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Anomalous transport model with axial magnetic fields

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ABSTRACT

The transport properties of massless fermions in $3+1$ spacetime dimension have been in the focus of recent theoretical and experimental research. New transport properties appear as consequences of chiral anomalies. The most prominent is the generation of a current in a magnetic field, the so-called chiral magnetic effect leading to an enhancement of the electric conductivity (negative magnetoresistivity). We study the analogous effect for axial magnetic fields that couple with opposite signs to fermions of different chirality. We emphasize local charge conservation and study the induced magneto-conductivities proportional to an electric field and a gradient in temperature. We find that the magnetoconductivity is enhanced whereas the magneto-thermoelectric conductivity is diminished. As a side result we interpret an anomalous contribution to the entropy current as a generalized thermal Hall effect.

1. Introduction and motivation

Chiral anomalies [1,2] and the specific transport phenomena induced by them such as the chiral magnetic and the chiral vortical effects have been extensively discussed in the recent years (see [3,4] for reviews).

In a theory of massless Dirac fermions the vector current $J^\mu = \bar\Psi\gamma^\mu\Psi$ and axial current $J_5^\mu = \bar\Psi\gamma_5\gamma^\mu\Psi$ can be defined. In such a theory the chiral magnetic effect (CME) describes the generation of an electric current in a magnetic field in the presence of an axial chemical potential

$$\vec{J} = \frac{\mu_5}{2\pi^2}\vec{B}\,, \tag{1}$$

where μ_5 is the axial chemical potential conjugate to the axial charge operator $Q_5 = \int d^3x\bar\Psi\gamma_5\gamma^0\Psi$.

This formula has to be interpreted with care. At first sight it predicts the generation of a current in equilibrium. It has been pointed out however that such an equilibrium current is forbidden by the so-called Bloch theorem. In relation to the CME this theorem has first been invoked in a condensed matter context in [5]. A recent discussion of the Bloch theorem has been given in [6]. The theorem can be formulated as

$$\int d^3x\,\vec{J}(x) = 0\,, \tag{2}$$

in thermal equilibrium. Seemingly this is violated by eq. (1) for a homogeneous magnetic field. The important point emphasized in [6] is that the Bloch theorem is valid only for exactly conserved currents. This allows to resolve the tension between eq. (1) and the Bloch theorem. More precisely eq. (1) holds only for the so-called covariant version of the current. This covariant current is not a truly conserved current but rather has the anomaly

$$\partial_\mu J^\mu = \frac{1}{8\pi^2}\epsilon^{\mu\nu\rho\lambda}F_{\mu\nu}F^5_{\rho\lambda}\,, \tag{3}$$

where one also introduces a axial field A^5_μ as source for insertions of the axial current J_5^μ. Similarly the covariant version of the axial anomaly is

$$\partial_\mu J_5^\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\rho\lambda}\left(F_{\mu\nu}F_{\rho\lambda} + F^5_{\mu\nu}F^5_{\rho\lambda}\right)\,. \tag{4}$$

In quantum field theory the currents are composite operators and have to be regularized. This regularization introduces certain ambiguities that have to be fixed by demanding certain classical properties of the currents to hold on the quantum level. One way to fix these ambiguities is to define J^μ and J_5^μ to be invariant objects under both vector- and axial-type gauge transformations [7]. The disadvantage of this definition is that it does not result in a conserved vector like current but rather leads to the anomaly in

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eq. (3). On the other hand one can insist on the vector like current to be exactly conserved $\partial_\mu \mathcal{J}^\mu = 0$. The relation between the two definitions of currents is

$$\mathcal{J}^\mu = J^\mu - \frac{1}{4\pi^2}\epsilon^{\mu\nu\rho\lambda} A^5_\nu F_{\rho\lambda} \,. \tag{5}$$

Due to axial anomaly the axial vector J^μ_5 is never conserved and therefore its source A^5_μ can not be interpreted as a true gauge field. Therefore the Chern–Simons current in (5) is a physical current in a completely analogous way as the Chern–Simons current appearing in the quantum Hall effect. This resolves the tension between the chiral magnetic effect and the Bloch theorem in the following manner. Thermal equilibrium is defined by the grand canonical ensemble with density matrix $\exp(-(H - \mu_5 Q_5)/T)$. This is equivalent to considering the theory in the background of a temporal component of the axial field $A^5_0 = \mu_5$. Now the chiral magnetic effect in the exactly conserved current $\mathcal{J}^\mu$ takes the form [8]

$$\vec{\mathcal{J}} = \frac{\mu_5}{2\pi^2}\vec{B} - \frac{A^5_0}{2\pi^2}\vec{B} \,, \tag{6}$$

where the second term stems from the Chern–Simons current in eq. (5). Since in strict equilibrium $A^5_0 = \mu_5$ this shows that the chiral magnetic effect for the conserved current (5) vanishes as demanded by the Bloch theorem. The importance of defining the coserved current has also been discussed in chiral kinetic theory in [9].

On the other hand the closely related chiral separation effect

$$\vec{J}_5 = \frac{\mu}{2\pi^2}\vec{B} \,, \tag{7}$$

does not suffer any such correction. Since the axial current is always affected by an anomaly there is no contradiction to the Bloch theorem as pointed out in [6].

There is however a third related effect if one allows for axial magnetic fields, $\vec{B}_5 = \vec{\nabla} \times \vec{A}_5$. This is a magnetic field that couples with opposite signs to fermions of different chirality. The axial magnetic effect takes the form

$$\vec{J} = \vec{\mathcal{J}} = \frac{\mu}{2\pi^2}\vec{B}_5 \,. \tag{8}$$

Formally it describes the generation of a vector-like current in the background of an axial magnetic field at finite (vector-like) chemical potential. Note that the formula holds for both the covariant and the conserved form of the currents. Therefore this formula seems to be in much greater tension with the Bloch theorem than the chiral magnetic effect. One might dismiss this tension on the grounds that so far at a fundamental level no axial fields seem to exist in nature. However, it has been argued that such fields can appear in the effective description of the electronics of advanced materials, the so-called Weyl semimetals [10–13]. A low energy field theoretical description of the electronics of these materials given by the Dirac equation

$$\gamma^\mu (i D_\mu + b_\mu \gamma_5)\Psi = 0 \,. \tag{9}$$

Here D_μ is the usual covariant derivative and the parameter b^μ enters just like the field A^5_μ coupling to the axial current. It has been argued that straining such materials can lead to spatial variation of the parameter b^μ and in consequence to the appearance of effective axial magnetic fields in eq. (9). The reason why there is no contradiction to the Bloch theorem in this case is as follows. The parameter b^μ exists only within the material and necessarily vanishes outside. If for definiteness we assume the axial magnetic field to be directed along the z direction and we compute the total axial flux at through a surface Ω at some fixed $z = z_0$

$$\Phi_5 = \int_\Omega dxdy B^5_z(x, y, z_0) = \int_{\partial\Omega} d\vec{S} \cdot \vec{b} = 0 \,, \tag{10}$$

since one can always take the boundary of the surface to lie entirely outside the material where $\vec{b} = 0$. Therefore the axial analogue of the chiral magnetic effect (8) can not induce a net current and this resolves the tension with the Bloch theorem since no net current can be generated [14,15].

We will take these considerations as motivation to study electro- and thermo-magnetotransport in the background of axial magnetic fields under the assumption that the Bloch theorem is implemented by a vanishing net axial magnetic flux (10). This implies that the net equilibrium electric current vanishes but as we will see upon applying an electric field (or equivalently a gradient in chemical potential) and a temperature gradient leads to anomaly induced net contributions to the currents.

2. Anomalous transport

We study a simple model of anomalous transport with coupled energy and charge transport. This means that in contrast to a full hydrodynamic model we assume that no significant collective flow parametrized by a flow velocity develops.[1] Not only is this a simpler model allowing to study the effects of anomalies on transport it might also be more directly relevant to systems where elastic scattering on impurities impedes the build up of collective flow.

We develop now a formal transport model based on the anomalous continuity equations

$$\dot{\epsilon} + \vec{\nabla} \cdot \vec{J}_\epsilon = \vec{E} \cdot \vec{J} \,, \tag{11}$$

$$\dot{\rho} + \vec{\nabla} \cdot \vec{J} = c\vec{E} \cdot \vec{B} \,, \tag{12}$$

where ϵ is the energy density and $\vec{J}_\epsilon$ is the energy current. Charge conservation is affected by an anomaly with anomaly coefficient c. The right hand side of equation (11) quantifies the energy injected into the system by an electric field (Joule heating) whereas (12) describes the (covariant) anomaly. So far this is not specific to axial magnetic fields but rather relies only on the presence of an anomaly in the current $J^\mu = (\rho, \vec{J})$.

To discuss transport we write down constitutive relations for $\vec{J}_\epsilon$, $\vec{J}$ and take as thermodynamic forces the gradients in the thermodynamic potentials and external electric and magnetic fields,

$$\begin{pmatrix} \vec{J}_\epsilon \\ \vec{J} \end{pmatrix} = L \cdot \begin{pmatrix} \vec{\nabla}\left(\frac{1}{T}\right) \\ \frac{\vec{E}}{T} - \vec{\nabla}\left(\frac{\mu}{T}\right) \end{pmatrix} + \begin{pmatrix} \hat{\sigma}_B \\ \sigma_B \end{pmatrix} \vec{B} \,. \tag{13}$$

The matrix L encodes response due to gradients in chemical potential and temperature. $\{\hat{\sigma}_B, \sigma_B\}$ describe response due to the magnetic field. In principle we could also allow an independent response due to the electric field. In our ansatz we have thus anticipated that positivity of entropy production is not compatible with such additional terms in the constitutive relations.

[1] This does not mean that the velocity or the variation of the velocity is zero, just that it cannot be determined by the conserved equations. Our transport model can not be obtained from hydrodynamics by setting the flow velocities to zero. Hydrodynamic flow (i.e. non vanishing velocity) appears already at zeroth order in derivatives and this imposes constraints on the first order transport coefficients that can appear in the constitutive relations [16]. Since for strong momentum relaxation flow is absent such relations are not present. This model has similarity to the treatment in the theory for incoherent metal in 2+1D [17].

The transport coefficients are constrained by the second law of thermodynamics. Using the thermodynamic relation $Tds = d\epsilon + \mu d\rho$ as guideline we define the entropy current [18] as

$$\vec{J}_s = \frac{1}{T}\vec{J}_\epsilon - \frac{\mu}{T}\vec{J} + \eta_B \vec{B}\,. \tag{14}$$

Up to the terms depending on the magnetic field this is the standard ansatz for coupled energy and charge transport [19].

Following [18,20] we impose the local form of the second law thermodynamics

$$\dot{s} + \vec{\nabla}\cdot\vec{J}_s \geq 0\,. \tag{15}$$

Using $T\dot{s} = \dot{\epsilon} + \mu\dot{\rho}$ this leads to

$$\frac{1}{T}\left(\frac{\partial\epsilon}{\partial t} + \vec{\nabla}\cdot\vec{J}\right) - \frac{\mu}{T}\left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J}\right) + \vec{\nabla}\left(\frac{1}{T}\right)\cdot\vec{J}_\epsilon$$
$$-\vec{\nabla}\left(\frac{\mu}{T}\right)\cdot\vec{J} + \vec{\nabla}\eta_B\cdot\vec{B} + \eta_B\nabla\cdot\vec{B} \geq 0\,. \tag{16}$$

We assume absence of magnetic monopoles and thus the last term vanishes. Using the constitutive relations we find the constraints $det(L) \geq 0$ and $L_{11} \geq 0$ and $L_{22} \geq 0$. Positivity of the entropy production also assures that the electric field does not give rise to additional response not already contained in L. Entropy is produced only by the symmetric part of the matrix L. For the magnetic conductivities one finds a set of one algebraic and two differential equations

$$\sigma_B - c\mu = 0\,, \tag{17}$$

$$\sigma_B + \frac{\partial\eta_B}{\partial\gamma_\rho} = 0\,, \tag{18}$$

$$\hat{\sigma}_B + \frac{\partial\eta_B}{\partial\gamma_\epsilon} = 0\,, \tag{19}$$

with $\gamma_\epsilon = 1/T$ and $\gamma_\rho = -\mu/T$. These equations are the coefficients of the terms $(\vec{E}\cdot\vec{B})$, $(\vec{\nabla}\gamma_\epsilon\cdot\vec{B})$ and $(\vec{\nabla}\gamma_\rho\cdot\vec{B})$. These terms can be either positive or negative and therefore their coefficients must vanish to guarantee the local form of the second law of thermodynamics. Since there is no further dimensionful parameter η_B must also fulfill $\gamma_\epsilon\partial\eta_B/\partial\gamma_\epsilon = -\eta_B$ as it has to have dimension one, where in our conventions (μ, T) have dimension one. The magnetic conductivities are almost completely determined

$$\sigma_B = c\mu\,,\quad \hat{\sigma}_B = c\frac{\mu^2}{2} + c_g T^2\,,\quad \eta_B = c\frac{\mu^2}{2T} + c_g T\,. \tag{20}$$

Up to ambiguities arising due to frame choice these are basically the same results as in hydrodynamics [18,20,21].

The priori undetermined integration constant c_g is related to (mixed) gravitational anomalies [22–30]. In holography it was also shown recently that the relation to the (mixed) gravitational anomaly is not modified by momentum relaxation in [31]. The intuition that dissipationless transport should not be affected by momentum relaxation together with the results of [31] and [21] (the case of weak momentum relaxation) we take as evidence that $c_g \neq 0$ also in the case of strong momentum relaxation and that it is related to the presence of (possibly global) gravitational anomalies. For theories containing only spin 1/2 particles and holographic theories this relation is $c_g = 32\pi^2\lambda$ where λ is the coefficient of the gravitational contribution to the anomaly $\partial_\mu J^\mu = \lambda\epsilon^{\mu\nu\rho\lambda}R^\alpha{}_{\beta\mu\nu}R^\beta{}_{\alpha\mu\nu}$. A single Weyl fermion has $\lambda = \pm\frac{1}{768\pi^2}$ and $c_g = \pm 1/24$ with the sign depending on the chirality. In the following we assume $c_g \neq 0$ to be related to the mixed axial gravitational anomaly as in the case without momentum relaxation and study its implications for thermo-electric transport in axial magnetic fields.

Using the results for the anomalous transport coefficients σ_B, $\hat{\sigma}_B$ and η_B the entropy current can be written as

$$\vec{J}_s = (1/T, -\mu/T)\cdot L\cdot\begin{pmatrix}\vec{\nabla}(1/T)\\ \frac{\vec{E}}{T} - \vec{\nabla}\left(\frac{\mu}{T}\right)\end{pmatrix} + 2c_g T\vec{B}\,. \tag{21}$$

Naively one might have expected that the anomalous transport does not contribute to the entropy current. It turns out however that the temperature dependence encoding the gravitational anomalies does contribute to entropy current. This has been previously observed in [21,32].

The previous considerations are general and assume only the presence of an anomaly in the charge current. We can now specialize to the case of the axial magnetic field. In this case the charge conservation takes the form

$$\dot{\rho} + \vec{\nabla}\cdot\vec{J} = \frac{N_f}{2\pi^2}(\vec{E}\cdot\vec{B}_5 + \vec{E}_5\cdot\vec{B})\,, \tag{22}$$

where N_f is the number of Dirac fermions. Rather than an anomaly in this case the right hand side should be interpreted as the divergence of the Chern–Simons current in eq. (5). Since we are mostly concerned with the effects of axial magnetic fields we will set $\vec{E}_5 = \vec{B} = 0$ in the following. In this case the conservation equations are precisely as in the general case before and we can take over the previous results by simply replacing $\vec{B}$ with $\vec{B}_5$ and setting $c = \frac{N_f}{2\pi^2}$ and $c_g = N_f/12$.

Now we want to give an interpretation for the anomalous contribution to the entropy current in eq. (21). We consider an axial magnetic field configuration of the form

$$\vec{B}_5(x) = \hat{e}_z\bar{\Phi}_5\big[\delta(x) - \delta(x-L)\big]. \tag{23}$$

According to our assumption of compatibility with Bloch's theorem the total axial magnetic flux along the z direction vanishes but the regions of positive and negative fluxes are well separated which for simplicity we model by delta-functions distribution localized in $x = 0$ and $x = L$ but spread out in the y direction. The first thing to notice is that according to (21) there is an anomalous entropy current localized at the locations of axial magnetic flux. If there is a temperature gradient such that $T(x=0) = T + \delta T$ and $T(x=L) = T$ a net entropy current flows along the z-direction

$$\delta\vec{I}_s = \int dx\vec{J}_s = 2c_g(\delta T)\Phi_5\hat{e}_z\,. \tag{24}$$

This current flows in a direction orthogonal to the temperature gradient. Heat is not a thermodynamic state variable still it can be defined as $\delta Q = T\delta S$ and in an analogous way we can define a heat current as $\delta\vec{I}_Q = T\delta\vec{I}_s$. This leads to the net heat current

$$\delta\vec{I}_Q = 2c_g T\delta T\bar{\Phi}_5\hat{e}_z\,. \tag{25}$$

We interpret this as anomalous thermal Hall effect. In this way the anomalous contribution to the entropy current in (21) can be understood as a generalization of the anomalous thermal Hall effect. Previous discussions of the relation between thermal Hall effect and gravitational anomalies are [33,34]. Let us also note that the very concept of heat current can be questioned on the grounds that heat is not a state variable [35]. In the context of anomalous transport there certainly arises the question if in the common definition of heat current $\vec{J}_Q = \vec{J}_\epsilon - \mu\vec{J}$ the current $\vec{J}$ should be taken to be the covariant or the conserved current. Defining the heat current as $\delta\vec{J}_Q = T\delta\vec{J}_s$ resolves this issue.

2.1. Induced conductivities

Let us now come to the main subject: the linear response of this system to a temperature gradient and an external electric field both aligned with the axial magnetic field. The continuity equations (11), (12) together with the constitutive relations (13) form a dynamical system that allows to compute current and charge distributions given some initial and boundary conditions. The effective response to an applied electric field and a temperature gradient can be computed by solving these equations.

Before studying the axial magnetic field case of interest it is worth to briefly recall how the chiral magnetic effect leads to negative magneto-resistivity [36–38]. One assumes a homogeneous magnetic field and a parallel electric field. Axial charge is not subject to an exact conservation law and thus it is natural to introduce an axial charge relaxation time τ_5. Non-conservation of axial charge is provided e.g. by a mass term in the Dirac equation or by inter-valley scattering the context of Weyl semimetals. The effective axial charge (non-)conservation is then

$$\dot{\rho}_5 = c\vec{E}\cdot\vec{B} - \frac{1}{\tau_5}\rho_5\,. \tag{26}$$

We note that if an external electric field is absent but instead a gradient of the chemical potential is induced the current has a non-vanishing gradient $\vec{\nabla}\cdot\vec{J}_5 = c\vec{\nabla}\mu\cdot\vec{B}$ which leads to en effectively equivalent equation for the time development of axial charge by replacing $\vec{E}\to-\vec{\nabla}\mu$. Axial charge is built up until a steady state is reached with $\delta\rho_5 = \tau_5 c\vec{E}\cdot\vec{B}$. The axial charge can be related to the axial chemical potential via $\chi_5\delta\mu_5 = \delta\rho_5$ where χ_5 is the axial susceptibility. Combining Ohmic and chiral magnetic currents leads to the enhanced current

$$\vec{J} = \sigma\vec{E} + \tau_5\frac{c^2(\vec{E}\cdot\vec{B})}{\chi_5}\vec{B}\,. \tag{27}$$

For infinite axial charge relaxation time the anomaly induced magnetoconductivity is formally infinite and this might be referred to as chiral magnetic superconductivity [39]. In nature fermions are however massive and effective chiral fermions in materials such as Weyl semimetals do not preserve there chirality at all energies due to the compact nature of the Brillouin zone.

In the case of the axial magnetic field the role of the axial chemical potential is played by the (electric) chemical potential μ. Electric charge is an exactly conserved quantity due to electromagnetic gauge invariance. Therefore it is not possible to introduce a relaxation time for electric charge without violating gauge invariance. If it were possible then to engineer homogeneous axial magnetic fields an analogous argument would lead necessarily to infinite axial magneto-conductivity. As we have argued however in the introduction the assumption of such a homogeneous axial magnetic field is by itself inconsistent with the Bloch theorem, which by itself is a consequence of gauge invariance [6]. Thus we are naturally lead to study induced electro- and thermo-axial magneto conductivity under the constraint of vanishing net axial magnetic flux. This makes the problem more complicated as diffusion from regions where charge is accumulated to regions with charge outflow has to be taken into account. It is this diffusion process that can lead to a stationary state and finite induced axial magneto-conductivities.

As external driving forces we assume a homogeneous electric field and a temperature gradient pointing in the z direction. We also assume an axial magnetic field directed along the z direction but inhomogeneous in the (x,y) plane and with zero net flux $\Phi_5 = \int dxdy B_5(x,y) = 0$. The dynamical variables are the chemical potential μ and the temperature T. We allow the system to adjust to the external forces by developing non-trivial profiles of chemical potential and temperature in the (x,y) plane around a constant background value. Thus our ansatz is

$$\vec{B}_5 = B_5(x,y)\hat{e}_z\,,\qquad \vec{E} = E\hat{e}_z\,, \tag{28}$$

$$\mu = \mu_0 + \delta\mu(x,y)\,,\quad T = T_0 + \delta T(x,y) + z\nabla_z T\,. \tag{29}$$

The response in $\delta\mu$ and δT to E and ∇T is now calculated in linear approximation.

Since the axial magnetic field is not uniform in the (x,y) plane the system will react to the local charge inflow induced by the anomalous Hall and axial magnetic effects by building up diffusion currents. Eventually a stationary state is reached. This stationary state can be obtained from the constitutive relations and the conservation equations by dropping the time derivative. We furthermore assume the matrix L to be spatially isotropic. Using (11), (12) the constitutive relations (13) with the anomalous transport coefficients (20) we find that the fluctuations δT and $\delta\mu$ have to fulfill a system of Poisson equations

$$L\cdot Y\Delta_\perp\begin{pmatrix}\delta T(x_\perp)\\ \delta\mu(x_\perp)\end{pmatrix} = \begin{pmatrix}-2c_g T_0 & c\mu_0\\ 0 & c\end{pmatrix}\cdot\begin{pmatrix}\nabla_z T\\ E\end{pmatrix}B_5(x_\perp)\,. \tag{30}$$

Here $\Delta_\perp$ is the two dimensional Laplace operator ($\Delta_\perp = \partial_x^2 + \partial_y^2$) and $Y = \frac{1}{T_0^2}\begin{pmatrix}-1 & 0\\ \mu_0 & -T_0\end{pmatrix}$ is the transformation matrix relating the thermodynamic forces $\delta(1/T)$ and $\delta(-\mu/T)$ to the fluctuations δT, $\delta\mu$. Once the fluctuations are determined they can be plugged into the *anomalous part* of constitutive relations (13) to find the anomaly induced contribution to the currents

$$\begin{pmatrix}J^z_\epsilon\\ J^z\end{pmatrix}_{\text{anom}} = -B_5(x_\perp)u(x_\perp)\Sigma\cdot\begin{pmatrix}\nabla^z(\frac{1}{T})\\ \frac{E}{T}-\nabla^z(\frac{\mu}{T})\end{pmatrix}, \tag{31}$$

with the conductivity matrix

$$\Sigma = \begin{pmatrix}2c_g T_0 & c\mu_0\\ 0 & c\end{pmatrix}\cdot(L\cdot Y)^{-1}\cdot\begin{pmatrix}-2c_g T_0 & c\mu_0\\ 0 & c\end{pmatrix}\cdot\begin{pmatrix}T_0^2 & 0\\ -T_0\mu & -T_0\end{pmatrix} \tag{32}$$

and the solution to the Poisson equation $\Delta_\perp u(x_\perp) = B_5(x_\perp)$, i.e.

$$u(x_\perp) = \int dx'_\perp G(x_\perp - x'_\perp)B_5(x'_\perp)\,. \tag{33}$$

We have written the induced conductivity matrix as acting on the naturally defined thermodynamic forces. This has the advantage that the Onsager reciprocity relations are automatically satisfied, i.e. Σ is symmetric,

$$\Sigma_{11} = \frac{1}{\det(L)}T^2\Big((L_{22}(2c_gT^2 + c\mu^2)^2 + c^2\mu^2 L_{11}) - 2c\mu(2c_gT^2 + c\mu^2)L_{12}\Big)\,, \tag{34}$$

$$\Sigma_{22} = \frac{1}{\det(L)}c^2T^2\Big(L_{11} - 2\mu L_{12} + \mu^2 L_{22}\Big)\,, \tag{35}$$

$$\Sigma_{12} = \Sigma_{21} = \frac{1}{\det(L)}cT^2\Big(2c_gT^2(\mu L_{22} - L_{12}) + c\mu(L_{11} - 2\mu L_{12} + \mu^2 L_{22})\Big)\,. \tag{36}$$

Using $a+b\geq 2\sqrt{ab}$ for $a,b\geq 0$ and the fact $L_{11}\geq 0$, $L_{22}\geq 0$, $\det(L)\geq 0$ one shows that Σ_{11} and Σ_{22} are positive. Furthermore the total current is proportional to the expression

$$-\int d^2x_\perp d^2x'_\perp B_5(x_\perp)u(x'_\perp) = \int \frac{d^2q}{(2\pi)^2}\frac{\tilde{B}_5(-q)\tilde{B}_5(q)}{q^2} \tag{37}$$

which for a real function $B_5(x_\perp)$ is positive definite. Thus the response matrix in the net current described by (32) has the same properties as L since its determinant is also positive

$$\det(\Sigma) = \frac{1}{\det(L)}4c^2c_g^2T^8\,. \tag{38}$$

The electric and thermoelectric conductivity is defined as $\vec{J} = \sigma\vec{E} - \alpha\vec{\nabla}T$. The thermoelectric conductivity α is non-vanishing only because of the contribution of the mixed axial-gravitational anomaly,

$$\sigma = \sigma_0 - uB_5\frac{\Sigma_{22}}{T}\,, \tag{39}$$

$$\alpha = \alpha_0\Big(1 + uB_5\frac{1}{\det(L)}2cc_gT^4\Big)\,, \tag{40}$$

with $\sigma_0 = L_{22}/T$ and $\alpha_0 = (L_{12} - \mu L_{22})/T^2$. Measuring therefore the total current induced by a temperature gradient in the background of an axial magnetic field is an experimental signature of the mixed axial-gravitational anomaly. Note however that the electric conductivity is enhanced whereas the thermoelectric conductivity gets diminished. This is in contrast to the anomaly induced thermoelectric conductivity in a usual magnetic field [40–43].

2.2. Example

Finally we would like to discuss a simple example demonstrating the finiteness of the total induced current. We assume the periodic axial magnetic field configuration of the form

$$\vec{B}_5 = \hat{e}_z\bar{B}_5\sin(2\pi x/L)\,. \tag{41}$$

Integrating over a period the net flux vanishes. The solution to the Poisson equation is now

$$u = -\bar{B}_5\frac{L^2}{4\pi^2}\sin(2\pi x/L)\,. \tag{42}$$

As boundary conditions we have imposed that no chemical potential is induced over one period. Now the net current density over one period of oscillation is proportional to

$$-\frac{1}{L}\int_0^L dxu(x)B_{5,z}(x) = (\bar{B}_5)^2\frac{L^2}{8\pi^2}\,. \tag{43}$$

The gradient in this field configuration is proportional to the inverse of the period L. The current density is therefore inversely proportional to the square of the field gradient as expected and diverges in the limit of homogeneous field $L \to \infty$. In this limit diffusion is not effective and since there is no relaxation of the exactly conserved electric charge we end up again with infinite conductivities.

3. Discussion

We have developed a simple model of coupled energy and charge transport for chiral fermions in the background of axial magnetic fields. Our study was motivated by considerations based on compatibility with the Bloch theorem that forbids net currents in thermal equilibrium. In order to circumvent this we assumed axial magnetic field configurations with vanishing net flux such that in equilibrium the integrated total current vanishes. The anomalous transport model was constructed demanding a positive definite entropy production. Even without assuming full hydrodynamics, i.e. assuming that no significant collective flow can develop we found that anomalies induce chiral magnetic charge and energy currents. The form of the chiral magnetic transport coefficients contain a priori undetermined integration constant depending on the temperature which can be related to the presence mixed gauge perturbative and global gravitational anomalies. As previously observed in [21,32] the entropy current contains somewhat unexpectedly an anomalous term. We gave a physical interpretation relating it to a generalized form of the thermal Hall effect, i.e. the generation of a heat current perpendicular to a temperature gradient.

Then we studied electro- and thermo-magneto conductivities. We found that the assumption of vanishing net axial magnetic flux activates the diffusion terms in the constitutive relations leading to finite induced conductivities. Despite the fact there is no net magnetic flux a net electric current is induced either by an external electric field or by a temperature gradient. The resulting net axial magneto conductivity is enhanced whereas the axial thermo-magneto conductivity is diminished and is proportional to the coefficient of the mixed axial-gravitational anomaly.

The focus in the previous literature is considers the effects due to anomalies in the presence of background magnetic fields. Anomaly related enhancement of electric and thermoelectric conductivities in magnetic fields have indeed been observed in [38, 43]. In contrast in this work we have concentrated on the observable effects in the presence of background axial magnetic fields. Our study differs in two important points from previous ones [11, 13] in that we take the Bloch theorem into account and also study the thermo-electric conductivity. We hope that the effects can be measured in the future and will enrich our current understanding of the role of chiral anomaly.

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