Coloring graphs by translates in the circle

Pablo Candela, Carlos Catalá, Robert Hancock, Adam Kabela,
Daniel Král', Ander Lamaison, Lluís Vena

Abstract

The fractional and circular chromatic numbers are the two most studied non-integral refinements of the chromatic number of a graph. Starting from the definition of a coloring base of a graph, which originated in work related to ergodic theory, we formalize the notion of a gyrocoloring of a graph: the vertices are colored by translates of a single Borel set in the circle group, and neighbouring vertices receive disjoint translates. The corresponding gyrochromatic number of a graph always lies between the fractional chromatic number and the circular chromatic number. We investigate basic properties of gyrocolorings. In particular, we construct examples of graphs whose gyrochromatic number is strictly between the fractional chromatic number and the circular chromatic number. We also establish several equivalent definitions of the gyrochromatic number, including a version involving all finite abelian groups.

1 Introduction

Graph coloring is one of the most studied topics in graph theory. In order to refine the basic notion of the chromatic number of a graph, various non-integral relaxations were introduced, in particular, to capture how close a graph is to being colorable with fewer colors. Among them, the two most intensively studied notions are the *circular chromatic number* and the *fractional chromatic number*. We build on the work of Avila and the first author [1] who introduced a notion of a coloring base of a graph in relation to applications of their new proof of a generalization of Rokhlin's lemma; this notion leads to a chromatic parameter of a graph which lies between the circular and fractional chromatic numbers. The

---

purpose of the present article is to introduce this parameter, which we refer to as the *gyrochromatic number* of a graph, in the context of graph coloring.

We begin by recalling the notions of circular and fractional colorings and fixing some notation. All graphs in this paper are finite and simple. If G is a graph, then $V(G)$ and $E(G)$ are its vertex and edge sets, and $|G|$ is the number of its vertices. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k for which there exists a mapping $f : V(G) \to [k]$ such that $f(u) \neq f(v)$ for every edge uv of G; we use $[k]$ to denote the set of the first k positive integers. The *circular chromatic number* $\chi_c(G)$ of a graph G is the smallest real z for which there exists a mapping $f : V(G) \to [0, z)$ such that $1 \leq |f(v) - f(u)| \leq z - 1$ for every edge uv (it can be shown that the minimum is always attained). The mapping f can be viewed as a mapping of the vertices of G to unit-length arcs of a circle of circumference z such that adjacent vertices are mapped to internally disjoint arcs. We remark that there are several equivalent definitions of the circular chromatic number [14], e.g., through homomorphisms to particular graphs or through balancing edge-orientations. In relation to the chromatic number, it is not hard to show that every graph G satisfies the following:

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G),$$

in particular $\chi(G) = \lceil \chi_c(G) \rceil$. The circular chromatic number was introduced by Vince [12] in the late 1980s and has been the main subject of many papers since then; we refer to the survey by Zhu [14] for a detailed exposition.

Coloring vertices of a graph G can be viewed as an integer program such that independent sets in G are assigned weights zero and one in such a way that every vertex belongs to independent sets whose total weight is (at least) one and the sum of the weights of all independent sets is minimized. Fractional coloring is a linear relaxation of this optimization problem: the *fractional chromatic number* $\chi_f(G)$ is the smallest real z for which there is an assignment of non-negative weights to independent sets in G such that the sum of their weights is z and each vertex belongs to independent sets whose total weight is at least one (it can be shown that the minimum is always attained). Equivalently, $\chi_f(G)$ can be defined as the smallest real z such that each vertex v of G can be assigned a Borel subset of $[0, z)$ of measure one in such a way that adjacent vertices are assigned disjoint subsets. It can be shown that the circular chromatic number lies between the fractional chromatic number and chromatic number for every graph G:

$$\chi_f(G) \leq \chi_c(G) \leq \chi(G).$$

Both inequalities can be strict, and the gap between the fractional chromatic number and chromatic number (and so the circular chromatic number) can be arbitrarily large. For instance, the chromatic number of Kneser graphs can be arbitrarily large and their fractional chromatic number arbitrarily close to two. We recall that the Kneser graph $K(n, k)$ has $\binom{n}{k}$ vertices that are viewed as

corresponding to k-element subsets of $[n]$; two vertices are adjacent if the corresponding subsets are disjoint. The chromatic number of $K(n,k)$ is $n - 2k + 2$ (if $n \geq 2k$) by the famous result of Lovász [9], and their fractional chromatic number is known to be equal to n/k. We refer to the book by Scheinerman and Ullman [11] for further results on fractional coloring and fractional graph parameters in general.

We next recall the notion of a coloring base of a graph introduced in [1, Definition 3.8], which is the starting point for the present discussion. Let G be a graph and Z be an abelian group. A set $A \subseteq Z$ is a *coloring Z-base* for G if there exists a function $f : V(G) \to A$ such that the sets $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G; we write just *coloring base* if Z is clear from the context. For a topological group Z equipped with a Haar probability measure μ, we define

$$\sigma_Z(G) = \sup\{\mu(A) : \ A \text{ is a Borel coloring } Z\text{-base for } G\}. \tag{1}$$

This notion is related to results in ergodic theory, and we refer the reader to [1] for the exposition of this relation. We will be particularly interested in the group $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and the groups $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ for $N \in \mathbb{N}$. The former can be viewed as given by the unit interval $[0, 1]$ with 0 and 1 identified and the usual Borel measure, and the latter is simply $\{0, \ldots, N-1\}$ with addition modulo N equipped with the uniform discrete probability measure.

The notion of a coloring base resembles equivalent definitions of circular and fractional chromatic number, which can be cast using the group $\mathbb{T}$ as follows. The circular chromatic number of a graph G is the inverse of the maximum $\mu(A)$ where A is a *connected* coloring $\mathbb{T}$-base for G (here, connected means as a subset of $\mathbb{T}$). The fractional chromatic number of a graph G is the inverse of the maximum z such that each vertex of G can be assigned a Borel subset of $\mathbb{T}$ with measure z and adjacent vertices are assigned disjoint subsets. Informally speaking, circular coloring assigns vertices of G to disjoint arcs and fractional coloring assigns vertices of G to Borel subsets. A coloring $\mathbb{T}$-base is tighter than fractional coloring since vertices must be assigned rotational copies of the same Borel subset, and it is looser than circular coloring since the Borel subset assigned to vertices need not be an arc. This leads us to the following definition: the *gyrochromatic number* $\chi_g(G)$ of a graph G is the inverse of $\sigma_{\mathbb{T}}(G)$. The equivalent definitions of the circular and fractional chromatic numbers given above yield that the gyrochromatic number of every graph G lies between its fractional and circular chromatic numbers:

$$\chi_f(G) \leq \chi_g(G) \leq \chi_c(G). \tag{2}$$

Similarly to the notions of the fractional and circular chromatic number, the definition of the gyrochromatic number of a graph is robust in the sense that it can be cast in several different ways. In Section 2, we show that the definition through coloring $\mathbb{T}$-bases is equivalent to its discrete variant using $\mathbb{Z}_N$-bases

3

(cf. Corollary 3). Coloring, circular coloring and fractional coloring of a graph can be defined by using homomorphisms to special classes of graphs: cliques, circular cliques and Kneser graphs. This holds also for the gyrochromatic number of a graph, which can be defined in terms of homomorphisms to circulant graphs (cf. Theorem 4). More interestingly, in Section 4 we prove that the definition stays the same when considering higher-dimensional-torus analogues of $\mathbb{T}$ (cf. Theorem 9), which implies that the discrete variant of the definition is the same when all finite abelian groups are considered instead of the groups $\mathbb{Z}_N$ only (cf. Corollaries 10 and 11). We note that the fractional chromatic number can be cast in a similar way but taking into account all finite (not necessarily abelian) groups (see the discussion after Corollary 11), i.e., the gyrochromatic number can be viewed as an analogue of the fractional chromatic number restricted to abelian groups. We believe that these properties show that the gyrochromatic number of a graph is a natural and robust parameter, which is likely to play an important role in providing a more detailed understanding of the structure of graphs whose circular and fractional chromatic numbers differ.

In addition to presenting several equivalent definitions of the gyrochromatic number in Sections 2 and 4, we also establish some of its basic properties and in particular construct graphs with gyrochromatic number strictly between the circular and fractional chromatic numbers (cf. Theorem 12). Finally, in Section 6 we show that, somewhat surprisingly, the supremum in the definition (1) need not be attained, which also implies that the infimum in its discrete variant (as given in Corollary 3) need not be attained.

2   Equivalent definitions

In this section, we present alternative definitions of the gyrochromatic number, and show that these are equivalent to the original definition stated in Section 1. We begin by giving another definition, which is analogous to circular coloring and provides the notion of a gyrocoloring. If z is a non-negative real, a *z-gyrocoloring* of a graph G is a mapping g from $V(G)$ to Borel subsets of $[0, z)$ such that the measure of each of $g(u)$, $u \in V(G)$, is one, the sets $g(u)$ and $g(v)$ are disjoint for every edge uv, and the sets $g(u)$ and $g(v)$ are rotationally equivalent for any two vertices u and v of G, i.e., there exists $x \in [0, z)$ such that

$$g(u) = (g(v) + x) \mod z = \{(y + x) \mod z : y \in g(v)\}.$$

The gyrochromatic number of G is the infimum of all z such that G has a z-gyrocoloring. The equivalence of this definition to the one given in Section 1 is rather easy to see. For completeness, we include a short proof.

Proposition 1. *Let G be a graph. For every positive real z, the graph G has a z-gyrocoloring if and only if G has a coloring $\mathbb{T}$-base of measure z^{-1}.*

Proof. Fix a graph G and a positive real z. Suppose that g is a z-gyrocoloring. Fix a vertex v_0 of G and let $x_v \in [0, z)$ be such that $g(v) = (g(v_0) + x_v) \mod z$; in particular, $x_{v_0} = 0$. The set

$$\{x/z : x \in g(v_0)\}$$

is a coloring $\mathbb{T}$-base; this can be seen by setting the function f from the definition of a coloring base to be $f(v) = g(v)/z$.

For the other direction, let A be a coloring $\mathbb{T}$-base of measure z^{-1}, and let f be the function from the definition of a coloring base. We can now define a z-gyrocoloring of the graph G as follows:

$$g(v) = \{(x + f(v)) \cdot z \mod z : x \in A\}$$

for every vertex $v \in V(G)$. $\qquad\square$

We next turn our attention to definitions where the equivalence is more complicated to see, and start with showing that the discrete and Borel variants of the definition of the gyrochromatic number are equivalent.

Theorem 2. *Let G be a graph. It holds that $\sigma_{\mathbb{Z}_N}(G) \leq \sigma_{\mathbb{T}}(G)$ for every $N \in \mathbb{N}$ and*

$$\sigma_{\mathbb{T}}(G) = \sup_{N \in \mathbb{N}} \sigma_{\mathbb{Z}_N}(G) = \lim_{N \to \infty} \sigma_{\mathbb{Z}_N}(G).$$

Proof. Fix a graph G. If A is a coloring $\mathbb{Z}_N$-base for G, then the set

$$\bigcup_{x \in A} \left[\frac{x - 1}{N}, \frac{x}{N} \right)$$

is a coloring $\mathbb{T}$-base. Hence, $\sigma_{\mathbb{T}}(G) \geq \sigma_{\mathbb{Z}_N}(G)$ for every $N \in \mathbb{N}$. We therefore have

$$\sigma_{\mathbb{T}}(G) \geq \sup_{N \in \mathbb{N}} \sigma_{\mathbb{Z}_N}(G) \geq \limsup_{N \to \infty} \sigma_{\mathbb{Z}_N}(G).$$

We now prove that

$$\lim_{N \to \infty} \sigma_{\mathbb{Z}_N}(G) = \sigma_{\mathbb{T}}(G),$$

which will complete the proof.

Choose any $\varepsilon > 0$ and let A be a coloring $\mathbb{T}$-base such that $\mu(A) \geq \sigma_{\mathbb{T}}(G) - \varepsilon$. We may assume without loss of generality that $\mu(A) > 0$. Let $f : V(G) \to \mathbb{T}$ be a mapping such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of the graph G.

Since the measure μ is regular, there exists a closed set $B \subseteq A$ such that $\mu(A \setminus B) < \varepsilon$. For $m \in \mathbb{N}$, let $B_m := B + \left[\frac{-1}{m}, \frac{1}{m} \right]$; note that

$$B = \bigcap_{m \in \mathbb{N}} B_m \text{ and } \lim_{m \to \infty} \mu(B_m) = \mu(B).$$

5

Hence, there exists an integer m such that $\mu(B_m \setminus B) < \varepsilon$, whence $\mu(B_m \triangle A) \le \mu(B_m \triangle B) + \mu(B \triangle A) < 2\varepsilon$; fix such m for the rest of the proof. Note that the measure of B_m is at least $\mu(A) - 2\varepsilon \ge \sigma_{\mathbb{T}}(G) - 3\varepsilon$.

Choose $N \in \mathbb{N}$ such that $m/N \le \varepsilon$ and define $f'(v) = \lfloor Nf(v) \rfloor / N$; here and in what follows, multiplications such as $Nf(v)$ mean that $f(v)$ is viewed as an element of $[0, 1]$ and is multiplied in $\mathbb{R}$ by N. Since the sets $A + f(u)$ and $B_m + f(u)$ differ on a set of measure at most 2ε for every vertex $u \in V(G)$, the measure of the intersection $B_m + f(u)$ and $B_m + f(v)$ is at most 4ε for every edge uv of G. Since the set B_m has at most $m/2$ connected components, i.e., B_m viewed as a subset of a circle consists of at most $m/2$ closed arcs, the measure of the intersection $B_m + f'(u)$ and $B_m + f'(v)$ is at most $4\varepsilon + m/N$. Choose $z \in \mathbb{T}$ randomly according to μ and define a set $A(z) \subseteq \mathbb{Z}_N$ to be the set containing all $i \in \{0, \dots, N - 1\}$ such that $z + i/N \in B_m$ and

$$z + i/N \notin B_m + f'(v) - f'(u)$$

for every edge uv of G. The probability that a particular i satisfies that $z + i/N \in B_m$ is equal to the measure of B_m. The probability that $z + i/N$ is both in B_m and in $B_m + f'(v) - f'(u)$ for a particular edge uv is equal to the measure of the intersection of $B_m + f'(u)$ and $B_m + f'(v)$, which is at most $4\varepsilon + m/N$. Hence, the probability that i is included in the set $A(z)$ is at least

$$\mu(A) - 2\varepsilon - (4\varepsilon + m/N)|E(G)| \ge \mu(A) - 2\varepsilon - 5\varepsilon|E(G)|.$$

The expected size of $A(z)$ is therefore at least

$$N(\mu(A) - 2\varepsilon - 5\varepsilon|E(G)|).$$

Fix z such that the size of $A(z)$ is at least the expected size.

We next show that $A(z)$ is a coloring $\mathbb{Z}_N$-base for G. Consider the function $f'' : V(G) \to \mathbb{Z}_N$ defined as $f''(v) = Nf'(v)$ and observe that the sets $A(z) + f''(u)$ and $A(z) + f''(v)$ are disjoint for every edge uv. Indeed, if the intersection of $A(z) + f''(u)$ and $A(z) + f''(v)$ were non-empty, there would exist $i, j \in A(z)$ such that $i + f''(u) = j + f''(v)$, which would imply that $z + i/N \in B_m$ and $z + j/N \in B_m$; since $z + i/N = z + j/N + f'(v) - f'(u)$, it would then follow that $z + i/N \in B_m + f'(v) - f'(u)$, contradicting the definition of $A(z)$. Hence $A(z)$ is a coloring $\mathbb{Z}_N$-base for G, so $\sigma_{\mathbb{Z}_N}(G) \ge \frac{|A(z)|}{N} = \mu(A) - 2\varepsilon - 5\varepsilon|E(G)|$.

We have thus proved that for all N sufficiently large (depending on ε) we have

$$\sigma_{\mathbb{T}}(G) \ge \sigma_{\mathbb{Z}_N}(G) \ge \sigma_{\mathbb{T}}(G) - 3\varepsilon - 5\varepsilon|E(G)|.$$

Since the choice of $\varepsilon > 0$ was arbitrary, we deduce that

$$\sigma_{\mathbb{T}}(G) = \lim_{N \to \infty} \sigma_{\mathbb{Z}_N}(G)$$

and the result follows. $\qquad\square$

Theorem 2 yields the following.

Corollary 3. *The gyrochromatic number of a graph G is equal to the infimum of K/N for which there exists a K-element set $A \subseteq \mathbb{Z}_N$ and a function $f : V(G) \to \mathbb{Z}_N$ such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G.*

We next turn our attention to a definition through homomorphisms to circulant graphs. Fix an integer N and a set $S \subseteq \mathbb{Z}_N$ such that $S = -S$ and $0 \notin S$; the *circulant graph* $C(N, S)$ is the graph with vertex set $\mathbb{Z}_N$ such that two vertices i and j of $C(N, S)$ are adjacent if $j - i \in S$. If G and H are two graphs, we say that G is *homomorphic* to H if there exists a mapping $h : V(G) \to V(H)$ such that $h(u)h(v)$ is an edge of H for every edge uv of G. The mapping h is a *homomorphism* from G to H. If G is homomorphic to H, we also write $G \to H$ and say that H *admits a homomorphism* from G.

Theorem 4. *The gyrochromatic number of a graph G is equal to*

$$\inf_{\substack{N \in \mathbb{N}, S \subseteq \mathbb{Z}_N \\ G \to C(N,S)}} \chi_f\left(C(N, S)\right) = \inf_{\substack{N \in \mathbb{N}, S \subseteq \mathbb{Z}_N \\ G \to C(N,S)}} \frac{N}{\alpha\left(C(N, S)\right)} \,,$$

i.e., is equal to the infimum of the fractional chromatic numbers of the circulant graphs that admit a homomorphism from G.

Proof. We will show that the following holds for every $N \in \mathbb{N}$:

$$\sigma_{\mathbb{Z}_N}(G) = \max_{\substack{S \subseteq \mathbb{Z}_N \\ G \to C(N,S)}} \frac{\alpha\left(C(N, S)\right)}{N}. \tag{3}$$

Since the graph $C(N, S)$ is vertex transitive for every choice of S, it holds that the fractional chromatic number of $C(N, S)$ is equal to $N/\alpha\left(C(N, S)\right)$. In particular, the statement of the theorem will follow from (3) and Theorem 2.

Fix $N \in \mathbb{N}$. We prove the equality in (3) as two inequalities, starting with showing that

$$\sigma_{\mathbb{Z}_N}(G) \leq \max_{\substack{S \subseteq \mathbb{Z}_N \\ G \to C(N,S)}} \frac{\alpha\left(C(N, S)\right)}{N}.$$

Let A be a coloring $\mathbb{Z}_N$-base for G and let $f : V(G) \to \mathbb{Z}_N$ be such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G. We define the set S as the set containing $f(u) - f(v)$ and $f(v) - f(u)$ for every edge uv. We claim that the set A is independent in the circulant graph $C(N, S)$: if A was not independent, then there would exist $i, j \in A$ such that $i = j + f(v) - f(u)$ for an edge uv, which would imply that the sets $A + f(u)$ and $A + f(v)$ are not disjoint ($i + f(u)$ would be their common element). Hence, the independence number of $C(N, S)$ is at least $|A|$. Since it holds that the vertices $f(u)$ and $f(v)$ of $C(N, S)$ are adjacent

7

for every edge uv of G, the mapping f is a homomorphism from G to $C(N,S)$ and the inequality follows.

We next prove that

$$\sigma_{\mathbb{Z}_N}(G) \geq \max_{\substack{S \subseteq \mathbb{Z}_N \\ G \to C(N,S)}} \frac{\alpha\left(C(N,S)\right)}{N}.$$

Fix a set $S \subseteq \mathbb{Z}_N$ such that there exists a homomorphism $f : V(G) \to \mathbb{Z}_N$ from G to the circulant graph $C(N,S)$, and let A be an independent set of $C(N,S)$ of size $\alpha(C(N,S))$. We claim that A is a coloring $\mathbb{Z}_N$-base for G. To establish this, it is enough to show that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G. Since A is an independent set in G, the sets A and $A + x$ are disjoint for every $x \in S$. Consider an edge uv of G. Since f is a homomorphism from G to $C(N,S)$, it follows that $f(v) - f(u) \in S$, in particular, the sets A and $A + f(v) - f(u)$ are disjoint. Hence, the sets $A + f(u)$ and $A + f(v)$ are disjoint. We conclude that A is a coloring $\mathbb{Z}_N$-base. $\qquad \square$

We remark that in Section 4 we establish a more general statement than that of Theorem 4; circulant graphs are Cayley graphs of the (abelian) group $\mathbb{Z}_N$ and we will show that the gyrochromatic number of a graph G is equal to the infimum of the fractional chromatic numbers of Cayley graphs of finite abelian groups that admit a homomorphism from G (cf. Corollary 11). We remark that the fractional chromatic number of G is equal to the minimum of the fractional chromatic numbers of Cayley graphs of finite groups that admit a homomorphism from G (see the discussion after Corollary 11).

3 Simple properties

In this section, we establish several simple properties of the gyrochromatic number; some of them will be used within our arguments later in the paper. We start with two properties related to products of graphs. Recall that the *Cartesian product* $G \square H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$ such that two vertices (v, v') and (w, w') of $G \square H$ are adjacent if either $v = w$ and $v'w'$ is an edge of H or $v' = w'$ and vw is an edge of G. Similarly to the chromatic number and the circular chromatic number, the gyrochromatic number is also preserved by the Cartesian product of a graph with itself.

Proposition 5. *For every graph G, it holds that $\chi_g(G) = \chi_g(G \square G)$.*

Proof. We show that $A \subseteq \mathbb{T}$ is a coloring base for G if and only if it is a coloring base for $G \square G$. If A is a coloring base for $G \square G$, then it is clearly a coloring base for G. Hence, we focus on proving the reverse implication. Let A be a coloring $\mathbb{T}$-base for G and $f : V(G) \to \mathbb{T}$ such that $A + f(v)$ and $A + f(w)$ are disjoint for

any edge vw of G. We define $g : V(G)^2 \to \mathbb{T}$ by setting $g(v, v')$ to be $f(v) + f(v')$, and show that if (v, v') and (w, w') are adjacent in $G \square G$, then $A + g(v, v')$ and $A + g(w, w')$ are disjoint.

Suppose that (v, v') and (w, w') are adjacent. It holds that either $v = w$ or $v' = w'$. In the former case, v' and w' are adjacent in G and so $A + f(v')$ and $A + f(w')$ are disjoint. Consequently, $A + g(v, v') = A + f(v) + f(v')$ and $A + g(w, w') = A + f(w) + f(w') = A + f(v) + f(w')$ are disjoint. Since the case $v' = w'$ is symmetric, A is a coloring base for the graph $G \square G$. $\qquad\square$

Proposition 5 does not generalize to Cartesian products of distinct graphs, however the following holds.

Proposition 6. *For any two graphs G and H, it holds that $\chi_g(G \square H) = \chi_g(G \cup H)$, where $G \cup H$ is the disjoint union of G and H.*

Proof. Since $G \square H$ is a component of the graph $(G \cup H) \square (G \cup H)$, it holds that $\chi_g(G \square H) \leq \chi_g(G \cup H)$ by Proposition 5. On the other hand, both G and H are subgraphs of $G \square H$, which implies that if $A \subseteq \mathbb{T}$ is a coloring base for $G \square H$, then it is also a coloring base of $G \cup H$. Indeed, if $f : V(G \square H) \to \mathbb{T}$ is a function such that $A + f(v)$ and $A + f(w)$ are disjoint for any edge vw of $G \square H$, then its restriction to the copies of G and H in $G \square H$ witness that A is a coloring base for $G \cup H$. Hence, $\chi_g(G \cup H) \leq \chi_g(G \square H)$. $\qquad\square$

In the next two paragraphs, we give an example of two (small) graphs G and H such that $\chi_g(G \square H) > \max\{\chi_g(G), \chi_g(H)\}$, which is equivalent to $\chi_g(G \cup H) > \max\{\chi_g(G), \chi_g(H)\}$ by Proposition 6. The two graphs G and H that we identify will also satisfy that $\chi_f(G \cup H) < \chi_g(G \cup H) < \chi_c(G \cup H)$; examples of connected graphs for which both inequalities are strict are given in Theorem 12.

Recall that the *lexicographic product* $G[G']$ of graphs G and G' is the graph whose vertex set is $V(G) \times V(G')$ and two vertices (v, v') and (w, w') are adjacent if either vw is an edge of G or $v = w$ and $v'w'$ is an edge of G'. Consider graphs $G = K_5$ and $H = K_2[C_5]$ and note that they both are circulant graphs (the graph $K_2[C_5]$ is isomorphic to the circulant graph $C(10, S)$ for $S = \{1, 3, 4, 5, 6, 7, 9\}$), and therefore their gyrochromatic number is equal to their fractional chromatic number, which is five for each of them. Clearly $\chi(G) = \chi_c(G) = 5$. Moreover, since the complement of H is the union of two disjoint copies of C_5, we can see that $\chi(H) = 6$, and also (since this complement is therefore disconnected) that $\chi_c(H) = \chi(H)$ by [14, Corollary 3.1]. We next observe that $\alpha(G \square H) \leq 9$: since G is a clique of size five, the independence number of $G \square H$ is the maximum total number of vertices assembled from five pairwise disjoint independent sets of H. As $\alpha(H) = 2$ and H is not 5-colorable, the maximum number of vertices of H contained in five (disjoint) independent sets of H is at most 9. We conclude that $\alpha(G \square H) \leq 9$, which implies that $\chi_f(G \square H) \geq 50/9$.

Since $\chi_g(G \square H) \geq 50/9$, Proposition 6 yields that $\chi_g(G \cup H) \geq 50/9 > \max\{\chi_g(G), \chi_g(H)\} = \max\{\chi_f(K_5), \chi_f(H)\} = 5$. On the other hand, observe
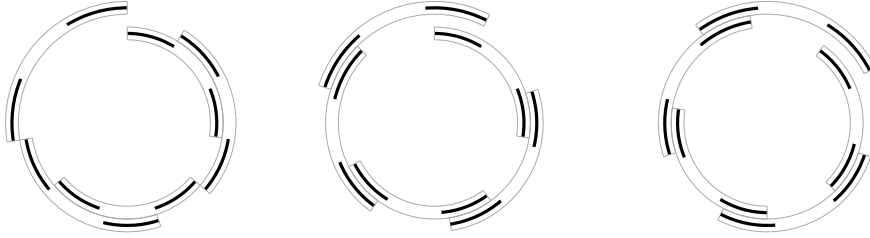
Figure 1: Outline of the gyrocoloring of $K_5 \cup K_2[C_5]$. The five sets depicted on the left-hand side correspond to the vertices of K_5 (the sets are pairwise disjoint). The middle correspond to the vertices of induced C_5, and similarly the right-hand side correspond to the other C_5 (the union of the sets in the middle is disjoint from the union of the sets on the right).

that $G \cup H$ admits a 40/7-gyrocoloring. The structure of such a coloring is outlined in Figure 1. Namely, consider the set $[0, \frac{1}{2}) \cup [\frac{15}{14}, \frac{22}{14})$ and assign its five copies shifted by 0, $\frac{1}{2}$, $\frac{29}{14}$, $\frac{36}{14}$ and $\frac{58}{14}$ to the vertices of G, its five copies shifted by $\frac{8i}{7}$ for $i = 0, 1, 2, 3, 4$ to the vertices of one copy of C_5 in H and its five copies shifted by $\frac{8i+4}{7}$ for $i = 0, \ldots, 4$ to the vertices of the other copy of C_5 in H. Furthermore, recalling the fractional and circular chromatic numbers of G and H, we get that $\chi_f(G \cup H) = 5$ and $\chi_c(G \cup H) = 6$ which yields that $\chi_f(G \cup H) < \chi_g(G \cup H) < \chi_c(G \cup H)$.

Considering lexicographic products, Gao and Zhu [4] showed that if $\chi_f(G) = \chi_c(G)$, then $\chi_c(G[H]) = \chi_c(G) \chi(H)$. In particular, $\chi_c(K_k[H]) = k \chi(H)$. Recall that if H is a circulant graph, then $\chi_f(K_k[H]) = \chi_g(K_k[H]) = k \chi_f[H]$. Hence, if H satisfies $\chi_f(H) < \chi(H)$, then $K_k[H]$ is an example of a graph such that the gyrochromatic and circular chromatic numbers differ; the graph $K_2[C_5]$ discussed above is a particular case of this more general argument.

We conclude this section with a lemma, which can be used to prove a lower bound on the gyrochromatic number of a graph that is larger than the fractional chromatic number. The lemma in particular yields an example of a graph such that its gyrochromatic number is strictly larger than its fractional chromatic number: consider the line graph G of the Petersen graph and note that $\omega(G) = \chi_f(G) = 3$ and $\chi(G) = 4$.

Lemma 7. *Let G be an n-vertex graph. If $\omega(G) < \chi(G)$, then $\chi_g(G) \geq \frac{n}{n-1}\omega(G)$.*

Proof. Let $k = \omega(H)$ and let $v_1, \ldots, v_k$ be vertices of a clique with k vertices in G. Suppose that A is a coloring $\mathbb{T}$-base for G with measure $\delta > 0$. Finally, let $f : V(G) \to \mathbb{T}$ be such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G. In particular, the sets $A + f(v_1), \ldots, A + f(v_k)$ are pairwise disjoint.

For $x \in \mathbb{T}$ and $i \in [k]$, define $V_i(x) \subseteq V(G)$ as the set of vertices v such that $x + f(v_i) \in A + f(v)$. Since the element $x + f(v_i)$ is contained in the set $A + f(v)$

10

for every $v \in V_i(x)$, each of the sets $V_i(x)$ is an independent set in G. In addition, the sets $V_1(x), \ldots, V_k(x)$ are pairwise disjoint for every $x \in \mathbb{T}$. Indeed, if a vertex v were contained in $V_i(x)$ and $V_j(x)$, $i \neq j$, $i, j \in [k]$, then both $x + f(v_i)$ and $x + f(v_j)$ would be contained in $A + f(v)$; this would imply that the element $x - f(v) + f(v_i) + f(v_j)$ is contained both in $A + f(v_i)$ and $A + f(v_j)$, which contradicts that the sets $A + f(v_i)$ and $A + f(v_j)$ are disjoint.

Since the graph G is not k-colorable, the union $V_1(x) \cup \cdots \cup V_k(x)$ does not contain all vertices of G for any $x \in \mathbb{T}$. Hence, for every $x \in \mathbb{T}$ we have

$$|V_1(x) \cup \cdots \cup V_k(x)| = \sum_{i=1}^{k} |V_i(x)| \leq n - 1,$$

which implies that $\int_{\mathbb{T}} \sum_{i=1}^{k} |V_i(x)| \, \mathrm{d}x \leq n - 1$. On the other hand it holds that

$$\int_{\mathbb{T}} \sum_{i=1}^{k} |V_i(x)| \, \mathrm{d}x = \sum_{v \in V(G)} \sum_{i=1}^{k} \mu\left(A + f(v) - f(v_i)\right) = kn\delta.$$

It follows that δ is at most $\frac{n-1}{kn}$. We conclude that $\sigma_{\mathbb{T}}(G) \leq \frac{n-1}{kn}$, which yields that the gyrochromatic number of G is at least $\frac{n}{n-1}k$. $\qquad\square$

4 Universality

In this section, we show that a higher-dimensional-torus analogue of the gyrochromatic number is equal to the (one-dimensional) gyrochromatic number and use this result to prove a generalization of Theorem 4 to all finite abelian groups. As the first step towards this result, we observe that the proof of Theorem 2 readily translates to higher dimensions; we formulate the corresponding statement as a lemma.

Lemma 8. *Let d be any positive integer. For every graph G, it holds that*

$$\sigma_{\mathbb{T}^d}(G) = \sup_{N \in \mathbb{N}} \sigma_{\mathbb{Z}_N^d}(G) = \lim_{N \to \infty} \sigma_{\mathbb{Z}_N^d}(G).$$

We next state and prove the main theorem of this section.

Theorem 9. *Let d be any positive integer. For every graph G, it holds that*

$$\sigma_{\mathbb{T}}(G) = \sigma_{\mathbb{T}^d}(G).$$

Proof. Fix an integer $d \geq 2$ and a graph G. Observe that if A is a coloring $\mathbb{T}$-base for G, then $A \times \mathbb{T}^{d-1}$ is a coloring $\mathbb{T}^d$-base for G; this implies that

$$\sigma_{\mathbb{T}}(G) \leq \sigma_{\mathbb{T}^d}(G).$$

11

The rest of the proof is devoted to establishing the reverse inequality. Choose $\varepsilon > 0$ arbitrarily and note that by Lemma 8, for all $N \in \mathbb{N}$ sufficiently large we have

$$\sigma_{\mathbb{Z}_N^d}(G) \geq \sigma_{\mathbb{T}^d}(G) - \varepsilon. \tag{4}$$

For any such N, since the group $\mathbb{Z}_N^d$ is finite, there exists a coloring $\mathbb{Z}_N^d$-base A for G with $\sigma_{\mathbb{Z}_N^d}(G) \cdot N^d$ elements. Next choose an integer $k \in \mathbb{N}$ such that $1 - \varepsilon \leq \left(\frac{k}{k+2}\right)^d$ and let N be large enough so that there are at least d distinct primes between $(k+1)N$ and $(k+2)N$ (the Prime Number Theorem implies that this holds for every N sufficiently large); let $P_1, \ldots, P_d$ be such primes.

Define A' to be the subset of $\mathbb{Z}_{P_1} \times \cdots \times \mathbb{Z}_{P_d}$ such that

$$A' = \{(x_1 + y_1 N, \ldots, x_d + y_d N) : (x_1, \ldots, x_d) \in A \text{ and } y_1, \ldots, y_d \in [k]\}.$$

Let $f : V(G) \to \mathbb{Z}_N^d$ be the function such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G. Observe that $A' + f(u)$ and $A' + f(v)$ are also disjoint for every edge uv of G when $f(u)$ and $f(v)$ are viewed as elements of $\mathbb{Z}_{P_1} \times \cdots \times \mathbb{Z}_{P_d}$ (here we use in particular that each of the primes $P_1, \ldots, P_d$ is at least $(k+1)N$, so that no addition needs to be done modulo P_i). Let $M = P_1 P_2 \cdots P_d$ and define a mapping $g : \mathbb{Z}_M \to \mathbb{Z}_{P_1} \times \cdots \times \mathbb{Z}_{P_d}$ as

$$g(x) = (x \mod P_1, \ldots, x \mod P_d).$$

Since $P_1, \ldots, P_d$ are distinct primes, the mapping g is an isomorphism of the groups $\mathbb{Z}_M$ and $\mathbb{Z}_{P_1} \times \cdots \times \mathbb{Z}_{P_d}$, in particular, g is a bijection.

We establish that the set $A'' = g^{-1}(A')$ is a coloring $\mathbb{Z}_M$-base for G. To do so, consider the function $f'' : V(G) \to \mathbb{Z}_M$ defined as $f''(v) = g^{-1}(f(v))$, $v \in V(G)$. Since g is an isomorphism of $\mathbb{Z}_M$ and $\mathbb{Z}_{P_1} \times \cdots \times \mathbb{Z}_{P_d}$, it holds that $A'' + f''(u)$ and $A'' + f''(v)$ are disjoint for every edge uv of G. It follows that A'' is a coloring $\mathbb{Z}_M$-base for G, which implies that

$$\sigma_{\mathbb{Z}_M}(G) \geq \frac{|A''|}{M} = \frac{k^d |A|}{M} \geq \frac{k^d |A|}{(k+2)^d N^d} \geq (1 - \varepsilon)\, \sigma_{\mathbb{Z}_N^d}(G). \tag{5}$$

The inequalities (4) and (5) yield that

$$\sigma_{\mathbb{Z}_M}(G) \geq (1 - \varepsilon)\,(\sigma_{\mathbb{T}^d}(G) - \varepsilon) \geq \sigma_{\mathbb{T}^d}(G) - 2\varepsilon.$$

Since the choice of ε was arbitrary, it follows that $\sigma_{\mathbb{Z}_M}(G)$ is at least $\sigma_{\mathbb{T}^d}(G)$ for every sufficiently large M. Theorem 2 now implies that $\sigma_{\mathbb{T}}(G) \geq \sigma_{\mathbb{T}^d}(G)$, which completes the proof of the theorem. $\qquad\square$

Theorem 9 yields the following corollary.

Corollary 10. *For every graph G, it holds that*

$$\sigma_{\mathbb{T}}(G) = \sup_{Z} \sigma_Z(G)$$

where the supremum is taken over all finite abelian groups Z.

Proof. Fix a graph G. Since every $\mathbb{Z}_N$, $N \in \mathbb{N}$, is a finite abelian group, Theorem 2 yields that the supremum given in the statement is at least $\sigma_{\mathbb{T}}(G)$. To establish the corollary, we need to show that $\sigma_{\mathbb{T}}(G) \geq \sigma_Z(G)$ for every finite abelian group Z.

Fix a finite abelian group Z; without loss of generality, we may assume that Z is $\mathbb{Z}_{M_1} \times \cdots \times \mathbb{Z}_{M_d}$. Let A be a coloring Z-base with $\sigma_Z(G)|Z|$ elements and let $f : V(G) \to Z$ be a function such that $A + f(u)$ and $A + f(v)$ are disjoint for every edge uv of G. Let N be the least common multiple of $M_1, \ldots, M_d$, and let π denote the natural homomorphism $\mathbb{Z}_N^d \to Z$, defined by $(x_1, \ldots, x_d) \mapsto (x_1 N/M_1, \ldots, x_d N/M_d)$. It is then easy to see that the preimage $A_N := \pi^{-1}(A)$ is a coloring $\mathbb{Z}_N^d$-base, with corresponding map $f' : V \to \mathbb{Z}_N^d$ where for each vertex v we let $f'(v)$ be any preimage of $f(v)$ under π. We deduce that $\sigma_{\mathbb{Z}_N^d}(G) \geq \sigma_Z(G)$. By Lemma 8 and Theorem 9, we get that $\sigma_{\mathbb{T}}(G) = \sigma_{\mathbb{T}^d}(G) \geq \sigma_{\mathbb{Z}_N^d}(G) \geq \sigma_Z(G)$, and the result follows. $\square$

We conclude this section with a generalization of Theorem 4 to finite abelian groups. Recall that if Z is an abelian group and S is a subset of Z such that $S = -S$ and $0 \notin S$, the *Cayley graph* $C(Z, S)$ is the graph with vertex set Z such that two vertices x and y of $C(Z, S)$ are adjacent if $y - x \in S$.

Corollary 11. *The gyrochromatic number of a graph G is equal to*

$$\inf_{\substack{Z, S \subseteq Z \\ G \to C(Z,S)}} \chi_f\left(C(Z,S)\right) = \inf_{\substack{Z, S \subseteq Z \\ G \to C(Z,S)}} \frac{|Z|}{\alpha\left(C(Z,S)\right)} \qquad (6)$$

where the infimum is taken over all finite abelian groups Z.

Proof. The reasoning given in the proof of Theorem 4 yields that the following holds for every abelian group Z:

$$\sigma_Z(G) = \max_{\substack{S \subseteq Z \\ G \to C(Z,S)}} \frac{\alpha\left(C(Z,S)\right)}{|Z|}. \qquad (7)$$

Since every Cayley graph is vertex-transitive, it holds for every abelian group Z and every $S \subseteq Z$:

$$\chi_f\left(C(Z,S)\right) = \max_{\substack{S \subseteq Z \\ G \to C(Z,S)}} \frac{\alpha\left(C(Z,S)\right)}{|Z|}. \qquad (8)$$

The corollary now follows from (7), (8) and Corollary 10. $\square$

We remark that if finite abelian groups Z in the infimum in (6) are replaced with all finite groups (with generating set S not containing the identity and satisfying $S = S^{-1}$), then the infimum is equal to the fractional chromatic number and is always attained. Indeed, it is well-known that the fractional chromatic number of a graph G is equal to the minimum fractional chromatic number of a Kneser graph that admits a homomorphism from G. Sabidussi's theorem [10] states that every vertex transitive graph (in particular, every Kneser graph) is a retract of (hence is homomorphically equivalent to) a Cayley graph, cf. [6, Theorem 3.1]. Hence, the gyrochromatic number can be viewed as a variant of the fractional chromatic number restricted to abelian groups.

5 Relation to circular and fractional colorings

We now identify graphs such that their gyrochromatic number is strictly between their fractional and circular chromatic numbers, and the difference with the latter number can be arbitrarily large.

Theorem 12. *There exists a sequence of graphs $(G_k)_{k \in \mathbb{N} \setminus \{1\}}$ such that $\chi_f(G_k) < \chi_g(G_k) \leq \chi_c(G_k) = k + 2$ and*

$$\lim_{k \to \infty} \chi_g(G_k) = 2.$$

Proof. We set G_k to be the Kneser graph $K(2k^3 + k, k^3)$. Since the circular chromatic number and the chromatic number coincide for Kneser graphs [2,3,8], it follows that the circular chromatic number $\chi_c(G_k)$ is equal to $k + 2$. Recall that the fractional chromatic number of G_k is $\frac{2k^3 + k}{k^3} = 2 + k^{-2}$, in particular, the limit of the fractional chromatic numbers of G_k is two.

We next show that the gyrochromatic numbers of the graphs G_k converge and their limit is also two. Since it holds that $\chi_f(G_k) \leq \chi_g(G_k)$ for every graph G_k, the limit (assuming that it exists) must be at least two. For the upper bound, consider the Cayley graph $H_k = C(\mathbb{Z}_2^{2k^3 + k}, S)$ with the generating set S consisting of those $x \in \mathbb{Z}_2^{2k^3 + k}$ that have exactly $2k^3$ entries equal to one. The graph H_k admits a homomorphism from G_k: indeed, each vertex of $K(2k^3 + k, k^3)$ corresponds to a k^3-element subset of $[2k^3 + k]$ and we map it to the characteristic vector of this set. This mapping is indeed a homomorphism from G_k to H_k since two vertices of $G_k = K(2k^3 + k, k^3)$ are adjacent if and only if their corresponding sets are disjoint, which happens if and only if the difference of their characteristic vectors (modulo two) has exactly $2k^3$ entries equal to one. Corollary 11 implies that

$$\limsup_{k \to \infty} \chi_g(G_k) \leq \limsup_{k \to \infty} \frac{|H_k|}{\alpha(H_k)}.$$

We next show that the right limit is at most two.

Let $I_k \subseteq V(H_k)$ be the set of those elements of H_k with fewer than k^3 entries equal to one. Since I_k is an independent set in H_k, it follows that

$$\alpha(H_k) \geq |I_k| = \sum_{i=0}^{k^3-1} \binom{2k^3 + k}{i} = 2^{2k^3+k} \cdot \left(\frac{1}{2} + o(1) \right),$$

which implies that

$$\limsup_{k\to\infty} \frac{|H_k|}{\alpha(H_k)} \leq \lim_{k\to\infty} \frac{2^{2k^3+k}}{|I_k|} = 2.$$

We conclude that the sequence $\chi_g(G_k)$ converges and its limit is equal to two.

To complete the proof of the theorem, we need to show that $\chi_f(G_k) < \chi_g(G_k)$. To do so, it suffices to show that $\chi_f(G_k) < \chi_f(G_k \square G_k)$. Indeed, if we show this, then by Proposition 5 and the fact that the fractional chromatic number of every graph is at most its gyrochromatic number, we will have $\chi_f(G_k) < \chi_f(G_k \square G_k) \leq \chi_g(G_k \square G_k) = \chi_g(G_k)$, as required.

Since the graphs G_k and $G_k \square G_k$ are vertex-transitive, it holds that

$$\chi_f(G_k) = \frac{|G_k|}{\alpha(G_k)} \qquad \text{and} \qquad \chi_f(G_k \square G_k) = \frac{|G_k \square G_k|}{\alpha(G_k \square G_k)} = \frac{|G_k|^2}{\alpha(G_k \square G_k)}.$$

Hence, it is enough to show that $\alpha(G_k \square G_k) < |G_k| \alpha(G_k)$. The Erdős-Ko-Rado Theorem yields that $\alpha(G_k) = \binom{2k^3+k-1}{k^3-1}$ and that for every independent set of G_k of this size, there exists $x \in [2k^3 + k]$ such that the vertices of the independent set correspond to the $\binom{2k^3+k-1}{k^3-1}$ k^3-element subsets of $[2k^3 + k]$ containing x. In particular, any two independent sets of the maximum cardinality in G_k have a non-empty intersection (here we use that $k \geq 2$).

Suppose that the graph $G_k \square G_k$ has an independent set of size $|G_k| \alpha(G_k)$, and let I be such an independent set. For every vertex v of G_k, the set $I_v = \{w : (v, w) \in I\}$ is an independent set in G_k (by definition of the Cartesian product). Since the set I contains $|G_k| \alpha(G_k)$ elements and $|I_v| \leq \alpha(G_k)$ for every vertex $v \in V(G_k)$, we obtain that $|I_v| = \alpha(G_k)$ for every $v \in V(G_k)$. Let v and v' be two adjacent vertices of G_k. It follows from the previous paragraph that I_v and I'_v have a common vertex w, i.e., the set I contains both (v, w) and (v', w), which contradicts that I is an independent set. Hence, the graph $G_k \square G_k$ has no independent set of size $|G_k| \alpha(G_k)$, i.e. $\alpha(G_k \square G_k) < |G_k| \alpha(G_k)$. This finishes the proof of the theorem. $\square$

Theorem 12 immediately implies that the gap between the gyrochromatic number and the circular chromatic number can be arbitrarily large and there exists a graph such that its gyrochromatic number is strictly between its fractional and its circular chromatic numbers.

Corollary 13. *For every $k \in \mathbb{N}$, there exists a graph G such that $\chi_f(G) < \chi_g(G) \leq \chi_c(G) - k$.*

6 Existence of optimal gyrocoloring

In this section, we establish the existence of a graph G such that there is no coloring $\mathbb{T}$-base for G of measure $\sigma_{\mathbb{T}}(G)$, i.e., the supremum in (1) is not attained. In other words, there exists a graph G with no $\chi_g(G)$-gyrocoloring.

Let G_5 be the graph with vertex set $\mathbb{Z}_5^2$ and two vertices (i, j) and (i', j') adjacent if $i' - i \in \{2, 3\}$ or $j' - j \in \{2, 3\}$ (calculations modulo five). In other words, G_5 is the Cayley graph on $\mathbb{Z}_5^2$ with generating set $\mathbb{Z}_5 \setminus \{-1, 0, 1\}^2$. Proposition 15 and Theorem 19, which we will prove in this section, yield that G_5 has no coloring $\mathbb{T}$-base with measure $\sigma_{\mathbb{T}}(G_5)$.

We begin by analyzing the structure of independent sets of G_5.

Proposition 14. *The independence number of G_5 is four and the only independent sets of size four are the following 25 sets:*

$$I_v = \{v, v + (0, 1), v + (1, 0), v + (1, 1)\}$$

where $v \in \mathbb{Z}_5^2$.

Proof. Let X be an independent set in G_5 and let $(i, j) \in \mathbb{Z}_5^2$ be any vertex of G_5 contained in X. Observe that the set X contains in addition to the vertex (i, j) at most one of the vertices $(i - 1, j)$ and $(i + 1, j)$, at most one of the vertices $(i, j - 1)$ and $(i, j + 1)$, at most one of the vertices $(i - 1, j - 1)$, $(i - 1, j + 1)$, $(i + 1, j - 1)$ and $(i + 1, j + 1)$, and no other vertex. In particular, the set X has size at most four. If the set X has size four, we can assume by symmetry that X contains the vertex $(i + 1, j)$ from the first pair and the vertex $(i, j + 1)$ from the second pair. If X contains the vertices (i, j), $(i + 1, j)$ and $(i, j + 1)$, then the only vertex from the last quadruple that it can contain is $(i + 1, j + 1)$. Hence, $X = I_v$ for $v = (i, j)$. $\square$

We next compute $\sigma_{\mathbb{T}}(G_5)$.

Proposition 15. *It holds that $\sigma_{\mathbb{T}}(G_5) = 4/25$.*

Proof. Since $\alpha(G_5) = 4$, it follows that $\chi_f(G_5) \geq 25/4$, which implies that $\sigma_{\mathbb{T}}(G_5) \leq 4/25$. On the other hand, the set $A = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$ is a coloring $\mathbb{Z}_5^2$-base for G_5: indeed, the sets $A + v$ and $A + w$ are disjoint for every edge vw of G_5. Hence, $\sigma_{\mathbb{Z}_5^2}(G_5) \geq 4/25$, which yields that $\sigma_{\mathbb{T}}(G_5) \geq 4/25$ by Corollary 10. $\square$

We next show that every coloring $\mathbb{T}$-base of G_5 of measure $4/25$ must have a very particular structure. In what follows, we will write $X \cong Y$ if two sets X and Y differ on a null set.

Lemma 16. *Let $A \subseteq \mathbb{T}$ be a coloring $\mathbb{T}$-base of the graph G_5 and let $f : V(G_5) \to \mathbb{T}$ be such that $A + f(v)$ and $A + f(w)$ are disjoint for every edge vw; let B_v, $v \in V(G_5)$ be the set $A + f(v)$. If the measure of A is $4/25$, then there exist disjoint measurable subsets $C_{i,j}$, $(i, j) \in \mathbb{Z}_5^2$, such that*

$$B_{i,j} \cong C_{i,j} \cup C_{i+1,j} \cup C_{i,j+1} \cup C_{i+1,j+1}$$

for every $(i, j) \in \mathbb{Z}_5^2$, and the measure of each set $C_{i,j}$, $(i, j) \in \mathbb{Z}_5^2$, is $1/25$.

Proof. Let $\mathcal{I}$ be the set containing all independent sets of vertices of G_5. For each $I \in \mathcal{I}$, we define the following measurable subset of $\mathbb{T}$:

$$D_I = \{x \in \mathbb{T} : x \in A + f(v) \text{ if and only if } v \in I\}.$$

Observe that every $x \in \mathbb{T}$ belongs to D_I for some $I \in \mathcal{I}$: indeed, the set containing the vertices v such that $x \in A + f(v)$ is independent (note that $\emptyset \in \mathcal{I}$). Hence, the sets D_I, $I \in \mathcal{I}$, partition $\mathbb{T}$. We also have from the definition of D_I that

$$A + f(v) = \bigcup_{I \in \mathcal{I} : I \ni v} D_I, \tag{9}$$

and it follows that $\sum_{I \in \mathcal{I}} |I| \cdot \mu(D_I) = \sum_{v \in V(G_5)} \mu(A + f(v))$. On the other hand, since the measure of A is $4/25$, we have $\sum_{v \in V(G_5)} \mu(A + f(v)) = 4$. Hence, using Proposition 14, we have

$$4 = \sum_{I \in \mathcal{I}} |I| \cdot \mu(D_I) = 4 \sum_{I \in \mathcal{I} : |I| = 4} \mu(D_I) + \sum_{I \in \mathcal{I} : |I| < 4} |I| \cdot \mu(D_I).$$

This implies that $\mu(D_I) = 0$ for all $I \in \mathcal{I}$ with $|I| < 4$ (using $\sum_{I \in \mathcal{I}} \mu(D_I) = 1$). Setting

$$C_{i,j} = D_{\{(i-1,j-1),(i-1,j),(i,j-1),(i,j)\}},$$

we have by (9) that $B_{i,j} \cong C_{i,j} \cup C_{i+1,j} \cup C_{i,j+1} \cup C_{i+1,j+1}$ for every $(i, j) \in \mathbb{Z}_5^2$. To complete the proof, we need to show each set $C_{i,j}$ has measure $1/25$.

Recalling the notation I_v from Proposition 14, let M be the matrix with rows and columns indexed by the elements of $\mathbb{Z}_5^2$ such that $M_{(i,j),(i',j')}$ is one if $(i, j) \in I_{(i',j')}$ and zero otherwise. Further, let x be the vector with entries indexed by the elements of $\mathbb{Z}_5^2$ such that $x_{(i',j')}$ is the measure of $D_{I_{(i',j')}}$. Observe (using (9) and that $\mu(D_I) = 0$ for $|I| < 4$) that Mx is the vector with all entries equal to $4/25$. We next show that the matrix M is invertible. This would imply that the vector x with all entries equal to $1/25$ is the only vector such that Mx is the vector with all entries equal to $4/25$, which would complete the proof.

Assume that the matrix M is singular, i.e., there exists a non-zero vector x such that Mx is the zero vector. The entries of x can be interpreted as numbers on the toroidal 5×5 grid such that each of the 25 quadruples of entries forming a square sums to zero; a "square" stands here for a translate of $\{(0,0), (1,0), (0,1), (1,1)\}$ in the grid.

We can assume that the first row of the grid is $\alpha, \beta_1, \ldots, \beta_4$ and the first column is $\alpha, \gamma_1, \ldots, \gamma_4$, which yields that the numbers are assigned to the grid as follows.

$$
\begin{array}{ccccc}
\alpha & \beta_1 & \beta_2 & \beta_3 & \beta_4 \\
\gamma_1 & -\alpha - \beta_1 - \gamma_1 & +\alpha - \beta_2 + \gamma_1 & -\alpha - \beta_3 - \gamma_1 & +\alpha - \beta_4 + \gamma_1 \\
\gamma_2 & +\alpha + \beta_1 - \gamma_2 & -\alpha + \beta_2 + \gamma_2 & +\alpha + \beta_3 - \gamma_2 & -\alpha + \beta_4 + \gamma_2 \\
\gamma_3 & -\alpha - \beta_1 - \gamma_3 & +\alpha - \beta_2 + \gamma_3 & -\alpha - \beta_3 - \gamma_3 & +\alpha - \beta_4 + \gamma_3 \\
\gamma_4 & +\alpha + \beta_1 - \gamma_4 & -\alpha + \beta_2 + \gamma_4 & +\alpha + \beta_3 - \gamma_4 & -\alpha + \beta_4 + \gamma_4
\end{array}
$$

Since there is a square containing β_1, β_2, $\alpha + \beta_1 - \gamma_4$ and $-\alpha + \beta_2 + \gamma_4$, we obtain $\beta_1 = -\beta_2$. By considering other squares wrapping around, we obtain that $\beta_1 = -\beta_2 = \beta_3 = -\beta_4$, $\gamma_1 = -\gamma_2 = \gamma_3 = -\gamma_4$ and $\beta_4 = -\gamma_4$. Hence, the table can be rewritten as follows.

$$
\begin{array}{ccccc}
\alpha & \beta_1 & -\beta_1 & \beta_1 & -\beta_1 \\
-\beta_1 & -\alpha & +\alpha & -\alpha & +\alpha \\
+\beta_1 & +\alpha & -\alpha & +\alpha & -\alpha \\
-\beta_1 & -\alpha & +\alpha & -\alpha & +\alpha \\
+\beta_1 & +\alpha & -\alpha & +\alpha & -\alpha
\end{array}
$$

Considering two of the squares containing the entry α in the top left corner, we obtain $2(\alpha + \beta_1) = 0$ and $2(\alpha - \beta_1) = 0$. Hence $\alpha = \beta_1 = 0$, i.e., the vector x is zero. We conclude that the matrix M is invertible, which completes the proof. $\qquad\square$

We are now ready to prove the main lemma of this section. To simplify our notation, we will understand the subscripts indexing sets $B_{i,j}$ and $C_{i,j}$ in Lemma 16 as pairs $(i, j) \in \mathbb{Z}_5^2$, which allows us to perform addition as with the elements of $\mathbb{Z}_5^2$, e.g., $C_{(0,1)+(1,2)}$ is the set $C_{(1,3)} = C_{1,3}$. In addition, we write $t_{v \to w}$ for $f(w) - f(v)$, when A is a coloring $\mathbb{T}$-base and $f : V(G) \to \mathbb{T}$ is a function such that $A + f(v)$ and $A + f(w)$ are disjoint for every edge vw.

Lemma 17. *Let $A \subseteq \mathbb{T}$ be a coloring $\mathbb{T}$-base of the graph G_5 with measure $4/25$ and let $C_{i,j} \subseteq \mathbb{T}$, $(i, j) \in \mathbb{Z}_5^2$, be the sets as in Lemma 16. For every $v \in \mathbb{Z}_5^2$ and $w \in \{(0,0), (0,1), (1,0), (1,1)\}$, it holds that*

$$C_{v+w} + t_{v \to v+(1,0)} \cong C_{v+w+(1,0)} \tag{10}$$

$$C_{v+w} + t_{v \to v+(0,1)} \cong C_{v+w+(0,1)}. \tag{11}$$

Proof. By symmetry, we will assume that $v = (0,0)$ in our presentation with the exception of equality (15), which we formulate in the general setting. Throughout the proof, we write $B_{i,j} \subseteq \mathbb{T}$ for the sets as in Lemma 16; to simplify our notation, we also write $t_{i,j}$ for $t_{(0,0) \to (i,j)}$.
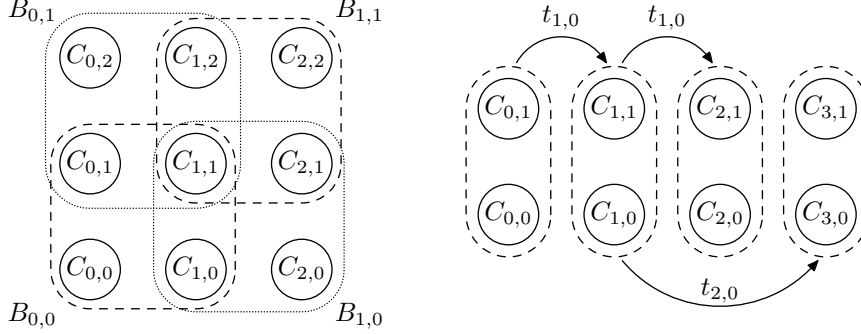
Figure 2: Visualization of the notation used in the proof of Lemma 17 and the equalities (12), (13) and (14).

Our first goal is to prove the following weaker statement, which is also visualized in Figure 2.

$$(C_{0,0} \cup C_{0,1}) + t_{1,0} \cong C_{1,0} \cup C_{1,1} \tag{12}$$

$$(C_{1,0} \cup C_{1,1}) + t_{1,0} \cong C_{2,0} \cup C_{2,1} \tag{13}$$

Since the vertices $(0,0)$ and $(2,0)$ are adjacent, the sets $B_{0,0}$ and $B_{2,0} = B_{0,0}+t_{2,0}$ are disjoint, and so are these sets shifted by $t_{1,0}$, i.e., the sets $B_{0,0}+t_{1,0} = B_{1,0}$ and $B_{0,0}+t_{2,0}+t_{1,0} = B_{1,0}+t_{2,0}$. In particular, the intersection of $C_{2,0} \cup C_{2,1} \subseteq B_{1,0}$ and $(C_{1,0} \cup C_{1,1})+t_{2,0} \subseteq B_{1,0}+t_{2,0}$ is empty. Since both $C_{2,0} \cup C_{2,1}$ and $(C_{1,0} \cup C_{1,1})+t_{2,0}$ are subsets of $B_{2,0}$ and the measure of each of them is half of the measure of $B_{2,0}$, it follows that the sets $(C_{1,0} \cup C_{1,1}) + t_{2,0}$ and $B_{2,0} \setminus (C_{2,0} \cup C_{2,1}) = C_{3,0} \cup C_{3,1}$ are the same (up to a null set), i.e.,

$$(C_{1,0} \cup C_{1,1}) + t_{2,0} \cong C_{3,0} \cup C_{3,1}. \tag{14}$$

We formulate (14) for an arbitrary vertex v since we need the statement later:

$$(C_{v+(1,0)} \cup C_{v+(1,1)}) + t_{v \to v+(2,0)} \cong C_{v+(3,0)} \cup C_{v+(3,1)}. \tag{15}$$

We next apply (15) with $v = (1,0)$ and $v = (3,0)$ as follows:

$$\begin{aligned}
(C_{2,0} \cup C_{2,1}) - t_{(0,0) \to (1,0)} &= (C_{2,0} \cup C_{2,1}) + t_{(1,0) \to (0,0)} \\
&= (C_{2,0} \cup C_{2,1}) + t_{(1,0) \to (3,0)} + t_{(3,0) \to (0,0)} \\
&\cong (C_{4,0} \cup C_{4,1}) + t_{(3,0) \to (0,0)} \\
&\cong C_{1,0} \cup C_{1,1},
\end{aligned}$$

which proves (13). Since both the sets $(C_{0,0} \cup C_{0,1} \cup C_{1,0} \cup C_{1,1}) + t_{1,0}$ and $C_{1,0} \cup C_{1,1} \cup C_{2,0} \cup C_{2,1}$ are equal to $B_{1,0}$, in particular, they are the same set, and all sets $C_{i,j}$, $(i,j) \in \mathbb{Z}_5^2$, are disjoint, the equality (12) also follows.

An argument symmetric to the one used to prove (12) and (13) yields the following.

$$(C_{0,0} \cup C_{1,0}) + t_{0,1} \cong C_{0,1} \cup C_{1,1} \qquad (16)$$

$$(C_{0,1} \cup C_{1,1}) + t_{0,1} \cong C_{0,2} \cup C_{1,2} \qquad (17)$$

Next suppose for a contradiction that the intersection of $C_{0,0} + t_{1,0}$ and $C_{1,1}$ has positive measure, and let X be the set $C_{0,0} \cap (C_{1,1} - t_{1,0})$. The equality (17) implies that $X + t_{1,0} + t_{0,1}$ is a subset of $C_{0,2} \cup C_{1,2}$ (up to a null set). On the other hand $X + t_{0,1} \subseteq C_{0,0} + t_{0,1}$ is a subset of $C_{0,1} \cup C_{1,1}$ by (16), and this is a subset of $B_{0,0}$, hence $(X + t_{0,1}) + t_{1,0} \subseteq B_{0,0} + t_{1,0} = B_{1,0}$. However $B_{1,0} \cong C_{1,0} \cup C_{1,1} \cup C_{2,0} \cup C_{2,1}$ has null intersection with $C_{0,2} \cup C_{1,2}$. We have thus deduced that $X + t_{1,0} + t_{0,1}$ is included (up to a null set) in the null set $B_{1,0} \cap (C_{0,2} \cup C_{1,2})$, which contradicts the assumption that $X + t_{0,1} + t_{1,0}$ has positive measure. We conclude that the intersection of $C_{0,0} + t_{1,0}$ and $C_{1,1}$ is null. Since all the sets $C_{0,0}$, $C_{0,1}$, $C_{1,0}$ and $C_{1,1}$ have the same measure, the equality (12) implies that

$$C_{0,0} + t_{1,0} \cong C_{1,0} \text{ and } C_{0,1} + t_{1,0} \cong C_{1,1}. \qquad (18)$$

A symmetric argument implies that

$$C_{0,0} + t_{0,1} \cong C_{0,1} \text{ and } C_{1,0} + t_{0,1} \cong C_{1,1}. \qquad (19)$$

We now prove that the intersection of the sets $C_{1,1} + t_{0,1}$ and $C_{0,2}$ is null. Assume the contrary, i.e., the set $X = C_{1,1} \cap (C_{0,2} - t_{0,1})$ has positive measure. The equality (15) applied with $v = (0, 1)$ implies that

$$(C_{1,1} \cup C_{1,2}) + t_{(0,1) \to (2,1)} \cong C_{3,1} \cup C_{3,2}.$$

Since it holds that $B_{0,1} + t_{(0,1) \to (2,1)} = B_{2,1}$, we get that

$$(C_{0,1} \cup C_{1,1} \cup C_{0,2} \cup C_{1,2}) + t_{(0,1) \to (2,1)} \cong C_{2,1} \cup C_{3,1} \cup C_{2,2} \cup C_{3,2}.$$

Since all the sets $C_{i,j}$ are disjoint, it follows that

$$(C_{0,1} \cup C_{0,2}) + t_{(0,1) \to (2,1)} \cong C_{2,1} \cup C_{2,2}.$$

Since $X + t_{0,1}$ is a subset of $C_{0,2}$, we obtain that

$$X + t_{2,1} = X + t_{0,1} + t_{(0,1) \to (2,1)} \subseteq C_{0,2} + t_{(0,1) \to (2,1)} \subseteq C_{2,1} \cup C_{2,2} \subseteq B_{1,1}. \quad (20)$$

On the other hand, it holds that $X \subseteq C_{1,1} \subseteq B_{1,1}$. Hence, the intersection of the sets $B_{1,1}$ and $B_{1,1} + t_{2,1}$ contains the set $X + t_{2,1}$, in particular, it has positive measure. However, this is impossible because $B_{1,1} = B_{0,0} + t_{1,1}$ and $B_{1,1} + t_{2,1} = B_{2,1} + t_{1,1}$ and the sets $B_{0,0}$ and $B_{2,1}$ are disjoint. We conclude that the intersection of sets $C_{1,1} + t_{0,1}$ and $C_{0,2}$ is null. Using (17), we obtain that

$$C_{0,1} + t_{0,1} \cong C_{0,2} \text{ and } C_{1,1} + t_{0,1} \cong C_{1,2},$$

and a symmetric argument yields that

$$C_{1,0} + t_{1,0} \cong C_{2,0} \text{ and } C_{1,1} + t_{1,0} \cong C_{2,1}.$$

The proof is now complete. $\square$

Our next step is to deduce that the elements $t_{v \to w}$ can be replaced by integer combinations of the elements $t_{(0,0) \to (1,0)}$ and $t_{(0,0) \to (0,1)}$.

Lemma 18. *Let $A \subseteq \mathbb{T}$ be a coloring $\mathbb{T}$-base of the graph G_5 with measure $4/25$ and let $C_{i,j} \subseteq \mathbb{T}$, $(i,j) \in \mathbb{Z}_5^2$, be the sets as in Lemma 16. It holds that*

$$C_{i \bmod 5, \ j \bmod 5} \cong C_{0,0} + it_{(0,0) \to (1,0)} + jt_{(0,0) \to (0,1)}$$

for any two non-negative integers i and j.

Proof. We proceed by induction on $i + j \in \mathbb{Z}$; all calculations with subscripts are done modulo five throughout the proof. The base of the induction is the case $i + j \in \{0,1\}$, which is implied by Lemma 17. For the rest of the proof fix i and j. By symmetry, we may assume that $j > 0$. Applying Lemma 17 with $v = (0,0), (1,0), \ldots, (i,0), (i,1), \ldots, (i,j-1)$, we obtain the following.

$$C_{i,j-1} \cong C_{0,0} + t_{(0,0) \to (1,0)} + \cdots + t_{(i-1,0) \to (i,0)} + t_{(i,0) \to (i,1)} + \cdots + t_{(i,j-2) \to (i,j-1)} \tag{21}$$

$$C_{i,j} \cong C_{0,1} + t_{(0,0) \to (1,0)} + \cdots + t_{(i-1,0) \to (i,0)} + t_{(i,0) \to (i,1)} + \cdots + t_{(i,j-2) \to (i,j-1)} \tag{22}$$

Since $C_{0,1} \cong C_{0,0} + t_{(0,0) \to (0,1)}$ by Lemma 17, we conclude using (21) and (22) that

$$C_{i,j} \cong C_{i,j-1} + t_{(0,0) \to (0,1)}. \tag{23}$$

On the other hand, the induction yields

$$C_{i,j-1} \cong C_{0,0} + it_{(0,0) \to (1,0)} + (j-1)t_{(0,0) \to (0,1)},$$

which combines with (23) to imply

$$C_{i,j} \cong C_{0,0} + it_{(0,0) \to (1,0)} + jt_{(0,0) \to (0,1)}.$$

$\square$

We are now ready to prove the main theorem of this section.

Theorem 19. *The graph G_5 has no coloring $\mathbb{T}$-base with measure $4/25$.*

Proof. Suppose that there exists a coloring $\mathbb{T}$-base $A \subseteq \mathbb{T}$ with measure $4/25$, and let $C_{i,j} \subseteq \mathbb{T}$, $(i,j) \in \mathbb{Z}_5^2$, be the sets as in Lemma 16. Further, let $\tau = t_{(0,0)\to(1,0)}$ and $\tau' = t_{(0,0)\to(0,1)}$. By Lemma 18 we have $C_{0,0} \cong C_{0,0}+5\tau n$ and $C_{0,0} \cong C_{0,0}+5\tau' n$ for any integer n. If 5τ were irrational, then the measure of $C_{0,0}$ would be either zero or one. Therefore, 5τ is rational. Similarly, $5\tau'$ is rational. It follows that both τ and τ' are also rational. Let p, q, r, p', q', r' be non-negative integers such that

$$\tau = \frac{p}{5^r q} \qquad \text{and} \qquad \tau' = \frac{p'}{5^{r'} q'},$$

p and $5^r q$ are coprime, p' and $5^{r'} q'$ are coprime, and neither q nor q' is divisible by five. By symmetry, we may assume that $r \le r'$. Let k be an integer such that $5^{r'-r}p$ and kp' are congruent modulo $5^{r'}$; note that such k exists since p' and $5^{r'}$ are coprime. Next observe that

$$q\tau = \frac{5^{r'-r}p}{5^{r'}} \qquad \text{and} \qquad kq'\tau' = \frac{kp'}{5^{r'}} = \frac{5^{r'-r}p}{5^{r'}} \quad \text{mod } 1. \tag{24}$$

By Lemma 18, we obtain that

$$C_{0,0} + q\tau \cong C_{0,0} + qt_{(0,0)\to(1,0)} \cong C_{q \bmod 5,\ 0}$$
$$C_{0,0} + kq'\tau' \cong C_{0,0} + kq't_{(0,0)\to(0,1)} \cong C_{0,\ kq' \bmod 5}.$$

Since $q \bmod 5 \neq 0$ and the sets $C_{i,j}$ are disjoint sets of measure $1/25$ by Lemma 16, we obtain that the intersection of the sets $C_{0,0}+q\tau \cong C_{q \bmod 5,\ 0}$ and $C_{0,0}+kq'\tau' \cong C_{0,\ kq' \bmod 5}$ is null. However, $q\tau$ and $kq'\tau'$ is the same element of $\mathbb{T}$ by (24), i.e., $C_{0,0} + q\tau \cong C_{0,0} + kq'\tau'$. This contradicts our assumption on the existence of a coloring $\mathbb{T}$-base with measure $4/25$. $\qquad\square$

7   Conclusion

We finish with giving three open problems that we find particularly interesting and briefly mentioning a relation of the gyrochromatic number to another graph parameter, the ultimate independence ratio of a graph. The *independence ratio* $i(G)$ of a graph G is the ratio $\alpha(G)/|V(G)|$; the *ultimate independence ratio* $I(G)$, which was introduced in [7], is the limit of the independence ratios of Cartesian powers of G:

$$I(G) = \lim_{k\to\infty} \frac{\alpha(G^k)}{|V(G^k)|},$$

where G^k is the Cartesian product of k copies of G. The inverse of this quantity is the *ultimate fractional chromatic number* $\chi_F(G)$ of a graph G and the following holds [5, 13]:

$$\chi_F(G) = \frac{1}{I(G)} = \lim_{k\to\infty} \chi_f(G^k).$$

Zhu [13, p. 236] related the ultimate fractional chromatic number to coloring bases of abelian groups (though he used different terminology), via the following inequality:

$$I(G) \geq \sup \left\{ \frac{\alpha(H)}{|V(H)|} \quad G \to H, H \text{ is a Cayley graph on a finite abelian group} \right\}.$$

Using Corollary 11, we conclude that $\chi_F(G) \leq \chi_g(G)$. Hence, we obtain that the following holds for every graph G:

$$\chi_f(G) \leq \chi_F(G) \leq \chi_g(G) \leq \chi_c(G) \leq \chi(G).$$

It seems plausible that $\chi_F(G)$ and $\chi_g(G)$ differ for some graphs G, and it would be interesting to give examples of such graphs.

Problem 1. *Construct a (connected) graph G such that $\chi_F(G) < \chi_g(G)$.*

We finish with two problems on the gyrochromatic number and its relation to fractional and circular chromatic numbers, which we believe to be of particular interest.

Problem 2. *Does there exist a function $f : \mathbb{R} \to \mathbb{R}$ such that $\chi_g(G) \leq f(\chi_f(G))$ for every graph G?*

Problem 3. *Does there exist a finite graph G such that the gyrochromatic number of G is not rational?*

Observe that a function f in Problem 2 exists for all graphs G if and only if it exists for Kneser graphs, i.e., such a function f exists if and only if the gyrochromatic number of Kneser graph $K(m, n)$ is at most $f(m/n)$. Also note that Theorem 12 implies that the circular chromatic number of a graph cannot be upper bounded by a function of its gyrochromatic number, i.e., a function f as in Problem 2 does not exist for the gyrochromatic and the circular chromatic number.

Acknowledgements

References

[1] A. Avila and P. Candela. Towers for commuting endomorphisms, and combinatorial applications. *Annales de l'Institut Fourier*, 66:1529–1544, 2016.

[2] G. J. Chang, D. D.-F. Liu, and X. Zhu. A short proof for Chen's alternative Kneser coloring lemma. *Journal of Combinatorial Theory, Series A*, 120:159–163, 2013.

[3] P.-A. Chen. A new coloring theorem of Kneser graphs. *Journal of Combinatorial Theory, Series A*, 118:1062–1071, 2011.

[4] G. Gao and X. Zhu. Star-extremal graphs and the lexicographic product. *Discrete Mathematics*, 152:147–156, 1996.

[5] G. Hahn, P. Hell, and S. Poljak. On the ultimate independence ratio of a graph. *European Journal of Combinatorics*, 16:253–261, 1995.

[6] G. Hahn and C. Tardif. Graph homomorphisms: structure and symmetry. *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, 497:107–166, 1997.

[7] P. Hell, X. Yu, and H. Zhou. Independence ratios of graphs powers. *Discrete Mathematics*, 127:213–220, 1994.

[8] D. D.-F. Liu and X. Zhu. A combinatorial proof for the circular chromatic number of Kneser graphs. *Journal of Combinatorial Optimization*, 32:765–774, 2016.

[9] L. Lovász. Kneser's conjecture, chromatic number, and homotopy. *Journal of Combinatorial Theory, Series A*, 25:319–324, 1978.

[10] G. Sabidussi. Vertex-transitive graphs. *Monatsh. Math.*, 68:426–438, 1964.

[11] E. R. Scheinerman and D. H. Ullman. *Fractional graph theory: a rational approach to the theory of graphs.* Courier Corporation, 2011.

[12] A. Vince. Star chromatic number. *Journal of Graph Theory*, 12:551–559, 1988.

[13] X. Zhu. On the bounds for the ultimate independence ratio of a graph. *Discrete Math.*, 156:229–236, 1996.

[14] X. Zhu. Circular chromatic number: a survey. *Discrete Mathematics*, 229:371–410, 2001.

P. Candela, Universidad Autónoma de Madrid, and ICMAT, Madrid 28049, Spain
    E-mail: pablo.candela@uam.es

C. Catalá, Universidad Autónoma de Madrid, Madrid 28049, Spain
    E-mail: ca.cataladlt@gmail.com

R. Hancock, Faculty of Informatics, Masaryk University, Botanická 68A, 602 00 Brno, Czech Republic
    E-mail: hancock@fi.muni.cz

A. Kabela, Faculty of Informatics, Masaryk University, Botanická 68A, 602 00 Brno, Czech Republic

E-mail: kabela@fi.muni.cz

D. Král', Faculty of Informatics, Masaryk University, Botanická 68A, 602 00 Brno, Czech Republic

Mathematics Institute, DIMAP and Department of Computer Science, University of Warwick, Coventry CV4 7AL, UK

E-mail: dkral@fi.muni.cz

A. Lamaison, Mathematics Department, Freie Universität, Arnimallee 3, 14195 Berlin

E-mail: lamaison@zedat.fu-berlin.de

L. Vena, Department of Applied Mathematics (KAM), Charles University, Malostranské náměstí 25, Praha 1, Czech Republic

E-mail: lluis.vena@gmail.com