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Economic cross-efficiency☆

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ABSTRACT

This paper introduces a series of new concepts under the name of Economic Cross-Efficiency, which is rendered operational through Data Envelopment Analysis (DEA) techniques. To achieve this goal, from a theoretical perspective, we connect two key topics in the efficiency literature that have been unrelated until now: economic efficiency and cross-efficiency. In particular, it is shown that, under input (output) homotheticity, the traditional bilateral notion of input (output) cross-efficiency for unit l, when the weights of an alternative counterpart k are used in the evaluation, coincides with the well-known Farrell notion of cost (revenue) efficiency for evaluated unit l when the weights of k are used as market prices. This motivates the introduction of the concept of Farrell Cross-Efficiency (FCE) based upon Farrell's notion of cost (revenue) efficiency. One advantage of the FCE is that it is well defined under Variable Returns to Scale (VRS), yielding scores between zero and one in a natural way, and thereby improving upon its standard cross-efficiency counterpart. To complete the analysis we extend the FCE to the notion of Nerlovian cross-inefficiency (NCI), based on the dual relationship between profit inefficiency and the directional distance function. Finally, we illustrate the new models with a recently compiled dataset of European warehouses.

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1. Introduction

Data Envelopment Analysis (DEA) is a data-driven approach for estimating a piece-wise linear frontier enveloping from above a cloud of points in a space with dimensions associated with variables categorized as inputs and outputs. DEA is classified as a non-parametric and multidimensional technique, which is based on a few postulates (mainly convexity, free disposability and minimum extrapolation), and is usually used for assessing relative efficiencies of a homogeneous set of Decision Making Units (DMUs). Due to its flexibility and other advantages, in recent times, DEA has become one of the most used methodologies by researchers, practitioners and scholars in Operations Research, Economics and Engineering to estimate best practice frontiers in many different contexts. In particular, this technique allows determining an efficiency score for each evaluated unit, calculated as the distance from each DMU to the fitted frontier (see, for example, [11]).

Regarding the determination of the distance to the frontier, it is worth mentioning that there exist in the DEA literature many different ways of implementing this idea of proximity; being the seminal and most used that associated with the radial models of Charnes et al. [9] and Banker et al. [6]. In these models, defined as fractional linear programming formulations in its basic ratio-form, the technique assigns a set of most favorable input and output weights that maximize the ratio of a weighted sum of outputs to a weighted sum of inputs. In this manner, the assessed DMU is evaluated in the best possible way and DEA provides a self-evaluation of the DMUs by using input and output weights that are unit-specific. Unfortunately, this flexibility that represents one of the distinctive landmarks of DEA makes it difficult to derive a suitable ordering of the units based on their efficiency score, as the best performing DMUs rank at the top with an equal value of one.

However, it is very common in real life that practitioners need to rank the set of assessed units with respect to their performance. One example is the famous Academic Ranking of World Universities (ARWU)—better known as the Shanghai Ranking, where over 1,200 universities are ranked according to six objective indicators every year. Other recent examples are the ranking of a list of journals using data from the Thomson Reuters Journal Citation Reports (JCR) (see [31]) or the ranking of countries participating in a sporting event as the Summer Olympic Games (see [19]). This

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need has motivated the introduction into the DEA literature of different approaches for ranking the set of DMUs [1].

One of the most popular approaches for ranking units in DEA is that known as Cross-Efficiency (CE) [10,33]. Cross-efficiency evaluation was originally introduced in Sexton et al. [37] and popularized by Doyle and Green [13]. While DEA provides a self-evaluation for each DMU, using unit-specific optimal input and output weights, the cross-efficiency evaluation provides a peer-appraisal of the DMUs in which each unit is also assessed using the optimal DEA weights of the remaining observations. From an economic perspective based on duality theory the optimal weights obtained from DEA can be interpreted as shadow prices defining the trade-offs between inputs and outputs from a technological perspective, e.g., marginal rates of substitution between inputs (see [15,27]). These trade-offs correspond to the reference hyperplanes obtained from DEA serving as benchmark for efficiency measurement. The cross-efficiency methodology relies on these reference hyperplanes, defined by the optimal weights or shadow prices obtained for each DMU, to calculate the so-called bilateral cross-efficiency scores, which again define as the usual ratio of a weighted sum of outputs to a weighted sum of inputs. The final (multilateral) cross-efficiency scores of the different units are the average of their (bilateral) cross-efficiencies, and such scores are used to rank the DMUs.

Whereas the ranking that we are determining through cross-efficiency is related to the notion of 'technical' efficiency, i.e., we are interested in evaluating the performance of a set of observations operating in a similar technological environment by comparing their activity with respect to the boundary enveloping the data; there exists another type of efficiency, with a more general meaning. We are referring to the concept of economic or overall efficiency, which is normally associated with the performance of 'for-profit' organizations when information on market prices are considered (e.g. firms operating within an industry). In market environments the measurement of, for example, cost efficiency is key to understand the competitiveness of firms, Aparicio et al. [2]. These units are usually interested in changing the relative amounts of inputs (input mix) if this adjustment leads to real economic gains (e.g., given revenue, more profit through less cost). In particular, cost efficiency may be defined as how close the firm is to the optimal (minimum) feasible cost of producing a given amount of output. In a similar manner, we can find in the literature analogous definitions of revenue efficiency and profit efficiency.

Farrell [17] was the first author in showing how to measure cost efficiency from the estimation of a best practice frontier, as the ratio between minimum cost and actual cost of a firm given input market prices. Additionally, he introduced a way of decomposing this overall measure into technical efficiency and allocative efficiency, as a means to understand what needs to be done to enhance the performance of the assessed unit. Technical efficiency measures how close the firm is to the frontier of the technology, whereas allocative efficiency measures the additional economic loss due to a sub-optimal input mix given market prices, once the firm is at the frontier. Moreover, under the Farrell approach, when the best practice frontier is estimated by DEA, the technical efficiency component coincides with the efficiency score linked to the (input-oriented) radial model by Charnes et al. [9], in the case of assuming a constant returns to scale (CRS) technology, and by Banker et al. [6], in the case of adopting variable returns to scale (VRS). It is worth mentioning that a revenue efficiency measure *à la* Farrell can be defined in an analogous way.

Following Farrell [17], the use of market prices as input and output weights leads to a market oriented ranking based on economic performance. However, in many instances, prices are unavailable, e.g. when studying public services such as health, education, etc. Then, based on the set of optimal multipliers (i.e., shadow prices), standard DEA provides a ranking with all the caveats previously mentioned. Yet, as this paper introduces, it is possible to perform a cross-efficiency analysis in the vein of Farrell, but using the obtained shadow prices. In this analysis, rather than calculating cost efficiency under (unavailable) market prices, this is performed with respect to the shadow prices obtained for each one of the observations (hence the name of Farrell cross-efficiency that we introduce later). As for the decomposition of this 'shadow' cost efficiency, units are first projected to the production frontier through their technical efficiency score, and then their relative allocative efficiency is determined with respect to the shadow prices. Our method inaugurates an alternative family of cross-efficiency models. Although our approach uses the same information yielded by the standard cross-efficiency DEA models, namely the optimal weights, we reinterpret them as shadow prices and, inspired by Farrell [17], offer a new method for cross-efficiency measurement.

While Farrell [17] introduced the notion of economic efficiency in the cost minimizing case, the interest of extending his ideas to profit efficiency resulted in the introduction of the so-called Nerlovian efficiency measure [7]. This approach defines profit inefficiency in an additive way and decomposes it into technical inefficiency and allocative inefficiency. Technical inefficiency is determined through the directional distance function, which is a graph measure in the sense that firms adjust both input and output quantities. As in the case of Farrell, the Nerlovian efficiency measure also uses the information of market prices to determine profit efficiency of each evaluated observation, and can be decomposed into mutually exclusive technical and allocative terms. Extending the new approach described above, in this study we also introduce the concept of Nerlovian Cross-Efficiency, comparing the economic (profit) performance of DMUs with respect to one another but, once again, rather than considering market prices, using the optimal multipliers or shadow prices yielded by the directional distance function DEA model.

In spite of input and output weights determined by models in DEA being interpreted as prices–i.e., as shadow prices specifically–cross-efficiency and economic efficiency are two independent topics in the literature that have evolved in parallel, without ever making a connection. Following this thread, this paper explores the existence of a common ground, providing a link between these two research fields by introducing the concept of *Economic Cross-Efficiency*, and its application through DEA. In particular, and to connect our new approach with the standard cross-efficiency methods, we show that under the customary assumption of input (output) homotheticity, the traditional bilateral notion of input (output) cross-efficiency for unit l, when the weights of unit k are used in the evaluation, coincides with the Farrell notion of cost (revenue) efficiency for unit l when the weights of unit k are used as market prices. This implies that, under homotheticity, the multilateral traditional cross-efficiency notion matches the arithmetic mean of n Farrell's cost efficiencies, where n denotes the sample size. Additionally, we show how to decompose the standard cross-efficiency into technical efficiency and (shadow) allocative efficiency.

The above result motivates the definition in a first instance of the concept of *Farrell Economic Cross-Efficiency* (*FCE*), based upon the notion of Farrell's cost efficiency. We prove that *FCE* coincides with standard cross-efficiency (*CE*) in the context of production functions, i.e., when only an output is produced, under restrictive assumptions. To complete the analytical framework, once the Farrell approach (*FCE*) has been introduced, we extend it to the wider case of profit inefficiency, by way of the notion of *'Nerlovian' Cross-Inefficiency* (*NCI*). This allows us to deal with the general situation of simultaneous output and input adjustments through the directional distance function.

In what constitutes a key advantage of the new *FCE* method, we show that it allows the extension of the concept of cross-efficiency to technologies characterized by variable returns to scale (*VRS*), obtaining scores always between zero and one in a natural way, something that contrasts with the standard cross-efficiency framework. This point is important in the context of cross-efficiency because the traditional cross-efficiency measure under VRS presents the problem of negative values for some DMUs, unamenable to sensible interpretation. However, many empirical situations require the assumption of VRS, for example when DMUs are of very different size (bank branches, universities, restaurants, etc.). This is the reason why some authors have tried to adapt the standard cross-efficiency to accommodate the need of using a VRS DEA model in order to avoid meaningless values (e.g., [26,39], and more recently, [23]). In the empirical section, resorting to example data from Wu et al., [39], we compare our new method to some of these proposals aimed at solving the problem of negative cross-efficiencies under VRS.

The paper is organized as follows. Section 2 is devoted to introduce the relationship between cross-efficiency and economic efficiency under homotheticity and to define the notion of Farrell (cost) cross efficiency under any returns to scale. In Section 3, we extend the Farrell cross-efficiency to the context of graph measures by introducing the Nerlovian economic (profit) cross-inefficiency measure. In Section 4 we compare our economic cross-efficiency method with other proposals in the literature solving the problem of negative scores under VRS, and illustrate the general feasibility of our economic cross-efficiency models for large datasets by applying the new approach to recently compiled data on European warehouses. Section 5 concludes.

2. The Farrell economic (cost) cross-efficiency

Let there be m inputs, the (non-negative) quantities of which are measured by a vector $X \equiv (x_1, \ldots, x_m)$, and s outputs, the (non-negative) quantities of which are measured by a vector $Y \equiv (y_1, \ldots, y_s)$. Given n observed observations or DMUs, we have the set of data denoted as $\{(X_k, Y_k), k = 1, \ldots, n\}$. The technology or production possibility set is defined, in general, as $T = \{(X, Y) \in R_+^{m+s} : X \text{ can produce Y}\}$.

Using Data Envelopment Analysis, T is characterized as $T_c = \{(X, Y) \in R_+^{m+s} : \sum_{j=1}^n \lambda_j x_{ij} \le x_i, \forall i, \sum_{j=1}^n \lambda_j y_{rj} \ge y_r, \forall r, \lambda_j \ge 0, \forall j\}$ under constants returns to scale (CRS) and as $T_v = \{(X, Y) \in R_+^{m+s} : \sum_{j=1}^n \lambda_j x_{ij} \le x_i, \forall i, \sum_{j=1}^n \lambda_j y_{rj} \ge y_r, \forall r, \sum_{j=1}^n \lambda_j = 1, \lambda_j \ge 0, \forall j\}$ under variable returns to scale (VRS) [6].

In DEA, for each DMU $k = 1, \ldots, n$ the radial input technical efficiency assuming CRS is calculated through the following linear fractional programing problem [9]:

$$
\begin{aligned}
ITE_c(X_k, Y_k) = \underset{U,V}{Max} \quad & \frac{\sum_{r=1}^{s} u_r y_{rk}}{\sum_{i=1}^{m} v_i x_{ik}} \\
s.t. \quad & \\
& \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}} \le 1, \quad j = 1, \ldots, n \quad (1.1) \\
& u_r \ge 0, \quad r = 1, \ldots, s \quad (1.2) \\
& v_i \ge 0, \quad i = 1, \ldots, m \quad (1.3)
\end{aligned}
\tag{1}
$$

$ITE_c(X_k, Y_k)$ always takes values between zero and one and its inverse coincides with the well-known Shephard input distance function in Economics [38]. Additionally, for computational purposes, model (1) can be easily linearized as:

$$
\begin{aligned}
ITE_c(X_k, Y_k) = \underset{U,V}{Max} \quad & \sum_{r=1}^{s} u_r y_{rk} \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{ik} = 1, & & (2.1) \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, & j = 1, \ldots, n \quad & (2.2) \\
& u_r \ge 0, & r = 1, \ldots, s \quad & (2.3) \\
& v_i \ge 0, & i = 1, \ldots, m \quad & (2.4)
\end{aligned}
\tag{2}
$$

Any optimal solution of model (2) is an optimal solution of model (1). Moreover, the optimal value of model (2) coincides with the optimal value of model (1). It is worth mentioning that the optimal value of the dual linear program corresponding to model (2), i.e. the well-known envelopment form in Data Envelopment Analysis, may be interpreted as a proportional reduction of inputs as well as a cost index measuring the relative contraction of cost at the most preferred virtual or shadow prices, which are the optimal virtual multipliers in (2).

As we aforementioned, one drawback of radial input technical efficiencies is that they exhibit a relevant shortcoming for ranking observations. To judge this, let (V_k^*, U_k^*) be one of the possible optimal solutions of problem (2) and, therefore, of model (1). In this way, the comparison of the scores ITE_c associated with two DMUs k and l involves not only their input and output quantities (as in standard bilateral productivity comparisons), but also two different profiles of shadow prices: (V_k^*, U_k^*) and (V_l^*, U_l^*).

$$
ITE_c(X_k, Y_k) \ge ITE_c(X_l, Y_l) \Leftrightarrow \frac{\sum_{r=1}^{s} u_{rk}^* y_{rk}}{\sum_{i=1}^{m} v_{ik}^* x_{ik}} \ge \frac{\sum_{r=1}^{s} u_{rl}^* y_{rl}}{\sum_{i=1}^{m} v_{il}^* x_{il}}. \tag{3}
$$

Since usually $(V_k^*, U_k^*) \ne (V_l^*, U_l^*)$, it is discouraged to compare the performance of the two units by direct comparison of their scores. Moreover, all technically efficient units receive a unitary score: $ITE_c(X_k, Y_k) = 1$, and therefore it is not possible to discriminate among them. Instead, a cross-evaluation strategy is suggested in the literature ([37], and [13]). In particular, the (bilateral) cross input technical efficiency of unit l with respect to unit k is defined by

$$
CITE_c(X_l, Y_l | k) = \frac{\sum_{r=1}^{s} u_{rk}^* y_{rl}}{\sum_{i=1}^{m} v_{ik}^* x_{il}}. \tag{4}
$$

$CITE_c(X_l, Y_l | k)$ takes values between zero and one and satisfies $CITE_c(X_l, Y_l | l) = ITE_c(X_l, Y_l)$ [P1].

Given the observed n units in the data sample, the literature on cross-efficiency suggests the aggregation of the bilateral cross input technical efficiencies of unit l with respect to all units k, $k = 1, \ldots, n$, through the arithmetic mean to obtain the multilateral notion of cross input technical efficiency of unit l:

$$
CITE_c(X_l, Y_l) = \frac{1}{n} \sum_{k=1}^{n} CITE_c(X_l, Y_l | k) = \frac{1}{n} \sum_{k=1}^{n} \frac{\sum_{r=1}^{s} u_{rk}^* y_{rl}}{\sum_{i=1}^{m} v_{ik}^* x_{il}}. \tag{5}
$$

This measure satisfies several properties:

[P2] The greater $CITE_c(X_l, Y_l)$, the better (meaning of efficiency);

[P3] $0 \le CITE_c(X_l, Y_l) \le 1$;

[P4] If $(v_k^*, u_k^*) = (v_l^*, u_l^*)$, $\forall k = 1, \ldots, n$, then $CITE_c(X_l, Y_l) = ITE_c(X_l, Y_l)$;

[P5] $CITE_c(X_l, Y_l)$ is units invariant.

Before bridging the gap between the above cross-efficiency methods and the economic efficiency literature, we need to briefly recall the latter through the classical Farrell approach [17]. We start considering the Farrell radial paradigm for measuring and decomposing cost efficiency. Therefore, for the sake of brevity, we state our discussion in the input space, defining the input requirement set $L(Y)$ as the set of non-negative inputs $X \in R^m_+$ that can produce non-negative output $Y \in R^s_+$, formally $L(Y) = \{X \in R^m_+ : (X, Y) \in T\}$, and the isoquant of $L(Y)$:$Isoq\, L(Y) = \{X \in L(Y) : \varepsilon < 1 \Rightarrow \varepsilon x \notin L(Y)\}$. Let us also denote by $C_L(Y, W)$ the minimum cost of producing the output level Y given the input market price vector $W \in R^m_+$: $C_L(Y, W) = \min\{\sum_{i=1}^m w_i x_i : X \in L(Y)\}$.

The standard (multiplicative) Farrell approach views cost efficiency as originating from technical efficiency and allocative efficiency. Specifically, Farrell quantified, and therefore defined, each of these terms as follows:

$$CE_L(X, Y) = \underbrace{\frac{C_L(Y, W)}{\sum_{i=1}^{m} w_i x_i}}_{\text{Cost Efficiency}} = \underbrace{\frac{1}{D_L(X, Y)}}_{\text{Technical Efficiency}} \cdot \underbrace{AE_L^F(X, Y; W)}_{\text{Allocative Efficiency}}, \qquad (6)$$

where $D_L(X, Y) = \sup\{\delta > 0 : X/\delta \in L(Y)\}$ is the Shephard input distance function [38], and allocative efficiency is defined residually as the ratio between cost efficiency and technical efficiency or, explicitly, as $AE_L^F(X, Y; W) = \frac{C_L(Y, W)}{\sum_{i=1}^m w_i \left(\frac{x_i}{D_L(X, Y)}\right)}$.

We use the subscript L to denote that we do not assume a specific type of returns to scale when characterizing $L(Y)$. Nevertheless, we will utilize $C_c(Y, W)$ and $D_c(X, Y)$ for CRS and $C_v(Y, W)$ and $D_v(X, Y)$ for VRS when needed. Additionally, as shown in (6), it is well-known in Data Envelopment Analysis that the inverse of $D_c(X, Y)$ coincides with $ITE_c(X_k, Y_k)$–program (1): $ITE_c(X_k, Y_k) = D_c(X, Y)^{-1}$. Now, if common market prices existed for all firms within an industry, then the natural way of comparing the performance of each one would be using the left-hand side in (6). We then could assess the obtained values for each firm using the same reference weights (prices) for all the observations, creating a market based ranking.

We are now ready to show that, under input homotheticity, the traditional bilateral notion of the cross input technical efficiency of unit l with respect to unit k, $CITE_c(X_l, Y_l|k)$, coincides with the Farrell notion of cost efficiency for unit l, i.e., the left-hand side in (6), when the input weights of unit k, $V_k^* = (v_{1k}^*, \dots, v_{mk}^*)$, i.e., its shadow prices, take the place of the market prices. For this purpose, we first recall the definition of input homotheticity [20].

Definition 1. The technology T is input homothetic if and only if $L(Y) = H(Y) \cdot L(1_s)$, where $H(Y) : R^s \to R_{++}$ and $1_s = (1, \dots, 1) \in R^s$.

Input homotheticity is customarily assumed in empirical applications measuring overall economic efficiency because it ensures that radial reductions of inputs can be rightly interpreted as technical improvements resulting in cost savings. This is because, whatever the allocative efficiency magnitude resulting from the first order conditions for cost minimization–i.e., summarized in the (in)equality of the marginal rates of substitution to the input price ratios, it does not change along the radial contraction path represented by the input distance function. This result stems from one remarkable technological property normally taken for granted in the literature by customarily assuming homotheticity, that the marginal rates of substitution among inputs are independent of the

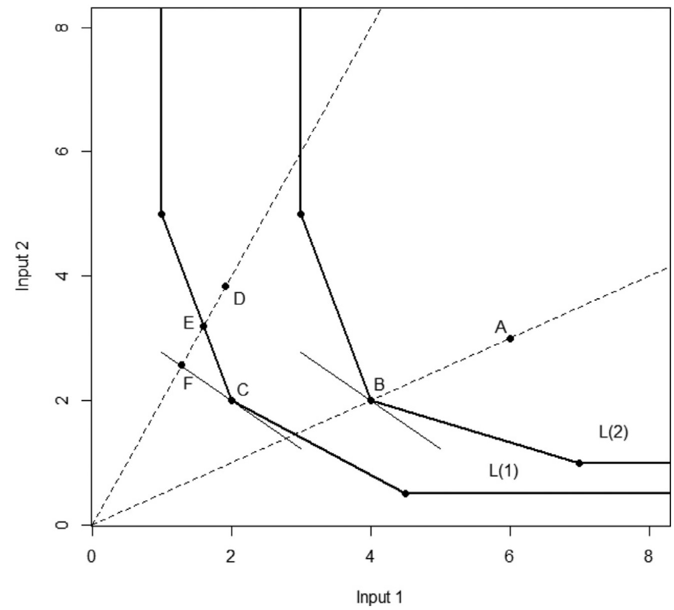

Fig. 1. Illustrating expression (16) and its decomposition.

output level, and therefore the radial contractions of input quantities leave allocative efficiency unchanged–see Proposition 2 in Aparicio and Zofío [[4]:137]. The geometric idea behind the notion of input homotheticity is that the input requirement sets for different output vectors along factor beams are "parallel" blown-ups (in contrast to Fig. 1 where the map of isoquants illustrates a non-homothetic technology).[1]

Indeed, the satisfaction of this property has relevant implications for this study in terms of the input requirement set and the separability of the cost function, which can be rewritten as follows (see [15]):

$$L(Y) = H(Y)L(1_s), \qquad (7)$$

$$C_L(Y, W) = H(Y)C_L(1_s, W). \qquad (8)$$

Färe and Mitchell [14] refine this notion of input-homotheticity for the case of multiple outputs, beyond that corresponding to the usual scalar single-valued production function. In the case of multiple outputs, these authors differentiate between input homotheticity and ray-homotheticity (understood as 'fixed output mix/input mix independent ray-homotheticity'). On the one hand, Definition 1 above simply recalls the standard definition of input-homotheticity for multiple outputs, implying both ray-homotheticity (linear expansion paths) and the separability of the cost function, which is the property required to prove the equivalence between the standard cross-efficiency model that assumes CRS and Farrell's cost efficiency model. On the other hand, if CRS are assumed, it implies that the technology is homogenous. Therefore, Theorem 1 below is presented under both input homotheticity and homogeneity.

In order to prove the result that relates the standard DEA cross-efficiency under CRS to Farrell's cost efficiency, we need to establish some previous results. Based on (7) and (8) we start showing the conventional linear programming model that is used in DEA to determine the minimum cost, given the output level Y_l and

[1] Given the advantages of assuming homotheticity among the most common technological properties, it comes as no surprise that it is routinely assumed by researchers, Chambers and Mitchell [8].

shadow prices V_k^* :[2]

$$C_c\left(Y_l, V_k^*\right) = \underset{\substack{\lambda_1,\dots,\lambda_n \\ x_1,\dots,x_m}}{Min} \quad \sum_{i=1}^{m} v_{ik}^* x_i$$
$$s.t.$$
$$-\sum_{j=1}^{n} \lambda_j x_{ij} + x_i \geq 0, \quad i = 1, \dots, m \quad (9.1)$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq y_{rl}, \quad r = 1, \dots, s \quad (9.2)$$
$$\lambda_j \geq 0, \quad j = 1, \dots, n \quad (9.3)$$
$$x_i \geq 0, \quad i = 1, \dots, m \quad (9.4)$$
$$(9)$$

Now, under input homotheticity, expression (8) holds, and the optimal cost may be also determined through model (10) or by its dual, model (11):

$$C_c\left(Y_l, V_k^*\right) = H(Y_l) \underset{\substack{\lambda_1,\dots,\lambda_n \\ x_1,\dots,x_m}}{Min} \quad \sum_{i=1}^{m} v_{ik}^* x_i$$
$$s.t.$$
$$-\sum_{j=1}^{n} \lambda_j x_{ij} + x_i \geq 0, \quad i = 1, \dots, m \quad (10.1)$$
$$\sum_{j=1}^{n} \lambda_j y_{rj} \geq 1, \quad r = 1, \dots, s \quad (10.2)$$
$$\lambda_j \geq 0, \quad j = 1, \dots, n \quad (10.3)$$
$$x_i \geq 0, \quad i = 1, \dots, m \quad (10.4)$$
$$(10)$$

$$C_c\left(Y_l, V_k^*\right) = H(Y_l) \underset{E,F}{Max} \quad \sum_{r=1}^{s} e_r$$
$$s.t.$$
$$\sum_{r=1}^{s} e_r y_{rj} - \sum_{i=1}^{m} f_i x_{ij} \leq 0, \quad j = 1, \dots, n \quad (11.1)$$
$$f_i \leq v_{ik}^*, \quad i = 1, \dots, m \quad (11.2)$$
$$E \geq 0_s, F \geq 0_m \quad (11.3)$$
$$(11)$$

Lemma 1. *Let (F^*, E^*) be an optimal solution of (11). Then, (V_k^*, E^*) is also an optimal solution of (11).*

Proof. See appendix A.1.

Corollary 1. There always exists an optimal solution of model (11), (F^*, E^*), with $F^* = V_k^*$.

Proof. This result is a direct consequence of Lemma 1. □

Corollary 1, and given that $H(Y_l)$ does not depend on the decision variables E and F, implies that $C_c(Y_l, V_k^*)$ can be computed as:

$$C_c\left(Y_l, V_k^*\right) = \underset{E}{Max} \quad H(Y_l) \sum_{r=1}^{s} e_r$$
$$s.t.$$
$$\sum_{r=1}^{s} e_r y_{rj} - \sum_{i=1}^{m} v_{ik}^* x_{ij} \leq 0, \quad j = 1, \dots, n \quad (12.1)$$
$$e_r \geq 0, \quad r = 1, \dots, s \quad (12.2)$$
$$(12)$$

Now, we are ready to prove a key result in this paper: if (V_k^*, U_k^*) is an optimal solution of model (2) then, under input-homotheticity, we have that the traditional (bilateral) cross input technical efficiency of unit l with respect to unit k coincides with Farrell notion of cost efficiency for unit l when V_k^* is considered as input prices, i.e., $CITE_c(X_l, Y_l|k) = \frac{\sum_{r=1}^{s} u_{rk}^* y_{rl}}{\sum_{i=1}^{m} v_{ik}^* x_{il}} = \frac{C_c(Y_l, V_k^*)}{\sum_{i=1}^{m} v_{ik}^* x_{il}}$.

Theorem 1. *Let (V_k^*, U_k^*) be an optimal solution of model (2). If T_c is input homothetic, then $CITE_c(X_l, Y_l|k) = \frac{C_c(Y_l, V_k^*)}{\sum_{i=1}^{m} v_{ik}^* x_{il}}$.*

Proof. See Appendix A.2.

Theorem 1 implies that, under input-homotheticity, the standard notion of multilateral cross input technical efficiency of unit l coincides with the arithmetic mean of n Farrell's cost efficiencies, i.e.,

$$CITE_c(X_l, Y_l) = \frac{1}{n} \sum_{k=1}^{n} CITE_c(X_l, Y_l|k) = \frac{1}{n} \sum_{k=1}^{n} \frac{C_c\left(Y_l, V_k^*\right)}{\sum_{i=1}^{m} v_{ik}^* x_{il}}. \quad (13)$$

In this way, cross-efficiency can be reinterpreted in terms of Farrell's overall economic efficiency. This also implies that cross-efficiency can be easily decomposed into two components by applying (6):

$$CITE_c(X_l, Y_l) = \frac{1}{n} \sum_{k=1}^{n} \frac{C_c\left(Y_l, V_k^*\right)}{\sum_{i=1}^{m} v_{ik}^* x_{il}} = \frac{1}{n} \sum_{k=1}^{n} \left[\frac{1}{D_c(X_l, Y_l)} \cdot AE_c^F\left(X_l, Y_l; V_k^*\right) \right]$$
$$= \frac{1}{D_c(X_l, Y_l)} \cdot \frac{1}{n} \sum_{k=1}^{n} AE_c^F\left(X_l, Y_l; V_k^*\right) = ITE_c(X_l, Y_l) \cdot \frac{1}{n} \sum_{k=1}^{n} AE_c^F\left(X_l, Y_l; V_k^*\right). \quad (14)$$

Hence, cross-efficiency of unit l can be seen as the technical efficiency of unit l times a 'correction' factor, associated with the arithmetic mean of n allocative efficiencies of unit l, each one calculated from the input shadow prices of unit k, $k = 1, \dots, n$.

Theorem 1 has also some interesting by-products. For example, in a DEA context where only an output is produced, i.e., when a production function is estimated, it can be proved that the standard notion of multilateral cross input technical efficiency always coincides with Farrell's notion of cost efficiency. This result, summarized in the next corollary, is verified because a single output DEA technology under CRS is always input homothetic.

Corollary 2. Let $s = 1$. Then,

$$CITE_c(X_l, y_l) = \frac{1}{n} \sum_{k=1}^{n} \frac{C_c\left(y_l, V_k^*\right)}{\sum_{i=1}^{m} v_{ik}^* x_{il}}. \quad (15)$$

Proof. Aparicio et al. [2] proved in their Proposition 3 that if $s = 1$ and constant returns to scale are assumed, as happens in the computation of traditional cross-efficiency, then input-homotheticity is satisfied. Finally, by Theorem 1, we have (15). □

The above discussion, which relates standard cross (input oriented) technical efficiency to a traditional measurement of economic (cost oriented) efficiency, serves as inspiration for defining next our new notion of cross-efficiency in DEA based on Farrell's cost efficiency, regardless of assuming or not input homotheticity. In this way, for a given set of shadow prices obtained from solving program (2), we define the bilateral Farrell cross-efficiency of unit l with respect to unit k as[3]

$$FCE_L(X_l, Y_l|k) = \frac{C_L\left(Y_l, V_k^*\right)}{\sum_{i=1}^{m} v_{ik}^* x_{il}}, \quad (16)$$

[2] In the model, the decision variables are $\lambda_1, \dots, \lambda_n, x_1, \dots, x_m$, while $v_{1k}^*, \dots, v_{mk}^*, x_{ij}$, y_{rj}, $\forall i = 1, \dots, m$, $\forall r = 1, \dots, s$, $\forall j = 1, \dots, n$ are data of the problem.

[3] Note that our method based on shadow prices can be adapted to other sets of prices such as market prices when they differ across units, or even imputed prices subjectively set by the researches, e.g. based on experts' judgements.

where $L\in\{c,v\}$ denote constant and variable returns to scale. Again, it is worth remarking that the expression in (16) for evaluating unit l is inspired by the formulation of Farrell's cost efficiency, i.e. the left-hand side of (6), substituting market prices by shadow prices, i.e., the optimal weights associated with the solution of the multipliers form of the input-oriented radial model (2) when DMU k is assessed.

As in (6), $FCE_L(X_l, Y_l|k) = \frac{1}{D_L(X_l,Y_l)} \cdot AE_L^F(X_l, Y_l; V_k^*)$. Therefore, Farrell cross-efficiency of unit l with respect to unit k corrects the usual technical efficiency, the inverse of Shephard distance function, through a term with meaning of allocative efficiency.

Fig. 1 illustrates expression (16) and its decomposition. Let us assume that unit l and unit k are represented by points D and A, respectively. Additionally, let us suppose that point D belongs to $L(1)$, while A belongs to $L(2)$. Then, first of all we need to solve the input-oriented radial model for point A in order to obtain its corresponding multipliers or shadow prices. In this case, the projection point on the isoquant of $L(2)$ corresponds to point B. The radial model also yields the marginal rate of input substitution defined as the ratio of the shadow prices: $-(v_{2A}^*/v_{1A}^*)$. Using the same rate of substitution, point C on the isoquant of $L(1)$ is determined. This is the optimal production plan incurring in the minimum cost on $L(1)$ according to $-(v_{2A}^*/v_{1A}^*)$, i.e. the cost minimizing benchmark for point D. In this way, (16) corresponds to the ratio of the cost of C to the cost of D. In Fig. 1, this ratio is 0F/0D. The score provided by (16) for unit D regarding unit A coincides with the traditional radial input technical efficiency, 0E/0D, whose calculation does not involve the (shadow) prices of unit A, modified by a correction term, which is 0F/0E, i.e., the corresponding (shadow) allocative efficiency.

One more time, given we have observed n units in the data sample, the traditional literature on cross-efficiency suggests to aggregate bilateral cross-efficiencies through the arithmetic mean to obtain the multilateral notion of cross efficiency. In this case, the aggregate Farrell cross-efficiency defines as:

$$FCE_L(X_l, Y_l) = \frac{1}{n}\sum_{k=1}^{n} FCE_L(X_l, Y_l|k) = \frac{1}{n}\sum_{k=1}^{n} \frac{C_L\left(Y_l, V_k^*\right)}{\sum_{i=1}^{m} v_{ik}^* x_{il}}. \tag{17}$$

Additionally, as in expression (14) under input-homotheticity, the general $FCE_L(X_l, Y_l)$ can be always decomposed (under any returns to scale) into (radial) technical efficiency and a correction factor defined as the arithmetic mean of n shadow allocative efficiency terms. I.e.,

$$\begin{aligned} FCE_L(X_l, Y_l) &= \frac{1}{n}\sum_{k=1}^{n} FCE_L(X_l, Y_l|k) = \frac{1}{n}\sum_{k=1}^{n} \frac{C_L\left(Y_l, V_k^*\right)}{\sum_{i=1}^{m} v_{ik}^* x_{il}} \\ &= \frac{1}{n}\sum_{k=1}^{n}\left[\frac{1}{D_L(X_l, Y_l)} \cdot AE_L^F\left(X_l, Y_l; V_k^*\right)\right] \\ &= \frac{1}{D_L(X_l, Y_l)} \cdot \frac{1}{n}\sum_{k=1}^{n} AE_L^F\left(X_l, Y_l; V_k^*\right) \\ &= ITE_L(X_l, Y_l) \cdot \frac{1}{n}\sum_{k=1}^{n} AE_L^F\left(X_l, Y_l; V_k^*\right), \end{aligned} \tag{18}$$

with $ITE_L(X_l, Y_l)$ and $AE_L^F(X_l, Y_l; V_k^*)$, $L\in\{c,v\}$, denoting constant and variable returns to scale technical and allocative efficiencies, respectively. Hence, $FCE_L(X_l, Y_l)$ can be interpreted as a value (cost efficiency) index resulting from the multiplication of a quantity index, represented by the technical efficiency measure $ITE_L(X_l, Y_l)$, and a price index, defined as the arithmetic average of n allocative efficiency terms, $\frac{1}{n}\sum_{k=1}^{n} AE_L^F(X_l, Y_l; V_k^*)$. This sheds lights on the dubious interpretation of the standard cross efficiency measure $CITE_c(X_l, Y_l)$ in expression (5) as the average of n productivity indices as discussed by Førsund [18]. This author notes that while each bilateral cross efficiency $CITE_c(X_l, Y_l|k)$ can be interpreted as a productivity index, their aggregation into $CITE_c(X_l, Y_l)$ by way of the arithmetic mean does not have a meaningful interpretation because the vectors of shadow prices (V_k^*, U_k^*) constitute different weights emanating from separate optimization problems. Comparing the aggregate cross efficiency scores for two units, e.g. $CITE_c(X_l, Y_l)$ and $CITE_c(X_k, Y_k)$, implies using of different sets of weights (shadow prices) and variables (input and outputs quantities), which renders the comparison meaningless—see expression (3). However, following our approach, the comparison of two cross-efficiency measures, $FCE_L(X_l, Y_l)$ and $FCE_L(X_k, Y_k)$, implies the contraposition of two aggregate n cost efficiency indices, which can be further decomposed into the comparison of their corresponding technical efficiency scores, $ITE_L(X_l, Y_l)$ and $ITE_L(X_k, Y_k)$, and two price indices aggregating the n allocative efficiencies for the set of shadow prices: $\frac{1}{n}\sum_{k=1}^{n} AE_L^F(X_l, Y_l; V_k^*)$ and $\frac{1}{n}\sum_{l=1}^{n} AE_L^F(X_k, Y_k; V_l^*)$; i.e., the cost excess in which units l and k incur after being projected to the production frontier through their technical efficiencies, by not demanding the optimal input amounts given the shadow prices of all units.[4]

Regarding the properties that this new notion of cross-efficiency satisfies, we next list the most important:

[P1] $FCE_L(X_l, Y_l|l) = \frac{1}{D_L(X_l,Y_l)}$;

[P2] The more $FCE_L(X_l, Y_l)$, the better (meaning of efficiency);

[P3] $0 \le FCE_L(X_l, Y_l) \le 1$;

[P4] If $(V_k^*, U_k^*) = (V_l^*, U_l^*)$, $\forall k = 1, \dots, n$, then $FCE_L(X_l, Y_l) = \frac{1}{D_L(X_l,Y_l)}$;

[P5] $FCE_L(X_l, Y_l)$ is units invariant;

[P6] Let (W, P) be a vector of input and output market prices. Then, if $(V_k^*, U_k^*) = (W, P)$, $\forall k = 1, \dots, n$, we have that $FCE_L(X_l, Y_l) = CE_L(X_l, Y_l)$, $\forall l = 1, \dots, n$.

Proposition 1. The Farrell cross-efficiency meets properties P1-P6.

Proof. See Appendix A.3

Probably, the most remarkable property is P3 since it means that cross-efficiency is well-defined regardless of the assumed returns to scale. As was noted in the Introduction, this issue is critical in the context of cross-efficiency in DEA because the standard cross-efficiency measure under VRS presents the problem of negative values for some DMUs, representing a meaningless result. Almost the totality of the empirical applications involves a VRS characterization of the technology; for example when the units to be evaluated are universities with very different sizes (number of students, number of professors, budget, etc.). This is the reason why some authors have adapted the standard cross-efficiency to accommodate the need of using a VRS DEA model in order to avoid odd values [26,39].

We illustrate this drawback associated with VRS through Fig. 2 representing a single input-single output technology under VRS. Let us assume that the multipliers form of the input-oriented BBC model [6] is solved for DMU C and that the obtained solution is related to the hyperplane graphically represented in Fig. 2 (with expression $\frac{3}{4}x - y + 3 = 0$). If one determines the cross-efficiency of unit D=(4,2) relying on the standard cross-efficiency methods by using the weights of unit C, then the calculation would be associated with the projection of D onto point D", which presents a negative value in the input. This is an extreme case of the bizarre projections outside the actual production possibility set that the use of standard cross-efficiency methods entails. However, as illustrated by Olesen [30], projections outside the

[4] It may be noted that in these expressions $AE_L^F(X_l, Y_l; V_l^*) = 1$ since a unit is allocative efficient with respect to its own shadow prices.

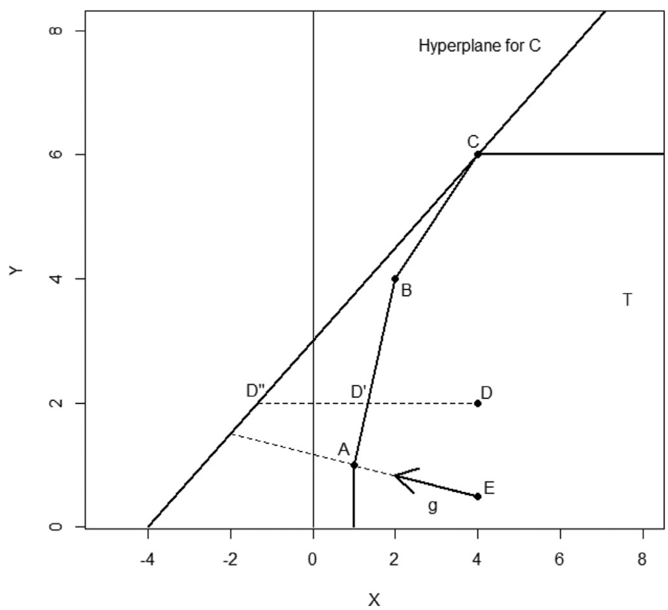

Fig. 2. Illustration of projections and negative cross-efficiencies under VRS.

actual technology are common regardless the orientation, although in this case the consequences are severe, resulting in negative cross-efficiencies. Under VRS, the formula to be applied would be $CITE_v(X_l, Y_l|k) = \frac{\sum_{r=1}^{s} u_{rk}^{*} y_{rl} - \alpha_k^{*}}{\sum_{i=1}^{m} v_{ik}^{*} x_{il}}$ (see, for example, [39]), where $(V_k^*, U_k^*, \alpha_k^*)$ is an optimal solution of the multipliers form of the input-oriented BBC model when DMU C is assessed. In the example, the standard cross-efficiency under VRS for DMU D through unit C would be $CITE_v(x_D, y_D|C) = \frac{1 \cdot 2 - 4}{0.75 \cdot 4} = -\frac{2}{3} < 0$. In our case, we do not need to adapt/modify the FCE to fit well to different types of returns to scale. It accommodates variable returns to scale in a natural way by its definition based on the Farrell cost efficiency index. If the Farrell cross-efficiency of unit D with respect to unit C is calculated under VRS in the graphical example, i.e. $FCE_v(x_D, y_D|C) = \frac{C_v(y_D, v_C^*)}{v_C^* x_D}$ following expression (16), we first need to determine the minimum cost of producing $y_D = 2$ when $v_C^* = \frac{3}{4}$, which by definition is $C_v(y_D, v_C^*) = \min\{\frac{3}{4}x : x \in L(y_D)\}$. For this instance, $L(y_D) = [\frac{4}{3}, +\infty)$. Note that the technical efficiency projection is D'=(4/3,2). Consequently, the minimum cost equals $\frac{3}{4} \cdot \frac{4}{3} = 1$, which is strictly positive. Secondly, we need to determine the cost $v_C^* x_D$, which in this simple example coincides with $\frac{3}{4} \cdot 4 = 3$. Thus, $FCE_v(x_D, y_D|C) = 1/3$, which is again strictly positive, as desired. Additionally, it is worth mentioning that in cross-efficiency is possible to get projections points with negative values even if we resort to another type of technical efficiency measures, as, for example, the directional distance function. This situation is shown also in Fig. 2 when DMU E is considered for evaluation by the direction vector g. This context will be studied in detail in Section 3.

Other important property is P6 since it means that, assuming for example perfect competition, the new approach collapses to the well-known Farrell measure of cost efficiency in (6), which should be the standard reference to be used for evaluating performance and ranking units when information on a common set of prices, in this case market prices, is available. This property is not satisfied by the traditional notion of cross input technical efficiency in the literature, as $CITE_c(X_l, Y_l) = \frac{1}{n}\sum_{k=1}^{n} \frac{\sum_{r=1}^{s} p_r y_{rl}}{\sum_{i=1}^{m} w_i x_{il}} = \frac{\sum_{r=1}^{s} p_r y_{rl}}{\sum_{i=1}^{m} w_i x_{il}}$, which is, in general, different from $CE_c(X_l, Y_l) = \frac{C_c(Y_l, W)}{\sum_{i=1}^{m} w_i x_{il}}$.

Next, we are going to prove another property, one that relates *FCE* and the traditional *CITE* under CRS, without assuming input homotheticity. The result states that $FCE_c(X_l, Y_l)$ is always an upper bound of $CITE_c(X_l, Y_l)$. To prove that, we first need to introduce some additional notions.

Given a vector of input and output prices $(W, P) \in R_+^{m+s}$, and a production possibility set T, the profit function Π is defined as $\Pi_T(W, P) = \max_{x,y}\{\sum_{r=1}^{s} p_r y_r - \sum_{i=1}^{m} w_i x_i : (X, Y) \in T\}$. In particular, let $\Pi_c(W, P)$ be the way of denoting the optimal profit given $(W, P) \in R_+^{m+s}$ and the technology T_c.

Now, we prove that if $(W, P) = (V_k^*, U_k^*)$, where (V_k^*, U_k^*) is an optimal solution of model (2), then $\Pi_c(W, P) = 0$.

Lemma 2. *Let* (V_k^*, U_k^*) *be an optimal solution of (2), then* $\Pi_c(V_k^*, U_k^*) = 0$.

Proof. See Appendix A.4.

Lemma 3. $CITE_c(X_l, Y_l|k) \le FCE_c(X_l, Y_l|k)$.

Proof. See Appendix A.5.

Now, applying Lemma 3, we get the desired result.

Proposition 2. $CITE_c(X_l, Y_l) \le FCE_c(X_l, Y_l)$.

Finally, it is worth mentioning that an analogous approach may be defined in a natural way for cross output technical efficiency from the Farrell definition of revenue efficiency.

3. The Nerlovian economic (profit) cross-inefficiency

In this section, we extend the newly proposed notion of economic cross-efficiency, presented through the concept of Farrell cross-efficiency in the previous section, to the case of graph measures that accommodate both input and output variations. In particular, we introduce the notion of *Nerlovian cross-inefficiency* based upon the dual relationship between the Nerlovian profit inefficiency and the directional distance function, as presented by Chambers et al. [7]. Luenberger [28] introduced the concept of benefit function as a representation of the amount that an individual is willing to trade, in terms of a specific reference commodity bundle g, for the opportunity to move from a consumption bundle to a utility threshold. Luenberger also defined a so-called shortage function ([28], p. 242, Definition 4.1), which basically measures the distance in the direction of a vector g of a production plan to the boundary of the production possibility set. In other words, the shortage function measures the amount by which a specific plan is short of reaching the frontier of the technology. In recent times, Chambers et al. [7] redefined the benefit function and the shortage function as efficiency measures, introducing to this end the so-called directional distance function.

We first need to recall some concepts and introduce the necessary notation. Profit inefficiency *à la Nerlove* for a DMU k is defined as optimal profit (i.e., the value of the profit function at the market prices) minus observed profit normalized by the value of a reference vector $(G_k^x, G_k^y) \in R_+^{m+s}$: $\frac{\Pi_T(W,P) - (\sum_{r=1}^{s} p_r y_{rk} - \sum_{i=1}^{m} w_i x_{ik})}{\sum_{r=1}^{s} p_r g_{rk}^y + \sum_{i=1}^{m} w_i g_{ik}^x}$. Additionally, Chamber et al. [7] showed that profit inefficiency may be decomposed into technical inefficiency and allocative inefficiency, where technical inefficiency is in particular the directional distance function $\vec{D}_T(X_k, Y_k; G_k^x, G_k^y) = \max\{\beta : (X_k - \beta G_k^x, Y_k + \beta G_k^y) \in T\}$:

$$\frac{\Pi_T(W,P) - \left(\sum_{r=1}^{s} p_r y_{rk} - \sum_{i=1}^{m} w_i x_{ik}\right)}{\sum_{r=1}^{s} p_r g_{rk}^y + \sum_{i=1}^{m} w_i g_{ik}^x} = \vec{D}_T\left(X_k, Y_k; G_k^x, G_k^y\right) + AI_T^N\left(X_k, Y_k; W, P; G_k^x, G_k^y\right). \quad (19)$$

We use the subscript T in $\Pi_T(W, P)$, $\vec{D}_T(X_k, Y_k; G_k^x, G_k^y)$ and $AI_T^N(X_k, Y_k; W, P; G_k^x, G_k^y)$ to denote that we do not assume a specific type of returns to scale. Nevertheless, we will utilize $\Pi_c(W,P)$, $\vec{D}_c(X_k,Y_k;G_k^x,G_k^y)$ and $AI_c^N(X_k,Y_k;W,P;G_k^x,G_k^y)$ for CRS and $\Pi_v(W,P)$, $\vec{D}_v(X_k,Y_k;G_k^x,G_k^y)$ and $AI_v^N(X_k,Y_k;W,P;G_k^x,G_k^y)$ for VRS.

In the case of DEA, when CRS is assumed, the directional distance function for DMU k is calculated through the following linear programming model:

$$\begin{aligned} \vec{D}_c\left(X_k,Y_k;G_k^x,G_k^y\right) = \max_{\beta,\,\lambda_1,\dots,\lambda_n} & \ \beta \\ s.t. & \\ & \sum_{j=1}^{n}\lambda_j x_{ij} \le x_{ik} - \beta g_{ik}^x, \qquad i=1,\dots,m, \\ & \sum_{j=1}^{n}\lambda_j y_{rj} \ge y_{rk} + \beta g_{rk}^y, \qquad r=1,\dots,s, \\ & \lambda \ge 0_n \end{aligned} \tag{20}$$

Additionally, when VRS is assumed, then the directional distance function is determined through (21) for evaluating unit k.

$$\begin{aligned} \vec{D}_v\left(X_k,Y_k;G_k^x,G_k^y\right) = \max_{\beta,\,\lambda_1,\dots,\lambda_n} & \ \beta \\ s.t. & \\ & \sum_{j=1}^{n}\lambda_j x_{ij} \le x_{ik} - \beta g_{ik}^x, \qquad i=1,\dots,m, \\ & \sum_{j=1}^{n}\lambda_j y_{rj} \ge y_{rk} + \beta g_{rk}^y, \qquad r=1,\dots,s, \\ & \sum_{j=1}^{n}\lambda_j = 1, \\ & \lambda \ge 0_n \end{aligned} \tag{21}$$

In the particular case of the directional distance function under VRS, we are interested in showing its corresponding (linear) dual program (22).

$$\begin{aligned} \underset{U,V,\alpha}{Min} & \ -\sum_{r=1}^{s}u_r y_{rk} + \sum_{i=1}^{m}v_i x_{ik} + \alpha \\ s.t. & \ \sum_{r=1}^{s}u_r y_{rj} - \sum_{i=1}^{m}v_i x_{ij} - \alpha \le 0, \qquad j=1,\dots,n, \\ & \ \sum_{r=1}^{s}u_r g_{rk}^y + \sum_{i=1}^{m}v_i g_{ik}^x = 1, \\ & \ U \ge 0_s,\ V \ge 0_m \end{aligned} \tag{22}$$

Let also denote one of the possible optimal solutions of problem (22) as $(\vec{V}_k^*,\vec{U}_k^*,\vec{\alpha}_k^*)$.

We are now in a position to define the Nerlovian cross-inefficiency of unit l with respect to unit k. We consider initially the case of variable returns to scale DEA technologies and, subsequently, constant returns to scale production possibility sets. In this way, and inspired in the Farrell cross-efficiency notion introduced in the previous section when dealing with input-oriented models, we now suggest to consider the shadow prices for inputs and outputs of each unit k,$(\vec{V}_k^*,\vec{U}_k^*)$, as reference prices for evaluating the performance of unit l through the left hand side of expression (19). So, we define the Nerlovian cross-inefficiency of unit l with respect to unit k as:

$$NCI_T\left(X_l,Y_l;G_l^x,G_l^y,G_k^x,G_k^y|k\right) = \frac{\Pi_T\left(\vec{V}_k^*,\vec{U}_k^*\right) - \left(\sum_{r=1}^{s}\vec{u}_{rk}^* y_{rl} - \sum_{i=1}^{m}\vec{v}_{ik}^* x_{il}\right)}{\sum_{r=1}^{s}\vec{u}_{rk}^* g_{rl}^y + \sum_{i=1}^{m}\vec{v}_{ik}^* g_{il}^x}. \tag{23}$$

Additionally, it is worth mentioning that $NCI_T(X_l,Y_l;G_l^x,G_l^y,G_k^x,G_k^y|k)$ always takes values greater than zero. By definition, $\Pi_T(\vec{V}_k^*,\vec{U}_k^*) = \max_{x,y}\{\sum_{r=1}^{s}\vec{u}_{rk}^* y_r - \sum_{i=1}^{m}\vec{v}_{ik}^* x_i : (X,Y)\in T\}$. Given that $(X_l,Y_l)\in T$, we have that $\Pi_T(\vec{V}_k^*,\vec{U}_k^*) \ge \sum_{r=1}^{s}\vec{u}_{rk}^* y_{rl} - \sum_{i=1}^{m}\vec{v}_{ik}^* x_{il}$. Consequently, $\Pi_T(\vec{V}_k^*,\vec{U}_k^*) - \sum_{r=1}^{s}\vec{u}_{rk}^* y_{rl} - \sum_{i=1}^{m}\vec{v}_{ik}^* x_{il} \ge 0$.

The next proposition allows us to understand (23) in more detail under variable returns to scale.

Proposition 3. *Let $(\vec{V}_k^*,\vec{U}_k^*,\vec{\alpha}_k^*)$ be an optimal solution of model (22). Then $\vec{\alpha}_k^* = \Pi_v(\vec{V}_k^*,\vec{U}_k^*)$.*

Proof. See Appendix A.6.

The above result implies that $\vec{\alpha}_k^*$ can be interpreted as shadow profit and, consequently, the Nerlovian cross-inefficiency for unit l with respect to unit k under VRS may be rewritten as

$$NCI_v\left(X_l,Y_l;G_l^x,G_l^y,G_k^x,G_k^y|k\right) = \frac{\vec{\alpha}_k^* - \left(\sum_{r=1}^{s}\vec{u}_{rk}^* y_{rl} - \sum_{i=1}^{m}\vec{v}_{ik}^* x_{il}\right)}{\sum_{r=1}^{s}\vec{u}_{rk}^* g_{rl}^y + \sum_{i=1}^{m}\vec{v}_{ik}^* g_{il}^x}. \tag{24}$$

The arithmetic mean of (23) over all observed units yields the final score for firm l:

$$\begin{aligned} NCI_T\left(X_l,Y_l;\left\{G_k^x,G_k^y\right\}_{k=1}^n\right) &= \frac{1}{n}\sum_{k=1}^{n}NCI_T\left(X_l,Y_l;G_l^x,G_l^y,G_k^x,G_k^y|k\right) \\ &= \frac{1}{n}\sum_{k=1}^{n}\frac{\Pi_T\left(\vec{V}_k^*,\vec{U}_k^*\right) - \left(\sum_{r=1}^{s}\vec{u}_{rk}^* y_{rl} - \sum_{i=1}^{m}\vec{v}_{ik}^* x_{il}\right)}{\sum_{r=1}^{s}\vec{u}_{rk}^* g_{rl}^y + \sum_{i=1}^{m}\vec{v}_{ik}^* g_{il}^x}. \end{aligned} \tag{25}$$

Invoking (19), we get that the Nerlovian cross-inefficiency of firm l is a 'correction' of the original directional distance function value for this unit, where the modification factor can be interpreted as (shadow) allocative inefficiency:

$$\begin{aligned} NCI_T\left(X_l,Y_l;\left\{G_k^x,G_k^y\right\}_{k=1}^n\right) &= \vec{D}_T\left(X_0,Y_0;G_l^x,G_l^y\right) \\ &+\frac{1}{n}\sum_{k=1}^{n}AI_T^N\left(X_l,Y_l;\vec{V}_k^*,\vec{U}_k^*;\left\{G_k^x,G_k^y\right\}_{k=1}^n\right). \end{aligned} \tag{26}$$

In Fig. 2, we illustrate graphically the notion of Nerlovian cross-inefficiency. Let us assume that we evaluate unit E by the directional vector g. From a conceptual viewpoint, our approach technically projects unit E onto A, which is a producible point in the technology, through the directional distance function. Then, a correction term that measures allocative or price inefficiency is calculated. The sum of these two terms leads to the value of the Nerlovian cross-inefficiency of unit E with respect to unit C. In particular, the allocative term in our approach measures, for technically efficient units (unit A in the graphical example), the loss due to being sub-optimal under given shadow input and output prices (the weights associated with unit C in the example).

Regarding the properties that the Nerlovian cross-inefficiency satisfies, we next list the most relevant ones (which can be proved along the lines of those for the Farrell cross-efficiency in the appendix).

[P1] $NCI_T(X_l,Y_l;G_l^x,G_l^y,G_l^x,G_l^y|l) = \vec{D}_T(X_l,Y_l;G_l^x,G_l^y)$;

[P2] The less $NCI_T(X_l,Y_l;\{G_k^x,G_k^y\}_{k=1}^n)$, the better (meaning of inefficiency);

[P3] $NCI_T(X_l,Y_l;\{G_k^x,G_k^y\}_{k=1}^n) \ge 0$;

[P4] If $(\vec{V}_k^*,\vec{U}_k^*) = (\vec{V}_l^*,\vec{U}_l^*)$, $\forall k=1,\dots,n$, then $NCI_T(X_l,Y_l;\{G_k^x,G_k^y\}_{k=1}^n) = \vec{D}_T(X_l,Y_l;G_l^x,G_l^y)$;

[P5] If (G_k^x,G_k^y), $k=1,\dots,n$, depends on data, then $NCI_T(X_l,Y_l;\{G_k^x,G_k^y\}_{k=1}^n)$ is units invariant;

[P6] If (G_k^x,G_k^y), $k=1,\dots,n$, depends on data, then $NCI_T(X_l,Y_l;\{G_k^x,G_k^y\}_{k=1}^n)$ is price invariant.

As we are aware, there is only an attempt to extend the notion of the traditional cross-efficiency $CITE_c(X_l, Y_l)$ to the world of non-oriented measures, i.e., which account for the inefficiency both in inputs and in outputs simultaneously. In particular, Ruiz [32] extended the cross-efficiency evaluation theory for use with the directional distance function. Specifically, this author considered the directional vector equal to the assessed observation, i.e., $(G_k^x, G_k^y) = (X_k, Y_k)$ for all $k = 1, \ldots, n$ in (20), and, as usual in cross-efficiency evaluation, he also assumed constant returns to scale,.

Ruiz [32], assuming CRS and $(G^x, G^y) = (X_k, Y_k)$,defined the cross DDF inefficiency of firm l with respect to firm k as ([32], Definition 1, p. 183):

$$CDDF_c(X_l, Y_l|k) = \frac{\sum_{i=1}^{m} \vec{v}_{ik}^* x_{il} - \sum_{r=1}^{s} \vec{u}_{rk}^* y_{rl}}{\sum_{i=1}^{m} \vec{v}_{ik}^* x_{il} + \sum_{r=1}^{s} \vec{u}_{rk}^* y_{rl}}. \quad (27)$$

As in the radial case, Ruiz [32] suggested averaging the n values of the $CDDF_c(X_l, Y_l|k)$, $k = 1, \ldots, n$, in order to define the DDF cross-efficiency of firm l:

$$CDDF_c(X_l, Y_l) = \frac{1}{n}\sum_{k=1}^{n} CDDF_c(X_l, Y_l|k) = \frac{1}{n}\frac{\sum_{i=1}^{m} \vec{v}_{ik}^* x_{il} - \sum_{r=1}^{s} \vec{u}_{rk}^* y_{rl}}{\sum_{i=1}^{m} \vec{v}_{ik}^* x_{il} + \sum_{r=1}^{s} \vec{u}_{rk}^* y_{rl}}. \quad (28)$$

Next, we show that the cross-efficiency based on the DDF under CRS, is a particular case of a more general approach based upon the Nerlovian inefficiency measure; notion related, by duality, to the DDF.

It is possible to define a Nerlovian cross-inefficiency measure under constant returns to scale resorting to expression (24). To do that, it is enough to substitute $\vec{\alpha}_k^*$ by zero in (24) since this is the value of the shadow profit under CRS (see Lemma 2). If, additionally, we fix $(G_k^x, G_k^y) = (X_k, Y_k)$ for all $k = 1, \ldots, n$, then we get:

$$\begin{aligned} NCI_c(X_l, Y_l; X_l, Y_l, X_k, Y_k|k) &= \frac{-\left(\sum_{r=1}^{s} \vec{u}_{rk}^* y_{rl} - \sum_{i=1}^{m} \vec{v}_{ik}^* x_{il}\right)}{\sum_{r=1}^{s} \vec{u}_{rk}^* y_{rl} + \sum_{i=1}^{m} \vec{v}_{ik}^* x_{il}} \\ &= \frac{\sum_{i=1}^{m} \vec{v}_{ik}^* x_{il} - \sum_{r=1}^{s} \vec{u}_{rk}^* y_{rl}}{\sum_{i=1}^{m} \vec{v}_{ik}^* x_{il} + \sum_{r=1}^{s} \vec{u}_{rk}^* y_{rl}} = CDDF_c(X_l, Y_l|k). \end{aligned} \quad (29)$$

And, finally, taking the mean over all the units in the sample, we obtain that the Nerlovian approach coincides with the cross-inefficiency defined by Ruiz [32] based on the directional distance function under CRS. Moreover, it can be decomposed likewise into the (directional) technical inefficiency and a correction factor defined as the arithmetic mean of n shadow allocative efficiency terms, as in expression (26). I.e.,

$$NCI_c\left(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n\right) = \vec{D}_c(X_l, Y_l; X_l, Y_l) + \frac{1}{n}\sum_{k=1}^{n} AI_c^N\left(X_l, Y_l; \vec{V}_k^*, \vec{U}_k^*; \{X_k, Y_k\}_{k=1}^n\right). \quad (30)$$

4. Empirical implementation of economic cross-efficiency

4.1. A comparison of methods dealing with negative cross-efficiency under VRS

In this subsection, we compare previous contributions dealing with cross-efficiency under VRS in Data Envelopment Analysis to our proposal. Given that the approach introduced in this paper is grounded on technical efficiency measures not based upon slacks, i.e., it focuses on radial and directional efficiency measures, the contributions selected for the comparison exercise will be those by Wu et al. [39] and Lim and Zhu [26], which resorted to the input-oriented BCC model in DEA [6]. Another related contribution, namely that by Kao and Liu [23], which is built from a slacks-based measure, is not considered at this time, as the corresponding comparison requires the extension of our current approach to this family of measures.

Table 1
A numerical example.

DMU	x_1	x_2	y	BCC score	v_1^*	v_2^*	u^*	α^*
A	2	2	2	1	0.2778	0.2222	0.0833	-0.8333
B	1	4	4	1	0.4000	0.1500	0.3750	0.5000
C	4	1	6	1	0.2286	0.0857	0.2143	0.2857
D	3	2	1	0.8571343	0.1429	0.2857	0.0000	-0.8571
E	4	6	8	1	0.1600	0.0600	0.1500	0.2000

Source: Own calculations using Wu et al.'s [39] data.

To carry out the comparison, we resort to a numerical example with two inputs and one output and only five DMUs taken from Wu et al. [39], where the problem of negative cross-efficiencies was illustrated. Table 1 shows the data for this example as well as the corresponding input-oriented BCC score and an optimal solution for each DMU.

Let us focus our attention on DMU D. Some traditional bilateral cross-efficiencies are negative for this unit, lacking a sensible interpretation. In particular, if $(V_k^*, U_k^*, \alpha_k^*)$ is an optimal solution of the input-oriented BCC model when DMU k is assessed, the traditional formula for (bilateral) cross-efficiency for evaluating unit (X_l, Y_l) would be $CITE_v(X_l, Y_l|k) = \frac{\sum_{r=1}^{s} u_{rk}^* y_{rl} - \alpha_k^*}{\sum_{i=1}^{m} v_{ik}^* x_{il}}$ (see, for example, [39]). In the case of D, $CITE_v(X_D, Y_D|k) = -0.083$ for units $k = B, C, E$.

Before showing the cross-efficiencies calculated from the methodologies proposed by Wu et al. [39], Lim and Zhu [26], and the Farrell cross-efficiency introduced in this paper, let us mention in passing the main theoretical idea which is behind each approach. The proposal by Wu et al. [39] is based on solving the optimization program corresponding to the input-oriented BCC model but incorporating non-negative constraints for the numerator: $\sum_{r=1}^{s} u_{rk}^* y_{rl} - \alpha_k^* \geq 0$, for all $l = 1, \ldots, n$. These constraints guarantee that $CITE_v(X_l, Y_l|k)$ is always greater or equal to zero. In contrast, Lim and Zhu [26] propose a Cartesian coordinate system translation and then applying the traditional input-oriented BCC model. Finally, the new approach is based on an economic reinterpretation of the classical cross-efficiency, resorting to the well-known Farrell's cost efficiency measure using (V_k^*, U_k^*) as shadow prices, which is always between zero and one regardless of the assumed returns to scale—as shown in [P3] above.

Next, we show in Table 2 the results associated with the three considered approaches for comparison purposes.

In particular, the 'bilateral' cross efficiency of unit D is zero for the Wu et al. [39] method when the weights of units B, C and E are used for the assessment. It equals 0.1875 in the case of the Lim and

Table 2
Cross-efficiencies calculated by different alternative methods: $CITE_v(X_l, Y_l)$

DMU	Wu et al. [39]	Lim & Zhu [26]	Farrell approach
A	0.5450	0.4813	0.9455
B	0.6967	0.8000	0.9333
C	1.0000	0.8000	1.0000
D	0.3429	0.1500	0.7280
E	0.5785	0.6828	1.0000

Source: Own calculations.

Table 3
Descriptive statistics for inputs and outputs, warehouse data, 2017.

	Inputs			Outputs			
	Floor space	FTEs	SKUs	Order lines	Error free	Order flexibility	Special processes
Minimum	500	50	100	54	1	12	2
Median	9,250	30	4,600	1,200	7	22	6
Average	18,244	59	21,088	4,931	6	21	6
Maximum	275,000	350	400,000	55,000	9	30	10
Stand. Dev.	32,414	74	57,393	9,815	2	4	2

Source: Kaps and de Koster [24].

Zhu [26] approach and 0.667 if we resort to the Farrell approach introduced in this paper, correcting in this way the negative value observed with the traditional approach.

Although the results are very different among the alternatives, some high correlations are observed among them. In particular, the correlation between the new approach and that by Wu et al. [39] equals 0.88, while the correlation with Lim and Zhu [26] equals 0.73. Obviously, we do not have any objective criterion to opt for one or another alternative in practice. However, in this example, the solution determined by the new approach is closer to the original input-oriented BCC efficiency score (see Table 2) in comparison with the other existing alternatives. Indeed, the correlation between the Farrell approach and the BCC efficiency score is 0.96, in contrast to the correlations with respect to the Wu et al. [39] and Lim and Zhu [26] methodologies, which present values of 0.67 and 0.88, respectively. In this example, the new methodology provides cross-efficiency values more similar to the BCC efficiency scores while, at the same time, guarantees the non-negativity of the cross-evaluation results. Nevertheless, we are aware that this issue merits further research considering alternative real or simulated datasets in diverse scenarios and comparing the different solutions under VRS.

4.2. Empirical application to warehousing data

To illustrate the new concept of economic cross-(in)efficiency and its empirical implementation, we rely on a database on 102 warehouses operating in the Benelux area in 2017. Following Johnson and McGinnis [21] and Balk et al [5], we characterize the production technology in terms of the following three inputs and four outputs.[5] Inputs are: I.1) Warehouse size in m2 (Floor space); I.2) Number of full time equivalent employees (FTEs); and I.3) Number of stock keeping units (SKUs). On the output side the following variables are considered: O.1) Number of order lines (Order lines shipped per day); O.2) Error-free order line percentage (Error free %); O.3) Order flexibility (per day); and O.4) Number of special processes (handled per day). Table 3 shows the descriptive statistics for all selected variables.

4.2.1. Farrell economic (cost) cross-efficiency

Table 4 reports the results for the original Farrell input oriented model that radially measures technical efficiency for warehouse l as in (1), $ITE_c(X_l,Y_l)$, its standard technical cross-efficiency measure (5), $CITE_c(X_l,Y_l)$, and the new Farrell cost cross-efficiency measure (17), $FCE_c(X_l,Y_l)$. Also, following the proposed decomposition, we also report the allocative efficiency associated to the new cross-efficiency measure, calculated as the ratio between $FCE_c(X_l,Y_l)$. and $ITE_c(X_l,Y_l)$; i.e., expression (18). The first set of results corresponds to the existing setting in the literature corresponding to constant returns to scale (CRS). These are grouped under that heading on the left hand side of Table 4.[6]

The results for the five best and worst performing warehouses are ranked using the values of the new Farrell economic cross-efficiency measure, $FCE_c(X_l,Y_l)$. First we focus on the comparison between this latter measure and the standard cross-efficiency measure $CITE_c(X_l,Y_l)$. The individual values show that both cross-efficiency measures have the capability of discriminating between radially efficient observations with $ITE_c(X_l,Y_l)= 1$. However, the ranking exhibits some variability. For example, warehouse #33, ranking first according to the cost cross-efficiency measure: $FCE_c(X_l,Y_l)= 0.960$, ranks below the fifth position according to $CITE_c(X_l,Y_l) = 0.667$. On the lower tail of the distribution there seems to be larger compatibility as the worst five performing warehouses exhibit the same ordering.

Throughout this empirical section we discuss the (dis)similarity between alternative cross-efficiency measures by studying their *ranking compatibility* by means of the Spearman correlation and, relying on kernel density estimations, by determining whether their *distributions are equal or not* according to the Li tests. When plotting the kernel density functions we follow the procedure proposed by Simar and Zelenyuk [36], which in short: (i) uses Gaussian kernels, (ii) employs the reflection method to overcome the issue of (radial) unitary or (directional) zero bounded supports for the cross-(in)efficiency scores [35], and (iii) determines the bandwidths using Sheather and Jones [34] method. Subsequently, once the kernel density functions are calculated we apply the nonparametric test developed by Li [25] to determine if they are statistically different. Here we again follow Simar and Zelenyuk [36] and use algorithm II with 1,000 replications, which computes the Li statistic on the bootstrapped estimates of the DEA scores, and where the unitary or null values of the efficient observations are smoothed by adding a small noise. These different dimensions will allow us to establish statistically to what extend the alternative cross-efficiency measures lead to equal or different results regarding warehouse performance.

We may now establish the similarity between the new Farrell cost cross-efficiency measure and its standard counterpart starting with their ranking compatibility. Their Spearman correlation is $\rho(CITE_c(X_l,Y_l),\ FCE_c(X_l,Y_l)) = 0.988$, which is significant at the 1% level. This result implies that beyond individual disparities, both series yield a very similar picture of the warehouse industry standing. This can be clearly visualized in Fig. 3 by comparing their kernel density functions, whose patterns closely follow each other, and is further corroborated by the Li-test comparing $CITE_c(X_l,Y_l)$ vs. $FCE_c(X_l,Y_l)$ (i.e., -0.735), whose result does not allow to reject the null hypothesis of the equality of distributions, as reported in Table 5.

The similarity of results confirmed in all three dimensions (Spearman correlation, density distributions and Li tests) is a

[5] The data set has been recently updated by Kaps and de Koster [24]. We are grateful to these authors for sharing them.

[6] For input-oriented radial measures, the greater the score the higher the efficiency.

Table 4
Farrell cost cross-efficiency decomposition, FCE_L. Expression (17), $L\in\{c,v\}$, CRS and VRS, respectively.

	Constant Returns to Scale, CRS (CCR model, input orientation)					Variable Returns to Scale, VRS (BCC model, input orientation)			
Ranking	Warehouse#	Standard ITE $ITE_c(X_l,Y_l)$ (1)	Standard Cross-Effic. $CITE_c(X_l,Y_l)$ (5)	New Farrell Cross- Effic. $FCE_c(X_l,Y_l)$ (17)	New Farrell Alloc. Efficiency $FCE_c(X_l,Y_l)/ITE_c(X_l,Y_l)$ (18)	Warehouse#	Standard ITE $ITE_v(X_l,Y_l)$	New Farrell Cross Effic. $FCE_v(X_l,Y_l)$ (17)	New Farrell Alloc. Effic. $FCE_v(X_l,Y_l)/ITE_v(X_l,Y_l)$ (18)
1	33	1.000	0.667	0.960	0.960	19	1.000	1.000	1.000
2	19	1.000	0.812	0.908	0.908	33	1.000	1.000	1.000
3	50	1.000	0.819	0.898	0.898	36	1.000	1.000	1.000
4	54	1.000	0.682	0.890	0.890	42	1.000	1.000	1.000
5	49	1.000	0.757	0.818	0.818	45	1.000	1.000	1.000
98	11	0.118	0.059	0.069	0.585	9	0.213	0.072	0.338
99	12	0.153	0.056	0.064	0.418	69	0.107	0.070	0.654
100	69	0.106	0.042	0.063	0.594	12	0.155	0.066	0.426
101	65	0.101	0.029	0.039	0.386	65	0.117	0.041	0.350
102	77	0.051	0.023	0.034	0.667	77	0.053	0.037	0.698
	Minimum	0.051	0.023	0.034	0.229	Minimum	0.053	0.037	0.223
	Median	0.411	0.180	0.224	0.605	Median	0.622	0.371	0.606
	Average	0.484	0.233	0.283	0.582	Average	0.662	0.443	0.631
	Maximum	1.000	0.819	0.960	0.960	Maximum	1.000	1.000	1.000
	Stand. Dev.	0.294	0.176	0.210	0.144	Stand. Dev.	0.314	0.306	0.202

Source: Own calculations.

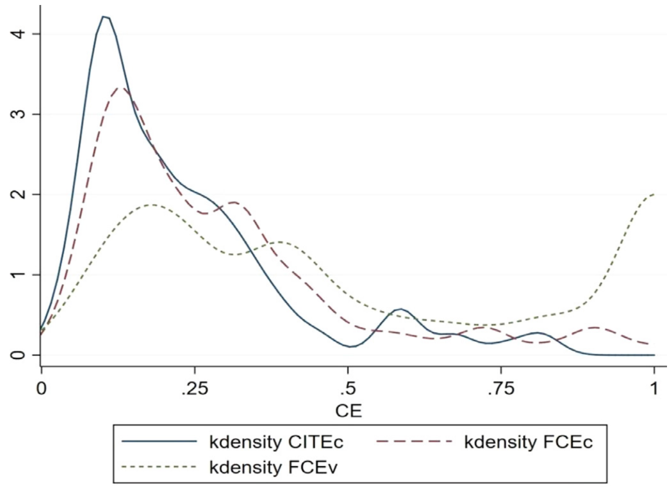

Fig. 3. Estimated kernel density distributions of the standard, $CITE_c(X_l,Y_l)-$ (5), and new Farrell economic cross-efficiencies: $FCE_c(X_l,Y_l)-$ (17) (under CRS), and $FCE_v(X_l,Y_l)-$ (17) (under VRS).

Table 5
Results of Simar and Zelenyuk [36] adapted Li test (test statistic and significance level).

	$CITE_c(X_l,Y_l)$ vs $FCE_c(X_l,Y_l)$	$CITE_c(X_l,Y_l)$ vs $FCE_v(X_l,Y_l)$	$FCE_c(X_l,Y_l)$ vs $FCE_v(X_l,Y_l)$	$NCI_v(X_l,Y_l;\{X_k,Y_k\}_{k=1}^n)$ vs $CDDF_c(X_l,Y_l)$
Statistic	-0.735	6.257*	2.056*	15.245*
p value	(0.220)	(0.000)	(0.000)	(0.000)

Source: Own calculations.
Notes: $CITE_c(X_l,Y_l)$, (5); $FCE_c(X_l,Y_l)$, (17)–under CRS; $FCE_v(X_l,Y_l)$, (17)–under VRS. $NCI_v(X_l,Y_l;\{X_k,Y_k\}_{k=1}^n)$, (25); $CDDF_c(X_l,Y_l)$, (28).
*Denotes statistically significant differences between models at the critical 1 percent level.

remarkable result that confirms the reliability of the new Farrell cost cross-efficiency measure when ranking performance (when compared to its traditional constant returns to scale counterpart), and endorses its use under variable returns to scale (VRS), for which a well-defined standard analogue does not exist.

However, before we comment on this additional set of results under VRS, we stress that for the warehouse industry, overall cost (in)efficiency can be almost equally blamed on faulty technical and allocative performance, with the latter having a marginally higher weight. While average technical efficiency is 0.484, allocative efficiency is 0.582. The median values being 0.411 and 0.605, respectively. We remark once again that this interpretation of cross-efficiency in economic terms, and its decomposition into both sources, as presented in (18), were unavailable until now. Finally, focusing still on the results under constant returns to scale, a second conclusion emerges. Despite the high similarly, the existence of large numerical differences at the individual level between the economic and standard cross-efficiency measures (in favor of the former as stated in Proposition 2, $CITE_c(X_l,Y_l)\leq FCE_c(X_l,Y_l)$) suggests that the warehouse production technology is non-homothetic. Indeed, according to Theorem 1, $CITE_c(X_l,Y_l)=FCE_c(X_l,Y_l)$ under input homotheticity, and therefore these disparities rule out its existence.

We comment now on the new economic cross-efficiency measure under variable returns to scale, $FCE_v(X_l,Y_l)$, presented on the right hand side of Table 4. An immediate critic that can be raised against it is that it does not solve the lack of discriminatory power of the standard Farrell input measure when observations are efficient: i.e., $ITE_c(X_l,Y_l)=1$. We can qualify this drawback of the new measure by remarking that as many as 37 warehouses are efficient under VRS (36.3% of the sample), while only 10 warehouses exhibit $FCE_v(X_l, Y_l)= 1$ (9.8%)—we also stress the fact that it is a well-defined measure not prone to negative values as the standard technical cross-efficiency under VRS. Hence, while full cost cross-efficiency under VRS is a feasible result likely to be observed (as opposed to its CRS counterpart $FCE_c(X_l, Y_l)$), its calculation is still quite useful from a managerial perspective, as it substantially increases discrimination among observation that are VRS efficient. As for the sources of cost (in)efficiency when decomposing $FCE_v(X_l, Y_l)$ according to (18), we note that the characterization of the reference technology by VRS does not change the relative weights of technical and allocative efficiencies, although the higher weight of the latter is now reversed. The average and median values of the technical efficiency are now 0.662 and 0.622, on a par with allocative efficiency whose values are 0.631 and 0.606, respectively.

One can compare the new Farrell cost cross-efficiencies calculated under both constant and variable returns to scale, i.e., $FCE_c(X_l, Y_l)$ vs. $FCE_v(X_l, Y_l)$, but the exercise requires further assumptions about the market structure. Generally only the technical side of the economic performance would have a valid interpretation as the usual measure of scale efficiency, defined as $ITE_c(X_l, Y_l)$ / $ITE_v(X_l, Y_l)$. The notion of a cost function defined under the restrictive case of constant returns to scale does not have any justification if the technology exhibits variable returns to scale or, from a weaker perspective, is non-homothetic. This is indeed the case for the usual DEA characterization of the production technologies, as in the current warehouse application–recalling Aparicio et al. [[2], 887; Proposition 3], these authors show that only in the restrictive case of a single output and CRS, the DEA technology is homothetic. Since virtually in all empirical applications the technology is characterized by VRS, our newly proposed Farrell cost cross-efficiency exhibits its full potential in its VRS definition (on top of its ability to provide an analytical framework that excludes negative values). Hence, the CRS definition of the Farrell cost cross-efficiency, $FCE_c(X_l, Y_l)$ (equal to its traditional counterpart $CITE_c(X_l, Y_l)$ under input homotheticity), from which our paper starts out theoretically, is only relevant for pedagogical purposes, presented so as to reinterpret the existing CRS (technical) cross efficiency measures in economic terms and, later on, move on to introduce the (empirically) relevant VRS definitions (17) and (18), i.e., $FCE_v(X_l, Y_l)$.

The only exception that would grant the assumption of CRS in studies where a distinct market structure can be considered, is the theoretical consideration of the perfectly competitive long run equilibrium, where the technology exhibits CRS, industry profits are zero and average cost is minimum, by definition. In this case, the difference between $FCE_v(X_l, Y_l|k)$ and $FCE_c(X_l, Y_l|k)$, compares the performance corresponding to the current short run situation (normally associated with a suboptimal scale size if scale inefficiency exists), and the hypothetical long run equilibrium–both measures evaluated at their respective optimal prices, (V_k^*, U_k^*). Arguably, the warehouse industry departs from the perfectly competitive framework in many ways, but if one were willing to assume it, then the comparison between the average values corresponding to $FCE_v(X_l, Y_l)$ and $FCE_c(X_l, Y_l)$ shows that the difference between both measures is noticeable, i.e., 0.443 and 0.233 (with a similar gap at the median). As for the ranking compatibility, it is relatively high: $\rho(FCE_c(X_l, Y_l),\ FCE_v(X_l, Y_l)) = 0.720$, also significant at the 1% level. However, in Fig. 2 the kernel density functions between the two follow different patterns with lower and higher density values for $FCE_v(X_l, Y_l)$ in the lower and upper tails, respectively. As seen in Table 5, this translates in a Li test result (2.056) that rejects the null hypothesis that both distributions are the same. Therefore, as expected, scale efficiency would play a big part in the assess-

Table 6
Nerlovian profit cross-inefficiency decomposition $NCI_T(X_l, Y_l; \{G_k^x, G_k^y\}_{k=1}^n)$. Expression (26), $T\in\{c,v\}$, −CRS and VRS, respectively.

Ranking	Variable Returns to Scale, VRS (DDF model, $(G_k^x, G_k^y) = (X_k, Y_k)$)				Constant Returns to Scale, CRS (DDF model, $(G_k^x, G_k^y) = (X_k, Y_k)$)			
	Warehouse#	Standard DDF $\vec{D}_v(X_l, Y_l; X_l, Y_l)$ (21)	New Nerlovian Cross-Ineffic. $NCI_v(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n)$ (25)	New Nerlovian Alloc. Inefficiency $NCI_v(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n) - \vec{D}_v(X_l, Y_l; X_l, Y_l)$ (26)	Warehouse#	Standard DDF $\vec{D}_c(X_l, Y_l; X_l, Y_l)$ (20)	New Nerlovian Cross Efficiency $NCI_c(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n) = CDDF_c(X_l, Y_l)$ (28)	New Nerlovian Alloc. Efficiency $NCI_c(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n) - \vec{D}_c(X_l, Y_l; X_l, Y_l)$ (30)
1	50	0.000	0.176	0.176	50	0.000	0.112	0.112
2	89	0.000	0.226	0.226	19	0.000	0.130	0.130
3	34	0.000	0.250	0.250	49	0.000	0.181	0.181
4	23	0.000	0.275	0.275	54	0.000	0.229	0.229
5	25	0.000	0.280	0.280	33	0.000	0.272	0.272
98	93	0.283	1.642	1.359	11	0.789	0.891	0.102
99	13	0.199	1.678	1.479	12	0.734	0.898	0.164
100	77	0.497	1.792	1.295	69	0.808	0.921	0.113
101	75	0.250	1.981	1.731	65	0.816	0.944	0.128
102	96	0.000	2.119	2.119	77	0.903	0.956	0.053
	Minimum	0.000	0.176	0.176	Minimum	0.000	0.112	0.053
	Median	0.045	0.550	0.444	Median	0.418	0.709	0.254
	Average	0.102	0.666	0.564	Average	0.398	0.671	0.273
	Maximum	0.568	2.119	2.119	Maximum	0.903	0.956	0.695
	Stand. Dev.	0.140	0.390	0.350	Stand. Dev.	0.259	0.187	0.140

Source: Own calculations.

ment of performance through the new economic efficiency measures, under alternative assumptions of market structures.[7]

4.2.2. Nerlovian economic (profit) cross-inefficiency

Table 6 presents our second set of results on the new Nerlovian profit cross-inefficiency measure based on the profit function and its duality with the directional distance function; i.e., expression (26). As normally assumed in the empirical literature we consider that the directional vector corresponds to the observed input and output quantities: $(G_k^x, G_k^y) = (X_k, Y_k)$, $k = 1, \dots, n$. Following the presentation in the theoretical section, we start our discussion considering the results obtained under the assumption of variable returns to scale. The first conclusion worth highlighting is that the ability to discriminate among VRS efficient observations is complete. Although once again a large set of warehouses are deemed efficient, with $\vec{D}_v(X_l, Y_l; X_l, Y_l) = 0$ (43, representing 42.2% of the sample), none of them are cross-efficient from an economic perspective: i.e., $NCI_v(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n) > 0$.[8] Consequently, the above criticism against the Farrell economic cross-efficiency measure cannot be raised on this occasion. This result also suggests that Nerlovian profit inefficiency, once the input and output dimensions are taken into consideration, is larger than in the Farrell case where inefficiency refers only to the input (cost) dimension.

Ultimately, the obtained Nerlovian cross-inefficiency values bear proof of the fact that, again for the first time, our model can effectively rank observations by appraising their profit performance against all remaining peers under VRS. Moreover, it is possible to decompose this relative economic performance following expression (26). The descriptive statistics show that the sources of Nerlovian profit cross-inefficiency substantially change with respect to those of the Farrell's cost approach. The average profit cross-inefficiency amounts 0.666, but now average technical inefficiency is a meager 0.102, while allocative inefficiency is 0.562; representing 15.3% and 84.7% of the overall profit inefficiency, respectively.

As recalled on expression (29), if constant returns to scale were assumed, the new Nerlovian profit cross-inefficiency coincides with the technical cross-inefficiency measure proposed by Ruiz [32]: $NCI_c(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n) = CDDF_c(X_l, Y_l)$. However, as previously discussed, the economic re-interpretation of the (technical) directional cross-efficiency under CRS would not be adequate unless its assumption is granted by the existence of a perfectly competitive market structure framed in the long run. In that case, although the average levels of profit cross-inefficiency are very similar, $NCI_v(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n) = 0.666$ vs. $NCI_c(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n) = 0.671$, its sources greatly differ since now technical and allocative inefficiencies represent 59.4% and 40.7% of the overall inefficiency. Despite the similar average values, the rank correlation between both series is relatively low at $\rho(NCI_v(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n), NCI_c(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n)) = 0.285$– significant at the 1% level, and implying that the choice of returns to scale is even more relevant when assessing industry performance than in the Farrell case. The disparity between both sets of results can be seen in Fig. 4, where the density of the CRS results peaks around one (rather than 0.5 under VRS), followed by its sudden fall and disappearance because no values beyond this threshold are observed. It comes, then, as no surprise that the associated Li test comparing $NCI_v(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n)$ and $CDDF_c(X_l, Y_l)$ (with a Li statistic equal to 15.245) rejects the null hypothesis of equality of distributions.

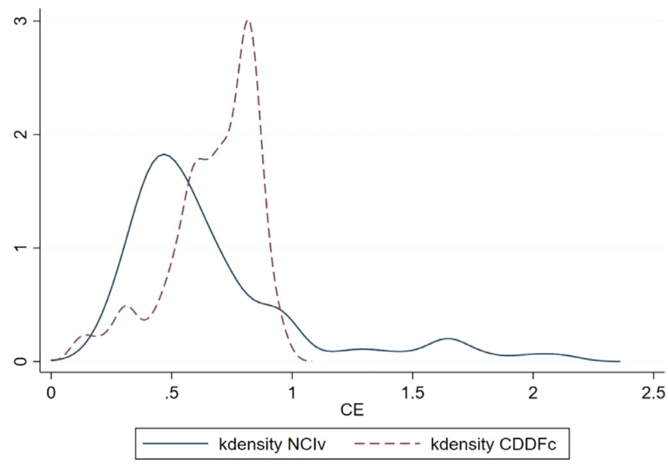

Fig. 4. Estimated kernel density distributions of the new Nerlovian economic cross-inefficiency, $NCI_v(X_l, Y_l; \{X_k, Y_k\}_{k=1}^n)$– (25), and the standard cross-inefficiency, $CDDF_c(X_l, Y_l)$– (28).

5. Summary and Conclusions

Despite the capability of cross-efficiency to yield a suitable ranking of observations based on the (shadow) prices associated with all the sample units when evaluating each observation, these techniques have developed without establishing any connection with the literature devoted to measuring economic efficiency when prices are present; i.e., relying on microeconomic theory. This paper makes the connection between the concepts of economic efficiency and cross-efficiency. Economic cross-(in)efficiency measures the performance of observations in terms of a set of reference prices that could correspond to either market prices, shadow prices or any other imputed prices. Hence, this economic cross-efficiency measure can be interpreted as the capability of firms to behave optimally by reaching minimum cost or maximum profit for a wide range of prices. The new methodology is particularly relevant in studies where market prices are not readily available because of the institutional framework (e.g., public services such as education, health, safety, etc.), but yet a robust ranking of observations based on their performance is demanded by decision makers and stakeholders.

Within the DEA framework we show that, under input homotheticity, the traditional bilateral notion of input cross-efficiency for unit l, when the weights of unit k are used in the evaluation, coincides with the well-known Farrell notion of cost efficiency for unit l when precisely unit k weights are taken as market prices. However, this result does not hold if the technology is not input homothetic. This motivates the introduction of the concept of bilateral Farrell cost cross-efficiency (FCE), corresponding to his notion of cost efficiency under *either* constant or variable returns to scale. We also extend this proposal based on the classic Farrell framework restricted to the input dimension to more recent developments corresponding to a complete representation of the economic objective of the firms through the profit function, and its dual characterization by way of the flexible directional distance function. This results in the introduction of the parallel concept of Nerlovian (profit) cross-inefficiency (NCI). In both cases, either *à la Farrell* or *à la Nerlove*, the new analytical framework allows us to further exploit the duality properties of the economic

[7] We conclude this subsection commenting on the disparity between $FCE_v(X_l, Y_l)$ and the standard technical cross-efficiency measure $CITE_c(X_l, Y_l)$, rather than $FCE_c(X_l, Y_l)$. The Li test, with a statistic equal to 6.257, also returns that both distributions are statistically different, as can be confirmed by visually inspecting them in Fig. 3. This result simply reinforces the previous one, i.e., between $FCE_v(X_l, Y_l)$ and $FCE_c(X_l, Y_l)$, as $CITE_c(X_l, Y_l)$ would be equal to the latter under input-homotheticity. Hence the numerical difference between both tests (columns 3 and 4 in Table 4), can be associated to the existence of a non-homothetic technology.

[8] For additive measures, the greater the score the higher the *in*efficiency; hence the change in denomination.

measures and decompose economic cross-(in)efficiency according to technological and allocative criteria.

We emphasize that a key advantage of the Farrell and Nerlovian cross-(in)efficiency measures is that they are well defined under variable returns to scale (VRS) by yielding scores that always lay between zero and one for the former and are always greater than zero for the latter. This solves a well-known weakness of the standard cross-efficiency methods, which may result in negative scores when the technology is characterized by VRS. As shown by solving a numerical example, the economic cross-(in)efficiency methodology solves this problem in a natural way, complementing ad-hoc methods such as those based on constraining the numerator of the BCC problem to be non-negative [39] or translating the data before solving it (e.g., [26]).

We illustrate the feasibility of the new models and associated measures using a recently compiled data set of European warehouses. We show that the economic cross-efficiency measures FCE and NCI are well defined under constant and variable returns to scale, and how they can be decomposed according to technical and allocative criteria. Moreover, the large rank correlation between the standard cross-efficiency values and the new Farrell cost cross-efficiency under constant returns to scale, suggests that these latter model can be extended to variable returns to scale with confidence. We compare the constant and variable returns to scale measures, and conclude through the visual inspection of their kernel density functions and associated Li tests that assuming alternative returns to scale does make a difference in the evaluation of economic performance, since results are statistically different. This is a remarkable conclusion because the numerical differences between the constant and variable returns to scale measures signal that warehouse operations are characterized by non-homothetic technologies (i.e., Theorem 1 does not hold), which further justifies the introduction of the new economic cross-efficiency models under variable returns and reinforces their use in empirical applications. How to interpret the difference between both sets of results in economic terms is harder than in the technological case associated to scale (in)efficiency, because different assumptions regarding the market structure need to be brought into the analysis (e.g., perfectly or imperfectly competitive markets, and long and short-run equilibria).

Next we identify some avenues for further follow-up research. First, we resorted in this paper to two specific approaches for measuring economic efficiency, and transpose them to the realm of what we term economic cross-efficiency evaluation. However, it seems natural to apply other alternative approaches like, for example, those related to the hyperbolic measure ([16], and [40]) or the weighted additive model ([12], and [3]). Second, there are contributions in the literature that study the measurement and decomposition of economic efficiency change over time when panel data are available (see, for example, [29], and [22]). A natural extension of the current paper would result in a model measuring how economic cross-efficiency rankings change over time. Third, there does not exist a notion of cross-efficiency in the parametric approach to efficiency analysis, where cost functions, for example, are estimated once a functional form has been specified, and depending on a set of parameters that must be estimated. In this respect, the introduced Farrell cost cross-efficiency measure could be determined parametrically, constituting a first application of cross-efficiency in the parametric framework for efficiency measurement. Fourth, one difficulty with traditional cross-efficiency evaluation is the possible existence of alternative optima in the DEA models providing the weights (first stage), resulting in different cross-efficiency scores (second stage). The approach that has been traditionally followed to address this issue is based on the use of secondary goals as criteria to choose a given set of weights among the alternative optimal solutions. The well-known benevolent and aggressive approaches proposed in Sexton et al. [37] and Doyle and Green [13] are among the most popular ones. All these proposals are relevant qualifications and natural extensions that would result in the consolidation and improvement of the new concept of economic cross-efficiency.

CRediT authorship contribution statement

Juan Aparicio: Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Data curation, Writing - original draft, Writing - review & editing. **José L. Zofío:** Conceptualization, Methodology, Software, Validation, Formal analysis, Investigation, Data curation, Writing - original draft, Writing - review & editing.

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Appendix A

A.1. Proof of Lemma 1

Proof. We first prove that (V_k^*, E^*) is a feasible solution of (11). Constraints (11.2) and (11.3) trivially hold. Regarding (11.1), $\sum_{r=1}^{s} e_r^* y_{rj} - \sum_{i=1}^{m} v_{ik}^* x_{ij} \leq \sum_{r=1}^{s} e_r^* y_{rj} - \sum_{i=1}^{m} f_i^* x_{ij} \leq 0$ since (F^*, E^*) satisfies (11.2) and (11.1). This implies that (V_k^*, E^*) is a feasible solution of (11). As for the value of the objective function of (11), evaluated at (V_k^*, E^*), $\sum_{r=1}^{s} e_r^*$, it coincides with the optimal value of (11). Therefore, (V_k^*, E^*) is an optimal solution of (11). □

A.2. Proof of Theorem 1

Proof. In particular, we need to prove that $\sum_{r=1}^{s} u_{rk}^* y_{rl} = C_c(Y_l, V_k^*)$. By (7), we have that $L(Y) = H(Y)L(1_s)$. Additionally, under Constant Returns to Scale, Färe and Primont [15] show that $L(\delta Y) = \delta L(Y)$, for all $\delta > 0$. Therefore, under both hypothesis, $L(Y) = L(H(Y)1_s)$. In this way, we have that $ITE_c(X_k, Y)^{-1} = D_c(X_k, Y) = \sup\{\delta > 0 : X_k/\delta \in L(Y)\} = \sup\{\delta > 0 : X_k/\delta \in L(H(Y)1_s)\} = ITE_c(X_k, H(Y)1_s)^{-1}$ for any $Y \in R_+^s$. This result also implies that when we evaluate the input vector X_k by means of the Shephard input distance function with respect to $L(Y)$, we get the same shadow prices than when we assess the input vector X_k by means of the Shephard input distance function with respect to $L(H(Y)1_s)$. Then, since we know that (V_k^*, U_k^*) are shadow prices for unit k, i.e, it is an optimal solution of model (2), we have that (V_k^*, U_k^*) is also an optimal solution of the following program:

$$\begin{aligned}
ITE_c(X_k, Y_k) = \quad & \underset{U,V}{Max} \quad H(Y_k) \sum_{r=1}^{s} u_r \\
& s.t. \quad \sum_{i=1}^{m} v_i x_{ik} = 1, \\
& \qquad \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, \qquad j = 1, \ldots, n \\
& \qquad u_r \geq 0, \qquad r = 1, \ldots, s \\
& \qquad v_i \geq 0, \qquad i = 1, \ldots, m
\end{aligned} \tag{A.1}$$

By the same reasoning, the following two programs are equivalent with respect to optimal solutions and the optimal value:

$$
\begin{aligned}
ITE_c(X_k, H(Y_l)1_s) = \underset{U,V}{Max} \quad & H(Y_l)\sum_{r=1}^{s} u_r \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{ik} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, && j = 1, \dots, n \\
& u_r \ge 0, && r = 1, \dots, s \\
& v_i \ge 0, && i = 1, \dots, m
\end{aligned}
\tag{A.2}
$$

$$
\begin{aligned}
ITE_c(X_k, Y_l) = \underset{U,V}{Max} \quad & \sum_{r=1}^{s} u_r y_{rl} \\
s.t. \quad & \sum_{i=1}^{m} v_i x_{ik} = 1, \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, && j = 1, \dots, n \\
& u_r \ge 0, && r = 1, \dots, s \\
& v_i \ge 0, && i = 1, \dots, m
\end{aligned}
\tag{A.3}
$$

Note that (A.1) and (A.2) are very similar. The difference is that $H(Y_k)$ has been substituted by $H(Y_l)$. Then, since the function $H(\cdot)$ does not depend on the decision variables U, V, we have that (V_k^*, U_k^*) is an optimal solution of (A.2) and, consequently, optimal solution of (A.3). This implies that $ITE_c(X_k, H(Y_l)1_s) = ITE_c(X_k, Y_l) = \sum_{r=1}^{s} u_{rk}^* y_{rl}$.

Finally, since (V_k^*, U_k^*) is an optimal solution of (A.2) and $\sum_{i=1}^{m} v_{ik}^* x_{ik} = 1$ by (2.1), we may compute (A.2) through (A.4).

$$
\begin{aligned}
\underset{U}{Max} \quad & H(Y_l)\sum_{r=1}^{s} u_r \\
s.t. \quad & \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_{ik}^* x_{ij} \le 0, && j = 1, \dots, n \\
& u_r \ge 0, && r = 1, \dots, s
\end{aligned}
\tag{A.4}
$$

Program (A.4) coincides with (12). Hence, $C_c(Y_l, V_k^*) = \sum_{r=1}^{s} u_{rk}^* y_{rl}$. $\square$

A.3. Proof of Theorem 1

Proof. [P1] First, we are going to prove that if V_k^* is V_l^*, then $\sum_{i=1}^{m} v_{il}^*(x_{il}/D_L(X_l, Y_l)) = C_L(Y_l, V_l^*)$. Seeking simplicity, we will assume Constant Returns to Scale but the proof for Variable Returns to Scale is analogous. Let $\theta_l = 1/D_L(X_l, Y_l)$, which is the optimal value of model (2) when (X_l, Y_l) is assessed, i.e. $\theta_l = ITE_c(X_l, Y_l)$. From (2), we can formulate the corresponding model for evaluating the point $(\theta_l X_l, Y_l)$:

$$
\begin{aligned}
ITE_c(\theta_l X_k, Y_k) = \underset{U,V}{Max} \quad & \sum_{r=1}^{s} u_r y_{rk} \\
s.t. \quad & \sum_{i=1}^{m} v_i \theta_l x_{ik} = 1, && && (23.1) \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \le 0, && j = 1, \dots, n && (23.2) \\
& u_r \ge 0, && r = 1, \dots, s && (23.3) \\
& v_i \ge 0, && i = 1, \dots, m && (23.4)
\end{aligned}
\tag{A.5}
$$

Let (V_l^*, U_l^*) be an optimal solution of model (2) when DMU (X_l, Y_l) is assessed. Then, it is easy to check that $(\frac{V_l^*}{\theta_l}, \frac{U_l^*}{\theta_l})$ is a feasible solution of model (A.5). Regarding the objective function value, we have $\sum_{r=1}^{s} (u_r^*/\theta_l) y_{rk} = (\sum_{r=1}^{s} u_r^* y_{rk})/\theta_l = \theta_l/\theta_l = 1$, which is the maximum value that the program can take. Thus, $(\frac{V_l^*}{\theta_l}, \frac{U_l^*}{\theta_l})$ is an optimal solution of (A.5) and $(\theta_l X_l, Y_l)$ is a point located onto the weakly efficient frontier. Consequently, $(\frac{V_l^*}{\theta_l}, \frac{U_l^*}{\theta_l})$ are its corresponding shadow prices and $\sum_{i=1}^{m} (v_{il}^*/\theta_l)(\theta_l x_{il}) = C_L(Y_l, \frac{V_l^*}{\theta_l})$ holds. The last expression is equivalent to $\sum_{i=1}^{m} v_{il}^* x_{il} = C_L(Y_l, \frac{V_l^*}{\theta_l}) = \frac{1}{\theta_l} C_L(Y_l, V_l^*)$ because the cost function is homogeneous of degree $+1$. Finally, we have that $\sum_{i=1}^{m} v_{il}^* \theta_l x_{il} = C_L(Y_l, V_l^*)$, which is the same that $\sum_{i=1}^{m} v_{il}^*(x_{il}/D_L(X_l, Y_l)) = C_L(Y_l, V_l^*)$, as we wanted to prove. By (6), $FCE_L(X_l, Y_l|k) = \frac{C_L(Y_l, V_k^*)}{\sum_{i=1}^{m} v_{ik}^* x_{il}} = \frac{1}{D_L(X_l, Y_l)} \cdot AE_L^F(X_l, Y_l; V_k^*)$. By Färe and Primont [[15], p. 61], $AE_L^F(X_l, Y_l; V_k^*) = \frac{C_L(Y_l, V_k^*)}{\sum_{i=1}^{m} v_{ik}^*(x_{il}/D_L(X_l, Y_l))}$. Therefore, $AE_L^F(X_l, Y_l; V_l^*) = \frac{C_L(Y_l, V_l^*)}{\sum_{i=1}^{m} v_{il}^*(x_{il}/D_L(X_l, Y_l))} = 1$. And, finally, we have that $FCE_L(X_l, Y_l|l) = \frac{C_L(Y_l, V_l^*)}{\sum_{i=1}^{m} v_{il}^* x_{il}} = \frac{1}{D_L(X_l, Y_l)}$. [P2] is a consequence of defining a cross-efficiency index by analogy with the Farrell cost efficiency index. [P3] $FCE_L(X_l, Y_l) = \frac{1}{n}\sum_{k=1}^{n} FCE_L(X_l, Y_l|k) = \frac{1}{n}\sum_{k=1}^{n} \frac{C_L(Y_l, V_k^*)}{\sum_{i=1}^{m} v_{ik}^* x_{il}}$. It is enough to prove that $C_L(Y_l, V_k^*) \le \sum_{i=1}^{m} v_{ik}^* x_{il}$ for all $k = 1, \dots, n$. By the definition of cost function, we have that $C_L(Y_l, V_k^*) = \min\{\sum_{i=1}^{m} v_{ik}^* x_i : X \in L(Y_l)\}$. We also have that $X_l \in L(Y_l)$ since (X_l, Y_l) has been observed in the data sample. Consequently, $C_L(Y_l, V_k^*) \le \sum_{i=1}^{m} v_{ik}^* x_{il}$, $\forall k = 1, \dots, n$. [P4] If $(V_k^*, U_k^*) = (V_l^*, U_l^*)$, $\forall k = 1, \dots, n$, then $FCE_L(X_l, Y_l|k) = \frac{C_L(Y_l, V_k^*)}{\sum_{i=1}^{m} v_{ik}^* x_{il}} = \frac{C_L(Y_l, V_l^*)}{\sum_{i=1}^{m} v_{il}^* x_{il}} = FCE_L(X_l, Y_l|l) = \frac{1}{D_L(X_l, Y_l)}$ by [P1]. Hence, $FCE_L(X_l, Y_l) = \frac{1}{n}\sum_{k=1}^{n} FCE_L(X_l, Y_l|l) = \frac{1}{D_L(X_l, Y_l)}$. [P5] is true because the original Farrell cost efficiency index is also units invariant. [P6] If $(V_k^*, U_k^*) = (W, P)$, $\forall k = 1, \dots, n$, then $FCE_L(X_l, Y_l) = \frac{1}{n}\sum_{k=1}^{n} FCE_L(X_l, Y_l|k) = \frac{1}{n}\sum_{k=1}^{n} \frac{C_L(Y_l, W)}{\sum_{i=1}^{m} w_i x_{il}} = \frac{1}{n}\sum_{k=1}^{n} CE_L(X_l, Y_l) = CE_L(X_l, Y_l)$, $\forall l = 1, \dots, n$, where the third equality is true by (6). $\square$

A.4. Proof of Lemma 2

Proof. Under constant returns to scale, $(0_m, 0_s) \in T_c$. Therefore, $\Pi_c(V_k^*, U_k^*)$ must be greater or equal than zero by its definition. Let us assume that $\Pi_c(V_k^*, U_k^*) > 0$. Then, there exists $(\hat{X}, \hat{Y}) \in T_c$ such that $\sum_{r=1}^{s} u_{rk}^* \hat{y}_r - \sum_{i=1}^{m} v_{ik}^* \hat{x}_i = \Pi_c(V_k^*, U_k^*) > 0$. Regarding $(\hat{X}, \hat{Y})$, by the definition of $T_c = \{(X, Y) \in R_+^{m+s} : \sum_{j=1}^{n} \lambda_j x_{ij} \le x_i, \forall i, \sum_{j=1}^{n} \lambda_j y_{rj} \ge y_r, \forall r, \lambda_j \ge 0, \forall j\}$, we know that there are $\hat{\lambda}_1, \dots, \hat{\lambda}_n \ge 0$ such that $\sum_{j=1}^{n} \hat{\lambda}_j x_{ij} \le \hat{x}_i$, $i = 1, \dots, m$, and $\sum_{j=1}^{n} \hat{\lambda}_j y_{rj} \ge \hat{y}_r$, $r = 1, \dots, s$. This implies that $\sum_{r=1}^{s} u_{rk}^* \hat{y}_r - \sum_{i=1}^{m} v_{ik}^* \hat{x}_i \le \sum_{r=1}^{s} u_{rk}^* (\sum_{j=1}^{n} \hat{\lambda}_j y_{rj}) - \sum_{i=1}^{m} v_{ik}^* (\sum_{j=1}^{n} \hat{\lambda}_j x_{ij}) = \sum_{j=1}^{n} \hat{\lambda}_j \underbrace{(\sum_{r=1}^{s} u_{rk}^* y_{rj} - \sum_{i=1}^{m} v_{ik}^* x_{ij})}_{\le 0 \text{by} (2.2)} \le 0$, which is a contradiction. Hence, $\Pi_c(V_k^*, U_k^*) = 0$. $\square$

A.5. Proof of Lemma 3

Proof. By the definitions of $CITE_c(X_l, Y_l|k)$ and $FCE_c(X_l, Y_l|k)$, $CITE_c(X_l, Y_l|k) \le FCE_c(X_l, Y_l|k)$ is equivalent to $C_c(Y_l, V_k^*) \ge \sum_{r=1}^{s} u_{rk}^* y_{rl}$. So, we are going to prove that this second inequality holds. In this respect, Färe and Primont [[15], p. 136] showed that $\Pi_T(W, P) + C_L(Y, W) \ge \sum_{r=1}^{s} p_r y_r$, for all $(W, P) \in R_+^{m+s}$ and

$Y \in R^s_+$. Let us assume CRS, $(W,P) = (V_k^*, U_k^*)$, where (V_k^*, U_k^*) is an optimal solution of model (2), and $Y = Y_l$. Then, we have that $\Pi_c(V_k^*, U_k^*) + C_c(Y_l, V_k^*) \geq \sum_{r=1}^s u_{rk}^* y_{rl}$. Finally, by Lemma 2, $C_c(Y_l, V_k^*) \geq \sum_{r=1}^s u_{rk}^* y_{rl}$. □

A.6. Proof of Proposition 6

Proof. (i) Let us first assume that $\vec{\alpha}_k^* > \Pi_v(\vec{V}_k^*, \vec{U}_k^*)$. Then, $\Pi_v(\vec{V}_k^*, \vec{U}_k^*) = \max_{x,y}\{\sum_{r=1}^s \vec{u}_{rk}^* y_r - \sum_{i=1}^m \vec{v}_{ik}^* x_i : (X,Y) \in T_v\} \geq \sum_{r=1}^s \vec{u}_{rk}^* y_{rj} - \sum_{i=1}^m \vec{v}_{ik}^* x_{ij}$ for all $j = 1, \dots, n$, since $(X_j, Y_j) \in T$. Therefore, $(\vec{V}_k^*, \vec{U}_k^*, \Pi_v(\vec{V}_k^*, \vec{U}_k^*))$ is a feasible solution for (22). Regarding the objective function in (22), we have that $-\sum_{r=1}^s \vec{u}_r^* y_{rk} + \sum_{i=1}^m \vec{v}_i^* x_{ik} + \Pi(\vec{V}_k^*, \vec{U}_k^*) < -\sum_{r=1}^s \vec{u}_r^* y_{rk} + \sum_{i=1}^m \vec{v}_i^* x_{ik} + \vec{\alpha}_k^*$, which is a contradiction with the fact that $(\vec{V}_k^*, \vec{U}_k^*, \vec{\alpha}_k^*)$ is an optimal solution of (22). (ii) Let us now assume that$\vec{\alpha}_k^* < \Pi_v(\vec{V}_k^*, \vec{U}_k^*)$. Then, $\sum_{r=1}^s \vec{u}_r^* y_{rj} - \sum_{i=1}^m \vec{v}_i^* x_{ij} \leq \vec{\alpha}_k^*$, $\forall j = 1, \dots, n$, by the first set of constraints in (22) . By the definition of the technology T_V, for all $(X,Y) \in T_V$ there exists a vector $(\lambda_1, \dots, \lambda_n) \in R^n_+$ with $\sum_{j=1}^n \lambda_j = 1$ such that $\sum_{r=1}^s \vec{u}_r^* y_r - \sum_{i=1}^m \vec{v}_i^* x_i \leq \sum_{r=1}^s \vec{u}_r^* (\sum_{j=1}^n \lambda_j y_{rj}) - \sum_{i=1}^m \vec{v}_i^* (\sum_{j=1}^n \lambda_j x_{ij}) = \sum_{j=1}^n \lambda_j (\sum_{r=1}^s \vec{u}_r^* y_{rj} - \sum_{i=1}^m \vec{v}_i^* x_{ij}) \leq \vec{\alpha}_k^* < \Pi_v(\vec{V}_k^*, \vec{U}_k^*)$.Consequently, the maximum profit at prices $(\vec{V}_k^*, \vec{U}_k^*)$, $\Pi_v(\vec{V}_k^*, \vec{U}_k^*)$, is not achieved by any point in T_V, which is a contradiction with polyhedral DEA technologies as is the case. □

References

[1] Aldamak A, Zolfaghari S. Review of efficiency ranking methods in data envelopment analysis. Measurement 2017;106:161–72.
[2] Aparicio J, Pastor JT, Zofio JL. How to properly decompose economic efficiency using technical and allocative criteria with non-homothetic DEA technologies. Eur J Oper Res 2015;240(3):882–91.
[3] Aparicio J, Pastor JT, Vidal F. The weighted additive distance function. Eur J Oper Res 2016;254(1):338–46.
[4] Aparicio J, Zofío JL. Revisiting the decomposition of cost efficiency for non-homothetic technologies: a directional distance function approach. J Prod Anal 2017;48(2):133–46.
[5] Balk BM, de Koster MBM, Kaps C, Zofío JL. An evaluation of cross-efficiency methods. Applied to measuring warehouse performance. ERIM report series research in management ERS-2017-015-LIS, The Netherlands: Erasmus University Rotterdam; 2017. https://repub.eur.nl/pub/103185.
[6] Banker RD, Charnes A, Cooper WW. Some models for estimating technical and scale inefficiencies in data envelopment analysis. Manage Sci 1984;30(9):1078–92.
[7] Chambers RG, Chung Y, Färe R. Profit, directional distance functions and Nerlovian efficiency. J Opt Theory Appl 1998;98(2):351–64.
[8] Chambers RG, Mitchell T. Homotheticity and non-radial changes. J Prod Anal 2001;15(1):31–9.
[9] Charnes A, Cooper WW, Rhodes E. Measuring the efficiency of decision making units. Eur J Oper Res 1978:429–44 2/6.
[10] Cook WD, Zhu J. DEA cross efficiency. In: Data envelopment analysis. Boston, MA: Springer; 2015. p. 23–43.
[11] Cooper WW, Pastor JT, Aparicio J, Borras F. Decomposing profit inefficiency in DEA through the weighted additive model. Eur J Oper Res 2011;212(2):411–16.
[12] Cooper WW, Seiford LM, Zhu J. Handbook on data envelopment analysis. 2nd ed. New York, NY: Springer Science+Business Media, LLC; 2011.
[13] Doyle J, Green R. Efficiency and cross-efficiency in DEA: derivations, meanings and uses. J Oper Res Soc 1994;45(5):567–78.
[14] Färe R, Mitchell T. Multiple outputs and homotheticity. South Econ J 1993;60(2):287–96.
[15] Färe R, Primont D. Multi-output production and duality: theory and applications. Kluwer Acad 1995.
[16] Färe R, Grosskopf S, Zaim O. Hyperbolic efficiency and return to the dollar. Eur J Oper Res 2002;136(3):671–9.
[17] Farrell MJ. The measurement of productive efficiency. J R Stat Soc, Ser A, Gen 1957;120:253–81.
[18] Førsund FR. Cross-efficiency: a critique. Data Envelop Anal J 2018;4(1):1–25.
[19] Jablonsky J. Ranking of countries in sporting events using two-stage data envelopment analysis models: a case of summer olympic games 2016. Cent Eur J Oper Res 2018:1–16.
[20] Jacobsen SE. Production correspondences. Econometrica 1970;38(5):754–68.
[21] Johnson A, McGinnis LF. Performance measurement in the warehousing industry. IIE Trans 2011;43:220–30.
[22] Juo JC, Fu TT, Yu MM, Lin YH. Profit-oriented productivity change. Omega 2015;57:176–87.
[23] Kao C, Liu ST. A slacks-based measure model for calculating cross efficiency in data envelopment analysis. Omega; 2020. in press.
[24] Kaps C, de Koster MBM. Warehouse data NL & BE, 2012 and 2017. The Netherlands: Erasmus University; 2019. doi:1025397/eur8279426.
[25] Li Q. Nonparametric testing of closeness between two unknown distribution functions. Econom Rev 1996;15:261–74.
[26] Lim S, Zhu J. DEA cross-efficiency evaluation under variable returns to scale. J Oper Res Soc 2015;66(3):476–87.
[27] Lovell CK, Grosskopf S, Ley E, Pastor JT, Prior D, Eeckaut PV. Linear programming approaches to the measurement and analysis of productive efficiency. Top 1994;2(2):175–248.
[28] Luenberger DG. New optimality principles for economic efficiency and equilibrium. J Opt Theory Appl 1992;75(2):221–64.
[29] Maniadakis N, Thanassoulis E. A cost Malmquist productivity index. Eur J Oper Res 2004;154(2):396–409.
[30] Olesen OB. Cross efficiency analysis and extended facets. Data Envelop Anal J 2018;4(1):27–65.
[31] Rosenthal EC, Weiss HJ. A data envelopment analysis approach for ranking journals. Omega 2017;70:135–47.
[32] Ruiz JL. Cross-efficiency evaluation with directional distance functions. Eur J Oper Res 2013;228(1):181–9.
[33] Ruiz JL, Sirvent I. Ranking decision making units: the cross-efficiency evaluation. In: Handbook of operations analytics using data envelopment analysis. Boston, MA: Springer; 2016. p. 1–29.
[34] Sheather S, Jones M. A reliable data-based bandwidth selection method for kernel density estimation. J R Stat Soc Ser B 1991;53:683–90.
[35] Silverman BW. Density estimation for statistics and data analysis. London: Chapman and Hall; 1986.
[36] Simar L, Zelenyuk V. On testing equality of distributions of technical efficiency scores. Econom Rev 2006;25(4):497–522.
[37] Sexton TR, Silkman RH, Hogan AJ. Data envelopment analysis: critique and extensions. In: Silkman RH, editor. Measuring efficiency: an assessment of data envelopment analysis. San Francisco, CA: Jossey-Bass; 1986. p. 73–105.
[38] Shephard RW. Cost and production functions. Princeton, NJ: Princeton Univ Press; 1953.
[39] Wu J, Liang L, Chen Y. DEA game cross-efficiency approach to olympic rankings. Omega 2009;37(4):909–18.
[40] Zofío JL, Prieto AM. Return to dollar, generalized distance function and the fisher productivity index. Spanish Econ. Rev. 2006;8(2):113–38.