UNIVERSIDAD AUTÓNOMA DE MADRID

# Quantum Foundation of Infrared Physics

Memoria de Tesis Doctoral realizada por

**Raoul Alexander Letschka**

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Tesis Doctoral dirigida por **Prof. Dr. César Gómez Lopez**,
Profesor Investigador del Instituto de Física Teórica

Instituto de Física Teórica UAM/CSIC

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# Abstract

In this thesis we will take a look on the infrared (IR) and collinear limit of QED and perturbative gravity. First we study the infrared regime with its connection to the memory effect and to the decoherence of density matrices. Lastly, we study the consistency of QED with massless electrons.

First we investigate the memory effects in scattering processes which are described in terms of the asymptotic retarded fields. These fields are completely determined by the scattering data and the zero mode part is set by the soft photon theorem. The dressed asymptotic states defining an infrared finite $S$-matrix for charged particles can be defined as quantum coherent states using the corpuscular resolution of the asymptotic retarded fields. Imposing that the net radiated energy in the scattering is zero leads to the new set of conservation laws for the scattering $S$-matrix which are equivalent to the decoupling of the soft modes. The actual observability of the memory requires a non-vanishing radiated energy and could be described using the infrared part of the differential cross section that only depends on the scattering data and the radiated energy. This is the IR safe cross section with any number of emitted photons carrying total energy equal to the energy involved in the actual memory detection.

Secondly, we investigate on possible decoherence of the density matrix in QED or perturbative gravity due to entanglement with soft radiated modes and use thereby the two standard approaches to cancel infrared divergences coming from soft loop contributions. In the inclusive way only rates that include emitted soft radiation are non-vanishing. Independently of detector resolution, finite observables can only be obtained after integrating over the IR-component of this radiation. This integration can lead to some loss of quantum coherence. We argue that it should in general not lead to full decoherence. Based on unitarity, we suggest a way to define non-vanishing off-diagonal pieces of the IR-finite density matrix. For this IR-finite density matrix, we estimate the dependence of the loss of quantum coherence,

i.e. of its purity, on the scattering kinematics. In the coherent state approach we dress the initial and final states with a cloud of infinite many photons to achieve a not fully decoherent density matrix. The inclusive way and the coherent state approach both yield the same IR-finite rates, but we point out that they are not equivalent since they encode different infrared scales. In particular, dressing states are independent of the resolution scale of radiation. Instead, they define radiative vacua in the von Neumann space. We present a combined formalism that can simultaneously describe both dressing and radiation. This unified approach allows us to tackle the problem of quantum decoherence due to tracing over unresolved radiation. We again obtain an IR-finite density matrix with non-vanishing off-diagonal elements and again estimate how its purity depends on scattering kinematics and the resolution scale. Along the way, we comment on collinear divergences as well as the connection of large gauge transformations and dressing.

Lastly, we work out in the forward limit and up to order $e^6$ in perturbation theory the collinear divergences. In this kinematical regime we discover new collinear divergences that we argue can be only cancelled using quantum interference with processes contributing to the gauge anomaly. This rules out the possibility of a quantum consistent and anomaly free theory with massless charges and long range interactions. We use the anomalous threshold singularities to derive a gravitational lower bound on the mass of the lightest charged fermion.

# Resumen

En esta tesis veremos el límite infrarrojo (IR) y colineal de QED y la gravedad perturbadora. Primero estudiamos el régimen infrarrojo con su conexión al efecto memoria y a la decoherencia de las matrices de densidad. Por último, se estudia la consistencia del QED con electrones sin masa.

Primero investigamos los efectos de memoria en los procesos de dispersión que se describen en términos de los campos asintóticos retardados. Estos campos están completamente determinados por los datos de dispersión y la pieza en modo cero es fijada por el teorema del fotón suave. Los estados asintóticos vestidos que definen una $S$-matriz infrarroja finita para partículas cargadas pueden ser definidos como estados cuánticos coherentes usando la resolución corpuscular de los campos asintóticos retardados. Imponer que la energía radiada neta en la dispersión es cero conduce al nuevo conjunto de leyes de conservación para la $S$-matriz de dispersión que son equivalentes al desacoplamiento de los modos blandos. La observabilidad real de la memoria requiere una energía radiada que no desaparece y podría describirse utilizando la parte infrarroja de la sección transversal diferencial que sólo depende de los datos de dispersión y de la energía radiada. Esta es la sección transversal segura del IR con cualquier número de fotones emitidos que transportan energía total igual a la energía involucrada en la detección de la memoria real.

En segundo lugar, investigamos la posible decoherencia de la matriz de densidad en QED o gravedad perturbadora debida al entrelazamiento con modos de radiación suave y utilizamos así los dos enfoques estándar para cancelar las divergencias infrarrojas procedentes de las contribuciones de bucle suave. De la manera inclusiva, sólo las tasas que incluyen la radiación suave emitida no desaparecen. Independientemente de la resolución del detector, los observables finitos sólo pueden obtenerse después de la integración sobre el componente IR de esta radiación. Esta integración puede llevar a una cierta pérdida de coherencia cuántica. Argumentamos que, en general, no debería conducir a la plena decoherencia. Basán-

donos en la unidad, sugerimos una forma de definir las piezas no desvanecedoras fuera de la diagonal de la matriz de densidad finita de infrarrojos. Para esta matriz de densidad finita IR, estimamos la dependencia de la pérdida de coherencia cuántica, es decir, de su pureza, de la cinemática de dispersión. En el enfoque de estado coherente vestimos los estados inicial y final con una nube de infinitos fotones para lograr una matriz de densidad no totalmente decoherente. La forma inclusiva y el enfoque de estado coherente producen las mismas tasas IR-finitas, pero señalamos que no son equivalentes ya que codifican diferentes escalas de infrarrojos. En particular, los estados de apósito son independientes de la escala de resolución de la radiación. En cambio, definen la vacua radiativa en el espacio von Neumann. Presentamos un formalismo combinado que puede describir simultáneamente tanto el apósito como la radiación. Este enfoque unificado nos permite abordar el problema de la decoherencia cuántica debida al rastreo de radiaciones no resueltas. De nuevo obtenemos una matriz de densidad fina por infrarrojos con elementos fuera de la diagonal que no se desvanecen y estimamos de nuevo cómo su pureza depende de la cinemática de dispersión y de la escala de resolución. A lo largo del camino, comentamos las divergencias colineales, así como la conexión de las transformaciones de gran calibre y el apósito.

Por último, se trabaja en el límite delantero y hasta un máximo de $e^6$ en teoría de la perturbación de las divergencias colineales. En este régimen cinemático descubrimos nuevas divergencias colineales que, en nuestra opinión, sólo pueden ser canceladas mediante interferencias cuánticas con los procesos que contribuyen a la anomalía del gálibo. Esto descarta la posibilidad de una teoría cuántica consistente y libre de anomalías con cargas sin masa e interacciones de largo alcance. Usamos las singularidades anómalas del umbral para derivar un límite gravitacional inferior sobre la masa del fermión más ligero cargado.

# Contents

# Publications and Authorship

This thesis is based on a series of papers [1–4]. They are the result of varying collaborations with Cesar Gomez and Sebastian Zell. All authors share first authorship and are sorted alphabetically. The main goal of this work is to present the above mentioned results in a wider context and to provide a unified picture. Therefore, the present thesis is to a large extent an ad verbatim reproduction (with respect to text, equations and figures) of the papers [1–4]. Unless indicated otherwise, the papers can be attributed to the chapters as follows.

Chapter 2 is based on [1], Chapter 3 and appendix A are based on [2], Chapter 4 and section 1.3.3 are based on [3] and Chapter 5 and appendices B,A, E and F are based on [4].

# Chapter 1

# Review of Infrared Physics

The infrared (IR) spectrum of a classical or quantum field theory (QFT) is purely determined by the massless or also called gapless particles in the theory. IR divergences appear when the massless particle runs in a virtual loop of a scattering process, where in quantum electrodynamics (QED) it's the photon and in perturbative gravity it's the graviton that is responsible for the IR divergences. The loop integrals have to be regulated with a parameter equal to a fictitious mass of that particle running in the loop. At the end physical measurable quantities, such as the cross section, have to be IR finite and must be independent of that mass regulator which goes to zero.

There are two standard procedures to deal with that divergences and in this review we will present them. The first is based on an inclusive summation of emitted infrared modes in the cross section, also called the Bloch-Nordsieck recipe [5–7], and the second is the so called coherent state approach by Faddeev and Kulish [8] and Chung [9], where the cancelation happens already at the level of $S$-Matrix.

We will start with a short review of the IR divergences coming from loop corrections and then move forward to the two standard approaches.

## 1.1 IR divergences by loop corrections

Let's consider an arbitrary tree level scattering process $\alpha \to \beta$, where the initial and the final states contain charged particles and photons/gravitons.

The matrix element of this process is described by $S_{\alpha,\beta}$ and does not contain loop corrections. If one takes into account the soft part of the loop corrections in this process, where the energy of the particles running in the loops are small compared to the energies of the scattered particles, their effect can be resummed and exponentiated so that we obtain [6, 7]:

$$S^{\text{loop}}_{\alpha,\beta} = \left(\frac{\lambda}{\Lambda}\right)^{B_{\alpha,\beta}/2} S_{\alpha,\beta}\ , \tag{1.1}$$

where $\lambda$ is the IR regulator or the fictitious mass of the photon/graviton running in the loop and $\Lambda > \lambda$ a UV-cutoff for the loop integration. The exponent $B_{\alpha,\beta}$, herein this thesis it is also called $B$-factor, is a non-negative number which depends on the kinematical data of the process $\alpha \to \beta$. For $B_{\alpha,\beta} \neq 0$, this IR correction leads to a vanishing amplitude in the limit $\lambda \to 0$.

In the case of QED, the exponent $B_{\alpha,\beta}$ is given by

$$\begin{aligned} B_{\alpha,\beta} &= \frac{1}{2(2\pi)^3}\int \mathrm{d}^2\Omega \left(\sum_{n\in\beta}\frac{e_n p_n^\mu}{p_n\cdot\hat{k}} - \sum_{n\in\alpha}\frac{e_n p_n^\mu}{p_n\cdot\hat{k}}\right)\eta_{\mu\nu}\left(\sum_{m\in\beta}\frac{e_m p_m^\nu}{p_m\cdot\hat{k}} - \sum_{m\in\alpha}\frac{e_m p_m^\nu}{p_m\cdot\hat{k}}\right) \\ &= -\frac{1}{8\pi^2}\sum_{\substack{n\in\alpha,\beta\\ m\in\alpha,\beta}} \eta_n\eta_m e_n e_m \beta_{nm}^{-1}\ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right)\ , \end{aligned} \tag{1.2}$$

where in the first line one integrates over the solid angle of the photon momentum, $\hat{k}^\mu$ is the unit 4-momentum of the photon and $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ being the Minkowski metric in 4 dimensions. In the second line both sums run over all external particles. Here $e_n$ marks the electric charge of a particle, $\eta_n = 1$ for outgoing particles and $\eta_n = -1$ for ingoing ones and $\beta_{nm}$ is the relative velocity:

$$\beta_{nm} = \left(1 - \frac{m_n^2 m_m^2}{(p_n\cdot p_m)^2}\right)^{1/2}\ , \tag{1.3}$$

where $m_n$ is the mass and $p_n^\mu$ the 4-momentum of the $n^{\text{th}}$ particle. In the case of gravity, we have[1]

$$
\begin{aligned}
B_{\alpha,\beta} &= \frac{8\pi G}{2(2\pi)^3}\int \mathrm{d}^2\Omega \left(\sum_{n\in\beta}\frac{p_n^\mu p_n^\nu}{p_n\cdot\hat{k}} - \sum_{n\in\alpha}\frac{p_n^\mu p_n^\nu}{p_n\cdot\hat{k}}\right)\Pi_{\mu\nu\sigma\rho}\left(\sum_{m\in\beta}\frac{p_m^\sigma p_m^\rho}{p_m\cdot\hat{k}} - \sum_{m\in\alpha}\frac{p_m^\sigma p_m^\rho}{p_m\cdot\hat{k}}\right) \\
&= \frac{G}{2\pi}\sum_{\substack{n\in\alpha,\beta\\ m\in\alpha,\beta}}\eta_n\eta_m m_n m_m \frac{1+\beta_{nm}^2}{\beta_{nm}(1-\beta_{nm}^2)^{1/2}}\ln\left(\frac{1+\beta_{nm}}{1-\beta_{nm}}\right),
\end{aligned}
\tag{1.4}
$$

where $G$ is the gravitational constant and $\Pi_{\mu\nu\sigma\rho} = \frac{1}{2}(\eta_{\mu\sigma}\eta_{\nu\rho} + \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\nu}\eta_{\sigma\rho})$. Both in gravity and QED, $B_{\alpha,\beta}$ is suppressed by the coupling constant. So it can only get big in a regime of strong coupling.[2]

As seen from the first line of equation (1.2) the $B$-factor is zero if and only if the angles of the outgoing and ingoing charges match antipodally, i.e. in forward scattering. In this case one can take the limit $\lambda \to 0$ in (1.1) and the amplitude is non-vanishing and there is no need for a procedure that cancel the regulator $\lambda$. The very unspectacular case of forward scattering is of course not very interesting so that we want to continue with the two standard approaches to cancel the divergence coming from the IR correction.

## 1.2 Bloch-Nordsieck recipe

The recipe by Bloch and Nordsieck [5] and later also worked out in more detail by Weinberg [7] and Yennie, Frautschi and Suura [6] is the standard approach to deal with the problem of the IR correction in the amplitude (1.1). The recipe is to include the effect of *soft IR emission* that is scattered off external lines, i.e. to consider a different process $\alpha \to \beta + \gamma$. Once the radiation is included one sums or integrates[3] in an *inclusive way* at the level of rates or differential cross sections over the IR radiation. It is very important to stress that this inclusive integration over IR radiation is not due to any practical limitation on resolution, that in reality obviously exists. Instead, it is needed to compensate for the problem created by the radiative loops. The IR finite quantities obtained, following this standard recipe, depend

[1] The massless limit of this expression is finite and was derived in [10].

[2] For example, in the case of scattering of two gravitons of ultra-planckian center of mass energy $s$ into an arbitrary number of final gravitons, we have $B_{\alpha,\beta} = 4\,c\,Gs/\pi$, where $0 \le c \le \ln 2$ [10]. The lower bound is reached if all final gravitons are collinear with the initial ones and the upper bound corresponds to all final gravitons orthogonal to the initial ones.

[3] In this thesis these two operations may sometimes be synonyms.

on an IR energy scale $\epsilon$ that sets the total amount of IR radiated energy as well as the upper bound on the energy of individual radiated quanta in the integration. In the literature it is common to set the total amount of radiated energy $\epsilon$ equal to the resolution scale of the detector. We start with the amplitude and according to the soft factorization theorems or soft photon theorem [6, 7], the resulting amplitude for radiating $N$ soft IR photons is

$$S_{\alpha,\,\beta\gamma} = \prod_{i=1}^{N} \frac{\mathcal{F}^{(l_i)}_{\alpha,\,\beta}(\boldsymbol{k}_i)}{\sqrt{|\boldsymbol{k}_i|}} S_{\alpha,\,\beta}\;, \tag{1.5}$$

where the energies of the emitted photons are small compared to the energies of charged particles in the process, which clarifies the description of *softness* of the photons. By *infrared photons* we mean only those photons that are scattered off external lines, see also figures 1.1a and 1.1b for more details. The amplitude $S_{\alpha,\,\beta}$ describes as above a scattering process $\alpha \to \beta$ without any soft IR photons and without the virtual corrections coming from the soft loops. The factor $\mathcal{F}^{(l_i)}_{\alpha,\,\beta}(\boldsymbol{k}_i)$ depends only on the kinematical data of the initial and final state and on the additional IR radiation. In the case of QED, it reads

$$\mathcal{F}^{(l_i)}_{\alpha,\,\beta}(\boldsymbol{k}_i) = \sum_{n\in\alpha,\,\beta} \frac{e_n \eta_n}{\sqrt{2(2\pi)^3}} \frac{p_n \cdot \varepsilon^{*}_{l_i,\boldsymbol{k}_i}}{p_n \cdot k_i}\;, \tag{1.6}$$

where $k_i^\mu$ is the on-shell 4-momentum of the $i^{\text{th}}$ soft photon and $\boldsymbol{k}_i$ its momentum. The sum runs over all externals lines, i.e. all in states $\alpha$ and all out states $\beta$. Moreover, $\varepsilon^{\mu}_{l,\boldsymbol{k}_i}$ is its polarization vector and $l$ labels the helicity. In this thesis shall hold that the expressions $a \cdot b$, $a^\mu b_\mu$ and $ab$ are all equivalent and represent the 4-vector scalar product in Minkowski space with the metric $\eta_{\mu,\nu}$.

In perturbative gravity, we obtain analogously for $N$ emitted soft IR gravitons

$$S_{\alpha,\,\beta h} = \prod_{i=1}^{N} \frac{\mathcal{F}^{(l_i)}_{\alpha,\,\beta}(\boldsymbol{k}_i)}{\sqrt{|\boldsymbol{k}_i|}} S_{\alpha,\,\beta}\;, \tag{1.7}$$

with

$$\mathcal{F}^{(l_i)}_{\alpha,\,\beta}(\boldsymbol{k}_i) = \sum_{n\in\alpha,\,\beta} \frac{\sqrt{8\pi G}\,\eta_n}{\sqrt{2(2\pi)^3}} \frac{p_n^\mu p_n^\nu \varepsilon^{*}_{\mu\nu,\,l_i,\boldsymbol{k}_i}}{p_n \cdot k_i}\;, \tag{1.8}$$

where $\varepsilon^{\mu\nu}_{l,\boldsymbol{k}}$ is the polarization tensor of the graviton with momentum $\boldsymbol{k}$ and helicity $l$.

## Lorentz invariance

Independently of its use to cancel the virtual loop contributions coming from soft loops, one interesting element of the theorem lies in the observation that the amplitude (1.5), which is divergent in the $\boldsymbol{k}_i = 0$ limit, does not satisfy Lorentz invariance unless the sum of incoming charges is equal to the sum of the outgoing ones. In other words, the soft theorem identifies what conservation law is needed in order to have a well-defined and Lorentz invariant soft limit of the amplitudes. Taking for simplicity only one soft photon with momentum $\boldsymbol{k}$ that scatters off external charged lines in the scattering process $\alpha \to \beta$. Under certain Lorentz transformations (little group transformations) the polarization vector for the photon transforms as [11]

$$\varepsilon^{\mu}_{l,\boldsymbol{k}} \to \varepsilon^{\mu}_{l,\boldsymbol{k}} + k^{\mu} \,. \tag{1.9}$$

Then this transforms the soft photon theorem (1.5) into

$$\mathcal{F}^{(l)}_{\alpha,\beta}(\boldsymbol{k}) S_{\alpha,\beta} \to \mathcal{F}^{(l)}_{\alpha,\beta}(\boldsymbol{k}) S_{\alpha,\beta} + \frac{1}{\sqrt{2(2\pi)^3|\boldsymbol{k}|}} \left( \sum_{m\in\beta} e_m - \sum_{n\in\alpha} e_n \right) \,. \tag{1.10}$$

Thus the amplitude (1.5) is only conserved under the Lorentz transformation (1.9) if the total charge is conserved in that scattering process [11, 12].

In the case of massless spin $2$ particles, i.e. gravity, the conservation law is the equivalence between inertial and gravitational mass. For only one soft graviton scattered from an external line the Lorentz transformation for the polarization tensor of the graviton with momentum $\boldsymbol{k}$ is given by [11]

$$\varepsilon^{\mu,\nu}_{l,\boldsymbol{k}} \to \varepsilon^{\mu,\nu}_{l,\boldsymbol{k}} + \Lambda^{\mu} k^{\nu} + \Lambda^{\nu} k^{\mu} + \Lambda k^{\mu} k^{\nu} \,, \tag{1.11}$$

for some parameter $\Lambda^{\mu}$ and $\Lambda$ that take care of the explicit way of how the Lorentz group acts. The amplitude for the soft graviton theorem (1.5) with only one out going soft graviton then transforms

$$\mathcal{F}^{(l)}_{\alpha,\beta}(\boldsymbol{k}) S_{\alpha,\beta} \to \mathcal{F}^{(l)}_{\alpha,\beta}(\boldsymbol{k}) S_{\alpha,\beta} + \sqrt{8\pi G} \left( 2\tilde{\Lambda}_{\mu} + \tilde{\Lambda} k^{\mu} \right) \left( \sum_{m\in\beta} p^{\mu}_m - \sum_{n\in\alpha} p^{\mu}_n \right) \,, \tag{1.12}$$

where, we define $\tilde{\Lambda}^\mu := \Lambda^\mu/\sqrt{2(2\pi)^3|\boldsymbol{k}|}$ and $\tilde{\Lambda} := \Lambda/\sqrt{2(2\pi)^3|\boldsymbol{k}|}$. For generic $\tilde{\Lambda}^\mu$ and $\tilde{\Lambda}$ the last term vanishes because of conservation of energy. Notice that if gravity would not couple universally to each particle the last term would not vanish. So that Lorentz invariance implies that gravity couples universally.

### 1.2.1 Finite rates

The main point of the soft photon theorem in this thesis will be the cancelation of the IR divergences coming from soft loops, which after resumming exponentiate and give the IR divergent amplitude (1.1). The Bloch-Nordsieck recipe tells us to trace the amplitude square over the radiation $\gamma$. In the sum over the soft photons, where each soft photon has an energy in the range $\lambda \leq |\boldsymbol{k}_i| < \epsilon$, the energy conservation forces us to put a term $\delta\left(\epsilon - \sum_i^N |\boldsymbol{k}_i|\right)$ at the level of differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon}$ or a Heaviside function $\mathcal{H}\left(\epsilon - \sum_i^N |\boldsymbol{k}_i|\right)$ at the level of the cross section $\sigma$ in each term of the sum, where $N$ again is the number of emitted IR photons.[4] The cross section will then be evaluated at $E_{\text{out}} = E_{\text{in}} - \epsilon$, where $E_{\text{out}}/E_{\text{in}}$ is the energy of all the non-IR particles in the out/in state and $\epsilon$ the total energy of all soft IR photons. So that the IR finite rate is

$$\begin{aligned}\Gamma_{\alpha,\beta} &= \sum_\gamma \left|S^{\text{loop}}_{\alpha,\beta\gamma}\right|^2 := \left|S^{\text{loop}}_{\alpha,\beta}\right|^2 \sum_N \frac{1}{N!} \prod_i^N \int_\lambda^\epsilon \frac{\mathrm{d}^3\boldsymbol{k}_i}{|\boldsymbol{k}_i|} \sum_{l_i} \left|\mathcal{F}^{(l_i)}_{\alpha,\beta}(\boldsymbol{k}_i)\right|^2 \mathcal{H}(\epsilon - \sum_{j=1}^N |\boldsymbol{k}_j|) \\ &= \left(\frac{\epsilon}{\lambda}\right)^{B_{\alpha,\beta}} f(B_{\alpha,\beta}) \left|S^{\text{loop}}_{\alpha,\beta}\right|^2 = \left(\frac{\epsilon}{\Lambda}\right)^{B_{\alpha,\beta}} f(B_{\alpha,\beta}) \left|S_{\alpha,\beta}\right|^2 ,\end{aligned} \tag{1.13}$$

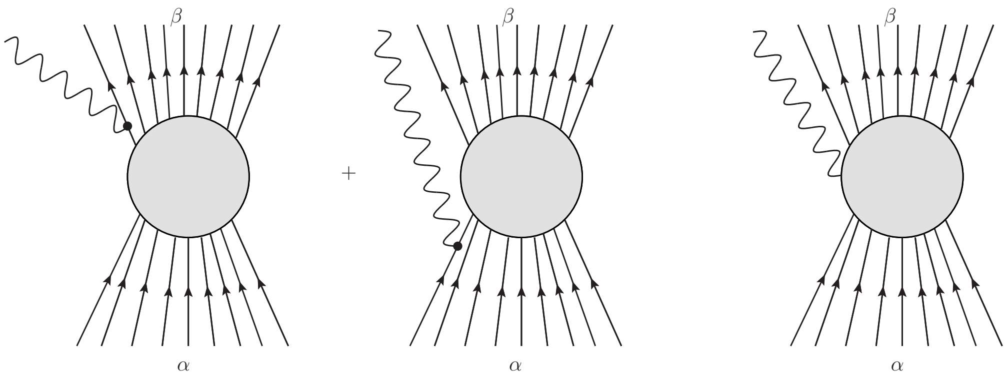
The additional factor $f(B_{\alpha,\beta})$ is due to energy conservation, i.e. the Heaviside function in each sum, and reads [7]

$$f(x) = \frac{\mathrm{e}^{-\gamma x}}{\Gamma(1+x)} , \tag{1.14}$$

where $\gamma$ is Euler's constant and $\Gamma$ is the gamma function. For small $x$, it can be approximated as

$$f(x) = 1 - \frac{\pi^2}{12} x^2 . \tag{1.15}$$

[4] Notice that the delta function is nothing else but the derivative of the Heaviside function, such as the differential cross section is the derivative of the cross section.




(a) IR radiation (b) Non-IR radiation

Figure 1.1: Comparison of IR radiation and non-IR radiation in the scattering process $\alpha \to \beta$. IR radiation is the IR divergent part of emission from external lines, i.e. from an incoming or an outgoing particle. In contrast, non-IR radiation comes from emission from internal lines and is IR finite.

For large $x$, it scales as

$$f(x) \sim \frac{1}{x!} \,. \tag{1.16}$$

Combining the effects of soft loops and soft emission, one obtains a rate which is independent of the IR regulator and in particular finite for $\lambda \to 0$.

We already mentioned this earlier but to make this even clearer: It is important to point out that the radiation $\gamma$ does not include all kinds of radiation with energies below or, in case of only one photon emitted, equal $\epsilon$ but only *IR radiation*, which is defined as the part of radiation which leads to a divergent amplitude for $\lambda \to 0$. Diagrammatically, IR radiation is due to emission from external legs (see Fig. 1.1a), whereas soft emission from internal lines is infrared-finite (see Fig. 1.1b), i.e. non-IR.

### 1.2.2 The Meaning of IR Symmetries

As a side note, we remark how the discussion of IR divergences could shed light on the recently discussed *soft symmetries*. In references [13–21] it has been suggested that in theories with long range forces there exist an infinite set of charges $Q_\epsilon$ that can be associated with the asymptotic states of scattering processes. Considered a process $\alpha \to \beta$, these charges relate the current in $\beta$ to the current in $\alpha$ at each angle, namely the incoming current should match the outgoing current antipodally. However, this is precisely the condition under which the corresponding kinematical soft factor $B_{\alpha,\beta}$ vanishes (1.2), i.e. we observe that

$$Q_\epsilon\,|\alpha\rangle = Q_\epsilon\,|\beta\rangle \qquad \Longleftrightarrow \qquad B_{\alpha,\beta} = 0\,. \tag{1.17}$$

Thus, imposing soft symmetries is equivalent to restricting to the special class of processes that are IR finite even without including soft IR emission. As we have reviewed, however, *all* scattering processes are IR finite after including the emission of soft IR radiation. So in general, there is no physical reason to restrict to final states that satisfy the constraint (1.17).

### 1.2.3 The role of the IR factors

In the limit where $B_{\alpha,\beta}$ is small the function $f(B_{\alpha,\beta})$ is close to one, thus the IR effect on the rate of a process is purely determined by $(\epsilon/\Lambda)^{B_{\alpha,\beta}}$, which is also of order $1$. Then the rate (1.13) is mostly determined by the scattering process with the amplitude $S_{\alpha,\beta}$.

In the high energy limit, where the mass of the charged particles goes to zero, the scattered charged particles radiate a huge amount of soft IR photons, where each photon has a small energy $\omega_i$, since the total energy of radiation $\epsilon$ stays constant. In that limit $B_{\alpha,\beta}$ is very small and goes as

$$B_{\alpha,\beta} \sim -\ln m\,. \tag{1.18}$$

for a charged particle with mass $m$. This extra divergence when taking the mass to zero, the so called *collinear* divergence, deserves its own treatment because it has a big impact on the physical consistency of the theory and we will come back to this topic later in chapter 5 in detail.

In that limit the factor coming from energy conservation $f(B_{\alpha,\beta})$ has a suppression effect to rate (1.13) since $f(B_{\alpha,\beta}) \sim \frac{1}{B_{\alpha,\beta}!} \approx \exp\left\{-B_{\alpha,\beta}\ln(B_{\alpha,\beta})\right\}$. The suppression coming from soft IR effects is then $\exp\left\{-B_{\alpha,\beta}[\ln(B_{\alpha,\beta}) + \ln(\Lambda/\epsilon)\right\}]$ which goes to zero in the high energy limit. Nevertheless, that doesn't mean that the full rate in (1.13) is zero since there are many other factors multiplying this factor which in general can compensate this convergence of the IR factor.

## 1.3 Coherent state approach

The second approach to address the problem of IR divergences coming from soft loops is to define asymptotic in and out states. In QED this has been mainly worked out in [8]. Other references which treat the same subject are [9,22–25]. The main difference in short is that the cancelation of the divergences happens already at the level of $S$-matrix, where one defines *asymptotic in and out states* that no longer live in the Fock space but in a much bigger so called *von Neumann space* [26].[5] These sates contain an infinite amount of zero-energy soft photons, when taking the mass regulator $\lambda$ to zero. Nevertheless, the asymptotic states have finite energy. The infinite amount of soft photons makes these states part of the von Neumann space.

In QED the asymptotic dynamics of these states in that large Hilbert space is described by asymptotic interaction operator $\hat{V}_{\text{as}}(t)$ in the limit $|t| \to \infty$ and is given by

$$\hat{V}_{\text{as}}(t) = \sum_l \int \hat{J}_{\text{as}}(t, \boldsymbol{k}) \cdot \left( \varepsilon^*_{l,-\boldsymbol{k}} \hat{a}^\dagger_{l,-\boldsymbol{k}} + \varepsilon_{l,\boldsymbol{k}} \hat{a}_{l,\boldsymbol{k}} \right) \frac{\mathrm{d}^3 \boldsymbol{k}}{|\boldsymbol{k}|} \,, \tag{1.19}$$

where $\hat{J}^\mu_{\text{as}}(t, \boldsymbol{k})$ is the asymptotic current operator and is given by

$$\hat{J}^\mu_{\text{as}}(t, \boldsymbol{k}) = -\frac{1}{\sqrt{2(2\pi)^3}} \int p^\mu \mathrm{e}^{\mathrm{i}\frac{pk}{p_0}t} \hat{\rho}(\boldsymbol{p}) \frac{\mathrm{d}^3 \boldsymbol{p}}{p_0} \,, \tag{1.20}$$

where $\hat{\rho}(\boldsymbol{p}) = e \sum_s \left( \hat{b}^\dagger_{s,\boldsymbol{p}} \hat{b}_{s,\boldsymbol{p}} - \hat{d}^\dagger_{s,\boldsymbol{p}} \hat{d}_{s,\boldsymbol{p}} \right)$ is the charge density operator for electrons and positrons. $\hat{b}^\dagger_{s,\boldsymbol{p}}/\hat{b}^\dagger_{s,\boldsymbol{p}}$ is the creation operator for an electron/positron of spin $s$ and momentum $\boldsymbol{p}$ and $\hat{a}^\dagger_{l,\boldsymbol{k}}$ is the creation operator of a photon with helicity $l$ and momentum $\boldsymbol{k}$. It is clear from the limit $|t| \to \infty$ that only the low energy modes with $|\boldsymbol{k}| \approx 0$ are responsible for the asymptotic dynamic of the states. In the interaction picture the asymptotic time evolution operator reads

$$Z(t) = \exp \left\{ R_{J_{\text{as}}}(t) \right\} \exp \left\{ \mathrm{i}\Phi(t) \right\} \,, \tag{1.21}$$

[5] We will come back to this special space later in section 1.3.3.

where

$$R_{J_{\text{as}}}(t) = -\mathrm{i}\int^t V^I_{\text{as}}(\tau)\mathrm{d}\tau \ , \tag{1.22}$$

and $\Phi(t)$ is a phase that will play an unimportant role in the IR discussion and will be therefore omitted. Once integrated $R_{J_{\text{as}}}(t)$ is

$$R_{J_{\text{as}}}(t) = \frac{1}{\sqrt{2(2\pi)^3}}\int\frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}}\sum_l\int\mathrm{d}^3\boldsymbol{p}\,\hat{\rho}(\boldsymbol{p})\left(\frac{p\cdot\varepsilon^*_{l,\boldsymbol{k}}}{p\cdot k}\,\hat{a}^\dagger_{l,\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\frac{pk}{p_0}t} - \text{h.c.}\right) \ . \tag{1.23}$$

Acting with the operator $\exp\left\{R_{J_{\text{as}}}(t)\right\}$ to an arbitrary state $|\alpha\rangle$ with certain amount of charged particles in the Fock space, or also called *dressing* the state, projects the state into the bigger von Neumann space, see section 1.3.3. Thus the in and out states of the scattering process get dressed by a coherent operator, which we define as

$$\hat{W}(t) := \exp\left\{R_{J_{\text{as}}}(t)\right\} = \exp\left\{\frac{1}{\sqrt{2(2\pi)^3}}\int\frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}}\sum_l\int\mathrm{d}^3\boldsymbol{p}\,\hat{\rho}(\boldsymbol{p})\left(\frac{p\cdot\varepsilon^*_{l,\boldsymbol{k}}}{p\cdot k}\,\hat{a}^\dagger_{l,\boldsymbol{k}}\mathrm{e}^{\mathrm{i}\frac{pk}{p_0}t} - \text{h.c.}\right)\right\} \ . \tag{1.24}$$

Alternatively it is though possible to define a different dressing operator which is equivalent to the one in (1.24) and maps the bare states into the *same equivalent class* in the sense of von Neumann:

$$\hat{W} := \exp\left\{\frac{1}{\sqrt{2(2\pi)^3}}\int\frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}}\sum_l\int\mathrm{d}^3\boldsymbol{p}\,\hat{\rho}(\boldsymbol{p})\left(\frac{p\cdot\varepsilon^*_{l,\boldsymbol{k}}}{p\cdot k}\varphi(\boldsymbol{k},\boldsymbol{p})\,\hat{a}^\dagger_{l,\boldsymbol{k}} - \text{h.c.}\right)\right\} \ , \tag{1.25}$$

where in the neighborhood of $\boldsymbol{k} = 0$ the convoluting function fulfills $\varphi(\boldsymbol{k},\boldsymbol{p}) = 1$, thus it can be neglected since the photons in the dressing operator (1.24) are infrared, i.e. $\boldsymbol{k} \approx 0$. It is the dressing operator defined in (1.25) that was used by [8, 9] to cancel the IR divergence in the $S$-matrix. For a bare state $|\alpha\rangle$ one gets the corresponding dressed state

$$|\alpha\rangle\rangle := \hat{W}\,|\alpha\rangle = |\alpha\rangle \otimes |D(\alpha)\rangle \ , \tag{1.26}$$

where

$$|D(\alpha)\rangle := \exp\left\{\int \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}} \sum_l \mathcal{F}_\alpha^{(l)}(\boldsymbol{k})\, \hat{a}^\dagger_{l,\boldsymbol{k}} - \text{h.c.}\right\} |0\rangle$$
$$= \exp\left\{-\frac{1}{2}\int \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|} \sum_l \left|\mathcal{F}_\alpha^{(l)}(\boldsymbol{k})\right|^2\right\} \exp\left\{\int \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}} \sum_l \mathcal{F}_\alpha^{(l)}(\boldsymbol{k})\, \hat{a}^\dagger_{l,\boldsymbol{k}}\right\} |0\rangle \ . \quad (1.27)$$

The form factor is given by

$$\mathcal{F}_\alpha^{(l)}(\boldsymbol{k}) = \sum_{n\in\alpha} \frac{e_n}{\sqrt{2(2\pi)^3}} \frac{p_n \cdot \varepsilon^*_{l,\boldsymbol{k}}}{p_n \cdot k} \ . \quad (1.28)$$

### 1.3.1 Coherent states in general

As in general for coherent states, the coherent state in (1.27) is an eigenstate of the photon annihilation operator $a_{l,\boldsymbol{k}}$ with the eigenvalue

$$\hat{a}_{m,\boldsymbol{q}}\, |D(\alpha)\rangle = \frac{\mathcal{F}_\alpha^{(m)}(\boldsymbol{q})}{\sqrt{|\boldsymbol{q}|}}\, |D(\alpha)\rangle \ , \quad (1.29)$$

and they are normalized $\langle D(\alpha)|D(\alpha)\rangle = 1$ so that the number of photons in that state is given by

$$\langle D(\alpha)|\, \hat{N}\, |D(\alpha)\rangle = \int \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|} \sum_l \left|\mathcal{F}_\alpha^{(l)}(\boldsymbol{k})\right|^2 \ , \quad (1.30)$$

where we used

$$\hat{N} = \sum_l \int \mathrm{d}^3\boldsymbol{k}\, \hat{a}^\dagger_{l,\boldsymbol{k}} \hat{a}_{l,\boldsymbol{k}} \ . \quad (1.31)$$

The scalar product or overlap of two different coherent states is[6]

$$\langle D(\beta)|D(\alpha)\rangle = \exp\left\{-\frac{1}{2}\int \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|} \sum_l \left|\mathcal{F}_\alpha^{(l)}(\boldsymbol{k})\right|^2 + \left|\mathcal{F}_\beta^{(l)}(\boldsymbol{k})\right|^2\right\}$$

[6] We use that $\sum_l \mathcal{F}_\alpha^{(l)*}(\boldsymbol{k})\mathcal{F}_\beta^{(l)}(\boldsymbol{k})$ is real.

$$\cdot \langle 0| \exp\left\{ \int \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}} \sum_l \mathcal{F}_\beta^{(l)*}(\boldsymbol{k}) \hat{a}_{l,\boldsymbol{k}} \right\} \exp\left\{ \int \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}} \sum_l \mathcal{F}_\alpha^{(l)}(\boldsymbol{k}) \hat{a}_{l,\boldsymbol{k}}^\dagger \right\} |0\rangle$$

$$= \exp\left\{ -\frac{1}{2} \int \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|} \sum_l \left| \mathcal{F}_{\alpha,\beta}^{(l)}(\boldsymbol{k}) \right|^2 \right\} , \tag{1.32}$$

where we used

$$\mathcal{F}_{\alpha,\beta}^{(l)}(\boldsymbol{k}) = \mathcal{F}_\beta^{(l)}(\boldsymbol{k}) - \mathcal{F}_\alpha^{(l)}(\boldsymbol{k}) = \sum_{n\in\alpha,\beta} \frac{e_n \eta_n}{\sqrt{2(2\pi)^3}} \frac{p_n \cdot \varepsilon_{l,\boldsymbol{k}}^*}{p_n \cdot k} \tag{1.33}$$

which is (1.6) for one photon with momentum $\boldsymbol{k}$ and polarization $l$.[7] The expectation value of the energy of a coherent state is

$$\bar{E}_{D(\alpha)} = \langle D(\alpha)| \hat{H}_0 |D(\alpha)\rangle = \int \mathrm{d}^3\boldsymbol{k} \sum_l \left| \mathcal{F}_{\alpha,\beta}^{(l)}(\boldsymbol{k}) \right|^2 , \tag{1.34}$$

where we used $\hat{H}_0 = \sum_l \mathrm{d}^3\boldsymbol{k}\, |\boldsymbol{k}|\ \hat{a}_{l,\boldsymbol{k}}^\dagger \hat{a}_{l,\boldsymbol{k}}$ for the free Hamilton operator.

### 1.3.2 Finite $S$-matrix

Chung showed in [9] that the $S$-matrix of a scattering process $\alpha \to \beta$, where the in and out states are dressed like in (1.26) and where the soft loop corrections of the tree level process are included, is IR finite to all orders in perturbation theory:

$$\langle\langle\beta| S_{\alpha,\beta}^{\text{loop}} |\alpha\rangle\rangle = \left(\frac{r}{\Lambda}\right)^{B_{\alpha,\beta}/2} S_{\alpha,\beta}\, \mathrm{e}^{\mathrm{i}\phi} , \tag{1.35}$$

where $\phi$ is an IR finite phase that vanishes after computing the cross section and $r$ is for the time being just a parameter with units of energy to fix the IR divergences coming from the virtual loops. We will come back to this parameter in more detail in chapter 4. Notice that here the factor that comes from the energy conservation $f(B_{\alpha,\beta})$ in the inclusive formalism is missing compared to (1.13). This is due to the fact that the dressed states defined in (1.26)

[7]Notice that it is possible to describe the same physical process with dressings that are shifted by a common function, $\mathcal{F}_\alpha^{(l)}(\boldsymbol{k}) \to \mathcal{F}_\alpha^{(l)}(\boldsymbol{k}) + C(\boldsymbol{k})$ and $\mathcal{F}_\beta^{(l)}(\boldsymbol{k}) \to \mathcal{F}_\beta^{(l)}(\boldsymbol{k}) + C(\boldsymbol{k})$. Such modifications of the dressing states have recently been considered in [18, 27].

contain an infinite number of zero-energy photons, which are defined in the von Neumann space. When Chung, as well as Faddeev and Kulish, showed that the coherent states lead to an IR finite $S$-matrix they kept the limits of integration in the coherent states throughout his calculations undefined just until they reached the final IR finite result. We want to address the question of the limits in the integration of the coherent states defined in (1.27) later in chapter 4 in more detail and clarify this point. Since this is a review chapter of the original prescription of the coherent state approach we keep it as it was originally presented and omit the boundaries of integrations as well.

### 1.3.3 Introduction to von Neumann Spaces

We begin by giving a brief review of how the Fock space can be constructed and what complications arise in a gapless theory. Our starting point are the Hilbert spaces in each momentum mode $\boldsymbol{k}$. So we are given well-defined Hilbert spaces $\mathcal{H}_{\boldsymbol{k}}$, which feature inner products $\langle\ ,\ \rangle_{\boldsymbol{k}}$ and creation and annihilation operators $\hat{a}^{\dagger}_{l,\boldsymbol{k}}$, $\hat{a}_{l,\boldsymbol{k}}$ that fulfill canonical commutation relations:

$$\left[\hat{a}_{l,\boldsymbol{k}}, \hat{a}^{\dagger}_{l',\boldsymbol{k}'}\right] = \delta_{ll'}\, \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\,, \qquad \left[\hat{a}_{l,\boldsymbol{k}}, \hat{a}_{l',\boldsymbol{k}'}\right] = \left[\hat{a}^{\dagger}_{l,\boldsymbol{k}}, \hat{a}^{\dagger}_{l',\boldsymbol{k}'}\right] = 0\,. \tag{1.36}$$

We already included the polarization $l$ since we will later be interested in photons. The problem lies in defining the tensor product $\bigotimes_{\boldsymbol{k}} \mathcal{H}_{\boldsymbol{k}}$ of the infinitely many Hilbert spaces corresponding to all possible momenta $\boldsymbol{k}$.

For this task we can rely on the seminal work by von Neumann [26], who defined the space $\mathcal{H}_{\text{VN}} \subset \bigotimes_{\boldsymbol{k}} \mathcal{H}_{\boldsymbol{k}}$. It consists of elements for which a scalar product can be defined. For $|\varphi\rangle\,, |\Psi\rangle \in \mathcal{H}_{\text{VN}}$, i.e. $|\varphi\rangle = \otimes_{\boldsymbol{k}} |\varphi\rangle_{\boldsymbol{k}}$ and $|\Psi\rangle = \otimes_{\boldsymbol{k}} |\Psi\rangle_{\boldsymbol{k}}$, it is given as

$$\langle\varphi|\Psi\rangle := \prod_{\boldsymbol{k}} \langle\varphi_{\boldsymbol{k}}|\Psi_{\boldsymbol{k}}\rangle_{\boldsymbol{k}}\ . \tag{1.37}$$

It is clear from this definition that the von Neumann space is very big. In particular, it contains any product of states that are normalizable in the individual $\mathcal{H}_{\boldsymbol{k}}$, i.e. $|\varphi\rangle = \otimes_{\boldsymbol{k}} |\varphi\rangle_{\boldsymbol{k}}$ such that $\langle\varphi_{\boldsymbol{k}}|\varphi_{\boldsymbol{k}}\rangle_{\boldsymbol{k}} = 1$ for all $\boldsymbol{k}$. Without loss of generality we will assume normalized states from now on.

This scalar product defines an equivalence relation in the von Neumann space given by

$$|\varphi\rangle \sim |\Psi\rangle \;:\Leftrightarrow\; \sum_{\boldsymbol{k}} \left|\langle\varphi_{\boldsymbol{k}}|\Psi_{\boldsymbol{k}}\rangle_{\boldsymbol{k}} - 1\right| \text{ convergent.} \tag{1.38}$$

The significance of this equivalence relation lies in the fact that elements from different *equivalence classes* are orthogonal,

$$|\varphi\rangle \nsim |\Psi\rangle \;\Rightarrow \langle\varphi|\Psi\rangle = 0\,. \tag{1.39}$$

Therefore, the equivalence classes constitute mutually disjoint subspaces in the von Neumann space. The physical implications of this construction were derived in [28]. First of all, a special role is played by the equivalence class of $|0\rangle := \otimes_{\boldsymbol{k}} |0\rangle_{\boldsymbol{k}}$, which we denote by $[0]$. In it, one has the standard representation of canonical commutation relations (1.36). Then we can write the particle number operator in equation (1.31) as

$$\hat{N} = \sum_{\boldsymbol{k},l} \hat{a}^{\dagger}_{l,\boldsymbol{k}} \hat{a}_{l,\boldsymbol{k}}\,, \tag{1.40}$$

where we write the $\boldsymbol{k}$ integral as sum. Thus, $\langle\varphi|\hat{N}|\varphi\rangle$ is finite for each $\varphi \in [0]$. Therefore, this equivalence class alone represents the whole Fock space.

One can also understand the other equivalence classes in terms of particle number [28]. Two states are in the same equivalence classes if and only if their difference in particle number is finite:

$$|\varphi\rangle \sim |\Psi\rangle \;\Leftrightarrow\; \langle\varphi|\hat{N}|\varphi\rangle - \langle\Psi|\hat{N}|\Psi\rangle < \infty\,, \tag{1.41}$$

where it is understood that the subtraction is performed before the sum over the momentum modes. Since one can moreover show that each equivalence class is isomorphic to the Fock space, it follows that the von Neumann space can be thought of as infinite product of Fock spaces with unitarily inequivalent representations of the commutation relations in each subspace. So in each equivalence class $[\alpha]$, we have:

$$\left[\hat{a}^{[\alpha]}_{l,\boldsymbol{k}}, \hat{a}^{[\alpha]\dagger}_{l',\boldsymbol{k}'}\right] = \delta^{(3)}(\boldsymbol{k}-\boldsymbol{k}')\delta_{ll'}\,, \qquad \left[\hat{a}^{[\alpha]}_{l,\boldsymbol{k}}, \hat{a}^{[\alpha]}_{l',\boldsymbol{k}'}\right] = \left[\hat{a}^{[\alpha]\dagger}_{l,\boldsymbol{k}}, \hat{a}^{[\alpha]\dagger}_{l',\boldsymbol{k}'}\right] = 0\,. \tag{1.42}$$

This immediately raises the question what subspace of $\mathcal{H}_{\text{VN}}$ is physically relevant. A reasonable requirement for any state to be physical is that it contains finite energy. Whenever a theory has a mass gap, the Fock space – defined by the requirement of finite particle number – is the only equivalence class with finite energy and therefore contains all physically reasonable states. So it makes sense to restrict oneself to the Fock space.

However, the situation is drastically different in a gapless theory. Then there can be states that contain an infinite amount of zero modes but nevertheless carry finite energy. Therefore, there are distinct equivalence classes with finite energy and there is no reason to restrict oneself to only one of them. In a gapless theory, states of different equivalence classes are therefore physically sensible. In fact, as already noticed in [28] and emphasized recently in [18], the $S$-matrix generically enforces the transition between different equivalence classes so that it is impossible to restrict oneself to a single equivalence class in an interacting system. We will elaborate in chapter 4 on how this comes about. The fact that states in different equivalence classes are – by definition – orthogonal will be crucial for our discussion of IR physics.

# Chapter 2

# Memory and the Infrared

## 2.1 Introduction

In this chapter we want to give a small introduction to the so calle memory effect in theories with long-ranged interactions and its connection to the infrared physics of the theory. Memory effects in gravitational scattering were first discussed in [29] and later on developed in [30,31]. Similar memory effects can be derived in classical electromagnetism (see [32,33] for a recent discussion). In a series of papers [20, 34, 35] a new understanding of memory effects in connection with soft theorems [7, 36, 37] has been put forward. The main result is that the infrared part of the Fourier transform of the classical memory effect is determined by the soft photon or graviton theorems. In addition it was shown in [14] that soft photon theorems can be interpreted as Ward identities relative to QED symmetries.

Related with this research the classic topic of infrared divergences in quantum field theories has been revisited. As pointed out in the review chapter 1 in theories like QED we have IR divergences due to virtual photons. These divergences can be resumed and regulated. In addition we have soft radiation and we can, at the level of the cross section or rate, sum over amplitudes square for different number of soft emitted photons. These two contributions, namely the one associated with virtual photons and the one coming from summing over different number of final infrared soft photons lead to infrared divergent pieces that cancel each other in the final cross section. What remains is an infrared finite cross section. The concrete form of this cross section or rate (1.13) depends on an infrared scale $\epsilon$, the total energy emitted by soft IR photons. More precisely the final differential cross section factorizes into a pure infrared part that only depends on the scattering data but not on the details of the scattering and a non-infrared part, see (1.13). This factorization of the cross section depends on the infrared scale used to define the upper limit (the total energy emitted $\epsilon$) on the energies for real infrared, measurable photons. The infrared part of the cross section, after IR divergences are cancelled, depends on the infrared scale $\epsilon$ in the general form $\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon} \sim \mathrm{e}^{F(\epsilon)}$ with $F(\epsilon) \sim \ln\left(\frac{\epsilon}{\Lambda}\right)$ (see (1.13) again).

How electromagnetic memory effects are related with this issue? The quick answer to this question is the following. In scattering processes among massive charged particles the charges and momentum of the in and out particles determine the non-radiative part of the asymptotic retarded field. These in and out scattering data are enough to extract the zero mode part, i.e. the photons with $\boldsymbol{k} = 0$ or based on the notation from previous chapter photons with energy $|\boldsymbol{k}| = \lambda$, where $\lambda$ goes to zero, of the interpolating retarded field and consequently they account for the information contained in the soft photon theorems. In contact with the coherent state approach to cancel IR divergences of the soft loop corrections, in the IR-finite $S$-matrix (1.35), these zero-energy modes are decoupled (see for example [38]) and moreover they don't lead to any *observable* (in a finite amount of time) memory effect.

The scattering data, although enough to derive the soft photon theorem, are not enough to fix the *radiative component* of the retarded field that depends on concrete information on how the scattering process is actually taking place, in particular (in the classical case) on the accelerations. This radiative part of the retarded field carries energy as well as radiative modes with typical frequencies of the order of the inverse of the time scale on which the scattering process is taking place. The *observability* of the memory effect using a physical detector depends crucially on this radiated energy. In QED this information is partially encoded in the infrared part of the differential cross section, namely on the dependence on the radiated energy $\epsilon$ that we can take as equal to the energy involved in the actual detection of the memory effect. In particular we shall associate to memory the infrared part of the cross section that only depends on the scattering data and where we consider an arbitrary number of emitted real infrared photons with total energy $\epsilon$ equal to the energy involved in the memory detection.

In this chapter we shall reduce the discussion of memory to the electromagnetic case and only at the end we will make few comments on similarities and differences with the gravitational case.

## 2.2 Classical Memory

For a given classical scattering where some initial charges $e_j$ with velocities $\boldsymbol{v}_j$ lead to a final state with charges $e_i$ and velocities $\boldsymbol{v}_i$ the electromagnetic memory is determined by the retarded field created by the currents $J^\mu$ defined by these scattering data. In four dimensions the retarded electromagnetic field at some observation point $O = (\boldsymbol{x}, t)$ is given by

$$A^\mu(\boldsymbol{x}, t) = \int \frac{J^\mu\left(\boldsymbol{x}', u\right)}{R}\, \mathrm{d}^3\boldsymbol{x}' \tag{2.1}$$

with $R^2 = (\boldsymbol{x} - \boldsymbol{x}')^2$ and $u = t - \frac{R}{c}$. For small velocities we can Taylor expand the current and to define the retarded field as a series in $1/c$. The field tensor $F_{\mu\nu}$ generated by the moving charges can be expanded in powers of $1/R$. It contains a piece that goes like $1/R^2$ that only depends on the velocities of the sources and a piece that goes like $1/R$ that accounts for the radiation emitted during the scattering process.

For an idealised point-like scattering taking place at the origin the radiative part of the retarded field has support on the $u = 0$ null hyper surface $t = \frac{R}{c}$. This simply reflects the fact that only at the origin the moving particles entering into the scattering are accelerated. At large distances $\boldsymbol{x} \gg \boldsymbol{x}'$ the retarded field field is given by

$$A^\mu(x) = \sum_{i\in\text{out}} \frac{\theta(u)}{R} \frac{q_i v_i^\mu}{1 - \boldsymbol{v}_i \hat{\boldsymbol{x}}} + \sum_{j\in\text{in}} \frac{\theta(-u)}{R} \frac{q_j v_j^\mu}{1 - \boldsymbol{v}_j \hat{\boldsymbol{x}}} \tag{2.2}$$

where $\hat{\boldsymbol{x}}$ is the norm vector of $\boldsymbol{x}$ and from now on we set the speed of light $c = 1$.

The field tensor is then given by

$$\begin{aligned} F_{\mu\nu} = &\sum_{i\in\text{out}} \frac{q_i \hat{x}_{[\mu} v_{\nu]i}}{\hat{x}_\alpha v_i^\alpha} \left[ \frac{1}{R} \frac{\delta(u)}{\hat{x}_\alpha v_i^\alpha} + \frac{1}{R^2} \frac{v_{i\beta} v_i^\beta}{\left(\hat{x}_\alpha v_i^\alpha\right)^2} \theta(u) \right] \\ &+ \sum_{j\in\text{in}} \frac{q_j \hat{x}_{[\mu} v_{\nu]j}}{\hat{x}_\alpha v_j^\alpha} \left[ -\frac{1}{R} \frac{\delta(u)}{\hat{x}_\alpha v_j^\alpha} + \frac{1}{R^2} \frac{v_{j\beta} v_j^\beta}{\left(\hat{x}_\alpha v_j^\alpha\right)^2} \theta(-u) \right] \end{aligned} \tag{2.3}$$

where

$$\hat{x}^\mu = \begin{pmatrix} 1 \\ \hat{\boldsymbol{x}} \end{pmatrix} \quad \text{and} \quad v_i^\mu = \begin{pmatrix} 1 \\ \boldsymbol{v}_i \end{pmatrix} \tag{2.4}$$

and indices are raised and lowered as usual in cartesian coordinates by the metric $\eta_{\mu\nu}$.

As we can see from the former expression the radiative $1/R$ part depends on the concrete classical modelling of the scattering. In this simple case in the form of an instantaneous change of the velocities taking place at the origin. The non-radiative part that goes as $1/R^2$ depends only on the in and out scattering data.

The *classical memory* effect associated with a given scattering process where we use as data the in and out 4-momentum of the scattered particles is given by the couple of *non-radiative fields* $F_{\text{in}}$ and $F_{\text{out}}$. However the actual detection of the memory is determined by the interaction of some charged detector with the interpolating *radiative field*. This effect on the memory detector is non-vanishing and *observable* due to the fact that the interpolating radiative field carries non-vanishing energy $\epsilon$.

### 2.2.1 Spectral resolution

For future convenience it would be important to work out the spectral decomposition of the asymptotic retarded fields defined by the in and out set of free moving charged particles. The Fourier modes of the retarded field are given by

$$A^\mu(t, \boldsymbol{k}) = \sum_{i \in \text{out}} \frac{4\pi e_i}{|\boldsymbol{k}| \left(1 + \hat{\boldsymbol{k}} \boldsymbol{v}_i\right)} \frac{p_i^\mu \mathrm{e}^{-\mathrm{i} \frac{\boldsymbol{k} \boldsymbol{p}_i}{E_i} t}}{p_i^\alpha k_\alpha} \Bigg|_{t>0} + \sum_{j \in \text{in}} \frac{4\pi e_j}{|\boldsymbol{k}| \left(1 + \hat{\boldsymbol{k}} \boldsymbol{v}_j\right)} \frac{p_j^\mu \mathrm{e}^{-\mathrm{i} \frac{\boldsymbol{k} \boldsymbol{p}_j}{E_j} t}}{p_j^\alpha k_\alpha} \Bigg|_{t<0} \tag{2.5}$$

where $p_i^\mu$ is the 4-momentum of the $i^{\text{th}}$ particle and $k^\mu = \left(|\boldsymbol{k}|, \boldsymbol{k}\right)$.

The important thing to be noticed is that the Fourier components of the retarded field created by a moving charge with constant velocity $\boldsymbol{v}_i$ are waves with wave vector $\boldsymbol{k}$ but frequency $\omega_i = \boldsymbol{k}\boldsymbol{v}_i$. These Fourier modes are obviously not real photons with the exception of the soft $\boldsymbol{k} = 0$ mode. Once we move into quantum field theory these modes will define *the quantum constituents* of the coherent state dressing of free moving charged particles.

### 2.2.2 Symmetries, Goldstones and Energy Conservation

Classically we can associate with a given scattering process among charged particles the non-radiative retarded fields defined by the in and out scattering data i.e. by the charges, masses and velocities of the incoming and outgoing particles. Let us generically denote $A_{\text{in}}$ and $A_{\text{out}}$ these retarded fields. Associated with these data we can formally define a transformation $T : A_{\text{in}} \to A_{\text{out}}$. This transformation is not a gauge transformation since $A_{\text{in}}$ and $A_{\text{out}}$ although satisfying the condition

$$\lim_{R\to\infty} R\left(F(A_{\text{in}}) - F(A_{\text{out}})\right) = 0 \tag{2.6}$$

have, at order $1/R^2$, different values of the corresponding stress tensor.

Let us now fix the asymptotic kinematical data for the incoming and outgoing charges in such a way that energy and momentum is conserved i.e $\sum_{j\in\text{in}} E_i = \sum_{i\in\text{out}} E_j$. In this case conservation of energy will implies that the only possible radiated mode is a zero-energy zero mode. In classical electrodynamics this constraint is not easy to impose. Indeed if we fix the scattering data and we use those kinematical data to derive the classical radiated field we will only achieve total energy conservation if in addition we take into account *the back reaction* of the radiated field i.e. the Abraham-Lorentz forces on the outgoing scattering data. As it is well known this problem cannot be fully solved in classical electrodynamics [39].

We can however formally impose the conservation of energy on the scattering data for the charged particles which is effectively equivalent to set the net amount of radiated energy to be equal to zero. To understand the physical meaning of this *zero radiated energy constraint* it can be illustrative to recall the attempt of Wheeler and Feynman (WF) [40, 41] to define in classical electrodynamics the radiative reaction on sources in the context of the absorber theory. Indeed if we think that all the radiation emitted is absorbed leading to zero radiated energy we get that asymptotically we can impose the WF condition:

$$F_{\text{ret}} = F_{\text{adv}} \tag{2.7}$$

for the radiative part of the total advanced and retarded fields[1]. Generically, although in Maxwell theory we have the advanced and retarded solution, only the retarded part of the

[1]Note that the condition (2.7) allows us to define the WF field associated with a moving charge as $1/2(F_{ret} + F_{adv})$.

radiative field is actually considered as physical. Thus the former condition makes sense if we have a formal *absorber* and no net radiation carrying non-zero energy is left free.

In scattering language we can think of the advanced field as associated with some incoming radiation and the retarded field as the outgoing radiation, so if we consider a scattering with in state defined by a set of *only* charged particles (and zero radiation) the former condition (2.7) only makes sense for the zero mode part that does not carry any energy.

The equality between retarded and advanced fields (2.7) leads to a set of conservation laws where the classical charges can be defined by the convolution of (2.7) with arbitrary test functions [42]. The so called soft charges can be defined as those determined by the zero mode part of the retarded and advanced fields.

In a scattering process among charged particles where we use as scattering data a set of in and out momenta for the charged matter satisfying conservation of energy i.e. with no net radiation, we can impose the condition (2.7) and these charges will act as symmetries of the $S$-matrix. Since there is not radiation the only relevant piece is the zero mode soft part. In this case any memory effect defined as the difference between the non-radiative part of the retarded fields created by the incoming and outgoing particles is physically *unobservable*. This unobservability becomes equivalent, in the $S$-matrix language, to the decoupling of the radiative zero mode [38, 43–45].

In summary the "new symmetries" of the QED $S$-matrix [14, 17] are a consequence of imposing what we can call the WF condition or in more physical terms the absence of any loos of energy in the form of radiated infrared photons. This condition is naturally implemented in any $S$-matrix formulation where in and out states are sets of charged particles. However in order to have observable memory effects a certain amount of energy should be radiated and in that case we need to work with the differential cross section. Once some energy is actually radiated we cannot impose (2.7) since this energy is only contained in the retarded part of the field.

In reality the probability that in a physical scattering we have zero net radiated energy is indeed zero,[2] so that these symmetries of the IR-finite $S$-matrix only account for the soft theorem part. We can think of the symmetries for zero energy radiation processes as being spontaneously broken with the $\boldsymbol{k} = 0$ soft mode as a Goldstone boson. However we would like to stress that whenever we have a real amount of energy radiated with no incoming radiation, which is actually always the case, the condition (2.7) can only be imposed for

[2] The rate (1.13) vanishes in the limit $\epsilon \to 0$ when $B_{\alpha,\beta} \neq 0$.

the zero mode part which is what, as we shall discuss in moment, you actually do in the definition of the IR-finite $S$-matrix.

## 2.3 Memory and infrared QED

### 2.3.1 IR-finite $S$-matrix

As already pointed out in section 1.3, a prescription to define an IR-finite $S$-matrix was partially developed. The key ingredient in this construction was to use the asymptotic dynamics in order to define new asymptotic states by dressing standard Fock matter states $|\alpha\rangle$ with the coherent state of photons sourced by the *asymptotic current* $J^{\mu}_{\text{as}}$. We can represent this dressing as in (1.26)

$$|\alpha\rangle \rightarrow \mathrm{e}^{R_{J_{\text{as}}}} |\alpha\rangle \ . \tag{2.8}$$

We can now easily identify the operator $R_{J_{\text{as}}}$. Using the spectral decomposition of the retarded field (2.5) created by the asymptotic free moving charges we can define the quantum resolution[3] of this field using as quantum constituents, quanta with momentum $\boldsymbol{k}$ and frequency

$$\omega_i = \boldsymbol{k}\boldsymbol{v}_i \ . \tag{2.9}$$

Denoting the creation annihilation operators for these quanta $\hat{c}_{l,\boldsymbol{k}}$ and $\hat{c}^{\dagger}_{l,\boldsymbol{k}}$ the corresponding coherent state will be defined by the operator

$$\exp\left\{\sum_l \int \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|}\left(\sum_{i\in\alpha} \frac{4\pi e_i}{\left(1+\hat{\boldsymbol{k}}\boldsymbol{v}_i\right)} \frac{p_i\cdot\varepsilon_{l,\boldsymbol{k}}}{p_i\cdot k}\, \hat{c}^{\dagger}_{l,\boldsymbol{k}}\, \mathrm{e}^{-\mathrm{i}\frac{\boldsymbol{k}\boldsymbol{p}_i}{E_i}t}\right)\right\} \tag{2.10}$$

acting on the vacuum defined by $\hat{c}_{l,\boldsymbol{k}}|0\rangle = 0$. If we want to use the creation annihilation operators $a_{l,\boldsymbol{k}}$ and $a^{\dagger}_{l,\boldsymbol{k}}$ of the Fock space of free photons with dispersion relation $\omega = |\boldsymbol{k}|$ we need to transform $\hat{c}_{l,\boldsymbol{k}}$ modes into $\hat{a}_{l,\boldsymbol{k}}$ photons. This leads to the FK expression derived from

[3] For other examples of the same technique see [46] and [47]

the asymptotic dynamics, namely (1.27)

$$\exp\left\{\sum_l \int \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}} \sum_{i\in\alpha} \frac{e_i}{\sqrt{2(2\pi)^3}} \frac{p_i\cdot\varepsilon_{l,\boldsymbol{k}}}{p_i\cdot k} \mathrm{e}^{\mathrm{i}\frac{p_i k}{p_i^0}t} \hat{a}^\dagger_{l,\boldsymbol{k}}\right\} . \tag{2.11}$$

The transformation of the $\hat{c}_{l,\boldsymbol{k}}$ modes into $\hat{a}_{l,\boldsymbol{k}}$ photons is a transformation in the same equivalence class, so that both modes describe the same asymptotic dynamics of the bare states. By construction on these coherent states the expectation value of the field operator $\hat{A}^\mu$ is given by the classical retarded field. Note that these coherent states contain an infinite number of $\boldsymbol{k} = 0$ photons because $\lambda \to 0$ in the integrals. If in the scattering process we impose zero energy radiated then the total number of modes in the in and out states will be conserved.

We can consider a more complicated coherent state of photons describing the whole radiative part of the retarded field and to think of this coherent state as a sort of *domain wall* interpolating between the asymptotic in and out retarded fields. The soft photon theorem accounts for the zero mode part of this *domain wall*. The radiated energy acting on the potential memory detector is roughly what we can interpret as the *mass* of this photonic domain wall.

The IR-finite $S$-matrix is defined by

$$\lim_{t\to\infty} \hat{W}(-t)\, S\, \hat{W}(t) , \tag{2.12}$$

where $\hat{W}(t)$ is defined in (1.24).[4] This $S$-matrix satisfies the decoupling of soft modes [38, 43, 44][5]

$$\lim_{\boldsymbol{k}\to 0}[S, \hat{a}_{l,\boldsymbol{k}}] = 0 . \tag{2.13}$$

The so defined $S$-matrix is IR-finite due to the fact that the former dressing factor cancels the infrared divergences (after resummation) coming from the virtual photon self energies.

Note that in this $S$-matrix we are imposing the zero-energy radiation condition (2.7) and consequently the $S$-matrix commutes with the charges defined by convoluting (2.7) with arbitrary test functions. These Ward identities are simply reflecting the kinematical constraints

[4]Notice that one can either define coherent states with the operator $\hat{W}(t)$ as in the review chapter or as in this case the IR-finite $S$-matrix. The main point is that the matrix elements are finite at the end in both prescriptions.

[5]For a more rigorous proof see [45]

we are imposing on the scattering states, namely vanishing net energy in the form of radiation for in states without real photons and are fully equivalent to the decoupling of soft photons. Note also that, in this case, the so called hard charges [14, 17, 38] are absorbed in the dressing.

It is important to stress that the decoupling of soft modes should not be confused with the absence of *observable* memory effects. Indeed as already stressed observable memory requires a certain amount of energy in the retarded field to be radiated in the form of infrared emitted photons and therefore does not satisfy the $S$-matrix matching condition for the charged kinematical data.

### 2.3.2 QED measure of memory

Given a scattering process in QED we can associate, as a way to characterize the memory, the differential cross section $\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon}$ for $\epsilon$ the radiated energy in form of infrared photons. The dependence of the cross section on $\epsilon$ is well known in QED [6]. We shall be interested only in the infrared part of the cross section i.e. in the part that only depends on scattering data

$$\left.\frac{\mathrm{d}\sigma}{\mathrm{d}\epsilon}\right|_{\mathrm{IR}} \sim A\mathrm{e}^{\ln\frac{\epsilon}{\Lambda}} \,, \tag{2.14}$$

with $A$ a finite coefficient depending only on the scattering data. This infrared part of the cross section corresponds to have arbitrary number of emitted infrared photons with total energy equal or less than $\epsilon$ i.e. is the cross section $\sigma(\alpha \to \beta + \gamma(E_{\text{soft}} \leq \epsilon))$ and is an IR safe quantity.

The important message of these cross sections is the dependence on the energy radiated. This is important for understanding the real nature of the memory. In fact we could think in nullifying the memory by pushing $\epsilon \to 0$. In this case the only remnant will be the zero mode part of the radiative mode that is actually decoupled. However the dependence of the cross section on $\epsilon$ is telling us that such a formal limit cannot be taken or equivalently that the actual probability to scatter without radiating is zero. In fact the only possibility that the cross section is non-zero in that limit if we have are in the forward limit scattering where the angles of the charged in and out particles matches antipodally. There the factor $B_{\alpha,\beta}$ vanishes and the limit $\epsilon \to 0$ can be taken. The interpolating radiative field measured by the memory detector contains energetic modes in addition to the Goldstone zero mode piece.

The actual interaction of the detector with these modes is what makes the memory effect, in scattering processes, actually observable in a finite amount of time.

# Chapter 3

# Infrared Divergences and Quantum Coherence

## 3.1 Introduction

As we have seen in section 1.2, we can define IR-finite rates by integrating over soft IR emission. A very interesting question raised in [48, 49] is if we can go one step further and define an IR-finite density matrix. Obviously, its diagonal elements are determined by the known IR-finite rates. So the task consists in determining the IR limit of the off-diagonal pieces of this density matrix. These elements contain the information about how much quantum coherence we lose by tracing over the soft IR radiation, which are entangled with the charged particles. This is an important question because this tracing is, as stressed in section 1.2, not due to any limit on detector resolution but – at least in the present formalism – a prerequisite for IR-finiteness.

The result of the calculation in [48, 49] was that almost all off-diagonal elements are zero in an arbitrary scattering process. This finding is surprising since it was obtained in the absence of any environment and does not depend on detector resolution. So it seems to imply that the requirement of IR-finiteness alone inevitably leads to an almost completely decohered density matrix. If this were true, we would find a very disturbing physical picture. Then the final state of an arbitrary scattering process would be a fully decohered mixed state and it would become impossible to have an IR-finite description of quantum interference phenomena. Concretely, we can perform a double-slit experiment with the products of a scattering event, i.e. we take the final state of scattering as initial state for the double-slit experiment. If the final scattering state were really fully decohered, it would never be able to lead to interference patterns in flagrant conflict with experimental evidence.

As it is well-known, decoherence is an omnipresent phenomenon in any experimental setup because of inevitable interaction with the environment. Nevertheless, this fact does not preclude quantum interference phenomena. The reason is that the experiment takes place in a time smaller than the time of decoherence. Even in the absence of an environment, it is conceivable that integration over IR radiation leads to decoherence depending on its entanglement with the hard scattering data. A natural estimate of the resulting decoherence time would be that it scales inversely with the energy in IR modes. Therefore, if one waits for an *infinite* amount of time, an arbitrarily small energy in IR modes could lead to full decoherence [48].

In contrast, our aim is to propose a density matrix that can describe the quantum coherence observed in experiments, i.e. that has non-vanishing off-diagonal elements. In doing so, we will achieve two goals. First, we point out why the off-diagonal elements vanish. The reason is a soft loop contribution, very similar to the one discussed in the section 1.1 that leads to vanishing amplitudes. Its origin lies in the contribution of zero-energy IR radiation. Secondly, we show that the requirement of IR-finiteness alone does not inevitably lead to full decoherence by providing an explicit example of a density matrix that is IR-finite but not fully decohered.

We will construct this density matrix using the optical theorem, i.e. we will split the $S$-matrix in its trivial and its non-trivial part $\hat{S} = \mathbb{1} + \mathrm{i}\hat{T}$. If the only IR-finite quantity were the rate, the optical theorem would only be informative about forward scattering. But if we are interested in obtaining non-trivial information about the imaginary part for generic amplitudes, we can solely achieve this with non-zero off-diagonal pieces of the density matrix. For generic states, non-trivial information about unitarity of the $S$-matrix can therefore only be obtained with IR-finite off-diagonal pieces of the density matrix. In what follows, we shall suggest a concrete definition of these off-diagonal pieces based on an IR-finite version of the optical theorem.

An important comment to be made at this point is that the so defined IR-finite and non-vanishing off-diagonal elements of the density matrix do not necessarily give rise to complete purification. After all, we are tracing over soft IR radiation in order to achieve IR-finiteness and this tracing, although needed by finiteness, can lead to some quantum decoherence. This decoherence, as already mentioned, is due to the entanglement between the soft IR radiation we integrate over and the rest of scattering products. One origin of this entanglement obviously is energy conservation. As we shall show, energy conservation leads to a dependence of the IR-finite density matrix on the IR kinematical factors, which scale with the coupling and are only sensitive to the initial and final scattering state. Thus, the von Neumann entropy of the IR-finite density matrix depends on these kinematical factors allowing us to study how IR decoherence depends on the scattering kinematics.

So far, we have only discussed tracing over soft IR radiation, which is required for IR-finiteness. In an experimental setup, there is obviously a second reason for tracing, namely a finite detector resolution. In that case, one traces over *all* soft radiation, i.e. also over non-IR radiation. Since this non-IR radiation is entangled with the scattering products and depends on the details of the scattering process, it is clear that decoherence occurs in that case. In this chapter, however, we will not consider the effect of a finite detector resolution but solely focus on IR-finiteness.

## 3.2 Finite Density Matrix

We consider the transition from an initial state $|\alpha\rangle$ to a final state $|\beta\rangle$, which is described by a generic tree level amplitude $T_{\alpha,\beta}$. Here $T_{\alpha,\beta}$ is the nontrivial part of the $S$-matrix, $S_{\alpha,\beta} = \delta_{\alpha,\beta} + \mathrm{i}T_{\alpha,\beta}$, and does not contain soft loops. If one takes into account soft loops in this process, their effect can be resummed and exponentiated so that we obtain for non-trivial part of the matrix element (1.1) in section 1.1:

$$T^{\text{loop}}_{\alpha,\beta} = \left(\frac{\lambda}{\Lambda}\right)^{B_{\alpha,\beta}/2} T_{\alpha,\beta}\,, \tag{3.1}$$

where $\lambda$ is an IR regulator and $\Lambda > \lambda$ a UV-cutoff for the loop integration and $B_{\alpha,\beta}$ is given in (1.2) or (1.4) for QED or perturbative gravity respectively. As a next step, we investigate the density matrix, which results from the initial state $|\alpha\rangle$ after computing the effect of soft loops and tracing over soft IR radiation $\gamma$. As said above we want to use the optical theorem later, which only contains the non-trivial part of the $S$-matrix, i.e. the amplitude $T_{\alpha,\beta}$. Therefore we want to treat in this chapter the density matrix

$$\hat{\rho}^{(\alpha),\,\mathrm{dec}} = \sum_{\beta\beta'\gamma} T^{\mathrm{loop}}_{\alpha,\,\beta\gamma} T^{\mathrm{loop}\,*}_{\alpha,\,\beta'\gamma} \left|\beta\right\rangle \left\langle\beta'\right| = \sum_{\beta\beta'} \rho^{(\alpha),\,\mathrm{dec}}_{\beta\beta'} \left|\beta\right\rangle \left\langle\beta'\right| \,, \tag{3.2}$$

where we already use the notation from the soft photon theorem (1.5). We can now use that soft factorization theorem to evaluate the matrix element of the density matrix:

$$\rho^{(\alpha),\,\mathrm{dec}}_{\beta\beta'} = T_{\alpha,\,\beta} T^*_{\alpha,\,\beta'} \left(\frac{\lambda}{\epsilon}\right)^{B_{\beta,\,\beta'}/2} \left(\frac{\epsilon}{\Lambda}\right)^{(B_{\alpha,\,\beta}+B_{\alpha,\,\beta'})/2} f\left(\frac{B_{\alpha,\,\beta}+B_{\alpha,\,\beta'}-B_{\beta,\,\beta'}}{2}\right) . \tag{3.3}$$

Even after tracing over soft IR emission, a non-negative power of $\lambda$ survives. Its exponent is the kinematical factor $B_{\beta,\,\beta'}$ of the hypothetical scattering process $\beta \to \beta'$. This means that $\lambda$ vanishes if and only if the currents in $\beta$ and $\beta'$ match antipodally at each angle. On the diagonal, this is the case so that we obtain

$$\rho^{(\alpha),\,\mathrm{dec}}_{\beta\beta} = \left|T_{\alpha,\,\beta}\right|^2 \left(\frac{\epsilon}{\Lambda}\right)^{B_{\alpha,\,\beta}} f(B_{\alpha,\,\beta})\,. \tag{3.4}$$

As it should, the diagonal gives the non-trivial part of the known rate (1.13). For generic states $\beta$ and $\beta'$, in which the currents do not match angle-wise, however, a positive power of $\lambda$ survives on the off-diagonal. Thus, the corresponding off-diagonal elements vanish in the limit $\lambda \to 0$. In a generic case, the resulting density matrix $\hat{\rho}^{(\alpha),\,\mathrm{dec}}$ therefore is mostly decohered, thereby justifying its superscript. This finding is independent of the specific process and the coupling strength.

#### Decoherence and Zero-Energy Modes

From the previous discussion it is easy to identify the root of the former decoherence of the density matrix. Since almost all off-diagonal elements vanish for $\lambda \to 0$, it follows from the form (3.2) of the density matrix that in this limit

$$\sum_{\gamma} T^{\mathrm{loop}}_{\alpha,\,\beta\gamma} T^{\mathrm{loop}\,*}_{\alpha,\,\beta'\gamma} \sim \delta_{\beta,\,\beta'}\,. \tag{3.5}$$

Thus, full decoherence after tracing over IR radiation is equivalent to maximal entanglement between the hard state $\left|\beta\right\rangle$ and the IR radiation. However, this behavior only occurs in the limit $\lambda \to 0$, independently of the values of the other scales $\epsilon$ and $\Lambda$. This shows that *only radiated quanta of zero energy are responsible for the decoherence* and immediately raises the question, whether the decoherence derived above really corresponds to a physical effect, because the actual decoupling of the zero-energy modes should lead to recovering quantum coherence, at least partially.

**Effect of Non-IR Radiation**

As an other remark, we want to point out that not all off-diagonal elements vanish. In particular, this happens if $\beta$ and $\beta'$ have the same current and only differ in their soft non-IR radiation. Moreover, since photons are uncharged, $\beta$ and $\beta'$ also yield a non-vanishing off-diagonal element in QED when they differ by hard photons. In a process in which a lot of radiation is produced, a sizable amount of off-diagonal elements therefore survives. For example, in the process of electron-positron annihilation in QED, all final states have the same electronic content (namely none) so that no decoherence at all takes place. In general, of course, taking into account soft non-IR radiation does not suffice to obtain an (approximately) pure density matrix since, as said above, the vanishing of all off-diagonal elements whose currents do not match angle-wise leads to a significant amount of decoherence. In particular, this is true if we consider a weakly coupled process, in which final states without hard radiation dominate. Our goal is to find a procedure that leads to an (approximately) pure final state also for those.

## 3.3 Proposal for IR-finite Density Matrix with Coherence

### 3.3.1 Modified Density Matrix from Optical Theorem

Thus, we have to change the procedure explained in section 3.2 so that the resulting density matrix is – at least approximately – pure. This means that we have to modify the off-diagonal elements without changing the diagonal. This will be achieved in three steps. First, we will show that the optical theorem relates the imaginary part of the amplitude for the process $\beta \to \beta'$ with the elements $\rho^{(\alpha)}_{\beta\beta'}$ of the density matrix discussed above. Secondly, we will derive an IR-finite version of the optical theorem. As third step, this IR-finite optical theorem will enable us to define a density matrix that possesses IR-finite off-diagonal elements.

For generic states $|\beta\rangle$ and $|\beta'\rangle$, the optical theorem reads

$$-\,\mathrm{i}\left(\langle\beta|\,\hat{T}\,|\beta'\rangle - \langle\beta'|\,\hat{T}\,|\beta\rangle^\star\right) = \sum_\sigma \langle\beta|\,\hat{T}\,|\sigma\rangle\,\langle\beta'|\,\hat{T}\,|\sigma\rangle^\star\;, \tag{3.6}$$

where the states $|\sigma\rangle$ form a complete set. In terms of the matrix elements $\rho^{(\sigma)}_{\beta\beta'}$ of the density matrix for the process $\sigma\to\beta$, one can write (3.6) as:

$$\mathcal{I}(\langle\beta|\,\hat{T}\,|\beta'\rangle) = \sum_\sigma \rho^{(\sigma)}_{\beta\beta'}\;, \tag{3.7}$$

where we introduced the abbreviation

$$\mathcal{I}(\langle\beta|\,\hat{T}\,|\beta'\rangle) := -\mathrm{i}\left(\langle\beta|\,\hat{T}\,|\beta'\rangle - \langle\beta'|\,\hat{T}\,|\beta\rangle^\star\right)\;. \tag{3.8}$$

Now we study the effect of soft modes in the optical theorem, i.e. we split the Hilbert space in IR radiation $\gamma$ and all other states $\alpha$:

$$\mathcal{I}(\langle\beta|\,\hat{T}\,|\beta'\rangle) = \sum_\alpha\sum_\gamma \langle\beta|\,\hat{T}\,|\alpha,\gamma\rangle\,\langle\beta'|\,\hat{T}\,|\alpha,\gamma\rangle^\star\;. \tag{3.9}$$

We can use that the contributions of IR radiation factorize. Since we have $B_{\alpha,\,\beta}=B_{\beta,\,\alpha}$ and moreover all soft correction factors are real, we get

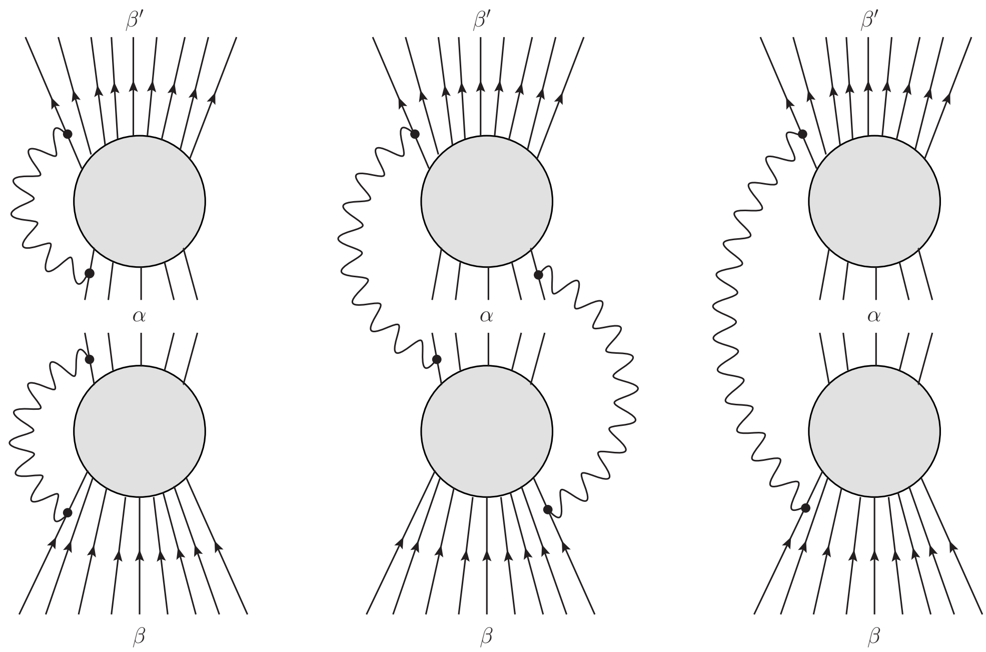
$$\mathcal{I}(\langle\beta|\,\hat{T}\,|\beta'\rangle) = \sum_\alpha \rho^{(\alpha),\,\mathrm{dec}}_{\beta\beta'}\;, \tag{3.10}$$

where the matrix elements $\rho^{(\alpha),\,\mathrm{dec}}_{\beta\beta'}$ of the decohered density matrix (3.2) appear. Plugging in the result (3.3), we obtain

$$\mathcal{I}(\langle\beta|\,\hat{T}\,|\beta'\rangle) = \left(\frac{\lambda}{\epsilon}\right)^{B_{\beta,\,\beta'}/2} \sum_\alpha T_{\alpha,\beta}T^*_{\alpha,\beta'}\left(\frac{\epsilon}{\Lambda}\right)^{(B_{\alpha,\,\beta}+B_{\alpha,\,\beta'})/2} f\left(\frac{B_{\alpha,\beta}+B_{\alpha,\beta'}-B_{\beta,\beta'}}{2}\right)\;. \tag{3.11}$$

It is crucial here that $B_{\beta,\,\beta'}$ does not depend on $\alpha$. This expression vanishes in the limit $\lambda\to 0$. However, this does not come as a surprise. By including IR radiation in the sum over all intermediate states, we effectively introduced soft loops in the process $\beta \to \beta'$. This is the reason why we obtain the factor $(\lambda/\epsilon)^{B_{\beta,\beta'}/2}$, which comes from including soft loops of energies below $\epsilon$ in the process $\beta \to \beta'$. So in the above computation, we should replace $\mathcal{I}(\langle\beta|\hat{T}|\beta'\rangle) \to \mathcal{I}^{\text{loop}}(\langle\beta|\hat{T}|\beta'\rangle)$.

That the density matrix $\rho^{(\alpha),\,\text{dec}}_{\beta\beta'}$ appears in the optical theorem also gives us the opportunity to better understand where its different contributions come from, as is illustrated in Fig. 3.1. In particular, we can identified the origin of the factor $(\lambda/\epsilon)^{B_{\beta,\beta'}/2}$, which leads to vanishing off-diagonal elements in the density matrix and therefore to decoherence: As already said, including soft IR emission in the processes $\alpha \to \beta$ and $\alpha \to \beta'$ effectively generates soft loops in the process $\beta \to \beta'$ (see Fig. 3.1c). Those loops lead to a vanishing matrix element $\rho^{(\alpha),\,\text{dec}}_{\beta\beta'}$ unless the currents in $\beta$ and $\beta'$ match antipodally at each angle.




(a) $(\lambda/\Lambda)^{(B_{\alpha,\beta}+B_{\alpha,\beta'})/2}$ (b) $(\epsilon/\lambda)^{(B_{\alpha,\beta}+B_{\alpha,\beta'})/2}$ (c) $(\lambda/\epsilon)^{B_{\beta,\beta'}/2}$

Figure 3.1: Diagrammatic representation of the different contributions to the density matrix element $\rho^{(\alpha),\,\text{dec}}_{\beta\beta'}$. The first contribution is due to soft loops. The second and the third one come from tracing over emitted IR radiation. The product of the first two contributions gives an IR-finite result, but because of the third one, most off-diagonal elements vanish for $\lambda \to 0$. The reason is that by tracing over IR emission in the processes $\alpha \to \beta$ and $\alpha \to \beta'$, we effectively introduce a soft loop from the perspective of the process $\beta \to \beta'$.

The second step is to derive an IR-finite version of the optical theorem. We remark that this is an important question on its own. Namely, if no IR-finite version existed for a generic process $\beta \to \beta'$, the optical theorem would become meaningless in a gapless theory except for the case $\beta \to \beta$ of forward scattering. Since we have concluded that soft loops are the reason why relation (3.11) is zero in the limit $\lambda \to 0$, it is clear how to obtain a non-trivial finite answer. Namely, we have to include soft IR emission in the process $\beta \to \beta'$. This means that we proceed in full analogy to (1.13) and define

$$\left|\mathcal{I}^{\text{full}}(\langle\beta|\,\hat{T}\,|\beta'\rangle)\right|^2 := \sum_{\gamma}\left|\mathcal{I}^{\text{loop}}(\langle\beta|\,\hat{T}\,|\beta',\gamma\rangle)\right|^2 . \tag{3.12}$$

Plugging in the definition (3.8) of $\mathcal{I}^{\text{loop}}(\langle\beta|\,\hat{T}\,|\beta'\rangle)$, we can generalize the standard computation displayed in (1.13) to obtain the simple result

$$\left|\mathcal{I}^{\text{full}}(\langle\beta|\,\hat{T}\,|\beta'\rangle)\right|^2 = \left(\frac{\epsilon}{\lambda}\right)^{B_{\beta,\,\beta'}} f(B_{\beta,\,\beta'})\left|\mathcal{I}^{\text{loop}}(\langle\beta|\,\hat{T}\,|\beta'\rangle)\right|^2 . \tag{3.13}$$

Since the soft factors that appeared due to the inclusion of the emission of IR modes are real and positive, there is a natural definition of the square root of the above equation that preserves all analytic properties of $\mathcal{I}^{\text{loop}}(\langle\beta|\,\hat{T}\,|\beta'\rangle)$:

$$\mathcal{I}^{\text{full}}(\langle\beta|\,\hat{T}\,|\beta'\rangle) = \left(\frac{\epsilon}{\lambda}\right)^{B_{\beta,\,\beta'}/2}\sqrt{f(B_{\beta,\,\beta'})}\,\mathcal{I}^{\text{loop}}(\langle\beta|\,\hat{T}\,|\beta'\rangle) . \tag{3.14}$$

Plugging in the explicit form (3.11) of $\mathcal{I}^{\text{loop}}(\langle\beta|\,\hat{T}\,|\beta'\rangle)$, we finally obtain

$$\begin{aligned}\mathcal{I}^{\text{full}}(\langle\beta|\,\hat{T}\,|\beta'\rangle) := & \sqrt{f(B_{\beta,\,\beta'})}\\ & \sum_{\alpha} T_{\alpha,\,\beta}T^{*}_{\alpha,\,\beta'}\left(\frac{\epsilon}{\Lambda}\right)^{(B_{\alpha,\,\beta}+B_{\alpha,\,\beta'})/2} f\left(\frac{B_{\alpha,\,\beta}+B_{\alpha,\,\beta'}-B_{\beta,\,\beta'}}{2}\right) .\end{aligned} \tag{3.15}$$

This is the result of applying the standard recipe [5–7] for dealing with infrared divergences to the optical theorem. While we believe that the definition (3.15) can in general give an IR-finite meaning to the optical theorem, the only important point for the present work is that we can derive an IR-finite density matrix from it.

Consequently, the third step is to use the IR-finite version (3.15) of the optical theorem to obtain a density matrix that is not fully decohered. We do so in an effective description in which IR modes are fully integrated out. Namely, we define the density matrix as the one that has to appear on the r.h.s. of the optical theorem (3.7) when full IR-finite quantities are used on the l.h.s.:

$$\mathcal{I}^{\text{full}}(\langle\beta|\,\hat{T}\,|\beta'\rangle) = \sum_{\alpha} \rho^{(\alpha),\,\text{coh}}_{\beta\beta'}\,, \tag{3.16}$$

where the sum over $\alpha$ no longer contains IR modes. Thus, we obtain the full IR-finite density matrix as

$$\rho^{(\alpha),\,\text{coh}}_{\beta\beta'} := T_{\alpha,\,\beta} T^*_{\alpha,\,\beta'} \left(\frac{\epsilon}{\Lambda}\right)^{(B_{\alpha,\,\beta}+B_{\alpha,\,\beta'})/2} \sqrt{f(B_{\beta,\,\beta'})}\, f\left(\frac{B_{\alpha,\,\beta}+B_{\alpha,\,\beta'}-B_{\beta,\,\beta'}}{2}\right)\,. \tag{3.17}$$

In comparison with the decohered density matrix (3.3), there are two changes: The factor $(\lambda/\epsilon)^{B_{\beta,\,\beta'}/2}$ was removed and the function $\sqrt{f(B_{\beta,\,\beta'})}$ was added. The first change is crucial since it alone suffices to avoid full decoherence. It is important to note that this form of the density matrix solely follows from requiring that the IR-finite description in terms of $\mathcal{I}^{\text{full}}(\langle\beta|\,\hat{T}\,|\beta'\rangle)$ and $\rho^{(\alpha),\,\text{coh}}$ is unitary in the sense that it fulfills the optical theorem in the form (3.16), which we obtained after integrating out IR modes.

As a side note, we remark how one can understand our approach in the diagrammatic representation in terms of Fig. 3.1. When we sum over all possible intermediate states in the optical theorem, we also include those where soft IR quanta are emitted or absorbed by the hard modes defining the intermediate state $|\alpha\rangle$, as is displayed in Fig. 3.1b. Additionally, however, there is the contribution of Fig. 3.1c, in which we sum over IR radiation that is emitted from $\beta$ and then absorbed in $\beta'$. It is fully insensitive to the intermediate state $|\alpha\rangle$ and the one that leads to the factor $(\lambda/\epsilon)^{B_{\beta,\,\beta'}/2}$, which is responsible for decoherence. Our recipe provides us with a concrete way to avoid decoherence by removing this contribution.

In summary, the logical flow of our approach can be described as follows. We consider a given scattering process $\beta \to \beta'$, whose amplitude is zero due to IR divergences. But if we IR regulate the amplitude, i.e. do not take $\lambda \to 0$, its imaginary part is still non-trivial if we have branch cuts reflecting the threshold of inelastic processes. Since the existence of these cuts is not affected by adding IR soft radiation, it should survive in the IR limit. In other words, in the same way that we know that the actual scattering $\beta \to \beta'$ is non-trivial once we add, in an appropriate way, soft IR radiation, we expect that the inelastic part of the scattering is equally non-vanishing after including soft IR radiation. What we have presented is a simple recipe to derive from this physics picture a natural characterization of quantum decoherence.

### 3.3.2 Resulting Entropy

We proceed to discuss the modified density matrix (3.17). First, we note that the diagonal is the same as for the decohered density matrix (3.3):

$$\rho_{\beta\beta}^{(\alpha),\,\mathrm{coh}} = \left|T_{\alpha,\,\beta}\right|^2 \left(\frac{\epsilon}{\Lambda}\right)^{B_{\alpha,\,\beta}} f(B_{\alpha,\,\beta})\,, \tag{3.18}$$

which follows from $B_{\beta,\,\beta} = 0$. Thus, we obtain the non-trivial part of the known rate (1.13). As it should be, our modification of the density matrix does not change the rates. The important question is how pure the density matrix (3.17) is. As a first step, we investigate what off-diagonal elements would be required to obtain a completely pure result. From the diagonal elements (3.18) it follows that we would need

$$\left(\rho_{\beta\beta'}^{(\alpha),\,\mathrm{coh}}\right)^{\mathrm{pure}} = T_{\alpha,\,\beta} T^*_{\alpha,\,\beta'} \left(\frac{\epsilon}{\Lambda}\right)^{B_{\alpha,\,\beta}/2 + B_{\alpha,\,\beta'}/2} \sqrt{f(B_{\alpha,\,\beta})}\sqrt{f(B_{\alpha,\,\beta'})}\,. \tag{3.19}$$

In that case, the density matrix would be pure since we could write it as

$$\left(\rho^{(\alpha),\,\mathrm{coh}}\right)^{\mathrm{pure}} = |\Psi\rangle\,\langle\Psi|\,, \tag{3.20}$$

where

$$|\Psi\rangle = \sum_\beta T_{\alpha,\,\beta} \left(\frac{\epsilon}{\Lambda}\right)^{B_{\alpha,\,\beta}/2} \sqrt{f(B_{\alpha,\,\beta})}\,|\beta\rangle\,. \tag{3.21}$$

Thus, only the functions $f(B)$, which arise due to energy conservation, lead to decoherence. We can parametrize the deviation from purity as the quotient of the factor in the full modified density matrix (3.17) and the factor required for purity:

$$c_{\beta,\,\beta'}^{(\alpha)} = \frac{\sqrt{f(B_{\beta,\,\beta'})}\, f(B_{\alpha,\,\beta}/2 + B_{\alpha,\,\beta'}/2 - B_{\beta,\,\beta'}/2)}{\sqrt{f(B_{\alpha,\,\beta}) f(B_{\alpha,\,\beta'})}}\,. \tag{3.22}$$

So the deviations of the $c_{\beta,\,\beta'}^{(\alpha)}$ from 1 determine the decoherence and full coherence corresponds to $c_{\beta,\,\beta'}^{(\alpha)} = 1$.

In order to study decoherence in more detail, we will restrict ourselves to the regime of weak coupling. In that case, all functions $f(B)$ are small, i.e. we can use the approximation (1.15). Then we get to leading order:

$$c^{(\alpha)}_{\beta,\beta'} = 1 + \frac{\pi^2}{48}\left((B_{\alpha,\beta} - B_{\alpha,\beta'})^2 + 2B_{\beta,\beta'}(B_{\alpha,\beta} + B_{\alpha,\beta'}) - 3B^2_{\beta,\beta'}\right) . \tag{3.23}$$

This shows that all $c^{(\alpha)}_{\beta,\beta'}$ are arbitrarily close to $1$ for sufficiently weak coupling, i.e. decoherence can be avoided by decreasing the coupling. We will make this statement quantitative, i.e. we determine an upper bound on the decoherence that can arise. We will estimate in terms of the von Neumann entropy $S = -\operatorname{Tr}\rho^{(\alpha),\,\mathrm{coh}} \ln \rho^{(\alpha),\,\mathrm{coh}}$. If all off-diagonal element were zero, the maximal entropy would be given by $S = \ln d_H$, where $d_H$ is the dimension of the hard Hilbert space. This maximal entropy would be reached if all final hard states were equally probable, i.e. all diagonal elements were equal. For our estimate, we will therefore restrict ourselves to a density matrix in which all diagonal elements are equal. Such a density matrix is pure if all elements, i.e. also the off-diagonal ones, are equal. In order to derive the upper bound on the entropy, we can consequently define $\Delta_{\text{max}} := \max_{\beta,\beta'} |1 - c^{(\alpha)}_{\beta,\beta'}|$ and then multiply the off-diagonal elements of the pure density matrix, in which all entries are equal, by the function $c := 1 - \Delta_{\text{max}}$.[1] In this setup, the eigenvalues of the density matrix are[2]

$$e_1 = \frac{1 + (d_H - 1)c}{d_H} \quad \text{and } e_i = \frac{1-c}{d_H} \text{ for } i \in [2,\, d_H] . \tag{3.24}$$

To leading order in $1-c$, this gives the bound

$$S < (1-c)\ln\left(\frac{d_H}{1-c}\right) . \tag{3.25}$$

As expected, we obtain $S = 0$, i.e. purity, for $c = 1$. Full decoherence can only be obtained in the limit $c = 0$. This confirms that the entropy is always small if the coupling is weak enough. So at least in the case of weak coupling, our formalism is able to describe the interference phenomena that we observe experimentally.

[1] At this point, one can wonder why we could not use $c := 1 + \Delta_{\text{max}}$ instead. The reason is that any $c > 1$ would lead to an unphysical density matrix with negative eigenvalues. Note that is it nonetheless not excluded that some $c^{(\alpha)}_{\beta,\beta'}$ are bigger than $1$.

[2] These are the eigenvalues of a quadratic matrix of dimension $d_H$ that has $1/d_H$ on the diagonal and $c/d_H$ on all off-diagonal elements. A linearly independent set of eigenvectors $v_i$ is given by the entries $(v_1)_k = 1$ and $(v_i)_k = \delta_{k1} - \delta_{ki}$ for $i \in [2,\, d_H]$.

Clearly, our estimate no longer works in the regime of strong coupling. However, from this fact it does not follow that a sizable amount of decoherence has to occur in that case. In particular, the fact that the hard amplitudes depend strongly on the final state in the strong coupling regime could prevent the generation of entropy. It would be interesting to investigate this question in a concrete setup, e.g. the process of $2 \to N$ scattering proposed in [50], whose infrared behavior was already studied in [10, 51, 52].

# Chapter 4

# The Scales of the Infrared

## 4.1 Introduction

In the previous chapter we proposed a procedure to obtain a coherent density matrix by using the optical theorem in combination with the Bloch-Nordsieck recipe. In this chapter we want to present a second approach to achieve a coherent density matrix with the other standard way to obtain IR-finite quantities, the coherent state recipe. In the review chapter 1.3 we mentioned that in the standard literature the limits of the integration in the dressed state $|D(\alpha)\rangle$ (see equation (1.27)) are never really treated well. To do so one needs to introduce a new energy scale, just as in the inclusive formalism, to define the dressing states, which we shall call $r$. Since its physical meaning is not immediately apparent, we will momentarily postpone its discussion. In the dressed formalism, one can use the soft theorems to obtain – up to subleading corrections – the same IR-finite rate as in the inclusive formalism, provided one sets $r = \epsilon$. As we will discuss, however, the reason for identifying these two scales is unclear.

Because of their infinite photon number, dressing states do not belong to the Fock space but can only be defined in the von Neumann space $\mathcal{H}_{\text{VN}}$ [26]. It consists of infinitely many subspaces, $\mathcal{H}_{\text{VN}} = \otimes_\alpha[\alpha]$, where each subspace $[\alpha]$, dubbed equivalence class, is isomorphic to the Fock space with an inequivalent representation of the creation and annihilation operator algebra [28]. So asymptotic dynamics defines the dressing operator $\hat{W}$ (1.25) that associates to a state of hard charged quanta, which we call as usual $|\alpha\rangle$, the photon state

$|D(\alpha)\rangle$ in the von Neumann equivalence class $[\alpha]$:

$$\hat{W} : |\alpha\rangle \to |\alpha\rangle\rangle := |\alpha\rangle \otimes |D(\alpha)\rangle \ . \tag{4.1}$$

As notation indicates, the equivalence class $[\alpha]$ is sensitive to the state of the charged particle $|\alpha\rangle$, i.e. the photon state $|D(\alpha')\rangle$ of a different charged particle $|\alpha'\rangle$ belongs to a different – and orthogonal – equivalence class $[\alpha']$. It is crucial to note that for the representation of the creation annihilation algebra in $[\alpha]$, the dressing state $|D(\alpha)\rangle$ is the vacuum. This reflects that fact that there is no radiation in the dressed formalism.

Since both the inclusive and the dressed formalism yield the same rate, the question arises if they are equivalent. This would come as a big surprise since the requirements of the two formalisms – emission of bremsstrahlung versus well-defined asymptotic states – are very different. Both requirements are, however, very reasonable and should be fulfilled. Therefore, we shall argue that both dressing and soft radiation should be present in a generic process. Thus, the first goal of this chapter is to present a concrete formalism that interpolates between the inclusive and the dressed formalism and makes the distinction between radiation and dressing explicit. We shall call it *combined formalism* and will derive it from first principles by applying the $S$-matrix, as operator in $\mathcal{H}_{\text{VN}}$, to the dressed initial state $|\alpha\rangle\rangle$. This gives

$$\hat{S}\,|\alpha\rangle\rangle = \sum_{\beta} \sum_{\gamma\in[\beta]} S_{\alpha,\beta\gamma}\,|\beta\rangle\rangle \otimes |\gamma(\alpha,\beta)_{D(\beta)}\rangle \ , \tag{4.2}$$

where $\beta$ sums over all possible final charged states. In turn, each of those determines an equivalence class $[\beta]$. The crucial novelty as compared to the dressed formalism is that a radiation state $|\gamma(\alpha,\beta)_{D(\beta)}\rangle$, which depends on both $|\alpha\rangle$ and $|\beta\rangle$, exists on top of the radiative vacuum defined by the dressing state $|D(\beta)\rangle$. Not surprisingly, it will turn out that also in the combined formalism, one obtains the same IR-finite rate as in the two known formalisms.

### IR-finite Density Matrix

This finding immediately raises the question about the relevance of our construction. However, one can go one step further than the rate and investigate the density matrix of the final state, as we did in the last chapter. Obviously, its diagonal is determined by the known IR-finite rates. So the task consists, as before, in determining the IR limit of the off-diagonal pieces of the density matrix. For a particular simplified setup, the density matrix in the presence of soft bremsstrahlung was already studied some time ago. In a framework of real time evolution [53–55], which goes beyond the $S$-matrix description, the result was that tracing over unresolvable soft radiation leads to some loss of coherence. But for realistic timescales, the decoherence is generically small. As it should be, it consequently does not spoil the interference properties that we observe in nature.

However, it was also derived in [53–55] how coherence depends on the timescale $t_{\text{obs}}$, after which the final state is observed: Albeit slowly, it decreases as the timescale increases. In the limit of infinite time, one obtains full decoherence. Since this is precisely the limit on which the definition of the $S$-matrix is based, it is immediately evident that it might be difficult to derive the density matrix from the $S$-matrix. In the inclusive formalism, this expectation turns out to be fulfilled. Tracing over soft radiation, which is required for IR-finiteness, leads to full decoherence [54]. In an independent line of research, this finding has recently received renewed interest in the context of a generic scattering process [2,48,49,56]. However, if it were not possible to improve this result, this would mean that the $S$-matrix is *in principle* unable to describe any interference phenomena in QED. While we have proposed a heuristic method to obtain IR-finite off-diagonal elements in the previous chapter, this finding is a clear indication that the inclusive formalism is insufficient to describe the density matrix of the final scattering state.

In the dressed formalism, the opposite situation is realized. The reason is that dressing photons are part of the definition of the asymptotic states and are independent of the scattering process. Therefore, there is no reason to trace over them. In fact, it is not even clear how to define the trace in the von Neumann space since it would amount to squeezing the infinite von Neumann subspaces into a single Fock space. This means that there is no tracing and no decoherence in the dressed formalism.[1] Also this finding is unsatisfactory since one expects some decoherence due to the emission of unresolvable soft bremsstrahlung.

The situation improves in the combined formalism that we propose. In it, the final state consists both of dressing, defined by the scale $r$, and of soft radiation, defined by the scale $\epsilon$. In order to obtain the density matrix of the final state, we have to trace over radiation but not over dressing. In this way, we avoid full decoherence. Since the purity of the density

[1] In [49, 56], the scales of radiation and dressing were identified, $r = \epsilon$, and a tracing over dressing states was performed. Since states in different equivalence classes are orthogonal, a similar result as in the inclusive formalism, i.e. a fully decohered density matrix of the final state, was obtained. As explained, however, the physical meaning of tracing over dressing states is unclear to us.

matrix depends on the scale $r$, the connection to [53–55] makes its meaning evident: It is set by the timescale after which the final state is observed, $r = t_{\text{obs}}^{-1}$. Thus, we obtain a sensible IR-finite density matrix, thereby continuing our work from the previous chapter. This is a clear indication of the physical relevance of the combined formalism. The second goal of this chapter therefore is to compute the density matrix of the final scattering state in the combined formalism and to estimate the amount of decoherence it exhibits.

In section 4.2, we want to come back to the discussion of properly defined limits of the integration in the coherent state operator and draw a clearer picture of the difference between dressing and radiation. In doing so, our goal is not to be mathematically rigorous, but to put well-known mathematical results in a physical context with the aim of making the distinction between dressing and radiation evident. In section 4.3, we will combine computations of the inclusive and dressed formalism to determine the final state of scattering in the combined formalism. Moreover, we will make additional comments about the inclusive formalism, collinear divergences as well as the connection of large gauge transformations and dressing. Then we will proceed in section 4.4 to calculate the density matrix and give a bound on its decoherence.

## 4.2 The Distinction of Dressing and Radiation

### 4.2.1 Well-Defined Asymptotic States

In the review section 1.3 we presented the original way of how the dressing was defined, where we omitted the boundaries of the integration in the definition of the dressing operator (1.25). This section restores these boundaries and explains its meanings. We start with the original operator $\hat{W}(t)$ from [8] in equation (1.24) and put the limits of the integration:

$$\hat{W}(t)_\lambda^r = \exp\left\{ \frac{1}{\sqrt{2(2\pi)^3}} \int_\lambda^r \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}} \sum_l \int \mathrm{d}^3\boldsymbol{p}\, \hat{\rho}(\boldsymbol{p}) \left( \frac{p\cdot\varepsilon_{l,\boldsymbol{k}}^*}{p\cdot k} \hat{a}_{l,\boldsymbol{k}}^\dagger \mathrm{e}^{\mathrm{i}\frac{pk}{p_0}t} - \text{h.c.} \right) \right\} . \quad (4.3)$$

The lower limit is an IR regulator $\lambda$. As long as we keep it finite, we can work in the Fock space and the operator (4.3) is well-defined there. In the end, however, $\lambda$ will go to zero and it will turn out that this forces us to work in the larger von Neumann space, as already pointed out in section 1.3. Whereas $\lambda$ is a regulator, it is clear that $r$ has to be non-vanishing since otherwise the operator (4.3) is trivial. So it is a finite and non-zero physical scale with units of energy. As already introduced in equation (1.26), we can define asymptotic states by applying $\hat{W}(t)^r_\lambda$ to a bare state $|\alpha\rangle$ of electrons and positrons:

$$|\alpha\rangle\rangle^r_\lambda := \hat{W}(t_{\text{obs}})^r_\lambda\,|\alpha\rangle \ , \tag{4.4}$$

where $t_{\text{obs}}$ is a so far arbitrary reference time. We will keep it finite for now, but follow [8, 9] and set it to zero for the computation. The reason we can do so is that the final result only depends on the divergent zero-mode part of the dressing state whereas the phase controlled by $t_{\text{obs}}$ only changes the finite part of non-zero modes.[2]

Definition (4.4) also depends on $r$ and is non-trivial only for $r$ non-zero. Although we will keep $r$ general in our computation, we shall briefly discuss its physical interpretation. If one wants to interpret $|\alpha\rangle\rangle^r_\lambda$ as initial or final state of scattering, it is most natural to think of $t_{\text{obs}}$ as the timescale after which the state will be measured. Once $t_{\text{obs}}$ is fixed, $r$ is no longer independent. The reason is the fact, noted in [8], that the phases wash out if $|\boldsymbol{k}|\,t_{\text{obs}}$ is sufficiently big, i.e. $\lim_{t\gg|\boldsymbol{k}|^{-1}} \exp\left(\mathrm{i}\frac{kp}{p_0}\,t\right)/pk \approx 0$. Therefore, all modes with $|\boldsymbol{k}| > t_{\text{obs}}^{-1}$ effectively disappear and do not contribute to the asymptotic dynamics any more:

$$\hat{W}(t_{\text{obs}})^r_\lambda \approx \hat{W}(t_{\text{obs}})^{t_{\text{obs}}^{-1}}_\lambda \ . \tag{4.5}$$

Thus, if we only want to consider the physical modes, we have to set

$$r = t_{\text{obs}}^{-1} \,, \tag{4.6}$$

i.e. we can identify $r$ with the timescale $t_{\text{obs}}$ after which the final state is measured. We can also justify the choice (4.6) from a more physical point of view. Namely it is crucial for the photons in the dressing state that they are decoupled. Since a photon of energy $r$ needs a timescale of $r^{-1}$ to interact, it only makes sense to consider $r < t_{\text{obs}}^{-1}$. While these arguments are heuristic, we will present a more precise justification for the choice (4.6) in section 4.4 by comparing the density matrix that we derive in our combined formalism with the result of [53–55] that was obtained in a framework of real time evolution. We note, however,

[2] Strictly speaking, one can even by more general and choose an arbitrary state in the equivalence class $[\alpha]$ [8]. But since only the zero-mode part of dressing matters, we can adapt the choice (4.4) of [8, 9]. We will further comment on this freedom in choosing a dressing state in section 4.3.2.

that our combined formalism is not tied to the physical interpretation of $r$ but works for an arbitrary choice.

Before we investigate the dressed states more closely, we want to mention that the $S$-matrix is not modified in the dressed formalism [8]. The reason is that in the limit of infinite time, relation (4.5) becomes

$$\lim_{t\to\infty} \hat{W}(t) = \mathbb{1}\,, \tag{4.7}$$

which follows from $\lim_{t\to\pm\infty} \exp\left(\mathrm{i}\frac{kp}{p_0}t\right)/pk = \pm\mathrm{i}\pi\delta(kp)$. For this reason, asymptotic dynamics do not contribute to the $S$-matrix but only modify the asymptotic states. Setting $t_{\text{obs}} = 0$, we get the asymptotic state (4.4):

$$|\alpha\rangle\rangle^r_\lambda = |\alpha\rangle \otimes |D(\alpha)\rangle^r_\lambda\,, \tag{4.8}$$

where again we explicitly indicated the limits of integration. The dressing $|D(\alpha)\rangle^r_\lambda$ is the well-known coherent state of soft photons described in section 1.3 and can be rewritten as

$$|D(\alpha)\rangle^r_\lambda = \exp\left\{-\frac{1}{2}B_\alpha \ln\frac{r}{\lambda}\right\} \exp\left\{\int_\lambda^r \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}} \sum_l \mathcal{F}^{(l)}_\alpha(\boldsymbol{k})\, \hat{a}^\dagger_{l,\boldsymbol{k}}\right\} |0\rangle\,, \tag{4.9}$$

where $\mathcal{F}^{(l)}_\alpha(\boldsymbol{k})$ is given by equation (1.28). The state is normalized, i.e. $\int_\lambda^r \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|} \sum\limits_l |\mathcal{F}^{(l)}_\alpha(\boldsymbol{k})|^2 = B_\alpha \ln\frac{r}{\lambda}$ with

$$B_\alpha = \frac{1}{2(2\pi)^3} \sum_{n,m\in\alpha} \int \mathrm{d}^2\Omega \frac{e_n e_m\, p_n \cdot p_m}{p_n \cdot \hat{k}\; p_m \cdot \hat{k}}\,, \tag{4.10}$$

where $\hat{k}$ denotes the normalized 4-momentum of the photon. When we investigate the particle number of the dressing state (1.30) with the proper limits for the integration,

$${}^{r}_{\lambda}\langle D(\alpha)|\hat{N}\,|D(\alpha)\rangle^r_\lambda = B_\alpha \ln\frac{r}{\lambda}\,, \tag{4.11}$$

it becomes evident that it contains an infinite number of zero-energy photons in the limit $\lambda \to$

0.[3] Thus, although the states possess the finite energy $B_\alpha r$,[4] they are not in the equivalence class $[0]$, i.e. in the Fock space. Note that varying $r$ does not change the equivalence class but only alters the energy of the dressing state. So the equivalence class only depends on the zero-momentum part of $\mathcal{F}_\alpha^{(l)}(\boldsymbol{k})$.

In order to investigate how many different equivalence classes we have, we compute the overlap of two different dressing states with formula (1.32):

$$ {}_\lambda^r\langle D(\alpha)|D(\beta)\rangle_\lambda^r = \exp\left\{-\frac{1}{2}\int_\lambda^r \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|}\sum_l \left|\mathcal{F}_{\alpha,\beta}^{(l)}(\boldsymbol{k})\right|^2\right\} = \left(\frac{\lambda}{r}\right)^{B_{\alpha,\beta}/2}, \tag{4.12} $$

where we used $\int_\lambda^r \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|}\sum_l \left|\mathcal{F}_{\alpha,\beta}^{(l)}(\boldsymbol{k})\right|^2 = B_{\alpha,\beta}\ln\frac{r}{\lambda}$, and the angular part is given by (1.2) or equally

$$ B_{\alpha,\beta} = \frac{1}{2(2\pi)^3}\sum_{n,\,m\in\alpha,\,\beta}\int \mathrm{d}^2\Omega \frac{\eta_n\eta_m e_n e_m\, p_n\cdot p_m}{p_n\cdot\hat{k}\; p_m\cdot\hat{k}}\,. \tag{4.13} $$

This is the kinematical factor that only depends on the initial and final state of scattering. It follows from (1.2) that $B_{\alpha,\beta}=0$ only if the currents in $|\alpha\rangle$ and $|\beta\rangle$ match at each angle [48]. If this is not the case, $|D(\alpha)\rangle_\lambda^r$ and $|D(\beta)\rangle_\lambda^r$ have overlap zero for $\lambda\to 0$ and therefore are in different equivalence classes. Thus, there is a different equivalence class for each charge distribution on the sphere. We can parametrize the equivalence classes as $[\alpha]$ in terms of the charged states $|\alpha\rangle$.

### 4.2.2 Equivalence Classes as Radiative Vacua

In gapless theories, we have seen that non-trivial asymptotic dynamics lead to dressing states (4.9), which – in the limit $\lambda\to 0$ – no longer belong to the Fock space because of an infinite number of zero-energy photons. However, as explained in section 1.3.3, each equivalence class of the von Neumann space is isomorphic to the Fock space. In particular, there is a representation of the commutation relations (1.42) in each of them [28]. We can formally

[3] When we talked in the previous chapters about the infinite number of zero-energy photons in the dressing but never really precisely gave an explanation, we meant exactly formula 4.11 in the limit $\lambda\to 0$! We postponed this intentionally until now, where we discussed the limits of integration in the dressing operator properly.

[4] We used formula (1.34), where the upper limit is $r$ and the lower limit is $\lambda$ in the integration and said $r\gg\lambda$.

relate them to the Fock space operators (1.36) via:

$$\hat{a}^{[\alpha]}_{l,\boldsymbol{k}} = \hat{W}(0)\hat{a}_{l,\boldsymbol{k}}\hat{W}^\dagger(0)\,. \tag{4.14}$$

For finite $\lambda$, this representation is unitarily equivalent whereas it is not for $\lambda \to 0$. From the perspective of the operators $\hat{a}^{[\alpha]}_{l,\boldsymbol{k}}$, the corresponding dressing state is a vacuum:

$$\hat{a}^{[\alpha]}_{l,\boldsymbol{k}}\,|\alpha\rangle\rangle^r_\lambda = 0\,. \tag{4.15}$$

So $\hat{a}^{[\alpha]\dagger}_{l,\boldsymbol{k}}$ represent excitations on top of the vacuum of the equivalence class $[\alpha]$, i.e. $\hat{a}^{[\alpha]\dagger}_{l,\boldsymbol{k}}$ corresponds to radiation on top of the dressing state defined by $|D(\alpha)\rangle$.

For $|\boldsymbol{k}| > r$, we have:

$$\hat{a}^{[\alpha]}_{l,\boldsymbol{k}} = \hat{a}_{l,\boldsymbol{k}}\,, \tag{4.16}$$

i.e. photons of energy above $r$ are insensitive to the dressing and can be treated as if they were defined in the Fock space. As it will turn out explicitly in the calculation, only those photons constitute physical radiation. In contrast, photons of smaller energy solely occur in the dressing states but do not exhibit dynamics on their own. This is in line with the well-known decoupling of soft photons [38,44,57,58]. We remark that this is moreover consistent with the identification (4.6) made above, $r = t_{\text{obs}}^{-1}$. Namely we expect that on the timescale $t_{\text{obs}}$, the softest radiation photons that can be produced have energy $t_{\text{obs}}^{-1}$, so all photons of smaller energy are decoupled.[5]

For our argument, however, the precise identification of the scale $r$ is inessential. The only important point is that $r$ splits the Hilbert space of photons in two parts. Photons below $r$ are part of the dressing. It is symmetric, i.e. initial and final states are analogously dressed. Moreover, the dressing of the initial state is only sensitive to the initial state, but not to the final states and likewise for the final state. Since the dressing states contain an infinite amount of photons, they are not in the Fock space, but can only be defined in the larger von Neumann space. In contrast, photons above $r$ are part of radiation. It is asymmetric since we can prepare an initial state without radiation, i.e. radiation only occurs in the final state but not in the initial state. In turn, it will become clear that it is sensitive to both the initial and the final state. In particular, it depends on the difference of initial and final state, i.e.

[5] That $t_{\text{obs}}^{-1}$ should correspond to an effective IR cutoff for physical radiation was also proposed in [53–55].

on the transfer momentum. The radiation state contains a finite number of photons and is well-defined in the Fock space. Thus, physical radiation is completely independent of the problems arising due to an infinite number of photons.

Radiation is characterized by a second scale $\epsilon$, which we can identify with the detector resolution. It is crucial to note that the scales $r$ and $\epsilon$ are in general independent since they contain different physical information. The energy $r$ describes the timescale after which the state is observed. In contrast, the scale $\epsilon$ corresponds to the resolution scale of the particular device used to measure the final state. As explained, the only requirement is that $r < \epsilon$. In fact, it will turn out that $r \ll \epsilon$ is needed for a well-defined separation of dressing and radiation. In this limit, the energy carried by the dressing states is negligible. So all energy is carried by the radiation state whereas the only significant contribution to the number of photons comes from the dressing. In total, we obtain the following hierarchy of scales:

$$\lambda < r < \epsilon < \Lambda\,, \tag{4.17}$$

where $\Lambda$ is the energy scale of the whole process, e.g. the center-of-mass energy. In the existing literature, the scales $\lambda$, $\epsilon$ and $\Lambda$ are well-known. However, there is no additional scale $r$. The reason is that – as we will show – all rates are independent of $r$. So the introduction of the scale $r$, which separates dressing from radiation, is unnecessary if one is solely interested in rates. In contrast, it will turn out that the final density matrix does depend on $r$. The reason is that unlike the rate, the density matrix depends on the timescale after which it is measured. Therefore, we have to keep the scale $r$ to derive an IR-finite density matrix.

Introducing the new scale $r$ amounts to interpolating between the well-known dressed and inclusive formalisms. We can consider the two limiting cases. For $r = \epsilon$, there is no radiation but all photons are attributed to dressing. This leads to Chung's calculation [9], but corresponds to the unsatisfactory situation that there is no soft emission and that the resolution scale $\epsilon$ appears in the dressing of the initial state. The opposite limiting case is to set $r = \lambda$. Then there is no dressing, in particular the initial state is bare, but the final state contains photons of arbitrarily low energies. This leads to the calculations by Yennie, Frautschi and Suura [6] as well as Weinberg [7]. However, this construction lacks well-defined asymptotic states. For these reasons, we will work in the combined formalism that realizes the general hierarchy (4.17). We will demonstrate that doing so leads to the well-known IR-finite rates, but additionally it will allow us to obtain a well-defined density matrix of the final state.

## 4.3 Combined Formalism

### 4.3.1 Calculation of Final State

We consider a generic process of scattering. In order to determine the final state, we only need two ingredients: a well-defined initial state and the $S$-matrix of QED. Having defined the initial state (4.8), it remains to apply the $S$-matrix to it. The first step it to insert an identity. As shown in appendix A, it can be split in three parts if we assume $\epsilon \gg r$. The first one consists of photons with energy below $r$. Those are contained in the dressing of the hard states. The second one corresponds to radiative soft IR photons, i.e. a state in which each single photon has an energy greater than $r$ but smaller than $\epsilon$. As explained, the definition of a radiation photon generically depends on the radiative vacuum on top of which it is defined. However, it follows from (4.14) that this distinction is inessential for photons of energy greater than $r$ and we can treat them as if they were defined in the usual Fock space. Finally, the third part consists of all remaining modes, i.e. electrons and possibly hard photons. As already introduced in Eq. (4.2), we therefore obtain a final state that consists both of dressed charged states and of radiation:

$$\hat{S}\,|\alpha\rangle\rangle^r_\lambda = \sum_\beta \sum_N \frac{1}{N!} \left( \prod_{i=1}^{N} \int_r^\epsilon \mathrm{d}^3\boldsymbol{k}_i \sum_{l_i} \right) \left( |\beta\rangle\rangle^r_\lambda \otimes |\gamma_N\rangle \right) \left( \langle\gamma_N| \otimes {}^r_\lambda\langle\langle\beta| \right) \hat{S}\,|\alpha\rangle\rangle^r_\lambda\,, \tag{4.18}$$

where as described in appendix A $|\gamma_N\rangle = \hat{a}^\dagger_{l_1,\boldsymbol{k}_1} \dots \hat{a}^\dagger_{l_N,\boldsymbol{k}_N} |0\rangle$ and $N$ sums over the number of soft photons. The matrix element $\left( \langle\gamma_N| \otimes {}^r_\lambda\langle\langle\beta| \right) \hat{S}\,|\alpha\rangle\rangle^r_\lambda$ is evaluated between dressed electron states and moreover contains radiation in the final state, i.e. $\langle\gamma_N|$. Now we can use that the soft photon theorem (1.5) holds in an arbitrary process to obtain

$$\left( \langle\gamma_N| \otimes {}^r_\lambda\langle\langle\beta| \right) \hat{S}\,|\alpha\rangle\rangle^r_\lambda = \prod_{i=1}^{N} \frac{\mathcal{F}^{(l_i)}_{\alpha,\beta}(\boldsymbol{k}_i)}{\sqrt{|\boldsymbol{k}_i|}}\, {}^r_\lambda\langle\langle\beta|\hat{S}\,|\alpha\rangle\rangle^r_\lambda\,, \tag{4.19}$$

where the soft factor $\mathcal{F}^{(l_i)}_{\alpha,\beta}(\boldsymbol{k}_i)$ is displayed in Eq. (1.6). Moreover, we follow Chung's computation to evaluate the contribution of the dressing photons (1.35):[6]

$$ {}^{r}_{\lambda}\langle\langle\beta|\hat{S}\,|\alpha\rangle\rangle^{r}_{\lambda} = \left(\frac{r}{\lambda}\right)^{B_{\alpha,\beta}/2} S^{\text{loop}}_{\alpha,\beta} = \left(\frac{r}{\Lambda}\right)^{B_{\alpha,\beta}/2} S_{\alpha,\beta}\ . \tag{4.20}$$

So we obtain

$$ \hat{S}\,|\alpha\rangle\rangle^{r}_{\lambda} = \sum_{\beta}\left(\frac{r}{\Lambda}\right)^{B_{\alpha,\beta}/2} S_{\alpha,\beta}\,|\beta\rangle\rangle^{r}_{\lambda} \otimes \sum_{n}\frac{1}{N!}\left(\prod_{i=1}^{N}\int_{r}^{\epsilon}\frac{\mathrm{d}^3\boldsymbol{k}_i}{\sqrt{|\boldsymbol{k}_i|}}\sum_{l_i}\mathcal{F}^{(l_i)}_{\alpha,\beta}(\boldsymbol{k}_i)\hat{a}^{\dagger}_{l_i,\boldsymbol{k}_i}\right)|0\rangle\ . \tag{4.21}$$

We can resum this final photon state:

$$ \hat{S}\,|\alpha\rangle\rangle^{r}_{\lambda} = \sum_{\beta}\left(\frac{\epsilon}{\Lambda}\right)^{B_{\alpha,\beta}/2} S_{\alpha,\beta}\left(|\beta\rangle\rangle^{r}_{\lambda}\otimes|\gamma(\alpha,\beta)\rangle^{\epsilon}_{r}\right)\ , \tag{4.22}$$

where

$$ \begin{aligned} |\gamma(\alpha,\beta)\rangle^{\epsilon}_{r} &= \left(\frac{r}{\epsilon}\right)^{B_{\alpha,\beta}/2} \mathrm{e}^{\int_r^{\epsilon}\frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}}\sum_{l}\mathcal{F}^{(l)}_{\alpha,\beta}(\boldsymbol{k})\,\hat{a}^{\dagger}_{l,\boldsymbol{k}}}\,|0\rangle \\ &= \mathrm{e}^{\int_r^{\epsilon}\frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}}\sum_{l}\left(\mathcal{F}^{(l)}_{\alpha,\beta}(\boldsymbol{k})\,\hat{a}^{\dagger}_{l,\boldsymbol{k}}-\text{h.c.}\right)}\,|0\rangle \end{aligned} \tag{4.23}$$

is a normalized coherent radiation state and we used the integral (4.13) to compute the norm.

Formula (4.22) makes the physics of the process very transparent. Both in the initial and in the final state, charged particles are dressed, as is required for well-defined asymptotic states. The dressings consist of photons of energy below $r$ and only depend on their respective state. This means that the dressing $|D(\alpha)\rangle^{r}_{\lambda}$ of the initial state only depends on $|\alpha\rangle$ and the dressing $|D(\beta)\rangle^{r}_{\lambda}$ of the final state only depends on $|\beta\rangle$. On top of the dressing, the final state (but not the initial state) also contains radiation. The radiation $|\gamma(\alpha,\beta)\rangle^{\epsilon}_{r}$ is made up of photons of energy above $r$ and depends both on the initial and on the final state of the hard electrons, and in particular on the momentum transfer between them.

As explained in the introduction 4.1, the main difficulty that arises from IR physics – which also seemingly leads to full decoherence – comes from the fact that the dressing states

[6] We omit the phase in (1.35), which is physically irrelevant.

are no longer in the Fock space due to the infinite number of zero-energy photons. For this reason, those states can only be defined in the much larger von Neumann space, which is isomorphic to an infinite product of Fock spaces. In our approach, we manage to separate this difficulty from the physical radiation. Namely only the dressing states $|D(\alpha)\rangle_\lambda^r$ and $|D(\beta)\rangle_\lambda^r$ contain an infinite number of photons, but these state do not correspond to physical radiation. Instead, they are part of the definition of asymptotic states. On top of the radiative vacuum defined by $|D(\beta)\rangle_\lambda^r$, the radiation state $|\gamma(\alpha,\beta)\rangle_r^\epsilon$ exists. Since it only contains a finite number of photons of energies above $r$, it can be treated as if they were part of the usual Fock space. Only the radiation is measurable and for $r \ll \epsilon$, only it carries a significant energy.

We can check that the amplitude (4.22) indeed gives the correct rate. To this end, we apply the Bloch-Nordsieck recipe and need to sum over all possible soft radiation in the final state, i.e. over all radiation states in which the sum of all photon energies is below $\epsilon$. Furthermore, we need to take into account the conservation of energy, i.e. put the Heaviside function as in (1.13). For $r \ll \epsilon$, we get

$$\begin{aligned}
\Gamma_{\alpha,\beta} &= \sum_N \frac{1}{N!}\left(\prod_{i=1}^N \int_r^\epsilon \mathrm{d}^3\boldsymbol{k}_i \sum_{l_i}\right) \mathcal{H}(\epsilon - \sum_{j=1}^N |\boldsymbol{k}_j|) \left| \left(\langle\gamma_N| \otimes {}_\lambda^r\langle\langle\beta|\right) \hat{S}\,|\alpha\rangle\rangle_\lambda^r \right|^2 \\
&= \left(\frac{r}{\Lambda}\right)^{B_{\alpha,\beta}} \sum_N \frac{1}{N!}\left(\prod_{i=1}^N \int_r^\epsilon \frac{\mathrm{d}^3\boldsymbol{k}_i}{|\boldsymbol{k}_i|} \sum_{l_i} \left|\mathcal{F}_{\alpha,\beta}^{(l_i)}(\boldsymbol{k}_i)\right|^2\right) \mathcal{H}(\epsilon - \sum_{j=1}^N |\boldsymbol{k}_j|)\, \left|S_{\alpha,\beta}\right|^2 \\
&= \left(\frac{\epsilon}{\Lambda}\right)^{B_{\alpha,\beta}} f(B_{\alpha,\beta})\, \left|S_{\alpha,\beta}\right|^2 \,, \qquad (4.24)
\end{aligned}$$

where first line the operators $\hat{a}_{l_i,\boldsymbol{k}_i}$ in the $\langle\gamma_N|$ state act on the coherent state $|\gamma(\alpha,\beta)\rangle_r^\epsilon$ and causes, see (1.29), the functions $\mathcal{F}_{\alpha,\beta}^{(l_i)}(\boldsymbol{k}_i)/\sqrt{|\boldsymbol{k}_i|}$ in the second line. The result is then the well-known result in the inclusive formalism (1.13). If we neglect the function $f(B_{\alpha,\beta})$, which is possible for weak coupling, the rate (4.24) is also identical to the result in the dressed formalism (1.35). In particular, it is clear that the answer that we obtain is IR-finite since the regulator $\lambda$ has dropped out. It is important to note that we never required IR-finiteness, but it simply arises as a consequence of applying the $S$-matrix to a well-defined initial state.

Moreover, we observe that the rate (4.24) is also independent of the scale $r$. As we have discussed, our approach interpolates between the dressed formalism, which corresponds to $r = \epsilon$, and the inclusive formalism, which we obtain for $r = \lambda$.[7] The fact that our result is independent of $r$ implies that not only dressed and inclusive formalism yield – except for $f(B_{\alpha,\beta})$ – the same rate, but that this is also true for our interpolation between them.

### 4.3.2 Additional Comments

#### IR-finite Amplitudes in Inclusive Formalism

Before we come to the study of the density matrix, we will briefly deviate from our main line of argument and make a few additional comments. First, we will for a moment take the limit $r = \lambda$, in which the dressings vanish and we obtain the inclusive formalism. Then formula (4.22) becomes

$$\hat{S}\,|\alpha\rangle = \sum_{\beta} \left(\frac{\epsilon}{\Lambda}\right)^{B_{\alpha,\beta}/2} S_{\alpha,\beta} \left(|\beta\rangle \otimes |\gamma(\alpha,\beta)\rangle_{\lambda}^{\epsilon}\right) , \tag{4.25}$$

where the electron states are not dressed. This leads to the IR-finite amplitude:

$$\left({}_{\lambda}^{\epsilon}\langle\gamma(\alpha,\beta)| \otimes \langle\beta|\right) \hat{S}\,|\alpha\rangle = \left(\frac{\epsilon}{\Lambda}\right)^{B_{\alpha,\beta}/2} S_{\alpha,\beta} \,. \tag{4.26}$$

So if we use as final state the correct state of radiation $|\gamma(\alpha,\beta)\rangle_{\lambda}^{\epsilon}$, which depends both on initial and final electrons, we get an IR-finite amplitude in the inclusive formalism. However, the price we pay is that on the one hand, we are not able to obtain the factor $f(B_{\alpha,\beta})$ that encodes energy conservation and on the other hand that now the radiation state $|\gamma(\alpha,\beta)\rangle_{\lambda}^{\epsilon}$ contains an infinite number of zero-energy photons and is no longer part of the Fock space. Nevertheless, it is a physically sensible state since it only contains a finite energy.

[7]Sending $\epsilon \to r$ for fixed $r$ corresponds to a situation in which no soft emission takes place. When we work with well-defined, i.e. dressed states, the rate of such a process is suppressed by the possibly small factor $\left(r/\Lambda\right)^{B_{\alpha,\beta}}$ but non-vanishing.

### Large Gauge Transformations and Dressing

Another interesting question is how gauge transformations act on dressed states. We can parametrize them as shift of the polarization tensor,[8]

$$\varepsilon^{\mu}_{l,\boldsymbol{k}} \to \varepsilon^{\mu}_{l,\boldsymbol{k}} + \lambda_l(\boldsymbol{k})k^{\mu}\,, \tag{4.28}$$

where $\lambda_l(\boldsymbol{k})$ is an arbitrary function. Since dressing is determined by photons with $|\boldsymbol{k}| < r$, small gauge transformations, for which $\lambda_l(\boldsymbol{k})$ vanishes for all $|\boldsymbol{k}| < r$, leave the dressing state invariant. Only large gauge transformations, for which $\lambda_l(\boldsymbol{k})$ has support below $r$, act non-trivially. With the definition $\tilde{\lambda}_l(\boldsymbol{k}) = \lambda_l(\boldsymbol{k})\sum_{n\in\alpha} e_n/\sqrt{2(2\pi)^3}$, those lead to the transformed dressed state

$$\begin{aligned}|\tilde{\alpha}\rangle\rangle^r_\lambda = &\exp\left\{-\frac{1}{2}\int_\lambda^r \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|}\sum_l \left|\mathcal{F}^{(l)}_\alpha(\boldsymbol{k}) + \tilde{\lambda}^*_l(\boldsymbol{k})\right|^2\right\}\\ &\cdot\exp\left\{\int_\lambda^r \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}}\sum_l \left(\mathcal{F}^{(l)}_\alpha(\boldsymbol{k}) + \tilde{\lambda}^*_l(\boldsymbol{k})\right)\hat{a}^\dagger_{l,\boldsymbol{k}}\right\}|\alpha\rangle\,. \end{aligned} \tag{4.29}$$

Thus, dressing states are not invariant under large gauge transformations [8, 22–25].[9]

Since the number of photons only changes by a finite amount, the equivalence class to which the dressing state belongs and consequently also the cancellation of IR divergences are left invariant. Instead, gauge transformations merely correspond to choosing a different representative of the equivalence class, i.e. to modifying the choice (4.4). However, the

[8] In a pure $S$-matrix formalism, invariance under the shift (4.28) can equivalently be derived from Lorentz invariance [12]. Then charge conservation follows from the soft theorem [12]: Plugging the shift (4.28) in the soft factor (1.33), we conclude that

$$\lambda^*_l(\boldsymbol{k})\sum_{n\in\alpha,\,\beta} e_n\eta_n = 0\,, \tag{4.27}$$

i.e. that the total incoming charge must be equal to the total outgoing charge. We note that this argument is completely independent of IR divergences that arise due to soft loops.

[9] In contrast, the radiation state (4.23) is manifestly gauge-invariant.

amplitude is not invariant under this transformation:

$$ {}^{r}_{\lambda}\langle\langle\beta|\hat{S}\,|\tilde{\alpha}\rangle\rangle^{r}_{\lambda} = {}^{r}_{\lambda}\langle\langle\beta|\hat{S}\,|\alpha\rangle\rangle^{r}_{\lambda} \left(1 - \frac{1}{2}\int_{\lambda}^{r} \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|} \sum_{l} \left|\tilde{\lambda}_l(\boldsymbol{k})\right|^2\right) , \tag{4.30} $$

where we restricted ourselves to the leading order in $\tilde{\lambda}_l(\boldsymbol{k})$. This effect is weak for sufficiently small $r\,\tilde{\lambda}_l(\boldsymbol{k})$ but generically non-zero. In order to restore full invariance, one has to apply the same shift (4.28) to both initial and final states: ${}^{r}_{\lambda}\langle\langle\tilde{\beta}|\hat{S}\,|\tilde{\alpha}\rangle\rangle^{r}_{\lambda} = {}^{r}_{\lambda}\langle\langle\beta|\hat{S}\,|\alpha\rangle\rangle^{r}_{\lambda}$. This shows that dressing states do not exhibit dynamics on their own, but that – in line with our previous discussion – the physical meaning of dressing is to decouple photons of energy below $r$.

There is an interesting interpretation of the gauge transformed dressing state (4.29). Up to an inessential phase factor, it can be written as

$$ |\tilde{\alpha}\rangle\rangle^{r}_{\lambda} \sim \exp\left\{-\frac{1}{2}\int_{\lambda}^{r} \frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|} \sum_{l} \left|\tilde{\lambda}_l(\boldsymbol{k})\right|^2\right\} \exp\left\{\int_{\lambda}^{r} \frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}} \sum_{l} \tilde{\lambda}_l^*(\boldsymbol{k})\,\hat{a}^{[\alpha]\dagger}_{l,\boldsymbol{k}}\right\} |\alpha\rangle\rangle^{r}_{\lambda} , \tag{4.31} $$

where $\hat{a}^{[\alpha]\dagger}_{l,\boldsymbol{k}}$ is defined in Eq. (4.14). It becomes evident that large gauge transformations correspond to adding photons that are not defined in the Fock space, but according to the representation of the commutation relations in the equivalence class $[\alpha]$. The important point is that the representations of the large gauge transformations in each equivalence class are not unitarily equivalent.

## 4.4 Reduced Density Matrix

### 4.4.1 Well-Defined Tracing

So far, we have rederived known results in a slightly different setting. Now we proceed to discuss the density matrix of the final state. The crucial question is how to define the trace, i.e. what states to trace over. But in our approach, in which we distinguish between dressing of asymptotic states and physical radiation, the answer is obvious. One should trace over soft radiation in the final state since it corresponds to physical states that are produced but not observed in a given setup. In contrast, the dressing is required for mere well-definedness of asymptotic states, so it makes no sense to trace over it. There is no asymptotic state without dressing. Once the trace refers to physical radiation, it is well-defined in the Fock space because radiation only contains a finite number of photons.

Before tracing, the density matrix of the final state (4.22) reads

$$\begin{aligned}\hat{\rho}^{\text{full}} &= \hat{S}\,|\alpha\rangle\rangle^r_\lambda\,{}^r_\lambda\langle\langle\alpha|\hat{S}\\ &= \sum_{\beta,\beta'}\left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}+B_{\alpha,\,\beta'}}{2}} S_{\alpha,\,\beta}S^*_{\alpha,\,\beta'}\left(|\beta\rangle\rangle^r_\lambda\otimes|\gamma(\alpha,\beta)\rangle^\epsilon_r\right)\left({}^\epsilon_r\langle\gamma(\alpha,\beta')|\otimes{}^r_\lambda\langle\langle\beta'|\right)\,. \quad (4.32)\end{aligned}$$

Obviously, it is pure since no tracing has happened yet. Using an arbitrary basis $\{|\gamma\rangle\}_\gamma$ of radiation, i.e. in the space of photons with energies above $r$ but below $\epsilon$, the trace is

$$\hat{\rho}^{\text{red}} = \sum_\gamma \mathcal{H}(\epsilon-\sum_{j=1}^N\left|\boldsymbol{k}_j\right|)\sum_{\beta,\beta'}\left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}+B_{\alpha,\,\beta'}}{2}} S_{\alpha,\,\beta}S^*_{\alpha,\,\beta'}\,|\beta\rangle\rangle^r_\lambda\,{}^r_\lambda\langle\langle\beta|\,\langle\gamma|\gamma(\alpha,\beta)\rangle^\epsilon_r\,{}^\epsilon_r\langle\gamma(\alpha,\beta')|\gamma\rangle\,, \tag{4.33}$$

where as in the computation of the rate, we imposed that the total energy $\sum_{j=1}^N\left|\boldsymbol{k}_j\right|$ in radiation is at most $\epsilon$. If we neglect energy conservation for a moment, the computation becomes particularly transparent:

$$\hat{\rho}^{\text{red}} \approx \sum_{\beta,\beta'}\left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}+B_{\alpha,\,\beta'}}{2}} S_{\alpha,\,\beta}S^*_{\alpha,\,\beta'}\,{}^\epsilon_r\langle\gamma(\alpha,\beta')|\gamma(\alpha,\beta)\rangle^\epsilon_r\;|\beta\rangle\rangle^r_\lambda\,{}^r_\lambda\langle\langle\beta|\,. \tag{4.34}$$

Thus, we only have to compute the overlap of coherent radiation states:

$$\begin{aligned}{}^\epsilon_r\langle\gamma(\alpha,\beta')|\gamma(\alpha,\beta)\rangle^\epsilon_r &= \mathrm{e}^{-\frac{1}{2}\int_r^\epsilon\frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|}\sum_l\left|\mathcal{F}^{(l)}_{\alpha,\,\beta}(\boldsymbol{k})\right|^2+\left|\mathcal{F}^{(l)}_{\alpha,\,\beta'}(\boldsymbol{k})\right|^2}\\ &\quad\cdot\langle 0|\,\mathrm{e}^{\int_r^\epsilon\frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}}\sum_l\mathcal{F}^{(l)*}_{\alpha,\beta'}(\boldsymbol{k})\,\hat{a}_{l,\boldsymbol{k}}}\,\mathrm{e}^{\int_r^\epsilon\frac{\mathrm{d}^3\boldsymbol{k}}{\sqrt{|\boldsymbol{k}|}}\sum_l\mathcal{F}^{(l)}_{\alpha,\beta}(\boldsymbol{k})\,\hat{a}^\dagger_{l,\boldsymbol{k}}}\,|0\rangle\\ &= \mathrm{e}^{-\frac{1}{2}\int_r^\epsilon\frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|}\sum_l\left|\mathcal{F}^{(l)}_{\alpha,\,\beta}(\boldsymbol{k})-\mathcal{F}^{(l)}_{\alpha,\,\beta'}(\boldsymbol{k})\right|^2}\\ &= \mathrm{e}^{-\frac{1}{2}\int_r^\epsilon\frac{\mathrm{d}^3\boldsymbol{k}}{|\boldsymbol{k}|}\sum_l\left|\mathcal{F}^{(l)}_{\beta,\,\beta'}(\boldsymbol{k})\right|^2}\end{aligned}$$

$$= \left(\frac{r}{\epsilon}\right)^{B_{\beta,\,\beta'}/2} , \tag{4.35}$$

where the kinematical factor for a hypothetical process $\beta \to \beta'$ appeared. In total, we obtain the element of the reduced density matrix:

$$\rho^{\text{red}}_{\beta\beta'} = \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}+B_{\alpha,\,\beta'}}{2}} \left(\frac{r}{\epsilon}\right)^{\frac{B_{\beta,\,\beta'}}{2}} S_{\alpha,\,\beta} S^{*}_{\alpha,\,\beta'} , \tag{4.36}$$

where it is understood that indices refer to dressed states. Clearly, this result is IR-finite. Had we taken into account energy conservation, we would have gotten the result (which is a generalization of the computation in [48]):

$$\rho^{\text{red}}_{\beta\beta'} = \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}+B_{\alpha,\,\beta'}}{2}} \left(\frac{r}{\epsilon}\right)^{\frac{B_{\beta,\,\beta'}}{2}} f\left(\frac{B_{\alpha,\,\beta} + B_{\alpha,\,\beta'} - B_{\beta,\,\beta'}}{2}\right) S_{\alpha,\,\beta} S^{*}_{\alpha,\,\beta'} . \tag{4.37}$$

As it should be, we observe that the diagonal terms reproduce the well-known rate (4.24), i.e. $\rho^{\text{red}}_{\beta\beta} = \Gamma_{\alpha,\,\beta}$.

In previous chapter, we already put forward one particular part of the density matrix to be IR-finite. We did so using an IR-finite version of the optical theorem and obtained a result similar to (4.37), but for $r = \epsilon$. Equipped with the arguments of the present chapter, we can conclude that the prescription proposed in chapter 3 corresponds to deriving the density matrix in the limit $r = \epsilon$, i.e. in the dressed formalism, in which no decoherence occurs.[10]

Finally, we can use the matrix element (4.37) to justify our choice (4.6) of $r$. In a framework of real time evolution, it was derived in [53–55] for a particular simplified setup that the off-diagonal elements of the density matrix scale as $(1/t_{\text{obs}})^{B_{\beta,\,\beta'}/2}$, where $t_{\text{obs}}$ is the timescale after which the final state is measured. Comparing this with (4.37), we conclude that the identification $r \sim t_{\text{obs}}^{-1}$ was indeed justified. In this way, we obtain the same behavior as in [53–55]: The longer we wait before we measure the final state, the smaller we have to choose $r$ and the more the off-diagonal elements of the density matrix get suppressed. We note, however, that our combined formalism does not rely on the identification (4.6) and holds for general $r$.

[10] In chapter 3, we observed the additional factor proportional to $f(B)$, which we do not obtain in the present treatment. This additional factor lead to some decoherence. As we shall show shortly, however, this effect is subleading since the resulting decoherence is much smaller than the one we observe now for $r \ll \epsilon$.

### 4.4.2 Generalization to Superposition as Initial State

As suggested in [56], it is interesting to study a situation in which the initial state $|\psi\rangle$ is not a momentum eigenstate:

$$|\psi\rangle\rangle_\lambda^r = \sum_\alpha f_\alpha^{(\psi)} \, |\alpha\rangle\rangle_\lambda^r \,, \tag{4.38}$$

where $\sum_\alpha |f_\alpha^{(\psi)}|^2 = 1$ and we used the linearity of the definition (4.4) of dressing. Generalizing the above calculations, we get

$$\begin{aligned}\rho^{\text{red}}_{\beta,\beta'} = \sum_{\alpha,\alpha'} f_\alpha^{(\psi)} f_{\alpha'}^{(\psi)*} \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}+B_{\alpha',\,\beta'}}{2}} \left(\frac{r}{\epsilon}\right)^{\frac{B_{\alpha,\,\beta}+B_{\alpha',\,\beta'}}{2}-B_{\alpha,\,\beta,\,\alpha',\,\beta'}} \\ \cdot f\left(B_{\alpha,\,\beta,\,\alpha',\,\beta'}\right) S_{\alpha,\,\beta} S^*_{\alpha',\,\beta'} \,,\end{aligned} \tag{4.39}$$

where

$$B_{\alpha,\,\beta,\,\alpha',\,\beta'} = \frac{1}{2(2\pi)^3} \sum_{\substack{n\in\alpha,\,\beta \\ m\in\alpha',\,\beta'}} \int \mathrm{d}^2\Omega \frac{\eta_n\eta_m e_n e_m \, p_n\cdot p_m}{p_n\cdot\hat{k}\; p_m\cdot\hat{k}} \,. \tag{4.40}$$

The density matrix (4.39) applies to the most general case and thereby constitutes the main result of this section.

Clearly, this density matrix avoids full decoherence. In order to further analyze our result, we can decompose the sums:

$$B_{\alpha,\,\beta,\,\alpha',\,\beta'} = \frac{B_{\alpha,\,\beta'} + B_{\alpha',\,\beta} - B_{\alpha,\,\alpha'} - B_{\beta,\,\beta'}}{2} \,. \tag{4.41}$$

This shows that if there is only one momentum eigenstate in the initial state, $f_\alpha^{(\psi)} = \delta_\alpha^{\alpha_0}$, the general density matrix (4.39) reduces to the result (4.37) obtained before. It is moreover interesting to analyze the rates that we obtain:

$$\rho^{\text{red}}_{\beta,\beta} = \sum_{\alpha,\alpha'} f_\alpha^{(\psi)} f_{\alpha'}^{(\psi)*} \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}+B_{\alpha',\,\beta}}{2}} \left(\frac{r}{\epsilon}\right)^{\frac{B_{\alpha,\,\alpha'}}{2}} f\left(\frac{B_{\alpha,\,\beta}+B_{\alpha',\,\beta}-B_{\alpha,\,\alpha'}}{2}\right) S_{\alpha,\,\beta} S^*_{\alpha',\beta} \,. \tag{4.42}$$

The $f(B)$-function is subleading since it follows from its definition (1.14) that it scales as $f(B) \sim 1 - B^2$ for small $B$ and additionally it is insensitive to the ratio $r/\epsilon$. Therefore, we can focus on the other two IR factors, $(\epsilon/\Lambda)^{(B_{\alpha,\beta}+B_{\alpha',\beta})/2}$ and $(r/\epsilon)^{B_{\alpha,\alpha'}/2}$. Clearly, both are always smaller than 1. In the case of constructive interference, they therefore always lead to a suppression of the rate. For destructive interference, however, they can work in both directions, i.e. they can also serve to diminish suppressing contributions and thereby increase the rate. The $r$-dependent contribution $(r/\epsilon)^{B_{\alpha,\alpha'}/2}$ is particularly interesting since it does not factorize, i.e. it cannot be absorbed in a redefinition of $S_{\alpha,\beta}$. These findings also hold for the off-diagonal elements. It is straightforward to show that the exponent of the $r$-dependent term in Eq. (4.39) is positive [56], so it also leads to a factor smaller than 1. As before, this means that it leads to a suppression of off-diagonal elements if there is constructive interference. In particular, this is always the case when the initial state is only a single momentum eigenstate. In contrast, it can cause both suppression and enhancement for destructive interference.

### 4.4.3 Estimate of Amount of Decoherence

In order to quantify decoherence and the corresponding loss of information due to tracing over soft radiation, we compute the entanglement entropy of hard and soft modes. It is defined as the von Neumann entropy of the density matrix $\rho^{\text{red}}$ obtained after tracing over soft radiation. To estimate it, we apply the procedure developed chapter 3, where it was shown that a bound on the decoherence of $\rho^{\text{red}}$ can be given in terms of its distance to the nearest pure density matrix. To this end, we assume that we are given a pure density matrix $\rho^{\text{pure}} = |\Psi\rangle\rangle^r_\lambda \, {}^r_\lambda\langle\langle\Psi|$ defined by a state

$$|\Psi\rangle\rangle^r_\lambda = \sum_\beta a_\beta \, |\beta\rangle\rangle^r_\lambda \,, \tag{4.43}$$

which fulfills $|a_\beta|^2 = \rho^{\text{red}}_{\beta,\beta}$ but can have arbitrary phases. Thus, $\rho^{\text{pure}}$ and $\rho^{\text{red}}$ have the same diagonal and therefore describe the same rates. In this situation, it was derived in chapter 3 that the entanglement entropy $S_{\text{soft}}$ is bounded by

$$\frac{S_{\text{soft}}}{S_{\text{max}}} \lesssim \max_{\beta,\beta'} |1 - c^{(\Psi)}_{\beta,\beta'}| \,, \tag{4.44}$$

where $S_{\text{max}}$ the maximal entropy that can exist in the Hilbert space and we defined

$$c^{(\Psi)}_{\beta,\,\beta'} = \frac{\rho^{\text{red}}_{\beta,\beta'}}{a_\beta a^*_{\beta'}}\,. \tag{4.45}$$

So the deviations of the $c^{(\Psi)}_{\beta,\,\beta'}$ from $1$ determine the decoherence and full coherence corresponds to $c^{(\Psi)}_{\beta,\,\beta'} = 1$.

To derive a concrete bound on the entanglement entropy, we have to choose $a_\beta$. As said, the absolute value is fixed by the requirement $|a_\beta|^2 = \rho^{\text{red}}_{\beta,\beta}$. To obtain a bound that is maximally sharp, we therefore have to set the phases such that $c^{(\Psi)}_{\beta,\,\beta'}$ is minimal. In the explicit computation of $c^{(\Psi)}_{\beta,\,\beta'}$, it turns out that a good choice is[11]

$$\begin{aligned} a_\beta = &\left(\sum_{\alpha,\alpha'} f^{(\psi)}_\alpha f^{(\psi)*}_{\alpha'} \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}+B_{\alpha',\,\beta}}{2}} \left(\frac{r}{\epsilon}\right)^{\frac{B_{\alpha,\,\alpha'}}{2}} f\left(\frac{B_{\alpha,\,\beta}+B_{\alpha',\,\beta}-B_{\alpha,\,\alpha'}}{2}\right) S_{\alpha,\,\beta} S^*_{\alpha',\,\beta}\right)^{1/2} \\ &\cdot \exp\left[i \arg \sum_\alpha f^{(\psi)}_\alpha \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}}{2}} S_{\alpha,\,\beta}\right]\,. \end{aligned} \tag{4.46}$$

Now we evaluate (4.44) in the regime of weak coupling where all kinematical factors become small, $B \ll 1$. Then we can expand the exponential and the $f(B)$-functions. In the regime $r \ll \epsilon$, in which we work throughout, the contribution of the $f(B)$-functions is, as already explained, subleading and we will ignore it. Then we obtain to leading order

$$\begin{aligned} \frac{S_{\text{soft}}}{S_{\text{max}}} \lesssim & \ln\frac{\epsilon}{r} \max_{\beta,\,\beta'} \frac{1}{2} \\ & \cdot \left| \frac{\sum_{\alpha,\alpha'} f^{(\psi)}_\alpha f^{(\psi)*}_{\alpha'} \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}+B_{\alpha',\,\beta'}}{2}} S_{\alpha,\,\beta} S^*_{\alpha',\,\beta'} \left(B_{\alpha,\,\beta}+B_{\alpha',\,\beta'}-2B_{\alpha,\,\beta,\,\alpha',\,\beta'}\right)}{\left(\sum_\alpha f^{(\psi)}_\alpha \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}}{2}} S_{\alpha,\,\beta}\right)\left(\sum_\alpha f^{(\psi)*}_\alpha \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta'}}{2}} S^*_{\alpha,\,\beta'}\right)} \right. \\ & \left. - \frac{\sum_{\alpha,\alpha'} f^{(\psi)}_\alpha f^{(\psi)*}_{\alpha'} \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}+B_{\alpha',\,\beta}}{2}} S_{\alpha,\,\beta} S^*_{\alpha',\,\beta} B_{\alpha,\,\alpha'}}{2\left|\sum_\alpha f^{(\psi)}_\alpha \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\,\beta}}{2}} S_{\alpha,\,\beta}\right|^2} \right. \end{aligned}$$

[11]The absolute value is fixed by rate (4.42). The phase is chosen such that it reproduces the density matrix (4.39) in the limit $r \to \epsilon$ and $f(B) \to 1$.

$$-\frac{\sum_{\alpha,\alpha'} f_\alpha^{(\psi)} f_{\alpha'}^{(\psi)*} \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\beta'}+B_{\alpha',\beta'}}{2}} S_{\alpha,\beta'} S^*_{\alpha',\beta'} B_{\alpha,\alpha'}}{2\left|\sum_\alpha f_\alpha^{(\psi)} \left(\frac{\epsilon}{\Lambda}\right)^{\frac{B_{\alpha,\beta'}}{2}} S_{\alpha,\beta'}\right|^2}\Bigg| . \tag{4.47}$$

Already at this point, the physical properties of this result become evident. First, decoherence depends logarithmically on the ratio $\epsilon/r$. This means that it gets big if the resolution gets worse, i.e. $\epsilon$ increases, or if one waits longer before measuring the final state, i.e. $r$ decreases.[12] In the limit of the best achievable resolution, $\epsilon = r$, there is no decoherence.[13] Moreover, we observe that the bound on the entanglement entropy scales with the kinematical factors $B$, i.e. becomes small for small $B$-factors.[14] Since they are proportional to $e^2$ and the momentum transfer, we conclude that decoherence scales with the coupling. Finally, it also depends on the kinematics of the scattering process. In the case in which the initial state is a single momentum eigenstates, this dependence becomes particularly transparent:

$$\frac{S_{\text{soft}}}{S_{\text{max}}} \lesssim \ln\frac{\epsilon}{r} \max_{\beta,\beta'} \frac{B_{\beta,\beta'}}{2} . \tag{4.48}$$

Since the kinematical factor $B_{\beta,\beta'}$ depends on the angle between the electrons in $\beta$ and $\beta'$, we conclude that decoherence scales with the angle between different final states, i.e. it gets bigger for bigger angles. This means that decoherence increases for final states whose bremsstrahlung is macroscopically different.

[12]That the entropy due to tracing over soft modes should scale logarithmically with the resolution was already suggested in [59] using a simpler argument.

[13]Our derivation of the density matrix (4.39) relies on $r \ll \epsilon$ and is no longer valid in the limit $r = \epsilon$, which corresponds to the dressed formalism. However, it is easy to rederive the density matrix in this case and one obtains (4.39) but without the $r$-dependent factor and without the $f(B)$-function. Therefore, the bound (4.47) also holds for $r = \epsilon$ and shows that there is no decoherence in this limit.

[14]An exception could occur in the case of fully destructive interference, i.e. when one of the denominators in Eq. (4.47) vanishes. However, as long as only a small fraction of the entries of the density matrix goes to zero, it is clear that the amount of decoherence is still small. In that case, one would have to employ a more sophisticated bound than the one that we use here.

# Chapter 5

# Masses and electric charges

In the last 3 chapter we treated the IR divergences, their physical nature and how to cancel them. Thereby we used the inclusive formalism as well the coherent state formalism and in chapter 4 we used even a combination of both formalisms. This chapter will treat an other divergence which occurs when going to the very high energy limit in a scattering process or equally when taking the mass of electrically charged particles to zero. As mentioned in the review chapter, these so called *collinear divergences* play an important role to understand the consistency of theories with long range gauge forces with massless charged particles. This is an old problem that has been considered from different angles along the years (see [60–63] for an incomplete list). As a matter of fact in nature we don't have any example of massless charged particles. In the Standard Model this is the case both for spin $1/2$ as well as for the spin $1$ charged vector bosons. In the particular case of charged leptons the potential inconsistency of a massless limit should imply severe constraints on the consistency of vanishing Yukawa couplings.

## 5.1 Introduction

Already when treating IR divergences in QED one is confronted with other divergences that take place when taking the mass of electrically charged particles such as the electron. In [6,7] it was mentioned that the $B$-factor for QED (1.2) goes as $\ln\left(\frac{2pp'}{m_e^2}\right)$, where they considered one electron with mass $m_e$ scattered off an external potential and $p\,/p'$ is the 4-momentum of the ingoing/outgoing electron. In order words, the $B$-factor diverges as $B_{\alpha,\beta} \sim -\ln(m_e)$ as the mass goes to zero. In this limit the rates (1.13) would be highly suppressed and strive to zero, which would make the theory obsolete or in first order perturbation theory inconsistent. In gravity however, the problem does not occur since there in the massless limit the $B$-factor (1.4) goes as $B_{\alpha,\beta} \sim -m_e^2 \ln(m_e)$ and thus goes to zero in the limit where the mass vanishes. Collinear divergences do not only appear in the $B$-factor but as we will see in processes where the scattered photons are neither IR photons nor soft, thus are scattered from internal lines and have comparable energies to the scattered charged particles.

It was found by Kinoshita [64] and Lee and Nauenberg [65] that these divergences can be canceled by a similar but more general procedure like the Bloch-Nordsieck recipe. The formulated *KLN theorem* is the generalized approach to address divergences in an inclusive way and incorporates ingoing radiation, disconnected diagrams as well as already mentioned any kind of radiation, i.e. non-IR radiation which we mentioned in the sections 1.2 and 3.2. This amplification of allowed diagrams makes the theorem less accepted in the literature. Nevertheless, it works outstandingly well in order to cancel all kinds of divergences so that we will assume that it holds.

In [65] a unifying picture to the problem was suggested on the basis of *degenerations*. The idea is to define, for a given amplitude $S_{\alpha,\beta}$ associated with a given scattering process $\alpha \to \beta$ an inclusive cross section formally defined as

$$\sum_{\alpha' \in D(\alpha)} \sum_{\beta' \in D(\beta)} \left| S_{\alpha',\beta'} \right|^2 , \tag{5.1}$$

where $D(\alpha)$ is the set of asymptotic states *degenerate* with the asymptotic state $\alpha$.

For the case of massless electrically charged particles the degeneration used in [65] for the case where the asymptotic state is a charged massless lepton with momentum $p^\mu$ is a state with the lepton having momentum $p-k$ and an additional on-shell photon with 4-momentum $k^\mu$ collinear to $p^\mu$.[1] The *very small angle* or *collinearity* between the two particle makes this state physically degenerate to the asymptotic state of a charged massless lepton.

At each order in perturbation theory the KLN recipe demands us to sum over all contributions at the same order in perturbation theory that we can build using degenerate incoming and/or outgoing states. Among these are specially interesting the *quantum interference effects* with disconnected diagrams where the additional photon entering into the definition of

[1]For a recent discussion of the KLN theorem for QED see [66] and references therein.

a degenerate incoming state is not interacting. In particular these interference effects play a crucial role to cancel collinear divergences in processes where the incoming electron emits a collinear photon, see for instance [65–67]. Therein it was for instance shown that the mass singularity of the $B$-factor can be resolved.

The main target of this chapter is to study the quantum consistency of the KLN prescription to define a quantum field theory of $U(1)$ massless charged particles. Our findings can be summarize in two main claims. On one side we shall argue that the consistency of the KLN prescription in the forward regime implies the existence of a non-vanishing gauge anomaly for the $U(1)$ gauge theory. This rules out the possibility of the existence of a theory with massless charges and long range interactions satisfying both KLN cancellation and being anomaly free. Secondly combining the *weak gravity conjecture* [68] and *anomalous thresholds* for form factors, we derive a gravitational lower bound on the mass of the lighter massless charged fermion and a qualitative upper bound on the total number of fermionic species with the same charge as the electron.

## 5.2 The KLN-theorem: degeneracies and energy dressing

Let us briefly review the key aspect of the KLN theorem [65]. In order to do that let us consider scattering theory for a given Hamiltonian $H = H_0 + gH_I$ and let us assume the Hamiltonian depends on a parameter $m$. Assuming a well defined scattering theory, the hamiltonian $H$ can be diagonalized using the corresponding Møller operators $U$. Let us denote $E_i(g, m)$ the corresponding eigenvalues. If for some value $m_c$ of the mass parameter we have degenerations i.e. $E_i(g, m_c) = E_j(g, m_c)$ then the perturbative expansion of $U_{i,j}$ becomes singular at each order in perturbation theory. However *at the same order* in perturbation theory the quantity $\sum_a U_{a,i}U^*_{j,a}$ where we sum over the set of states degenerate with the state $a$ is free of singularities in the limit $m = m_c$ leading to the prescription (5.1) for finite cross sections. The former result is true provided $\Delta_a(g, m) = (H_0 - E)_{aa}$ has a good finite limit for $m = m_c$.

The quantum field theory meaning of $\Delta$ is the difference of energy between the bare and the dressed state. The theorem works if for fixed and finite UV cutoff the limit of this dressing effect is finite in the degeneration limit.

For the case of QED and for $m_e$ the mass of the electron, degeneracies appear in the limit $m_e \to 0$. As stressed in [65] in this case the limit of $\Delta$ for $m_e \to 0$ and fixed UV cutoff is not finite. The problem is associated with the well known behavior of the renormalization constant $Z$ for the photon field which goes as

$$Z = 1 - \frac{e^2}{6\pi^2} \ln \frac{\Lambda}{m_e} \, . \tag{5.2}$$

The origin of the problem is well understood. Using Källen-Lehmann-representation to extract the value of $Z$ from the imaginary part of the bubble amplitude i.e. Im$(D(k^2))$ for the photon propagator $D(k^2)$, we get for massless electrons a branch cut singularity in the physical sheet for the threshold $k^2 = 0$ where the on-shell photon can go into a collinear pair of on-shell electron and positron.

In [65] this problem was explicitly addressed and the suggested solution was to keep $m_e = 0$ but to add a mass scale in the definition of $Z$ (see [69]) associated with a parameter $\delta$ that guarantees collinear finiteness and is often set to the resolution scale of a detector. The logic of this argument is to assume a collinear correction of (5.2) where effectively $m_e$ is replaced by $\delta$ and to use this corrected $Z$ to define a $\Delta$ non-singular in the limit $m_e \to 0$.

Note that the singular limit of $Z$ in the massless limit is the IR version of the famous Landau pole problem for QED. In this case we are not considering the limit where we send the UV cutoff to infinity but instead the limit $m_e \to 0$. In the massless limit there are contributions to the Källen-Lehmann-function coming from processes in which the on-shell photon with energy $\omega$ produces a pair of electron positron both collinear and on-shell. Incidentally note that in principle we have contributions of amplitudes where the on-shell photon decays into a set of a large number $n$ of electron-positron-pairs and photons where all of them are on-shell and collinear. The approach of the KLN program is to assume that after taking all these IR contributions into account the resulting $Z$, for fixed UV cutoff, is finite in the limit $m_e \to 0$. This does not imply solving the UV problem or avoiding the standard Landau pole, that depends on the sign in (5.2) and that now will become dependent on the added resolution scale $\delta$. It simply means that for fixed UV cutoff the limit $m_e \to 0$ *could be* non-singular. In section 5.4 we will revisit the consistency of the limit $m_e \to 0$ from a different point of view.

Can we check the consistency of the KLN proposal perturbatively? To the best of our knowledge the KLN program of finding a redefinition of $Z$ where the cancellation can be defined in an effective way has not been developed. Thus we should expect that perturbative violations of the KLN theorem could appear whenever we work in the kinematical regime where originally appears the singularity responsible for the former behaviour of $Z$, namely in the forward regime, where the 4-momentum of the electron before and after the scattering is equal and $q^2 = 0$.

## 5.3 Degeneracies and anomalous thresholds

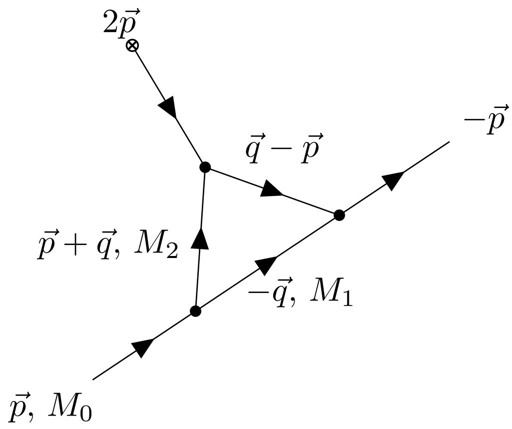



Figure 5.1: Anomalous threshold in Breit frame.

In scattering theory the existence of *anomalous thresholds* for form factors of bound states is well known (see [70–72]). The idea is simply to consider the triangular contribution to the form factor of a particle $A$ by some external potential. If the particle $A$ can decay into a pair of particles $N$ and $B$ where only $N$ interacts with the external potential we get the triangular amplitude depicted in figure 5.1. If we now impose all the internal lines to be on-shell we can find a critical transfer momentum for which the corresponding amplitude has a leading Landau singularity in the physical sheet. This transfer momentum defines the anomalous threshold. This leads to a logarithmic contribution to the amplitude and to a non-vanishing absorptive part forbidden by standard unitarity. The simplest way to set when this singularity is physical is using the Coleman-Grossman-theorem [73] that dictates that the singularity is physical if the triangular diagram can be interpreted as a space-time physical process with energy momentum conservation in all vertices and with the internal lines on-shell i.e. as a Landau-Cutkosky-diagram.

Let us now consider the degenerations as formally representing the massless electron as a composite state of electron and collinear photon. In this case we can consider the triangular contribution in figure 5.2 to the form factor where the electron in the triangle interacts with the external potential. In this case it is easy to see that an anomalous threshold can appear only in the forward limit when the transfer momentum $q^2$ is zero (see appendix B).

From the KLN theorem point of view we can associate these kinematical conditions to the degeneration defined by the absorption and emission process of a collinear photon with the same value of the 4-momentum $k^\mu$ and with $k^\mu$ collinear to $q^\mu$. In this case the logarithmic divergence $\ln(m_e)$ of the anomalous threshold can be canceled with the corresponding KLN sum.

However the KLN prescription in this forward limit allows us to have different 4-momentum $k^\mu$ and $k'^\mu$ for the absorbed and emitted photon. If this amplitude is logarithmically divergent it cannot be trivially canceled by a one loop contribution to the form factor. Next we shall see that this is indeed the case and that the only possible cancellation leading to a consistent theory of massless charged particles is using quantum interference with processes controlled by the triangular graph defining the gauge anomaly of the underlying gauge theory.

## 5.4 The KLN anomaly

In this section we shall consider the absorption emission process in the forward limit with $k^\mu \neq k'^\mu$, pictured in the diagrams of figure 5.3. An electron with 4-momentum $p^\mu$ is *scattered forward* from a transfer momentum $q^\mu$ and we have an incoming and outgoing photon with 4-momentum $k^\mu$ and $k'^\mu$ attached to the external lines of the ingoing and outgoing electron. We fix the data $p^\mu$ and $q^\mu$ so that because of energy and momentum conservation $k'^\mu = k^\mu + q^\mu$ holds. Let us denote the amplitude $S(p, q)$. For these data the KLN prescription requires to define the sum

$$\sum_{n_\alpha, n_\beta} \left| S(p, q; n_\beta, n_\alpha) \right|^2 , \tag{5.3}$$

where $n_\alpha$ and $n_\beta$ denote the different degenerate states contributing to the process that are characterized by the number $n_\alpha$ of absorbed collinear photons attached to the incoming line and the number $n_\beta$ of emitted collinear photons attached to the outgoing line. All of these photons are assumed to have energies bigger than the IR energy resolution scale set by the

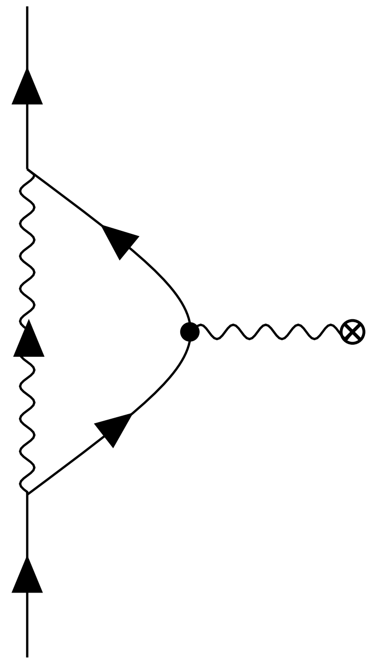


Figure 5.2: Landau Cutkosky diagram associated with the anomaly.

Bloch-Nordiesk-recipe. Generically each term in the sum (5.3) involves the integral over the 3-momentum of the collinear photons within a given angular resolution scale. The amplitudes in (5.3) contain internal lines with the corresponding propagators being on-shell.

In what follows we shall be interested in the *forward corner* of phase space characterized by vanishing transfer momentum, i.e.

$$q^2 = 0 \,. \tag{5.4}$$

In the forward regime the first absorption emission process contributing to the sum contains one absorbed photon and one emitted photon. This process is characterized by the following set of kinematical conditions $pq \approx kp \approx k'p' \approx 0$. This implies that in this corner of phase space the two propagators entering into the amplitude are on-shell. This after integration leads to a collinear divergence. Moreover in these kinematical conditions we have

$$q^\mu + k^\mu = k'^\mu \tag{5.5}$$

and, as mentioned, in the forward limit the outgoing electron has the same momentum as the incoming one, i.e. $p^\mu = p'^\mu$. Since for this amplitude both the absorbed and the emitted photons are collinear to the incoming and outgoing electron respectively, the KLN recipe indicates that this divergence should be canceled by the collinear contribution of virtual photons running in the loop.

In what follows we shall show that in the forward limit emission absorption processes with $k^\mu \neq k'^\mu$ lead to logarithmic divergences. The diagrams that lead to the collinear term are given in figure 5.3. We work in the chiral basis and choose the kinematics for the electron to run in z-direction. In the appendix C we explain the details and the notations used in the calculation. We omit all terms that will not lead to a collinear divergence. In these conditions we get for the amplitudes for a forward scattered right-/left-handed elctron

$$\begin{aligned} \mathrm{i}M^R =& -\mathrm{i}e^3 \frac{\sqrt{2}\theta\left[\omega(\omega+\omega_q) + (2E+\omega\lambda)(2E+(\omega+\omega_q)\lambda')\right]}{E\omega\omega_q\left(\theta^2+\frac{m^2}{E^2}\right)} \\ &+ \mathrm{i}e^3 \frac{\sqrt{2}\theta\left[-\omega\omega_q + (2E+\omega\lambda)(2E+\omega_q\lambda_q)\right]}{E\omega(\omega+\omega_q)\left(\theta^2+\frac{m^2}{E^2}\right)}\,, \end{aligned} \tag{5.6}$$

$$\begin{aligned} \mathrm{i}M^L =& -\mathrm{i}e^3 \frac{\sqrt{2}\theta\left[\omega(\omega+\omega_q) + (2E-\omega\lambda)(2E-(\omega+\omega_q)\lambda')\right]}{E\omega\omega_q\left(\theta^2+\frac{m^2}{E^2}\right)} \\ &+ \mathrm{i}e^3 \frac{\sqrt{2}\theta\left[-\omega\omega_q + (2E-\omega\lambda)(2E-\omega_q\lambda_q)\right]}{E\omega(\omega+\omega_q)\left(\theta^2+\frac{m^2}{E^2}\right)}\,. \end{aligned} \tag{5.7}$$

In order to use the KLN-theorem we need to perform the integration over photon momenta $\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}$, and taking into account the constraint $k'^\mu = q^\mu + k^\mu$, coming from the conservation of energy and momentum. The interesting part of the integral is the one over the small angle $\theta$, since there the collinear divergence shows up. In the collinear limit $\omega' = \omega_q + \omega$ and $\theta' = \frac{\omega_q\theta_q}{\omega+\omega_q}$ (see appendix C). Including these constraints, and integrating over the phase space $\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}$ with small angle $\theta$ gives

$$\begin{aligned} &\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega} \frac{1}{4} \sum_{\text{spins}} |\mathrm{i}M|^2 \\ =& \int \frac{\mathrm{d}\omega}{(2\pi)^2} \frac{e^6}{4E^2\omega} \ln\left(\frac{E\delta}{m}\right) \left[\frac{-\omega\omega_q + (2E+\omega\lambda)(2E+\omega_q\lambda_q)}{(\omega+\omega_q)}\right. \end{aligned}$$

$$
- \frac{\omega(\omega+\omega_q) + (2E+\omega\lambda)(2E+(\omega+\omega_q)\lambda')}{\omega_q} \Bigg]^2
$$

$$
+ \left(\lambda \to -\lambda,\, \lambda' \to -\lambda',\, \lambda_q \to -\lambda_q\right) \ , \tag{5.8}
$$

where $\delta$ is a small angular regulator, just like the energy regulator $\epsilon$ in the Bloch-Nordsieck recipe, in order to define collinearly finite rates. The details of the calculations can be seen in the appendix C.

In summary for generic $q^\mu$ and for emission absorption processes we get a double pole for $k^\mu = k'^\mu$ that can interfere with a disconnected diagram where the photon is not interacting. For $q^2 = 0$ we have a double pole on the kinematical sub manifold defined by $k^\mu - k'^\mu = q^\mu$ that leads, for fixed $q^\mu$ and after integration over $k^\mu$, to a collinear divergence that don't interfere with disconnected diagrams where the photon is not interacting. Thus we have obtained an additional collinear divergent contribution from the KLN-theorem (5.1), which is not canceled by any known loop factors. We will refer to this contributions as a KLN anomaly.

## 5.5 The KLN anomaly and the triangular anomaly

From a perturbative point of view a crucial ingredient of anomalies in four dimensions are triangle Feynman diagrams with currents inserted at the vertices. This is the case for the original ABJ anomaly [74,75] as well as for gauge anomalies. The difference lies in the type of currents we insert in the vertices.

The analytic properties of triangular graph amplitudes were extensively studied in the early 60's using Landau equations [76, 77] and Cutkosky rules [78]. As already mentioned it was first observed in [79] the existence, for triangular graphs, of singularities associated with non-unitary cuts. These singularities are the *anomalous thresholds* [72] (see appendix B for the relevant formulae).

In reference [80] it was first pointed out the connection of the anomaly with the IR singularities of the corresponding triangular graph amplitude. This approach was further developed in [81] and [73] in the context of t'Hooft's anomaly matching conditions [82].

Let us first briefly recall the analytic structure of anomalies. In a nutshell, given a triangular amplitude $\Gamma^{\mu\nu\rho}$ for three chiral currents let us denote $\Gamma(q^2)$ the invariant part of the

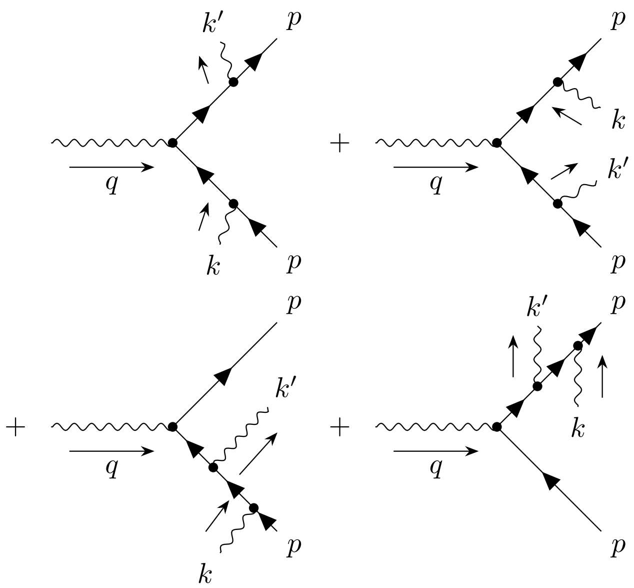



Figure 5.3: These diagrams lead to a mass divergence once (5.1) is applied, with the kinematics $q^\mu + k^\mu = k'^\mu$. There are two topologically equal diagrams which are not shown here. The reason is that they don't lead to any collinear divergent terms, which is explained in more detail in appendix D.

amplitude for $q^2$ the relevant transfer momentum (see figure 5.4). The anomaly is defined as the residue of $\Gamma(q^2)$ at $q^2 = 0$, i.e.

$$q^2\Gamma(q^2) = \mathcal{A}\,, \tag{5.9}$$

for $\mathcal{A}$ the c-number setting the anomaly. Standard dispersion relations connect (5.9) with the imaginary part of $\Gamma(q^2)$, namely

$$\text{Im}\Gamma(q^2) \sim \delta(q^2)\,. \tag{5.10}$$

The physical meaning of the singularity underlying the anomaly requires to understand the analytic properties of the full amplitude.

As already mentioned for the triangular graph we can have normal threshold singularities as well as the anomalous threshold singularities that correspond to the leading Landau singularity. In the language of Landau equations the normal threshold corresponds to the

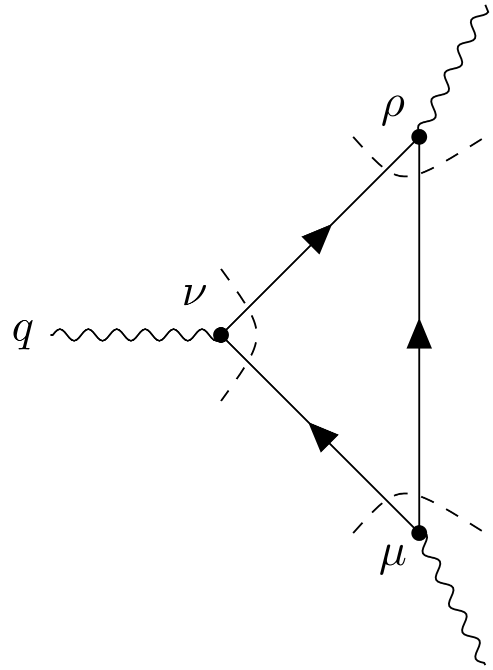



Figure 5.4: The anomaly diagram with the non-unitary cuts.

reduced graph where the Feynman parameter $x$ of one of the three lines is equal to zero. In what follows we shall discuss the anomalous threshold.[2] This corresponds to put the three lines of the triangle on-shell. The threshold is determined by the value of transfer momentum $q^2$ at which the corresponding diagram with all the internal lines on-shell and with external real photons is kinematically allowed. For massless particles running in the triangle this anomalous threshold exists and it is given by $q^2 = 0$. The corresponding discontinuity is determined by Cutkosky rules as

$$\int \mathrm{d}^4 p \prod \theta(p_i^0)\delta(p_i^2) \prod C_i \, , \tag{5.11}$$

where the $C_i$ are the *physical values* of the three amplitudes determined by the non-unitary

[2]Normal thresholds are relevant for the study of chiral anomalies in two dimensions. In this case the leading singularity for the corresponding two point diagram represents the $\eta'$ [83].

cut (see figure 5.4).[3] As shown in [73] the discontinuity of the triangular amplitude goes as

$$q\delta(q^2) \tag{5.12}$$

and it is non-vanishing. Let us now look at this discontinuity as an anomalous threshold. The physical process associated with this discontinuity can be understood as a real incoming photon that for massless charges decays into a pair of collinear on-shell electron and positron. One piece of the pair interacts with the external potential with some transfer momentum and finally the pair annihilates giving rise to a massless photon. Note that the discontinuity for the anomaly graph relies on the fact that for massless charges the photon can decay into a pair of on-shell collinear charged particles. If we fix one chirality for the running electron this discontinuity gives us the anomaly. To cancel the gauge anomaly for $U(1)$ we need to have real representations i.e. to add both chiralities in the loop.

The decay of the photon into a pair of collinear massless fermions can be formally interpreted as a degeneracy between the photon and a pair of collinear massless charged particles. From this point of view the anomaly is just the anomalous threshold associated with this formal compositeness of the photon. In more precise terms what makes the anomaly *anomalous* is *the existence of an absorptive part of the triangular amplitude that is expected, from standard unitarity (only one cut), to vanish.*[4]

Let us now relate the KLN anomaly and the triangular anomaly. As discussed the KLN anomaly appears whenever $k^\mu \neq k'^\mu$ with zero transfer momentum (5.4). From the KLN theorem point of view the contribution computed in the former section should cancel with some contribution to the form factor of the electron.

Since we are working at order $e^6$ we need, in principle, to include all loop diagrams to this order in perturbation theory contributing to the form factor. The interference term of two-loop diagrams and the tree-level diagram and the interference term of an one-loop diagram with one incoming collinear photon and a tree-level diagram with also one incoming collinear photon are of order $e^6$. We treat these diagrams and its collinear divergent contribution to the amplitude in the appendix E and F. The two-loop contribution treated in F goes like $\ln^2(m_e)$ and therefore can not cancel the KLN anomaly. The interference term treated

[3] In (5.11) we have formally included in the $C_i$ the propagator factors distinguishing bosons from fermions in the cuted lines.

[4] The anomaly matching [82] reflects that the discontinuity of the triangular graph is the same for the IR and UV physical spectrum running in the triangle.

in E is of order $\ln(m_e)$ but will not cancel the KLN anomaly as shown in appendix E. Thus, the $\ln$-divergent term in the amplitude square (5.8) can't be canceled. This is intuitively clear from the fact that the KLN anomaly appears when $k^\mu \neq k'^\mu$.

However, in this case we have the possibility of defining an interference term at this order in perturbation theory. Namely, we can think a diagram where we have the electron non-interacting and where the companion collinear photon is interacting *through the triangular anomaly* with the external source. This allows $k^\mu \neq k'^\mu$ in the forward limit where $k^\mu$ and $k'^\mu$ are both collinear to the momentum $p^\mu$ of the electron. The role of the triangular anomaly graph is to account for the difference between $k^\mu$ and $k'^\mu$ and to provide the needed logarithmic singularity. Thus for $k^\mu \neq k'^\mu$ the only possible contribution will come from the interference with the anomaly diagram in figure 5.5. Therefore for fixed chirality of the electron in figure 5.3 the only possibility to cancel the KLN anomaly is to assume a non-vanishing value for the triangular graph. However, this is only possible if the corresponding gauge theory is anomalous. In fact once we sum over all chiralities in the triangle we get a zero contribution to a form factor with $k^\mu \neq k'^\mu$.

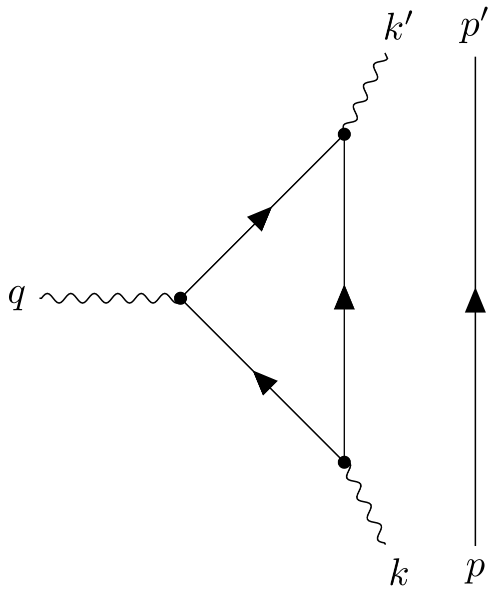



Figure 5.5: Anomaly triangle diagram with disconnected electron contributing to the amplitude square.

In short we have shown that *the KLN anomaly can be only canceled if the gauge theory is anomalous*.

Consequently we conclude that the the KLN anomaly can be only canceled effectively adding a mass for the charged fermions.

In summary the KLN anomaly in the forward limit with $k^\mu \neq k'^\mu$ corresponds to an anomalous threshold in the form factor where it is the photon, the one that interacts with the external potential. This can only take place through the triangular graph and it is only non-vanishing if the theory is anomalous with respect to the underlying gauge symmetry.

Before finishing this section let us make two brief comments that could help to clarify the argument. First of all note that in [65] processes as the ones in figure 5.3 were considered. For generic $q^\mu$ this produces logarithmic divergences only in the case $k^\mu = k'^\mu$ and these are compensated using a disconnected diagram where the companion photons is not interacting. In the particular case of $q^2 = 0$ we have a collinear divergence even for $k^\mu \neq k'^\mu$ and the corresponding disconnected diagram is now the one in figure 5.5 where we need to include the triangular anomaly in the photon line. The second comment concerns the recent discussion of symmetries in massless QED [14]. In the symmetry language this could be interpreted as indicating that KLN recipe is violating these symmetries. Actually a potential way to interpret our result is that in the massless case the collinear dressing in the forward limit $q^2 = 0$ is actually incompatible with non-anomalous gauge invariance.[5]

In case the origin of the transfer momentum is gravitational the situation is more interesting and richer. In fact in this case although we keep the same electromagnetic degeneration due to collinear electromagnetic radiation the external field, once it is assumed to be gravitational, can contribute to the form factor due to the graviton photon vertex.

## 5.6 A lower bound for the electron mass

In the former section we have argued that a quantum theory of massless charged fermions is inconsistent. The core of the argument is that consistency requires to cancel the KLN anomaly and that is only possible if the theory has non-vanishing $U(1)$ gauge anomaly i.e. if the theory is by itself inconsistent.

In what follows we shall put forward the following conjecture:

[5] In [84] it is argued that non-vanishing gravitational topological susceptibility implies the absence of massless fermions.

*In a theory with minimal length scale $L$ the minimal mass of a $U(1)$ charged fermion, for instance the electron, is given by*

$$m_e \geq \frac{\hbar}{L} e^{-\frac{1}{e^2 \nu}} , \tag{5.13}$$

*where $e^2$ is the corresponding coupling and $\nu$ is the number of fermionic species with charge equal to the electron charge.*

Before sketching the argument let us make explicit the logic underlying this conjecture. The bound (5.13) can be naively obtained from the perturbative expression (5.2) as the minimal mass of the electron consistent with pushing the perturbative Landau pole to $\frac{\hbar}{L}$. To argue in that way will force us to assume that the perturbative result for $Z$ already rules out the consistency of a theory of massless electrons. This will contradict the basic assumption of the KLN theorem of the potential redefinition of $Z$ with a well defined $m_e \to 0$ limit. Thus our approach to set a bound on the electron mass will consist in looking for some anomalous threshold singularity depending on the electron mass and to set the bound by analyzing the limit $m_e \to 0$ of these contributions to form factors.

In order to look for the appropriated form factor we shall use the constraints on the charged spectrum coming from the weak gravity conjecture [68]. This conjecture is equivalent to say that in absence of SUSY extremal electrically charged black holes are unstable. This leads to the existence in the spectrum of a particle with mass satisfying

$$m_e^2 \leq e^2 M_P^2 . \tag{5.14}$$

Once we accept the instability of charged black holes in absence of SUSY we can compute the effect *of this instability to the form factor of the charged black hole in the presence of an external electric potential*. Denoting $m_e$ the mass of the minimally charged particle we have again the anomalous threshold contribution where the black hole interaction with the external potential is mediated by the charged particle through the corresponding triangular graph. Assuming all the particles in the process *to be on-shell* the anomalous threshold is given by

$$t_0 = 4m_e^2 - \frac{\left(M_{bh}^2 - M'^2_{bh} - m_e^2\right)^2}{M'^2_{bh}} , \tag{5.15}$$

where we think the instability as the decay of a black hole of mass $M$ and charge $q^\mu$ into a smaller black hole of mass $M'$ and a particle with mass $m_e$ and minimal charge $e$ that we

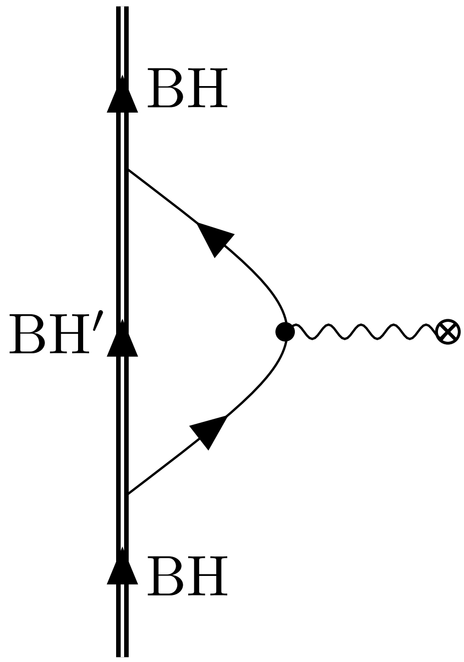



Figure 5.6: Anomalous threshold for the form factor of a RN black hole.

will call the electron (see figure 5.6).

As shown in appendix B for the typical gravitational binding energy that we expect for a black hole the anomalous threshold contribution to the corresponding amplitude will go as

$$\ln\left(\frac{M_P}{m_e}\right) , \tag{5.16}$$

where we have used as UV cutoff the Planck mass. The imaginary part of this amplitude can be interpreted as an anomalous threshold to the absorptive part of the form factor of the charged black hole in the presence of an external electromagnetic field. Now we have what we were looking for, namely a *physical amplitude* that depends on the electron mass in a way that is singular in the massless limit. In order to avoid the singular limit $m_e \to 0$ we can impose, on the basis of unitarity, that the corresponding amplitude is smaller than one. If we do that we get

$$\nu C^2 \ln\left(\frac{M_P}{m_e}\right) \leq 1 , \tag{5.17}$$

where $C$ represents the physical decay amplitude of the black hole to emit an electron. If we assume this amplitude to be proportional to the electromagnetic coupling we get the lower bound above (5.13). Here $\nu$ is the number of charged fermionic species with equal charge to the electron.[6] Taking seriously the former bound on the electron mass leads to an upper bound on the number of fermionic species with the electron charge of the order of 11 species.[7] The key point to be stressed here is that in deriving this bound we don't use the perturbative Landau pole but instead the anomalous threshold singularity we get assuming the gravitational instability of RN extremal black holes.

A different way to understand the anomalous threshold is as follows. For the case of standard black holes with entropy $N$ we should expect that the threshold for an absorptive part should be $t_0 \sim O(1/N)$ in Planck units i.e. absorption of one information bit. The existence of massless charged particles pushes down this threshold to the anomalous value $O(m_e^2)$ and therefore we could expect a lower *information bound* for the mass of the electron $m_e \sim 1/N$ in Planck units for the largest possible black hole. Thus and using a cosmological bound for the largest black hole we could conclude that the lower bound on the mass of electrically charged fermions is given, in Planck units, by $\frac{1}{\sqrt{N_H}}$ with $N_H$ determined by the Hubble radius of the Universe as $\frac{R_H^2}{L_P^2}$.

To end let us make a comment on (5.14). For equality this can be written as $e^2 = \frac{m_e^2}{M_P^2}$. Thinking in a diagram representing an energetic Planckian photon decaying into a set of $n$ on-shell pairs and estimating $n \sim \frac{M_P}{m_e}$ the former relation (5.14) simply express the *criticality condition* [87] $e^2 \sim \frac{1}{n}$ typical of classicalization.

Before ending we would like to make a very general comment on black hole physics intimately related with the former discussion. In [88] it was put forward a *constituent portrait* of black holes. The most obvious consequence of this model is the prediction of anomalous thresholds in the corresponding form factors at small angle. On the other hand these anomalous thresholds define a canonical example of in principle observable *quantum hair*.

[6]The role of electrically charged species is analogous to the one suggested originally by Landau [85] to lower the Landau pole.

[7]Adding the effect of gravitational species [86] will multiply the former bound by a global suppression factor $\frac{1}{\sqrt{N_g}}$.

## 5.7 Final comment

It looks like that nature abhors massless charged particles whenever the charge is associated with a long range force as electromagnetism. This is not a serious problem for confined particles but it is certainly a problem for charged leptons. Taken seriously, it will means that the limit with vanishing Yukawa couplings should be quantum mechanically inconsistent. In string theory we count with a geometrical interpretation of Yukawa couplings in terms of intersections [89] and in some constructions based on brane configurations in terms of world sheet instanton contributions. It looks like that a consistency criteria for string compactifications should prevent the possibility of massless charged leptons and consequently of vanishing Yukawa couplings. The problem of a consistent massless limit of leptons is on the other hand related with the problem of *naturalness* in t'Hooft's sense [82]. Naively the symmetry enhancement that will make natural the massless limit is chiral symmetry. What we have observed in this chapter can be read from this point of view. The IR collinear divergences, if canceled in the way suggested by the KLN-theorem, prevent the realization of this chiral symmetry indicating the unnatural condition of the massless limit of charged leptons. A hint in that direction was the observation of [65] about the existence for massless QED of non-vanishing helicity changing amplitudes in the absence of any supporting instanton like topology. Thus, it looks that the existence of a fundamental lower bound on the mass of charged leptons is inescapable.

# Chapter 6

# Conclusion

## 6.1 English

In this thesis we have taken a closer look to the IR structure of quantum field theories, especially of QED but so some extend also to perturbative gravity. The first chapter serves as a small review of the known findings from the 60's, where the IR divergences of QED and perturbative gravity were first treated and resolved by the two standard procedures, the Bloch-Nordsieck recipe and the coherent state approach.

In chapter 2 we treated the memory effect in QED. The measurable or observable electromagnetic memory effect is the change of velocities of the ingoing due to some radiation emitted during the scattering process. This radiation is in part given by emitted infrared photons and has non-vanishing energy $\epsilon$. We clarified the notation of observable and non-observable memory. The unobservable memory effect is given by the zero-energy modes, which decouple, of the scattering. In the classical theory it is harder to realize a process where zero energy is emitted since it would require to impose that the radiative parts of the retarded and advanced electromagnetic fields are equal. This is nothing else but, what we called, the WF condition, where we have to take into account the back reaction of the radiated field. This can be done by introducing a formal absorber which brings up the notion of an unphysical incoming radiation physical. In the quantum theory this is the same as pushing the energy $\epsilon$ of the radiated IR photons to zero. The probability of such process with no net radiation goes then to zero except if the massive charged particles scatter forwardly where the $B$-factor vanishes. Since the unobservable memory effect comes from the zero-energy modes, it is clear that the dressing of bare states does not account for an observable memory effect. We don't even know how this could be even possible since the dressed states contain an infinite number of zero-energy photons and thus do not belong to the usual Fock space any more, where the real radiation is contained. Similarly the discussion holds for gravity as well. Furthermore, we emphasize that the WF condition or equally having no net energy radiated defines the so called soft charges, which are determined by the zero-mode part of the retarded and advanced field.

The chapters 3 and 4 dealt with a very important and interesting question. The question of how much decoherence of a density matrix do we expect from infrared divergences and infrared radiation and how big the resulting entanglement entropy is. In the chapter 3 we took a closer look to the non-trival part of the density matrix of the outer product of all possible out sates resulting from generic scattering (3.2). We identified the source that is responsible for most of the decoherence in the limit where the IR regulator $\lambda$ goes to zero. The term that produces this decoherence comes from effective soft loops that where introduced when tracing over the emitted IR radiation in the out states. Since we treated the non-trival part of the density matrix we could use the optical theorem to define a not fully decoherent density matrix. The imaginary part of a scattering amplitude of an arbitrary scattering is in general non-zero, where the density matrix describes this imaginary part. So we used the Bloch-Nordsieck recipe to get an IR-finite imaginary part of a generic amplitude and proposed a definition of a not fully decoherent density matrix through the optical theorem. As it should be we reproduced the non-IR divergent rates (1.13), which are the diagonal elements of the density matrix. Since we traced over IR radiation, which is entangled with the charged hard particles through their scattering data and through energy conservation, we have some decoherence appearing. With the entanglement entropy, we estimated that this decoherence in the weak coupling regime is small, which is conform of what we see in experiments.

In chapter 4 we used the out states from acting with the $S$-matrix on a coherent initial state to define the coherent density matrix. In doing so we clarified the actual notion of coherent state, where we introduced the limits of integration in the dressing operator, which was often omitted in the literature. The new scale $r$ separates the dressing from real radiation and can be understood as the inverse of the time after which the final state is measured. Since the dressing operator is a state of infinite zero-energy photons, but nevertheless has finite energy, it is defined in the bigger von Neumann space. Each single dressed state belongs to its very own equivalence class in that von Neumann space and is fully determined by the scattering data of the charged particles in that single state, i.e. initial or final state, whereas radiation states are determined by the scattering data of the charged particles in the initial and final states. We combined the formalism of dressing and radiation by splitting it in different spaces and showed that the resulting density matrix, either from a single initial state or a superposition of initial states, is coherent. Therein we emphasized that tracing is only meaningfully defined in the Fock space where the radiation is defined. The tracing over the photons in the coherent state makes no sense since dressing is only needed for the well-definedness of asymptotic states in the von Neumann space. Again we estimated the resulting coherence with entanglement entropy and came to the similar result as in chapter 3, that the occurring coherence is small in the weak coupling limit. We have to make a final comment on the density matrices in the two chapters. Since we used two different approaches to define a coherent density matrix in the chapters 3 and 4 it is not surprising that the density matrices are different. Which of these two in the end is realized in nature is a task yet to accomplish by experiments. The theoretically treatment of the two different approaches is though very interesting on its own.

The chapter 5 took a slight detour away from the IR regime to the collinear regime of QED. We saw that not only problems occur when the fictitious photon mass $\lambda$ goes to zero but also when the electron mass goes to zero. We introduced the KLN theorem which is a similar but more powerful theorem than the Bloch-Nordsieck recipe for IR divergences because it is based on the degeneracies of the initial and final states. This is independent of the renormalization schema one uses since it is possible to renormalize QED even if the electron mass is zero and it also incorporates disconnected diagrams and allows incoming radiation in order to cancel all kind of non-UV divergences. We then looked at a very special order $e^6$ scattering process where an electron is forward scattered from a momentum transfer $q^\mu$ with $q^2 = 0$. To built up the $e^6$ order we included one incoming and one outgoing photon scattered from the external lines of the electron. The two key points were that the angles between the photons and the massless electron are small and that the momenta of the ingoing and outgoing photons were not equal. We found a non-vanishing logarithmic divergence, after summing/integrating over the degenerate state, which we called KLN anomaly. We argued that this $\ln$-divergence could only be canceled by the triangular anomaly diagrams of the $U(1)$ group of QED. We claim this inclusively because we couldn't find any other contribution to oder $e^6$ that could possibly cancel the KLN anomaly, which we putted forward in the calculations and discussions of the appendix. So that we either have QED with massless electrons with anomalies that cancel that KLN anomaly or, way more likely, a consistent anomaly-free theory of electrons with non-vanishing masses. In the collinear limit all particles, even those running in the loops, are on-shell in the triangular anomaly process, which brought us to the well-known treatment of the anomaly threshold, where one uses the Cutkosky cutting rules to fish out the singularities of the triangular diagram. When we treat an other anomalous threshold triangular process, where a black hole emits a collinear, nearly massless charged particle that scatters off on a external potential and then falls into the black hole again, the appearing anomaly threshold singularities together with the requirement of unitarity gave us the opportunity to formulate a conjecture saying that there exists a minimal mass for the lightest charged particle. It also predicts approximately 11 species of charged particles with electric charges equal to the electron charge when taking the electron mass as lightest mass.

In summary we hope to shed light on infrared and collinear side of QED and in parts on perturbative gravity by answering some interesting questions on the memory effect, decoherence of density matrices and the consistency of massless QED. Though, there are still intriguing open questions like how much information could be encoded in the IR modes of the infrared gravitational memory or how the different equivalence classes of the von Neumann space acquire a meaning in the context of black holes. Or is there a possibility to understand the purification of the black hole evaporation process in terms of non-IR radiation? We hope to come back to these questions in future works.

## 6.2 Epañol

En esta tesis hemos mirado más de cerca la estructura IR de las teorías de campo cuántico, especialmente de la QED, pero algunas se extienden también a la gravedad perturbadora. El primer capítulo sirve como una pequeña revisión de los hallazgos conocidos de los años 60, donde las divergencias IR de QED y la gravedad perturbadora fueron tratadas y resueltas primero por los dos procedimientos estándar, la receta Bloch-Nordsieck y el enfoque de estado coherente.

En el capítulo 2 tratamos el efecto memoria en QED. El efecto de memoria electromagnética medible u observable es el cambio de velocidad de la entrada debido a alguna radiación emitida durante el proceso de dispersión. Esta radiación es en parte dada por fotones infrarrojos emitidos y tiene una energía que no desaparece $\epsilon$. Aclaramos la notación de memoria observable e inobservable. El efecto de memoria inobservable viene dado por los modos de energía cero, que desacoplan, de la dispersión. En la teoría clásica es más difícil realizar un proceso en el que se emite energía cero, ya que requeriría imponer que las partes radiativas de los campos electromagnéticos retardados y avanzados sean iguales. Esto no es otra cosa que, lo que llamamos, la condición WF, en la que tenemos que tener en cuenta la reacción posterior del campo radiado. Esto se puede hacer introduciendo un absorbedor formal que hace que la noción de una radiación entrante no física sea física. En la teoría cuántica esto es lo mismo que empujar a cero la energía de los fotones IR radiados. La probabilidad de un proceso de este tipo sin radiación neta es cero, excepto que las partículas cargadas masivas se dispersan hacia adelante, donde el factor $B$ desaparece. Dado que el efecto de memoria inobservable proviene de los modos de energía cero, está claro que el apósito de los estados desnudos no tiene en cuenta un efecto de memoria observable. Ni siquiera sabemos cómo puede ser posible, ya que los estados vestidos contienen un número infinito de fotones de energía cero y, por lo tanto, ya no pertenecen al espacio Fock habitual, donde se contiene la radiación real. Del mismo modo, el debate también es válido para la gravedad. Además, enfatizamos que la condición de WF o igualmente el no tener energía neta irradiada define las llamadas cargas blandas, las cuales son determinadas por la parte de modo cero del campo retardado y avanzado.

Los capítulos 3 y 4 trataron una cuestión muy importante e interesante. La pregunta de cuánta decoherencia de una matriz de densidad esperamos de las divergencias infrarrojas y la radiación infrarroja y cuán grande es la entropía de enredo resultante. En el capítulo 3 observamos más de cerca la parte no trivalente de la matriz de densidad del producto exterior de todos los posibles estados externos resultantes de la dispersión genérica (3.2). Identificamos la fuente responsable de la mayor parte de la decoherencia en el límite donde el regulador IR $\lambda$ va a cero. El término que produce esta decoherencia proviene de bucles suaves efectivos que se introducen al trazar sobre la radiación IR emitida en los estados de salida. Dado que tratamos la parte no trivalente de la matriz de densidad, podríamos utilizar el teorema óptico para definir una matriz de densidad no totalmente decoherente. La parte imaginaria de una amplitud de dispersión de una dispersión arbitraria es en general distinta de cero, donde la matriz de densidad describe esta parte imaginaria. Así que usamos la receta Bloch-Nordsieck para obtener una parte imaginaria finita IR de una amplitud genérica y propusimos una definición de una matriz de densidad no totalmente decoherente a través del teorema óptico. Como debe ser, reproducimos las tasas divergentes no IR, que son los elementos diagonales de la matriz de densidad. Como hemos trazado la radiación IR, que se enreda con las partículas duras cargadas a través de sus datos de dispersión y a través de la conservación de energía, tenemos algo de decoherencia que aparece. Con la entropía de enredo, estimamos que esta decoherencia en el régimen de acoplamiento débil es pequeña, lo cual es conforme a lo que vemos en los experimentos.

En el capítulo 4 utilizamos los estados de salida de actuar con la matriz $S$ sobre un estado inicial coherente para definir la matriz de densidad coherente. Al hacerlo, aclaramos la noción real de estado coherente, en la que introdujimos los límites de la integración en el operario del apósito, que a menudo se omitió en la literatura. La nueva escala $r$ separa el apósito de la radiación real y puede entenderse como el inverso del tiempo después del cual se mide el estado final. Puesto que el operador del apósito es un estado de infinitos fotones de energía cero, pero sin embargo tiene energía finita, se define en el espacio más grande de von Neumann. Cada estado vestido individualmente pertenece a su propia clase de equivalencia en ese espacio von Neumann y está totalmente determinado por los datos de dispersión de las partículas cargadas en ese único estado, es decir, estado inicial o final, mientras que los estados de radiación están determinados por los datos de dispersión de las partículas cargadas en los estados inicial y final. Combinamos el formalismo del vestir y la radiación dividiéndola en diferentes espacios y demostramos que la matriz de densidad resultante, ya sea a partir de un único estado inicial o de una superposición de estados iniciales, es coherente. Allí enfatizamos que el trazado sólo se define de manera significativa en el espacio Fock donde se define la radiación. El trazado sobre los fotones en el estado coherente no tiene sentido ya que el apósito sólo es necesario para la definición de los estados asintóticos en el espacio von Neumann. Nuevamente estimamos la coherencia resultante con la entropía de enredo y llegamos al resultado similar como en el capítulo 3, que la coherencia que ocurre es pequeña en el límite de acoplamiento débil. Tenemos que hacer un comentario final sobre las matrices de densidad en los dos capítulos. Dado que utilizamos dos enfoques diferentes para definir una matriz de densidad coherente en los capítulos 3 y 4, no es sorprendente que las matrices de densidad sean diferentes. Cuál de estos dos al final se realiza en la naturaleza es una tarea que aún no se ha logrado mediante experimentos. El tratamiento teórico de los dos enfoques diferentes es, sin embargo, muy interesante por sí solo.

El capítulo 5 tomó un pequeño desvío del régimen IR al régimen colineal de QED. Vimos que no sólo ocurren problemas cuando la masa fotónica ficticia $\lambda$ va a cero, sino también cuando la masa del electrón va a cero. Introdujimos el teorema del KLN, que es un teorema similar pero más poderoso que la receta de Bloch-Nordsieck para las divergencias IR porque se basa en las degeneraciones de los estados inicial y final. Es independiente del esquema de renormalización que se utilice, ya que es posible renormalizar el QED aunque la masa del electrón sea cero y además incorpora diagramas desconectados y permite la entrada de radiación para anular todo tipo de divergencias no UV. Luego observamos un proceso de dispersión muy especial de $e^6$ en el que un electrón se dispersa a partir de una transferencia de momento $q^\mu$ con $q^2 = 0$. Para construir el orden de $e^6$ incluimos un fotón entrante y otro saliente dispersos de las líneas externas del electrón. Los dos puntos clave fueron que los ángulos entre los fotones y el electrón sin masa son pequeños y que el momento de los fotones entrantes y salientes no era igual. Encontramos una divergencia logarítmica no desvaneciente, después de sumar/integrar sobre el estado degenerado, lo que llamamos anomalía del KLN. Argumentamos que esta divergencia sólo podía ser cancelada por los diagramas triangulares de anomalías del grupo $U(1)$ de QED. Afirmamos esto inclusivamente porque no pudimos encontrar ninguna otra contribución a más de $e^6$ que pudiese cancelar la anomalía del KLN, que propusimos en los cálculos y discusiones del apéndice. De modo que tenemos QED con electrones sin masa con anomalías que cancelan esa anomalía KLN o, lo que es más probable, una teoría consistente libre de anomalías de electrones con masas que no se desvanecen. En el límite colineal, todas las partículas, incluso las que corren en los bucles, se encuentran en la cáscara en el proceso de anomalía triangular, lo que nos llevó al conocido tratamiento del umbral de anomalía, donde se utilizan las reglas de corte de Cutkosky para pescar las singularidades del diagrama triangular. Cuando tratamos otro proceso triangular de umbral anómalo, en el que un agujero negro emite una partícula cargada colineal, casi sin masa, que se dispersa en un potencial externo y luego cae de nuevo en el agujero negro, las singularidades del umbral de anomalía que aparecen junto con el requisito de la unidad nos dieron la oportunidad de formular una conjetura que dice que existe una masa mínima para la partícula cargada más ligera. También predice aproximadamente 11 especies de partículas cargadas con cargas eléctricas iguales a la carga del electrón cuando se toma la masa del electrón como la masa más ligera.

En resumen, esperamos arrojar luz sobre el lado infrarrojo y colineal del QED y, en parte, sobre la gravedad perturbadora, respondiendo a algunas preguntas interesantes sobre el efecto memoria, la decoherencia de las matrices de densidad y la consistencia del QED sin masa. Sin embargo, todavía hay preguntas abiertas intrigantes como cuánta información podría ser codificada en los modos IR de la memoria gravitacional infrarroja o cómo las diferentes clases de equivalencia del espacio de von Neumann adquieren un significado en el contexto de los agujeros negros. ¿Existe la posibilidad de entender la purificación del proceso de evaporación del agujero negro en términos de radiación no infrarroja? Esperamos volver sobre estas cuestiones en futuros trabajos.

# Appendix A

# Split of Identity in Photon Sector

Our goal is to derive Eq. (4.18). It originates from multiplying $\hat{S}\,|\alpha\rangle\rangle^r_\lambda$ with an identity that is decomposed as a tensor product of three factors. The first one, which we shall denote by $D$ and which will correspond to dressing, consists of all possible photons states composed of quanta with an energy below $r$. Analogously, the second one, which we shall call $\gamma$ and which will represent soft radiation, contains all possible photon states in which each photon has an energy above $r$ but below $\epsilon$. Finally, the third factor $\beta$ is composed of all remaining states, i.e. photons with energy above $\epsilon$ and all other excitations, in particular charged particles. So we obtain:

$$\hat{S}\,|\alpha\rangle\rangle^r_\lambda = \sum_{\substack{D \\ (\lambda<E_D<r)}} \sum_{\substack{\gamma \\ (r<E_\gamma<\epsilon)}} \sum_{\substack{\beta \\ (\epsilon<E_\beta)}} \Big(|\beta\rangle \otimes |\gamma\rangle \otimes |D\rangle\Big)\Big(\langle D| \otimes \langle\gamma| \otimes \langle\beta|\Big)\hat{S}\,|\alpha\rangle\rangle^r_\lambda\,. \quad \text{(A.1)}$$

We will first turn to the sum over $D$. From Chung's computation [9] we know that $\Big(\langle D(\beta)| \otimes \langle\gamma| \otimes \langle\beta|\Big)\hat{S}\,|\alpha\rangle\rangle^r_\lambda \neq 0$, i.e. when we take the appropriate dressing $|D(\beta)\rangle$ of the final state $|\beta\rangle$, we obtain an IR-finite amplitude. (From the point of view of this computation, $|\gamma\rangle$ is a hard state.) This implies that any state $|D\rangle$ that belongs to a different equivalence class than $|D(\beta)\rangle$ has zero overlap with $\hat{S}\,|\alpha\rangle\rangle^r_\lambda$. In other words, the state in the mode $\boldsymbol{k} = 0$, in which the number of photons is infinite, is fixed. In the identity, one would nevertheless have to perform independent sums over photons in the modes $0 < |\boldsymbol{k}| < r$.[1] However, if we

[1] In other words, as is discussed in section (1.3), one can replace $\mathcal{F}^{(l)}_\alpha(\boldsymbol{k})$ by $\mathcal{F}^{(l)}_\alpha(\boldsymbol{k})\varphi(\boldsymbol{k})$, where $\varphi(\boldsymbol{k})$ is an arbitrary function that fulfills $\varphi(\boldsymbol{k}) = 1$ in a neighborhood of $\boldsymbol{k} = 0$. Then neglecting the sum over modes $0 < |\boldsymbol{k}| < r$ corresponds to setting $\varphi(\boldsymbol{k}) = 1$ everywhere.

take $r$ small enough, those mode do not change the result of $\hat{S}\,|\alpha\rangle\rangle^r_\lambda$ and we can proceed as for Eq. (4.4) and fix them by the state $|D(\beta)\rangle$. For $r \ll \epsilon$, we therefore obtain

$$\sum_{\substack{D \\ (\lambda<E_D<r)}} |D\rangle\,\langle D| \approx |D(\beta)\rangle\,\langle D(\beta)|\ . \tag{A.2}$$

This means that the dressing is not independent but fixed by the hard state $|\beta\rangle$. In [9], the same approximation is used, i.e. the modes $0 < |\boldsymbol{k}| < r$ are not treated as independent.

In contrast, we will not neglect any states in the sum over radiation. Writing it out explicitly, we get

$$\sum_{\substack{\gamma \\ (r<E_\gamma<\epsilon)}} |\gamma\rangle\,\langle\gamma| = \sum_n \frac{1}{n!}\left(\prod_{i=1}^n \int_r^\epsilon \mathrm{d}^3\boldsymbol{k}_i \sum_{l_i}\right)\Big(\hat{a}^\dagger_{l_1,\boldsymbol{k}_1}\dots\hat{a}^\dagger_{l_n,\boldsymbol{k}_n}\,|0\rangle\Big)\Big(\langle 0|\,\hat{a}_{l_1,\boldsymbol{k}_1}\dots\hat{a}_{l_n,\boldsymbol{k}_n}\Big)\,, \tag{A.3}$$

where $1/n!$ comes from the normalization of the photon states. We will not resolve the third sum over hard modes $\beta$. In total, we obtain

$$\hat{S}\,|\alpha\rangle\rangle^r_\lambda = \sum_\beta\sum_n \frac{1}{n!}\left(\prod_{i=1}^n \int_r^\epsilon \mathrm{d}^3\boldsymbol{k}_i \sum_{l_i}\right)\Big(|\beta\rangle\rangle^r_\lambda \otimes |\gamma_n\rangle\Big)\Big(\langle\gamma_n| \otimes {}^r_\lambda\langle\langle\beta|\Big)\hat{S}\,|\alpha\rangle\rangle^r_\lambda\,, \tag{A.4}$$

where we introduced the notation $|\gamma_n\rangle = \hat{a}^\dagger_{l_1,\boldsymbol{k}_1}\dots\hat{a}^\dagger_{l_n,\boldsymbol{k}_n}\,|0\rangle$.

# Appendix B

# Anomalous threshold kinematics

Let us consider the leading Landau singularity for the diagram in figure 5.1. This corresponds to have all the internal lines of the diagram on-shell satisfying energy momentum conservation in the three vertices. Following [72] the diagram is presented in Breit frame. The transfer momentum is given by $-4\boldsymbol{p}^2$, the normal threshold is given by $4M_2^2$ where $M_2$ is the mass of the particle in the triangle interacting with the external source. The anomalous threshold associated with the leading Landau singularity is given by

$$t_0 = 4M_2^2 - \frac{M_0^2 - M_1^2 - M_2^2}{M_1^2} \,, \tag{B.1}$$

where $t_0 = -4\boldsymbol{p}_0^2$. This is the minimum momentum where all the particles in figure 5.1 can be on-shell. Here also the scattering angles have to be below a small threshold which in our case refer to the resolution scale angle $\delta$. Note that this anomalous threshold is independent on the energy of the process. The reason for calling it anomalous is that it is smaller than the normal threshold given by standard unitarity.

As discussed in the text the discontinuity associated with this singularity can be computed using the Cutkosky rules for the diagram. The corresponding amplitude contains a term proportional to $\ln\left(1 - \frac{t}{t_0}\right)$. For the diagram in figure 5.2 where we use the degeneration between the electron and a pair electron and collinear photon (both on-shell) the anomalous threshold gives the $\ln(m_e)$ terms in the amplitude.

In order to get a clearer picture of the underlying kinematics we can compute the relative velocity $v$ between the two particles $1$ and $2$. This is given by the so called Källen-function

$$v = A(M_0^2 - M_1^2 - M_2^2) \,, \tag{B.2}$$

with $A^2 = M_0^4 + M_1^4 + M_2^4 - 2M_0^2M_1^2 - 2M_0^2M_2^2 - 2M_1^2M_2^2$. In the degenerate case with $M_0 = M_2 = m_e$ and $M_1 = m_\gamma$ the mass of a photon we get the limit $v = i\infty$ corresponding to particles $1$ and $2$ moving collinearly i.e. they remain coincident.

Introducing a *binding energy* as $M_0 + B = M_1 + M_2$ we observe that for $M_0 < M_1 + M_2$ the velocity $u$ defined above is imaginary reaching collinearity in the limit $B \to 0$. Moreover in the limit where $M_1$ is much larger than $M_2$ the anomalous threshold can be approximated by:

$$t_0 \approx 4Bm_e \left(2 - \frac{B}{m_e}\right) \,. \tag{B.3}$$

In the gravitational case $t_0$ goes from zero in the limit $B \to 0$ to the normal threshold $4m_e^2$ in the limit of maximal gravitational binding energy.

# Appendix C

# Notation and calculation for the amplitudes of the KLN-anomaly

We set the kinematics of the forward scattered right- or left-handed electron in such way that the electrons runs with momentum $|\boldsymbol{p}|$ along the z-axes, i.e. for the 4-momentum of the electron we have

$$p^{\mu} = \begin{pmatrix} E \\ 0 \\ 0 \\ |\boldsymbol{p}| \end{pmatrix} , \tag{C.1}$$

where $E$ is the energy of the electron. The Dirac spinor for the right-/left-handed electron in chiral representation is given by

$$u_p^R := \begin{pmatrix} 0 \\ u_R \end{pmatrix} \text{ and } u_p^L := \begin{pmatrix} u_L \\ 0 \end{pmatrix} , \tag{C.2}$$

In the limit where the mass $m_e$ of the electron goes to $0$

$$u_L = \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ and } u_R = \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{C.3}$$

and

$$p^\mu \approx \begin{pmatrix} E\left(1+\frac{m_e^2}{2E^2}\right) \\ 0 \\ 0 \\ E \end{pmatrix} , \tag{C.4}$$

holds. We work in Weyl (chiral) basis where the gamma matrices are given by

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} , \tag{C.5}$$

with $\sigma^\mu = (1, \sigma^i)$ and $\bar{\sigma}^\mu = (1, -\sigma^i)$, where $\sigma^i$ are the standard Pauli matrices.

We start with the first amplitude of the diagrams in figure 5.3. The notation will be $\mathrm{i}M^R$/$\mathrm{i}M^L$ is the amplitude where the electron is right-/left-handed before and after the scattering. We keep the electrons helicity before and after the scattering equal in the process since we are not interested in helicity flipping processes. We are also only interested in the part that will be collinearly divergent, so we omit the mass term in the electron propagator, i.e. an electron propagator with momentum $p^\mu$ is $\frac{\not{p}}{p^2}$. The first amplitude in figure 5.3 for a right-handed electron scattered is then given by

$$\mathrm{i}M_1^R = \frac{-\mathrm{i}e^3}{(2pk)(2pk')} \bar{u}_p^R \, \not{\varepsilon}'^* (\not{p} + \not{k}') \not{\varepsilon}_q (\not{p} + \not{k}) \not{\varepsilon} \, u_p^R \, . \tag{C.6}$$

Writing this and the other amplitudes in terms of 2x2 matrices, using (C.2) and (C.5), figure 5.3 gives the amplitudes

$$\mathrm{i}M_1^R = \frac{-\mathrm{i}e^3}{(2pk)(2pk')} \left( u_R^\dagger \, \varepsilon'^* \cdot \sigma \, (p+k') \cdot \bar{\sigma} \, \varepsilon_q \cdot \sigma \, (p+k) \cdot \bar{\sigma} \, \varepsilon \cdot \sigma \, u_R \right) \, , \tag{C.7}$$

$$\mathrm{i}M_1^L = \frac{-\mathrm{i}e^3}{(2pk)(2pk')} \left( u_L^\dagger \, \varepsilon'^* \cdot \bar{\sigma} \, (p+k') \cdot \sigma \, \varepsilon_q \cdot \bar{\sigma} \, (p+k) \cdot \sigma \, \varepsilon \cdot \bar{\sigma} \, u_L \right) \, , \tag{C.8}$$

$$\mathrm{i}M_2^R = \frac{-\mathrm{i}e^3}{(2pk)(2pk')} \left( u_R^\dagger \, \varepsilon \cdot \sigma \, (p-k) \cdot \bar{\sigma} \, \varepsilon_q \cdot \sigma \, (p-k') \cdot \bar{\sigma} \, \varepsilon'^* \cdot \sigma \, u_R \right) \, , \tag{C.9}$$

$$\mathrm{i}M_2^L = \frac{-\mathrm{i}e^3}{(2pk)(2pk')} \left( u_L^\dagger \, \varepsilon \cdot \bar{\sigma} \, (p-k) \cdot \sigma \, \varepsilon_q \cdot \bar{\sigma} \, (p-k') \cdot \sigma \, \varepsilon'^* \cdot \bar{\sigma} \, u_L \right) \, , \tag{C.10}$$

and

$$\mathrm{i}M_3^R = \frac{\mathrm{i}e^3}{(2pk)(2pq)} \left( u_R^\dagger \, \varepsilon_q \cdot \sigma \, (p-q) \cdot \bar{\sigma} \, \varepsilon'^* \cdot \sigma \, (p+k) \cdot \bar{\sigma} \, \varepsilon \cdot \sigma \, u_R \right) \, , \tag{C.11}$$

$$\mathrm{i}M_3^L = \frac{\mathrm{i}e^3}{(2pk)(2pq)} \left( u_L^\dagger \, \varepsilon_q \cdot \bar{\sigma} \, (p-q) \cdot \sigma \, \varepsilon'^* \cdot \bar{\sigma} \, (p+k) \cdot \sigma \, \varepsilon \cdot \bar{\sigma} \, u_L \right) \, , \tag{C.12}$$

$$\mathrm{i}M_4^R = \frac{\mathrm{i}e^3}{(2pk)(2pq)} \left( u_R^\dagger \, \varepsilon \cdot \sigma \, (p-k) \cdot \bar{\sigma} \, \varepsilon'^* \cdot \sigma \, (p+q) \cdot \bar{\sigma} \, \varepsilon_q \cdot \sigma \, u_R \right) \, , \tag{C.13}$$

$$\mathrm{i}M_4^L = \frac{\mathrm{i}e^3}{(2pk)(2pq)} \left( u_L^\dagger \, \varepsilon \cdot \bar{\sigma} \, (p-k) \cdot \sigma \, \varepsilon'^* \cdot \bar{\sigma} \, (p+q) \cdot \sigma \, \varepsilon_q \cdot \bar{\sigma} \, u_L \right) \, , \tag{C.14}$$

where we omitted the terms proportional to the electron mass because they will give no collinear divergent term in the limit $m \to 0$ and $a \cdot b$ is the normal scalar product in 4d Minkowski space. The notation is $\varepsilon^\mu = \varepsilon^\mu(\lambda, \theta, \phi)$, $\varepsilon'^\mu = \varepsilon^\mu(\lambda', \theta', \phi')$, ${\varepsilon_q}^\mu = \varepsilon^\mu(\lambda_q, \theta_q, \phi_q)$ with

$$\varepsilon^\mu(\lambda, \theta, \phi) = \frac{1}{\sqrt{\cos^2(\theta) + 1}} \begin{pmatrix} 0 \\ \exp(-\mathrm{i}\lambda\phi)\cos(\theta) \\ \mathrm{i}\lambda\exp(-\mathrm{i}\lambda\phi)\cos(\theta) \\ -\sin(\theta) \end{pmatrix} \, , \tag{C.15}$$

and $k^\mu = k^\mu(\omega, \theta, \phi)$, $k'^\mu = k^\mu(\omega', \theta', \phi')$, $q^\mu = k^\mu(\omega_q, \theta_q, \phi_q)$ with

$$k^\mu(\omega, \theta, \phi) = \omega \begin{pmatrix} 1 \\ \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix} \, , \tag{C.16}$$

where $\omega$ is the energy, $\lambda$ the polarization and $\theta$ and $\phi$ the scattering angles of the corresponding photon. For example, a photon with polarization vector $\varepsilon^\mu$ and $\lambda = +1/-1$ is an incoming right-/left-handed photon.

In the collinear limit the angles $\theta$, $\theta'$ and $\theta_q$ appearing in the calculations are small, i.e. $\cos\theta \approx 1 - \frac{\theta^2}{2}$, $\cos\theta' \approx 1 - \frac{\theta'^2}{2}$ and $\cos\theta_q \approx 1 - \frac{{\theta_q}^2}{2}$. So that together with (C.4) we can approximate

$$2pk \approx E\omega \left( \frac{m_e^2}{E^2} + \theta^2 \right) \,, \tag{C.17}$$

$$2pk' \approx E\omega \left( \frac{m_e^2}{E^2} + \theta'^2 \right) \,, \tag{C.18}$$

$$2pq \approx E\omega \left( \frac{m_e^2}{E^2} + {\theta_q}^2 \right) \,. \tag{C.19}$$

A simple Taylor expansion to first order in $\theta$ and matrix multiplication shows that in general for the right-handed electron

$$(p \pm k(\omega,\theta,\phi)) \cdot \bar{\sigma}\, \varepsilon(\lambda,\theta,\phi) \cdot \sigma\, u_R \approx \sqrt{2}\theta \left( E \pm \frac{\omega}{2}(1+\lambda) \right) u_R \,, \tag{C.20}$$

$$u_R^\dagger\, \varepsilon(\lambda,\theta,\phi) \cdot \sigma\, (p \pm k(\omega,\theta,\phi)) \cdot \bar{\sigma} \approx \sqrt{2}\theta \left( E \pm \frac{\omega}{2}(1-\lambda) \right) u_R^\dagger \,, \tag{C.21}$$

holds and for the left-handed electron

$$(p \pm k(\omega,\theta,\phi)) \cdot \sigma\, \varepsilon(\lambda,\theta,\phi) \cdot \bar{\sigma}\, u_L \approx \sqrt{2}\theta \left( E \pm \frac{\omega}{2}(1-\lambda) \right) u_L \,, \tag{C.22}$$

$$u_L^\dagger\, \varepsilon(\lambda,\theta,\phi) \cdot \bar{\sigma}\, (p \pm k(\omega,\theta,\phi)) \cdot \sigma \approx \sqrt{2}\theta \left( E \pm \frac{\omega}{2}(1+\lambda) \right) u_L^\dagger \,, \tag{C.23}$$

holds. These identities (also see [65]) will be used in the amplitudes (C.7) to (C.14). Interesting is that there is no $\phi$ or $\phi'$ dependence in the expressions (C.20) to (C.23). Furthermore, for a small arbitrary angle $\theta$ we have

$$u_R^\dagger\, \varepsilon(\lambda,\theta,\phi) \cdot \sigma\, u_R = u_L^\dagger\, \varepsilon(\lambda,\theta,\phi) \cdot \bar{\sigma}\, u_L \approx \sqrt{2}E\theta \,. \tag{C.24}$$

The amplitudes from (C.7) to (C.14) simplify then to

$$\mathrm{i}M_1^R = -\mathrm{i}e^3 \frac{\theta\theta'\theta_q \big(2E + \omega(1+\lambda)\big) \big(2E + \omega'(1+\lambda')\big)}{\sqrt{2}E\omega\omega' \left(\theta^2 + \frac{m_e^2}{E^2}\right) \left(\theta'^2 + \frac{m_e^2}{E^2}\right)} \,, \tag{C.25}$$

$$\mathrm{i}M_1^L = -\mathrm{i}e^3 \frac{\theta\theta'\theta_q \big(2E + \omega(1-\lambda)\big) \big(2E + \omega'(1-\lambda')\big)}{\sqrt{2}E\omega\omega' \left(\theta^2 + \frac{m_e^2}{E^2}\right) \left(\theta'^2 + \frac{m_e^2}{E^2}\right)} \,, \tag{C.26}$$

$$\mathrm{i}M_2^R = -\mathrm{i}e^3 \frac{\theta\theta'\theta_q \left(2E - \omega(1-\lambda)\right)\left(2E - \omega'(1-\lambda')\right)}{\sqrt{2}E\omega\omega'\left(\theta^2 + \frac{m^2}{E^2}\right)\left(\theta'^2 + \frac{m^2}{E^2}\right)}\,, \tag{C.27}$$

$$\mathrm{i}M_2^L = -\mathrm{i}e^3 \frac{\theta\theta'\theta_q \left(2E - \omega(1+\lambda)\right)\left(2E - \omega'(1+\lambda')\right)}{\sqrt{2}E\omega\omega'\left(\theta^2 + \frac{m^2}{E^2}\right)\left(\theta'^2 + \frac{m^2}{E^2}\right)}\,, \tag{C.28}$$

and

$$\mathrm{i}M_3^R = \mathrm{i}e^3 \frac{\theta\theta'\theta_q \left(2E + \omega(1+\lambda)\right)\left(2E - \omega_q(1-\lambda_q)\right)}{\sqrt{2}E\omega\omega_q\left(\theta^2 + \frac{m^2}{E^2}\right)\left({\theta_q}^2 + \frac{m^2}{E^2}\right)}\,, \tag{C.29}$$

$$\mathrm{i}M_3^L = \mathrm{i}e^3 \frac{\theta\theta'\theta_q \left(2E + \omega(1-\lambda)\right)\left(2E - \omega_q(1+\lambda_q)\right)}{\sqrt{2}E\omega\omega_q\left(\theta^2 + \frac{m^2}{E^2}\right)\left({\theta_q}^2 + \frac{m^2}{E^2}\right)}\,, \tag{C.30}$$

$$\mathrm{i}M_4^R = \mathrm{i}e^3 \frac{\theta\theta'\theta_q \left(2E - \omega(1-\lambda)\right)\left(2E + \omega_q(1+\lambda_q)\right)}{\sqrt{2}E\omega\omega_q\left(\theta^2 + \frac{m^2}{E^2}\right)\left({\theta_q}^2 + \frac{m^2}{E^2}\right)}\,, \tag{C.31}$$

$$\mathrm{i}M_4^L = \mathrm{i}e^3 \frac{\theta\theta'\theta_q \left(2E - \omega(1+\lambda)\right)\left(2E + \omega_q(1-\lambda_q)\right)}{\sqrt{2}E\omega\omega_q\left(\theta^2 + \frac{m^2}{E^2}\right)\left({\theta_q}^2 + \frac{m^2}{E^2}\right)}\,. \tag{C.32}$$

Notice that the amplitudes of the left-handed electron just differ by exchanging all polarizations of the photons to minus the polarizations, i.e.

$$\mathrm{i}M_i^L(\lambda, \lambda', \lambda_q) = \mathrm{i}M_i^R(-\lambda, -\lambda', -\lambda_q)\,. \tag{C.33}$$

Thus, apart from now we will write down only the amplitudes with the right-handed electron and get the amplitudes of the left-handed electron by this simple relation (C.33).

We are interested in a very special corner of the phase space where $\theta'^2$ and ${\theta_q}^2$ are very small but still bigger than $\frac{m^2}{E^2}$, i.e. $\theta'^2 \gg \frac{m^2}{E^2}$ and ${\theta_q}^2 \gg \frac{m^2}{E^2}$. On the other side we allow $\theta^2$ to be of the order of $\frac{m^2}{E^2}$. Thus, $\theta' \gg \theta$ and $\theta_q \gg \theta$ holds as well. Furthermore, the phase space gets more restricted by the fact that the electron is forward scattered, i.e. $p^\mu_{\text{in}} = p^\mu_{\text{out}}$. The constraint from energy and momentum conservation is then $k'^\mu = q^\mu + k^\mu$. This constraint in the collinear limit gives

$$\omega' = \omega_q + \omega\,,\; \theta' = \theta_q \frac{\omega_q}{\omega + \omega_q} \text{ and } \phi' = \phi_q\,. \tag{C.34}$$

The constraint $\phi' = \phi_q$ isn't important since these angles don't appear in the amplitudes (C.25) to (C.32). Inserting the constraints and using the special corner of phase space one gets for the amplitudes

$$\mathrm{i}M_1^R = -\mathrm{i}e^3 \frac{\theta\left(2E+\omega(1+\lambda)\right)\left(2E+(\omega+\omega_q)(1+\lambda')\right)}{\sqrt{2}E\omega\omega_q\left(\theta^2+\frac{m^2}{E^2}\right)}\,, \tag{C.35}$$

$$\mathrm{i}M_2^R = -\mathrm{i}e^3 \frac{\theta\left(2E-\omega(1-\lambda)\right)\left(2E-(\omega+\omega_q)(1-\lambda')\right)}{\sqrt{2}E\omega\omega_q\left(\theta^2+\frac{m^2}{E^2}\right)}\,, \tag{C.36}$$

$$\mathrm{i}M_3^R = \mathrm{i}e^3 \frac{\theta\left(2E+\omega(1+\lambda)\right)\left(2E-\omega_q(1-\lambda_q)\right)}{\sqrt{2}E\omega(\omega+\omega_q)\left(\theta^2+\frac{m^2}{E^2}\right)}\,, \tag{C.37}$$

$$\mathrm{i}M_4^R = \mathrm{i}e^3 \frac{\theta\left(2E-\omega(1-\lambda)\right)\left(2E+\omega_q(1+\lambda_q)\right)}{\sqrt{2}E\omega(\omega+\omega_q)\left(\theta^2+\frac{m^2}{E^2}\right)}\,, \tag{C.38}$$

where we only kept the terms that will lead to collinear divergent terms after the phase space integration $\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}$ of the incoming collinear photon. Then summing up the amplitudes gives

$$\begin{aligned}\mathrm{i}M^R = \sum_{i=1}^{4} \mathrm{i}M_i^R = &-\mathrm{i}e^3 \frac{\sqrt{2}\theta\left[\omega(\omega+\omega_q)+(2E+\omega\lambda)(2E+(\omega+\omega_q)\lambda')\right]}{E\omega\omega_q\left(\theta^2+\frac{m^2}{E^2}\right)} \\ &+ \mathrm{i}e^3 \frac{\sqrt{2}\theta\left[-\omega\omega_q+(2E+\omega\lambda)(2E+\omega_q\lambda_q)\right]}{E\omega(\omega+\omega_q)\left(\theta^2+\frac{m^2}{E^2}\right)}\,.\end{aligned} \tag{C.39}$$

Clearly the collinear divergence comes from when one integrates the amplitude square over the $\theta$ since this is proportional to $\int_0^\delta \mathrm{d}\theta\, \theta \frac{\theta^2}{\left(\theta^2+\frac{m^2}{E^2}\right)^2} \propto \ln\left(\frac{E\delta}{m}\right)$. We are interested in the full amplitude $\mathrm{i}M$, where we want to sum over the electron polarizations. In general holds for a generic amplitude $\bar{u}'^s \mathcal{M} u^r$ with an outgoing electron with spinor $u'$ and spin $s$ and an ingoing electron with spinor $u$ and spin $r$

$$\begin{aligned}\frac{1}{4}\sum_{s,r=\pm\frac{1}{2}} \left|\bar{u}'^s \mathcal{M} u^r\right|^2 &= \frac{1}{4}\sum_{s,r=\pm\frac{1}{2}} \bar{u}'^s \mathcal{M} u^r \bar{u}^r \mathcal{M}^\dagger u'^s \\ = \frac{1}{4}\Big(\bar{u}'^{\frac{1}{2}} \mathcal{M} u^{\frac{1}{2}} \bar{u}^{\frac{1}{2}} \mathcal{M}^\dagger u'^{\frac{1}{2}} &+ \bar{u}'^{-\frac{1}{2}} \mathcal{M} u^{-\frac{1}{2}} \bar{u}^{-\frac{1}{2}} \mathcal{M}^\dagger u'^{-\frac{1}{2}}\end{aligned}$$

$$+\bar{u}'^{\frac{1}{2}}\mathcal{M}u^{-\frac{1}{2}}\bar{u}^{-\frac{1}{2}}\mathcal{M}^\dagger u'^{\frac{1}{2}} + \bar{u}'^{-\frac{1}{2}}\mathcal{M}u^{\frac{1}{2}}\bar{u}^{\frac{1}{2}}\mathcal{M}^\dagger u'^{-\frac{1}{2}}\Big) \ . \tag{C.40}$$

The two terms in the last line describe the spin-flipping process of the amplitude or the helicity-flipping process in the collinear limit. Helicity-flipping processes don't possess collinear divergences, see e.g. [65]. Thus interesting for us are the two terms in the second line of (C.40). Then, the unpolarized amplitude that will produce collinear divergences is given by

$$\frac{1}{4}\sum_{\text{spins}}|\mathrm{i}M|^2 = \frac{1}{4}\left(\left|\mathrm{i}M^R\right|^2 + \left|\mathrm{i}M^L\right|^2\right) \ , \tag{C.41}$$

where $\mathrm{i}M^L = \sum_{i=1}^4 \mathrm{i}M_i^L$.

Now we can apply the KLN-theorem and integrate over the phase space of the incoming photon $\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}$, which is in the collinear limit given by $\int\frac{\mathrm{d}\omega\mathrm{d}\theta\mathrm{d}\phi\,\omega^2\sin\theta}{(2\pi)^3 2\omega} = \int\frac{\mathrm{d}\omega\,\omega}{(2\pi)^2}\int_0^\delta \mathrm{d}\theta\,\theta$, where we integrated $\int_0^{2\pi}\mathrm{d}\phi = 2\pi$ since the amplitudes do not depend on $\phi$. Then the collinear part of the unpolarized amplitude is

$$\begin{aligned}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}\frac{1}{4}\sum_{\text{spins}}|\mathrm{i}M|^2 &= \frac{1}{(2\pi)^3}\int\mathrm{d}\omega\,\omega\int_0^\delta\mathrm{d}\theta\,\theta\int_0^{2\pi}\mathrm{d}\phi\,\frac{1}{4}\sum_{\text{spins}}|\mathrm{i}M|^2\\
&= \int\frac{\mathrm{d}\omega\,\omega}{(2\pi)^2}\frac{e^6}{4E^2\omega^2}\ln\left(\frac{E\delta}{m}\right)\left[\frac{-\omega\omega_q + (2E+\omega\lambda)(2E+\omega_q\lambda_q)}{(\omega+\omega_q)}\right.\\
&\qquad\left. - \frac{\omega(\omega+\omega_q) + (2E+\omega\lambda)(2E+(\omega+\omega_q)\lambda')}{\omega_q}\right]^2\\
&\qquad + \left(\lambda\to-\lambda,\,\lambda'\to-\lambda',\,\lambda_q\to-\lambda_q\right) \ ,\end{aligned} \tag{C.42}$$

which is the result we present in (5.8).

# Appendix D

# Two missing diagrams

The reader may have noticed that in figure 5.3 we omit two diagrams which are of the same topology, which are displayed in figure D.1. The reason is that these extra two diagrams have $1/(pk')(pq)$ propagators which won't lead to collinear divergence in this special corner of the phase space after applying the KLN theorem.

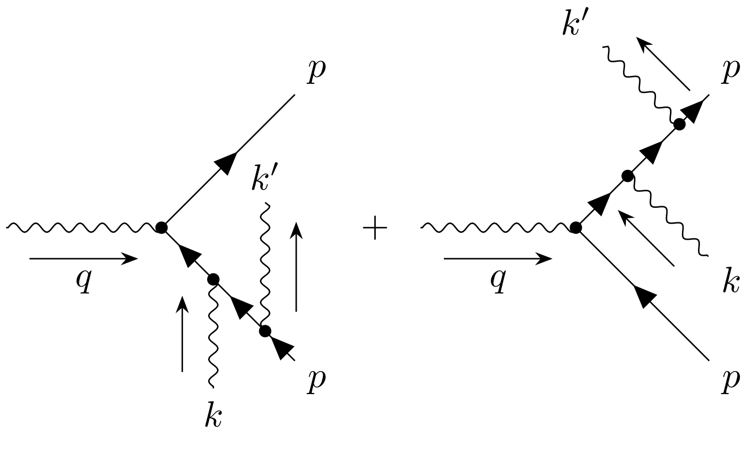



Figure D.1: The two diagrams that have the same topology as in figure 5.3.

The amplitude in figure D.1 is given by

$$\frac{-\mathrm{i}e^3}{(2pk')(2pq)}\bar{u}_p^R \left[\not{\epsilon}_q(\not{p}-\not{q})\not{\epsilon}(\not{p}-\not{k}')\not{\epsilon}'^* + \not{\epsilon}'^*(\not{p}+\not{k}')\not{\epsilon}(\not{p}+\not{q})\not{\epsilon}_q\right] u_p^R \,. \tag{D.1}$$

As in appendix C the term in the nominator will be proportional to $\theta\theta'\theta_q$ and in the collinear limit the denominator will be $E^2\omega'\omega_q(\theta'^2 + \frac{m^2}{E^2})({\theta_q}^2 + \frac{m^2}{E^2})$. Since we work in the very special corner of phase space where only $\theta^2$ is of the order $\frac{m^2}{E^2}$, i.e. $\theta'^2 \gg \frac{m^2}{E^2}$, ${\theta_q}^2 \gg \frac{m^2}{E^2}$ and $\theta^2 \sim \mathcal{O}\left(\frac{m^2}{E^2}\right)$, the denominator is $E^2\omega'\omega_q\theta'^2{\theta_q}^2$. This means that once the amplitude (D.1) is integrated over the small angle $\theta$ it will not give a collinear divergent contribution to the entire process we consider in chapter 5.

# Appendix E

# One-loop amplitude interfered with tree-level amplitude

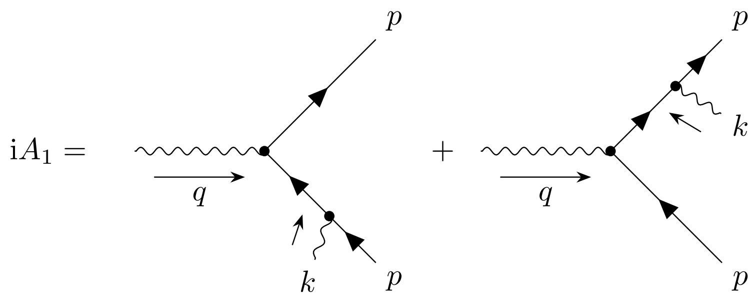



Figure E.1: Tree-level diagrams with a collinear, incoming photon at order $e^2$.

An other term that contributes to the order $e^6$ in perturbation theory to the process considered in chapter 5 is the interference of the amplitudes in figure E.1 and E.2. The process in figure E.1 describes an electron that scatters with two incoming photons, one with momentum $q^\mu$ which is the transfer momentum and of course $q^2 = 0$ holds as before and one with momentum $k^\mu$ which is a collinear photon. In order to possibly contribute to the cancelation process of the KLN anomaly in section 5.4 the electron has to be forward scattered, thus $p^\mu_{\text{in}} = p^\mu_{\text{out}}$. Then from the conservation of energy and momentum we get the constraint $q^\mu = -k^\mu$, which means in the notation of appendix C: $\omega_q = -\omega$, $\theta_q = \theta$ and $\phi_q = \phi$. The same constraint holds also for the amplitudes in the diagrams of figure E.2. The amplitudes from figure E.2 are one-loop diagrams with a collinear incoming photon. We will change from now on the notation a little bit and name the amplitudes now $\mathrm{i}A$ instead of $\mathrm{i}M$ in order to keep it easier to distinguish but nevertheless keep the rest of the notations in appendix C the same. Of course when ever a photon runs in a loop it is no longer on-shell, i.e. $k_{\text{loop}}^2 \neq 0$. Other than in the appendix above we will write down the following amplitudes non-approximatively, i.e. not in collinear limit, and just later when apply the KLN theorem we will Taylor expand the amplitudes in the collinear limit.

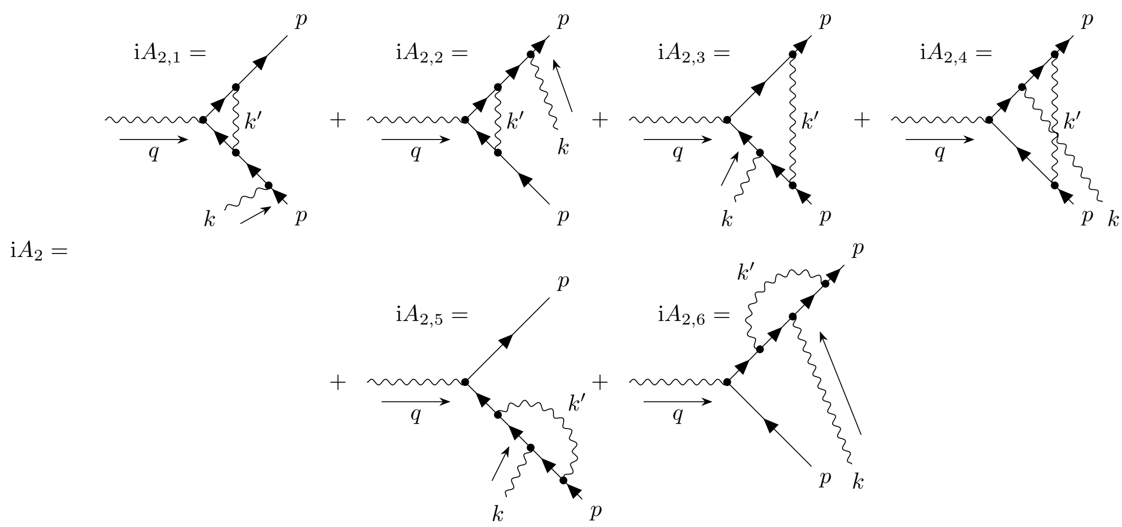



Figure E.2: one-loop diagrams with a collinear, incoming photon at order $e^4$.

## E.1 The tree-level amplitude at order $e^2$

At this point we want to anticipate that the relation (C.33) holds here and in the following as well, which can't be seen directly but was found during the calculations for this appendix with the program Mathematica, i.e.

$$\mathrm{i}A^L(\lambda, \lambda_q) = \mathrm{i}A^R(-\lambda, -\lambda_q) \; . \tag{E.1}$$

So we begin with the amplitude $\mathrm{i}A_1^R$ which is given by the diagrams in figure E.1 and gives

$$\begin{aligned}\mathrm{i}A_1^R &= -\mathrm{i}e^2\bar{u}_p^R\left[\frac{\not{\varepsilon}_q(\not{p}+\not{k})\not{\varepsilon}}{2pk} - \frac{\not{\varepsilon}(\not{p}-\not{k})\not{\varepsilon}_q}{2pk}\right]u_p^R \\ &= -\mathrm{i}e^2u_R^\dagger\left[\frac{\varepsilon_q\cdot\sigma\,(p+k)\cdot\bar{\sigma}\,\varepsilon\cdot\sigma}{2pk} - \frac{\varepsilon\cdot\sigma\,(p-k)\cdot\bar{\sigma}\,\varepsilon_q\cdot\sigma}{2pk}\right]u_R\,. \end{aligned} \tag{E.2}$$

for the amplitude where a right-handed electron is forward scattered. Then the matrix multiplication in equation (E.2) can be done by hand or using Mathematica and gives

$$\mathrm{i}A_1^R = -\mathrm{i}e^2\frac{4E(1-\lambda\lambda_q\cos^2\theta)\sin^2\left(\frac{\theta}{2}\right)}{(1+\cos^2\theta)(E-|\boldsymbol{p}|\cos\theta)}\,. \tag{E.3}$$

The amplitude for the left-handed electron is the same as the for the right-handed one, i.e. $\mathrm{i}A_1^L = \mathrm{i}A_1^R$, since we have a multiplication of two polarizations $\lambda\lambda_q$. This is the first amplitude of the interference term that could cancel the KLN anomaly.

## E.2 One-loop amplitudes

The one-loop amplitudes of the diagrams in figure E.1 are given by

$$iA_{2,1}^{R/L} = \frac{-e^4}{(2\pi)^4}\int\frac{\mathrm{d}^4k'}{k'^2}\frac{\bar{u}_p^{R/L}\,\gamma^\mu(\not{p}-\not{k}')\not{\varepsilon}_q(\not{p}-\not{k}'+\not{k})\gamma_\mu(\not{p}+\not{k})\not{\varepsilon}\,u_p^{R/L}}{[(p-k'+k)^2-m^2][(p-k')^2-m^2][(p+k)^2-m^2]} \tag{E.4}$$

$$iA_{2,2}^{R/L} = \frac{-e^4}{(2\pi)^4}\int\frac{\mathrm{d}^4k'}{k'^2}\frac{\bar{u}_p^{R/L}\,\not{\varepsilon}(\not{p}-\not{k})\gamma^\mu(\not{p}-\not{k}'-\not{k})\not{\varepsilon}_q(\not{p}-\not{k}')\gamma_\mu\,u_p^{R/L}}{[(p-k'-k)^2-m^2][(p-k')^2-m^2][(p-k)^2-m^2]} \tag{E.5}$$

$$\mathrm{i}A_{2,3}^{R/L} = \frac{-e^4}{(2\pi)^4}\int\frac{\mathrm{d}^4k'}{k'^2}\frac{\bar{u}_p^{R/L}\,\gamma^\mu(\not{p}-\not{k}')\not{\varepsilon}_q(\not{p}-\not{k}'+\not{k})\not{\varepsilon}(\not{p}-\not{k}')\gamma_\mu\,u_p^{R/L}}{[(p-k'+k)^2-m^2][(p-k')^2-m^2]^2} \tag{E.6}$$

$$\mathrm{i}A_{2,4}^{R/L} = \frac{-e^4}{(2\pi)^4}\int\frac{\mathrm{d}^4k'}{k'^2}\frac{\bar{u}_p^{R/L}\,\gamma^\mu(\not{p}-\not{k}')\not{\varepsilon}(\not{p}-\not{k}'-\not{k})\not{\varepsilon}_q(\not{p}-\not{k}')\gamma_\mu\,u_p^{R/L}}{[(p-k'-k)^2-m^2][(p-k')^2-m^2]^2} \tag{E.7}$$

$$\mathrm{i}A_{2,5}^{R/L} = \frac{-e^4}{(2\pi)^4}\int\frac{\mathrm{d}^4k'}{k'^2}\frac{\bar{u}_p^{R/L}\,\not{\varepsilon}_q(\not{p}+\not{k})\gamma^\mu(\not{p}-\not{k}'+\not{k})\not{\varepsilon}(\not{p}-\not{k}')\gamma_\mu\,u_p^{R/L}}{[(p-k'+k)^2-m^2][(p-k')^2-m^2][(p+k)^2-m^2]} \tag{E.8}$$

$$\mathrm{i}A_{2,6}^{R/L} = \frac{-e^4}{(2\pi)^4}\int\frac{\mathrm{d}^4k'}{k'^2}\frac{\bar{u}_p^{R/L}\,\gamma^\mu(\not{p}-\not{k}')\not{\varepsilon}(\not{p}-\not{k}'-\not{k})\gamma_\mu(\not{p}-\not{k})\not{\varepsilon}_q\,u_p^{R/L}}{[(p-k'-k)^2-m^2][(p-k')^2-m^2][(p-k)^2-m^2]} \tag{E.9}$$

As Weinberg mentioned in chapter III of [7] collinear divergences appear because the denominator of the type $pk$ vanish if $k^2$ is zero. So the interesting part of the one-loop amplitude is the one coming from the poles $pk' = 0$ with $k'^2 = 0$. The pole $1/k'^2$ gives a contribution $\mathrm{i}\pi\delta(k'^2)$. As in [6,7] shown the integral of $\int \mathrm{d}\omega'^0$ in the amplitudes (E.4) to (E.9) sets the loop-photon with momentum $k'$ on-shell. We use the standard $\gamma$-matrices identity $\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\nu\gamma_\mu = -2\gamma^\nu\gamma^\beta\gamma^\alpha$ and for the amplitudes $\mathrm{i}A_{2,3}$ and $\mathrm{i}A_{2,4}$ we use a formula that can be easily can verified and only holds for the specific choice of spinors (C.3) and using $\gamma^\mu$ in Weyl representation: $\bar{u}_p^{R/L}\,\gamma^\mu[...]\gamma_\mu\,u_p^{R/L} = -2\bar{u}_p^{L/R}[...]u_p^{L/R}$, where [...] stands for any set of $\gamma$-matrices.

Then the amplitudes (E.4) to (E.9) are given by

$$\mathrm{i}A_{2,1}^{R/L} = \frac{-\mathrm{i}e^4}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}'}{\omega'}\frac{\bar{u}_p^{R/L}\,(\not{p}-\not{k}'+\not{k})\not{\varepsilon}_q(\not{p}-\not{k}')(\not{p}+\not{k})\varepsilon\,u_p^{R/L}}{[-2pk'+2pk-2kk'](2pk')(2pk)} \tag{E.10}$$

$$\mathrm{i}A_{2,2}^{R/L} = \frac{\mathrm{i}e^4}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}'}{\omega'}\frac{\bar{u}_p^{R/L}\,\not{\varepsilon}(\not{p}-\not{k})(\not{p}-\not{k}')\not{\varepsilon}_q(\not{p}-\not{k}'-\not{k})\,u_p^{R/L}}{[-2pk'-2pk+2kk'](2pk')(2pk)} \tag{E.11}$$

$$\mathrm{i}A_{2,3}^{R/L} = \frac{\mathrm{i}e^4}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}'}{\omega'}\frac{\bar{u}_p^{L/R}\,(\not{p}-\not{k}')\not{\varepsilon}_q(\not{p}-\not{k}'+\not{k})\not{\varepsilon}(\not{p}-\not{k}')\,u_p^{L/R}}{[-2pk'+2pk-2kk'](2pk')^2} \tag{E.12}$$

$$\mathrm{i}A_{2,4}^{R/L} = \frac{\mathrm{i}e^4}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}'}{\omega'}\frac{\bar{u}_p^{L/R}\,(\not{p}-\not{k}')\not{\varepsilon}(\not{p}-\not{k}'-\not{k})\not{\varepsilon}_q(\not{p}-\not{k}')\,u_p^{L/R}}{[-2pk'-2pk+2kk'](2pk')^2} \tag{E.13}$$

$$\mathrm{i}A_{2,5}^{R/L} = \frac{-\mathrm{i}e^4}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}'}{\omega'}\frac{\bar{u}_p^{R/L}\,\not{\varepsilon}_q(\not{p}+\not{k})(\not{p}-\not{k}')\not{\varepsilon}(\not{p}-\not{k}'+\not{k})\,u_p^{R/L}}{[-2pk'+2pk-2kk'](2pk')(2pk)} \tag{E.14}$$

$$\mathrm{i}A_{2,6}^{R/L} = \frac{\mathrm{i}e^4}{(2\pi)^3}\int\frac{\mathrm{d}^3\boldsymbol{k}'}{\omega'}\frac{\bar{u}_p^{R/L}\,(\not{p}-\not{k}'-\not{k})\not{\varepsilon}(\not{p}-\not{k}')(\not{p}-\not{k})\not{\varepsilon}_q\,u_p^{R/L}}{[-2pk'-2pk+2kk'](2pk')(2pk)} \tag{E.15}$$

**Comment on the amplitudes $\mathrm{i}A_{2,3}^R$ and $\mathrm{i}A_{2,4}^R$.** The formulas (E.12) and (E.13) for the amplitudes $\mathrm{i}A_{2,3}^R$ and $\mathrm{i}A_{2,4}^R$ show that the incoming photon with momentum $k^\mu$ is non-IR absorption in these two amplitudes, since this photon is attached to an internal line so that there is no propagator with $1/pk$ in the amplitudes (see chapter 5).

### E.2.1 Symmetries between the amplitudes $\mathrm{i}A_{2,i}^{R/L}$

If one takes a closer look to the amplitudes (E.10) to (E.15) one can manifest some symmetries between them. $\mathrm{i}A_{2,3}$ and $\mathrm{i}A_{2,4}$ are related to each other as well as the other 4 amplitudes.

In the following we will omit the right-/left-labelling of the amplitudes to keep it shorter. Writing the amplitudes as functions of the polarisations of the photons $\lambda$, $\lambda_q$ and the energy of the incoming collinear photon $\omega$ then

$$\mathrm{i}A_{2,4}(\lambda,\lambda_q,\omega) = \mathrm{i}A_{2,3}(\lambda_q,\lambda,-\omega)\ , \tag{E.16}$$

$$\mathrm{i}A_{2,4}(\lambda,\lambda_q,\omega) = \mathrm{i}A^{\dagger}_{2,3}(-\lambda,-\lambda_q,-\omega) = \mathrm{i}A_{2,3}(-\lambda,-\lambda_q,-\omega)\ , \tag{E.17}$$

holds for $\mathrm{i}A_{2,4}$. For the other amplitudes one can show that

$$\mathrm{i}A_{2,2}(\lambda,\lambda_q,\omega) = \mathrm{i}A^{\dagger}_{2,1}(-\lambda,-\lambda_q,-\omega) = \mathrm{i}A_{2,1}(-\lambda,-\lambda_q,-\omega)\ , \tag{E.18}$$

$$\mathrm{i}A_{2,5}(\lambda,\lambda_q,\omega) = \mathrm{i}A_{2,2}(\lambda_q,\lambda,-\omega) = \mathrm{i}A_{2,1}(-\lambda_q,-\lambda,\omega)\ , \tag{E.19}$$

$$\mathrm{i}A_{2,6}(\lambda,\lambda_q,\omega) = \mathrm{i}A_{2,1}(\lambda_q,\lambda,-\omega)\ , \tag{E.20}$$

holds.

We want to anticipate that the integrals in (E.10) to (E.15) are real, which can't be seen directly since the numerators have terms proportional to $\mathrm{e}^{\pm \mathrm{i}n(\phi'-\phi)}$, with $n = 0, 1, 2$, but this showed up during the calculations for this appendix.

Together with relation (E.1) one only has to calculate $\mathrm{i}A^R_{2,1}$ and $\mathrm{i}A^R_{2,3}$ to get the complete amplitude $\mathrm{i}A^{R/L}_2 = \sum_{i=1}^6 \mathrm{i}A^{R/L}_{2,i}$. There are two other symmetry in this amplitude. The first is changing $\lambda \to -\lambda$, $\lambda_q \to -\lambda_q$ and $\omega \to -\omega$, which is nothing else but having outgoing photons in figure E.2. The second is changing $\lambda \to -\lambda_q$ and $\lambda_q \to -\lambda$, which is exchanging the ingoing photon with momentum $k^\mu/q^\mu$ to an outgoing photon with momentum $q^\mu/k^\mu$. In other words, the amplitudes of the diagrams with outgoing photons instead of ingoing once are the same as the amplitudes in figure E.2. This is the same behaviour as already seen for the IR case in [7, 65, 66]. The symmetries can be seen in

$$\begin{aligned}
\mathrm{i}A_2(\lambda,\lambda_q,\omega) &= \sum_{i=1}^{6} \mathrm{i}A_{2,i}(\lambda,\lambda_q,\omega) \\
&= \mathrm{i}A_{2,1}(\lambda,\lambda_q,\omega) + \mathrm{i}A_{2,1}(-\lambda,-\lambda_q,-\omega) + \mathrm{i}A_{2,1}(-\lambda_q,-\lambda,\omega) + \mathrm{i}A_{2,1}(\lambda_q,\lambda,-\omega) \\
&\quad + \mathrm{i}A_{2,3}(\lambda,\lambda_q,\omega) + \mathrm{i}A_{2,3}(\lambda_q,\lambda,-\omega) \qquad \text{(E.21)} \\
&= \mathrm{i}A_{2,1}(\lambda,\lambda_q,\omega) + \mathrm{i}A_{2,1}(-\lambda,-\lambda_q,-\omega) + \mathrm{i}A_{2,1}(-\lambda_q,-\lambda,\omega) + \mathrm{i}A_{2,1}(\lambda_q,\lambda,-\omega) \\
&\quad + \mathrm{i}A_{2,3}(\lambda,\lambda_q,\omega) + \mathrm{i}A_{2,3}(-\lambda,-\lambda_q,-\omega)\ , \qquad \text{(E.22)}
\end{aligned}$$

where the last equal sign comes from the relations (E.16) and (E.17).

The two determining amplitudes are (E.10) and (E.12) which can be simplified to

$$\mathrm{i}A^R_{2,1} = \frac{-\mathrm{i}e^4}{(2\pi)^3}\int \frac{\mathrm{d}^3\boldsymbol{k}'}{\omega'} \frac{u^\dagger_R\,(p-k'+k)\cdot\sigma\varepsilon_q\cdot\bar\sigma(p-k')\cdot\sigma(p+k)\cdot\bar\sigma\varepsilon\cdot\sigma\,u_R}{[-2pk'+2pk-2kk'](2pk')(2pk)}\,, \tag{E.23}$$

$$\mathrm{i}A^R_{2,3} = \frac{\mathrm{i}e^4}{(2\pi)^3}\int \frac{\mathrm{d}^3\boldsymbol{k}'}{\omega'} \frac{u^\dagger_L\,(p-k')\cdot\bar\sigma\varepsilon_q\cdot\sigma(p-k'+k)\cdot\bar\sigma\varepsilon\cdot\sigma(p-k')\cdot\bar\sigma\,u_L}{[-2pk'+2pk-2kk'](2pk')^2}\,. \tag{E.24}$$

We can now move forward to perform the integral $\int\frac{\mathrm{d}^3\boldsymbol{k}'}{\omega'} = \int \mathrm{d}\omega'\mathrm{d}\theta'\mathrm{d}\phi'\omega'\sin\theta'$, where we will see that in fact the amplitudes $A_{2,i}$ are real after the integration.

### E.2.2 Details of the $\int_0^{2\pi}\mathrm{d}\phi'$ integral

In the denominator $\phi'$ appears only in the $kk' = k\cdot k'$, where $k^\mu$ and $k'^\mu$ are on-shell. The nominators of $\mathrm{i}A^R_{2,1}$ and $\mathrm{i}A^R_{2,3}$ are calculated by Mathematica and have terms that go like $\mathrm{e}^{\pm\mathrm{i}n(\phi'-\phi)}$, with $n = 0, 1, 2$. Then there are integrals of the form

$$\int_0^{2\pi} \frac{\mathrm{e}^{\pm\mathrm{i}n(\phi'-\phi)}}{a+b\cos(\phi'-\phi)}\mathrm{d}\phi' = \int_0^{2\pi}\frac{\cos(n(\phi'-\phi))\pm\mathrm{i}\sin(n(\phi'-\phi))}{a+b\cos(\phi'-\phi)}\mathrm{d}\phi'\,, \tag{E.25}$$

where $a$ and $b$ are independent of $\phi'$ and $\phi$. All integrals including $\sin(n(\phi'-\phi))$ vanish, as well as the one with $n = 0$, i.e. the one with a constant term. The non-vanishing integrals are

$$\int_0^{2\pi}\mathrm{d}\phi'\frac{\cos(\phi'-\phi)}{a+b\cos(\phi'-\phi)} = \frac{2\pi}{b}\,, \tag{E.26}$$

$$\int_0^{2\pi}\mathrm{d}\phi'\frac{\cos(2(\phi'-\phi))}{a+b\cos(\phi'-\phi)} = -\frac{4\pi a}{b^2}\,. \tag{E.27}$$

In the integration the variables are $a = -2pk' + 2pk - \omega\omega'(1-\cos\theta\cos\theta')$ and $b = 2\omega\omega'\sin\theta\sin\theta'$. We won't write down the amplitudes $\mathrm{i}A^R_{2,1}$ and $\mathrm{i}A^R_{2,3}$ after the $\int_0^{2\pi}\mathrm{d}\phi'$ integration, since these terms are quite long and there is no greater benefit from knowing these formulas. So that we go on to the next integration.

### E.2.3 Details of the $\int \mathrm{d}\theta'$ integral

The next step is the $\int \mathrm{d}\theta'$ integral, where the $\ln(m_e)$ divergences will appear. In the following we will keep terms that are finite after the $\int \mathrm{d}\theta'$ integration and the terms that are logarithmic divergent. What we will omit are terms that go as the mass of the electron $m_e$ since they will vanish in the collinear limit where $m_e \to 0$. In the logarithmic divergent terms we will have to use an angle regulator $\delta$ which is a small angle between the electron and the collinear photon in the loop.[1] In the finite terms we can perform the full integration, meaning integrate over $\theta'$ from $0$ to $\pi$. In the following we will skip some intermediate steps and write down the amplitude $\mathrm{i}A_2^R$ after the two angle integrations $\int \mathrm{d}\theta' \mathrm{d}\phi' \sin\theta'$, since the formula for the full amplitude $\mathrm{i}A_2^R$ is shorter than all the single amplitudes $\mathrm{i}A_{2,i}^R$. Then the integrals are performed by Mathematica and give the result

$$
\begin{aligned}
\mathrm{i}A_2^R = \frac{\mathrm{i}e^4}{(2\pi)^3} \int \mathrm{d}\omega' & \frac{4\pi}{\omega^2 (E - |\boldsymbol{p}|\cos\theta)(1+\cos^2\theta)(1+\cos\theta)} \\
& \Big\{ \omega' \cos\theta \left[ 2E(1 + (2+\lambda\lambda_q)\cos\theta) - \omega(\lambda-\lambda_q)(1-\cos^2\theta) \right] \\
& + \omega \sin^2\theta \left[ E(\lambda-\lambda_q)\cos\theta + 2\omega(1-\lambda\lambda_q\cos^2\theta) \right] \ln\left(\frac{E\delta}{m}\right) \Big\} \\
+ \frac{\mathrm{i}e^4}{(2\pi)^3} \int \mathrm{d}\omega' \frac{2\pi}{E\omega(1+\cos^2\theta)} & \Bigg\{ \left[ E(\lambda-\lambda_q)\cos\theta + \omega(1-\lambda\lambda_q\cos^2\theta) \right] \left( 1 - 2\ln\left(\frac{E\delta}{m}\right) \right) \\
& + \frac{4E\omega'}{\omega}(1+\lambda\lambda_q)\cot^2\theta \Bigg\} \ . \qquad \text{(E.28)}
\end{aligned}
$$

The first three lines come from $\mathrm{i}A_{2,1}^R + \mathrm{i}A_{2,2}^R + \mathrm{i}A_{2,5}^R + \mathrm{i}A_{2,6}^R$, which can be seen from the factor $pk = \omega(E - |\boldsymbol{p}|\cos\theta)$ in the denominator, and the fourth and the fifth line come from $\mathrm{i}A_{2,3}^R + \mathrm{i}A_{2,4}^R$. What also can be seen is that there is no IR divergence in the one-loop amplitude after performing the $\int \mathrm{d}\omega'$ integration, which is conform with [6, 7] since the $B$-factor vanishes in the forward scattering.

[1] As it was the case in appendix C, where we also had to put a regulator for the angle between the electron and the absorbed collinear photon connected to the external line of the electron.

# E.3 The interference term

In order to see if there is a cancelation with (5.8) to the order $e^6$ one has to apply the KLN theorem to the unpolarized interference term of the amplitude $\mathrm{i}A_1$ and $\mathrm{i}A_2$, i.e. $\mathrm{i}A_1^R\left(\mathrm{i}A_2^R\right)^* + \mathrm{i}A_1^L\left(\mathrm{i}A_2^L\right)^* + \text{h.c.}$, where we used (C.41). The contribution is given by

$$\frac{1}{4}\int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}\left[\mathrm{i}A_1^R\left(\mathrm{i}A_2^R\right)^* + \mathrm{i}A_1^L\left(\mathrm{i}A_2^L\right)^* + \text{h.c.}\right]\,. \tag{E.29}$$

In the following we again just calculate the contribution coming from the $\mathrm{i}A_1^R\left(\mathrm{i}A_2^R\right)^*$ since we can apply the relation (E.1) to get the contribution coming from the amplitude with the left-handed electron.

## E.3.1 Small angle approximation

A Taylor expansion for small $\theta$ of the expressions in (E.3) and (E.28) gives

$$\mathrm{i}A_1^R \approx -\,\mathrm{i}e^2(1-\lambda\lambda_q)\frac{\theta^2}{\theta^2+\frac{m^2}{E^2}}\,, \tag{E.30}$$

$$\begin{aligned}\mathrm{i}A_2^R \approx \frac{\mathrm{i}e^4}{(2\pi)^2}\int\mathrm{d}\omega'\frac{1}{\omega^2}\Bigg\{&2\omega'\frac{3+\lambda\lambda_q}{\theta^2+\frac{m^2}{E^2}} + \frac{\theta^2}{\theta^2+\frac{m^2}{E^2}}\Bigg[-\frac{\omega'}{2E}\left(2\omega(\lambda-\lambda_q)+E(1+\lambda\lambda_q)\right)\\ &+\frac{\omega}{E}\left(2\omega(1-\lambda\lambda_q)+E(\lambda-\lambda_q)\right)\ln\left(\frac{E\delta}{m}\right)\Bigg]\Bigg\}\\ &\frac{\mathrm{i}e^4}{(2\pi)^2}\int\mathrm{d}\omega'\left[\frac{\lambda-\lambda_q}{\omega}+\frac{1-\lambda\lambda_q}{E}\right]\left(\frac{1}{2}-\ln\left(\frac{E\delta}{m}\right)\right)+\frac{\theta^2(1+\lambda\lambda_q)}{4}\left[\frac{1-2\ln\left(\frac{E\delta}{m}\right)}{E}+\frac{22}{15}\frac{\omega'}{\omega^2}\right]\\ &+\frac{\omega'(1+\lambda\lambda_q)}{3\omega^2}-\frac{2\omega'(1+\lambda\lambda_q)}{\omega^2\theta^2}\Bigg\}\,.\end{aligned} \tag{E.31}$$

where as usual in the collinear limit (see [65]) $2pk = 2\omega(E-|\boldsymbol{p}|\cos\theta) \approx \omega E(\theta^2+m^2/E^2)$. From the Taylor expansion of the two amplitudes we can see that once the KLN theorem is applied the interference term will have collinear divergences coming from the term proportional to $\theta^2/(\theta^2+m^2/E^2)^2$. And there will be terms that are collinearly divergent coming only from the loop integration from the previous section E.2.3.

The interference term of $\mathrm{i}A_1^R$ and $\left(\mathrm{i}A_2^R\right)^*$ involves the following multiplication of terms with the polarisations of the photons

$$(1-\lambda\lambda_q)(3+\lambda\lambda_q) = 2(1-\lambda\lambda_q)\,, \tag{E.32}$$

$$(1-\lambda\lambda_q)(1+\lambda\lambda_q) = 0\,, \tag{E.33}$$

$$(1-\lambda\lambda_q)(\lambda-\lambda_q) = 2(\lambda-\lambda_q)\,, \tag{E.34}$$

$$(1-\lambda\lambda_q)(1-\lambda\lambda_q) = 2(1-\lambda\lambda_q)\,, \tag{E.35}$$

where $\lambda^2 = 1$ and $\lambda_q^2 = 1$ is used. The integral of interest is then

$$\begin{aligned}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}\left(\mathrm{i}A_1^R\left(\mathrm{i}A_2^R\right)^* + \text{h.c.}\right) &= \int \mathrm{d}\omega\,\omega \int_0^\delta \mathrm{d}\theta\,\theta \int_0^{2\pi} \mathrm{d}\phi \left(\mathrm{i}A_1^R\left(\mathrm{i}A_2^R\right)^* + \text{h.c.}\right)\\ \approx -\frac{e^6}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}\omega' \int_0^\delta \mathrm{d}\theta \frac{\theta}{\omega} &\left\{ 2\omega' \frac{2(1-\lambda\lambda_q)\theta^2}{\left(\theta^2+\frac{m^2}{E^2}\right)^2} + \frac{\theta^4}{\left(\theta^2+\frac{m^2}{E^2}\right)^2}\left[-2\frac{\omega\omega'}{E}(\lambda-\lambda_q)\right.\right.\\ &\left.\left. +2\frac{\omega}{E}\left(2\omega(1-\lambda\lambda_q)+E(\lambda-\lambda_q)\right)\ln\left(\frac{E\delta}{m}\right)\right]\right\}\\ +\frac{e^6}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}\omega' \int_0^\delta \mathrm{d}\theta\,\theta \frac{\theta^2}{\theta^2+\frac{m^2}{E^2}} &\left[\lambda-\lambda_q+\frac{\omega}{E}(1-\lambda\lambda_q)\right]\left(1-2\ln\left(\frac{E\delta}{m}\right)\right)\,.\end{aligned} \tag{E.36}$$

Then one can use the following identities

$$\int_0^\delta \mathrm{d}\theta\,\theta \frac{\theta^2}{(\theta^2+\frac{m^2}{E^2})^2} = -\frac{1}{2}+\ln\left(\frac{E\delta}{m}\right)\,, \tag{E.37}$$

$$\int_0^\delta \mathrm{d}\theta\,\theta \frac{\theta^4}{(\theta^2+\frac{m^2}{E^2})^2} = \frac{\delta^2}{2}\,, \tag{E.38}$$

$$\int_0^\delta \mathrm{d}\theta\,\theta \frac{\theta^2}{\theta^2+\frac{m^2}{E^2}} = \frac{\delta^2}{2}\,, \int_0^\delta \mathrm{d}\theta\,\theta \frac{\theta^4}{\theta^2+\frac{m^2}{E^2}} = \frac{\delta^4}{4}\,, \tag{E.39}$$

to simplify the interference term

$$\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}\left(\mathrm{i}A_1^R\left(\mathrm{i}A_2^R\right)^* + \text{h.c.}\right) \approx$$

$$\frac{e^6}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}\omega' \left\{ -2\frac{\omega'}{\omega}(1-\lambda\lambda_q)\left(1-2\ln\left(\frac{E\delta}{m}\right)\right) + \delta^2\left[-\frac{\omega'}{E}(\lambda-\lambda_q)\right.\right.$$

$$\left.\left. +\frac{1}{E}\left(2\omega(1-\lambda\lambda_q)+E(\lambda-\lambda_q)\right)\ln\left(\frac{E\delta}{m}\right)\right]\right\}$$

$$+\frac{e^6}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}\omega'\, \delta^2\left[\lambda-\lambda_q+\frac{\omega}{E}(1-\lambda\lambda_q)\right]\left(\frac{1}{2}-\ln\left(\frac{E\delta}{m}\right)\right), \tag{E.40}$$

which is more simplified

$$\int_0^\delta \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}\left(\mathrm{i}A_1^R\left(\mathrm{i}A_2^R\right)^* + \text{h.c.}\right) \approx$$

$$-\frac{e^6}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}\omega' \left\{ 2\frac{\omega'}{\omega}(1-\lambda\lambda_q)\left(1-2\ln\left(\frac{E\delta}{m}\right)\right)\right.$$

$$\left. -\frac{\delta^2}{2}\left[\frac{\omega}{E}(1-\lambda\lambda_q)\left(1+2\ln\left(\frac{E\delta}{m}\right)\right)+\left(1-2\frac{\omega'}{E}\right)(\lambda-\lambda_q)\right]\right\}. \tag{E.41}$$

The $\ln\left(\frac{E\delta}{m}\right)$ term in the second line of (E.41) is coming from the phase space integration $\int \mathrm{d}^3\boldsymbol{k}/\omega$. There are no $\ln^2\left(\frac{E\delta}{m}\right)$ terms and the IR divergent term is coming from the $\int \mathrm{d}\omega/\omega$ integration not from the loop integral $\int \mathrm{d}\omega'$. This is conform with what we said in chapter 3 and [6, 7], since the IR term comes from the interference term of the amplitudes $\mathrm{i}A_1$ and $\mathrm{i}A_{2,1}+\mathrm{i}A_{2,2}+\mathrm{i}A_{2,5}+\mathrm{i}A_{2,6}$, which are the amplitudes where the incoming photon is attached to a external electron line.

### E.3.2 Full interference term

The unpolarized contribution is given by (E.29) which is now

$$\frac{1}{4}\int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3 2\omega}\left[\mathrm{i}A_1^R(\lambda,\lambda_q)\left(\mathrm{i}A_2^R(\lambda,\lambda_q)\right)^* + \mathrm{i}A_1^R(-\lambda,-\lambda_q)\left(\mathrm{i}A_2^R(-\lambda,-\lambda_q)\right)^* + \text{h.c.}\right]$$

$$= -\frac{e^6}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}\omega' \left\{ 4\frac{\omega'}{\omega}(1-\lambda\lambda_q)\left(1-2\ln\left(\frac{E\delta}{m}\right)\right)\right.$$

$$-\delta^2 \left[\frac{\omega}{E}(1-\lambda\lambda_q)\left(1-2\ln\left(\frac{E\delta}{m}\right)\right)\right]\Bigg\} \ . \tag{E.42}$$

The collinear divergent part is given by

$$\frac{e^6}{8\pi^4}\int \mathrm{d}\omega \int \mathrm{d}\omega' \left\{4\frac{\omega'}{\omega} - \delta^2\frac{\omega}{E}\right\}(1-\lambda\lambda_q)\ln\left(\frac{E\delta}{m}\right) \ . \tag{E.43}$$

This term does not cancel the KLN anomaly in equation (5.8).

# Appendix F

# Two-loop amplitude

There are also possible cancelations of the KLN anomaly in chapter 5 with two-loop diagrams, since an interference term of an amplitude of order $e$ and the two-loop amplitude in figure F.1 (order $e^5$) is again of order $e^6$.

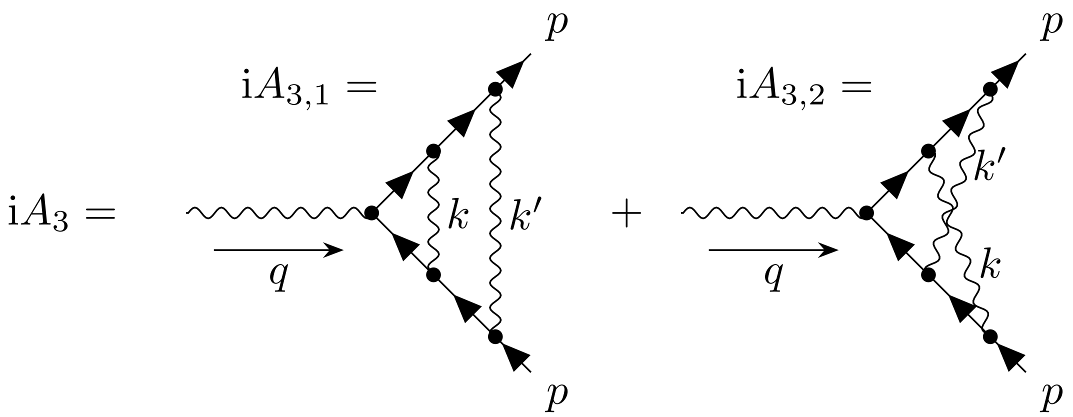



Figure F.1: Two-loop diagrams at order $e^5$ for a forward scattered electron with 4-momentum $p^\mu$.

# F.1 Amplitude $\mathrm{i}A_{3,1}$

From the diagram in figure F.1 one can read off

$$\mathrm{i}A_{3,1}^{R/L} = \frac{\mathrm{i}e^5}{(2\pi)^8}\int \frac{\mathrm{d}^4k\,\mathrm{d}^4k'}{k^2\,k'^2}\frac{\bar{u}_p^{R/L}\gamma^\mu(\not{p}-\not{k}')\gamma^\nu(\not{p}-\not{k}'-\not{k})\varepsilon_q(\not{p}-\not{k}'-\not{k})\gamma_\nu(\not{p}-\not{k}')\gamma_\mu u_p^{R/L}}{\left[(p-k'-k)^2-m^2\right]^2\left[(p-k')^2-m^2\right]^2}\,. \tag{F.1}$$

To simplify the amplitude we use the same steps and identities of section E.2. Then the amplitude $\mathrm{i}A_{4,1}^{R/L}$ is simplified to

$$\mathrm{i}A_{4,1}^{R/L} = \frac{-\mathrm{i}e^5}{(2\pi)^6}\int \frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{k}'}{\omega\omega'}\frac{\bar{u}_p^{L/R}(\not{p}-\not{k}')(\not{p}-\not{k}'-\not{k})\not{\varepsilon}_q(\not{p}-\not{k}'-\not{k})(\not{p}-\not{k}')u_p^{L/R}}{\left[-2pk'-2pk+2kk'\right]^2(2pk')^2}\,. \tag{F.2}$$

As before we will look first at the $\int_0^{2\pi}\mathrm{d}\phi'$ integration. The integration is a bit different since the denominator is of the form $\left[a+b\cos(\phi'-\phi)\right]^2$, where $a$ and $b$ are again independent of $\phi$ and $\phi'$. Mathematica calculates the nominator inside the integrals of the amplitude (F.2) and what we get are terms proportional to the exponential functions $\mathrm{e}^{\pm\mathrm{i}(\phi-\phi_q)}$, $\mathrm{e}^{\pm\mathrm{i}(\phi'-\phi_q)}$, $\mathrm{e}^{\pm\mathrm{i}(\phi'-\phi)}$ and $\mathrm{e}^{\pm\mathrm{i}(2\phi-\phi'-\phi_q)}$.[1] The amplitude $\mathrm{i}A_{3,1}^{R/L}$ vanishes, since

$$\int_0^{2\pi}\mathrm{d}\phi'\,\frac{\mathrm{e}^{\pm\mathrm{i}(\phi-\phi_q)}}{\left[a+b\cos(\phi'-\phi)\right]^2} = 0\,, \tag{F.3}$$

$$\int_0^{2\pi}\mathrm{d}\phi'\,\frac{\mathrm{e}^{\pm\mathrm{i}(\phi'-\phi_q)}}{\left[a+b\cos(\phi'-\phi)\right]^2} = 0\,, \tag{F.4}$$

$$\int_0^{2\pi}\mathrm{d}\phi'\,\frac{\mathrm{e}^{\pm\mathrm{i}(\phi'-\phi)}}{\left[a+b\cos(\phi'-\phi)\right]^2} = 0\,, \tag{F.5}$$

$$\int_0^{2\pi}\mathrm{d}\phi'\,\frac{\mathrm{e}^{\pm\mathrm{i}(2\phi-\phi'-\phi_q)}}{\left[a+b\cos(\phi'-\phi)\right]^2} = 0\,. \tag{F.6}$$

Thus, there can only be a contribution from the other diagram in figure F.1.

[1] Notice that the angles of $\theta$ and $\phi$ are the ones of the on-shell photon that runs in the loop with momentum $k$.

## F.2 Amplitude $\mathrm{i}A_{3,2}$

From the figure F.1 one can read off that the amplitude is given by

$$\mathrm{i}A_{3,2}^{R/L} = \frac{\mathrm{i}e^5}{(2\pi)^8}\int \frac{\mathrm{d}^4k\,\mathrm{d}^4k'}{k^2\,k'^2}\frac{\bar{u}_p^{R/L}\gamma^\mu(\not{p}-\not{k}')\gamma^\nu(\not{p}-\not{k}'-\not{k})\not{\varepsilon}_q(\not{p}-\not{k}'-\not{k})\gamma_\mu(\not{p}-\not{k})\gamma_\nu u_p^{R/L}}{\left[(p-k'-k)^2-m^2\right]^2\left[(p-k')^2-m^2\right]\left[(p-k)^2-m^2\right]}\,. \tag{F.7}$$

Notice the difference to (F.1). The two last $\gamma$-matrices are exchanged and in the denominator there is a propagator $pk$. This differences makes it impossible to directly apply the identities of section E.2. Before using them we will apply an other $\gamma$-matrix identity, $\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho+\eta^{\nu\rho}\gamma^\mu-\eta^{\mu\rho}\gamma^\nu-\mathrm{i}\varepsilon^{\sigma\mu\nu\rho}\gamma_\sigma\gamma^5$, where $\eta^{\mu\nu}$ is the metric tensor in Minkowski spacetime, $\varepsilon^{\sigma\mu\nu\rho}$ is the Levi-Civita symbol in 4d. With that we rewrite the middle part of the nominator to

$$(\not{p}-\not{k}'-\not{k})\varepsilon_q(\not{p}-\not{k}'-\not{k}) = 2(p-k'-k)\cdot\varepsilon_q\,(\not{p}-\not{k}'-\not{k})-(p-k'-k)^2\,\not{\varepsilon}_q\,. \tag{F.8}$$

The term with the Levi-Civita symbol is zero since there is a summation of 2 equal terms, i.e. $\varepsilon^{\alpha\beta\mu\nu}a_\alpha a_\beta b_\mu c_\nu = 0$. Once this identity is used one can also apply the identities of section E.2. Putting all together then the amplitude (F.7) is

$$\begin{aligned}\mathrm{i}A_{3,2}^{R/L} =& -\frac{\mathrm{i}e^5}{(2\pi)^6}\int\frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{k}'}{2\omega\,\omega'}(p-k'-k)\cdot\varepsilon_q\frac{\bar{u}_p^{R/L}\gamma^\mu(\not{p}-\not{k}')\gamma^\nu(\not{p}-\not{k}'-\not{k})\gamma_\mu(\not{p}-\not{k})\gamma_\nu u_p^{R/L}}{\left[-2pk'-2pk+2kk'\right]^2(2pk')(2pk)}\\ &+\frac{\mathrm{i}e^5}{(2\pi)^6}\int\frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{k}'}{2\omega\,2\omega'}\frac{\bar{u}_p^{R/L}\gamma^\mu(\not{p}-\not{k}')\gamma^\nu\not{\varepsilon}_q\gamma_\mu(\not{p}-\not{k})\gamma_\nu u_p^{R/L}}{\left[-2pk'-2pk+2kk'\right](2pk')(2pk)}\,,\end{aligned} \tag{F.9}$$

where in the second line one $-2pk'-2pk+2kk'$ propagator is canceled by the $(p-k'-k)^2 = -2pk'-2pk+2kk'$ term in the identity (F.8). Now we use the identities $\gamma^\mu\gamma^\alpha\gamma^\beta\gamma^\nu\gamma_\mu = -2\gamma^\nu\gamma^\beta\gamma^\alpha$ and $\gamma^\mu\gamma^\alpha\gamma^\beta\gamma_\mu = 4\eta^{\alpha\beta}$ to get

$$\begin{aligned}\mathrm{i}A_{3,2}^{R/L} =& \frac{\mathrm{i}e^5}{(2\pi)^6}\int\frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{k}'}{\omega\,\omega'}2(p-k'-k)\cdot\varepsilon_q\frac{\bar{u}_p^{R/L}(\not{p}-\not{k}'-\not{k})u_p^{R/L}}{\left[-2pk'-2pk+2kk'\right](2pk')(2pk)}\\ &-\frac{\mathrm{i}e^5}{(2\pi)^6}\int\frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{k}'}{\omega\,\omega'}\frac{\bar{u}_p^{R/L}\not{\varepsilon}_q u_p^{R/L}}{(2pk')(2pk)}\,.\end{aligned} \tag{F.10}$$

A small matrix calculation by hand or using Mathematica for the nominators gives

$$\begin{aligned}
\mathrm{i}A_{3,2}^{R/L} =& \frac{\mathrm{i}e^5}{(2\pi)^6}\frac{4E}{\sqrt{1+\cos^2\theta_q}}\int\frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{k}'}{\omega\,\omega'}\Big[\omega'\sin\theta'\cos\theta_q\mathrm{e}^{\mathrm{i}\lambda_q(\phi'-\phi_q)}+\omega\sin\theta\cos\theta_q\mathrm{e}^{\mathrm{i}\lambda_q(\phi-\phi_q)} \\
&+E\sin\theta_q-\omega'\cos\theta'\sin\theta_q-\omega\cos\theta\sin\theta_q\Big]\frac{2E-\omega(1+\cos\theta)-\omega'(1+\cos\theta')}{[-2pk'-2pk+2kk']\,(2pk')(2pk)} \\
&-\frac{\mathrm{i}e^5}{(2\pi)^6}\frac{2E\sin\theta_q}{\sqrt{1+\cos^2\theta_q}}\int\frac{\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{k}'}{\omega\,\omega'}\frac{1}{(2pk')(2pk)}\,. \qquad \text{(F.11)}
\end{aligned}$$

The $\int_0^{2\pi}\mathrm{d}\phi\int_0^{2\pi}\mathrm{d}\phi'$ integration of the first integral of (F.11) will make this term 0. This can be seen by the following integrals, where as before the denominator is of the form $a+b\cos(\phi'-\phi)$:

$$\int_0^{2\pi}\mathrm{d}\phi\int_0^{2\pi}\mathrm{d}\phi'\frac{\mathrm{e}^{\mathrm{i}\lambda_q(\phi'-\phi_q)}}{a+b\cos(\phi'-\phi)}=0\,, \qquad \text{(F.12)}$$

$$\int_0^{2\pi}\mathrm{d}\phi\int_0^{2\pi}\mathrm{d}\phi'\frac{\mathrm{e}^{\mathrm{i}\lambda_q(\phi-\phi_q)}}{a+b\cos(\phi'-\phi)}=0\,, \qquad \text{(F.13)}$$

$$\int_0^{2\pi}\mathrm{d}\phi\int_0^{2\pi}\mathrm{d}\phi'\frac{1}{a+b\cos(\phi'-\phi)}=0\,. \qquad \text{(F.14)}$$

What is left over is the second line of (F.11) as result for the two-loop amplitude $\mathrm{i}A_4$:

$$\mathrm{i}A_3^R=\mathrm{i}A_3^L=\mathrm{i}A_{3,2}^R=\mathrm{i}A_{3,2}^L=-\frac{\mathrm{i}e^5}{(2\pi)^6}\frac{2}{E}\frac{\sin\theta_q}{\sqrt{1+\cos^2\theta_q}}\ln^2\left(\frac{E\delta}{m}\right)\left(\int\mathrm{d}\omega\right)^2\,, \qquad \text{(F.15)}$$

where we performed the angular integration of $\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}^3\boldsymbol{k}'$ and we abbreviate $\int\mathrm{d}\omega\int\mathrm{d}\omega'=\left(\int\mathrm{d}\omega\right)^2$. Thus, there is again no cancelation of the KLN anomaly in equation (5.8).

This means that the two-loop amplitude $\mathrm{i}A_3$ from the diagrams in figure F.1 has only a $\ln^2(m_e)$ divergence. Notice that this result (F.15) is neither IR divergent, which is conform with [7], because in the forward limit the $B$-factor is 0, nor there are single $\ln(m_e)$ divergent terms.

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