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The Laurent-Horner method for validated evaluation of Chebyshev expansions

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Abstract

We develop a simple two-step algorithm for enclosing Chebyshev expansions whose cost is linear in terms of the polynomial degree. The algorithm first transforms the expansion from Chebyshev to the Laurent basis and then applies the interval Horner method. It outperforms the existing eigenvalue-based methods if the degree is high or the evaluation point is close to the boundaries of the domain.

Keywords: Chebyshev expansions, Joukowski map, interval arithmetic
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1. Introduction

Let

$$p(x) = \sum_{k=0}^{n} c_k T_k(x) \tag{1}$$

be a finite Chebyshev expansion in which

$$T_k(x) = \cos(k \cos^{-1}(x))$$

is the k-th Chebyshev polynomial of the first kind defined on $[-1, 1]$, and c_k is the k-th constant real coefficient. Recently, several techniques are developed in [1] for validated evaluation of p. We refer the reader to [2, 3, 4] for the basics of interval arithmetic, in particular the enclosure property, dependency issue, the wrapping effect, and how directed roundings are employed with floating point arithmetic to efficiently compute results which are guaranteed to be mathematically correct. Enclosing Chebyshev expansions has application in computer-assisted existence

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proofs of the spherical t-designs [5], ultra arithmetic, Chebyshev models [6] and automatic a posteriori forward error analysis of floating point evaluation of Chebyshev expansions; see [1] and references therein.

In this paper we develop a new algorithm which outperforms the eigenvalue-based methods explored in [1] if the degree n is high or the evaluation points x are close to the boundaries of the domain $[-1,1]$. The new algorithm has two main steps: We first employ the inverse Joukoswki map to convert the problem from the Chebyshev basis into that of Laurent. Then, we apply the interval Horner method to enclose the polynomial in the new basis. Like most of the techniques investigated in [1], the number of basic arithmetic operations involved in the new algorithm is $\mathcal{O}(n)$.

The most well-known algorithm for evaluation of the polynomial p in *floating point arithmetic* is the Clenshaw recurrence [7] which defines the quantities b_k as:

$$\begin{cases} b_{n+2} = b_{n+1} := 0, \\ b_k := 2xb_{k+1} - b_{k+2} + c_k, \quad k = n, n-1, \ldots, 0, \end{cases} \tag{2}$$

so that $p(x) = b_0 - b_1 x$. On the other hand, an important category of interval arithmetic techniques for enclosing Chebyshev expansions include the parallelepiped and Lohner's QR decomposition methods and two eigenvalue-based algorithms of [1]. The basic idea behind all of these techniques is to reformulate the Clenshaw recurrence (2) in terms of the following discrete dynamical system

$$\hat{b}_k = M\hat{b}_{k+1} + \hat{c}_k \tag{3}$$

with $M \in \mathbb{R}^{2\times 2}$ and $\hat{b}_k, \hat{b}_{k+1}, \hat{c}_k \in \mathbb{R}^2$ where

$$\underbrace{\begin{pmatrix} b_k \\ b_{k+1} \end{pmatrix}}_{\hat{b}_k} = \underbrace{\begin{pmatrix} 2x & -1 \\ 1 & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} b_{k+1} \\ b_{k+2} \end{pmatrix}}_{\hat{b}_{k+1}} + \underbrace{\begin{pmatrix} c_k \\ 0 \end{pmatrix}}_{\hat{c}_k}, \; k = n, \ldots, 1, 0.$$

Unfortunately, the matrix-vector multiplications in (3), when performed in interval arithmetic, cause a severe amount of overestimation called the wrapping effect [8]. To alleviate these overestimations, the eigenvalue-based methods of [1] employ the spectral transformation $M = VDV^{-1}$ where

$$V = \begin{pmatrix} x + i\sqrt{1-x^2} & x - i\sqrt{1-x^2} \\ 1 & 1 \end{pmatrix}, \quad V^{-1} = \tfrac{-i}{2\sqrt{1-x^2}} \begin{pmatrix} 1 & i\sqrt{1-x^2} - x \\ -1 & i\sqrt{1-x^2} + x \end{pmatrix},$$

$$D = \begin{pmatrix} x + i\sqrt{1-x^2} & 0 \\ 0 & x - i\sqrt{1-x^2} \end{pmatrix}. \tag{4}$$

Then (3) is equivalent to the transformed iteration

$$\check{b}_k = D\check{b}_{k+1} + \check{c}_k, \tag{5}$$

where $\check{b}_k := V^{-1}\hat{b}_k$, and $\check{c}_k := V^{-1}\hat{c}_k$. In practice, interval matrices $\boldsymbol{D} \ni D$, $\boldsymbol{V} \ni V$, and $\boldsymbol{I}_V \ni V^{-1}$ are used so that rounding errors in the computation of

V, D and V^{-1} are taken care of. Note that the amount of overestimation, when performing iterations like (5) and (3) in interval arithmetic, is governed by the spectral radius of the absolute value of the iteration matrix. As discussed in [1], the reason the transformed iteration (5) outperforms the original iteration (3) in interval arithmetic is that $\rho(|D|) = 1$ for every $x \in [-1, 1]$ whereas $1 \le \rho(|M|) \le 1 + \sqrt{2}$; see [1, Fig. 3]. On the other hand, it is proved that [1]

$$\kappa_2(V) := \|V^{-1}\|\|V\| = \begin{cases} \sqrt{\frac{1-x}{1+x}}, & -1 \le x \le 0, \\ \sqrt{\frac{1+x}{1-x}}, & 0 \le x \le 1. \end{cases}$$

Therefore, the eigenvector matrix V is numerically ill-conditioned for $x \approx \pm 1$. This makes entries of the interval matrix $\boldsymbol{I}_V$ wide; see e.g., [9] and [10, p. 346]. Consequently, the computed enclosure for $p(x)$ becomes wide. Hence, as discussed in [1], one cannot expect the transformed iteration (5) to give very narrow enclosures at $x \approx \pm 1$. So, here is the question: can we develop a transformation of (3) that takes advantage of the fact that $\rho(|D|) = 1$ while avoiding multiplications by the eigenvector matrix V and its inverse? It turns out that the following method is what we are looking for.

2. The Laurent-Horner method

The Joukowski map $x := J(z) = \frac{1}{2}(z + \frac{1}{z})$ is a popular conformal map in approximation theory and complex analysis. It transforms origin-centered circles to ellipses with foci at $\{-1, 1\}$ which are known as *Bernstein ellipses*. In particular, it maps the unit circle to the unit interval $[-1, 1]$. Since $J(z) = J(z^{-1})$, there is a 2-to-1 correspondence between z on the unit circle and x on the real interval $[-1, 1]$. In other words, the quadratic equation corresponding to the map has two solutions $z = x \pm i\sqrt{1 - x^2}$. Notice that these are the eigenvalues of M in (3) as can be observed from the spectral decomposition (4).

Another basic fact that is important for the development of the new method is that the k-th Chebyshev polynomial is the real part of the function z^k on the unit circle [11], i.e.,

$$T_k(x) = \frac{z^k + z^{-k}}{2} = \frac{z^k + \bar{z}^k}{2},$$

where $\bar{z}$ denotes the complex conjugate. Hence, the Chebyshev expansion (1) can be converted to the following *Laurent* polynomial

$$p(x) = \frac{1}{2} \sum_{k=0}^{n} c_k (z^k + z^{-k}) = \text{ real } \Big(\sum_{k=0}^{n} c_k z^k \Big). \tag{6}$$

To take care of rounding errors in the conversion from x to z, we compute an interval $\boldsymbol{z}$ containing the exact value of z.

The second step of our enclosure method simply applies the Horner's rule in interval arithmetic to (6); see e.g., [12]. The interval Horner method is a straightforward extension of the standard Horner's nested multiplication form to interval arithmetic and can be used to enclose the range of polynomials with a linear complexity in terms of its degree.

While *both* the existing eigenvalue-based methods and the Laurent-Horner method inherit possible ill-conditioning of the transformation from x to z, computations in the Laurent-Horner method do not involve the matrices V and V^{-1}. Therefore, the new method might be considered as an eigenvector-avoiding "spectral" transformation of (3) and can be expected to outperform the eigenvalue-based methods of [1] especially at $x \approx \pm 1$.

3. Numerical experiments

To compare the new algorithm with the older ones, we illustrate the time needed (in seconds) together with $\operatorname{rad} \boldsymbol{p}(x)$; the radius of the computed enclosures. The radii are employed also for obtaining the average number of correct digits for enclosures over all points x computed as $\operatorname{mean}(-\log_{10}(\operatorname{rad} \boldsymbol{p}(x)))$. Our numerical results are generated using INTLAB [13].

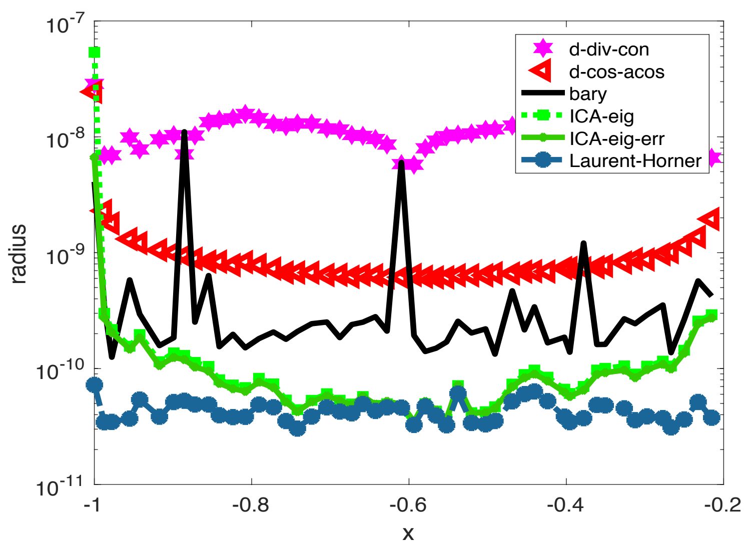

Figure 1: Radius of enclosures for a degree 9150 interval polynomial at 1000 intervals. Only 50 points are depicted to make the curves easier to distinguish.

We consider a Chebyshev expansion of degree 9150 corresponding to the random smooth function `randfun(0.0007)` [14] where its real coefficients are inflated to be intervals of a radius of about 2×10^{-15}. The same Chebyshev expansion is used in Example 6.6 of [1]. We compute enclosures for the value of the interval polynomial at 1000 random intervals $\boldsymbol{x}$ whose radii are again of the order of 10^{-15}. Figures 1 and 2 contain our results for six methods. Here, `ICA-eig` and `ICA-eig-err` denote the two eigenvalue-based methods of [1] which rely on (4) and (5). Also, `d-cos-acos` and `d-div-con` denote two

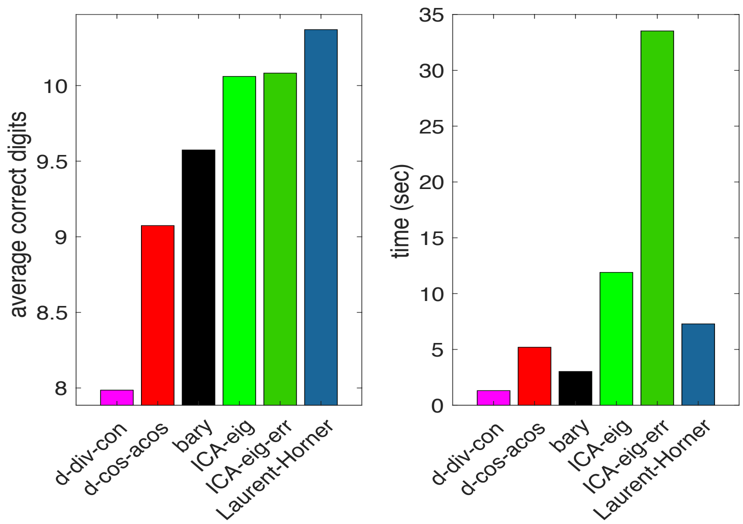

Figure 2: Average number of correct digits (left) and computing time (right) of different methods for bounding the range of a Chebyshev expansion of degree 9150 at 1000 points.

direct methods which typically give narrow enclosures for low-degree Chebyshev expansions. Moreover, `bary` refers to the extension of barycentric formula to interval arithmetic.

It was shown in [1] that in the case of high-degree Chebyshev expansions, `ICA-eig` and `ICA-eig-err` typically give narrowest enclosures among the vectorized techniques. Nevertheless, we observe that in this example the Laurent-Horner method not only outperforms those techniques with respect to speed, but also computes narrowest enclosures among *all* the techniques.

In a second variant of the above experiment, narrowest-possible enclosures for c_k and x are used instead of those with a width of order 10^{-15}. This time, the most accurate methods *on average* are `ICA-eig-err` and Laurent-Horner, while `ICA-eig-err` is three times slower than the Laurent-Horner.

Let us end this paper with a note concerning the conversion from real to complex interval arithmetic in the new method as well as in `ICA-eig` and `ICA-eig-err`. The conversion has a speed penalty observed for the three slower methods in Figure 2 (right). However, moving to the complex plane, while avoiding wrappings caused by the eignevector transformations are the main reasons the new method gives the narrowest enclosures. Avoiding these multiplications also makes the new method the fastest among those that use complex arithmetic. Note also that INTLAB employs midpoint-radius representation in its implementation of complex as well as real machine interval arithmetic [15].

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