FINITE MORPHISMS AND SIMULTANEOUS REDUCTION OF THE MULTIPLICITY

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ABSTRACT. Let X be a singular algebraic variety defined over a perfect field k, with quotient field $K(X)$. Let $s \geq 2$ be the highest multiplicity of X and $F_s(X)$ the set of points of multiplicity s. If $Y \subset F_s(X)$ is a regular center and $X \leftarrow X_1$ is the blow up at Y, then the highest multiplicity of X_1 is less than or equal to s. A sequence of blow ups at regular centers $Y_i \subset F_s(X_i)$, say $X \leftarrow X_1 \leftarrow \cdots \leftarrow X_n$, is said to be a *simplification* of the multiplicity if the maximum multiplicity of X_n is strictly lower than that of X, that is, if $F_s(X_n)$ is empty. In characteristic zero there is an algorithm which assigns to each X a unique simplification of the multiplicity. However, the problem remains open when the characteristic is positive.

In this paper we will study finite dominant morphisms between singular varieties $\beta : X' \to X$ of generic rank $r \geq 1$ (i.e., $[K(X') : K(X)] = r$). We will see that, when imposing suitable conditions on β, there is a strong link between the strata of maximum multiplicity of X and X', say $F_s(X)$ and $F_{rs}(X')$ respectively. In such case, we will say that the morphism is strongly transversal. When $\beta : X' \to X$ is strongly transversal one can obtain information about the simplification of the multiplicity of X from that of X' and vice versa. Finally, we will see that given a singular variety X and a finite field extension L of $K(X)$ of rank $r \geq 1$, one can construct (at least locally, in étale topology) a strongly transversal morphism $\beta : X' \to X$, where X' has quotient field L.

CONTENTS

1. INTRODUCTION

Let X be an algebraic variety defined over a perfect field k. Along these notes, all varieties will be assumed to be reduced and irreducible. The multiplicity along points of X defines an upper semi-continuous function, say mult : $X \to \mathbb{N}$, which assigns to each point $\xi \in X$ the multiplicity of the local ring $\mathcal{O}_{X,\xi}$ with respect to its maximal ideal. Under the previous hypotheses, the multiplicity function along points of X has the following property: X is regular at a point ξ if and only if mult$(\xi) = 1$. Thus, one can characterize the set of singular points of X by

$$\text{Sing}(X) = \{\xi \in X \mid \text{mult}(\xi) \geq 2\}.$$

Assume that X is singular, i.e., that max mult$(X) \geq 2$. Set $s = \max \text{mult}(X)$ and let $F_s(X)$ denote the set of points of maximum multiplicity of X. It can be proved that the multiplicity does not increase when blowing up along regular centers contained in $F_s(X)$ (for a proof of this fact see [27]; there it is mentioned that the statement was first proved by Dade in [10]). Namely, given a closed regular center $Y \subset F_s(X)$ and the blow up of X along Y, say $\pi_1 : X_1 = \text{Bl}_Y(X) \to X$, one has that

$$\text{mult}(\xi_1) \leq \text{mult}(\pi_1(\xi_1))$$

for every $\xi_1 \in X_1$. Thus max mult$(X_1) \leq$ max mult(X). A blow up of this form will be called a F_s-permissible blow up. Attending to Dade's result, it is natural ask whether there exists a sequence of F_s-permissible blow ups, say

$$X = X_0 \xleftarrow{\pi_1} X_1 \xleftarrow{\pi_2} \ldots \xleftarrow{\pi_l} X_l,$$

such that

$$\max \text{mult}(X_0) = \cdots = \max \text{mult}(X_{l-1}) > \max \text{mult}(X_l).$$

A sequence with this property, whenever it exists, will be called a *simplification of $F_s(X)$*. Note that, if one could find a simplification of $F_s(X)$ for every $s \geq 2$ and for every X with max mult$(X) = s$, then one could also find a resolution of singularities of X by iterating this process (the resolution would be achieved when the maximum multiplicity of the scheme, after a finite number of blow ups, lowers to 1). This approach differs from Hironaka's proof of resolution of singularities, as he uses the Hilbert-Samuel function instead of the multiplicity. We refer here to [7] where the semicontinuity of this invariant along points of a singular scheme is studied in the first chapter (Theorem 1.34), and the behavior under blowups is carefully treated in the second chapter.

The existence of a simplification of $F_s(X)$ has already been proved for the case in which X is a singular algebraic variety defined over a field k of characteristic zero (cf. [34]). However, the problem remains open in the case of positive characteristic (some references about recent progress in positive characteristic are [7], [8], [9], [18], [19]).

In this work we will study finite morphisms between singular algebraic varieties, $\beta : X' \to X$. We will show that, under suitable conditions, there is a link between the stratum of maximum multiplicity of X, say $F_s(X)$, and that of X', say $F_{s'}(X')$. In this setting we present a notion of *blow up of a finite morphism* so that if $Y \subset F_{s'}(X')$ is regular, then $\beta(Y) \subset F_s(X)$ is regular too, and there is a commutative diagram of blow ups and finite morphisms:

$$\begin{array}{ccc} X' & \longleftarrow & X'_1 \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta_1} \\ X & \longleftarrow & X_1. \end{array}$$

Notice that this parallels the situation in which X is embedded in some scheme V and $Y \subset X$ is a center; then after the blow up at Y, there is again an embedding of the strict transform of X in the transform of V, say $X_1 \subset V_1$ and a commutative diagram of blow ups and immersions.

Now the idea is that, given a singular algebraic variety X with maximum multiplicity $s \geq 2$, one can construct a suitable algebraic variety X' with maximum multiplicity $s' \geq 2$, endowed with

a suitable finite morphism $\beta : X' \to X$, in such way that one can obtain information about the process of simplification of $F_s(X)$ from that of $F_{s'}(X')$. These ideas are developed in more detail in the following lines.

Transversal morphisms. Let $\beta : X' \to X$ be a finite and dominant morphism of singular algebraic varieties over a perfect field k. Set $s = \max \text{mult}(X)$ and let $r = [K(X') : K(X)]$ denote the generic rank of the morphism β. Using Zariski's multiplicity formula for finite projections (see Theorem 2.2) one can prove that the maximum multiplicity on X' is bounded by rs. Moreover, if $\max \text{mult}(X') = rs$, one can show that the image of $F_{rs}(X')$ sits inside $F_s(X)$ and that $F_{rs}(X')$ is mapped homeomorphically to its image in $F_s(X)$ (see section 2). When the previous equality holds, i.e., when $\max \text{mult}(X') = rs$, we will say that the morphism $\beta : X' \to X$ is *transversal*. Firstly we study the stability of transversality after blowing up at an $F_{sr}(X')$-permissible center:

Theorem 4.4. *Let X be an algebraic variety with maximum multiplicity s and let $\beta : X' \to X$ be a transversal morphism of generic rank r. Then:*

i) *An F_{rs}-permissible center on X', say $Y' \subset F_{rs}(X')$, induces an F_s-permissible center on X, say $Y = \beta(Y') \subset F_s(X)$, and a commutative diagram of blow ups of X at Y, say $X \leftarrow X_1$, and of X' at Y', say $X' \leftarrow X_1'$, as follows,*

$$
\begin{array}{ccc}
X' & \longleftarrow & X_1' \\
\downarrow{\beta} & & \downarrow{\beta_1} \\
X & \longleftarrow & X_1,
\end{array}
$$

where β_1 is finite of generic rank r. In addition, if $F_{rs}(X_1') \neq \emptyset$, then $F_s(X_1) \neq \emptyset$, and the morphism β_1 is transversal.

ii) *Any sequence of F_{rs}-permissible blow ups on X', $X' \leftarrow X_1' \leftarrow \cdots \leftarrow X_{N-1}' \leftarrow X_N'$, induces a sequence of F_s-permissible blow ups on X, and a commutative diagram as follows,*

$$
\begin{array}{ccccccccc}
X' & \longleftarrow & X_1' & \longleftarrow & \cdots & \longleftarrow & X_{N-1}' & \longleftarrow & X_N' \\
\downarrow{\beta} & & \downarrow{\beta_1} & & & & \downarrow{\beta_{N-1}} & & \downarrow{\beta_N} \\
X & \longleftarrow & X_1 & \longleftarrow & \cdots & \longleftarrow & X_{N-1} & \longleftarrow & X_N,
\end{array}
$$

where each β_i is finite of generic rank r. Moreover, if $F_{rs}(X_N') \neq \emptyset$, then $F_s(X_N) \neq \emptyset$, and the morphism β_N is transversal.

Next we will study conditions under which, given a transversal morphism $\beta : X' \to X$, the set $F_{rs}(X')$ is mapped surjectively onto $F_s(X)$, in such a way that $F_{rs}(X')$ and $F_s(X)$ are homeomorphic. In addition, we will require this condition to be preserved by sequences of F_s-permissible blow ups. Namely, we will require that any sequence of F_s-permissible blow ups over X, say

$$
X = X_0 \xleftarrow{\ \pi_1\ } X_1 \xleftarrow{\ \pi_2\ } \ldots \xleftarrow{\ \pi_l\ } X_l
$$

such that $\max \text{mult}(X_i) = \max \text{mult}(X_0)$ for $i < l$, induces a sequence of F_{rs}-permissible blow ups over X' and a commutative

$$
\begin{array}{ccccccccc}
X' & \longleftarrow & X_1' & \longleftarrow & \cdots & \longleftarrow & X_{l-1}' & \longleftarrow & X_l' \\
\downarrow{\beta} & & \downarrow{\beta_1} & & & & \downarrow{\beta_{l-1}} & & \downarrow{\beta_l} \\
X & \longleftarrow & X_1 & \longleftarrow & \cdots & \longleftarrow & X_{l-1} & \longleftarrow & X_l,
\end{array}
$$

with the following properties:

i) $\beta_i : X_i' \to X_i$ is transversal for $i \leq l - 1$;

ii) $F_{rs}(X_i')$ is homeomorphic to $F_s(X_i)$ for $i \leq l - 1$; and

iii) $\max \text{mult}(X_l') < rs$ if and only if $\max \text{mult}(X_i) < s$.

When a transversal morphism $\beta : X' \to X$ satisfies these properties for every sequence of F_s-permissible blow ups over X, we shall say that β is *strongly transversal* (or that $F_{sr}(X')$ and $F_s(X)$ are *strongly homeomorphic*). In section 6 we will show that, if two algebraic varieties X and X' are linked by a strongly transversal morphism, say $\beta : X' \to X$, then, when the characteristic is zero, the processes of simplification of $F_s(X)$ and $F_{rs}(X')$ are equivalent, in the sense that any simplification of $F_s(X)$ induces a simplification of $F_{rs}(X')$ and vice-versa (see also Remark 7.9).

Characterization of strong transversality. Let X be a singular algebraic variety defined over a perfect field k with maximum multiplicity $s \geq 2$. There is a graded $\mathcal{O}_X$-algebra that one can naturally attach to the stratum of maximum multiplicity of X, which we shall denote by $\mathcal{G}_X$. In characteristic zero, this algebra encodes information about the simplification process of $F_s(X)$ (see Remark 7.9). Moreover, given a transversal morphism $\beta : X' \to X$, there is a natural inclusion of $\mathcal{G}_X$ into $\mathcal{G}_{X'}$ (see [1, Proposition 6.3]). In section 7 we will show that strong transversality can be characterized in terms of the algebras $\mathcal{G}_X$ and $\mathcal{G}_{X'}$. More precisely, the following theorem is proven:

Theorem 7.2. *Let $\beta : X' \to X$ be a transversal morphism of generic rank r between two singular algebraic varieties defined over a perfect field k. Then:*

(1) *If $\beta : X' \to X$ is strongly transversal then the inclusion $\mathcal{G}_X \subset \mathcal{G}_{X'}$ is finite;*
(2) *If k is a field of characteristic zero, then the converse holds. Namely, if the inclusion $\mathcal{G}_X \subset \mathcal{G}_{X'}$ is finite, then $\beta : X' \to X$ is strongly transversal.*

Note that in the case of positive characteristic, the characterization is just partial, as strong transversality implies that $\mathcal{G}_{X'}$ is integral over $\mathcal{G}_X$ but the converse fails in general. In Example 7.5 we give a counterexample for the latter case.

Construction of strongly transversal morphisms. The last section is devoted to the construction of strongly transversal morphisms. Consider a singular algebraic variety X over a perfect field k with maximum multiplicity $s \geq 2$ and field of fractions K. Given a field extension $L \supset K$ of rank $r = [L : K]$, we would like to find a singular variety X' with field of fractions L endowed with a strongly transversal morphism $\beta : X' \to X$. Although we do not know whether it is possible to find such X' with a strongly transversal morphism $\beta : X' \to X$ in general, at least we can prove that it possible to do a similar construction locally in étale topology, as explained below.

Let $\xi \in F_s(X)$ be a fixed point in the stratum of maximum multiplicity of X. For an étale neighborhood of X at ξ, say $\widetilde{X} \to X$, let $\widetilde{K}$ denote the field of fractions of $\widetilde{X}$, and set $\widetilde{L} = L \otimes_K \widetilde{K}$ (as it will be explained, when X normal, $\widetilde{X}$ can be assumed to be irreducible). Note that, in this setting, $\widetilde{L}$ might not be a field. However, as by base change it is étale over L, it is reduced and, as it is finite (of rank r) over $\widetilde{K}$, it should be a direct sum of fields over $\widetilde{K}$. Then we show:

Theorem 8.2. *Let X be a singular algebraic variety over a perfect field k with maximum multiplicity $s \geq 2$. Suppose that X is normal with field of fractions K and let L be a finite field extension of K of rank r. Then, for every point of $\xi \in F_s(X)$, there exist an étale neighborhood of X at ξ, say $\widetilde{X} \to X$, together with a scheme $\widetilde{X}'$ and a strongly transversal morphism $\beta : \widetilde{X}' \to \widetilde{X}$ of generic rank r such that $\widetilde{L} = L \otimes_K \widetilde{K}$, where $\widetilde{K}$ and $\widetilde{L}$ denote the total quotient rings of $\widetilde{X}$ and $\widetilde{X}'$ respectively.*

To conclude, we mention that, for technical reasons, to prove this last result we will assume that X is normal. Actually, the construction can also be done without this assumption: in the general case, however, some details are rather technical while the main ideas of the construction remain the same.

On the organization of the paper. The paper is organized in three parts. In Part I we study transversal morphisms. Section 2 is devoted to presenting the notion of transversal morphisms;

properties of such morphisms are studied in Proposition 2.7 and Corollary 2.8. The main purpose of section 3 is to give a local presentation of transversal morphisms. This is the content of Lemma 3.6. In Section 4 we study the compatibility of transversalitiy with blow ups at regular equimultiple centers (the *blow up of a finite morphism*). The main result is Theorem 4.4, which leads us naturally to the notion of strongly transversal morphism (see Definition 4.8).

Part II is entirely devoted to present the basic results on Rees algebras that we need to state and proof Theorems 7.2 and 8.2. Rees algebras will allow us to work with equations with weights: this turns out to be a useful language to describe the maximum multiplicity locus of an algebraic variety defined over a perfect field k.

To conclude, strongly transversal morphisms are studied in Part III. Here Theorems 7.2 and 8.2 are proven.

We thank A. Benito, D. Sulca, and S. Encinas for stimulating discussions on this subject.

Part I. Transversality

2. Transversality and finite morphisms

Let (R, M) be a local Noetherian ring, and let $\mathfrak{a} \subset R$ be an M-primary ideal. Observe that, for $n > 0$, the quotient ring $R/\mathfrak{a}^n$ has finite length when regarded as an R-module. Let us denote this length by $\lambda(R/\mathfrak{a}^n)$. It can be shown that, for $n \gg 0$, the value of $\lambda(R/\mathfrak{a}^n)$ is given by a polynomial in n with rational coefficients, say $\lambda(R/\mathfrak{a}^n) = c_d \cdot n^d + c_{d-1} \cdot n^{d-1} + \cdots + c_0 \in \mathbb{Q}[n]$, where $d = \dim(R)$, which is known as the Hilbert-Samuel polynomial of R with respect to $\mathfrak{a}$. In addition, it can be shown that $c_d = \frac{e}{d!}$ for some $e \in \mathbb{N}$. The integer e is called the *multiplicity* of R with respect to $\mathfrak{a}$, and we shall denote it by $e_{\mathfrak{a}}(R)$. The *multiplicity* of a Noetherian scheme X at a point $\xi \in X$ is defined as that of the local ring $\mathcal{O}_{X,\xi}$ with respect to its maximal ideal.

2.1. General setting and notation. Let X and X' be algebraic varieties over a perfect field k with quotient fields $K = K(X)$ and $L = K(X')$ respectively. Suppose that $\beta : X' \to X$ is a finite dominant morphism of *generic rank* $r := [K(X') : K(X)]$. We will be assuming that the maximum multiplicity of X is $s \geq 1$, and will denote by $F_s(X)$ the (closed) set of points of maximum multiplicity s of X. As we will see, under these conditions, the maximum multiplicity at points of X' is at most rs, and we denote by $F_{sr}(X')$ the closed set (possibly empty) of points of multiplicity sr of X'. Since our arguments will be of local nature, in several parts of this paper we may assume that both, X and X' are affine, say $X = \operatorname{Spec}(B)$ and $X' = \operatorname{Spec}(B')$, and that there is a finite extension $B \subset B'$. In this case we will use $F_s(B)$ (resp. $F_{sr}(B')$) to denote $F_s(X)$ (resp. $F_{sr}(X')$). Moreover, we will see that the previous discussion also applies to the case in which X' is an equidimensional scheme of finite type over k with total ring of fractions $K(X')$.

Theorem 2.2 (Zariski's multiplicity formula for finite projections [36, Theorem 24, p. 297])**.** *Let* $(B, \mathfrak{m})$ *be a local domain, and let* B' *be a finite extension of* B. *Let* K *denote the quotient field of* B, *and let* $L = K \otimes_B B'$. *Let* $M_1, \ldots, M_r$ *denote the maximal ideals of the semi-local ring* B', *and assume that* $\dim B'_{M_i} = \dim B$, $i = 1, \ldots, r$. *Then*

$$e_B(\mathfrak{m})[L : K] = \sum_{1 \leq i \leq r} e_{B'_{M_i}}(\mathfrak{m}B'_{M_i})[k_i : k],$$

where k_i *is the residue field of* B'_{M_i}, k *is the residue field of* $(B, \mathfrak{m})$, *and* $[L : K] = \dim_K L$.

As a consequence of Zariski's formula, if P is a prime ideal in B' and $\mathfrak{p} = P \cap B$, one has that

$$e_{B'_P}(PB'_P) \leq e_{B'_P}(\mathfrak{p}B'_P) \leq e_{B_\mathfrak{p}}(\mathfrak{p}B_\mathfrak{p})[L : K]. \tag{2.2.1}$$

That is, the maximum of the multiplicity at points of $\operatorname{Spec}(B') = X'$ is bounded by the generic rank of the projection times the maximum of the multiplicity at points of $\operatorname{Spec}(B) = X$. Namely,

$$\max \operatorname{mult}(X') \leq [L : K] \cdot \max \operatorname{mult}(X). \tag{2.2.2}$$

2.3. The (*) condition. With the same notation and hypotheses as in Theorem 2.2 we will say that *condition (*) holds at* $P \in Spec(B')$ if:

$$(*) \ e_{B'_P}(PB'_P) = e_{B_{\mathfrak{p}}}(\mathfrak{p}B_{\mathfrak{p}})[L : K].$$

Hence, condition (*) is satisfied at P if and only the following three conditions hold simultaneously:

(i) P is the only prime in B' dominating $\mathfrak{p}$ (i.e., $B'_P = B' \otimes_B B_{\mathfrak{p}}$);
(ii) $B_{\mathfrak{p}}/\mathfrak{p}B_{\mathfrak{p}} = B'_P/PB'_P$;
(iii) $e_{B'_P}(\mathfrak{p}B'_P) = e_{B'_P}(PB'_P)$.

In particular, condition (*) necessarily holds for all primes $P \subset B'$ where the multiplicity is sr, where s is the maximum multiplicity in $\mathrm{Spec}(B)$, and $r = [L : K]$.

Remark 2.4. Suppose that B and B' are formally equidimensional locally at any prime. Then, as we will see in Section 3, condition (iii) is equivalent to saying that $\mathfrak{p}B'_P$ is a reduction of PB'_P, i.e., that the ideal PB'_P is integral over $\mathfrak{p}B'_P$ (cf. Theorem 3.2).

Definition 2.5. Let X be an algebraic variety over a perfect field k, and let X' be either an algebraic variety or a reduced equidimensional scheme of finite type over k. Let K be the field of rational functions of X and let L be the total ring of fractions of X'. We will say that a k-morphism $\beta : X' \to X$ is *transversal* if it is finite and dominant, and the following equality holds:

$$\max \mathrm{mult}(X') = [L : K] \cdot \max \mathrm{mult}(X).$$

We will say that an extension of k-algebras of finite type $B \subset B'$, where B is a domain, and B' is reduced and equidimensional, is *transversal* if the corresponding morphism of schemes, say $\beta : \mathrm{Spec}(B') \to \mathrm{Spec}(B)$, is so.

Remark 2.6. By definition, if $\beta : X' \to X$ is a transversal morphism of generic rank $r = [L : K]$ and X has maximum multiplicity $s \geq 1$, then X' has maximum multiplicity rs and the equality is attained in (2.2.1) for all points in $F_{rs}(X')$. Thus condition (*) holds at all points of $F_{rs}(X')$.

Proposition 2.7. *Let $\beta : X' \to X$ be a transversal morphism of generic rank r and let $s = \max \mathrm{mult}(X)$. Then:*

(1) $\beta(F_{sr}(X')) \subset F_s(X)$;
(2) $F_{sr}(X')$ is homeomorphic to $\beta(F_{sr}(X')) \subset F_s(X)$.

Proof. It suffices to argue locally, so we may assume that $X = \mathrm{Spec}(B)$ and $X' = \mathrm{Spec}(B')$. Now it follows from the discussion in 2.3 that the maximum multiplicity at points of $\mathrm{Spec}(B')$ is rs, and that condition (*) necessarily holds for all primes in $F_{sr}(B')$. Furthermore, if $Q \in F_{sr}(B')$ and $\mathfrak{q} = Q \cap B$ then necessarily $e_{B_{\mathfrak{q}}}(\mathfrak{q}B_{\mathfrak{q}}) = s$, so $\beta(F_{sr}(X')) \subset F_s(X)$. Therefore β induces an injective morphism, say $\beta : F_{sr}(X') \to F_s(X)$, which is proper and hence it induces an homeomorphism $\beta : F_{sr}(X') \to \beta(F_{sr}(X))(\subset F_s(X))$.

Corollary 2.8. *Suppose that the extension $B \subset B'$ is transversal of generic rank r. Let s be the maximum multiplicity at points of $\mathrm{Spec}(B)$, let $Q \in F_{sr}(B')$ and let $\mathfrak{q} := B \cap Q$; so Q and $\mathfrak{q}$ are in correspondance according to the homeomorphism of Proposition 2.7, in particular Q is the unique prime dominating $\mathfrak{q}$ (i.e., Q is the radical of $\mathfrak{p}B'$). Then:*

(1) The extension $B/\mathfrak{q} \to B'/Q$ is finite;
(2) The field of fractions of $B/\mathfrak{q}$, say $K(\mathfrak{q})$, equals the field of fractions of B'/Q, say $K(Q)$;
(3) If $P \subset B'$ is a prime ideal containing Q, $\mathfrak{p} := P \cap B$, $\overline{P} = P/Q$, and $\overline{\mathfrak{p}} = \mathfrak{p}/\mathfrak{q}$, then

$$e_{B/\mathfrak{q}}(\overline{\mathfrak{p}}) = e_{B/\mathfrak{q}}(\overline{\mathfrak{p}})[K(Q) : K(\mathfrak{q})] = e_{B'/Q}(\overline{P}).$$

In other words, the finite extension $B/\mathfrak{q} \to B'/Q$ is birational and the morphism $\mathrm{Spec}(B'/Q) \to \mathrm{Spec}(B/\mathfrak{q})$ is bijective. In addition, the multiplicity at corresponding points is the same. In particular, B'/Q is regular if and only if $B/\mathfrak{q}$ is regular, and in that case necessarily $B/\mathfrak{q} = B'/Q$ as regular rings are normal.

Proof. The statement in (1) is clear and (2) follows from condition (ii) of 2.3. As for (3) note here that $\mathrm{Spec}(B'/Q) \subset F_{sr}(B')$, and the conditions (i), (ii) and (iii) of 2.3, which hold for all P containing Q, are inherited by $\overline{P} = P/Q(\subset B'/Q)$.

2.9. Summarizing. With the same notation and conventions as 2.1, if $\beta : X' \to X$ is transversal, then from Proposition 2.7 and Corollary 2.8 it follows that:

(1) $\beta(F_{sr}(X')) \subset F_s(X)$;
(2) $F_{sr}(X')$ is homeomorphic to $\beta(F_{sr}(X'))$;
(3) If $Y \subset F_{sr}(X')$ is an irreducible regular closed subscheme, then $\beta(Y) \subset F_s(X)$ is an irreducible regular closed subscheme;
(4) If $Z \subset F_s(X)$ is an irreducible closed regular subscheme, and if $\beta^{-1}(Z)_{\mathrm{red}} \subset F_{sr}(X')$, then $\beta^{-1}(Z)_{\mathrm{red}}$ is regular.

3. Presentations of transversal morphisms

The notion of multiplicity is closely related to that of integral closure of ideals. Given ideals $I \subset J$ in a Noetherian ring B, there are several (equivalent) formulations for J to be integral over I, or say for I to be a reduction of J. Northcott and Rees introduced the notion of reduction. Given ideals $I \subset J$ as above, I is a reduction of J if $IJ^n = J^{n+1}$ for some $n \geq 0$ or, equivalently, if the inclusion $B[IW] \subset B[JW]$ is a finite extension of subrings in $B[W]$ (here W is an indeterminate). In this case, if $X \to \mathrm{Spec}(B)$ denotes the blow up at I and $X' \to \mathrm{Spec}(B)$ is the blow up at J, there is a factorization $X' \to X$ induced by this finite extension, which is also a finite morphism.

The notion of reduction of an ideal J in B will appear naturally when studying the fibers of the blow up, which is a notion that relates to that of the integral closure of J.

3.1. A first connection of integral closure with the notion of multiplicity is given by the following Theorem of Rees. A Noetherian local ring (A, M) is said to be formally equidimensional (quasi-unmixed in Nagata's terminology) if $\dim(\hat{A}/\hat{\mathfrak{p}}) = \dim(\hat{A})$ at each minimal prime ideal $\hat{\mathfrak{p}}$ in the completion $\hat{A}$.

Theorem 3.2 (Rees Theorem, [29])**.** *Let (A, M) be a formally equidimensional local ring, and let $I \subset J$ be two M-primary ideals. Then I is a reduction of J if and only if $e_A(I) = e_A(J)$.*

3.3. *A generalization of Rees Theorem.* Theorem 3.2 can be generalized to the case of non-necessarily M-primary ideals, see Theorem 3.4 below. Before stating the theorem, we will review some of the terms that appear among the hypotheses. Let I denote an ideal in a local ring (A, M). Let $f : Z \to \mathrm{Spec}(A)$ be the blow up at I, and let $f_0 : Z_0 \to \mathrm{Spec}(A/I)$ be the proper morphism induced by restriction, where Z_0 denotes the closed subscheme determined by $I\mathcal{O}_Z$ in Z. Northcott and Rees defined the *analytic spread* as:

$$l(I) = \dim(A/M \otimes_A \mathrm{Gr}_I(A)) = \delta + 1,$$

where $\mathrm{Gr}_I(A)$ denotes the graded ring $\oplus_{n=0}^{\infty}(I^n/I^{n+1})$ (i.e., $Z_0 = \mathrm{Proj}\,\mathrm{Gr}_I(A)$), and δ is the dimension of the fiber of f over the closed point of $\mathrm{Spec}(A)$, or equivalently, the dimension of the fiber of f_0 over the closed point. Note that $l(I) = \dim(A)$ when I is M-primary.

The height of I, say $ht(I)$, is $\min(\dim A_{\mathfrak{q}})$ as $\mathfrak{q}$ runs through all primes containing I, and we claim that $l(I) \geq ht(I)$, with equality if and only if all fibers of f_0 have the same dimension (see [21, §2]). The inequality holds because the dimension of the fibers of $f_0 : Z_0 \to \mathrm{Spec}(A/I)$ is an upper semi-continuous function on primes of A/I (this result is due to Chevalley, cf. [13, Theorem 13.1.3]). In addition, if $\mathfrak{q}$ is minimal containing I, the dimension of the fiber over $\mathfrak{q}$ is $\dim A_{\mathfrak{q}}$.

Finally, let $I \subset J$ be a reduction in a Noetherian ring B, and let $X \to \mathrm{Spec}(B)$ and $X' \to \mathrm{Spec}(B)$ denote the blow ups at I and J (respectively). Since there is a factorization $X' \to X$ which is finite, $l(IB_P) = l(JB_P)$ at any prime P in B.

Theorem 3.4 (Böger, cf. [21, Theorems 2 and 3]). *Let (A, M) be a formally equidimensional local ring. Fix an ideal $I \subset A$ so that $ht(I) = l(I)$. Consider another ideal $J \subset A$ so that $I \subset J \subset \sqrt{I}$. Then I is a reduction of J if and only if $e_{A_\mathfrak{q}}(IA_\mathfrak{q}) = e_{A_\mathfrak{q}}(JA_\mathfrak{q})$ for each minimal prime $\mathfrak{q}$ of I.*

Along these notes we will consider and study the blow ups along regular centers. The following theorem relates the notion of equimultiplicity with that of the fiber dimension at the blow up (analytic spread), studied by Dade, Hironaka, and Schickhoff.

Theorem 3.5 (Hironaka-Schickhoff, [21, Corollary 3, p. 121]). *Let (A, M) be a formally equidimensional local ring, and let $\mathfrak{p} \subset A$ be a prime ideal so that $A/\mathfrak{p}$ is regular. Then $ht(\mathfrak{p}) = l(\mathfrak{p})$ in A if and only if the local rings A and $A_\mathfrak{p}$ have the same multiplicity.*

Lemma 3.6 (Presentation of transversal extensions). *Let B be a domain of finite type over a perfect field k with maximum multiplicity s ($s \geq 1$), and let $B \subset B'$ be a transversal extension of generic rank r (see 2.1 and Definition 2.5). Let $P \in F_{sr}(B')$ be a prime, and let $\mathfrak{p} = B \cap P$. Then:*
  (1) *The extension of local rings $B_\mathfrak{p} \to B'_P$ is finite. Moreover, there are elements $\theta_1, \ldots, \theta_m \in B'_P$ integral over $\mathfrak{p}B'_P$ such that $B'_P = B_\mathfrak{p}[\theta_1, \ldots, \theta_m]$;*
  (2) *If in addition B'/P is a regular ring, then the $\theta_1, \ldots, \theta_m$ can be chosen in P, and P is the integral closure of $\mathfrak{p}B'$ in B'.*

Proof. (1) The statement follows from an argument similar to one used in [34, Proposition 5.7] but we include the proof here for self-containement. Since $P \in F_{sr}(P)$, by 2.3 (i) one has that $B' \otimes_B B_\mathfrak{p}$ is local and equal to B'_P. Thus $B_\mathfrak{p} \to B'_P$ is finite and as a consequence there are elements $\omega_1, \ldots, \omega_m \in B'_P$ so that $B'_P = B_\mathfrak{p}[\omega_1, \ldots, \omega_m]$. The quotient field of $B_\mathfrak{p}$, say $K(\mathfrak{p})$, is equal to the quotient field of B'_P, say $K(P)$. Hence there are elements $\alpha_1, \ldots, \alpha_m \in B_\mathfrak{p}$ such that $\overline{\alpha_i} = \overline{\omega_i} \in K(\mathfrak{p}) = K(P)$. Setting $\theta_i := \omega_i - \alpha_i$ one has that $\theta_i \in PB'_P$ and $B'_P = B_\mathfrak{p}[\theta_1, \ldots, \theta_m]$. Since PB'_P is integral over $\mathfrak{p}B'_P$ each θ_i is integral over $\mathfrak{p}B'_P$.

(2) Now, suppose that B'/P is regular and assume that $B' = B[\zeta_1, \ldots, \zeta_l]$. By condition (3) of Corollary 2.8, $B'/P = B/\mathfrak{p}$. Analogous to before $\zeta_1, \ldots, \zeta_l$ can be modified by taking $\zeta_1 - \gamma_1, \ldots, \zeta_l - \gamma_l$, $\gamma_i \in B$, and we may assume that they are in P.

The discussion in 2.3, more precisely in condition (iii), applies for the local rings $B_\mathfrak{p} \subset B'_P$, and therefore $\mathfrak{p}B'_P$ is a reduction of PB'_P or say that PB'_P is the integral closure of $\mathfrak{p}B'_P$. Our claim now is that P is the integral closure of $\mathfrak{p}B'$ at the ring B'. It suffices to check that this claim holds locally at every maximal ideal $M \subset B'$ containing P. We will apply Theorem 3.4 at B'_M to prove this latter claim: Let $\mathfrak{m} = M \cap B$, as B'/P is regular and equal to $B/\mathfrak{p}$, it follows that $B_\mathfrak{m}/\mathfrak{p}B_\mathfrak{m}$ is a regular local ring. Under these conditions, as B is a domain and the multiplicity of $B_\mathfrak{m}$ coincides with that of $B_\mathfrak{p}$, Theorem 3.5 asserts that $l(\mathfrak{p}) = ht(\mathfrak{p})$ in $B_\mathfrak{m}$.

Now, since the extension $B_\mathfrak{m} \to B'_M$ is finite, we have that $ht(\mathfrak{p}B'_M) = ht(\mathfrak{p})$, and the blow up of B'_M at $\mathfrak{p}B'_M$ is finite over the blow up of $B_\mathfrak{m}$ at $\mathfrak{p}B_\mathfrak{m}$. This implies that the fibers over the closed points have the same dimension. Thus $l(\mathfrak{p}B'_M) = l(\mathfrak{p})$, and therefore $l(\mathfrak{p}B'_M) = ht(\mathfrak{p}B'_M)$ in B'_M. Recall that $e_{B'_P}(PB'_P) = e_{B'_P}(\mathfrak{p}B'_P)$ by condition (iii) of 2.3 and Theorem 3.2. Now, since PB'_M is the only minimal prime of $\mathfrak{p}B'_M$, the claim follows from Theorem 3.4 (taking $A = B'_M$, $I = \mathfrak{p}B'_M$, and $J = PB'_M$). $\square$

4. Transversality and blow ups

Let X be an equidimensional reduced scheme of finite type over a perfect field k with maximum multiplicity s. As we will see, there are some types of transformations that play an important role in the study of the multiplicity. We will say that a morphism $X \leftarrow X_1$ is an F_s-*local transformation* if it is of one of the following types:
  i) The blow up of X along a regular center Y contained in $F_s(X)$. This will be called an F_s-*permissible blow up*, or simply a *permissible blow up* when there is no confusion with s. In this case we will also say that Y is an F_s-*permissible center*.

ii) An open restriction, i.e., X_1 is an open subscheme of X. In order to avoid trivial transformations, we will always require $X_1 \cap F_s(X) \neq \emptyset$.

iii) The multiplication of X by an affine line, say $X_1 = X \times \mathbb{A}^1_k$.

Note that, in either case, as indicated in the introduction, $\max \operatorname{mult}(X) \geq \max \operatorname{mult}(X_1)$. We will say that a sequence of transformations, say

$$X = X_0 \xleftarrow{\varphi_1} X_1 \xleftarrow{\varphi_2} \ldots \xleftarrow{\varphi_m} X_m \, ,$$

is an F_s-*local sequence on X* if φ_i is an F_s-local transformation of X_{i-1} for $i = 1, \ldots, m$, and

$$s = \max \operatorname{mult}(X_0) = \ldots = \max \operatorname{mult}(X_{m-1}) \geq \max \operatorname{mult}(X_m).$$

The notion of local transformation, essentially due to Hironaka, is the starting point for the definition of some of the fundamental invariants in his Theorem of Resolution of Singularities (the role of transformations of type (i)-(iii) are the key point in the so called *Hironaka's trick* see [12, §7.1]).

In the following lines we will study the behavior of transversality under local sequences. The main result is Theorem 4.4.

Remark 4.1 (multiplicity and étale morphisms)**.** The behavior of the multiplicity is naturally compatible with étale topology. Namely, given an equidimensional scheme X of finite type over k and an étale morphism $\varepsilon : \widetilde{X} \to X$, one has that $\operatorname{mult}(\tilde{\xi}) = \operatorname{mult}(\varepsilon(\tilde{\xi}))$ for every $\tilde{\xi} \in \widetilde{X}$. Moreover, every F_s-local sequence over X, say

$$X \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_l,$$

induces by base change an F_s-local sequence over $\widetilde{X}$, say

$$\widetilde{X} \longleftarrow \widetilde{X}_1 \longleftarrow \cdots \longleftarrow \widetilde{X}_l,$$

and a commutative diagram,

$$
\begin{array}{ccccccc}
\widetilde{X} & \longleftarrow & \widetilde{X}_1 & \longleftarrow & \cdots & \longleftarrow & \widetilde{X}_l \\
\downarrow{\scriptstyle \varepsilon} & & \downarrow{\scriptstyle \varepsilon_1} & & & & \downarrow{\scriptstyle \varepsilon_l} \\
X & \longleftarrow & X_1 & \longleftarrow & \cdots & \longleftarrow & X_l,
\end{array}
$$

where each ε_i is étale.

Consider a finite morphism of schemes, say $\beta : X' \to X$, a closed center $Y' \subset X'$, and its image in X, say $Y = \beta(Y')$. In general, there is no natural map from the blow up of X' at Y' to the blow up of X at Y. The next lemma provides a condition under which such map exists, and moreover it is finite.

Lemma 4.2. *Let $\gamma : B \to B'$ be a homomorphism of Noetherian rings. Consider two ideals, say $I \subset B$, and $J \subset B'$, such that $\gamma(I)B'$ is a reduction of J. Let $X \to \operatorname{Spec}(B)$ be the blow up at I and let $X' \to \operatorname{Spec}(B')$ be the blow up at J. Then there exists a unique morphism $\beta_1 : X' \to X$ which makes the following diagram commutative:*

$$
\begin{array}{ccc}
\operatorname{Spec}(B') & \longleftarrow & X' \\
\downarrow & & \downarrow{\scriptstyle \beta_1} \\
\operatorname{Spec}(B) & \longleftarrow & X.
\end{array}
\tag{4.2.1}
$$

In addition, if B' is finite over B, then β_1 is also finite.

Proof. Here J is integral over $\gamma(I)B'$, or equivalently, $\gamma(I)J^n = J^{n+1}$ for some $n \geq 0$ (see [23, p. 156], and [20, Lemma 1.1, p. 792]). Therefore $\gamma(I)J^n\mathcal{O}_{X'} = J^{n+1}\mathcal{O}_{X'}$. Observe that, by definition, $J\mathcal{O}_{X'}$ is an invertible sheaf of ideals over X'. Hence $J^n\mathcal{O}_{X'}$ is also invertible, and, in this way, $\gamma(I)J^n\mathcal{O}_{X'} = J^{n+1}\mathcal{O}_{X'}$ implies $\gamma(I)\mathcal{O}_{X'} = J\mathcal{O}_{X'}$. In particular, this means that $\gamma(I)\mathcal{O}_{X'}$

is invertible over X'. Thus, by the universal property of the blow up at I, $X \to \mathrm{Spec}(B)$, there exists a unique morphism of schemes, say $\beta_1 : X' \to X$, making (4.2.1) commutative.

For the second claim, fix a set of generators of I over B, say $I = \langle f_1, \ldots, f_r \rangle$. Recall that $X = \mathrm{Proj}(B[IW])$, and $X' = \mathrm{Proj}(B'[JW])$. From the existence of β_1, we deduce that X' is covered by the affine charts associated to $\gamma(f_1), \ldots, \gamma(f_r)$, i.e., those given by the rings $\left[(B'[JW])_{\gamma(f_i)W}\right]_0$ (the homogeneous part of degree 0 of $(B'[JW])_{\gamma(f_i)W}$). Since B' is finite over B, and $\gamma(I)B'$ is a reduction of J, the graded algebra $B'[JW]$ is finite over $B[IW]$. Hence $\left[(B'[JW])_{\gamma(f_i)W}\right]_0$ is finite over $[(B[IW])_{f_iW}]_0$ for $i = 1, \ldots, r$, and therefore β_1 is finite.

Remark 4.3. Let $B \subset B'$ be a finite extension of Noetherian domains, and consider two ideals $I \subset B$ and $J \subset B'$ so that J is integral over IB'. Let $X' \to \mathrm{Spec}(B')$ be the blow up at J and let $X \to \mathrm{Spec}(B)$ be the blow up at I. Fix generators of I and J, say $I = \langle x_1, \ldots, x_r \rangle$, and $J = \langle x_1, \ldots, x_r, \theta_1, \ldots, \theta_s \rangle$, where $\theta_1, \ldots, \theta_s$ are integral over IB'. Under these hypotheses, the Lemma says there is a natural finite map $X' \to X$. Note that X can be covered by r affine charts of the form $\mathrm{Spec}(B_1), \ldots, \mathrm{Spec}(B_r)$, with $B_i = B\left[\frac{x_1}{x_i}, \ldots, \frac{x_r}{x_i}\right]$. Moreover, from the second part of the proof it follows that X' can be covered by r affine charts of the form $\mathrm{Spec}(B'_1), \ldots, \mathrm{Spec}(B'_r)$, where $B'_i = B'\left[\frac{x_1}{x_i}, \ldots, \frac{x_r}{x_i}, \frac{\theta_1}{x_i}, \ldots, \frac{\theta_s}{x_i}\right]$, and the map $X' \to X$ is locally given by the extension $B_i \subset B'_i$, which is finite.

Theorem 4.4. *Let X be an algebraic variety with maximum multiplicity s and let $\beta : X' \to X$ be a transversal morphism of generic rank r. Then:*

i) *An F_{rs}-permissible center on X', say $Y' \subset F_{rs}(X')$, induces an F_s-permissible center on X, say $Y = \beta(Y') \subset F_s(X)$, and a commutative diagram of blow ups of X at Y, say $X \leftarrow X_1$, and of X' at Y', say $X' \leftarrow X'_1$, as follows,*

$$
\begin{array}{ccc}
X' & \longleftarrow & X'_1 \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta_1} \\
X & \longleftarrow & X_1,
\end{array}
$$

where β_1 is finite of generic rank r. In addition, if $F_{rs}(X'_1) \neq \emptyset$, then $F_s(X_1) \neq \emptyset$, and the morphism β_1 is transversal.

ii) *Any sequence of F_{rs}-permissible blow ups on X', $X' \leftarrow X'_1 \leftarrow \cdots \leftarrow X'_{N-1} \leftarrow X'_N$, induces a sequence of F_s-permissible blow ups on X, and a commutative diagram as follows,*

$$
\begin{array}{ccccccccc}
X' & \longleftarrow & X'_1 & \longleftarrow & \cdots & \longleftarrow & X'_{N-1} & \longleftarrow & X'_N \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta_1} & & & & \downarrow{\scriptstyle\beta_{N-1}} & & \downarrow{\scriptstyle\beta_N} \\
X & \longleftarrow & X_1 & \longleftarrow & \cdots & \longleftarrow & X_{N-1} & \longleftarrow & X_N,
\end{array}
$$

where each β_i is finite of generic rank r. Moreover, if $F_{rs}(X'_N) \neq \emptyset$, then $F_s(X_N) \neq \emptyset$, and the morphism β_N is transversal.

Proof. Property ii) follows from i) by induction on N. Thus we just need to prove i).

Note that the center $Y = \beta(Y')$ is F_s-permissible by 2.9. The varieties X and X' can be locally covered by affine charts of the form $\mathrm{Spec}(B)$ and $\mathrm{Spec}(B')$, with the morphism β being given by a finite extension $B \subset B'$. Let $P \subset B$ and $P' \subset B'$ denote the ideals of definition of Y and Y' respectively. By Lemma 3.6, P' is integral over the extended ideal PB'. Then, under these hypotheses, Lemma 4.2 says that there is natural finite morphism $\beta_1 : X'_1 \to X_1$ which makes the diagram in i) commutative. Since the blow up of an integral scheme along a proper center is birational, the generic rank of β_1 coincides with that of β. That is, β_1 is a finite morphism of generic rank r, which proves the first part of i).

For the second part of the claim, recall first that the multiplicity does not increase when blowing up along permissible centers, and hence $\max \mathrm{mult}(X_1) \leq s$, and $\max \mathrm{mult}(X'_1) \leq rs$. In addition,

the map $\beta : X_1' \to X_1$ has generic rank r. Thus $F_{rs}(X_1') \neq \emptyset$ implies $F_s(X_1) \neq \emptyset$. In particular, if $F_{rs}(X_1') \neq \emptyset$, then the morphism $\beta : X_1' \to X_1$ is transversal.

Remark 4.5. Consider a transversal morphism $\beta : X' \to X$ as in the theorem. As pointed out at the beginning of the section, we are interested in studying the behavior of $F_{rs}(X')$ and $F_s(X)$. Since the set $F_{rs}(X')$ is homeomorphic to its image in $F_s(X)$ via β, for any open subset $U' \subset X'$, one can find an open subset $U \subset X$ satisfying $U' \cap F_{rs}(X') = \beta^{-1}(U) \cap F_{rs}(X')$. In this setting we will say that $U' \subset X'$ is an F_{rs}-permissible restriction if there is an open subset $U \subset X$ so that $U' = \beta^{-1}(U)$. Thus, as a consequence of Theorem 4.4, one readily checks that every F_{rs}-local sequence on X', say $X' \leftarrow X_1' \leftarrow \cdots \leftarrow X_N'$, induces an F_s-local sequence on X, and a commutative diagram as follows,

$$
\begin{array}{ccccccccc}
X' & \longleftarrow & X_1' & \longleftarrow & \cdots & \longleftarrow & X_{N-1}' & \longleftarrow & X_N' \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta_1} & & & & \downarrow{\scriptstyle\beta_{N-1}} & & \downarrow{\scriptstyle\beta_N} \\
X & \longleftarrow & X_1 & \longleftarrow & \cdots & \longleftarrow & X_{N-1} & \longleftarrow & X_N,
\end{array}
$$

where each β_i is finite of generic rank r. In addition, if $F_{rs}(X_N') \neq \emptyset$, then β_N is transversal.

Normalization and transversal morphisms. Let X be a singular algebraic variety over a perfect field k with maximum multiplicity $s \geq 2$. The normalization of X, say $\overline{X}$, is endowed with a natural finite morphism $\beta : \overline{X} \to X$ which is dominant and birational. Hence, by Zariski's formula (Theorem 2.2), $\max \mathrm{mult}(\overline{X}) \leq \max \mathrm{mult}(X)$. In addition, if the equality holds, then $\beta : \overline{X} \to X$ is transversal, and $F_s(\overline{X})$ is mapped homeomorphically to $\beta(F_s(\overline{X})) \subset F_s(X)$ (see Proposition 2.7 and 2.9).

Assume that the morphism $\beta : \overline{X} \to X$ is transversal, i.e., that $\max \mathrm{mult}(\overline{X}) = \max \mathrm{mult}(X)$. In this case, Theorem 4.4 says that any sequence of blow ups along regular equimultiple centers (of maximum multiplicity s) on $\overline{X}$ induces a sequence of blow ups on X. As a consequence of this result, one can also establish a relation between sequences of normalized blow ups on $\overline{X}$ and sequences of blow ups on X, as long as transversality is preserved. Recall that the normalized blow up of $\overline{X}$ along a closed center $\overline{Y} \subset \overline{X}$ is the normalization of the blow up of $\overline{X}$ along $\overline{Y}$.

Corollary 4.6. *Let X_0 be an algebraic variety over a perfect field k, and let $\overline{X}_0$ denote the normalization of X_0. Assume that $\max \mathrm{mult}(X_0) = \max \mathrm{mult}(\overline{X}_0) = s$. Let $\overline{X}_0 \leftarrow \overline{X}_1 \leftarrow \cdots \leftarrow \overline{X}_{l-1} \leftarrow \overline{X}_l$ be a sequence of normalized blow ups along closed regular centers $\overline{Y}_i \subset F_s(\overline{X}_{i-1})$, such that*

$$\max \mathrm{mult}(\overline{X}_0) = \cdots = \max \mathrm{mult}(\overline{X}_{l-1}) \geq \max \mathrm{mult}(\overline{X}_l).$$

Then there is a natural sequence of blow ups on X_0, say $X_0 \leftarrow X_1 \leftarrow \cdots \leftarrow X_{l-1} \leftarrow X_l$, along closed regular centers $Y_i \subset F_s(X_i)$, so that

$$\max \mathrm{mult}(X_0) = \cdots = \max \mathrm{mult}(X_{l-1}) \geq \max \mathrm{mult}(X_l),$$

and $\overline{X}_i$ is the normalization of X_i for $i = 1, \ldots, l$.

Proof. It suffices to prove the case $l = 1$ (the general case follows by induction on l). Since $\max \mathrm{mult}(X_0) = \max \mathrm{mult}(\overline{X}_0)$, the natural morphism $\beta_0 : \overline{X}_0 \to X_0$ is transversal, and hence $Y_0 = \beta_0(\overline{Y}_0) \subset F_s(X_0)$ defines a regular center in X_0. Let $X_1 \to X$ be the blow up at Y_0 and let $\widetilde{X}_1 \to \overline{X}_0$ be the blow up at $\overline{Y}_0$. Then by Theorem 4.4, there is a birational map, say $\widetilde{X}_1 \to X_1$, which is finite. Let $\overline{X}_1$ be the normalization of $\widetilde{X}_1$. Then it follows that $\overline{X}_1$ is the normalization of X_1. Moreover, if $\max \mathrm{mult}(\overline{X}_1) = \max \mathrm{mult}(\overline{X}_0)$, then $\max \mathrm{mult}(X_1) = \max \mathrm{mult}(X_0)$, and the morphism $\beta_1 : \overline{X}_1 \to X_1$ is again transversal.

Strongly transversal morphisms. Let $\beta : X' \to X$ be transversal morphism of generic rank r and suppose that X has maximum multiplicity $s \geq 2$. Under these hypotheses, Theorem 4.4 says that any sequence of F_{rs}-permissible blow ups on X' induces a sequence of F_s-permissible blow ups on X. In general, the converse to this theorem, by blowing up centers over X, fails because β does

not map $F_{rs}(X')$ surjectively to $F_s(X)$ (see Example 4.7 below). Hence not every center contained in $F_s(X)$ induces a center in $F_{rs}(X')$. This observation motivates Definition 4.8.

Example 4.7. Let k be a perfect field. Consider $X = \operatorname{Spec}(k[x, y, z]/\langle y^2 - z^2 x\rangle)$ and $X' = \operatorname{Spec}(k[x, y, z, t]/\langle y^2 - z^2 x, t^3 - xyz\rangle)$. It can be checked that the extension $k[x, y, z]/\langle y^2 - z^2 x\rangle \subset k[x, y, z, t]/\langle y^2 - z^2 x, t^3 - xyz\rangle$ is transversal and the maximum multiplicity of X' is 6. In addition one can check $F_6(X')$ is the closed point determined by $\langle x, y, z, t\rangle$ in X' while $F_2(X)$ is the one dimensional closed subscheme determined by $\langle y, z\rangle$ in X. Thus $F_6(X')$ does not map surjectively to $F_2(X')$.

Definition 4.8. We will say that a transversal morphism of generic rank r, $\beta : X' \to X$, is *strongly transversal* if $F_{rs}(X')$ is homeomorphic to $F_s(X)$ via β, and every F_{rs}-local sequence over X', say $X' \leftarrow X'_1 \leftarrow \cdots \leftarrow X'_N$, induces an F_s-local sequence over X in the sense of Remark 4.5, and a commutative diagram as follows,

$$
\begin{array}{ccccccccc}
X' & \longleftarrow & X'_1 & \longleftarrow & \cdots & \longleftarrow & X'_{N-1} & \longleftarrow & X'_N \\
\downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta_1} & & & & \downarrow{\scriptstyle\beta_{N-1}} & & \downarrow{\scriptstyle\beta_N} \\
X & \longleftarrow & X_1 & \longleftarrow & \cdots & \longleftarrow & X_{N-1} & \longleftarrow & X_N,
\end{array}
\tag{4.8.1}
$$

where each β_i is finite of generic rank r and induces a homeomorphism between $F_{rs}(X'_i)$ and $F_s(X_i)$. In this case we will also say that $F_{rs}(X')$ is *strongly homeomorphic* to $F_s(X)$. Note in particular that this definition yields $F_{rs}(X'_N) = \emptyset$ if and only if $F_s(X_N) = \emptyset$.

Remark 4.9. This definition is equivalent to saying that any F_s-local sequence on X induces an F_{rs}-local sequence on X', and a commutative diagram like (4.8.1). In particular, note that the homeomorphism between $F_{rs}(X')$ and $F_s(X)$ is preserved by transformations.

Remark 4.10. Note that, in virtue of 2.9, checking that a finite morphism $\beta : X' \to X$ of generic rank r is strongly transversal is equivalent to showing that $\beta : X' \to X$ maps $F_{rs}(X')$ surjectively onto $F_s(X)$ (where s denotes the maximum multiplicity of X), and that this property is preserved by any F_s-local sequence.

From the point of view of resolution of singularities, $\beta : X' \to X$ is strongly transversal if and only if the processes of lowering the maximum multiplicity of X' and X are equivalent. In the case of varieties over a field of characteristic zero, it is possible to give a characterization of strong transversality in terms of Rees algebras (see Theorem 7.2 in Section 7). In the next two sections we introduce the theory of Rees algebras and elimination.

Part II. **Rees algebras and elimination**

5. Rees algebras and local presentations of the multiplicity

Let X be a d-dimensional algebraic variety defined over a perfect field k. When seeking a resolution of singularities of X we may start by constructing a sequence of blow ups along closed regular equimultiple centers, say $X \leftarrow X_1 \leftarrow \ldots \leftarrow X_{m-1} \leftarrow X_m$, so that

$$\max \operatorname{mult}(X) = \max \operatorname{mult}(X_1) = \cdots = \max \operatorname{mult}(X_{m-1}) > \max \operatorname{mult}(X_m).$$

In general, the maximum multiplicity locus of a variety X, say $\underline{\operatorname{Max}} \operatorname{mult}(X)$, does not define a regular center. Thus the multiplicity function has to be refined in order to obtain regular centers at each stage of the process. It is in this setting that the machinery provided by Rees algebras comes in handy. For a more detailed introduction to Rees algebras and their use in resolution of singularities we refer to [6] or [11] (we will not go into these detalis here). In this paper we will use Rees algebras in the formulation and the proof of Theorem 7.2 and in the proof of Theorem 8.2. Thus, we will devote the present section and the next to review the main definitions and results on Rees algebras that we will be needing in sections 7 and 8.

Definition 5.1. Let B be a Noetherian ring, and let $\{I_n\}_{n \in \mathbb{N}}$ be a sequence of ideals in B satisfying the following conditions: $I_0 = B$; and $I_k \cdot I_l \subset I_{k+l}$ for all $l, k \in \mathbb{N}$. The graded subring $\mathcal{G} = \bigoplus_{n \geq 0} I_n W^n$ of the polynomial ring $B[W]$ is said to be a *B-Rees algebra*, or a Rees algebra over B, if it is a finitely generated B-algebra. A Rees algebra can be described by giving a finite set of generators, say $\{f_1 W^{n_1}, \ldots, f_s W^{n_s}\}$, with $f_i \in B$ for $i = 1 \ldots, s$, in which case we will write $\mathcal{G} = B[f_1 W^{n_1}, \ldots, f_s W^{n_s}] \subset B[W]$.

The notion of Rees algebra extends naturally to schemes. Consider a non-necessarily affine scheme V. We will say that a quasi-coherent subsheaf $\mathcal{G} \subset \mathcal{O}_V[W]$ is a Rees algebra over V, or simply an $\mathcal{O}_V$-Rees algebra, if V can be covered by affine charts of the form $U = \mathrm{Spec}(B)$, where $\Gamma(U, \mathcal{G})$ is a Rees algebra over B. Note that, in this way, there is a natural one-to-one correspondence between the Rees algebras defined over a ring B, and those defined over the affine scheme $\mathrm{Spec}(B)$. Thus sometimes we will abuse our notation, and we will make no distinction between them.

5.2. The singular locus of a Rees algebra. [32, 1.2] Let $\mathcal{G} = \bigoplus_{n \in \mathbb{N}} \mathcal{I}_n W^n$ be a Rees algebra over a *regular* scheme V. The *singular locus* of $\mathcal{G}$ is defined by

$$\mathrm{Sing}(\mathcal{G}) := \bigcap_{n \geq 1} \{\xi \in V \mid \nu_\xi(\mathcal{I}_n) \geq n\},$$

where $\nu_\xi(\mathcal{I}_n)$ denotes the order of the ideal sheaf $\mathcal{I}_n$ at ξ. When V is excellent, $\mathrm{Sing}(\mathcal{G})$ turns out to be a closed subset of V. In addition, when V is affine, put $V = \mathrm{Spec}(S)$, and $\mathcal{G}$ is generated by elements $f_1 W^{N_1}, \ldots, f_s W^{N_s}$, one has that (see [11, Proposition 1.4]):

$$\mathrm{Sing}(\mathcal{G}) = \bigcap_{i=1}^{s} \{\xi \in V \mid \nu_\xi(f_i) \geq N_i\}.$$

5.3. If X is an algebraic variety over a perfect field k, then, locally in an (étale) neighborhood U of a point of maximum multiplicity, there is an embedding in a smooth variety V over k and an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ such that $\underline{\mathrm{Max}}\,\mathrm{mult}(X) \cap U = \mathrm{Sing}\,\mathcal{G}$, where $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ denotes the (closed) set of points of maximum multiplicity of X. In fact, this equality holds in a very strong sense, to be defined in 5.7. This issue will be treated along this section (see Theorem 5.10, and the discussion in 5.12 for more precise details).

5.4. Transforms of Rees algebras by local sequences. *Transforms by permissible blow ups.* Let V be a regular scheme, and let $\mathcal{G}$ be a Rees algebra on V. A regular closed subscheme $Y \subset V$ is said to be *permissible* for $\mathcal{G} = \bigoplus_n J_n W^n \subset \mathcal{O}_V[W]$ if $Y \subset \mathrm{Sing}\,\mathcal{G}$. A *permissible blow up* is the blow up at a permissible center, $V \leftarrow V_1$. If $H_1 \subset V_1$ denotes the exceptional divisor, then for each $n \in \mathbb{N}$, one has that $J_n \mathcal{O}_{V_1} = I(H_1)^n J_{n,1}$ for some sheaf of ideals $J_{n,1} \subset \mathcal{O}_{V_1}$. The *transform of $\mathcal{G}$* in V_1 is then defined as $\mathcal{G}_1 := \bigoplus_n J_{n,1} W^n$ (see [11, Proposition 1.6]).

Transforms by smooth morphisms. Let $V \leftarrow V_1$ be either the restriction to some open subset V_1 of V or the multiplication by an affine space, say $V_1 = V \times_k \mathbb{A}_k^n$. Then we define the transform $\mathcal{G}_1$ of $\mathcal{G}$ in V_1 as the pull-back of $\mathcal{G}$ in V_1.

A *$\mathcal{G}$-local sequence over V* is a sequence of transformations over V,

$$(V = V_0, \mathcal{G} = \mathcal{G}_0) \xleftarrow{\pi_0} (V_1, \mathcal{G}_1) \xleftarrow{\pi_1} \cdots \xleftarrow{\pi_{n-1}} (V_n, \mathcal{G}_n), \tag{5.4.1}$$

where each π_i is either a permissible monoidal transformation for $\mathcal{G}_i \subset \mathcal{O}_{V_i}[W]$ (and $\mathcal{G}_{i+1}$ is the transform of $\mathcal{G}_i$ in the sense of 5.4), or a smooth morphism as above (and $\mathcal{G}_{i+1}$ is the pull-back of $\mathcal{G}_i$ in V_{i+1}).

Definition 5.5. Let V be a regular scheme, and $\mathcal{G}$ an $\mathcal{O}_V$-Rees algebra. A *resolution* of $\mathcal{G}$ consists of a sequence of permissible blow ups, say

$$(V = V_0, \mathcal{G} = \mathcal{G}_0) \longleftarrow (V_1, \mathcal{G}_1) \longleftarrow \cdots \longleftarrow (V_m, \mathcal{G}_m),$$

so that $\mathrm{Sing}(\mathcal{G}_m) = \emptyset$.

Local presentations of the multiplicity

In 5.4 and Definition 5.5 we discussed about local sequences of transformations of a Rees algebra $\mathcal{G}$ over a regular scheme V. In the next definition we will fix a singular variety X and will consider local transformations on X.

Definition 5.6. We will say that a sequence of transformations $X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_m$ is a $\mathrm{mult}(X)$-*local sequence* if for $i = 0, \ldots, m - 1$, $X_i \leftarrow X_{i+1}$ is either a blow at a smooth center $Y \subset \underline{\mathrm{Max}}\,\mathrm{mult}(X_i)$, or $X_{i+1} = X_i \times \mathbb{A}^1_k$ and $X_i \leftarrow X_{i+1}$ is the smooth morphism defined by the natural projection, or $X_i \leftarrow X_{i+1}$ is the restriction to some Zariski open subset of X_i.

Definition 5.7. We will say that $\mathrm{mult}(X)$ *is locally representable for a scheme X*, if for each point $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}\,X$ there is an open (étale) neighborhood (which we denote again by X to ease the notation), an embedding in some regular scheme, say $X \subset V$, and a $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ so that the following conditions hold:
  (1) There is an equality, $\underline{\mathrm{Max}}\,\mathrm{mult}(X) = \mathrm{Sing}(\mathcal{G})$, of closed sets;
  (2) Every $\mathrm{mult}(X)$-local sequence as in Definition 5.6, say $X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_m$ with

$$\max\mathrm{mult}(X) = \max\mathrm{mult}(X_0) = \cdots = \max\mathrm{mult}(X_{m-1}) \geq \max\mathrm{mult}(X_m)$$

  induces a $\mathcal{G}$-local sequence

$$(V = V_0, \mathcal{G} = \mathcal{G}_0) \longleftarrow (V_1, \mathcal{G}_1) \longleftarrow \cdots \longleftarrow (V_m, \mathcal{G}_m), \tag{5.7.1}$$

  and: $\underline{\mathrm{Max}}\,\mathrm{mult}(X_i) = \mathrm{Sing}(\mathcal{G}_i)$ for $i = 1, \ldots, m - 1$; if $\max\mathrm{mult}(X_{m-1}) = \max\mathrm{mult}(X_m)$, then $\underline{\mathrm{Max}}\,\mathrm{mult}(X_m) = \mathrm{Sing}(\mathcal{G}_m)$; if $\max\mathrm{mult}(X_{m-1}) > \max\mathrm{mult}(X_m)$, then $\mathrm{Sing}(\mathcal{G}_m) = \emptyset$.
  (3) Conversely, any $\mathcal{G}$-local sequence as (5.7.1) with $\mathrm{Sing}(\mathcal{G}_i) \neq \emptyset$ for $i = 0, \ldots, m - 1$, induces an $\mathrm{mult}(X)$-local sequence $X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_m$ with

$$\max\mathrm{mult}(X) = \max\mathrm{mult}(X_0) = \cdots = \max\mathrm{mult}(X_{m-1}) \geq \max\mathrm{mult}(X_m),$$

  and: $\underline{\mathrm{Max}}\,\mathrm{mult}(X_i) = \mathrm{Sing}(\mathcal{G}_i)$ for $i = 1, \ldots, m - 1$; $\max\mathrm{mult}(X_{m-1}) = \max\mathrm{mult}(X_m)$ if $\underline{\mathrm{Max}}\,\mathrm{mult}(X_m) = \mathrm{Sing}(\mathcal{G}_m) \neq \emptyset$, and, if $\mathrm{Sing}(\mathcal{G}_m) = \emptyset$, then $\max\mathrm{mult}(X_{m-1}) > \max\mathrm{mult}(X_m)$.

5.8. It can be proved that the previous condition is fulfilled for algebraic varieties defined over a perfect field, at least locally in étale topology. Namely, given an algebraic variety X over a perfect field k, and a point $\xi \in X$, there exists an étale neighborhood of X at ξ, say U, a closed immersion of U into a regular variety V, and an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ satisfying conditions (1), (2) and (3) (see [34, §7]). This local embedding, together with the Rees algebra $\mathcal{G}$, will be called a *local presentation* of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$.

Remark 5.9. Hereafter, when we refer to a local presentation of X, we shall simply write a pair $(V, \mathcal{G})$, assuming that a closed immersion $X \hookrightarrow V$ has already been fixed, and omitting the fact that $X \hookrightarrow V$ might by defined in an étale neighborhood.

Theorem 5.10. [34, §5] *Let X be an equidimensional scheme of finite type over a perfect field k, and let $s = \max\mathrm{mult}(X)$. Then, $\mathrm{mult}(X)$ is locally representable.*

5.11. Hilbert-Samuel vs. Multiplicity. In [16] Hironaka proves that the Hilbert-Samuel function of an algebraic variety X, or more precisely the stratum of points having the same Hilbert-Samuel function, is locally representable. There, instead of $\mathrm{mult}(X)$-local sequences, one has to consider HS_X-local sequences: set $\underline{\mathrm{Max}}\,\mathrm{HS}_X$ as the closed set of points with maximum Hilbert-Samuel function on X. We will say that $X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_m$ is an HS_X-local sequence if for $0 \leq i \leq m - 1$, the morphism $X_i \leftarrow X_{i+1}$ is either a blow up at a smooth center $Y \subset \underline{\mathrm{Max}}\,\mathrm{HS}_{X_i}$ (this is normal flatness), or $X_i \leftarrow X_{i+1}$ is a smooth morphism with $X_{i+1} = X_i \times \mathbb{A}^1_k$, or $X_i \leftarrow X_{i+1}$ is the restriction to some Zariski open subset of X_i.

By considering the Hilbert-Samuel function, or the multiplicity function, we ultimately want to produce a resolution of singularities of X. In the first case, we seek a resolution by blowing up at normally flat centers; in the second case, one uses equimultiple centers. Over fields of characteristic zero, in either of these two approaches, Rees algebras can be used in order to produce such centers (as was done in 5.8 for the case of the multiplicity).

5.12. Some ideas behind the proof of Theorem 5.10. Here we seek for local representations of the multiplicity on X in the sense of Definition 5.7. Suppose that X is affine, say $X = \mathrm{Spec}(B)$, where B is an equidimensional algebra of finite type over a perfect field k. Fix a closed point $\xi \in F_s(X)$. After replacing X by a suitable étale neighborhood of ξ, we may assume that there exists a regular domain $S \subset B$ so that B is a finite extension of S of generic rank s, which necessarily satisfies property (*) from 2.3 at ξ. Then, choose $\theta_1, \ldots, \theta_m \in B$ so that $B = S[\theta_1, \ldots, \theta_m]$. These elements induce a surjective morphism $S[Z_1, \ldots, Z_m] \to B = S[\theta_1, \ldots, \theta_m]$, where $Z_1, \ldots, Z_m$ represent variables, and we can take $V = \mathrm{Spec}(S[Z_1, \ldots, Z_m])$. Observe that this morphism induces a natural embedding of X into V. Moreover, since θ_i is integral over S, it has a minimal polynomial over K, the field of fractions of S. Denote it by $f_i(Z_i) \in K[Z_i]$, and set $d_i = \deg(f_i)$. It can be shown that the coefficients of these polynomials actually belong to S, i.e., that $f_i(Z_i) \in S[Z_i]$. Finally, (see [34, §7.1] for details), a local presentation of $F_s(X)$ is given by the Rees algebra:

$$\mathcal{G}_V = \mathcal{O}_V[f_1(Z_1)W^{d_1}, \ldots, f_m(Z_m)W^{d_m}] \subset \mathcal{O}_V[W].$$

Comparing different presentations: weak equivalence and canonical representatives. Consider a reduced equidimensional scheme X of finite type over a perfect field k and fix a point $\xi \in \underline{\mathrm{Max}}\,\mathrm{mult}(X)$. In principle, there may be many presentations of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ (locally at ξ) coming from different immersions and Rees algebras. As we want to use local presentations to find a resolution of singularities of X one would like to show that two different presentations lead to the same process of resolution (at least in the case of characteristic zero in which we know how to resolve singularities). Assume that a closed immersion $X \hookrightarrow V$ has been fixed, and that we have two presentations of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ given by two $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ and $\mathcal{G}'$. Then, we would like that $\mathcal{G}$ and $\mathcal{G}'$ lead to the same process of resolution. As both of them represent $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$, we have that $\mathrm{Sing}(\mathcal{G}) = \underline{\mathrm{Max}}\,\mathrm{mult}(X) = \mathrm{Sing}(\mathcal{G}')$, and this equality is preserved by local sequences. This motivates the following definition.

Definition 5.13. Two Rees algebras $\mathcal{G}$ and $\mathcal{G}'$ over a regular scheme V are *weakly equivalent* if:
  (1) $\mathrm{Sing}(\mathcal{G}) = \mathrm{Sing}(\mathcal{G}')$.
  (2) Any $\mathcal{G}$-local sequence on V, say $V \leftarrow V_1 \leftarrow \cdots \leftarrow V_m$, is a $\mathcal{G}'$-local sequence, and vice-versa.
  (3) For any local sequence as that in (2), if $\mathcal{G}_i$ and $\mathcal{G}'_i$ denote the transforms of $\mathcal{G}$ and $\mathcal{G}'$ on V_i respectively, we have that $\mathrm{Sing}(\mathcal{G}_i) = \mathrm{Sing}(\mathcal{G}'_i)$.

It is easy to see that weak equivalence defines an equivalence relation on the family of $\mathcal{O}_V$-Rees algebras, and thus each $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ has a class of equivalence. By definition, if an $\mathcal{O}_V$-Rees algebra represents $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ (via a fixed immersion $X \hookrightarrow V$), any other element of its class also do so. It can be proved that two weakly equivalent $\mathcal{O}_V$-Rees algebras induce the same invariants, and hence they share the same process of resolution of Rees algebras (over fields of characteristic zero). In this way one can see that the process of simplification of the maximum multiplicity of a singular variety X is independent of the local presentations chosen (see [6, §27] or [11]).

Remark 5.14. Consider a Rees algebra $\mathcal{G}$ defined over a smooth variety V. One can consider a tree with $(V, \mathcal{G})$ as its root and consisting on all the $\mathcal{G}$-local sequences as in (5.4.1). This tree will be called the *tree of permissible transformations* of $\mathcal{G}$, and we shall denote it by $\mathcal{F}_V(\mathcal{G})$. Given another Rees algebra $\mathcal{G}'$ over V, if every $\mathcal{G}$-local sequence induces a $\mathcal{G}'$-local sequence, then we shall say that $\mathcal{F}_V(\mathcal{G}) \subset \mathcal{F}_V(\mathcal{G}')$. Note that, with this notation, $\mathcal{G}$ is weakly equivalent to $\mathcal{G}'$ if and only if $\mathcal{F}_V(\mathcal{G}) \subset \mathcal{F}_V(\mathcal{G}')$ and $\mathcal{F}_V(\mathcal{G}') \subset \mathcal{F}_V(\mathcal{G})$ and, in such case, we shall write $\mathcal{F}_V(\mathcal{G}) = \mathcal{F}_V(\mathcal{G}')$. Observe that, in addition, the tree $\mathcal{F}_V(\mathcal{G})$ also defines a tree of closed sets: the ones defined by

the singular locus of $\mathcal{G}$ and the singular loci of all its transforms after considering all the possible $\mathcal{G}$-local sequences over V.

5.15. Integral closure. Consider a regular domain S, and a S-Rees algebra $\mathcal{G} \subset S[W]$. We define the integral closure of $\mathcal{G}$, which we shall denote by $\overline{\mathcal{G}}$, as that of $\mathcal{G}$ regarded as a ring, inside its field of fractions. Since S is normal, $S[W]$ is normal, and hence $\overline{\mathcal{G}} \subset S[W]$. In addition, if S is excellent, $\overline{\mathcal{G}}$ is finitely generated over S (see [13, 7.8.3 ii) and vi)]). Thus $\overline{\mathcal{G}}$ is again a Rees algebra over S. As integral closure is a concept of local nature, this notion extends naturally to non-affine schemes. In this way, given a regular excellent scheme V, and an $\mathcal{O}_V$-Rees algebra $\mathcal{G}$, we may talk unambiguously about the integral closure of $\mathcal{G}$, say $\overline{\mathcal{G}}$, which is again an $\mathcal{O}_V$-Rees algebra.

Remark 5.16. Observe that the regularity of S (resp. V) does not play an essential role on the previous discussion. Thus the concept of integral closure extends to any Rees algebra defined over a normal excellent domain (resp. normal excellent integral scheme).

Lemma 5.17 (cf. [11, Proposition 5.4]). *Let V be a smooth scheme of finite type over a perfect field k, and $\mathcal{G}$ an $\mathcal{O}_V$-Rees algebra. Then the integral closure of $\mathcal{G}$, say $\overline{\mathcal{G}}$, is weakly equivalent to $\mathcal{G}$.*

5.18. Differential Rees algebras. Let S be a smooth algebra over a perfect field k. For any non-negative integer n, denote by $Diff_{S|k}^n$ the module of differential operators of order at most n of S over k. A Rees algebra over S, say $\mathcal{G} = \bigoplus_{i \in \mathbb{N}} I_i W^i$, is said to be *differentially saturated* (with respect to k) if the following condition holds: for any homogeneous element $f W^N \in \mathcal{G}$, and any differential operator $\Delta \in Diff_{S|k}^n$, with $n < N$, we have that $\Delta(f) W^{N-n} \in \mathcal{G}$. In particular, $I_{i+1} \subset I_i$, since $Diff_{S|k}^0 \subset Diff_{S|k}^1$. Note that this notion extends naturally to Rees algebras defined on a smooth variety over k, say V: in this case, we denote by $Diff_{V|k}^n$ the sheaf of differential operators of V over k, and the condition of being differentially saturated is defined locally.

Given a smooth variety over a perfect field k, and an arbitrary $\mathcal{O}_V$-Rees algebra $\mathcal{G}$, there is a natural way to construct the smallest differentially saturated algebra containing $\mathcal{G}$ (see [33, Theorem 3.4]), which we will denote by $Diff(\mathcal{G})$. It can be checked that $Diff(\mathcal{G})$ is again finitely generated over $\mathcal{O}_V$ (cf. [33, Proof of Theorem 3.4]), and hence it is a Rees algebra. Moreover $Diff(\mathcal{G})$ is weakly equivalent to $\mathcal{G}$ (see Giraud's Lemma [11, Theorem 4.1]).

Theorem 5.19 ([17], [4, Theorem 3.10]). *Let V be a smooth scheme of finite type over a perfect field k. Then two $\mathcal{O}_V$-Rees algebra $\mathcal{G}$ and $\mathcal{G}'$ are weakly equivalent if and only if $\overline{Diff(\mathcal{G})} = \overline{Diff(\mathcal{G}')}$.*

In particular, this theorem says that $\overline{Diff(\mathcal{G})}$ is the canonical representative of the class of $\mathcal{G}$. A generalization of this result to the case of varieties over non-perfect fields can be found in [2, Theorem 6.6.8]. See also [22] for an alternative approach to equivalence of algebras.

5.20. Relative Differential Rees algebras. Let $\beta : V \to V'$ be a smooth morphism of smooth schemes defined over a perfect field k with $\dim V > \dim V'$. Then, for any integer s, the sheaf of relative differential operators of order at most s, $Diff_{V/V'}^s$, is locally free over V ([13, (4) § 16.11]). We will say that a sheaf of $\mathcal{O}_V$-Rees algebras $\mathcal{G} = \oplus_n I_n W^n$ is a *β-relative differential Rees algebra* or simply a *β-differential Rees algebra* if there is an affine covering $\{U_i\}$ of V, such that for every homogeneous element $f W^N \in \mathcal{G}$ and every $\Delta \in Diff_{V/V'}^n(U_i)$ with $n < N$, we have that $\Delta(f) W^{N-n} \in \mathcal{G}$ (in particular, $I_{i+1} \subset I_i$ since $Diff_{V/V'}^0 \subset Diff_{V/V'}^1$). Given an arbitrary Rees algebra $\mathcal{G}$ over V there is a natural way to construct a β-relative differential algebra with the property of being the smallest containing $\mathcal{G}$, and we will denote it by $Diff_{V/V'}(\mathcal{G})$ (see [31, Theorem 2.7]). Relative differential Rees algebras will play a role in the definition of the so called *elimination algebras* that will be treated in Section 6.

The Rees algebra associated to $\underline{\operatorname{Max}} \operatorname{mult}(X)$.

Theorem 5.21. [1, Theorem 5.3] *Let X be a singular algebraic variety defined over a perfect field k. Suppose that there are two immersions $X \subset V_1$ and $X \subset V_2$ together with two Rees algebras $\mathcal{G}_1$ and $\mathcal{G}_2$ over V_1 and V_2 respectively such that $(V_1, \mathcal{G}_1)$ and $(V_2, \mathcal{G}_2)$ represent the stratum of maximum multiplicity of X (see Definition 5.7). Then the $\mathcal{O}_X$-algebras $\mathrm{Diff}(\mathcal{G}_1)|_X$ and $\mathrm{Diff}(\mathcal{G}_2)|_X$ are equal up to integral closure.*

Remark 5.22. Recall that a situation as in the previous theorem can always be achieved locally in étale topology (see 5.8). Moreover, it can be shown that if $X \leftarrow \widetilde{X}_1$ and $X \leftarrow \widetilde{X}_2$ are two étale neighborhoods together with embedings $\widetilde{X}_1 \subset V_1$ and $\widetilde{X}_2 \subset V_2$ where two Rees algebras attached to the maximum multiplicity locus are defined, then there is an étale (common) neighborhood $\widetilde{X}$ of $X, \widetilde{X}_1$, and $\widetilde{X}_2$, and two embeddings, $\widetilde{X} \subset \widetilde{V}_1$ and $\widetilde{X} \subset \widetilde{V}_2$ together with two Rees algebras, say $\mathcal{G}_1$ and $\mathcal{G}_2$ defined on $\widetilde{V}_1$ and $\widetilde{V}_2$ respectively, representing the maximum multiplicity of $\widetilde{X}$. This is the sense in which the previous theorem should be interpreted (see [6, §29]).

Definition 5.23. Let X be a singular algebraic variety over a perfect field k with maximum multiplicity $s > 1$. Given a local presentation of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ as in Definition 5.7, say $(V, \mathcal{G})$, we define the *Rees algebra attached to* $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ to be $\mathcal{G}_X := \overline{\mathrm{Diff}(\mathcal{G})|_X}$.

Remark 5.24. Given a singular algebraic variety X over a perfect field k, local presentations of $\underline{\mathrm{Max}}\,\mathrm{mult}(X)$ exist only in étale topology (see 5.8 and 5.9). Thus $\mathcal{G}_X$ is only defined locally in étale topology. Theorem 5.21 says that $\mathcal{G}_X$ does not depend on the choice of the local presentation $(V, \mathcal{G})$.

6. Elimination algebras

In the previous section we discussed about local presentations of the multiplicity for a given algebraic variety X via some pair $(V, \mathcal{G})$. This has been useful to prove algorithmic resolution of singularities in characteristic zero using the multiplicity as main invariant. In this context, one advantage of using the multiplicity instead of the Hilbert-Samuel function (the invariant used by Hironaka in [14]) is that, at least when the characteristic is zero, it can be shown that the lowering of the maximum multiplicity of X is equivalent to resolving a Rees algebra defined in some d-dimensional regular scheme, where d is the dimension of X. This follows from the fact that the local presentation of the multiplicity is found by considering a suitable finite projection from X to some regular scheme Z (see 5.12 and Example 6.21 below).

The previous discussion leads to the following question. Once a pair $(V, \mathcal{G})$ is given, one may wonder whether there is another $(V', \mathcal{G}')$ with $\dim V' < \dim V$ somehow *equivalent* to $(V, \mathcal{G})$. That is, we would like, (1) that $\mathrm{Sing}(\mathcal{G})$ be homeomorphic to $\mathrm{Sing}(\mathcal{G}')$, and (2), that this homeomorphism be preserved by local sequences. This would mean that finding a resolution of $\mathcal{G}$ is *equivalent* to finding a resolution of $\mathcal{G}'$, the latter being *less complex* since the problem concerns an ambient space of lower dimension. This would allow us to *resolve* Rees algebras by using an inductive argument. This is the motivation behind the theory of *elimination algebras* to be carefully discussed in this section (further details about elimination algebras can be found in [5], [6], [32], and [31]).

To define an elimination algebra for a given one, the starting point will be a pair $(V, \mathcal{G})$ as before, and some smooth morphism to a (smooth) scheme of lower dimension $\beta : V \to V'$. To be able to talk about elimination we will require some additional conditions on β (which will be made precise in Definition 6.2 below). Now, as we will see, the dificulty here is that if we are given a $\mathcal{G}$-permissible sequence over V we would like to be able to define (in a natural way) another over V' together with smooth morphisms β_i and commutative diagrams:

$$
\begin{array}{ccccccc}
V & \longleftarrow & V_1 & \longleftarrow & \ldots & \longleftarrow & V_l \\
\beta \downarrow & & \beta_1 \downarrow & & & & \beta_l \downarrow \\
V' & \longleftarrow & V_1' & \longleftarrow & \ldots & \longleftarrow & V_l'.
\end{array}
$$

However this requires a little bit of care when the permissible transformations involve permissible blow ups. This motivates the following discussion which we hope will help to clarify the meaning of Definition 6.2.

6.1. On the compatibility of permissible blow ups with smooth projections. Let $\beta : V \to V'$ be a smooth morphism of regular varieties over a perfect field k with $\dim V \geq \dim V'$. Let $\mathcal{G} \subset \mathcal{O}_V[W]$ and $\mathcal{G}' \subset \mathcal{O}_{V'}[W]$ be Rees algebras. Suppose that $\mathrm{Sing}(\mathcal{G})$ is homeomorphic to $\mathrm{Sing}(\mathcal{G}')$ via β, and let $Y \subset \mathrm{Sing}\,\mathcal{G}$ be a permissible center. If $Z' = \beta(Y) \subset \mathrm{Sing}(\mathcal{G}')$ is also a permissible center, then we can consider the blow ups of V and V' at Y and $Z = \beta(Y)$ respectively, say

$$
\begin{array}{ccc}
V & \longleftarrow & V_1 \\
{\scriptstyle\beta}\downarrow & & \\
V' & \longleftarrow & V_1'.
\end{array}
$$

Let $\mathcal{G}_1'$ be the transform of $\mathcal{G}'$ in V_1' and let $\mathcal{G}_1$ be the transform of $\mathcal{G}$ in V_1. In general, β cannot be lifted to a morphism $\beta_1 : V_1 \to V_1'$ to complete the square. However for our purposes it would be enough if we could guarantee that β can be lifted to a smooth morphims from some open subset $U_1 \subset V_1$ with $\mathrm{Sing}(\mathcal{G}_1) \subset U_1$,

$$
\begin{array}{ccc}
V & \longleftarrow & U_1 \\
{\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta_1} \\
V' & \longleftarrow & V_1'.
\end{array}
\tag{6.1.1}
$$

In such case $\beta_1(\mathrm{Sing}(\mathcal{G}_1))$ can be defined and we may wonder whether it is homeomorphic to $\mathrm{Sing}(\mathcal{G}_1')$ or not. The situation is quite similar if instead we consider a permissible center $Z \subset \mathrm{Sing}(\mathcal{G}')$ and consider the blow ups of V' at Z and of V at $\beta^{-1}(Z) \cap \mathrm{Sing}(\mathcal{G})$. This motivates the formulation of part (2) of the following definition. As we will see, for a smooth morphism $\beta : V \to V'$ to be $\mathcal{G}$-*admissible* (see Definition 6.2), we will only ask for the existence of a commutative diagram like (6.1.1) after a $\mathcal{G}$-permissible blow up (see Remark 6.5: there we describe the open set U_1 after a $\mathcal{G}$-permissible blow up for a given $\mathcal{G}$-admissible projection).

Thus, in forthcoming discussions whenever we define local sequences over V' and V as above we will be assuming that the blow ups at permissible centers will be restricted to suitable open subsets of the transforms of V that contain the singular locus of the transforms of $\mathcal{G}$ so that commutative diagrams as (6.1.1) can be considered.

Definition 6.2. Let $V^{(n)}$ an n-dimensional smooth variety over a perfect field k, and let $\mathcal{G}^{(n)}$ be a Rees algebra over $V^{(n)}$. We will say that a smooth morphism from $V^{(n)}$ to a smooth k-variety $V^{(n-e)}$ of dimension $(n-e)$, say $\beta : V^{(n)} \to V^{(n-e)}$, is a $\mathcal{G}^{(n)}$-*admissible projection* if the following conditions hold:

(1) β maps $\mathrm{Sing}(\mathcal{G}^{(n)})$ homeomorphically to its image in $V^{(n-e)}$. Moreover, it is required that, for any closed subscheme $Y \subset \mathrm{Sing}(\mathcal{G}^{(n)})$, Y is regular if and only if $\beta(Y) \subset V^{(n-e)}$ is so;

(2) For any $\mathcal{G}^{(n)}$-local sequence of transformations on $V^{(n)}$, say

$$
\begin{array}{cccc}
\mathcal{G}^{(n)} = \mathcal{G}_0^{(n)} & \mathcal{G}_1^{(n)} & & \mathcal{G}_l^{(n)} \\
V^{(n)} = V_0^{(n)} \xleftarrow{\;\varphi_1\;} & V_1^{(n)} \xleftarrow{\;\varphi_2\;} & \cdots \xleftarrow{\;\varphi_l\;} & V_l^{(n)},
\end{array}
\tag{6.2.1}
$$

there exists a collection of open subschemes, say $U_i^{(n)} \subset V_i^{(n)}$ for $i = 1, \ldots, l$, together with a sequence of transformations over $V^{(n-e)}$, say

$$
V^{(n-e)} = V_0^{(n-e)} \longleftarrow V_1^{(n-e)} \longleftarrow \cdots \longleftarrow V_l^{(n-e)},
\tag{6.2.2}
$$

and a commutative diagram, say

$$
\begin{array}{ccccccccc}
V^{(n)} & \xleftarrow{\varphi_1} & V_1^{(n)} & \xleftarrow{\varphi_2} & V_2^{(n)} & \xleftarrow{\varphi_3} & \cdots & \xleftarrow{\varphi_l} & V_l^{(n)} \\
& & \uparrow & & \uparrow & & & & \uparrow \\
\beta \downarrow & & U_1^{(n)} & \longleftarrow & U_2^{(n)} & \longleftarrow & \cdots & \longleftarrow & U_l^{(n)} \\
& & \downarrow \beta_1 & & \downarrow \beta_2 & & & & \downarrow \beta_l \\
V^{(n-e)} & \longleftarrow & V_1^{(n-e)} & \longleftarrow & V_2^{(n-e)} & \longleftarrow & \cdots & \longleftarrow & V_l^{(n-e)},
\end{array}
$$

(6.2.3)

with the following properties:

i) $\varphi_i\big(U_i^{(n)}\big) \subset U_{i-1}^{(n)}$, in such a way that $U_{i-1}^{(n)} \leftarrow U_i^{(n)}$ is the natural restriction of φ_i to $U_i^{(n)}$ for every $i = 2, \ldots, l$;

ii) $\mathrm{Sing}\big(\mathcal{G}_i^{(n)}\big) \subset U_i^{(n)}$ for every $i = 1, \ldots, l$;

iii) Each $\beta_i : U_i^{(n)} \to V_i^{(n-e)}$ is a smooth morphism mapping $\mathrm{Sing}\big(\mathcal{G}_i^{(n)}\big)$ homeomorphically to its image in $V_i^{(n-e)}$. Moreover, it is required that, for any closed subscheme $Y_i \subset \mathrm{Sing}\big(\mathcal{G}_i^{(n)}\big)$, Y_i is regular if and only if $\beta_i(Y_i) \subset V_i^{(n-e)}$ is so;

iv) The sequence in (6.2.2) is the sequence naturally induced by (6.2.1) in the following sense: for each $i = 1, \ldots, l$, if $V_i^{(n)}$ is the blow up of $V_{i-1}^{(n)}$ at a closed center $Y_{i-1} \subset \mathrm{Sing}\big(\mathcal{G}_{i-1}^{(n)}\big)$, then $V_i^{(n-e)}$ is the blow up of $V_{i-1}^{(n-e)}$ at $\beta_{i-1}(Y_{i-1})$; if $V_i^{(n)}$ is an open restriction or the multiplication by an affine line of $V_{i-1}^{(n)}$, then so is $V_i^{(n-e)}$ with respect to $V_{i-1}^{(n-e)}$.

Given $V^{(n)}$ and $\mathcal{G}^{(n)}$, the question of whether there exists a $\mathcal{G}^{(n)}$-admissible projection to some smooth space of dimension $(n-e)$ arises. The existence of such admissible projections can be characterized in terms of the τ-invariant, an invariant attached to each closed point $\xi \in \mathrm{Sing}\,\mathcal{G}^{(n)}$.

Definition 6.3. Let $V^{(n)}$ be a regular variety over a perfect field k and let $\mathcal{G}^{(n)}$ be a Rees algebra over $V^{(n)}$. Fix a closed point $\xi \in \mathrm{Sing}(\mathcal{G}^{(n)})$, and let $\mathrm{Gr}_{m_\xi}(\mathcal{O}_{V^{(n)},\xi})$ denote the graded ring of $\mathcal{O}_{V^{(n)},\xi}$. Consider the tangent ideal of $\mathcal{G}^{(n)}$ at ξ, say $\mathrm{In}_\xi\,\mathcal{G}^{(n)} \subset \mathrm{Gr}_{m_\xi}(\mathcal{O}_{V^{(n)},\xi})$, defined as the homogeneous ideal generated by $\mathrm{In}_\xi(I_n) := \big(I_n + m_\xi^{n+1}\big)/m_\xi^{n+1}$ for all $n \geq 1$. Note that $\mathrm{Gr}_{m_\xi}(\mathcal{O}_{V^{(n)},\xi}) \simeq k'[Z_1, \ldots, Z_n]$, where $Z_1, \ldots, Z_n$ is a basis of the subspace of linear forms of $\mathrm{Gr}_{m_\xi}(\mathcal{O}_{V^{(n)},\xi})$, say $\mathrm{Gr}^1_{m_\xi}(\mathcal{O}_{V^{(n)},\xi})$, and k' is the residue field at ξ. The τ-invariant is the minimum integer so that there exists a basis of $Gr^1_{m_\xi}(\mathcal{O}_{V^{(n)},\xi})$, say $Y_1, \ldots, Y_\tau$, satisfying that $\mathrm{In}_\xi\,\mathcal{G}^{(n)}$ can be generated by polynomials in $k'[Y_1, \ldots, Y_\tau]$ (see [3]). That is, so that

$$
\mathrm{In}_\xi\,\mathcal{G}^{(n)} = \Big(\mathrm{In}_\xi\,\mathcal{G}^{(n)} \cap k'[Y_1, \ldots, Y_\tau]\Big) \cdot \mathrm{Gr}_{m_\xi}(\mathcal{O}_{V^{(n)},\xi}).
$$

The τ-invariant appears in [14], and has been thoroughly studied in [24], [25] and [26]. See also [15], [18] and [3].

Lemma 6.4 (cf. [31, §4], [5, §8]). *Let $V^{(n)}$ be a smooth variety of dimension n over a perfect field k, and let $\mathcal{G}^{(n)}$ be a Rees algebra over $V^{(n)}$. Let $\xi \in \mathrm{Sing}(\mathcal{G}^{(n)})$ be a closed point. Then, for $e \leq \tau_{\mathcal{G}^{(n)},\xi}$, one can construct a $\mathcal{G}^{(n)}$-admissible projection (locally in étale topology) to some smooth k-variety of dimension $(n-e)$, $\beta : V^{(n)} \to V^{(n-e)}$.*

Remark 6.5. If $\beta : V^{(n)} \to V^{(n-e)}$ is a $\mathcal{G}^{(n)}$-admissible projection (e.g., as the ones constructed in [31] or [5]) and if $Y \subset \mathrm{Sing}(\mathcal{G}^{(n)})$ is a permissible center, then it can be shown that there is a well defined open subset $U_1^{(n)} \subset V_1^{(n)}$ of the blow up of $V^{(n)}$ at Y so that the following diagram of

smooth projections and blow ups at Y and at $Z := \beta(Y)$ commutes:

$$
\begin{array}{ccc}
V^{(n)} & \longleftarrow & U_1^{(n)} \\
\beta \downarrow & & \beta_1 \downarrow \\
V^{(n-e)} & \longleftarrow & V_1^{(n-e)}.
\end{array}
\tag{6.5.1}
$$

In fact, the open $U_1^{(n)}$ can be defined as follows. Set $\widetilde{Z} = \beta^{-1}(Z)$ and let $\widetilde{Z}_1$ be the strict transform of $\widetilde{Z}$ in V_1. Define $U_1^{(n)} := V_1^{(n)} \setminus \widetilde{Z}_1$. Then it can be checked that $\mathrm{Sing}(\mathcal{G}_1^{(n)}) \subset U_1^{(n)}$ and that β can be lifted to a smooth morphism from $U_1^{(n)}$ (see [31, §6], [5, Theorem 9.1]). Just to ease the notation when we write a diagram like (6.5.1) we will write $V_1^{(n)}$ instead of $U_1^{(n)}$.

Roughly speaking, Lemma 6.4 says that, under some assumptions, the tree of closed sets defined by $\mathcal{G}^{(n)}$ can be interpreted as a tree of closed subsets in some smooth scheme of lower dimension. The next question is whether one can find a Rees algebra over $V^{(n-e)}$ which determines this tree of closed sets.

Definition 6.6. Let $\mathcal{G}^{(n)}$ be a Rees algebra over a smooth variety $V^{(n)}$, and let $\beta : V^{(n)} \to V^{(n-e)}$ be a $\mathcal{G}^{(n)}$-admissible projection. We will say that a Rees algebra $\mathcal{G}^{(n-e)}$ over $V^{(n-e)}$ is *strongly linked* to $\mathcal{G}^{(n)}$ via β if the following conditions hold:

(1) $\mathrm{Sing}(\mathcal{G}^{(n-e)}) = \beta\big(\mathrm{Sing}(\mathcal{G}^{(n)})\big)$ (note that, since β is required to be $\mathcal{G}^{(n)}$-admissible, this implies that a closed center $Y \subset \mathrm{Sing}(\mathcal{G}^{(n)})$ is regular if and only if the corresponding closed subscheme, say $\beta(Y) \subset \mathrm{Sing}(\mathcal{G}^{(n-e)})$, is so);

(2) Any $\mathcal{G}^{(n)}$-permissible sequence on $V^{(n)}$, say

$$
\begin{array}{cccc}
\mathcal{G}^{(n)} = \mathcal{G}_0^{(n)} & \mathcal{G}_1^{(n)} & & \mathcal{G}_l^{(n)} \\
V^{(n)} = V_0^{(n)} & \longleftarrow V_1^{(n)} & \longleftarrow \cdots \longleftarrow & V_l^{(n)},
\end{array}
\tag{6.6.1}
$$

induces a $\mathcal{G}^{(n-e)}$-permissible sequence on $V^{(n-e)}$, say

$$
\begin{array}{cccc}
\mathcal{G}^{(n-e)} = \mathcal{G}_0^{(n-e)} & \mathcal{G}_1^{(n-e)} & & \mathcal{G}_l^{(n-e)} \\
V^{(n-e)} = V_0^{(n-e)} & \longleftarrow V_1^{(n-e)} & \longleftarrow \cdots \longleftarrow & V_l^{(n-e)},
\end{array}
\tag{6.6.2}
$$

and a commutative diagram as (6.2.3) in Definition 6.2 in such a way that β_i is a $\mathcal{G}_i^{(n)}$-admissible projection and

$$
\mathrm{Sing}(\mathcal{G}_i^{(n-e)}) = \beta_i\left(\mathrm{Sing}(\mathcal{G}_i^{(n)})\right)
$$

for every $i = 1, \ldots, l$ (recall that, by condition (2) ii) of Definition 6.2, one has that $\mathrm{Sing}(\mathcal{G}_i^{(n)}) \subset U_i^{(n)}$ for all i).

Remark 6.7. Let $\mathcal{G}^{(n-e)}$ be a Rees algebra which is strongly linked to $\mathcal{G}^{(n)}$ as in the previous definition via the morphism $\beta : V^{(n)} \to V^{(n-e)}$. Then any closed regular center $Z \subset \mathrm{Sing}(\mathcal{G}^{(n-e)})$ induces a closed regular center $Y \subset \mathrm{Sing}(\mathcal{G}^{(n)})$. Thus, any $\mathcal{G}^{(n-e)}$-permissible sequence of transformations like (6.6.2) induces a $\mathcal{G}^{(n)}$-permissible sequence like (6.6.1) and a commutative diagram like (6.2.3). In particular $\mathrm{Sing}(\mathcal{G}_l^{(n-e)}) = \emptyset$ if and only if $\mathrm{Sing}(\mathcal{G}_l^{(n)}) = \emptyset$. Thus it turns out that, if $\mathcal{G}^{(n-e)}$ is strongly linked to $\mathcal{G}^{(n)}$, then a resolution of $\mathcal{G}^{(n)}$ induces a resolution $\mathcal{G}^{(n-e)}$, and vice versa.

Given a Rees algebra $\mathcal{G}^{(n)}$ over a smooth variety $V^{(n)}$ and a $\mathcal{G}^{(n)}$-admissible projection $\beta : V^{(n)} \to V^{(n-e)}$, we will see that, if the characteristic is zero, there is always a Rees algebras over $V^{(n-e)}$ that is srongly linked to $\mathcal{G}^{(n)}$. However, if the the characteristic is positive, it may occur that one

cannot find algebras over $V^{(n-e)}$ which are strongly linked to $\mathcal{G}^{(n)}$ (see Example 6.18). This leads us to introduce a weaker notion:

Definition 6.8. [31, 1.25, Definitions 1.42, 4.10, Theorem 4.11] Let $V^{(n)}$ be an n-dimensional smooth variety over a perfect field k, and let $\mathcal{G}^{(n)}$ be a Rees algebra over $V^{(n)}$. Consider a $\mathcal{G}^{(n)}$-admissible projection to a smooth variety $V^{(n-e)}$ of dimension $(n-e)$, say $\beta : V^{(n)} \to V^{(n-e)}$ and assume that $\mathcal{G}^{(n)}$ is a β-relative differential Rees algebra (see 5.20). Then the $\mathcal{O}_{V^{(n-e)}}$-Rees algebra algebra

$$\mathcal{G}^{(n-e)} := \mathcal{G}^{(n)} \cap \mathcal{O}_{V^{(n-e)}}[W], \tag{6.8.1}$$

or any other Rees algebra over $V^{(n-e)}$ with the same integral closure is called *an elimination algebra* of $\mathcal{G}^{(n)}$ via β.

Remark 6.9. Note that, if an algebra $\mathcal{G}^{(n)}$ is (absolutely) differential, then it is relative differential for any $\mathcal{G}^{(n)}$-admissible projection $\beta : V^{(n)} \to V^{(n-e)}$. Hence, in this case,

$$\mathcal{G}^{(n-e)} = \mathcal{G}^{(n)} \cap \mathcal{O}_{V^{(n-e)}}[W] \tag{6.9.1}$$

is also an elimination algebra of $\mathcal{G}^{(n)}$. However, it may occur that the elimination algebra defined in (6.9.1) is not integral over the one defined in 6.8 (observe that in Definition 6.8 we only require that $\mathcal{G}^{(n)}$ be β-relative differential, see 5.20). In other words, given a Rees algebra $\mathcal{G}^{(n)}$ in $V^{(n)}$ and a $\mathcal{G}^{(n)}$-admissible projection, $\beta : V^{(n)} \to V^{(n-e)}$, in general the $\mathcal{O}_{V^{(n-e)}}$-Rees algebras

$$Diff(\mathcal{G}^{(n)}) \cap \mathcal{O}_{V^{(n-e)}}[W] \quad \text{and } Diff_{V/V'}(\mathcal{G}^{(n)}) \cap \mathcal{O}_{V^{(n-e)}}[W]$$

do not share the same integral closure in $\mathcal{O}_{V^{(n-e)}}[W]$, and, in general, they may not be weakly equivalent either as it can be checked in the following example.

Example 6.10. Suppose that k is a perfect field of characteristic 2. Consider the curve $X = \mathrm{Spec}(k[x,y]/\langle y^2 - x^3 \rangle)$, endowed with its natural immersion in $V^{(2)} = \mathrm{Spec}(k[x,y])$. One can check that $\mathcal{O}_{V^{(2)}}\left[(y^2 - x^3)W^2\right]$ is a Rees algebra over $V^{(2)}$ which represents $F_2(X)$. Now define:

$$\mathcal{G}^{(2)} := Diff\left(\mathcal{O}_{V^{(2)}}\left[(y^2 - x^3)W^2\right]\right) = \mathcal{O}_{V^{(2)}}\left[x^2 W, (y^2 - x^3)W^2\right]$$

and

$$\mathcal{H}^{(2)} := Diff_{V^{(2)}/V^{(1)}}\left(\mathcal{O}_{V^{(2)}}\left[(y^2 - x^3)W^2\right]\right) = \mathcal{O}_{V^{(2)}}\left[(y^2 - x^3)W^2\right].$$

Observe that the inclusion $k[x] \subset k[x,y]$ induces a smooth projection of $V^{(2)}$ to $V^{(1)} = \mathrm{Spec}(k[x])$, say $\beta : V^{(2)} \to V^{(1)}$ which is both, $\mathcal{G}^{(2)}$-admissible and $\mathcal{H}^{(2)}$-admissible. Now note that

$$\mathcal{G}^{(1)} = \mathcal{G}^{(2)} \cap \mathcal{O}_{V^{(1)}}[W] = \mathcal{O}_{V^{(2)}}\left[x^2 W, (y^2 - x^3)W^2\right] \cap \mathcal{O}_{V^{(1)}}[W] = \mathcal{O}_{V^{(1)}}[x^2 W],$$

while

$$\mathcal{H}^{(1)} = \mathcal{H}^{(2)} \cap \mathcal{O}_{V^{(1)}}[W] = \mathcal{O}_{V^{(2)}}\left[(y^2 - x^3)W^2\right] \cap \mathcal{O}_{V^{(1)}}[W] = 0.$$

Thus $\mathcal{G}^{(1)}$ and $\mathcal{H}^{(1)}$ are not weakly equivalent.

Remark 6.11. Given a Rees algebra $\mathcal{G}^{(n)}$, and a $\mathcal{G}^{(n)}$-admissible projection $\beta : V^{(n)} \to V^{(n-e)}$, for geometrical reasons we will be interested in the case in which $\beta(\mathrm{Sing}(\mathcal{G}^{(n)})) \subsetneq V^{(n-e)}$. In fact, if $\beta(\mathrm{Sing}(\mathcal{G}^{(n)})) = V^{(n-e)}$ then it can be checked that a resolution of $\mathcal{G}^{(n)}$ can be achieved in a trivial way (see [5, Lemma 13.2]). However, in this paper, we will always be in the case in which $\beta(\mathrm{Sing}(\mathcal{G}^{(n)})) \subsetneq V^{(n-e)}$ (this will follow because of the hypotheses under we will be considering the admissible projections and the elimination algebras).

6.12. Elimination algebras and singular loci of Rees algebras. [30, Theorem 2.9], [31, Corollary 4.12] Let $V^{(n)}$ be an n-dimensional smooth variety over a perfect field k, and let $\mathcal{G}^{(n)}$ be a Rees algebra over $V^{(n)}$. Consider a $\mathcal{G}^{(n)}$-admissible projection to a smooth variety $V^{(n-e)}$ of dimension $(n-e)$, say $\beta : V^{(n)} \to V^{(n-e)}$, and suppose that $\mathcal{G}^{(n)}$ is a β-differential Rees algebra. Define an elimination algebra $\mathcal{G}^{(n-e)}$ as in (6.8.1). Then:

(i) If the characteristic of the base field k is zero, $\beta(\mathrm{Sing}(\mathcal{G}^{(n)})) = \mathrm{Sing}(\mathcal{G}^{(n-e)})$;

(ii) If the characteristic of the base field k is positive, there is a containment $\beta(\mathrm{Sing}(\mathcal{G}^{(n)})) \subseteq \mathrm{Sing}(\mathcal{G}^{(n-e)})$ which may be strict. (see Example 6.13). On the other hand, if $\mathcal{G}^{(n)}$ is a differential Rees algebra, then it can be shown that $\beta(\mathrm{Sing}(\mathcal{G}^{(n)})) = \mathrm{Sing}(\mathcal{G}^{(n-e)})$. However, in general, this equality is not stable after considering local sequences; this issue will be discussed in the following paragraphs (see also Example 6.18).

Example 6.13. Going back to Example 6.10, we have that $\beta(\mathrm{Sing}(\mathcal{G}^{(2)})) = \mathrm{Sing}(\mathcal{G}^{(1)})$. On the other hand, $\beta(\mathrm{Sing}(\mathcal{H}^{(2)}) \subsetneq \mathrm{Sing}(\mathcal{H}^{(1)}) = \mathrm{Spec}(k[x])$.

6.14. Elimination algebras and permissible transformations. [30, Lemma 1.7], [31, §6], [5, §8, Theorem 9.1] Let $V^{(n)}$ be an n-dimensional smooth variety over a perfect field k, and let $\mathcal{G}^{(n)}$ be a Rees algebra over $V^{(n)}$. Consider a $\mathcal{G}^{(n)}$-admissible projection to a smooth variety $V^{(n-e)}$ of dimension $(n - e)$, say $\beta : V^{(n)} \to V^{(n-e)}$, and suppose that $\mathcal{G}^{(n)}$ is a β-differential Rees algebra. Define an elimination algebra $\mathcal{G}^{(n-e)}$ as in (6.8.1). Let $Y \subset \mathrm{Sing}(\mathcal{G}^{(n)})$ be a permissible center. Then it can be shown that $\beta(Y) \subset \mathrm{Sing}(\mathcal{G}^{(n-e)})$ is a $\mathcal{G}^{(n-e)}$-permissible center, and there is a commutative diagram of blow ups, smooth morphisms and transforms of Rees algebras

$$
\begin{array}{ccccccc}
\mathcal{G}^{(n)} & & V^{(n)} & \longleftarrow & V_1^{(n)} & & \mathcal{G}_1^{(n)} \\
& & \beta\downarrow & & \beta_1\downarrow & & \\
\mathcal{G}^{(n-e)} & & V^{(n-e)} & \longleftarrow & V_1^{(n-e)} & & \mathcal{G}_1^{(n-e)}
\end{array}
\tag{6.14.1}
$$

(see Remark 6.5), and:

(i) The projection $\beta_1 : V_1^{(n)} \to V_1^{(n-e)}$ is $\mathcal{G}_1^{(n)}$-admissible;

(ii) The Rees algebra $\mathcal{G}_1^{(n)}$ is a β_1-differential Rees algebra, therefore an elimination algebra can be defined as in (6.8.1);

(iii) The transform of $\mathcal{G}^{(n-e)}$ in $V_1^{(n-e)}$, $\mathcal{G}_1^{(n-e)}$, is an elimination algebra of $\mathcal{G}_1^{(n)}$, i.e., up to integral closure,

$$
\mathcal{G}_1^{(n-e)} = \mathcal{G}_1^{(n)} \cap \mathcal{O}_{V_1^{(n-e)}}[W].
$$

Remark 6.15. With the same setting and notation as in 6.14 above, we would like to point out that, even if we assume that the Rees algebra $\mathcal{G}^{(n)}$ is a differential Rees algebra, after a permissible blow up, its transform $\mathcal{G}_1^{(n)}$, in general, is not a differential Rees algebra; however it is always a β_1-relative differential Rees algebra.

6.16. Elimination algebras and local sequences. [6, §16] Let $V^{(n)}$ be an n-dimensional smooth algebraic variety over a perfect field k, and let $\mathcal{G}^{(n)}$ be a Rees algebra over $V^{(n)}$. Consider a $\mathcal{G}^{(n)}$-admissible projection to a smooth variety $V^{(n-e)}$ of dimension $(n - e)$, say $\beta : V^{(n)} \to V^{(n-e)}$ and suppose that $\mathcal{G}^{(n)}$-is a β-differential Rees algebra. Define an elimination algebra $\mathcal{G}^{(n-e)}$ as in (6.8.1). Then it can be checked that any $\mathcal{G}^{(n)}$-local sequence induces a $\mathcal{G}^{(n-e)}$-local sequence and a commutative diagram of smooth morphisms and transforms:

$$
\begin{array}{ccccccc}
(V^{(n)}, \mathcal{G}^{(n)}) & \longleftarrow & (V_1^{(n)}, \mathcal{G}_1^{(n)}) & \longleftarrow \cdots \longleftarrow & (V_l^{(n)}, \mathcal{G}_l^{(n)}) \\
\downarrow\beta & & \downarrow\beta_1 & & \downarrow\beta_l \\
(V^{(n-e)}, \mathcal{G}^{(n-e)}) & \longleftarrow & (V_1^{(n-e)}, \mathcal{G}_1^{(n-e)}) & \longleftarrow \cdots \longleftarrow & (V_l^{(n-e)}, \mathcal{G}_l^{(n-e)})
\end{array}
\tag{6.16.1}
$$

where each β_i is $\mathcal{G}_i^{(n)}$-admissible, $\mathcal{G}_i^{(n-e)}$ is an elimination algebra of $\mathcal{G}^{(n)}$, and $\beta_i(\mathrm{Sing}\,\mathcal{G}_i^{(n)}) \subseteq \mathrm{Sing}\,\mathcal{G}_i^{(n-e)}$.

When the characteristic of the base field is zero, then the equality $\beta_i\big(\mathrm{Sing}\,\mathcal{G}_i^{(n)}\big) = \mathrm{Sing}\,\mathcal{G}_i^{(n-e)}$ holds for $i = 1, \ldots, n$, and therefore, in this case the elimination algebra $\mathcal{G}^{(n-e)}$ is stronly linked to $\mathcal{G}^{(n)}$. As a consequence, a resolution of $\mathcal{G}^{(n)}$ induces a resolution of $\mathcal{G}^{(n-e)}$ and vice versa (see Remark 6.7).

If the charateristic of the base field is positive, the elimination algebra $\mathcal{G}^{(n-e)}$ might not be strongly linked to $\mathcal{G}^{(n)}$. Moreover, given $\mathcal{G}^{(n)}$ and $\beta : V^{(n)} \to V^{(n-e)}$ as above, it may occur that there is no Rees algebra over $V^{(n-e)}$ which is strongly linked to $\mathcal{G}^{(n)}$ (see Example 6.18 below). Still, it can be shown that $\mathcal{G}^{(n-e)}$ is the largest algebra over $V^{(n-e)}$ satisfying the containment

$$\mathcal{F}_{V^{(n)}}\big(\mathcal{G}^{(n)}\big) \subset \mathcal{F}_{V^{(n)}}\big(\beta^*(\mathcal{G}^{(n-e)})\big). \tag{6.16.2}$$

(see Remark 6.17 below). Thus, an elimination algebra of $\mathcal{G}^{(n)}$ can also be regarded as a maximal Rees algebra over $V^{(n-e)}$ with the previous property.

Remark 6.17. Let $\mathcal{G}^{(n)}$ be a Rees algebra over a regular algebraic variety $V^{(n)}$ defined over a perfect field k, and consider a $\mathcal{G}^{(n)}$-admissible projection, say $\beta : V^{(n)} \to V^{(n-e)}$. Assume in addition that $\mathcal{G}^{(n)}$ is a differential Rees algebra, and consider the corresponding elimination algebra on $V^{(n-e)}$. Now suppose that $\mathcal{K}^{(n-e)}$ is a Rees algebra over $V^{(n-e)}$ which is strongly linked to $\mathcal{G}^{(n)}$. In such case, $\mathcal{K}^{(n-e)}$ is, necessarily, an elimination algebra of $\mathcal{G}^{(n)}$. To see this, observe first that if $\mathcal{K}^{(n-e)}$ is strongly linked to $\mathcal{G}^{(n)}$, by Theorem 5.19 it can be assumed that the pull-back of $\mathcal{K}^{(n-e)}$ to $V^{(n)}$ is contained in $\mathcal{G}^{(n)}$; hence $\mathcal{K}^{(n-e)} \subset \mathcal{G}^{(n)} \cap \mathcal{O}_{V^{(n-e)}}[W]$ the latter being an elimination algebra of $\mathcal{G}^{(n)}$, say $\mathcal{G}^{(n-e)}$. Then there is a chain of containments $\mathrm{Sing}(\mathcal{K}^{(n-e)}) \supseteq \mathrm{Sing}(\mathcal{G}^{(n-e)}) \supseteq \beta(\mathrm{Sing}(\mathcal{G}^{(n)}))$. Finally, as $\mathcal{K}^{(n-e)}$ is assumed to be strongly linked to $\mathcal{G}^{(n)}$ one has that the previous containments are equalities, and furthermore, such equalities are preserved after considering any $\mathcal{G}^{(n)}$-local sequence (or any $\mathcal{K}^{(n-e)}$-local sequence). From here it follows that $\mathcal{K}^{(n-e)}$ is weakly equivalent to $\mathcal{G}^{(n-e)}$ by Theorem 5.19 (see also Theorem 6.19 and Corollary 6.20 below).

The following example illustrates that, in positive characteristic, an elimination algebra might not be strongly linked to the original Rees algebra.

Example 6.18. We go back to Example 6.10. Consider the blowing up of $V^{(2)}$ and $V^{(1)}$ at the origin. Then one can check that $\mathcal{G}^{(1)}$ is not strongly linked to $\mathcal{G}^{(2)}$. To see this recall how Rees algebras transform after a permissible blow up (see 5.4). Observe that after the blow up, $V^{(2)} \leftarrow V_1^{(2)}$, at the $k\left[x, \frac{y}{x}\right]$-affine chart,

$$\mathcal{G}_1^{(2)} = k\left[x, \frac{y}{x}\right]\left[xW, \left(\left(\frac{y}{x}\right)^2 - x\right)W^2\right],$$

and therefore, here $\mathrm{Sing}(\mathcal{G}_1^{(2)}) = \emptyset$. On the other hand, after the blow up $V^{(1)} \leftarrow V_1^{(1)}$ (which is an isomorphism on $V^{(1)}$) $\mathcal{G}_1^{(1)} = k[x][xW]$, and hence $\mathrm{Sing}(\mathcal{G}_1^{(1)}) \neq \emptyset$. On the other hand, if there was an $\mathcal{O}_{V^{(1)}}$-Rees algebra strongly linked to $\mathcal{G}^{(2)}$, say $\mathcal{K}^{(1)}$, then, by Remark 6.17 necessarily $\mathcal{K}^{(1)}$ should be weakly equivalent to $\mathcal{G}^{(1)}$. This shows that there is no $\mathcal{O}_{V^{(1)}}$-Rees algebra strongly linked to $\mathcal{G}^{(2)}$.

Now suppose that $\mathcal{G}^{(n)}$ and $\mathcal{H}^{(n)}$ are two weakly equivalent Rees algebras over $V^{(n)}$, and that $\beta : V^{(n)} \to V^{(n-e)}$ is $\mathcal{G}^{(n)}$-admissible. Then it can be shown that β is also $\mathcal{H}^{(n)}$-admissible (see [3, §5, 6]), and it is natural to ask about the relation between the elimination algebras. The next results address these questions:

Theorem 6.19 (cf. [31, Theorem 4.11]). *Let $V^{(n)}$ be a regular algebraic variety over a perfect field k, and let $\mathcal{G}^{(n)}$ be a Rees algebra over $V^{(n)}$. Consider a $\mathcal{G}^{(n)}$-admissible projection $\beta : V^{(n)} \to V^{(n-e)}$, and assume that $\mathcal{G}^{(n)}$ is β-differential. Let $\mathcal{K}^{(n)}$ be another Rees algebra over $V^{(n)}$, so that $\mathcal{G}^{(n)} \subset \mathcal{K}^{(n)}$ and so that the inclusion is finite. Then $\mathcal{K}^{(n)} \cap \mathcal{O}_{V^{(n-e)}}[W]$ is finite over $\mathcal{G}^{(n)} \cap \mathcal{O}_{V^{(n-e)}}[W]$.*

Corollary 6.20 (cf. [31, Corollary 4.14]). *Let $V^{(n)}$ be a regular algebraic variety over a perfect field k, and let $\mathcal{G}^{(n)}$ be a Rees algebra over $V^{(n)}$. Consider a $\mathcal{G}^{(n)}$-admissible projection $\beta : V^{(n)} \to V^{(n-e)}$ and suppose that $\mathcal{G}^{(n)}$ is a differential Rees algebra. Then the elimination algebra*

$$\mathcal{G}^{(n-e)} = \mathcal{G}^{(n)} \cap \mathcal{O}_{V^{(n-e)}}[W].$$

is also a differential Rees algebra.

The previous results together with the discussion in 6.16 lead us to the following conclusion: when the characteristic of the base field is zero, if two Rees algebras are weakly equivalent, after considering an admissible projection their respective elimination algebras are weakly equivalent; in positive characteristic this fails to hold in general.

Example 6.21 (Representation and elimination in characteristic zero). Let $X = \mathrm{Spec}(B)$ and V be as in 5.12. Set $V^{(d+m)} := V$. With the same notation and setting as in 5.12, the Rees algebra

$$\mathcal{G}^{(d+m)} = \mathcal{O}_{V^{(d+m)}}[f_1(Z_1)W^{d_1}, \ldots, f_m(Z_m)W^{d_m}] \subset \mathcal{O}_{V^{(d+m)}}[W]$$

represents $F_s(X)$. Observe that this means that $F_s(X)$ is *represented* by $\bigcap_{i=1}^{m} F_{d_i}(f_i(Z_i))$ (i.e., $F_s(X)$ is the intersection of the maximum multiplicity loci of the hypersurfaces defined by the $f_i(Z_i)$ -at least in an étale neighborhood of ξ). In this setting, it can be proved that the natural projection $\beta : V^{(d+m)} \to V^{(d)} = \mathrm{Spec}(S)$ is $\mathcal{G}^{(d+m)}$-admissible. Moreover, an elimination algebra of $\mathcal{G}^{(d+m)}$ can be obtained by considering suitable functions on the coefficients of the polynomials $f_1(Z_1), \ldots, f_m(Z_m)$. Namely, when $\mathrm{char}(k) = 0$, one can find elements $a_1, \ldots, a_m \in S$ so that, after taking the change of variables $Z_i' = Z_i + a_i$,

$$f_i(Z_i) = (Z_i')^{d_i} + b_{i,2}(Z_i')^{d_i-2} + \cdots + b_{i,d_i},$$

with $b_{i,j} \in S$ for all i, j. In this case, the S-algebra generated by the elements $b_{i,j}W^j$, with $1 \le i \le m$ and $2 \le j \le d_i$, say $\mathcal{G}^{(d)} = S[b_{i,j}W^j] \subset S[W]$ is an elimination algebra of $\mathcal{G}^{(d+m)}$ (see [31] and [6, Remark 16.10] for further details).

Part III. **Strong transversality**

7. Characterization of strongly transversal morphisms

Let k be a perfect field, and let $\beta : X' \to X$ be a transversal morphism of singular algebraic varieties over k. Recall that this is a finite and dominant morphism such that $\max \mathrm{mult}(X') = r \cdot \max \mathrm{mult}(X)$, where r is the generic rank of $\beta : X' \to X$ (see Definition 2.5). According to Theorem 5.21 and Definition 5.23, one can attach an intrinsic Rees algebra to the stratum of maximum multiplicity of X, say $\mathcal{G}_X$. Recall that this algebra is defined in étale topology. Similarly, one can attach an intrinsic algebra to the stratum of maximum multiplicity of X', say $\mathcal{G}_{X'}$ (see Remark 7.3 below). As Proposition 7.1 shows, there a relation between $\mathcal{G}_X$ and $\mathcal{G}_{X'}$. Along this section we will study the connection between the transversal morphism β and the algebras $\mathcal{G}_X$ and $\mathcal{G}_{X'}$. The main result is Theorem 7.2.

Proposition 7.1 ([1, Proposition 6.3]). *Let $\beta : X' \to X$ be a finite morphism of singular algebraic varieties defined over a perfect field k. Let $\mathcal{G}_X \subset \mathcal{O}_X[W]$ and $\mathcal{G}_{X'} \subset \mathcal{O}_{X'}[W]$ denote the intrinsic Rees algebras attached to the strata of maximum multiplicity of X and X' respectively. If β is transversal, then there is an inclusion $\mathcal{G}_X \subset \mathcal{G}_{X'}$.*

Theorem 7.2. *Let $\beta : X' \to X$ be a transversal morphism of generic rank r between two singular algebraic varieties defined over a perfect field k. Then:*

   (1) If $\beta : X' \to X$ is strongly transversal then the inclusion $\mathcal{G}_X \subset \mathcal{G}_{X'}$ is finite;
   (2) If k is a field of characteristic zero, then the converse holds. Namely, if the inclusion $\mathcal{G}_X \subset \mathcal{G}_{X'}$ is finite, then $\beta : X' \to X$ is strongly transversal.

The proof of this theorem, which will be addressed in 7.8, requires some preliminary technical results.

Remark 7.3. Recall that $\mathcal{G}_X$ and $\mathcal{G}_{X'}$ are only defined locally in étale topology. However, as we will explain below, given a point $\xi \in F_s(X)$, it is possible to find a suitable étale neighborhood of X at ξ, say $\widetilde{X} \to X$, so that one can construct the intrinsic algebra $\mathcal{G}_{\widetilde{X}}$ associated to $\widetilde{X}$, as well as the intrinsic algebra $\mathcal{G}_{\widetilde{X}'}$ associated to $\widetilde{X}' = X' \times_X \widetilde{X}$. It is in this setting that there is an inclusion $\mathcal{G}_{\widetilde{X}'} \subset \mathcal{G}_{\widetilde{X}}$, and in which we will compare these algebras.

Remark 7.4. In case that char $k = 0$, the theorem says that $\beta : X' \to X$ is strongly transversal if and only if the inclusion $\mathcal{G}_X \subset \mathcal{G}_{X'}$ is finite. That is, we can characterize the strong transversality of β by means of the intrinsic algebras $\mathcal{G}_X$ and $\mathcal{G}_{X'}$. On the other hand, when char $k > 0$, property (1) holds as well, but the condition on the characteristic is necessary in (2), as the following example shows.

Example 7.5. Let k be a perfect field of characteristic 2. Consider the algebraic varieties $X = \mathrm{Spec}(B)$ and $X' = \mathrm{Spec}(B')$, where

$$B = k[t, x, y]/\langle y^4 - x^{13} \rangle \quad \text{and} \quad B' = k[t, x, y, z]/\langle z^2 - x^5, y^4 - x^{13} \rangle.$$

Note that there is a natural finite and dominant morphism $\beta : X' \to X$ given by the inclusion $B \subset B'$, which has generic rank 2. One can check that $\max \mathrm{mult}(X) = 4$ and $\max \mathrm{mult}(X') = 8$ (both values are attained at the origin), and thus $\beta : X' \to X$ is transversal (see Definition 2.5).

Set $S = k[t, x]$, $V = \mathrm{Spec}(k[t, x, y])$, and $V' = \mathrm{Spec}(k[t, x, y, z])$. Since $S \subset B$ is a finite extension of generic rank 4, the closed set $F_4(X)$ is represented in V by the differential Rees algebra
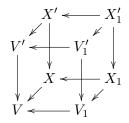
$$\mathcal{G} = k[t, x, y][x^{12}W^3, (y^4 - x^{13})W^4] \subset k[t, x, y][W],$$

and $\mathcal{G}_X = \overline{\mathcal{G}|_X}$ (see 5.12 and Definition 5.23). Similarly, the differential algebra

$$\mathcal{G}' = k[t, x, y, z][x^4W, (z^2 - x^5)W^2, (y^4 - x^{13})W^4] \subset k[t, x, y, z][W]$$

represents $F_8(X')$ in V', and $\mathcal{G}_{X'} = \overline{\mathcal{G}'|_{X'}}$. Note that the map $\mathcal{G} \to \mathcal{G}|_X$ sends the element $(y^4 - x^{13})W^4$ to zero. Hence $\mathcal{G}|_X$ is generated by the class of $x^{12}W^3$ in $B[W]$. Similarly, one readily checks that $\mathcal{G}'|_{X'}$ is generated by the class of x^4W in $B'[W]$. Since $(x^4W)^3 = x^{12}W^3$, it follows that $\mathcal{G}'|_{X'}$ is integral over $\mathcal{G}|_X$. Thus we see that $\mathcal{G}_{X'} = \overline{\mathcal{G}'|_{X'}}$ is finite over $\mathcal{G}_X = \overline{\mathcal{G}|_X}$.

Next we will show that $\beta : X' \to X$ is not strongly transversal. Consider the blow ups of $X' \subset V'$ and $X \subset V$ at the origin, i.e., at the closed points defined by $\langle t, x, y \rangle$ and $\langle t, x, y, z \rangle$ respectively. These blow ups induce a commutative diagram of inclusions and finite morphisms as follows:

One can see that the strict transforms of X and X' on the t-chart of these blow ups are given by

$$X_1 = \mathrm{Spec}\big(k[t_1, x_1, y_1]/\langle y_1^4 - t_1^9 x_1^{13} \rangle\big), \quad \text{and} \quad X_1' = \mathrm{Spec}\big(k[t_1, x_1, y_1, z_1]/\langle z_1^2 - t_1^3 x_1^5, y_1^4 - t_1^9 x_1^{13} \rangle\big),$$

where $t_1 = t; x_1 = x/t, y_1 = y/t$ and $z_1 = z/t$ (see [35, Remark 5.3] to justify that X_1 and X_1' are defined by the equations indicated above after the blow up, or more precisely, that the scheme X_1' is defined by the strict transforms of both hypersurfaces, the ones defined by $z^2 - x^5$ and $y^4 - x^{13}$ in V'). Next, consider the centers $Y_1 \subset F_4(X_1)$ and $Y_1' \subset F_8(X_1')$ defined by $\langle t_1, y_1 \rangle$ and $\langle t_1, y_1, z_1 \rangle$

respectively. Note that Y_1' sits on Y_1 via the finite morphism $X_1' \to X$. The blow ups of $X_1 \subset V_1$ and $X_1' \subset V_1'$ along these centers induce a commutative diagram

Moreover, on the t_1-chart, X_2 and X_2' are given by

$$\mathrm{Spec}\big(k[t_2, x_2, y_2]/\langle y_2^4 - t_2^5 x_2^{13}\rangle\big), \quad \text{and} \quad \mathrm{Spec}\big(k[t_2, x_2, y_2, z_2]/\langle z_2^2 - t_2 x_2^5, y_2^4 - t_2^5 x_2^{13}\rangle\big)$$

respectively (here $t_2 = t_1, x_2 = x_1/t_1, y_2 = y_1/t_1$ and $z_2 = z_1/t_1$). Next observe that the (non-closed) point $\xi_2 \in X_2$ defined by the ideal $\langle t_2, y_2\rangle$ belongs to $F_4(X_2)$. However, the (non-closed) point $\xi_2' \in X_2'$ defined by $\langle t_2, y_2, z_2\rangle$, which sits on ξ_2, does not belong to $F_8(X_2')$, since x_2 is a unit in $k[t_2, x_2, y_2, z_2]_{\langle t_2, y_2, z_2\rangle}$. This shows that $F_8(X_2')$ is not homeomorphic to $F_4(X_2)$, and therefore X' is not strongly transversal to X.

In order to prove Theorem 7.2 we need two preliminary results. The next theorem concerns the elimination of one variable, whereas the Corollary concerns the elimination of several variables.

Theorem 7.6 (cf. [31, Theorem 4.11]). *Let k be a perfect field, let S be a d-dimensional smooth k-algebra, and let $\mathcal{G}^{(d+1)} \subset S[Z][W]$ be a differential Rees algebra over $S[Z]$. Suppose that there is a monic polynomial of degree n, say $f(Z) \in S[Z]$, such that $f(Z)W^n \in \mathcal{G}^{(d+1)}$. Set $B = S[Z]/\langle f(Z)\rangle$. Then the smooth morphism $\beta : \mathrm{Spec}(S[Z]) \to \mathrm{Spec}(S)$ is $\mathcal{G}^{(d+1)}$-admissible, and*

$$\mathcal{G}^{(d)} := \mathcal{G}^{(d+1)} \cap S[W]$$

is an elimination algebra of $\mathcal{G}^{(d+1)}$. Moreover, there is an inclusion of Rees algebras, say $\mathcal{G}^{(d)} \subset \mathcal{G}^{(d+1)}_{|_B}$, which is a finite extension of graded algebras.

Corollary 7.7. *Let k be a perfect field, let S be a smooth k-algebra of dimension d. Let $Z_1, \ldots, Z_h$ denote variables and, for $i = 1, \ldots, h$, let $f_i(Z_i) \in S[Z_i]$ be a monic polynomial of degree l_i, and set*

$$B := S[Z_1, \cdots, Z_h]/\langle f_1(Z_1), \cdots, f_h(Z_h)\rangle.$$

Let $\mathcal{G}^{(d+h)}$ be a differential Rees algebra over $S[Z_1, \ldots, Z_h]$ containing $f_1(Z_1)W^{l_1}, \ldots, f_h(Z_h)W^{l_h}$. Then the natural morphism $\mathrm{Spec}(S[Z_1 \ldots, , Z_h]) \to \mathrm{Spec}(S)$ is $\mathcal{G}^{(d+h)}$-admissible and

$$\mathcal{G}^{(d)} := \mathcal{G}^{(d+h)} \cap S[W].$$

is an elimination algebra of $\mathcal{G}^{(d+h)}$. Furthermore, there is an inclusion of Rees algebras, say

$$\mathcal{G}^{(d)} \subset \mathcal{G}^{(d+h)}_{|_B}, \tag{7.7.1}$$

that is finite. Moreover, as a consequence, there is another inclusion of Rees algebras over S, say

$$\mathcal{G}^{(d)} \subset \left(\mathcal{G}^{(d+h)}_{|_B} \cap S[W]\right), \tag{7.7.2}$$

which is also finite.

Proof. The result follows from the following observation by an inductive argument. Suppose that $h = 2$. Recall that $\mathcal{G}^{(d+2)}$ is a differential algebra containing $f_1(Z_1)W^{l_1}$ and $f_2(Z_2)W^{l_2}$. Then

one can check that $\tau_{\mathcal{G}^{(d+2)}} \geq 2$ at all $\xi \in \operatorname{Sing}\mathcal{G}^{(d+2)}$ and that $\operatorname{Spec}(S[Z_1, Z_2]) \to \operatorname{Spec}(S[Z_1])$ is $\mathcal{G}^{(d+2)}$-admissible (see [5, §8]). Set $\mathcal{G}^{(d+1)} = \mathcal{G}^{(d+2)} \cap S[Z_1][W]$, and consider the diagram

$$
\begin{array}{ccc}
\mathcal{G}^{(d+2)} & S[Z_1, Z_2] \longrightarrow B_2 = S[Z_1][Z_2]/\langle f_2(Z_2)\rangle \longrightarrow B = S[Z_1, Z_2]/\langle f_1(Z_1), f_2(Z_2)\rangle \\
& \uparrow \nearrow \nearrow \\
\mathcal{G}^{(d+1)} & S[Z_1] \longrightarrow B_1 = S[Z_1]/\langle f_1(Z_1)\rangle \\
& \uparrow \nearrow \\
\mathcal{G}^{(d)} & S
\end{array}
$$

Recall that, by Corollary 6.20, $\mathcal{G}^{(d+1)}$ is differential, and it is an elimination algebra of $\mathcal{G}^{(d+2)}$ over $S[Z_1]$. Similarly, $\mathcal{G}^{(d)} := \mathcal{G}^{(d+1)} \cap S$ is also an elimination algebra of $\mathcal{G}^{(d+1)}$ over S (again, it can be checked that $\operatorname{Spec}(S[Z_1]) \to \operatorname{Spec}(S)$ is $\mathcal{G}^{(d+1)}$-admissible and Corollary 6.20 applies). Now (7.7.1) follows from Theorem 7.6, as we have that:

- There is an inclusion $\mathcal{G}^{(d+1)} \subset \mathcal{G}^{(d+2)}_{|B_2}$ that is finite, hence $\mathcal{G}^{(d+1)}_{|B_1} \subset \mathcal{G}^{(d+2)}_{|B}$ is also finite;
- There is an inclusion, $\mathcal{G}^{(d)} \subset \mathcal{G}^{(d+1)}_{|B_1}$ that is finite.

To conclude, observe that, since the extension $S \subset B$ is finite, all the following extensions are finite:

$$
\mathcal{G}^{(d)} \subset \left(\mathcal{G}^{(d)}_{|B} \cap S[W]\right) \subset \left(\mathcal{G}^{(d+1)}_{|B} \cap S[W]\right) \subset \left(\mathcal{G}^{(d+2)}_{|B} \cap S[W]\right).
$$

This proves (7.7.2).

7.8. Proof of Theorem 7.2. It suffices to prove the theorem in the affine case. Hence we may assume that $X = \operatorname{Spec}(B)$, $X' = \operatorname{Spec}(B')$, and that $\beta : X' \to X$ is the morphism induced by a finite inclusion $B \subset B'$. Set $d = \dim B = \dim B'$.

After replacing B by some étale extension we may assume that there is a finite inclusion $S \subset B$ with S smooth as in 5.12. This étale extension of B induces an étale extension of B' which we will denote by B' again, and the induced finite inclusion $S \subset B'$ is in the same setting as that of 5.12. Observe, that, by the hypotheses, $F_{rs}(B')$ is homeomorphic to its image by β, which necessarily sits inside $F_s(B)$. Then there is a commutative diagram of inclusions and finite morphisms,

$$
\begin{array}{ccc}
\mathcal{G}^{(d+n')} & C' = S[Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n'}] \longrightarrow A' \longrightarrow B' = S[\theta_1, \ldots, \theta_n, \theta_{n+1}, \ldots, \theta_{n'}] \\
& \uparrow \qquad \uparrow \qquad \uparrow \\
\mathcal{G}^{(d+n)} & C = S[Z_1, \ldots, Z_n] \longrightarrow A \longrightarrow B = S[\theta_1, \ldots, \theta_n] \\
& \uparrow \nearrow \nearrow \\
& S
\end{array}
$$

where $A = S[Z_1, \ldots, Z_n]/\langle f_1(Z_1), \ldots, f_n(Z_n)\rangle$,

$$
A' = S[Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n'}]/\langle f_1(Z_1), \ldots, f_n(Z_n), f_{n+1}(Z_{n+1}), \ldots, f_{n'}(Z_{n'})\rangle,
$$

and where each monic polynomial $f_i(Z_i) \in S[Z_i]$ of degree d_i is the minimum polynomial of θ_i over S, for $i = 1, \ldots, n, \ldots, n'$, as in 5.12. Let $\mathcal{G}^{(d+n)}$ be the differential C-Rees algebra generated by $f_1(Z_1)W^{d_1}, \ldots, f_n(Z_n)W^{d_n}$, and $\mathcal{G}^{(d+n')}$ the differential C'-Rees algebra generated by
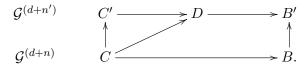
$$
f_1(Z_1)W^{d_1}, \ldots, f_n(Z_n)W^{d_n}, f_{n+1}(Z_{n+1})W^{d_{n+1}} \ldots, f_{n'}(Z_{n'})W^{d_{n'}}.
$$

Then $\mathcal{G}^{(d+n)}$ represents $F_s(B)$ in $\operatorname{Spec}(C)$, and $\mathcal{G}^{(d+n')}$ represents $F_{rs}(B')$ in $\operatorname{Spec}(C')$. By definition, $\mathcal{G}_B = \overline{\mathcal{G}^{(d+n)}_{|B}} \subset B[W]$ while $\mathcal{G}_{B'} = \overline{\mathcal{G}^{(d+n')}_{|B'}} \subset B'[W]$ (see Definition 5.23).

Next we prove (1). Suppose that β is strongly transversal. In such case $\mathcal{G}^{(d+n')}$ is strongly linked to $\mathcal{G}^{(d+n)}$, and hence $\mathcal{G}^{(d+n)}$ is an elimination algebra of $\mathcal{G}^{(d+n')}$ (see Remark 6.17). Set

$$D = S[Z_1, \ldots, Z_n, Z_{n+1}, \ldots, Z_{n'}]/\langle f_{n+1}(Z_{n+1}), \ldots, f_{n'}(Z_{n'})\rangle.$$

The algebra D fits into the previous diagram as follows:

$$
\begin{array}{cccccc}
\mathcal{G}^{(d+n')} & & C' & \longrightarrow & D & \longrightarrow & B' \\
& & \uparrow & \nearrow & & & \uparrow \\
\mathcal{G}^{(d+n)} & & C & \longrightarrow & & & B.
\end{array}
$$

By construction, the extension of $\mathcal{G}^{(d+n)}$ to C' is contained in $\mathcal{G}^{(d+n')}$, hence there is an inclusion $\mathcal{G}^{(d+n)} \subset \mathcal{G}^{(d+n')} \cap C[W]$. Now, this inclusion is finite by Remark 6.17 and Definition 6.8. By Corollary 7.7 there is an inclusion $\mathcal{G}^{(d+n')} \cap C[W] \subset \mathcal{G}^{(d+n')}_{|D}$ which is finite, and hence the inclusion $\mathcal{G}^{(d+n)} \subset \mathcal{G}^{(d+n')}_{|D}$ is finite. From here it follows that the inclusion $\mathcal{G}^{(d+n)} \subset \mathcal{G}^{(d+n')}_{|B'}$ is finite, as B' is a quotient of D. Thus (1) follows because $\mathcal{G}_B = \overline{\mathcal{G}^{(d+n)}_{|B}} \subset B[W]$ and $\mathcal{G}_{B'} = \overline{\mathcal{G}^{(d+n')}_{|B'}} \subset B'[W]$.

To prove (2) we will restrict to characteristic zero. Assume that $\mathcal{G}^{(d+n)}_{|B} \subset \mathcal{G}^{(d+n')}_{|B'}$ is a finite extension of Rees algebras. Set

$$\mathcal{H} = \mathcal{G}^{(d+n)} \cap S[W], \quad \text{and} \quad \mathcal{H}' = \mathcal{G}^{(d+n')} \cap S[W].$$

By Corollary 6.20, $\mathcal{H}$ and $\mathcal{H}'$ are two elimination algebras of $\mathcal{G}^{(d+n)}$ and $\mathcal{G}^{(d+n')}$ over S respectively. Since we are assuming characteristic zero, Remark 6.17 says that $\mathcal{H}$ is strongly linked to $\mathcal{G}^{(d+n)}$, and $\mathcal{H}'$ is strongly linked to $\mathcal{G}^{(d+n')}$. Thus, in order to prove that $\beta : X' \to X$ is strongly transversal, it suffices to show that $\mathcal{H}$ and $\mathcal{H}'$ are weakly equivalent. To this end, observe that we have a commutative diagram of inclusions of Rees algebras as follows:

$$
\begin{array}{ccc}
\mathcal{G}^{(d+n)}_{|B} & \hookrightarrow & \mathcal{G}^{(d+n')}_{|B'} \\
\uparrow & & \uparrow \\
\mathcal{H} & \hookrightarrow & \mathcal{H}'.
\end{array}
$$

The vertical arrows are finite by Corollary 7.7, and the top horizontal arrow is finite by hypothesis. Hence $\mathcal{G}^{(d+n')}_{|B'}$ is finite over $\mathcal{H}$, and therefore $\mathcal{H}'$ is also finite over $\mathcal{H}$. This latter condition shows that $\mathcal{H}$ and $\mathcal{H}'$ are weakly equivalent and, as a consequence, that $\beta : X' \to X$ is strongly transversal.

Remark 7.9. Assume that the characteristic is zero, so that one can apply an algorithm of resolution of Rees algebras (see Definition 5.5). A feature of algorithmic resolution of algebras is that it does not distinguish between weakly equivalent algebras (see Definition 5.13). Namely, if $\mathcal{H}$ and $\mathcal{H}'$ are two weakly equivalent algebras over a regular variety $V^{(d)}$ of characteristic zero, then the algorithm produces the same resolution for $\mathcal{H}$ and $\mathcal{H}'$.

Consider a strongly transversal morphism of singular algebraic varieties of characteristic zero, say $\beta : X' \to X$, with $s = \max \operatorname{mult}(X)$ and $rs = \max \operatorname{mult}(X')$ (see Definition 4.8). Fix the setting as in 7.8. Since the characteristic is zero, the simplification of $F_s(X)$ and $F_{rs}(X')$ can be reduced to the resolution of two Rees algebras, say $\mathcal{H}$ and $\mathcal{H}'$ respectively, over a regular ambient space $\operatorname{Spec}(S)$. Since $\beta : X' \to X$ is strongly transversal, $\mathcal{H}$ and $\mathcal{H}'$ turn out to be weakly equivalent. Thus one can see that the algorithmic resolution of $F_s(X)$ induces that of $F_{rs}(X')$, and vice versa.

8. On the construction of strongly transversal morphisms

In the previous sections we have studied conditions under which a finite morphism of singular algebraic varieties, say $\beta : X' \to X$, is strongly transversal. Along this section we proceed the other way around. Namely, we start with a singular algebraic variety X with field of fractions K and,

given a finite field extension L/K, we ask whether there exists a singular variety X' with field of fractions L endowed with a strongly transversal morphism $\beta : X' \to X$. We do not know how to construct such X' in general. However, as we will explain in the following lines, we can achieve such result locally in étale topology. The main result of this section is Theorem 8.2.

Remark 8.1. Let X be a normal algebraic variety with field of fractions K and maximum multiplicity $s \geq 2$. For a fixed point $\xi \in F_s(X)$, consider an étale neighborhood of X at ξ, say $\widetilde{X} \to X$. Note that $\widetilde{X}$ might not be irreducible. However, as X is normal, $\widetilde{X}$ is a disjoint union of irreducible components, each one being normal (see [28, Proposition VII.2.2, p. 75]). Thus, after replacing $\widetilde{X}$ by one of its irreducible components, we may assume that $\widetilde{X}$ is irreducible. Note also that, if L is a finite extension of K of rank r and $\widetilde{K}$ denotes the field of fractions of $\widetilde{X}$, then $\widetilde{L} = L \otimes_K \widetilde{K}$ is a finite extension of $\widetilde{K}$ of generic rank r.

Theorem 8.2. *Let X be a singular algebraic variety over a perfect field k with maximum multiplicity $s \geq 2$. Suppose that X is normal with field of fractions K and let L be a finite field extension of K of rank r. Then, for every point of $\xi \in F_s(X)$, there exist an étale neighborhood of X at ξ, say $\widetilde{X} \to X$, together with a scheme $\widetilde{X}'$ and a strongly transversal morphism $\beta : \widetilde{X}' \to \widetilde{X}$ of generic rank r such that $\widetilde{L} = L \otimes_K \widetilde{K}$, where $\widetilde{K}$ and $\widetilde{L}$ denote the total quotient rings of $\widetilde{X}$ and $\widetilde{X}'$ respectively.*

Remark 8.3. If $\widetilde{X}$ is not irreducible, then it is not possible to construct a strongly transversal morphism $\beta : \widetilde{X}' \to \widetilde{X}$, as transversal and strongly transversal morphisms have been defined just for the case in which $\widetilde{X}$ is a variety.

It is possible to extend the notions of transversal morphism and strongly transversal morphism to include the case in which $\widetilde{X}$ is an equidimensional reduced scheme of finite type over a perfect field k. In this way, one could also generalize the previous theorem to the case in which X is non-necessarily normal. These generalizations would not involve new ideas. However, they would require to introduce more notation and the proof of Theorem 8.2 would be more technical. Thus, for the shake of simplicity and readability, we have chosen to keep the definitions of transversal morphism and strongly transversal morphism and Theorem 8.2 as they are.

Nevertheless note that, from the point of view of resolution of singularities, there is no loss of generality on replacing a variety X by its normalization, say $\overline{X}$, as any resolution of singularities of X should factor through $\overline{X}$.

Proof. After replacing X by a suitable étale neighborhood, assume that X is an affine algebraic variety, say $X = \mathrm{Spec}(B)$, whose intrinsic algebra is given by $\mathcal{G}_X = \mathcal{G}_B = \bigoplus_i J_i W^i \subset B[W]$. Let $\overline{B}$ denote the integral closure of the domain B in L, and consider the integral closure of $\mathcal{G}_B$ in $\overline{B}[W]$, say

$$\overline{\mathcal{G}_B} = \overline{B} \oplus \widetilde{J}_1 W \oplus \widetilde{J}_2 W^2 \oplus \cdots = \bigoplus_i \widetilde{J}_i W^i \subset \overline{B}[W].$$

Note that the integral closure of the extended ideal $J_i \overline{B}$ must be contained in the ideal $\widetilde{J}_i \subset \overline{B}$ but, in general, this inclusion is strict (see Example 8.4).

As $\widetilde{J}_1$ is an ideal of $\overline{B}$ and $\overline{B}$ is finite over B, the ideal $\widetilde{J}_1$ can also be regarded as a finite module over B. Fix a family of generators of $\widetilde{J}_1$ regarded as a B-module, say $\widetilde{J}_1 = \langle \theta_1, \ldots, \theta_m \rangle$. Set $B' = B[\theta_1, \ldots, \theta_m] = B[\widetilde{J}_1] \subset L$. We claim that $X' = \mathrm{Spec}(B')$ is a singular variety with field of fractions L, and that the induced morphism $X' \to X$ is strongly transversal.

To prove the claim, first note that $\widetilde{J}_1$ is a non-zero ideal in $\overline{B}$, which is contained in B'. Hence B' has L as its field of fractions. On the other hand observe that, by construction, the elements $\theta_1 W, \ldots, \theta_m W \in B'[W]$ are integral over $\mathcal{G}_B$. Then the strong transversality will follow from Proposition 8.7 and Proposition 8.8 below.

Example 8.4. Let k be a field of characteristic zero, and consider the curve $X = \mathrm{Spec}(B)$ given by $B = k[x,y]/\langle y^3 + x^3 y + x^7 \rangle$. Let us denote by $\bar{x}$ and $\bar{y}$ the residue classes of x and y in B respectively. Note that X has maximum multiplicity 3, and this value is reached at the origin. Following the notation of the proof, set $S = k[\bar{x}]$ and $B = S[\bar{y}]$. The maximum multiplicity locus of $X \subset \mathrm{Spec}(S[T])$ is represented by the Rees algebra

$$\mathcal{G} = S[T][(T^3 + \bar{x}^3 T + \bar{x}^7)W^3] \subset S[T][W],$$

and a simple computation shows that $Diff(\mathcal{G}) = S[T][TW, \bar{x}^2 W, \bar{x}^3 W^2]$. Therefore, the Rees algebra $\mathcal{H} = Diff(\mathcal{G}) \cap S[W]$ is given by $\mathcal{H} = S[\bar{x}^2 W, \bar{x}^3 W^2] \subset S[W]$ and, in this case, $J_1 = \langle \bar{x}^2 \rangle \subset S$.

Next, consider the field extension $L = K[z]/\langle z^2 - x^3 \rangle$, where K represents the field of fractions of B. Let $\overline{\mathcal{H}} = \bigoplus_l \tilde{J}_l W^l$ denote the integral closure of $\mathcal{H}$ in $L[W]$. Since

$$(\bar{z}W)^2 - \bar{x}^3 W^2 = (\bar{z}^2 - \bar{x}^3)W^2 = 0,$$

with $\bar{x}^3 W^2 \in J_2$, it follows that $\bar{z}W$ is integral over $\mathcal{H}$. That is, $\bar{z} \in \tilde{J}_1$. However, we claim that $\bar{z}$ is not integral over $J_1 \overline{S}$. Indeed, according to [36, Appendix 4, Theorem 1, p. 350], $\bar{z}$ is integral over $J_1 \overline{S}$ if and only if, for every valuation ring $R \subset L$ so that $S \subset R$, one has that $\bar{z} \in J_1 R$. Consider the subalgebra $k[\frac{\bar{z}}{\bar{x}}] \subset L$ (note that $k[\frac{\bar{z}}{\bar{x}}]$ is the ring obtained by blowing-up $k[\bar{x}, \bar{z}]$ along $\langle \bar{x}, \bar{z} \rangle$). Observe that $(\frac{\bar{z}}{\bar{x}})^2 = \bar{x}$ in L. Hence $S \subset k[\frac{\bar{z}}{\bar{x}}]$. Set $R_0 = k[\frac{\bar{z}}{\bar{x}}]_{\langle \frac{\bar{z}}{\bar{x}} \rangle}$, and let $L_0 \subset L$ denote its field of fractions. As R_0 is a regular local ring of dimension 1, it is a discrete valuation ring with parameter $\frac{\bar{z}}{\bar{x}}$. In this way, since $\bar{z} = \bar{x} \cdot \frac{\bar{z}}{\bar{x}} = (\frac{\bar{z}}{\bar{x}})^3$, one readily checks that

$$\bar{z} \notin J_1 R_0 = \bar{x}^2 \; R_0 = \left\langle \frac{\bar{z}}{\bar{x}} \right\rangle^4 R_0. \tag{8.4.1}$$

By [36, Theorem VI.5, p. 12], R_0 can be extended to a valuation ring in L, say $R \subset L$, with parameter $\frac{\bar{z}}{\bar{x}}$. Thus (8.4.1) implies that $\bar{z} \notin J_1 R$, and therefore $\bar{z}$ is not integral $J_1 \overline{S}$.

Lemma 8.5. *Let B be a singular domain of finite type over a perfect field k with maximum multiplicity $s \geq 2$. Let $\mathcal{G}_B \subset B[W]$ denote the intrinsic algebra attached to $F_s(B)$. Consider a regular subalgebra of B, say S, so that the extension $S \subset B$ is finite of generic rank $s = \max \mathrm{mult}(B)$. That is, so that the induced morphism $\eta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is transversal. Then the S-Rees algebra $\mathcal{H} := \mathcal{G}_B \cap S[W]$ has the following properties:*

(1) The extension $\mathcal{H} \subset \mathcal{G}_B$ is finite.
(2) The morphism β maps $F_s(X)$ homeomorphically to $\mathrm{Sing}(\mathcal{H})$, i.e., $\beta(F_s(X)) \cong \mathrm{Sing}(\mathcal{H})$.

Proof. Following the argument exposed in 5.12 set $V^{(d+n)} = \mathrm{Spec}(S[T_1, \ldots, T_n])$, where the $T_1, \ldots, T_n$, are indeterminates and let $f_1(T_1), \ldots, f_n(T_n) \in S[T_i]$ of degree d_i for $i = 1, \ldots, n$ such that

$$\mathcal{G}^{(d+n)} = \mathcal{O}_{V^{(d+n)}} \left[f_1(T_1)W^{d_1}, \ldots, f_n(T_n)W^{d_n} \right] \subset \mathcal{O}_{V^{(d+n)}}[W].$$

represents $F_s(B)$ in $V^{(d+n)}$. Moreover, there is a commutative diagram, say

$$
\begin{array}{ccc}
\mathcal{G}^{(d+n)} & V^{(d+n)} \xleftarrow{\hspace{2em}} \mathrm{Spec}(B) & \hspace{3em} (8.5.1) \\
& \varphi \downarrow \quad\swarrow \eta & \\
& \mathrm{Spec}(S). &
\end{array}
$$

Next, consider the Rees algebras

$$\mathcal{K}^{(d+n)} := \overline{Diff\left(\mathcal{G}^{(d+n)}\right)} \subset \mathcal{O}_{V^{(d+n)}}[W], \quad \text{and} \quad \mathcal{K}^{(d)} := \mathcal{K}^{(d+n)} \cap S[W].$$

Recall that, by Definition 5.23,

$$\mathcal{G}_B = \overline{\left(\mathcal{K}^{(d+n)}_{|B} \right)} \subset B[W].$$

Thus there are inclusions $\mathcal{K}^{(d)} \subset \mathcal{H} \subset \mathcal{G}_B$. Moreover, Corollary 7.7 says that $\mathcal{G}_B$ is finite over $\mathcal{K}^{(d)}$, and hence $\mathcal{K}^{(d)} \subset \mathcal{H}$ is also a finite extension of subalgebras of $S[W]$. Since $\mathcal{K}^{(d+n)}$ is integrally closed, $\mathcal{K}^{(d)}$ is integrally closed. Therefore $\mathcal{H} = \mathcal{K}^{(d)}$, and (1) follows from Corollary 7.7.

On the other hand, since $\mathcal{K}^{(d+n)}$ is weakly equivalent to $\mathcal{G}^{(d+n)}$, $\mathcal{K}^{(d+n)}$ represents $F_s(B)$ over $V^{(d+n)}$. As $\mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is transversal, the commutativity of (8.5.1) implies that φ is a $\mathcal{K}^{(d+n)}$-admissible projection, and $\mathcal{H} = \mathcal{K}^{(d)}$ is an elimination algebra of $\mathcal{K}^{(d+n)}$ (see Definition 6.8). Then $\mathrm{Sing}(\mathcal{H}) \cong \mathrm{Sing}\,(\mathcal{K}^{(d+n)}) = F_s(B)$ (see 6.12), which proves (2).

Lemma 8.6. *Let $S \subset B$ be a finite extension of domains of finite type over a perfect field k of generic rank $s = \max \mathrm{mult}(B)$, where S is regular and B is singular, and let $\mathcal{H} := \mathcal{G}_B \cap S[W]$ be as in Lemma 8.5. Consider a prime ideal $P \in F_s(B)$ such that B/P is regular, and set $\mathfrak{p} = P \cap S \in \mathrm{Sing}(\mathcal{H})$. Recall that, in this setting, there is a natural commutative diagram as follows,*

$$
\begin{array}{ccc}
\mathrm{Spec}(B) & \longleftarrow & X_1 = \mathrm{Bl}_P(B) \\
\downarrow & & \downarrow \\
\mathcal{H} \quad \mathrm{Spec}(S) & \longleftarrow & Z_1 = \mathrm{Bl}_{\mathfrak{p}}(S) \quad \mathcal{H}_1,
\end{array}
$$

where the vertical arrows are finite morphisms, and $\mathcal{H}_1 \subset \mathcal{O}_{Z_1}[W]$ denotes the transform of $\mathcal{H}$ (in the sense of 5.4). Assume that $F_s(X_1) \neq \emptyset$ (i.e., $\max \mathrm{mult}(X_1) = \max \mathrm{mult}(X) = s$), and let $\mathcal{G}_{X_1} \subset \mathcal{O}_{X_1}[W]$ denote the intrinsic algebra attached to $F_s(X_1)$. Then there is an inclusion of algebras $\mathcal{H}_1 \subset \mathcal{G}_{X_1}$ (which in general is not finite).

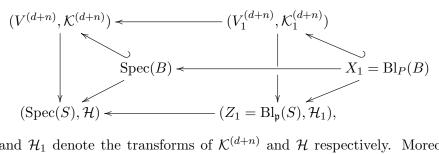*Proof.* Following the proof of Lemma 8.5, one can construct a commutative diagram as follows,

$$
\begin{array}{ccc}
\mathcal{K}^{(d+n)} \quad V^{(d+n)} & \longleftarrow & \mathrm{Spec}(B) \\
& \varphi \downarrow \quad \swarrow{\scriptstyle \eta} & \\
\mathcal{H} \quad \mathrm{Spec}(S), &
\end{array}
$$

where $\mathcal{K}^{(d+n)}$ is a differential and integrally closed Rees algebra representing $F_s(B)$ in $V^{(d+n)}$, and $\mathcal{H} = \mathcal{K}^{(d+n)} \cap S[W] = \mathcal{G}_B \cap S[W]$ is an elimination algebra of $\mathcal{K}^{(d+n)}$ over S.

Note that P defines a closed regular center contained in $F_s(B)$, say Y, which can also be regarded as regular center contained in $\mathrm{Sing}(\mathcal{K}^{(d+n)}) \subset V^{(d+n)}$. Moreover, $\eta(Y) \subset \mathrm{Sing}(\mathcal{H})$ is the regular center defined by the prime $\mathfrak{p}$ mentioned in the lemma. In this way, after blowing up $\mathrm{Spec}(B)$, $V^{(d+n)}$ and $\mathrm{Spec}(S)$ along these centers, we obtain a commutative diagram

$$
\begin{array}{ccc}
(V^{(d+n)}, \mathcal{K}^{(d+n)}) & \longleftarrow & (V_1^{(d+n)}, \mathcal{K}_1^{(d+n)}) \\
& & \\
\downarrow \quad \swarrow & & \downarrow \quad \swarrow \\
\mathrm{Spec}(B) & \longleftarrow & X_1 = \mathrm{Bl}_P(B) \\
\downarrow \quad \swarrow & & \downarrow \quad \swarrow \\
(\mathrm{Spec}(S), \mathcal{H}) & \longleftarrow & (Z_1 = \mathrm{Bl}_{\mathfrak{p}}(S), \mathcal{H}_1),
\end{array}
$$

where $\mathcal{K}_1^{(d+n)}$ and $\mathcal{H}_1$ denote the transforms of $\mathcal{K}^{(d+n)}$ and $\mathcal{H}$ respectively. Moreover, locally at points of $\mathrm{Sing}(\mathcal{K}_1^{(d+n)})$, there is an inclusion $\mathcal{O}_{Z_1} \subset \mathcal{O}_{V_1^{(d+n)}}$, and it can be checked that $\mathcal{H}_1 \subset \mathcal{K}_1^{(d+n)}$. In addition, if $F_s(X_1) \neq \emptyset$, then $\mathcal{K}_1^{(d+n)}$ represents $F_s(X_1)$ in $V_1^{(d+n)}$. Finally, $\mathcal{H}_1 \subset \mathcal{G}_{X_1}$ because by Definition 5.23, we have that $\mathcal{G}_{X_1} = \overline{Diff(\mathcal{K}_1^{(d+n)})}|_{X_1}$.

Proposition 8.7. *Let B be a singular domain of finite type over a perfect field k with maximum multiplicity $s > 1$, and let $\mathcal{G}_B \subset B[W]$ denote the intrinsic Rees algebra attached to $F_s(B)$ in the sense of Definition 5.23. Let $B' = B[\theta_1, \ldots, \theta_m]$ be a finite and dominant extension of B of generic rank r. If $\theta_1 W, \ldots, \theta_m W \in B'[W]$ are integral over $\mathcal{G}_B \subset B[W]$, then $\max \mathrm{mult}(B') = rs$, and*

therefore the morphism $\beta : \mathrm{Spec}(B') \to \mathrm{Spec}(B)$ *is transversal. Moreover, in such case,* β *induces a natural homeomorphism between* $F_{rs}(B')$ *and* $F_s(B)$.

Proof. As $B \subset B'$ has generic rank r, Zarisiki's formula (Theorem 2.2) says that $\max \mathrm{mult}(B') \leq r \cdot \max \mathrm{mult}(B) = rs$. Thus, in order to check the transversality of β, we just need to show that $F_{rs}(B') \neq \emptyset$. Consider a prime ideal $P \in F_s(B)$. Since the extension $B \subset B'$ is finite and dominant, β is surjective, and therefore there exists at least a prime $P' \subset B'$ sitting on P. We shall show that $P' \in F_{rs}(B')$. Note that this property implies that:

(1) $F_{rs}(B') \neq \emptyset$, and hence β is transversal;
(2) As β is transversal, $F_{rs}(B')$ is mapped homeomorphically onto $\beta(F_{rs}(B'))$, and $\beta(F_{rs}(B')) \subset F_s(B)$ (see Corollary 2.7);
(3) $F_{rs}(B')$ maps surjectively onto $F_s(B)$, as the previous argument holds for any $P \in F_s(B)$;
(4) As a consequence of (2) and (3), the set $F_{rs}(B')$ is mapped homeomorphically to $F_s(B)$.

After replacing B by a suitable étale neighborhood, we may assume that there is a regular k-algebra contained in B, say S, so that $S \subset B$ is a finite extension of generic rank $s = \max \mathrm{mult}(B)$ (see 5.12). In other words, the induced map $\eta : \mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is transversal. Consider the Rees algebra $\mathcal{H} := \mathcal{G}_B \cap S[W]$. By Lemma 8.5, we have that $\mathcal{G}_B$ contains $\mathcal{H}$ and the inclusion $\mathcal{H} \subset \mathcal{G}_B$ is finite. Set $\mathcal{H} = \bigoplus_l J_l W^l \subset S[W]$. As $\theta_1 W, \ldots, \theta_m W \in B'[W]$ are integral over $\mathcal{G}_B$, and $\mathcal{G}_B$ is integral over $\mathcal{H}$, then $\theta_1 W, \ldots, \theta_m W$ are integral over $\mathcal{H}$. Hence each θ_i must satisfy a relation of integral dependence of the form

$$\theta_i^N + a_1 \theta_i^{N-1} + \cdots + a_N = 0; \quad a_j \in J_j. \tag{8.7.1}$$

Consider the prime ideal $\mathfrak{p} = P \cap S$. Since $P \in F_s(B)$ and $\mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is a finite transversal morphism, P is the unique prime of $\mathrm{Spec}(B)$ sitting on $\mathfrak{p}$, and hence $B_P = B \otimes_S S_{\mathfrak{p}}$. Thus we may assume without loss of generality that $S = S_{\mathfrak{p}}$, and $B = B_P$. By Lemma 8.5, we have that $\mathfrak{p} \in \mathrm{Sing}(\mathcal{H})$. Therefore, as each a_j in (8.7.1) belongs to J_j, we deduce that $\nu_{\mathfrak{p}}(a_j) \geq j$ for all j. In this way, (8.7.1) can also be regarded as a relation of integral dependence of θ_i over the ideal $\mathfrak{p}B' \subset B'$. Thus, if a prime ideal $Q \subset B'$ contains $\mathfrak{p}B'$, then $\theta_1, \ldots, \theta_m \in Q$. Note also that

$$B'/(PB' + \langle \theta_1, \ldots, \theta_m \rangle) = B/P.$$

Since B/P is a domain, we deduce that $PB' + \langle \theta_1, \ldots, \theta_m \rangle$ is a prime ideal in B', and hence $P' = PB' + \langle \theta_1, \ldots, \theta_m \rangle$. As a consequence, we have that: (i) P' is the unique prime of B' dominating P; (ii) P' is rational over P; and (iii) $PB'_{P'}$ is a reduction of $P'B'_{P'}$. In virtue of 2.3, this implies that $P' \in F_{rs}(P')$, which proves the lemma.

Proposition 8.8. *Under the same hypotheses of Proposition 8.7, the morphism* $\beta : \mathrm{Spec}(B') \to \mathrm{Spec}(B)$ *is strongly transversal.*

Proof. Proposition 8.7 says that β is transversal, and $\beta(F_{rs}(B')) = F_s(B)$. Thus we just need to check that the conditions of the lemma are preserved (locally) after blowing up.

Fix a prime ideal $P \in F_s(B)$ which defines a regular center in $\mathrm{Spec}(B)$, and let $P' \in F_{rs}(B')$ be the unique prime in B' sitting on P. After replacing B and B' by suitable étale neighborhoods, we may assume that B contains a regular subalgebra, say S, so that $S \subset B$ is a finite extension of generic rank $s = \max \mathrm{mult}(B)$. That is, so that the morphism $\mathrm{Spec}(B) \to \mathrm{Spec}(S)$ is transversal. Set $\mathfrak{p} = P \cap S$, and consider the Rees algebra $\mathcal{H} := \mathcal{G}_B \cap S[W]$. By Lemma 8.5, we have that $\mathfrak{p} \in \mathrm{Sing}(\mathcal{H})$, and hence it defines a permissible center for $\mathcal{H}$. Moreover, by Lemma 4.2, there are finite morphisms $\mathrm{Bl}_{P'}(B') \to \mathrm{Bl}_P(B) \to \mathrm{Bl}_{\mathfrak{p}}(S)$.

Fix a collection of generators of the ideal $\mathfrak{p} \subset S$, say $\mathfrak{p} = \langle x_1, \ldots, x_t \rangle$. Then $\mathrm{Bl}_{\mathfrak{p}}(S)$ can be covered by t affine charts of the form $\mathrm{Spec}(S_i)$, with $S_i = S\left[\frac{x_1}{x_i}, \ldots, \frac{x_t}{x_i}\right]$. In addition, by Remark 4.3, $\mathrm{Bl}_P(B)$ and $\mathrm{Bl}_{P'}(B')$ can be covered by affine charts of the form $\mathrm{Spec}(B_i)$ and $\mathrm{Spec}(B'_i)$ respectively, where B_i contains and is finite over S_i, and $B'_i = B_i\left[\frac{\theta_1}{x_i}, \ldots, \frac{\theta_m}{x_i}\right]$. In order to check that the conditions of

the lemma are preserved, we shall show that, if $F_s(B_i) \neq \emptyset$, then $\frac{\theta_1}{x_i}, \ldots, \frac{\theta_m}{x_i} \in B_i'[W]$ are integral over $\mathcal{G}_{B_i}$ (the intrinsic Rees algebra attached to $F_s(B_i)$).

Let $\mathcal{H}_1$ denote the transform of $\mathcal{H}$ over the chart $\mathrm{Spec}(S_i) \subset \mathrm{Bl}_{\mathfrak{p}}(S)$, and suppose that $\mathcal{H} = \bigoplus_l J_l W^l \subset S[W]$. By Lemma 8.5, there is a finite inclusion $\mathcal{H} \subset \mathcal{G}_B$. Since $\theta_1 W, \ldots, \theta_m W \in B'[W]$ are integral over $\mathcal{G}_B$, they are also integral over $\mathcal{H}$. Hence each θ_j satisfies an equation of integral dependence of the form $\theta_j^N + a_1 \theta_j^{N-1} + \cdots + a_N = 0$ with $a_l \in J_l$. That is, with $a_l W^l \in \mathcal{H}$. Then one readily checks that $\frac{a_1}{x_i} W, \ldots, \frac{a_N}{x_i^N} W^N \in \mathcal{H}_1$. Hence $\frac{\theta_j}{x_i} \in B_i'$ satisfies an equation of integral dependence of the form

$$\left(\frac{\theta_j}{x_i}\right)^N + \frac{a_1}{x_i}\left(\frac{\theta_j}{x_i}\right)^{N-1} + \cdots + \frac{a_N}{x_i^N} = 0,$$

which implies that $\frac{\theta_j}{x_i} W \in B_i'[W]$ is integral over $\mathcal{H}_1$. In addition, if $F_s(B_i) \neq \emptyset$, then $\mathcal{H}_1 \subset \mathcal{G}_{B_i}$ by Lemma 8.6. This proves that $\frac{\theta_j}{x_i} W \in B_i'[W]$ is integral over $\mathcal{G}_{B_i}$, and hence the conditions of the lemma are locally preserved.

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