,

$$\left|\partial_y^l\mathcal{P}_yf(t,x)\right| = \left|C\int_{\mathbb{R}^{n+1}}\partial_y\mathcal{P}_y(\tau,x,z)\Big|_{y/2}\partial_y^k\mathcal{P}_yf(t-\tau,x-z)\Big|_{y/2}d\tau dz\right|$$
$$\leq Cy^{-k+\alpha}\int_{\mathbb{R}^{n+1}}\left|\partial_y\mathcal{P}_y(\tau,x,z)\Big|_{y/2}\right|d\tau dz\leq Cy^{-(k+1)+\alpha}.$$

For the converse, Remark 3.80 allows the integration $\partial_y^k\mathcal{P}_yf(t,x)=\int_y^\infty\partial_z^{k+1}\mathcal{P}_zf(t,x)dz$, that gives the result. □

Corollary 3.82. *Let $\alpha>0$. If $f\in\Lambda_\alpha^{\mathcal{P}}$, then for every $0<\beta<\alpha$, $f\in\Lambda_\beta^{\mathcal{P}}$.*

Proof. Suppose that $f\in\Lambda_\alpha^{\mathcal{P}}$ and let $k_\alpha=[\alpha]+1$. We have (for $y<1$)

$$\left\|\partial_y^{k_\alpha}\mathcal{P}_yf\right\|\leq A_{k_\alpha}\|f\|_{\Lambda_\alpha^{\mathcal{P}}}y^{-k_\alpha+\alpha}\leq A_{k_\alpha}\|f\|_{\Lambda_\alpha^{\mathcal{P}}}y^{-k_\alpha+\beta}.$$

Then, Lemma 3.81 gives the result.

□

Lemma 3.83. *Let $\alpha > 0$, $f \in \Lambda_\alpha^{\mathcal{P}}$ and $k = [\alpha] + 1$.*

1. *For every $\gamma \geq 0$ and $m, j \in \mathbb{N}_0$ such that $\gamma + m + j \geq k$ there exists a constant $C_{\gamma,m,j}$ such that $\||\cdot|^\gamma \partial_y^m \partial_{x_i}^j \mathcal{P}_y f\|_\infty \leq C_{\gamma,m,j}\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-(\gamma+m+j)+\alpha}$.*
2. *For every m such that $m+2 \geq k$, there exists a constant C_m such that $\|\partial_y^m \partial_t \mathcal{P}_y f\|_\infty \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-(m+2)+\alpha}$.*

Proof. Observe that the case $\gamma = j = 0$ follows from the definition of the space $\Lambda_\alpha^{\mathcal{P}}$, so we will exclude it in the following. Let us analyze first the case when $m \geq k$. By the semigroup property and integration by parts we have

$$\begin{aligned}
\left||x|^\gamma \partial_y^m \partial_{x_i}^j \mathcal{P}_y f(t,x)\right| &= \left|C|x|^\gamma \partial_{x_i}^j \int_{\mathbb{R}^{n+1}} \mathcal{P}_{y/2}(\tau,x,z)\partial_y^m \mathcal{P}_y f(t-\tau,x-z)\big|_{y/2} d\tau dz\right| \\
&= \left|C|x|^\gamma \int_{\mathbb{R}^{n+1}} (\partial_{x_i} + \partial_{z_i})^j \mathcal{P}_{y/2}(\tau,x,z)\partial_y^m \mathcal{P}_y f(t-\tau,x-z)\big|_{y/2} d\tau dz\right| \\
&\leq C\|\partial_y^m \mathcal{P}_y f\big|_{y/2}\|_\infty \sum_{p+q=j} \int_{\mathbb{R}^{n+1}} |x|^\gamma |\partial_{z_i}^p \partial_{x_i}^q \mathcal{P}_{y/2}(\tau,x,z)| d\tau dz \\
&\leq C_{\gamma,m,j}\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-(\gamma+m+j)+\alpha}, \quad (t,x) \in \mathbb{R}^{n+1}.
\end{aligned}$$

In the last inequality we have used the hypothesis on f and Lemma 3.78 (ii) in each summand. We have chosen $s = j + \gamma$ in the case $p - q + \gamma \leq 0$, and $s = 2q$ in the case $p - q + \gamma > 0$.

Now we prove epigraph 2 for $m \geq k$. By the semigroup property, the hypothesis on f and Lemma 3.78 (iv) we have, for $(t,x) \in \mathbb{R}^{n+1}$,

$$\begin{aligned}
\left|\partial_y^m \partial_t \mathcal{P}_y f(t,x)\right| &= \left|C \int_{\mathbb{R}^{n+1}} \partial_\tau \mathcal{P}_{y/2}(\tau,x,z)\partial_y^m \mathcal{P}_y f(t-\tau,x-z)\big|_{y/2} d\tau dz\right| \\
&\leq C\|\partial_y^m \mathcal{P}_y f\big|_{y/2}\|_\infty \int_{\mathbb{R}^{n+1}} |\partial_\tau \mathcal{P}_{y/2}(\tau,x,z)| d\tau dz \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-(m+2)+\alpha}.
\end{aligned}$$

In both cases, for $m < k$ we start from the above estimates for the case $m = k$ and then we perform an $k - m$ iterated integration. ☐

Remark 3.84. *Observe that the previous Lemma remains valid when we substitute $\partial_{x_i}^j$ by $\partial_{x_1}^{\delta_1} \dots \partial_{x_n}^{\delta_n}$, where $\delta_i \in \mathbb{N}_0$ and $\delta_1 + \dots + \delta_n = j$.*

3.2.2 Proof of main results.

At first we shall show that, for $0 < \alpha < 1$, the pointwise Definition 3.65 is equivalent to the Definition 3.67 given by using of Poisson semigroup.

Theorem 3.85. *Let $0 < \alpha < 1$. Then*

$$C_{t,\mathcal{H}}^{\alpha/2,\alpha} = \Lambda_\alpha^{\mathcal{P}},$$

with equivalence of norms.

Proof. For $f \in C^{\alpha/2,\alpha}_{t,\mathcal{H}}(\mathbb{R}^{n+1})$, we write

$$\begin{aligned} y\partial_y\mathcal{P}_yf(t,x) &= \int_{\mathbb{R}^{n+1}} y\partial_y\mathcal{P}_y(\tau,x,z)(f(t-\tau,x-z)-f(t,x))d\tau dz \\ &\quad + f(t,x)\int_{\mathbb{R}^{n+1}} y\partial_y\mathcal{P}_y(\tau,x,z)d\tau dz = I_1+I_2. \end{aligned}$$

By Lemma 3.78 (i) we have

$$\begin{aligned} |I_1| &\le \int_{\mathbb{R}^{n+1}} |y\partial_y\mathcal{P}_y(\tau,x,z)||f(t-\tau,x-z)-f(\tau,x)|dz \\ &\le C\|f\|_{C^{\alpha/2,\alpha}_{t,\mathcal{H}}}\int_{\mathbb{R}^n}\int_0^\infty \frac{ye^{-\frac{y^2+|z|^2}{c\tau}}(\tau^{1/2}+|z|)^\alpha}{\tau^{\frac{n+3}{2}}}d\tau dz \le C\|f\|_{C^{\alpha/2,\alpha}_{t,\mathcal{H}}}y^\alpha. \end{aligned}$$

Regarding I_2, as $\displaystyle\int_0^\infty y\partial_y(ye^{-y^2/4\tau})\frac{d\tau}{\tau^{3/2}}=0$ we can write

$$I_2 = f(t,x)\frac{1}{2\sqrt{\pi}}\int_0^\infty y\partial_y(ye^{-y^2/4\tau})\Big(e^{-\tau\mathbb{H}}1(t,x)-1\Big)\frac{d\tau}{\tau^{3/2}}.$$

Since $\left|y\partial_y(ye^{-\frac{y^2}{4\tau}})\right| \le Cye^{-\frac{y^2}{c\tau}}$, we can proceed as in the proof of Lemma 3.77 (3) to get

$$|I_2| \le C[f]_{M^\alpha}y^\alpha.$$

Conversely, suppose that $f\in\Lambda^{\mathcal{P}}_\alpha$. We can write, for every $y>0$, $t,\tau\in\mathbb{R}$ and $x,z\in\mathbb{R}^n$,

$$\begin{aligned} &f(t+\tau,x+z)-f(t,x) \\ &= (\mathcal{P}_yf(t+\tau,x+z)-\mathcal{P}_yf(t,x))+(f(t+\tau,x+z)-\mathcal{P}_yf(t+\tau,x+z))+(\mathcal{P}_yf(t,x)-f(t,x)). \end{aligned}$$

Let $y=\tau^{1/2}+|z|$. For the second summand we have

$$\begin{aligned} \sup_{(t,x)\in\mathbb{R}^{n+1}}|f(t+\tau,x+z)-\mathcal{P}_yf(t+\tau,x+z)| &= \sup_{(t,x)\in\mathbb{R}^{n+1}}\left|-\int_0^y\partial_{y'}\mathcal{P}_{y'}f(t+\tau,x+z)dy'\right| \\ &\le C\|f\|_{\Lambda^{\mathcal{P}}_\alpha}\int_0^y y'^{-1+\alpha}dy' = C\|f\|_{\Lambda^{\mathcal{P}}_\alpha}y^\alpha = C\|f\|_{\Lambda^{\mathcal{P}}_\alpha}(\tau^{1/2}+|z|)^\alpha. \end{aligned}$$

A similar estimate can be performed for the third summand. On the other hand, by the Mean Value Theorem and Lemma 3.83, we have

$$\begin{aligned} |\mathcal{P}_yf(t+\tau,x+z)-\mathcal{P}_yf(t,x)| &\le |\mathcal{P}_yf(t+\tau,x+z)-\mathcal{P}_yf(t+\tau,x)|+|\mathcal{P}_yf(t+\tau,x)-\mathcal{P}_yf(t,x)| \\ &\le |\nabla_x\mathcal{P}_yf(t+\tau,x+\theta z)||z|+|\partial_t\mathcal{P}_yf(t+\lambda\tau,x)||\tau| \\ &\le C\|f\|_{\Lambda^{\mathcal{P}}_\alpha}y^{-1+\alpha}|z|+C\|f\|_{\Lambda^{\mathcal{P}}_\alpha}y^{-2+\alpha}|\tau| \\ &= C\|f\|_{\Lambda^{\mathcal{P}}_\alpha}(\tau^{1/2}+|z|)^\alpha. \end{aligned}$$

Finally, we shall see that $(1+|x|)^\alpha f \in L^\infty(\mathbb{R}^{n+1})$. Let $k=[\alpha]+1=1$. From Lemma 3.83 we know that $\||\cdot|^k\mathcal{P}_y f\|_\infty \le C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-k+\alpha}$. Then, for $t\in\mathbb{R}$ and $x\in\mathbb{R}^n$ such that $|x|>1$, we have

$$\begin{aligned}|x|^\alpha|f(t,x)| &\le |x|^\alpha \sup_{0<y<\frac{1}{|x|}} |\mathcal{P}_y f(t,x)| \le |x|^\alpha \sup_{0<y<\frac{1}{|x|}} \Big(|\mathcal{P}_y f(t,x)-\mathcal{P}_{\frac{1}{|x|}}f(t,x)| + |\mathcal{P}_{\frac{1}{|x|}}f(t,x)|\Big)\\ &\le |x|^\alpha \sup_{0<y<\frac{1}{|x|}} \left|\int_y^{\frac{1}{|x|}} \partial_{z_1}\mathcal{P}_{z_1}f(t,x)dz_1\right| + |x|^{\alpha-k}|x|^k|\mathcal{P}_{\frac{1}{|x|}}f(t,x)|\\ &\le |x|^\alpha\|f\|_{\Lambda_\alpha^{\mathcal{P}}} \sup_{0<y<\frac{1}{|x|}} \int_y^{\frac{1}{|x|}} z_1^{-1+\alpha}dz_1 + C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} \le C\|f\|_{\Lambda_\alpha^{\mathcal{P}}}<\infty.\end{aligned}$$

□

Elliptic Hermite setting.

Let g an $L^\infty(\mathbb{R}^n)$ function. Consider the function $f(t,x)=g(x)$. It is clear that $g\in C^\alpha_{\mathcal{H}}$ if and only if $f\in C^{\alpha/2,\alpha}_{t,\mathcal{H}}$. Moreover, as $\mathcal{P}_y f(t,x) = P^{\mathcal{H}}_y g(x)$, $g\in\Lambda_\alpha^{P^{\mathcal{H}}}$ if and only if $f\in\Lambda_\alpha^{\mathcal{P}}$. Hence, for $0<\alpha<1$, Proposition 3.66 and Theorem 3.85 have as consequences the continuity of the functions $g\in\Lambda_\alpha^{P^{\mathcal{H}}}$ and the identity $\Lambda_\alpha^{P^{\mathcal{H}}} = C^\alpha_{\mathcal{H}}(\mathbb{R}^n)$, $0<\alpha<1$.

Proposition 3.86. *Let $\alpha>0$ and $f\in\Lambda_\alpha^{\mathcal{P}}$. Then $|x|^\alpha f\in L^\infty(\mathbb{R}^{n+1})$.*

Proof. Since f is bounded we only have to prove the result for $|x|>1$. If α is not an integer we can use the same argument as in the proof of Theorem 3.85.

Suppose $\alpha=1$. By using Lemma 3.83 we have that

$$\Big||x|^2\mathcal{P}_y f(t,x)\Big| \le C\frac{\|f\|_{\Lambda_1^{\mathcal{P}}}}{y},\ (y<1), \quad \Big||x|\partial_v\mathcal{P}_v f(t,x)\Big|_{v=\frac{1}{|x|}} \le C\|f\|_{\Lambda_1^{\mathcal{P}}}|x|$$

and $\Big||x|\mathcal{P}_{1/|x|}f(t,x)\Big| \le C\|f\|_{\Lambda_1^{\mathcal{P}}}$.

Then, by using that $\partial_{z_1}\mathcal{P}_{z_1}f(t,x) = -\int_{z_1}^{\frac{1}{x}} \partial^2_{z_2}\mathcal{P}_{z_2}f(t,x)dz_2 + \partial_v\mathcal{P}_v f(t,x)\Big|_{v=\frac{1}{|x|}}$, we have

$$\begin{aligned}&||x|f(t,x)|\\ &\le |x| \sup_{0<y<\frac{1}{|x|}} \left| -\int_y^{\frac{1}{|x|}} \Big(\int_{z_1}^{\frac{1}{|x|}} \partial^2_{z_2}\mathcal{P}_{z_2}f(t,x)dz_2 + \partial_v\mathcal{P}_v f(t,x)\Big|_{v=\frac{1}{|x|}}\Big)dz_1\right| + \Big||x|\mathcal{P}_{1/|x|}f(t,x)\Big|\\ &\le |x| \sup_{0<y<\frac{1}{|x|}} \int_y^{\frac{1}{|x|}}\int_{z_1}^{\frac{1}{|x|}} \left|\partial^2_{z_2}\mathcal{P}_{z_2}f(t,x)\right| dz_2dz_1 + C\|f\|_{\Lambda_1^{\mathcal{P}}} \sup_{0<y<\frac{1}{|x|}} |x|\left(\frac{1}{|x|}-y\right) + C\|f\|_{\Lambda_1^{\mathcal{P}}}\\ &\le C\|f\|_{\Lambda_1^{\mathcal{P}}}|x| \sup_{0<y<\frac{1}{|x|}} \int_y^{\frac{1}{|x|}}\int_{z_1}^{\frac{1}{|x|}} z_2^{-1}dz_2dz_1 + C\|f\|_{\Lambda_1^{\mathcal{P}}}.\end{aligned}$$

Since for every $0 < y < \frac{1}{|x|}$ we have

$$\begin{aligned}|x| \int_y^{\frac{1}{|x|}} \int_{z_1}^{\frac{1}{|x|}} z_2^{-1} dz_2 dz_1 &= |x| \int_y^{\frac{1}{|x|}} \left(\log\left(\frac{1}{|x|}\right) - \log z_1 \right) dz_1 \\ &= |x| \left[\log\left(\frac{1}{|x|}\right) \left(\frac{1}{|x|} - y\right) - \left(\frac{1}{|x|} \log \frac{1}{|x|} - \frac{1}{|x|} - y \log y + y \right) \right] \\ &= |x| y \log(|x| y) + |x| \left(\frac{1}{|x|} - y \right) \le C,\end{aligned}$$

we conclude that $|x||f(t,x)| \le C\|f\|_{\Lambda_1^{\mathcal{P}}}$.

For the cases in which α is an integer bigger that 1, we have to write $\partial_{z_1} \mathcal{P}_{z_1} f$ in terms of the integral of the derivative of order k, where $k = [\alpha] + 1$, and proceed analogously. We leave the details to the interested reader. □

Proof of Theorem 3.68.

Proof of epigraph 1 in Theorem 3.68.

Suppose that f satisfies (3.2) and $(1 + |x|)^\alpha f \in L^\infty(\mathbb{R}^{n+1})$. Let $k = [\alpha] + 1$. Since $\int_{\mathbb{R}^{n+1}} \partial_y^k \mathcal{P}_y(\tau, x, z) f(t - \tau, x + z) d\tau dz = \int_{\mathbb{R}^{n+1}} \partial_y^k \mathcal{P}_y(\tau, x, -z) f(t - \tau, x - z) d\tau dz$, and $\int_0^\infty \partial_y^k \left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \right) d\tau = 0$ we have

$$\begin{aligned}\partial_y^k \mathcal{P}_y f(t,x) &= \frac{1}{2} \int_{\mathbb{R}^{n+1}} \partial_y^k \mathcal{P}_y(\tau, x, z)(f(t-\tau, x-z) + f(t-\tau, x+z) - 2f(t,x)) d\tau dz \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^{n+1}} \left(\partial_y^k \mathcal{P}_y(\tau, x, z) - \partial_y^k \mathcal{P}_y(\tau, x, -z) \right) f(t-\tau, x-z) d\tau dz \qquad (3.14) \\ &\quad + \frac{f(t,x)}{2\sqrt{\pi}} \int_0^\infty \partial_y^k \left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \right) \left(e^{-\tau \mathbb{H}} 1(t,x) - 1 \right) d\tau \\ &= I_1 + I_2 + I_3.\end{aligned}$$

By Lemma 3.78, $|I_1| \le C \int_{\mathbb{R}^n} \int_0^\infty \frac{e^{-\frac{y^2 + |z|^2}{c\tau}} (\tau^{1/2} + |z|)^\alpha}{\tau^{\frac{n+k}{2}}} \frac{d\tau}{\tau} dz \le C y^{\alpha - k}$. For I_3 we use that $(1 + |x|)^\alpha f \in L^\infty(\mathbb{R}^{n+1})$ and the proof of Lemma 3.77 (3) to get

$$|y^k I_3| = \left| \frac{f(t,x)}{2\sqrt{\pi}} \int_0^\infty y^k \partial_y^k \left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}} \right) \left(e^{-\tau \mathbb{H}} 1(t,x) - 1 \right) d\tau \right| \le C[f]_{M^\alpha} y^\alpha.$$

Regarding I_2, we have

$$
\begin{aligned}
2I_2 &= \int_{\mathbb{R}^{n+1}} (\partial_y^k \mathcal{P}_y(\tau,x,z) - \partial_y^k \mathcal{P}_y(\tau,x,-z)) f(t-\tau,x-z)\,d\tau dz \\
&= \int_{\mathbb{R}^n}\int_0^\infty \partial_y^k \left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}}\right) \frac{e^{-\frac{|z|^2}{4}\coth\tau}}{2\sqrt{\pi}(2\pi\sinh(2\tau))^{n/2}} \left(e^{-\frac{|2x-z|^2}{4}\tanh\tau} - e^{-\frac{|2x+z|^2}{4}\tanh\tau}\right) \\
&\times f(t-\tau,x-z)\,d\tau dz.
\end{aligned}
$$

By the Mean Value Theorem applied to the function $e^{-\frac{|2x-z|^2}{4}\tanh\tau}$ we get

$$
\begin{aligned}
|2I_2| &\le C\int_0^\infty \partial_y^k\left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}}\right)\int_{\mathbb{R}^n} \frac{e^{-\frac{|z|^2}{4}\coth\tau}}{(\sinh(2\tau))^{n/2}} (\tanh\tau)^{1/2}|z||f(t-\tau,x-z)|\,dz d\tau \\
&\underbrace{\le}_{\frac{z\sqrt{\coth\tau}}{2}=w} C\|f\|_\infty \int_0^\infty \partial_y^k\left(\frac{y e^{-\frac{y^2}{4\tau}}}{\tau^{3/2}}\right)\int_{\mathbb{R}^n} \frac{e^{-|w|^2}|w|(\tanh\tau)^{1/2}}{(\sinh(2\tau))^{n/2}(\coth\tau)^{\frac{n+1}{2}}}\,dw d\tau \\
&\le C_k\|f\|_\infty \int_0^\infty \frac{e^{-\frac{y^2}{c\tau}}}{\tau^{k/2+1}} \frac{(\tanh\tau)^{1/2}}{(\sinh(2\tau))^{n/2}(\coth\tau)^{\frac{n+1}{2}}}\,d\tau \\
&\le C_k\|f\|_\infty \int_0^\infty \frac{e^{-\frac{y^2}{c\tau}}}{\tau^{k/2}}\tau^{\alpha/2}\frac{d\tau}{\tau} \le C_k\|f\|_\infty y^{-k+\alpha}.
\end{aligned}
$$

We conclude that $f \in \Lambda_\alpha^{\mathcal{P}}$.

Now we prove the converse. Assume $f \in \Lambda_\alpha^{\mathcal{P}}$, $0<\alpha<2$. If $0<\alpha<1$, the result is a consequence of Theorem 3.85. If $1 \le \alpha < 2$, by Corollary 3.82 and Theorem 3.85, $f \in \Lambda_{\alpha'}^{\mathcal{P}} = C_{t,\mathcal{H}}^{\alpha'/2,\alpha'}$, for some $\alpha' < 1$. Therefore, $\|y\partial_y\mathcal{P}_y f\|_{L^\infty(\mathbb{R}^{n+1})} \to 0$, as $y \to 0^+$. On the other hand, by the proof of Proposition 3.66 we know that $\|\mathcal{P}_y f - f\|_{L^\infty(\mathbb{R}^{n+1})} \to 0$, as $y \to 0^+$. Hence, we can write

$$
f(t,x) = \int_0^y y'\partial_{y'}^2 \mathcal{P}_{y'} f(t,x)\,dy' - y\partial_y \mathcal{P}_y f(t,x) + \mathcal{P}_y f(t,x).
$$

We only do computations for $g(t,x) = \mathcal{P}_y f(t,x)$. For the other cases we have to follow the same path. By using Lemma 3.83 we have, for $y = \tau^{1/2} + |z|$,

$$
\begin{aligned}
|g(t-\tau,x+z)&+g(t-\tau,x-z)-2g(t,x)| \\
&\le |\nabla_x g(t-\tau,x+\theta z) - \nabla_x g(t-\tau,x-\lambda z)||z| + 2|\partial_t g(t-\eta\tau,x)|\tau \\
&\le \left|D_x^2 g(t-\tau,x+\nu z)\right|(\theta+\lambda)|z|^2 + 2|\partial_t \mathcal{P}_y f(t-\eta\tau,x)|\tau \\
&\le C\|f\|_{\Lambda_\alpha^{\mathcal{P}}}(\tau^{1/2}+|z|)^{-2+\alpha}(|z|^2+\tau) \le C\|f\|_{\Lambda_\alpha^{\mathcal{P}}}(\tau^{1/2}+|z|)^{\alpha},
\end{aligned}
$$

where $0<\theta,\lambda<1$, $-1<\nu<1$.

The fact that $(1+|x|)^\alpha f \in L^\infty(\mathbb{R}^{n+1})$ follows from Proposition 3.86.

□

For the proof of epigraph 2 in Theorem 3.68, we shall prove the following theorem.

Theorem 3.87. *Suppose that* $\alpha > 2$. *Then,* $f \in \Lambda_\alpha^{\mathcal{P}}$ *if and only if* $f \in L^\infty(\mathbb{R}^{n+1})$,

$$\partial_{x_i} f, x_i f \in \Lambda_{\alpha-1}^{\mathcal{P}},\ \ i = 1, \dots, n, \quad \text{and} \quad \partial_t f \in \Lambda_{\alpha-2}^{\mathcal{P}}.$$

In this case the following equivalence holds

$$\|f\|_{\Lambda_\alpha^{\mathcal{P}}} \sim \|f\|_\infty + \sum_{i=1}^n \left(\|\partial_{x_i} f\|_{\Lambda_{\alpha-1}^{\mathcal{P}}} + \|x_i f\|_{\Lambda_{\alpha-1}^{\mathcal{P}}} \right) + \|\partial_t f\|_{\Lambda_{\alpha-2}^{\mathcal{P}}}.$$

For the reader's convenience, the proof of this Theorem 3.87 will be divided in several steps.

Proposition 3.88. *Suppose that* $f \in \Lambda_\alpha^{\mathcal{P}}$ *with* $\alpha > 2$. *Then,* $\partial_t f \in \Lambda_{\alpha-2}^{\mathcal{P}}$.

Proof. Let $2 < \alpha < 3$. By Lemma 3.83 we have

$$\|\partial_y \partial_t \mathcal{P}_y f\|_\infty \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-3+\alpha}. \tag{3.15}$$

If $0 < y < 1$, we have $\partial_t \mathcal{P}_y f = \int_y^1 \partial_z \partial_t \mathcal{P}_z f dz + \partial_t \mathcal{P}_y f\Big|_{y=1}$ and this implies that $\partial_t \mathcal{P}_y f$ is in $L^\infty(\mathbb{R}^{n+1})$ uniformly on y. Moreover, since $|\partial_t \mathcal{P}_{y'} f - \partial_t \mathcal{P}_y f| \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} \int_y^{y'} z^{-3+\alpha} dz \to 0$ as $(y', y) \to 0$, then $\partial_t \mathcal{P}_y f$ converges uniformly when $y \to 0$. As $\mathcal{P}_y f$ converges uniformly to f when $y \to 0$, we conclude that $\partial_t f$ exists and it is the uniform limit of $\partial_t \mathcal{P}_y f = \mathcal{P}_y \partial_t f$. Hence, $\partial_y \mathcal{P}_y \partial_t f = \partial_y \partial_t \mathcal{P}_y f$. The last identity together with inequality (3.15) implies $\partial_t f \in \Lambda_{\alpha-2}^{\mathcal{P}}$.

If $\alpha \geq 3$, by Corollary 3.82, the function $f \in \Lambda_\beta^{\mathcal{P}}$ for some $\beta < 1$. Hence by the thoughts developed before, $\partial_t f$ exists and $\partial_t \mathcal{P}_y f = \mathcal{P}_y \partial_t f$. The proof for the other values of α is analogous.

□

Proposition 3.89. *Suppose that* $f \in \Lambda_\alpha^{\mathcal{P}}$ *with* $\alpha > 1$. *Then,* $\partial_{x_i} f \in \Lambda_{\alpha-1}^{\mathcal{P}},\ i = 1, \dots, n$.

Proof. Let $1 < \alpha < 3$. By Lemma 3.83 we have $\left\|\partial_y^2 \partial_{x_i} \mathcal{P}_y f\right\|_\infty \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-3+\alpha}$. For $y < 1$, an integration gives

$$\left|\partial_y \partial_{x_i} \mathcal{P}_y f(t, x)\right| \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-2+\alpha} + C\left\|\partial_y \partial_{x_i} \mathcal{P}_y f\Big|_{y=1}\right\|_\infty.$$

We can proceed as in the proof of Proposition 3.88 and we get that $\partial_{x_i} f$ does exist and $\|\partial_{x_i} f\|_\infty \leq C$. To prove that $\partial_{x_i} f \in \Lambda_{\alpha-1}^{\mathcal{P}}$, we shall see that $\|\partial_y^2 \mathcal{P}_y(\partial_{x_i} f)\|_\infty \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-3+\alpha}$. Observe that

$$\partial_y^2 \mathcal{P}_y(\partial_{x_i} f)(t, x) = \partial_{x_i} \partial_y^2 \mathcal{P}_y f(t, x) - \int_{\mathbb{R}^{n+1}} \partial_{x_i} \partial_y^2 \mathcal{P}_y(\tau, x, z) f(t - \tau, x - z) d\tau dz = I + II.$$

By Lemma 3.83 we have that $|I| \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-3+\alpha} = C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-2+(\alpha-1)}$. As $f \in \Lambda_\alpha^{\mathcal{P}}$, $1 < \alpha < 3$, by Proposition 3.86 we know that $|x|f \in L^\infty(\mathbb{R}^{n+1})$. Hence, by Lemma 3.78 (iii) we get that $|II| \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-3+\alpha} = C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-2+(\alpha-1)}$.

Suppose now $3 \le \alpha < 5$. By Corollary 3.82, $f \in \Lambda^{\mathcal{P}}_{\beta}$ for all $\beta < 3$. Then, the result just proved says that $\partial_{x_i} f \in \Lambda^{\mathcal{P}}_{\gamma}$, for all $\gamma < 2$ and $\partial^2_{x_i} f \in \Lambda^{\mathcal{P}}_{\delta}$, for all $\delta < 1$. We shall see that $\|\partial^4_y \mathcal{P}_y(\partial_{x_i} f)\|_\infty \le C\|f\|_{\Lambda^{\mathcal{P}}_\alpha} y^{-4+(\alpha-1)}$. As $\mathcal{P}_y(\partial_{x_i} f)$ satisfies (3.8), it is enough to prove that $\|\partial^2_y(-\sum_{j=1}^n \partial^2_{x_j} + |x|^2 + \partial_t)\mathcal{P}_y(\partial_{x_i} f)\|_\infty \le C\|f\|_{\Lambda^{\mathcal{P}}_\alpha} y^{-5+\alpha}$.

Observe that

$$\begin{aligned}
\partial^2_y \partial^2_{x_j} \mathcal{P}_y(\partial_{x_i} f)(t,x) &= \partial^2_y \partial^2_{x_j} \partial_{x_i} \mathcal{P}_y f(t,x) - \partial^2_{x_j} \int_{\mathbb{R}^{n+1}} \partial^2_y \partial_{x_i} \mathcal{P}_y(\tau,x,z) f(t-\tau,x-z)\,d\tau dz \\
&= \partial^2_y \partial^2_{x_j} \partial_{x_i} \mathcal{P}_y f(t,x) - \int_{\mathbb{R}^{n+1}} \partial^2_y \partial^2_{x_j} \partial_{x_i} \mathcal{P}_y(\tau,x,z) f(t-\tau,x-z)\,d\tau dz \\
&\quad - 2\int_{\mathbb{R}^{n+1}} \partial^2_y \partial_{x_i} \partial_{x_j} \mathcal{P}_y(\tau,x,z) \partial_{x_j} f(t-\tau,x-z)\,d\tau dz \\
&\quad - \int_{\mathbb{R}^{n+1}} \partial^2_y \partial_{x_i} \mathcal{P}_y(\tau,x,z) \partial^2_{x_j} f(t-\tau,x-z)\,d\tau dz.
\end{aligned}$$

The first summand is bounded by $C\|f\|_{\Lambda^{\mathcal{P}}_\alpha} y^{-5+\alpha}$ because of Lemma 3.83. As f and $\partial_{x_i} f$ are bounded functions, by using Lemma 3.78 (ii) we get the desired boundedness for the second and third summand. Finally, Lemma 3.78 (iii) says that the forth summand is bounded by $Cy^{-(1-\nu+s)}$, where $\nu < 1$ and $s > 0$, then by choosing ν and s with $s - \nu = 4 - \alpha$ we get the estimate.

To prove that $\||\cdot|^2 \partial^2_y \mathcal{P}_y(\partial_{x_i} f)\|_\infty \le C\|f\|_{\Lambda^{\mathcal{P}}_\alpha} y^{-5+\alpha}$, we write

$$|x|^2 \partial^2_y \mathcal{P}_y(\partial_{x_i} f)(t,x) = |x|^2 \partial_{x_i} \partial^2_y \mathcal{P}_y f(t,x) - |x|^2 \int_{\mathbb{R}^{n+1}} \partial_{x_i} \partial^2_y \mathcal{P}_y(\tau,x,z) f(t-\tau,x-z)\,d\tau dz$$

By Lemma 3.83 we know that the first summand is bounded by $C\|f\|_{\Lambda^{\mathcal{P}}_\alpha} y^{-5+\alpha}$. For the second summand we have

$$\begin{aligned}
|x|^2 &\int_{\mathbb{R}^{n+1}} |\partial_{x_i} \partial^2_y \mathcal{P}_y(\tau,x,z) f(t-\tau,x-z)|\,d\tau dz \\
&\qquad \le C \int_{\mathbb{R}^{n+1}} |\partial_{x_i} \partial^2_y \mathcal{P}_y(\tau,x,z)|(|x-z|^2 + |z|^2)|f(t-\tau,x-z)|\,d\tau dz,
\end{aligned}$$

and by Lemma 3.78 (iii) applied to $|x|^2 f$ and Lemma 3.78 (ii) we get the desired bound $C\|f\|_{\Lambda^{\mathcal{P}}_\alpha} y^{-5+\alpha}$.

To get the estimate for $\|\partial^2_y \partial_t \mathcal{P}_y(\partial_{x_i} f)\|_\infty$, we write

$$\partial^2_y \partial_t \mathcal{P}_y(\partial_{x_i} f)(t,x) = \partial^2_y \partial_{x_i} \partial_t \mathcal{P}_y f(t,x) - \int_{\mathbb{R}^{n+1}} \partial_{x_i} \partial^2_y \mathcal{P}_y(\tau,x,z) \partial_t f(t-\tau,x-z)\,d\tau dz.$$

By Proposition 3.88 we know that $\partial_t f \in \Lambda^{\mathcal{P}}_{\alpha-2}$, $1 \le \alpha - 2 < 3$. Hence, as $\partial^2_y \partial_{x_i} \partial_t \mathcal{P}_y f(t,x) = \partial^2_y \partial_{x_i} \mathcal{P}_y(\partial_t f)(t,x)$, by applying epigraph 1 of Lemma 3.83 we get that the first summand is bounded by $C\|f\|_{\Lambda^{\mathcal{P}}_\alpha} y^{-5+\alpha}$, and by Lemma 3.78 (iii) applied to $\partial_t f$ we get the same bound for the second summand.

The rest of the cases, $2m+1 \le \alpha < 2m+3$, can be handled analogously by estimating the norms $\|\partial^2_y(-\sum_j \partial^2_{x_j} + |x|^2 + \partial_t)^m \mathcal{P}_y(\partial_{x_i} f)\|_\infty$. We leave the details to the reader. □

Proposition 3.90. *Suppose that $f \in \Lambda_\alpha^{\mathcal{P}}$ with $\alpha > 1$. Then, $x_i f \in \Lambda_{\alpha-1}^{\mathcal{P}}$, $i = 1, \dots, n$.*

Proof. Consider the case $1 < \alpha < 2$. By Proposition 3.86 we know that $x_i f \in L^\infty(\mathbb{R}^{n+1})$. In addition, we can write

$$\partial_y \mathcal{P}_y(x_i f)(t,x) = x_i \partial_y \mathcal{P}_y f(t,x) - \int_{\mathbb{R}^{n+1}} z_i \mathcal{P}_y(\tau,x,z) f(t-\tau, x-z)\, d\tau dz,$$

and by using Lemma 3.83 (1) for the first summand and Lemma 3.78 together with the boundedness of f for the second summand, we get that $\|\partial_y \mathcal{P}_y(x_i f)\|_\infty \le C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-2+\alpha}$.

Let $2 < \alpha < 3$. We have to prove that $\|\partial_y^2 \mathcal{P}_y(x_i f)\|_\infty \le C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-3+\alpha}$. As $\mathcal{P}_y(x_i f)$ satisfies (3.8), we have

$$\begin{aligned}\left\|\partial_y^2 \mathcal{P}_y(x_i f)\right\|_\infty &= \left\|\left[\partial_t - \sum_{j=1}^n \partial_{x_j}^2 + |x|^2\right] \mathcal{P}_y(x_i f)\right\|_\infty \\ &\le \|\partial_t \mathcal{P}_y(x_i f)\|_\infty + \sum_{j=1}^n \|\partial_{x_j}^2 \mathcal{P}_y(x_i f)\|_\infty + \||\cdot|^2 \mathcal{P}_y(x_i f)\|_\infty.\end{aligned}$$

As $\partial_t f$ is well defined and bounded, see Proposition 3.88,

$$\begin{aligned}\partial_t \mathcal{P}_y(x_i f)(t,x) &= \int_{\mathbb{R}^{n+1}} \mathcal{P}_y(\tau,x,z)(x_i - z_i)\partial_t f(t-\tau, x-z)\, d\tau dz \\ &= x_i \mathcal{P}_y(\partial_t f)(t,x) - \int_{\mathbb{R}^{n+1}} z_i \mathcal{P}_y(\tau,x,z)\partial_t f(t-\tau,x-z)\, d\tau dz.\end{aligned}$$

Therefore, by using Proposition 3.88 and epigraph 1 of Lemma 3.83 for $\partial_t f$, we get that the first summand is bounded by $C\|\partial_t f\|_{\Lambda_{\alpha-2}^{\mathcal{P}}} y^{-1+(\alpha-2)} \le C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-3+\alpha}$. For the second summand we use that $\partial_t f \in L^\infty(\mathbb{R}^{n+1})$ and Lemma 3.78 (ii).

Now we shall get the bound for $\||\partial_{x_j}^2 \mathcal{P}_y(x_i f)\|_\infty$, for $j = 1, \dots, n$. We can write, for every $j \in \{1, \dots, n\}$,

$$\begin{aligned}\partial_{x_j}^2 \mathcal{P}_y(x_i f)(t,x) &= \partial_{x_j}^2 \int_{\mathbb{R}^{n+1}} \mathcal{P}_y(\tau,x,z)(x_i - z_i) f(t-\tau,x-z)\, d\tau dz \\ &= \int_{\mathbb{R}^{n+1}} [\partial_{x_j}^2 \mathcal{P}_y(\tau,x,z)(x_i - z_i) + 2\partial_{x_j} P_y(\tau,x,z)\delta_{ij}] f(t-\tau,x-z)\, d\tau dz \\ &\quad + 2\int_{\mathbb{R}^{n+1}} [\partial_{x_j} \mathcal{P}_y(\tau,x,z)(x_i - z_i) + P_y(\tau,x,z)\delta_{ij}] \partial_{x_j} f(t-\tau,x-z)\, d\tau dz \\ &\quad + \int_{\mathbb{R}^{n+1}} \mathcal{P}_y(\tau,x,z)(x_i - z_i)\partial_{x_j}^2 f(t-\tau,x-z)\, d\tau dz,\end{aligned}$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ if $i \ne j$. For the first two integrals we apply Lemma 3.78 (ii), since f and $\partial_{x_i} f$ are bounded functions, to get the bound $Cy^{-3+\alpha}$. Regarding the last summand, since $\partial^2_{x_j} f \in \Lambda^{\mathcal{P}}_{\alpha-2}$, $0 < \alpha - 2 < 1$, by using Lemma 3.78 (ii) we get

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^{n+1}} \mathcal{P}_y(\tau, x, z)(x_i - z_i)\partial^2_{x_j} f(t-\tau, x-z)d\tau dz \Big| \\
&\quad \le \int_{\mathbb{R}^{n+1}} |\mathcal{P}_y(\tau, x, z)||x-z|^{1-(\alpha-2)}|x-z|^{\alpha-2}|\partial^2_{x_j} f(t-\tau, x-z)|d\tau dz \\
&\quad \le C[\partial^2_{x_j} f]_{M^{\alpha-2}} \int_{\mathbb{R}^{n+1}} |\mathcal{P}_y(\tau, x, z)|(|x|^{1-(\alpha-2)} + |z|^{1-(\alpha-2)})d\tau dz \\
&\quad \le C[\partial^2_{x_j} f]_{M^{\alpha-2}} y^{-1+(\alpha-2)} = C[\partial^2_{x_j} f]_{M^{\alpha-2}} y^{-3+\alpha}.
\end{aligned}$$

In remains the case $\||\cdot|^2 \mathcal{P}_y(x_i f)\|_\infty$. Observe that,

$$\begin{aligned}
|x|^2|\mathcal{P}_y(x_i f)(t,x)| &\le C \int_{\mathbb{R}^{n+1}} (|x-z|^2 + |z|^2)|\mathcal{P}_y(\tau, x, z)||x_i - z_i||f(t-\tau, x-z)|d\tau dz \\
&\le C \int_{\mathbb{R}^{n+1}} |x-z|^{3-\alpha}|\mathcal{P}_y(\tau, x, z)||x-z|^{\alpha}|f(t-\tau, x-z)|d\tau dz \\
&\quad + C \int_{\mathbb{R}^{n}} |z|^2|\mathcal{P}_y(\tau, x, z)||x-z||f(t-\tau, x-z)|d\tau dz \\
&\le C(\||x|^\alpha f\|_\infty + \||x| f\|_\infty) \int_{\mathbb{R}^{n+1}} (|x|^{3-\alpha} + |z|^{3-\alpha} + |z|^2)|\mathcal{P}_y(\tau, x, z)|d\tau dz \\
&\le C\|f\|_{\Lambda^{\mathcal{P}}_\alpha} y^{-3+\alpha}.
\end{aligned}$$

In the last inequality we have used Lemma 3.78 (ii).

For the cases $2m+1 < \alpha < 2m+3$, with $m \ge 1$, we get the result by following the same kind of reasonings, that is, by estimating the norms $\|(-\sum_j \partial^2_{x_j} + |x|^2 + \partial_t)^{m+1}\mathcal{P}_y(x_i f)\|_\infty$. We leave the details for the interested reader.

It remains the cases where α is even. Suppose $\alpha = 2$. We can write

$$\partial^2_y \mathcal{P}_y(x_i f) = x_i \partial^2_y \mathcal{P}_y f + \int_{\mathbb{R}^{n+1}} \partial^2_y \mathcal{P}_y(\tau, x, z) z_i f(t-\tau, x-z)d\tau dz.$$

By using Lemma 3.83, the first summand is bounded by Cy^{-1}. The same bound for the second summand follows from the boundedness of f and Lemma 3.78 (ii).

In the general, for $\alpha = 2\ell$, $\ell \in \mathbb{N}$, we write $\partial^{2\ell}_y \mathcal{P}_y(x_i f) = \partial^2_y \left(\partial_t - \sum_j \partial^2_{x_j} + |x|^2\right)^{\ell-1} \mathcal{P}_y(x_i f)$ and we proceed as in the previous cases.

□

Proposition 3.91. *Let $\alpha > 2$ and $f \in L^\infty(\mathbb{R}^{n+1})$ and suppose that $\partial_{x_i} f, x_i f \in \Lambda^{\mathcal{P}}_{\alpha-1}$, $i = 1, \dots, n$, and $\partial_t f \in \Lambda^{\mathcal{P}}_{\alpha-2}$. Then $f \in \Lambda^{\mathcal{P}}_\alpha$.*

Proof. Consider the case $2 < \alpha < 4$. We want to see that $\|\partial^4_y \mathcal{P}_y f\|_\infty \le Cy^{-4+\alpha}$, and as $\mathcal{P}_y f$ satisfies (3.8), we have that $\partial^4_y \mathcal{P}_y f(t,x) = \left(\partial_t - \sum_{j=1}^n \partial^2_{x_j} + |x|^2\right)^2 \mathcal{P}_y f(t,x)$. Hence, it is sufficient to prove that

a) $\|\partial^4_{x_j} \mathcal{P}_y f\|_\infty \le Cy^{-4+\alpha}$ and $\|\partial^2_{x_j}\partial^2_{x_l} \mathcal{P}_y f\|_\infty \le Cy^{-4+\alpha}$, $l \ne j$,

b) $\|\partial^2_{x_j}(|x|^2\mathcal{P}_y f)\|_\infty \le Cy^{-4+\alpha}$,

c) $\||x|^2\partial^2_{x_j}\mathcal{P}_y f\|_\infty \le Cy^{-4+\alpha}$, $j\in\{1,\dots,n\}$, and

d) $\||x|^4\mathcal{P}_y f\|_\infty \le Cy^{-4+\alpha}$.

e) $\||\partial_t^2\mathcal{P}_y f\|_\infty \le Cy^{-4+\alpha}$.

f) $\|\partial_t|x|^2\mathcal{P}_y f\|_\infty \le Cy^{-4+\alpha}$.

g) $\||\partial_t\partial^2_{x_j}\mathcal{P}_y f\|_\infty \le Cy^{-4+\alpha}$.

Integration by parts gives

$$\begin{aligned}\partial^4_{x_j}\mathcal{P}_y f(t,x) &= \partial^3_{x_j}\int_{\mathbb{R}^{n+1}}\mathcal{P}_y(\tau,x,z)\partial_{x_j}f(t-\tau,x-z)d\tau dz\\ &+\int_{\mathbb{R}^{n+1}}\partial^3_{x_j}(\partial_{x_j}+\partial_{z_j})\mathcal{P}_y(\tau,x,z)f(t-\tau,x-z)d\tau dz\\ &+2\int_{\mathbb{R}^{n+1}}\partial^2_{x_j}(\partial_{x_j}+\partial_{z_j})\mathcal{P}_y(\tau,x,z)\partial_{x_j}f(t-\tau,x-z)d\tau dz\\ &+\int_{\mathbb{R}^{n+1}}\partial_{x_j}(\partial_{x_j}+\partial_{z_j})\mathcal{P}_y(\tau,x,z)\partial^2_{x_j}f(t-\tau,x-z)d\tau dz.\end{aligned}$$

As $\partial_{x_j}f\in\Lambda^{\mathcal{P}}_{\alpha-1}$, by Lemma 3.83 we get that the first summand is bounded by $C\|\partial_{x_j}f\|_{\Lambda^{\mathcal{P}}_{\alpha-1}}y^{-4+\alpha}$. For the rest of the summands we apply Lemma 3.78 together of the boundedness of the functions $f,\partial_{x_j}f$ and $\partial^2_{x_j}f$. In an analogous way it is proved the same bound for $\|\partial^2_{x_j}\partial^2_{x_l}\mathcal{P}_y f\|_\infty$, $l\ne j$. To prove b), we write

$$\begin{aligned}\partial^2_{x_j}(|x|^2\mathcal{P}_y f)(t,x) &= 2\mathcal{P}_y f(t,x)+4x_j\partial_{x_j}\mathcal{P}_y f(t,x)+|x|^2\partial^2_{x_j}\mathcal{P}_y f(t,x) \qquad (3.16)\\ &= 2\mathcal{P}_y f(t,x)+4\int_{\mathbb{R}^{n+1}}x_j\partial_{x_j}\mathcal{P}_y(\tau,x,z)f(t-\tau,x-z)d\tau dz\\ &+4\int_{\mathbb{R}^{n+1}}\mathcal{P}_y(\tau,x,z)x_j\partial_{x_j}f(t-\tau,x-z)d\tau dz\\ &+\int_{\mathbb{R}^{n+1}}|x|\partial_{x_j}\mathcal{P}_y(\tau,x,z)|x|\partial_{x_j}f(t-\tau,x-z)d\tau dz\\ &+\int_{\mathbb{R}^{n+1}}|x|^2\partial^2_{x_j}\mathcal{P}_y(\tau,x,z)f(t-\tau,x-z)d\tau dz+|x|^2\partial_{x_j}\mathcal{P}_y(\partial_{x_j}f)(t,x).\end{aligned}$$

As the functions f and $|x|\partial_{x_j}f$ are bounded, Lemma 3.78 (ii) takes care of the first five summands. The bound of last summand follows from the fact that $\partial_{x_j}f\in\Lambda^{\mathcal{P}}_{\alpha-1}$ and Lemma 3.83.

Moreover, observe that the boundedness of *c*) follows from the estimate of the third summand in the first identity of (3.16).

To see d), we use that $|x|^\alpha f \in L^\infty(\mathbb{R}^{n+1})$ (because $x_i f \in \Lambda^{\mathcal{P}}_{\alpha-1}$) and Lemma 3.78 to get

$$\begin{aligned}||x|^4\mathcal{P}_y f(t,x)| &\le C\int_{\mathbb{R}^{n+1}} \mathcal{P}_y(\tau,x,z)(|x-z|^{4-\alpha}|x-z|^\alpha + |z|^4)|f(t-\tau,x-z)|d\tau dz\\ &\le C(\||x|^\alpha f\|_\infty + \|f\|_\infty)\int_{\mathbb{R}^{n+1}} \mathcal{P}_y(\tau,x,z)(|x|^{4-\alpha} + |z|^{4-\alpha} + |z|^4)d\tau dz\\ &\le C(\||x|^\alpha f\|_\infty + \|f\|_\infty)y^{-4+\alpha}.\end{aligned}$$

Finally, for the estimates e)-g) observe that

$$\|\partial_t^2\mathcal{P}_y f\|_\infty = \|\partial_t\mathcal{P}_y(\partial_t f)\|_\infty, \quad \|\partial_t\partial^2_{x_j}\mathcal{P}_y f\|_\infty = \|\partial^2_{x_j}\mathcal{P}_y(\partial_t f)\|_\infty,$$

and $\||x|^2\partial_t\mathcal{P}_y f\|_\infty = \||x|^2\mathcal{P}_y(\partial_t f)\|_\infty$. Hence, by using that $\partial_t f \in \Lambda^{\mathcal{P}}_{\alpha-2}$ and Lemma 3.83 we get the result.

For the rest of the values of α we proceed analogously. We leave the details for the interested reader.

□

Propositions 3.88, 3.89, 3.90, 3.91 show the validity of Theorem 3.87. Therefore we have proved Theorem 3.68, epigraph 2.

The proof of Theorem 3.68 is now complete. As a consequence of it we get the characterization of the spaces of Krylov's type introduced in Definition 3.65.

Proof of Therem 3.69. The case $0 < \alpha < 1$ was proved in Theorem 3.85. Consider $1 < \alpha < 2$. Suppose that $f \in \Lambda^{\mathcal{P}}_\alpha$. By epigraph 1 of Theorem 3.68 we know that (3.2) holds, an by taking $z = 0$ in this inequality we get that $f(\cdot,x) \in C^{\alpha/2}(\mathbb{R})$ uniformly on x. In addition, by Propositions 3.89 and 3.90 we have that $(\partial_{x_i} \pm x_i)f \in \Lambda^{\mathcal{P}}_{\alpha-1} = C^{\frac{\alpha-1}{2},\alpha-1}_{t,\mathcal{H}}$. Thus, we get that $f \in C^{\frac{\alpha}{2},\alpha}_{t,\mathcal{H}}$. Conversely, suppose that $f \in C^{\frac{\alpha}{2},\alpha}_{t,\mathcal{H}}$. Then, we have that $(\partial_{x_i} \pm x_i)f(t,\cdot) \in C^{\alpha-1}_{\mathcal{H}}$ uniformly on t and $f(\cdot,x) \in C^{\alpha/2}(\mathbb{R})$ uniformly on x. Hence, $(1+|x|)^\alpha f \in L^\infty(\mathbb{R}^{n+1})$ and

$$\begin{aligned}&|f(t-\tau,x-z) + f(t-\tau,x+z) - 2f(t,x)|\\ &\qquad\le |f(t-\tau,x-z) + f(t-\tau,x+z) - 2f(t-\tau,x)| + 2|f(t-\tau,x) - f(t,x)|\\ &\qquad\le C|\nabla_x f(t-\tau,x+\theta z) - \nabla_x f(t-\tau,x-\lambda z)||z| + C\tau^{\alpha/2}\\ &\qquad\le C|\theta+\lambda|^{\alpha-1}|z|^{\alpha-1}|z| + C\tau^{\alpha/2} \le C(\tau^{1/2} + |z|)^\alpha.\end{aligned}$$

By epigraph 1 of Theorem 3.68 we conclude that $f \in \Lambda^{\mathcal{P}}_\alpha$. The case $\alpha > 2$ is a corollary of Theorem 3.87. □

Elliptic Hermite setting.

Observe that when the function f does not depend on t, that is, $f(t,x) = g(x)$, $x \in \mathbb{R}^n$, epigraph 1 of Theorem 3.68 gives the proof of epigraph 1 of Theorem 3.71.

Moreover, Propositions 3.89 and 3.90 are also valid for $\Lambda^{P^{\mathcal{H}}}_\alpha$ spaces. Indeed, the proofs are almost the same but we have to use that $P^{\mathcal{H}}_y f$ satisfies the equation $\partial_y^2 P^{\mathcal{H}}_y f = (-\Delta_x +$

$|x|^2)P_y^{\mathcal{H}}f$, instead of (3.8) (so there is not any estimate depending on the derivatives of t). In addition, we can also prove the version of Proposition 3.91 for the spaces $\Lambda_\alpha^{P^{\mathcal{H}}}$, $\alpha > 1$.

Proposition 3.92. *Let $\alpha > 1$, $g \in L^\infty(\mathbb{R}^n)$ and suppose that $\partial_{x_i}g, x_ig \in \Lambda_{\alpha-1}^{P^{\mathcal{H}}}$, $i = 1, \dots, n$. Then $g \in \Lambda_\alpha^{P^{\mathcal{H}}}$.*

Proof. Observe that for $\alpha > 2$, we can proceed as in the proof of Proposition 3.91 but by using that $P_y^{\mathcal{H}}g$ satisfies $\partial_y^2 P_y^{\mathcal{H}}g = (-\Delta_x + |x|^2)P_y^{\mathcal{H}}g$, instead of (3.8), so the estimates depending on the derivatives of t do not appear in this case.

Suppose $1 < \alpha \le 2$. It is enough to prove that $\|\partial_y^3 P_y^{\mathcal{H}}g\|_\infty \le Cy^{-3+\alpha}$. Since $P_y^{\mathcal{H}}g$ satisfies the equation $\partial_y^2 P_y^{\mathcal{H}}g = (-\Delta_x + |x|^2)P_y^{\mathcal{H}}g$, we shall prove that $\|\partial_y(-\sum_i \partial_{x_i}^2 + |x|^2)P_y^{\mathcal{H}}g\|_\infty \le Cy^{-3+\alpha}$.

On the one hand, observe that

$$\begin{aligned}\partial_y\partial_{x_i}^2 P_y^{\mathcal{H}}g(x) &= \partial_y\partial_{x_i}\left(\int_{\mathbb{R}^n} P_y^{\mathcal{H}}(x,z)\partial_{x_i}g(x-z)dz + \int_{\mathbb{R}^n} \partial_{x_i}P_y^{\mathcal{H}}(x,z)g(x-z)dz\right)\\ &= \partial_y\partial_{x_i}P_y^{\mathcal{H}}(\partial_{x_i}g)(x) + \int_{\mathbb{R}^n} \partial_y\partial_{x_i}P_y^{\mathcal{H}}(x,z)\partial_{x_i}g(x-z)dz + \int_{\mathbb{R}^n} \partial_y\partial_{x_i}^2 P_y^{\mathcal{H}}(x,z)g(x-z)dz.\end{aligned}$$

By Lemma 3.83, we have that

$$|\partial_y\partial_{x_i}P_y^{\mathcal{H}}(\partial_{x_i}g)(x)| \le C\|\partial_{x_i}g\|_{\Lambda_{\alpha-1}^{P^{\mathcal{H}}}}y^{-2+(\alpha-1)} = C\|\partial_{x_i}g\|_{\Lambda_{\alpha-1}^{P^{\mathcal{H}}}}y^{-3+\alpha}.$$

The same bound for the other two integrals follows from the boundedness of g and $\partial_{x_i}g$ and Lemma 3.78 (ii).

It remains to prove that $\||x|^2\partial_y P_y^{\mathcal{H}}g\|_\infty \le Cy^{-3+\alpha}$. Observe that the fact $x_ig \in \Lambda_{\alpha-1}^{P^{\mathcal{H}}}$ implies that $|x|^\alpha g \in L^\infty(\mathbb{R}^n)$. Therefore, by using this and Lemma 3.78 (ii) we get

$$\begin{aligned}||x|^2\partial_y P_y^{\mathcal{H}}g(x)| &\le C\int_{\mathbb{R}^n} |\partial_y P_y^{\mathcal{H}}(x,z)|(|x-z|^2 + |z|^2)|g(x-z)|dz\\ &= C\int_{\mathbb{R}^n} |\partial_y P_y^{\mathcal{H}}(x,z)|(|x-z|^{2-\alpha}|x-z|^\alpha|g(x-z)|dz\\ &\quad + C\int_{\mathbb{R}^n} |\partial_y P_y^{\mathcal{H}}(x,z)||z|^2|g(x-z)|dz\\ &\le C[g]_{M^\alpha}\int_{\mathbb{R}^n} |\partial_y P_y^{\mathcal{H}}(x,z)|(|x|^{2-\alpha} + |z|^{2-\alpha})dz + C\|g\|_\infty y^{-3+\alpha}\\ &\le C([g]_{M^\alpha} + \|g\|_\infty)y^{-3+\alpha}.\end{aligned}$$

□

Thus, Propositions 3.89 and 3.90 and 3.92 give the proof of epigraph 2 in Theorem 3.71. Moreover, by proceeding as in the proof of Theorem 3.69 we get that Theorem 3.72 is also true.

Remark 3.93. *There exists a function* $g \in \Lambda_1^{P^{\mathcal{H}}}(\mathbb{R})$*, but so that* $\sup_{\{x:x\in[0,1],z\in[0,1]\}} |g(x+z)-g(x)| \le Cz$ *fails for all* C.

Consider the functions h *and* φ *as follows.* $h(x) = \sum_{k=1}^{\infty} 2^{-k}\cos(2\pi 2^k x)$ *and* φ *is a positive differentiable function, with continuous derivative, such that* $\varphi(x) = 1$ *when* $x \in [-3,3]$*, and for any* x *there exists a constant* C *with* $(1+|x|)\varphi(x) \le C$ *and* $|\varphi'(x)| \le C$*. It is clear that* $|h(x)| \le 1$*. Moreover, it can be checked, see [98, Theorem 4.9], that* $\|h(x+z)+h(x-z)-2h(x)\|_\infty \le A|z|$.

Now we choose the function $g(x) = h(x)\varphi(x)$*, then by the properties of* h *and* φ *we have* $|(1+|x|)g(x)| \le C$*. On the other hand, by the Mean Value Theorem we have*

$$\Big|g(x+z)+g(x-z)-2g(x)\Big| \le \Big|\left(h(x+z)+h(x-z)-2h(x)\right)\varphi(x+z)\Big| + \Big|h(x-z)\left(\varphi(x-z)-\varphi(x+z)\right)\Big| + 2\Big|h(x)\left(\varphi(x+z)-\varphi(x)\right)\Big| \le C|z|.$$

Now assume that g *satisfies* $|g(x+z)-g(x)| \le C|z|$*. Hence, for* $x,z \in [0,1]$ *we would have* $|h(x+z)-h(x)| \le C|z|$*. But it is well known that the Weierstrass function does not satisfy the Lipschitz condition, see [98, Theorem 4.9].*

3.3 Hermite Lipschitz spaces via the heat semigroup

All the definitions and results in this section will be given for the elliptic operator $\mathcal{H}$ but they can be established in the parabolic case parallely, as we did in Section 3.2 for the Poisson case.

As we mentioned at the begining of this chapter, all the results of Chapter 2 regarding Schrödinger Lipschitz spaces defined through the heat semigroup $e^{-y\mathcal{L}}$ apply for $\mathcal{L} = \mathcal{H}$ on $\mathbb{R}^n$, $n \ge 3$. However, we can take advantage of our knowledge about the heat semigroup, $e^{-y\mathcal{H}}$, and its estimates, to get better results in $\mathbb{R}^n$, $n \ge 1$, as we did for the Poisson semigroup in the previous section.

Definition 3.94. *Let* $e^{-y\mathcal{H}} = W_y^{\mathcal{H}}$ *be the heat semigroup associated to* $\mathcal{H}$*. For* $\alpha > 0$*, we define the spaces* $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ *as*

$$\Lambda_{\alpha/2}^{W^{\mathcal{H}}} = \Big\{ f \in L^\infty(\mathbb{R}^n) : \left\|\partial_y^k W_y^{\mathcal{H}} f\right\|_{L^\infty(\mathbb{R}^n)} \le C_k y^{-k+\alpha/2},\ k = [\alpha/2]+1 \Big\}.$$

When we defined in Chapter 2 Lipschitz spaces adapted to a general Schrödinger operator $\mathcal{L}$, we imposed the "natural" growth condition $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$. However, in the Hermite case, if f satisfies $\rho(\cdot)^{-\alpha} f \in L^\infty(\mathbb{R}^n)$, with $\rho(x) = \frac{1}{1+|x|}$, then f is in particular bounded. We will prove that, in fact, in our definitions of Hermite Lipschitz spaces we can consider any of both conditions, either $f \in L^\infty(\mathbb{R}^n)$ or $(1+|\cdot|)^\alpha f \in L^\infty(\mathbb{R}^n)$, see Remark 3.106.

In Section 3.2 we proved that the spaces $\Lambda_\alpha^{P^{\mathcal{H}}}$ coincide with the spaces $C_{\mathcal{H}}^\alpha$ for $\alpha \notin \mathbb{N}$. Our first main result of this section is the following. It will be proved in Subsection 3.3.2.

Theorem 3.95. *Let* $\alpha > 0$*. Then, the spaces* $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ *and* $\Lambda_\alpha^{P^{\mathcal{H}}}$ *do coincide in the sense of normed spaces.*

As a consequence, by using the coincidence of $\Lambda_\alpha^{P^{\mathcal{H}}}$ and $C^\alpha_{\mathcal{H}}$, see Theorem 3.72, we have next corollary.

Corollary 3.96. *If $\alpha > 0$ is not an integer, $\Lambda_{\alpha/2}^{W^{\mathcal{H}}} = \Lambda_\alpha^{P^{\mathcal{H}}} = C^\alpha_{\mathcal{H}}$, where the identities are understood in the sense of normed spaces.*

3.3.1 Previous results.

Analogously to the Poisson semigroup case, we need some estimates on the heat kernel and its derivatives.

Lemma 3.97. *Let $\gamma, \nu \geq 0, s \geq 0$. For each $\ell, k, m \in \mathbb{N} \cup \{0\}$, there exists a constant $C_{\gamma,\nu,k,\ell,m,s} > 0$ such that, for every $x \in \mathbb{R}^n$ and $y > 0$,*

$$\int_{\mathbb{R}^n} |x|^\gamma |z|^\nu |\partial_y^k \partial_{z_i}^m \partial_{x_j}^\ell W_y^{\mathcal{H}}(x,z)| dz \leq \begin{cases} C_{\gamma,\nu,\ell,k,m,s}\, y^{-(k+\frac{m+\gamma}{2}-\frac{\ell+\nu}{2}+s)}, & \text{if } s \geq 0, \zeta > 0, \\ C_{\gamma,\nu,\ell,k,m,s}\, y^{-s}, & \text{if } s > 0, \zeta \leq 0, \end{cases}$$

for $\zeta = k + \frac{m+\gamma}{2} - \frac{\ell+\nu}{2}$ and $i = 1, \ldots, n,\ j = 1, \ldots, n$.

Proof. Along this proof will use Remark 3.76 and the estimates:

$$|\partial_y^k W_y^{\mathcal{H}}(x,z)| \leq C_k \frac{e^{-\frac{|2x-z|^2 \tanh y}{c}} e^{-\frac{|z|^2 \coth y}{c}}}{(\sinh(2y))^{n/2} y^k}, \quad \left|\partial_{x_i}^\ell \left(e^{-\frac{|2x-z|^2 \tanh y}{4}}\right)\right| \leq C_\ell e^{-\frac{|2x-z|^2 \tanh y}{8}} (\tanh y)^{\ell/2}$$

and $|\partial_{z_i}^m e^{-\frac{|z|^2 \coth y}{4}}| \leq C_m e^{-\frac{|z|^2 \coth y}{8}} (\coth y)^{m/2}$.

By using these estimates we get that

$$|\partial_y^k \partial_{z_i}^m \partial_{x_j}^\ell W_y^{\mathcal{H}}(x,z)| \leq \frac{C e^{-\frac{|z|^2 \coth y}{c}} e^{-c|x-\frac{z}{2}|^2 \tanh y}}{(\sinh(2y))^{n/2} y^k} (\coth y)^{m/2} (\tanh y)^{\ell/2}.$$

Therefore, by Remark 3.76, for every $\lambda \geq 0$ we get

$$\begin{aligned} &\int_{\mathbb{R}^n} |x|^\gamma |z|^\nu |\partial_y^k \partial_{z_i}^m \partial_{x_j}^\ell W_y^{\mathcal{H}}(x,z)| dz \\ &\leq C \int_{\mathbb{R}^n} \frac{(|x-\frac{z}{2}|^\gamma + |\frac{z}{2}|^\gamma)|z|^\nu e^{-\frac{|z|^2 \coth y}{C}} e^{-|x-\frac{z}{2}|^2 \tanh y}}{(\sinh(2y))^{n/2} y^k} (\coth y)^{m/2} (\tanh y)^{\ell/2} dz \\ &\leq \frac{C}{y^{k+\frac{m+\gamma-\ell-\nu}{2}+\lambda}}. \end{aligned}$$

The constant C depends on γ, ν, ℓ, k, m and λ. The result follows by choosing $\lambda = s$ in the case $k + \frac{m+\gamma-\ell-\nu}{2} > 0$ and $\lambda = -(k + \frac{m+\gamma-\ell-\nu}{2}) + s$ in the case $k + \frac{m+\gamma-\ell-\nu}{2} \leq 0$. ☐

Remark 3.98. *Observe that the previous Lemma remains valid when we substitute $\partial_{x_i}^\ell$ and $\partial_{x_i}^m$ by $\partial_{x_1}^{\delta_1} \ldots \partial_{x_n}^{\delta_n}$ and $\partial_{z_1}^{\mu_1} \ldots \partial_{z_n}^{\mu_n}$, respectively, where $\delta_i, \mu_i \in \mathbb{N}_0$ and $\delta_1 + \cdots + \delta_n = \ell$ and $\mu_1 + \cdots + \mu_n = m$.*

Remark 3.99. *Observe that for bounded functions f, Lemma 3.97 assures that* $\left\|\partial_y^k W_y^{\mathcal{H}} f\right\|_{L^\infty(\mathbb{R}^n)} \leq C\|f\|_\infty y^{-k}$. *Therefore we can assume in the definition of* $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ *that* $y < 1$.

The following two results are completely analogous to the ones we proved for the space $\Lambda_\alpha^{P^{\mathcal{H}}}$ in Section 3.2, so we will omit their proofs.

Lemma 3.100. *Let $f \in L^\infty(\mathbb{R}^n)$, $\alpha > 0$, and k, l integers bigger than $\alpha/2$. Then, for $y > 0$, the following conditions are equivalent:*

(a) $\left\|\partial_y^k W_y^{\mathcal{H}} f\right\|_{L^\infty(\mathbb{R}^n)} \leq A_k y^{-k+\alpha/2}$

(b) $\left\|\partial_y^l W_y^{\mathcal{H}} f\right\|_{L^\infty(\mathbb{R}^n)} \leq A_l y^{-l+\alpha/2}$,

where A_k and A_l are positive constants with $A_k \sim A_l$.

Corollary 3.101. *Let $\alpha > 0$. If $f \in \Lambda_{\alpha/2}^{W^{\mathcal{H}}}$, then for every $0 < \beta < \alpha$, $f \in \Lambda_{\beta/2}^{W^{\mathcal{H}}}$.*

Lemma 3.102. *Let $\alpha > 0$, $f \in \Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ and $k = [\alpha/2] + 1$. For every $\gamma \geq 0$ and $m, j \in \mathbb{N}_0$ such that $m + \gamma/2 + j/2 \geq k$, there exists a constant $C_{\gamma,m,j}$ such that $\||\cdot|^\gamma \partial_y^m \partial_{x_i}^j W_y^{\mathcal{H}} f\|_\infty \leq C_{\gamma,m,j}\|f\|_{\Lambda_{\alpha/2}^{W^{\mathcal{H}}}} y^{-(m+\gamma/2+j/2)+\alpha/2}$.*

Proof. Observe that the case $\gamma = j = 0$ follows from the definition of the space $\Lambda_{\alpha/2}^{W}$, so we will exclude it in the following. Let us consider the case $m \geq k$. By the semigroup property and integration by parts we have that, for every $x \in \mathbb{R}^n$,

$$\begin{aligned} \left||x|^\gamma \partial_y^m \partial_{x_i}^j W_y^{\mathcal{H}} f(x)\right| &= C\left||x|^\gamma \partial_{x_i}^j \int_{\mathbb{R}^n} W_{y/2}^{\mathcal{H}}(x,z) \partial_y^m W_y^{\mathcal{H}} f(x-z)\big|_{y/2} dz\right| \\ &= C\left||x|^\gamma \int_{\mathbb{R}^n} (\partial_{x_i} + \partial_{z_i})^j W_{y/2}^{\mathcal{H}}(x,z) \partial_y^m W_y^{\mathcal{H}} f(x-z)\big|_{y/2} dz\right| \\ &\leq C\|\partial_y^m W_y^{\mathcal{H}} f\big|_{y/2}\|_\infty \sum_{p+q=j} \int_{\mathbb{R}^n} |x|^\gamma |\partial_{z_i}^p \partial_{x_i}^q W_{y/2}^{\mathcal{H}}(x,z)| dz \\ &\leq C_{\gamma,m,j}\|f\|_{\Lambda_{\alpha/2}^{W^{\mathcal{H}}}} y^{-(m+\gamma/2+j/2)+\alpha/2}. \end{aligned}$$

In the last inequality we have use Lemma 3.100 and Lemma 3.97 in each summand, where we have chosen $s = j/2 + \gamma/2$ in the case $p - q + \gamma \leq 0$ and $s = q$ in the case $p - q + \gamma > 0$.

If $m < k$, we perform above estimate with k derivatives on y and then we do $k - m$ iterated integrations. □

Remark 3.103. *Observe that the previous Lemma remains valid when we substitute $\partial_{x_i}^j$ by $\partial_{x_1}^{\delta_1} \ldots \partial_{x_n}^{\delta_n}$, where $\delta_i \in \mathbb{N}_0$ and $\delta_1 + \cdots + \delta_n = j$.*

Following the same steps used to prove Theorem 2.57 we have the following result.

Theorem 3.104. *Let $\alpha > 0$. If $f \in \Lambda_{\alpha/2}^{W^{\mathcal{H}}}$, then $f \in \Lambda_\alpha^{P^{\mathcal{H}}}$.*

As a consequence of Theorem 3.104 and Proposition 3.86 we get the following.

Proposition 3.105. *Let $\alpha > 0$. If $f \in \Lambda_{\alpha/2}^{W^{\mathcal{H}}}$, then $|x|^\alpha f \in L^\infty(\mathbb{R}^n)$.*

Remark 3.106. *It is clear that if f is a function such that $(1+|\cdot|)^\alpha f \in L^\infty(\mathbb{R}^n)$, then $f \in L^\infty(\mathbb{R}^n)$. Therefore, Proposition 3.105 establishes that in the definition of $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$, we can consider indistinctly $f \in L^\infty(\mathbb{R}^n)$ or $(1+|x|)^\alpha f \in L^\infty(\mathbb{R}^n)$.*

3.3.2 Proof of the main result.

To prove Theorem 3.95 we need two previous results.

Theorem 3.107. *Let $\alpha > 1$. If f is a bounded function such that $\partial_{x_i} f$ and $x_i f$ belong to $\Lambda_{\frac{\alpha-1}{2}}^{W^{\mathcal{H}}}$, for every $i = 1, \dots, n$, then $f \in \Lambda_{\alpha/2}^{W^{\mathcal{H}}}$.*

Proof. We prove first that $\partial_{x_i} f, x_i f \in \Lambda_{\frac{\alpha-1}{2}}^{W^{\mathcal{H}}}$ implies $f \in \Lambda_{\alpha/2}^{W^{\mathcal{H}}}$, for every $\alpha > 1$. We only do the case $1 < \alpha < 3$. For the other cases the proof is completely analogous. We want to see that $\|\partial_y^2 W_y^{\mathcal{H}} f\|_\infty \le C y^{-2+\alpha/2}$. Since $W_y^{\mathcal{H}} f$ satisfies the heat equation, this is equivalent to prove that

$$\left\| \left(\sum_{j=1}^n -\partial_{x_j}^2 + x_j^2 \right)^2 W_y^{\mathcal{H}} f \right\|_\infty \le C y^{-2+\alpha/2}.$$

Therefore, it would be enough to prove that

a) $\|\partial_{x_j}^4 W_y^{\mathcal{H}} f\|_\infty \le C y^{-2+\alpha/2}$ and $\|\partial_{x_j}^2 \partial_{x_i}^2 W_y^{\mathcal{H}} f\|_\infty \le C y^{-2+\alpha/2}$, $i \ne j$,

b) $\|\partial_{x_j}^2 (|x|^2 W_y^{\mathcal{H}} f)\|_\infty \le C y^{-2+\alpha/2}$,

c) $\||x|^2 \partial_{x_j}^2 W_y^{\mathcal{H}} f\|_\infty \le C y^{-2+\alpha/2}$, $j \in \{1, \dots, n\}$, and

d) $\||x|^4 W_y^{\mathcal{H}} f\|_\infty \le C y^{-2+\alpha/2}$.

On the one hand,

$$|\partial_{x_j}^4 W_y^{\mathcal{H}} f(x)| = \partial_{x_j}^3 W_y^{\mathcal{H}} (\partial_{x_j} f)(x) - \partial_{x_j}^3 \int_{R^n} \partial_{x_j} W_y^{\mathcal{H}}(x,z) f(x-z) dz.$$

By the hypothesis and Lemma 3.102, the first summand is bounded by $C\|\partial_{x_j} f\|_{\Lambda_{\frac{\alpha-1}{2}}^{W^{\mathcal{H}}}} y^{-\frac{3}{2}+\frac{\alpha-1}{2}}$. Partial integration allows us to write the second summand as

$$\begin{aligned} \partial_{x_j}^3 \int_{\mathbb{R}^n} \partial_{x_j} W_y^{\mathcal{H}}(x,z) f(x-z) dz &= \int_{\mathbb{R}^n} \partial_{x_j}^4 W_y^{\mathcal{H}}(x,z) f(x-z) dz \\ &\quad + 3 \int_{\mathbb{R}^n} \partial_{x_j}^3 W_y^{\mathcal{H}}(x,z) \partial_{x_j} f(x-z) dz + 3 \int_{\mathbb{R}^n} \partial_{x_j}^2 \partial_{z_j}^2 W_y^{\mathcal{H}}(x,z) f(x-z) dz \\ &\quad + \int_{\mathbb{R}^n} \partial_{x_j} \partial_{z_j}^2 W_y^{\mathcal{H}}(x,z) \partial_{x_j} f(x-z) dz, \end{aligned}$$

and by using the boundedness of f and $\partial_{x_j} f$ and Lemma 3.97 we get the desired estimate. The estimate for $\|\partial^2_{x_j}\partial^2_{x_i} W^{\mathcal{H}}_y f\|_\infty$ is completely analogous.

To prove b), we write

$$\begin{aligned}\partial^2_{x_j}(|x|^2 W^{\mathcal{H}}_y f)(x) &= 2W^{\mathcal{H}}_y f(x) + 4\int_{\mathbb{R}^n} x_j \partial_{x_j} W^{\mathcal{H}}_y(x,z) f(x-z)dz \\ &+ 4\int_{\mathbb{R}^n} W^{\mathcal{H}}_y(x,z) x_j \partial_{x_j} f(x-z)dz + \int_{\mathbb{R}^n} |x|\partial_{x_j} W^{\mathcal{H}}_y(x,z)|x|\partial_{x_j} f(x-z)dz \\ &+ \int_{\mathbb{R}^n} |x|^2 \partial^2_{x_j} W^{\mathcal{H}}_y(x,z) f(x-z)dz + |x|^2 \partial_{x_j} W^{\mathcal{H}}_y(\partial_{x_j} f)(x).\end{aligned} \tag{3.17}$$

As the functions f and $|x|\partial_{x_j} f$ are bounded, Lemma 3.97 takes care of the four first summands. The bound of last summand in (3.17) follows from the fact that $\partial_{x_j} f \in \Lambda^{W^{\mathcal{H}}}_{\frac{\alpha-1}{2}}$ and Lemma 3.102.

Observe that c) is consequence of the estimates done in (3.17).

To see d), we use that $|x|^\alpha f \in L^\infty(\mathbb{R}^n)$ and Lemma 3.97 to get

$$\begin{aligned}||x|^4 W^{\mathcal{H}}_y f(x)| &\le C\int_{\mathbb{R}^n} W^{\mathcal{H}}_y(x,z)(|x-z|^{4-\alpha}|x-z|^\alpha + |z|^4)|f(x-z)|dz \\ &\le C(\||x|^\alpha f\|_\infty + \|f\|_\infty)\int_{\mathbb{R}^n} W^{\mathcal{H}}_y(x,z)(|x|^{4-\alpha} + |z|^{4-\alpha} + |z|^4)dz \\ &\le C(\||x|^\alpha f\|_\infty + \|f\|_\infty) y^{-2+\alpha/2}.\end{aligned}$$

For the rest of the values of α we would proceed analogously. We leave the details for the interested reader.

□

Remark 3.108. *The converse of the previous result is also true and can be proved directly in an analogous way as we did in the previous section for the space* $\Lambda^{P^{\mathcal{H}}}_\alpha$. *However, we will obtain it as a consequence of Theorem 3.95 and epigraph 2 of Theorem 3.71.*

The following theorem is a particular case of Theorem 2.47, when $n \ge 3$. Now we give a proof for $n \in \mathbb{N}$.

Theorem 3.109. *Let* $0 < \alpha < 2$. *The following are equivalent:*

(1). $(1+|\cdot|)^\alpha f \in L^\infty(\mathbb{R}^n)$ *and* $\sup_{|z|>0} \frac{\|f(\cdot+z)+f(\cdot-z)-2f(\cdot)\|_\infty}{|z|^\alpha} < \infty$.

(2). $f \in \Lambda^{W^{\mathcal{H}}}_{\alpha/2}$.

(3). $f \in \Lambda^{P^{\mathcal{H}}}_\alpha$.

Proof. The equivalence between (1). and (3). was proved in Theorem 3.71, epigraph 1. Moreover, Theorem 3.104 establishes that (2). implies (3). Therefore, it only remains to prove that (1). implies (2). Suppose that f is a function that satisfies the conditions in (1).

Let $y<1$. By using that $\int_{\mathbb{R}^n}\partial_y W_y^{\mathcal{H}}(x,z)f(x+z)dz=\int_{\mathbb{R}^n}\partial_y W_y^{\mathcal{H}}(x,-z)f(x-z)dz$, we can write

$$\begin{aligned}\partial_y W_y^{\mathcal{H}}f(x)&=\frac{1}{2}\int_{\mathbb{R}^n}\partial_y W_y^{\mathcal{H}}(x,z)(f(x-z)+f(x+z)-2f(x))dz\\&+\frac{1}{2}\int_{\mathbb{R}^n}\Big(\partial_y W_y^{\mathcal{H}}(x,z)-\partial_y W_y^{\mathcal{H}}(x,-z)\Big)f(x-z)dz+f(x)\partial_y e^{-y\mathcal{H}}1(x)\\&=I+II+III.\end{aligned}$$

On the one hand, by using Remark 3.76 and Lemma 3.97 we have that

$$|I|\le C\int_{\mathbb{R}^n}\frac{e^{-\frac{|z|^2\coth y}{c}}e^{-\frac{|2x-z|^2\tanh y}{c}}|z|^\alpha}{(\sinh(2y))^{n/2}y}dz\le Cy^{-1+\alpha/2}.$$

Regarding II, observe that

$$\begin{aligned}\partial_y W_y^{\mathcal{H}}(x,z)-\partial_y W_y^{\mathcal{H}}(x,-z)&=\partial_y\left(\frac{e^{-\frac{|z|^2\coth y}{4}}}{(2\pi\sinh(2y))^{n/2}}\left[e^{-\frac{|2x-z|^2\tanh y}{4}}-e^{-\frac{|2x+z|^2\tanh y}{4}}\right]\right)\\
&=\partial_y\left(\frac{e^{-\frac{|z|^2\coth y}{4}}}{(2\pi\sinh(2y))^{n/2}}\right)\left[e^{-\frac{|2x-z|^2\tanh y}{4}}-e^{-\frac{|2x+z|^2\tanh y}{4}}\right]\\
&+\frac{e^{-\frac{|z|^2\coth y}{4}}}{(2\pi\sinh(2y))^{n/2}}\partial_y\left[e^{-\frac{|2x-z|^2\tanh y}{4}}-e^{-\frac{|2x+z|^2\tanh y}{4}}\right]\\
&\le Ce^{-\frac{|z|^2\coth y}{4}}\left(\frac{|z|^2}{(\sinh(y))^2(\sinh(2y))^{n/2}}+\frac{\coth(2y)}{(\sinh(2y))^{n/2}}\right)\left|e^{-\frac{|2x-z|^2\tanh y}{4}}-e^{-\frac{|2x+z|^2\tanh y}{4}}\right|\\
&+\frac{e^{-\frac{|z|^2\coth y}{4}}}{(2\pi\sinh(2y))^{n/2}}\left|\int_{-1}^1\partial_\theta\partial_y\left(e^{-\frac{|2x-\theta z|^2\tanh y}{4}}\right)d\theta\right|\\
&=II_a+II_b.\end{aligned}$$

Observe that

$$\begin{aligned}\left|e^{-\frac{|2x-z|^2\tanh y}{4}}-e^{-\frac{|2x+z|^2\tanh y}{4}}\right|&=\left|\int_{-1}^1\partial_\theta e^{-\frac{|2x-\theta z|^2\tanh y}{4}}d\theta\right|=\left|\int_{-1}^1\nabla_z(e^{-\frac{|2x-\theta z|^2\tanh y}{4}})\cdot z\,d\theta\right|\\
&=\left|\int_{-1}^1 e^{-\frac{|2x-\theta z|^2\tanh y}{4}}(\frac{\theta\tanh y}{2}(2x-\theta z)\cdot z)d\theta\right|\\
&\le C|z|(\tanh y)^{1/2}.\end{aligned}$$

Therefore, by using Remark 3.76 we have that

$$\begin{aligned}|II_a|&\le Ce^{-\frac{|z|^2\coth y}{4}}\left(\frac{|z|^3(\tanh y)^{1/2}}{(\sinh(y))^2(\sinh(2y))^{n/2}}+\frac{\coth(2y)|z|(\tanh y)^{1/2}}{(\sinh(2y))^{n/2}}\right)\\
&\le Ce^{-\frac{|z|^2}{cy}}\left(\frac{|z|^3}{y^{3/2+n/2}}+\frac{|z|}{y^{1/2+n/2}}\right)\le C\frac{e^{-\frac{|z|^2}{cy}}}{y^{n/2}}.\end{aligned}$$

On the other hand, since

$$\begin{aligned}\Big|\int_{-1}^{1}\partial_\theta\partial_y\left(e^{-\frac{|2x-\theta z|^2\tanh y}{4}}\right)d\theta\Big| &= \Big|\int_{-1}^{1}\nabla_z\partial_y\left(e^{-\frac{|2x-\theta z|^2\tanh y}{4}}\right)\cdot z\,d\theta\Big|\\ &= \Big|\int_{-1}^{1}\partial_y\left(e^{-\frac{|2x-\theta z|^2\tanh y}{4}}\frac{\theta\tanh y}{2}(2x-\theta z)\cdot z\right)d\theta\Big|\\ &= \Big|\int_{-1}^{1}e^{-\frac{|2x-\theta z|^2\tanh y}{4}}\left(-\frac{\theta\tanh y}{2}\frac{|2x-\theta z|^2}{4\cosh^2(y)}(2x-\theta z)\cdot z+\frac{\theta(2x-\theta z)\cdot z}{2\cosh^2 y}\right)d\theta\Big|\\ &\le C\frac{|z|}{(\tanh y)^{1/2}\cosh^2 y},\end{aligned}$$

we have that $|II_b| \le C\frac{e^{-\frac{|z|^2\coth y}{4}}}{(2\pi\sinh(2y))^{n/2}}\frac{|z|}{(\tanh y)^{1/2}\cosh^2 y} \le C\frac{e^{-\frac{|z|^2}{cy}}}{y^{n/2}}$.

Estimates II_a and II_b and the fact that $y<1$ allow us to get $|II| \le C\|f\|_\infty y^{-1+\alpha/2}$.

Finally, by using Remark 3.76 and Lemma 3.77 (2) we get

$$\begin{aligned}|III| &\le C|f(x)|(1+|x|^2)\frac{e^{-\frac{\tanh(2y)|x|^2}{2}}}{(\cosh(2y))^{n/2}} \le C|f(x)|(1+|x|^2)e^{-cy|x|^2}\\ &\le C([f]_{M^\alpha}+\|f\|_\infty)y^{-1+\alpha/2}.\end{aligned}$$

□

Proof of Theorem 3.95.

From Theorems 3.104 and 3.109, it only remains to prove that $\Lambda_\alpha^{P^{\mathcal{H}}} \subset \Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ for $\alpha \ge 2$. Let $2 \le \alpha < 3$ and suppose $f \in \Lambda_\alpha^{P^{\mathcal{H}}}$. By epigraph 2 of Theorem 3.71 we have that $\partial_{x_i}f, x_if \in \Lambda_{\alpha-1}^{P^{\mathcal{H}}}$ and, by Theorem 3.109, this is equivalent to $\partial_{x_i}f, x_if \in \Lambda_{(\alpha-1)/2}^{W^{\mathcal{H}}}$. Therefore, Theorem 3.107 gives that $f \in \Lambda_{\alpha/2}^{W^{\mathcal{H}}}$. Therefore, we have established that $\Lambda_{\alpha/2}^{W^{\mathcal{H}}} = \Lambda_\alpha^{P^{\mathcal{H}}}$, for $0<\alpha<3$. The rest of the proof follows by iterating the previous arguments. □

3.4 Applications: regularity results for fractional operators and maximum principle.

In this subsection we shall prove regularity results for operators associated to $\mathbb{H}$ and $\mathcal{H}$ when acting over the classes defined in this chapter and we shall see that the solutions of the fractional equation satisfy a maximum principle.

3.4.1 Fractional operators and regularity results.

As we have noticed in (3.5), the infinitesimal generator of the semigroup $e^{-\tau\mathbb{H}}$, $\mathbb{H}$, is not positive. This forced us to use some complex variable techniques in order to give a sense to the powers of the operator $\mathbb{H}$. Given a non necessarily positive operator $\mathbb{L}$, formulas to define $\mathbb{L}^{\pm\alpha}$, where $0<\alpha<1$, were considered in [8], [84] and [85].

Given $\beta > 0$, we recall the following two integrals related with the Gamma function:

$$\tilde{c}_\beta = \int_0^\infty e^{-t} t^\beta \, \frac{dt}{t}, \quad c_\beta = \int_0^\infty \left(e^{-t} - 1\right)^{[\beta]+1} \frac{dt}{t^{1+\beta}}. \tag{3.18}$$

It is well known that $\tilde{c}_\beta = \Gamma(\beta)$ for all $\beta > 0$ and $c_\beta = \Gamma(-\beta)$ for $0 < \beta < 1$. The following Lemma was proved in [8].

Lemma 3.110. *Let $0 < \beta < 1$ and $-\pi/2 \le \varphi_0 \le \pi/2$. Consider the ray in the complex plane* $\mathrm{ray}_{\varphi_0} := \{z = re^{i\varphi_0} : 0 < r < \infty\}$. *Then*

$$\Gamma(\beta) = \int_{\mathrm{ray}_{\varphi_0}} e^{-z} z^\beta \, \frac{dz}{z}, \quad \text{and} \quad \Gamma(-\beta) = \int_{\mathrm{ray}_{\varphi_0}} \left(e^{-z} - 1\right) \frac{dz}{z^{1+\beta}}.$$

For $0 < \beta < 1$, the absolutely convergent integrals in (3.18) can be interpreted as integrals of the functions $F(t) = e^{-t} t^{\beta-1}$ and $G(t) = (e^{-t} - 1)/t^{1+\beta}$ along the "complex" path $\{z = t : 0 < t < \infty\}$. The proof of the Lemma is based in the Cauchy Integral Theorem applied to the functions $F(z) = e^{-z} z^{\beta-1}$ and $G(z) = (e^{-z} - 1)/z^{1+\beta}$. Both functions are analytic for $z \neq 0$. For the integrals defined in (3.18) we could state a parallel Lemma to 3.110, by taking $F(z)$ as before and $H(z) = (e^{-z} - 1)^{[\beta]+1}/z^{1+\beta}$, $\beta > 0$. The proof follows the same steps. We leave the details to the reader. We have the following Corollary.

Corollary 3.111. *Let $\beta > 0$ and λ a complex number with $\Re\lambda \ge 0$. Then*

$$\lambda^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\lambda t} t^\beta \, \frac{dt}{t}, \quad \text{and} \quad \lambda^\beta = \frac{1}{c_\beta} \int_0^\infty \left(e^{-\lambda t} - 1\right)^{[\beta]+1} \frac{dt}{t^{1+\beta}}.$$

We use the last Corollary to define the negative and positive fractional powers of the operator $\mathbb{H}$ as

$$\mathbb{H}^\beta f(t,x) = \frac{1}{c_{2\beta}} \int_0^\infty \left(e^{-\tau \mathbb{H}^{1/2}} - I\right)^{[2\beta]+1} f(t,x) \frac{d\tau}{\tau^{1+2\beta}},$$

where $c_{2\beta} = \int_0^\infty \left(e^{-\tau} - 1\right)^{[2\beta]+1} \frac{d\tau}{\tau^{1+2\beta}}$, and also, for $\beta > 0$,

$$\mathbb{H}^{-\beta} f(t,x) = \frac{1}{\Gamma(2\beta)} \int_0^\infty e^{-\tau \mathbb{H}^{1/2}} f(t,x) \frac{d\tau}{\tau^{1-2\beta}}.$$

Observe that, for good enough functions,

$$\mathcal{F}(\mathbb{H}^{\pm\beta} f)(\rho,\mu) = (i\rho + 2\mu + n)^{\pm\beta} \mathcal{F}(f)(\rho,\mu), \ \rho \in \mathbb{R}, \text{ and } \mu \in \mathbb{N}^n.$$

In addition, for $\beta > 0$ we define the modified Bessel potentials, $\mathcal{J}_{\mathbb{H}}^\beta$, as

$$\mathcal{J}_{\mathbb{H}}^\beta f(t,x) = (I + \mathbb{H}^{1/2})^{-\beta} f(t,x) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\tau(I + \mathbb{H}^{1/2})} f(t,x) \frac{d\tau}{\tau^{1-\beta}}.$$

Operators in the elliptic Hermite setting.

Given $g \in L^\infty(\mathbb{R}^n)$, consider the function $f(t,x) = g(x)$, then $P_y^{\mathcal{H}} g(x)$ is the Poisson semigroup associated to the operator $\mathcal{H} = -\Delta_x + |x|^2$. The thoughts developed along this section show that:

- Let $\beta > 0$, if is g good enough, then

$$\mathcal{H}^\beta g(x) = \frac{1}{c_{2\beta}} \int \left(e^{-\tau\mathcal{H}^{1/2}} - I \right)^{[2\beta]+1} g(x) \frac{d\tau}{\tau^{1+2\beta}}$$

is well defined and $\widehat{\mathcal{H}^\beta g}(\mu) = (2|\mu| + n)^\beta \hat{g}(\mu)$, $\mu \in \mathbb{N}^n$, with $\hat{g}(\mu) = \int_{\mathbb{R}^n} g(x) h_\mu(x) dx$.

- Let $\beta > 0$, if g is good enough, then

$$\mathcal{H}^{-\beta} g(x) = \frac{1}{\Gamma(2\beta)} \int_0^\infty e^{-\tau\mathcal{H}^{1/2}} g(x) \frac{d\tau}{\tau^{1-2\beta}}$$

is well defined and $\widehat{\mathcal{H}^{-\beta} g}(\mu) = (2|\mu| + n)^{-\beta} \hat{g}(\mu)$, $\mu \in \mathbb{N}^n$.

- For $\beta > 0$, the *modified Bessel potentials* of order β are defined by

$$\mathcal{J}_{\mathcal{H}}^\beta g(x) := (I + \sqrt{\mathcal{H}})^{-\beta} g(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty e^{-\tau(I+\mathcal{H}^{1/2})} g(x) \frac{d\tau}{\tau^{1-\beta}}.$$

From the previous thoughts we can also define the following operators by means of the heat semigroup $e^{-t\mathcal{H}}$.

- The *Bessel potential of order* $\beta > 0$,

$$(Id + \mathcal{H})^{-\beta/2} f(x) = \frac{1}{\Gamma(\beta/2)} \int_0^\infty e^{-t} e^{-t\mathcal{H}} f(x) t^{\beta/2} \frac{dt}{t}.$$

- The multiplier operator of the Laplace transform type $\tilde{m}(\mathcal{H})$,

$$\tilde{m}(\mathcal{H}) f(x) = \mathcal{H} \left(\int_0^\infty e^{-s\mathcal{H}} f(x) a(s) ds \right),$$

where a be a measurable bounded function on $[0, \infty)$.

They can also be defined analogously in the parabolic case, but for simplicity here we will prove the boundedness of these operators in the spaces defined through the heat semigroup associated to $\mathcal{H}$, $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$. The proofs in the parabolic case are completely analogous.

Now we can present and prove our results. We shall prove the boundedness in the adapted Lipschitz spaces of positive and negative powers of the operators $\mathbb{H}$ and $\mathcal{H}$, as well as Riesz transforms Bessel potentials and multipliers of Laplace transform type. We shall prove first the results regarding the operator $\mathbb{H}$ and at the end of the subsection we shall make the corresponding remarks for the results concerning $\mathcal{H}$.

Theorem 3.112 (Hölder estimates). *Let* $0 < 2\beta < \alpha$ *and* $f \in \Lambda_\alpha^{\mathcal{P}}$ *(respectively* $g \in \Lambda_\alpha^{P^{\mathcal{H}}}$*), then* $\mathbb{H}^\beta f \in \Lambda_{\alpha-2\beta}^{\mathcal{P}}$ *(respectively* $\mathcal{H}^\beta g \in \Lambda_{\alpha-2\beta}^{P^{\mathcal{H}}}$*) and*

$$\|\mathbb{H}^\beta f\|_{\Lambda_{\alpha-2\beta}^{\mathcal{P}}} \le C \|f\|_{\Lambda_\alpha^{\mathcal{P}}}, \ \textit{(respectively } \|\mathcal{H}^\beta g\|_{\Lambda_{\alpha-2\beta}^{P^{\mathcal{H}}}} \le C \|g\|_{\Lambda_\alpha^{P^{\mathcal{H}}}}\textit{)}.$$

Theorem 3.113 (Schauder estimates)**.** *Let $\alpha, \beta > 0$.*

(i) Given $f \in \Lambda_\alpha^{\mathcal{P}}$ (respectively $g \in \Lambda_\alpha^{P^{\mathcal{H}}}$), then $\mathbb{H}^{-\beta} f \in \Lambda_{\alpha+2\beta}^{\mathcal{P}}$ (respectively $\mathcal{H}^{-\beta} g \in \Lambda_{\alpha+2\beta}^{P^{\mathcal{H}}}$) and

$$\|\mathbb{H}^{-\beta} f\|_{\Lambda_{\alpha+2\beta}^{\mathcal{P}}} \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}}, \text{ (respectively } \|\mathcal{H}^{-\beta} g\|_{\Lambda_{\alpha+2\beta}^{P^{\mathcal{H}}}} \leq C\|g\|_{\Lambda_\alpha^{P^{\mathcal{H}}}}).$$

(ii) If $f \in L^\infty(\mathbb{R}^{n+1})$ (respectively $g \in L^\infty(\mathbb{R}^n)$), then $\mathbb{H}^{-\beta} f \in \Lambda_{2\beta}^{\mathcal{P}}$ (respectively $\mathcal{H}^{-\beta} g \in \Lambda_{2\beta}^{P^{\mathcal{H}}}$) and

$$\|\mathbb{H}^{-\beta} f\|_{\Lambda_{2\beta}^{\mathcal{P}}} \leq C\|f\|_\infty, \text{ (respectively } \|\mathcal{H}^{-\beta} g\|_{\Lambda_{2\beta}^{P^{\mathcal{H}}}} \leq C\|g\|_\infty).$$

To prove Theorems 3.112 and 3.113 we need the following lemma.

Lemma 3.114. *Let α, β positive real numbers.*

(a) If $0 < 2\beta < \alpha$ and $f \in \Lambda_\alpha^{\mathcal{P}}$, then we have $|\mathbb{H}^\beta f(t,x)| \leq C < \infty$, $(t,x) \in \mathbb{R}^{n+1}$.

(b) If $f \in L^\infty(\mathbb{R}^{n+1})$ we have $|\mathbb{H}^{-\beta} f(t,x)| \leq C < \infty$, for all $(t,x) \in \mathbb{R}^{n+1}$.

Proof. By Corollary 3.82, it suffices to consider the case $2\beta < \alpha < [2\beta] + 1 = \ell$. Then

$$\begin{aligned}\|(\mathcal{P}_\nu - I)^{[2\beta]+1} f\|_{L^\infty(\mathbb{R}^{n+1})} &= \left\| \int_0^\nu \underbrace{\dots}_{\ell} \int_0^\nu \partial_{y_1} \dots \partial_{y_\ell} \mathcal{P}_{y_1+\dots+y_\ell} f(t,x) dy_\ell \dots dy_1 \right\|_\infty \\ &\leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} \int_0^\nu \underbrace{\dots}_{\ell} \int_0^\nu (y_1 + \dots + y_\ell)^{-\ell+\alpha} dy_\ell \dots dy_1 \leq C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} \nu^\alpha.\end{aligned}$$

Then, as $0 < 2\beta < \alpha$, and $f, \mathcal{P}_\nu f \in L^\infty(\mathbb{R}^{n+1})$ we have

$$|\mathbb{H}^\beta f(t,x)| \leq C_\beta \|f\|_{\Lambda_\alpha^{\mathcal{P}}} \int_0^1 \frac{\nu^\alpha}{\nu^{1+2\beta}} d\nu + C_\beta \|f\|_\infty \int_1^\infty \frac{1}{\nu^{1+2\beta}} \leq C_\beta < \infty. \tag{3.19}$$

To prove (b), we use the boundedness of f for $\nu < 1$, and Lemma 3.78 (ii) with $s > 2\beta$, when $\nu > 1$. Thus,

$$\mathbb{H}^{-\beta} f(t,x) = \frac{1}{\Gamma(2\beta)} \int_0^\infty \mathcal{P}_\nu f(t,x) \frac{d\nu}{\nu^{1-2\beta}} \leq C_\beta \|f\|_\infty \left(\int_0^1 \frac{d\nu}{\nu^{1-2\beta}} + \int_1^\infty \frac{d\nu}{\nu^{1+s-2\beta}} \right) \leq C_\beta.$$

□

Proof of Theorem 3.112. Let $m = [\alpha - 2\beta] + 1$ and $\ell = [2\beta] + 1$. Then, $m + \ell = [\alpha - 2\beta] + 1 + [2\beta] + 1 > \alpha - 2\beta + 2\beta = \alpha$ and as $m + \ell \in \mathbb{N}$ we get $m + \ell \geq [\alpha] + 1$.

Previous Lemma 3.114 and Fubini's Theorem allow us to write

$$
\begin{aligned}
\left|\partial_y^m \mathcal{P}_y(\mathbb{H}^\beta f)\right| &= \left|c_{2\beta}\int_0^\infty \partial_y^m\mathcal{P}_y\Big(\underbrace{\int_0^\nu \dots \int_0^\nu}_{\ell=[2\beta]+1} \partial_w^\ell \mathcal{P}_w|_{w=s_1+\dots+s_\ell} f ds_1\dots ds_\ell\Big)\frac{d\nu}{\nu^{1+2\beta}}\right| \\
&= \left|c_{2\beta}\int_0^\infty \Big(\underbrace{\int_0^\nu \dots \int_0^\nu}_{\ell=[2\beta]+1} \partial_w^{m+\ell} \mathcal{P}_w|_{w=y+s_1+\dots+s_\ell} f ds_1\dots ds_\ell\Big)\frac{d\nu}{\nu^{1+2\beta}}\right| \\
&\le C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}}\int_0^\infty \Big(\underbrace{\int_0^\nu \dots \int_0^\nu}_{\ell=[2\beta]+1} (y+s_1+\dots s_\ell)^{-(m+\ell)+\alpha} ds_1\dots ds_\ell\Big)\frac{d\nu}{\nu^{1+2\beta}} \\
&= C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}}\int_0^y (\dots)\frac{d\nu}{\nu^{1+2\beta}} + C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}}\int_y^\infty (\dots)\frac{d\nu}{\nu^{1+2\beta}} = I + II,
\end{aligned}
$$

where in the last inequality we have used that $m+\ell \ge [\alpha]+1 > \alpha$. Now we shall estimate I and II.

$$
\begin{aligned}
|I| &\le C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-m+\alpha}\int_0^y \underbrace{\int_0^{\nu/y} \dots \int_0^{\nu/y}}_{\ell=[2\beta]+1} (1+s_1+\dots s_\ell)^{-(m+\ell)+\alpha} ds_1\dots ds_\ell \frac{d\nu}{\nu^{1+2\beta}} \\
&\le C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-m+\alpha}\int_0^y \left(\frac{\nu}{y}\right)^\ell \frac{d\nu}{\nu^{1+2\beta}} = C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-m+\alpha-\ell}\int_0^y \frac{d\nu}{\nu^{1+2\beta-\ell}} \le C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-m+\alpha-2\beta}.
\end{aligned}
$$

Notice that in the last inequality we have used that $1+2\beta-\ell = 2\beta-[2\beta] < 1$. On the other hand,

$$
|II| \le C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}}\int_y^\infty \Big((y+\nu)^{-m+\alpha} + y^{-m+\alpha}\Big)\frac{d\nu}{\nu^{1+2\beta}}.
$$

If $-m+\alpha \le 0$, we have $|II| \le C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}}\int_y^\infty y^{-m+\alpha}\frac{d\nu}{\nu^{1+2\beta}} = C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-m+\alpha-2\beta}$. In the case $-m+\alpha > 0$, as $m-\alpha+2\beta+1 = [\alpha-2\beta]+1-\alpha+2\beta+1 > 1$, we get $|II| \le C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}}\int_y^\infty \nu^{-m+\alpha}\frac{d\nu}{\nu^{1+2\beta}} \le C_\beta\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-m+\alpha-2\beta}$.

□

Proof of Theorem 3.113. Assume that $f \in \Lambda_\alpha^{\mathcal{P}}$ and let $\ell = [\alpha+2\beta]+1 \ge [\alpha]+1 > \alpha$. Fubini Theorem together with Lemma 3.114 allow us to get

$$
\begin{aligned}
\|\partial_y^\ell \mathcal{P}_y(\mathbb{H}^{-\beta} f)(t,x)\|_{L^\infty(\mathbb{R}^{n+1})} &= \left\|\int_0^\infty \partial_y^\ell \mathcal{P}_y\mathcal{P}_\nu f(t,x)\frac{d\nu}{\nu^{1-2\beta}}\right\|_\infty \\
&\le C\|f\|_{\Lambda_\alpha^{\mathcal{P}}}\int_0^\infty (y+\nu)^{-\ell+\alpha}\frac{d\nu}{\nu^{1-2\beta}} \le C\|f\|_{\Lambda_\alpha^{\mathcal{P}}} y^{-\ell+\alpha-2\beta}.
\end{aligned}
$$

Assume now that $f \in L^\infty(\mathbb{R}^{n+1})$ and let $\ell = [2\beta]+1$. By applying Lemma 3.78 (ii) we have $|\partial_y^\ell \mathcal{P}_y\mathcal{P}_\nu f(t,x)| \le C\frac{\|f\|_\infty}{y^\ell}$. Then we can proceed as before. □

Theorem 3.115. *Let $\alpha, \beta > 0$.*

(i) *If* $f \in \Lambda^{\mathcal{P}}_\alpha$ *(respectively* $g \in \Lambda^{P^{\mathcal{H}}}_\alpha$*), then* $\mathcal{J}^\beta_{\mathbb{H}} f \in \Lambda^{\mathcal{P}}_{\alpha+\beta}$ *(respectively* $\mathcal{J}^\beta_{\mathcal{H}} g \in \Lambda^{P^{\mathcal{H}}}_{\alpha+\beta}$*) and*

$$\|\mathcal{J}^\beta_{\mathbb{H}} f\|_{\Lambda^{\mathcal{P}}_{\alpha+\beta}} \le C\|f\|_{\Lambda^{\mathcal{P}}_\alpha}, \text{ (respectively } \|\mathcal{J}^\beta_{\mathcal{H}} g\|_{\Lambda^{P^{\mathcal{H}}}_{\alpha+\beta}} \le C\|g\|_{\Lambda^{P^{\mathcal{H}}}_\alpha}).$$

(ii) *If* $f \in L^\infty(\mathbb{R}^{n+1})$ *(respectively* $g \in L^\infty(\mathbb{R}^n)$*), then* $\mathcal{J}^\beta_{\mathbb{H}} f \in \Lambda^{\mathcal{P}}_\beta$ *(respectively* $\mathcal{J}^\beta_{\mathcal{H}} g \in \Lambda^{P^{\mathcal{H}}}_\beta$*) and*

$$\|\mathcal{J}^\beta_{\mathbb{H}} f\|_{\Lambda^{\mathcal{P}}_\beta} \le C\|f\|_\infty, \text{ (respectively } \|\mathcal{J}^\beta_{\mathcal{H}} g\|_{\Lambda^{P^{\mathcal{H}}}_\beta} \le C\|g\|_\infty).$$

Observe that the proof of Theorem 3.115 follows from the proof of Theorem 3.113.

Theorem 3.116. *Consider the Parabolic Hermite Riesz transforms of order* $m \ge 1$ *defined by*

$$R_\nu = (A^{\nu_1}_{\pm 1} A^{\nu_2}_{\pm 2} \dots A^{\nu_n}_{\pm n})\mathbb{H}^{-m/2} \text{ and } R_m = \partial^m_t \mathbb{H}^{-m}$$

where $\nu_i \ge 0, i = 1, \dots, n$ *and* $|\nu| = \nu_1 + \cdots + \nu_n = m$*. Let* $\alpha > 0$*, then* R_ν *and* R_m *are bounded from* $\Lambda^{\mathcal{P}}_\alpha$ *into itself. A parallel result holds for the operators* $(A^{\nu_1}_{\pm 1} A^{\nu_2}_{\pm 2} \dots A^{\nu_n}_{\pm n})\mathcal{H}^{-m/2}$ *when acting on the spaces* $\Lambda^{P^{\mathcal{H}}}_\alpha$.

See [82], [86] and [89] and the references there in for more information about the Hermite Riesz transforms $A_j \mathcal{H}^{-1/2}$.

The proof of Theorem 3.116 is a direct consequence of Theorems 3.113 and 3.68.

We also get the boundedness of the multiplier operator of the Laplace transform type on the spaces $\Lambda^{\mathcal{P}}_\alpha$ and $\Lambda^{P^{\mathcal{H}}}_\alpha$. We recall to the reader that the imaginary powers $\lambda^{i\gamma}$ are examples of multipliers of Laplace transform type.

Theorem 3.117. *Let* a *be a bounded function on* $[0, \infty)$ *and consider*

$$m(\lambda) = \lambda^{1/2} \int_0^\infty e^{-s\lambda^{1/2}} a(s)ds, \ \ \lambda > 0.$$

Then, for every $\alpha > 0$*, the multiplier operator of the Laplace transform type* $m(\mathbb{H})$ *(respectively* $m(\mathcal{H})$*) is bounded from* $\Lambda^{\mathcal{P}}_\alpha$ *(respectively* $\Lambda^{P^{\mathcal{H}}}_\alpha$*) into itself.*

Proof of Theorem 3.117. Observe that for $f \in L^\infty(\mathbb{R}^{n+1})$, Lemma 3.78 (i) and (ii), give $\Big| \int_0^\infty e^{-s\mathbb{H}^{1/2}} f(t,x) a(s) ds \Big| \le C\|a\|_\infty \|f\|_\infty \int_0^\infty \min(1, s^{-2}) ds < \infty$. Moreover, if $f \in \Lambda^{\mathcal{P}}_\alpha(\mathbb{R}^{n+1})$, $\alpha > 0$ and $\ell = [\alpha+1]+1 > \alpha+1$, by Fubini's Theorem we have

$$\begin{aligned}\Big|\partial^\ell_y \mathcal{P}_y\Big(\int_0^\infty \mathcal{P}_s f(t,x) a(s) ds\Big)\Big| &= \Big|\int_0^\infty \partial^\ell_w \mathcal{P}_w f(t,x)\Big|_{w=y+s} a(s) ds\Big| \\ &\le C\|a\|_\infty \|f\|_{\Lambda^{\mathcal{P}}_\alpha} \int_0^\infty (y+s)^{-\ell+\alpha} ds \le C\|a\|_\infty \|f\|_{\Lambda^{\mathcal{P}}_\alpha} y^{-\ell+\alpha+1}.\end{aligned}$$

We have proved that the operator $f \longrightarrow \int_0^\infty e^{-s\mathbb{H}^{1/2}} f a(s) ds$ maps $\Lambda^{\mathcal{P}}_\alpha(\mathbb{R}^{n+1})$ into $\Lambda^{\mathcal{P}}_{\alpha+1}(\mathbb{R}^{n+1})$. Then, Theorem 3.112 gives the result. □

Proofs in the elliptic Hermite setting.

As we did in the previous sections, if $f(t,x) = g(x)$, then it can be easily checked that $\mathcal{H}^{\pm\beta}g(x) = \mathbb{H}^{\pm\beta}f(t,x)$ and $m(\mathcal{H}) = m(\mathbb{H})$. Hence, the Hermite's version of Theorems 3.112, 3.113, 3.115, 3.117 and 3.116 hold. Moreover, due to Theorem 3.95 and the regularity results proved in $\Lambda_\alpha^{P^{\mathcal{H}}}$ for the operators $\mathcal{H}^{\pm\beta}$, we get the corresponding results in the $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ classes.

Finally, we prove the results regarding the operators defined through the heat semigroup.

Theorem 3.118. *Let $\alpha, \beta > 0$. Then, the Bessel potential satisfies*

(i) $\|(Id + \mathcal{H})^{-\beta/2} f\|_{\Lambda^{W^{\mathcal{H}}}_{\frac{\alpha+\beta}{2}}} \le C\|f\|_{\Lambda^{W^{\mathcal{H}}}_{\alpha/2}}$.

(ii) $\|(Id + \mathcal{H})^{-\beta/2} f\|_{\Lambda^{W^{\mathcal{H}}}_{\beta/2}} \le C\|f\|_\infty$.

Proof of Theorem 3.118.

Since $\|W_y^{\mathcal{H}} f\|_\infty \le C\|f\|_\infty$ and $\|\partial_y^l W_y^{\mathcal{H}} f\|_\infty \le C\frac{\|f\|_\infty}{y^l}$ for $l \in \mathbb{N}$, we can apply Fubini's Theorem and we can introduce the derivatives inside the integral in both cases.

Let $f \in \Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ and $\ell = [\alpha/2 + \beta/2] + 1$. By using Lemma 3.100 we have

$$\begin{aligned}
|\partial_y^\ell W_y^{\mathcal{H}}((Id + \mathcal{H})^{-\beta/2} f(x))| &= \left| \frac{1}{\Gamma(\beta/2)} \int_0^\infty e^{-t} \partial_y^\ell W_y^{\mathcal{H}} (W_t^{\mathcal{H}} f)(x) t^{\beta/2} \frac{dt}{t} \right| \\
&\le C \int_0^\infty e^{-t} (\partial_w^\ell W_w^{\mathcal{H}} f(x)\Big|_{w=y+t}) t^{\beta/2} \frac{dt}{t} \\
&\le C\|f\|_{\Lambda^{W^{\mathcal{H}}}_{\alpha/2}} \int_0^\infty e^{-t} (y+t)^{-\ell+\alpha/2} t^{\beta/2} \frac{dt}{t} \\
&\overset{\frac{t}{y}=u}{\le} C\|f\|_{\Lambda^{W^{\mathcal{H}}}_{\alpha/2}} y^{\alpha/2+\beta/2-\ell} \int_0^\infty \frac{u^{\beta/2} e^{-yu}}{(1+u)^{\ell-\alpha/2}} \frac{du}{u} \\
&\le C\|f\|_{\Lambda^{W^{\mathcal{H}}}_{\alpha/2}} y^{\alpha/2+\beta/2-\ell}.
\end{aligned}$$

When $f \in L^\infty(\mathbb{R}^n)$ we proceed analogously by using that, for $\ell = [\beta/2]+1$, $\|\partial_y^\ell W_y^{\mathcal{H}} W_\nu^{\mathcal{H}} f\|_\infty \le C\frac{\|f\|_\infty}{y^\ell}$.

□

Theorem 3.119. *For $\alpha > 0$, the multiplier operator of the Laplace transform type $\tilde{m}(\mathcal{H})$ is bounded from $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ into itself.*

Proof of Theorem 3.119.

Assume $f \in \Lambda_{\alpha/2}^{W^{\mathcal{H}}}$, $\alpha > 0$. Since $f \in L^\infty(\mathbb{R}^n)$, by using Lemma 3.97 we have

$\left| \int_0^\infty e^{-s\mathcal{H}} f(x) a(s) ds \right| \le C\|f\|_\infty \|a\|_\infty \int_0^\infty \min(1, s^{-2}) ds \le C\|a\|_\infty \|f\|_\infty.$

Let $\ell = [\alpha/2 + 1] + 1$, by Fubini's Theorem we have

$$\begin{aligned}
\left|\partial_y^\ell W_y^{\mathcal{H}} \Big(\int_0^\infty W_s^{\mathcal{H}} f(x) a(s) ds \Big) \right| &= \left| \int_0^\infty \partial_w^\ell W_w^{\mathcal{H}} f(x)\Big|_{w=y+s} a(s) ds \right| \\
&\le C\|f\|_{\Lambda^{W^{\mathcal{H}}}_{\alpha/2}} \int_0^\infty (y+s)^{-\ell+\alpha/2} ds \le C\|f\|_{\Lambda^{W^{\mathcal{H}}}_{\alpha/2}} y^{-\ell+\alpha/2+1}.
\end{aligned}$$

We have proved that the operator $f \longrightarrow \int_0^\infty e^{-s\mathcal{H}} fa(s)ds$ maps $\Lambda_{\alpha/2}^{W^{\mathcal{H}}}$ into $\Lambda_{\alpha/2+1}^{W^{\mathcal{H}}}$. Then, Theorem 3.95 and Theorem 3.71 (2) give the result.

□

Observe that Theorem 3.95 establishes that the previous two results are valid for the spaces $\Lambda_\alpha^{P^{\mathcal{H}}}$.

3.4.2 Maximum principle.

Apart from the above regularity results, our semigroup language allows us to get some maximum principle.

Theorem 3.120 (Maximum principle). *Let $0 < \beta < 1$, $\alpha > 2\beta$ and $f \in \Lambda_\alpha^{\mathcal{P}}$. Suppose that*

1. *$f(t_0, x_0) = 0$ for some $(t_0, x_0) \in \mathbb{R}^{n+1}$, and*
2. *$f(t, x) \geq 0$ for $t \leq t_0$, $x \in \mathbb{R}^n$.*

Then $\mathbb{H}^\beta f(t_0, x_0) \leq 0$. Moreover, $\mathbb{H}^\beta f(t_0, x_0) = 0$ if and only if $f(t, x) = 0$ for $t \leq t_0$ and $x \in \mathbb{R}^n$.

Proof of Theorem 3.120. Observe that $c_{2\beta} > 0$ for $[2\beta] + 1$ odd and $c_{2\beta} < 0$ for $[2\beta] + 1$ even. On the other hand as the kernel $\mathcal{P}_\nu(\tau, x, z)$ is always positive we have $\mathcal{P}_\nu f(t, x) \geq 0$, $t \leq t_0$. If $0 < \beta < 1/2$, $\mathbb{H}^\beta f(t_0, x_0) = \frac{1}{c_{2\beta}} \int_0^\infty \mathcal{P}_\nu f(t_0, x_0) \frac{d\nu}{\nu^{1+2\beta}}$, then $\mathbb{H}^\beta f(t_0, x_0) \leq 0$. If $1/2 \leq \beta < 1$, then $\mathbb{H}^\beta f(t_0, x_0) = \frac{1}{c_{2\beta}} \int_0^\infty (\mathcal{P}_{2\nu} f(t_0, x_0) - 2\mathcal{P}_\nu f(t_0, x_0)) \frac{d\nu}{\nu^{1+2\beta}}$, and as $(\mathcal{P}_{2\nu} f(t_0, x_0) - 2\mathcal{P}_\nu f(t_0, x_0)) \leq 0$, we obtain that $\mathbb{H}^\beta f(t_0, x_0) \leq 0$.

□

Chapter 4

Parabolic equations in the Bessel setting

Along this chapter we consider the parabolic equations

$$\frac{\partial u(t,x)}{\partial t} = \Delta_\mu u(t,x) + f(t,x), \quad (t,x) \in \mathbb{R} \times (0,\infty) \ \ or \ \ (t,x) \in (0,\infty) \times (0,\infty), \tag{4.1}$$

and the corresponding Cauchy problems where, for every $\mu > -1$, Δ_μ represents the Bessel operator defined by $\Delta_\mu = \partial_x^2 + (\frac{1}{4} - \mu^2)x^{-2}$. We establish weighted and mixed weighted Sobolev type inequalities for solutions of Bessel parabolic equations. We use singular integrals techniques in a parabolic setting.

The content of this chapter **corresponds to [12]** and it is motivated by [73].

The Bessel operator Δ_μ can be seen as a one dimensional Schrödinger operator with the singular potential $V_\mu(x) = (\frac{1}{4} - \mu^2)x^{-2}$, $x \in (0,\infty)$. Singular integrals associated with parabolic Schrödinger operators $\partial_t - \Delta + V$ in $\mathbb{R}^{n+1}$ have been investigated in [24], [41], [57] and [69]. Our potentials V_μ, $\mu > -1$, are not included in the class of potentials considered in the above mentioned papers. There, the potentials V are nonnegative and in $L^1_{\text{loc}}(\mathbb{R}^{n+1})$ and they belong to the parabolic reverse Hölder classes.

Let $\mu > -1$. For every $\phi \in C_c^\infty((0,\infty))$, the space of smooth functions with compact support on $(0,\infty)$, the Hankel transform $h_\mu(\phi)$ of ϕ is defined by

$$h_\mu(\phi)(x) = \int_0^\infty \sqrt{xy} J_\mu(xy)\phi(y)dy, \quad x \in (0,\infty),$$

where J_μ denotes the Bessel function of the first kind and order μ. h_μ can be extended to $L^2((0,\infty))$ as an isometry (see [18] and [90]) and $h_\mu^{-1} = h_\mu$ on $L^2((0,\infty))$. For every $\phi \in C_c^\infty((0,\infty))$, we have that (see [97, Lemma 5.4-1]),

$$h_\mu(\Delta_\mu \phi)(x) = -x^2 h_\mu(\phi)(x), \quad x \in (0,\infty).$$

We extend the definition of the operator Δ_μ as follows. We define the domain of $\boldsymbol{\Delta}_\mu$, $D(\boldsymbol{\Delta}_\mu)$, by $D(\boldsymbol{\Delta}_\mu) = \{\phi \in L^2((0,\infty)) : \ x^2 h_\mu(\phi) \in L^2((0,\infty))\}$ and, for every $\phi \in D(\boldsymbol{\Delta}_\mu)$, $\boldsymbol{\Delta}_\mu \phi = -h_\mu(x^2 h_\mu(\phi))$. According to [97, Theorem 5.4-1], $C_c^\infty((0,\infty)) \subset D(\boldsymbol{\Delta}_\mu)$ and $\boldsymbol{\Delta}_\mu \phi = \Delta_\mu \phi$,

$\phi \in C_c^\infty((0,\infty))$. Note that, for every $\mu \in (-1,1)$, $\Delta_\mu \phi = \Delta_{-\mu}\phi$, $\phi \in C_c^\infty((0,\infty))$, and $\boldsymbol{\Delta}_\mu \neq \boldsymbol{\Delta}_{-\mu}$.

The operator $-\boldsymbol{\Delta}_\mu$ is positive and selfadjoint on $L^2((0,\infty))$. Moreover, $-\boldsymbol{\Delta}_\mu$ generates a semigroup $\{W_t^\mu\}_{t>0}$ of operators in $L^2((0,\infty))$ where, for every $t>0$ and $\phi \in L^2((0,\infty))$,

$$W_t^\mu(\phi)(x) = \int_0^\infty W_t^\mu(x,y)\phi(y)dy, \quad x \in (0,\infty). \tag{4.2}$$

Here, $W_t^\mu(x,y) = \frac{(xy)^{1/2}}{2t} I_\mu\left(\frac{xy}{2t}\right) e^{-\frac{x^2+y^2}{4t}}$, $x,y,t \in (0,\infty)$, where I_μ represents the modified Bessel function of the first kind and order μ. $\{W_t^\mu\}_{t>0}$ is usually called the heat semigroup associated with the Bessel operator $\boldsymbol{\Delta}_\mu$.

If, for every $t>0$, W_t^μ is given as in (4.2), $\{W_t^\mu\}_{t>0}$ also defines a semigroup of operators on $L^p((0,\infty))$, for each $1<p<\infty$ when $\mu > -1/2$ and for each $1<p<\infty$ such that $-\mu - 1/2 < \frac{1}{p} < \mu + 3/2$, when $-1 < \mu \leq -1/2$.

Harmonic analysis associated with Bessel operator (Riesz transforms, maximal operators, Littlewood-Paley functions, fractional Bessel operators, Hardy spaces,..) has been developed in the last years ([9], [10], [13], [14], etc) although the first results about this topic had been obtain by Muckenhoupt and Stein ([64]) in the sixties of the last century.

Along this chapter we will use some properties of the modified Bessel function I_ν that can be found in the Lebedev's monograph ([56]) and we recall now. For every $\nu > -1$, the modified Bessel function I_ν is defined by

$$I_\nu(z) = \sum_{k=0}^\infty \frac{z^{2k+\nu}}{2^{2k+\nu} k! \Gamma(\nu+1)}, \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

The following properties hold

$$\lim_{z\to 0} \frac{I_\nu(z)}{z^\nu} = \frac{1}{2^\nu \Gamma(\nu+1)}. \tag{4.3}$$

$$I_\nu(z) = \frac{e^z}{\sqrt{2\pi z}} \left(\sum_{k=0}^n (-1)^k [\nu,k](2z)^{-k} + O(|z|^{-n-1}) \right), \; z \in \mathbb{C}, \; |Arg(z)| < \frac{\pi}{4}. \tag{4.4}$$

where $[\nu, 0] = 1$ and

$$[\nu,k] = \frac{(4\nu^2-1)(4\nu^2-9)...(4\nu^2-(2k-1)^2)}{2^{2k}\Gamma(k+1)}, \quad k \in \mathbb{N} \text{ and } k \geq 1,$$

and

$$\frac{d}{dz}(z^{-\nu} I_\nu(z)) = z^{-\nu} I_{\nu+1}(z), \quad z \in \mathbb{C} \setminus (-\infty, 0]. \tag{4.5}$$

4.1 Results for the solutions in the space $\mathbb{R} \times (0,\infty)$.

In this section we show our results concerning to the solutions of (4.1) in the whole space $\mathbb{R} \times (0,\infty)$.

4.1.1 Classical solvability.

In this subsection we will prove the following result.

Theorem 4.121. *Assume that $f \in L^\infty(\mathbb{R} \times (0,\infty))$ has compact support on $\mathbb{R} \times (0,\infty)$. Then, for $\mu > -1$, the function $u(t,x)$, $(t,x) \in \mathbb{R} \times (0,\infty)$, given by*

$$u(t,x) = \int_0^\infty \int_0^\infty W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau, \quad (t,x) \in \mathbb{R} \times (0,\infty),$$

is defined by an absolutely convergent integral, for every $(t,x) \in \mathbb{R} \times (0,\infty)$. Moreover, if f is also in $C^2(\mathbb{R} \times (0,\infty))$, then, for every $(t,x) \in \mathbb{R} \times (0,\infty)$, $\frac{\partial u(t,x)}{\partial t} = \Delta_\mu u(t,x) + f(t,x)$, being

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \int_0^\infty \frac{\partial}{\partial \tau} W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau + f(t,x)\\ &= \lim_{\epsilon \to 0^+} \int_{\Omega_\epsilon(x)} \frac{\partial}{\partial \tau} W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau + Af(t,x), \quad t,x \in (0,\infty),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 u(t,x)}{\partial x^2} &= \lim_{\epsilon \to 0^+} \int_\epsilon^\infty \int_0^\infty \frac{\partial^2}{\partial x^2} W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau\\ &= \lim_{\epsilon \to 0^+} \int_{\Omega_\epsilon(x)} \frac{\partial^2}{\partial x^2} W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau - (1-A)f(t,x)\end{aligned}$$

where, for every $\epsilon, x \in (0,\infty)$, $\Omega_\epsilon(x) = \{(\tau,y) \in (0,\infty)^2 : \ \tau^{1/2} + |x-y| > \epsilon\}$, and $A = \frac{1}{\sqrt{\pi}} \int_0^1 e^{-\frac{w^2}{4}} dw$.

Proof. Suppose that $f \in L^\infty(\mathbb{R} \times (0,\infty))$ is a complex function such that supp$f$ is compact on $\mathbb{R} \times (0,\infty)$. Since $\int_0^\infty W_\tau^\mu(x,y) y^{\mu+1/2} dy = x^{\mu+1/2}$, $\tau, x \in (0,\infty)$, we can write

$$\int_0^\infty \int_0^\infty W_\tau^\mu(x,y)|f(t-\tau,y)|dyd\tau \leq C\|f\|_\infty x^{\mu+1/2}, \quad x \in (0,\infty) \text{ and } t \in \mathbb{R}.$$

Here $C > 0$ depends on the support of f. Hence, the integral defining

$$u(t,x) = \int_0^\infty \int_0^\infty W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau, \quad x \in (0,\infty) \text{ and } t \in \mathbb{R},$$

is absolutely convergent.

Assume now that $f \in C^1(\mathbb{R} \times (0,\infty))$ and it has compact support. By proceeding as above we can prove that

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= \int_0^\infty \int_0^\infty W_\tau^\mu(x,y) \frac{\partial}{\partial t} f(t-\tau,y)dyd\tau\\ &= -\int_0^\infty \int_0^\infty W_\tau^\mu(x,y) \frac{\partial}{\partial \tau} f(t-\tau,y)dyd\tau, \quad x \in (0,\infty), \text{ and } t \in \mathbb{R},\end{aligned}$$

where the integrals are absolutely convergent.

Here and in the sequel we denote by

$$W_\tau(z) = \frac{1}{\sqrt{4\pi\tau}} e^{-|z|^2/(4\tau)}, \quad \tau > 0 \text{ and } z \in \mathbb{R},$$

the classical heat kernel.

To get the expressions of the statement we will compare the heat kernel $W_\tau^\mu(x,y)$ with $W_\tau(x-y)$ near the singularity. Thus, we can write

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= -\int_{x/2}^{2x}\int_0^\infty (W_\tau^\mu(x,y) - W_\tau(x-y))\frac{\partial}{\partial\tau} f(t-\tau,y)d\tau dy \\ &- \int_0^{x/2}\int_0^\infty W_\tau^\mu(x,y)\frac{\partial}{\partial\tau} f(t-\tau,y)d\tau dy - \int_{2x}^\infty\int_0^\infty W_\tau^\mu(x,y)\frac{\partial}{\partial\tau} f(t-\tau,y)d\tau dy \\ &- \int_{x/2}^{2x}\int_0^\infty W_\tau(x-y)\frac{\partial}{\partial\tau} f(t-\tau,y)d\tau dy, \quad x,t \in (0,\infty).\end{aligned} \tag{4.6}$$

According to (4.3) and (4.4) ([14, Lemma 3.1]), we have that

$$0 \le W_\tau^\mu(x,y) \le C\left(1 + \left(\frac{xy}{\tau}\right)^{\mu+1/2}\right)\frac{e^{-\frac{(x-y)^2}{\tau}}}{\sqrt{\tau}}, \quad x,y,\tau \in (0,\infty). \tag{4.7}$$

Then,

$$W_\tau^\mu(x,y) \le C\left(1 + \left(\frac{xy}{\tau}\right)^{\mu+1/2}\right)\frac{e^{-cx^2/\tau}}{\sqrt{\tau}}, \quad 0 < y < x/2 \text{ and } \tau \in (0,\infty).$$

By partial integration we get

$$\int_0^{x/2}\int_0^\infty W_\tau^\mu(x,y)\frac{\partial}{\partial\tau} f(t-\tau,y)d\tau dy = -\int_0^{x/2}\int_0^\infty \frac{\partial}{\partial\tau} W_\tau^\mu(x,y) f(t-\tau,y)d\tau dy, \tag{4.8}$$

with $t,x \in (0,\infty)$. Also, we have that

$$\int_{2x}^\infty\int_0^\infty W_\tau^\mu(x,y)\frac{\partial}{\partial\tau} f(t-\tau,y)d\tau dy = -\int_{2x}^\infty\int_0^\infty \frac{\partial}{\partial\tau} W_\tau^\mu(x,y) f(t-\tau,y)d\tau dy, \tag{4.9}$$

$t,x \in (0,\infty)$. By using (4.4), it follows that

$$\begin{aligned}|W_\tau^\mu(x,y) - W_\tau(x-y)| &= \left|\frac{(xy)^{1/2}}{2\tau} e^{-\frac{x^2+y^2}{4\tau}} I_\mu\left(\frac{xy}{2\tau}\right) - \frac{1}{2\sqrt{\pi}}\frac{e^{-(x-y)^2/4\tau}}{\sqrt{\tau}}\right| \\ &\le C\frac{\sqrt{\tau}}{xy} e^{-(x-y)^2/(4\tau)}, \quad \tau,x,y \in (0,\infty).\end{aligned}$$

Partial integration leads to

$$\begin{aligned}&\int_{x/2}^{2x}\int_0^\infty (W_\tau^\mu(x,y) - W_\tau(x-y))\frac{\partial}{\partial\tau} f(t-\tau,y)d\tau dy \\ &\quad = -\int_{x/2}^{2x}\int_0^\infty \frac{\partial}{\partial\tau}(W_\tau^\mu(x,y) - W_\tau(x-y)) f(t-\tau,y)d\tau dy, \ t,x \in (0,\infty).\end{aligned} \tag{4.10}$$

We are going to see that the integrals on the right hand side of (4.8), (4.9) and (4.10) are absolutely convergent. By (4.3), (4.4), and (4.5) ([14, pages 128-131]) we have that, for every $0 < y < x/2$,

$$\left|\frac{\partial}{\partial \tau} W_\tau^\mu(x,y)\right| \leq C \begin{cases} \frac{1}{\tau^{3/2}} e^{-cx^2/\tau}, & 0 < \tau < xy, \\ \frac{(xy)^{\mu+1/2}}{\tau^{\mu+2}} e^{-cx^2/\tau}, & \tau \geq xy. \end{cases} \tag{4.11}$$

Let $x \in (0,\infty)$ and $t \in \mathbb{R}$. Since supp$f$ is compact, there exist $0 < a < x/2$, $2x < b$ and $c > 0$, such that $f(t-\tau, y) = 0$, $(\tau, y) \notin (-\infty, c) \times (a,b)$. Then,

$$\int_0^{x/2} \int_0^\infty \left|\frac{\partial}{\partial \tau} W_\tau^\mu(x,y)\right| |f(t-\tau,y)| d\tau dy \leq C \int_a^{x/2} \left(\int_0^{xy} \frac{d\tau}{x^2\sqrt{\tau}} d\tau + \int_{xy}^\infty \frac{(xy)^{\mu+1/2}}{\tau^{\mu+2}} d\tau \right) dy < \infty.$$

In a similar way we can see that

$$\int_{2x}^\infty \int_0^\infty \left|\frac{\partial}{\partial \tau} W_\tau^\mu(x,y)\right| |f(t-\tau,y)| d\tau dy < \infty.$$

Again, according to (4.3), (4.4) and (4.5) ([14, pages 128-130]) we have that

$$\left|\frac{\partial}{\partial \tau} W_\tau^\mu(x,y) - \frac{\partial}{\partial \tau} W_\tau(x-y)\right| \leq C \frac{e^{-c\frac{(x-y)^2}{\tau}}}{xy\sqrt{\tau}} \leq \frac{C}{xy\sqrt{\tau}}, \quad \tau, x, y \in (0,\infty). \tag{4.12}$$

Then

$$\int_{x/2}^{2x} \int_0^\infty \left|\frac{\partial}{\partial \tau} W_\tau^\mu(x,y) - \frac{\partial}{\partial \tau} W_\tau(x-y)\right| |f(t-\tau,y)| d\tau dy \leq C \int_a^b \int_0^c \frac{d\tau}{xy\sqrt{\tau}} dy < \infty.$$

On the other hand,

$$\begin{aligned} \int_{x/2}^{2x} \int_0^\infty W_\tau(x-y) \frac{\partial f}{\partial \tau}(t-\tau,y) d\tau dy &= \lim_{\epsilon \to 0^+} \int_{x/2}^{2x} \int_\epsilon^\infty W_\tau(x-y) \frac{\partial f}{\partial \tau}(t-\tau,y) d\tau dy \\ = -\lim_{\epsilon \to 0^+} &\left(\int_{x/2}^{2x} W_\epsilon(x-y) f(t-\epsilon,y) dy + \int_{x/2}^{2x} \int_\epsilon^\infty \frac{\partial}{\partial \tau} W_\tau(x-y) f(t-\tau,y) d\tau dy \right) \\ = -\lim_{\epsilon \to 0^+} &\int_\epsilon^\infty \int_{x/2}^{2x} \frac{\partial}{\partial \tau} W_\tau(x-y) f(t-\tau,y) d\tau dy - f(t,x), \quad t \in \mathbb{R} \text{ and } x \in (0,\infty). \end{aligned} \tag{4.13}$$

In the last equality we have taken into account that

$$\lim_{s \to 0^+} \int_{x/2}^{2x} W_s(x-y) f(t-s,y) dy = f(t,x), \quad t \in \mathbb{R} \text{ and } x \in (0,\infty).$$

Indeed, let $t \in \mathbb{R}$ and $x \in (0,\infty)$. Since $f \in C^1(\mathbb{R}\times(0,\infty))$ with compact support, by using mean value theorem we deduce that $|f(t-s,y)-f(t,y)| \leq Cs$, $s,y \in (0,\infty)$. Then, we can write

$$\left|\int_{x/2}^{2x} W_s(x-y)f(t-s,y)dy - \int_{x/2}^{2x} W_s(x-y)f(t,y)dy\right| \leq Cs\int_{x/2}^{2x} W_s(x-y)dy \leq Cs.$$

On the other hand, for a certain $a>0$ such that $2/a < x < a/2$ and $f(t,y)=0$, $y \notin (1/a,a)$. It follows, with the obvious extension of f, that

$$\left|\int_{-\infty}^{x/2} W_s(x-y)f(t,y)dy\right| \leq C\int_{1/a}^{x/2} W_s(x-y)dy \leq Cs^{-1/2}e^{-cx^2/s}, \quad s>0,$$

and

$$\left|\int_{2x}^{\infty} W_s(x-y)f(t,y)dy\right| \leq C\int_{2x}^{a} W_s(x-y)dy \leq Cs^{-1/2}e^{-cx^2/s}, \quad s>0.$$

Moreover, it is well known that

$$\lim_{s\to 0^+}\int_{-\infty}^{\infty} W_s(x-y)f(t,y)dy = f(t,x).$$

Putting together all the above estimates we obtain

$$\begin{aligned}
&\left|\int_{x/2}^{2x} W_s(x-y)f(t-s,y)dy - f(t,x)\right| \\
&\quad\leq \left|\int_{x/2}^{2x} W_s(x-y)f(t-s,y)dy - \int_{x/2}^{2x} W_s(x-y)f(t,y)dy\right| \\
&\quad + \left|\int_{x/2}^{2x} W_s(x-y)f(t,y)dy - \int_{-\infty}^{\infty} W_s(x-y)f(t,y)dy\right| \\
&\quad + \left|\int_{-\infty}^{\infty} W_s(x-y)f(t,y)dy - f(t,x)\right| \\
&\quad\leq C\Big(s + s^{-1/2}e^{-cx^2/s}\Big) + \left|\int_{-\infty}^{\infty} W_s(x-y)f(t,y)dy - f(t,x)\right|, \quad s>0.
\end{aligned}$$

We conclude that

$$\lim_{s\to 0^+}\int_{x/2}^{2x} W_s(x-y)f(t-s,y)dy = f(t,x).$$

From (4.6), (4.8), (4.9), (4.10) and (4.13) we deduce that

$$\frac{\partial}{\partial t}u(t,x) = \lim_{\epsilon\to 0^+}\left(\int_\epsilon^\infty \int_{x/2}^{2x} \frac{\partial}{\partial \tau}(W_\tau^\mu(x,y) - W_\tau(x-y))f(t-\tau,y)dyd\tau \right.$$
$$+ \int_\epsilon^\infty \int_0^{x/2} \frac{\partial}{\partial \tau}W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau$$
$$+ \int_\epsilon^\infty \int_{2x}^{\infty} \frac{\partial}{\partial \tau}W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau$$
$$\left. + \int_\epsilon^\infty \int_{x/2}^{2x} \frac{\partial}{\partial \tau}W_\tau(x-y)f(t-\tau,y)dyd\tau\right) + f(t,x), \;\; t,x>0.$$

We conclude that

$$\partial_t u(t,x) = \lim_{\epsilon\to 0^+}\int_\epsilon^\infty \int_0^\infty \frac{\partial}{\partial \tau}W_\tau^\mu(x-y)f(t-\tau,y)dyd\tau + f(t,x), \;\; t,x\in(0,\infty). \tag{4.14}$$

Assume now that $f \in C^2(\mathbb{R}\times(0,\infty))$ and it has compact support. We consider the function

$$H(t,x) = \int_0^\infty \int_0^\infty (W_\tau^\mu(x,y) - W_\tau(x-y))f(t-\tau,y)dyd\tau, \;\; t,x\in(0,\infty).$$

Note that there exists $0<a<b<\infty$ such that

$$H(t,x) = \int_0^\infty \int_a^b (W_\tau^\mu(x,y) - W_\tau(x-y))f(t-\tau,y)dyd\tau, \;\; t,x\in(0,\infty).$$

Let $t\in\mathbb{R}$. There exists $\tau_0\in(0,\infty)$ for which

$$H(t,x) = \int_0^{\tau_0} \int_a^b (W_\tau^\mu(x,y) - W_\tau(x-y))f(t-\tau,y)dyd\tau, \;\; x\in(0,\infty).$$

By (4.5) we have that

$$\frac{\partial}{\partial x}W_\tau^\mu(x,y) = \frac{e^{-\frac{x^2+y^2}{4\tau}}}{(2\tau)^{1/2}}\left(x\left(\frac{y}{2\tau}\right)^2\left(\frac{xy}{2\tau}\right)^{-1/2} I_{\mu+1}\left(\frac{xy}{2\tau}\right) - \frac{x}{2\tau}\left(\frac{xy}{2\tau}\right)^{1/2} I_\mu\left(\frac{xy}{2\tau}\right)\right.$$
$$\left. + (\mu+1/2)y(xy)^{-1/2}\frac{1}{\sqrt{2\tau}}I_\mu\left(\frac{xy}{2\tau}\right)\right), \;\; \tau,x,y\in(0,\infty). \tag{4.15}$$

Then, from (4.3) it follows that

$$\left|\frac{\partial}{\partial x}W_\tau^\mu(x,y) - \frac{\partial}{\partial x}W_\tau(x-y)\right| \le C|x-y|\frac{e^{-\frac{(x-y)^2}{4\tau}}}{\tau^{3/2}}$$
$$+ C\frac{e^{-\frac{x^2+y^2}{4\tau}}}{\tau^{3/2}}\left(\frac{xy}{2\tau}\right)^\mu\left(\left(\frac{xy}{2\tau}\right)^{3/2}y + \left(\frac{xy}{2\tau}\right)^{1/2}x + \left(\frac{xy}{2\tau}\right)^{-1/2}y\right)$$
$$\le C\frac{e^{-\frac{x^2+y^2}{4\tau}}}{\tau^{3/2}}\left(x+y+\left(\frac{xy}{2\tau}\right)^{-1/2}x + \left(\frac{xy}{2\tau}\right)^{-3/2}y\right), \;\; \tau,x,y\in(0,\infty),\; xy\le\tau, \tag{4.16}$$

and (4.4) implies that

$$\begin{aligned}
&\left|\frac{\partial}{\partial x}W_\tau^\mu(x,y)-\frac{\partial}{\partial x}W_\tau(x-y)\right|\\
&=\left|\frac{e^{-\frac{(x-y)^2}{4\tau}}}{\sqrt{4\pi\tau}}\left\{\left(x\left(\frac{y}{2\tau}\right)^2\left(\frac{xy}{2\tau}\right)^{-1}-\frac{x}{2\tau}+(\mu+1/2)y(xy)^{-1}\right)\left(1+O\left(\frac{\tau}{xy}\right)\right)\right.\right.\\
&\quad\left.\left.+\frac{x-y}{2\tau}\right\}\right|\\
&\leq Ce^{-\frac{(x-y)^2}{4\tau}}\left(\frac{1}{x\sqrt{\tau}}+\frac{1}{y\sqrt{\tau}}\right),\ \ \tau,x,y\in(0,\infty),\ xy\geq\tau.
\end{aligned}\tag{4.17}$$

From (4.16) and (4.17) it follows that

$$\begin{aligned}
&\int_0^{\tau_0}\int_a^b\left|\frac{\partial}{\partial x}W_\tau^\mu(x,y)-\frac{\partial}{\partial x}W_\tau(x-y)\right||f(t-\tau,y)|dyd\tau\\
&\leq C\int_a^b\int_0^{xy}e^{-\frac{(x-y)^2}{4\tau}}\left(\frac{1}{x\sqrt{\tau}}+\frac{1}{y\sqrt{\tau}}\right)d\tau dy\\
&+\int_a^b\int_{xy}^{\max\{xy,\tau_0\}}\frac{e^{-\frac{x^2+y^2}{4\tau}}}{\tau^{3/2}}\left(x+y+\left(\frac{xy}{\tau}\right)^{-1/2}x+\left(\frac{xy}{\tau}\right)^{-3/2}y\right)d\tau dy<\infty,
\end{aligned}$$

for $x\in(0,\infty)$.

Hence,

$$\frac{\partial}{\partial x}H(t,x)=\int_0^\infty\int_0^\infty\left(\frac{\partial}{\partial x}W_\tau^\mu(x,y)-\frac{\partial}{\partial x}W_\tau(x-y)\right)f(t-\tau,y)dyd\tau,$$

for $x\in(0,\infty)$, and the last integral is absolutely convergent.

On the other hand, for every $x,y\in(0,\infty)$,

$$\frac{\partial^2}{\partial x^2}[W_\tau^\mu(x,y)-W_\tau(x-y)]=\frac{\mu^2-1/4}{x^2}W_\tau^\mu(x,y)+\frac{\partial}{\partial\tau}[W_\tau^\mu(x,y)-W_\tau(x-y)]\tag{4.18}$$

By proceeding as above we get that

$$\frac{\partial^2}{\partial x^2}H(t,x)=\int_0^\infty\int_0^\infty\frac{\partial^2}{\partial x^2}[W_\tau^\mu(x,y)-W_\tau(x-y)]f(t-\tau,y)dyd\tau,\ \ x\in(0,\infty),$$

and the last integral is absolutely convergent.

We now consider the function

$$\mathcal{H}(t,x)=\int_0^\infty\int_0^\infty W_\tau(x-y)f(t-\tau,y)dyd\tau,\ \ t,x\in(0,\infty).$$

Note that, by extending f in the obvious way,

$$\begin{aligned}
\mathcal{H}(t,x)&=\int_0^\infty\int_{-\infty}^\infty W_\tau(x-y)f(t-\tau,y)dyd\tau\\
&=\int_0^\infty\int_{-\infty}^\infty W_\tau(y)f(t-\tau,x-y)dyd\tau,\ \ t,x\in(0,\infty).
\end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial}{\partial x}\mathcal{H}(t,x) &= \int_0^\infty \int_{-\infty}^\infty W_\tau(y)\frac{\partial}{\partial x}f(t-\tau,x-y)dyd\tau \\ &= -\int_0^\infty \int_{-\infty}^\infty W_\tau(y)\frac{\partial}{\partial y}f(t-\tau,x-y)dyd\tau, \quad t,x\in(0,\infty),\end{aligned}$$

and the last integral is absolutely convergent.

Partial integration leads to

$$\begin{aligned}\frac{\partial}{\partial x}\mathcal{H}(t,x) &= -\lim_{\epsilon\to0^+}\int_\epsilon^\infty \int_{-\infty}^\infty W_\tau(y)\frac{\partial}{\partial y}f(t-\tau,x-y)dyd\tau \\ &= \lim_{\epsilon\to0^+}\int_\epsilon^\infty \int_{-\infty}^\infty \frac{\partial}{\partial y}W_\tau(y)f(t-\tau,x-y)dyd\tau, \ t,x\in(0,\infty).\end{aligned}$$

In a similar way we can see that

$$\frac{\partial^2}{\partial x^2}\mathcal{H}(t,x) = \lim_{\epsilon\to0^+}\int_\epsilon^\infty \int_{-\infty}^\infty \frac{\partial^2}{\partial y^2}W_\tau(y)f(t-\tau,x-y)dyd\tau, \quad t,x\in(0,\infty).$$

We conclude that, for $i=1,2$,

$$\frac{\partial^i}{\partial x^i}u(t,x) = \lim_{\epsilon\to0^+}\int_\epsilon^\infty \int_0^\infty \frac{\partial^i}{\partial x^i}W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau, \quad t,x\in(0,\infty). \tag{4.19}$$

By combining (4.14) and (4.19) we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}u(t,x) - \frac{\partial^2}{\partial x^2}u(t,x) + \frac{\mu^2-1/4}{x^2}u(t,x) \\ &= \lim_{\epsilon\to0^+}\int_\epsilon^\infty \int_0^\infty \left(\partial_\tau - \partial_x^2 + \frac{\mu^2-1/4}{x^2}\right)W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau + f(t,x) \\ &= f(t,x), \quad t,x\in(0,\infty).\end{aligned}$$

The other representations of the derivatives of u as principal values can be proved by proceeding as above and by taking into account [73, Theorem 1.3,(A)].

□

4.1.2 Boundedness of Bessel Riesz transforms on (weighted) L^p spaces.

Let $\mu>-1$. The Bessel operator can be written as $\Delta_\mu = \delta_\mu^*\delta_\mu$, where $\delta_\mu = x^{\mu+1/2}\frac{d}{dx}x^{-\mu-1/2}$, and $\delta_\mu^* = x^{-\mu-1/2}\frac{d}{dx}x^{\mu+1/2}$ represents the formal adjoint of δ_μ. This decomposition of Δ_μ suggests, according to Stein's ideas ([80]), defining the Riesz transform $\mathfrak{R}_\mu$ associated with Δ_μ by $\mathfrak{R}_\mu = \delta_\mu\Delta_\mu^{-1/2}$. The main L^p-boundedness properties of $\mathfrak{R}_\mu$ can be found in [11] and [15].

We now consider the operator L_μ defined by

$$(L_\mu f)(t,x) = \int_0^\infty \int_0^\infty W_s^\mu(x,y) f(t-s,y) dyds,$$

being f a measurable complex function defined on $\mathbb{R} \times (0,\infty)$, provided that the last integral exists. In Theorem 4.121 we have established that if $f \in C^2(\mathbb{R} \times (0,\infty))$ and has compact support, then $(\partial_t - \Delta_\mu)L_\mu(f) = f$. In a similar way we can see that $L_\mu((\partial_t - \Delta_\mu)f) = f$, provided that $f \in C^2(\mathbb{R} \times (0,\infty))$ with compact support. Thus, L_μ can be seen as an inverse of $\partial_t - \Delta_\mu$. Keeping in mind Stein's ideas ([80]), we define Riesz transforms associated with the parabolic operator $\partial_t - \Delta_\mu$ as follows: for every $f \in C^2(\mathbb{R} \times (0,\infty))$ with compact support,

$$R_\mu(f) = \delta_{\mu+1}\delta_\mu L_\mu(f) \text{ and } \widetilde{R_\mu}(f) = \partial_t L_\mu(f).$$

Note that, according to Theorem 4.121, if $f \in C^2(\mathbb{R} \times (0,\infty))$ with compact support, the above definitions of $R_\mu(f)$ and $\widetilde{R_\mu}(f)$ have sense because the derivatives of $L_\mu(f)$ do exist. Moreover, we can write, for every $f \in C^2(\mathbb{R} \times (0,\infty))$ with compact support,

$$R_\mu(f)(t,x) = \lim_{\epsilon \to 0^+} \int_{\Omega_\epsilon(t,x)} K_\mu(t,x;\tau,y) f(\tau,y) d\tau dy + f(t,x) \frac{1}{\sqrt{\pi}} \int_1^\infty e^{-s^2/4} ds \tag{4.20}$$

and

$$\widetilde{R_\mu}(f)(t,x) = \lim_{\epsilon \to 0^+} \int_{\Omega_\epsilon(t,x)} \widetilde{K_\mu}(t,x;\tau,y) f(\tau,y) d\tau dy + f(t,x) \frac{1}{\sqrt{\pi}} \int_0^1 e^{-s^2/4} ds, \tag{4.21}$$

with $(t,x) \in \mathbb{R} \times (0,\infty)$, where

$$K_\mu(t,x;\tau,y) = \delta_{\mu+1}\delta_\mu W_{t-\tau}^\mu(x,y)\chi_{(0,\infty)}(t-\tau), \quad x,y \in (0,\infty), \quad t,\tau \in \mathbb{R},$$
$$\widetilde{K_\mu}(t,x;\tau,y) = -\partial_\tau W_{t-\tau}^\mu(x,y)\chi_{(0,\infty)}(t-\tau), \quad x,y \in (0,\infty), \quad t,\tau \in \mathbb{R},$$

and $\Omega_\epsilon(t,x) = \{(\tau,y) \in (0,\infty) \times (0,\infty) : \max\{|t-\tau|^{1/2}, |x-y|\} > \epsilon\}$, for $\epsilon, x \in (0,\infty)$ and $t \in (0,\infty)$.

Next we establish L^p-boundedness properties of the Riesz transforms. If m denotes the Lebesgue measure on $\mathbb{R} \times (0,\infty)$ and d represents the parabolic metric defined by $d((t,x),(\tau,y)) = |t-\tau|^{1/2} + |x-y|$, $t,\tau \in \mathbb{R}$ and $x,y \in (0,\infty)$, the triple $(\mathbb{R} \times (0,\infty), m, d)$ is a space of homogeneous type in the sense of Coifman and Weiss ([28]). We represent, for every $1 \leq p < \infty$, by $A_p^*(\mathbb{R} \times (0,\infty))$ the class of Muckenhoupt weigths in the space of homogeneous type $(\mathbb{R} \times (0,\infty), m, d)$.

Theorem 4.122. *(1) If $\mu > -1$, the Riesz transformations R_μ and $\widetilde{R_\mu}$ are bounded from $L^2(\mathbb{R} \times (0,\infty))$ into itself.*

(2) Suppose that $\mu > 1/2$ or $\mu = -1/2$. The Riesz transformations R_μ and $\widetilde{R_\mu}$ can be extended from $L^2(\mathbb{R} \times (0,\infty)) \cap L^p(\mathbb{R} \times (0,\infty),\omega)$ to $L^p(\mathbb{R} \times (0,\infty),\omega)$ as bounded operators from $L^p(\mathbb{R} \times (0,\infty),\omega)$

- *into* $L^p(\mathbb{R}\times(0,\infty),\omega)$*, for every* $1<p<\infty$ *and* $\omega\in A_p^*(\mathbb{R}\times(0,\infty))$.
- *into* $L^{1,\infty}(\mathbb{R}\times(0,\infty),\omega)$*, for* $p=1$ *and* $\omega\in A_1^*(\mathbb{R}\times(0,\infty))$.

(3) If $\mu>-1/2$*, the Riesz transformations* R_μ *and* $\widetilde{R_\mu}$ *can be extended from* $L^2(\mathbb{R}\times(0,\infty))\cap L^p(\mathbb{R}\times(0,\infty))$ *to* $L^p(\mathbb{R}\times(0,\infty))$ *as bounded operators from* $L^p(\mathbb{R}\times(0,\infty))$

- *into* $L^p(\mathbb{R}\times(0,\infty))$*, for every* $1<p<\infty$.
- *into* $L^{1,\infty}(\mathbb{R}\times(0,\infty))$*, for* $p=1$.

(4) If $-1<\mu\leq-1/2$*, then the Riesz transformation* $\widetilde{R_\mu}$ *can be extended from* $L^2(\mathbb{R}\times(0,\infty))\cap L^p(\mathbb{R}\times(0,\infty))$ *to* $L^p(\mathbb{R}\times(0,\infty))$ *as a bounded operator from* $L^p(\mathbb{R}\times(0,\infty))$ *into itself, provided that* $-\mu-1/2<1/p<\mu+3/2$ *and* $1<p<\infty$.

(5) If $-1<\mu\leq-1/2$*, then the Riesz transformation* R_μ *can be extended from* $L^2(\mathbb{R}\times(0,\infty))\cap L^p(\mathbb{R}\times(0,\infty))$ *to* $L^p(\mathbb{R}\times(0,\infty))$ *as a bounded operator from* $L^p(\mathbb{R}\times(0,\infty))$ *into itself, provided that* $p>\frac{1}{\mu+3/2}$ *and* $1<p<\infty$.

Moreover, when $\mu>-1/2$ *in all these cases the extensions of the operators* R_μ *and* $\widetilde{R_\mu}$ *are defined by* (4.20) *and* (4.21)*, respectively, where the limit exist a.e.* $(t,x)\in\mathbb{R}\times(0,\infty)$ *and the equalities are understood also in a.e.* $(t,x)\in\mathbb{R}\times(0,\infty)$.

Proof of Theorem 4.122. (1).

Assume that $\mu>-1$. Suppose that $f\in C^2(\mathbb{R}\times(0,\infty))$ and it has compact support. According to [56, page 134] we have that $|\sqrt{z}J_\nu(z)|\leq C$, $z\in(1,\infty)$, and $|\sqrt{z}J_\nu(z)|\leq Cz^{\nu+1/2}$, $z\in(0,1)$, when $\nu>-1$. Let $z\in(0,\infty)$ and $t\in\mathbb{R}$. There exist $0<a<b<\infty$ and $c>0$ such that

$$\int_0^\infty\int_0^\infty\int_0^\infty|\sqrt{xz}J_\mu(xz)||W_\tau^\mu(x,y)||f(t-\tau,y)|dyd\tau dx$$
$$\leq C\int_a^b\int_0^c\int_0^\infty|\sqrt{xz}J_\mu(xz)||W_\tau^\mu(x,y)|dxd\tau dy.$$

Let $\mu>-1/2$. From (4.3) and (4.4) we deduce that

$$W_\tau^\mu(x,y)\leq C\frac{1}{\sqrt{\tau}}e^{-\frac{(x-y)^2}{4\tau}},\;\; x,\,y,\,\tau\in(0,\infty).$$

We also have that $|\sqrt{xz}J_\mu(xz)|\leq C$, $x,z\in(0,\infty)$. Then, we get

$$\int_0^\infty|\sqrt{xz}J_\mu(xz)||W_\tau^\mu(x,y)|dx\leq C\int_{\mathbb{R}}\frac{1}{\sqrt{\tau}}e^{-\frac{(x-y)^2}{4\tau}}dx\leq C,\;\;\tau>0.$$

Hence,

$$\int_0^\infty\int_0^\infty\int_0^\infty|\sqrt{xz}J_\mu(xz)||W_\tau^\mu(x,y)||f(t-\tau,y)|dyd\tau dx<\infty.$$

Assume that now $-1<\mu\leq-1/2$. By using again (4.3) and (4.4) we obtain that

$$W_\tau^\mu(x,y)\leq C\begin{cases}\frac{1}{\sqrt{\tau}}e^{-c(x-y)^2/\tau}, & 0<\tau<xy,\\[2ex] \frac{(xy)^{\mu+1/2}}{\tau^{\mu+1}}e^{-c(x^2+y^2)/\tau}, & \tau\geq xy.\end{cases}$$

Then, it follows that

$$\int_0^\infty |\sqrt{xz}J_\mu(xz)|W_\tau^\mu(x,y)dx$$
$$\le C\left(\int_0^1 (xz)^{\mu+1/2}W_\tau^\mu(x,y)dx + \int_1^\infty W_\tau^\mu(x,y)dx\right)$$
$$\le C\left(\int_0^{min\{1,\tau/y\}} \frac{(xz)^{\mu+1/2}(xy)^{\mu+1/2}}{(x^2+y^2)^{\mu+1}}dx\right.$$
$$+\int_{min\{1,\tau/y\}}^1 (xz)^{\mu+1/2}\frac{e^{-c(x-y)^2/\tau}}{\sqrt{\tau}}dx$$
$$\left.+\int_1^{max\{1,\tau/y\}} \frac{(xy)^{\mu+1/2}}{(x^2+y^2)^{\mu+1}}dx + \int_{max\{1,\tau/y\}}^\infty \frac{e^{-c(x-y)^2/\tau}}{\sqrt{\tau}}dx\right)$$
$$\le C\left(\int_0^1 x^{2\mu+1}dx + \frac{1}{\sqrt{\tau}}\int_0^1 x^{\mu+1/2}dx + 1 + \int_{\mathbb{R}} \frac{e^{-c(x-y)^2/\tau}}{\sqrt{\tau}}dx\right)$$
$$\le C\left(1+\frac{1}{\sqrt{\tau}}\right), \quad y\in(a,b) \text{ and } \tau\in(0,c).$$

Hence,

$$\int_0^\infty\int_0^\infty\int_0^\infty |\sqrt{xz}J_\mu(xz)||W_\tau^\mu(x,y)||f(t-\tau,y)|dyd\tau dx<\infty.$$

This fact justifies the interchanges in the orders of integration to get

$$h_\mu((L_\mu f)(t,x);x\to z) = \int_0^\infty h_\mu\left(\int_0^\infty W_t^\mu(x,y)f(t-\tau,y)dy;x\to z\right)d\tau$$
$$= \int_0^\infty e^{-z^2\tau}h_\mu(f(t-\tau,y);y\to z)d\tau,$$

because (see [92, p. 195])

$$W_\tau^\mu(x,y) = \int_0^\infty e^{-z^2\tau}\sqrt{xz}J_\mu(xz)\sqrt{yz}J_\mu(yz)dz, \quad x,y,\tau\in(0,\infty).$$

Also, we have that, for certain $0<a<b<+\infty$ and $-\infty<c<d<+\infty$,

$$\int_{\mathbb{R}}\int_0^\infty\int_0^\infty e^{-z^2\tau}|\sqrt{yz}J_\mu(yz)||e^{-it\rho}||f(t-\tau,y)|dyd\tau dt$$
$$\le C\int_0^\infty e^{-z^2\tau}\int_{c+\tau}^{d+\tau}\int_a^b |f(t-\tau,y)|dydtd\tau<\infty, \quad z\in(0,\infty) \text{ and } \rho\in\mathbb{R}.$$

Note that, fixed $z\in(0,\infty)$, $|\sqrt{yz}J_\mu(yz)|\le C$, $y\in(a,b)$.

We denote, as usual, by $\mathcal{F}$ the Fourier transformation defined by, for every $\phi\in L^1(\mathbb{R})$, by

$$\mathcal{F}(\phi)(\rho) = \int_{\mathbb{R}} e^{-i\rho t}\phi(t)dt, \quad \rho\in\mathbb{R}.$$

Then,

$$\mathcal{F}(h_\mu((L_\mu f)(t,x);x\to z),t\to\rho)=\int_0^\infty e^{-(z^2+i\rho)\tau}d\tau\mathcal{F}(h_\mu(f(t,x),x\to z);t\to\rho)$$
$$=\frac{1}{z^2+i\rho}\mathcal{F}(h_\mu(f)(t,x);x\to z);t\to\rho),\quad z\in(0,\infty)\text{ and }\rho\in\mathbb{R}.$$

We define the space of functions S_μ as follows. A smooth function f on $\mathbb{R}\times(0,\infty)$ is in S_μ if and only if, for every $m,k,l\in\mathbb{N}$,

$$\sup_{t\in\mathbb{R},x\in(0,\infty)}(1+x^2)^m(1+t^2)^m\left|\frac{\partial^k}{\partial t^k}\left(\frac{1}{x}\frac{\partial}{\partial x}\right)^l(x^{-\mu-1/2}f(t,x))\right|<\infty.$$

By proceeding as above we can see that if $f\in S_\mu$ then the integral defining $L_\mu(f)(t,x)$ is absolutely convergent, for every $x\in(0,\infty)$ and $t\in\mathbb{R}$, and

$$\mathcal{F}(h_\mu((L_\mu f)(t,x);x\to z);t\to\rho)=\frac{1}{z^2+i\rho}\mathcal{F}(h_\mu(f(t,x),x\to z);t\to\rho),$$

with $z\in(0,\infty)$, and $\rho\in\mathbb{R}$. We consider the function space $C^\infty_{c,0}(\mathbb{R})$ that consists of all those $C^\infty(\mathbb{R})$-functions ϕ such that $\operatorname{supp}\phi$ is compact and $\phi(t)=0$, $t\in(-r,r)$, for some $r>0$. $C^\infty_{c,0}(\mathbb{R})$ is a dense subspace of $L^2(\mathbb{R})$. We define $Z=\mathcal{F}(C^\infty_{c,0}(\mathbb{R}))$ and

$$Z\otimes C^\infty_c(0,\infty)=\left\{\sum_{i=1}^n\alpha_i\beta_i,\;\;\alpha_i\in Z,\beta_i\in C^\infty_c(0,\infty),\;i=1,\dots,n,\;n\in\mathbb{N}\right\}.$$

Since the Fourier transform $\mathcal{F}$ is an isometry on $L^2(\mathbb{R})$, Z is a dense subspace of $L^2(\mathbb{R})$. Then $Z\otimes C^\infty_c(0,\infty)$ is a dense subset of $L^2(\mathbb{R}\times(0,\infty))$. If $\alpha\in Z$ and $\beta\in C^\infty_c(0,\infty)$, for a certain $r>0$,

$$\left|\frac{1}{z^2+i\rho}\mathcal{F}(\alpha)(\rho)h_\mu(\beta)(z)\right|\le\frac{1}{r}|\mathcal{F}(\alpha)(\rho)h_\mu(\beta)(z)|,\quad\rho\in\mathbb{R}\text{ and }z\in(0,\infty),$$

and hence

$$\frac{1}{z^2+i\rho}\mathcal{F}(\alpha)(\rho)h_\mu(\beta)(z)\in L^2(\mathbb{R}\times(0,\infty))\cap L^1(\mathbb{R}\times(0,\infty)).$$

It follows that, for every $f\in Z\otimes C^\infty_c(0,\infty)$,

$$(L_\mu f)(t,x)=\mathcal{F}^{-1}(h_\mu(\frac{1}{z^2+i\rho}\mathcal{F}(h_\mu(f(s,y);y\to z);s\to\rho);z\to x);\rho\to t).$$

According to [97] we have that, for every $\beta\in H_\mu$, $\delta_\mu h_\mu(\beta)=-h_{\mu+1}(z\beta)$, where $\delta_\mu=x^{\mu+1/2}\frac{d}{dx}x^{-\mu-1/2}$. Here H_μ denotes the space introduced by Zemanian [97, Chapter 5] consisting of all those $\phi\in C^\infty(0,\infty)$ such that, for every $m,k\in\mathbb{N}$,

$$\sup_{x\in(0,\infty)}\left|(1+x^2)^m\left(\frac{1}{x}\frac{d}{dx}\right)^k(x^{-\mu-1/2}\phi(x))\right|<\infty.$$

Since $z\beta \in H_{\mu+1}$, for every $\beta \in H_\mu$, we can write

$$\begin{aligned}
&\delta_{\mu+1}\delta_\mu L_\mu(f)(t,x)\\
&\quad = \mathcal{F}^{-1}\left(h_{\mu+2}\left(\frac{z^2}{z^2+i\rho}\mathcal{F}(h_\mu(f(s,y);y\to z);s\to\rho);z\to x\right);\rho\to t\right),
\end{aligned}$$

with $t\in\mathbb{R}$, $x\in(0,\infty)$, for each $f\in Z\otimes C_c^\infty(0,\infty)$.

We define the Riesz transformation R_μ by $R_\mu f = \mathcal{F}^{-1}h_{\mu+2}\left(\frac{z^2}{z^2+i\rho}\mathcal{F}h_\mu(f)\right)$, $f\in L^2(\mathbb{R}\times(0,\infty))$.

Thus, $R_\mu f = \delta_{\mu+1}\delta_\mu L_\mu f$, $f\in Z\otimes C_c^\infty(0,\infty)$, and R_μ is bounded from $L^2(\mathbb{R}\times(0,\infty))$ into itself.

Also, for every $f\in Z\otimes C_c^\infty(0,\infty)$, we have that

$$\begin{aligned}
&\partial_t L_\mu(f)(t,x)\\
&\quad = \mathcal{F}^{-1}\left(h_\mu\left(\frac{-i\rho}{z^2+i\rho}\mathcal{F}(h_\mu(f(s,y);y\to z);s\to\rho);z\to x\right);\rho\to t\right),
\end{aligned}$$

with $t\in\mathbb{R}$, $x\in(0,\infty)$. We define the Riesz transformation $\widetilde{R_\mu}$ by $\widetilde{R_\mu}f = \mathcal{F}^{-1}h_\mu\left(\frac{-i\rho}{z^2+i\rho}\mathcal{F}h_\mu(f)\right)$, $f\in L^2(\mathbb{R}\times(0,\infty))$. Thus, $\widetilde{R_\mu}f = \partial_t L_\mu f$, $f\in Z\otimes C_c^\infty(0,\infty)$, and $\widetilde{R_\mu}$ is bounded from $L^2(\mathbb{R}\times(0,\infty))$ into itself.

Suppose that $f(t,x)=\alpha(t)\beta(x)$, $t\in\mathbb{R}$ and $x\in(0,\infty)$, where $\alpha\in Z$ and $\beta\in C_c^\infty(0,\infty)$. We have that

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\int_0^\infty\int_0^\infty W_\tau(x-y)\alpha(t-\tau)\beta(y)\,dy\,d\tau\\
&\quad = \frac{\partial^2}{\partial x^2}\int_{-\infty}^x\int_0^\infty W_\tau(y)\alpha(t-\tau)\,d\tau\,\beta(x-y)\,dy\\
&\quad = \int_{-\infty}^x\int_0^\infty W_\tau(y)\alpha(t-\tau)\,d\tau\,\frac{\partial^2}{\partial x^2}\beta(x-y)\,dy,\ \ t\in\mathbb{R}\ \text{and}\ x\in(0,\infty),
\end{aligned}$$

and the last integral is absolutely convergent. Then, we can write, for every $t\in\mathbb{R}$ and $x\in(0,\infty)$,

$$\begin{aligned}
&\frac{\partial^2}{\partial x^2}\int_0^\infty\int_0^\infty W_\tau(x-y)\alpha(t-\tau)\beta(y)\,dy\,d\tau\\
&\qquad\qquad = \lim_{\varepsilon\to0^+}\int_{\Omega_\varepsilon} W_\tau(y)\alpha(t-\tau)\frac{\partial^2}{\partial x^2}\beta(x-y)\,dy\,d\tau,
\end{aligned}$$

where $\Omega_\varepsilon = \{(\tau,y)\in(0,\infty)\times\mathbb{R} : |y|+\sqrt{\tau}>\varepsilon\}$. By partial integration as in the proof of

[73, Theorem 2.3, (B)] we obtain

$$\frac{\partial^2}{\partial x^2}\int_0^\infty\int_0^\infty W_\tau(x-y)\alpha(t-\tau)\beta(y)dyd\tau$$
$$= \lim_{\varepsilon\to 0^+}\int_{\Omega_\varepsilon}\frac{\partial^2}{\partial y^2}W_\tau(y)\alpha(t-\tau)\beta(x-y)dyd\tau$$
$$+ f(t,x)\frac{1}{\sqrt{\pi}}\int_1^\infty e^{-w^2/4}dw,\ \ t\in\mathbb{R}\ \text{ and }\ x\in(0,\infty).$$

By proceeding as in the proof of Theorem 4.121 we can see that, for every $f\in Z\otimes C_c^\infty(0,\infty)$,

$$R_\mu(f)(t,x) = \lim_{\varepsilon\to 0^+}\int_{\Omega_\varepsilon(x)}\delta_{\mu+1}\delta_\mu W_\tau(x,y)\alpha(t-\tau)\beta(y)dyd\tau$$
$$+ f(t,x)\frac{1}{\sqrt{\pi}}\int_1^\infty e^{-w^2/4}dw,\ \ t\in\mathbb{R}\ \text{ and }\ x\in(0,\infty),$$

where $\Omega_\varepsilon(x)=\{(\tau,y)\in(0,\infty)\times(0,\infty):\ |y-x|+\sqrt{\tau}>\varepsilon\}$, for every $x\in(0,\infty)$.

In a similar way we can show that, for every $f\in Z\otimes C_c^\infty(0,\infty)$,

$$\widetilde{R_\mu}(f)(t,x) = \lim_{\varepsilon\to 0^+}\int_{\Omega_\varepsilon(x)}\frac{\partial}{\partial\tau}W_\tau^\mu(x,y)\alpha(t-\tau)\beta(y)dyd\tau$$
$$+ f(t,x)\frac{1}{\sqrt{\pi}}\int_0^1 e^{-w^2/4}dw,\ \ t\in\mathbb{R}\ \text{ and }\ x\in(0,\infty).$$

□

Calderón-Zygmund theory on the space of homogeneous type.

In order to prove Theorem 4.122 (2), we use *Calderón-Zygmund theory on the space of homogeneous type* $(\mathbb{R}\times(0,\infty),m,d)$, where m and d denote the Lebesgue measure and the parabolic distance, respectively, on $\mathbb{R}\times(0,\infty)$. We now recall the definitions and results that will be useful in the sequel. We describe now Calderón-Zygmund theory in the more general vectorial setting because we will use it in the proof of Theorem 4.127.

Suppose that X and Y are Banach spaces. By $\mathcal{L}(X,Y)$ we denote the space of bounded operators from X to Y. If $1\le p<\infty$ we represent by $L^p(\mathbb{R}\times(0,\infty),X)$ and $L^{p,\infty}(\mathbb{R}\times(0,\infty),X)$ the Bochner Lebesgue L^p-space and weak Bochner Lebesgue $L^{p,\infty}$-space. Assume that T is a bounded operator from $L^p(\mathbb{R}\times(0,\infty),X)$ into $L^p(\mathbb{R}\times(0,\infty),Y)$, for some $1<p<\infty$, satisfying that

$$T(f)(t,x)=\int_{\mathbb{R}\times(0,\infty)}K(t,x;s,y)(f(s,y))dsdy,\ \ (t,x)\notin\operatorname{supp} f, \tag{4.22}$$

for every $f\in S$, where S represents a linear space that is dense in $L^q(\mathbb{R}\times(0,\infty),X)$, for every $1\le q<\infty$. Here

$$K:[(\mathbb{R}\times(0,\infty)\times(\mathbb{R}\times(0,\infty)]\setminus D\longrightarrow\mathcal{L}(X,Y),$$

is a strongly measurable function, being

$$D = \{(t, x; s, y) \in (\mathbb{R} \times (0, \infty)) \times (\mathbb{R} \times (0, \infty)) : (t, x) = (s, y)\}.$$

We say that K is a **standard Calderón-Zygmund kernel** in $(\mathbb{R} \times (0, \infty), m, d)$ when the following properties hold

(a)$\|K(t, x; s, y)\|_{\mathcal{L}(X,Y)} \leq \frac{C}{d((t,x),(s,y))^3}$, $(t, x) \neq (s, y)$.

(b) Provided that $d((t, x), (s_0, y_0)) > d((s, y), (s_0, y_0))$,

$$\|K(t, x; s, y) - K(t, x; s_0, y_0)\|_{\mathcal{L}(X,Y)} + \|K(s, y; t, x) - K(s_0, y_0; t, x)\|_{\mathcal{L}(X,Y)} \leq C \frac{d((s, y), (s_0, y_0))}{d((t, x), (s_0, y_0))^4}.$$

If $1 < p < \infty$, a weight w on $\mathbb{R} \times (0, \infty)$ is in the Muckenhoupt class $A_p^*(\mathbb{R} \times (0, \infty))$ when there exists $C > 0$ such that

$$\frac{1}{|B|} \int_B w(t, x) dt dx \left(\frac{1}{|B|} \int_B w(t, x)^{1/(1-p)} dt dx \right)^{p-1} \leq C,$$

for every ball (with respect to d) in $\mathbb{R} \times (0, \infty)$.

A weight w is in $A_1^*(\mathbb{R} \times (0, \infty))$ when there exists $C > 0$ such that, for a.e. $(t, x) \in \mathbb{R} \times (0, \infty)$,

$$\frac{1}{|B|} \int_B w(s, y) ds dy \leq C w(t, x),$$

for every ball B (with respect to d) containing (t, x).

The Calderón-Zygmund Theorem says that if T satisfies the above properties where K in (4.22) is a standard Calderón-Zygmund kernel, then the operator T can be extended,

(a) for every $1 < q < \infty$ and $w \in A_q^*(\mathbb{R} \times (0, \infty))$, from $L^p(\mathbb{R} \times (0, \infty), X) \cap L^q(\mathbb{R} \times (0, \infty), w, X)$ to $L^q(\mathbb{R} \times (0, \infty), w, X)$ as a bounded operator from $L^q(\mathbb{R} \times (0, \infty), w, X)$ into $L^q(\mathbb{R} \times (0, \infty), w, Y)$;

(b) for every $w \in A_1^*(\mathbb{R} \times (0, \infty))$, from $L^p(\mathbb{R} \times (0, \infty), X) \cap L^1(\mathbb{R} \times (0, \infty), w, X)$ to $L^1(\mathbb{R} \times (0, \infty), w, X)$ as a bounded operator from $L^1(\mathbb{R} \times (0, \infty), w, X)$ into $L^{1,\infty}(\mathbb{R} \times (0, \infty), w, Y)$.

Moreover, the maximal operator given by

$$T^*(f)(t, x) = \sup_{\varepsilon > 0} \left\| \int_{d((t,x),(s,y)) > \varepsilon} K(t, x; s, y)(f(s, y)) ds dy \right\|_Y,$$

defines a bounded operator from

(a) $L^q(\mathbb{R} \times (0, \infty), w, X)$ into $L^q(\mathbb{R} \times (0, \infty), w)$, for every $1 < q < \infty$ and $w \in A_q^*(\mathbb{R} \times (0, \infty))$;

(b) $L^1(\mathbb{R} \times (0, \infty), w, X)$ into $L^{1,\infty}(\mathbb{R} \times (0, \infty), w)$, for every $w \in A_1^*(\mathbb{R} \times (0, \infty))$.

A complete study about vector valued Calderón-Zygmund theory on spaces of homogeneous type can be encountered in [75], [76] and [77].

We have that, for every $f \in Z \otimes C_c^\infty(0,\infty)$,

$$R_\mu f(t,x) = \int_0^\infty \int_0^\infty \delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y) f(t-\tau,y)d\tau dy, \quad (t,x) \notin \text{supp} f,$$

and

$$\widetilde{R_\mu} f(t,x) = \int_0^\infty \int_0^\infty \partial_\tau W_\tau^\mu(x,y) f(t-\tau,y)d\tau dy, \quad (t,x) \notin \text{supp} f.$$

We consider the kernel functions defined as follows

$$K_\mu(t,x;\tau,y) = \delta_{\mu+1}\delta_\mu W_{t-\tau}^\mu(x,y)\chi_{(0,\infty)}(t-\tau), \quad x,y \in (0,\infty), \quad t,\tau \in \mathbb{R},$$

and

$$\widetilde{K_\mu}(t,x;\tau,y) = -\partial_\tau W_{t-\tau}^\mu(x,y)\chi_{(0,\infty)}(t-\tau), \quad x,y \in (0,\infty), \quad t,\tau \in \mathbb{R}.$$

It is clear that, for every $f \in Z \otimes C_c^\infty(0,\infty)$,
$R_\mu f(t,x) = \int_\mathbb{R} \int_0^\infty K_\mu(x,t;y,\tau)f(\tau,y)dyd\tau$ and $\widetilde{R_\mu} f(t,x) = \int_\mathbb{R} \int_0^\infty \widetilde{K_\mu}(x,t;y,\tau)f(\tau,y)dyd\tau$, $(t,x) \notin \text{supp} f$.

We remark that $d((t,x),(s,y)) = |x-y| + \sqrt{|t-s|}$, $t,s \in \mathbb{R}$ and $x,y \in (0,\infty)$.

The following result will imply Theorem 4.122 (2).

Proposition 4.123. *Let $\mu > 1/2$ or $\mu = -1/2$. The kernels K_μ and $\widetilde{K_\mu}$ are standard Calderón-Zygmund with respect to the homogeneous type space $(\mathbb{R} \times (0,\infty), m, d)$.*

Proof. Firstly we analyze K_μ. We consider the function,

$$\mathbb{K}_\mu(x,y,s) = \delta_{\mu+1}\delta_\mu W_s^\mu(x,y)\chi_{(0,\infty)}(s), \quad x,y \in (0,\infty), \text{and} \quad s \in \mathbb{R}.$$

According to (4.5) we have that

$$\begin{aligned}
\mathbb{K}_\mu(x,y,s) &= x^{\mu+5/2}\left(\frac{1}{x}\frac{\partial}{\partial x}\right)^2 \left(\left(\frac{xy}{2s}\right)^{-\mu} I_\mu\left(\frac{xy}{2s}\right)e^{-\frac{x^2+y^2}{4s}}\right)\frac{y^{\mu+1/2}}{(2s)^{\mu+1}} \\
&= \frac{x^{\mu+5/2}y^{\mu+1/2}}{(2s)^{\mu+1}}\left(\frac{1}{x}\frac{\partial}{\partial x}\right)\Big[\Big(\frac{y}{2sx}\left(\frac{xy}{2s}\right)^{-\mu} I_{\mu+1}\left(\frac{xy}{2s}\right) - \frac{1}{2s}\left(\frac{xy}{2s}\right)^{-\mu} I_\mu\left(\frac{xy}{2s}\right)\Big)e^{-\frac{x^2+y^2}{4s}}\Big] \\
&= \frac{x^{\mu+5/2}y^{\mu+1/2}}{(2s)^{\mu+1}}\Bigg[\left(\frac{y}{2s}\right)^3\left(\frac{xy}{2s}\right)^{-\mu-1} I_{\mu+2}\left(\frac{xy}{2s}\right)\frac{1}{x} - \frac{y}{4s^2x}\left(\frac{xy}{2s}\right)^{-\mu} I_{\mu+1}\left(\frac{xy}{2s}\right) \\
&\quad - \frac{1}{2s}\left(\frac{y}{2sx}\left(\frac{xy}{2s}\right)^{-\mu} I_{\mu+1}\left(\frac{xy}{2s}\right) - \frac{1}{2s}\left(\frac{xy}{2s}\right)^{-\mu} I_\mu\left(\frac{xy}{2s}\right)\right)\Bigg]e^{-\frac{x^2+y^2}{4s}} \\
&= \frac{x^{\mu+5/2}y^{\mu+1/2}}{(2s)^{\mu+3}}\left(\frac{y^3}{2sx}\left(\frac{xy}{2s}\right)^{-\mu-1} I_{\mu+2}\left(\frac{xy}{2s}\right) - \frac{2y}{x}\left(\frac{xy}{2s}\right)^{-\mu} I_{\mu+1}\left(\frac{xy}{2s}\right)\right. \\
&\quad \left. + \left(\frac{xy}{2s}\right)^{-\mu} I_\mu\left(\frac{xy}{2s}\right)\right)e^{-\frac{x^2+y^2}{4s}}, \quad x,y,s \in (0,\infty). \qquad (4.23)
\end{aligned}$$

By taking into account (4.3) we obtain

$$\begin{aligned}|\mathbb{K}_\mu(x,y,s)| &\leq C\left(\frac{xy^3}{s^{7/2}}+\frac{xy}{s^{5/2}}+\frac{x^2}{s^{5/2}}\right)e^{-\frac{x^2+y^2}{4s}}\\ &\leq \frac{C}{(\sqrt{s}+|x-y|)^3},\quad x,y,s\in(0,\infty),\quad xy\leq s. \end{aligned}\tag{4.24}$$

On the other hand, we can write

$$\begin{aligned}\mathbb{K}_\mu(x,y,s) = \frac{x^{\mu+5/2}y^{\mu+1/2}}{(2s)^{\mu+2}}&\left(\frac{y^3}{4s^2x}\left(\frac{xy}{2s}\right)^{-\mu-3/2}\sqrt{\frac{xy}{2s}}I_{\mu+2}\left(\frac{xy}{2s}\right)\right.\\ &\left.-\frac{y}{sx}\left(\frac{xy}{2s}\right)^{-\mu-1/2}\sqrt{\frac{xy}{2s}}I_{\mu+1}\left(\frac{xy}{2s}\right)+\frac{1}{2s}\left(\frac{xy}{2s}\right)^{-\mu-1/2}\sqrt{\frac{xy}{2s}}I_\mu\left(\frac{xy}{2s}\right)\right)e^{-\frac{x^2+y^2}{4s}},\end{aligned}$$

for $x,y,s\in(0,\infty)$. By (4.4) we deduce that

$$\begin{aligned}\mathbb{K}_\mu(x,y,s) &= \frac{x^{\mu+5/2}y^{\mu+1/2}}{\sqrt{2\pi}(2s)^{\mu+2}}\left(\frac{xy}{2s}\right)^{-\mu}\left(\frac{y^3}{4s^2x}\left(\frac{xy}{2s}\right)^{-3/2}-\frac{y}{sx}\left(\frac{xy}{2s}\right)^{-1/2}\right.\\ &\left.+\frac{1}{2s}\left(\frac{xy}{2s}\right)^{-1/2}\right)\left(1+O\left(\frac{s}{xy}\right)\right)e^{-\frac{(x-y)^2}{4s}}\\ &=\frac{1}{\sqrt{2\pi}(2s)^2}\left(\frac{y^2-2yx+x^2}{(2s)^{1/2}}+O\left(\frac{y\sqrt{s}}{x}\right)+O(\sqrt{s})+O\left(\frac{x\sqrt{s}}{y}\right)\right)e^{-\frac{(x-y)^2}{4s}}\\ &=\frac{1}{\sqrt{2\pi}(2s)^{3/2}}\left(\frac{(y-x)^2}{2s}+O\left(\frac{y}{x}\right)+O(1)+O\left(\frac{x}{y}\right)\right)e^{-\frac{(x-y)^2}{4s}},\end{aligned}\tag{4.25}$$

for $x,y,s\in(0,\infty)$. Hence, if $s,x,y\in(0,\infty)$, and $xy\geq s$, then

$$|\mathbb{K}_\mu(x,y,s)|\leq\frac{Ce^{-\frac{c(x-y)^2}{s}}}{s^{3/2}}\leq\frac{C}{(\sqrt{s}+|x-y|)^3},\quad x/2<y<2x,\tag{4.26}$$

and

$$\begin{aligned}|\mathbb{K}_\mu(x,y,s)| &\leq \frac{C}{s^{3/2}}\left(1+\frac{x^2+y^2}{xy}\right)e^{-c\frac{(x-y)^2}{s}}\\ &\leq\frac{C}{s^{3/2}}\left(1+\frac{\max\{x,y\}^2}{s}\right)e^{-c\frac{\max\{x,y\}^2}{s}}e^{-c\frac{(x-y)^2}{s}}\\ &\leq\frac{C}{(\sqrt{s}+|x-y|)^3},\quad 0<y<x/2,\ \text{ or }\ 2x<y<\infty.\end{aligned}\tag{4.27}$$

We conclude that

$$|K_\mu(x,t;y,\tau)|\leq\frac{C}{(\sqrt{|t-\tau|}+|x-y|)^3},\quad x,y\in(0,\infty),\text{and }\ t,\tau\in\mathbb{R}.$$

According to (4.25) we have that $\lim_{s\to0^+}\mathbb{K}_\mu(x,y,s)=0$, $x,y\in(0,\infty)$, $x\neq y$. Hence, K_μ is a continuous function on $[((0,\infty)\times\mathbb{R})\times((0,\infty)\times\mathbb{R})]\setminus D$.

By using that $\frac{\partial}{\partial x}=\delta_{\mu+2}+\frac{\mu+5/2}{x}$ and (4.5), from (4.23) it follows that

$$\begin{aligned}
\partial_x\mathbb{K}_\mu(x,y,s)&=\frac{x^{\mu+7/2}y^{\mu+1/2}}{(2s)^{\mu+3}}\Bigg(-\frac{y^3}{4s^2x}\left(\frac{xy}{2s}\right)^{-\mu-1}I_{\mu+2}\left(\frac{xy}{2s}\right)+\frac{y}{sx}\left(\frac{xy}{2s}\right)^{-\mu}I_{\mu+1}\left(\frac{xy}{2s}\right)\\
&-\frac{1}{2s}\left(\frac{xy}{2s}\right)^{-\mu}I_\mu\left(\frac{xy}{2s}\right)+\frac{y^5}{(2s)^3x}\left(\frac{xy}{2s}\right)^{-\mu-2}I_{\mu+3}\left(\frac{xy}{2s}\right)-\frac{y^3}{2s^2x}\left(\frac{xy}{2s}\right)^{-\mu-1}I_{\mu+2}\left(\frac{xy}{2s}\right)\\
&+\frac{y}{2sx}\left(\frac{xy}{2s}\right)^{-\mu}I_{\mu+1}\left(\frac{xy}{2s}\right)\Bigg)e^{-\frac{x^2+y^2}{4s}}+\frac{\mu+\frac{5}{2}}{x}\frac{x^{\mu+5/2}y^{\mu+1/2}}{(2s)^{\mu+3}}\Bigg(\frac{y^3}{2sx}\left(\frac{xy}{2s}\right)^{-\mu-1}I_{\mu+2}\left(\frac{xy}{2s}\right)\\
&-\frac{2y}{x}\left(\frac{xy}{2s}\right)^{-\mu}I_{\mu+1}\left(\frac{xy}{2s}\right)+\left(\frac{xy}{2s}\right)^{-\mu}I_\mu\left(\frac{xy}{2s}\right)\Bigg)e^{-\frac{x^2+y^2}{4s}}\\
&=\frac{x^{\mu+5/2}y^{\mu+1/2}}{(2s)^{\mu+3}}e^{-\frac{x^2+y^2}{4s}}\Bigg[\frac{y^5}{(2s)^3}\left(\frac{x}{2s}\right)^{-\mu-2}I_{\mu+3}\left(\frac{xy}{2s}\right)+\left(\frac{xy}{2s}\right)^{-\mu-1}I_{\mu+2}\left(\frac{xy}{2s}\right)\left(-\frac{3y^3}{4s^2}+\frac{(\mu+\frac{5}{2})y^3}{2sx^2}\right)\\
&+\left(\frac{xy}{2s}\right)^{-\mu}I_{\mu+1}\left(\frac{xy}{2s}\right)\left(\frac{3y}{2s}-2\frac{(\mu+\frac{5}{2})y}{x^2}\right)+\left(\frac{xy}{2s}\right)^{-\mu}I_\mu\left(\frac{xy}{2s}\right)\left(-\frac{x}{2s}+\frac{\mu+\frac{5}{2}}{x}\right)\Bigg],\quad x,y,s\in(0,\infty),
\end{aligned}\tag{4.28}$$

and since $\partial_y=\delta_\mu+\frac{\mu+1/2}{y}$,

$$\begin{aligned}
\partial_y\mathbb{K}_\mu(x,y,s)=&\frac{x^{\mu+5/2}y^{\mu+1/2}}{(2s)^{\mu+2}}\Bigg[\frac{4y^3}{(2s)^3}\left(\frac{xy}{2s}\right)^{-\mu-2}I_{\mu+2}\left(\frac{xy}{2s}\right)+\frac{y^4x}{(2s)^4}\left(\frac{xy}{2s}\right)^{-\mu-2}I_{\mu+3}\left(\frac{xy}{2s}\right)\\
&-\frac{y}{s^2}\left(\frac{xy}{2s}\right)^{-\mu-1}I_{\mu+1}\left(\frac{xy}{2s}\right)-\frac{y^2x}{4s^3}\left(\frac{xy}{2s}\right)^{-\mu-1}I_{\mu+2}\left(\frac{xy}{2s}\right)\\
&+\frac{x}{(2s)^2}\left(\frac{xy}{2s}\right)^{-\mu}I_{\mu+1}\left(\frac{xy}{2s}\right)+\left(\frac{y^4}{(2s)^3}\left(\frac{xy}{2s}\right)^{-\mu-2}I_{\mu+2}\left(\frac{xy}{2s}\right)-\frac{y^2}{2s^2}\left(\frac{xy}{2s}\right)^{-\mu-1}I_{\mu+1}\left(\frac{xy}{2s}\right)\right.\\
&\left.+\frac{1}{2s}\left(\frac{xy}{2s}\right)^{-\mu}I_\mu\left(\frac{xy}{2s}\right)\right)\left(-\frac{y}{2s}+\frac{\mu+1/2}{y}\right)\Bigg]e^{-\frac{x^2+y^2}{4s}}
\end{aligned}\tag{4.29}$$

$$\begin{aligned}
=&\frac{x^{\mu+5/2}y^{\mu+1/2}}{(2s)^{\mu+3}}e^{-\frac{x^2+y^2}{4s}}\Bigg[\frac{xy^4}{(2s)^3}\left(\frac{xy}{2s}\right)^{-\mu-2}I_{\mu+3}\left(\frac{xy}{2s}\right)\\
&+\left(\frac{xy}{2s}\right)^{-\mu-1}I_{\mu+2}\left(\frac{xy}{2s}\right)\left(\frac{4y^2}{2sx}-\frac{xy^2}{2s^2}+\frac{y^3}{2sx}\left(-\frac{y}{2s}+\frac{\mu+1/2}{y}\right)\right)\\
&+\left(\frac{xy}{2s}\right)^{-\mu}I_{\mu+1}\left(\frac{xy}{2s}\right)\left(-\frac{4}{x}+\frac{x}{2s}-\frac{2y}{x}\left(-\frac{y}{2s}+\frac{\mu+1/2}{y}\right)\right)\\
&+\left(\frac{xy}{2s}\right)^{-\mu}I_\mu\left(\frac{xy}{2s}\right)\left(-\frac{y}{2s}+\frac{\mu+1/2}{y}\right)\Bigg],\quad x,y,s\in(0,\infty).
\end{aligned}$$

It is clear that

$$\partial_x\mathbb{K}_\mu(x,y,s)=\partial_y\mathbb{K}_\mu(x,y,s)=0,\quad x,y\in(0,\infty),\quad s\in(-\infty,0].\tag{4.30}$$

Now, we estimate $\partial_x \mathbb{K}_\mu(x,y,s)$. By (4.3), (4.28) leads that

$$|\partial_x \mathbb{K}_\mu(x,y,s)| \leq C\frac{x^{\mu+5/2}y^{\mu+1/2}}{s^{\mu+3}}e^{-\frac{x^2+y^2}{4s}}\left(\frac{y^5}{(2s)^3}\frac{xy}{2s}+\left(\frac{y^3}{s^2}+\frac{y^3}{sx^2}\right)\frac{xy}{2s}\right.$$
$$\left.+\left(\frac{y}{s}+\frac{y}{x^2}\right)\frac{xy}{2s}+\frac{x}{s}+\frac{1}{x}\right)$$
$$\leq C\left(\frac{y^{\mu+13/2}x^{\mu+7/2}}{(s+x^2+y^2)^{\mu+7}}+\frac{y^{\mu+9/2}x^{\mu+7/2}}{(s+x^2+y^2)^{\mu+6}}+\frac{y^{\mu+9/2}x^{\mu+3/2}}{(s+x^2+y^2)^{\mu+5}}\right.$$
$$\left.+\frac{y^{\mu+5/2}x^{\mu+7/2}}{(s+x^2+y^2)^{\mu+5}}+\frac{y^{\mu+5/2}x^{\mu+3/2}+y^{\mu+1/2}x^{\mu+7/2}}{(s+x^2+y^2)^{\mu+4}}+\frac{y^{\mu+1/2}x^{\mu+3/2}}{(s+x^2+y^2)^{\mu+3}}\right)$$
$$\leq \frac{C}{(\sqrt{s}+x+y)^4} \leq \frac{C}{(\sqrt{s}+|x-y|)^4}, \quad s,x,y\in(0,\infty) \text{ and } \frac{xy}{s}\leq 1.$$

On the other hand, (4.4) allows us to deduce from (4.28) that

$$\partial_x \mathbb{K}_\mu(x,y,s) = \frac{x^2 e^{-\frac{(x-y)^2}{4s}}}{(2s)^{5/2}\sqrt{2\pi}}\left[\left\{\frac{y^5}{(2s)^3}\left(\frac{xy}{2s}\right)^{-2}+\left(\frac{xy}{2s}\right)^{-1}\left(-\frac{3y^3}{4s^2}+\frac{\mu+5/2}{2s}\frac{y^3}{x^2}\right)\right.\right.$$
$$\left.\left.+\frac{3y}{2s}-\frac{2(\mu+5/2)y}{x^2}-\frac{x}{2s}+\frac{\mu+5/2}{x}\right\}\left(1+O\left(\frac{s}{xy}\right)\right)\right]$$
$$= \frac{x^2 e^{-\frac{(x-y)^2}{4s}}}{(2s)^{5/2}\sqrt{2\pi}}\left[\frac{y^3}{2sx^2}-\frac{3y^2}{2sx}+\frac{3y}{2s}-\frac{x}{2s}+(\mu+5/2)\left(\frac{y^2}{x^3}-\frac{2y}{x^2}+\frac{1}{x}\right)\right]O(1)$$
$$= \frac{x^2}{(2s)^{5/2}}\frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{2\pi}}\left[(\mu+5/2)\frac{(x-y)^2}{x^3}+\frac{(y-x)^3}{2sx^2}\right]O(1),$$

for $x,y,s\in(0,\infty)$, $\frac{xy}{s}\geq 1$.

Then, $|\partial_x \mathbb{K}_\mu(x,y,s)| \leq C\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{5/2}}\left(\frac{(x-y)^2}{x}+\frac{|x-y|^3}{s}\right)$, $x,y,s\in(0,\infty)$ and $\frac{xy}{s}\geq 1$. We have that, for every $x,y,s\in(0,\infty)$, and $xy\geq s$,

$$\frac{1}{x}\leq C\begin{cases}\frac{1}{\sqrt{xy}}\leq\frac{C}{\sqrt{s}}, & x>y/2,\\ \frac{y}{s}\leq\frac{|y-x|+x}{s}\leq\frac{2|y-x|}{s}, & 0<x<\frac{y}{2}.\end{cases} \tag{4.31}$$

Hence, we get

$$|\partial_x \mathbb{K}_\mu(x,y,s)| \leq Ce^{-\frac{(x-y)^2}{4s}}\left(\frac{(x-y)^3}{s^{7/2}}+\frac{(x-y)^2}{s^3}\right)\leq\frac{C}{(\sqrt{s}+|x-y|)^4},$$

for $x,y,s\in(0,\infty)$, $xy\geq s$. We conclude that

$$|\partial_x \mathbb{K}_\mu(x,y,s)| \leq \frac{C}{(\sqrt{s}+|x-y|)^4}, \quad x,y,s\in(0,\infty). \tag{4.32}$$

We now estimate $\partial_y \mathbb{K}_\mu(x,y,s)$. By (4.3), from (4.29) we deduce that

$$\begin{aligned}|\partial_y \mathbb{K}_\mu(x,y,s)| \le Ce^{-\frac{x^2+y^2}{4s}} \Big(&\frac{x^{\mu+9/2}y^{\mu+4/2}}{s^{\mu+7}} \\ &+ \frac{x^{\mu+5/2}y^{\mu+7/2}}{s^{\mu+5}} + \frac{x^{\mu+5/2}y^{\mu+11/2}}{s^{\mu+6}} + \frac{x^{\mu+5/2}y^{\mu+3/2}}{s^{\mu+4}} \\ &+ \frac{x^{\mu+9/2}y^{\mu+3/2}}{s^{\mu+5}} + \frac{x^{\mu+9/2}y^{\mu+7/2}}{s^{\mu+6}} + (\mu+\frac{1}{2})\frac{x^{\mu+5/2}y^{\mu-1/2}}{s^{\mu+3}} \Big) \\ &\le \frac{C}{(\sqrt{s}+|x-y|)^4}, \quad s,x,y \in (0,\infty), \; xy \le s.\end{aligned}$$

Note that in the last estimate we use that $\mu > 1/2$ or $\mu = -1/2$.

By (4.29) we can write

$$\begin{aligned}\partial_y \mathbb{K}_\mu(x,y,s) = e^{-\frac{x^2+y^2}{4s}} \frac{x^2}{(2s)^{5/2}} \Big(&\sqrt{\frac{xy}{2s}} I_{\mu+3}\left(\frac{xy}{2s}\right) \frac{y^2}{2sx} \\ &+ \sqrt{\frac{xy}{2s}} I_{\mu+2}\left(\frac{xy}{2s}\right) \left(\frac{4y}{x^2} - \frac{y}{s} + \frac{y^2}{x^2}\left(-\frac{y}{2s} + \frac{\mu+1/2}{y}\right)\right) \\ &+ \sqrt{\frac{xy}{2s}} I_{\mu+1}\left(\frac{xy}{2s}\right) \left(-\frac{4}{x} + \frac{x}{2s} - \frac{2y}{x}\left(-\frac{y}{2s} + \frac{\mu+1/2}{y}\right)\right) \\ &+ \sqrt{\frac{xy}{2s}} I_{\mu}\left(\frac{xy}{2s}\right) \left(-\frac{y}{2s} + \frac{\mu+1/2}{y}\right)\Big), \quad x,y,s \in (0,\infty).\end{aligned}$$

Then, (4.4) leads to

$$\begin{aligned}\partial_y \mathbb{K}_\mu(x,y,s) = &\frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{2\pi}} \frac{x^2}{(2s)^{5/2}} \left(1 + O\left(\frac{s}{xy}\right)\right) \left(\frac{y^2}{2sx} + \frac{4y}{x^2} - \frac{y}{s} - \frac{y^3}{x^2 2s} \right. \\ &\left. - \frac{4}{x} + \frac{x}{2s} + \frac{y^2}{xs} - \frac{y}{2s} + (\mu+1/2)\left(\frac{y}{x^2} - \frac{2}{x} + \frac{1}{y}\right)\right) \\ = &e^{-\frac{(x-y)^2}{4s}} O(1) \frac{x^2}{s^{5/2}} \left[(\mu+1/2)\frac{(y-x)^2}{x^2 y} + \frac{(x-y)^3}{x^2 2s} + \frac{4(y-x)}{x^2}\right],\end{aligned}$$

for $s,x,y \in (0,\infty)$, $xy \ge s$. Hence, by (4.31) we obtain

$$\begin{aligned}|\partial_y \mathbb{K}_\mu(x,y,s)| &\le Ce^{-\frac{(x-y)^2}{4s}} \left(\frac{|y-x|^2}{s^{5/2}y} + \frac{|x-y|^3}{s^{7/2}} + \frac{|y-x|}{s^{5/2}}\right) \\ &\le Ce^{-\frac{(x-y)^2}{4s}} \left(\frac{|y-x|^2}{s^3} + \frac{|x-y|^3}{s^{7/2}} + \frac{|y-x|}{s^{5/2}}\right) \\ &\le \frac{C}{(\sqrt{s}+|x-y|)^4}, \quad x,y,s \in (0,\infty), \quad xy \ge s.\end{aligned}$$

We conclude that

$$|\partial_y \mathbb{K}_\mu(x,y,s)| \le \frac{C}{(\sqrt{s}+|x-y|)^4}, \quad x,y,s \in (0,\infty). \tag{4.33}$$

Now we estimate $\partial_s \mathbb{K}_\mu(x,y,s)$. Let $x,y \in (0,\infty)$. We define the function $\varphi_{x,y}(z) = \mathbb{K}(x,y,z)$, $z \in \mathbb{C}$, $\mathrm{Re}\, z > 0$. Thus, $\varphi_{x,y}$ is an holomorphic function in $\{z \in \mathbb{C} : Re\, z > 0\}$. Note that, if $a > 0$, $Arg\, \frac{a}{z} = -Arg(z)$ and $Re\left(\frac{a}{z}\right) = \frac{a}{|z|^2} Re\, z$, $z \in \mathbb{C}$. Note that $Rez \geq \frac{\sqrt{2}}{2}|z|$ provided that $|Argz| \leq \frac{\pi}{4}$. Hence, $\left|e^{-\frac{(x-y)^2}{4z}}\right| = e^{\frac{-Rez(x-y)^2}{4|z|^2}} \leq e^{-\frac{\sqrt{2}}{8}\frac{(x-y)^2}{|z|}}$, and $\left|e^{-\frac{x^2+y^2}{4z}}\right| \leq e^{-\frac{\sqrt{2}}{8}\frac{x^2+y^2}{|z|}}$, when $|Argz| \leq \frac{\pi}{4}$.

According (4.3) and (4.4) as in (4.24) and (4.25), we can obtain

$$|\varphi_{x,y}(z)| \leq C|z|^{-3/2} e^{-cRe\frac{|x-y|^2}{z}} \leq C|z|^{-3/2} e^{-c\frac{|x-y|^2 Re\, z}{|z|^2}}$$
$$\leq C|z|^{-3/2} e^{-c\frac{|x-y|^2}{|z|}}, \quad z \in \mathbb{C}, \quad |Arg(z)| < \frac{\pi}{4}.$$

By using Cauchy integral formula we get

$$|\partial_s \mathbb{K}_\mu(x,y,s)| \leq C \frac{1}{s^{5/2}} e^{-c\frac{(x-y)^2}{s}}, \quad s \in (0,\infty).$$

Here C and c do not depend on $x,y \in (0,\infty)$. Then, we obtain

$$|\partial_s \mathbb{K}_\mu(x,y,s)| \leq \frac{C}{(\sqrt{s} + |x-y|)^5}, \quad x,y,s \in (0,\infty). \tag{4.34}$$

Hence, for every $x,y \in (0,\infty)$, $x \neq y$, $\lim_{s\to 0^+} \partial_s \mathbb{K}_\mu(x,y,s) = 0$.

Then, we deduce that $\mathbb{K}_\mu$ is in $C^1((0,\infty) \times (0,\infty) \times \mathbb{R} \setminus \{(x,x,0) : x \in (0,\infty)\})$, and K_μ is in $C^1((\mathbb{R} \times (0,\infty) \times \mathbb{R} \times (0,\infty)) \setminus D)$.

According to (4.32), (4.33) and (4.34), we have that

$$|\partial_t K_\mu(x,t;y,\tau)| = |\partial_\tau K_\mu(x,t;y,\tau)| \leq \frac{C}{(\sqrt{|t-\tau|} + |x-y|)^5},$$

$$|\partial_x K_\mu(x,t;y,\tau)| \leq \frac{C}{(\sqrt{|t-\tau|} + |x-y|)^4},$$

and

$$|\partial_y K_\mu(x,t;y,\tau)| \leq \frac{C}{(\sqrt{|t-\tau|} + |x-y|)^4},$$

for every $(x,t;y,\tau) \in [((0,\infty) \times \mathbb{R}) \times ((0,\infty) \times \mathbb{R})] \setminus D$, where $D = \{(x,t;x,t) : x \in (0,\infty) \text{ and } t \in \mathbb{R}\}$.

Let now $x,y,y_0 \in (0,\infty)$ and $t,\tau,\tau_0 \in \mathbb{R}$ such that $d((x,t);(y_0,\tau_0)) = \sqrt{|t-\tau_0|} + |x-y_0| > 2(\sqrt{|\tau-\tau_0|} + |y-y_0|) = 2d((y,\tau);(y_0,\tau_0))$. Then, $s(x,t;y_0,\tau_0) + (1-s)(x,t;y,\tau) \notin D$, for every $s \in (0,1)$. Indeed, suppose that $s \in (0,1)$ and that $s(x,t;y_0,\tau_0) + (1-s)(x,t;y,\tau) \in D$. We have that $x = sy_0 + (1-s)y$ and $t = s\tau_0 + (1-s)\tau$. It follows that

$$\sqrt{|t-\tau_0|} + |x-y_0| = \sqrt{(1-s)|\tau_0-\tau|} + |1-s||y-y_0| \leq \sqrt{|\tau_0-\tau|} + |y-y_0|,$$

and this is not possible.

By using the mean value theorem, we can write

$$\begin{aligned} K_\mu(x,t;y,\tau) - K_\mu(x,t;&y_0,\tau_0) \\ &= \partial_z K_\mu(x,t;z,\theta)_{\big| \substack{z=sy+(1-s)y_0 \\ \theta=s\tau+(1-s)\tau_0}} (y-y_0) \\ &+ \partial_\theta K_\mu(x,t;z,\theta)_{\big| \substack{z=sy+(1-s)y_0 \\ \theta=s\tau+(1-s)\tau_0}} (\tau-\tau_0), \end{aligned}$$

for certain $s \in (0,1)$. Then,

$$\begin{aligned} |K_\mu(x,t;y,\tau) - &K_\mu(x,t;y_0,\tau_0)| \\ &\leq C\Bigg(\frac{|y-y_0|}{\left(\sqrt{|t-s\tau-(1-s)\tau_0|} + |x-sy-(1-s)y_0|\right)^4} \\ &+ \frac{|\tau-\tau_0|}{\left(\sqrt{|t-s\tau-(1-s)\tau_0|} + |x-sy-(1-s)y_0|\right)^5}\Bigg). \end{aligned}$$

Note that $|\tau-\tau_0| \leq \sqrt{|\tau-\tau_0|}(\sqrt{|\tau-\tau_0|}+|y-y_0|) < \frac{1}{2}\sqrt{|\tau-\tau_0|}(\sqrt{|t-\tau_0|}+|x-y_0|)$, and

$$\begin{aligned} d((x,t),(z,\theta)) = |x-z| + \sqrt{|t-\theta|} &\geq |x-y_0| - |z-y_0| + \sqrt{|t-\tau_0|} - \sqrt{|\theta-\tau_0|} \\ &= |x-y_0| + \sqrt{|t-\tau_0|} - (s|y-y_0| + \sqrt{s}\sqrt{|\tau-\tau_0|}) \\ &\geq |x-y_0| + \sqrt{|t-\tau_0|} - (|y-y_0| + \sqrt{|\tau-\tau_0|}) \\ &> \frac{1}{2}(|x-y_0| + \sqrt{|t-\tau_0|}) = \frac{1}{2} d((x,t);(y_0,\tau_0)). \end{aligned}$$

We get $|K_\mu(x,t;y,\tau) - K_\mu(x,t;y_0,\tau_0)| \leq C\frac{|y-y_0|+\sqrt{|\tau-\tau_0|}}{(\sqrt{|t-\tau_0|}+|x-x_0|)^4}$.

By proceeding in a similar way we also obtain $|K_\mu(y,\tau;x,t) - K_\mu(y_0,\tau_0;x,t)| \leq C\frac{|y-y_0|+\sqrt{|\tau-\tau_0|}}{(\sqrt{|t-\tau_0|}+|x-x_0|)^4}$. We have just proved that K_μ is a standard Calderón-Zygmund kernel with respect to the homogeneous type space $(\mathbb{R}\times(0,\infty), m, d)$.

Now we prove that $\widetilde{K_\mu}$ is a standard Calderón-Zygmund kernel with respect to the homogeneous type space $(\mathbb{R}\times(0,\infty), m, d)$.

We define

$$\mathcal{K}_\mu(x,y,s) = \frac{\partial}{\partial s}(W_s^\mu(x,y))\chi_{(0,\infty)}(s), \ \ s \in \mathbb{R} \ \ and \ \ x,y \in (0,\infty).$$

By (4.5) we get

$$\begin{aligned} \mathcal{K}_\mu(x,y,s) = (xy)^{\mu+1/2} e^{-\frac{x^2+y^2}{4s}} \Bigg(&- \left(\frac{xy}{2s}\right)^{-\mu} I_{\mu+1}\left(\frac{xy}{2s}\right) \frac{2xy}{(2s)^{\mu+3}} \\ + \left(\frac{xy}{2s}\right)^{-\mu} I_\mu\left(\frac{xy}{2s}\right) \frac{2(\mu+1)}{(2s)^{\mu+2}} &+ \left(\frac{xy}{2s}\right)^{-\mu} I_\mu\left(\frac{xy}{2s}\right) \frac{x^2+y^2}{(2s)^{\mu+3}} \Bigg), \end{aligned}$$

$x, y, s \in (0, \infty)$. According to (4.3), we can write

$$\begin{aligned}\mathcal{K}_\mu(x,y,s) &\leq C\left(\frac{(xy)^2}{s^{\mu+4}} + \frac{1}{s^{\mu+2}} + \frac{x^2+y^2}{s^{\mu+3}}\right)(xy)^{\mu+1/2}e^{-\frac{x^2+y^2}{4s}} \\ &\leq C\left(\frac{xy}{s}\right)^{\mu+1/2}\left(\frac{x^2+y^2}{(s+x^2+y^2)^{5/2}} + \frac{1}{(s+x^2+y^2)^{3/2}}\right) \\ &\leq \frac{C}{(\sqrt{s}+|x-y|)^3}, \quad s,x,y \in (0,\infty) \text{ and } \frac{xy}{s} \leq 1.\end{aligned}$$

Note that $\mu \geq -1/2$.

Also, by (4.4), we have that

$$\begin{aligned}\mathcal{K}_\mu(x,y,s) &= \left[-\frac{2xy}{(2s)^{5/2}}\left(\frac{xy}{2s}\right)^{1/2} I_{\mu+1}\left(\frac{xy}{2s}\right) - \frac{2(\mu+1)}{(2s)^{3/2}}\left(\frac{xy}{2s}\right)^{1/2} I_\mu\left(\frac{xy}{2s}\right)\right. \\ &\quad \left. + \left(\frac{xy}{2s}\right)^{1/2} I_\mu\left(\frac{xy}{2s}\right)\frac{x^2+y^2}{(2s)^{5/2}}\right]e^{-\frac{x^2+y^2}{4s}} \\ &= \frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{2\pi}}\left[\frac{(x-y)^2}{(2s)^{5/2}} - \frac{2(\mu+1)}{2s^{3/2}} + O\left(\frac{1}{s^{3/2}}\right)\right], \qquad (4.35)\end{aligned}$$

for $s, x, y \in (0, \infty)$, $xy \geq s$. Then,

$$|\mathcal{K}_\mu(x,y,s)| \leq \frac{C}{(s+|x-y|^2)^{3/2}} \leq \frac{C}{(\sqrt{s}+|x-y|)^3}, \ s,x,y \in (0,\infty), \ xy \geq s.$$

We conclude that

$$|\mathcal{K}_\mu(x,y,s)| \leq \frac{C}{(\sqrt{s}+|x-y|)^3}, \quad s,x,y \in (0,\infty).$$

By proceeding in a similar way, (4.3) and (4.4) lead to

$$|\mathcal{K}_\mu(x,y,z)| \leq C\frac{e^{-c\frac{(x-y)^2}{|z|}}}{|z|^{3/2}}, \quad x,y \in (0,\infty), \ |Arg z| \leq \frac{\pi}{4}. \qquad (4.36)$$

By using Cauchy integral formula we deduce that

$$\left|\frac{\partial}{\partial s}\mathcal{K}_\mu(x,y,s)\right| \leq C\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{5/2}}, \quad s,x,y \in (0,\infty).$$

Then, we obtain

$$\left|\frac{\partial}{\partial s}\mathcal{K}_\mu(x,y,s)\right| \leq \frac{C}{(\sqrt{s}+|x-y|)^5}, \quad s,x,y \in (0,\infty).$$

On the other hand, by using (4.5) (see (4.15)), we have that

$$\frac{\partial}{\partial x}W_s^\mu(x,y) = \frac{\partial}{\partial x}\left(\left(\frac{xy}{2s}\right)^{-\mu} I_\mu\left(\frac{xy}{2s}\right)\left(\frac{xy}{2s}\right)^{\mu+1/2}\frac{1}{\sqrt{2s}}e^{-\frac{x^2+y^2}{4s}}\right)$$
$$= \left[\frac{y}{2s}\left(\frac{xy}{2s}\right)^{-\mu} I_{\mu+1}\left(\frac{xy}{2s}\right)\left(\frac{xy}{2s}\right)^{\mu+1/2}\frac{1}{\sqrt{2s}}\right.$$
$$+\left(\frac{xy}{2s}\right)^{-\mu} I_\mu\left(\frac{xy}{2s}\right)\frac{1}{(2s)^{\mu+1}}y^{\mu+1/2}(\mu+1/2)x^{\mu-1/2}$$
$$\left.-\frac{x}{2s}\left(\frac{xy}{2s}\right)^{-\mu} I_\mu\left(\frac{xy}{2s}\right)\left(\frac{xy}{2s}\right)^{\mu+1/2}\frac{1}{\sqrt{2s}}\right]e^{-\frac{x^2+y^2}{4s}},\quad x,y,s\in(0,\infty).$$

Then, (4.3) implies that

$$\left|\frac{\partial}{\partial x}W_s^\mu(x,y)\right| \leq C\left(\frac{y}{s^{3/2}}+\frac{x}{s^{3/2}}\right)e^{-\frac{x^2+y^2}{4s}} \leq \frac{C}{s}e^{-c\frac{(x-y)^2}{s}},$$

for $s,x,y\in(0,\infty),\ xy\leq s$, provided that $\mu>1/2$ or $\mu=-1/2$. Also, (4.4) leads to

$$\frac{\partial}{\partial x}W_s^\mu(x,y) = \left(\frac{y}{(2s)^{3/2}}+\frac{1}{x\sqrt{2s}}-\frac{x}{(2s)^{3/2}}\right)\left(1+O\left(\frac{s}{xy}\right)\right)e^{-\frac{(x-y)^2}{4s}}$$
$$= \frac{x-y}{(2s)^{3/2}}e^{-\frac{(x-y)^2}{4s}}+\left(\frac{1}{x\sqrt{s}}+\frac{1}{y\sqrt{s}}\right)e^{-\frac{(x-y)^2}{4s}}O(1),\ s,x,y\in(0,\infty),\ xy\geq s.$$

We obtain,

$$\left|\frac{\partial}{\partial x}W_s^\mu(x,y)\right| \leq C\left(\frac{1}{s}+\frac{1}{\sqrt{xys}}\right)e^{-\frac{(x-y)^2}{4s}} \leq \frac{C}{s}e^{-c\frac{(x-y)^2}{s}},$$

when $xy\geq s$, and $x/2\leq y\leq 2x$, and, by taking $z=\max\{x,y\}$,

$$\left|\frac{\partial}{\partial x}W_s^\mu(x,y)\right| \leq C\frac{y+x}{s^{3/2}}e^{-\frac{(x-y)^2}{4s}} \leq C\frac{z}{s^{3/2}}e^{-\frac{cz^2}{s}} \leq \frac{C}{s}e^{-c\frac{z^2}{s}}$$
$$\leq \frac{C}{s}e^{-c\frac{(x-y)^2}{s}},\quad xy\geq s,\ \text{ and } 0<y<x/2 \text{ or } 2x<y.$$

We conclude that

$$\left|\frac{\partial}{\partial x}W_s^\mu(x,y)\right| \leq C\frac{e^{-c\frac{(x-y)^2}{s}}}{s},\quad x,y,s\in(0,\infty).$$

The same arguments, by using again (4.3), (4.4) and (4.5), allows us to obtain

$$\left|\frac{\partial}{\partial x}W_s^\mu(x,y)\right| \leq C\frac{e^{-c\frac{(x-y)^2}{|z|}}}{|z|},\quad x,y\in(0,\infty),\ |Arg z|\leq\frac{\pi}{4},$$

and Cauchy integral formula leads to

$$\left|\frac{\partial}{\partial s}\frac{\partial}{\partial x}W_s^\mu(x,y)\right| \leq C\frac{e^{-c\frac{(x-y)^2}{s}}}{s^2},\quad x,y,s\in(0,\infty). \tag{4.37}$$

Symmetries imply that

$$\left|\frac{\partial}{\partial s}\frac{\partial}{\partial y}W_s^\mu(x,y)\right| \leq C\frac{e^{-c\frac{(x-y)^2}{s}}}{s^2}, \quad x,y,s\in(0,\infty).$$

We get

$$\left|\frac{\partial}{\partial x}\mathcal{K}_\mu(x,y,s)\right| + \left|\frac{\partial}{\partial y}\mathcal{K}_\mu(x,y,s)\right| \leq \frac{C}{(|x-y|+\sqrt{s})^4}, \quad x,y,s\in(0,\infty).$$

Putting together the above estimates and proceeding as in the K_μ-case we can prove that $\widetilde{K_\mu}$ is a standard Calderón-Zygmund kernel with respect to the homogeneous type space $(\mathbb{R}\times(0,\infty),m,d)$.

Thus, the proof of this proposition is finished.

□

Proof of Theorem 4.122 (2).

The result holds from the previous proposition and Calderón-Zygmund Theorem.

□

In order to prove the parts (3), (4) and (5) in Theorem 4.122 we use a procedure which is different from the one employed to prove Theorem 4.122 (2). As it was mentioned in the introduction, Ping, Stinga and Torrea [73] investigated L^p-boundedness properties of the Riesz transformations associated with the parabolic equation for the Laplace operator. They studied, when a one dimensional spatial variable is considered, the following two operators

$$R(f)(t,x) = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \partial^2_{yy}W_s(y)f(t-s,x-y)dsdy \tag{4.38}$$

and

$$\widetilde{R}(f)(t,x) = \lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon} \partial_s W_s(y)f(t-s,x-y)dsdy, \tag{4.39}$$

where, for every $\varepsilon>0$, $\Omega_\varepsilon=\{(s,y)\in(0,\infty)\times\mathbb{R}, \sqrt{s}+y>\varepsilon\}$. Our procedure consists, roughly speaking, in studying the L^p-boundedness properties of the difference operators $R_\mu - R$ and $\widetilde{R}_\mu - \widetilde{R}$. Then, L^p-boundedness properties of R_μ and $\widetilde{R}_\mu$ are deduced from the corresponding ones of R and $\widetilde{R}$, respectively, established in [73, Theorem 2.3, (B)].

Calderón-Zygmund Theorem employed in the proof of Theorem 4.122 (2) allows us to consider weighted L^p-spaces but the parameter μ is restricted to $\mu=-1/2$ or $\mu>1/2$. This comparative approach applies to the full range of values of $\mu>-1$.

Proof of Theorem 4.122 (3)**,** (4) **and** (5).

A) Proofs concerning the Riesz transformation $\widetilde{R_\mu}$.

We consider the operator

$$\widetilde{R_\mu}(f)(t,x) = \lim_{\epsilon\to 0^+}\int_{\Omega_\epsilon(x)} \mathcal{K}_\mu(s,x,y)f(t-s,y)dsdy, \quad f\in C_c^\infty(\mathbb{R}\times(0,\infty)),$$

where we name $\mathcal{K}_\mu(s,x,y)=\partial_s W_s^\mu(x,y)$, $s,x,y\in(0,\infty)$.

We shall also fix our attention in the operator

$$\widetilde{R}(f)(t,x) = \lim_{\epsilon\to 0^+} \int_{\Omega_\epsilon(x)} \mathcal{K}(s,x,y) f(t-s,y)\,dsdy, \quad f \in C_c^\infty(\mathbb{R}^2),$$

where $\mathcal{K}(s,x,y) = \partial_s W_s(x-y)$, $s \in (0,\infty)$ and $x,y \in \mathbb{R}$.

Ping, Stinga and Torrea in [73] studied L^p-boundedness properties for the operator $\widetilde{R}$. Our objective is to prove L^p-boundedness properties for the operator $\widetilde{R_\mu}$ by using the corresponding ones for $\widetilde{R}$.

According to (4.12) we have that

$$\mathcal{K}_\mu(s,x,y) - \mathcal{K}(s,x,y) = e^{-\frac{(x-y)^2}{16s}} O\left(\frac{1}{\sqrt{sxy}}\right), \quad s,x,y \in (0,\infty) \text{ and } s \le xy. \tag{4.40}$$

By (4.11) we obtain

$$\begin{aligned} |\mathcal{K}_\mu(s,x,y) - \mathcal{K}(s,x,y)| &\le |\mathcal{K}_\mu(s,x,y)| + |\mathcal{K}(s,x,y)| \\ &\le C\left(\frac{(xy)^{\mu+1/2}}{s^{\mu+2}} + \frac{1}{s^{3/2}}\right) e^{-\frac{x^2+y^2}{8s}}, \quad s,x,y \in (0,\infty), \quad xy \le s. \end{aligned} \tag{4.41}$$

From (4.40) and (4.41) we deduce that

$$\begin{aligned} &\int_0^\infty \int_{x/2}^{3x/2} |\mathcal{K}_\mu(s,x,y) - \mathcal{K}(s,x,y)| dyds \\ &\le C\int_{x/2}^{3x/2}\int_0^{xy} \frac{e^{-c\frac{(x-y)^2}{s}}}{\sqrt{sxy}} dsdy + C\int_{x/2}^{3x/2}\int_{xy}^{\infty} e^{-c\frac{x^2+y^2}{s}} \left(\frac{1}{s^{3/2}} + \frac{(xy)^{\mu+1/2}}{s^{\mu+2}}\right) dsdy \\ &\le C\Bigg(\int_{x/2}^{3x/2} \frac{\sqrt{xy}}{xy} dy + \int_{x/2}^{3x/2} \left(\frac{1}{(xy)^{1/2}} + \frac{(xy)^{\mu+1/2}}{(xy)^{\mu+1}}\right) dy\Bigg) \le C, \quad x \in (0,\infty). \end{aligned}$$

By proceeding as in (4.11) we get

$$|\mathcal{K}_\mu(s,x,y)| \le C\frac{(xy)^{\mu+1/2}}{s^{\mu+2}} e^{-c\frac{x^2+y^2}{s}}, \quad s,x,y \in (0,\infty), \; s \ge xy.$$

Also, from (4.36), it follows that

$$|\mathcal{K}_\mu(s,x,y)| \le C\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}}, \quad s,x,y \in (0,\infty), \quad s \le xy.$$

Then,

$$\begin{aligned}
&\int_0^\infty \int_0^{x/2} |\mathcal{K}_\mu(s,x,y)| dyds \\
&\leq C\Big(\int_0^{x/2} \int_0^{xy} \frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}} dsdy + \int_0^{x/2}\int_{xy}^\infty \frac{(xy)^{\mu+1/2}}{s^{\mu+2}} e^{-c\frac{x^2+y^2}{s}} dsdy \Big) \\
&\leq C\left(\int_0^{x/2} \int_0^{xy} \frac{e^{-c\frac{x^2}{s}}}{s^{3/2}} dsdy + \int_0^{x/2} \frac{(xy)^{\mu+1/2}}{(xy)^{\mu+1}} dy \right) \\
&\leq C\left(\int_0^{x/2} \frac{(xy)^{1/2}}{x^2} dy + 1 \right) \leq C, \quad x \in (0,\infty),
\end{aligned} \tag{4.42}$$

and when $\mu > -1/2$,

$$\begin{aligned}
&\int_0^\infty \int_{3x/2}^\infty |\mathcal{K}_\mu(s,x,y)| dyds \\
&\quad \leq C\left(\int_{3x/2}^\infty \int_0^{xy} \frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}} dsdy + \int_{3x/2}^\infty\int_{xy}^\infty \frac{(xy)^{\mu+1/2}}{s^{\mu+2}} e^{-c\frac{x^2+y^2}{s}} dsdy \right) \\
&\quad \leq C\left(\int_{3x/2}^\infty \int_0^{xy} \frac{e^{-c\frac{y^2}{s}}}{s^{3/2}} dsdy + \int_{3x/2}^\infty\int_{xy}^\infty \frac{(xy)^{\mu+1/2}}{s^{\mu+2}} e^{-c\frac{x^2+y^2}{s}} dsdy \right) \\
&\quad \leq C\left(\int_{3x/2}^\infty \int_0^{xy} \frac{y^{-2}}{s^{1/2}} dsdy + \int_{3x/2}^\infty\int_{xy}^\infty \frac{(xy)^{\mu+1/2}}{s^{\mu+2}} \frac{s^{\frac{2\mu+3}{4}}}{(x^2+y^2)^{\frac{2\mu+3}{4}}} dsdy \right) \\
&\quad \leq C\left(x^{1/2} \int_{3x/2}^\infty \frac{dy}{y^{3/2}} + x^{\frac{2\mu+1}{4}} \int_{3x/2}^\infty \frac{dy}{y^{\frac{2\mu+5}{4}}} \right) \leq C, \quad x \in (0,\infty).
\end{aligned} \tag{4.43}$$

Observe that this estimate can not be improved for $-1 < \mu \leq -1/2$.

We now suppose that g is a complex valued continuous function with compact support in $(0,\infty)$. We define g_0 as the odd extension of g to $\mathbb{R}$. We can write

$$\begin{aligned}
&\int_{\mathbb{R}} \partial_t W_t(x-y) g_0(y) dy = \int_0^\infty \partial_t W_t(x-y) g(y) dy + \int_{-\infty}^0 \partial_t W_t(x-y) g(-y) dy \\
&\quad = \int_0^\infty \partial_t (W_t(x-y) - W_t(x+y)) g(y) dy, \quad x \in \mathbb{R}, \text{ and } t \in (0,\infty).
\end{aligned}$$

This fact and the following estimate will be useful in the sequel.

Note that

$$\begin{aligned}
&\partial_t (W_t(x-y) - W_t(x+y)) = \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{4\pi t}} \left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}} \right) \right] \\
&\quad = \frac{\partial}{\partial t}\left[W_t(x-y)\left(1 - e^{-\frac{xy}{t}}\right) \right] \\
&\quad = \frac{\partial}{\partial t}\left(W_t(x-y) \right)\left(1 - e^{-\frac{xy}{t}}\right) + W_t(x-y) \frac{xy}{t^2} e^{-\frac{xy}{t}}, \quad t,x,y \in (0,\infty).
\end{aligned}$$

Then,

$$|\partial_t(W_t(x-y)-W_t(x+y))| \leq C\left(|\partial_t W_t(x-y)|\,\frac{xy}{t}+|W_t(x-y)|\frac{xy}{t^2}\right)$$
$$\leq C\frac{e^{-c\frac{(x-y)^2}{t}}}{t^{5/2}}xy \leq C\frac{e^{-c\frac{x^2+y^2}{t}}}{t^{5/2}}xy,\quad t,x,y\in(0,\infty),\quad xy\leq t. \tag{4.44}$$

Let $f \in C_c^\infty(\mathbb{R}\times(0,\infty))$. We define

$$f_0(t,x)=\begin{cases} f(t,x), & t\in\mathbb{R}, x\geq 0,\\ f(t,-x), & t\in\mathbb{R}, x<0.\end{cases}$$

Thus, $f_0 \in C_c^\infty(\mathbb{R}^2)$. We can write

$$\int_0^\infty\int_{\mathbb{R}}\partial_s W_s(x-y)f_0(t-s,y)dyds$$
$$=\int_0^\infty\int_0^\infty(\partial_s W_s(x-y)-\partial_s W_s(x+y))f(t-s,y)dyds,\ (t,x)\notin \text{supp} f_0. \tag{4.45}$$

This fact suggests the following analysis. From (4.44) we deduce that

$$\int_0^\infty\int_0^{x/2}|\partial_s W_s(x-y)-\partial_s W_s(x+y)|dyds$$
$$\leq C\Big(\int_0^{x/2}\int_0^{xy}\left(|\partial_s W_s(x-y)|+|\partial_s W_s(x+y)|\right)dsdy$$
$$+\int_0^{x/2}\int_{xy}^\infty\frac{xye^{-c\frac{x^2+y^2}{s}}}{s^{5/2}}dsdy\Big)$$
$$\leq C\left(\int_0^{x/2}\int_0^{xy}\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}}dsdy+\int_0^{x/2}\int_{xy}^\infty\frac{xy}{s^{5/2}}dsdy\right)$$
$$\leq C\left(\int_0^{x/2}\int_0^{xy}\frac{e^{-c\frac{x^2}{s}}}{s^{3/2}}dsdy+\int_0^{x/2}\frac{1}{(xy)^{1/2}}dy\right)\leq C,\quad x\in(0,\infty), \tag{4.46}$$

and

$$\begin{aligned}
&\int_0^\infty \int_{3x/2}^\infty |\partial_s W_s(x-y) - \partial_s W_s(x+y)| dyds \\
&\qquad \leq C \left(\int_{3x/2}^\infty \int_0^{xy} \frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}} dsdy + \int_{3x/2}^\infty \int_{xy}^\infty \frac{xye^{-c\frac{x^2+y^2}{s}}}{s^{5/2}} dsdy \right) \\
&\qquad \leq C \left(\int_{3x/2}^\infty \int_0^{xy} \frac{e^{-c\frac{y^2}{s}}}{s^{3/2}} dsdy + \int_{3x/2}^\infty \int_{xy}^\infty \frac{xys}{s^{5/2}(x^2+y^2)} dsdy \right) \\
&\qquad \leq \left(\int_{3x/2}^\infty \int_0^{xy} \frac{1}{y^3} dsdy + \int_{3x/2}^\infty \frac{\sqrt{xy}}{y^2} dy \right) \leq C, \; x \in (0,\infty). \qquad (4.47)
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
\int_0^\infty \int_{x/2}^{3x/2} |\partial_s W_s(x+y)| dyds &\leq \int_{x/2}^{3x/2} \int_0^\infty \frac{e^{-c\frac{(x+y)^2}{s}}}{s^{3/2}} dsdy \leq C \int_{x/2}^{3x/2} \frac{dy}{x+y} \\
&\leq C(\log(x+3x/2) - log(x+x/2)) \leq C, \quad x \in (0,\infty). \qquad (4.48)
\end{aligned}$$

Fix $\mu > -1/2$. Let $f \in C_c^\infty(\mathbb{R} \times (0,\infty))$. We define f_0 as above. According to (4.45) and [73, Theorem 2.3, (B)], we can write, for every $t \in \mathbb{R}$ and $x \in (0,\infty)$,

$$\begin{aligned}
\widetilde{R_\mu}(f)(t,x) - \widetilde{R}(f_0)(t,x) = \lim_{\epsilon \to 0^+} \Big(&\int_{\Omega_\epsilon(x)} \mathcal{K}_\mu(\tau,x,y) f(t-\tau) d\tau dy \\
&- \int_{W_\epsilon(x)} \mathcal{K}(\tau,x,y)) f(t-\tau,y) d\tau dy \Big),
\end{aligned}$$

where $W_\varepsilon(x) = \{(\tau,y) \in (0,\infty) \times \mathbb{R} : \sqrt{\tau} + |x-y| > \varepsilon\}$, for every $x \in \mathbb{R}$ and $\varepsilon > 0$.

Then,

$$\begin{aligned}
&\widetilde{R_\mu}(f)(t,x) - \widetilde{R}(f_0)(t,x) \\
&= \lim_{\epsilon \to 0^+} \Bigg[\int_0^\infty \int_{x/2,\ (\tau,y)\in\Omega_\epsilon(x)}^{2x} (\mathcal{K}_\mu(\tau,x,y) - \mathcal{K}(\tau,x,y)) f(t-\tau,y) dyd\tau \\
&+ \int_0^\infty \int_{0,\ (\tau,y)\in\Omega_\epsilon(x)}^{x/2} \mathcal{K}_\mu(\tau,x,y) f(t-\tau,y) dyd\tau \\
&+ \int_0^\infty \int_{3x/2,\ (\tau,y)\in\Omega_\epsilon(x)}^{\infty} \mathcal{K}_\mu(\tau,x,y) f(t-\tau,y) dyd\tau \\
&- \int_0^\infty \int_{0,\ (\tau,y)\in\Omega_\epsilon(x)}^{x/2} (\mathcal{K}(\tau,x,y) - \mathcal{K}(\tau,x,-y)) f(t-\tau,y) dyd\tau \\
&+ \int_0^\infty \int_{x/2,\ (\tau,y)\in\Omega_\epsilon(x)}^{2x} \mathcal{K}(\tau,x,-y) f(t-\tau,y) dyd\tau \\
&- \int_0^\infty \int_{3x/2,\ (\tau,y)\in\Omega_\epsilon(x)}^{\infty} (\mathcal{K}(\tau,x,y) - \mathcal{K}(\tau,x,-y)) f(t-\tau,y) dyd\tau \Bigg],
\end{aligned}$$

$t \in \mathbb{R}$, $x \in (0,\infty)$. Note that from (4.42), (4.43), (4.44), (4.46), (4.47) and (4.48) we deduce that the last six integrals are absolutely convergent. We get

$$\widetilde{R_\mu}(f)(t,x) - \widetilde{R}(f_0)(t,x) = \int_0^\infty \int_0^\infty T_\mu(\tau,x,y)f(t-\tau,y)dyd\tau,\ t \in \mathbb{R},\ x \in (0,\infty),$$

where $T_\mu(t,x,y) = \mathcal{K}_\mu(t,x,y) - \mathcal{K}(t,x,y) + \mathcal{K}(t,x,-y)$, $t \in \mathbb{R}$ and $x, y \in (0,\infty)$.

Note that

$$\sup_{x\in(0,\infty)} \int_0^\infty \int_0^\infty |T_\mu(\tau,x,y)|d\tau dy < \infty. \tag{4.49}$$

We consider the operator $\widetilde{T_\mu}(f)(t,x) = \int_0^\infty \int_0^\infty T_\mu(\tau,x,y)f(t-\tau,y)d\tau dy$, $t \in \mathbb{R}$, $x \in (0,\infty)$.

According to (4.49) we obtain

- There exists a constant $C > 0$ such that, for every $f \in L^1(\mathbb{R}\times(0,\infty))$,

$$\begin{aligned}\|\widetilde{T_\mu}(f)\|_{L^1(\mathbb{R}\times(0,\infty))} &= \int_{\mathbb{R}}\int_0^\infty \left|\int_0^\infty \int_0^\infty T_\mu(\tau,x,y)f(t-\tau,y)d\tau dy\right| dxdt \\ &\leq \int_{\mathbb{R}}\int_0^\infty \int_{\mathbb{R}}\int_0^\infty |T_\mu(t-\tau,x,y)|\,|f(\tau,y)|\chi_{(0,\infty)}(t-\tau)dyd\tau dxdt \\ &= \int_{\mathbb{R}}\int_0^\infty |f(\tau,y)| \int_0^\infty \int_0^\infty |T_\mu(u,x,y)|\,dxdud\tau dy \\ &\leq C\|f\|_{L^1(\mathbb{R}\times(0,\infty))}.\end{aligned}$$

  We have used that $T_\mu(u,x,y) = T_\mu(u,y,x)$, $u, x, y \in (0,\infty)$.

- There exists a constant $C > 0$ such that, for every $f \in L^\infty(\mathbb{R}\times(0,\infty))$,

$$\|\widetilde{T_\mu}(f)\|_{L^\infty(\mathbb{R}\times(0,\infty))} \leq C\|f\|_{L^\infty(\mathbb{R}\times(0,\infty))}.$$

Marcinkiewicz interpolation theorem allows us to conclude that $\widetilde{T_\mu}$ is a bounded operator from $L^p(\mathbb{R}\times(0,\infty))$ into itself, for every $1 \leq p \leq \infty$.

By [73, Theorem 2.3, (B)] we deduce that, for every $1 \leq p < \infty$, the operator $\widetilde{R_\mu}$ can be extended to $L^p(\mathbb{R}\times(0,\infty))$ as a bounded operator from $L^p(\mathbb{R}\times(0,\infty))$ into itself when $1 < p < \infty$ and from $L^1(\mathbb{R}\times(0,\infty))$ into $L^{1,\infty}(\mathbb{R}\times(0,\infty))$.

As it was established in [73, Theorem 2.3, (B)], the maximal opertor

$$\tilde{R}_*(f)(t,x) = \sup_{\epsilon>0}\left|\int_{\Omega_\epsilon(x)} \tilde{R}(\tau,x,y)f(t-\tau,y)d\tau dy\right|,\quad t, x \in \mathbb{R},$$

is bounded from $L^p(\mathbb{R}^2)$ into itself, for every $1 < p < \infty$, and from $L^1(\mathbb{R}^2)$ into $L^{1,\infty}(\mathbb{R}^2)$. Then, the above results imply that the maximal operator

$$\widetilde{R_{\mu,*}}(f)(t,x) = \sup_{\epsilon>0}\left|\int_{\Omega_\epsilon(x)} \widetilde{R_\mu}(\tau,x,y)f(t-\tau,y)d\tau dy\right|,\ t \in \mathbb{R},\ x \in (0,\infty),$$

is bounded from $L^p(\mathbb{R}\times(0,\infty))$ into itself, for every $1<p<\infty$, and from $L^1(\mathbb{R}\times(0,\infty))$ into $L^{1,\infty}(\mathbb{R}\times(0,\infty))$.

Since the principal value $\lim_{\epsilon\to 0^+}\int_{\Omega_\epsilon(x)}\widetilde{R_\mu}(\tau,x,y)f(t-\tau,y)d\tau dy$ exists, for every $f\in C_c^\infty(\mathbb{R}\times(0,\infty))$ and $(t,x)\in\mathbb{R}\times(0,\infty)$, and as $C_c^\infty(\mathbb{R}\times(0,\infty))$ is a dense subspace of $L^p(\mathbb{R}\times(0,\infty))$, $1\le p<\infty$, we have that, for every $f\in L^p(\mathbb{R}\times(0,\infty))$, $1\le p<\infty$, there exists the principal value $\lim_{\epsilon\to 0^+}\int_{\Omega_\epsilon(x)}\widetilde{R_\mu}(\tau,x,y)f(t-\tau,y)d\tau dy$ for almost all $(t,x)\in\mathbb{R}\times(0,\infty)$. Also, the operator $\widehat{R_\mu}$ defined on $L^p(\mathbb{R}\times(0,\infty))$, $1\le p<\infty$, as follows

$$\widehat{R_\mu}(f)(t,x)=\lim_{\epsilon\to 0^+}\int_{\Omega_\epsilon(x)}\widetilde{R_\mu}(\tau,x,y)f(t-\tau,y)d\tau dy,\ \text{ a.e. } (t,x)\in\mathbb{R}\times(0,\infty),$$

is bounded from $L^p(\mathbb{R}\times(0,\infty))$ into itself, for every $1<p<\infty$, and from $L^1(\mathbb{R}\times(0,\infty))$ into $L^{1,\infty}(\mathbb{R}\times(0,\infty))$.

Our objective is to study the L^p-boundedness properties for $\widetilde{R_\mu}$ when $-1<\mu\le -1/2$. We have that

$$\begin{aligned}\widetilde{R_\mu}(f)&=\mathcal{F}^{-1}h_\mu\left(\frac{-i\rho}{z^2+i\rho}\mathcal{F}h_\mu(f)\right)=h_\mu h_{\mu+2}\mathcal{F}^{-1}h_{\mu+2}\left(\frac{-i\rho}{z^2+i\rho}\mathcal{F}h_{\mu+2}h_{\mu+2}h_\mu(f)\right)\\&=S_\mu\widetilde{R_{\mu+2}}S_\mu^*f,\ \ f\in L^2(\mathbb{R}\times(0,\infty)),\end{aligned}$$

where $S_\mu=h_\mu h_{\mu+2}$ and $S_\mu^*=h_{\mu+2}h_\mu$.

The composition operators $h_\mu h_\nu$ are named transplantation operators associated with Hankel transforms.

According to [66, Theorem 2.1], if $\mu>-1$, $1<p<\infty$ and v is a nonnegative measurable function such that

$$\sup_{r>0}\left(\int_0^r v(x)^p x^{p(\mu+1/2)}dx\right)^{1/p}\left(\int_r^\infty v(x)^{-p'}x^{-p'(\mu+3/2)}dx\right)^{1/p'}<\infty$$

then, the operator S_μ can be extended to $L^p(v)$ as a bounded operator from $L^p(v)$ into itself. Here, as usual, p' denotes the conjugated of p, that is, $p'=\frac{p}{p-1}$.

Since

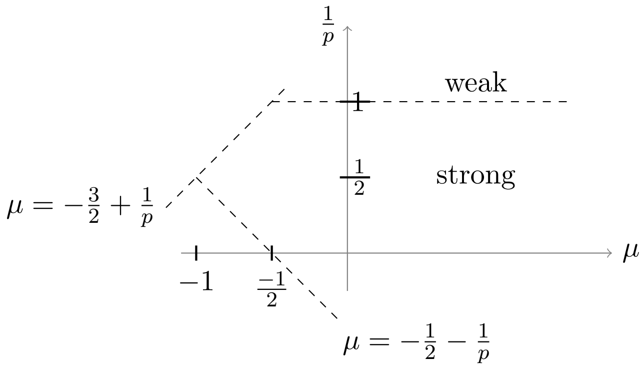
$$\begin{aligned}\left(\int_0^r x^{p(\mu+1/2)}dx\right)^{1/p}&\left(\int_r^\infty x^{-p'(\mu+3/2)}dx\right)^{1/p'}\\&=\frac{r^{\frac{1}{p}(p(\mu+1/2)+1)+\frac{1}{p'}(1-p'(\mu+3/2))}}{(p(\mu+1/2)+1)(-1+p'(\mu+3/2))}\\&=\frac{1}{(p(\mu+1/2)+1)(-1+p'(\mu+3/2))},\ \ r>0,\end{aligned}$$

provided that $p(\mu+1/2)+1>0$ and $1-p'(\mu+3/2)<0$, S_μ defines a bounded operator from $L^p((0,\infty))$ into itself when $1<p<\infty$ and $\mu>-1/2-1/p$ and $\mu>-1$. Then S_μ^* is bounded from $L^p((0,\infty))$ into itself when $1<p<\infty$, $\mu>-1$ and $\mu>-1/2-(1-1/p)=-3/2+1/p$.

Hence, since $\widetilde{R_{\mu+2}}$ is bounded from $L^p((0,\infty))$ into itself when $1<p<\infty$ and $\mu>-1$, $\widetilde{R_\mu}$ defines a bounded operator from $L^p(\mathbb{R}\times(0,\infty))$ into itself provided that $-1<\mu\le 1/2$ and $-1/2-\mu<1/p<3/2+\mu$.

Therefore, we have proved that

- $\widetilde{R_\mu}$ is bounded from $L^p(\mathbb{R}\times(0,\infty))$ into itself when
  1. $\mu>-1/2$ and $1<p<\infty$.
  2. $-1<\mu\le -1/2$ and $-1/2-\mu<1/p<3/2+\mu$.
- $\widetilde{R_\mu}$ is bounded from $L^1(\mathbb{R}\times(0,\infty))$ into $L^{1,\infty}(\mathbb{R}\times(0,\infty))$, when $\mu>-1/2$.



B) Proofs concerning the Riesz transformation R_μ.

We consider the operator R_μ defined by

$$R_\mu(f)(t,x)=\lim_{\varepsilon\to 0}\int_{\Omega_\varepsilon(x)}\mathbb{K}_\mu(x,y,s)f(t-s,y)dsdy,$$

for every $f\in C_c^\infty((\mathbb{R}\times(0,\infty))$, where $\mathbb{K}_\mu(x,y,s)=\delta_{\mu+1}\delta_\mu W_s^\mu(x,y),\ \ s,x,y\in(0,\infty)$.

We also write

$$\mathbb{K}(x,y,s)=\partial^2_{xx}W_s(x-y),\ \ x,y\in\mathbb{R}\ \text{ and }\ s\in(0,\infty).$$

According to (4.23) we have that

$$\begin{aligned}\mathbb{K}_\mu(x,y,s)=&\frac{x^{\mu+5/2}y^{\mu+1/2}}{(2s)^{\mu+3}}\left(\frac{y^3}{2sx}\left(\frac{xy}{2s}\right)^{-\mu-1}I_{\mu+2}\left(\frac{xy}{2s}\right)\right.\\&\left.-\frac{2y}{x}\left(\frac{xy}{2s}\right)^{-\mu}I_{\mu+1}\left(\frac{xy}{2s}\right)+\left(\frac{xy}{2s}\right)^{-\mu}I_\mu\left(\frac{xy}{2s}\right)\right)e^{-\frac{x^2+y^2}{4s}},\ x,y,s\in(0,\infty).\end{aligned}$$

Also, we get

$$\mathbb{K}(x,y,s)=\frac{\partial^2}{\partial x^2}\left(\frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{4\pi s}}\right)=\frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{4\pi s}}\left(\frac{(x-y)^2}{(2s)^2}-\frac{1}{2s}\right),$$

for $x, y \in \mathbb{R},\ s \in (0,\infty)$. We define $T_\mu(x,y,s) = \mathbb{K}_\mu(x,y) - \mathbb{K}(x,y,s) + \mathbb{K}(x,-y,s),\ s,x,y \in (0,\infty)$.

Our first objective is to study when

$$\sup_{x\in(0,\infty)} \int_0^\infty \int_0^\infty |T_\mu(x,y,s)| dyds < \infty, \tag{4.50}$$

is true.

We decompose the proof of (4.50) in several steps. According to (4.3), we can write

$$\begin{aligned} |\mathbb{K}_\mu(x,y,s)| &\le C \frac{x^{\mu+5/2} y^{\mu+1/2}}{s^{\mu+2}} \left(\frac{y^4}{s^3} + \frac{y^2}{s^2} + \frac{1}{s} \right) e^{-\frac{x^2+y^2}{4s}} \\ &\le C \left(\frac{xy}{s} \right)^{\mu+1/2} \frac{x^2}{s^{5/2}} \left(\frac{y^4}{s^2} + \frac{y^2}{s} + 1 \right) e^{-\frac{x^2+y^2}{4s}},\ s,x,y \in (0,\infty) \text{ and } xy \le s. \end{aligned} \tag{4.51}$$

By (4.4) we get

$$\begin{aligned} |\mathbb{K}_\mu(x,y,s)| &\le C \frac{x^{\mu+5/2} y^{\mu+1/2}}{s^{\mu+2}} \left(\frac{y^2}{sx^2} + \frac{y}{sx} + \frac{1}{s} \right) \left(\frac{xy}{s} \right)^{-\mu-1/2} e^{-\frac{(x-y)^2}{4s}} \\ &\le C \frac{x^2}{s^{5/2}} e^{-\frac{(x-y)^2}{4s}} \left(\frac{y^2}{x^2} + \frac{y}{x} + 1 \right),\quad s,x,y \in (0,\infty),\ xy \ge s. \end{aligned} \tag{4.52}$$

From (4.52) it follows that, for every $s,x,y \in (0,\infty)$ such that $xy \ge s$,

$$|\mathbb{K}_\mu(x,y,s)| \le C \frac{x^2}{s^{5/2}} \left(\frac{y^2}{x^2} + \frac{y}{x} + 1 \right) e^{-c\frac{\max\{x,y\}^2}{s}},\quad 0 < y < x/2 \text{ or } y > 2x. \tag{4.53}$$

By using (4.51) and (4.53) we obtain

$$\begin{aligned} &\int_0^{x/2} \int_0^\infty |\mathbb{K}_\mu(x,y,s)| dsdy \le C \Bigg(\int_0^{x/2} \int_0^{xy} \frac{x^2}{s^{5/2}} \left(\frac{y^2}{x^2} + \frac{y}{x} + 1 \right) e^{-c\frac{x^2}{s}} dsdy \\ &+ \int_0^{x/2} \int_{xy}^\infty \frac{(xy)^{\mu+1/2}}{s^{\mu+3}} x^2 \left(\frac{y^4}{s^2} + \frac{y^2}{s} + 1 \right) e^{-\frac{x^2+y^2}{4s}} dsdy \Bigg) \\ &\le Cx^2 \Bigg(\int_0^{x/2} \int_0^{xy} \frac{e^{-c\frac{x^2}{s}}}{s^{5/2}} dsdy \\ &+ \int_0^{x/2} (xy)^{\mu+1/2} \int_{xy}^\infty \left(\frac{y^4}{(x^2+y^2)^2} + \frac{y^2}{x^2+y^2} + 1 \right) \frac{1}{s^{\mu+3}} \left(\frac{s}{x^2+y^2} \right)^{\frac{2\mu+5}{4}} dsdy \Bigg) \\ &\le Cx^2 \left(\int_0^{x/2} \int_{x/y}^\infty \frac{e^{-cu}\sqrt{u}}{x^3} dudy + x^{\mu+1/2} \int_0^{x/2} \frac{y^{\mu+1/2}}{(x^2+y^2)^{\frac{2\mu+5}{4}} (xy)^{\mu/2+3/4}} dy \right) \\ &\le C \frac{1}{x} \int_0^{x/2} dy \int_0^\infty e^{-cu}\sqrt{u} du + x^{2+\mu+1/2-\frac{2\mu+5}{2}-\frac{\mu}{2}-3/4} \int_0^{x/2} y^{\mu+1/2-\mu/2-3/4} dy \\ &\le C \left(1 + x^{-\mu/2-3/4} \int_0^{x/2} y^{\mu/2-1/4} dy \right) \le C,\quad x \in (0,\infty), \end{aligned} \tag{4.54}$$

and

$$\int_{3x/2}^{\infty}\int_0^{\infty}|\mathbb{K}_\mu(x,y,s)|dsdy \leq C\Big(\int_{3x/2}^{\infty}\int_0^{xy}\frac{x^2}{s^{5/2}}\left(\frac{y^2}{x^2}+\frac{y}{x}+1\right)e^{-c\frac{y^2}{s}}dsdy$$
$$+\int_{3x/2}^{\infty}\int_{xy}^{\infty}\frac{(xy)^{\mu+1/2}}{s^{\mu+3}}x^2\left(\frac{y^4}{s^2}+\frac{y^2}{s}+1\right)e^{-\frac{x^2+y^2}{4s}}dsdy\Big)$$
$$\leq C\int_{3x/2}^{\infty}y^2\int_0^{xy}\frac{e^{-c\frac{y^2}{s}}}{s^{5/2}}dsdy$$
$$+C\int_{3x/2}^{\infty}(xy)^{\mu+1/2}x^2\int_{xy}^{\infty}\left(\frac{y^4}{s^2}+\frac{y^2}{s}+1\right)\frac{e^{-\frac{y^2}{4s}}}{s^{\mu+3}}dsdy$$
$$\leq C\left(\int_{3x/2}^{\infty}y^2\int_{y/x}^{\infty}\frac{e^{-cu}\sqrt{u}}{y^3}dudy+\int_{3x/2}^{\infty}(xy)^{\mu+1/2}\frac{x^2}{(xy)^{\mu+2}}dy\right)$$
$$\leq C\left(x\int_{3x/2}^{\infty}\frac{y}{2x}\frac{e^{-c\frac{y}{2x}}}{y^2}dy+1\right)\leq C,\quad x\in(0,\infty). \tag{4.55}$$

We have that

$$\mathbb{K}(x,y,s)-\mathbb{K}(x,-y,s)=\frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{4\pi}(2s)^{3/2}}\left(1-e^{-\frac{(x+y)^2-(x-y)^2}{4s}}\right)\left(\frac{(x-y)^2}{2s}-1\right)$$
$$-\frac{4xye^{-\frac{(x+y)^2}{4s}}}{\sqrt{4\pi}(2s)^{5/2}}$$
$$=\frac{1}{\sqrt{4\pi}(2s)^{3/2}}e^{-\frac{(x-y)^2}{4s}}\left(1-e^{-\frac{xy}{s}}\right)\left(\frac{(x-y)^2}{2s}-1\right)-\frac{4xye^{-\frac{(x+y)^2}{4s}}}{\sqrt{4\pi}(2s)^{5/2}},$$

for $s,x,y\in(0,\infty)$. Then,

$$\big|\mathbb{K}(x,y,s)-\mathbb{K}(x,-y,s)\big|\leq C\frac{xye^{-c\frac{(x-y)^2}{s}}}{s^{5/2}},\quad s,x,y\in(0,\infty),\ xy\leq s, \tag{4.56}$$

and

$$\big|\mathbb{K}(x,y,s)-\mathbb{K}(x,-y,s)\big|\leq C\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}},\quad s,x,y\in(0,\infty). \tag{4.57}$$

From (4.56) and (4.57) we deduce that

$$\int_0^{x/2}\int_0^{\infty}\big|\mathbb{K}(x,y,s)-\mathbb{K}(x,-y,s)\big|dsdy\leq C\int_0^{x/2}\int_0^{\infty}\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}}dsdy\leq C, \tag{4.58}$$

for $x\in(0,\infty)$, and

$$\begin{aligned}
&\int_{3x/2}^{\infty}\int_0^{\infty}\big|\mathbb{K}(x,y,s)-\mathbb{K}(x,-y,s)\big|dsdy\\
&\leq C\Big(\int_{3x/2}^{\infty}\int_0^{xy}\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}}dsdy+\int_{3x/2}^{\infty}\int_{xy}^{\infty}\frac{xye^{-c\frac{(x-y)^2}{s}}}{s^{5/2}}dsdy\Big)\\
&\leq C\left(\int_{3x/2}^{\infty}\frac{1}{y}\int_{y/x}^{\infty}\frac{e^{-cu}}{\sqrt{u}}du+\int_{3x/2}^{\infty}xy\int_0^{y/x}\frac{e^{-cu}\sqrt{u}}{y^3}dudy\right)\\
&\leq C\left(\int_{3x/2}^{\infty}\frac{e^{-c\frac{y}{x}}}{y}\int_0^{\infty}\frac{e^{-cu}}{\sqrt{u}}du+x\int_{3x/2}^{\infty}\frac{1}{y^2}\int_0^{\infty}e^{-cu}\sqrt{u}dudy\right)\\
&\leq Cx\int_{3x/2}^{\infty}\frac{dy}{y^2}\leq C,\quad x\in(0,\infty).
\end{aligned}\tag{4.59}$$

On the other hand, we have that

$$\big|\mathbb{K}(x,-y,s)\big|\leq C\frac{e^{-c\frac{(x+y)^2}{s}}}{s^{3/2}},\quad s,x,y\in(0,\infty).$$

We can write

$$\begin{aligned}
\int_{x/2}^{3x/2}\int_0^{\infty}|\mathbb{K}(x,-y,s)|dsdy&\leq C\int_{x/2}^{3x/2}\int_0^{\infty}\frac{e^{-c\frac{(x+y)^2}{s}}}{s^{3/2}}dsdy\\
&\leq C\int_{x/2}^{3x/2}\frac{1}{x+y}\int_0^{\infty}\frac{e^{-u}}{\sqrt{u}}dudy\leq C\quad x\in(0,\infty).
\end{aligned}\tag{4.60}$$

Finally we are going to estimate $\displaystyle\int_{x/2}^{3x/2}\int_0^{\infty}\big|\mathbb{K}_\mu(x,y,s)-\mathbb{K}(x,y,s)\big|dsdy.$

According to (4.51) we obtain

$$\begin{aligned}
&\int_{x/2}^{3x/2}\int_{xy}^{\infty}\big|\mathbb{K}_\mu(x,y,s)-\mathbb{K}(x,y,s)\big|dsdy\leq\int_{x/2}^{3x/2}\int_{xy}^{\infty}\Big|\mathbb{K}_\mu(x,y,s)\Big|dsdy\\
&\quad+\int_{x/2}^{3x/2}\int_{xy}^{\infty}\Big|\mathbb{K}(x,y,s)\Big|dsdy\\
&\leq C\Big(\int_{x/2}^{3x/2}\int_{xy}^{\infty}x^2(xy)^{\mu+1/2}\Big(\frac{y^4}{(x^2+y^2)^2}+\frac{y^2}{x^2+y^2}+1\Big)\frac{e^{-c\frac{x^2+y^2}{s}}}{s^{\mu+3}}dsdy\\
&\quad+\int_{x/2}^{3x/2}\int_{xy}^{\infty}\frac{e^{-c\frac{x^2+y^2}{s}}}{s^{3/2}}dsdy\Big)\\
&\leq C\left(x^2\int_{x/2}^{3x/2}\frac{(xy)^{\mu+1/2}}{(xy)^{\mu+2}}dy+\int_{x/2}^{3x/2}\frac{dy}{\sqrt{xy}}\right)\leq C,\quad x\in(0,\infty).
\end{aligned}\tag{4.61}$$

By using (4.25) we get

$$\left|\mathbb{K}_\mu(x,y,s)-\mathbb{K}(x,y,s)\right|\leq C\frac{e^{-\frac{(x-y)^2}{8s}}}{s^{3/2}},\quad 0<s\leq xy\ \text{ and }\ x/2<y<3x/2,$$

but at this moment it is not sufficient. We need to improve the last estimates.

Since $I_\nu(z)=\frac{e^z}{\sqrt{2\pi z}}\left(1+\frac{1-4\nu^2}{8z}+O\left(\frac{1}{z^2}\right)\right)$, for $z>0$ and $\nu>-1$ (see (4.4)), we can write

$$\begin{aligned}
\mathbb{K}_\mu(x,y,s)&=\frac{x^2}{\sqrt{2\pi}(2s)^{3/2}}\left(\frac{y^3}{(2s)^2x}\frac{2s}{xy}\left(1+\frac{1-4(\mu+2)^2}{4}\frac{s}{xy}+O\left(\left(\frac{s}{xy}\right)^2\right)\right)\right.\\
&-\frac{y}{sx}\left(1+\frac{1-4(\mu+1)^2}{4}\frac{s}{xy}+O\left(\left(\frac{s}{xy}\right)^2\right)\right)\\
&\left.+\frac{1}{2s}\left(1+\frac{1-4\mu^2}{4}\frac{s}{xy}+O\left(\left(\frac{s}{xy}\right)^2\right)\right)\right)e^{-\frac{(x-y)^2}{4s}}\\
&=\frac{x^2}{\sqrt{2\pi}(2s)^{3/2}}\left(\frac{y^2}{2sx^2}-\frac{y}{sx}+\frac{1}{2s}+\frac{1-4\mu^2}{4}\left(\frac{y}{2x^3}-\frac{1}{x^2}+\frac{1}{2xy}\right)\right.\\
&\left.-2\mu\left(\frac{y}{x^3}-\frac{1}{x^2}\right)-\frac{2y}{x^3}+\frac{1}{x^2}+O\left(\frac{s}{x^4}\right)+O\left(\frac{s}{x^3y}\right)+O\left(\frac{s}{x^2y^2}\right)\right)e^{-\frac{(x-y)^2}{4s}}\\
&=\frac{x^2}{\sqrt{2\pi}(2s)^{3/2}}e^{-\frac{(x-y)^2}{4s}}\left(\frac{(y-x)^2}{2sx^2}+\frac{1-4\mu^2}{4}\frac{(y-x)^2}{2x^3y}-2\mu\frac{y-x}{x^3}\right.\\
&\left.-\frac{2y-x}{x^3}+O\left(\frac{s}{x^4}\right)\right),\quad 0<x/2<y<3x/2\text{ and }\ 0<s\leq xy.
\end{aligned}$$

We get

$$\begin{aligned}
\mathbb{K}_\mu(x,y,s)-\mathbb{K}(x,y,s)&=\frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{2\pi}}\left[\frac{(y-x)^2}{(2s)^{5/2}}+\frac{1-4\mu^2}{8}\frac{(y-x)^2}{(2s)^{3/2}xy}\right.\\
&\left.-2\mu\frac{y-x}{(2s)^{3/2}x}-\frac{2y-x}{(2s)^{3/2}x}-\frac{(x-y)^2}{(2s)^{5/2}}+\frac{1}{(2s)^{3/2}}+O\left(\frac{1}{x^2\sqrt{s}}\right)\right]\\
&=\frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{2\pi}}\left[\frac{1-4\mu^2}{8}\frac{(y-x)^2}{(2s)^{3/2}xy}-2(\mu+1)\frac{y-x}{(2s)^{3/2}x}\right.\\
&\left.+O\left(\frac{1}{x^2\sqrt{s}}\right)\right],\quad x/2<y<3x/2,\ \text{ and }\ 0<s\leq xy.
\end{aligned}$$

By using the last equality we deduce that

$$\int_{x/2}^{3x/2}\int_0^{xy}\Big|\mathbb{K}_\mu(x,y,s)-\mathbb{K}(x,y,s)\Big|dsdy \leq C\Big(\int_{x/2}^{3x/2}\int_0^{xy}\frac{e^{-c\frac{(x-y)^2}{s}}}{xy\sqrt{s}}dsdy$$

$$+\int_{x/2}^{3x/2}\int_0^{xy}\frac{e^{-c\frac{(x-y)^2}{s}}\sqrt{|x-y|}}{s^{5/4}x}dsdy+\int_{x/2}^{3x/2}\int_0^{xy}\frac{e^{-c\frac{(x-y)^2}{s}}}{x^2\sqrt{s}}dsdy\Big)$$

$$\leq C\int_{x/2}^{3x/2}\frac{\sqrt{xy}}{xy}dy+C\int_{x/2}^{3x/2}\frac{\sqrt{|x-y|}}{x}\int_0^\infty\frac{e^{-u}}{u^{3/4}\sqrt{|x-y|}}dudy$$

$$+C\int_{x/2}^{3x/2}\frac{\sqrt{xy}}{x^2}dy\leq C\int_{x/2}^{3x/2}\frac{dy}{x}\leq C,\ x\in(0,\infty). \tag{4.62}$$

By combining (4.54), (4.55), (4.58), (4.59), (4.60), (4.61) and (4.62) we obtain

$$\sup_{x\in(0,\infty)}\int_0^\infty\int_0^\infty|T_\mu(x,y,s)|dsdy<\infty.$$

This property implies that the operator

$$T_\mu(f)(t,x)=\int_0^\infty\int_0^\infty T_\mu(x,y,s)f(t-s,y)dsdy$$

is bounded from $L^\infty(\mathbb{R}\times(0,\infty))$ into itself, for every $\mu>-1$.

The operator R_μ is bounded from $L^2(\mathbb{R}\times(0,\infty))$ into itself and the operator R is bounded from $L^2(\mathbb{R}^2)$ into itself, see [73, Theorem 2.3, (B)]. From these facts we deduce that T_μ is bounded from $L^2(\mathbb{R}\times(0,\infty))$ into itself. Hence, interpolation theorem implies that T_μ is bounded from $L^p(\mathbb{R}\times(0,\infty))$ into itself, for every $2\leq p\leq\infty$. By using again [73, Theorem 2.3, (B)] we conclude that R_μ defines a bounded operator from $L^p(\mathbb{R}\times(0,\infty))$ into itself for every $2\leq p<\infty$ and $\mu>-1$.

It is remarkable that the operator R_μ is not selfadjoint in $L^2(\mathbb{R}\times(0,\infty))$.

To simplify we now consider the function

$$M_\mu(x,y,s)=\frac{y^{\mu+5/2}x^{\mu+1/2}}{(2s)^{\mu+2}}\Big(\frac{x^3}{4s^2y}\Big(\frac{xy}{2s}\Big)^{-\mu-1}I_{\mu+2}\Big(\frac{xy}{2s}\Big)-\frac{x}{sy}\Big(\frac{xy}{2s}\Big)^{-\mu}I_{\mu+1}\Big(\frac{xy}{2s}\Big)$$

$$+\frac{1}{2s}\Big(\frac{xy}{2s}\Big)^{-\mu}I_\mu\Big(\frac{xy}{2s}\Big)\Big)e^{-\frac{x^2+y^2}{4s}}-\frac{\partial^2}{\partial x^2}\left(\frac{e^{-\frac{(x-y)^2}{4s}}}{\sqrt{4\pi s}}\right),\quad x,y,s\in(0,\infty).$$

We are going to see that

$$\sup_{x\in(0,\infty)}\int_0^\infty\int_0^\infty|M_\mu(x,y,s)|dyds<\infty.$$

We define

$$K_\mu(x,y,s)=\frac{y^{\mu+5/2}x^{\mu+1/2}}{(2s)^{\mu+2}}\Big(\frac{x^3}{4s^2y}\Big(\frac{xy}{2s}\Big)^{-\mu-1}I_{\mu+2}\Big(\frac{xy}{2s}\Big)$$

$$-\frac{x}{sy}\Big(\frac{xy}{2s}\Big)^{-\mu}I_{\mu+1}\Big(\frac{xy}{2s}\Big)+\frac{1}{2s}\Big(\frac{xy}{2s}\Big)^{-\mu}I_\mu\Big(\frac{xy}{2s}\Big)\Big)e^{-\frac{x^2+y^2}{4s}},$$

for $x, y, s \in (0, \infty)$. From (4.51), (4.52) and (4.53) we deduce that

$$|K_\mu(x,y,s)| \leq C\Big(\frac{xy}{s}\Big)^{\mu+1/2}\frac{y^2}{s^{5/2}}\left(\frac{x^4}{s^2}+\frac{x^2}{s}+1\right)e^{-\frac{x^2+y^2}{4s}}, \tag{4.63}$$

for $s, x, y \in (0, \infty)$, such that $xy \leq s$;

$$|K_\mu(x,y,s)| \leq C\frac{y^2}{s^{5/2}}e^{-\frac{(x-y)^2}{4s}}\left(\frac{x^2}{y^2}+\frac{x}{y}+1\right), \; s,x,y \in (0,\infty), \; xy \geq s; \tag{4.64}$$

and, for every $x, y, s \in (0, \infty)$ and $xy \geq s$,

$$|K_\mu(x,y,s)| \leq C\frac{y^2}{s^{5/2}}e^{-c\frac{\max\{x,y\}^2}{s}}\left(\frac{x^2}{y^2}+\frac{x}{y}+1\right), \; 0<y<x/2, \text{ or } y>3x/2. \tag{4.65}$$

By (4.63) and (4.65) it follows that

$$\begin{aligned}
&\int_0^{x/2}\int_0^\infty |K_\mu(x,y,s)|dsdy \leq C\Big(\int_0^{x/2}\int_0^{xy}\frac{y^2}{s^{5/2}}\left(\frac{x^2}{y^2}+\frac{x}{y}+1\right)e^{-c\frac{x^2}{s}}dsdy\\
&+\int_0^{x/2}\int_{xy}^\infty \frac{(xy)^{\mu+1/2}}{s^{\mu+3}}y^2\left(\frac{x^4}{s^2}+\frac{x^2}{s}+1\right)e^{-\frac{x^2+y^2}{4s}}dsdy\Big)\\
&\leq C\Big(\int_0^{x/2}(x^2+xy+y^2)\int_0^{xy}\frac{e^{-c\frac{x^2}{s}}}{s^{5/2}}dsdy\\
&+\int_0^{x/2}\left(\frac{x^4}{(x^2+y^2)^2}+\frac{x^2}{x^2+y^2}+1\right)\frac{(xy)^{\mu+1/2}y^2}{(x^2+y^2)^{\mu+2}}dy\int_0^\infty e^{-u}u^{\mu+1}du\Big)\\
&\leq C\Big(\int_0^{x/2}\frac{x^2+xy+y^2}{x^3}\int_0^\infty e^{-u}\sqrt{u}du\\
&\quad + x^{\mu+1/2}\int_0^{x/2}\frac{y^{\mu+5/2}}{(x+y)^{2\mu+4}}dy\int_0^\infty e^{-u}u^{\mu+1}du\Big)\\
&\quad\leq C\left(\frac{1}{x}\int_0^{x/2}dy+\frac{1}{x^{\mu+7/2}}\int_0^{x/2}y^{\mu+5/2}dy\right)\leq C, \; x\in(0,\infty);
\end{aligned} \tag{4.66}$$

and

$$\int_{3x/2}^{\infty}\int_0^{\infty}|K_\mu(x,y,s)|dsdy \leq C\Big(\int_{3x/2}^{\infty}\int_0^{xy}\frac{y^2}{s^{5/2}}\left(\frac{x^2}{y^2}+\frac{x}{y}+1\right)e^{-c\frac{y^2}{s}}dsdy$$

$$+\int_{3x/2}^{\infty}\int_{xy}^{\infty}\frac{(xy)^{\mu+1/2}}{s^{\mu+3}}y^2\left(\frac{x^4}{s^2}+\frac{x^2}{s}+1\right)e^{-\frac{x^2+y^2}{4s}}dsdy\Big)$$

$$\leq C\Big(\int_{3x/2}^{\infty}\frac{x^2+xy+y^2}{y^3}\int_{y/x}^{\infty}e^{-cu}\sqrt{u}dudy$$

$$+\int_{3x/2}^{\infty}\int_{xy}^{\infty}\frac{x^{\mu+1/2}y^{\mu+5/2}}{s^{\mu+3}}e^{-c\frac{x^2+y^2}{s}}dsdy\Big)$$

$$\leq C\left(x\int_{3x/2}^{\infty}\frac{e^{-c\frac{y}{x}}}{y^2}dy+x^{\mu+1/2}\int_{3x/2}^{\infty}\frac{dy}{y^{\mu+3/2}}\right)\leq C,\quad x\in(0,\infty), \tag{4.67}$$

provided that $\mu>-1/2$.

Also, for $x\in(0,\infty)$, we have

$$\int_{x/2}^{3x/2}\int_{xy}^{\infty}|K_\mu(x,y,s)|dsdy \leq C\int_{x/2}^{3x/2}\int_{xy}^{\infty}y^2(xy)^{\mu+1/2}\frac{e^{-c\frac{x^2+y^2}{s}}}{s^{\mu+3}}dsdy$$

$$\leq C\int_{x/2}^{3x/2}y^2\frac{(xy)^{\mu+1/2}}{(xy)^{\mu+2}}dy\leq C, \tag{4.68}$$

and, as in (4.62),

$$\int_{x/2}^{3x/2}\int_0^{xy}\Big|K_\mu(x,y,s)-\mathbb{K}(x,y,s)\Big|dsdy\leq C\Big(\int_{x/2}^{3x/2}\int_0^{xy}\frac{e^{-\frac{(x-y)^2}{8s}}}{\sqrt{s}xy}dsdy$$

$$+\int_{x/2}^{3x/2}\int_0^{xy}\frac{e^{-\frac{(x-y)^2}{8s}}\sqrt{|x-y|}}{s^{5/4}y}dsdy+\int_{x/2}^{3x/2}\int_0^{xy}\frac{e^{-\frac{(x-y)^2}{8s}}}{y^2\sqrt{s}}dsdy\Big)\leq C. \tag{4.69}$$

From (4.66), (4.67), (4.68) and (4.69) and by taking into account the other above estimates, we deduce that

$$\sup_{x\in(0,\infty)}\int_0^{\infty}\int_0^{\infty}|M_\mu(x,y,s)|dsdy<\infty.$$

Then, the operator T_μ is bounded from $L^1(\mathbb{R}\times(0,\infty))$ into itself, provided that $\mu>-1/2$.

By invoking interpolation theorem we infer that T_μ is bounded from $L^p(\mathbb{R}\times(0,\infty)$ into itself, for every $1\leq p\leq\infty$ when $\mu>-1/2$. By using again [73, Theorem 2.3, (B)] we conclude that R_μ defines, for every $1<p<\infty$, a bounded operator from $L^p(\mathbb{R}\times(0,\infty))$ into itself and from $L^1(\mathbb{R}\times(0,\infty))$ into $L^{1,\infty}(\mathbb{R}\times(0,\infty))$ provided that $\mu>-1/2$.

By following the same argument as in the previous case, the use of the maximal operator associated to the singular integral R_μ and [73, Theorem 2.3, (B)] allow us to conclude that, for every $f\in L^p(\mathbb{R}\times(0,\infty))$, $1\leq p<\infty$, the limit

$$\lim_{\varepsilon\to0^+}\int_{\Omega_\varepsilon(x)}\mathbb{K}(x,y,s)f(t-s,y)dsdy,$$

exists, for a.e. $(t,x) \in \mathbb{R} \times (0,\infty)$, when $\mu > -1/2$. Moreover, the operator $\mathbb{R}_\mu$ defined by

$$\mathbb{R}_\mu(f)(t,x) = \lim_{\varepsilon \to 0^+} \int_{\Omega_\varepsilon(x)} \mathbb{K}(x,y,s) f(t-s,y)\,ds\,dy,$$

is bounded, for every $1 < p < \infty$, from $L^p(\mathbb{R} \times (0,\infty))$ into itself and from $L^1(\mathbb{R} \times (0,\infty)$ into $L^{1,\infty}(\mathbb{R} \times (0,\infty)$ provided that $\mu > -1/2$.

Our next objective is to complete the study of the boundedness of the operator R_μ when $-1 < \mu \leq -1/2$.

We have that, for every $f \in L^2(\mathbb{R} \times (0,\infty))$, $R_\mu f = \mathcal{F}^{-1} h_{\mu+2}\left(\frac{z^2}{z^2+i\rho} \mathcal{F} h_\mu(f)\right)$.

Then, the adjoint R_μ^* of R_μ is given by

$$R_\mu^* f = \mathcal{F}^{-1} h_\mu \left(\frac{z^2}{z^2 - i\rho} h_{\mu+2} \mathcal{F} f\right) = \left[\mathcal{F}^{-1} h_\mu \left(\frac{z^2}{z^2 + i\rho} h_{\mu+2} \mathcal{F} \tilde{f}\right)\right]^{\sim},$$

$f \in L^2(\mathbb{R} \times (0,\infty))$, where $\tilde{f}(t,x) = f(-t,x)$, $t \in \mathbb{R}$ and $x \in (0,\infty)$.

We consider the operator $\mathcal{H}_\mu f = \mathcal{F}^{-1} h_\mu \left(\frac{z^2}{z^2+i\rho} h_{\mu+2} \mathcal{F} f\right)$, $f \in L^2(\mathbb{R} \times (0,\infty))$. Since $h_\alpha^2 = I$ in $L^2((0,\infty))$, for every $\alpha > -1$ we can write

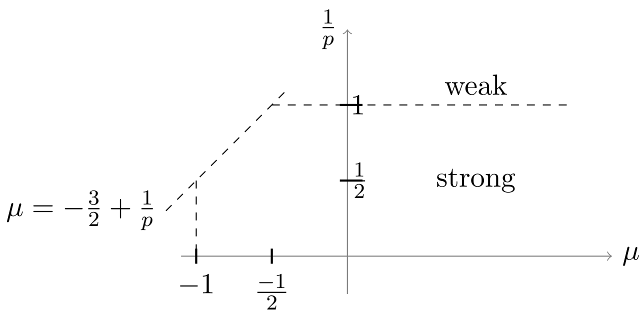
$$\mathcal{H}_\mu = h_\mu h_{\mu+2} \mathcal{F}^{-1} h_{\mu+2} \left(\frac{z^2}{z^2 + i\rho} h_\mu \mathcal{F} h_\mu h_{\mu+2}\right) = S_\mu R_\mu S_\mu,$$

where $S_\mu = h_\mu h_{\mu+2}$.

According to [66, Theorem 2.1], S_μ defines a bounded operator from $L^p((0,\infty))$ into itself when $1 < p < \infty$, $\mu > -1/2 - 1/p$ and $\mu > -1$. By taking into account that R_μ is bounded from $L^p(\mathbb{R} \times (0,\infty))$ into itself when $2 \leq p < \infty$, we conclude that R_μ^* is bounded from $L^p(\mathbb{R} \times (0,\infty))$ into itself provided that $\mu > -1/2 - 1/p$ and $2 \leq p < \infty$. Duality implies that R_μ defines a bounded operator from $L^p(\mathbb{R} \times (0,\infty))$ into itself provided that $1 < p \leq 2$ and $\mu > 1/p - 3/2$.

Therefore, we have proved that

- R_μ is bounded from $L^p(\mathbb{R} \times (0,\infty))$ into itself when
  1. $\mu > -1/2$ and $1 < p < \infty$.
  2. $-1 < \mu \leq -1/2$ and $p > \frac{1}{\mu+3/2}$.
- R_μ is bounded from $L^1(\mathbb{R} \times (0,\infty))$ into $L^{1,\infty}(\mathbb{R} \times (0,\infty))$, when $\mu > -1/2$.



Remark 4.124. *L^p-boundedness properties for the Riesz transforms established in Theorem 4.122 can be seen as Sobolev estimates in our parabolic Bessel setting. Note that the auxiliar operator δ_μ plays the role of derivatives to define correct Sobolev spaces in the Bessel setting (see [17]). On the other hand,* (4) *and* (5) *in Theorem 4.122 remember the so called pencil phenomenon that appears related to the L^p-boundedness properties of harmonic analysis operators in Laguerre settings (see [45], [61], [62], and [65]).*

4.1.3 Boundedness of Bessel Riesz transforms on mixed weighted L^p spaces.

Our next aim is to prove mixed weighted norm inequalities for the Riesz transforms R_μ and $\widetilde{R_\mu}$ (Theorem 4.127). The main tool to prove these results is the **vector-valued Calderón-Zygmund theory** (see [76]). For our purposes, first we have to prove some previous results.

We firstly consider the Riesz transform R_μ defined by

$$\begin{aligned} R_\mu(f)(t,x) &= \lim_{\epsilon\to 0^+}\int_{\Omega_\epsilon(x)} \mathbb{K}_\mu(s,x,y)f(t-s,y)dsdy \\ &\quad + f(t,x)\frac{1}{\sqrt{\pi}}\int_1^\infty e^{-s^2/4}ds, \ \ a.e.\ (t,x)\in\mathbb{R}\times(0,\infty), \end{aligned}$$

for every $f\in L^p(\mathbb{R}\times(0,\infty))$ $1\le p<\infty$, where $\mathbb{K}_\mu(s,x,y)=\delta_{\mu+1}\delta_\mu W_s^\mu(x,y)$, $s,x,y\in(0,\infty)$. We have that, for each $f\in L^p(\mathbb{R}\times(0,\infty))$, $1\le p<\infty$,

$$R_\mu(f)(t,x)=\int_0^\infty\int_0^\infty \mathbb{K}_\mu(s,x,y)f(t-s,y)dsdy, \ \ (t,x)\notin \operatorname{supp}(f).$$

Proposition 4.125. *Let $\mu>-1/2$ and $1<p<\infty$. The operator R_μ can be extended to $L^q(\mathbb{R},L^p((0,\infty)))$ as a bounded operator from $L^q(\mathbb{R},L^p((0,\infty)))$ into itself, for $1<q<\infty$ and from $L^1(\mathbb{R},L^p((0,\infty)))$ into $L^{1,\infty}(\mathbb{R},L^p((0,\infty)))$.*

Proof. We define, for every $s\in(0,\infty)$,

$$\mathbb{T}_s(F)(x)=\int_0^\infty \mathbb{K}_\mu(s,x,y)F(y)dy, \ \ x\in(0,\infty),$$

for every $F\in L^p((0,\infty))$.

Note that, according to (4.24), (4.26) and (4.27),

$$|\mathbb{K}_\mu(s,x,y)|\le C\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}}, \ \ s,x,y\in(0,\infty). \tag{4.70}$$

Then,

$$\int_0^\infty |\mathbb{K}_\mu(s,x,y)|dy\le C\int_{\mathbb{R}}\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}}dy\le\frac{C}{s}, \ \ s,x\in(0,\infty).$$

and also

$$\int_0^\infty |\mathbb{K}_\mu(s,x,y)|dx \leq \frac{C}{s}, \quad s,y \in (0,\infty).$$

It follows that, for every $s \in (0,\infty)$, $\mathbb{T}_s$ defines a bounded operator from $L^p((0,\infty))$ into itself and $\|\mathbb{T}_s\|_{p\to p} \leq \frac{C}{s}$.

For every $t,s \in \mathbb{R}$, we define

$$\mathbb{H}(t,s)(F) = \begin{cases} \mathbb{T}_{t-s}(F), & t > s \\ 0, & t \leq s, \end{cases}$$

for $F \in L^p((0,\infty))$.

Thus, for every $t,s \in \mathbb{R}$, $\mathbb{H}(t,s) \in \mathcal{L}(L^p((0,\infty)))$ and $\|\mathbb{H}(t,s)\|_{p\to p} \leq \frac{C}{|t-s|}$.

We consider, for every $g \in C_c^\infty(\mathbb{R}\times(0,\infty)) \subset L^p(\mathbb{R}, L^p((0,\infty)))$,

$$\beta_\mu(g)(t) = \int_{\mathbb{R}} \mathbb{H}(t,s)(g(s))ds, \quad t \notin \text{supp}(g).$$

Note that if $g \in C_c^\infty(\mathbb{R}\times(0,\infty))$ and $t \notin \text{supp}(g)$, then

$$\begin{aligned}\int_{\mathbb{R}} \|\mathbb{H}(t,s)(g(s))\|_{L^p((0,\infty))}ds &\leq C\int_{\text{supp}(g)} \frac{\|g(s)\|_{L^p((0,\infty))}}{t-s}ds \\ &\leq C\left(\int_{\text{supp}(g)} \frac{ds}{|t-s|^{q'}}\right)^{1/q'} \|g\|_{L^q(\mathbb{R},L^p((0,\infty)))} < \infty, \end{aligned} \tag{4.71}$$

and the integral $\int_{\mathbb{R}} \mathbb{H}(t,s)(g(s))ds$ converges in the $L^q(\mathbb{R})$-Bochner sense.

We established in Theorem 4.122 that R_μ is bounded from $L^p(\mathbb{R}\times(0,\infty)) = L^p(\mathbb{R}, L^p((0,\infty)))$ into itself.

Let $g \in C_c^\infty(\mathbb{R}\times(0,\infty))$ and $\ell \in (L^p((0,\infty)))' = L^{p'}((0,\infty))$. According to the well-known properties of Bochner integrals we have that

$$\begin{aligned}\langle \ell, \int_{\mathbb{R}} \mathbb{H}(t,s)(g(s))ds\rangle &= \int_{\mathbb{R}} \langle \ell, \mathbb{H}(t,s)(g(s))\rangle ds \\ &= \int_{-\infty}^t \int_0^\infty \ell(x) \int_0^\infty \mathbb{K}_\mu(t-s,x,y)g(s,y)dydxds \\ &= \int_0^\infty \ell(x) \int_{-\infty}^t \int_0^\infty \mathbb{K}_\mu(t-s,x,y)g(s,y)dydsdx, \ t \notin \text{supp} g.\end{aligned}$$

The interchange of the order of integration is justified by (4.71). Note that

$$\begin{aligned}&\left\|\int_{-\infty}^t \int_0^\infty |\mathbb{K}_\mu(t-s,x,y)||g(s,y)|dyds\right\|_{L^p((0,\infty))} \\ &\leq C\left(\int_{\text{supp} g(\cdot)} \frac{ds}{|t-s|^{q'}}\right)^{\frac{1}{q'}} \|g\|_{L^q(\mathbb{R},L^p((0,\infty)))}, \quad t \notin \text{supp}(g).\end{aligned}$$

We conclude that, for every $t \notin \operatorname{supp}(g)$,

$$\left(\int_{\mathbb{R}} \mathbb{H}(t,s)(g(s))ds\right)(x) = \int_0^\infty \int_0^\infty \mathbb{K}_\mu(s,x,y)g(t-s,y)dyds, \text{ a.e } x \in (0,\infty).$$

Suppose that $t_1, t_2, s \in (0,\infty)$, being $|t_1 - s| > 2|t_1 - t_2|$. Then, $s > \max\{t_1, t_2\}$ or $s < \min\{t_1, t_2\}$. Let $g \in L^p((0,\infty))$. Assume that $s < \min\{t_1, t_2\}$. We have that

$$[\mathbb{H}(t_1,s)(g) - \mathbb{H}(t_2,s)(g)](x) = \int_0^\infty (\mathbb{K}_\mu(t_1 - s,x,y) - \mathbb{K}_\mu(t_2 - s,x,y))g(y)dy,$$

for $x \in (0,\infty)$. Note that $\mathbb{H}(t_1,s)(g) - \mathbb{H}(t_2,s)(g) = 0$ when $s > \max\{t_1,t_2\}$. According to (4.34) we get

$$\begin{aligned}|\mathbb{K}_\mu(t_1 - s,x,y) - \mathbb{K}_\mu(t_2 - s,x,y)| &\leq \left|\frac{\partial}{\partial u}\mathbb{K}_\mu(u,x,y)\right| |t_1 - t_2| \\ &\leq C\frac{|t_1 - t_2|}{(\sqrt{u} + |x-y|)^5}, \quad x,y \in (0,\infty),\end{aligned}$$

for some $u \in (\min\{t_1,t_2\} - s, \max\{t_1,t_2\} - s)$. Suppose that $t_1 < t_2$. Then, $u > t_1 - s$ and $t_2 - s = t_2 - t_1 + t_1 - s < 3(t_1 - s)/2 < 3u/2$. We obtain

$$|\mathbb{K}_\mu(t_1 - s,x,y) - \mathbb{K}_\mu(t_2 - s,x,y)| \leq C\frac{|t_1 - t_2|}{(\sqrt{t_i - s} + |x-y|)^5}, \; i = 1,2, \; x,y \in (0,\infty). \tag{4.72}$$

It follows that

$$\begin{aligned}&\int_0^\infty |\mathbb{K}_\mu(t_1 - s,x,y) - \mathbb{K}_\mu(t_2 - s,x,y)|dy \\ &\quad \leq C|t_1 - t_2| \int_0^\infty \frac{dy}{(\sqrt{t_1 - s} + |x-y|)^5} \leq C\frac{|t_1 - t_2|}{|t_1 - s|^2}, \quad x \in (0,\infty).\end{aligned}$$

Also,

$$\int_0^\infty |\mathbb{K}_\mu(t_1 - s,y,x) - \mathbb{K}_\mu(t_2 - s,y,x)|dy \leq C\frac{|t_1 - t_2|}{|t_1 - s|^2}, \quad x \in (0,\infty).$$

We conclude that $\mathbb{H}(t_1,s) - \mathbb{H}(t_2,s) \in \mathcal{L}(L^p((0,\infty)))$ and

$$\|\mathbb{H}(t_1,s) - \mathbb{H}(t_2,s)\|_{L^p((0,\infty)) \to L^p((0,\infty))} \leq C\frac{|t_1 - t_2|}{|t_1 - s|^2}.$$

By using vector valued Calderón-Zygmund theory we deduce that the operator R_μ can be extended to $L^q(\mathbb{R}, L^p((0,\infty)))$ as a bounded operator from

- $L^q(\mathbb{R}, L^p((0,\infty)))$ into itself for every $1 < q < \infty$,
- $L^1(\mathbb{R}, L^p((0,\infty)))$ into $L^{1,\infty}(\mathbb{R}, L^p((0,\infty)))$.

□

On the other hand, recall that Riesz transform $\widetilde{R_\mu}$ is defined for every $f \in L^p(\mathbb{R}\times(0,\infty))$, $1 \le p < \infty$, by

$$\widetilde{R_\mu}(f)(t,x) = \lim_{\epsilon\to 0^+} \int_{\Omega_\epsilon(x)} \mathcal{K}_\mu(s,x,y) f(t-s,y)dyds + f(t,x)\frac{1}{\sqrt{\pi}}\int_0^1 e^{-s^2/4}ds,$$

a.e. $(t,x) \in \mathbb{R}\times(0,\infty)$, where $\mathcal{K}_\mu(s,x,y) = \frac{\partial}{\partial s}(W_s^\mu(x,y))$, $s,x,y \in (0,\infty)$.

We first prove the following result.

Proposition 4.126. *Let $\mu > -1/2$ and $1 < p < \infty$. The operator $\widetilde{R_\mu}$ can be extended to $L^q(\mathbb{R}, L^p((0,\infty)))$ as a bounded operator from $L^q(\mathbb{R}, L^p((0,\infty)))$ into itself, for $1 < q < \infty$ and from $L^1(\mathbb{R}, L^p((0,\infty)))$ into $L^{1,\infty}(\mathbb{R}, L^p((0,\infty)))$.*

Proof. This result can be proved by proceeding as in the proof of Proposition 4.125. We define, for every $s \in (0,\infty)$,

$$\mathcal{T}_s(F)(x) = \int_0^\infty \mathcal{K}_\mu(s,x,y)F(y)dy, \quad x \in (0,\infty).$$

By (4.36), we have that

$$|\mathcal{K}_\mu(s,x,y)| \le C\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}}, \quad s,x,y \in (0,\infty). \tag{4.73}$$

Then, for every $s \in (0,\infty)$, $\mathbb{T}_s$ defines a bounded operator from $L^p((0,\infty))$ into itself and $\|\mathbb{T}_s\|_{L^p((0,\infty))\to L^p((0,\infty))} \le \frac{C}{s}$.

For every $t,s \in \mathbb{R}$, we define

$$\mathcal{H}(t,s)(F) = \begin{cases} \mathcal{T}_{t-s}(F), & t > s \\ 0, & t \le s, \end{cases}$$

for $F \in L^p((0,\infty))$.

Thus, for every $t,s \in \mathbb{R}$, $\mathcal{H}(t,s) \in \mathcal{L}(L^p((0,\infty)))$ and $\|\mathcal{H}(t,s)\|_{p\to p} \le \frac{C}{|t-s|}$.

Let $1 < q < \infty$, We consider, for every $g \in L^q(\mathbb{R}, L^p((0,\infty)))$,

$$\gamma_\mu(g)(t) = \int_{\mathbb{R}} \mathcal{H}(t,s)(g(s))ds, \quad t \notin \operatorname{supp}(g).$$

In Theorem 4.122 we proved that the Riesz transformation $\widetilde{R_\mu}$ is bounded $L^p(\mathbb{R}\times(0,\infty)) = L^p(\mathbb{R}, L^p((0,\infty)))$.

We have that, for every $t \notin \operatorname{supp}(g)$,

$$\left(\int_{\mathbb{R}} \mathcal{H}(t,s)(g(s))ds\right)(x) = \int_0^\infty\int_0^\infty \mathcal{K}_\mu(s,x,y)g(t-s,y)dyds, \text{a.e } x \in (0,\infty).$$

Also, if $t_1,t_2,s \in (0,\infty)$, being $|t_1 - s| > 2|t_1 - t_2|$, $\mathcal{H}(t_1,s) - \mathcal{H}(t_2,s) \in \mathcal{L}(L^p((0,\infty)))$.

$$\|\mathcal{H}(t_1,s) - \mathcal{H}(t_2,s)\|_{p\to p} \le C\frac{|t_1-t_2|}{|t_1-s|^2}.$$

By invoking vector-valued Calderón-Zygmund theory we prove that the operator $\widetilde{R_\mu}$ can be extended to $L^q(\mathbb{R}, L^p((0,\infty)))$ as a bounded operator from

- $L^q(\mathbb{R}, L^p((0,\infty)))$ into itself for every $1 < q < \infty$,
- $L^1(\mathbb{R}, L^p((0,\infty)))$ into $L^{1,\infty}(\mathbb{R}, L^p((0,\infty)))$.

□

Now we can establish the following mixed weighted norm inequalities for Riesz transforms R_μ and $\widetilde{R_\mu}$. For every $1 \le p < \infty$, we denote the classical classes of Muckenhoupt weights by $A_p(\Omega)$, where $\Omega = (0,\infty)$ or $\Omega = \mathbb{R}$.

Theorem 4.127. *Assume that $\mu > 1/2$ or $\mu = -1/2$. If $1 < p < \infty$ and $v \in A_p((0,\infty))$, then the Riesz transforms R_μ and $\widetilde{R_\mu}$ can be extended from $L^2(\mathbb{R}\times(0,\infty)) \cap L^q(\mathbb{R}, u, L^p((0,\infty), v))$ to $L^q(\mathbb{R}, u, L^p((0,\infty), v))$ as a bounded operator from $L^q(\mathbb{R}, u, L^p((0,\infty), v))$ into itself, provided that $1 < q < \infty$ and $u \in A_q(\mathbb{R})$; and, for every $u \in A_1(\mathbb{R})$, from $L^2(\mathbb{R} \times (0,\infty)) \cap L^1(\mathbb{R}, u, L^p((0,\infty), v))$ to $L^1(\mathbb{R}, u, L^p((0,\infty), v))$ as a bounded operator from $L^1(\mathbb{R}, u, L^p((0,\infty), v))$ into $L^{1,\infty}(\mathbb{R}, u, L^p((0,\infty), v))$.*

Proof of Theorem 4.127.
We prove first the results regarding R_μ. Let $s \in (0,\infty)$. We consider again the operator

$$\mathbb{T}_s(F)(x) = \int_0^\infty \mathbb{K}_\mu(s,x,y)F(y)dy, \;\; x \in (0,\infty), \; F \in L^p((0,\infty)).$$

According to (4.70) we get

$$|\mathbb{K}_\mu(s,x,y)| \le C\frac{e^{-c\frac{(x-y)^2}{s}}}{s^{3/2}} \le \frac{C}{s|x-y|}, \;\; x \neq y, \;\; s,x,y \in (0,\infty),$$

Also by (4.32) and (4.33) we obtain

$$\begin{aligned}|\partial_x \mathbb{K}_\mu(x,y,s)| + |\partial_y \mathbb{K}_\mu(x,y,s)| &\le \frac{C}{(\sqrt{s}+|x-y|)^4}\\ &\le \frac{C}{s|x-y|^2}, \;\; x \neq y, \;\; s,x,y \in (0,\infty),\end{aligned}$$

provided that $\mu > 1/2$ or $\mu = -1/2$.

Since $\mathbb{T}_s$ is bounded from, for instance, $L^2((0,\infty))$ into itself and $\|\mathbb{T}_s\|_{2\to 2} \le \frac{C}{s}$, Calderón-Zygmund theory implies that, for every $1 < p < \infty$ and $\omega \in A_p((0,\infty))$, the operator $\mathbb{T}_s$ can be extended to $L^p((0,\infty),\omega)$ as a bounded operator from $L^p((0,\infty),\omega)$ into itself and $\|\mathbb{T}_s\|_{L^p((0,\infty),\omega)\to L^p((0,\infty),\omega)} \le \frac{C}{s}$, provided that $\mu > 1/2$ or $\mu = -1/2$.

In the sequel we assume that $\mu > 1/2$ or $\mu = -1/2$.

Let $1 < p < \infty$ and $\omega \in A_p((0,\infty))$. We define as above, for every $t, s \in \mathbb{R}$,

$$\mathbb{H}(t,s)(F) = \begin{cases} \mathbb{T}_{t-s}(F), & t > s \\ 0, & t \le s, \end{cases}$$

for $F \in L^p((0,\infty))$. Also, for every $g \in L^p(\mathbb{R}, L^p((0,\infty),\omega))$, we consider

$$\beta_\mu(g)(t) = \int_{\mathbb{R}} \mathbb{H}(t,s)(g(s))ds, \;\; t \notin \operatorname{supp}(g).$$

We have that, for every $1 < p < \infty$ and $\omega \in A_p((0,\infty))$,

$$\|\mathbb{H}(t,s)\|_{L^p((0,\infty),\omega)\to L^p((0,\infty),\omega)} \leq \frac{C}{|t-s|}, \quad t,s \in (0,\infty), \ t \neq s.$$

Then, we infer that, for every $g \in C_c^\infty(\mathbb{R}\times(0,\infty))$ and $t \notin supp\, g$,

$$[\beta_\mu(g)(t)](x) = R_\mu(g)(t,x), \quad a.e. \ x \in (0,\infty).$$

According Theorem 4.122 (2), R_μ is bounded from $L^p(\mathbb{R}\times(0,\infty),W) = L^p(\mathbb{R}, L^p((0,\infty),\omega))$ into itself, where $W(t,x) = \omega(x)$, $(t,x) \in \mathbb{R}\times(0,\infty)$.

Suppose that $t_1, t_2, s \in (0,\infty)$, being $|t_1 - s| > 2|t_1 - t_2|$. We are going to see that,

$$\|\mathbb{H}(t_1,s) - \mathbb{H}(t_2,s)\|_{L^p((0,\infty),\omega)\to L^p((0,\infty),\omega)} \leq C\frac{|t_1-t_2|}{|t_1-s|^2}.$$

We have proved that,

$$\|\mathbb{H}(t_1,s) - \mathbb{H}(t_2,s)\|_{L^p((0,\infty))\to L^p((0,\infty))} \leq C\frac{|t_1-t_2|}{|t_1-s|^2}.$$

From (4.72) we deduce that

$$\begin{aligned} |\mathbb{K}_\mu(t_1-s,x,y) - \mathbb{K}_\mu(t_2-s,x,y)| &\leq C\frac{|t_1-t_2|}{(\sqrt{t_1-s}+|x-y|)^5} \\ &\leq C\frac{|t_1-t_2|}{|t_1-s|^2}\frac{1}{|x-y|}, \quad x,y \in (0,\infty), \quad x \neq y. \end{aligned}$$

Our objective is to see that, for $x, y \in (0,\infty), \ x \neq y$,

$$\left|\frac{\partial}{\partial x}\left(\mathbb{K}_\mu(t_1-s,x,y) - \mathbb{K}_\mu(t_2-s,x,y)\right)\right| \leq C\frac{|t_1-t_2|}{|t_1-s|^2}\frac{1}{|x-y|^2}$$

and

$$\left|\frac{\partial}{\partial y}\left(\mathbb{K}_\mu(t_1-s,x,y) - \mathbb{K}_\mu(t_2-s,x,y)\right)\right| \leq C\frac{|t_1-t_2|}{|t_1-s|^2}\frac{1}{|x-y|^2}.$$

Assume that $s < \min\{t_1,t_2\}$. We can write

$$\left|\frac{\partial}{\partial x}\left(\mathbb{K}_\mu(t_1-s,x,y) - \mathbb{K}_\mu(t_2-s,x,y)\right)\right| \leq C\left|\frac{\partial^2}{\partial u \partial x}\mathbb{K}_\mu(u,x,y)\right||t_1-t_2|,$$

for some $u \in (\min\{t_1,t_2\}-s, \max\{t_1,t_2\}-s)$.

We have that (see (4.28))

$$\partial_x \mathbb{K}_\mu(s,x,y) = \frac{x^{\mu+5/2}y^{\mu+1/2}}{(2s)^{\mu+3}} e^{-\frac{x^2+y^2}{4s}} \left(\frac{xy^6}{(2s)^4} \left(\frac{xy}{2s}\right)^{-\mu-3} I_{\mu+3}\left(\frac{xy}{2s}\right) \right.$$
$$+ \left(\frac{xy}{2s}\right)^{-\mu-2} I_{\mu+2}\left(\frac{xy}{2s}\right) \frac{xy}{2s} \left(-\frac{3y^3}{(2s)^2} + \frac{\mu+5/2}{2s}\frac{y^3}{x^2}\right)$$
$$+ \left(\frac{xy}{2s}\right)^{-\mu-1} I_{\mu+1}\left(\frac{xy}{2s}\right) \frac{xy}{2s} \left(\frac{3y}{2s} - \frac{2(\mu+5/2)y}{x^2}\right)$$
$$\left. + \left(\frac{xy}{2s}\right)^{-\mu} I_{\mu}\left(\frac{xy}{2s}\right) \left(-\frac{x}{2s} + \frac{\mu+5/2}{x}\right)\right), \quad x,y,s \in (0,\infty).$$

Let $x, y \in (0,\infty)$. We define the function

$$F_{x,y}(z) = \frac{x^{\mu+5/2}y^{\mu+1/2}}{(2z)^{\mu+3}} e^{-\frac{x^2+y^2}{4z}} \left(\frac{xy^6}{(2z)^4} \left(\frac{xy}{2z}\right)^{-\mu-3} I_{\mu+3}\left(\frac{xy}{2z}\right) \right.$$
$$+ \left(\frac{xy}{2z}\right)^{-\mu-2} I_{\mu+2}\left(\frac{xy}{2z}\right) \frac{xy}{2z} \left(-\frac{3y^3}{(2z)^2} + \frac{\mu+5/2}{2z}\frac{y^3}{x^2}\right)$$
$$+ \left(\frac{xy}{2z}\right)^{-\mu-1} I_{\mu+1}\left(\frac{xy}{2z}\right) \frac{xy}{2z} \left(\frac{3y}{2z} - \frac{2(\mu+5/2)y}{x^2}\right)$$
$$\left. + \left(\frac{xy}{2z}\right)^{-\mu} I_{\mu}\left(\frac{xy}{2z}\right) \left(-\frac{x}{2z} + \frac{\mu+5/2}{x}\right)\right), \quad z \in \mathbb{C} \setminus (-\infty, 0].$$

$F_{x,y}$ is holomorphic in $\mathbb{C} \setminus (-\infty, 0]$.

According to (4.3) and (4.4), by proceeding as in the proof of (4.32), we obtain

$$|F_{x,y}(z)| \leq C \frac{e^{-c\frac{(x-y)^2}{|z|}}}{|z|^2}, \quad |Arg z| \leq \frac{\pi}{4}.$$

By using Cauchy integral formula we obtain that

$$\left|\frac{d}{dt} F_{x,y}(t)\right| \leq \frac{C}{t^3} e^{-\frac{(x-y)^2}{32t}} \leq \frac{C}{(\sqrt{t} + |x-y|)^6}, \quad t > 0.$$

By proceeding as above we get

$$|\partial_x (\mathbb{K}_\mu(t_1 - s, x, y) - \mathbb{K}_\mu(t_2 - s, x, y))| \leq C \frac{|t_1 - t_2|}{|t_1 - s|^2} \frac{1}{|x-y|^2},$$

and

$$|\partial_y (\mathbb{K}_\mu(t_1 - s, x, y) - \mathbb{K}_\mu(t_2 - s, x, y))| \leq C \frac{|t_1 - t_2|}{|t_1 - s|^2} \frac{1}{|x-y|^2},$$

for $x, y \in (0,\infty)$, $x \neq y$.

Calderón-Zygmund theory leads to

$$\|\mathbb{H}(t_1, s) - \mathbb{H}(t_2, s)\|_{L^p((0,\infty),\omega) \to L^p((0,\infty),\omega)} \leq C \frac{|t_1 - t_2|}{|t_1 - s|^2}.$$

In a similar way we can see that

$$\|\mathbb{H}(t,s_1)-\mathbb{H}(t,s_2)\|_{L^p((0,\infty),\omega)\to L^p((0,\infty),\omega)}\leq C\frac{|s_1-s_2|}{|s_1-t|^2},$$

provided $|s_1-t|\geq 2|s_1-s_2|$.

Calderón-Zygmund theory implies that, for every $\omega\in A^p((0,\infty))$, $1<p<\infty$, R_μ defines a bounded operator

- from $L^q(\mathbb{R},v,L^p((0,\infty),\omega))$ into itself, for every $1<q<\infty$ and $v\in A^q(\mathbb{R})$,
- from $L^1(\mathbb{R},v,L^p((0,\infty),\omega))$ into $L^{1,\infty}(\mathbb{R},v,L^p((0,\infty),\omega))$, for every $v\in A^1(\mathbb{R})$,

provided that $\mu=-1/2$ and $\mu>1/2$.

In particular, for every $1<p<\infty$, $v\in A^p(\mathbb{R})$ and $\omega\in A^p((0,\infty))$, R_μ defines a bounded operator from $L^p(\mathbb{R}\times(0,\infty),v\omega)$ into itself, when $\mu=-1/2$ or $\mu>1/2$.

We had proved that, for every $1<p<\infty$ and $W\in A^p_*(\mathbb{R}\times(0,\infty))$, R_μ defines a bounded operator from $L^p(\mathbb{R}\times(0,\infty),W)$ into itself.

The proof of Theorem 4.127 for the Riesz transformation R_μ is finished.

Our next objective is to get mixed norm inequalities for $\widetilde{R_\mu}$. Suppose now that $\mu>1/2$ or $\mu=-1/2$.

By (4.73), we get

$$|\mathcal{K}_\mu(s,x,y)|\leq\frac{C}{s|x-y|},\quad s,x,y\in(0,\infty),\ \text{ and }\ x\neq y.$$

From (4.37), we have that

$$\left|\frac{\partial}{\partial x}\mathcal{K}_\mu(s,x,y)\right|\leq C\frac{e^{-c\frac{(x-y)^2}{s}}}{s^2}\leq\frac{C}{s|x-y|^2},\quad s,x,y\in(0,\infty).$$

By symmetry, we also have that

$$\left|\frac{\partial}{\partial y}\mathcal{K}_\mu(s,x,y)\right|\leq\frac{C}{s|x-y|^2},\quad s,x,y\in(0,\infty).$$

By using Calderón-Zygmund theory we deduce that, for every $1<p<\infty$ and $\omega\in A^p((0,\infty))$, $\mathcal{T}_s$ defines a bounded operator from $L^p((0,\infty),\omega)$ into itself and $\|\mathcal{T}_s\|_{L^p((0,\infty),\omega)\to L^p((0,\infty),\omega)}\leq\frac{C}{s}$, for each $s\in(0,\infty)$.

We consider again, for every $t,s\in\mathbb{R}$,

$$\mathcal{H}(t,s)=\begin{cases}\mathcal{T}_{t-s}, & t>s\\ 0, & t\leq s.\end{cases}$$

Let $1<p<\infty$ and $\omega\in A^p((0,\infty))$. For every $t,s\in\mathbb{R}$, $\mathcal{H}(t,s)$ defines a bounded operator from $L^p((0,\infty),\omega)$ into itself and $\|\mathcal{H}(t,s)\|_{L^p((0,\infty),\omega)\to L^p((0,\infty),\omega)}\leq\frac{C}{|t-s|}$.

Let $1 < q < \infty$. We consider, for every $g \in L^q(\mathbb{R}, L^p((0,\infty)),\omega))$,

$$\gamma_\mu(g)(t) = \int_{\mathbb{R}} \mathcal{H}(t,s)(g(s))ds, \quad t \notin \text{supp}(g).$$

The last integral is absolutely convergent in the $L^p((0,\infty),\omega)$-Bochner sense. Moreover, by proceeding as above we get that, if $g \in L^q(\mathbb{R}, L^p((0,\infty)),\omega)$ and $t \notin \text{supp}(g)$, then

$$[\gamma_\mu(g)(t)](x) = \int_0^\infty \int_0^\infty \mathcal{K}_\mu(s,x,y)g(t-s,y)dyds, \quad a.e.\ x \in (0,\infty).$$

Suppose that $t_1, t_2, s \in \mathbb{R}$, being $|t_1 - s| > 2|t_1 - t_2|$. Then, we have that $\mathcal{H}(t_1,s) - \mathcal{H}(t_2,s)$ defines a bounded operator from $L^p((0,\infty),\omega)$ into itself and

$$\|\mathcal{H}(t_1,s) - \mathcal{H}(t_2,s)\|_{L^p((0,\infty),\omega)\to L^p((0,\infty),\omega)} \le C\frac{|t_1-t_2|}{|s-t_1|^2}.$$

Moreover, according to Theorem 4.122 (2), the operator $\widetilde{R_\mu}$ is bounded from $L^p(\mathbb{R} \times (0,\infty), W) = L^p(\mathbb{R}, L^p((0,\infty),\omega))$, where $W(t,x) = \omega(x)$, $(t,x) \in \mathbb{R} \times (0,\infty)$, because $W \in A_p^*(\mathbb{R} \times (0,\infty))$.

Again, according to Calderón-Zygmund theory we deduce that $\widetilde{R_\mu}$ defines, for every $1 < q < \infty$ and $v \in A^q(\mathbb{R})$, a bounded operator from $L^q(\mathbb{R}, v, L^p((0,\infty),\omega))$ into itself and from $L^1(\mathbb{R}, v, L^p((0,\infty),\omega))$ into $L^{1,\infty}(\mathbb{R}, v, L^p((0,\infty),\omega))$, for every $v \in A^1(\mathbb{R})$.

The same remark at the end of the study of the mixed norm inequalities for R_μ is now in order with respect to $\widetilde{R_\mu}$. □

Remark 4.128. *Note that from Theorem 4.127 we can deduce that, if $\mu > 1/2$ or $\mu = -1/2$, R_μ and $\widetilde{R_\mu}$ define bounded operators from $L^p(\mathbb{R} \times (0,\infty), uv)$ into itself, for every $1 < p < \infty$, $u \in A_p(\mathbb{R})$ and $v \in A_p((0,\infty))$. Moreover, $uv \in A_p^*(\mathbb{R} \times (0,\infty))$ provided that $u \in A_p(\mathbb{R})$ and $v \in A_p((0,\infty))$, but $A_p^*(\mathbb{R} \times (0,\infty)) \neq A_p(\mathbb{R}) \cdot A_p((0,\infty))$, when $1 < p < \infty$. Hence, Theorem 4.122 (2) is not a special case of strong type results in Theorem 4.127.*

4.2 Results for the solutions in the space $(0,\infty) \times (0,\infty)$.

We now consider the following Cauchy problem associated with (4.1):

$$\begin{cases} \partial_t u(t,x) = \Delta_\mu u(t,x) + f(t,x), & (t,x) \in (0,\infty) \times (0,\infty), \\ u(0,x) = g(x), & x \in (0,\infty). \end{cases} \tag{4.74}$$

4.2.1 Classical solvability.

Theorem 4.129. *Let $\mu > -1$. Assume that $f \in L^\infty((0,\infty) \times (0,\infty))$ with compact support and $g \in L^\infty((0,\infty))$ with compact support. We define*

$$u(t,x) = \int_0^t \int_0^\infty W_\tau^\mu(x,y) f(t-\tau,y)dyd\tau + \int_0^\infty W_t^\mu(x,y)g(y)dy, \ t,x \in (0,\infty). \tag{4.75}$$

Then, the last integrals are absolutely convergent for every $t, x \in (0, \infty)$*. Moreover, if* f *is also in* $C^2((0,\infty) \times (0,\infty))$*, then the function* u *defined by* (4.75) *is a classical solution of* (4.74) *and*

$$\frac{\partial u(t,x)}{\partial t} = \lim_{\epsilon \to 0^+} \int_0^{t-\epsilon} \int_0^\infty \frac{\partial}{\partial \tau} W_\tau^\mu(x,y) f(t-\tau,y) dy d\tau + \int_0^\infty \frac{\partial}{\partial \tau} W_t^\mu(x,y) g(y) dy$$

and

$$\frac{\partial^2 u(t,x)}{\partial x^2} = \lim_{\epsilon \to 0^+} \int_0^{t-\epsilon} \int_0^\infty \frac{\partial^2}{\partial x^2} W_\tau^\mu(x,y) f(t-\tau,y) dy d\tau + \int_0^\infty \frac{\partial^2}{\partial x^2} W_t^\mu(x,y) g(y) dy,$$

with $t, x \in (0, \infty)$.

Proof. Suppose initially that $f \in L_c^\infty((0,\infty) \times (0,\infty))$ and $g \in L_c^\infty((0,\infty))$. We define

$$\begin{aligned} u(t,x) &= \int_0^t \int_0^\infty W_\tau^\mu(x,y) f(t-\tau,y) dy d\tau + \int_0^\infty W_t^\mu(x,y) g(y) dy \\ &= u_1(t,x) + u_2(t,x), \quad t, x \in (0,\infty). \end{aligned}$$

According to (4.7) the integrals defining u_1 and u_2 are absolutely convergent for every $t, x \in (0, \infty)$.

Assume now that $f \in C_c^2((0,\infty) \times (0,\infty))$. By (4.11), (4.16), and (4.19), we have that $\partial_t u_2(t,x) = \Delta_\mu u_2(t,x)$, $(t,x) \in (0,\infty) \times (0,\infty)$. Moreover,

$$\partial_t u_2(t,x) = \int_0^\infty \partial_t W_t^\mu(x,y) g(y) dy, \ \ (t,x) \in (0,\infty) \times (0,\infty),$$

and

$$\partial_x^i u_2(t,x) = \int_0^\infty \partial_x^i W_t^\mu(x,y) g(y) dy, \ \ (t,x) \in (0,\infty) \times (0,\infty) \ \text{ and } \ i = 1,2.$$

By [14, Theorem 2.1] we have that $\lim_{t \to 0} u_2(t,x) = g(x)$, for a.e. $x \in (0,\infty)$ and for every $x \in (0,\infty)$ provided that g is continuous on $(0,\infty)$.

By proceeding as in the first section of this chapter and by taking into account [73, (2.17)] we can obtain that, for $i = 1, 2$,

$$\partial_x^i u_1(t,x) = \lim_{\epsilon \to 0^+} \int_\epsilon^t \int_0^\infty \partial_x^i W_\tau^\mu(x,y) f(t-\tau,y) dy d\tau, \ \ (t,x) \in (0,\infty) \times (0,\infty).$$

By using parametric derivation we get

$$\begin{aligned}
\partial_t u_1(t,x) &= \int_0^t \int_0^\infty W_\tau^\mu(x,y)\partial_t f(t-\tau,y)dyd\tau = -\int_0^t \int_0^\infty W_\tau^\mu(x,y)(\partial_\tau f)(t-\tau,y)dyd\tau \\
&= -\int_0^t \int_0^\infty (W_\tau^\mu(x,y) - W_\tau(x-y))(\partial_\tau f)(t-\tau,y)dyd\tau \\
&\quad - \int_0^t \int_0^\infty W_\tau(x-y)(\partial_\tau f)(t-\tau,y)dyd\tau \\
&= \int_0^t \int_0^\infty \partial_\tau(W_\tau^\mu(x,y) - W_\tau(x-y))f(t-\tau,y)dyd\tau - \int_0^t \int_0^\infty W_\tau(x-y)(\partial_\tau f)(t-\tau,y)dyd\tau \\
&= \lim_{\epsilon\to 0^+} \left(\int_\epsilon^t \int_0^\infty \partial_\tau(W_\tau^\mu(x,y) - W_\tau(x-y))f(t-\tau,y)dyd\tau \right. \\
&\quad \left. - \int_\epsilon^t \int_0^\infty W_\tau(x-y)(\partial_\tau f)(t-\tau,y)dyd\tau \right) \\
&= \lim_{\epsilon\to 0^+} \int_\epsilon^t \int_0^\infty \partial_\tau W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau + f(t,x), \;\; (t,x) \in (0,\infty)\times(0,\infty).
\end{aligned}$$

Putting together the above equalities we get

$$\partial_t u_1(t,x) = \Delta_\mu u_1(t,x) + f(t,x), \;\; (t,x) \in (0,\infty)\times(0,\infty).$$

Moreover, by (4.7) since f has compact support, we can find $a > 0$ such that, for every $x \in (0,\infty)$, there exists $C > 0$ for which

$$|u_1(t,x)| \leq C \int_0^t \int_0^\infty \frac{e^{-c\frac{|x-y|^2}{\tau}}}{\sqrt{\tau}} dyd\tau \leq Ct, \;\; 0 < t < a.$$

Then, $\lim_{t\to 0^+} u_1(t,x) = 0, \;\; x \in (0,\infty)$. Thus, we prove that the function u is a classical solution of (4.74). □

4.2.2 Boundedness of Bessel Riesz transforms on (weighted) and mixed L^p spaces.

For every $f \in C^2((0,\infty)\times(0,\infty))$ with compact support we define

$$\mathbf{R}_\mu(f)(t,x) = \lim_{\epsilon\to 0^+} \int_0^{t-\epsilon} \int_0^\infty \delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau, \;\; t,x \in (0,\infty), \qquad (4.76)$$

and

$$\widetilde{\mathbf{R}_\mu}(f)(t,x) = \lim_{\epsilon\to 0^+} \int_0^{t-\epsilon} \int_0^\infty \partial_\tau W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau, \;\; t,x \in (0,\infty). \qquad (4.77)$$

Note that the above limits do exist.

Theorem 4.130. *1. Suppose that $\mu > 1/2$ or $\mu = -1/2$. The Riesz transformations $\mathbf{R}_\mu$ and $\widetilde{\mathbf{R}_\mu}$ can be extended to $L^p((0,\infty)\times(0,\infty),\omega)$ as bounded operators from $L^p((0,\infty)\times(0,\infty),\omega)$*

- *into* $L^p((0,\infty)\times(0,\infty),\omega)$, *for every* $1<p<\infty$ *and* $\omega\in A_p^*(\mathbb{R}\times(0,\infty))$.
- *into* $L^{1,\infty}((0,\infty)\times(0,\infty),\omega)$, *for* $p=1$ *and* $\omega\in A_1^*(\mathbb{R}\times(0,\infty))$.

2. *If* $\mu>-1/2$, *the Riesz transformations* $\mathbf{R}_\mu$ *and* $\widetilde{\mathbf{R}_\mu}$ *can be extended to* $L^p((0,\infty)\times(0,\infty))$ *as bounded operators from* $L^p((0,\infty)\times(0,\infty))$

- *into* $L^p((0,\infty)\times(0,\infty))$, *for every* $1<p<\infty$.
- *into* $L^{1,\infty}((0,\infty)\times(0,\infty))$, *for* $p=1$.

Moreover, the extensions of $\mathbf{R}_\mu$ *and* $\widetilde{\mathbf{R}_\mu}$ *to* $L^p((0,\infty)\times(0,\infty),\omega)$ *are defined as the principal value integral operators in* (4.76) *and* (4.77), *respectively, where the limits exist a.e.* $(t,x)\in(0,\infty)\times(0,\infty)$.

Proof. Assume now that $\mu>-1/2$. It was established in (4.24), (4.25), (4.26), and (4.27) that
$|\delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y)|\leq C\frac{e^{-c\frac{(x-y)^2}{\tau}}}{\tau^{3/2}}$, $\tau,x,y\in(0,\infty)$.

We can write

$$\begin{aligned}\delta_{\mu+1}\delta_\mu u_1(t,x)&=\frac{1}{x^2}(\mu+3/2)(\mu+5/2)u_1(t,x)-2\frac{(\mu+1)}{x}\frac{\partial}{\partial x}u_1(t,x)+\frac{\partial^2}{\partial x^2}u_1(t,x)\\&=\lim_{\epsilon\to0^+}\int_\epsilon^t\int_0^\infty\delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y)f(t-\tau,y)dyd\tau,\quad t,x\in(0,\infty).\end{aligned}$$

By proceeding as in [73, page 11] we get

$$\begin{aligned}&|(\chi_{\Omega_\epsilon(x)}(\tau,y)-\chi_{\{\tau>\epsilon^2\}}(\tau))\chi_{\{\tau<t\}}(\tau)\delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y)|\chi_{\{\tau>0\}}(\tau)\\&\leq C\chi_{\{|x-y|>\epsilon\}}(y)\chi_{(0,\epsilon^2)}(\tau)\frac{e^{-c\frac{(x-y)^2}{\tau}}}{\tau^{3/2}}\\&\leq C\chi_{\{|x-y|>\epsilon\}}(y)\chi_{(0,\epsilon^2)}(\tau)\frac{\tau}{|x-y|^5}\\&\leq C\chi_{\{|x-y|>\epsilon\}}(y)\chi_{(0,\epsilon^2)}(\tau)\frac{\tau}{(|x-y|+\sqrt{\tau})^5}\\&\leq C\chi_{\{|x-y|+\sqrt{\tau}>\epsilon\}}(y)\frac{\epsilon^2}{(|x-y|+\sqrt{\tau})^5}\\&\leq C\frac{1}{\epsilon^3}\psi\Big(\frac{|x-y|+\sqrt{\tau}}{\epsilon}\Big),\quad\epsilon,\tau,x,y\in(0,\infty),\end{aligned}$$

where $\psi(z)=\chi_{(\epsilon,\infty)}(z)z^{-5}$, $z\in(0,\infty)$.

Suppose that f is a measurable function on $\mathbb{R}\times(0,\infty)$ such that $f(s,y)=0$, $s<0$. We can write

$$\begin{aligned}
&\left|\iint_{\Omega_\epsilon(x)} \delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y)f(t-\tau,y)d\tau dy - \int_{\epsilon^2}^t\int_0^\infty \delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y)f(t-\tau,y)d\tau dy\right| \\
&\leq C\int_{|x-y|>\epsilon}\int_{0<\tau<\epsilon^2} \frac{\epsilon^2}{(\sqrt{\tau}+|x-y|^2)^5}|f(t-\tau,y)|d\tau dy \\
&\leq C\sum_{k=0}^\infty \int_{2^k\epsilon<|x-y|+\sqrt{|\tau|}<2^{k+1}\epsilon} \frac{\epsilon^2}{(\sqrt{|\tau|}+|x-y|^2)^5}|f(t-\tau,y)|d\tau dy \\
&\leq C\sum_{k=0}^\infty \frac{1}{2^{2k}}\frac{1}{(2^k\epsilon)^3}\int_{|x-y|+\sqrt{|\tau|}<2^{k+1}\epsilon} |f(t-\tau,y)|d\tau dy \\
&\leq C\mathcal{M}(f)(x,t), \quad \epsilon, x, t\in(0,\infty),
\end{aligned}$$

where $\mathcal{M}$ denotes the centered maximal function in our parabolic setting.

We consider the maximal operator T_* defined by

$$T_*(f)(t,x) = \sup_{0<\epsilon<\sqrt{t}}\left|\left(\int_{\Omega_\epsilon(x)} - \int_{\epsilon^2}^t\int_0^\infty\right)\delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y)f(t-\tau)d\tau dy\right|,$$

for $t,x\in(0,\infty)$. According to the boundedness properties of the maximal function $\mathcal{M}$ we deduce that the operator T_* is bounded from $L^p(\mathbb{R}\times(0,\infty),\omega)$

- into $L^p(\mathbb{R}\times(0,\infty),\omega)$, for every $1<p<\infty$ and $\omega\in A_p^*(\mathbb{R}\times(0,\infty))$.
- into $L^{1,\infty}(\mathbb{R}\times(0,\infty),\omega)$, for $p=1$ and $\omega\in A_1^*(\mathbb{R}\times(0,\infty))$.

Let $f\in L^p((0,\infty)^2,\omega)$ with $1\leq p<\infty$ and $\omega\in A_*^p(\mathbb{R}\times(0,\infty))$. We define $\tilde{f}$ by

$$\tilde{f}(s,x) = \begin{cases} f(s,x), & s,x>0,\\ 0, & s\leq 0,\ x>0.\end{cases}$$

If $\mu>1/2$ or $\mu=-1/2$, by Theorem 4.122, we know that there exists the limit

$$\lim_{\epsilon\to0^+}\int_{\Omega_\epsilon(x)} \delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y)\tilde{f}(t-\tau,y)d\tau dy,\ a.e.\ (t,x)\in(0,\infty)^2.$$

Then, a wellknown argument allows us to obtain the existence of the limit

$$S_\mu(f)(t,x) = \lim_{\epsilon\to0^+}\int_\epsilon^t\int_0^\infty \delta_{\mu+1}\delta_\mu W_\tau^\mu(x,y)\tilde{f}(t-\tau,y)d\tau dy,\ a.e.(t,x)\in(0,\infty)^2.$$

By using again Theorem 4.122 we can also prove that the above limit exists for almost everywhere $(t,x)\in(0,\infty)^2$, when $\mu\in(-1/2,1/2]$ and $f\in L^p((0,\infty)^2)$.

From Theorem 4.122 and the boundedness properties of the operator T_* we deduce that

1. If $\mu>1/2$ or $\mu=-1/2$. The operator S_μ is bounded from $L^p(\mathbb{R}\times(0,\infty),\omega)$

- into $L^p(\mathbb{R}\times(0,\infty),\omega)$, for every $1<p<\infty$ and $\omega\in A_p^*(\mathbb{R}\times(0,\infty))$.
- into $L^{1,\infty}(\mathbb{R}\times(0,\infty),\omega)$, for $p=1$ and $\omega\in A_1^*(\mathbb{R}\times(0,\infty))$.

2. If $\mu>-1/2$, The operator S_μ is bounded from $L^p(\mathbb{R}\times(0,\infty))$
   - into $L^p(\mathbb{R}\times(0,\infty))$, for every $1<p<\infty$.
   - into $L^{1,\infty}(\mathbb{R}\times(0,\infty))$, for $p=1$.

According to (4.11), we have that

$$|\partial_\tau W_\tau^\mu(x,y)|\leq C\frac{e^{-c|x-y|^2/\tau}}{\tau^{3/2}},\quad x,y,\tau\in(0,\infty).$$

By proceeding as above we can prove that the desired properties for the Riesz transforms $\widetilde{\mathbf{R}}_\mu$.

□

Suppose that X is a Banach space and $A:D(A)\subset X\to X$ is an operator. If $1<p<\infty$, we say that A has **maximal L^p- regularity** when there exists a constant $C>0$ such that, for every $f\in L^p((0,\infty),X)$ there exists a unique $u_f\in L^p((0,\infty),D(A))$ solution of the Cauchy problem

$$\begin{cases}\frac{\partial}{\partial t}u(t)+Au(t)=f(t), & t\in(0,\infty),\\ u(0)=0\end{cases}\tag{4.78}$$

satisfying

$$\left\|\frac{\partial}{\partial t}u_f\right\|_{L^p((0,\infty),X)}+\|Au_f\|_{L^p((0,\infty),X)}\leq C\|f\|_{L^p((0,\infty),X)}.$$

If the operator $-A$ generates a semigroup $\{T_t\}_{t\geq 0}$ of operators on X, the solution of (4.78) can be written as

$$u(t)=\int_0^t T_{t-s}(f(s))ds,\quad t\geq 0,$$

and A has maximal L^p-regularity when the operator

$$R(f)(t)=\int_0^t\frac{\partial}{\partial t}(T_{t-s})(f(s))ds$$

is bounded from $L^p((0,\infty),X)$ into itself. Note that $\frac{\partial}{\partial t}T_t=-AT_t$, $t>0$. This fact leads, from the point of view of harmonic analysis, to replace the property of maximal L^p-regularity by the L^p-boundedness of certain Banach space valued singular integrals. If suitable Gaussian bounds hold for the semigroup generated by $-A$, then A has maximal L^p-regularity (see [27] and [44]).

Theorem 4.131. *Let $\mu\geq -1/2$. Assume that $1<p,q<\infty$. Then, the Bessel operator Δ_μ has maximal L^p-regularity on $L^q((0,\infty))$.*

Note that Theorem 4.131 actually establishes mixed norm estimates for $\widetilde{\mathbf{R}_\mu}$.

Proof of Theorem 4.131.

We now consider the Cauchy problem

$$\begin{cases} \partial_t u(t) = \Delta_\mu u(t) + f(t), & t \in (0,\infty), \\ u(0) = 0, \end{cases}$$

with $\mu \geq -1/2$ and $f \in L^p((0,\infty), L^q((0,\infty)))$, being $1 < p,q < \infty$. The operator Δ_μ generates the semigroup $\{W_t^\mu\}_{t>0}$ in $L^q((0,\infty))$, where, for every $t > 0$,

$$W_t^\mu(g)(x) = \int_0^\infty W_t^\mu(x,y)g(y)dy, \quad g \in L^q((0,\infty)).$$

By (4.3) and (4.4) we have that

$$|W_z^\mu(x,y)| \leq C\frac{e^{-c\frac{|x-y|^2}{|z|}}}{|z|^{1/2}}, \quad |Argz| < \frac{\pi}{4}, \quad x,y \in (0,\infty).$$

Then, according to [63, Theorem 3.3], Δ_μ has the maximal L^p-regularity property in $L^q((0,\infty))$, that is, there exists $C > 0$ such that

$$\|\mathcal{R}_\mu f\|_{L^p((0,\infty),L^q((0,\infty)))} \leq C\|f\|_{L^p((0,\infty),L^q((0,\infty)))}, \tag{4.79}$$

where

$$\mathcal{R}_\mu f(t) = \int_0^t \Delta_\mu W_{t-s}(f(s))ds = \int_0^t \partial_t W_{t-s}(f(s))ds,$$

with $f \in L^p((0,\infty), L^q((0,\infty)))$. Note that (4.79) is a mixed-norm inequality.

□

Conclusions

Along this thesis we have shown that the semigroup language can be quite useful to deal with different problems that arise in analysis and PDE. Moreover, the ideas developed here can also be used in others problems that have not been studied yet. For example, there is nothing in the literature related with Lipschitz spaces adapted to Bessel and Laguerre operators. On the other hand, the regularity results obtained in Chapter 1 in the discrete Hölder spaces may be applied in numerical analysis. In addition, as we have done in the case of discrete fractional derivatives, if the constants obtained in the regularity estimates do not depend on the step length h, approximation theorems can be derived from those results.

Conclusiones

A lo largo de esta tesis hemos mostrado que el lenguaje de semigrupos puede ser muy útil para tratar con diferentes problemas que surgen en análisis y ecuaciones en derivadas parciales. Además, las ideas que se han desarrollado aquí pueden ser utilizadas en otros problemas que aún no han sido estudiados. Por ejemplo, no hay nada en la literatura relacionado con espacios Lipschitz (o Hölder) adaptados a los operadores de Bessel o de Laguerre. Por otra parte, los resultados de regularidad obtenidos en el Capítulo 1 en los espacios Hölder discretos pueden aplicarse en análisis numérico. Además, del mismo modo que hicimos en el caso de las derivadas fraccionarias discretas, si las constantes que se obtienen en las estimaciones de regularidad no dependen de la longitud de paso h, pueden obtenerse teoremas de aproximación a partir de esos resultados.

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